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MICROMERITICS:

THE TECHNOLOGY OF FINE PARTICLES

MICROMERITICS

THE TECHNOLOGY of FINE PARTICLES

by

J. M. DALLAVALLE

SECOND EDITION



PITMAN PUBLISHING CORPORATION
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To

My Mother

PREFACE TO THE SECOND EDITION

The success of the first edition of MICROMERITICS has encouraged both me and my publishers to enlarge on the material so as to render the book more generally useful. As a matter of fact much of the material now offered was intended for the first edition but shortages created by the war made it imperative to limit the size of the text at the time (1943). Thus, many phases of the subject could only be mentioned briefly in that edition. We now feel that the time has come to revise the subject matter completely and to offer a text which we hope will be considered a standard reference in the field of fine-particle technology.

The acceptance of the first edition, which went through two printings, has been gratifying. I feel that the science of micromeritics is now recognized and that the day will not be long in coming when certain portions of the subject will form part of the curricula in schools of engineering. I shall not dwell here on my concepts regarding the importance of the subject. I have done this in another publication (1944). It is only necessary to point out that there is no phase of engineering or applied science where some knowledge of particulate technology is not required.

As in the first edition, the symbols have given me much trouble. In a text covering so many diverse fields it is very difficult indeed to harmonize the symbols and at the same time avoid conflicts. Thus, S has been used to denote slope or grade and particle surface per unit volume. There are occasions where both may appear in the same equation and the resulting confusion would be embarrassing. In such cases I have used the reciprocal of particle diameter or S_v for the specific surface (volume) to avoid the difficulty. But there have been other more serious troubles with symbols, not so obvious as the one cited, and I have had to avoid the use of many symbols advocated by the American Standards Association. This has been done as sparingly as possible and on the whole the symbols in the text conform with those of the ASA.

Failure to mention the electron microscope as a method of measuring very small particles may appear to some as a serious omission. However, the procedure for measurement is similar to that of the ordinary microscope using photographs instead of a micrometer eyepiece. There are, on the other hand, many useful applications of the electron

microscope which could have been mentioned, but unfortunately at the present time there is little definite information available about them. These concern the ability of the electron microscope to reveal the surface characteristics of very small particles.

On the basis of work done with the ordinary microscope under high power we have assumed that as particles become finer in size they tend toward a greater degree of sphericity. Harkins and his associates at the University of Chicago have done much work on the subject, but it has been directed solely to the study of adsorption phenomena. While indications thus far sustain the belief that particles tend to become spherical as their size decreases, the evidence is of indirect nature, being based on the fact that diameters computed by various adsorption methods conform with those obtained by the electron microscope. Yet, in the first instance a diameter based on the surface per unit volume or weight is used, and in the case of the electron microscope the particles themselves are directly measured. There may be similarities in order of magnitude, but it is possible that on a size-frequency basis the differences will be significantly different. There is a need to examine this problem more closely than heretofore.

Five new chapters have been added to this edition and a rather extensive rearrangement has been made of the text of the first edition. On the whole, I believe the material is better unified and is a more complete discussion of the physical behavior of fine particles. I sincerely hope that the usefulness of the book has been increased.

A few words about the new subject matter are perhaps in order. The chapter on behavior of particles under pressure deals only with cohesionless material. I have spent much time attempting to bring together pertinent material on the subject. It might be presumed that the widespread applications of powder metallurgy would make a great mass of information available. However, such has not been the case, and for reasons hard to explain the information at hand is limited. I have relied a great deal on the contributions of soil mechanics, especially the work of Rankine (1857); but the whole subject, though indispensable to the design of earth foundations, is also complex and there are many diverse views regarding the nature of pressure distributions in cohesionless soil. Therefore it seemed best to restrict the presentation to the elements of the subject. The reader is referred to the extensive literature on soil mechanics for a more complete and authoritative discussion.

The subject of particle dispersion by eddy diffusion has been enlarged to a complete chapter, in order to lay the groundwork for a later chapter dealing with smoke clouds and sand storms. But entirely apart from this purpose is the importance of the subject by itself. Merely to give

the dynamics of a particle without regard to the basis of its behavior in a flowing fluid would be as unfortunate as omitting any reference to particle size. The development I have given to eddy diffusion is as simple as it can possibly be made. Some confusion may result in interpreting the significance of the root-mean-square and mean. The former is used to obtain a measure of intensity of forces or conditions which fluctuate with time or distance, the terms being squared so as to be proportional to energy; while the latter is usually an arithmetic average. The difference between the two is not great and they may be used interchangeably in most circumstances. In the text a symbol with a bar above it signifies a root-mean-square value only when it is so identified. Otherwise, it signifies any average we choose which is a proper measure of the fluctuations.

In Chapter 8 I have utilized the Poissonian distribution to arrive at the phase relationships which determine the magnitude of an eddy. This seemed the simplest procedure consistent with Taylor's concept of turbulence. As to the reasons for placing other material between the closely related Chapters 8 and 19, I have been motivated by the thought that theoretical material should form the first portion of the text and practical material the second portion. As I have stated, it has been difficult to decide how to assemble so much apparently diverse material. The present arrangement is essentially one of personal appeal and I agree readily that there may be others which are better. In preparing Chapter 19, dealing with smoke clouds and sand storms, I have taken most of my material from the writings of British authors. The work of Bosanquet and Pearson with regard to the dispersion of smoke from chimneys has shown how fundamentally simple are the laws of behavior of what otherwise would be considered an overwhelming subject. We have not in this country utilized the basic information set forth by these authors, but unquestionably, as the problems of smoke pollution and pollution from stacks in general become more important, we will have to recognize the value of the contributions made

The subject of sand storms and the formation of sand dunes and other land forms by wind has been treated in a remarkable manner by Bagnold (1943). I have been much influenced by Bagnold and the contribution he has made to a subject which bears on the whole history of mankind. The war undoubtedly reduced the amount of discussion that he would have given to many phases of the subject, but the clear style of his book more than compensates for the condensation. Bagnold's treatment of sand storms and their effects is most interesting and his analysis so simple and direct that it is well within the comprehension of the non-mathematical layman. In reducing Bagnold's contribution to the

pages of a short chapter I have had to contend with what is already a brief and clear presentation.

Bagnold's discussion of "singing sands" is an engrossing subject. The nature of the sounds experienced by desert travelers is, I think, susceptible of simpler explanation than surmised by Bagnold and others. I am of the opinion that the key to the problem lies in investigations of Buckingham (1904) with regard to barometric pressure waves in the soil. The movement of the barometric wave front alters the pressure equilibrium in the layers and causes a release of air from the stratum below, and as the air moves out through the capillaries of the stratum it creates a sound whose nature is not unlike that of the reed of an organ pipe. The average pitch and intensity of the sound should be a function of the capillary diameters of the sand bed and these in turn are functions of the degree of packing and the size of the particles composing it. In other words, the previous history of the locale in which the singing sands are found has much to do with the occurrence of the sounds heard. The suddenness of their occurrence and their movement across a desert terrain as described by Bagnold seem to make for strong support of the Buckingham barometric wave front.

The growing importance of adsorption phenomena has compelled me to devote more space to a discussion of surface energy, with which the subject is closely allied. The presentation which I have given is, I believe, for the most part new. As much as possible I have tried to unify the treatment and make it specific. I have avoided controversial subjects as, for example, the current discussion of mono- versus polymolecular layer theories in adsorption. Involvement in problems of microcapillarity, while important, did not seem to me pertinent. The reader is referred to the writings of Langmuir, Harkins, and Brunauer if he desires more complete information on prevailing theories.

There is finally a chapter dealing with the theory of sampling. I did not wish to present in the chapter more than the basic elements needed elsewhere in the text. For this reason a great deal of what might normally be considered within the scope of sampling has been avoided. There are available many complete books on the subject, which in itself can be said to be as vast in its detail as any phase of engineering. Practically every textbook on statistics deals with sampling but as concerns the subject of micromeritics, the contributions of Landry (1944) and Shewhart (1931) are outstanding. I believe Landry is the first to deal with the theory of sampling of particulate substances in its broader aspects. Shewhart, on the other hand, is concerned with the significant measures of product control.

In the chapter on sampling I have dealt at some length with the theory

of probes used for obtaining samples. This is a somewhat complicated subject and there is need for much more work than has been done thus far. Lapple and Shepherd (1940) have examined the basis of sampling particles in streamline motion and have deduced correction factors to allow for differences in the velocity of sampling and the velocity of the suspended solids. Unfortunately, the corrections do not apply generally. The developments which I present are, I think, wholly new and offer a means for re-examining the problem of probe sampling which I consider a most important subject.

The subject matter of many chapters in the first edition has been revised and extended. I have almost completely rewritten Chapter 2 on dynamics. It is now much longer and includes a discussion of many special problems. In my treatment on the motion of a particle in a viscous medium I have attempted to utilize an empirical relation between the resistance factor C_R and Reynolds number. It has always seemed to me that such a procedure is as logical and correct as the arbitrary assumption of the point where streamline motion ends and turbulent motion begins. In fact, there is a good deal of question regarding the exact curve relating the variables. My position is, of course, open to criticism by those who prefer an analytical treatment of each kind of motion, but the procedure I have adopted is, I believe, much the easier and the more convenient.

I have expanded on the behavior of particles in an electric field. The growing importance of particle-sizing by use of an electrostatic field, such applications as cleaning particulate matter by utilizing dielectric differences in the various components, and dust removal from air have made the subject a very important one. The discussion contained in the text is probably more complete than is to be found in any other representative American text.

There are other changes which are too numerous to mention and I shall close by indicating my indebtedness to many who have written me about errors in context. I regret that even now there may be new errors of which I am not aware. Both editions have been prepared under trying circumstances. I have often had to write parts of the text in places where adequate library and consultation facilities were lacking.

I am especially obligated to Professor Charles F. Bonilla of Johns Hopkins University, who went over much of the text of the first edition critically, and I believe that all of the errors he indicated have been corrected. I would appreciate having errors in the present edition called to my attention. I am also indebted to Wm. Morrow & Co. for the use of material by Bagnold, included in Chapter 19.

As in the first edition, my wife, Marion Clare DallaValle, has done most of the editorial work and the bibliography. She has constantly encouraged me from the time the first line was set on paper through the reading of the final proofs and preparation of the index. She is a co-author in every sense of the word.

J. M. D.

Bogotá, Colombia

PREFACE TO THE FIRST EDITION

This text is intended as a guide to the general subject of the behavior and characteristics of small particles. It brings together a mass of widely scattered information on methods of particle-measurement, size-distributions, packing arrangements, and a general theory concerning the physical properties of finely divided substances. Industrial applications of the subject matter are also included and every attempt has been made to present the material in a useful form.

Particular attention was given to the fundamental aspects of particle-measurement, and to physical and chemical behavior. Too little appreciation has been given the significance of methods used to measure particles and their special properties. Therefore, it is hoped that by bringing together known and established facts concerning them some progress will be made to standardize procedures in the future. While a rich source of material has accumulated, much time and expense have been wasted through failure to recognize the need for appraising the significant developments made in particulate technology. For this reason I have endeavored to bring together all outstanding contributions in the bibliography. Needless to say, much material had to be passed over, but an effort was made to include all original and pertinent contributions.

As much as possible the symbols used in the text follow the recommended practice of the American Standards Association. However, special symbols had to be introduced, the idea of avoiding confusion being the criterion. In selecting the letter d to represent particle-size, rather than r the radius of a particle, I was motivated by considerations of particle-measurement. This step naturally resulted in some difficulty since it can be confused with the differential d. Wherever the two appear together, their separate meanings have been denoted by writing particle-diameter as (d); thus, $d(d)$ means the differential of the particle-diameter. The letter r for the radius of a particle was retained in only one instance, in order to conform to accepted procedure.

I have refrained from using the term "mechanical analysis" except once or twice in the first part of the text. The use of this term to describe a size-frequency distribution of particles is unfortunate and should be discouraged because its meaning is too limited.

Some users of this text may feel that it touches upon too many apparently unrelated fields. This may be true, but the inclusion of such fields was made purposely to indicate the wide applications of a subject which

should receive recognition as deserving a special place in the engineering sciences. For this reason some thought was given to an all-embracing term, and at the suggestion of my friend Mr. Godfrey Schmidt, I am proposing the word *Micromeritics* as the science of small particles. This is derived from the Greek *μικρός* meaning *small* and *μέρος* meaning *part*. I know that many will not welcome the proposal of another "scientific" name in the engineering field, but I do so in anticipation that small-particle technology will soon attain greater importance because of its many practical applications.

I am indebted to the following journals and firms for permission to reproduce certain figures and tables: Journal of the American Society for Testing Materials, Journal of the American Chemical Society, and The W. S. Tyler Company, Cleveland, Ohio. These are acknowledged in the text. Especial acknowledgment must be made to my wife, for her assistance and encouragement in the preparation of the text Whatever success this book may achieve is in large measure due to her untiring efforts.

I also desire to express my thanks to the editors of the Pitman Publishing Corporation, for their kind interest and assistance.

J. M. D.

New York, N. Y.

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PRINCIPAL SYMBOLS

USED IN TEXT AND THEIR UNITS

Mass [M], Length [L], Time [T]. Temperature [θ], Numeric [N], Dielectric constant [k_0]

A	Amplitude (Units specified in text)	
	Area	$[L^2]$
a	Constant or variable	
B	Exponent of Furnas' equation	[N]
	Gas constant	$[ML^2T^{-2}\theta^{-1}]$
B	Bagnold number	[N]
b	Constant or variable	
C	Heat conductivity per deg change in temperature gradient	$[MLT^{-3}\theta]$
	Number of contacts	[N]
	Concentration per volume	$[ML^{-3}]$
C_R	Coefficient of resistance	[N]
C_s	Concentration per area	$[ML^{-2}]$
C_t	Emission concentration rate	MT^{-1}
c	Constant or variable	
ϕ	Degree of circularity	[N]
D	Diameter of tube or sieve opening	[L]
d	Diameter	[L]
$\mathfrak{D}$	Diffusion coefficient	$[L^2T^{-1}]$
E	Voltage gradient (esu)	$[M^{1/2}L^{-1/2}T^{-1}] \cdot [k_0^{-1/2}]$
	Modulus of elasticity	$[ML^{-1}T^{-2}]$
e	2.718...	
	Coefficient of restitution	[N]
$\exp(\pm x)$	$e^{\pm x}$	$e = 2.718...$
F	Force	$[MLT^{-2}]$
F	Froude's number	[N]
f	Friction factor	[N]
G	Rate of silt movement	$[MT^{-1}]$
	Geometric mean (Units specified in text)	
	Mass velocity	$[ML^{-2}T^{-1}]$
g	Gravitational acceleration constant	$[LT^{-2}]$
H	Height	[L]
	Harmonic mean (Units specified in text)	
h	Liquid level	[L], $[ML^{-1}T^{-2}]$
I	Light intensity (energy—usually expressed as ratio I/I_0)	
	Index of inflammability	[N]
	Current (esu)	$[M^{1/2}L^{3/2}T^{-2}] \cdot [k_0^{1/2}]$
	Current density (esu)	$[M^{1/2}L^{-1/2}T^{-2}] \cdot [k_0^{1/2}]$

K	Specific heat	$[L^2T^{-2}\theta]$
	Constant	
k	Constant	
k_0	von Kármán's turbulence constant	[N]
	Dielectric constant	$[k_0]$
L	Length	[L]
log	log base 10 (Briggs)	
ln	log base e (Napierian)	
M	Moisture absorbed per volume	$[ML^{-3}]$
	Moisture equivalent	[N]
	Molecular weight	[N], [M]
	Momentum	$[MLT^{-1}]$
m	Mass	[M]
N	Number particles per unit weight	$[M^{-1}]$
n	Frequency	$[T^{-1}]$
	Manning's coefficient of roughness	[N]
	Number of particles	[N]
n_s	Roughness coefficient	[N]
P	Total pressure	$[MLT^{-2}]$
P_{ox}	Partial pressure of oxygen	$[ML^{-1}T^{-2}]$
p	Probability of success	[N]
	Unit pressure	$[ML^{-1}T^{-2}]$
$\mathcal{P}$	Absolute permeability	$[L^2]$
$\mathcal{P}_0$	Darcy's permeability	$[M^{-1}L^3T]$
$\mathcal{P}_c$	Capillary permeability	$[MT^{-2}]$
$\mathcal{P}_c'$	Modified capillary permeability	$[LT^{-2}]$
Q	Volume of flow	$[L^3T^{-1}]$
	Volume of gas absorbed	$[L^3]$
q	Flow through unit area $= v$	$[LT^{-1}]$
	Probability of failure	[N]
q_0	Initial oxidation rate	$[L^3M^{-1}]$
$\mathcal{Q}$	Electric charge	$[M^{1/2}L^{3/2}T^{-1}] \cdot [k_0^{1/2}]$
	Sedimentation parameter	
R	Radius	[L]
	Hydraulic radius	[L]
R	Reynolds number	[N]
$\mathcal{R}$	Resistance	$[MLT^{-2}]$
r	Radial distance	[L]
S	Slope	[N]
	Surface	$[L^2]$
S_v	Surface per unit volume	$[L^{-1}]$
S_u	Surface of particles per unit weight	$[L^2M^{-1}]$
s	Surface	$[L^2]$
T	Temperature	$[\theta]$
	Tyndall reading (Relative energy)	
t	Time	$[T]$
u	Ion mobility	$[M^{1/2}L^{1/2}T^{-2}] \cdot [k_0^{-1/2}]$
V	Voltage (esu)	$[M^{1/2}L^{1/2}T^{-1}] \cdot [k_0^{-1/2}]$
	Volume	$[L^3]$

v	Velocity	$[LT^{-1}]$
$\mathbf{v}$	Vectorial velocity	$[LT^{-1}]$
$\bar{v}$	Root mean square velocity (mean velocity)	$[LT^{-1}]$
$*v$	Drag velocity	$[LT^{-1}]$
W	Weight	$[MLT^{-2}]$
	Work	$[ML^2T^{-2}]$
W	Weber number	[N]
w	weight	$[MLT^{-2}]$
	Work of adhesion	$[MT^{-2}]$
x	Coordinate	[L]
Y	Ratio of velocities	[N]
y	Coordinate	[L]
z	Coordinate	[L]
α (alpha)	Dissociation constant	[N]
	Shape factor	[N]
B (beta)	Ratio of concentrations	[N]
β	Ratio, constant exponent	
γ (gamma)	Wall factor	[N]
Δ (delta)	Increment of variation	
δ	Thickness	[L]
ϵ (epsilon)	Crushing efficiency	[N]
	Fractional excess air	[N]
	Unit electric charge $\mathcal{Q}/n$	$[\mathcal{Q}/n]$
ζ (zeta)	Relative humidity	[N]
θ (theta)	Ratio, angle	[N]
ϑ	Voids	[N]
λ (lambda)	Capillary conductivity (Buckingham)	$[LT^2]$
	Capillary conductivity (Gardner)	$[L^3T^{-1}]$
	Capillary conductivity (Richards)	[T]
	Mixing length	[L]
	Wave length	[L]
μ (mu)	Viscosity	$[ML^{-1}T^{-1}]$
μ'	Viscosity of plastic fluid	$[ML^{-1}T^{-1}]$
ν (nu)	Coefficient of variation	[N]
	Kinematic viscosity	$[L^2T^{-1}]$
ξ (xi)	Constant, coefficient	
$\bar{\xi}$	Root mean square displacement	[L]
Π (pi)	Ratio of concentrations	[N]
π	3.1416	
	Free surface energy	$[MT^{-2}]$
ρ (rho)	Density	$[ML^{-3}]$
Σ (sigma)	Adhesion tension	$[MT^{-2}]$
	Sign of summation	
σ	Standard deviation (units conform to object measured)	
	Surface tension	$[MT^{-2}]$
τ (tau)	Shear	$ML^{-1}T^{-2}$

Φ (phi)	Ratio of concentrations	[N]
	Pressure gradient	$[ML^{-2}T^{-2}]$
	Probability integral	[N]
ϕ	Angle of repose	[N]
	Correlation coefficient	[N]
	Maximum value of $C_R\mathbf{R}^2$	[N]
	Phase angle	[N]
φ	Density of suspension	$[ML^{-3}]$
ψ (psi)	Capillary potential (Buckingham)	$[L^2T^{-2}]$
	Degree of sphericity	[N]
Ω (omega)	Ratio of concentrations	[N]
ω	Rotational speed	$[T^{-1}]$

MICROMERITICS:

THE TECHNOLOGY OF FINE PARTICLES

CHAPTER 1

INTRODUCTION

NEXT to the molecule in order of dimension are those minute particles commonly called colloids. To some extent colloids obey the gas laws, but in general they follow other very specific laws. Colloid chemistry attempts to explain the basis of ultramicroscopic particle-behavior. It includes the results of investigations on diffusivity, viscosity, optical and electrical properties, and other special properties of practical interest. But beyond the particulate sizes studied in colloid chemistry are those whose limits reach from microscopic resolution to visibility by the naked eye. The behavior of these particles is entirely different from that of molecules and colloids. Furthermore, these particles not only have wide industrial application but are related to natural phenomena of common experience as, for example, erosion of soils, silting of rivers, sandstorms, and the color of the sky at twilight.

A great mass of literature on various phases of particles larger than colloids has accumulated, but unfortunately there has been no systematic or comprehensive treatment of the subject as a whole. The laws governing behavior and physical properties of small particles are still not well known, and much research remains to be done on such elementary matters as uniform methods for size-measurement and determination of surface. Even adequate descriptions of particle-shapes and mode of packing are lacking. While there are other equally important angles for investigation, these are fundamental to all laws of particle-behavior. and until they are resolved progress is necessarily hampered.

ORDER OF MAGNITUDE

In the present work we consider only particles within a definite range of size. Using the micron ($1\mu = 0.001$ mm) as the unit, this text for the most part is limited to particles ranging from 10^{-1} to 10^{5} microns. These include submicroscopic, microscopic, and relatively macroscopic sizes. The particle-size range covered and its relation to molecular and colloid dimensions are shown in Figure 1.

From this figure it is clear that we are considering a fairly wide range of particle-sizes. In fact, at one end of the scale we start with the range of colloid particles, and in the direction of increasing size we include

particles up to an inch in diameter. Some concept of the differences in masses may be gained by considering the extremes in the size range. Thus, the volume of a cube with sides of 10^{-1} μ is 10^{-3} cu μ, and that of a cube with sides of 10^{4} μ is 10^{12} cu μ; or, 10^{-15} and 1.0 cu cm, respectively. This is indeed a wide span and, as might be expected, the particles within it behave according to different laws. The smallest particles, approximately the size of tobacco smoke, diffuse almost like a gas and settle under gravity according to Cunningham's modification of Stokes' law. Above this range of sizes and even within the range of microscopic visibility (0.2–50 μ) the particles settle according to Stokes' law, without correction (that is, the force retarding them is due entirely to the viscous nature of the fluid). And finally, in the upper range of sizes viscosity is of less importance, the resistance encountered in falling being due almost entirely to the density difference between particle and fluid. Moreover, the rate of fall will no longer vary as the square of the particle-diameter as in the case of the smaller particles, but directly as the square root of the diameter.

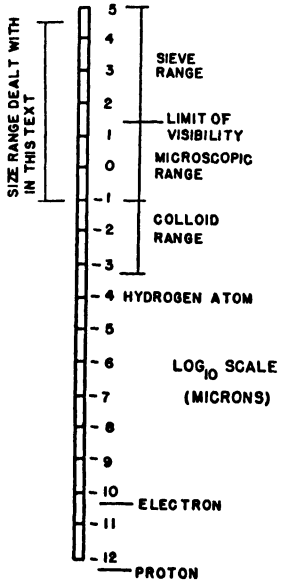


FIGURE 1. SCALE OF MAGNITUDES.

PARTICLE-SIZE MEASUREMENT

After selecting the kind of particles with which we shall deal, it is natural to inquire into the methods used for their measurement. As a rule particles are irregular in shape, and certain distinctions must be made with regard to the meaning of particle-size. Obviously this must resolve itself into a matter of convention since from purely geometrical considerations we can deal only with regular shapes. Such conventions will be discussed in Chapters 3 and 4, and for the moment we shall confine ourselves to the implications of this difficult problem. Consider any fine granular material whose size-distribution is to be reproduced, and assume that it is the residue retained on the finer of two sieves. This is all that is known. If all the material originally came from a single source no difficulty would be experienced, but if some is from another source we may be dealing with grains of different shape, say, more needle-like than the original. While this difference may not be readily apparent to the eye, the product, when screened, may have a size different

from that desired. The needle-like structure of the material would prevent it from passing through the interstices (apparently larger than the particles) with the result that the retained material would actually be somewhat different from that desired. Moreover, the properties of the material would be different. Without knowing the shape and size distribution of the two products concerned, we might be likely to consider them the same simply because they are chemically alike and otherwise handled similarly. While this case may seem an exaggeration, it does illustrate a situation of common occurrence. For instance, a certain food may possess different tastes depending upon the degree or size to which it is ground. One batch may grind in a certain way, while another will fracture differently. Both will be sieved alike, yet the products will differ in taste. In a measure the cause will be due to inherent differences in composition, but dissimilarity in particle-shapes will often prove to be of equal importance. This and similar difficulties can be overcome if we know something of the size-distribution in the materials concerned. As will be shown, sieves in themselves are not so selective in sizing materials as might ordinarily be assumed.

Unfortunately, no set procedure for measuring aggregates of particles has ever been developed; in fact, it is probable that no single general method can ever be developed. Considering the range of sizes involved —from the submicroscopic to the macroscopic—several methods must be used, each having virtues suited to the size-range studied. But even within a single size-range different approaches have been used by different investigators. For instance, in the size-range 1 μ to 35 μ, which is relatively narrow, there are two methods worth considering, namely microscopic measurements and sedimentation analysis. In the first we deal with individual measurement of each particle in a random sample according to certain prescribed techniques, while in the second we utilize a gravimetric procedure based on sampling a suspension of particles at a set distance from the surface and at a given time. The amount determined is then said to consist of particles less than a specified size depending on laws of settling. Whichever method is used, it is assumed that the particles are spherical. As far as can be determined no comprehensive study of the difference between the two methods has been made, and consequently size-distributions determined by them are not strictly comparable even when equivalence to spheres is assumed. Thus, it is desirable to state not only the average size of a distribution, but also the method by which it has been derived. Whatever the method selected, it must give reproducible results on the same sample by different observers.

It is important that the methods of particle-size analysis (frequently

called mechanical analysis) be thoroughly understood. Much more than accuracy and technique is involved. We must realize the manifold limitations of each method in order to appreciate the true significance of the measurements made. Unfortunately, this subject has never been studied critically and more information is sorely needed. *The characteristics and behavior of small particles can be understood satisfactorily only when the methods of determining size-distribution are placed on a firm basis.*

One aspect of particle-size distributions requires the special attention of the student. It is obvious that, gravimetrically, small particles in a distribution contribute little to the mass as a whole. One cubic particle of a substance 0.1 cm to a side will have a volume of 0.001 cu cm, while another, just one-tenth the size (0.01 cm to a side), will have a volume 0.000001 cu cm. Hence, if the densities of both particles are the same, it will take 1000 of the smaller particles to equal the weight of the larger one. On the other hand, when particle-surface is considered, the amount of surface contained in the 1000 particles is ten times the surface of the single large particle of equivalent weight. This example brings out the important fact that different methods for calculating weight and surface in a mass of irregular particles must be used, so that in one case the largest particles forming the mass are credited, and in the other the smallest particles which contribute most to the total amount of surface are duly weighted. This is of great importance in pigment technology which is concerned primarily with particle-surface rather than particle-volume.

PACKINGS

Next in importance for a proper understanding of size-distribution and particle-measurement is the manner in which fine material packs. It is well known that a column filled with sand may be shaken so that it will occupy less space. Similarly, certain soils may be pressed or tamped to become denser. Compaction is achieved at the expense of the void space, which may vary from a theoretical minimum of 26 percent to as high as 48 percent for spheres. Unfortunately there is no adequate method of describing a packing in terms of particle-orientation. We can only deal with it in terms of the free space present or the ease with which a liquid flows through it. Except in a statistical sense it is never possible to make two packings precisely identical unless we make a systematic arrangement of spheres.

The importance of packings is shown by the relative ease with which a liquid flows through them. In Chapter 13 it will be shown that a small change in void space greatly affects the permeability of the pack-

ing. Thus, if it is possible to predict simply in terms of voids what a packing will do under any stage of compaction, a great contribution will have been made to such important subjects as sand filtration of water, pressure-drop through blast furnaces, absorption towers composed of particulate matter, separation of dust from air, and capacities of ground water and oil fields. These are matters of practical importance in many lines of endeavor.

Packings are inherently very complex and there is still much to be learned about them. For example, we may pose the problem of how to select the proper sizes and the correct proportions of a given material so that the larger voids are filled with smaller particles, and these voids in turn filled with still smaller particles, and so on. Theoretically we should be able to obtain a very low percentage of voids in this way; nevertheless, both theoretically and practically this problem remains unsolved.

APPLICATIONS

Soil Physics—This subject concerns the behavior of soil—particulate matter—as to mineral composition, moisture capacity, and plant growth. Only moisture capacity is of interest here. This comprises a number of interrelated variables including capillarity, infiltration, and evaporation. We are indebted to soil physicists not only for first appreciating the importance of particulate behavior, but also for making most of the contributions concerned in this text. The work of outstanding investigators as Atterberg in Sweden; Russell, Keene, and Haines in England; and Hilgard, Buckingham, and Briggs in this country cannot be overemphasized.

If we delve deeply into the subject of soil physics, particularly into capillarity, infiltration, and evaporation, we discover that the fundamentals set forth have far-reaching applications. Not only do they supply an explanation of why certain soils support growth, but indirectly they also explain the vastness and importance of the water beneath the surface, how it reaches the roots of plants, and eventually comes to the surface where it evaporates. We have barely begun to study the physical aspects of soils and their influence on climate. Indeed, from a study of the soil and its granular elements we shall learn much concerning droughts and floods. While reliance should always be placed upon weather cycles as an explanation of climatic events, the cause of these cycles themselves may be learned from an understanding of the soil structure and behavior. This requires a knowledge of soil temperature gradients and movement of moisture. In turn these are dependent upon soil structure itself—the behavior of the particles composing it.

A constant struggle for equilibrium exists between the soil and the upper elements.

Mineral Physics—In few fields of engineering is a knowledge of the characteristics and behavior of small particles of greater importance than in mineral physics. Crushing, grinding, and separation of minerals must be done scientifically. Ores are crushed or ground to prescribed sizes, such that the rich and poor fractions are separate and clean-cut. By various methods of floating and settling, these fractions may be separated. In this way mines which once were uneconomical to operate because of poor grade ore are now quite valuable. Sizing and grading of ores may be said to constitute the foundations for treatment and *benefaction* of ores and minerals.

The mining engineer must be familiar with all kinds of crushing and grinding, not only from the operating standpoint but also with regard to the character of product desired. Given a material to be ground, he must control the equipment in order to obtain the size fraction desired. Too many fines may be both undesirable and uneconomical. Thus, he must size and control the material so that the maximum amount of the required size range is obtained. Here again we stress the importance of fully appreciating the methods used to measure or size material.

With the development of powder metallurgy a vast new field has recently been opened in which the value of particle-size measurements and packing behavior will play an important role. Selection of the proper combination of sizes and their relation to void structure will be fundamental considerations. A knowledge of the electrical and gas adsorption properties of fine powders should also prove of inestimable value in this field.

Chemical Engineering—Powdered, crystalline, or granular materials are encountered in almost every phase of chemical engineering. The sizing and characteristics of every product are constantly matters of concern. How does a material bulk? Is it hygroscopic? What makes it stick? These are all questions asked many times. Small particles behave in many peculiar ways, not often understood. They behave variously due to electric charge or heat. They may adsorb vapors and gases, or their surfaces may be such that they are highly active chemically. All these factors and many others affect the products made by the chemical engineer, who attempts to maintain uniformity under all types of conditions.

The engineer is frequently called upon to design special apparatus for handling gases, carrying out chemical reactions, or conveying slurries. The first case concerns the design of towers packed with irregularly shaped material for the purpose of transferring heat or adsorbing gases.

In such towers pumps for moving gases must be provided. Since power calculations are dependent upon moving a known volume of gas against the resistance of the tower, some basis must be provided for determining the latter in terms of constants of the packing. The capacity of a gas to transfer its heat to the packing or to receive heat is also determinable from packing constants so that the engineer is in a position to estimate the performance of a tower fairly closely from easily determined factors.

The second topic within the scope of chemical engineering applications concerns the relation of particle-size to chemical reaction—such as rates of solution and oxidation. This is a subject complicated by many indeterminate factors. In general, however, the main item concerned is particle-surface. As is well known, where solids are involved, reactions of a chemical nature are greatly influenced by the amount of surface exposed.

Finally there is the third topic—muds and slurries. This includes not only their characteristics under static conditions but also their movement through pipes. Most muds and slurries act like plastic materials when flowing in pipes. The prediction of their flow properties in terms of readily determinable constants has important engineering applications.

Although these three main topics were selected as coming within the scope of chemical engineering, obviously many others might also be considered. Nor can the fundamental physical and chemical properties of fine particles be forgotten. For example, it is not generally known that fine particles, when wetted, generate heat and because of their vast surface have the ability to adsorb large quantities of different gases. In the field of electrostatic separation particles react in various ways. These properties, together with many others, are useful in explaining many industrial phenomena.

Geology—Ground Water and Petroleum—The science of sedimentary geology is founded for the most part on the study of consolidated and unconsolidated sands. In this text our attention will be directed only to the retentivity of unconsolidated sands in regard to water and oil. While consolidated sands are of great geological importance, we know very little concerning the sizes of the particles composing them. Methods have been devised for crushing sandstone and measuring random samples, but it is never certain that all particles are distinct and separate. If thin sections are possible, random measurements can be made.

In unconsolidated sands a relationship exists between the particles and the pore space where water may be stored. The classic works of King and Slichter in this country near the end of the Nineteenth Century form the basic information on this subject. The work on ground

water during the past forty years has followed along lines established by these workers. Incidental to determination of the laws of flow through sands was Slichter's analysis of void construction by systematic piling of spheres. Considering the interesting geometrical aspects of the problem of sphere arrangement, it is difficult to understand why no attempt was made to study this subject before Slichter's studies (1899). While his theoretical equation for the flow of fluids through packings is no longer used, Slichter was the first to appreciate the significance of particle diameter and voids, and to explain their interrelationship on a rational basis. His extensions into the field of ground water generally, and the variables involved in well drilling, did much to advance the practice of gauging ground water and oil supplies.

Unfortunately, it is not possible to determine satisfactorily the composition of the whole subsoil structure bearing water or oil. This can only be estimated indirectly from the behavior of wells, although something concerning the extent and capacity of the field may be determined by borings and plug samples. The methods outlined in this text pertaining to the permeability of the subsoil structure indicate a means of checking on the void space, and are thus of value. Development of more extensive equations governing the continued behavior of a well is beyond the scope of this text.

Hydrology—Silting of Streams—Each year the rivers of the world carry enormous quantities of silt into the sea. This silt may be distributed over a wide area of the ocean bottom, or confined to relatively narrow areas, forming deltas. The best known deltas are those of the Mississippi and the Nile rivers. Every year when the Nile overflows its banks hundreds of square miles of lowland are covered with a deposit. After the river returns to its normal channels this silt forms one of the most fertile areas in the world. This is a definite attribute of silt-laden streams. But silting may also be detrimental; streams blocked by dams may convey so much silt that the reservoirs they form become filled and their capacities lowered. Again, in navigation and irrigation canals, silting must be prevented if their usefulness is to be maintained.

It should be remembered that silt carried by rivers is the product of soil erosion and scouring of their channels by the rivers themselves. The elements causing both conditions are well known; our greatest interest is to prevent the loss of valuable surface soils and minerals. This cannot be done scientifically unless we know the physical factors causing movement of small particles. Many notable contributions have been made in this field, here and abroad. Some concept of the magnitude of the engineering problems involved may be estimated from an excellent discussion of the Yellow River Problem by Todd and Eliassen

(1938). The silt volume at Shanchow (624 miles above the mouth) amounted to almost 45,000,000,000 cu ft in 1934 alone. This is equivalent to a cube about 0.7 mile on each side. The authors state:

"There is perhaps no important river in the world so intimately related to every phase of conservancy work as the Yellow River. Whether it is flood control, irrigation, hydroelectric development, or navigation, the question as to how to overcome the evil effects of the silt overshadows all other technical considerations. This is due to the staggering proportions of the load carried. On other rivers a silt content of 1 percent or 2 percent by weight is considered a heavy and obnoxious charge which causes much trouble; but where the Yellow River enters the plain the silt load reaches 40 percent by weight at times. On several of its tributaries as high as 50 percent by weight have been observed."

As in the case of soil physics, we note that in comprehending this phenomenon we must learn the behavior of small particles in flowing media. What are the forces causing their movement, and what are their magnitudes? These are questions bearing upon particle-dynamics, although considerably complicated.

Wind Storms—While we have stressed the amount of silt moved by rivers, and the losses by water-erosion of land surface, the effect of wind movement on dry loose particles is of even greater importance in many respects. In recent times the great droughts of the Plains States have caused large shifts in surface contours and have raised dust storms of considerable proportions. The case of the Nubian and Libyan deserts, where sand storms are constantly shifting prominences almost 1000 ft in height as well as the whole surrounding terrain, illustrates the magnitude of the forces at work. Moreover, the continuance of storms over loose surface grains not only results in an expansion of desert and drought areas, but may even alter weather conditions and the accompanying ecology of surrounding country. There is good reason to suppose that the great desert of North Africa is gradually extending itself by these methods.

The author estimates that in any one day during the great droughts of the late 1930's in the United States as much as 10^7 tons of fine particles remained suspended in the air and were moved to areas far removed from the Dust Bowl of their origin. In the Dust Bowl itself, the estimated movement within a few inches of the surface probably ranged from 0.2 to 0.5 ton per yd per hr across the direction of wind motion. Samples of settled dust taken in the Dust Bowl and analyzed as to composition and size frequency indicated a sharp differentiation as to the size of

organic and inorganic constituents, the former being much the finer.*

In the paragraphs to follow we shall borrow an observation from the studies on blown sand and desert dunes by Bagnold (1943). His contributions cannot be overemphasized. Aside from presenting a comprehensive theory of particle movement by wind, he has indicated the vast scope of a problem (only briefly dealt with in this section) and has staked out a field of research which will have a great bearing on the future of lands and peoples. Of the scope of the desert problem from the geomorphic point of view, Bagnold has written:

". . . it is possible to study in the further interiors of the great deserts the free interplay of wind and sand, uncomplicated by the effects of moisture, vegetation, or of fauna, and to observe the results of that interplay extended over great periods of time.

"Here, instead of finding chaos and disorder, the observer never fails to be amazed at a simplicity of form, an exactitude of repetition and a geometric order unknown in nature on a scale larger than that of crystalline structure. In places vast accumulations of sand weighing millions of tons move inexorably, in regular formation, over the surface of the country, growing, retaining their shape, even breeding, in a manner which, by its grotesque imitation of life, is vaguely disturbing to an imaginative mind. Elsewhere the dunes are cut to another pattern—lined up in parallel ranges, peak following peak in regular succession like the teeth of a monstrous saw for scores, even hundreds of miles, without a break and without a change of direction, over a landscape so flat that their formation cannot be influenced by any local geographical features. Or again we find smaller forms, rare among the coastal sand hills, consisting of rows of coarse-grained ridges even more regular than the dunes. Over large areas of accumulated sand the loose, dry, uncemented grains are so firmly packed that a loaded lorry driven across the surface makes tracks less than an inch in depth. Then, without the slightest visual indication of a change, the substance only a few inches ahead is found to be a dry quicksand through which no vehicle can force its way. At times, especially on a still evening after a windy day, the dunes emit, suddenly, spontaneously, and for many minutes, a low-pitched sound so penetrating that normal speech can be heard only with difficulty."

Other Applications—A knowledge of the behavior and characteristics of small particles can find a field of application in almost any industry. Particle-size distributions are important in pigment technology

* *Unpublished data* Atmospheric samples taken during the summer of 1936 after the dust clouds from the Dust Bowl reached Washington, D. C., showed concentrations as high as 250 mg per cu cm at breathing level. Settled dust samples from various localities in Kansas and Nebraska were furnished the author by Dr. James Leake of the U. S Public Health Service. The samples were air-elutriated and fractions submitted for both petrographic and chemical analysis.

in connection with covering power and other physical factors; in the manufacture of abrasives to produce wheels of various grinding capacities; and the field of ceramics, including clay products and cement, is primarily concerned with small particles from the standpoint of product control. Certain physical properties, often highly desirable, are dependent upon size of the material used. As already mentioned, the methods used to measure particle-surface or particle-volume have great influence on the results achieved.

CHAPTER 2

DYNAMICS OF SMALL PARTICLES

UNITS

THE fundamental units of dynamics are those of mass, length, and time. These units are denoted by the symbols [M], [L], and [T], respectively. The magnitude of these units may be fixed arbitrarily; but all other units are derived from them, and depend upon them alone. The various derivative units are developed simply. For example, unit-density is unit-mass contained in a unit-volume. In terms of the fundamental units it is therefore $[M/L^3]$ or $[ML^{-3}]$. Again, unit-velocity is unit-distance divided by unit-time, or [L/T]. By means of these units the terms in a given equation may be checked, inasmuch as all terms added to a given expression must be of the same kind and therefore of the same dimensions. If, for illustration, it is desired to determine the units of surface tension, σ, from the capillary equation

$$P = \frac{\sigma}{R}$$

the procedure is as follows: The units of the pressure P (force per unit-area) are $[MLT^{-2}][L^{-2}]$ and for R (radius of curvature of the meniscus) simply [L]. Hence

$$[\sigma] = [MLT^{-2}] \cdot [L^{-2}] \cdot [L] = [MT^{-2}]$$

or the force per unit-length. However, results are not always so easily interpreted, as may be seen in determining the units of the universal gravitational constant, G, in Newton's equation

$$F = \frac{G \times m_1 \times m_2}{l^2}$$

$$[MLT^{-2}] = [GM^2L^{-2}], \text{ and } [G] = [M^{-1}L^3T^{-2}]$$

These are the correct units for gravitation. However, owing to our ignorance of the mechanism of gravitation, the units are unintelligible. If it were possible to express G in terms of some effect the difficulty

of interpretation would undoubtedly disappear. The dynamic units of frequent occurrence in this text are given in Table 1.

TABLE 1—UNITS OF TERMS FREQUENTLY OCCURRING IN TEXT

Term	Units
Mass	[M]
Length	[L]
Time	[T]
Surface	[L^2]
Volume	[L^3]
Density	[ML^{-3}]
Velocity	[LT^{-1}]
Acceleration	[LT^{-2}]
Force	[MLT^{-2}]
Pressure	[$ML^{-1}T^{-2}$]
Viscosity	[$ML^{-1}T^{-1}$]
Surface tension	[MT^{-2}]
Work (energy)	[ML^2T^{-2}]
Power	[ML^2T^{-3}]

DIMENSIONAL ANALYSIS

Each term of an equation possesses specific units and these must be the same as the combined units of all the other terms (properly transposed). A use for dimension theory is to predetermine physical relationships. Suppose, for example, it is desired to determine the resistance $\mathfrak{R}$ of a particle in motion, and that the resistance involves three independent quantities: the diameter of the particle d, its velocity v, and the viscosity of the medium μ.

$$\mathfrak{R} = d^a v^b \mu^c$$

where a, b, and c are undetermined exponents. In terms of units

$$[MLT^{-2}] = [L^a] \cdot [L^b T^{-b}] \cdot [M^c L^{-c} T^{-c}]$$

Equating indices of corresponding terms, since the equation must be dimensionally correct, gives

$$\begin{aligned} 1 &= c \\ 1 &= a + b - c \\ -2 &= -b - c \end{aligned}$$

whence, $c = 1$, $b = 1$, $a = 1$, and

$$\begin{aligned} \mathfrak{R} &\propto dv\mu \\ \mathfrak{R} &= kdv\mu \end{aligned} \qquad \text{Eq (2-1)}$$

This is Stokes' law for small particles or particles moving in streamline motion. The constant k for spheres has a value 3π.

If the resistance is now assumed to vary as the density of the medium ρ_0 rather than as the viscosity, and is proportional to the quantities concerned

$$\mathcal{R} \propto d^r v^s \rho_0{}^t$$

where r, s, and t are undetermined exponents. In terms of the units of the variables

$$[\mathrm{MLT}^{-2}] = [\mathrm{L}^r] \cdot [\mathrm{L}^s\mathrm{T}^{-s}] \cdot [\mathrm{M}^t\mathrm{L}^{-3t}]$$

Equating the indices of like quantities

$$\begin{aligned} 1 &= t \\ 1 &= r + s - 3t \\ -2 &= -s \end{aligned}$$

Solving, we obtain $t = 1$, $s = 2$, and $r = 2$, or

$$\begin{aligned} \mathcal{R} &\propto d^2 v^2 \rho_0 \\ \mathcal{R} &= k' d^2 v^2 \rho_0 \end{aligned} \qquad \text{Eq (2-2)}$$

This is Newton's law for bodies in turbulent motion. For spheres the value of $k = \pi/16$, approximately.

It is obvious that this procedure has many extensions in determining laws of physical phenomena. No difficulty is encountered as long as only three quantities are concerned. When four quantities or variables are involved explicitly, as for example, a particle whose resistance in motion depends on d, v, ρ_0, and μ simultaneously, there are three equations and four unknowns. Suppose that

$$\begin{aligned} \mathcal{R} &\propto d^m v^n \mu^p \rho_0{}^q \\ [\mathrm{MLT}^{-2}] &= [\mathrm{L}^m] \cdot [\mathrm{L}^n\mathrm{T}^{-n}] \cdot [\mathrm{M}^p\mathrm{L}^{-p}\mathrm{T}^{-p}] \cdot [\mathrm{M}^q\mathrm{L}^{-3q}] \end{aligned}$$

then

$$\begin{aligned} 1 &= p + q \\ 1 &= m + n - p - 3q \\ -2 &= -n - p \end{aligned}$$

Hence, solving for m, p, and q in terms of n

$$\mathcal{R} = k'' d^n v^n \mu^{2-n} \rho_0{}^{n-1} \qquad \text{Eq (2-3)}$$

Note that when $n = 1$, this equation is the same as Eq (2-1) and when $n = 2$ it is identical with Eq (2-2).

Similar relations containing four or more terms result in much confusion and one must resort to a theorem due to Buckingham (1921*a*),

which can best be illustrated by utilizing the material previously developed.

We note first that Eq (2-1) and Eq (2-2) when thrown into the forms

$$\left[\frac{\mathfrak{R}}{dv\mu}\right] = [K]$$

$$\left[\frac{\mathfrak{R}}{d^2v^2\rho_0}\right] = [K']$$

are dimensionless. For the present we desire to develop Eq (2-3) in a similar dimensionless form, without undetermined exponents. Take μ = constant and we have

$$\mathfrak{R} = d^r v^s \rho_0{}^t$$

or as in Eq (2-2)

$$\mathfrak{R} \propto d^2v^2\rho_0$$

Now, make $\mathfrak{R}$ constant, and we have

$$\mu \propto d^a v^s \rho_0$$

which by previously established procedures gives

$$\mu \propto dv\rho_0$$

or

$$\left[\frac{\mu}{dv\rho_0}\right] = [K'']$$

a dimensionless quantity.

Since $\mathfrak{R}$ and μ may vary at the same time, both $[K']$ and $[K'']$ must be dimensionless simultaneously. This reasoning may be extended so that if there are n different quantities involved in any physical relationship there must be $(n - 3)$ dimensionless quantities, $[K_1]$, $[K_2]$, $[K_3]$ ... $[K_{n-3}]$ and these quantities must all be dimensionless simultaneously Thus for n, physical quantities, we have the implicit relation

$$\psi\{[K_1], [K_2], [K_3], \ldots . [K_{n-3}]\} = 1$$

or explicitly

$$[K_1] = \varphi\{[K_1], [K_3] \ldots . [K_{n-3}]\}$$

where ψ and φ are functions whose forms must be determined experimentally.

For the equation in question we have the five quantities $\mathfrak{R}$, μ, r, v, ρ_0 so that $(n - 3) = 2$. Hence

$$\psi\left\{\left[\frac{\mathfrak{R}}{d^2v^2\rho_0}\right], \left[\frac{\mu}{dv\rho_0}\right]\right\} = 1$$

or

$$\left[\frac{\mathfrak{R}}{d^2v^2\rho_0}\right] = \varphi\left[\frac{\mu}{dv\rho_0}\right] = \varphi\left[\frac{\nu}{dv}\right]$$

where ν is written for μ/ρ_0, the kinematic viscosity. The importance of putting the equation in this form lies in the fact that if the form of function ψ or φ is such that the bracketed terms are connected by a single relation, then this relation will be independent of the units of measurement, and will apply to any scale model.

MODELS

Models are simply miniatures of the prototypes they represent. The physical parts of the model are in an arbitrary fixed ratio to corresponding parts of the prototype. In so far as statical problems are concerned, the ratio between corresponding parts is all that is desired. But if we consider a fluid flowing in an open channel, the depths of flow at corresponding points of the model and prototype will not necessarily be in the same ratio as the other dimensions. The moment we establish the depths of flow in two different points of a channel we also establish the flow itself and in addition to geometric similarity we require an accounting of the forces affecting the flow. This being the case it becomes necessary to set up within the model the forces acting on a given portion of the liquid and to observe that the same ratio exists for each corresponding force in the prototype. This amounts to a force similarity, usually referred to as "dynamic" similarity, since the corresponding force polygons in the model and prototype are similar.

The forces dealt with in flow problems are more complicated than those usually encountered in statical problems. Flow is affected by inertial, gravitational, frictional, and capillary forces. The inertial forces are characteristic of any kind of flow, as are also frictional forces; gravitational forces are always present but affect the flow only when there is a free surface such as a channel or canal; capillary forces do not find extensive use in hydrodynamic problems except when dealing with a thin sheet of water flow. All these forces have the same dimension $[MLT^{-2}]$ but do not affect flow in the same manner. Thus, with a change of velocity, gravitational and capillary forces do not change at all, but inertial and frictional forces change according to two different laws. If we are to build a model dynamically similar to the prototype it is necessary to express these forces in terms of the variables characteristic of them. Expressions for the various forces mentioned, in terms characteristic of them, are as follows·

$$\text{inertial forces} = \rho_0 v^2/L$$
$$\text{frictional forces} = \mu v/L^2$$
$$\text{gravitational forces} = \rho_0 g$$
$$\text{capillary forces} = c/L^2$$

where v is the velocity, L is any definite linear dimension, μ the coefficient of viscosity, g the acceleration of gravity, c the capillary constant, ρ_0 the density and L a linear dimension. Observe that all the above are forces and possess the same units; hence, if we establish ratios of these forces they will be dimensionless. Let us take the following ratios:

$$\frac{\text{inertial forces}}{\text{frictional forces}} = \frac{\rho_0 v^2}{L} \Big/ \frac{\mu v}{L^2} = \frac{\rho_0 L v}{\mu} = \mathbf{R} \qquad \text{Eq (2-4)}$$

$$\frac{\text{inertial forces}}{\text{gravitational forces}} = \frac{\rho_0 v^2}{L} \Big/ \rho_0 g = v^2/gL = \mathbf{F} \qquad \text{Eq (2-5)}$$

$$\frac{\text{inertial forces}}{\text{capillary forces}} = \frac{\rho_0 v^2}{L} \Big/ \frac{c}{L^2} = \frac{\rho_0 v^2 L}{c} = \mathbf{W} \qquad \text{Eq (2-6)}$$

The letters **R, F,** and **W** stand for so-called Reynolds, Froude, and Weber numbers, respectively; these are dimensionless numbers, as indicated. For example, if we make the Reynolds number the same in model and prototype, using the same fluid, the dimension of length is smaller in the model and hence the velocity v will have to be greater. In other words, the water would have to flow faster in the model. If we now consider the Froude number as the same in model and prototype, and that the same fluid is used in both, we see that the velocity would have to be less in the model than in the prototype. This may be regarded as two contradictory demands on the model. Theoretically, by using a different fluid in the model (thus changing ρ_0 and μ), it is possible to eliminate the difficulty. The root of the difficulty is the fact that the numbers are derived for two entirely different kinds of flow. In a fluid system without a free surface, dynamic similarity requires only that the *Reynolds number* be the same in model and prototype; the Froude number does not enter into the problem. If we consider the flow in an open channel, then the *Froude number* must be the same in model and prototype.

Note that Eqs (2-4), (2-5), and (2-6) are expressions of the frictional forces considered above.

In previous paragraphs, we obtained groups of dimensionless variables by means of Buckingham's theorem. In the development of Reynolds, Froude, and Weber numbers we utilized the concept of force ratios although the same numbers can be produced by means of Buckingham's theorem.

BEHAVIOR OF PARTICLES IN A FLUID

Resistance to Motion—It has been demonstrated experimentally that Eq (2-3) represents fairly accurately the general law of resistance of bodies to motion. Resistance in the case of streamline or turbulent mo-

tion is determined by the exponent n, which in the first case is equal to 1, and in the second case equal to 2.

No sharp boundary exists between streamline and turbulent flow. In fact there is a rather extensive region where resistance to motion is a function of both the density and viscosity of the medium. For this intermediate region Allen (1900) found the value of n in Eq (2-3) to be $^3/_2$, giving

$$\begin{aligned} \mathfrak{R} &= k''\rho_0\nu^{1/2}d_0^{3/2}v^{3/2} \\ &= k''\rho_0^{1/2}\mu^{1/2}d_0^{3/2}v^{3/2} \end{aligned} \qquad \text{Eq (2-7)}$$

where d_0 is written for d since, as will be explained in another paragraph, the particle-diameter includes a constant defining its lower region of applicability.

Terminal Velocities—If the density of the particle and fluid are ρ and ρ_0, respectively, and if the diameter of the particle is some arbitrary linear measurement, the gravitational pull F on the particle is

$$F = \alpha_v d^3(\rho - \rho_0)g \qquad \text{Eq (2-8)}$$

where $\alpha_v d^3$ measures the volume of the particle and g is the acceleration due to gravitation. When the particle has reached its maximum or "terminal" velocity, $\mathfrak{R} = F$, so that for

$$\text{streamline motion, } v_m = K_S\left(\frac{\rho - \rho_0}{\rho_0}\right)d^2\nu^{-1} \qquad \text{Eq (2-9)}$$

$$\text{turbulent motion, } v_m = K_T\left(\frac{\rho - \rho_0}{\rho_0}\right)^{1/2}d^{1/2} \qquad \text{Eq (2-10)}$$

$$\text{intermediate region, } v_m = K_I\left(\frac{\rho - \rho_0}{\rho_0}\right)^{2/3}\nu^{-1/3}d_0 \qquad \text{Eq (2-11)}$$

where the K's are constants. Values of K_S and K_T are given in Table 2 for spheres, crushed quartz, and coke. The data for crushed quartz

TABLE 2—TERMINAL-VELOCITY CONSTANTS OF DIFFERENT-SHAPED PARTICLES IN STREAMLINE AND TURBULENT MOTION

Shape	K_S	K_T
Sphere	54.5	24.5
Irregular (quartz)	36.0	50 0
Irregular (coke)	35.0	48.0

are those obtained by G. Martin (1926–1927*a*), and those for coke by Burke and Plummer (1928). The constants are in cgs units.

The value of d_0 in Eq (2-11) is composed of a true diameter d and a diameter $\mathbf{d}$ which is the uppermost limit of size to which Stokes' equation applies.

$$d_0 = d - \zeta \mathbf{d}$$

where d is the true diameter, $\mathbf{d}$ the limiting diameter to which Stokes' equation applies, and ζ is a constant depending upon the shape of the particle. The value of $\mathbf{d}$ was obtained by Allen.

$$\mathbf{d} = \sqrt[3]{\frac{36\ \mu^2}{g\rho_0(\rho - \rho_0)}}$$

Using cgs units the value of ζ for spheres is 0.4, and for irregular quartz particles as determined by Martin is 0.279. In general, it is worth noting that the terminal velocities of a particle as given by Eqs (2-9) to (2-11) is as the square, the first power, and the square root of the particle diameter depending on the motion. If quartz is used as an example, it is found that the terminal velocities are according to Stokes' equation for particles below 85 μ in diameter. Between 85 μ and approximately 2000 μ the terminal velocities vary directly as the diameter, and above this diameter particles of quartz follow Newton's parabolic law. The nature of the relationships existing between terminal velocity and average diameter is shown in Figure 2.

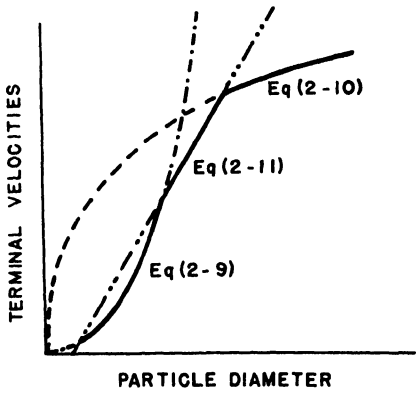


FIGURE 2. LAWS OF FREE-FALLING PARTICLES.

Lifting Velocities—The velocities required to lift particles are identical with those given by Eqs (2-9) to (2-11). Thus, for particles of various sizes, the velocity required to lift and convey all of them must exceed the velocity required for the largest particle. More general formulas for lifting velocities are given in a later section of this chapter.

As has been pointed out, there are three formulas to consider since the size of the particle, its speed, and the properties of the fluid determine the nature of the motion. Often it is difficult to determine precisely the boundary that distinguishes one type of movement from another, so that if a single curve could be drawn through the whole range of sizes, from boulders to clay and silt, and an equation derived for it, some labor would

be saved This, in fact, was done by Rubey (1933*b*) whose procedure was as follows: The total force holding a spherical (or other) particle in suspension at a constant height in a rising current of a given fluid may be conceived as the sum of two forces—viscous resistance and fluid impact,

$$\frac{\pi d^3}{6}(\rho - \rho_0)g = 3\pi d\mu v + \frac{\pi d^2}{4}\rho_0 v^2$$

Solving for v, we obtain a quadratic which, on retaining the positive root, becomes

$$v = -\frac{6\mu}{\rho_0 d} + \sqrt{\frac{36\mu^2}{\rho_0^2 d^2} + \frac{2}{3}g\left(\frac{\rho - \rho_0}{\rho_0}\right)d}$$

Rubey states that this equation closely fits the entire range of particle-size. For quartz in water at 16 deg C

$$v = \frac{6.6}{d}\left(\sqrt{247.5d^3 + 1} - 1\right)$$

Cunningham's Correction—In addition to the upper limit of particle size, Stokes' equation also has a lower limit. When the particles are small compared with the mean free path of the molecules, the particles may slip between the molecules. Thus, less resistance is encountered and the particles develop a higher velocity. For diameters of 0.1 μ and less we must apply a correction to the velocity determined by Stokes' equation, Eq (2-9), which is due to Cunningham (1910)

$$v = \frac{1}{18}gd^2\left(\frac{\rho - \rho_0}{\mu}\right)\left(1 + k\frac{\lambda}{d}\right)$$

where λ is the mean free path of the molecules, and k a constant, which for air is approximately 0.86.

Ladenburg's Correction for Falling Spheres—When a sphere falls in a cylinder whose dimensions are of the same order of magnitude as the sphere, Stokes' equation must be modified to take account of the wall and bottom effects. This is necessary since the displacement action of the sphere interferes with its motion. Ladenburg (1907) supplied the necessary corrections. For the wall-effect, Ladenburg gave the correction as

$$v\left(1 + 2.4\frac{d}{D}\right) = v_\infty$$

where d/D is the ratio of the diameter of the sphere to that of the cylinder, v the velocity of the sphere, and v_∞ the velocity for which Stokes' equation applies (that is, the velocity of the sphere in an infinite me-

dium). The correction is only approximate for large values of d/D. Ladenburg's second correction, for end-effect, is

$$v\left(1 + 1.7\,\frac{d}{L}\right) = v_\infty$$

where L is the height of the liquid through which the sphere moves. The corrected Stokes' equation is therefore

$$v = \frac{d^2 g(\rho - \rho_0)}{18\mu(1 + 2.4d/D)(1 + 1.7d/L)}$$

Relation between Resistance and Reynolds Number—The terminal velocity varies with the shape of the particle, its mass, and with Reynolds number. Wadell (1934*a*, *b*) collated existing data for spheres. His results are shown in Figure 3 where coefficient of resistance is plotted against Reynolds number. Reynolds number **R** has been shown to be nondimensional and equal to $\rho_0 dv/\mu$, where d is the diameter of the sphere. The coefficient of resistance C_R is by definition contained in the equation

$$\mathcal{R} = \frac{1}{2}\rho_0 v^2 C_R A \qquad \text{Eq (2-12)}$$

where A is the cross-sectional area of the particle.

Reference to Figure 3 shows that in the streamline region

$$C_R = f(\mathbf{R}) = \frac{k\mu}{dv\rho_0} = \frac{k}{\mathbf{R}} \qquad \text{Eq (2-13)}$$

where k is a constant of proportionality and is equal to 24. Also we note that in the turbulent region C_R = constant = 0.4 roughly

The complete span of the curve is represented to a fair degree of accuracy by the expression

$$\sqrt{C_R} = 0.63 + \frac{4.8}{\sqrt{\mathbf{R}}} \qquad \text{Eq (2-14)}$$

This equation will be used to obtain a general solution in the case of a particle moving in the absence of a gravitational field, *i.e.*, for horizontal motion. For a particle moving in a gravitational field (for vertical motion), we shall use three equations which cover the span of the curve given in Figure 3. Thus, for

$$\left.\begin{array}{ll} (a)\ \text{streamline motion} & 10^{-4} < \mathbf{R} < 2,\ C_R = \dfrac{24}{\mathbf{R}} \\ (b)\ \text{intermediate motion} & 2 < \mathbf{R} < 500,\ C_R = 0.4 + \dfrac{40}{\mathbf{R}} \\ (c)\ \text{turbulent motion} & 500 < \mathbf{R} < 10^5,\ C_R = 0.44 \end{array}\right\} \qquad \text{Eq (2-15)}$$

These relations offer considerable simplification of the equation for vertical motion, as we shall see. However, note that Eq (2-14) approximates closely the expressions given by Eq (2-15) in the specified ranges of Reynolds numbers.

The importance of the relationship between C_R and **R**, given by the above equations and Figure 3, is readily apparent when we consider that

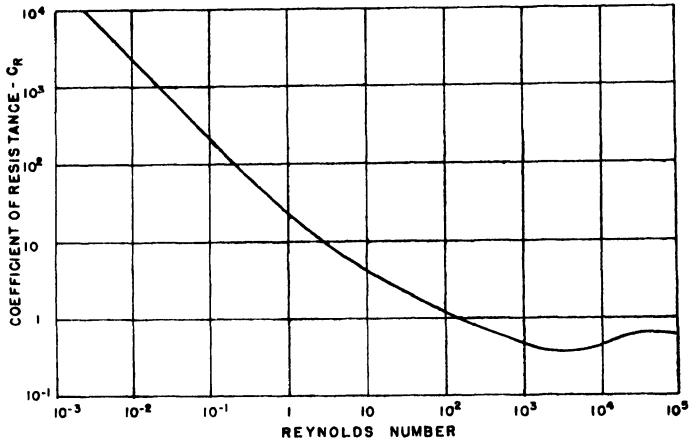


FIGURE 3. COEFFICIENT OF RESISTANCE AS A FUNCTION OF REYNOLDS NUMBER

particles in motion accelerate, decelerate, or reach constant terminal velocities depending on the forces imposed and the nature of the motion. Therefore a particle may cover a rather large span of Reynolds numbers and be subjected to different types of resistance to motion depending on its speed in relation to the medium. In the case of a particle projected in a nongravitational field, it may pass through turbulent, intermediate, and streamline motions. The exact solution of the problem is far more difficult than solving a simple equation to the effect that the resistance encountered by a particle through its entire motion varies simply as the square of the velocity. In utilizing Reynolds number we take into consideration the density and viscosity of the medium.

EQUATIONS OF MOTION

TWO-DIMENSIONAL MOTION

Because it is easier to develop special cases for particle motion from the equations covering two-dimensional motion, we shall begin by discussing the motion of a particle in a plane with rectangular coordinate axes x, y.

We shall consider that motion to the right of the origin, horizontally, is positive, and that motion upward and parallel to the y-axis is negative. We follow the latter procedure since we deal generally with the fall rather than with the rise of particles. Let the particle possess a mass m and density ρ, and assume that the resisting force $\mathfrak{R}$ along the path of motion makes an angle α with the tangent to the path at any particular time t and position x, y. Denote horizontal and vertical velocities by the symbols v_x and v_y. Then

$$m\frac{d\mathrm{v}_x}{dt} = -\mathfrak{R}\cos\alpha \qquad \text{Eq (2-16)}$$

$$m\frac{d\mathrm{v}_y}{dt} = mg\left(\frac{\rho - \rho_0}{\rho}\right) - \mathfrak{R}\sin\alpha \qquad \text{Eq (2-17)}$$

where ρ_0 is the density of the fluid and g is the gravitational acceleration constant. Also we have

$$\cos\alpha = \frac{\mathrm{v}_x}{v}$$

$$\sin\alpha = \frac{\mathrm{v}_y}{v}$$

$$v^2 = v_x^2 + v_y^2$$

$$\mathfrak{R} = \frac{1}{2}\rho_0 v^2 C_R A$$

Substituting these values in the above, we obtain

$$\frac{d\mathrm{v}_x}{dt} = -\frac{\rho_0 C_R A v \mathrm{v}_x}{2m} \qquad \text{Eq (2-18)}$$

$$\frac{d\mathrm{v}_y}{dt} = g\left(\frac{\rho - \rho_0}{\rho}\right) - \frac{\rho_0 C_R A v \mathrm{v}_y}{2m} \qquad \text{Eq (2-19)}$$

These are the equations for two-dimensional motion. We see that the acceleration along either axis comprises the actual vector velocity along the corresponding axis and the actual velocity along the tangent to the point v. Eqs (2-18) and (2-19) cannot be solved explicitly and we must resort to a method of approximations, an example of which will be given. Write the equation in incremental form, as follows:

$$\Delta\mathrm{v}_x = -\frac{\rho_0 C_R A v \mathrm{v}_x}{2m}\,\Delta t$$

$$\Delta\mathrm{v}_y = \left[g\left(\frac{\rho - \rho_0}{\rho}\right) - \frac{\rho_0 C_R A v \mathrm{v}_y}{2m}\right]\Delta t$$

and assume that we are dealing with heavy spherical particles in air.

Thus if $\rho >> \rho_0$, $A = \pi d^2/4$ and $m = \pi d^3\rho/6$

$$\left.\begin{aligned} \Delta v_x &= -\frac{3\rho_0 C_R v v_x}{4\rho d}\,\Delta t \\ \Delta v_y &= \left(g - \frac{3\rho_0 C_R v v_y}{4\rho d}\right)\Delta t \end{aligned}\right\}$$

Since air is the fluid medium, let $\rho_0 = 0.067$ lb per cu ft and $\mu = 1.3 \times 10^{-5}$ lb per ft sec; and for the particle, let $d = 0.001$ ft, and $\rho = 200$ lb per cu ft. Since g = 32.2 we have, on inserting these values:

$$\left.\begin{aligned} \Delta v_x &= -0.25\, C_R v v_x \Delta t \\ \Delta v_y &= (32.2 - 0.25\, C_R v v_y)\,\Delta t \end{aligned}\right\} \qquad \text{Eq (2-20)}$$

We have shown that C_R is determinable from Reynolds number, $\mathbf{R} = \rho_0 v d/\mu$, in each case corresponding to v. Hence,

$$\mathbf{R} = \frac{0.067 \times 0.001 v}{1.3 \times 10^{-5}} = 5v$$

The last three equations, together with Eq (2-14) comprise all that is required for calculating the motion. If the particle is given an initial horizontal velocity of 68 ft per sec ($v_x = 68$), we may proceed with the calculations by assuming successive increments of time. The complete procedure is outlined in Table 3. It will be seen that an initial value of v is computed; then $\mathbf{R}$, C_R (from Eq 2-14), and the other variables are

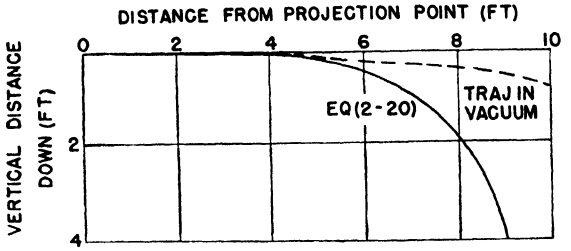


FIGURE 4. THEORETICAL TRAJECTORY OF A PARTICLE 0.001 FT IN DIAMETER HAVING INITIAL VELOCITY OF 68 FT PER SEC. BROKEN CURVE GIVES TRAJECTORY WHEN NO AIR RESISTANCE IS ENCOUNTERED. (SEE TEXT.)

as indicated. From the incremented velocities Δv_x and Δv_y the first approximations to the velocities at the end of the time interval are made ($\Delta t = 0.01$ sec). The average of the incremented velocities at the start and end of the time interval is then obtained and is used to make a second

approximation to velocities at the end of the same time interval. This procedure is repeated until the differences between calculated average incremental velocities approximate zero.

TABLE 3 –COMPUTATION OF PARTICLE TRAJECTORY FROM EQ. (2-20)

t	Δt	v_x	v_y	v	R	C_R	mv_x	$-\Delta v_x$	mv_y	Δv_y
0 00	0 01	68	0	68	340	0 79	4620	9 13	0	0 32
		60	0 32	60	300	0 85	3600	7 65	19 2	0 28
0 01		59 6	0 30	59 6	298	0 85	3560	7 58	17 9	0 28
	0 01	52	0 58	52	260	0 87	2700	6 85	30 0	0 26
		46 1	0 58	46 1	230	0 90	2180	4 8	26 8	0 26
0 02		46 7	0 58	46 7	232	0 90	2190	4 9	27 1	0 26
	0 01	42	0 84	42	210	0 92	1760	4 1	35 3	0 24
0 03		42 6	0 84	42 6	213	0 92	1820	4 2	35 8	0 23
	0 01	38	1 1	38	190	0 96	1440	3 5	41 8	0 22
0 04		39	1 1	39	195	0.96	1520	3 6	43	0 22
	0 01	35 4	1 3	35 4	177	1 00	1250	3 1	46	0 21
	0 01	32	1 5	32	160	1 00	1020	2 6	48	0 20
0 06		33	1 5	33	165	1 00	1090	2 7	50	0 20
0 07	0 01	30	1 7	30	150	1 02	902	2 3	51	0 19
	0 02	27	2 0	27	135	1 03	729	3 8	54	0 36
0 09		26 5	2 1	26 5	132	1 03	704	3 6	56	0 34
	0 02	22 8	2 5	22 8	114	1 17	520	3 2	57	0 31
0 11		23 3	2 4	23 3	116	1 17	544	3 3	56	0 32
0 13	0 02	20	2 7	20	100	1 23	400	2 5	54	0 31
	0 02	17 5	3 0	17 7	89	1 30	310	2 0	54	0 29
0 15		18	3 0	18 2	91	1 30	328	2 2	55	0 29
	0 05	15 7	3 3	16	80	1 37	251	4 3	53	0 70
0.2		14	3 7	14 7	74	1 42	206	3 7	54	0 65
	0 05	10	4 5	11	55	1 64	110	2 3	50	0 59
		11.7	4 5	12 6	63	1 54	145	2 8	57	0 51
0 25		11	4 3	11 8	59	1 57	130	2 6	51	0 61
	0 05	8 4	4 9	9 7	49	1.78	82	1 9	47 5	0 56
0 3		9	4 9	10 2	51	1 72	92	2 0	50	0 51
	0 05	7	5 5	8 9	45	1 70	62	1 5	49	0 43
0 35		7 5	5 3	9 3	47	1 77	70	1 6	49	0 51
0 4	0 05	5 9	5 8	8 3	42	1 88	49	1 2	48	0 48
0 5	0 1	4 7	6 3	7 9	40	1 93	37	1 8	50	0 8
0 6		4 1	6 6	7 7	39	1.94	32	1 6	51	0 72
	0 1	2 4	7 4	7 7	39	1 94	18 5	0 9	57	0 45
0 7		3 2	7.1	7.8	39	1 94	25	1 2	55	0 56
		3 0	7 1	7 7	39	1 94	23	1 1	55	9 56
	0 1	1 9	7 7	7 9	40	1 93	15	0 7	59	0 37
0 8		2 3	8.1	8 4	83	1 88	19	0 45	68	0 0

t (sec)	Δt (sec)	Δs_x (ft)	s_x (ft)	Δs_y (ft)	s_y (ft)
0 00	0	0	...	0	..
0 03	0 03	1 52	1.52	0 011	0 011
0 06	0 03	1 31	2 83	0 034	0 045
0.10	0 04	1 14	3 97	0 071	0 12
0.20	0 1	1.82	5 79	0 27	0 39
0 30	0 1	1 10	6 89	0 43	0 82
0.40	0.1	0 73	7.62	0 53	1 35
0 50	0.1	0 54	8.16	0.61	1 96
0 60	0 1	0 41	8 57	0 69	2 65
0.70	0 1	0 32	8 89	0 75	3 40

When the velocity-time data have been obtained it is a simple matter to plot them and obtain distance-time values by graphical integration, and hence the trajectory of the particle. This procedure has been used to compile Table 3 and Figure 4, and the result is compared with the trajectory obtained by considering the motion of the particles as if they were moved in a frictionless fluid, *i.e.*, $C_R = 0$. The equations for this condition are

$$\mathbf{v}_x = \mathbf{v}_{ox}$$

$$\mathbf{s}_x = \mathbf{v}_{ox}t$$

$$\mathbf{v}_y = \mathbf{v}_{oy} + g\left(\frac{\rho - \rho_0}{\rho}\right)t$$

$$\mathbf{s}_y = \mathbf{v}_{oy}t + \frac{1}{2}g\left(\frac{\rho - \rho_0}{\rho}\right)t^2$$

where the subscripts *ox* and *oy* denote initial velocity conditions.

Horizontal Motion

When $\mathbf{v}_x$ is large as compared to $\mathbf{v}_y$, which corresponds to motion in the absence of a gravitational field, we have only to deal with a solution for Eq (2-18). For this case $\mathbf{v}_x = \mathbf{v} = v$, so that the equation simplifies to

$$\frac{dv}{dt} = -\frac{\rho_0 C_R A v^2}{2m} \qquad \text{Eq (2-21)}$$

Substituting Reynolds number for v, *i.e.*, $v = \mathbf{R}\mu/\rho_0 d$, and transposing terms, we get

$$\frac{d\mathbf{R}}{C_R\mathbf{R}^2} = -\frac{\mu A}{2md}\,dt \qquad \text{Eq (2-22)}$$

When we substitute C_R as a function of Reynolds number as given in Eq (2-14), and integrate between the limits $\mathbf{R}_0$ and $\mathbf{R}$ ($\mathbf{R}_0 > \mathbf{R}$) and $t = 0$ and $t = t$, Eq (2-22) becomes

$$\int_{\mathbf{R}}^{\mathbf{R}_0} \frac{d\mathbf{R}}{C_R\mathbf{R}^2} = \frac{1}{23}\left[\ln\frac{C_R\mathbf{R}_0}{C_{R_0}\mathbf{R}} - 13.5\tan^{-1}\frac{0.71\sqrt{\mathbf{R}_0} - \sqrt{\mathbf{R}}}{0.90 + (0.80\sqrt{\mathbf{R}_0} + 6)^2}\right]$$

$$= \frac{\mu A t}{2md} \qquad \text{Eq (2-23)}$$

where C_R and C_{R_0} are written for values of C_R corresponding to Reynolds numbers $\mathbf{R}$ and $\mathbf{R}_0$, and

$$\mathbf{R}C_R = 0.4\mathbf{R} + 6\sqrt{\mathbf{R}} + 23 \qquad \text{Eq (2-24)}$$

Over the usual range of Reynolds numbers met in practice ($\mathbf{R} < 10^4$)

$$\tan^{-1} \frac{0.71\sqrt{\mathbf{R}_0} - \sqrt{\mathbf{R}}}{0.90 + (0.80\sqrt{\mathbf{R}_0} + 6)^2} = \frac{0.71\sqrt{\mathbf{R}_0} - \sqrt{\mathbf{R}}}{0.90 + (0.80\sqrt{\mathbf{R}_0} + 6)^2}$$

If we deal with spherical particles, the last term of Eq (2-23) reduces to

$$\frac{\mu A t}{2md} = \frac{3\mu t}{4\rho d^2}$$

since $A = \pi d^2/4$ and $m = \pi d^3\rho/6$.

Eq (2-23) covers the whole range of C_R and $\mathbf{R}$. If the motion is known to be streamline or turbulent, it is easy to obtain simple expressions, since for the former $C_R = 24/\mathbf{R}$ and for the latter $C_R = 0.44$. If these values of C_R are inserted in Eq (2-22) we have, for streamline motion

$$t = \frac{md}{12\mu A} \ln \frac{v_0}{v}$$

or

$$v = v_0 \exp(-12\mu A t/md) \qquad \text{Eq (2-25)}$$

Also, since $v = ds/dt$, the path traversed from time $t = 0$ to time $t = t$ is

$$s = \frac{mv_0 d}{12\mu A}\left[1 - \exp\left(-\frac{24\mu A t}{md}\right)\right] \qquad \text{Eq (2-26)}$$

From this equation we note that for infinite values of t, $s = mv_0 d/12\mu A$, which is the total distance a particle can traverse if its motion is constantly viscous.

For turbulent motion, $C_R = 0.44$, the equations for t and s are

$$t = \frac{5.5m}{\rho_0 A}\left(\frac{1}{v} - \frac{1}{v_0}\right) \qquad \text{Eq (2-27)}$$

$$s = \frac{5.5m}{\rho_0 A}\left(\ln \frac{0.22 v_0 \rho A t}{m} + 1\right) \qquad \text{Eq (2-28)}$$

Vertical Motion

When the motion of the particle is strictly vertical, possessing no horizontal component, as in the case of a particle falling under gravity in still air, we have $v_y = \mathrm{v} = v$ and we are required to solve Eq (2-19). However, before proceeding, we must realize that our equations were developed vectorially and hence a reversal in the direction of motion involves a change of sign. Using Eq (2-19), and substituting for the

v's the expression in terms of Reynolds number ($v = \mathbf{R}\mu/\rho_0 d$), we obtain for terminal conditions, $t = 0$, $t = t$, and $\mathbf{R} = \mathbf{R}_0$, $\mathbf{R} = \mathbf{R}$.

$$\frac{\mu A t}{2md} = \int_{\mathbf{R}_0}^{\mathbf{R}} \frac{d\mathbf{R}}{\phi - C_R \mathbf{R}^2} \qquad \text{Eq (2-29)}$$

when the motion is downward. In this equation,

$$\phi = \frac{2g\rho_0(\rho - \rho_0)md^2}{\rho\mu^2 A} \qquad \text{Eq (2-30)}$$

This equation is the maximum value of $C_R\mathbf{R}^2$ which a particle of diameter d can attain when falling in a gravitational field. For spherical particles

$$\phi = \frac{4g\rho_0(\rho - \rho_0)d^3}{3\mu^2} \qquad \text{Eq (2-31)}$$

Eq (2-29) is easily integrated for both streamline, intermediate, and turbulent motions by using the expressions contained in Eq (2-15). Eq (2-14) can be used for the whole range but the resultant expression is quite complicated. Thus, for streamline motion downward, Eq (2-28) gives after substitution of Eq (2-15a)

$$\frac{\mu A t}{2md} = \frac{1}{24} \ln \left(\frac{\phi - 24\mathbf{R}_0}{\phi - 24\mathbf{R}} \right) \qquad \text{Eq (2-32)}$$

This is easily converted in terms of the initial and terminal velocities. Since $\phi = C_R\mathbf{R}_m{}^2 = 24\mathbf{R}_m = 24\rho_0 v_m d/\mu$, where $\mathbf{R}_m$ and v_m are maximum Reynolds number and velocity corresponding to ϕ, we obtain from Eq (2-24)

$$v_m = \frac{g(\rho - \rho_0)md}{12\rho\mu A}$$

and from Eq (2-26)

$$t = \frac{md}{12\mu A} \ln \left(\frac{v_m - v_0}{v_m - v} \right) \qquad \text{Eq (2-33)}$$

The distance traveled by the particle in the time t is

$$s = v_m t - (v_m - v_0)\left(\frac{md}{12\mu A}\right)\left[1 - \exp\left(-\frac{12\mu A t}{md}\right)\right]$$

If the motion is upward and streamline, so that v is negative, we have the integral

$$\int_{\mathbf{R}}^{\mathbf{R}_0} \frac{d\mathbf{R}}{\phi + 24\mathbf{R}}, \quad \mathbf{R}_0 > \mathbf{R}$$

for which the solution is

$$\frac{1}{24}\ln\left(\frac{\phi + 24\mathbf{R}_0}{\phi + 24\mathbf{R}}\right) \qquad \text{Eq (2-34)}$$

In terms of v_m, v_0, and v

$$t = \frac{md}{12\mu A}\ln\left(\frac{v_m + v_0}{v_m + v}\right) \qquad \text{Eq (2-35)}$$

and

$$s = v_m t + \frac{md}{12\mu A}(v_m + v_0)\left[\exp\left(-\frac{12\mu A t}{md}\right) - 1\right] \qquad \text{Eq (2-36)}$$

Substituting the value of t in Eq (2-35) in this expression we obtain, on setting $v = 0$, the maximum height to which the particle will rise if the motion is streamline, or

$$s_{max} = \frac{md}{12\mu A}\left[v_0 - v_m \ln\left(\frac{v_m - v_0}{v_m}\right)\right] \qquad \text{Eq (2-37)}$$

Considering now the region of intermediate motion, it seems simplest to utilize the expression given by Eq (2-15b). This region is not well defined and the expression we use is as good a fit as any available at the present time. It results in considerable simplification and undoubtedly is much easier to use than the methods of graphical integration so often employed.

Using Eq (2-15b), we have for downward motion

$$\begin{aligned} t &= \frac{2md}{\mu A}\int_{\mathbf{R}_0}^{\mathbf{R}} \frac{d\mathbf{R}}{\phi - C_R\mathbf{R}^2} \\ &= \frac{2md}{\mu A}\int_{\mathbf{R}_0}^{\mathbf{R}} \frac{d\mathbf{R}}{[(\phi - 40) - 0.4\mathbf{R}]\mathbf{R}} \\ &= \frac{2md}{\mu A(\phi - 40)}\ln\left[\frac{(\phi - 40 - 0.4\mathbf{R})\mathbf{R}}{(\phi - 40 - 0.4\mathbf{R})\mathbf{R}_0}\right] \end{aligned} \qquad \text{Eq (2-38)}$$

and for upward motion, $\mathbf{R}_0 > \mathbf{R}$

$$t = \frac{2md}{\mu A(\phi + 50)}\ln\frac{(\phi + 40 + 0.4\mathbf{R}_0)\mathbf{R}}{(\phi + 40 + 0.4\mathbf{R})\mathbf{R}_0} \qquad \text{Eq (2-39)}$$

These equations may be used in the range of $2 < \mathbf{R} < 500$, a very important range in many practical investigations. No simplification is gained in solving in terms of v, which is easier to do directly from the Reynolds number identity, $\mathbf{R} = \rho_0 v d/\mu$. The calculation of the distance traveled by a particle in intermediate motion is somewhat involved, but

offers no serious difficulties in so far as an explicit relation between s and t is concerned.

We come finally to the solution of the case of vertical motion in the turbulent region. Here $C_R = 0.44$, and on integrating for *downward* motion and development in terms of the values involved, Eq (2-29) becomes

$$t = \frac{1.5md}{\mu A\sqrt{C_R\phi}} \ln\left[\left(\frac{v_m + v}{v_m - v}\right)\left(\frac{v_m - v_0}{v_m + v_0}\right)\right] \qquad \text{Eq (2-40)}$$

$$s = \frac{4.5m}{\rho_0 A} \ln\left[\frac{(v_m + v_0)\exp\left(\frac{0.44\,\rho_0 A v_m t}{m}\right) + (v_m - v_0)}{2v_m}\right] - v_m t \qquad \text{Eq (2-41)}$$

Integrating for negative (upward) motion and reducing as before gives

$$t = \frac{4.5m}{\rho_0 A v_m} \tan^{-1}\left[\frac{(v_0 - v)v_m}{v_0 v + {v_m}^2}\right] \qquad \text{Eq (2-42)}$$

$$s = \frac{4.5m}{\rho_0 A} \ln\left[\frac{v_m \cos(0.44\,\rho_0 A v_m t/2m) + v_0 \sin(0.44\,\rho_0 A v_m t/2m)}{v_m}\right] \qquad \text{Eq (2-43)}$$

In Eqs (2-40) to (2-43) the value of v_m is obtained from ϕ by setting it equal to $0.44\mathbf{R}^2 = 0.44\,\rho_0 v_m d/\mu$, so that

$$v_m = \sqrt{\frac{g(\rho - \rho_0)m}{0.22\,\rho\rho_0 A}}$$

The maximum height of rise can be determined from Eqs (2-42) and (2-43), by setting $v = 0$ whence

$$s_{max} = \frac{4.5m}{\rho_0 A} \ln\sqrt{1 + \left(\frac{v_0}{v_m}\right)^2} \qquad \text{Eq (2-44)}$$

Let us discuss one other case. Suppose a particle has attained its terminal velocity and is then suddenly subjected to a horizontal flow of fluid, the particle first having fallen through relatively motionless fluid. If the particles are large and if the horizontal velocity of fluid after the particle has reached its terminal velocity is much less than the horizontal velocity of the fluid (for air $v_x < v_m/3$), then Eqs (2-18) and (2-19) may be integrated, so that

$$s_x = v_x t - \left(\frac{2mv_x}{\rho_0 C_R A v_m}\right)\left[1 - \exp\left(-\frac{\rho C_1 A v_m t}{2m}\right)\right]$$

$$s_y = v_m t$$

CENTRIFUGAL MOTION

The force acting on a particle moving in a circle about a fixed point is

$$F = \alpha_r d^3(\rho - \rho_0)R\omega^2$$

where R is the radius of the circular path and ω the angular velocity of the particle expressed in radians per unit time. Equating this to the resistance of particles obeying Stokes' equation, Eq (2-1), we obtain the centrifugal settling velocity in the streamline zone, v_c,

$$v_c = \frac{d^2}{k}\left(\frac{\rho - \rho_0}{\mu}\right) R\omega^2 \qquad \text{Eq (2-45)}$$

The ratio of this equation to Eq (2-9), which is the settling velocity under gravity, gives the following result:

$$v_c = \frac{R\omega^2}{g} v$$

Since $\omega = v_t/R$ where v_t is the tangential velocity, we obtain

$$v_c = \frac{v_t^2}{Rg} v \qquad \text{Eq (2-46)}$$

The term v_t^2/Rg is called the "separation factor." It indicates the amount of increase in settling obtained by centrifugal action as against gravity. Eq (2-46) indicates that for a given velocity of fluid the separation factor increases with decreasing radius of path.

If we are concerned with the position of the particle at any point R from the axis of spin, then

$$\alpha_v d^3(\rho - \rho_0)\omega^2 R = k\mu d \frac{dR}{dt}$$

For spheres, $\alpha_v = \pi/6$ and $k = 3\pi$. Hence, we have

$$\frac{1}{R}\frac{dR}{dt} = \frac{\alpha_v d^2 (\rho - \rho_0)\omega^2}{k\mu}$$

Integrating between R_1 and R_2

$$\ln \frac{R_2}{R_1} = \frac{\alpha_v d^2(\rho - \rho_0)\omega^2 t}{k\mu} \qquad \text{Eq (2-47)}$$

where R_1 and R_2 are the initial and extreme positions of the particle. This gives a method for computing diameters of exceedingly small particles and is fundamental to the theory of the ultracentrifuge.

The above equations apply to particles in streamline motion. The complete equations describing the motion of a particle moving about a fixed center are easily determined by vector analysis. Referring to Figure 5, let $\mathbf{v}_t$ be a unit vector along the tangent and $\mathbf{v}_r$ the unit vector directed along the radius R. Then

$$\mathbf{v} = \mathbf{v}_t v$$

and

$$\frac{d\mathbf{v}}{dt} = v_t \frac{dv}{dt} + \frac{d\mathbf{v}_t}{dt} v$$

Referring to the figure, we note that $d\mathbf{v}_t$ has the direction of $\mathbf{v}_r$ and magnitude $d\theta$, so that

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}_t \frac{dv}{dt} + \mathbf{v}_r v \frac{d\theta}{dt}$$

The distance traveled by the particle in the time dt is vdt; hence

$$Rd\theta = vdt$$

and, substituting above,

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}_t \frac{dv}{dt} + \mathbf{v}_r \frac{v^2}{R} \qquad \text{Eq (2-48)}$$

Since the motion is everywhere tangential, $v = v_t$,

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}_t \frac{dv_t}{dt} + \mathbf{v}_r \frac{v_t^2}{R}$$

If the particle moves with a fluid having a tangential velocity v_{0t}, then the relative velocity of the particle with respect to the fluid is $v_{ot} = \mathbf{v}_t$

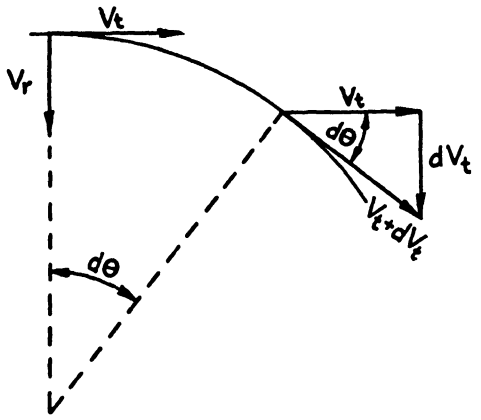


FIGURE 5. RELATION OF VELOCITY VECTORS FOR CENTRIFUGAL MOTION.

$= v_t$. Hence, since we consider v_{ot} constant, the last equation above becomes

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v}_t \frac{d\mathbf{v}_t}{dt} + \mathbf{v}_r \frac{v_t^2}{R}$$

The second term on the right is not reduced relative to the fluid since the

radial acceleration depends only upon the absolute tangential velocity of the particle. But $dv_t/dt = v_r d\theta/dt = -v_r v_t/R$ and

$$\frac{dv}{dt} = v_t\left(\frac{v_r v_t}{R}\right) + v_r\frac{v_t^2}{R}$$

where the terms in parentheses denote the accelerations in the direction of the corresponding unit vectors.

If we now take into account the resistance of the fluid to the motion of the particle, remembering that the relative velocity of the particle with respect to the fluid is contained in the relation

$$v_f^2 = v_r^2 + (v_{0t} - v_t)^2$$

we obtain for the equations of circular motion of a particle resolved along the radius and tangent:

$$\frac{dv_r}{dt} = \frac{v_t^2}{R}\left(\frac{\rho v_t^2 - \rho_0 v_{0t}^2}{\rho v_t^2}\right) - \frac{\rho C_R A v_f v_r}{2m} \qquad \text{Eq (2-49)}$$

$$\frac{dv_t}{dt} = -\frac{v_t v_r}{R} + \frac{\rho C_R A(v_{0t} - v_t)}{2m} \qquad \text{Eq (2-50)}$$

For streamline motion and for spherical particles the above equations reduce to

$$\frac{dv_r}{dt} = \frac{v_t^2}{R}\left(\frac{\rho v_t^2 - \rho_0 v_{0t}^2}{\rho v_t^2}\right) - \frac{18\mu v_r}{\rho d^2} \qquad \text{Eq (2-51)}$$

$$\frac{dv_t}{dt} = -\frac{v_t v_r}{R} + \frac{18\mu(v_{0t} - v_t)}{\rho d^2} \qquad \text{Eq (2-52)}$$

These equations as well as Eqs (2-49) and (2-50) are difficult to solve, but following Lapple and Shepherd (1940) we may obtain a solution for a special case by assuming that the particle moves at the same speed as the fluid ($v_t = v_t = v_{0t}$) and that the radial acceleration $dv_r/dt = 0$. Thus $v_f = v_r$ and Eq (2-49) reduces to

$$v_r^2 = \frac{2mv_t^2(\rho - \rho_0)}{\rho\rho_0 R C_R A}$$

In the case of streamline motion, for a spherical particle this equation becomes

$$v_r = \frac{v_t^2(\rho - \rho_0)d^2}{18R\mu}$$

which is Stokes' law for a constant settling rate in a centrifugal field

If ω represents the angular velocity, then for the last equation we may write

$$v_r = \frac{R\omega^2(\rho - \rho_0)d^2}{18\mu} \qquad \text{Eq (2-53)}$$

which is identical to Eq (2-45).

Continuous Centrifuge—Let us consider the motion of particles in a long vertical rotating cylinder constantly fed from the bottom with a suspension of fine particles. As the cylinder spins at high speeds, the suspension moves so as to form a layer next to the cylinder wall, leaving an air column of radius R formed along the axis. Dimensional and other details of such a rotating cylinder, constituting what is essentially a Sharples supercentrifuge, are shown in Figure 6.

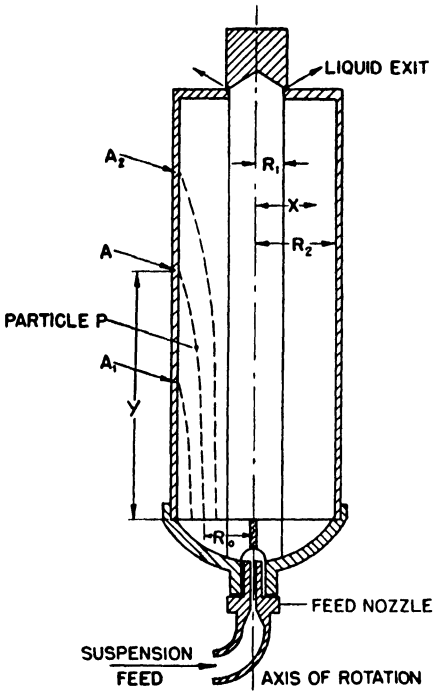


FIGURE 6. SECTION OF A SUPERCENTRIFUGE.

Particles entering at the bottom will in general follow an expanding helical path; the heavier particles are separated near the inlet to the rotating cylinder and the finer ones graded in smallness until the suspension reaches the overflow weir at the top. At this point fines are cut off which have not had time to reach the cylinder wall, or to pass through any outlet especially provided for them along the wall.

If the upward motion of the suspension is very slow and the radial motion of the particles is streamline, the equations governing the behavior of particles in the suspension can be derived by a method due to Hauser and Lynn (1940). Since the particles are distributed over a thickness of $R_2 - R_1$, our problem is somewhat complicated with regard to the initial location of any particle and its absolute path through the concentric ring of suspension. However, while sharp fractionation by size as determined by a measured position in the cylinder (all particles assumed of the same density) cannot be obtained, positions along the

y-axis designated by A_1 and A_2 may well represent the paths of identical particles. For practical purposes we shall regard the size dispersion as limited to narrower extremes, a condition possible to obtain by mechanical means.

The vertical motion of a suspended particle for a volumetric feed of Q is determined by the equation*

$$\frac{dy}{dt} = \frac{KQ}{\pi(R_2^2 - R_1^2)}\left(R_1^2 \ln \frac{R}{R_2} + \frac{R_2 - R^2}{2}\right) \qquad \text{Eq (2-54)}$$

where dimensions correspond to those shown in Figure 6, and K is a function of the bowl and is represented by the relation

$$K = \frac{R_2^2 - R_1^2}{\frac{3}{4}R_1^4 + \frac{R_2^4}{2} - R_1^2R_2^2 - R_2^4 \ln \frac{R_1}{R_2}}$$

The radial motion of the suspended particle is that given by Eq (2-53) that is,

$$\frac{dR}{dt} = \frac{\omega^2 R(\rho - \rho_0)d^2}{18\mu} \qquad \text{Eq (2-55)}$$

where the terms are as defined elsewhere in this chapter.

Dividing Eq (2-54) by Eq (2-55), we obtain

$$\frac{dy}{dR} = \frac{18KQ\mu}{\pi(R_2^2 - R_1^2)d^2\omega^2 R(\rho - \rho_0)}\left(R_1^2 \ln \frac{R}{R_2} + \frac{R_2 - R^2}{2}\right)$$

For constant values of Q, ω, μ, and $\rho - \rho_0$, we have, on integrating between the limits $y = 0$ when $R = R_0$, and $y = y$ when $R = R_2$,

$$y = \frac{18KQ\mu}{\pi(R_2^2 - R_1^2)d^2\omega^2(\rho - \rho_0)}\left[R_2^2 \ln \frac{R_2}{R_0} - \frac{R_1^2}{2}\left(\ln \frac{R_2}{R_0}\right) + \frac{R_0^2 - R_2^2}{4}\right] \qquad \text{Eq (2-56)}$$

In this equation R_0 may be regarded as that position at entrance, measured from the axis of rotation, such that all particles of diameter d will reach the point A on the cylinder. Note that while $y = f(R_0, d)$, it is clear that d is itself not an independent variable, but is also a function of R_0. Hence

$$y = f(R_0)$$

If we denote the bracketed term in Eq (2-56) by χ (called the centrifuge factor), and let

$$\xi = \frac{\pi(R_2^2 - R_1^2)\omega^2(\rho - \rho_0)}{18KQ\mu} \qquad \text{Eq (2-57)}$$

* See Lamb, H., *Hydrodynamics*, 3d ed., Camb. Univ. Press, 1906.

which may be called the modulus of sedimentation. Eq (2-56) then becomes

$$y = \chi/\xi d^2 \qquad \text{Eq (2-58)}$$

Also, since $Q \propto d^2$,

$$\frac{Q_2}{Q_1} = \frac{d_2^2}{d_1^2}$$

That is, for separating a given size at a fixed y-position of the cylinder all that is required is to vary the flow of suspension through the cylinder.

Hauser and Lynn developed the curves showing the relation between y and χ and y and R_0. Thus, for any value of y, we may calculate the diameter of a particle separated:

$$d = \sqrt{\chi/\xi y}$$

Motion of Particles in a Vibrating Field

This problem is encountered in the theory of coagulation of particles in sonic fields.

Consider a stationary sound wave. It is composed of a progressive wave traveling from right to left, and a reflected wave traveling in the reverse direction. Let the amplitude of the wave be A, n its frequency, a any point along the horizontal axis connecting two nodes, λ the wave length, and t any instant of time for which a displacement x is to be measured. Then for the first wave,

$$x = A \cos 2\pi(nt + a/\lambda)$$

and for the reflected wave,

$$x = -A \cos 2\pi(nt - a/\lambda)$$

Hence, combining the two waves, the resultant displacement is

$$\begin{aligned} 2x &= A \cos 2\pi(nt + a/\lambda) - A \cos 2\pi(nt - a/\lambda) \\ &= -2A \sin 2\pi\, nt \cdot \sin 2\pi a/\lambda \end{aligned} \qquad \text{Eq (2-59)}$$

The rate of change of this displacement will then be (in an absolute sense)

$$\frac{dx}{dt} = 2\pi An \cos 2\pi nt \cdot \sin \frac{2\pi a}{\lambda} \qquad \text{Eq (2-60)}$$

Now consider a particle having a diameter d and vibrating with an amplitude equal to that of the stationary wave; let its motion be in accordance with Stokes' law. The resistance to motion will then be

$$\mathfrak{R} = 3\pi\mu d\,\Delta v \qquad \text{Eq (2-61)}$$

where Δv denotes the relative velocity between the wave and the particle.

If dx_p/dt denotes the velocity of a particle of mass m, then from Eq (2-60) and Eq (2-61),

$$m \frac{d^2x_p}{dt^2} = 3\pi\mu d\left(2\pi An \cos 2\pi nt \sin \frac{2\pi a}{\lambda} - \frac{dx_p}{dt}\right)$$

Considering a spherical particle, $m = \pi\rho d_x/6$, and solving for x_p, we obtain

$$x_p = \frac{A \sin (2\pi a/\lambda) \sin (2\pi nt - \phi)}{\sqrt{(\pi\rho d^2 n/9\mu)^2 + 1}} \qquad \text{Eq (2-62)}$$

where ϕ is the phase relation between the particle and the gas element. Hence, combining Eq (2-59) in a positive sense with Eq (2-62), and assuming $\phi = 0$, that is, the motions of gas and particle are in phase

$$\frac{x_p}{x} = \frac{1}{\sqrt{(\pi\rho d^2 n/9\mu)^2 + 1}} \qquad \text{Eq (2-63)}$$

Thus we see that the ratio amplitudes for the particle and the sound wave (or gas element), for constant values of ρ and μ, depend upon the term nd^2, so that for a particle having a diameter d the ratio corresponds to one and only one frequency n. Therefore, we conclude that nd^2 is a critical function determining the behavior of a particle in a vibrating field.

In passing it is of interest to point out that Reynolds' criterion for turbulent motion of vibrating particles has been found by Andrade (1936) to occur when

$$2\pi nAd\mu/\rho_0 > 4.6$$

where the terms are as before defined. Andrade states that this equation has been checked experimentally for spheres above 0.25 mm in diameter, and for frequencies of almost 1200 cycles per sec. He also gives reasons why the expression is valid at higher frequencies including the supersonic, provided the d's are consistent with the equation proposed.

Relative Motion

An interesting situation arises with reference to the application of particle dynamics to fluids and may be stated as follows: Suppose a fluid moves with a velocity v in a duct; the motion of this fluid is determined in accordance with the variables stated in Eq (2-4); now inject into this moving fluid a particle of diameter d having an initial velocity v_{po}. What is the motion of the particle?

We observe that Reynolds' criterion for pipes which distinguishes

turbulent from streamline flow has a value of 2500, whereas the value of the criterion for particles is about 1.0, as may be seen in Figure 3. But we must not be led astray by such criteria. Turbulent motion of a particle in a still fluid is a localized condition, while turbulence in a fluid in motion is general. As we have shown, turbulence denotes a destruction of parallel shearing elements in the fluid so that motion becomes a function of fluid density only. Hence, a particle injected into a turbulent fluid behaves as though its Reynolds number, $\mathbf{R}_p$, is greater than 1.0, even though its true value may actually be less.

In general we may summarize the above in two statements as follows:

(1) If a fluid is in streamline motion, the motion of a particle injected into it may be streamline or turbulent, depending upon whether the *relative* velocity between the fluid and the particle is greater or less than 1.0. Thus,

$$\text{for streamline motion} \quad \mathbf{R}_p = \frac{\Delta v \mu d}{\rho_0} < 1.0$$

$$\text{for turbulent motion} \quad \mathbf{R}_p = \frac{\Delta v \mu d}{\rho_0} > 1.0$$

$$\Delta v = |v_p - v|$$

(2) If the fluid is in turbulent motion, the motion of a particle injected into it will be turbulent, regardless of the relative velocity between the particle and fluid.

Problems

1. A particle of sand ($\rho = 2.65$) having a diameter 0.1 cm is dropped from rest at the surface of a flume of water moving at a velocity of 300 cm per min. Assuming that the flume is of great depth and the velocity of the water is uniform throughout, describe the path of the particle.

2. Show that particles of sand having diameters of 0.1, 0.2, and 0.3 cm, which enter the flume in the above problem at a fixed point, have different paths and that they can be separated according to size at moderate depths below the surface.

3. A sphere 0.7 cm in diameter having a density of 11.0 is projected into air in a horizontal direction from a nozzle with an initial velocity of 1000 ft per sec. Calculate the distance required for the sphere to come to rest.

4. A spherical particle having a diameter of 200 μ and a specific gravity of 2 0 is placed in a suitable horizontal screen. Air is blown through the screen vertically. Assuming a temperature of 20 deg C, calculate the velocity required to just lift the particle.

5. Calculate the settling rates for particles having diameters of 0.01, 0.03, and 0.05 μ with and without Cunningham's correction. Assume settling in air at 20 deg C and a specific gravity relative to air of 8.3

6. Discuss the application of models in the study of the behavior of particles in a fluid stream. See paper by Kenneth D. Nichols, "Observed Effects of Geometric Distortion in Hydraulic Models," *Trans. Am. Soc. Civil Eng.*, **104**, 1488–1509 (1939).

CHAPTER 3

SHAPE AND SIZE-DISTRIBUTION OF PARTICLES

VOLUMES and surface-areas of regular geometrical solids are readily determinable. It is only when we are required to consider irregular particles that our problem becomes difficult. Indeed, no known method is available for defining an irregular particle in geometric terms. We cannot assign to it length, breadth, and thickness in a manner permitting accurate determination of volume or surface-area, as in the case of a regular parallelopiped.

The problem of volume and surface measurement would seem to be even more formidable in the case of an aggregate of irregular particles. However, this is not necessarily true, since statistical methods are available for evaluating the surface-area and volume of aggregates. These methods are based on the measurement of a single dimension, referred to as the "diameter" of the particle. If this diameter is measured in a consistent manner for a large number of particles, the values obtained may be averaged in several ways. The surface-area and volume of the aggregate are based on the number or weight of particles having the "average diameter" thus determined. The theory of dimensions requires that the surface-area be proportional to the square, and the volume proportional to the cube, of the dimension used.

EQUIVALENT PARTICLE-SHAPES

Characteristics of Single Irregular Particles—For any irregular particle we may substitute an equivalent particle: a cube, a sphere, or other geometrical figure having the same volume or surface. For statistical purposes and for reasons of convenience it is customary to reduce a particle to an equivalent sphere, using a single parameter—the distance between the extreme limits of a particle along a fixed direction. Determination of the nominal sphere is as follows: the volume is determined by displacement in water, and this volume is equated to the volume of a hypothetical sphere having an equivalent diameter.

The volume of a sphere is known to be $\pi d^3/6$, hence the nominal or equivalent diameter, d_n, of the particle is that given by the equation

$$d_n = \left(\frac{6 \times \text{volume of particle}}{\pi}\right)^{1/3} \qquad \text{Eq (3-1)}$$

In other words the nominal diameter of any particle is the diameter of a sphere having the same volume. If there are N particles per gram, and the density of the particles is ρ, then

$$d_n = \sqrt[3]{\frac{6}{\pi \rho N}} \qquad \text{Eq (3-2)}$$

where d_n is the nominal diameter of the particles.

An indirect method may also be utilized in obtaining a measure of the particle-diameter. This method may be based either on Stokes' law of a falling body, Eq (2-9), or upon Newton's law, Eq (2-10), depending upon whether the particle is large or small. The procedure for obtaining the average diameter by this method is as follows: The time required for the particle to pass two fixed points in a suitable liquid medium is observed. This permits computation of the average velocity of the particle between the fixed points. Since Newton's or Stokes' laws relate the diameter of the particle to its velocity in any given medium, we may at once obtain a so-called "effective" diameter d_e for the particle. This diameter is the diameter of a sphere which takes the same length of time to fall between two fixed points in the medium as does the irregular particle. The effective diameter has become increasingly important through development of the Odén balance and sedimentation studies, discussed in later chapters of this text. In addition to difficulties in its determination, the effective diameter is influenced not only by the density and size of the particle, but also by its configuration and the condition of its surface. Thus, two particles having the same nominal diameter and density may have different effective diameters.

The "nominal" and "effective" diameters described above are widely used. However, many other diameters are also possible. For example, we may outline the configuration of the particle with a camera lucida. The area of the configuration may then be obtained by a planimeter or other device, and the diameter of a circle having an equivalent area determined. This diameter may be applied to the particle with suitable corrections for magnification. The precise value of this approach is difficult to determine. Again, we may take the average of the longest and shortest dimensions of the particle with a camera lucida, or the geometric mean of these dimensions. In fact the parameters possible for expressing the diameter of a particle are and must be a matter of choice since any suitable consistent procedure of measurement is related to the volume or surface of the particle. An irregular sand particle does not possess a definite diameter; it merely possesses a definite volume and a

definite surface-area which, in accordance with the theory of dimensions vary, respectively, as the cube and as the square of the dimension, thus we are required only to choose some consistent method of selecting a diameter.

Circularity and Sphericity—It is only natural that some measurement of circularity or sphericity should have been evolved for classifying particles as to shape. Wadell (1934*c*) developed measurements for sphericity and circularity which proved to have valuable hydrodynamic characteristics. He defined the degree of true sphericity as

$$S'/S = \psi = \text{degree of true sphericity}$$

where S' is the surface-area of a sphere having the same volume as the particle, and S the actual surface-area of the particle; the limiting value for ψ is 1.0, which occurs when the particle is a sphere. Since measurement of sphericity is exceedingly difficult, Wadell developed a closely correlated measurement which he called circularity, that is,

$$C'/C = \phi = \text{degree of circularity}$$

where C' is the circumference of a circle having the same cross-sectional area as the particle (a camera-lucida image), and C is the actual perimeter of the cross section. Wadell then proceeded to show that when movement of irregular shapes through a fluid medium is considered, the circularity of these figures (as above defined) can be correlated with resistance to flow and Reynolds number. Wadell found that measurement of circularity or sphericity could be obtained by dividing the nominal diameter d_n by the diameter d_{sm} of the smallest sphere circumscribing the particle. By putting $\Psi = d_n/d_{sm}$, Wadell determined experimentally that for values of $d_n/d_{sm} > 0.8$, $\psi = \Psi$ approximately, and that for values of $d_n/d_{sm} < 0.8$, $\psi = \Psi + 0.1$. To a close degree of approximation ψ may be replaced by ϕ.

DETERMINATION OF MEAN DIAMETERS

Average Diameter—The arithmetic average of various particle-diameters is simply the sum of the diameters of the separate particles divided by the number of particles. Let n_1, n_2, n_3, etc., be the number of particles having diameters d_1, d_2, d_3, etc., respectively, then

$$d_{av} = \frac{n_1d_1 + n_2d_2 + \ldots + n_nd_n}{n_1 + n_2 + \ldots + n_n} = \frac{\Sigma(nd)}{\Sigma n} \qquad \text{Eq (3-3)}$$

where Σ denotes the sum of the terms indicated. The method for computing the arithmetic average of a group of measured particles is shown

in Table 4. The computation method is self-evident. The size-groups are in intervals of one μ (Column 1); the average mid-size of the group is given in the second column; the third column gives the number of particles within the size-groups shown in the first column.

TABLE 4—METHOD OF DETERMINING THE AVERAGE DIAMETER OF 245 PARTICLES MEASURED BY MEANS OF AN OPTICAL MICROMETER

Size-group (microns) (1)	Mean of size-group (d) (microns) (2)	Number of partices in each size (3)	Value of nd (2) × (3)
0- 4.9	2.5	2	5.0
5- 9.9	7.5	10	75 0
10-14.9	12.5	56	700.0
15-19.9	17.5	82	1435.0
20-24.9	22.5	35	787.5
25-29.9	27.5	22	605.0
30-34.9	32.5	26	845.0
35-39.9	37.5	7	262.5
40-44.9	42.5	5	212.5
		245 (Σn)	4927.5 (Σnd)

$$d_{av} = \frac{4927.5}{245} = 20\ 1\ \mu$$

Geometric Mean—The geometric mean d_g is the nth root of the product of the n particles measured

$$d_g = \sqrt[n]{d_1 d_2 d_3 \ldots . d_n}$$

or in logarithmic form to the base 10

$$\log d_g = \frac{n_1 \log d_1 + n_2 \log d_2 + \ldots . + n_n \log d_n}{n_1 + n_2 + \ldots + n_n} = \frac{\Sigma\ (n \log d)}{\Sigma n} \qquad \text{Eq (3-4)}$$

where the n's are as previously defined. The geometric mean is always less than the arithmetic mean. The method for computing the logarithmic mean is similar to that used for the arithmetic mean. The data given in Columns (1) to (3) of Table 4 are again presented in Table 5, for computing the geometric mean.

Harmonic Mean—Another average of some usefulness is the harmonic mean. This mean is the reciprocal of the diameters measured, and is given by the equation

$$\frac{n}{d_h} = \frac{1}{d_1} + \frac{1}{d_2} + \ldots . + \frac{1}{d_n}$$

where d_h is called the harmonic mean diameter. If there are n_1, n_2, . . . n_n particles corresponding to the diameters d_1, d_2, ... d_n, then the equation takes the general form

$$\frac{1}{d_h} = \frac{1}{\Sigma n} \cdot \Sigma \frac{n}{d} \qquad \text{Eq (3-5)}$$

Mean values determined by this equation are always lower than the geometric mean (and hence the arithmetic mean) of the same quantities.

TABLE 5—METHOD OF DETERMINING THE GEOMETRIC MEAN DIAMETER OF 245 PARTICLES MEASURED BY MEANS OF AN OPTICAL MICROMETER

Size-group (microns) (1)	Mean of size-group (d) (microns) (2)	Number of particles in each size (3)	Log d (4)	n log d (3) × (4)
0– 4.9	2.5	2	0.3979	0.80
5– 9.9	7.5	10	0.8751	8.75
10–14.9	12.5	56	1.0969	61.43
15–19.9	17.5	82	1.2430	101.93
20–24.9	22.5	35	1.3522	47.33
25–29.9	27.5	22	1.4393	31.66
30–34.9	32.5	26	1.5119	39.31
35–39.9	37.5	7	1.5740	11.02
40–44.9	42.5	5	1.6385	8.10
		245		310.42

$$\log d_g = \frac{310.42}{245} = 1\ 267$$

$$\text{antilog } 1.267 = d_g = 18.5\ \mu$$

The harmonic mean is related to the specific surface. Thus, if N is the number of particles per unit-weight considered as spheres, ρ the density of the particles, and d their mean diameter,

$$\frac{\pi}{6} d^3 N \rho = 1$$

and since the specific surface (surface of the particles per unit-weight) is $N\pi d^2 = S_w$, substituting this value of S_w in the above equation we have

$$S_w = \frac{6}{\rho} \cdot \frac{1}{d} \qquad \text{Eq (3-6)}$$

Median Diameter—The median is that diameter for which 50 percent of the particles measured are less than stated size. The median may be determined approximately by reference to Column 3 of Table 4. It will be noted that 123 particles are smaller than 20 μ, and that the

precise 50 percent value is between 15.0 and 20 μ. The exact value may be determined by summation as shown in Table 6, or by plotting the data as in Figure 7. The 50 percent value is then easily interpolated,

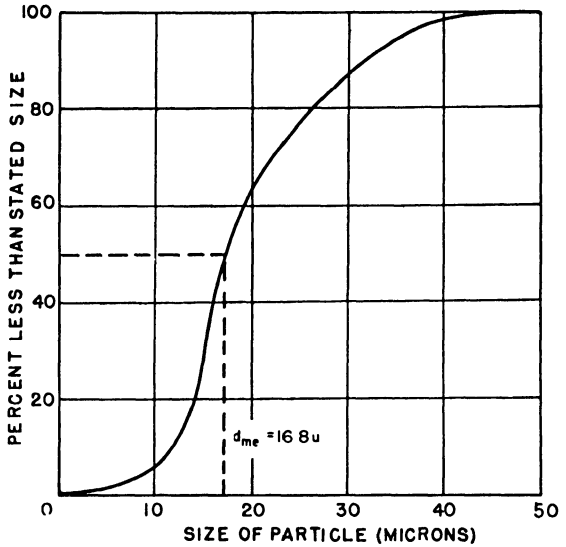


FIGURE 7. DETERMINATION OF MEDIAN SIZE.

and is 16.8 μ, which compares favorably with the value obtained in Table 6.

TABLE 6—METHOD OF SUMMATION FOR DETERMINING THE MEDIAN DIAMETER OF 245 PARTICLES MEASURED BY MEANS OF AN OPTICAL MICROMETER

Size-group (microns)	Number of particles in each size	Number less than maximum size of groups	Percent of particles in each size-group	Percent less than maximum size of group
0– 4.9	2	2	0.8	0.8
5– 9.9	10	12	4.1	4.9
10–14.9	56	68	22.9	27.8
15–19.9	82	150	33.5	61.3
20–24.9	35	185	14.2	75.5
25–29.9	22	207	9.0	84.5
30–34.9	26	233	10.6	95.1
35–39.9	7	240	3.0	98.1
40–44.9	5	245	1.9	100 0
	245		100.0	

Interpolating:

$$\frac{61.3 - 50}{61.3} = \frac{20 - d_{me}}{20}, \text{ and } d_{me} = 16.3\ \mu$$

Statistical Diameters—In calculating the average size of measured particles attention should always be given to their ultimate use. The arithmetic or geometric mean and the median have no physical significance. It is desirable to devise other averages relating to definite physical properties, such as surface or volume. For example, if the surface of an aggregate of particles is to be determined, more weight should be given to finer particles than would be the case if the arithmetic-average size were used. The surface of a unit-weight of material increases as the particle-size decreases; hence the surface offered by each size-group must be given due weight. Similarly, if the volume of particles is a primary consideration, the average diameter must be based on the volume of particles in each size-group. The use of other kinds of average diameters for the computation of a specific property is of great practical importance in the study of pigments, fillers, and other industrial materials.

Perrott and Kinney (1923) and Henry Green (1927) stressed the importance of the above factors in determining average size, and gave suitable formulas for this purpose. Their formulas follow:

$$\text{mean diameter (length)} = d_l = \frac{\Sigma nd^2}{\Sigma nd} \qquad \text{Eq (3-7)}$$

$$\text{mean volume diameter} = d_v = \sqrt[3]{\frac{\Sigma nd^3}{\Sigma n}} \qquad \text{Eq (3-8)}$$

$$\text{mean surface diameter} = d_s = \sqrt{\frac{\Sigma nd^2}{\Sigma n}} \qquad \text{Eq (3-9)}$$

$$\text{mean volume-surface diameter} = d_{vs} = \frac{\Sigma nd^3}{\Sigma nd^2} \qquad \text{Eq (3-10)}$$

$$\text{weight mean diameter} = d_w = \frac{\Sigma nd^4}{\Sigma nd^3} \qquad \text{Eq (3-11)}$$

Eq (3-7) represents the summation of the surface-areas divided by the summation of the diameters. It gives a mean based on the surface observed, and the volume or total surface of the particle does not enter into the calculation. As an average it is comparable to the arithmetic and geometric means.

Eq (3-8) is based on the average volume and may be said to be that diameter whose corresponding volume divided into the total volume gives the total number of particles. Thus, if d is the particle diameter, and V = the total volume, then

$$V = \Sigma nd^3$$

Now d_v^3 is a measure of the volume of an average particle, and Σn is the total number of particles. Hence

$$\Sigma n = \frac{\Sigma n d^3}{d_v^3}$$

$$d_v = \sqrt[3]{\frac{\Sigma n d^3}{\Sigma n}}$$

The value of d_v thus determined is larger than the average diameter computed with Eq (3-7).

The diameter d_s of a hypothetical particle having an average surface-area is given by Eq (3-10). The total surface is measured by $\Sigma n d^2$ where, as before, n is the number of particles having a diameter d. Since Σn is the total number of particles, then

$$d_s = \sqrt{\frac{\Sigma n d^2}{\Sigma n}}$$

On the other hand, if the total surface of a unit-weight of the material (*i.e.*, specific surface) is desired, we may utilize the average diameter as calculated by Eq (3-10). If the particles are regarded as spheres or rectangular parallelopipeds, then the diameter may be determined by means of Eq (3-6).

$$d = \frac{6}{S_w \rho}$$

If y is the weight percent of particles of diameter d and specific surface S_w'

$$y = \frac{nd^3}{\Sigma n d^3}$$

$$S_w = \Sigma y S_w' = \frac{\Sigma n d^3 S_w'}{\Sigma n d^3}$$

If d_{vs} is the average diameter, as in Eq (3-10)

$$d_{vs} = \frac{6}{S_w \rho} = \frac{6}{\rho \Sigma y S_w'} = \frac{6}{\rho}\left(\frac{1}{\frac{\Sigma n d^3 S_w'}{\Sigma n d^3}}\right)$$

where n is the number of particles of diameter d. Hence, substituting for S_w' its value $6/d\rho$, we obtain

$$d_{vs} = \frac{6}{\rho}\left(\frac{1}{\frac{\Sigma n d^3 \cdot 6/d\rho}{\Sigma n d^3}}\right) = \frac{\Sigma n d^3}{\Sigma n d^2}$$

Thus, since the average diameter is a linear dimension independent of weight and specific gravity, the average diameter calculated by Eq (3-10) may be considered as based on specific surface per unit-volume. The average diameter obtained by this method, in the opinion of Perrott and Kinney, represents the diameter which should be used in the computation of specific volume-surface. Thus, if we are concerned with estimating the particle-surface of a given volume of particulate material we should use the average diameter obtained by application of Eq (3-10).

The formula given by Eq (3-11) is based on the surface of a unit-weight of the particles and gives an average diameter larger than those calculated by other methods. The basis of the equation follows: if y is the percent of particles having a diameter d, figured as a percentage of the total volume, we have

$$d_w = \Sigma y d$$

$$y = \frac{nd^3}{\Sigma nd^3}$$

so that, substituting the value of y,

$$d_w = \Sigma\left(\frac{nd^3}{\Sigma nd^3} \cdot d\right) = \frac{\Sigma nd^4}{\Sigma nd^3}$$

In the computation, the fourth moment of each diameter is multiplied by the number of particles corresponding to it, and the cube of each diameter is multiplied by the number of particles corresponding to it.

The computations entailed in determining the average statistical diameters just discussed are given in Table 7. The computations are for a distribution of sizes given in Table 4. It is obvious from this table that values of d_{av} are about a third smaller than the next largest statistical diameter d_w. The use of a correct diameter giving proper weight to the physical property to be measured is apparent.

With uniform particles the difference between the statistical diameters diminishes. When all particles are the same size, the statistical diameters are the same.

Correction for Thickness of Particles—Weigel (1924) indicated certain corrections which must be applied for flat or tubular material. He assumed that the particles under consideration were cubes having a side d and surface $6d^2$. If the cube is split one way, the surface becomes $6d^2 + 2d^2$, and if split three ways the surface becomes $6d^2 + 6d^2$, or $12d^2$. Thus, if the cube is cut by a number of planes parallel to the observed side, the increased surface is equal to twice the number of resulting parallelopipeds less one, multiplied by the square of the diameter. If the average thickness of the particles is expressed as a percentage of

TABLE 7—COMPUTATION OF STATISTICAL DIAMETERS OF A DISTRIBUTION OF 245 PARTICLES MEASURED BY MEANS OF AN OPTICAL MICROMETER

Mean of size group (d) (microns)	Number of particles in each size (n)	(nd)	(nd^2)	(nd^3)	(nd^4)
2.5	2	5.0	12.5	30.3	75.6
7.5	10	75.0	562.5	4,218.8	31,640.6
12.5	56	700.0	8,750.0	109,375.0	1,367,187.5
17.5	82	1435.0	25,112.5	439,488.8	7,691,053.1
22.5	35	787.5	17,718.8	398,671.9	8,970,117.2
27.5	22	605.0	16,647.5	457,806.3	12,589,672.0
32.5	26	845.0	27,462.5	892,531.3	29,007,265.6
37.5	7	262.5	9,843.8	369,140.6	13,842,773.4
42.5	5	212.5	9,031.3	383,828.1	16,312,696.2
	245 (Σn)	5127.5 (Σnd)	115,141.4 (Σnd^2)	3,055,091.1 (Σnd^3)	89,782,481.2 (Σnd^4)

$d_{av} = \frac{\Sigma nd}{\Sigma n} = 20.9\ \mu$ Eq (3-3)

$d_s = \sqrt{\frac{\Sigma nd^2}{\Sigma n}} = 21.7\ \mu$ Eq (3-9)

$d_l = \frac{\Sigma nd^2}{\Sigma nd} = 22.2\ \mu$ Eq (3-7)

$d_{vs} = \frac{\Sigma nd^3}{\Sigma nd^2} = 26.5\ \mu$ Eq (3-10)

$d_v = \sqrt[3]{\frac{\Sigma nd^3}{\Sigma n}} = 23.0\ \mu$ Eq (3-8)

$d_w = \frac{\Sigma nd^4}{\Sigma nd^3} = 29.4\ \mu$ Eq (3-11)

the observed diameter, the cube so subdivided will have a new surface, as follows:

$$\text{surface} = 6d^2 + 2\left(\frac{100}{\delta} - 1\right)d^2$$

where δ is the thickness of the particles in percent of the observed diameter. If $100/\delta$ be represented by c

$$\text{surface} = d^2\,(4 + 2c)$$

and the ratio of the new surface to the old is $(4 + 2c)/6$. The new specific surface then becomes

$$S_w = \frac{4 + 2c}{6} \cdot \frac{6}{d\rho}$$

and the corrected diameter, d_{cor} is

$$d_{cor} = \frac{6}{S_w\rho} = \frac{6}{\frac{6\rho(4 + 2c)}{6d\rho}} = \frac{6d}{4 + 2c} \qquad \text{Eq (3-12)}$$

In practice the average diameter is calculated in the usual way (see Table 4). Then the thickness of enough particles to provide a fair

average from the smallest to the largest is measured. The thickness of each size is calculated as a percentage for that size, and the resulting percentages are averaged and used in Eq (3-12).

This method of taking into account the thickness of particles can only be regarded as approximate. It depends on careful focusing from the top to the bottom of the particle. If the particles are fairly large, good results may be obtained, but if they are small and exceedingly thin the results may be seriously in error. Thus far no satisfactory method has been developed for the study of flat or fibrous particles. Indeed, it is questionable whether the methods for measuring particles outlined above can be applied to conditions other than those in which the particles are more or less uniform as to sphericity, or to length, breadth, and width.

SIZE-FREQUENCY CURVES

When the percent frequencies of various particles are plotted against the mean of size-groups, we obtain what is called the "size-frequency" curve. Figure 8 shows the size-frequency curve for the particular example given in previous tables. This curve was obtained by plotting the frequencies given in the fourth column of Table 6 against the mid-group diameters given in the second column of Table 4. It will be seen from the figure that the curve is triangular in shape and somewhat asymmetrical. The arithmetic mean and median are indicated. Note that the arithmetic mean divides the curve into two areas of equal size.

It is evident that the median and average particle-size are influenced by the shape of the size-frequency curve. Two widely differing types of distribution may have the same average size or the same median. In fact there is an infinite variety of size-frequency curves having a given mean or median. Thus, it is clear that parameters other than the median or average are necessary to define a size-distribution.*

It is to be noted that the curves of Figures 7 and 8 are closely related. The summation curve of Figure 7 is the sum of frequencies below a stated diameter d. Mathematically, the relation may be expressed as follows:

$$F(d) = \frac{dq}{d(d)}$$

where $F(d)$ is a function defining the frequency distribution of particles having diameters d_1, d_2, d_3, etc., the successive differences being infinitesimal. The term q is defined as the sum of particles greater or smaller than a stated size d, for which the proper sign is applied. If $q(d)$ is the

* See Chapter 23.

function defining the summation curve, then between any two particles with diameters d_1 and d_2 we have*

$$q(d_1) - q(d_2) = \int_{d_2}^{d_1} \frac{dq}{d(d)}\, d(d) = \int_{d_2}^{d_1} F(d)\, d(d) \qquad \text{Eq (3-13)}$$

Extending these definitions we have further that $F(d)$ denotes the ordinates of the distribution function, and $F(d)\cdot d(d)$ the number of particles in the interval $d_2 - d_1 = \Delta d$. If the data include a wide range of sizes,

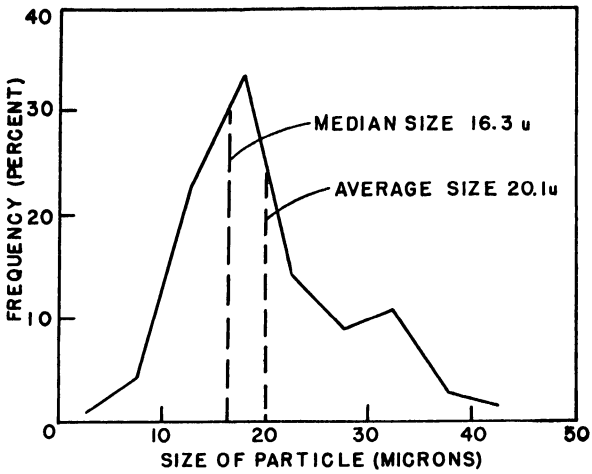


FIGURE 8. SIZE-FREQUENCY DISTRIBUTION FOR EXAMPLES IN TEXT.

it is often best to plot frequency against the logarithm of size. The term d now becomes ln d and

$$dq = F(d)\cdot d(d) = \frac{1}{d} F(d)\cdot d\cdot d(d) = d(\ln d)\cdot d\cdot F(d)$$

so that

$$\frac{dq}{d(\ln d)} = d\cdot F(d) \qquad \text{Eq (3-14)}$$

It is usual to plot $d\cdot F(d)$ against ln d, instead of $F(d)$ against d, for distributions extending over a wide range of sizes.

No restrictions are placed on the form of $F(d)$, but for most particle-

* In order to avoid confusion between the *differential* of d and the *diameter* d, the latter is enclosed in parentheses.

size determinations the size-frequency curves obtained follow the probability law. The usual normal-probability equation applies only to distributions which are symmetrical about a vertical axis. Since most

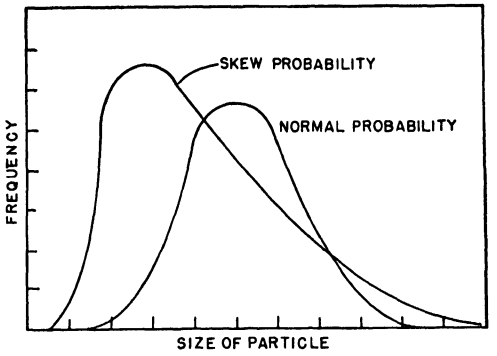


FIGURE 9. NORMAL- AND SKEWED-PROBABILITY FUNCTIONS.

distributions are asymmetrical or "skewed," the normal law does not apply. Normal and asymmetric probability curves are shown in Figure 9. Fortunately, in most instances the asymmetrical frequency curves can be made symmetrical and thus follow the normal-probability law, if the logarithms of the sizes are substituted for the sizes. This means simply that if frequencies are plotted arithmetically, and the corresponding sizes logarithmically, the resulting curve is symmetrical. The importance of this fact was shown by Hatch and Choate (1929) who adapted the normal-probability curve and utilized its parameters to define the most common types of particle-size distributions. The data plotted in Figure 8 are shown replotted in logarithmic grid in Figure 10. It is seen that this method of plotting results in a symmetrical curve.

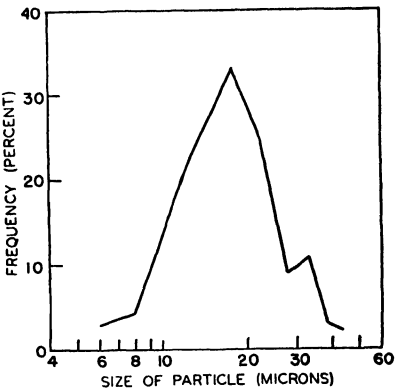


FIGURE 10. LOGARITHMIC PLOT OF DATA SHOWN IN FIGURE 8.

The equation of the normal-probability curve, as applied to size-frequency distributions, is

$$F(d) = n = \frac{\Sigma n}{\sigma\sqrt{2\pi}} \cdot \exp\left[- \frac{(d - d_{av})^2}{2\sigma^2} \right] \qquad \text{Eq (3-15)}$$

where $F(d)$ is here defined as the frequency of observations of diameter d, Σn the total number of observations, d_{av} the arithmetic average of observations (Eq 3-3) and σ the standard deviation, given by the following expression

$$\sigma = \sqrt{\frac{\Sigma[n(d - d_{av})^2]}{\Sigma n}} \qquad \text{Eq (3-16)}$$

The constants d_{av} and σ therefore completely define the frequency-distribution of a series of observations.

If we now consider an asymmetrical distribution, and for d_{av} and σ substitute the logarithms of their measurements, we have

$$F(d) = \frac{\Sigma n}{\ln\ \sigma_g\sqrt{2\pi}} \cdot \exp\left[- \frac{(\ln\ d - \ln\ d_g)^2}{2 \ln^2 \sigma_g} \right] \qquad \text{Eq (3-17)}$$

where d_g now refers to the geometric mean, Eq (3-4), and σ_g is obtained from the equation

$$\log \sigma_g = \sqrt{\frac{\Sigma[n(\log d - \log d_g)^2]}{\Sigma n}} \qquad \text{Eq (3-18)}$$

The terms d_g and σ_g are called geometric mean diameter and geometric standard deviation, respectively.

One other parameter regarding size-distribution may be mentioned, namely the mode, or the value of d for which the size-frequency curve is a maximum. For example, in Figure 8, this value occurs for $d = 17.5$. If the distribution were symmetrical, that is, if the frequencies were evenly distributed about a line passing vertically through the mode (see Figure 9), the mode, mean, and median of the size-frequency distribution would be the same. If the distribution were moderately asymmetrical or skewed, then the relation between these averages would be given by the equation (see Yule, 1927)

$$\text{mode} = \text{mean} - 3(\text{mean} - \text{median})$$

This equation applies to a smooth curve through the points, and is therefore more accurate when applied to data obtained with small size-intervals.

Other Size-Frequency Distributions—Since it is conceivable that there is a smallest particle below which no other particles are found, and also a largest particle above which no larger particles are found, the size-distribution curve is limited at two extremes. It is probable that a most general type of particle-distribution occurring in nature is a

combination of what may be termed the hyperbolic and the skew-probability distributions denoted by *A* and *B* in Figure 11. In general, a random sample of particles measured along a single direction by means

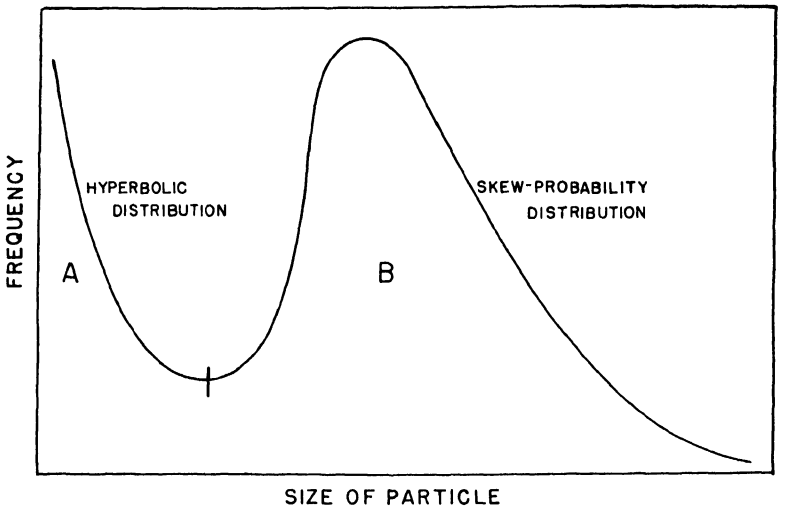


FIGURE 11. GENERAL TYPE OF SIZE-FREQUENCY DISTRIBUTION.

of a microscope follows a probability distribution. There will be one size which occurs more frequently than any other, and as we proceed to sizes right or left of this maximum the frequencies decrease.

It will be shown in another chapter that a possible type of distribution generated by certain processes is of the hyperbolic type (see Chapter 22). The number of particles increases with decreasing size until we reach the dimensions of molecules, atoms, and electrons. There appears to be no definite limit to which material may be subdivided As the particles increase in size they become relatively fewer in number. The hyperbolic curve is a close approximation of the subdivision of matter in the universe.

PLOTTING OF SIZE-FREQUENCY DATA

Hazen (1914) developed a special grid for plotting size-frequency data so that the resulting curve is a straight line. This grid consists of a system of coordinates based on a probability scale, that is, a scale based on the probability integral. On this scale, which may be considered as the ordinate of the system, are plotted the cumulative percent oversize of

undersize. The ordinates consist of either an arithmetic or logarithmic scale on which are plotted the diameters of the particles measured. Thus the coordinates of a point $P(x, y)$ denote the total percentage of the number or weight of particles (y) above or below the diameter (x). Whether the ordinates are plotted as larger or smaller than a stated size is a matter of choice.

If the size-frequency data fit Eq (3-15) and are plotted on arithmetic-probability grid, the resulting summation curve is a straight line. Similarly, if the curve is asymmetric so that Eq (3-17) applies, then the data plot as a straight line on log-probability grid. As an example of this method the data of Figure 7 are shown replotted in Figure 12.

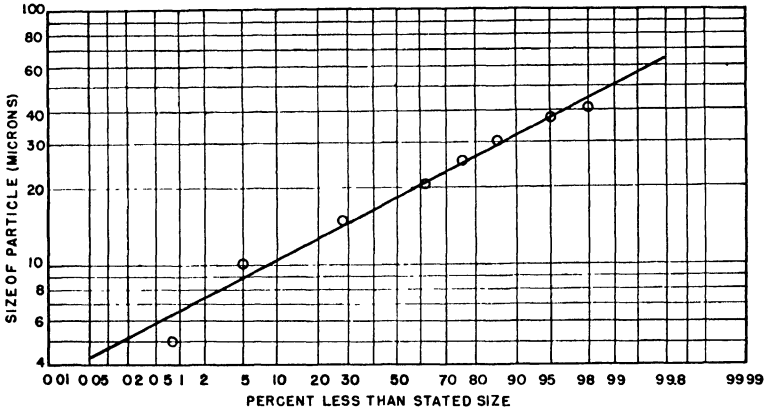


FIGURE 12. LOG-PROBABILITY RE-PLOT OF DATA ILLUSTRATED IN FIGURE 7.

Now, it has been found that most particles of a sample measured at random obey either of the two probability laws given. However, one matter of some importance must be held in mind. When plots are made on either probability grid, the distributions must be asymptotic on both extremes. For practical reasons the particles measured have a smallest particle and a largest particle. Therefore, the distribution is not asymptotic and plots on probability grid often depart from the straight line at the extremes. This would be of some concern if it were not for the fact that the areas extending from the extremes to infinity are negligible compared with the area contained under the distribution curve between the largest and smallest particles measured. Although Eqs (3-15) and (3-17) apply only to asymptotic distributions, they represent fairly well most conditions encountered in practice.

The amount of labor involved in computing mean diameter and standard deviation is reduced by using probability grids. The median

value for both types of grid is obtained by reading the size corresponding to the value of 50 percent on the probability scale. In the case of an arithmetic-probability plot, the mean is a simple arithmetic average, d_{av}, while from a log-probability plot the 50 percent size is the geometric mean, d_g. The latter is also the median of the distribution if the log of the variate d normalizes the distribution.

The standard deviations may also be obtained easily from the grids based on relationships derivable from the probability equation. Thus for arithmetic-probability plots

$$\begin{aligned}\sigma &= 84.13 \text{ percent size} - 50 \text{ percent size} \\ &= 50 \text{ percent size} - 15.87 \text{ percent size}\end{aligned}$$

and

$$\sigma_g = \frac{84.13 \text{ percent size}}{50 \text{ percent size}} = \frac{50 \text{ percent size}}{15.87 \text{ percent size}}$$

UNIFORMITY

The standard deviation of a distribution of particle-sizes is a measure of particle-uniformity. This measure does not pertain to the shape of the particles themselves, but only to the fact that the particles are closely or openly sized; that is, whether the particles are more or less of the same size, or have widely different sizes. It may be shown by statistical theory that approximately 68 percent of the observations fall between $d_{av} \pm \sigma$, and approximately 95 percent of the observations fall between $d_{av} \pm 2\sigma$.

This method of obtaining the measure of uniformity is generally useful when particles are measured with an ocular micrometer. While there are many different methods of expressing uniformity, we shall confine ourselves to the three most widely used.

(I) *Uniformity Coefficient*—The uniformity coefficient is used in sieving aggregates, and is determined by dividing the size opening which will pass 60 percent of a sample being screened by the size which will just pass 10 percent. Both the 60 percent size and the 10 percent size may be estimated by plotting a summation curve. When the uniformity coefficient is low, particles are more or less uniform in size. On the other hand, when the uniformity coefficient is high the sizes are widely distributed. The "effective" size,* which has been defined as the opening which will just pass 10 percent of the particles (by weight), is often taken in the same sense as the median or average. A great amount of experience has accumulated on the use of effective size since the idea

* Not to be confused with the effective diameter determined by Stokes' law discussed earlier.

was developed by Hazen (1892). It is generally applied to water-filtration practice.

(II) *Measure of Asymmetry*—A measure of the asymmetry of a distribution (its departure from the normal-probability curve) is widely used among statisticians. This measure is as follows:

$$\text{asymmetry} = \frac{\text{mean} - \text{mode}}{\text{standard deviation}}$$

Thus, when the mean and mode coincide, asymmetry is zero and the curves are symmetrical. There is no upper limit to the ratio, but for most particle-size distributions ordinarily encountered in practice the value does not exceed unity. The mode may be replaced by its approximate value, *mean* − 3(*mean* − *median*), if desired.

(III) *Kramer's Modulus*—Kramer (1935), in studies on tractive forces involved in model channels, developed an expression for measuring uniformity which depends on the ratio of the areas above and below the 50 percent line, as indicated in Figure 13. Kramer's uniformity modulus is the ratio of the area below to the area above this line. The modulus possesses the following characteristics: (1) for a uniform grain size its value is 1; (2) for a uniform distribution $^1/_3$; (3) the addition of fine or coarse materials to a given mixture tends to reduce its value.

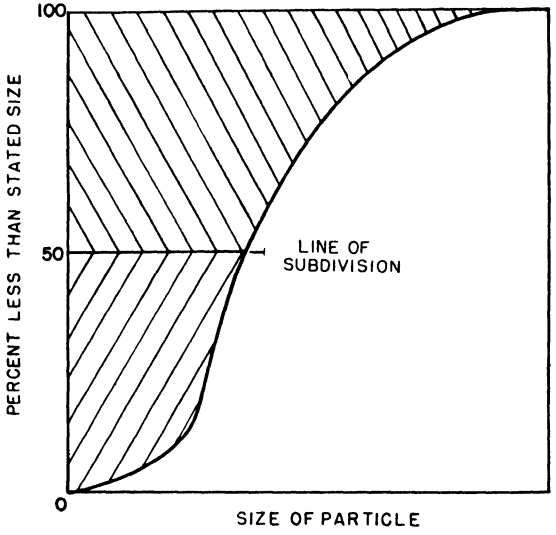


FIGURE 13. SCHEMATIC-SUMMATION CURVE SHOWING METHOD OF DETERMINING KRAMER'S MODULUS.

The modulus is also said to be a measure of the voids ratio. It has a maximum value of 1 for uniform material, and smaller values if finer or coarser materials are added. When the voids between the large particles are filled with smaller particles the modulus decreases. However, it is hard to conceive that Kramer's modulus is any more suited to such representation than the standard deviation.

For any general curve of the type shown in Figure 13 and when the percentages less than stated size are expressed on a weight basis, we have

$$\text{mean grain diameter} = \frac{\sum_{w=0}^{w=100} d\,\Delta w}{\sum_{w=0}^{w=100} w}$$

and

$$\text{Kramer's modulus} = K = \frac{\sum_{w=0}^{w=50} d\,\Delta w}{\sum_{w=50}^{w=100} d\,\Delta w}$$

PARAMETERS FOR DETERMINATION OF AVERAGE PARTICLE-SIZE

Hatch-Choate Equations—It was shown that the usual types of size-frequency distributions may be summed and plotted on log-probability grid. The mathematical relationships applying to such distributions were shown to be given by Eqs (3-15) and (3-17). Furthermore, we indicated how an average diameter of a distribution may be obtained, and demonstrated that there are many other kinds of averages to suit the different physical properties to be measured. However, following Hatch and Choate (1929) we now show how Eqs (3-3), (3-8), and (3-9) may be combined with Eq (3-17) (for an asymmetrical distribution) to obtain diameters indicative of (*a*) a theoretical particle having an average linear dimension, (*b*) a theoretical particle having an average surface area, and (*c*) a theoretical particle having an average volume; thus

$$d_{av} = \frac{\Sigma(nd)}{\Sigma n} = \frac{\Sigma n}{\Sigma n \cdot \ln\ \sigma_g\sqrt{2\pi}} \int_0^\infty d \cdot \exp\left[-\frac{(\ln d - \ln d_g)^2}{2 \ln^2 \sigma_g}\right] d \ln d$$

Eq (3-19)

$$d_s^2 = \frac{\Sigma(nd^2)}{\Sigma n} = \frac{\Sigma n}{\Sigma n \cdot \ln\ \sigma_g\sqrt{2\pi}} \int_0^\infty d^2 \cdot \exp\left[-\frac{(\ln d - \ln d_g)^2}{2 \ln^2 \sigma_g}\right] d \ln d$$

Eq (3-20)

$$d_v^3 = \frac{\Sigma(nd^3)}{\Sigma n} = \frac{\Sigma n}{\Sigma n \cdot \ln\ \sigma_g\sqrt{2\pi}} \int_0^\infty d^3 \cdot \exp\left[-\frac{(\ln\ d\ -\ \ln\ d_g)^2}{2\ \ln^2\ \sigma_g}\right] d\ \ln\ d \qquad \text{Eq (3-21)}$$

These expressions may be integrated and the following equations obtained:

$$\frac{\Sigma(nd)}{\Sigma n} = \exp\ (\ln\ d_g + 0.5\ \ln^2\ \sigma_g) \qquad \text{Eq (3-22)}$$

$$\frac{\Sigma(nd^2)}{\Sigma n} = \exp\ (2\ \ln\ d_g + 2\ \ln^2\ \sigma_g) \qquad \text{Eq (3-23)}$$

$$\frac{\Sigma(nd^3)}{\Sigma n} = \exp\ (3\ \ln\ d_g + 4.5\ \ln^2\ \sigma_g) \qquad \text{Eq (3-24)}$$

Reduced to log base 10, we have

$$\log\ d_{av} = \log\ d_g + 1.151\ \log^2\ \sigma_g \qquad \text{Eq (3-25)}$$

$$\log\ d_s^2 = \log\ d_g^2 + 4.605\ \log^2\ \sigma_g \qquad \text{Eq (3-26)}$$

$$\log\ d_v^3 = \log\ d_g^3 + 10.362\ \log^2\ \sigma_g \qquad \text{Eq (3-27)}$$

From these equations we see that once the geometric-mean size and the standard deviation are determined, we are in position to compute the diameter of a particle having an average surface area and an average volume.* Hatch and Choate tested these equations and found general agreement with laboratory measurements.

We note in passing that Eq (3-23) and Eq (3-24) may be combined, giving

$$\frac{\Sigma(nd^3)}{\Sigma(nd^2)} = d_{vs} = \exp\ (\log\ d_g + 2.5\ \log^2\ \sigma_g) \qquad \text{Eq (3-28)}$$

Taking logs to the base 10

$$\log\ d_{vs} = \log\ d_g + 5.757\ \log^2\ \sigma_g \qquad \text{Eq (3-29)}$$

which is a kind of specific surface diameter. It may also be shown that the harmonic mean

$$d_h = \left(\frac{1}{\Sigma n} \cdot \Sigma\ \frac{n}{d}\right)^{-1} = \log\ d_g\ -\ 1.151\ \log^2\ \sigma_g \qquad \text{Eq (3-30)}$$

Roller's Equations—Roller (1937*b*) recently developed a law of size-

* If the frequency curve follows the normal-probability law, Eq (3-15) may be developed in the same manner as for the asymmetric case. The results are more complicated. Thus

$$d_s^2 = d_{av}^2 + \sigma^2 + 4\sigma d_{av}/\sqrt{2\pi}$$

$$d_v^3 = d_{av}^3 + 6d_{av}^2\sigma/\sqrt{2\pi} + 3\sigma^2 d_{av} + 4\sigma^3/\sqrt{2\pi}$$

distribution which relates the weight percent and the size of finely divided materials. He developed the following equation:

$$w = a\sqrt{d}\cdot\exp\,(-\,b/d) \qquad \text{Eq (3-31)}$$

where w is the weight percent of all material having diameters less than d, and a and b are constants. Eq (3-31) may be differentiated with respect to d to obtain the frequencies in weight percent per unit of diameter.

$$\frac{dw}{d(d)} = a\left(\frac{1}{2\sqrt{d}} + \frac{b}{\sqrt{d^3}}\right)\cdot\exp\,(-\,b/d) \qquad \text{Eq (3-32)}$$

The equation may also be written as a number-frequency law by noting the relation between weight and number, as follows:

$$dw = \rho d^3\cdot dn$$

where n is the number of particles, ρ the density, and d the diameter of the particles By substituting dw from this equation into Eq (3-32) the frequency distribution is obtained.

$$\frac{dn}{d(d)} = \frac{a}{\rho}\left(\frac{1}{2\sqrt{d^7}} + \frac{b}{\sqrt{d^9}}\right)\cdot\exp\,(-\,b/d) \qquad \text{Eq (3-33)}$$

The equation for mass distribution, Eq (3-32), represents a bell-shaped curve which has a maximum at a diameter $d = d_m$. The value of d_m may be determined analytically by differentiating Eq (3-32) and setting the derivative equal to 0. The value of d_m so determined is $d_m = 0.828b$. Eq (3-33) may also be differentiated and set equal to 0, and we find that $d_m' = 0.228b$. The values of the constants a and b may be determined graphically from the experimental data. Thus, Eq (3-31) may be written in the form

$$\log\frac{w}{\sqrt{d}} = \log a - \frac{1}{2.303}\cdot\frac{b}{d} \qquad \text{Eq (3-34)}$$

If $w/\sqrt{d}$ is plotted on a log-scale against $1/d$, the resulting curve is called the characteristic line. The intercept of the curve (assuming it is a straight line on semi-log grid) is log a, while the slope is equal to $-\,b/2.303$.

Roller extended his basic equation to determine the specific surface and the number of particles per gram. He has shown that the specific surface is given by the equation

$$S_w = \frac{1064}{\rho}\cdot\frac{a}{\sqrt{b}}\left[1 - 0.56\left(\frac{\left(\frac{a}{100}\right)^2 b}{1 + 2\left(\frac{a}{100}\right)^2 b}\right)^{1/2}\right] \qquad \text{Eq (3-35)}$$

where a and b are determined as defined above. Roller's equation for the number of particles N per gram is as follows:

$$N = \frac{4}{\rho} \frac{a}{\sqrt{b^5}} \times 10^{10} \qquad \text{Eq (3-36)}$$

Roller obtained good agreement with an extensive amount of experimental data. However, the application of his equation is dependent on straight-line plots between log $w/\sqrt{d}$ and $1/d$ in order that a and b may be determined. Roller's equations appear to have no distinct advantage over the Hatch-Choate equations.

Rosin-Rammler Equation—Rosin and Rammler (1933) proposed an equation which in some respects is similar to Roller's basic equation, Eq (3-32). Their equation fits size-frequency data of friable material such as coal, but otherwise does not have the wide application of the Hatch-Choate and Roller equations. The Rosin-Rammler equation is

$$n = 100 \cdot \exp\left(-\frac{d}{c}\right)^b \qquad \text{Eq (3-37)}$$

where n is the number percent of particles having a diameter greater than d, and c and b are constants. This equation obviously plots as a straight line when n is plotted on a log-log scale, and the size of the particle d on a log-scale. It should be noted that when $c = d$ in Eq (3-37), $n = 100/\exp = 36.8$ percent. If the data are plotted so that n is the percent of particles having a diameter smaller than d, then the value of n corresponding to $c = d$ is $(100 - 36.8) = 63.2$ percent. The value of d corresponding to this value of n on the plotted data gives the value of c. The constant b is analogous to the standard deviation. In other words, if b is large the particles are closely grouped, and if b is small the particles are distributed over a relatively wide range. The constant b, of course, is the slope of the curve plotted as outlined above.

Eq (3-37) may be written in a different form. Let $n/100 = n_0$ and $(1/c)^b = a$, so that

$$n_0 = \exp(-ad^b)$$

and

$$\frac{dn_0}{d(d)} = -abd^{b-1} \cdot \exp(-ad^b)$$

which gives the frequency-distribution. The constant a is easily derived. If, for example, the weighted average diameter of a sieve analysis is d_{ws}, we have

$$d_{ws} = \int_{n_0=0}^{n_0=1} (d)dn_0 = \int_{d=0}^{d=\infty} abd^b \cdot \exp(-ad^b) \cdot d(d)$$

Integrating and solving for a,

$$d_{ws} = \left(\frac{1}{a}\right)^{1/b} \cdot \left(\frac{1}{b}!\right)$$

or

$$a = \frac{1}{d_{ws}{}^b} \cdot \left(\frac{1}{b}!\right)^b$$

Substituting above

$$n_0 = \exp\left[- \left(\frac{1}{b}!\right)\frac{d}{d_{ws}}\right]^b$$

and

$$\frac{dn_0}{d(d/d_{ws})} = - b\left(\frac{1}{b}!\right)^b\left(\frac{d}{d_{ws}}\right)^{b-1} \cdot \exp\left[- \left(\frac{1}{b}!\right)\frac{d}{d_{ws}}\right]^b$$

which give the summation and distribution equations in terms of the weighted diameter.

Combination of Distributions—Frequently two size-distributions are given from which it is necessary to determine the average size of the combination and the standard deviation. The procedure for obtaining the new parameters consists in weighting the arithmetic average of the means. Thus if d_x and d_y denote the average diameters of two distributions, and σ_x and σ_y the corresponding standard deviations, then the average diameters and standard deviations of the combined distributions are

$$d_{x+y} = \frac{n_x d_x + n_y d_y}{n_x + n_y} \qquad \text{Eq (3-38)}$$

$$\sigma_{x+y} = \sqrt{\frac{n_x(\sigma_x{}^2 + d_x{}^2) + n_y(\sigma_y{}^2 + d_y{}^2)}{n_x + n_y} - d_{x+y}{}^2} \qquad \text{Eq (3-39)}$$

where n_x and n_y are the number of particles in each distribution.

Special-Distribution Equations—There are many other types of equations to fit the asymmetric distributions of the type discussed above. Slade (1936) developed a special type of asymmetric probability function which is very general. His developments have since been extended by A. Fisher (1935). In addition there are frequency curves developed by Pearson and other statisticians, and the reader is referred to other sources for information pertaining to them. (See Elderton, 1938.) These functions are involved and form a subject of considerable intricacy. For the general purposes of obtaining information on the parameters determining size-distribution, it will be found that the best distribution is one which gives the simplest equation. If the distribu-

tion is similar to the type shown in Figure 8, we may utilize either arithmetic-probability or log-probability grid to determine the nature of the distribution.* If the curve is symmetrical about its maximum then we may use a simple arithmetic-probability grid. The equations developed in this chapter have the widest application and are most useful in determining simple parameters.

PARTICLE-SHAPE FACTORS

Henry Green (1927) and G. Martin *et al* (1923) have shown that in irregularly shaped particles of a material having a statistical diameter d_v and a volume V, V/d_v^3 = a constant. The various particles of the sample to which d_v applies will possess different individual volumes but the total sum of the individual volumes for a distribution of particles having the same average diameter will be constant. If the particles are individual spheres of different sizes, the value of the constant will be $\pi/6$ However, as Martin and others have shown, the value of this constant is less than $\pi/6$. This reasoning may also be extended to surfaces, so that if S is the aggregate surface-area of a group of particles possessing an average diameter d_s, then S/d_s^2 = constant. The value of this constant for irregular particles will be less than π. If the volume constant be denoted by α_v and the surface constant by α_s, then

$$V = \alpha_v d_v^3 \qquad \text{Eq (3-40)}$$

$$S = \alpha_s d_s^2 \qquad \text{Eq (3-41)}$$

The terms α_v and α_s are called volume and surface shape-factors, respectively. Some interesting derivations to which the Hatch-Choate equations may be applied are deducible from these relationships. Let N be the number of particles per unit-weight of material having a density ρ. Then

$$V = \frac{1}{\rho N} = \alpha_v d_v^3 \qquad \text{Eq (3-42)}$$

If S_w is the specific surface, that is, the surface area per unit-weight of material

$$S_w = SN = \frac{1}{\rho} \cdot \frac{\alpha_s}{\alpha_v} \cdot \frac{d_s^2}{d_v^3} \qquad \text{Eq (3-43)}$$

Values of α_v have been obtained for several materials by different investi-

* Berg (1940) criticized this procedure and proposed a multiparameter distribution function. Berg's equation is more general, but computation of the parameters is laborious.

gators and are shown in Table 8. Because of the irregularity of particles and the use of statistical diameters, the value of α_v and α_s may vary with different samples of the same material. It has been shown by

TABLE 8—VALUES OF VOLUME SHAPE-FACTOR FOR DIFFERENT MATERIALS

Material	Mean diameter	Shape-factor (α_v)	Source
Quartz	1.7–64.2[a]	0.140	Hatch and Choate (1929)
Calcite	1.2–69.0[a]	0.135	Hatch and Choate (1929)
Granite	1.5–72.0[a]	0.140	Hatch and Choate (1929)
Crushed quartz	0.089[b]	0.15	DallaValle and Goldman (1939)
Crushed quartz	0.19[b]	0.17	Martin (1923)
Crushed quartz	0.22[b]	0.28	Martin (1923)
Crushed quartz	0.26[c]	0.28	Martin (1923)
Crushed quartz	0.30[c]	0.27	Martin (1923)
Crushed quartz	0.36[c]	0.27	Martin (1923)

[a] Geometric mean in microns.
[b] Geometric mean in centimeters.
[c] Average volume diameter, $d_v = \sqrt[3]{\Sigma n d^3/\Sigma n}$.

DallaValle and Goldman (1939) that for representative samples having the same size-distribution the probable error of α_v may exceed 10 percent.

Applying the Hatch-Choate equations for d_s and d_v to Eqs (3-42) and (3-43) we obtain

$$\log N = \log \frac{1}{\rho \alpha_v} - \log d_g^3 - 10.362 \log^2 \sigma_g \qquad \text{Eq (3-44)}$$

$$\log S_w = \log \frac{\alpha_s}{\rho \alpha_v} - \log d_g - 5.757 \log^2 \sigma_g \qquad \text{Eq (3-45)}$$

If now we write S_v to represent the surface of a unit-volume of particles, then by means of Eq (3-30) it may be shown that

$$\log S_v = \log \frac{\alpha_s}{\alpha_v} - \log d_g - 1.151 \log^2 \sigma_g \qquad \text{Eq (3-46)}$$

Thus, the surface-area and volume of irregular particles are seen to be functions of the statistical parameters d_g and σ_g. The relationships established are particularly important since they take into account the shape of the size-frequency curve.

The ratio of surface to volume factors is a useful measure of particle-shape, as shown by Fair and Hatch (1933). Thus, for rounded particles α_s/α_v is approximately 6.1; for worn particles, 6.4; for sharp particles 7.0; and for angular particles 7.7. (For spheres $\alpha_s/\alpha_v = 6$.) A similar

and almost identical classification for sand particles was also obtained by the Corps of Engineers (1935*a*). Its classification was as follows: well-rounded, rounded, sub-rounded, sub-angular, and angular. It may be assumed that as a close approximation the well-rounded and rounded particles correspond to the factor 6.1 given by Fair and Hatch. The classification "sub-rounded," developed by the Corps of Engineers, corresponds to the Fair and Hatch classification of "worn."

Specific Shape-Factor—The shape-factor may be defined in another way. Let there be N particles of weight w, and density ρ. Then the diameter of the particles is

$$d = \left(\frac{1}{\alpha_v}\right)^{1/3}\left(\frac{w}{N\rho}\right)^{1/3}$$

The surface will be $\alpha_s d^2$, so that

$$\text{surface} = \left[\alpha_s\left(\frac{1}{\alpha_v^2}\right)^{1/3}\right]\left(\frac{w}{N\rho}\right)^{2/3} = \alpha_{sv}\left(\frac{w}{N\rho}\right)^{2/3} \quad \text{Eq (3-47)}$$

The term in brackets is also a measure of particle-shape and may be termed the specific shape-factor of a particle. For spheres ($\alpha_s = \pi$, $\alpha_v = \pi/6$), we have $\alpha_{sv} = 4.85$, and for a cube $\alpha_{sv} = 6$. The specific shape-factor for irregular particles is usually greater than 6.0.

Problems

1. The following is a size-frequency distribution of fly-ash:

Size microns	*Frequency*	*Size microns*	*Frequency*
40	3	180	24
60	11	200	13
80	20	220	9
100	22	240	5
120	29	260	6
140	41	280	3
160	25		

From this distribution calculate the average diameter d_{av}. Plot the size-frequency curve.

2. From the distribution given in Problem 1 determine the median size.

3. What is the skewness factor of the distribution in Problem 1?

4. Plot the summation curve of the data given in Problem 1 on log-probability grid and obtain the logarithmic standard deviation. What is the value of the uniformity modulus?

5. Using the distribution given in Problem 1, calculate the diameters d_v, d_s, and d_w.

6. What is the shape-factor for a regular dodecahedron, one of whose faces is a-units in size?

7. Two samples are taken from two bins containing fine material. Four hundred particles are measured from the sample taken from one bin, and the average size and standard deviation found to be 5.16 μ and $\pm$1 23 μ, respectively; from the other bin,

275 particles are measured, the mean being 5.72 μ and the standard deviation ± 1.06 μ. The products contained in the two bins are then mixed. What will be the average diameter and standard deviation of the mixture, assuming that the number of particles in each bin is approximately as the number represented by the samples?

From the data of the above problem calculate the standard error of the difference of their means. *Note:* The standard error of the difference of the means is given by the equation

$$\sigma^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

where n_1 and σ_1 are the number of particles and standard deviation of the sample from the first bin, and n_2 and σ_2 the same for the second bin. This equation shows whether the two universes from which samples were drawn differ in their means, apart from any difference in dispersion.

8. Given a sample of particles of different mineral composition. A representative portion of these particles is placed under a petrographic microscope and each individual grain in a number of fields is identified and measured. If the densities of each of the components is known, as well as the shape-factor of one of them, indicate the method to be used in obtaining the percentage by weight of each component. (See Goldman and DallaValle, 1939.)

9. Several of the important distribution functions are special cases of the formula

$$F(d) \cdot d(d) = ad^m \cdot \exp(-bd^n) \cdot d(d)$$

where $F(d)$ denotes a distribution by weight. Some of these special cases are known by the names of the investigators who first studied them.

Thus:

When exponents are:		*The equation is known as*
m	n	
3	1	Martin's law
3	n	Heywood's law
m	2	Weinig's law
$n - 1$	n	Rosin-Rammler's law
m	0	Gaudin's law

Discuss these possible distributions as they relate to the example given in the text of this chapter.

10. Fagerholt (1945) gives as a general parameter for the size distribution of ground products

$$f(d) = a \cdot \ln d + bd + c \qquad (A)$$

an equation which involves the determination of three parameters, a, b, and c Corresponding to this are the weight and surface distribution.

$$F(d) \cdot d(d) = \left(\frac{a}{d} + b\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(a \ln d + bd + c)^2}{2}\right] d(d) \qquad (B)$$

$$dS = \frac{\alpha_s}{\rho} \cdot \frac{1}{d}\left(\frac{a}{d} + b\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(a \ln d + bd + c)^2}{2}\right] d(d) \qquad (C)$$

the functions $F(d)$ and dS applying to a size interval d and $d + d(d)$—the integrated value of the last equation being the surface per gram of material.

Differentiate equation (B) and show that two maxima are possible for the distribution curve.

CHAPTER 4

METHODS OF PARTICLE-SIZE MEASUREMENT

VARIOUS methods are available for determination of average size. The method selected depends upon the kind of material to be measured. If particles are confined to narrow limits of size, screens or microscopic methods of direct measurement may be used. When particles are distributed over a wide range of sizes we must choose indirect methods such as sedimentation or centrifuging. There is no simple method of measurement in either case, and the results are not always susceptible of interpretation unless the composition of the material is known. This will be even more evident when we consider sedimentation methods applied to particles varying widely not only in size but also in density.

DIRECT METHODS OF PARTICLE-MEASUREMENT

Sieves—The simplest method of measuring average size of particles is to use standard sieves. These sieves are composed of wire-mesh cloth with openings of known size. By passing a sample of the material to be measured through two sieves, the particles retained on the second sieve are said to have a diameter equal to the arithmetic or geometric average of openings in the two sieves. If three successive standard sieve sizes are used, the size of the material contained on the second sieve is said to be the arithmetic or geometric average of the first and third sieves. However, this is not generally the preferred procedure. It is customary to use as many sieves of a standard series as are necessary to size all the particles; this is explained in Chapter 5. The average size of sieved material thus obtained may be determined by any of the methods given in Chapter 5.

Microscopic Measurements—The average size or diameter of a distribution of particles may be obtained by measuring the particles at random along a given fixed line. With large particles this procedure may be followed with dividers or a rule. Fine particles require the use of a microscope having an eyepiece fitted with a filar micrometer. The sample to be measured is mounted on a glass slide and examined under ordinary magnifications or under oil immersion. It is placed under the

objective of the microscope, and each particle in the field of vision measured by an optical micrometer.

The optical micrometer for direct measurement of particles consists of a movable cross-hair built into a standard ocular. This cross-hair is actuated by a calibrated micrometer drum. In use, the hair is moved until it touches one edge of a particle. The reading on the micrometer drum scale is noted. The hair is next moved across the particle to its other extreme, and the micrometer reading again noted. The difference between the two micrometer readings is a measure of the particle-length. Since the micrometer drum may be calibrated with a stage micrometer the readings may be expressed in any units desired. In actual practice it is usually simpler to work in terms of the divisions of the drum rather than in microns or millimeters. The constants of the distribution curve, which are the only items concerned here, may ultimately be converted into the units desired. In using the filar micrometer all the particles in a field are measured and counted. Figure 14 illustrates the method of measuring particles. The distance measured is that traversed by the moving hair or filar in the micrometer from one extreme edge of the particle to the other. The significance of this type of measurement, which is termed statistical, has been explained by Martin *et al* (1923):

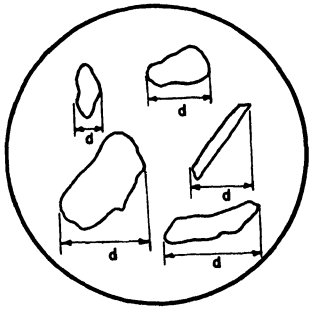


FIGURE 14. METHOD OF PARTICLE-SIZE MEASUREMENT.

"Suppose we select at random out of all these particles, say 1000 of precisely the same apparent diameter *d*, and then place them in a box. Then these 1000 particles thus sorted out would form a grade whose *statistical diameter* is called *d*.

"As the 1000 particles lie in all sorts of haphazard directions, their diameters are measured in the long run with equal frequency in every direction and therefore . . . . they may be considered as replaceable by 1000 spherical particles of the same diameter *d*."

As a general rule 1000 particles are not required to obtain a statistical distribution unless the range of sizes varies over wide limits. For most measurements 200 particles are sufficient. The use of the filar micrometer is tedious and incurs serious eye-strain. To overcome this difficulty particles may be projected on a large ruled screen or grid. The particles will then be sufficiently large to be measured by eye. The use of photographs is also quite common.

For many types of granular material the camera lucida has been found useful. This device consists of a prism arrangement on the eyepiece of a microscope so that the particles are projected downward to one side of the table on which the microscope rests. By means of the camera lucida the observer obtains an enlargement of the particles so each may be outlined with pencil and paper. Of the various methods thus far developed, a projector has proved most satisfactory, unless very high magnifications are used. In the latter instance the outline of the particles is blurred since resolution of the lens system may be such that sharp focus of the boundaries of the particle cannot be obtained.

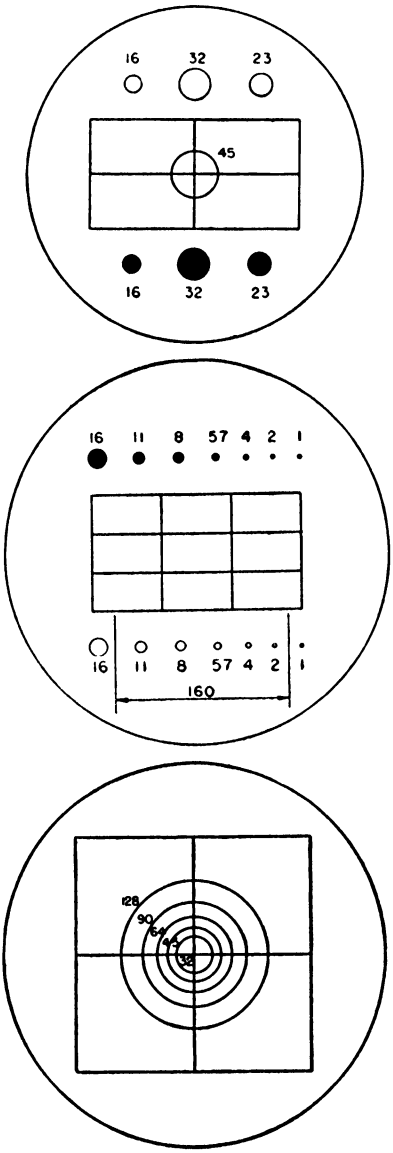


FIGURE 15. FORMS OF GRATICULES DESIGNED BY FAIRS (1943).

Fairs (1943) has criticized the method of linear measurements above described. He points out that the diameter so measured does not correspond with the Stokes or effective diameter d_e, but is usually greater. The importance of avoiding some "shape" factor to convert d to d_e is obvious; but over and above this lies the fact that a linear measure is of statistical interest only, as already inferred. A diameter to be useful must be related to measures of mass or surface. Schweyer (1942) in his comprehensive analysis of particle size techniques (about which more will be said later) has dealt with this subject in detail.

Fairs' investigation of microscopic measurement consisted of two parts: (1) Type of illumination for improving the resolution of the microscope system, and (2) the development of a series of graticules for the direct comparison of the projected particle

areas. In the first instance it was found that the use of a sodium-vapor lamp gave better definition to the finer particles, so that sizes as small as 0.5 μ could be measured. Fairs also found that more particles less than 1 μ in size could be observed with sodium-vapor illumination than with ordinary light and that the eyestrain was lessened. Other experiments using a zinc-vapor lamp improved the resolution to 0.36 μ, but in this case the eyestrain was severe and further experiments were abandoned.

The second part of Fairs' study consisted of an improvement in the graticule system first used by Patterson and Cawood (1936). Instead of a single graticule (inserted in a $\times$10-eyepiece of the microscope), Fairs developed three such graticules. In this way, using different objectives but retaining the $\times$10-eyepiece, he was able to examine a wide range of particle sizes. The circles of the three graticules arranged as shown in Figure 15 covered a range 128:1. The size intervals between the various circles, except for the two lowest, were arranged in $\sqrt{2}$ progression. Such an interval between successive circle diameters is highly desirable and there is little to be gained by decreasing it.

It will be noted that rectangles are provided with each graticule. These are subdivided in a convenient manner so as to simplify coverage of the field. Thus, in measuring, the particles in each rectangle are compared visually with the graticule system and the number of the circle which best fits it is recorded. Fairs recommends a special procedure for estimating the number of areas which should be used to obtain a representative size distribution. This procedure is important when the magnification is changed, although if the size range is such that change is not necessary, good size distributions should be obtainable with comparison of about 250 particles, provided that all particles in the rectangle are measured. Otherwise the following procedure is used: Suppose the A-graticule is used with a 24-mm objective and that for the moment we are concerned with the number of particles fitting the [11]-circle in n areas examined. Let the number be N. Then the frequency per sq mm is N/a_{24} where a_{24} denotes the area of the rectangle. If now the diameter of the [11]-circle corresponds to the [16]-circle when a 16-mm objective is used, and if N' is the count of particles fitting this circle in n' areas,

$$\frac{N}{a_{24}n} = \frac{N'}{a_{16}n'}$$

where a_{16} is the area of the rectangle when the 16-mm objective is used Placing $N = N'$, the number of areas n' to be used with the 16-mm objective is

$$n' = \frac{a_{24}n}{a_{16}}$$

This procedure should be followed in cases where a change of magnification is required to cover the size range. Table 9 illustrates the overlapping of the *B*-graticule with different objectives.

TABLE 9 —EFFECT OF MAGNIFICATION ON SIZE OF CIRCLES AND RECTANGULAR AREAS OF FAIRS' *B*-GRATICULE, FIGURE 15

Objective (mm)	Area of rectangle (sq mm)	Circle designation and diameter of circles (microns)[a] [16]	[11]	[8]	[5 7]	[4]	[2]	[1]
24	0.32	63	43	32	22	16	8	4
16	0.13	40	28	20	14	10	5	1 5
4	0.005	8	5 5	4	3	2	1	0.5

[a] Calibration made with a stage micrometer.

Fairs' technique appears best suited to suspensions, although reasonable success should be achieved with dried specimens. Hemocytomatic cells are preferable in the former instance because of the thinness of the field examined, thus permitting a sharper focus on the plane of the particles.

A comparison of the procedure outlined for determining particle-size indicates that the diameters so obtained correspond with the d_e diameters of sedimentation procedures. Fairs attributes this correspondence to areal matching inherent in his technique. However, agreement with surface diameters determined by turbidimeter was not wholly satisfactory.

As compared with the linear microscopic technique, two facts stand out with regard to Fairs' contribution: (1) That a uniform dispersion of particles is necessary, and (2) that complete fields should be examined rather than random particles. Usually, linear measurements are made on dry samples, which do not have a uniform dispersion of particulate matter and are also quite dense. The use of the filar, of course, tends to cause the technician to be selective, thus favoring the larger particles.

INDIRECT METHODS

The indirect methods depend upon the use of sedimentation, elutriation, or centrifuging. The sedimentation method can be applied in numerous ways to the determination of average particle-diameter. These will be developed in subsequent paragraphs. So far the elutriation method has not been used extensively as a method of particle-size

measurement, although it has been applied industrially in many ways. Centrifuging was developed in recent years for the separation of particles of extremely small size. For particles of the order of 0.5 μ it has been used successfully in a number of instances.

(I) Sedimentation Methods

Theory of Sedimentation Methods—Development of the sedimentation method and generalization of all possible techniques was done by Odén (1915, 1921–1922, 1925). Sedimentation techniques depend on Stokes' law, as developed in Chapter 2, Eq (2-9). It will be noted that for a given density and viscosity of fluid the equation reduces to

$$d_s = K\sqrt{v} = K\sqrt{\frac{x}{t}} \qquad \text{Eq (4-1)}$$

where v is the velocity of the particle at a point x, at a given time t. Thus to solve this equation we need only observe the time required for a particle to move a given distance. However, this procedure becomes exceedingly tedious when applied to a large number of particles, especially if the particles are fine. The mathematical developments given hereafter may be explained as follows: Suppose the particles to be measured are evenly distributed at the start of the experiment and have the same density. Assume further that the fall of each particle is independent of the others—a condition which will hold true if the concentration of the particles is not high. (It is customary to utilize concentrations which are not greater than 2 percent, although as much as 5 percent is frequently used.) If concentrations are too high, the particles begin to interfere with each other and some coalescence and coagulation may take place. Now if all the particles were of the same size they would sink with the same velocity, and the particle-concentration in the medium would be constant at any point x until the top layer of particles began moving past this point. This takes place when particles having a velocity $v = x/t$ (which at the start were in the top layer) have passed the point x. When the particles are not uniform but of various sizes, they fall with different velocities. The size-distribution of particles varies all along the length of the suspension and the concentration decreases as the sedimentation process continues. Fixing our attention on such a suspension of particles at a distance x from the surface, at a given time t from the beginning of sedimentation, some of the particles will have passed below this level and approached the bottom. These particles have a velocity greater than x/t. Our problem is to develop a suitable equation which will determine the precise size-distribution of the particles at any point and at any time. If $F(d)$ defines a frequency function

(weight frequency of particles having a diameter d), the value of the summation function q is then determined by the equation

$$F(d) = -\frac{dq}{d(d)}$$

The sign in the present instance is negative if we regard q as the number of particles greater than a given diameter d. We are not concerned with the form of either q or $F(d)$ for the time being. The technique involved requires only that we determine the size-distribution graphically. With sedimentation methods the form of frequency function is important mathematically only in so far as it explains the relationships of the variables measured

From the discussion previously given we have that at a depth x and a time t all particles whose velocity v is greater than x/t will be descending through this layer and will be in the same concentration as at the time t. Let ρ be the density of the solid material, ρ_0 the density of the fluid, and C_0 the initial concentration of the suspension (weight of particles per unit-volume of suspension); the density of the suspension at any point x and any time t is then given by the equation*

$$\varphi(x, t) = \rho_0 + C_0 \frac{\rho - \rho_0}{\rho} \int_0^d F(d) \cdot d(d) \qquad \text{Eq (4-2)}$$

where φ is the density of the suspension and d is defined by Stokes' equation, Eq (4-1). From these two equations may be derived the possible methods for determining particle-size distribution.

Pipette Method—A widely used method for determining size-distribution consists in drawing samples of the suspension at various times from a fixed depth or from different depths at a fixed time.† Such samples may be drawn with a calibrated pipette. The samples are then evaporated to dryness and the residue weighed. Of the two possible methods for using the pipette, the one based on a fixed depth is used. The basic equations involved in the final computations of the data collected are given below.

Assume it is desired to find the variation of specific gravity and concentration with time at a given distance from the surface. In Eq (4-2) we have t variable and x constant. Hence, substituting the value of d as given in Eq (4-1)

$$\frac{d\varphi}{dt} = -\frac{K}{2}\sqrt{\frac{x}{t^3}} \cdot C_0 \frac{\rho - \rho_0}{\rho} \cdot F(d)$$

* For the analysis of this equation, as it applies to various techniques of particle-size measurements, I am indebted to the writings of Sven Odén

† The effect of pipette tube on the representatives of the sample drawn is discussed in Chapter 23.

From Eq (4-1)

$$K\sqrt{x} = d_e\sqrt{t}$$

so that substituting this value of $K\sqrt{x}$ in the above equation and solving for $F(d)$

$$F(d) = -\frac{2\rho}{\rho - \rho_0}\cdot\frac{1}{C_0}\cdot\frac{t}{d_e}\cdot\frac{d\varphi}{dt}$$

when $t = 0$, Eq (4-2) becomes

$$\varphi_0 = \rho_0 + C_0\frac{\rho - \rho_0}{\rho} \qquad \text{Eq (4-3)}$$

so that

$$F(d) = -\frac{2}{\varphi_0 - \rho_0}\cdot\frac{t}{d_e}\cdot\frac{d\varphi}{dt} \qquad \text{Eq (4-4)}$$

where φ_0 is the initial density of the suspension.

Now if C denotes the average concentration at the point in question at a stated time t, and φ_t is the density of the suspension at that time, that is

$$\varphi_t = \rho_0 + C\frac{\rho - \rho_0}{\rho}$$

then combining this equation with Eq (4-3), we obtain

$$\frac{\varphi_t - \rho_0}{\varphi_0 - \rho_0} = \frac{C}{C_0} = \Phi$$

Hence Eq (4-2) becomes

$$F(d) = -\frac{2t}{d_e}\cdot\frac{d\Phi}{dt} \qquad \text{Eq (4-5)}$$

If the pipette sample is now taken at different depths of the suspension at a definite time t (t = constant) Eq (4-2) becomes

$$\frac{d\varphi}{dt} = K\sqrt{\frac{1}{tx}}\cdot C_0\frac{\rho - \rho_0}{\rho}\cdot F(d)$$

or substituting $d_e/\sqrt{x}$ for $K\sqrt{1/t}$ and the value of C_0 in Eq (4-3)

$$F(d) = \frac{2}{\varphi_0 - \rho_0}\cdot\frac{x}{d_e}\cdot\frac{d\varphi}{dx} \qquad \text{Eq (4-6)}$$

As in the previous analysis, if now

$$\varphi_x = \rho_0 + C\frac{\rho - \rho_0}{\rho}$$

defines the density of the suspension at x, and C is the corresponding concentration, so that

$$\frac{\varphi_x - \rho_0}{\varphi_0 - \rho_0} = \frac{C}{C_0} = \Phi$$

then

$$F(d) = \frac{2x}{d_e} \cdot \frac{d\Phi}{dx} \qquad \text{Eq (4-7)}$$

We present here two procedures for applying the pipette method: The first used by the U. S. Department of Agriculture (Olmstead *et al*, 1930), and the other described by Keen (1931). Both of these methods pertain to the determination of silt and colloids in soil suspensions, rather than to a size-distribution. However, the descriptions given indicate the manner of using the pipette.

In the Department of Agriculture method, the sedimentation chamber contains only fine material with low settling velocities. The suspension in the sedimentation chamber is stirred and an aliquot is quickly taken with a 25-cu cm automatic pipette. After the suspension has stood 77 min at 20 deg C and again after 8 hrs, aliquots are taken at a depth of 10 cm, the material in the chamber being stirred thoroughly each time at the beginning of the sedimentation period. The portion drawn at the end of 77 min represents particles of less than 0.05 mm and consists of what is known as the silt-and-clay portion. The size determined after 8 hrs represents particles of clay having diameters less than 0.005 mm. The difference between these quantities gives the amount of silt which, according to the Department of Agriculture classification,* ranges from 0.05 to 0.005 mm in diameter.

The method widely used in England and described by Keen is as follows: A cylinder of suspension equivalent to 2 percent of the sample is carefully stirred. The first pipette sample is taken at a 10-cm depth after 4 min 48 sec. The pipette with top closed is lowered about 20 sec before this time and 20 cu cm of suspension are drawn off and put into a suitable tared dish. The sample is evaporated to dryness in a water bath, oven-dried at 105 deg C, and weighed. The weight of the material times five gives the concentration in the suspension at the point sampled. Let this concentration be C. Since the original concentration was 2 percent, the percentage of material having a velocity less than $10/288 = 0.0347$ cm per sec is given by $100 \times C/2$. As a percentage, this represents the sum of clay plus silt. The contents of the cylinder are again shaken for about 1 min and allowed to settle 8 hrs. Sampling is again

* See the newer classification explained later in this chapter

made at 10 cm below the new level and we proceed as before. In this way we obtain the percentage of clay. Thus, in the first step we obtained the percentage of clay plus silt, and in the second step the percentage of clay. The percentage of silt is obtained by subtraction. The size-fractions as determined by this method are: Silt 0.04–0.002 mm, and clay < 0.002 mm.

Pressure-Change Method—As in the previous case, this method may be used to measure the change in pressure at a definite distance from the surface at varying times, or at varying distances at a fixed time. If h denotes the pressure at a point x (fixed) at a time t, it may be shown that the size-function varies with time in accordance with the following equation:

$$F(d) = \frac{2\rho}{\rho - \rho_0} \cdot \frac{1}{C_0} \cdot \frac{t^2}{x d_e} \cdot \frac{d^2h}{dt^2} \qquad \text{Eq (4-8)}$$

Also, if h_0 is the initial pressure at x ($t = 0$)

$$\left.\begin{aligned} h_0 &= \rho x + C_0 x \cdot \frac{\rho - \rho_0}{\rho} \\ h_t &= \rho x + C x \cdot \frac{\rho - \rho_0}{\rho} \end{aligned}\right\} \qquad \text{Eq (4-9)}$$

but when $t = \infty$ the pressure at x at this time

$$h_\infty = \rho x \qquad \text{Eq (4-10)}$$

Hence combining Eqs (4-9) and (4-10)

$$\frac{h_t - h_\infty}{h_0 - h_\infty} = \frac{C}{C_0} = \Pi$$

and

$$F(d) = \frac{2t^2}{d_e} \cdot \frac{d^2\Pi}{dt^2} \qquad \text{Eq (4-11)}$$

The corresponding equation giving the variation in pressure at a definite time is

$$F(d) = \frac{2x}{d_e} \frac{d^2\Pi}{dx^2} \qquad \text{Eq (4-12)}$$

where Π is defined as follows:

$$\Pi = \frac{h}{\rho - \rho_0}; \quad \rho = \rho_0 + C_0 \frac{\rho - \rho_0}{\rho}$$

Methods based upon the measurement of pressures at various points, at

a definite time, have never proved successful because of the experimental difficulties involved.

On the other hand many methods have been devised for measuring pressure changes with time at a given point in the sedimentation tube. These methods are based on the so-called Wiegner (1918) sedimentation tube. This tube consists of a cylinder with relatively large diameter. At a point x-distant from the top is attached a smaller tube provided with a stopcock. This tube is bent upward parallel to the sedimentation tube. The smaller tube is filled with sedimentation fluid, free of particles, to the level of the suspension itself in the sedimentation tube. When the stopcock is opened the liquid in the smaller tube rises; as sedimentation progresses, the liquid in the smaller tube decreases.

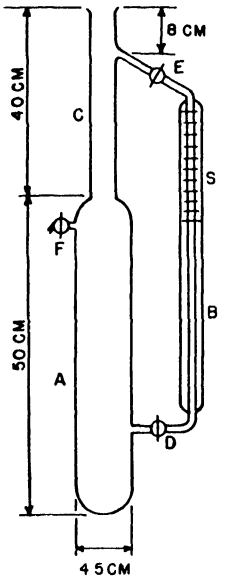


FIGURE 16. GOODHUE AND SMITH SEDIMENTATION TUBE FOR PARTICLE-SIZE DETERMINATION.

A variation of the Wiegner sedimentation tube, producing greater sensitivity, was developed by Goodhue and Smith (1936) and is shown in Figure 16. The chief difference between the Goodhue and Smith apparatus and that of Wiegner consists in the use of a differential manometer utilizing immiscible liquids. By means of the former, differences in pressure may be increased from 50 to 100 times that of Wiegner's original apparatus. Goodhue and Smith's apparatus consists of a large tube A, 50 cm in length and 4.5 cm in diameter, fitted to a smaller tube C approximately 40 cm in length and 2.5 cm in diameter. The manometer tube B has a 4-mm inside diameter and is joined through a 3-mm bore stopcock D, 13 cm from the bottom of the large tube, and through a similar stopcock E, 12 mm from the top of the apparatus. The scale S is inserted behind the upper half of the manometer tube inside the jacket. The manometer is jacketed and evacuated in order to reduce temperature effects. In use, a 7-g sample is mixed to a thick paste with 50 percent alcohol until smooth. The manometer tube is carefully cleaned and rinsed with alcohol, filled to the stopcock F with the sedimentation medium (about 600 cu cm) and from 3 to 6 drops of concentrated ammonium hydroxide are added. After thorough mixing the manometer is filled by tilting.

The level in the manometer is adjusted by adding a sufficient quantity of dilute alcohol so as to overflow at F. The stopcocks are then closed.

The suspension is stirred with a plunger and placed in a water bath until it reaches a temperature of 25 deg C. The suspension is stirred again after 20 min and a second liquid consisting of decahydronaphthalene is added by permitting it to run down the inside wall of *C*. (The decahydronaphthalene is adjusted to a density of 0.9000 by adding trichlorobenzene.) A colored oil-soluble dye is added and the mixture is extracted three times with a slightly ammoniacal 50 percent alcohol solution. This extraction process is necessary in order to remove soluble substances in the liquid and to saturate it with dilute alcohol. This same liquid may be used as often as desired and the density adjusted by suitable addition of either component.

When the second liquid is added, care should be taken to have it at the same temperature as the suspension medium. The stop watch, which was started after the last stirring, is returned to zero after 30 sec, because settling does not start until all eddying has stopped. With practice, less than a minute is required to introduce the top liquid. The two stopcocks connected to the manometer at *E* and *D* are then opened. The upper liquid rapidly replaces the dilute alcohol contained in part of the manometer tube and equilibrium is reached in approximately 2 min. The zero point, or point denoting complete settling, is obtained either at the end of the experiment or by running a blank determination. However, this zero point is not necessary for routine work. Figure 17 shows a typical set of distributions for three different substances obtained by this technique. Computation of various physical constants with this device is not difficult. Let the density of the suspension be φ, that of the suspending medium ρ_0, that of the top liquid ρ_0', and assume the settling height to be x. The rise produced in the manometer by the suspension of the particles is therefore

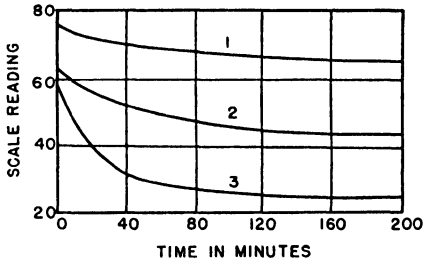


FIGURE 17. SETTLING CURVES OBTAINED WITH GOODHUE AND SMITH APPARATUS: (*1*) CALCIUM CARBONATE; (*2*) GEORGIA CLAY; (*3*) SULFUR.

$$\Delta h = \frac{(\varphi - \rho_0)x}{\rho_0 - \rho_0'}$$

and

$$\varphi = \rho_0 + \frac{\Delta h}{x}(\rho_0 - \rho_0')$$

by which the density of the suspension may be calculated. Again if C_0 denotes the initial concentration of the particles and φ_0 the initial density of the suspension, then the density of these particles ρ is obtained from the simple expression

$$\rho = \rho_0 \frac{C_0}{C_0 - (\varphi_0 - \rho_0)}$$

In so far as the nature of the frequency function given by Eq (4-11) is concerned, it is to be noted that $\Pi = \Delta h / \Delta h_0$, where Δh_0 is the observed initial rise of the liquid in the manometer.

With practice much may be learned from the shape of the sedimentation curves shown in Figure 17. Thus, a curve with a rapidly changing slope indicates a poly-dispersed sample, while a curve approaching a straight line indicates mono-dispersion. Two samples have approximately the same particle-size distribution when their slopes at corresponding times are the same. The curves are also valuable since they give a graphical comparison of the density. The relative densities of various materials sized in this manner are in accordance with the point of ordinate interception.

Hydrometer Method—The constant variation in density of a suspension permits the use of sensitive hydrometers for size-analysis. A hydrometer device was used by Pratolongo (1917); Bouyoucos (1927*a*, *b*) made extensive experiments with a special hydrometer and found it to be a valuable and convenient device for sizing soil particles. The form of frequency function for the hydrometer is closely analogous to Eq (4-11). If ω_0 is the original reading of the hydrometer, ω the reading at the time t, and ω_∞ the reading when t is infinite, we have

$$\frac{\omega - \omega_0}{\omega_\infty - \omega_0} = \frac{C}{C_0} = \Omega$$

so that

$$F(d) = -\frac{2t^2}{d_e} \cdot \frac{d^2\Omega}{dt^2}$$

In using the hydrometer, it must be remembered that the change of density recorded by the hydrometer is not of a particular cross section of the sedimentation cylinder, but depends upon the depth to which the hydrometer is immersed. The hydrometer method can be applied only to suspensions which are of very small size since rapid changes in density of the suspension cannot be read on the hydrometer.*

The procedure for using a hydrometer consists in measuring the den-

* See also A.S.T.M. Designation D 422 - 39.

sity of the suspension contained in a graduate at stated intervals of time. The usual time intervals are 1, 3, 10, and 30 min, and 1, 2, 5, and 24 hrs. A 3 to 5 percent suspension is made up. Higher concentrations alter the viscosity of the fluid and the necessary computations are somewhat complicated. From 10 to 20 sec before each reading, the hydrometer is lowered carefully into the suspension to minimize turbulence. The density is read on the hydrometer scale immediately after it comes to rest. The hydrometer is not permitted to remain in the suspension since particles of the suspended material settle on it and affect its accuracy. After removal, the hydrometer is wiped clean and immersed in a graduate of distilled water. It is preferable to keep both the graduate containing the suspension and that containing distilled water in a constant-temperature bath.

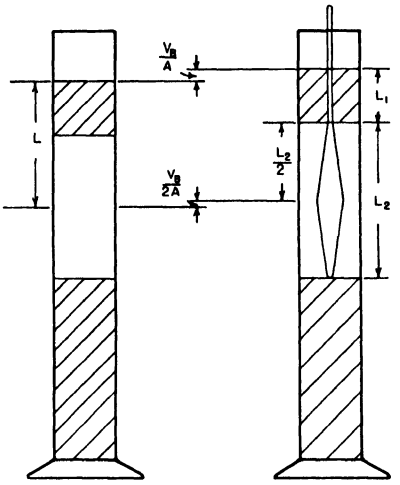


FIGURE 18. RELATIONS FOR COMPUTING EFFECTIVE-DEPTH FORMULA.

If relatively high concentrations of suspended material are used, the viscosity of the suspension should be determined by an appropriate viscosimeter. Care should also be taken to reduce the effect of the suspension's solubility since this may affect the final results seriously. Note that the hydrometer method of particle-size determination is useful for sizes usually below 200 U. S.-sieve sizes. Above this range the material may be sized by sieves, although certainty should be made that agglomerations do not exist.

The accuracy of the hydrometer method for particle-size measurement is obviously dependent upon some mean distance along its bulb which is taken as a reference point. Along the entire submerged stem and bulb the density of the concentration and the size-distribution of particles are constantly changing. Their reference point may be determined as follows: Let L be the distance from the surface of the suspension to the volume center of the hydrometer; then if the submerged part of the stem is neglected

$$L = L_1 + \frac{1}{2}\left(L_2 - \frac{V_B}{A}\right) \qquad \text{Eq (4-13)}$$

where L is the distance from the top of the suspension to the top of the hydrometer bulb, L_2 the length of the hydrometer bulb, V_B the volume of the hydrometer bulb, and A the cross-sectional area of the graduate. The significance of these terms is shown in Figure 18. Note that L_1 is the distance from the top of the hydrometer bulb to the bottom of the specific-gravity scale plus the length of the submerged portion of the scale. Thus, we have

$$L_1 = \left\{\begin{array}{l}\text{distance from}\\ \text{top of bulb to}\\ \text{the bottom of}\\ \text{scale (cm)}\end{array}\right\} + \frac{\text{length of scale (cm)}}{\text{specific gravity range of hydrometer}} \left[\begin{array}{l}\text{maximum}\\ \text{reading}\\ \text{possible}\end{array}\right\} - \left\{\begin{array}{l}\text{specific}\\ \text{gravity}\\ \text{of sus-}\\ \text{pension}\end{array}\right] = a - b\varphi_0$$

where a and b are constants, and φ_0 the specific gravity of the suspension. Hence after substituting this value of L_1 in Eq (4-13) we obtain a new equation of the form

$$L = a' - b\varphi_0 \qquad \text{Eq (4-14)}$$

where a' is a new constant. This value of L is called the "effective" depth of the hydrometer.

In Chapter 3 there is a discussion of the equivalent diameter of a particle. This is the diameter as obtained by application of Stokes' law. In cgs units the equivalent diameter adjusted to the conditions of the present problem is

$$d_e = 0.175\sqrt{\frac{\mu}{\rho - \rho_0}} \cdot \sqrt{\frac{L}{t}} \qquad \text{Eq (4-15)}$$

This is a form of Eq (4-1). The terms have the following meanings. The term d_e is the equivalent diameter in mm, μ the viscosity of the *suspension* in poises, ρ and ρ_0 the density of particles and fluid, respectively, t the time in min, and L a function of the density of the suspension as given by Eq (4-14).

If it is desired to obtain the weight percent of solids remaining in suspension at the effective depth L after t min, we have

$$\varphi_0 = C + \rho_0\left(1 - \frac{C}{\rho}\right)$$

where C is the weight concentration of particles per unit-suspension; hence

$$C = \frac{\rho}{\rho - \rho_0}(\varphi_0 - \rho_0) \qquad \text{Eq (4-16)}$$

Thus, if W represents the amount of solids in the original sample and the various densities are expressed in g per cu cm, then the percentage of solids C' at depth L at time t for a 1-liter graduate is

$$\frac{C'}{100} = \frac{1000}{W} \cdot \frac{\rho}{\rho - \rho_0} (\varphi_0 - \rho_0) \qquad \text{Eq (4-17)}$$

In using the hydrometer method for determining particle-size, special care should be taken to make due provision for temperature change. The amount of labor involved may be reduced by compiling tables of the constant term in Eq (4-15) and the term $\rho/(\rho - \rho_0)$ in Eq (4-16). Tables of $\rho/(\rho - \rho_0)$ and μ for water are given in the appendix.

For accurate work it is necessary to take into account certain correction factors, namely the meniscus correction and the change in φ_0. Hydrometers are usually calibrated at the bottom of the meniscus. In practice, because of the presence of suspended material, it is necessary to make readings at the top of the meniscus. Hence, there must be added to the latter a small quantity to correct for the procedure. In the case of water suspensions this correction is about 0.0003, and this figure is generally used.

The second correction involves a correction for temperature. If ρ_{0c} denotes the density of water at calibration temperature, T_c (20 deg C as a rule), and h is the hydrometer reading at any temperature T, then it may be shown that the value of φ_0 at this temperature is

$$\varphi_0 = \frac{h\rho_{0c}}{1 + (T - T_c)\overline{\lambda}} \qquad \text{Eq (4-18)}$$

where λ is the coefficient of cubical expansion of glass, values for which are to be found in the appendix.

In order to illustrate the method of using the above equations, we here give the following example:

A hydrometer and graduate having the dimensions given below are used to determine the particle size of a suspension consisting of 45 g dry weight of particulate matter.

Length of hydrometer bulb	17.5 cm
Volume of hydrometer bulb	40.3 cu cm
Distance from top of bulb to maximum reading of scale	2.6 cm
Length of specific gravity scale	6.9 cm
Specific gravity range	1.050–1.002
Cross-sectional area of graduate	28 sq cm
Volume of graduate	1000 cu cm

If the hydrometer is used in a room having a temperature of 27 deg C, and particles in the above suspension have a specific gravity of 2.65 g

per cu cm, compute the particle size for the following times and corresponding hydrometer readings, no meniscus correction having been applied.

Time of sedimentation (min)	*Hydrometer reading*
1/2	1.0200
8	1.0235
120	1.0075
1440	1.0035

The computations proceed as follows: (1) Add the meniscus correction 0.0003; (2) compute the value of φ_0 in Eq (4-18); (3) compute corresponding values of C' in Eq (4-17); calculate the corresponding L in Eq (4-13); and (4) calculate the corresponding diameter, if necessary, from Eq (4-15).

Let us note at once that the values of μ and ρ_0 for water at 27 deg C (see appendix) are 0.008545 poises and 0.9965451 g per cu cm, respectively. Hence

$$\sqrt{\frac{\mu}{\rho - \rho_0}} = \sqrt{\frac{0.0085}{2.65 - 0.996}} = 0.072$$

$$\frac{\rho - \rho_0}{\rho} = \frac{2.65}{1.65} = 1.61 \ (\textit{nearly})$$

Arranging our computations in tabular form, we finally obtain:

Time of sedimentation, t (min)	Original hydrometer reading[a]	Hydrometer reading with meniscus correction, h' (+0 0003)	Value of φ_0 Eq (4-18), at 27 deg C (λ = 0 000025)	Value of C', Eq (4-17) (percent)	Effective depth L, Eq (4-13) (cm)	Diameter of particles, d_e, Eq (4-15) (cm)
1/2	1.0200	1 0203	1.0180	64 5	14 12	0 0726
8	1 0135	1 0138	1 0110	39 8	15 02	0 0188
120	1 0075	1 0078	1 0050	17 9	16 00	0 0188
1440	1.0035	1 0038	1 0010	3 58	16 55	0 0047

[a] Calibration reading at 20 deg C is $\rho_{oc} = 0.9982$.

Odén Balance Method—This method consists of a tray which fits closely inside a sedimentation flask and is suspended from one arm of a delicate balance. The change of weight due to particles accumulating on the pan may be determined in various ways. Many difficulties are encountered in the use of the Odén balance. The particles do not appear to distribute themselves evenly on the pan. Many different designs of the pan were used by Odén to overcome this difficulty, without success.

The frequency distribution is again of the same form as in previous equations. Let β be the accumulation weighed in water and β_∞ the weight accumulated after complete sedimentation. If A_B is the area of the pan and the other variables are as previously defined,

$$\beta = \beta_\infty - A_B(h - \rho_0 x)$$

Hence

$$\frac{d^2\beta}{dt^2} = - A_B \cdot \frac{d^2h}{dt^2}$$

Substituting in Eq (4-8)

$$F(d) = - \frac{2\rho}{\rho - \rho_0} \cdot \frac{1}{C_0} \cdot \frac{t^2}{A_B d_e x} \cdot \frac{d^2\beta}{dt^2}$$

But

$$\beta_\infty = \frac{\rho - \rho_0}{\rho} \cdot C_0 A_B x$$

and if $\beta/\beta_\infty = \mathrm{B}$

$$F(d) = - \frac{2t^2}{d_e} \cdot \frac{d^2\,\mathrm{B}}{dt^2} \qquad \text{Eq (4-19)}$$

Cummings Sedimentation Method—Cummings (1929) developed a method which is useful for obtaining size-fractions of particulate substances. This method consists of the following: The sample of small particles is thoroughly dried and then passed through suitable sieves for separating particles larger than the upper limit of size desired. Twenty-five grams of the material are then spread evenly over the bottom of a large glass bacteriological culture dish 22 cm in diameter and 8 cm deep. Then 125 cu cm of 95 percent alcohol are diluted to 250 cu cm and this mixture is slowly poured into the dish. The particles are carefully worked into a smooth paste. To this suspension is added a liter of water and stirring is continued until a uniform suspension is obtained. At the moment agitation stops the time is observed and the particles allowed to settle for $^1/_2$ min. The supernatant liquid is carefully but quickly decanted into a second container of the same size. The time is noted when this decantation is completed, and the suspension is allowed to settle for 1 min. The process is repeated by decanting the suspension into a third container and allowing it to settle for 2 min. Successive decantations are made following settling periods of 4, 8, 16, 32, etc., min. The process may be discontinued after any decantation, depending upon the size-distribution desired Particles remaining in suspension after a period of four days will generally be in Brownian motion and remain suspended for an indefinite period.

Each fraction must be cleared of fines, which may be accomplished by an "ashing" process. The various decantations are brought into suspension in a liter of water-alcohol mixture and the process above described repeated for each fraction. The supernatant liquid of the first decantation, which settled for $1^1/_2$ min, is included with the material which had been obtained after settling out for 1 min The supernatant liquid of each fraction is passed on to the next succeeding fraction, and the process may be repeated as often as necessary to obtain a clear supernatant fluid in the settling interval. For best results the sedimentation vessels should be kept free from temperature and vibration changes. This method is only approximate but capable of producing fractions of material having a limited range of sizes. It differs from the technique and analysis developed by Odén, which may be classed as cumulative, in that it gives a weight-size distribution directly. Cummings found that the theoretical sizes calculated by Stokes' law for the various fractions do not agree with those obtained by direct measurement.

Graphical Determination of Size-Frequency Curves—The usual sedimentation curves may be said to be historical in the sense that they picture the state of the suspension at any time t. However, by graphical means we may obtain the distribution function $F(d)$ which gives the relation between the percentage weights of the suspended material and the corresponding diameters. Let $\mathcal{Q}$ be the particular sedimentation parameter observed. Note that

$$\frac{d\mathcal{Q}}{dt} = \frac{\mathcal{Q}_2 - \mathcal{Q}_1}{t_2 - t_1}$$

Thus, by taking t_1 and t_2 as close together as desired we obtain the shape of the sedimentation curve at any point between t_1 and t_2. Since we are interested in the second derivative of this function, let us take $d\mathcal{Q}/dt$, set it equal to $\mathcal{Q}'$ and plot values of $\mathcal{Q}'$ against t. Then

$$\frac{d\mathcal{Q}'}{dt} = \frac{d^2\mathcal{Q}}{dt^2} = \frac{\mathcal{Q}_2' - \mathcal{Q}_1'}{t_2 - t_1}$$

That is, by taking t_2 as close to t_1 as we choose, we obtain from the second curve the second derivative of $\mathcal{Q}$ with respect to t at any time t. Since d is defined by Eq (4-1) the value of $F(d)$ for the sedimentation equations previously given is completely determined.

(II) Elutriation Method

Sizing by elutriation is the reverse of the technique used for sedimentation. While any fluid may be used for elutriating particles, air has been found most satisfactory thus far. The basis of elutriation methods is

the provision of velocities capable of supporting only particles smaller than a given size. Size-distributions obtained by elutriation are therefore cumulative in character, as was the case for sedimentation. However, one exception must be noted. In the use of sedimentation methods

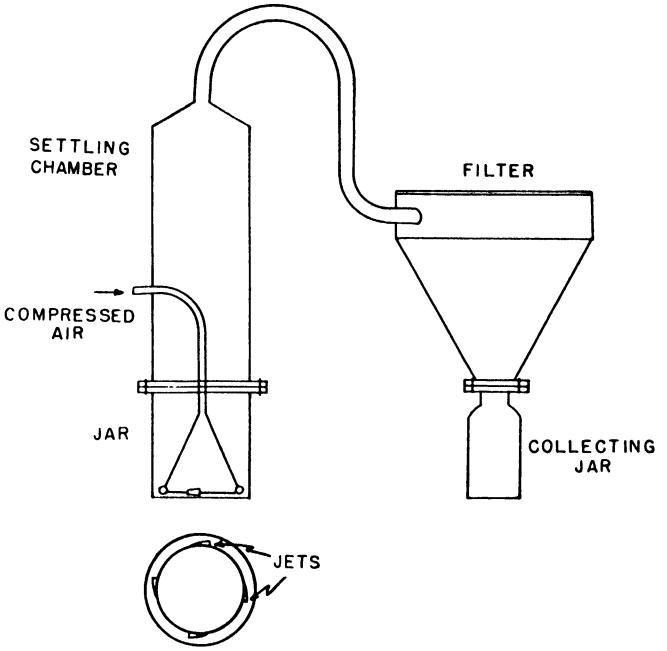


FIGURE 19. ELUTRIATOR FOR SIZING SMALL PARTICLES.

we plot the weight or frequency of particles *larger* than a stated size, whereas in elutriation we plot frequencies of particles *less* than a stated size. However, this entails no difficulties in so far as ultimate interpretation of the data is concerned. The bases of elutriation techniques are included in the various equations developed in the second chapter, namely Eqs (2-9) and (2-10). To obtain particle-size distributions in practice, recourse may be had to the Roller (1931) elutriator or to a design developed by the author. Roller's elutriator consists of a U-tube in which is placed a small amount of the material to be sized. Near this material is placed an air jet. The U-tube connects with a cylinder which may vary in height, but which has a known cross-sectional area. The extreme end of the cylinder, which is vertical, is connected to a suitable filter for arresting the material. In use, the volume of air supplied by the jet is measured, and with the known area of the vertical cylinder it is possible

to compute the velocity of the moving air. Fractions of the sized material may be obtained by varying the quantity of air passed through the jet or by altering the size of the cylinder, as desired. The author's method* is shown diagrammatically in Figure 19. A known volume of air is introduced by tangential jets at the bottom of a cone. The purpose of this cone is to permit the material always to come in contact with the jets. The cycloning action causes a spiral flow of air to the lower part of the cylinder. This tends to centrifuge the larger particles and to create a uniform velocity distribution across the cylinder. The material is collected in a manner similar to that of the Roller elutriator.

The essential differences between the two types of elutriators are as follows: Roller's type requires a small quantity of dust which must be vibrated continually in the U-tube so as to bring it in contact with the air jet. The author's technique permits the use of large quantities of material and secures more rapid separation. Both devices can produce sharply defined size-distributions.

Elutriation methods can be used successfully with sub-sieve sizes and for particles which are not much below 1 μ in diameter. The effectiveness of elutriation is determined by the type of filter used to capture the dust. Smaller particles could undoubtedly be obtained if an electrical precipitator, instead of cloth filters, were used for the dust passing the elutriation cylinder.

(III) Centrifuging Method

Centrifuging methods for obtaining particle-size distribution are based on the use of Eq (2-47). Full details on centrifuging for the determination of extremely small sizes were given by Svedberg (1928). For the general purpose of determining particles which are very slow in settling but which do not go below $^1/_2$ μ in size, the following procedure according to S. W. Martin (1939) is used. The centrifuge must be free from vibration and should operate under temperature conditions as nearly constant as possible. Martin used a centrifuge capable of 1300 rpm. The centrifuge tubes were flat-bottom vials having a diameter of 15 mm and length of 35 mm; the tubes were of equal length and each was equipped with a scale such that 29 mm of suspension were indicated for each sedimentation interval.

If the particles measured are considered as spheres the value of d is determined from Eq (2-41)

$$d = \frac{6}{\omega}\sqrt{\frac{\mu \ln R_2/R_1}{2(\rho - \rho_0)t}} \qquad \text{Eq (4-20)}$$

* Unpublished

The value of R_2 is taken as the distance from the axis of rotation to the bottom of the centrifuge tube and R_1 is the distance from the axis of rotation to the meniscus of the suspension. The values of R_2 and R_1 are usually fixed in experimental work, so that if the constants of the liquid and particulate material are known, Eq (4-20) reduces to

$$d = K/\sqrt{t} \qquad \text{Eq (4-21)}$$

where K is a constant.

Centrifuge methods should be used only for particles which are extremely fine. Suspensions should be allowed to settle for some time before being centrifuged to eliminate larger particles as much as possible.

Martin used a glycerol-water mixture having a specific gravity of 1.2385 and a viscosity coefficient of 2.2 poises. In determining the particle-size distribution for titanium oxide having a specific gravity of 3.9, Eq (4-21) becomes $d\ (\mu) = 22.1/\sqrt{t}$ (t in min). Martin's technique is extremely useful for pigments, the only requirement being the selection of a suitable dispersion medium. Good agreement was obtained when this method was checked against measurements made under high magnification.

COMPARISON OF PARTICLE-SIZE METHODS

Schweyer (1942) compared various methods of particle-size measurement (except centrifuging). He found excellent agreement between pipette and hydrometer methods. He considers the former the best method for determining the particle-size distribution of sub-sieve material by sedimentation, and prefers the hydrometer as a rapid control procedure.

The microscopic method was used in only one set of experiments, *viz.*, for a material whose median effective diameter was 25 μ. Good agreement was found between pipette, hydrometer, and microscopic methods for sizes ranging from 25 μ to the upper limit of the experiment, 100 μ. Below 25 μ, both sedimentation methods gave similar results, but there was a marked difference with the microscopic method. Thus, at 10 μ, the percent found undersize by the sedimentation methods was 25 percent and that found undersize by the microscopic method was 12 percent. This difference can be attributed in part to difficulties in preparing samples for measurement, but undoubtedly the failure to compute shape factors can be regarded as the chief source of divergencies.

Schweyer also found that elutriation gave results corresponding to sedimentation methods. This is a reasonable expectation in view of the fact that both procedures depend upon application of the fundamental laws of settling described in Chapter 2.

It is not always easy to determine what procedure to follow in making a particle-size distribution. Obviously, if the particles are soluble in water or any other fluid, sedimentation procedures must be applied with caution. It is likewise clear that any sedimentation technique is markedly affected by the shape of the particles used, and that results are subject to interpretation. In other words, determinations depending upon sedimentation (and elutriation) are merely equivalent measures of spheres having the same rate of settling. Greatest reliance naturally applies to that size range whose motion is known to be specified by Stokes' law.

Microscopic methods have certain advantages provided that a representative distribution of particles can be prepared for examination. Using refined techniques, sizes as small as 0.5 μ are readily measured with ordinary microscopes. Electron microscopes permit resolution to sizes thousands of times smaller, and indeed, this method is at present the only one which can be used on discrete particles of extremely fine size.* The two-dimensional aspects of microscopic measurements often render this technique unsatisfactory. Furthermore, it is not always possible to obtain necessary shape factors to yield accurate volume and surface computations.

PREPARATION OF SAMPLES

When particle-size is obtained by direct or indirect measurement, careful consideration must be given to proper preparation of the samples. No difficulty is encountered with ordinary sieve-sizes. It is only when we are considering conglomerate material which consists of fine and coarse particles closely adhering to each other that difficulties arise. Likewise, some materials are of such a nature that they are not easily wetted and fail to disperse properly. The most essential feature of any particle-size determination is to disperse the material so that any small segment will contain sufficient particles to give a true representation of the whole.

Preparation of Slides—Most material may be dispersed easily in xylene or balsam. When these liquids dry the material remains firmly adherent to the glass slide. Dunn (1930) recommends the use of a stiff brush when mounting small particles with xylene or balsam. He claims the use of a brush in smearing the slide tends to produce a uniform distribution. Cover glasses may be used on the slide, but it must be remembered that under high magnification the use of a cover glass interferes

* In Chapter 11 we shall consider adsorption procedures for measuring fine particles. However, such methods do not yield size distributions, but only an average of the mass based on total surface per unit weight or volume.

with the microscopic setup. Henry Green (1921) developed a method useful for materials which are difficult to disperse. Green's technique is essentially as follows: Approximately 1 mg of the material is placed in the center of a microscope slide. To this small quantity is added a drop of redistilled turpentine. The slide is held at one end and a glass rod with smooth, straight edges is brought into contact with the glass, the turpentine and fine particles being rubbed into a very thin layer. This is best accomplished by a to-and-fro motion of the rod along the length of the slide, care being taken to confine the material to the central area. Green states that the rubbing must cease at a certain critical stage when sufficient unevaporated turpentine still remains to prevent the dispersed material from becoming sticky. When this critical point is reached, a quick flourish of the rod on the last stroke will draw the particles in a wedge shape so that one extreme of the smear is denser than the other. Thus, when the slide is examined a suitable area may be selected for study. After the material has been dispersed properly the remaining turpentine is evaporated by placing the slide on a hot plate. The plate should be at a temperature which causes complete evaporation within 40 or 50 sec. When the turpentine is completely evaporated, particles will be cemented to the glass, and will remain in this condition as long as care is exercised in handling the slide.

Silverman and Franklin (1942) criticized Green's method of preparing slides, pointing out that the vigorous treatment given the particles causes them to break. Several examples of this effect were discussed. These investigators state that evaporation methods are more representative of the correct size-distribution and are to be preferred.

Preparation of Suspensions—Particle-size determination by sedimentation methods requires that the particles be discrete. This is frequently attained with water which has been treated with a wetting agent. Materials containing no clumps or conglomerate particles and which do not coagulate may be used without further treatment. In the case of soils containing conglomerate material as well as organic matter, the sample must be subjected to extensive treatment. The type of treatment required has been outlined by Olmstead *et al* (1930); the technique is as follows:

1. The sample for analysis is mixed with a spatula and quartered. The quarter reserved for analysis is rolled with a wooden rolling pin and then passed through a sieve with 2-mm round holes. Two samples of each soil are weighed out at the same time, a 10-g sample for analysis and a 5-g sample for moisture determination. The latter is dried at 105 deg C for 16 hrs, cooled in a desiccator, and weighed. The 10-g sample is placed in a 250-cu cm pyrex electrolytic beaker.

2. To the 10-g sample are added 40 cu cm of 6 percent hydrogen peroxide, to oxidize the organic matter. The beaker is shaken frequently until the reaction has quieted down. More peroxide may be added if required, and the beaker is then left on a slow steam bath overnight. If more organic matter is present at the end of this period, full-strength hydrogen peroxide is added. In case manganese dioxide is known to be present, a small quantity of acetic acid is added. About 50 mg of glacial acetic acid are usually sufficient for a soil containing 0.5 percent manganese dioxide.

3. If acid treatment is desired, the sample is cooled and then treated with 10 cu cm of normal hydrochloric acid, made up to a volume of 100 cu cm, and allowed to stand overnight. More acid may be added if it is desired to remove all carbonates. In this case, care should be taken to determine precisely the amount of carbonates present in the soil samples.

4. The sample, with or without acid treatment, is now ready for washing. This is done by removing the solution in a beaker with a short Pasteur-Chamberland suction filter, the lower 12 cm of the filter being sawed off and fitted with a removable stopper. The liquid is removed from the hollow filter core and replaced with distilled water. To remove adhering soil, back pressure is applied by means of a rubber pump. The sample is stirred with 125 cu cm of distilled water and the solution again removed by filtration. Six such washings are usually sufficient.

5. After cleaning and removing the suction filter the soil sample is evaporated to dryness on a steam bath and dried in an oven at 105 deg C for 16 hrs. Upon cooling in a desiccator the sample is quickly weighed so that reabsorption of moisture is kept at a minimum. The sample is then soaked for a few minutes with 25 cu cm of water, stirred with a rubber policeman, and transferred to a 250-cu cm nursing bottle. The sample contained in the beaker is again dried and weighed. The difference in weight is equal to the portion of the sample left after organic matter and solution loss. This is the portion of the sample upon which the percentage of size classes is based. To the shaking bottle are added 10 cu cm of $^1/_2$-normal sodium oxalate and the volume is made up to 150 cu cm. This bottle is shaken in a reciprocating shaker overnight.

6. The sample is wet-sieved through a Tyler 300-mesh phosphor-bronze twilled-wire-cloth sieve which passes material finer than 50 μ. All the clay and the finer portion of the silt are washed through the sieve into a 1-liter sedimentation chamber. The silt remaining with the sands is dry-sifted through another 300-mesh Tyler sieve, added to a series of sand sieves.

7. The fines passing the screen consist of silt and clay and are subject to sedimentation analysis, as outlined in the earlier sections of this chapter.

SOIL-CLASSIFICATION SYSTEMS

Many attempts have been made to classify particles according to size. This is particularly important with regard to soils whose particle-sizes may range from as low as 0.0002 to as high as 2 mm in diameter. Par-

ticle-size, as shown elsewhere, greatly affects the reaction of the particles to temperature and other effects. Thus, very fine clay particles tend to obey the gas laws and are subject to Brownian movement. Larger particles are subject to different laws and behavior. Moreover, there is no simple method for obtaining size-distribution of particles whose size-range is comparable to that in soils, and for this reason it is necessary to size small fractions of the sample. By sizing is meant the separation of the fines from the larger particles, in various steps. There have been many systems of classification proposed. These systems are conditioned by custom and to some extent by the kind of materials used. There are three systems of classification in general use: (*a*) The United States De-

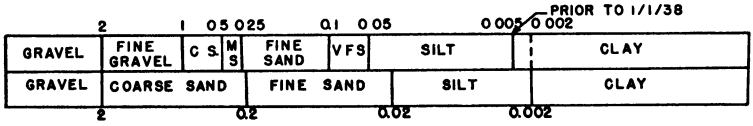


FIGURE 20. U. S. DEPT. OF AGRICULTURE (UPPER) AND INTERNATIONAL (LOWER) SOIL CLASSIFICATIONS.

partment of Agriculture system; (*b*) the Atterberg (1908) or International system; and (*c*) the Krumbein (1937) scale. These systems apply to the mechanical analysis of soil.

A comparison of the U. S. Department of Agriculture system and the Atterberg system with respect to the size-limits included and their designations is shown in Figure 20. With modification of the pipette method reported by Shaw and Miles (1939), the Department of Agriculture system can easily be made to conform to the Atterberg system. The importance of providing a degree of uniformity lies in the fact that the Atterberg system has been adopted as the standard classification of the International Society of Soil Science since 1934.

It will be noted that the size-groups are simpler and more uniform in the Atterberg system. Atterberg states that particles above 2 mm in diameter retain little if any water, while those below this size retain appreciable amounts. Fine sand in the Atterberg system includes most of the fine sand, very fine sand, and part of the coarser silt of the Department of Agriculture system. Truog *et al* (1936) note that there is little reason for several fractions in the region in question, namely 0.2 to 0.02 mm, since the particles are too coarse to be active chemically. Atterberg states that the fine sand of his system has considerable water-holding capacity and is significant when considering capillary movement. With regard to silt, Atterberg says that the larger particles of this component are just below visibility to the naked eye, but that when the same

silt is compacted the interstices are too small to permit penetration of root hairs or the ready movement of capillary water. The upper limit of particle-size for clay in the Atterberg system is 0.002 and was chosen because it is the upper limit of Brownian movement. Recent changes adopted by the Department of Agriculture make the lower limit of their classification conform to the Atterberg classification.

Krumbein's phi and zeta scales have a certain mathematical interest since they help simplify certain calculations. The phi scale is based on the natural logarithm of the diameters expressed in mm, that is

$$d = 2^{-\phi}$$

Krumbein's zeta scale is especially applicable to Atterberg's system of classification. The diameter of a particle expressed in mm is denoted by the expression

$$d = 2 \times 10^{-\zeta}$$

The Atterberg classification and zeta scale are shown in Table 10.

TABLE 10—THE ATTERBERG CLASSIFICATION AND THE ZETA SCALE

Atterberg's diameters (mm)	ζ
2000	−3
200	−2
20	−1
2	0
2	1
0.2	2
0.02	3
0.002	4

Problems

1. Using the data given in Problem 1 of the previous chapter plot the corresponding size-frequency distributions.

2. Assuming a particle-density of 1.25, determine the air velocities necessary to separate the size-fraction 120 μ to 180 μ in Problem 1 of Chapter 3. Discuss the method used for obtaining the fraction.

3. If the total weight of the sample being elutriated in Problem 2 is 100 g, calculate the approximate weight of the elutriated sample, assuming that the separation is complete.

4. A water suspension of soil is prepared at 10 A.M. At 12 M. a sample is drawn from the suspension at a depth of 5 cm. What will be the largest particle removed by a pipette at this depth if the density of the soil is taken as 2.5?

5. Determine the effective depth of a hydrometer used for particle-size measurement, given the following information:

Length of hydrometer bulb	17.0 cm
Cross-sectional area of graduate	28.0 sq cm
Volume of hydrometer bulb	38.1 cu cm
Distance from top of bulb to maximum reading of scale	2.1 cm
Length of specific gravity scale	6.8 cm
Specific gravity range	1.050–1.025

6. Given the following information pertaining to determination of particle-size by means of a hydrometer:

Specific gravity of original suspension	1.15	
Density of original sample	71.8	lb per cu ft
Volume of original sample for analysis	210.9	cu cm
Weight of solids in sample (110 deg C)	50.0	g
Specific gravity of solids	2.67	
Viscosity of suspension	1.2	centipoises
Temperature of suspension	70	deg C

HYDROMETER OBSERVATIONS

Elapsed time (min)	Observed specific gravity of suspension	Effective depth (cm)	Suspension weight density
1	1.0242	12.71	1.0245
3	1.0219	13.41	1.0222
10	1.0172	8.50	1.0175
30	1.0159	8.82	1.0162
60	1.0149	9.06	1.0152
120	1.0109	10.02	1.0112
300	1.0041	11.66	1.0044
440	1.0010	12.41	1.0013

Compute the size-distribution of the suspension, assuming 87.8 percent is less than 200-mesh Tyler sieve.

CHAPTER 5

THEORY OF SIEVING AND GRADING OF MATERIALS

THE most rapid and convenient method for size separation of particles is by sieves. Sieves are constructed of wire cloth with definitely spaced openings so that material may be graded uniformly. Any size of opening may be obtained by proper choice of wire size and appropriate spacing, although it is customary to use openings and wire sizes which, on the basis of experience, have been proved to give satisfactory size separation. In this chapter we are concerned chiefly with the theory of sieving. Sieve-sizes, where stated, refer to standard series. The results apply generally to test sieves, but it is obvious that they have wide application to industrial conditions.

All crushing and pulverizing equipment has a point of maximum efficiency which is a function of particle-size. If we exceed this size or fall below it, we fail to obtain the highest capacities of the equipment. Moreover, it is desirable so to utilize such equipment that the maximum amount of material produced is of the size desired. This is but one example; other examples are the necessary fineness of cement and sizing of aggregates to produce concrete mixtures for maximum strength, the production of uniform-grade ceramic ware, and the manufacture of abrasives with definite cutting qualities.

It is customary to use square openings in the various types of sieves for industrial and test purposes. However, round and rectangular openings are frequently used, the former in cylindrical-type screens. As will be shown later, neither the square opening nor the round opening is capable of sharply differentiating sizes. Much depends upon the shape of the particles and how they pass through the openings. The average size of particles passing one screen and retained upon another (next in size) can be determined only approximately from the size of the openings. As already explained, this is partly due to the shape of the particles; however, to some degree it is also a function of the shape of the opening itself. Thus, given a square opening of a side equal to l, any needle-shaped particle can be made to pass diagonally if it has dimensions less than $l\sqrt{2}$. Sieves and screens have contributed much to industrial progress even though particle-size differences do occur in carefully graded

TABLE 11—THE TYLER STANDARD SCREEN SCALE SIEVES

This screen scale has as its base an opening of 0.0029 in. which is the opening in 200-mesh 0.0021-in. wire, the standard sieve, as adopted by the Bureau of Standards of the United States Government, the openings increasing in the ratio of the square root of 2 or 1.414.

Where a closer sizing is required Column 5 shows the Tyler Standard Screen Scale with intermediate sieves In this series the sieve openings increase in the ratio of the fourth root of 2 or 1.189.

1	2	3	4	5	6	7	8	9
Tyler Standard Screen Scale $\sqrt{2}$ or 1 414 openings (in.)	Every other sieve from 0 0029 in to 0.742 in Ratio of 2 to 1	Every other sieve from 0 0041 in. to 1.050 in Ratio of 2 to 1	Every 4th sieve from 0.0029 in to 0 742 in. Ratio of 4 to 1	For closer sizing sieves from 0.0029 in to 1 050 in. Ratio $\sqrt[4]{2}$ or 1.189	Open-ings (mm)	Open-ing in frac-tions of inch (ap-prox.)	Mesh	Diam-eter of wire
1.050	....	1.050	....	1.050	26.67	1	...	0.148
....	....	....	....	0.883	22.43	7/8	...	0.135
0.742	0.742	....	0.742	0.742	18.85	3/4	...	0.135
....	....	....	....	0.624	15.85	5/8	...	0.120
0.525	....	0.525	....	0.525	13.33	1/2	...	0.105
....	....	....	....	0.441	11.20	7/16	...	0.105
0.371	0.371	...	....	0.371	9.423	3/8	...	0.092
....	....	....	...	0.312	7.925	5/16	2 1/2	0.088
0.263	....	0.263	....	0.263	6.680	1/4	3	0.070
....	....	....	....	0.221	5.613	7/32	3 1/2	0.065
0.185	0.185	....	0.185	0.185	4.699	3/16	4	0.065
....	....	....	....	0.156	3.962	5/32	5	0.044
0.131	....	0.131	....	0.131	3.327	1/8	6	0.036
....	....	....	....	0.110	2.794	7/64	7	0.0328
0.093	0.093	....	....	0.093	2.362	3/32	8	0.032
....	....	....	....	0.078	1.981	5/64	9	0.033
0.065	....	0.065	....	0.065	1.651	1/16	10	0.035
....	....	....	....	0.055	1.397	..	12	0.028
0.046	0.046	....	0.046	0.046	1.168	3/64	14	0.025
....	....	....	....	0.0390	0.991	..	16	0.0235
0.0328	....	0.0328	....	0.0328	0.833	1/32	20	0.0172
....	....	....	....	0.0276	0.701	..	24	0.0141
0.0232	0.0232	....	....	0.0232	0.589	..	28	0.0125
....	....	....	....	0.0195	0.495	..	32	0.0118
0.0164	....	0.0164	....	0.0164	0.417	1/64	35	0.0122
....	....	....	....	0.0138	0.351	..	42	0.0100
0.0116	0.0116	....	0.0116	0.0116	0.295	..	48	0.0092
....	....	....	....	0.0097	0.246	..	60	0.0070
0.0082	....	0.0082	....	0.0082	0.208	..	65	0.0072
....	....	....	....	0.0069	0.175	..	80	0.0056
0.0058	0.0058	....	....	0.0058	0.147	..	100	0.0042
....	....	....	....	0.0049	0.124	..	115	0.0038
0.0041	....	0.0041	....	0.0041	0.104	..	150	0.0026
....	....	....	....	0.0035	0.088	..	170	0.0024
0.0029	0.0029	....	0.0029	0.0029	0.074	..	200	0.0021

material. For large particles and aggregates there is no simpler method of grading material than sieving. Even the production of fine materials, of the order of 70 μ and less, depends on the use of wire-cloth screens. In fact no industry, whether it be road building, ceramics, or chemical engineering, can succeed without the necessary sieve equipment to size materials. In spite of great reliance on sieves for grading of materials, little is known about the rate of sieving for a mass of material of different sizes through a given screen.

TYPES OF SIEVES

Tyler Sieve Series—Because of the economic importance of sieving operations, it was necessary to standardize sizes of sieve openings. In this country recognition is given two types of wire-cloth screen: The so-called "Tyler" sieves and the U. S. Sieve Series. British practice calls for use of the British Engineering Standards Association (B.S.A) Series. The Tyler Standard Screen Scale consists of openings whose sizes are in the fixed ratio of $\sqrt{2}$. These screens were brought out by the W. S. Tyler Company in 1910, and for many years were the only standardized sieves used in the United States. Various bases have been proposed as the starting point of screen scales; some advocated the use of 1 in., and others of 1-mm openings. The Tyler series is based on the size of the opening in 200-mesh wire cloth. The standard Tyler 200-mesh sieve is made from 0.001-in. wire and has an opening of 0.0029 in. Table 11 gives the Tyler Standard Screen Scale. It will be noted that the openings vary from 1.050 to 0.0029 in. Screens larger and smaller than those given in the table are available, but for sizing of materials, laboratory practice, and industrial grading according to a specified scale, the sizes shown in Table 11 are most practicable.

U. S. Sieve Series—A series of sieves, differing but slightly from the Tyler Standard Screen Scale, was proposed in 1919 by the National Bureau of Standards. This has been called the U. S. Sieve Series and in general the sizes advance in the same ratio as the Tyler Screen Scale, but a 1-mm opening is used as a base rather than the opening for the 200-mesh cloth in the Tyler series. There appears to be no reason for preferring one type of series over the other. Table 12 shows the various characteristics of the U. S. Sieve Series adopted by the A.S.T.M. which may be contrasted with those of the Tyler Screen Series given in Table 11.

The I.M.M. Series and British Engineering Standards Association Series—The I.M.M. Series was developed in 1907 by the Institute of Mining and Metallurgy of England. This series is mainly for laboratory use, and was developed on the theory that the diameter of the wire

should be equal to the size of the screen opening, keeping as nearly as possible to 25 percent opening and 75 percent dead space. The object

TABLE 12—NOMINAL DIMENSIONS, PERMISSIBLE VARIATIONS, AND LIMITS FOR WOVEN WIRE CLOTH OF STANDARD SIEVES, U. S. SERIES. A.S T.M. STANDARD[a]

Size or sieve designation Micron	No	Sieve opening Mm	Inch (approx. equivalents)	Permissible variations in av. opening (percent ±)	Permissible variations in max opening (percent ±)	Wire diameter Mm	Inch (approx equivalents)
5660	3½	5.66	0.233	3	10	1.28 -1.90	0.050 -0.075
4760	4	4.76	0.187	3	10	1.14 -1.68	0.045 -0.066
4000	5	4.00	0.157	3	10	1.00 -1.47	0.039 -0.058
8360	6	3.36	0.132	3	10	0.87 -1.32	0.034 -0.052
2330	7	2.83	0.111	3	10	0.80 -1.20	0.031 -0.047
2380	8	2.38	0.0937	3	10	0.74 -1.10	0.0291-0.0433
2000	10	2.00	0.0787	3	10	0.68 -1.00	0.0268-0.0394
1680	12	1.68	0.0661	3	10	0.62 -0.90	0.0244-0.0354
1410	14	1.41	0.0555	3	10	0.56 -0.80	0.0220-0.0315
1190	16	1.19	0.0469	3	10	0.50 -0.70	0.0197-0.0276
1000	18	1.00	0.0394	5	15	0.43 -0.62	0.0169-0.0244
840	20	0.84	0.0331	5	15	0.38 -0.55	0.0150-0.0217
710	25	0.71	0.0280	5	15	0.33 -0.48	0.0130-0.0189
590	30	0.59	0.0232	5	15	0.29 -0.42	0.0114-0.0165
500	35	0.50	0.0197	5	15	0.26 -0.37	0.0102-0.0146
420	40	0.42	0.0165	5	25	0.23 -0.33	0.0091-0.0130
350	45	0.35	0.0138	5	25	0.20 -0.29	0.0079-0.0114
297	50	0.297	0.0117	5	25	0.170-0 253	0 0067-0.0100
250	60	0 250	0.0098	5	25	0.149-0.220	0.0059-0.0087
210	70	0.210	0.0083	5	25	0.130-0.187	0.0051-0.0074
177	80	0 177	0.0070	6	40	0.114-0.154	0.0045-0.0061
149	100	0.149	0.0059	6	40	0.096-0.125	0.0038-0.0049
125	120	0.125	0.0049	6	40	0.079-0.103	0.0031-0.0041
105	140	0.105	0.0041	6	40	0.063-0.087	0.0025-0.0034
88	170	0.088	0.0035	6	40	0.054-0.073	0.0021-0.0029
74	200	0.074	0.0029	7	60	0.045-0.061	0.0018-0.0024
62	230	0.062	0.0024	7	90	0.039-0.052	0.0015-0.0020
53	270	0.053	0.0021	7	90	0.035-0.046	0.0014-0.0018
44	325	0.044	0.0017	7	90	0.031-0.040	0.0012-0.0016
37	400	0.037	0.0015	7	90	0.023-0.035	0.0009-0.0014

[a] For sieves from the 1000-micron (No. 18) to the 37-micron (No. 400) size, inclusive, not more than 5 percent of the openings shall exceed the nominal opening by more than one-half of the permissible variation in the maximum opening.

of maintaining this relation between wire diameter and sieve opening seems to have been that it would prevent the wires from shifting. In

modern sieving practice there is no need for concern about shifting or slipping of wires since crimping forces them to stay in place. Due to selection of openings in terms of wire diameters, the I.M.M. series is made of heavy wire and is not always suitable for test sieves. Some question also arises on the desirability of limiting the open area of the sieve to a ratio of 1:3. In spite of these objections, however, the screen has found wide use in England for more than 30 years. The chief characteristics of this type of series are shown in Table 13.

TABLE 13—THE BRITISH ENGINEERING STANDARDS ASSOCIATION AND I.M.M. SIEVE SERIES

B. S. A. sieve	Nominal size of aperture	Nominal diameter of wire	Equivalent Tyler mesh	Equivalent I.M.M. sieve
5	0.1320	0.068	6	...
6	0.1107	0.056	7	...
7	0.0949	0.048	8	...
8	0.0810	0.044	9	...
10	0.0660	0.034	10	...
12	0.0553	0.028	12	...
14	0.0474	0.024	14	...
16	0.0395	0.023	16	...
18	0 0336	0.022	20	...
22	0 0275	0.018	24	...
25	0.0236	0.0164	28	...
30	0 0197	0.0136	32	...
36	0.0166	0.0112	35	30
44	0.0139	0.0088	42	...
52	0.0116	0.0076	48	...
60	0 0099	0.0068	60	50
72	0 0083	0.0056	65	60
85	0.007	0.0048	80	70
100	0.006	0.004	100	80
120	0.0049	0.0034	115	100
150	0.0041	0.0026	150	120
170	0.0035	0.0024	170	...
200	0.003	0.002	200	150
240	0.0026	0.0016	250	200

The I.M.M. Series has been superseded by the British Engineering Standards Association Series (1932). This sieve series is also shown in Table 13. It will be noted that the openings follow closely those of the Tyler series, the differences being due to the desire of specifying British Standard Wire diameters.

German and French Series—Because foreign literature often lists experiments in which powdered material is sized by sieves peculiar to a

given country, for completeness we also list in Table 14 the screens used in France and Germany. Those for France are designated AFNOR (*Association française de normalisation*, 1938). The German series, DIN, refers to the *Deutsche Industrie-Normen*, 1171, 1934. It is very much similar to the French series in many respects.

TABLE 14.—FRENCH AND GERMAN STANDARD SIEVE OPENINGS (MM)

French Series AFNOR (1938)	German series DIN (1934)	Equivalent Tyler mesh number	Equivalent U. S standard mesh number
..	...	...	. .
5 00	5.00		...
4 00	4.00	5	5
3.15	3.00	.	...
2 50	2 50		...
2 00	2 00	9	10
1 60	1 50 (4)		.
1 25	...	.	.
	1 20 (5)	14	16
1 00	1 00 (6)	16	18
0 80	..	.	.
	0 75 (8)	.	..
0 63			.
.	0 60 (10)	28	30
0 50	0 50 (12)	32	35
0 40	0 43 (14)	35	40
	0 40 (16)	.	...
0 315		. .	
	0 30 (20)	48	50
0 25	0 25 (24)	60	60
0.16	..		.
	0 15 (40)	100	100
0 125	0 12 (50)	115	120
0.10	0 10 (60)	150	140
...	0.09 (70)	170	170
0 08	..		...
...	0 075 (80)	200	200
0 063	0.060 (100)	250	230
0 05	...	270	270

Large-Opening Screens—For industrial sizing, screens may be more than 4 in. in size and may have either square openings, or round openings which are approximately equivalent to the stated sizes of square openings in testing sieves. The relation between square and round open-

ings is shown in Table 15. The numbered sieves are those of the U. S. Standard Sieve Series. Typical uses for the sizes listed are given in Table 16. The size numbers and nominal size square openings are those developed by the U. S. Bureau of Standards.

Size Ratio of Openings—In this country the ratio between the different sieve openings has been taken as $\sqrt{2}$, a practice recommended by Rittinger (1867). This, of course, is only approximately true in the I.M.M. Series, as indicated. In this latter series, the ratio is determined entirely by the proper selection of wire diameters for the size of opening to be produced. In the Tyler series, developed from a 200-mesh sieve as the base (0.0029-in. opening), the width of opening in each successive screen is $\sqrt{2}$ times the opening in the preceding sieve. The choice of this ratio makes the area of each successive opening double that of the

TABLE 15.—APPROXIMATELY EQUIVALENT ROUND- AND SQUARE-OPENING TESTING SCREENS

(This table shows the sizes of round openings which are approximately equivalent to the stated sizes of square openings in testing sieves. Numbered sieves are those of the United States Standard Sieve Series.)

Square openings	Round openings (in.)	Square openings (in.)	Round openings (in.)
No. 8	1/8	1 1/4	1 1/2
No. 4	1/4	1 1/2	1 3/4
3/8 in.	1/2	1 3/4	2
1/2 in	5/8	2	2 3/8
3/8 in.	3/4	2 1/4	2 3/4
5/8 in.	7/8	2 1/2	3
7/8 in.	1	3	3 1/2
1 in.	1 1/4	3 1/2	4 1/4
1 1/8 in	1 3/8	4	4 3/4

next finer, or one-half that of the next coarser sieve. Another advantage of the Tyler series, due to selection of the $\sqrt{2}$ ratio, is that by omitting every other screen a ratio width of 2 to 1 is obtained. Skipping two sizes the ratio is approximately 3 to 1, and by skipping three sizes the ratio is 4 to 1, etc. Thus, for many important industrial operations it is possible to obtain $\sqrt{2}$, 2, 3, or 4 to 1 ratios of openings from the Tyler series.

The U. S. Sieve Series does not progress by the same ratio as the Tyler series in every instance. This may be checked by reference to Tables 11 and 12. The U. S. Sieve Series has been widely used in this country, and accepted as the standard by the American Society for Testing Materials.

TABLE 16—TYPICAL APPLICATIONS OF SIEVES FOR SIZING COARSE AGGREGATES

Use	Size number and nominal size[a]																			
Size number	1	2	24	3	357	4	467	5	57	6	67	68	7	79	8	9	10	G1	G2	G3
Nominal size	3½ to 1½	2½ to 1½	2½ to ¾	2 to 1	2 to No 4	1½ to ¾	1½ to No 4	1 to ⅜	1 to No 4	¾ to ⅜	¾ to No 4	¾ to No 8	½ to No 4	½ to No 8	⅜ to No 8	No. 4 to No. 16	No. 4 to 0	1½ to No. 50	1½ to No 8	1½ to No. 4
Water-bound macadam:																				
Coarse aggregate	×	×		×																
Filler																	×			
Bituminous macadam, penetration method:																				
Coarse aggregate	×	×		×																
Choke								×		×	×			×						
Seal															×	×				
Bituminous plant mixes																				
Base or surface courses:																				
Base, open mix				×		×														
Base, closed mix					×		×		×											
Binder course									×		×									
Surface course, coarse grading								×	×		×									
Surface course, fine grading															×	×				
Seal															×	×				
Bituminous road mix:																				
Mixing course								×			×									
Choke															×	×				
Seal															×	×				
Drag leveling course:																				
Leveling course											×	×		×						
Seal															×	×				
Bituminous surface treatment										×		×		×	×	×				
Seal for airport construction																×				
Portland cement concrete	×	×		×	×	×	×		×		×		×							
Railroad ballast:																				
Stone or slag			×	×		×		×												
Gravel																		×	×	×
Roofing															×					

[a] In inches, except where otherwise indicated. numbered sieves are those of the United States Standard Sieve Series.

For closer sizing it is possible to provide a ratio between sieve openings of $\sqrt[4]{2}$. The ratio between sizes of the openings is then $\sqrt[4]{2}$ or 1.189, and the difference in areas between two successive openings is about $1^1/_2$ times instead of 2, as with the Tyler and U. S. series. Reference to Table 11 shows the refinements possible with the use of the $\sqrt[4]{2}$ series

In addition to the $\sqrt{2}$ and $\sqrt[4]{2}$ series, it is also possible to utilize a series whose successive differences are in the ratio of $\sqrt[3]{3}$. This series includes both the English inch and the metric millimeter, and should have had a certain amount of international acceptance. Unfortunately, it has never been practical to work out a series of such meshes and sizes of wire that could be woven satisfactorily. In addition to this objection, the cube root of the series does not include any of the sieves now in general use.

SIEVING EFFICIENCY

Factors Affecting Efficiency—To obtain a gradation of material passing from one sieve to another or to a container, some motion must be given to the particles contained on the screens so they will pass through the openings. This motion may be imparted by hand or by some apparatus which either vibrates or shakes the material. The method chosen should be one in which a maximum amount of material will pass a given screen in the shortest period of time. There are several mechanical shaker devices available such as the Tyler Ro-Tap, vibrators of the Hummer electric type, or the Leahy No-Blind screen, which depend on a high-frequency vibration imparted to the material being screened. Fahrenwald and Stockdale (1929) made a careful study of the effect of sieve motion on sieving efficiency. These investigators point out that 100 percent screening efficiency is unattainable inasmuch as the element of time is an important factor. Thus, a single difficult grain placed on a vibrating sieve may pass the sieve almost at once, but on a second trial may remain for a considerable time. The mechanics of sieving were aptly described by Stephenson (1926–1927), who compared sieving to floating timber logs down a river. The logs pile up against boulders and jam, then break loose and travel on for some time before jamming again. He indicated this periodicity in the case of sieving quartz. The quartz was originally passed through a 183-mesh silk screen and then made to pass a 200-mesh copper screen The data obtained in a test are shown in Figure 21.

Effect of Screen Motion—It will be seen that from the first to the fourth minute the amount passed ranged from 2.272 to 2.314 g per min, but after the seventeenth minute there was considerable variation in the

percentages which passed. The nature of the jamming is clearly indicated by the inconsistent character of the amounts passing from minute to minute. Stephenson did not describe the method by which the sieving was done, but in later paragraphs this will be shown to be extremely important. However, assuming that whatever the method used it was constant throughout the experiment, the results indicate clearly the importance of the time element in sieving. The experiments of Fahrenwald and Stockdale were more extensive than those described by Stephenson. These investigators utilized two types of materials: Pure

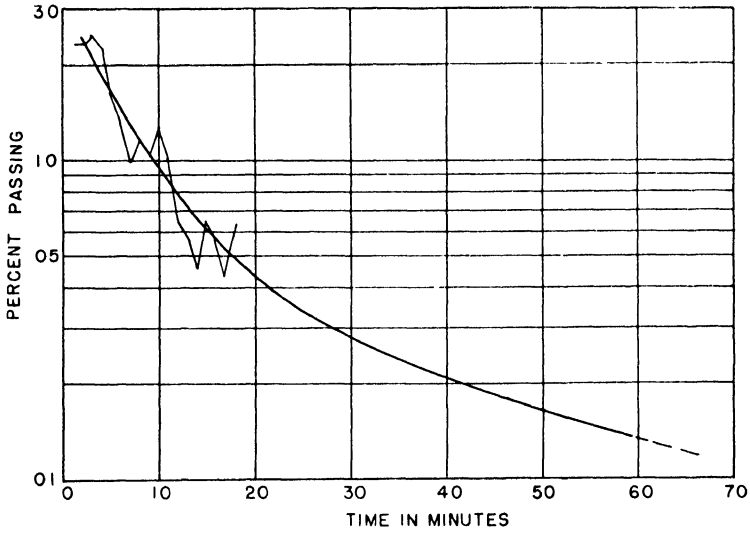


FIGURE 21. RATE-OF-SIEVING CURVE

quartz (specific gravity 2.65) crushed to pass a 10-mesh sieve, and galena (specific gravity 7.5) prepared in a like manner. The quartz and galena were screened into six sizes ranging from 14/20-mesh to 65/100-mesh Tyler sieve. The sieving was done with the Tyler Ro-Tap sieve shaker, and shaken by hand until practically no more material passed the limiting sieve.

The following sieve motions were employed:

1. Horizontal rotary motion produced by a Ro-Tap sieve shaker, without the aid of the hammer or bumper.
2. Horizontal rotary motion combined with a vertical bumping or jarring motion such as produced by the Ro-Tap shaker with the aid of the hammer, which struck the screen frame about 155 times per min.

3. Vertical jarring motion produced by striking the bottom of a screen, held in one hand, with the palm of the other hand, using sufficient force to make the particles jump about.
4. Horizontal jarring or shaking motion produced by tapping with the palm of one hand, while holding a screen in the other—these taps were imparted to the screen at the rate of about 240 per min.
5. High-frequency vibrating motion similar to that produced by the Hummer electric, or the Leahy No-Blind screen.

Efficiency tests with sand and galena were made in the intermediate-size sieves of the Tyler $\sqrt[4]{2}$ series. In this way the average of the limiting sizes (Tyler $\sqrt{2}$ series) was almost equal to the size of the opening of the testing sieve, and the ratio of undersize to oversize was about one to one. Thus the condition of the feed in each test was made as difficult as possible.

In their experiments with high-frequency vibrating sieves, Fahrenwald and Stockdale constructed a special apparatus with which the frequency and amplitude of vibration could be adjusted at will. The frequencies used in the experiments ranged from 500 to 2500 vibrations per min, while the amplitude of vibrations extended from 0.1 mm to approximately 0.7 mm.

Tests were also made with thin and thick beds of material, with the motions previously described. In the thin-bed tests conducted sufficient material was weighed out to cover the screen surface 1 particle in depth. For quartz this was approximately 25 g and for galena approximately 75 g. The sample was placed in the testing sieve and the sieve was shaken until all the material was scattered evenly over the screen surface. The material that passed through the screen in this operation was returned to the sieve surface. The sieve was then subjected to the motions previously described. When tests on deep beds of material were made, 250-g samples were used, giving a depth of approximately $^1/_4$ in. The material was subjected to the same motions as for the thin bed. Typical results obtained from various experiments are shown in Figure 22. It will be noted that the maximum amount of material sieved through each series of screens during a given interval of time does not follow any definite law. The figures also indicate that in so far as the relative efficiencies of the various methods are concerned, vibratory motion is first (provided the best combination of amplitude and frequency is used), followed successively by side-tap (hand) motion, bottom-tap (hand) motion, Ro-Tap (with hammer) motion, and Ro-Tap (without hammer) motion.

The curves of Figure 22 obtained with thick beds of material are not consistent with those obtained for thin beds of material. No data are

available on the effect of varying the load on the screens. Blinding is excessive in each motion used, except with the vibrator, in which case it is at a minimum. The data show further that with large loads the

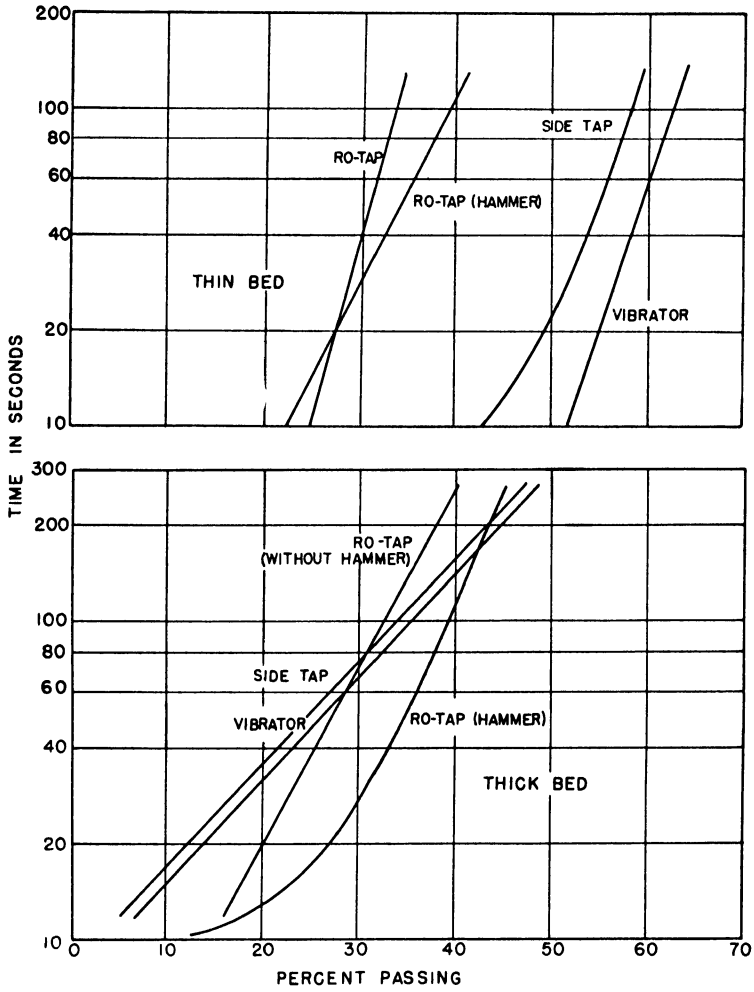


FIGURE 22. EFFECT OF VARIOUS MOTIONS ON RATE OF SIEVING.

amount of material passing a given series of sieves is approximately 25 percent less than with smaller amounts. Even after sieving for a relatively long period of time the amount of material passing for a given screen motion is never equal to that for the smaller bed loads. In general

the data suggest that if a set of screens can be standardized for a given load, the time required to screen additional material is almost proportional to the amount of material placed upon the screen. In other words if 2 min are required to screen 10 g of a given material, 100 g will require approximately 20 min. This, of course, assumes that the materials concerned are similar and that the sieve motions are the same in each case.

The results of studies made by varying the frequency and amplitude of the vibration are shown in Figure 23. The curves of the figure show that amplitude of vibration has a greater effect on the rate of screening than frequency. A change in amplitude of approximately 0.20 mm

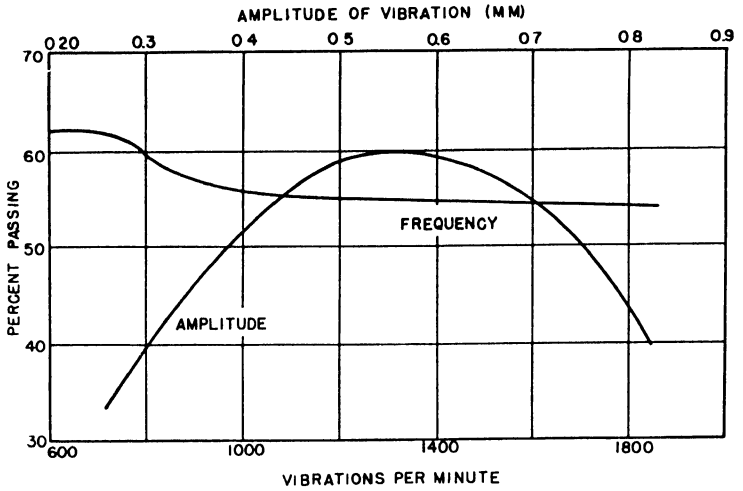


FIGURE 23 RELATION OF FREQUENCY AND AMPLITUDE OF VIBRATION TO EFFECTIVENESS OF SIEVING

produces a greater change in the screening rate than a change in frequency of 300 or 400 vibrations per min. The condensed data showing amplitudes of vibration and frequencies, taken from the peaks of the amplitude and frequency curves for different sizes of quartz particles, are shown in Table 17.

The "efficiency" of the various sieve motions may be expressed by the formula

$$\text{efficiency} = \frac{\text{percentage material actually passing}}{\text{total percentage capable of passing}}$$

Fahrenwald and Stockdale considered the total material capable of passing a given sieve as that amount of material passing a sieve in 120

TABLE 17—COMBINATION OF AMPLITUDES AND FREQUENCIES OF VIBRATION FOUND MOST EFFICIENT FOR QUARTZ OF VARIOUS SIZES

Tyler mesh	Amplitude of vibration (mm)	Frequency of vibration (vpm)
14/20	0.60	580
20/28	0.40	1100
28/35	0.35	1350
35/48	0.30	1350
48/65	0.30	1500
65/100	0.20	1700

sec when subjected to vibratory motion. In comparing the effectiveness of various sieve motions, the total amount passing in this time was used as a criterion. Data using this criterion are presented in Table 18.

TABLE 18—EFFECTIVENESS OF VARIOUS SIEVE MOTIONS

Material used	Size mesh	Ro-Tap (without hammer) motion (percentage through)	Ro-Tap (with hammer) motion (percentage through)	Bottom-Tap (hand) motion (percentage through)	Side-Tap (hand) motion (percentage through)	Vibratory motion (percentage through)	Maximum difference (percentage through)
Quartz	14/20	30.2	36.7	55.9	56.2	54.8	26.0
Quartz	20/28	46.8	45.4	76.3	73.4	76.0	29.5
Quartz	28/35	43.8	37.0	60.2	62.7	64.0	27.0
Quartz	35/48	41.3	45.9	56.3	67.8	67.1	26.5
Quartz	48/65	32.6	35.3	32.5	40.9	43.0	10.5
Quartz	65/100	43.8	54.8	55.6	56.8	57.4	13.6
Galena	14/20	44.2	50.5	..	60.0	63.2	19.0
Galena	20/28	32.3	36.0	..	53.6	54.7	22.4
Galena	28/35	25.1	25.6	..	39.6	43.1	18.0
Galena	35/48	34.3	37.7	..	52.3	53.1	18.8
Galena	48/65	27.1	35.3	..	42.3	43.3	16.2
Galena	65/100	33.4	38.1	..	42.6	44.7	11.3

Lister (1924) found that the rapidity of screening through round-opening screens varied approximately in proportion to the square root of the diameters of the holes. Expressed on a weight basis Lister gives the following formula

$$w = k\sqrt{D} \qquad \text{Eq (5-1)}$$

where w is the weight of material passing in a unit-time, D the diameter of the screen opening, and k a constant to be determined by experiment. Lister gave no reasoning for the derivation of this formula.

Fagerholt's Analysis—Fagernolt (1945) gives the following with regard to the laws governing sieving action. If we assume that the weight of particles of a given size passing a sieve in a unit time is proportional to the weight of the particles of the same size remaining on the sieve, we have

$$\frac{dw(d,t)}{dt} = -w'(d)w(d,t)$$

where $w(d,t)$ denotes the weight of particles which are retained on the sieve at the time t, and $w'(d)$ is a constant greater than zero. Integrating from time $t = t_0$ to $t = t$

$$w(d,t) = w(d,t_0) \exp [-w'(d)]t$$

If d_{max} denotes the diameter of the largest particle capable of passing the sieve, then

$$w'(d) \rightarrow 0 \text{ when } d \rightarrow d_{max}$$
$$w'(d) = 0 \text{ when } d \geqq d_{max}$$

Thus, $w'(d)$ is a measure of the probability of passage.

The probability p of a particle passing a sieve, assuming the particle is thrown at right angles to the opening and that it does not touch the wire, has been determined by Mogensen (cited by Fagerholt) as

$$p = \frac{(l - d)^2}{(l + \delta)^2}$$

where l is the width of the mesh, d the diameter of the particle, and δ the thickness of the mesh. For a given sieve we may write

$$p = \alpha(l - d)^2 = \beta(d_{max} - d)^2$$

or as indicated above

$$w'(d) = \gamma(d_{max} - d)^2$$

where α, β and γ are s uitable constants. Thus, for values of $d \leqq d_{max}$

$$w(d,t) = w(d,t_0) \exp [-\gamma t(d_{max} - d)^2]$$

For values of $d \geqq d_{max}$

$$w(d,t) = w(d,t_0)$$

If $C(d_{max})$ denotes the weight of particles capable of passing a given sieve and $R(t)$ denotes the residue remaining at the time t, then

$$C(d_{max}) = 1 - R(t)$$

If $F(d)$ denotes the weight-frequency distribution of the particles being sieved, then the residue $R(t)$ will be expressed by

$$R(t) = \int_0^{\infty} F(d) \frac{w(d,t)}{w(d,t_0)} \cdot d(d)$$

$$= \int_0^{d_{max}} F(d) \cdot \exp[-Ct(d_{max} - d)^2]dt + \int_{max}^{\infty} F(d) \cdot d(d)$$

$$= \int_0^{d_{max}} F(d) \cdot \exp[-Ct(d_{max} - d)^2]dt + 1 - C(d_{max})$$

Since for practical purposes we may say that between two sieve sizes, the particle-weight distribution is linear:

$$C(d) = a + bd = a + F(d) \cdot d$$

and that particles smaller than d_0 have disappeared from the residue in time t. Thus

$$b \int_0^{d_{max}} \exp\,[-\gamma t(d - d_{max})^2] \cdot d(d) =$$

$$b \int_0^{d_0} \exp\,[-\gamma t(d - d_{max})^2]d(d) + b \int_{d_0}^{d_{max}} \exp\,[\gamma t(d - d_{max})^2] \cdot d(d)$$

and following the assumption just made the first integral on the right is approximately zero. Hence we have

$$R(t) = 1 - C(d_{max}) + b \int_{d_0}^{d_{max}} \exp\,[-\gamma t(d_{max} - d)^2] \cdot d(d)$$

and writing $\Phi(d)$ for the probability integral

$$R(t) = 1 - C(d_{max}) + \frac{b\sqrt{\pi}}{\sqrt{\gamma t}}\{\Phi(0) - \Phi[(d_0 - d_{max})\sqrt{2\gamma t}]\}$$

If $(d_{max} - d_0)\sqrt{2\gamma t}$ is very large the corresponding Φ-function can be discarded, so that

$$R(t) = 1 - C(d_{max}) + \frac{b}{2}\sqrt{\frac{\pi}{\gamma t}} \quad \text{(approximately)}$$

Replacing $1 - C(d_{max})$ by $R(\infty)$ which denotes the residue corresponding to complete separation

$$R(t) = R(\infty) + \frac{b}{2}\sqrt{\frac{\pi}{\gamma t}}$$

The weight-distribution being linear between two sieve sizes, we may write

$$b[d_{max} - dR(t)] = \frac{b}{2}\sqrt{\frac{\pi}{\gamma t}}$$

$$dR(t) = d_{max} - \frac{1}{2}\sqrt{\frac{\pi}{\gamma t}}$$

where $dR(t)$ denotes a particle size corresponding to the residue obtained by sieving for a time t and is dependent on particle shape and conditions of sieving. The value of d_{max} cannot be ascertained directly, though reasonable approximations can be made as to its value.

Fagerholt has shown that the average weight diameter d_w of particles may be determined from the weights of residues after two sieving periods t_1 and t_2, $(t_2 > t_1)$, and from corresponding expressions of the number of particles passing. Thus, he obtains

$$d_w = d_{max} - \frac{1}{\sqrt{\pi\gamma}}\left(\frac{1}{\sqrt{t_1}} + \frac{1}{\sqrt{t_2}}\right)$$

$R(t_1) - R(t_2)$ = weight of particles passing. For determining $dR(t)$ above, it is convenient to have the sieving time t_2 adjusted so that

$$d_{R(t_1)} = d_w$$

so that

$$\frac{1}{2}\sqrt{\frac{\pi}{\gamma t_1}} = \frac{1}{\sqrt{\pi\gamma}}\left(\frac{1}{\sqrt{t_1}} + \frac{1}{\sqrt{t_2}}\right)$$

$$\sqrt{\frac{t_1}{t_2}} = \frac{\pi}{2} - 1$$

or

$$t_2 = 3t_1 \quad (approximately)$$

Thus, the particle size $dR(t)$ corresponding to the sieving residue $R(t)$ obtained by sieving in the time t is determined as the average particle

size d_w of the fraction passing the sieve by continued sieving under the same conditions up to the time $3t$.

PARTICLE-SIZE DETERMINATION

Average Size Based on Openings—The use of screens for particle-size determination was mentioned briefly in Chapter 4. When material is passed through one sieve and retained on the next smaller, the size assigned to the particles retained is a matter of choice. We may assign a diameter to the particles which is equal to the size of the opening in the larger or smaller sieve, the arithmetic or geometric mean of the openings of the two sieves, or a statistical average diameter determined by microscopic or other methods of size-measurement for particles retained on the smaller sieve. Averages obtained by the last-mentioned

TABLE 19—SIZE-WEIGHT DISTRIBUTION OF 435.2 G SAMPLE OF CRUSHED QUARTZ (TYLER SIEVE SERIES)

Sieve numbers (passing, retained) (1)	Arithmetic-mean size of openings (mm) (2)	Weight of quartz retained on smaller sieve (g) (3)	Percent retained on smaller sieve (4)	Weight-size (2) × (4) (5)
4/6	4.013	8.7	2.0	8.0
6/8	2.845	36.6	8.4	23.9
8/10	2.014	65.3	15 0	30.5
10/14	1.422	103.1	23.7	33.7
14/20	1.000	91.0	20 9	20.9
20/28	0.711	77.5	17 8	12 7
28/35	0.503	33.4	7.7	3.9
35/48	0 356	19 6	4.5	1 6
		435.2	100.0	135.2

Arithmetic mean = 135 2/100 = 1 35 mm

method are dependent upon the character of the material sieved as well as the size of the sieve openings In practice, it is customary to express the size of sieved material in terms of sieve-sizes used. Particular attention should always be given to stating which method is used in deriving the average size of sieved material. If, for example, a given material passes a 14-mesh and is retained on 20-mesh Tyler sieves (expressed symbolically 14/20), it should be stated clearly whether the average diameter of the particles is based on (*a*) the opening size of the 14-mesh sieve, (*b*) arithmetic average of 14- and 20-mesh sieve openings, or (*c*)

geometric average of these openings. The opening of a 14-mesh sieve is 1.17 mm, that of a 20-mesh 0.83 mm, and the average of the two sieves $(1.17 + 0.83)/2 = 1.00$ mm. The geometric mean is $\sqrt{1.17 \times 0.83} = 0.97$ mm. The differences are significant, thus emphasizing the need for stating clearly how particle-size averages of screened materials are derived.

Weight-Size Distribution—When a given material is passed through a series of sieves much as the Tyler or U. S. series, and the amount of material retained on each sieve determined, a weight-size distribution of the material is obtained. An example of such a size-analysis is shown in Columns 1, 2, and 3 of Table 19. The table also illustrates the method of computing the average diameter of the material. The geometric mean or the median may be obtained by methods outlined in Chapter 3, when weight of material passing or retained is substituted for number of particles.

TABLE 20—MOMENT METHOD FOR COMPUTING AVERAGE MESH SIZE OF CRUSHED QUARTZ. (TYLER SIEVE SERIES)

Sieve numbers (passing, retained) (1)	Percent quartz passing (g) (2)	Weight assigned to sieve (3)	Moment (2) × (3)
4/6	2.0	1	2.0
6/8	8.4	2	16.8
8/10	15.0	3	45.0
10/14	23.7	4	94.8
			4 4[a]
14/20	20.9	5	104.5
20/28	17.8	6	106.8
28/35	7.7	7	53.9
35/48	4.5	8	20.0
	100.0		443.8

Mesh size = sieve corresponding to weight 4.4
= mesh between numbers 14 and 20 sieves.

[a] Position of mean mesh.

A more simple method of sizing material, based on determining average "mesh-size," was developed by Coghill (1928). Since, as already indicated, standard-sieve series progress in size according to a fixed ratio, each sieve starting with the largest or smallest may be assigned a weight 1, the next in size 2, the next 3, and so on. The method of application is shown in Table 20 for data given in the preceding table. The weight assigned to each sieve is multiplied by the corresponding weight percent of the material. These are simply the moments of the weights.

The sum of the moments divided by the total weight (100 in the present case) is the mean "mesh-size."

RELATION BETWEEN COUNT AND WEIGHT DISTRIBUTIONS

T. Hatch (1933) extended the basic size-distribution equation (Eq 3-17) to apply to weight-distribution curves in sieve-analysis. Thus

$$\rho\alpha_v nd^3 = \frac{\Sigma\rho\alpha_v nd^3}{\log \sigma_g'\sqrt{2\pi}} \cdot \exp\left[-(\ln d - \ln d_g')^2/2\ln^2 \sigma_g'\right] \qquad \text{Eq (5-2)}$$

where ρ, d, α_v, and n are the density, diameter, volume-shape factor, and number of particles retained on each sieve, respectively, and d_g' and σ_g' are the geometric mean diameter and geometric standard deviation of the curve, by weight. The total weight of the sample is of course the term $\Sigma\rho\alpha_v nd^3$ on the right of the equation. Hence $\rho\alpha_v nd^3$ is the weight retained on each sieve.

The arithmetic mean diameter is the sum of the weights retained on each sieve multiplied by the corresponding diameter and divided by the total weight, thus

$$d_w = \Sigma(\rho\alpha_v nd^3)d/\Sigma\rho\alpha_v nd^3 = \frac{\Sigma nd^4}{\Sigma nd^3} \qquad \text{Eq (5-3)}$$

Eq (5-2) may be developed by methods outlined in Chapter 3, so that

$$\ln d_w = \ln \frac{\Sigma nd^4}{\Sigma nd^3} = \ln d_g' + 0.5\ln^2 \sigma_g' \qquad \text{Eq (5-4)}$$

But by Eq (3-24)

$$\ln d_v^3 = \ln\left(\frac{\Sigma nd^3}{\Sigma n}\right) = 3\ln d_g + 4.5\ln^2 \sigma_g \qquad \text{Eq (5-5)}$$

where d_g and σ_g are the statistical parameters of the size-frequency curve, by count. Adding Eqs (5-4) and (5-5)

$$\ln\left(\frac{\Sigma nd^4}{\Sigma n}\right) = \ln d_g' + 0\,5\ln^2 \sigma_g' + 3\ln d_g + 4.5\ln^2 \sigma_g \qquad \text{Eq (5-6)}$$

Again using the development given in Chapter 3, Eq (5-6) may be expressed by the equation

$$\ln\left(\frac{\Sigma nd^4}{\Sigma n}\right) = 4\ln d_g + 8\ln^2 \sigma_g \qquad \text{Eq (5-7)}$$

Substituting in Eq (5-6)

$$\ln d_g' = \ln d_g + 3.5 \ln^2 \sigma_g - 0.5 \ln^2 \sigma_g' \qquad \text{Eq (5-8)}$$

The values of σ_g and σ_g' are the same, that is, the geometric standard deviations by count and weight are equal. This is easily proved by substitution of Eq (5-8) and the value of n given by the size-frequency distribution by count in Eq (5-2). Hence,

$$\ln d_g = \ln d_g' - 3 \ln^2 \sigma_g$$

or using logs to the base 10

$$\log d_g = \log d_g' - 6.908 \log^2 \sigma_g \qquad \text{Eq (5-9)}$$

This relation permits us to transfer size-distributions from a weight to a size basis.

Corresponding formulas for other statistical diameters discussed in Chapter 3 are as follows:

$$\log d_{av} = \log d_g' - 5.7565 \log^2 \sigma_g \qquad \text{Eq (5-10)}$$

$$\log d_h = \log d_g' - 8.0591 \log^2 \sigma_g \qquad \text{Eq (5-11)}$$

$$\log d_s = \log d_g' - 4.6052 \log^2 \sigma_g \qquad \text{Eq (5-12)}$$

$$\log d_v = \log d_g' - 3.4539 \log^2 \sigma_g \qquad \text{Eq (5-13)}$$

$$\log d_{rs} = \log d_g' - 1.1513 \log^2 \sigma_g \qquad \text{Eq (5-14)}$$

The above equations apply only when the weight-in-size distribution is known to follow the log-probability distribution. Their application to a complete sieve-analysis also depends upon "calibration" of each sieve of the series used as explained in subsequent paragraphs.

CALIBRATION OF SIEVES

Hatch Method of Calibration—For the production of materials with uniform size, it is essential that sieve openings be calibrated. Specifications for many industrial products require that sizes be within specified limits. Unless screen openings are of uniform dimensions, specified limits may be exceeded. When fine material is made to pass through two sieves, the size of the material retained on the sieve with smaller openings is usually taken as the arithmetic or the geometric mean of the openings of the two screens. Actually the material is widely distributed as to size, as may be seen by reference to Figure 24, which shows material passing a 65-mesh Tyler sieve and retained on a 100-mesh. The arith-

metic or geometric mean of the openings does not indicate the true mean size of the material. Figure 24 illustrates the fact that particles of a given diameter may be found on several sieves. Clean-cut and definite size separation, therefore, is not possible by use of sieves. It will be seen in Figure 24 that the median size (count basis = 290 μ) is closer to the opening size of a 48-mesh sieve than the 65-mesh Tyler sieve through which the material passed. Material obtained from any single source should always give the same distribution, but other materials may give much different distributions.

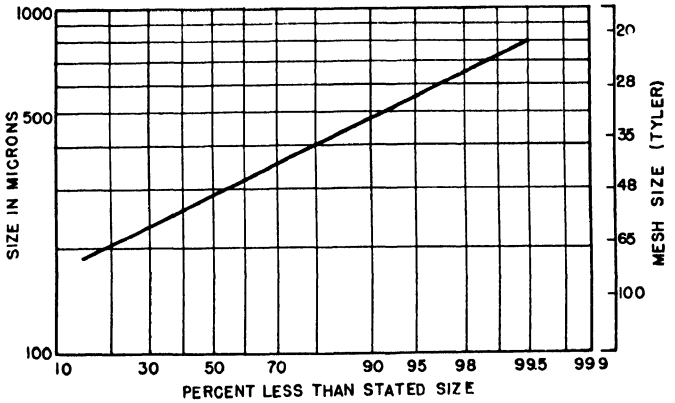


FIGURE 24. SUMMATION CURVE OF MATERIAL PASSING 65- AND RETAINED ON 100-MESH TYLER SIEVES.

T. Hatch (1933) applied this type of calibration to samples of lime stone. The particles were measured microscopically as outlined in Chapter 4, and the median diameter *by weight* determined from Eq (5-9). The calibrated size of a sieve was taken as the median or geometric average size *by weight* of the material retained on the sieve. Thus, applying Eq (5-9) to the data plotted in Figure 24, we obtain the calibrated size of the sieve as follows:

$$\begin{aligned} \log d_g' &= \log d_g + 6.098 \log^2 \sigma_g \\ &= 2.462 + 6.098 \times 0.162 \times 0.162 \\ &= 2.478 \end{aligned}$$

so that

$$d_g' = 302\ \mu$$

Calibration sizes of four sieves, determined by Hatch for crushed lime-

stone, are given in Table 21. The data presented in the table show that the calibrated sizes are larger than the sieve openings.

TABLE 21—HATCH'S (1933) CALIBRATION OF FOUR SIEVES (TYLER SERIES) USING CRUSHED LIMESTONE

Material retained on (mesh)	Size of sieve opening (μ)	Calibration size (μ)
100	147	282
150	104	199
200	74	136
325	37	100

Application of the Hatch equations to a screen analysis is as follows. A summation curve is plotted on log-probability grid. The "percentage less than *calibrated* sieve-size" is plotted instead of the usual "percentage less than stated sieve-size." If the summation curve is a straight line, Eq (5-9) applies. The method is useful where precise information is desired on particle-size of a screened product

Hatch extended his method of analysis to size-distribution curves ranging from coarse-screen analysis through fine particles measured microscopically. While excellent results were obtained by using this technique on laboratory samples, the method cannot be generalized to cover all types of distributions encountered in practice. As already explained in Chapter 3, size-frequency distributions may assume a variety of shapes. The Hatch development applies only to distributions which follow the normal or log-probability law. When size-distributions are hyperbolic in the lower extremes and follow normal log-probability laws in the upper extremes, the Hatch analysis must necessarily fail Nevertheless, the relationships developed by Hatch have a far-reaching practical importance.

Weber and Moran Method of Calibration—Another method of calibrating sieves consists in measuring random openings in the sieve and obtaining size-frequency distributions as outlined in Chapter 3. The openings are measured by filar micrometer or by projection methods. Two screens are then identical if the mean size of the openings and the standard deviations are the same. Weber and Moran (1938) use the coefficient of variation

$$\nu = 100\sigma/l_{av} \qquad \text{Eq (5-15)}$$

as a measure of sieve equivalence, where σ is the standard deviation and l_{av} the arithmetic average of the sieve openings. Their data indicate that

the difference between the average opening of a sieve with a high coefficient of variation and that of a sieve having the same retention but a smaller coefficient increases as the standard of the former increases above 6 percent. Sieves with high coefficients of variation behave as if their average opening were larger than that calculated in terms of the average of sieve openings. This difference is due to the relatively larger portion of oversized openings in screens possessing high coefficients of variation. However, since shaking time is also an important variable, it too must be considered. With longer shaking times a relatively small number of oversize openings would tend to increase the average size of material passed. Weber and Moran have proposed the following empirical formula for determining the effective opening of a sieve in terms of statistical constants and shaking time

$$l_e = l_{av}\left[1 + 0.002t\left(\frac{\nu - 6}{0.06}\right)^{1/2}\right] \qquad \text{Eq (5-16)}$$

where l_e is the effective opening for a shaking time of t min and ν is the coefficient of variation as defined by Eq (5-15).

FINENESS MODULUS

The fineness modulus was developed in an effort to correlate strength of concrete in terms of its component sizes. The fineness modulus is simply the sum of the percentages coarser in a given sieve-analysis divided by 100. In concrete work, fineness modulus is applied to the aggregate and the following Tyler Standard Scale sieves are used: 100, 48, 28, 14, 8, 4, and $^3/_8$-in., $^3/_4$-in., and $1^1/_2$-in. mesh. By using this series of sieves each sieve has a clear width opening double that of the next finest sieve, or one-half that of the next coarsest sieve.

Problems

1 The following is a sieve-analysis of a certain sand:

Sieve-size (Tyler)	Weight (percent)
6	0.5
8	3.5
10	12.6
14	25.3
20	38.4
28	48.9
35	65.3

Obtain the geometric mean size by plotting on a log-probability grid, and by means of Eq (5-10) determine the average size on a count basis.

2. Calculate the calibration size (weight basis) for a sand ($\rho = 2.64$) passing a 200-mesh and retained on a 325-mesh Tyler sieve, given the following size-frequency distribution:

Average group size (μ)	Percent less than stated size
60	0.7
70	5.0
80	30.0
90	50.0
100	82.0
110	90.3
120	96.1
130	99 1
140	99.5
150	99 8

3. Given a screen which is vibrated and assume that N of the particles contained on it are capable of passing with equal facility. If Φ denotes the probability of a single particle passing in a given interval of time, and assuming that Φ remains unchanged during the screening process, show that after ν time intervals the number of particles which has passed the screen is given by the expression

$$N\Phi[1 + (1 - \Phi)\ (1 - \Phi)^2 + \qquad (1 - \Phi)^{\nu - 1}$$

or, what is equivalent, that the amount passing during the νth interval is

$$N\Phi(1 - \Phi)^{\nu - 1}$$

Hence also prove that the number of particles of the same kind left on the screen after the νth interval of time is

$$N(1 - \Phi)^{\nu}$$

4. In a study of screening capacity, Herried (1935) obtained the following relation expressing certain screening variables when separating dry coke fines in $^1/_4$-in. square opening screen

$$[(L/0.038) - 0.88]\ (\omega - 868) = 1020$$

where L is the length of the stroke and ω the shaft speed in rpm. The formula gives the maximum capacity of the screen. Plot the curve and discuss possible limitations of the formula. Note: The value of L should range from $^1/_{32}$ in. to $^{21}/_{32}$ in.

5. Given a screen of n-squares, each of area a. If the whole area of the screen is A, determine the probability of a particle whose largest dimension is less than $\sqrt{a}$ falling through the screen.

Note: The chance of a particle falling through a given opening a_1 is determined

as follows: Since there are n equally probable openings through which the particle may fall, the probability p_1 for this opening is

$$p_1 = a_1/A,$$

and since the whole probability is $p_1 + p_2 + \ldots + p_n$,

$$p = \frac{1}{n} \cdot \frac{a_1}{A}$$

For squares of equal size $a = a_1 = a_2 = \text{etc.}$

$$p = a/A = 1/n$$

6 Given a screen with openings $2a$ by $2b$ on which is thrown at random a rodlike particle of length $2c$ less than either side of the opening, determine the probability that the particle will pass without touching the sides of the opening.

Note: The solution given by LaPlace for the probability q of crossing one side of the rectangular opening is

$$q = (c/\pi ab)\ (2a + 2b - c)$$

Hence the probability of not crossing either side is

$$p = 1 - (c/\pi ab)(2a + 2b - c)$$

If one of the sides, say b, is infinite, then the solution reduces to Buffon's problem of parallel lines on which a rod of length c less than the distance between the lines is thrown In particular, when the openings of a screen are square

$$p = 1 - 4c/\pi a - c^2/\pi a^2$$

Thus, p is a minimum when $c = a$, or $p = 0.045$. (Compute p for various values of c/a.) For a wire screen, the wire thickness being δ, Fagerholt (1940) cites Andreasen as having determined the probability of a particle of diameter d passing through the opening as $[(a - d)/(a + \delta)]^2$.

7 Particles are screened on two sieves the size of whose openings are D_1 and D_2. Show that the average weight diameter of the particles passing the larger and retained on the smaller is

$$d_w = \alpha \sqrt[3]{\frac{2D_1^2D_2^2}{D_1 + D_2}}$$

Note: Write the average particle passing D_1 and D_2 as $d_1 = \alpha D_1$ and $d_2 = \alpha D_2$, where the α's are shape factors. If the $C(d)$ denotes the weight of particles less than d_w and if in the interval d_1 and d_2 the size distribution is linear

$$C(d) = a + bd$$

The constant b is therefore the weight of the particles. If there are $N(d)$ particles of size d_w

$$dN(d) = bd^{-3}d(d)/\rho$$

$$N(d_1) - N(d_2) = b(d_1^{-2} - d_2^{-2})/2\rho$$

Since the weight distribution of these particles may be denoted by $C(d)$

$$C(d_1) - C(d_2) = b(d_2 - d_1)$$

$$d_w = \sqrt{\frac{2\bar{d}_1^2\bar{d}_2^2}{\bar{d}_1 + \bar{d}_2}} = \alpha \sqrt{\frac{2\bar{D}_1^2\bar{D}_2^2}{\bar{D}_1 + \bar{D}^2}}$$

CHAPTER 6

CHARACTERISTICS OF PACKINGS

SYSTEMATIC ARRANGEMENT OF SPHERES

The study of problems concerning the flow of fluids through beds of granular material has many applications. It is indispensable to the theory of industrial filtration and water purification. The flow of ground water is controlled by soil structure and the most elementary phases of granular bed structure are fundamental to soil mechanics and soil physics.

In practice, we are generally concerned with the behavior of irregularly shaped particles when packed in different ways. The arrangements assumed by such particles when dropped haphazardly into a container cannot be predetermined. However, it is possible to establish criteria for the behavior of irregular particles, provided that most of the particles concerned are neither disk-like nor plate-like in form; these shapes tend to destroy homogeneity of packing structure. The mathematical treatment of packings of irregular particles requires consideration of the structural layers as uniform and homogeneous throughout. Moreover, when discussing the behavior of packings under pressure, we must regard these packings as isotropic.

While it is possible to describe the characteristics of packings in terms of voids, no clear-cut means has been found to relate these voids to particle-size and other superficial features of the particles themselves. In other words, no comprehensive relationship between variables describing a packing arrangement has thus far been possible. In the chapter on infiltration the relation of pore-space and particle-size to flow will be developed, but a complete description of a packing in anything but hydrodynamic terms is wanting. Therefore, we are forced to use empirical and experimental approaches in evaluating packings. There is need for research on packing arrangements and, indeed, we shall never comprehend the behavior of particulate matter in the aggregate until we learn more about its mathematical representation. Even the treatment of spheres in a fixed pattern or arrangement, stable or otherwise, gives us nothing more than a simple means of predetermining pore-spaces or voids. In random arrangements of uniform spheres we are concerned only with the limits of porosity and we may perhaps agree on some aver-

age of these limits as satisfying our immediate problem. As will be shown in Chapter 15 the trouble is that slight changes in porosity (which in turn is dependent upon the arrangement of spheres, and hence of particles) affect the behavior of packings tremendously. For the time being, therefore, we must regard the hydrodynamic behavior of a packing as giving the most satisfactory description of its internal arrangement, subject of course, to a knowledge of its porosity, particle-size, particle-size distribution, shape-factors, and surface characteristics.

Theoretically, no equation can completely describe the physical properties of a packing without taking into account particle-diameter, size-distribution, and other particulate parameters already mentioned. The reason for their omission in some equations is threefold: (*a*) The equation may contain arbitrary constants related to particle-characteristics constituting the packing; (*b*) the derivation of the equation may be in terms of an ideal or isotropic medium, and (*c*) the equations may be empirical. So far, statistical analysis seems to have played only a small role in studies of packing problems, and probably will continue to do so until the particulate properties of various materials are better understood and made subject to mathematical treatment.

Piles of Uniform Spheres—When spheres are piled in the form of a triangular pyramid, the top sphere rests in the hollow formed by the three immediately below it. The three in the second row rest on a course of six in the third row, these six on a course of ten, and so on. This forms the series $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \ldots + (1 + 2 + \ldots + n)$, so that if n is the number of spheres in the side of the nth course, then the number of spheres in the pile including the nth course is

$$N = \frac{n(n+1)(n+2)}{6} \qquad \text{Eq (6-1)}$$

If the pile is a pyramid with a square base, then the top sphere rests on four spheres, which in turn rest on nine, then 16, and so on. The number of spheres to the nth row is therefore the sum of the series $1^2 + 2^2 + 3^2 + \ldots + n^2$, or

$$N = \frac{n(n+1)(2n+1)}{6} \qquad \text{Eq (6-2)}$$

where n, as before, is the number of spheres on the bottom side, or th nth course.

When the pile has a base which is rectangular, the top of the pile will consist of a single row of spheres. Let n_0 be the number of spheres in the top row, then the second row will consist of $2(n_0 + 1)$ spheres, the third

row of $3(n_0 + 2)$, and the nth row of $n(n_0 + n - 1)$. The series to be summed to the nth row will be

$$n_0 + 2n_0 + 2 + 3n_0 + 6 + \ldots + n(n_0 + n - 1)$$

the sum of which is

$$N = \frac{n}{6}(n + 1)(3n_0 + 2n - 2) \qquad \text{Eq (6-3)}$$

Systematic Packing of Spheres—A number of special arrangements of spheres in packings is possible. A representative section of spheres, piled in a three-sided pyramidal piling as discussed in the preceding paragraph, gives one type of arrangement—called rhombohedral. The arrangement of spheres in this type of packing is shown in Figure 25. It is interesting to note that the first study of arrangement of spheres was made by Slichter (1899), with particular reference to the flow of water

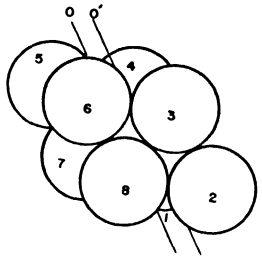


FIGURE 25 RHOMBOHEDRAL ARRANGEMENT OF SPHERES.

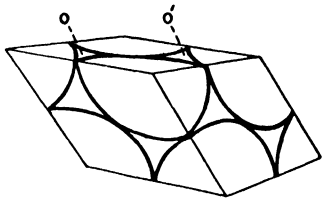


FIGURE 26. UNIT-CELL OF RHOMBOHEDRAL PACKING OF SPHERES.

through soil. He was led to the study of sphere arrangements in order to reduce the hydraulics of a complex soil to conditions of an "ideal" soil. Thus, no matter how complex the soil under consideration, Slichter believed its hydraulic properties could be simulated by an ideal soil of uniformly spherical grains.

In order to clarify terms which will occur frequently in later portions of this text, their definitions are given at this point. These definitions are taken from a study of systematic packings made by Graton and Fraser (1935), to which the reader is referred for an excellent analysis of packing problems.

(*a*) A unit-solid is a sphere of diameter d.

(*b*) A unit-cell is the smallest portion of a packing which gives a complete picture of the manner of packing; Figure 26 shows the unit-cell structure for a rhombohedral packing.

(*c*) Voids are defined as the ratio of open-space volume to the combined open-space and solid-material volume in a packing.

(*d*) Porosity is used synonymously with voids and is usually expressed as a percentage.

"Closest" Packing—The closest arrangement for a packing of uniform spheres is that shown in Figures 25 and 26. In this arrangement the face angles of the unit solid are inclined at angles of 90 deg, 60 deg,

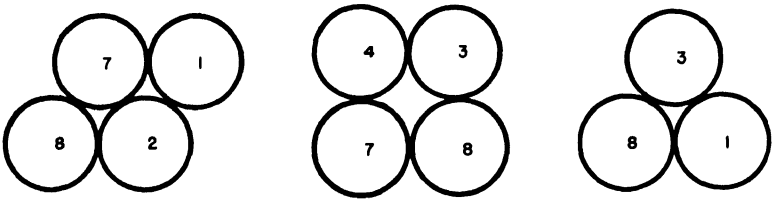


FIGURE 27. PORE ARRANGEMENT FOR SPHERES IN RHOMBOHEDRAL ARRAY SHOWN IN FIGURE 25.

and 120 deg. This has been termed the "rhombohedral" type of packing and possesses the following characteristics, d being the diameter of the spheres:

Volume of unit cell	$0.71d^3$
Volume of voids	$0.18d^3$
Porosity	25.95 percent

Several points with regard to the arrangement are worth mentioning. Referring to Figure 25, it will be seen that the pores of the rhombohedron are formed by various combinations of spheres. These are identified by Figure 27 as the rhombus formed in the spheres 1, 2, 7, 8, the square formed by 3, 4, 7, 8, and the triangle formed by 1, 3, 8. Slichter was concerned only with the continuous triangular pore shown by the lines O and O' in Figure 25. In capillarity or lysimetry, however, we are concerned with *all* the pores since they share in the filling and draining of the voids. The relative frequency of the various types of pores for a rhombohedral packing are

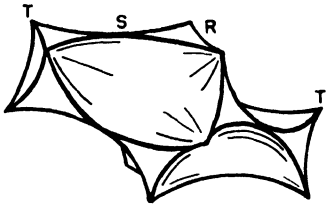


FIGURE 28. CENTRAL VOID OF UNIT-CELL FOR SPHERES IN "CLOSEST" PACKING.

$$\text{rhombus pore : square pore : triangle pore}$$
$$= n_R : n_S : n_T = 3 : 3 : 2$$

where the n's refer to the frequency of the various pores. Other frequencies may be expected for other types of packing, but in general it is

customary to base calculations on this type because it is the most stable.

The shape of the void in "closest" packing is shown in Figure 28, traced from a wax model developed by Slichter. This void must be studied with regard to the significance of various salient points of discontinuity. The points are generated by two kinds of pores or cells, best illustrated by the plan of spherical arrangements producing a system of "closest" packing (Figure 85). The circles (formed of broken arcs) represent spheres, each of which rests in the hollow formed by three spheres immediately below (shown by solid lines). Each of the spaces marked T is the hollow or depression into which a sphere of the layer above rests. Each cell thus formed is a closed tetrahedral cell. There is also a second set of tetrahedral cells, T_1 (similar to those marked T), formed by spheres in the first or bottom layer, fitting into three spheres immediately above in the second layer. The cells marked R are cells formed by a group of six spheres. These cells are rhombohedral and are unlike the tetrahedral cells in that they are not closed. These form the continuous pores shown by the lines O and O' in Figure 25. In any rhombohedral packing ("closest" packing) there are twice as many tetrahedral as rhombohedral cells.

"Most Open" Packing—When spheres are piled so that each layer lies immediately above the other on a rectangular-base arrangement, we have the most open type of packing possible. Under these conditions the characteristics of the packing (called cubical) are as follows:

Volume of unit-cell.....	d^3
Volume of unit-void.....	$0/48\, d^3$
Porosity.....	47/64 percent

where d again is the diameter of the sphere.

Intermediate Arrangements—Between the two extreme types of packing there are an infinite number of possible arrangements, giving any porosity between 26.95 and 47.64 percent (the values for rhombohedral and cubical packings, respectively). This was recognized by Slichter, who stated that the porosity of an ideal soil depends upon the acute face angles of the rhombohedron. Graton and Fraser reinvestigated the theory of systematic packings, and distinguished six stable arrangements. The characteristics of the various packings, including the two above discussed, are shown in Table 22. The data given in the table are of geometrical interest, and probably include all the types of arrangements encountered in any stable packing of uniform particles. However, while these arrangements are capable of producing porosities to the extent indicated, namely 25.95 and 47.65 percent, it will be shown that with sized particles almost any degree of porosity is possible.

Nearly all loose or compact granular materials encountered in practice possess a porosity between the values of approximately 30 and 50 percent.

TABLE 22—CHARACTERISTICS OF INTERMEDIATE-TYPE PACKINGS (GRATON AND FRASER, 1935)

Case	Face angles	Volume of unit-cell	Volume of void	Porosity (percent)	Remarks
1	90°, 90°, 90°	d^3	$0.48d^3$	47.64	Cubic
2	60°, 90°, 90°	$0.87d^3$	$0.34d^3$	39.54	Orthorhombic
3	60°, 90°, 120°	$0.71d^3$	$0.18d^3$	25.95	Rhombohedral
4	90°, 60°, 90°	$0.87d^3$	$0.34d^3$	39.54	Orthorhombic
5	60°, 60°, 116° 34′	$0.75d^3$	$0.23d^3$	30.19	Tetragonalspheroidal
6	60°, 60°, 90°	$0.71d^3$	$0.18d^3$	25.95	Rhombohedral (different orientation)

Contacts—The number of contacts experienced by a sphere in systematic packings is of considerable importance in the theory of capillary action of the soil. If we consider the arrangement in closest packing, it is evident that every sphere touches twelve other spheres. In the most open packing, contact is established with only six adjoining spheres.

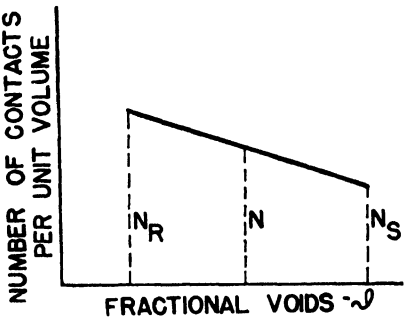


FIGURE 29. LINEAR INTERPOLATION FOR THE NUMBER OF CONTACTS PER UNIT VOLUME OF PACKING.

Referring now to the shape of the void as shown in Figure 28, the rhombohedral and tetrahedral cells are denoted by the letters R and T. The T-spaces correspond to the voids between four spheres in closest packing with one sphere nested in the hollow formed by the other three. In this space there are six points of contact between spheres and four lines or branches. These branches connect one cell with another.

The rhombohedral cell is the space confined by six spheres in closest packing. In this configuration there are twelve points of S-contact and eight triangular contacts

W. O. Smith (1933) suggested the following method of interpolation for securing the number of contacts between spheres of equal diameter. Let ϑ denote the fractional voids, and assume that a linear relationship exists between the number of contacts, N, and the voids. Referring to Figure 29, if N_s denotes the number of contacts per unit volume for a

most open arrangement (square pore) and N_R those for a rhombohedral arrangement, by interpolation we have

$$\frac{N - N_S}{0.476 - \vartheta} = \frac{N_R - N_S}{0.476 - 0.259}$$

Whence, solving for N

$$N = N_R\vartheta + N_S(1 - \vartheta)$$
$$\alpha = (0.476 - \vartheta)/0.217$$

However, since from simple geometry

$$N_R = 6\sqrt{2}/d_3{}^3$$
$$N_S = 3/d_3{}^3$$

we obtain for the number of contacts per unit volume of uniform sphere

$$N = 3(1 + 1.828\vartheta)/d^3 \qquad \text{Eq (6-4)}$$

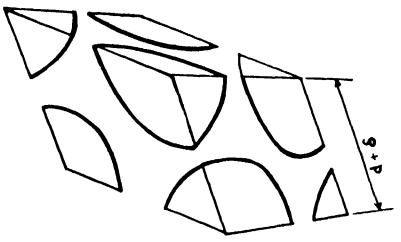


FIGURE 30. SCHEMATIC ARRANGEMENT OF UNIT-CELL OF A RHOMBOHEDRAL PACKING TO ACCOUNT FOR VOIDS.

Special Treatment of Voids—An interesting concept of voids in a regular array of spherical particles has been presented by W. O. Smith *et al* (1929). Assume a rhombohedral array such as shown in Figure 30, where the spacing is $d + \delta$ between the particle centers, and δ is adjusted to the observed porosity. From geometry, the number of spheres per unit-volume is equal to $\sqrt{2}/(d + \delta)^3$ and the total sphere volume is this quantity multiplied by $(\pi/6)d^3$. Hence, if the observed void is ϑ

$$(d + \delta)^3 = \frac{\sqrt{2}\pi d^3}{6(1 - \vartheta)} = 0.74\,\frac{d^3}{1 - \vartheta} \qquad \text{Eq (6-5)}$$

This equation is particularly useful for mixtures of particles of various sizes for which only average diameter and voids are available, and if it is desired to determine the ultimate degree to which material is packed. But the treatment of pore-space developed by Smith *et al* is capable of further extension. Assume that the three types of pores shown in Figure 27 are composed of spheres separated by a distance δ. The area contained in each pore is then

$$\text{Rhombus-pore area} = A_R = \frac{\sqrt{3}}{2}(d + \delta)^2 - \frac{\pi d^2}{4} \quad \text{(Frequency 3)}$$

$$\text{Square-pore area} = A_S = (d + \delta)^2 - \frac{\pi d^2}{4} \quad \text{(Frequency 3)}$$

$$\text{Triangle-pore area} = A_T = \frac{\sqrt{3}}{4}(d + \delta)^2 - \frac{\pi d^2}{8} \quad \text{(Frequency 2)}$$

But from Eq (6-5)

$$d + \delta = 0.90d/\sqrt[3]{1 - \vartheta}$$

so that after substitution in the above equations

$$A_R = \frac{\sqrt{3}}{2} \cdot \frac{0.81d^2}{(1 - \vartheta)^{2/3}} - \frac{\pi d^2}{4} = d^2\left[\frac{0.71}{(1 - \vartheta)^{2/3}} - 0.79\right]$$

$$A_S = \frac{0.81d^2}{(1 - \vartheta)^{2/3}} - 0.79d^2 = d^2\left[\frac{0.81}{(1 - \vartheta)^{2/3}} - 0.79\right]$$

$$A_T = \frac{\sqrt{3}}{4}(d + \delta)^2 - \frac{\pi d^2}{8} = 0.39d^2\left[\frac{0.912}{(1 - \vartheta)^{2/3}} - 1\right]$$

The weighted average of all these pores will be

$$A = \frac{3A_R + 3A_S + 2A_T}{8}$$

$$= 0.75(d + \delta)^2 - 0.69d^2$$

$$= d^2\left[\frac{0.61}{(1 - \vartheta)^{2/3}} - 0.69\right]$$

In the closest packing of spheres of equal diameter ($\vartheta = 0.26$), the narrow opening formed by contact of three spheres has an area equal to $0.0403d^2$, where d is the diameter of the sphere. In problems concerned with capillarity, the pore-space plays an important role. However, in addition to the area at the throat, or narrowest point, it is often desirable to derive the average area of the void formed by contact of the spheres (along the line indicated in Figure 25) in terms of the voids. While we can solve this problem only for a regular packing, the general result is indicative of the order of magnitude with which we are concerned in practice. Consider a three-sphere pore as in Figure 27. The area A_T of the triangle is $(\sqrt{3}/4)(d + \delta)^2$ (assuming spheres are separated by distance δ) less the area of the portions of the included spheres, $\pi/2$-$[(d^2/4) - y^2]$, where y is the distance between the given section and the plane centers. Thus

$$A_T = \frac{\sqrt{3}}{4}(d + \delta)^2 - \frac{\pi}{2}\left(\frac{d^2}{4} - y^2\right)$$

and the average area

$$A_{av} = \frac{4}{d}\int_0^{d/4} A_T dy = \sqrt{3}(d + \delta)^2 - \frac{11\pi d^2}{48}$$

and from the fact that $(d + \delta)^3 = \sqrt{2}\pi d^3/6(1 - \vartheta)$ we obtain

$$A_{av} = 0.36d^2\left[\frac{0.985}{(1 - \vartheta)^{2/3}} - 1\right] \qquad \text{Eq (6-6)}$$

which is independent of δ. For closest packing, $\vartheta = 0.26$, $A_{av} = 0.074d^2$, and the diameter of a circle having this area is $0.31d$.

In computing the average area, it is necessary to picture the centers of the spheres of two adjacent pores as forming a tetrahedron, with the center of the common sphere as a vertex. Hence the integration from the plane of centers to a section one-fourth the diameter of a sphere above this.

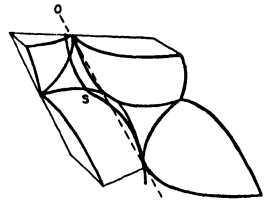


FIGURE 31. COURSE OF PORE IN RHOMBOHEDRAL PACKING

Length of Pore—The course of the pore for which we have just calculated the average area is tortuous, as shown in Figure 31. However, if the pore is considered straight and parallel to an edge of the unit-cell, as in Figure 26, the length of the mean course is given by

$$L_{av} = \sqrt{\frac{3}{2}}L \qquad \text{Eq (6-7)}$$

where L is the height of the column. The actual length of the course, corrected for curvature, has been shown by Slichter to be $1.065L_{av}$. Hence, the true length of the path is

$$L_{av}' = 1.065L_{av} = 1.304L \qquad \text{Eq (6-8)}$$

HETEROGENEOUS SYSTEMS

Computation of Voids—Except for ideal arrangements of spheres, voids can be determined only experimentally. Calculations of voids are in terms of the true and "apparent" specific densities of the material used in the packing, in accordance with the formula

$$\text{voids} = 1 - \frac{\text{apparent density of material}}{\text{true density of material}} \qquad \text{Eq (6-9)}$$

or if ϑ is written for the voids, ρ_a for the apparent density, and ρ for the true density

$$\vartheta = \rho_a \left(\frac{1}{\rho_a} - \frac{1}{\rho} \right) \qquad \text{Eq (6-10)}$$

On a volume basis, Eqs (6-9) and (6-10) are equivalent to

$$\vartheta = 1 - \frac{\text{volume of material (actual)}}{\text{volume of material and voids}} \qquad \text{Eq (6-11)}$$

If the voids are to be expressed on a volume basis per unit-weight of material,

$$\text{volume of voids per unit-weight of bulk} = \vartheta_w = \frac{1}{\rho_a} - \frac{1}{\rho}$$

The apparent density is the weight of a unit-volume of the packing. Of course, the apparent density will vary, depending upon the treatment to which the packing is subjected. A packing constructed at random will be loose, while one which is vibrated or tapped will attain a relatively high degree of compactness. In experimental work, determination of apparent density must be accomplished *in situ*. This is possible if the weight and volume occupied by the packing are known. The porosity of the packing is defined as the percentage of voids in a packing, that is, 100 × voids.

McGeorge Method—Recently McGeorge (1941) used a method for determining voids which is useful for soil studies in areas affected by floods and river silts. The voids were determined by stirring the mixtures in water and allowing them to settle. While this process causes a size-gradation in the settling cylinder used, the data given in Table 23 nevertheless indicate the increase in voids which takes place as the percentage of silt is increased. The size-analyses of the sand and silt used in the above experiments are shown in Table 24.

TABLE 23—RELATION BETWEEN MIXTURES OF SAND AND SILT AND VOIDS (MCGEORGE 1941)

Proportion of sand and silt		Apparent specific gravity	
Sand	Silt	(g/cc)	Voids
100	0	1.72	0.34
90	10	1.66	0.36
80	20	1.61	0.38
70	30	1.66	0.36
60	40	1.61	0.37
50	50	1.56	0.39
40	60	1.56	0.39
30	70	1.51	0.41
20	80	1.47	0.42
10	90	1.42	0.44
0	100	1.37	0.45

Effective Free Area—The effective free area of a packing expresses the cross-sectional open area in any plane of the packing. Various formulas have been devised to express this area, but as Furnas (1929) stated, no definite relationship can be found when the particles are heterogeneous. It has been found that the effective free area of spherical packings in ordered arrangements ranges from 9.3 percent for spheres in closest packing, to approximately 21.5 percent for the most open packing, and is independent of the diameter of the spheres. For most packings of irregularly shaped particles, the effective free area ranges from 10 to about 20 percent and as a general rule the smaller the particles the less the effective free area.

TABLE 24—SIZE-ANALYSES OF SAND AND SILT STUDIED BY MCGEORGE (1941)

Size	Sand (percent)	Silt No. 2 (percent)
Percent sand (0.005–1.0 mm)	88	27
Percent silt (0.005–0.05 mm)	6	22
Percent clay (0.005 mm)	6	51

Number of Particles in a Packing—From the average diameter d of the particles, it is possible to compute the number of particles present in a unit-volume of the packing. The solid material in a unit-volume is clearly $(1 - \vartheta)$, where ϑ represents the voids. Hence, if α_v represents the volume shape-factor and N the number of particles per unit-volume (see Chapter 3), then

$$\alpha_v d^3 N = 1 - \vartheta$$

or

$$N = (1 - \vartheta)/\alpha_v d^3 \qquad \text{Eq (6-12)}$$

When the particles are spheres, $\alpha_v = \pi/6$. Likewise, the average volume occupied by a single particle is given by

$$1/N = \alpha_v d^3/(1 - \vartheta) \qquad \text{Eq (6-13)}$$

If the particles are sufficiently large for counting, and ϑ can be determined, the nominal diameter (diameter of a sphere of equal volume) can be calculated at once. In this case $\alpha_v = \pi/6$, so that the nominal diameter d_n is given by the expression

$$d_n = 1.24 \sqrt[3]{\frac{1 - \vartheta}{N}} \qquad \text{Eq (6-14)}$$

This is a convenient and simple method for calculating average diameter of large particles.

Apparent Density—Apparent density (bulk weight of a unit-volume of a packing) depends on the shape of the particles, their density, and their arrangement in the measuring container. Table 25 gives the true and apparent densities of some of the more common materials. Obviously, the values given in the table can be considered only approximate since there is no known method for consistently achieving identical arrangements of even a single material.

TABLE 25—TRUE AND APPARENT DENSITIES OF SOME COMMON LOOSE MATERIALS

Material	Specific gravity (water = 1)	Apparent density (lb/cu ft)
Portland cement	1.5	90–120
Coal, anthracite	1.4–1.8	90–120
Coal, bituminous	1 2–1.5	80–90
Earth, dry, loose	1.2	76
Earth, dry, packed	1.5	95
Earth, moist, loose	1.3	78
Earth, moist, packed	1.6	96
Limestone	...	90–110
Sand, gravel, dry, loose	1.4–1.7	90–105
Sand, gravel, dry, packed	1.6–1.9	100–120
Sand, gravel, wet	1.9–2.2	126

Mixtures of Spheres of Different Sizes—The voids in any packing of regular spheres may be occupied by smaller spheres, thus affecting the density of a unit-volume. First let us examine a "normal" packing of a binary system, composed of spheres with varying ratios of diameters. By "normal packing" is meant the arrangement assumed by spheres (or particles) when they are dropped singly in a container. This will not give a most stable arrangement as far as the packing is concerned, but the procedure will result in a more or less constant percentage of voids (porosity). Furnas studied the arrangement of a binary system (systems composed of particles or spheres of two sizes) and found that the porosity of such a packing is less than the voids in packings with separate components; that is, if the normal voids are not the same, the mixture possesses a porosity less than the weighted average of the constituents. This fact has an important bearing on the density of aggregates. Figure 32 shows the voids experimentally determined by Furnas in a binary packing composed of particles of various size-ratios, when normal voids of the single components are 0.50. It is evident from the figure that in a binary system combinations are possible in which the voids will be at a minimum. Thus, in a system composed of particles whose initial porosity is 40 percent, Furnas has shown that this will occur when the

percentage of large constituents ranges from 65 to 70 percent by volume; for 50 percent porosity when the percentage of large constituents varies from 55 to 67 percent; and for 60 percent porosity when the percentage of large constituents varies from 55 to 62.5 percent.

Systems of Maximum Density—In practice, binary systems of particles are not as important as systems with three or more component sizes. This section attempts to give Furnas' solution to determination of the proper quantities of various substances producing mixtures of maximum density. The discussion furnishes data on which approxima-

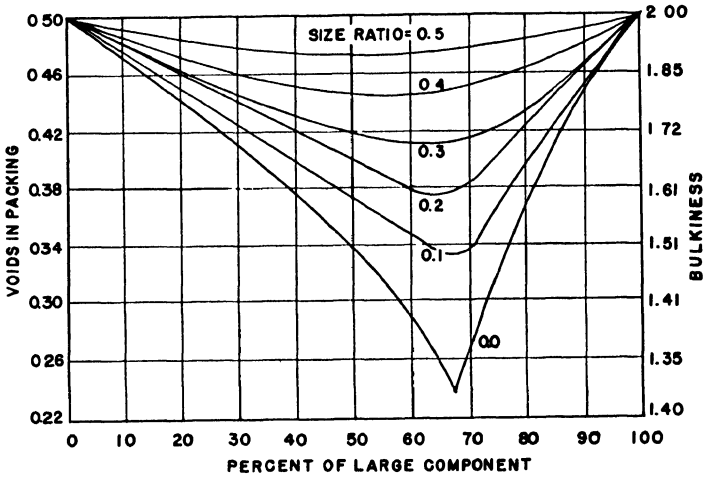


FIGURE 32. RELATION BETWEEN VOIDS AND SIZE COMPOSITION IN TWO-COMPONENT SYSTEMS OF BROKEN SOLIDS WHEN THE VOIDS OF SINGLE COMPONENTS ARE 0.5.

tions may be made for mixtures of not more than four component parts We shall begin by outlining the basic formula for a two-component system of maximum density.

Assume a packing of large particles containing a fixed amount of voids, ϑ_1. If small particles (small compared to the large particles) are introduced into the packing, the volume of the packing will not increase. We may obtain a mixture such that all the original voids among the large particles are filled with the smaller component. But the smaller component also has voids, equivalent say to ϑ_2, no matter how small the particles; when all the large voids are thus filled, we have $W_1 = (1 - \vartheta_1)\rho_1$, for a unit-cell where W_1 is the weight of the larger constituent, and ρ_1 its true specific gravity. Similarly, for the smaller component, $W_2 = \vartheta_1(1 - \vartheta_2)\rho_2$ where W_2 and ρ_2 correspond, respectively, to the

weight and true specific gravity of the small constituent. The degree to which the large particles are saturated by the small ones is then

$$\varphi_1 = \frac{W_1}{W_1 + W_2} = \frac{(1 - \vartheta_1)\rho_1}{(1 - \vartheta_1)\rho_1 + \vartheta_1(1 - \vartheta_2)\rho_2} \qquad \text{Eq (6-15)}$$

and when $\rho_1 = \rho_2$ and $\vartheta_1 = \vartheta_2 = \vartheta$, we have a condition of maximum density

$$\varphi = \frac{1}{1 + \vartheta} \qquad \text{Eq (6-16)}$$

This equation holds as long as the particles have the same shape, and the voids are the same for each component.

The argument developed in the preceding paragraph indicates a possible procedure for filling all the voids without changing the total volume. Actually, in practice there is a small increase in packing volume with the introduction of small particles, although the maximum apparent density is still attained by the ratio $1/(1 + \vartheta)$, as long as the diameters of the particles do not approach each other in size. In Eq (6-16), ϑ is the fraction of voids in a packing of sized materials, and φ is the absolute volume occupied by the larger particles in a two-component system of maximum density. This is based on an actual absolute volume of both components being unity. Let d_1 be the diameter of the larger particles. The amount of material having a diameter d_2 which will fill all the interstices of d_1 is $1 - \varphi$; similarly, a third component of still finer particles can be put into the interstices of the second component. The process may be repeated indefinitely, and we obtain the total absolute volume of the solids, Φ, as a geometric series with a constant inter-term ratio equal to $(1 - \varphi)/\varphi$

$$\Phi = \varphi + (1-\varphi) + (1-\varphi)\frac{(1 - \varphi)}{\varphi} + \cdots \qquad \text{Eq (6-17)}$$

Since $(1 - \varphi)/\varphi = \vartheta$, this equation becomes

$$\Phi = \frac{1}{1 + \vartheta} + \frac{\vartheta}{1 + \vartheta} + \frac{\vartheta^2}{1 + \vartheta} + \cdots \text{ to } n \text{ components} \qquad \text{Eq (6-18)}$$

Each term of the series applies to a single component and there are n-terms for n-components. Eq (6-18) holds only when the voids of each component are the same. Now, if each term of the series is divided by Φ and multiplied by 100, we obtain the percentage volume of each component in a mixture to produce the minimum voids when the voids in each component are ϑ. Furnas (1931) computed possible minimum voids in packings of two- to four-component systems when the initial voids of

each component are known. Figure 33 shows these minimum voids for initial packing voids of 0.4 and 0.6. In conjunction with this figure, Table 26 shows the composition of packings for minimum voids.

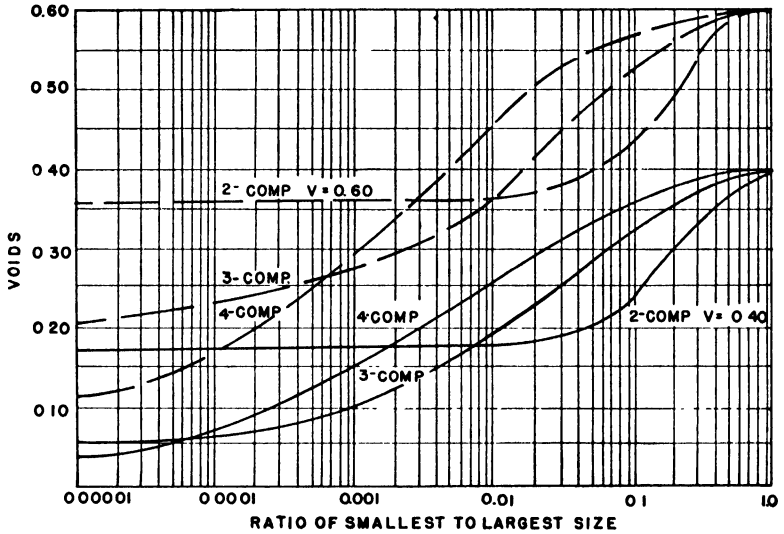


FIGURE 33. MINIMUM VOIDS FOR VARIOUS COMPONENT SYSTEMS FOR INITIAL VOIDS OF 0.40 AND 0.60.

It will be noted that the abscissae in Figure 33 are in terms of the ratio of diameters of the smallest to the largest sizes. The density of a packing will be affected by the number of components and by the size of the smallest and largest particles. Furnas also computed the number of components required to produce maximum density (minimum voids) for extreme size-ratios ranging from 0.00001 to 0.1. In general, values of the ratios of diameters greater than 0.003 require two components; those from 0.0001 to 0.003 require three components; and those less than 0.0001 require four components.

TABLE 26—COMPOSITION OF PACKINGS FOR MINIMUM VOIDS

Initial voids in packing of uniformly sized particles (percent)	Number of components	Volume (percent of each component) d_1	d_2	d_3	d_4
40	2	71.5	28.5	...	...
	3	64.2	25.6	10.2	...
	4	61.7	24.6	9.8	3.9
60	2	62.5	37.5	...	...
	3	51.0	30.6	18.4	...
	4	46.0	27.6	16.5	9.9

In using Figure 33 which applies only to substances having the same densities and voids, we may proceed as indicated in the following example. Assume a three-component system, the diameters of the respective components being 1 in., 0.03 in., and 0.001 in. The ratio of the smallest to the largest size is therefore 0.001/1 = 0.001. If the initial voids are taken as 0.40, by reference to Figure 33 we find that the possible minimum voids for a three-component system ($\vartheta = 0.40$) are 0.10. To obtain these voids, however, we must have the following composition by volume of the various components: of the 1 in., 64.2 percent; of the 0.03 in., 25.6 per cent; and of the 0.001 in., 10.2 percent. For initial void values between 40 and 60 percent, we may interpolate in Figure 33 and Table 26.

In practice, it is best to have sized material (material with a constant fixed ratio between sizes) and to confine the number of components within the limits indicated by the discussion above. The data presented are of little value if we depart from these conventions.

By extending the development given earlier for a binary system of maximum density it is possible to solve the general case of varying voids and densities. For a three-component system subject to these conditions we have

$$\Phi = \varphi_1 + (1 + \varphi_1) + (1 - \varphi_1)\left(\frac{1 - \varphi_2}{\varphi_2}\right) \qquad \text{Eq (6-19)}$$

$$\varphi_1 = \frac{(1 - \vartheta_1)\rho_1}{(1 - \vartheta_1)\rho_1 + \vartheta_1(1 - \vartheta_2)\rho_2} \qquad \text{Eq (6-20)}$$

$$\varphi_2 = \frac{(1 - \vartheta_2)\rho_2}{(1 - \vartheta_2)\rho_2 + \vartheta_2(1 - \vartheta_3)\rho_3} \qquad \text{Eq (6-21)}$$

and the ϑ's and ρ's are, respectively, the voids and true specific gravities of the components, identified by subscripts. As before, the value of φ_1 represents the degree to which the large particles are saturated by the second component, and φ_2 the degree to which the second component is saturated by the third. The process may be repeated for any number of components. The weight of each component for densest packing will be as follows:

For the first component	φ_1
For the second component	$1 - \varphi_1$
For the third component	$(1 - \varphi_1)\left(\frac{1 - \varphi_2}{\varphi_2}\right)$

Each of these quantities divided by Φ will then give the proportion by weight of each component for densest packing. After the proportions by weight have been determined, the proportions by volume may then

be obtained by dividing the weight by the apparent specific gravity. In using these equations, the same precautions must be taken with regard to sizing of the components. An example will explain the use of these equations.

Assume three materials with the following characteristics:

Component	*Size*	*Voids*	*Specific gravity*
1	1.0	0.50	2.33
2	0.03	0.45	2.65
3	0.001	0.55	2.95

Size-ratio = 0.001/1 = 0.001.

$$\varphi_1 = \frac{(1 - 0.50) \times 2.33}{(1 - 0.50) \times 2.33 + 0.50\,(1 - 0.45) \times 2.95} = 0.62$$

$$\varphi_2 = \frac{(1 - 0.45) \times 2.65}{(1 - 0.45) \times 2.65 + 0.45\,(1 - 0.55) \times 2.95} = 0.71$$

Hence, the total weight, from Eq (6-19) is

$$\Phi = 0.62 + 0.38 + 0.16 = 1.16$$

Since the apparent specific gravity = (1 − voids) (true specific gravity), we have for the case under discussion:

Apparent density, component 1 = 0.50 × 2.33 = 1.17
Apparent density, component 2 = 0.55 × 2.65 = 1.46
Apparent density, component 3 = 0.45 × 2.95 = 1.33

The total volume will then be:

$$\text{Total volume} = \frac{0.62}{1.17} + \frac{0.38}{1.46} + \frac{0.16}{1.33} = 0\,91$$

and the proportion by volume as follows:

Volume of component 1 = 58.4 percent
Volume of component 2 = 28.6 percent
Volume of component 3 = 13.2 percent

The weighted average of the voids in packings of sized material of this system is 0.46. We may now interpolate in Figure 33 and determine the minimum voids. The voids thus determined are 0.14.

While the discussion above seems reasonably logical and has received a certain amount of experimental confirmation by Furnas, it cannot be regarded as adequate. The initial voids of the separate components cannot influence the final voids of the mixture itself. It must be remembered that Furnas is dealing with "normal" packings, and under the conditions stipulated for them the respective voids of each component will not differ much from any of the others. The analysis given is ingenious

and, within the limitations to which it applies, is singularly remarkable. On the other hand, it might well lead one astray in so far as the real nature of packings of maximum density are concerned. Obviously, as has been shown in earlier chapters, "graded" particles themselves possess rather extended size-frequency distributions so that particles of one component may overlap in size those of another. Furnas' method should give best results only when the components differ widely as to size, which must be clear from our previous discussion.

From geometrical considerations, using either the basic rhombic or cubic packing arrangements, we may determine the number and sizes of particles necessary to fill the interstices. In this way, the degree of void reduction possible by using components of various sizes and quantities can be determined. While this procedure is more or less idealized, investigations by White and Walton (1937) have indicated that voids as low as 15 per cent are possible. The following geometrical development is due to these investigators.

Consider the section x–y in the rhombohedral arrangement of spheres having a diameter d as shown in Figure 34, the dotted lines indicating

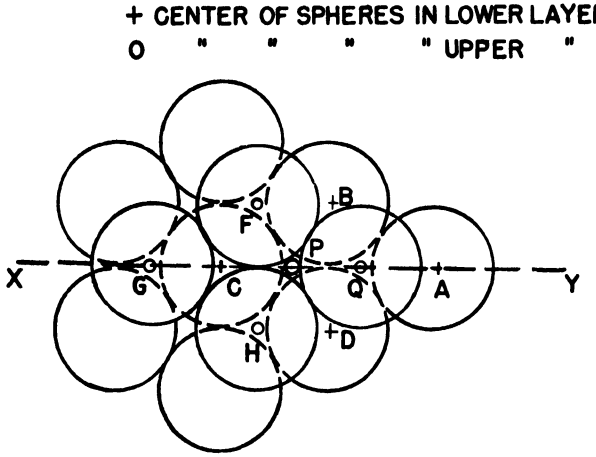


FIGURE 34. GENERAL ARRANGEMENT OF RHOMBOHEDRAL PACKING.

spheres not cut by the section. Let the height between the layers be ET, and let P be the point of contact between the two spheres B and D; then for the primary spheres (Figure 35A), $AP = d\sqrt{3}/2$, $AE = d$, $AT = d/\sqrt{3}$, $(ET)^2 = (AE)^2 - (AT)^2 = 2d^2/3$ or $ET = d\sqrt{2/3}$.

Within the unit cell is the void (Figure 28). The dimensions of the

largest sphere which can be inserted in this void are shown in Figure 35A. This is called the secondary sphere. Referring to Figure 34 it is seen that this sphere is bounded by six spheres having centers B, C, D, E, F, and H, and its center will be on the line CE. By geometry, we have from Figure 35A that $CT = CP + PT = d\sqrt{3}/2 + (d/6)\sqrt{3} = 2d\sqrt{3}$; $ET = d\sqrt{2}/3$; $(CE)^2 = (CT)^2 + (ET)^2 = 2d^2$, and $CE = d\sqrt{2}$. Therefore, if d_2 represents the diameter of the secondary sphere, $d + d_2 = CE$ and $d_2 = d(\sqrt{2} - 1) = 0.414d$.

Referring again to Figure 34, we note that the void formed by the spheres A, B, D, and E is symmetrical about any of six planes. The point K in Figure 35B will be the center of a tertiary sphere, and this lies in the plane of EA and the contact P between the spheres B and D. The center must also lie on the plane of BD and the contact Q of the

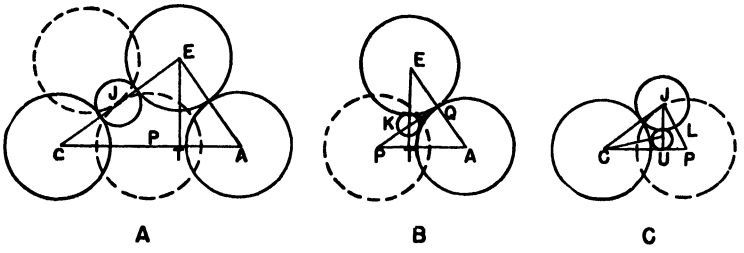


FIGURE 35. LOCATION AND RELATIVE SIZES OF (*A*) SECONDARY, (*B*) TERTIARY, AND (*C*) QUATERNARY SPHERES IN A RHOMBOHEDRAL PACKING (WHITE AND WALTIN, 1937).

spheres A and E. This plane is represented by the line PQ. In addition, the center of K must lie vertically below E on the line ET. Note that there will be two tertiary spheres formed within a unit configuration of eight primary spheres.

To compute the diameter of the tertiary sphere, we have triangles AET and EKQ similar and as a consequence $EK/EQ = AE/ET$; whence $EK = EQ{\cdot}AE/ET = d/2\sqrt{2/3}$. But $EK = (d/2) + (d_3/2)$ where d_3 is the diameter of the tertiary sphere. Hence, $d_3 = 0.225d$.

The secondary and tertiary spheres are major insertions in the voids of a rhombohedral arrangement of eight primary spheres. It is clear that the secondary sphere does not completely fill the center cavity, but allows for eight spheres of smaller diameter to be inserted about it. These spheres may be termed (following White and Walton) quaternary spheres. The axis of one of these spheres is the vertical line JU (Figure 35C) and its center L will lie on this axis and make contact with spheres

C and J. If d_4 denotes the diameter of the quaternary sphere, we have from Figure 35c and other geometrical facts

$$JU = ET/2 = d/2\sqrt{2/3}$$
$$JL = 0.207d + d_4/2$$
$$CL = d/2 + d_4/2$$
$$CU = d\sqrt{2/3}/2$$
$$LU = d\sqrt{3}/3 - 0.207d - d_4/2 = 0.201d - d_4/2$$
$$(CU)^2 = (CL)^2 - (LU)^2 = 0.209d^2 + 0.702dd_4$$

Hence, $d_4 = 0.495d/2.804 = 0.177d$ and we note that the edge of the quaternary sphere is $0.024d$ above the line CP.

The computation for quinary spheres is more involved. There are apparently many spaces, especially in the void about the tertiary spheres which may be filled. It is not difficult to compute the diameter of the quinary sphere to fit these voids. The center will lie on the axis ET and the diameter of the sphere d_5 will be $0.155d$. Unfortunately, these spheres (four for each tertiary sphere) are too large. If we fit the quinary spheres between the tertiary and quaternary spheres, we obtain a value of $d_5 = 0.116$ and this is the maximum size which can fit within the rhombohedral arrangement under discussion.

The effect of distributing the gradations just computed in a primary rhombohedral arrangement is shown in Table 27. We see that the voids are reduced by such gradations to less than 15 percent. By using material finer than $0.116d$, it is conceivable that voids of lesser amounts can be obtained.

TABLE 27—EFFECT OF SIZE GRADATIONS ON PROPERTIES OF A RHOMBOHEDRAL PACKING (WHITE AND WALTON, 1937)

Property	Mixture with—				
	Primary	Secondary	Tertiary	Quaternary	Quinary
Diameter	d	$0.414d$	$0.225d$	$0.177d$	$0.116d$
Relative number	1	1	2	8	8
Volume of space	$0.524d^3$	$0.037d^3$	$0.006d^3$	$0.0026d^3$	$0.0008d^3$
Volume of spheres added	$0.524d^3$	$0.037d^3$	$0.012d^3$	$0.021d^3$	$0.0064d^3$
Total solid volume of spheres added	$0.524d^3$	$0.561d^3$	$0.573d^3$	$0.595d^3$	$0.602d^3$
Fractional voids in mixtures	0.2595	0.207	0.190	0.158	0.149
Weight of spheres in final mixture (percent)	77.08	5.47	1.75	3.31	0.97
Total surface area of spheres in mixture	$3.14d^2$	$3.68d^2$	$4.00d^2$	$4.77d^2$	$5.11d^2$

Had we used a cubical arrangement to begin with, the decrease in porosity would have been greater through addition of the first sphere, and thereafter the other spheres would have had about the same effect as in the case of the cubical cell above. It is noted from this discussion that the decrease in porosity is not very marked with the addition of fine particles, unless they are small enough to fill the smallest interstices. It is also to be noted that had the original void been filled only with fine particles of one size the decrease in porosity would not have been as great as the above. This is evident if we refer to Eq (6-12) and with certain arbitrary assumptions regarding values of ϑ and d calculate the number of particles to fill the unit-void in question. The computation of ϑ of the mixture may then be made and it will not be found as low as expected. In passing it may be mentioned that no method is available for distributing the various components above determined so as to achieve the practical maximum density with particles of the diameters computed. The discussion merely serves to point out that not only a definite grading of sizes, but also a frequency pattern, are necessary to obtain maximum densities. The extremely low values of voids obtainable by Furnas' theory are indeed noteworthy, in view of the requirements outlined.

BULKINESS

Bulkiness of a packing is defined as the reciprocal of the apparent density, that is, in terms of Eq (6-10)

$$\text{bulkiness} = \frac{1}{\rho_a} = \frac{1}{\rho(1 - \vartheta)} \qquad \text{Eq (6-22)}$$

In a preceding paragraph it was pointed out that in a binary system it is possible to increase the apparent density without increasing the volume of the system as a whole. Since specific volume and voids are related by Eq (6-20), the ordinates of Figure 32 may be changed with no alteration of the curves. The curves pertain to a binary system having voids of 0.50, both components having the same specific gravity. As an example, it is seen that if the size-ratio of spheres is 0.02, and the percentage of fines is 35 percent, there would be a volume saving equal to

$$\frac{2.00 - 1.39}{2.00} \times 100 = 30.5 \text{ percent}$$

While the curves shown in Figure 32 exemplify conditions in a binary system of spheres, they are, nevertheless, indicative of what takes place in any similar type of system, even though composed of irregular particles.

Systems of more than two components will possess a slightly greater degree of bulkiness except, as might be expected, when most of the material is extremely fine. In general, it may be said that mixtures of different-sized materials greatly increase the efficient use of packing space. The problem of proper proportions to obtain a minimum bulkiness can be approximated by methods previously outlined.

The subject of bulkiness, particularly with regard to particle-size, has received but scant attention. It is important in view of its commercial aspects in packaging powders. The principles of bulkiness are partially explained in the foregoing paragraphs, in so far as proportioning in accordance with particle-size distributions is concerned. However, there are other characteristics of packings of fine particles which were investigated by Cocking (1921) and Roller (1930).

It is an observed phenomenon that bulkiness increases with decreasing particle-size. Cocking utilized this fact to standardize certain properties of pharmaceuticals, and indicated the degree to which tapping affected bulkiness. Roller subsequently investigated the relation of

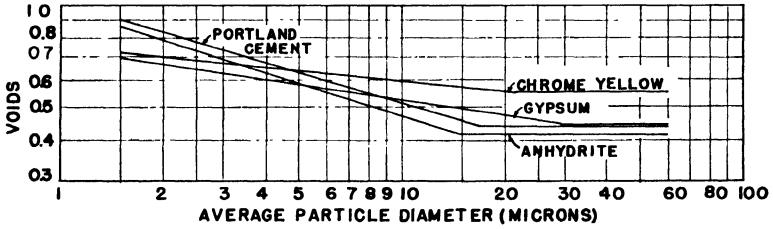


FIGURE 36. BULKING PROPERTIES OF VARIOUS POWDERS.

particle-size to bulkiness in four powders: Anhydrite, gypsum, Portland cement, and chrome-yellow powders. These powders were carefully sized by elutriation and the surface mean diameters determined in accordance with Eq (3-13). Roller states that his selection of this measure of diameter is more significant than the usual arithmetic diameter. The latter merely accentuates the fact that the smaller particles are more frequent in number and that the arithmetic means for the powders studied will be approximately the same. However, it is difficult to justify this reasoning since bulkiness is determined by the superincumbent weight of powder and the shape of the particles. Surface considerations appear to be minor. For these reasons, although not essential, it would seem that a volume mean diameter is indicated.

Roller investigated the effect of varying container size and amount of material required in determining bulkiness. Bulkiness was determined by tapping each type container until no more material was required to fill it to a predetermined level. These studies indicated that cylindrical

containers were satisfactory, provided the ratio of height of the powder to the diameter of the tube was 6 or greater. This is necessary in order to reduce the dispersive jogging effect of tapping. The results of Roller's studies are shown in Figure 36 and it is evident that the bulkiness (as well as the voids) increases as the particle-size decreases. The relationship between voids and particle-size may be considered as consisting of two parts, one which may be expressed by the equation

$$\vartheta = K(1/d)^n \qquad \text{Eq (6-23)}$$

for values of d less than a critical diameter d_c; and the other

$$\vartheta = constant = C$$

for values greater than this. Combining these two equations, we may calculate the critical diameter

$$d_c = (K/C)^{1/n} \qquad \text{Eq (6-24)}$$

The critical diameter varies with the different powders and ranges from 14.3 to 28.5 μ. Data pertaining to the values of constants K, n, C, and d_c, are shown in Table 28.

TABLE 28—BULKINESS CONSTANTS OF DIFFERENT POWDERS

Powder	K	n	C	d_c
Anhydrite	0.98	0.320	0.42	14.3
Gypsum	0.75	0.160	0.44	28.5
Portland cement	1.01	0.290	0.43	16.2
Chrome-yellow	0.76	0.10	0.55	21.4

RELATION BETWEEN VOIDS AND PORE-SIZE

The development given for determining pore-size in terms of particle-diameter applies only to an array of spheres of equal size. When the spheres are of different sizes the solution is possible only by involved statistical procedures. However, the average pore-diameter may be determined experimentally by means of the Poiseuille equation (see Chapter 15). Traxler and Baum (1936) used this method in the following manner: The Poiseuille equation is

$$q = \frac{Q}{t} = \frac{\pi D_c^4 P}{128 \mu L} \qquad \text{Eq (6-25)}$$

where q is the rate of flow in a unit-time, D_c the diameter of the capillary, P the pressure producing flow, L the length of the capillary, and μ the viscosity of the liquid. If there are N capillaries, then

$$q = \frac{Q}{t} = \frac{\pi N D_c^4 P}{128 \mu L} \qquad \text{Eq (6-26)}$$

The cross-sectional area A_c of the sample which is not filled with solids is given to a close degree of approximation by the equation $A_c = A\vartheta$, where A is the area of the packing and ϑ the voids. Also, if the area A_c is composed of capillaries with diameter D_c, $A_c = \pi N D_c^2/4$. From these two equations we obtain the equation $N = 4A\vartheta/\pi D_c^2$. Substituting this value of N in Eq (6-26) and solving for D_c

$$D_c = \sqrt{\frac{32q\mu L}{A\vartheta P}} \qquad \text{Eq (6-27)}$$

If a gas is made to flow through the packing and the pressure-drop is $P = P_1 - P_2$, where P_1 and P_2 are observed terminal pressures (absolute), the value of q must be corrected as follows:

$$q_{cor} = \frac{2(\text{barometric pressure})}{P_1 + P_2} q$$

Using Eq (6-27), Traxler and Baum obtained the following equation expressing the relation between pore-diameter and voids

$$\ln D_c = b + 0.019\vartheta \qquad \text{Eq (6-28)}$$

where b is a constant which must be determined for each material. For particles of the order of 3.5 μ, b has a value of approximately 0.1. In terms of average particle-diameter D_c is approximately $2d$ for particles less than 2 μ; $5d$ for particles from 2 to 3 μ; $7d$ for particles 3 to 5 μ, and $9d$ for particles >5 μ. It is assumed, of course, that the particles are more or less uniform in diameter and that the density of the packing is homogeneous.

Reduction in Volume Due to Settling—Let ϑ_1 be the voids in a given random packing of particles, and ϑ_2 the voids in the same packing after tapping or vibration. Also, let the height of the original packing be H_1, and the final height H_2. Then the decrease in the height of the packing will be

$$\Delta H = H_1 - H_2 = H_1(\vartheta_1 - \vartheta_2) \qquad \text{Eq (6-29)}$$

The volume decrease, if the packing is in a container of uniform cross-sectional area, A, will then be given by the expression

$$\text{volume decrease} = A\,\Delta H = AH_1(\vartheta_1 - \vartheta_2) \qquad \text{Eq (6-30)}$$

Wall Effect—When particulate matter is packed into a cylinder or column, the voids along the wall will be greater than in the body of the bed. Furnas (1929) examined the nature of these voids for a circular container, and obtained the relation for the wall voids,

$$\vartheta_{wa} = [\vartheta + \xi(1 - \vartheta)]\left(\frac{1 + 2\xi d}{D}\right) - \frac{2\xi d}{D} \qquad \text{Eq (6-31)}$$

where ϑ is the voids in a volume of the center portion of the packing (approximately the over-all voids), ξ a factor experimentally determined, d the average diameter of the particles, and D the diameter of the container. The approximate value of ξ was found by Furnas to be 0.3. The area of the ring to which Eq (6-30) is applicable is

$$A_{wa} = \pi Dd/2 \qquad \text{Eq (6-32)}$$

When the diameter of the particles is very small in comparison with the diameter of the container, the wall effect may be appreciable. The method of accounting for the difference in packing voids with regard to flow of fluids is discussed in Chapter 13.

Influence of Particle-Shape and Moisture on Porosity—Graton and Fraser (1935) examined the influence of particle-shape on porosity under dry and wet conditions. All samples studied by these investigators passed an 18- and were retained on 35-mesh Tyler screens. Their results for several materials are shown in Table 29. The loose state was the arrangement assumed by the particles when poured into the test container. The compact state was achieved by tapping the test container. In the wet state the material was first wetted and poured into the measuring container filled with water. After the desired packing was obtained, the material was dried at 110 deg C, collected, then weighed and its volume determined. It is to be noted in Table 29 that wet materials pack more loosely than dry materials.

TABLE 29—INFLUENCE OF PARTICLE-SHAPE ON POROSITY

Material	Sp gr.	Dry		Wet	
		Loose	Compacted	Loose	Compacted
Spherical lead shot	11.21	40.6	37.18	42.40	38.89
Spherical sulfur shot	2.024	43.38	37.35	44.14	38.24
Marine sand	2 681	38.52	34.78	42.96	35.04
Beach sand	2.658	41.17	36 55	46.55	38.46
Dune sand	2 681	41.17	37.60	44.93	39.34
Crushed calcite	2.665	50.50	40.76	54.50	42.74
Crushed quartz	2.650	48.13	41.20	53.88	43.96
Crushed halite	2.180	52.05	43.51	...	...
Crushed mica	2.837	93.53	86.62	92.38	87.28

STUDY OF PACKING STRUCTURE

We have defined normal packings as packings formed by dropping particles singly into a container. The arrangement assumed by the particles determines the voids. In addition, the arrangement of particles together with the general shape of the particle materially affect the resistance to flow of fluids through the packing. Therefore, it is of inter-

est to investigate the general relationship of particles to each other. For this purpose we may resort to either of two methods, depending upon the size of material dealt with. In the case of large solids, the particles may be set in a thin mixture of plaster of Paris, and the packing then sectioned. The tracings shown in Figure 37 are from photographs of such sections made by Furnas (1929). It will be noted that there is a relatively large amount of open area with both broken solids and lead shot.

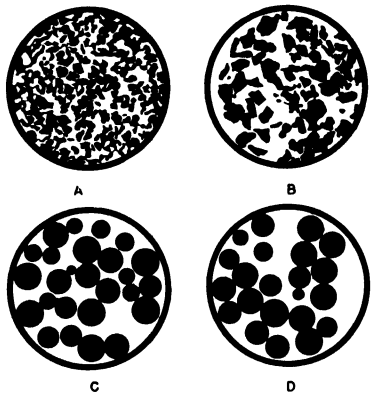


FIGURE 37. CROSS SECTION OF PACKINGS CONTAINING SPHERES AND IRREGULAR MATERIALS. (*A*) IRON ORE, 6/8-MESH; (*B*) IRON ORE, 3/4-MESH; (*C*) 3-0 SHOT; (*D*) 4-0 SHOT. (REDUCED ONE-HALF.)

When arrangements of fine particles are to be examined, recourse must be had to the microscopic technique used by sedimentary petrologists (Milner, 1940). This is a so-called "thin-section" technique which requires considerable skill in manipulation. H. Harper *et al* (1936) described methods for examination of soils which can be used with most fine materials. Since the sample to be studied must be consolidated with a minimum disturbance of the pore-space, Harper studied several consolidating substances including sodium silicate, colorless and black lacquer, paraffin, naphthalene, white enamel, Canada balsam, and many other materials. Most successful results were obtained with thin lacquer (about 1 part lacquer to 2 parts thinner). This mixture is applied to the packing and dried. The packing may then be sectioned, or one end ground to a suitable depth for examination. Harper recommends that the end being ground should receive frequent applications of lacquer to prevent disintegration.

Problems

1. There are 210,000 particles per cu cm in a packing whose voids by measurement amount to 34.6 percent. Calculate the nominal diameter of the particles in mm units.

2. The packing of Problem 1 is shaken until the percentage of voids is decreased to 31.3 percent. Compute the amount of shrinkage in volume.

3. Determine the pore-size in Problems 1 and 2, assuming that the value of b of Eq (6-28) is 1.0.

4. Given fine metallic particles averaging 0.01, 0 1, and 1 mm in diameter, $\vartheta = 0.4$ and $\rho = 5.0$. Determine the percentages of each which will give a mixture of maximum density.

CHAPTER 7

BEHAVIOR OF PARTICLES UNDER PRESSURE

THE whole problem of computing pressure distributions in particulate packings is one of great complexity. In addition to the fact that we are unable to deal with a material whose apparent density is not uniform, we must consider added difficulties such as diffusion, sliding friction, deformation of individual particles, cohesive forces, and perhaps others. The quantitative relationships of these factors to particle size must remain empirical for the time being. In the paragraphs to follow we shall be concerned only with a limited theory of the problem of particles under pressure.

THE RECIPROCAL THEOREM

Consider an assemblage of rigid particles in contact with each other and assume that a point pressure is applied to one of the particles inside the assemblage; then the pressure will be transmitted in diminishing amounts to *all* particles immediately in contact with it and there will be a varying degree of displacement of these particles, assuming that there is no permanent deformation or slippage of the particles themselves. In such a case the following theorem, known as the reciprocal theorem, holds: *If at a point A in an assemblage of particles a pressure P_A is applied causing a displacement δ_A in the direction R_A, and if as a result a displacement δ_B and pressure P_B in the direction R_B is noted for a particle at point B, then a pressure P_A at the point B in the direction R_B will cause displacement δ_A of particle A in the direction R_A.* A proof of this theorem is perhaps best demonstrated graphically

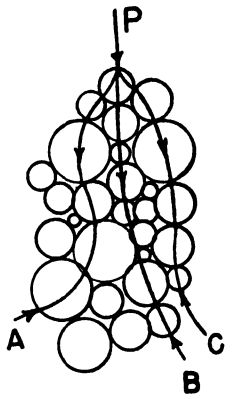


FIGURE 38. LINES OF PRESSURE TRANSMISSION IN A PACKING

Figure 38 shows a section (not necessarily plane) of a group of particles all in contact with each other. The points A and B are designated, as are also the pressures and their direction.

Without loss of generality we may assume that the lines of pressure transmission are I and II or I, II, and III, or whatever we may desire. As long as the resultant is P_B in the direction R_B, the number of paths and their directions to B are immaterial. We must show that if conditions at P and B are reversed, the displacements will also be reversed. That such must be the case is obvious since the lines of pressure in either case are the same as are their component directions, the only change being in the magnitude of the pressures along the paths.

The foregoing is a fundamental theorem of great importance and holds only so long as (*a*) no slippage occurs between the particles, and (*b*) there is no permanent deformation of individual particles—that is, the particles are perfectly elastic, as we noted before. The theorem is particularly valuable in establishing boundary conditions where multiple pressures are applied to an extended packing of particles.

ANGLE OF REPOSE

As already noted, the reciprocal theorem does not include specifically conditions where sliding between particles takes place. Geometrical alterations between granular material affect reciprocity of pressure and displacement.

By itself the stability of a mass of particles depends upon the mutual friction between grains. This applies in the absence of adhesive forces. Water, for example, tends to hold many particles by surface tension. Therefore we shall consider our particles dry and separate grains, and that internal and external stability of the mass are functions of mutual friction. Rankine (1858) stated the general law regarding behavior of particles as follows: The resistance to displacement by sliding along a given plane in a loose granular mass is equal to the normal pressure exerted between the parts of the mass on either side of that plane, multiplied by a specific constant. This constant is called the coefficient of friction and is the tangent of the angle of repose. This angle is simply the maximum angle at which a granular material, when piled, will not slide. Its value varies with size and kind of material but rarely exceeds 48 deg and is usually taken as 34 deg.

Mathematically stated: if p_n denotes the normal pressure per unit area and p_t the resistance to sliding per unit area, then the angle of repose, ϕ, is simply given by the expression

$$\phi = \tan^{-1} \frac{p_t}{p_n} \qquad \text{Eq (7-1)}$$

As a corollary of the principle just given we may state that a necessary and sufficient condition for the stability of a mass of particulate matter re-

quires that the direction of the pressure between the portions into which it is divided by any plane should not at any point make an angle with the normal to that plane exceeding the angle of repose.

MASS OF PARTICLES LOADED WITH THEIR OWN WEIGHT

This problem has been solved by Rankine and applied to the pressure of the earth loaded with its own weight. Obviously, in this case gravity causes the vertical pressure which in turn causes the mass of material to spread laterally. The vertical pressure is proportional to the depth. If the surface is inclined at an angle θ, and w is the weight of a unit volume of particles, then the pressure at a depth z is

$$p_z = wz \cos \theta \qquad \text{Eq (7-2)}$$

The pressure parallel to the surface, p_θ, was shown by Rankine to be given by the expression

$$p_\theta = \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \phi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \phi}} \cdot p_z \qquad \text{Eq (7-3)}$$

where ϕ is the angle of repose mentioned above. For a horizontal surface, $\theta = 0$, and if we consider the horizontal pressure p_x as parallel to the x-axis, that is, perpendicular to p_z,

$$p_z = wz \qquad \text{Eq (7-4)}$$

and

$$p_x = wz \cdot \frac{1 - \sin \phi}{1 + \sin \phi} \qquad \text{Eq (7-5)}$$

For the special case $\theta = \phi$

$$p_z = p_x = wz \cos \phi \qquad \text{Eq (7-6)}$$

Now suppose that a mass of particles inclined at an angle θ ($\theta < \phi$) is confined within a vertical retaining wall (Figure 39) and we desire to determine the total pressure P_θ exerted by the particles against this wall per unit of width.

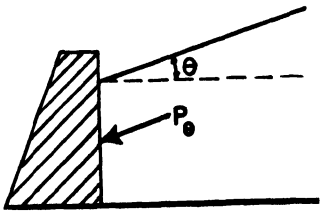


FIGURE 39. SECTION THROUGH AN ABUTMENT.

It can be shown that the pressure is given by the expression

$$P_\theta = \frac{wz^2}{2} \cos \theta \cdot \frac{p_\theta}{p_z} \qquad \text{Eq (7-7)}$$

where z is the distance from the top surface to the base of the wall and p_z and p_θ are as defined in Eqs (7-2) and (7-3). The center of pressure

measured from the top surface at the retaining wall is simply $^2/_3(z)$.

When the surface is horizontal, $\theta = 0$, and

$$P_x = \frac{wz^2}{2} \cdot \frac{1 - \sin \phi}{1 + \sin \phi} \qquad \text{Eq (7-8)}$$

and when $\theta = \phi$, (Eq(7-7))

$$P_x = \frac{wz^2}{2} \cos \phi \qquad \text{Eq (7-9)}$$

EFFECT OF PRESSURE ON POROSITY

When pressure is applied to a mass of irregular particles confined in a chamber, the particles tend to consolidate and reduce the porosity. A marked decrease in porosity has been observed with regard to oil sands and shales far below the surface of the ground. In this connection, Athy (1930) has shown that the porosity of compact material below the earth's surface is given by the formula:

$$\vartheta = \vartheta_0 \cdot \exp(-bz) \qquad \text{Eq (7-10)}$$

where ϑ is the porosity at depth z below the surface, ϑ_1 the porosity of surface clays, and b a constant ranging from 3×10^{-4} to 3.5×10^{-4}, when z is given in feet. On the basis of his studies Athy found that while the porosity of surface clays is 45 to 50 percent, that of shale 6000 ft below the earth's surface is approximately 5 percent; compaction is therefore considerable. On the other hand, it must be remembered that this compaction resulted not only from a rearrangement of the grains, but also from possible recrystallization. The Athy equation cannot be applied to an experimental packing under pressure without some modification since, in addition to rearrangement of the particles, crushing may take place without any eventual consolidation, and perhaps with only a comparatively slight change in porosity. In application of the Athy equation, Hedberg (1926) has shown that the porosities of shales at the same depth in any specific locality are practically the same. Hedberg claims that the porosity at any given depth may be taken as a measure of the compaction which the shale has undergone.

PRESSURE DISTRIBUTION

Point Load—The general solution of the pressure-distribution in a medium, generated by a point load applied to a packing of unlimited depth and extent, is due to Boussinesq (1876, 1885)

$$p_z = (3P/2\pi z^2) \cos^5 \theta = \frac{3}{2\pi} \frac{1}{[1 + (r/z)^2]^{5/2}} \cdot \frac{P}{z^2} \qquad \text{Eq (7-11)}$$

where p_z is the pressure intensity measured at a point defined by the coordinates r and z or θ as shown in Figure 40. The origin of the coordinate system is taken at the point where the pressure is applied. This formula, as in the case of the previous formulas, is independent of the characteristics of the material (assumed elastic and isotropic), and in value depends only on geometrical constants. The general validity of Boussinesq's pressure-distribution equation is substantiated in part by experiment. The units of p_z are determined by the choice of units for P and z. Thus, if P is in tons, and z in feet, then p_z is given in tons per square foot.

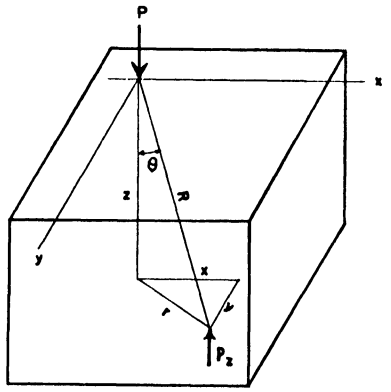


FIGURE 40. COORDINATE SYSTEM APPLYING TO EQ (7-11).

In addition to Eq (7-11) we have two other equations of interest. The horizontal pressure intensity p_r is given by the expression

$$p_r = \frac{P}{2\pi}\left\{(1-2\nu)\left[\frac{1}{r^2} - \frac{z}{r^2(r^2+z^2)^{1/2}}\right] - \frac{3r^2z}{(r^2+z^2)^{5/2}}\right\} \quad \text{Eq (7-12)}$$

and the vertical shear at r by

$$\tau_{rz} = \frac{3P}{2\pi}\cdot\frac{rz^2}{(r^2+z^2)^{5/2}} \quad \text{Eq (7-13)}$$

where ν is Poisson's ratio and other terms are as in Eq (7-11). The nature of these forces is shown in Figure 41.

If we divide Eq (7-11) by Eq (7-13) we obtain the relation

$$\frac{p_z}{\tau_{rz}} = \frac{z}{r}$$

indicating that the resultant stress passes through the origin. If this is denoted by p_R, we have

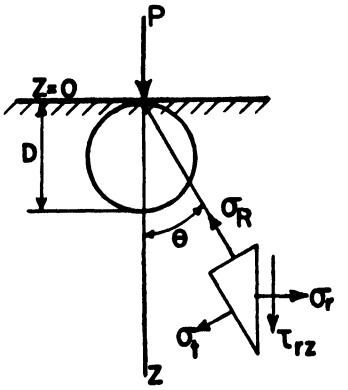


FIGURE 41. UNIT PRESSURE COMPONENTS IN A PACKING DUE TO A POINT PRESSURE ON THE SURFACE.

$$p_r = \sqrt{p_z^2 + \tau_{rz}^2} = \frac{3P}{2\pi}\cdot\frac{z^2}{(r^2+z^2)^2} = \frac{3P}{2\pi}\,\frac{\cos^2\theta}{r^2+z^2} \quad \text{Eq (7-14)}$$

A further important and simple relation may now be discussed. Let a sphere of diameter D be placed with its center on the z-ordinate through P and tangent to the plane $z = 0$. For every point on the surface of this sphere

$$r^2 + z^2 = D^2 \cos^2 \theta$$

Thus, substituting this expression in Eq (7-14) we conclude that for all points on the sphere the total stress is constant and equal to $3P/2\pi D^2$.

As we have stated, the Boussinesq equations are based on the assumption that the material dealt with is elastic and isotropic. Of course, this is not the case, although there is evidence that the form of pressure distribution obtained from the above equation conforms to experiments at considerable distance from the point of pressure application. In view of the importance of pressure distribution in granular material, it is remarkable that Boussinesq's derivations were not subjected to further study until comparatively recently. Weiskopf (1945) directed attention to the obvious limitations of elastic constants on which the original Boussinesq derivations depend, and presented a wholly new approach to solution of the problem of pressure distributions. Let us turn our attention to Figure 42 which shows—as in Figure 40—the application of a point pressure P to a semi-infinite mass of granular material. R_1 and R_2 indicate radii to a vertical line at a distance r from P. These radii are chosen so that

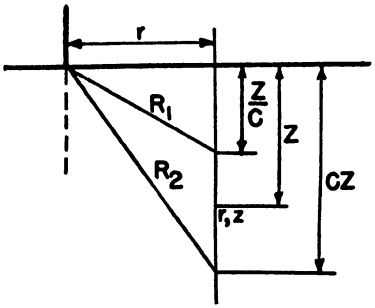


FIGURE 42. WEISKOPF COORDINATES FOR A POINT PRESSURE.

$$R_1^2 = (z/c)^2 + r^2$$
$$R_2^2 = (cz)^2 + r^2$$

where c is termed a particulate constant which for irregular particles has a value of 4.5. Weiskopf then proves the following relations to hold (without regard to sign):

$$p_z = \frac{Pcz}{\pi(c^2 - 1)}\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) \qquad \text{Eq (7-15)}$$

$$p_r = \frac{Pcz}{\pi(c^2 - 1)}\left(\frac{1}{c^2 R_1^2} - \frac{c^2}{R_2^2}\right) \qquad \text{Eq (7-16)}$$

$$p_{rz} = \frac{Pcr}{\pi(c^2 - 1)}\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) \qquad \text{Eq (7-17)}$$

Note that Eqs (7-15) and (7-17) also give

$$\frac{p_z}{\tau_{rz}} = \frac{z}{r}$$

as above.

Distributed Load—If the pressure is distributed over a beam of length $2b$ and unit width, as shown in Figure 43, with a uniform loading of P_0 per unit length, then the stress intensity p_z at any point (x, z) is given by either of the equations:

$$p_z = \frac{P_0}{\pi} [\alpha - \sin \alpha \cos (\alpha_1 + \alpha_2)]$$

or

$$p_z = \frac{P_0}{\pi} \left[\sin^{-1} \frac{2bz}{r_1 r_2} + \frac{2bz}{r_1^2 r_2^2} (z^2 - x^2 + b^2) \right] \quad \text{Eq (7-18)}$$

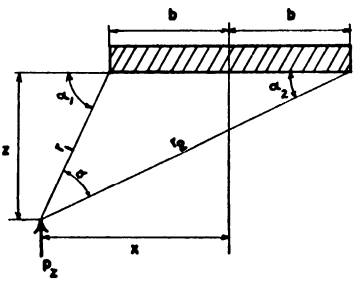


FIGURE 43. COORDINATE SYSTEM APPLYING TO EQ (7-18).

where $r_1^2 = z^2 + (x - b)^2$ and $r_2^2 = z^2 + (x + b)^2$

Along the center line (the z-axis), the pressures are given by

$$p_z = \frac{P_0}{\pi} (\alpha - \sin \alpha) = \frac{P_0}{\pi} \left(\sin^{-1} \frac{2u}{1 + u^2} + \frac{2u}{1 + u^2} \right) = kP_0 \quad \text{Eq (7-19)}$$

where $u = b/z$. Values of k for various values of u are given in Figure 44.

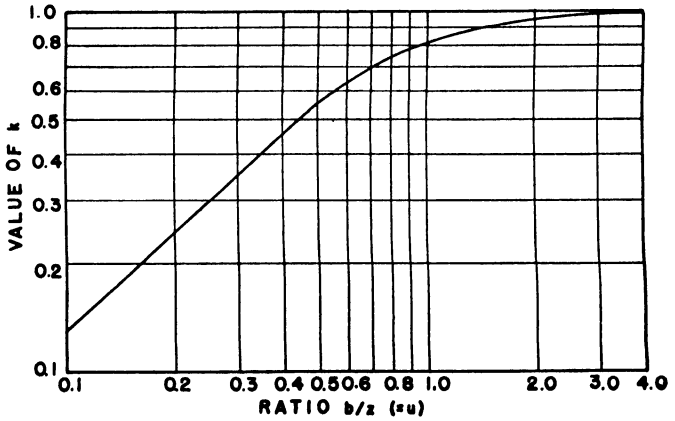


FIGURE 44. VALUES OF CONSTANT k IN EQ (7-19).

Weiskopf's corresponding solutions for a distributed load of unit width and length $2a$ in accordance with the coordinate system shown in Figure 45 are given below:

$$p_z = \frac{P_0}{\pi(c^2 - 1)} (c^2\theta_1 - \theta_2) \qquad \text{Eq (7-20)}$$

$$p_x = \frac{P_0}{\pi(c^2 - 1)} (\theta_1 - c^2\theta_2) \qquad \text{Eq (7-21)}$$

$$\tau_{xz} = \frac{P_0 c}{\pi(c^2 - 1)} \ln \frac{R_1 R_2'}{R_1' R_2} \qquad \text{Eq (7-22)}$$

where P_0 is the load per unit length.

Pressure Distribution Due to a Loaded Disc—We may solve the pressure distribution along the axis of a loaded disc from the following equation. If P_0 denotes the pressure intensity on the disc and D is the diameter of the disc, then

$$p_z = \frac{P_0 z^3}{(D^2 + z^2)^{3/2}} - P_0 \qquad \text{Eq (7-23)}$$

Limitation of Pressure Equations—Eqs (7-11) to (7-23) are useful as first approximations in determining the nature of pressure-distributions in a packing (as soil, for example) when loaded at a point or along a given area. Of course, the pressures computed from these equations can be applied only for distances considerably removed from the point of application. The criterion for determining this critical distance was discussed by Froelich (1933) who developed an equation giving the radius of a hemisphere, with center at the point of application of the pressure within which plastic flow takes place. The equation was developed for cohesionless sand and defines the limits within which the Boussinesq equation (Eq 7-11) does not hold, namely,

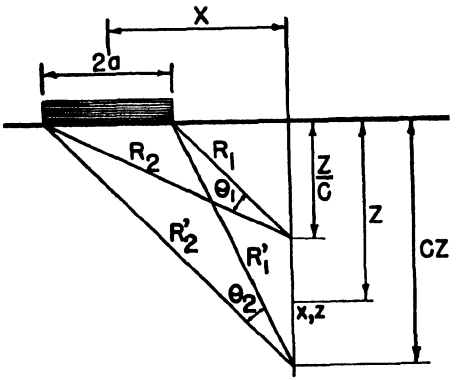


FIGURE 45. WEISKOPF COORDINATES FOR A DISTRIBUTED PRESSURE

$$r = \sqrt[3]{\frac{3}{4} \frac{P}{\pi\rho} \left(\frac{1 - k}{k}\right)} \qquad \text{Eq (7-24)}$$

where P is the weight applied, ρ the density of the packing, and k a

constant ($k = \sin \phi$, where ϕ represents the internal friction ranging from 20 deg to 35 deg).

Similarly, for a distributed load Froelich has shown that a "plasticity-condition" equation (applying to Eqs 7-18 and 7-19) may be derived "and a limiting value P_0' may be found, which the surface load, P_0, must not exceed if valid readings are to be found near the surface." This equation is

$$P_0' = \pi P_0 / [\cot \phi - (\pi/2 - \phi)] \qquad \text{Eq (7-25)}$$

where the terms are as previously defined.

Experimental Data—The Boussinesq equations, as has been pointed out, apply to solids which are infinite in extent. For mathematical reasons, these solids must necessarily be isotropic and elastic. Therefore, application of the equations to particulate packings is subject to confirmation by test. Unfortunately, experimental tests are rather limited. Some work has been done by Kögler and Scheidig (1927-1929) in Europe, confirming the limitations of the Boussinesq equation in a so-called "region of disturbance" (approximately as given by Eqs 7-24 and 7-25). Exterior to this region, the Boussinesq equation appears to hold. It must be evident, however, that the nature of the load and the special characteristics of the soil, especially its motion when subjected to pressure, are matters of great importance.

In this country, Goldbeck (1925) and Moran (1925) studied the distribution of pressure in packings of loose sand. Goldbeck applied a load of 5000 lb to a $13^1/_2$-in. plate and studied the distribution of pressures through a bed of sand. Moran carried out experiments of similar nature, and arrived at the following conclusions:

1. The amount of elastic compression and crushing is so small as to be negligible when referred to the total movement of the plate.
2. The sand immediately below the plate and inside a surface, which is approximately a 45-deg inverted cone having the plate as its base, moved downward with the plates.
3. All the sand exterior to the inverted cone had a component of motion away from the axis.
4. All the sand exterior to the inverted cone had a component of perpendicular motion which appeared to be *downward* for all the sand exterior to the 45-deg lines drawn downward and outward from the edge of the plate, and *upward* for the sand exterior to these lines.

Kögler and Scheidig's results for a 63 cm diameter disc load are shown in Figure 46. The pressure distribution is characteristic and contours of constant pressure intensity may be termed Boussinesq pressure bulbs. It will be noted from the figure that there is an upper region where no

vertical pressure exists and as a consequence pressures are necessarily greater near the center. This condition is contrary to the Boussinesq formula for multiple pressures, derivable from Eq (7-11) by superposition (see Timoshenko, 1934), and may be attributed to the non-cohesive-

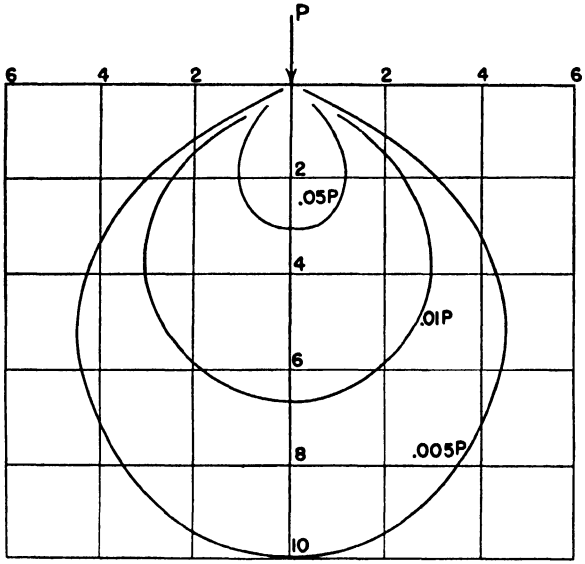


FIGURE 46. BOUSSINESQ PRESSURE BULBS FOR POINT PRESSURE.

ness of the particulate matter used. The unstressed material is confined to a region starting at the edge of the disc and extending in a direction about 35 deg from the vertical. At increasing distances from the disc the angle increases until all bulbs conform to Boussinesq's theory, in the present instance at about 180 cm below the disc.

The differences between Goldbeck's results and those of Kögler and Scheidig emphasize the difficulties of determining pressure distributions accurately in non-cohesive packings. Unquestionably, if the packings were wetted, so that particulate friction alone were not responsible for transmitting pressure, the Boussinesq formulas would prove satisfactory.

RESISTANCE OF PACKINGS TO IMPACT

When a pile is driven into unconsolidated material which is infinite in extent, a relation may be developed giving the resistance encountered by methods of elementary mechanics. The common velocity v between

two perfectly elastic bodies after impact, and before restitution takes place, is given by

$$v = (1 + e)\frac{mv_0}{m + m'} \qquad \text{Eq (7-26)}$$

where m is the mass of the hammer applied to the pile, and m' the mass of the pile. The velocity of the hammer before impact is v_0, and the coefficient of restitution e. Since $m = W/g$ and $m' = W'/g$, where W and W' are the weights of hammer and pile, respectively, and g the acceleration of gravity

$$v = (1 + e)\frac{v_0}{(1 + W'/W)}$$

Squaring both sides of the equation and multiplying by $W'/2g$, we obtain

$$\frac{W'v^2}{2g} = \frac{W'v_0^2}{2g}\left(\frac{1 + e}{1 + W'/W}\right)^2$$

Now $v_0^2 = 2gH$, where H is the height of fall of the hammer, and $W'v^2/2g$ is equal to the work done by the hammer. The work done is equal to the resistance $\mathfrak{R}$ encountered by the pile times the distance traversed by the pile and hammer together after impact, say l. Thus

$$\mathfrak{R}l = W'H\left(\frac{1 + e}{1 + W'/W}\right)^2 \qquad \text{Eq (7-27)}$$

If the pile is driven to a total depth L after successive hammer blows, the total work will then be

$$\frac{1}{2}\mathfrak{R}L = W'H\left(\frac{1 + e}{1 + W'/W}\right)^2 \qquad \text{Eq (7-28)}$$

The value e varies as follows:

For wood piling	0.15–0.2
For concrete piling	0.2 –0.3
For steel piling	0.5 –0.6

Eqs (7-27) and (7-28) take no account of the shortening of the pile due to the resistance encountered. However, this may be accounted for by the equation

$$\Delta L' = \epsilon\mathfrak{R}L'/AE \qquad \text{Eq (7-29)}$$

where $\Delta L'$ is the amount of shortening, L' the length of the pile, A is cross-sectional area, E the modulus of elasticity, and ϵ a coefficient. The value of ϵ depends upon whether the resistance encountered is due to wall friction or to the point of bearing. The usual values taken for

ϵ are 1.0 for full point bearing, and $\epsilon = {}^1/_2$ for uniform friction. If the friction increases uniformly with depth, ϵ is taken as ${}^2/_3$.

Eq (7-27) assumes that H, the distance from the top of the pile to its starting position, is constant. The equation, however, does not allow for the force expended in deforming the pile. Zesiger (cited by Plummer and Dore, 1940) derived a complete expression to allow for this deformation, as well as the factors above discussed.

$$\mathfrak{R} = \frac{W'H\left(\dfrac{1+e}{1+W'/W}\right)^2}{l + \sqrt{\dfrac{W'H}{2} \cdot \dfrac{k + \epsilon L/AE}{1 + W'/W}}} \qquad \text{Eq (7-30)}$$

where k has the value 2 or 3×10^{-7} in./lb. if the pile is driven in sand and gravel. Other units, namely ϵ and e, are dimensionless.

It is noteworthy that this formula is independent of the diameters of particles and depends only on the values of ϵ and k.

DIFFUSION UNDER PRESSURE

In Chapter 6 we gave a brief account of the normal manner of filling voids. By such a procedure we may reduce the void space to very small amounts. In powder metallurgy the void space is reduced partly by a deformation of particles and partly by a diffusion through slippage of the finer particles to the void spaces. This may be termed particle-pressure diffusion.

There is no established relationship between variables affecting the above diffusion. However, if we assume that the slippage forces are negligible as compared to the pressures applied to the particulate mass, and if we confine the mass within an immovable boundary, by means of dimensional analysis we may arrive at a sort of possible diffusion relationship. Let the diffusion $\mathfrak{D}$ be regarded as a function of some average particle diameter d, the apparent density of the mass ρ_a and the pressure intensity p, that is

$$\mathfrak{D} = \varphi(d, \rho_a, p)$$

Then

$$[L^2T^{-1}] = k \cdot [L]^a \cdot [ML^{-3}]^b [ML^{-1}T^{-2}]^c$$

$$0 = \quad b + c$$

$$2 = a - 3b - c$$

$$-1 = \quad - 2c$$

From which, $c = {}^1/_2$, $b = -{}^1/_2$, $a = 1$, and

$$\mathfrak{D} = kd\sqrt{\frac{\bar{p}}{\rho_a}} \qquad \text{Eq (7-31)}$$

Thus we see that the diffusion coefficient varies directly as the average particle size and the square root of the pressure intensity, and inversely as the square root of the apparent density. This equation appears to be in a reasonable form, although it is obvious that a wide dispersion of particle size is needed to cause diffusion. If particles are all of the same size, the diffusion should be a minimum. The term under the radical sign has the dimensions of velocity squared so that $\mathfrak{D}$ may be said to be directly proportional to the diameter of the particles and the rate at which the pressure is applied.

Some further consideration of Eq (7-31) leads us to the following alterations. For we may suppose that the diffusion does not begin until a certain critical pressure intensity p_c is reached, and that no diffusion proceeds beyond an ultimate pressure p_u. Furthermore, we observe that the apparent density increases with pressure in such a way that $\rho_a' = \rho_a \cdot \exp(-\alpha p/p_c)$ where α is a constant. There is reason to suppose that this is a proper form of function as we have seen above. Hence a suitable equation describing diffusion under pressure is

$$\mathfrak{D} = k'd\sqrt{\frac{p_u - \bar{p}}{\rho_a}} \cdot \exp(-\alpha p/2p_c) \qquad \text{Eq (7-32)}$$

where k' is a new adjusting constant and ρ_a is the *initial* apparent density. Thus, $\mathfrak{D}$ is not constant, and depends upon p, other things being equal. In general, the ultimate mixture is most homogeneous with materials having a high p_u and a low p_c, both of which are observable and measurable parameters.

In powder metallurgy, the pressure is generally applied slowly, so that the reorientation of fine particles is given sufficient time. Impact pressures—that is, those which are applied suddenly—curtail reorientation and the diffusion is of limited extent. As a rule pressure should be applied slowly, and the larger the mass of material the more slowly should the pressure be applied. When diffusion is complete, the particles may be deformed and the interstices filled by extrusion. The pressure curve for forming a given article from an apparent density ρ_a to one ρ_a' may therefore be considered as in three steps: (*a*) Compaction by pure slippage; (*b*) diffusion of fine particles; and (*c*) deformation. In the last-mentioned case, the usual elastic constants of a homogeeonus mass may be assumed to hold.

Particle-diffusion may be accelerated by use of a suitable lubricant thus reducing internal sliding friction.

VOID PRESSURE

When metal powders are compressed, numerous small cells are sealed off. The air within these cells often approximates the pressure to which the powder is subjected in the process of giving it cohesiveness and shape. The behavior of these cells when the compressed powder is heated is important from a general practical viewpoint, and also in regard to formation of certain alloys to which powder metallurgy is adapted. It might be expected that the process of heating sealed cells would tend to disrupt the void structure, and injure the strength of the compressed powder as a whole. That this is improbable can be shown in an elementary way by considering what happens in a single cell. We shall attribute to this cell a spherical shape and a uniform thickness of shell structure, continuous throughout. This procedure is an over-simplification, though as an average of conditions in a packing it is not unreasonable.

Let the diameter of the spherical cell be D_v; then if p is the intensity of air pressure within it and δ is the thickness of the shell

$$\frac{\pi D_v^2 p}{4} = \pi D_v \delta E \qquad \text{Eq (7-33)}$$

where E is the tension in the shell due to the pressure. Hence

$$E = \frac{pD_v}{4\delta} \qquad \text{Eq (7-34)}$$

As the temperature changes, both the volume of the cell and the air pressure within it change. If the diameter of the cell changes by an amount dD_v, then energy relationships lead us to the following:

$$E \cdot \pi D_v^2 \cdot dD_v = pdV \qquad \text{Eq (7-35)}$$

where dV represents the change in volume of the air. From the equation for an ideal gas,

$$pV = BT \qquad \text{Eq (7-36)}$$

we have

$$pdV + Vdp = BdT$$

and on substitution into Eq (7-35)

$$E \cdot \pi D_v^2 \cdot \frac{dD_v}{dT} = B\left(1 - \frac{T}{p}\frac{dp}{dT}\right)$$

From Eq (7-35)

$$\frac{dD_v}{dT} = -\frac{4\delta E}{p^2} \cdot \frac{dp}{dT}$$

so that

$$-64\pi\delta^3 \frac{E^4}{p^4} \cdot \frac{dp}{dT} = B\left(1 - \frac{T}{p}\frac{dp}{dT}\right)$$

But

$$\frac{E^3}{p^3} = \frac{D_v^3}{64\delta^3} = \frac{6V}{64\pi\delta^3} = \frac{6BT}{64\pi\delta^3 p}$$

and on further reduction and substitution of the value of E in Eq (7-34), we obtain

$$\frac{T}{p}\frac{dp}{dT}\left(1 - \frac{3}{2}\frac{D_v}{\delta}\right) = 1$$

The solution of this equation is obtainable by direct integration, and is

$$Tp^{[3/2\ (D_v/\delta)\ -1]} = constant \qquad \text{Eq (7-37)}$$

Thus, we see that the relationship between temperature and pressure in the cell depends upon the cell diameter and the shell thickness. If p_0 and T_0 indicate the initial pressure and temperature (absolute) existing in a given cell, then

$$\left(\frac{p}{p_0}\right)^{[3/2\ (D_v/\delta)\ -1]} = \frac{T_0}{T} \qquad \text{Eq (7-38)}$$

As a general rule D_v is always greater than δ. Hence an increase in temperature is accompanied by a decrease in the air pressure in the cell, and the converse if the temperature is decreased. Heating, therefore, does not tend to disrupt cells, unless the stresses brought into being destroy the cohesiveness of the shell. Cases can also arise, of course, when $\delta > D_v$ so that the pressure is directly proportional to some fractional power of T. Only when $\delta >> D_v$ is the pressure proportional to the temperature, a situation which occurs infrequently in the compression of metal powders. Note in the special case of $\delta > D_v$ that the pressure decreases very rapidly as the temperature increases.

Problems

1. A point load of 1 ton is applied to a homogeneous soil. Determine the pressure-distribution contour equivalent to 0.25 ton.

2. Given a mass of particles of uniform height, determine the pressure at any depth z.

Note: Here the pressure p per unit area is proportional to the density of the mass ρ_a

$$p = k\rho_a$$

Hence, due to the action of gravity, at any depth z

$$dp/p = (g/k)dz; \quad p = C\exp(gz/k)$$

where C and k are constants and g the acceleration due to gravity.

CHAPTER 8

DIFFUSION OF PARTICLES

It is often observed that dust in still air diffuses throughout the space surrounding its point of origin This diffusion is due in part to trajectorial forces at the source although the diffusion, especially in the case of fine particles, is similar in some respects to molecular diffusion. The larger particles, generally greater than 5 μ in diameter, settle fairly rapidly. When the particles are of colloidal or molecular dimensions, and issue from a source in a motionless fluid, the diffusion constant differs in many respects from the constant used in the case of turbulence. Moreover, in practical cases, being more concerned with suspensions in turbulent media we must develop an approach to a study of diffusion phenomena somewhat different from that which has been used in gas or colloidal theory. In our case we are not greatly concerned with the mean free path of particles, since their inertia is quite large and temperature alone cannot set them in motion, but rather with statistical measures of the turbulent fluid elements and their motion. This turbulence may be due to the motion of the fluid itself, to convection currents in the fluid caused by temperature differences between its various elements, or to motion from other causes to be discussed later. In the following we give only a brief analysis of colloidal diffusion. Later we shall consider the essential features of the theory of turbulence and special investigations by various workers who have given a rigorous analysis of the physico-mathematical theory concerning diffusion of particles coming within the scope of this discussion.

THERMAL DIFFUSION

Brownian Motion—Extremely small particles suspended in a fluid are in constant motion due to bombardment by molecules of the fluid. The motion of these small particles is commonly referred to as Brownian motion. It may be shown that when the particles are suspended in a gas they follow the relationship $PV = (n/N)BT$, where P is the pressure, V the volume, n the number of particles into which 1 gram-molecule of the disperse phase is disintegrated, N the number of gas molecules (Avogadro's number), B the gas constant, and T the absolute temperature. The length of path λ_B followed by a particle in any given direc-

tion at a given time t was obtained by A. Einstein (1906) and is expressed by the following equation:

$$\frac{\lambda_B^2}{2t} = \frac{BT}{\mathfrak{R}N} \qquad \text{Eq (8-1)}$$

where $\mathfrak{R}$ is the frictional resistance of the fluid and is equivalent to Stokes' equation for the resistance of a sphere in streamline motion, $3\pi\mu d$. Hence

$$\lambda_B = \sqrt{\frac{2BT}{N} \cdot \frac{t}{3\pi\mu d}}$$

Gibbs (1924) has given the following essential information with regard to particles in Brownian motion: (1) the particle moves with uniform velocity; (2) smaller particles move more rapidly than larger ones; (3) particles in high concentration move more rapidly than those in dilute concentrations; (4) particles move more rapidly through media of lower viscosity; (5) for constant viscosity the amplitude of the motion is directly proportional to the absolute temperature; (6) due to gravity, particles gradually arrange themselves so that their concentration is greatest at the lowest layer.

The diffusion of small particles depends upon many factors. In addition to Brownian motion, we must consider the effect of gravity and the motion of the fluid in which the particles reside. Ordinary diffusion as understood in colloid chemistry must be modified considerably when we deal with turbulence. However, we still retain the usual definition of diffusion, namely that it is the number or mass of particles passing a unit cross section of the fluid in unit-time and unit-concentration gradient. That is, if dw particles (or mass) move through an area A in time dt and dC/dx is the concentration increase in the x-direction, then

$$\frac{1}{A} \cdot \frac{dw}{dt} = \frac{\mathfrak{D}dC}{dx} \qquad \text{Eq (8-2)}$$

where $\mathfrak{D}$ is the diffusion coefficient and is equal to λ_B^2/t, as given in Eq (8-1). Thus, for particles in Brownian motion

$$\mathfrak{D} = \frac{\lambda_B^2}{2t} = \frac{BT}{N} \cdot \frac{1}{3\pi\mu d} \qquad \text{Eq (8-3)}$$

The dimensions of $\mathfrak{D}$ are obviously $[L^2T^{-1}]$ and are the same as those of kinematic viscosity $\nu = \mu/\rho_0$.

By means of the hypsometric formula it is not difficult to determine the diffusion coefficient of a suspension of colloidal particles at any height.

The hypsometric formula relating the decrease in density of a gas with increasing height L is

$$\frac{BT}{N} \ln \frac{P_0}{P} = \frac{\pi g}{6} d^3(\rho - \rho_0)L$$

where d in this case is taken as the diameter of the gas molecule, ρ its density, P_0 the pressure at the lowest point, and P the pressure at height L. Now, these pressures are proportional to the number of molecules (or particles, in the case of a suspension) so that we may write

$$\frac{BT}{N} \ln \frac{C_0}{C} = \frac{\pi g}{6} d^3(\rho - \rho_0) L \qquad \text{Eq (8-4)}$$

where the C's are the number or mass of particles per unit volume corresponding to the P's. Using Eq (8-3) we have $BT/N = 3\pi\mu\mathfrak{D}d$, so that

$$\mathfrak{D} = \frac{gLd^2(\rho - \rho_0)}{18\mu \ln \frac{C_0}{C}} \qquad \text{Eq (8-5)}$$

From this equation we may also determine the length of path λ_B at any time t at height L, since $\mathfrak{D} = \lambda_B^2/2t$.

The diffusion equations thus far developed assume that the particles are colloidal and not affected by any motion of the fluid itself which is regarded as stationary. If we limit our discussion to particles in the size range from 0.5 to 5 μ, which remain in suspension for rather long periods of time, and if these particles are emitted from a point source and not subject to disturbance by the surrounding fluid, it is obvious that the concentration of particles at any point must be proportional to the diffusion constant and inversely proportional to the square of the distance from the source. Let C be the concentration per unit time, passing a point at any distance R from the source, then

$$C \propto \frac{\mathfrak{D}}{R} \qquad \text{Eq (8-6)}$$

since the distribution is over a spherical surface with the point source as center. Moreover, it has been shown that in large-scale diffusion phenomena such as cloud billows and smoke from tall stacks, when influenced by a slight wind the particles proceed downstream and diffuse in accordance with Eq (8-6). In other words, if the particles remain in suspension and if the scale of the phenomenon is large and not interfered with by terrain, the cross-sectional expansion of the cloud or smoke is inversely as the square of the distance along a line perpendicular to a

fixed plane or line taken as the origin. In the former the plane referred to is that passing through the center of the mass.

DIFFUSION DUE TO TURBULENCE

In this section we shall discuss the simplest forms of turbulence, and following Dryden (1939) we shall consider two types, namely, isotropic and non-isotropic. Isotropic turbulence, as it applies to a given fluid, may be defined as one in which the eddying motion is so distributed that the components of motion in different directions have the same *average* magnitude and no correlation exists between the components in different directions. Such a form of turbulence is produced by wire screens and honeycombs or may be found in a fluid along the axis of the duct in which it is flowing. It is also found at heights above the ground where the motion of the wind is uniform.

Non-isotropic turbulence is less uniform and applies to conditions such as the motion of the wind near the ground or in a fluid at some distance from the axis of the duct in which it flows. It is characteristic of conditions of flow where the motion of the mass of fluid along any given direction is not uniform from one layer to another. For example, a mass of air flowing along the surface of the ground encounters frictional resistance and there is a progressive decrease in its motion as we approach the surface. The effect of this resistance is transmitted upwards for hundreds of feet and the change in velocity along the direction of flow sets up shearing stresses between successive layers. This shearing causes an upward and downward displacement of small components of the mass, each tending to preserve the horizontal motion originally imparted to it. Thus if one of these components of the air mass moves upward it tends to retard the horizontal motion in the layers above, and conversely if a component moves downward.

A complete theory of turbulence is still lacking, so we must restrict our discussions to two general cases of interest to us: (*a*) the diffusion of particles from point or line sources where the turbulence may be said to be isotropic, and (*b*) the behavior of particles near large land surfaces—as for example, dust storms. We shall begin our discussion with an explanation of the meaning of eddy-diffusion, which is characteristic of the conditions to be more fully discussed later.

The diffusion constant of Eq (8-2) applies to a still fluid, and to particles of very small size. Large particles can diffuse only if the medium is in a state of agitation or turbulence. A moving fluid such as air or water produces the necessary conditions for diffusion, especially near boundary surfaces. As we have stated, the gradient from layer to layer

as we move from such a surface creates shearing stresses causing upward and downward movements of many fluid elements.

It is obvious that the dimensions of the diffusion constant in any theory of diffusion must be the same as those of Eq (8-2), Thus, if $\mathfrak{D}_e$ is written for the coefficient of turbulent or eddy diffusion,

$$\mathfrak{D} \equiv \mathfrak{D}_e = [L^2T^{-1}]$$

Now the units of shear are $[ML^{-1}T^{-2}]$, and for a fluid moving in the x-direction with a velocity v_x

$$\tau_{xy} = \mu \frac{dv_x}{dz} \qquad \text{Eq (8-7)}$$

where τ_{xy} denotes the shearing force in the xy-plane and is the viscosity of the fluid. It is easily shown that the shearing force is also given by the expression

$$\tau_{xy} = \mathfrak{D}\rho_0 \frac{dv_x}{dz} \qquad \text{Eq (8-8)}$$

since $\mu = \mathfrak{D}\rho_0$, ρ_0 being the density of the fluid. The value of $\mathfrak{D}$ is therefore that of the kinematic viscosity μ/ρ_0. The term μ may be identified with μ_e, the coefficient of eddy viscosity, and $\mathfrak{D}$ with $\mathfrak{D}_e$ for the case of turbulent mixing.

DIFFUSION DUE TO MOTION OF A FLUID MASS ABOVE A PLANE SURFACE

The computation of μ_e and hence $\mathfrak{D}_e$ can be accomplished by means of a second equation for the shearing stress, τ_{xy}. Let the *mean* values of the velocity components in the x-, y-, z-directions be denoted by $\bar{v}_x$, $\bar{v}_y$, and $\bar{v}_z$, and let the instantaneous deviations from these values be denoted by v_x', v_y', and v_z'. Hence, if v_x, v_y, and v_z are the actual velocities we must have that $v_x = \bar{v}_x + v_x'$, $v_y = \bar{v}_y + v_y'$, and $v_z = \bar{v}_z + v_z'$. If the flow of fluid is assumed to take place in the x-direction, the velocities to the right and left and up and down at any point counterbalance each other, so that $\bar{v}_y$ and $\bar{v}_z$ must be zero. Hence, $v_y = v_y'$ and $v_z = v_z'$.

Consider the volume of fluid passing up or down through a unit area in any horizontal plane. The mass of this fluid in time dt is simply $\rho_0 v_z dt$. The momentum of the same mass having a velocity v_x (in the x-direction) is $\rho_0 v_x v_z dt$ and this is transferred vertically upward through the horizontal plane. If $M\uparrow$ denotes the transfer of momentum vertically upward

$$M\uparrow = \rho_0 v_x v_z dt$$

so that substituting the values of v_x and v_z above

$$M\uparrow = \rho_0 v_x v_z dt = \rho_0(\bar{v}_x + v_x')(\bar{v}_z + v_z')dt$$
$$= \rho_0(\overline{v_x v_z} + \bar{v}_x v_z' + v_x'\bar{v}_z + v_x' v_z + v_x' v_z')dt$$

But since the flow is in the x-direction and the upward and downward components of velocity counterbalance each other, the mean value of these components is zero. Hence

$$M\uparrow = \rho_0(\bar{v}_x v_z' + v_x' v_z')dt$$

The above equations apply to time dt. Over an extended period of time the average momentum transferred is obtained from the integral

$$M\uparrow = \overline{\int_0^t \rho_0(\bar{v}_x v_z' + v_x' v_z')dt} \qquad \text{Eq (8-9)}$$

where the bar, as before, denotes a time-average. Also we may write

$$M\uparrow = \overline{\int_0^t \rho_0 v_x v_z dt}$$

and this represents the *average* momentum transfer per unit time, or

$$M\uparrow = \rho_0 \overline{v_x v_z} \qquad \text{Eq (8-10)}$$

where the term $\overline{v_x v_z}$ is the *average* value of the product. Returning to Eq (8-9), we then have

$$M\uparrow = \rho_0 \overline{\bar{v}_x v_z'} + \rho_0 \overline{v_x' v_z'} \qquad \text{Eq (8-11)}$$

Note that although the mean value of v_z' is zero ($\bar{v}_z' = 0$) it does not follow that the mean value of $v_x' v_z'$ is zero. This disparity is due to a negative correlation between v_x' and v_z' arising from the fact that the horizontal component of wind velocity increases with height, v_z, the vertical velocity being positive. The average vertical velocity $\bar{v}_z$ is zero, hence v_z' is positive. In rising the fluid element tends to retain its initial horizontal velocity, v_x, and since this is less than the average horizontal velocity $\bar{v}_x$ at the new level, v_x' is negative. When the fluid moves to a lower level and v_z' is negative, the horizontal velocity v_x is greater than $\bar{v}_x$ in the lower adjoining layer and v_x is positive. Hence the instantaneous deviations v_x' and v_z' are necessarily of opposite signs; when one is negative the other is positive, and conversely. Hence, $v_x' v_z'$ must always be negative.

On the other hand the product $\overline{\bar{v}_x v_z'}$ is zero because $\bar{v}_x$ in any given situation is constant. Hence, Eq (8-11) becomes

$$M\uparrow = \rho_0 \overline{v_x' v_z'} \qquad \text{Eq (8-12)}$$

also $M\uparrow = -M\downarrow$, where $M\downarrow$ is the force exerted by the upper air on the lower. Thus for τ_{xz} of Eq (8-7)

$$\tau_{xz} = -\rho_0 \overline{v_x' v_z'} \qquad \text{Eq (8-13)}$$

is the average transfer of momentum and represents the tangential force per unit area due to the turbulent motion of the fluid. More generally, combining the effect of viscosity

$$\tau_{xz} = \mu\frac{dv_x}{dz} - \rho_0\overline{v_x'v_z'}$$

where the first term predominates only when the fluid is in streamline motion, as near boundary surfaces.

In the above equations it is not usually possible to measure the instantaneous deviations v_x' and v_z'. The average velocity $\bar{v}_x$ is easily obtained, however. If an element of the air moves through a distance λ' upward, it is obvious that

$$v_x' = \bar{v}_x - \left(\bar{v}_x + \lambda'\frac{d\bar{v}_x}{dz}\right) = -\lambda'\frac{d\bar{v}_x}{dz}$$

We now make an assumption which is important and is borne out by experiment. It is that *on the average* in turbulent motion there are equal amounts of kinetic energy associated with the three component velocity deviations v_x', v_y', v_z'. Hence v_z' must, on the average, be proportional to the distance the turbulent eddy travels, λ', and to $d\bar{v}_x/dz$. Hence

$$v_z' \propto \lambda'\frac{d\bar{v}_x}{dz}$$

Remembering that v_x' and v_z' are of opposite signs, and writing τ for τ_{xz} and λ^2 for the constant of proportionality, we get

$$\tau = \rho_0\lambda^2\left(\frac{d\bar{v}_x}{dz}\right)^2 \qquad \text{Eq (8-14)}$$

The term λ is called the mean mixing length. It may be regarded as the distance traveled by the eddies from their point of origin to the point where they lose their identity and mix with the turbulent fluid surrounding them. It is related to the size of the eddy and in general the larger the eddy, the further it travels before disappearing. It is borne out by observation that the size of an eddy is directly proportional to its distance from a boundary surface. Hence, with a chimney emitting smoke, or in dust storms, λ is proportional, in either case, to the height, z, of the cloud above the ground

$$\lambda = k_0 z \qquad \text{Eq (8-15)}$$

The constant k_0 is called von Kármán's turbulence constant. Using this value of λ

$$\tau = (k_0 z)^2\left(\frac{d\bar{v}_x}{dz}\right)^2 \qquad \text{Eq (8-16)}$$

Returning now to our fundamental concepts of eddy diffusion and eddy viscosity, as given by Eq (8-7), we have that

$$\mathfrak{D}_e = (k_0 z)^2 \frac{d\bar{v}_x}{dz}$$

and

$$\mu_e = \rho_0 (k_0 z)^2 \frac{d\bar{v}_x}{dz} \qquad \text{Eq (8-17)}$$

Solving for $d\bar{v}_x/dz$ in Eq (8-16), we proceed:

$$\frac{d\bar{v}_x}{dz} = \frac{1}{k_0 z} \sqrt{\frac{\bar{\tau}}{\rho_0}}$$

and if τ is considered as constant, we may integrate between corresponding limits v_{x1} and v_{x2}, and z_1 and z_2, obtaining after rearranging terms

$$\sqrt{\frac{\bar{\tau}}{\rho_0}} = k_0 \frac{\bar{v}_{x2} - \bar{v}_{x1}}{\ln \frac{z_2}{z_1}} \qquad \text{Eq (8-18)}$$

The term $\sqrt{\bar{\tau}/\rho_0}$ is identified as having the units of velocity and is usually called the "drag" velocity. If $_*v$ denotes this velocity, then $_*v = \sqrt{\bar{\tau}/\rho_0}$. We see, therefore, from the above equation that average velocity measurements at two given heights suffice to determine $_*v$. We shall have more to say in this connection when we consider sand storms.

The effect of fluid turbulence, in so far as a suspension of particles is concerned, is to cause the particles to be agitated in conformity with the motion of the fluid. This effect applies generally to small particles which do not settle rapidly, and to concentrations not so great as to interfere with the free motion of turbulence in the fluid. Under these conditions, in the case of a fluid moving over a bed of fine materials, we may write that

$$C_{os} = -\rho_0 (k_0 z)^2 \frac{d\bar{v}_x}{dz} \cdot \frac{d\bar{C}}{dz} = \rho_0 \mathfrak{D}_e \frac{d\bar{C}}{dz} \qquad \text{Eq (8-19)}$$

which is the analogue of Eq (8-16) and where C_{os} now is the amount of material picked up from the bed and $d\bar{C}/dz$ is its concentration gradient in the fluid. From this equation, using the developments given for the case of fluid turbulence, it is not difficult to show that

$$C_{os} = \frac{\rho_0 k_0^2 (C_1 - C_2)(\bar{v}_{x2} - \bar{v}_{x1})}{\ln^2 (z_2/z_1)} \qquad \text{Eq (8-20)}$$

Of course this equation is subject to the limitations set forth above. The significance and the units of the various terms are discussed in Chapter 19 (pp. 398–407).

DIFFUSION FROM A POINT SOURCE

Following Taylor (1921, 1935), we may now modify slightly the basic ideas presented in the previous paragraphs and consider the nature of diffusion of particles originating from a point source and moving with the fluid in a specified direction. Before proceeding, however, it is necessary to state that while heretofore we have spoken of average velocities and concentrations, they may be replaced by root mean square values if observations on these quantities are made at stipulated intervals, rather than recorded continuously. The use of root mean square values for various quantities in the following paragraphs in no way alters the basic concepts of our treatment of turbulence.

The essential feature of Taylor's theory of turbulence is that it is unnecessary to stipulate anything about the motion of individual portions of a fluid in order to investigate such properties as composition and momentum. It is only necessary to assume that such statistical calculations as exist between the velocities and other specified properties of an element are functions of time or space differences. If v_t is the velocity of an element of fluid in some specified direction at time t, and $v_t - t'$ and $v_t + t'$ its velocity an instant earlier and an instant later, respectively, the correlation coefficient ϕ_v may be written as follows:

$$\phi_v = \frac{\overline{\bar{v}_t v_{t+t'}}}{\sqrt{\bar{v}_t^2}\sqrt{\overline{v_{t+t'}^2}}}$$

If the turbulence is uniform with respect to space and time,

$$\sqrt{\bar{v}_t^2} = \sqrt{\overline{v_{t+t'}^2}} = \sqrt{\overline{v_{t-t'}^2}}$$

so that

$$\phi_v = \frac{\overline{v_t\, v_{t-t'}}}{\bar{v}_t^2} \qquad \text{Eq (8-21)}$$

In the above equations the bars, as before, indicate the time average and $\bar{v}^2$ is the mean square value of v which fluctuates because of eddies. Integrating the product of the average velocities $\bar{v}_t$ and $\overline{v_{t-t'}}$ gives

$$\int_0^t \overline{v_t v_{t-t'}}\, dt' = \int_0^t \phi_v \bar{v}_t^2\, dt' = \bar{v}_t^2 \int_0^t \phi_v\, dt'$$

We have also that

$$\int_0^t \overline{v_t v_{t-t'}}\, dt' = \overline{v_t \int_0^t v_{t-t'}\, dt'} = \bar{v}_t \bar{\xi} = \frac{1}{2}\frac{d\bar{\xi}^2}{dt}$$

or essentially

$$\frac{d\bar{\xi}^2}{dt} = 2\bar{v}_t^2 \int_0^t \phi_v\, dt' \qquad \text{Eq (8-22)}$$

where $\bar{\xi}$ is the average displacement of the fluid element in the time t. The term $\bar{\xi}^2$ may be regarded as the mean square displacement of many particles. Solving for $\bar{\xi}^2$ at any time t_1,

$$\bar{\xi}^2 = 2\bar{v}^2 \int_0^{t_1} \int_0^t \phi_v dt dt'$$

where $\bar{v}^2$ is written for $\bar{v}_t{}^2$. This equation offers a means for solving for the intensity of turbulence $\sqrt{\bar{v}^2}$ since at points close to the source the elements are closely correlated at short time intervals and $\phi_v = 1$. Hence

$$\bar{\xi}^2 = \bar{v}^2 t_1{}^2$$

or

$$\sqrt{\bar{v}^2} = \sqrt{\bar{\xi}^2}/t_1$$

If $\bar{v}_x$ denotes the mean velocity in the x-direction, $\bar{v}_x t_1 = x$ (x being very small) and

$$\sqrt{\bar{v}^2} = \sqrt{\bar{\xi}^2} \cdot \frac{\bar{v}_x}{x}$$

In the case of linear flow, $\bar{v}$ is the mean velocity at right angles to v_x. Also, it is clear from Eq (8-22) that

$$\frac{d\bar{\xi}^2}{dx} = \frac{2\bar{v}^2 x}{\bar{v}_x{}^2} \qquad \text{Eq (8-23)}$$

At great distances from the source of turbulence there is no correlation between velocities of the same element and hence $\phi_v = 0$. Let the interval of time be t_2; then $\int_0^{t_2} \phi_v dt'$ is finite and

$$\bar{v}^2 \int_0^{t_2} \phi_v dt' = \bar{v}^2 \lambda = \text{constant}$$

for all values of $t > t_2$ and $x > \lambda$. The term t_2 may be said to be the time required for the element of fluid (or particle) to lose its velocity identity in the surrounding fluid. The term λ_0 is a length factor which determines the scale of turbulence and corresponds to λ as determined in Eq (8-14). Again using Eq (8-22)

$$\frac{d\bar{\xi}^2}{dx} = \frac{2\bar{v}^2 \lambda_0}{\bar{v}_x{}^2} \qquad \text{Eq (8-24)}$$

Note that from the relation $\bar{\xi} = \bar{v}t$ the integrations of the various functions discussed can also be carried out between 0 and x.

Using Eq (8-24) we may obtain the eddy diffusion constant $\mathfrak{D}_e$, for obviously

$$\frac{\bar{v}_x}{2} \cdot \frac{d\bar{\xi}^2}{dx} = \frac{\bar{v}^2 \lambda_0}{v_x} = \mathfrak{D}_e \qquad \text{Eq (8-25)}$$

which is dimensionally correct and is similar to Eq (8-3). Integrating Eq (8-25), we obtain

$$\bar{\xi}^2 = \frac{2\mathfrak{D}_e}{\bar{v}_x}(x - \lambda_0) \qquad \text{Eq (8-26)}$$

which applies to the diffusion of particles for large values of x. The value of $\bar{\xi}^2$ may be obtained from observation, and hence $\mathfrak{D}_e$ and λ_0 may be calculated, the latter really being a factor defining the extent of turbulence in so far as diffusion itself is concerned. Hence, if $\mathfrak{D}_e$ and λ_0 are known, then from Eq (8-25)

$$\bar{v}^2 = \frac{\mathfrak{D}_e \bar{v}_x}{\lambda_0}$$

which determines the intensity of turbulence.

Significance of Correlation Coefficient—Let ϕ_v denote the correlation coefficient between velocities at points in an element of turbulent fluid at times differing by t, and let ϕ_c denote the correlation coefficient between deviations of the concentration C of a suspended material from the average of the surrounding concentrations at different points in the element in the same time difference.

The function ϕ merely expresses a probability relationship. Of course, there are infinite possibilities between coincidence or orthogonality of v_t and v_t', $(0 \leqq \phi_v \leqq 1)$, but the actual probability of any event within this defined range is quite small. Hence, we may apply the Poisson relationship. The probability $\phi(n)$ of n-coincidences in a space of time, t, is

$$\phi(n) = \frac{1}{n!}(at)^n \exp(-kt)$$

where k is a function of time. Hence the chance that no coincidences occur in this interval of time is

$$\phi(0) = \phi = \exp(-kt) \qquad \text{Eq (8-27)}$$

and

$$\phi_v = \exp(-k_v t) \qquad \text{Eq (8-28)}$$

$$\phi_c = \exp(-k_c t) \qquad \text{Eq (8-29)}$$

where k_v and k_c are constants having special significance. These assumptions are important since they are substantiated by experiment, as will be shown later. If, for the moment, diffusion is determined solely by correlation of velocities at different points of an element in the time interval t, we have by application of the above relation to Eq (8-22)

$$\frac{d\bar{\xi}^2}{dt} = 2\bar{v}^2 \int_0^t \phi_v dt = 2\bar{v}^2 \int_0^t \exp(-k_v t)dt$$

$$= 2\bar{v}^2 \int_0^{x/v_x} \exp(-k_v x/\bar{v}_x)\, dx/\bar{v}_x$$

$$= \frac{2\bar{v}^2}{k_v \bar{v}_x}\left[1 - \exp\left(-\frac{k_v x}{\bar{v}_x}\right)\right] \qquad \text{Eq (8-30)}$$

This equation is the general eddy diffusion equation for a fluid. Thus, when $\bar{v}_x/k_v$ is small in comparison with x,

$$[1 - \exp(-k_v x/v_x)] = k_v x/\bar{v}_x$$

$$\frac{d^2\bar{\xi}^2}{dx} = \frac{2\bar{v}^2 x}{\bar{v}_x{}^2}$$

which is identical with Eq (8-23), and when $\bar{v}_x/k_v$ is large as compared with x,

$$\frac{d\bar{\xi}^2}{dx} = \frac{2\bar{v}^2}{\bar{v}_x{}^2}\left(\frac{\bar{v}_x}{k_v}\right) = \frac{2\bar{v}^2}{\bar{v}_x{}^2} \cdot \lambda_0 \qquad \text{Eq (8-31)}$$

where $\bar{v}_x/k_v$ is now identified as λ_0 (Eq 3-24), and has the dimensions of length. Hence, k_v has dimensions $[T^{-1}]$, as of course it must.

Mention may be made here that in the case of a pipe in turbulent flow, the intensity of turbulence along the axis is about 3 percent of the mean

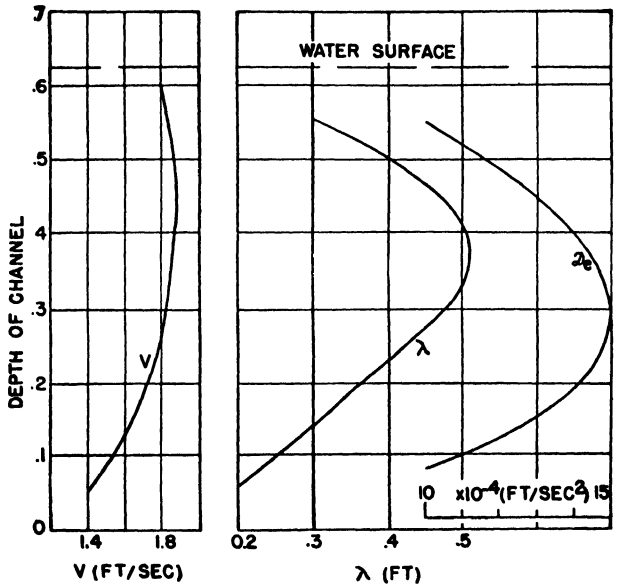


FIGURE 47. RELATIONSHIP BETWEEN $\mathfrak{D}_s$, λ_0, AND v_z FOR WATER FLOWING IN A CHANNEL ONE FT WIDE (KALINSKI AND PIEN, 1944).

velocity, and λ_0 is about 0.34 times the diameter of the pipe. At $^1/_6$ the radius, the value of $\sqrt{\bar{v}^2}$ is twice that at the axis.

Experimental Investigations—Kalinski and Pien (1944) have presented interesting evidence of the relationships between $\mathfrak{D}_e$, λ_0, and $\bar{v}_x$. Some of their results as applied to a water channel 1 ft wide are shown in Figure 47. Note the correlation between λ_0 and $\mathfrak{D}_e$. The data shown apply to the injection of an easily determined fluid into the stream.

Work by the same investigators showed that plots of values of $\bar{v}_x\bar{\xi}^2/2\mathfrak{D}_e\lambda_0$ against x/λ_0 all lay on the same curve, regardless of channel width and flow speeds, provided the latter were in the turbulent region. The curve obtained was essentially a straight line for values of $\bar{v}_x\bar{\xi}^2/2\mathfrak{D}_e\lambda_0 > 1$ and $x/\lambda_0 > 2$. From such a curve ϕ_v may readily be calculated, for, using Eq (8-22) we have

$$\frac{d\bar{\xi}^2}{dt} = 2\bar{v}_t^2 \int_0^t \phi_v dt_n$$

whence

$$\frac{d^2\bar{\xi}^2}{dx^2} = \frac{2\mathfrak{D}_e\phi_v}{\bar{v}_x\lambda_0}$$

and

$$\phi_v = \frac{\bar{v}_x\lambda_0}{2\mathfrak{D}_e}\left(\frac{d^2\bar{\xi}^2}{dx^2}\right)$$

Hence, ϕ_v is as the second derivative of the curve described. Plots of values of ϕ_v thus obtained lie on the curve $\phi_v = \exp(-x/\lambda_0)$.

DIMENSIONAL APPROACH

If we regard $\bar{v}$ and $\bar{v}_x$ in Eq (8-31) as instantaneous velocities in an element of turbulent fluid, then $\bar{v}/v_x$ is essentially equal to ϕ_v. Since $d\bar{\xi}^2/dt = \mathfrak{D}_e$, it is clear from Eq (8-30) that $\mathfrak{D}_e$ is proportional to the following quantities:

$$\mathfrak{D}_e \propto \frac{\bar{v}^2}{k_v\bar{v}_x}, \frac{\bar{v}\phi_v}{k_v}, \frac{\bar{v}_x\phi_v^2}{k_v}, \lambda_0\phi_v^2 \qquad \text{Eq (8-32)}$$

The term ϕ_v of course is dimensionless. Note especially that $\mathfrak{D}_e$ is proportional to $\bar{v}$ and $\bar{v}_x$ and hence $\bar{\xi}$ and x, and that therefore $\bar{v}$ is proportional to $\bar{v}_x$ and $\bar{\xi}$ to x. Hence, we conclude that the spreading of a mass of particles from a point source depends upon the distance traveled and not on the time taken to travel the distance.

It is clear from the foregoing that $\mathfrak{D}_e$ is proportional to a number of variables, some of which are proportional to each other. From a dimensional standpoint, the following identity holds:

$$\mathfrak{D}_e = k\bar{v}_x\bar{\xi} \qquad \text{Eq (8-33)}$$

Similar identities of different variables also hold, but the above is perhaps the most useful and will be used in preference to others. The term $\bar{v}_x$ is the velocity of the fluid in the x-direction and $\bar{\xi}$ is the mean displacement of the elements of a diffusing mass of particles from a plane passing through the center of gravity and normal to the direction in which the diffusion takes place. If the mass originates close to a horizontal boundary surface, then the mean height, $\bar{z}$, of this mass must be nearly proportional to $\bar{\xi}$. Hence, in general, for vertical diffusion above a plane surface, we may write

$$\mathfrak{D}_e = (k_1 z)v \qquad \text{Eq (8-34)}$$

where k_1 is a constant. The product $k_1 z$ is identified as being proportional to the mixing length, λ, and as it applies to the equation set forth, it is equivalent to $k_1'(k_0 z)$ where k_1' is another constant and k_0 is von Kármán's constant.

Let us now consider Eq (8-25) to obtain the significance of k in Eq (8-33). It is clear from Eq (8-25) that

$$\bar{v}_x \bar{\xi} \frac{d\bar{\xi}}{dx} = \frac{\bar{v}^2 \lambda_0}{\bar{v}_x} = \mathfrak{D}_e$$

or

$$\frac{d\bar{\xi}}{dx} = \frac{\mathfrak{D}_e}{\bar{v}_x \bar{\xi}}$$

Hence the ratio $d\bar{\xi}/dx$ is a constant and must be equal to k of Eq (8-33), and

$$\xi = kx \qquad \text{Eq (8-35)}$$

This constant differs slightly in numerical value from k_1 since it applies to lateral diffusion. The constant k_1 of Eq (8-33) applies to vertical diffusion. Actually, in regions remote from boundaries, $k = k_1$, but near horizontal plane surfaces, infinite in extent, as for example smoke coming from chimneys and moving with the wind over the earth's surface, k is slightly larger than k_1.

In connection with the diffusion of smoke from chimneys by a wind, values of k and k_1 have been determined by several investigators. These values vary considerably, but if an average value of $k_1 = 0.05$ is used, $k = 0.08$.

Problems

1. Determine the influence of diffusion on a suspension of fine particles in a fluid

Note: It may be shown that the force of diffusion (Berg, 1940) amounts to $(-BT/N)d \ln C_{Ai}/dh$, where B is the gas constant, T the absolute temperature, N

Avogadro's number, and C_{ht} the concentration at depth h and time t. Hence, we may write

$$3\pi\rho_0\mu v = (\pi d^3/6)(\rho - \rho_0)g - (BT/N)d \ln C_{ht}/dh.$$

Since $\mathfrak{D} = BT/3\pi\mu N$, a possible solution of the problem, omitting boundary effects, is

$$C_{ht} = \frac{C_0}{2}\left[1 - \frac{2}{\sqrt{\pi}}\int^{y} \exp(-y^2)dy\right].$$

where C_0 is the concentration of the homogeneous suspension and $y = (h' - h)/2\sqrt{\mathfrak{D}t}$. Here h' is the distance through which particles of a given size have sedimented during the time t. The integral is the probability integral for which tables are readily available for its computation. If we denote the integral and its constant by $\Phi(y)$, we have

$$\Phi(y) = 0 \text{ when } y = 0 \text{ and } y = 0 \text{ when } h' - h = 0$$

Hence the concentration at depth $h' = {}^1/_2C_0$, and

$$y = (h' - h)(3/4\pi N\mu d/BTt).$$

2. The fact that the velocity fluctuation within a cloud of suspended particles is greater as the mass of the suspension increases is not readily apparent. Yet this is easily demonstrated by means of a remarkable theorem due to Poincaré, which may be stated as follows: If we have given a suspension of discrete particles each of mass m moving under no forces except their own natural attraction (gravitation, electrical or vortical), then if T represents the total kinetic energy of the system and W the potential energy of the system arising from their mutual attraction,

$$2T + W = 0.$$

If we now write $T = {}^1/_2\Sigma m\bar{v}^2$, where $\bar{v}^2$ is the mean square velocity of a particle of mass m, and $W = -{}^1/_2\Sigma m\Phi$ where Φ may be regarded as the Newtonian potential, then Poincaré's theorem takes the form

$$\Sigma m(\bar{v}^2 - {}^1/_2\Phi) = 0.$$

Hence the *average* value of $\bar{v}^2$ averaged over all the separate masses is equal to the average value of ${}^1/_2\Phi$.

Suppose we now consider a cloud of smoke particles having a total mass M and a mean radius R. The average value of ${}^1/_2\Phi$ will have an order of magnitude GM/R where G is the gravitational constant. Thus, on the *average*

$$v^2 \simeq GM/R$$

and we see that to a first degree of approximation,

$$\bar{v} \propto \sqrt{M/R} \propto \sqrt{\rho v/R} \propto \sqrt{\rho R^3/R} \propto R$$

where V is the volume of the cloud suspension. Hence, the internal velocity of agitation is proportional to the size of the cloud. This important relation holds for a steady state, that is, when no new particles are added or old ones removed from the cloud of suspension. It may be shown that the growth of the cloud due to agitation does not disturb its dynamic equilibrium.

In the case of a cloud of charged particles, let the totality of charge be $\mathfrak{Q}$, and k the specific inductive capacity, then

$$v^2 \simeq \mathfrak{Q}/kR.$$

As an illustration of the amount of agitation to be expected in a cloud suspension, assume the following:

Average radius of cloud $= R = 0.5$ km $= 5 \times 10^4$ cm.
Average concentration of particles $= 10^2$ per cu cm.
Average diameter of particles $= 100\mu = 10^{-2}$ cm.
Density of particles $= \rho = {}^1/_2$ g per cu cm.

If $G = 6.66 \times 10^{-8}$, calculate the order of magnitude of the velocity of agitation.

CHAPTER 9

ELECTRICAL, OPTICAL, AND SONIC PROPERTIES

Much has been written on the basic behavior of particles in electrical, optical, and sonic fields, and it is not possible here to cover completely the mass of information available. We can only indicate briefly how small particles behave when subjected to the fields mentioned. The properties outlined have many important applications. Some of these are indicated, but in general the material presented covers the theoretical factors which explain the behavior of particles to give as wide an application as possible.

ELECTRIC AND MAGNETIC PROPERTIES

Electrification of Particles*

Charge on Atmospheric Dust—The extent to which atmospheric dust is charged has never been satisfactorily determined. Some work has been done on dust storms, such as investigations by Rudge (1912) in the Sahara Desert. Because of the dryness of the climate such storms are accompanied by strong electric charges. Rudge states that during a Sahara Desert dust storm the voltage gradient, which normally seldom exceeds 200 volts per meter, may be reversed in polarity (that is, becomes negative) and exceed 500 volts per meter. Voltage gradients may even exceed 10,000 volts per meter. An interesting observation was that both the atmospheric dust and the earth were negative at a given place. Usually the atmospheric dust is positively charged. Small lightning discharges from clouds are frequently reported.

Charge Taken by Particles—Since 1900 several investigations have been made on the nature of dust electrification. Knoblauch (1901) determined the charge on particles of many different materials placed on plates of platinum, sulfur, glass, and paraffin. This dust was then dropped on a grounded copper plate and the charge noted with an electroscope connected to the different plates. Knoblauch's experiments were carefully made. He found that platinum and paraffin

* Voltages given in this section are electrostatic unless otherwise specified. A discussion of electrical units will be found later in this chapter.

plates as a rule were positively charged with acid substances, and negatively charged with alkaline substances. Except with a few acid substances, sulfur plates were usually positive. Knoblauch assumed that the charges were developed in a molecular layer of water which was absorbed on all surfaces. This molecular layer contained dissolved ions from the surface beneath. Thus, for glass the solution was alkaline; for platinum and paraffin the solution was neutral; and for sulfur it was acid. In the case of organic substances, the surface layer was acid or basic, depending on the nature of the ionization.

Later work by Rudge (1913) resulted in classifying the charge assumed by dust clouds formed by air currents. His experiments were made with a 2-cm × 20-cm brass tube to which was added a similar length of brass gauze. This ensemble was mounted on an insulated base and the gauze end connected to an electroscope. Dust to be tested was placed in the free end of the solid-brass tube and a puff of air from a bellows directed obliquely against the dust. As the dust moved through the tube it imparted its charge to the gauze. From numerous tests made with this apparatus, Rudge concluded that the charge acquired by dust particles is in part determined by their chemical composition. Thus, confining ourselves to air, Rudge has shown that: (*a*) Nonmetallic dust becomes positively charged; (*b*) metallic dust becomes negatively charged; (*c*) dust of acid-forming oxides becomes positively charged; (*d*) the dust of basic-forming oxides becomes negatively charged; (*e*) with salts, the charge depends upon the relative strength of the acid and basic ions. This classification applies to compositions of distinct chemical character. In many organic compounds, such as plastics, as well as in many minerals, the division is not so well defined. Rudge was convinced that the charges observed were actually carried by the dust particles and not by the air, which was believed to carry an opposite charge. The charge upon the air usually observed was considered due to extremely fine particles which do not settle readily or which move with such slow velocities that they escape capture by the charged electrodes.

The observations resulting in this conclusion were stated by Rudge as follows:

1. A dust-and-air mixture blown through a gauze and thence upon a collector imparted to one obstruction a negative charge and to the other a positive charge.
2. Other gases than air generally caused no difference in results.
3. Air blown over various dusts cemented to electrodes showed no charge unless particles of the dust were detached; air passing gently through a tube containing dust carried no charge unless some of the dust also passed through cotton-wool plugs.

4. Oppositely charged electrodes placed in a dust-laden atmosphere each attracted a certain amount of dust. In most cases there was a marked difference in the amount of dust collected on the electrodes and the greater this difference the greater the charge carried away by the air. The charge carried away by the air was always opposite to that upon the electrode which carried the greater quantity of dust.

These observations led to the conclusion that the charges are not due to surface contact or friction since insertion of various plugs or filters in the apparatus did not affect the charge.

Of the many dusts tested, Rudge found that mercuric sulfide (HgS) developed the highest charge. An amount of less than $^1/_2$ mg when blown into the air caused a rise in potential of 6 to 8 volts. The density of the cloud was approximately 5×10^{-9} g per cc.

V. E. Whitman (1926) extended Rudge's investigations by using a polarity recorder. In this device the particles fell between electrically charged plates and their motion was photographed. Whitman's studies show the presence of positive, negative, and neutral particles in all dust clouds, even of very pure substances. The ratio of positive to negative electrification was found to change as the larger particles settled out. However, no evidence was found that large particles carry an opposite charge from small particles in a given cloud. The range of particles studied was from 0.1 mm to submicroscopic dimensions.

TABLE 30—NET CHARGE ACQUIRED BY BLOWING FUSED QUARTZ THROUGH TUBES OF DIFFERENT MATERIALS WHITMAN (1926)

Tube material	Weight of dust used (g)	Charge on tube (esu)	Charge on cloud (esu/g)
Aluminum	0.13	+78	−600
Copper	0.13	+55	−425
Platinum	0 13	+89	−685
Steel	0 13	+61	−470
Nickel	0.13	+54	−415

TABLE 31—NET CHARGE ACQUIRED BY BLOWING EQUAL VOLUME OF VARIOUS SUBSTANCES (0.24 CC) THROUGH A BRASS TUBE

Material	Weight (g)	Charge on tube (esu)	Charge on cloud (esu/g)
Borax	0.15	+1.5	−10
Sodium bicarbonate	0 17	−2.0	+12
Lime (CaO)	0 07	−8.1	+115
Fused quartz (SiO_2)	0 13	+11.5	−88
Ammonium chloride	0.19	+2.1	−11
Sodium carbonate	0 12	+0.1	−0.8
Sodium fluoride	0 13	+5.9	+5

A most important observation by Whitman, which substantiates Knoblauch's earlier work, was that the net charge acquired by a dust-cloud depends on the nature of the surface from which it is raised. By blowing ground fused quartz and other substances through tubes made of different materials, Whitman found that the net amount of charge on the dust per gram was by no means constant. Some of Whitman's results are given in Tables 30 and 31.

The investigations of Böning (1927) deserve special mention. Until his work was published it had been assumed that electrification of particles was produced by friction or contact with dissimilar substances. Although Rudge (1912), as mentioned above, observed that during sandstorms the charge on the dust was negative and that this charge increased so as to produce voltage gradients up to 10,000 volts per meter, he did not recognize that such increases could be produced by the dust itself. Böning demonstrated that charges could also be produced by friction and contact between the particles themselves, even though similar in composition. On the basis of his experiments Böning advanced the theory that the charge developed in a dust-cloud was the result of differences in size and mass of the particles. He assumed that all particles carry a rather loosely bound layer of *negatively* charged particles of the nature of "free" electrons. On contact or collision two particle masses behave like true elastic spheres, the resultant charging of the particles being accomplished as follows:

1. If the masses of the two particles are equal, there will be an equal exchange of negative charge.
2. If a large particle collides with a small particle, the latter accelerates so fast that it loses its attached charge, and the amount of charge lost is gained by the larger particle.
3. If a small particle collides with a large particle, the former rebounds and loses its attached charge to the larger particle.

Böning also stated that in the general case of different-sized particles of the same or different material, the charge is determined by the net gain or loss during collisions. When dusts of various materials suspended in air were blown into a grounded metal tube containing irregular material having the same composition as the dust, and thence to a collector separated from the ensemble, the collector was found to be positively charged. The negative charge remained in the tube where the collisions occurred.

Charge Produced by Dust in Moving Air—When dust particles are moved in the air stream of a duct, the particles become charged. Where inorganic dusts (and certain metal dusts, such as aluminum) are moved a discharge may cause the dust to explode. Nuckolls (1939) reported a

series of tests using grain, wheat, starch, zinc oxide, lycopodium, and cotton lint, moved in an exhaust system. No charge was detected on the system when it was grounded, but when a copper plate or screen was exposed to the dust and connected to a voltmeter, charges ranging from 200 to 7500 volts were produced. Ungrounded systems showed charges in many instances exceeding 10,000 volts. In these experiments from 5 to 20 oz of material were used in a 3-in. sheet-metal duct. The air velocities used did not exceed 316 ft. per min. Nuckolls' data indicate that there is some relation between the quantity of dust carried in the system and its rate of movement. Thus, 6 oz of grain conveyed at 136 ft per min produced 500 volts, while the same amount when moved at a velocity of 316 ft per min produced 2500 volts. Comparative amounts of wheat, starch, and lycopodium produced over 6000 volts. Similar work done earlier by Blactin and Robinson (1931), using coal dust, showed that with an air speed of 450 ft per min a given feed developed 4000 volts; when the rate of feed was doubled, the voltage increased to 5800; and on quadrupling, the voltage increased to 7000. When the velocity of the dust cloud was 2600 ft per min, potentials as high as 20,000 volts were obtained.

Effect of Humidity—Nuckolls found no charge on dust moved by an air stream when the relative humidity exceeded 65 percent. Walther and Franke (1929) performed a number of experiments with particles of brown coal (similar to lignite). Their investigations consisted of admitting dust at a known rate in a horizontal glass tube. The material was conveyed a distance of about one meter through the tube, and thence to a short length of copper tube connected to an electrometer. Copper wire was wound around the glass tube for most of its length and grounded. The dust in all tests was negative and the tube (inside) was positive. By varying the humidity and temperature of the air and other gases,

TABLE 32—EFFECT OF TEMPERATURE AND HUMIDITY ON DUST PARTICLES OF BROWN COAL IN A MOVING AIR STREAM. (COAL INITIALLY DRY). WALTHER AND FRANKE (1929)

Gas	Temperature (deg C)	Relative humidity (percent)	Voltage
Air	100	90	5000
Air	100	0	7930
Air	Room	90	2000
Air	Room	50	2300
Air	Room	0	4460
Nitrogen	Room	90	900
Nitrogen	Room	0	3150
Carbon dioxide	Room	90	450
Carbon dioxide	Room	0	2740

Walther and Franke obtained the results given in Table 32. It will be noted that the voltage increases with temperature and decreases with increasing humidity. By replacing the air with inert gas, the voltage generated was much less than with air.

Properties of an Electric Field

Mobility of Ions—If we have two planes perpendicular to the x-axis at a potential difference V such that ions are produced, and if the ions of one charge have a mean velocity v_i, then

$$\frac{v_i}{dV/dx} = \frac{v_i}{E} = u \qquad \text{Eq (9-1)}$$

where E is the force due to the electric field acting on the ion and u is called the mobility of the ion. Since E has the units of electric pressure, u may be said to be the average velocity acquired by an ion per unit of electric field intensity. More generally, using the electrostatic system of units, the mobility of an ion is expressed as the average velocity acquired by an ion due to a unit voltage gradient, that is, as cm per sec per volt per cm. The value of u is independent of E in the range employed in practice. However, it is about 24 percent greater for negative ions than positive ions in air. As regards mobility of ions at different pressures, it may be said that in general the mobility of positive and negative ions varies inversely as the pressure.

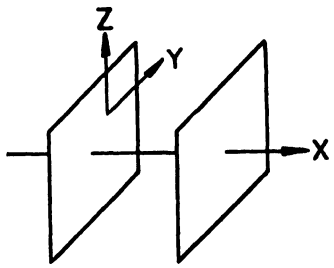


Figure 48. Coordinate System for Charged Plates

Now, electric current is defined as the time-rate of change of a charge. Suppose that a given charge $\mathcal{Q}$ is made up of N unit charges ϵ, then $\mathcal{Q} = N\epsilon$. If these charges are streaming in the direction of x (Figure 48), the total current I may be obtained as follows:

$$I = \frac{d\mathcal{Q}}{dt} = \frac{d(N\epsilon)}{dx} \cdot v_i$$

where we have written v_i for dx/dt. Hence, using the value for v_i in Eq (9-1),

$$I = \frac{d(N\epsilon)}{dx} \cdot Eu$$

If we write n for the number of charges per unit volume, and i denotes the current flowing across a unit area

$$i = un\epsilon \cdot E \qquad \text{Eq (9-2)}$$

and if the numbers of both positive and negative charges per unit volume are each equal to n, and their respective mobilities are denoted by u_+ and u_-

$$i = n\epsilon(u_+ + u_-) \cdot E \qquad \text{Eq (9-3)}$$

Diffusion of Ions—Ions diffuse in accordance with Eq (8-2) where for w and C we may use N and n, respectively. The diffusion coefficient $\mathfrak{D}$ retains its significance as given in Chapter 8. Values of $\mathfrak{D}$ vary with ions of different kind and size, and with the condition of the air. Thus, positive small ions in air possess an average diffusion coefficient of 0.030, and negative small ions of 0.043. For moist air the diffusion coefficient of both kinds of ions is about 0.033.

The effect of pressure on diffusion is easily ascertained. Writing our diffusion equation as above outlined, we have that the rate of passage of ions through a unit area is

$$\frac{1}{A}\frac{dN}{dt} = \mathfrak{D}\frac{dn}{dx}$$

We may consider this equation as signifying the passage of particles with an average velocity v_i in the x-direction defined by the expression.

$$v_i = \frac{1}{n}\mathfrak{D}\frac{dn}{dx}$$

Since the ions are gaseous molecules, they produce a partial pressure p, which at constant temperature is proportional to n so that

$$v_i = \frac{1}{p}\mathfrak{D}\frac{dp}{dx}$$

The term dp/dx is recognized as the force acting upon a unit volume of the gas in the x-direction, so that if this is regarded as unity, v_i is simply equal to $\mathfrak{D}/p$.

In case we consider a unit volume having a charge $n\epsilon$ in a field whose intensity is E,

$$v_i = E \cdot n\epsilon \cdot (\mathfrak{D}/p)$$

Identifying v_i/E as n/p in Eq (9-1) and noting that n/p is the same for all gases at the same temperature,

$$u = (n/p) \cdot \epsilon \cdot \mathfrak{D}$$

Writing N_A for Avogadro's number at a pressure p', we have $n/p = N_A/p'$, so that

$$u = (N_A/p') \cdot \epsilon \cdot \mathfrak{D}$$

Conduction of Electricity through Gases—There is a theorem of electrostatics due to Gauss which states that if a volume of charge resides in a body, the space rate of change of electric intensity is equal to 4π times the volume density of charge. Let there be n_+ and n_- ions carrying each a unit of charge ϵ in a unit of volume between parallel plates perpendicular to the x-axis (Figure 48). If the electric intensity is E, by Gauss' theorem we have therefore

$$\frac{dE}{dx} = 4\pi(n_+ - n_-) \cdot \epsilon \qquad \text{Eq (9-4)}$$

and if the respective mobilities of n_+ and n_- are u_+ and u_-, the current flow through a unit area i is from Eq (9-3)

$$i = n_+ \cdot u_+ \cdot \epsilon E + n_- \cdot u_- \cdot \epsilon E$$

Solving for the volume charges, we then have from these two equations

$$\epsilon n_+ = \frac{1}{u_+ + u_-}\left(\frac{i}{E} + \frac{u_-}{4\pi} \cdot \frac{dE}{dx}\right)$$

$$\epsilon n_- = \frac{1}{u_+ + u_-}\left(\frac{i}{E} - \frac{u_+}{4\pi} \cdot \frac{dE}{dx}\right)$$

Some simplification of these equations is possible if for $n_+ - n_-$ we write n as the excess number of charges per unit volume of either sign; we have using Eqs (9-2) and (9-4) subject to these changes, when the intensity is uniform $dE/dx = E/x$

$$n\epsilon = \frac{E}{4\pi x} \qquad \text{Eq (9-5)}$$

and

$$i = \frac{E^2 \mu}{4\pi x} \qquad \text{Eq (9-6)}$$

The units are electrostatic, but they are easily converted to practical units by use of Appendix IV.

Small and Large Ions—The ordinary ions we have thus far discussed are single charges residing on molecules. There are, in addition to these, so-called Langevin or large ions. These are to be found in the atmosphere and are actually fine dust particles of the order of tobacco smoke in size (0.05 μ). On them reside multiple charges. The mobility of these ions is from $^1/_{1000}$ to $^1/_{3000}$ that of ordinary ions and easily measured by means of appropriate apparatus.

In a field with a constant ionizing source, the number of large ions depends upon the number of particles and is not affected by the number

of ordinary ions; for when there is equilibrium, the number of dust particles which take on a charge (say positive) must equal the number which lose such a charge. A positively charged dust particle may lose its charge by combining with one or more negative ions, or by coalescing with another dust particle which is negatively charged. The loss in the second case must, however, be negligible compared with the first since the large ions are less active. Even with relatively high concentrations of dust particles the number of charged molecules remains infinitely great by comparison. De Broglie (1909) estimates that in a state of equilibrium about 10 percent of the dust particles would be charged. Furthermore, where a feeble or strong source of ionization is used, the number of charged particles is the same when equilibrium is reached.

Some further light may be cast upon the above discussion by the following analysis. Assume that there are n uncharged particles in a unit volume of air, and let n_+ and n_- be the number of particles charged as indicated. If now n'_+ and n'_- represent the number of ordinary ions in the same unit volume, the number of uncharged dust particles which acquire a positive charge in a unit of time is $\alpha n n'_+$ where α is a constant. The number losing their positive charge is $\beta n_+ n'_-$, where β is another constant. When the state of equilibrium is reached we must have

$$\alpha n n'_+ = \beta n_+ n'_-$$

For negative ions, an analogous condition holds, such that

$$\alpha' n n'_- = \beta' n_- n'_+$$

Therefore, we see that the proportion of charged and uncharged particles depends only upon the ratio of n'_+ to n'_-, and not upon the absolute magnitude of these quantities. This is an important theorem due to J. J. Thomson.

Rate of Charge of Particles in an Electric Field—The process of charging small particles in an electric field is of considerable practical importance. A particle in an electric field is not completely charged instantly but assumes a charge which is dependent upon the dielectric, the diameter of the particle, and other variables. Pauthenier *et al.* (1932, 1934), assuming that the particle being charged is so large that the effects due to diffusion of the ions toward it can be neglected and that each ion striking the surface of the particle gives up its charge, obtained the following relation

$$\mathfrak{Q} = \left[1 + 2\left(\frac{k-1}{k+2}\right)\right] \cdot \frac{Ed^2}{4} \cdot \left(\frac{\pi u n e t}{1 + \pi u n e t}\right) \qquad \text{Eq (9-7)}$$

where $\mathfrak{Q}$ is the charge acquired by the particle in the time t, d the diameter of the particle, k the dielectric constant of the particle, E the

strength of the external field, n the number of ions per cc, u the mobility of the ions, and ϵ the electronic charge. For a conducting sphere the term $[1 + 2\,(k - 1)/(k + 2)]$ is equal to 3. Eq (9-7) may be converted into more convenient units by substituting i/E for $un\epsilon$, where i is the current density, and E the intensity of the field at a given point, thus obtaining

$$\mathcal{Q} = \left[1 + 2\left(\frac{k-1}{k+2}\right)\right] \cdot \frac{\pi d^2 ti}{4} \cdot \left(\frac{1}{1 + \pi ti/E}\right) \qquad \text{Eq (9-8)}$$

In air the mobility of ions may be taken to have a value of approximately 1.8 cm per sec per volt per cm (about 600 esu). If we are concerned with a particle inside a cylindrical electric precipitator the value of n in Eq (9-7) may be calculated by Eq (9-5)

$$n = \frac{E}{4\pi R\epsilon}$$

where R is the distance of the particle from the axis of the cylinder. By substituting i/E in this equation for $un\epsilon$, we may easily calculate the mobility u (Eq 9-6)

$$u = \frac{4\pi iR}{E^2}$$

where i and E are in electrostatic units.

By means of Eq (9-8) we can calculate the relation between the charge and the diameter of a particle for any instant of time, or for any current density and field intensity. The data computed by Eq (9-8) have been verified experimentally by Fuchs *et al* (1936) for particles 0.5 to 3 μ in size.

When t is very great, note that Eq (9-8) reduces to the following equation

$$\mathcal{Q}_0 = \left[1 + 2\left(\frac{k-1}{k+2}\right)\right] \cdot \frac{d^2 E}{4} \qquad \text{Eq (9-9)}$$

and

$$\mathcal{Q} = \mathcal{Q}_0 \cdot \frac{\pi ti}{E}\left(\frac{1}{1 + \pi ti/E}\right) \qquad \text{Eq (9-10)}$$

From this expression the amount of charge received by a particle at any time t may be derived.*

Units of Electrostatics—We will devote some space for a few remarks relative to electrostatic units.

* Computations pertaining to motion in an electric field are given in Chapter 21

We cannot assign the units of mass, length, or time directly to the electric charge because we do not know the nature of the dielectric constant k. However, the force exerted by two charged particles is a real force measurable in terms of mass, length, and time, and is as definitely an observable physical fact as, for example, the tension on a string holding a ball. The electrostatic system of units is based on Coulomb's law which states that the force of attraction or repulsion F between charges $\mathcal{Q}$ and $\mathcal{Q}'$ is proportional to their product and inversely proportional to the square of their distance apart,

$$F = \frac{1}{k} \cdot \frac{\mathcal{Q}\mathcal{Q}'}{R^2}$$

In terms of dimensional units

$$[\mathrm{MLT}^{-2}] = [k_0]^{-1} \cdot [\mathcal{Q}^2] \cdot [\mathrm{L}^{-2}]$$

in which the unit of k is written $[k_0]$; hence we derive

$$[\mathcal{Q}] = [\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}] \cdot [k_0^{1/2}]$$

The units of ϵ are the same, of course, and this is a difference only in magnitude.

Electromotive force or voltage, V, is simply the work done in moving a unit electric charge ($\mathcal{Q}' = 1$) from a point R to R_1; since the voltage or potential is zero when $R_1 = \infty$, we may write by definition

$$V = \frac{\mathcal{Q}}{kR}, \text{ or in units}$$

$$[V] = [\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}] \cdot [k_0^{-1/2}]$$

Otherwise, voltage is always a difference of potential, though in either case the units are the same.

Electrostatic intensity or field strength is force per unit charge

$$[E] = [F\mathcal{Q}^{-1}] = [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}] \cdot [k_0^{-1/2}]$$

The above are sufficient for the rapid computation of all the quantities encountered in electrostatic theory. Thus, using the equation for ion mobility, $u = v_i/E$, where v_i is the velocity of the ion,

$$[u] = [\mathrm{LT}^{-1}] \cdot [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}] \cdot [k_0^{-1/2}]$$

$$= [\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-2}] \cdot [k_0^{-1/2}]$$

Electric current I is given by the expression $\mathcal{Q}/t$, and

$$[I] = [\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}] \cdot [k_0^{1/2}]$$

also current density, $i = I/A$, where A is the area through which the current flows

$$[i] = [\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-2}] \cdot [k_0^{1/2}]$$

Finally, with regard to capacity, $C = \mathcal{Q}/V$, and

$$[C] = [L] \cdot [k_0]$$

The magnitude of the units above are obviously dynamic if for $[k_o]$ we write $[k_o] = 1$. Since the attraction or repulsion of charges is measured in air, k is taken as 1 and $[k_o]$ is considered dimensionless. In the cgs we may define this attraction or repulsion as the force of 1 dyne due to two equal charges which are 1 cm apart in air. The amount of charge giving rise to this force under these conditions is called a stat coulomb. In general, the names of the various quantities developed above are the same as the practiced units with the prefix *stat*. Thus, the electrostatic unit of current is the *stat* ampere, or capacity the *stat* farad, etc. However, in this text we prefer to designate magnitudes as electrostatic, writing after them the letters esu (electrostatic units). Conversion to practical units is accomplished by factors given in Appendix IV.

Magnetization of Particles

Interest in magnetization of particles is centered on a special group of magnetite materials, Fe_3O_4. Magnetic properties of these materials are important in metallurgy from the standpoint of iron-ore separation and copper losses caused by the formation of magnetite in reverberatory and converter slags.

The properties of finely divided magnetite in a magnetic field were first studied by F. C. Thompson (1916). A general theory has since been outlined by Dean *et al* (1934), and later by Gottschalk and Wartman (1935). The latter investigators assumed that each individual particle, as well as the finest physical subdivisions of each particle, are small magnets or magnetons. In a magnetic field the elements will "line up" and remain in position until thrown into a random orientation by the application of heat. The technique of measurement of the magnetic properties of finely divided substances was outlined by Gottschalk (1935*a*). Briefly, the method used consisted of two identical cylindrical capsules with secondary windings and soft-iron end-plates. The material to be tested was placed in one capsule while the other capsule neutralized the galvanometer throw due to the field. The design features as well as the method of calibration used are given in detail by Gottschalk and Davis (1935).

Experimental Results—Gottschalk and Wartman (*loc cit*) investigated the properties of a number of magnetite powders. Magnetites from four sources were purified and sized, each size-fraction then being analyzed for Fe_3O_4. Only size-fractions below 100-U. S. mesh were studied. Sizes below 350-mesh were air elutriated by the Roller method.

Thus, the mean particle-sizes studied ranged from 180 to less than 5 μ. For most of the magnetites it was found that the Fe_3O_4 content did not change much with particle-size. However, definite relationships were found to exist between permeability and particle-size for various field strengths At low field strengths, there was a marked decrease in permeability as the particle-size decreased. Some data from Gottschalk and Wartman's investigations are shown in Figure 49.

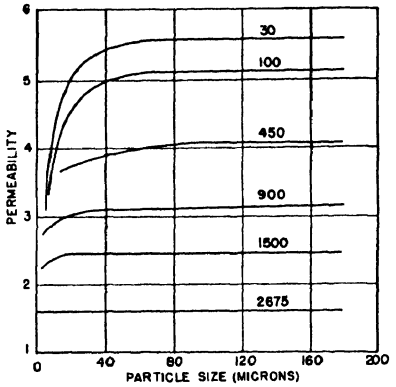


FIGURE 49. PERMEABILITY OF URAL-MOUNTAIN MAGNETITE AT VARIOUS FIELD STRENGTHS. VARIATION WITH PARTICLE SIZE.

The magnetic properties of finely divided material should be affected by the degree of packing. By varying the degree of packing in the above-mentioned capsule, Gottschalk and Wartman obtained the results shown in Figure 50. The coercive force, which has been plotted as the ordinates of the graph, is for the hysteresis loop at saturation and indicates the tenacity with which the material will hold its magnetism; that is, it indicates the strength of the field which must be applied in order to make the magnetization vanish.

Figure 51 shows the straight-line relationship which generally occurs

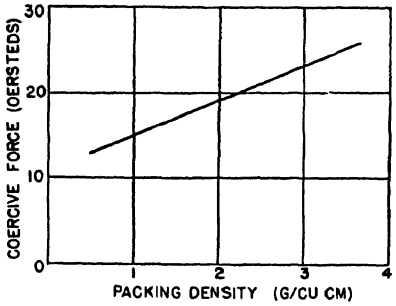


FIGURE 50. VARIATION OF COERCIVE FORCE WITH PACKING DENSITY.

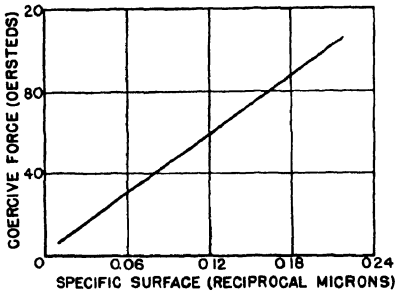


FIGURE 51. VARIATION OF COERCIVE FORCE WITH SPECIFIC SURFACE (MAGNETITE.)

when the coercive force is plotted against the surface of the particles. In the figure, the reciprocal of the diameter, expressed in reciprocal micron units, is plotted. (The diameters in this instance were measured

with a micrometer eyepiece.) Gottschalk (1935*b*) concluded that the linear relation shown in the figure was real and was not affected by packing density or by magnetic contact between particles. There appears to be no satisfactory reason for this relationship.

Later, Dean (1938) tested the existence of dipoles at the surface of metallic substances. His studies were based on the premise that if dipoles exist, then metallic powders should absorb energy in an alternating field having a frequency of the same order as their natural frequency. He estimated that this natural frequency should occur between 0 and 100 kilocycles, and very close to 50 kilocycles. Using magnetite, Dean found that when the losses through a core made of the particles were plotted against frequency (ohms per henry per cycle versus frequency) the losses observed were due to eddy currents and independent of particle-size. He concluded that no space existed for the dipoles to oscillate at the points of contact between the particles. However, when the magnetite was mixed with sand, there was a definite departure from the previous straight-line relationship and a sharp upturn in losses at the expected natural frequency of the dipoles (about 50 kilocycles). He assumed that the sand permitted the dipoles to oscillate by furnishing an internal open surface.

Work was also done with pastilles of aluminum (30-mesh), antimony (150- and 325-mesh), galena (200-mesh), and copper (325-mesh). The method of making these pastilles is described in the next section. The results obtained showed that the D.C. resistance (0-frequency) was high. The resistance decreased rapidly as the frequency approached the natural frequency of the dipoles and then increased. Dean's conclusions were as follows:

"The behavior of powdered metallic substances in high-frequency electrical circuits is in accord with the conception of electrical dipoles at metallic surfaces. These dipoles are anchored to the lattice at one end but have the other end vibrating freely around the electron pair as a center under the influence of gravity and thermal agitation."

Conductivity of Small Particles

The electrical conductivity of small particles depends upon their method of packing, that is, the number of contacts and the nature of the contacts between particles. If a given packing of conducting material is placed between two electrodes and voltage is applied, the total heat produced in a given time t will be

$$H = JI^2\mathcal{R}t \qquad \text{Eq (9-11)}$$

where J is the mechanical equivalent of heat, I is the current, and $\mathcal{R}$ the

resistance The rate of heat increase will be

$$dH/dt = JI^2\mathcal{R} \qquad \text{Eq (9-12)}$$

From Newton's law of cooling

$$\frac{dT}{dt} = k(T - T_0) \qquad \text{Eq (9-13)}$$

where T is the temperature of the body, T_0 the temperature of the surroundings, and k a constant. But, the specific heat of a substance is defined as $K_p = dH/dT$, where K_p is the specific heat at constant pressure. Hence, using this relationship as well as those given above

$$\frac{dH}{dt} = K_p \frac{dT}{dt} = K_p k(T - T_0) = JI^2\mathcal{R}$$

Hence

$$\mathcal{R} = \frac{K_p k(T - T_0)}{JI^2} \qquad \text{Eq (9-14)}$$

This relation was developed by Dean and Koster (1935) to determine whether galena (lead-sulfide ore) followed Ohm's law at low current densities. Experiments were made using particles of various sizes made into pastilles. These pastilles were made by placing a 3-g sample in a specia holder and tapping in a uniform manner until it occupied a minimum volume The specimen was then subjected to a pressure of 175 kg per sq cm for 3 hrs, thus forming the pastilles. The particles used in these tests were measured by means of a camera lucida. The projected areas of from 50 to 150 particles were measured with a planimeter. The square root of the mean thus obtained for each group of screen-sized material was taken as the mean diameter. Experimental results obtained by Dean and Koster are shown in Figures 52 and 53. The first figure shows the relation between current and voltage ($\mathcal{R} = E/I$) for pastilles made of 100/200-mesh galena. If the galena obeyed Ohm's law all of the lines would pass through the origin. The relationship between voltage and current was found to be as follows:

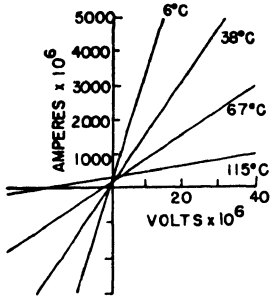


FIGURE 52. TEST OF OHM'S LAW FOR PASTILLES MADE OF 100/200-MESH LEAD SULFIDE.

$$E = I\mathcal{R} + E' \qquad \text{Eq (9-15)}$$

where E' is the intercept on the ordinate axis expressed in volts. This tends to show that the specimens used developed a self-electromotive

force. This effect was observed by Dean and Koster even before current was passed through the specimen, so that it could not be attributed to chemical polarization. The value of E' was found to be dependent upon temperature and grain size

$$E' = cT + C \qquad \text{Eq (9-16)}$$

where c and C are constants depending on particle-size. The cause of the self-electromotive force was not determined, although it was pointed

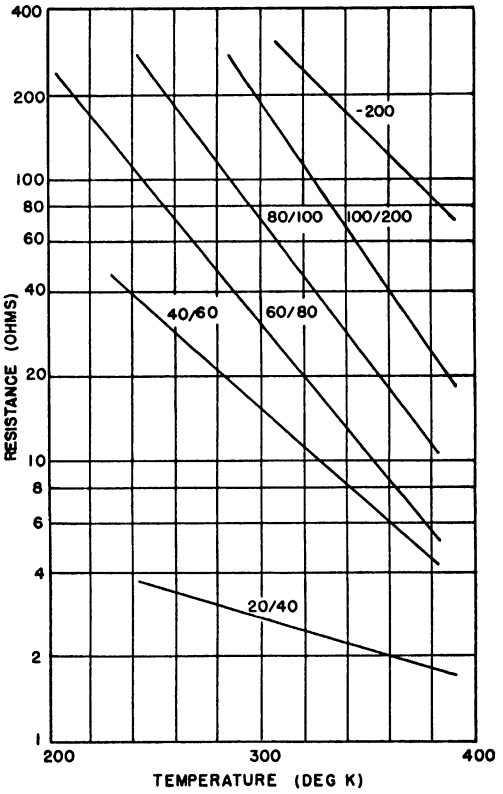


FIGURE 53. VARIATION OF RESISTANCE WITH TEMPERATURE FOR VARIOUS SIEVE SIZES OF LEAD SULFIDE.

out that it could not be thermal since such a force would reverse at low temperatures. It is probable that the method of packing causes a difference of potential among its parts. Some sections of the packing were undoubtedly subjected to more work (and hence deformation) during

pressing than others. The self-electromotive force may also be due to a gradual slow expansion of the pastille as it tends to attain a state of equilibrium.

In regard to Figure 53, it will be seen that the relation between resistance and temperature (deg K) is given by the equation

$$\mathcal{R} = a \cdot \exp\,(b/T) \qquad \text{Eq (9-17)}$$

where a and b are constants. Now, combining Eqs (9-15) to (9-17), we obtain

$$E = aI \cdot \exp\,(b/T) + cT - C \qquad \text{Eq (9-18)}$$

Values of the constants given in this equation are shown in Table 33.

TABLE 33—VALUES OF CONSTANTS IN EQ (9-18) FOR GALENA PASTILLES

Screen size (1)	Mean projected camera-lucida area (sq mm) (2)	Mean diameter col. (2) (mm) (3)	Value of a (ohm) (4)	Value of b (deg K) (5)	Value of c (volts $\times 10^6$) (6)	Value of C (volts $\times 10^6$) (7)
<200	..	...	0.122	2436	5 20	1511
100/200	0 0129	0.144	0.0294	2638	3.10	913
80/100	0.0369	0.192	0.0164	2477	0.76	229
60/80	0.0587	0.242	0.0110	2390	0.73	221
40/60	0 177	0 421	0.0334	1839	0.35	101
20/40	0 617	0 786	0.375	601	0.20	60

OPTICAL PROPERTIES

DIFFUSION OF SUNLIGHT

Dust suspended at high altitudes tends to diffuse sunlight. The particles absorb, scatter, and deflect the sunlight in all directions, thus illuminating the air itself as well as objects in the shade not receiving direct sunlight. Suspended particles are also partly responsible for the intensity and duration of twilight. When the sun's rays fall on the upper air obliquely, the light is scattered and irradiates the earth with a faint, soft light. Gibbs (1924) states that after the eruption of Krakatoa in 1883 a faint, reddish corona of large radius was observed surrounding the sun; the inner edge of the corona was 12 deg and its outer edge 23 deg from the sun. From the observed angular diameters of this ring the deflecting particles were calculated to be 0.002 mm in diameter.

Dust and smoke particles have a capacity for absorbing and radiating heat. This may affect the atmospheric temperature in several ways, (1) The heat rays may be absorbed in part by the dust particles and thus tend to warm the surrounding air by conduction, (2) the intensity of the

original rays is diminished so that the normal temperature rise at ground level is also diminished, and (3) the radiant energy from the ground and from the dust particles is checked. This principle is employed for the protection of orchards and garden crops from frost. Smudge pots for producing smoke clouds are placed at frequent intervals, thus checking the loss of surface heat from the ground. To date no determination has been made on the relative effect of atmospheric dust and smoke particles in regard to atmospheric temperature. Considering that the mean free path of particles in normal air is many times the diameters of the particles themselves, it is doubtful if the influence is ordinarily very remarkable. It is only when the particles are in exceedingly high concentration, as in the case of smudge pots placed several feet apart in an orchard, that temperature effects would be noticeable.

Intensity of Scattered and Reflected Light

Small Particles—The optical properties of suspended particles depend on a number of variables. In his important contributions on the color of the sky, Rayleigh (1871) set forth the equation governing the intensity of light scattered by small spherical particles of uniform size in a gaseous medium. This equation for a single particle is as follows:

$$\frac{I_s}{I_0} = \frac{4\pi^2(\mu_n - 1)^2 \sin^2 \beta}{N^2 r^2 \lambda^4} \cdot \exp\left(- \frac{32\pi^3(\mu_n - 1)^2 l}{3N\lambda^4} \right)$$

where I_s is the intensity of the scattered ray in a plane normal to the line of vision to the particle, I_0 the intensity in the plane normal to the incident ray, μ_n the refractive index of the medium, β the angle between the incident and scattered ray, l the length of path of the incident ray through the medium to the particle in question, N the number of particles per unit-volume, r the distance of the observer from the particle, and λ the wave length of the incident light. The equation assumes that (*a*) the particles are identical, (*b*) they are equally numerous throughout the medium, (*c*) they are not too close together, and (*d*) they are small in comparison to the wave length of light used. The exponential term on the right side of the equation is the extinction factor. Some simplification of Rayleigh's equation is possible by noting the following: For a volume of medium V_0 the number of particles will be NV_0 so that the total scattering is obtained by multiplying the right-hand side of the equation by this quantity. Also $(\mu_n - 1)/N$ is proportional to the density of the medium, and hence to the volume of each

particle, say V. Substituting these equivalents in Rayleigh's equation, we obtain

$$\frac{I_s}{I_0} = \frac{k_1 V_0 N V^2}{r^2 \lambda^4} \cdot \exp\left(-\frac{k_2 N V^2 l}{\lambda^4}\right)$$

where k_1 and k_2 are constants. The shape of the volume V_0 need not be specified. Differentiating I_s/I_0 with respect to λ, we see that the ratio is a maximum for $\lambda = k_2 V^2 l$. If the subscript max denotes the maximum value of the ratio I_s/I_0 and λ, then

$$\frac{I_s}{I_0} = \left(\frac{I_s}{I_0}\right)_{max} \left(\frac{\lambda_{max}}{\lambda}\right)^4 \cdot \exp\left[1 - \left(\frac{\lambda_{max}}{\lambda}\right)^4\right]$$

Hence the wave length of the dominating color increases as l increases ($\lambda_{max} = k_2 V^2 l$) which explains the variation in color occurring at twilight. We also note from the extinction term of Rayleigh's equation that light of very short wave length is rapidly lessened on passing through the medium, while the scatter term shows that very little light of long wave length is diffused by the small particles. The dominant color of the sky therefore lies somewhere between the most scattered and least scattered radiation, or in other words, between the shortest and longest wave length of visible light, which is in the blue of the spectrum.

Utilizing the scatter term of Rayleigh's equation as simplified above we have

$$\frac{I_s}{I_0} = k_1 \frac{\pi V_0^2 N d^6}{36 r^2 \lambda^4} = \frac{k C d^3}{r^2 \lambda^4} \qquad \text{Eq (9-19)}$$

where d is the diameter of the particle, C the concentration of the particles in the medium, and k a suitable constant.

In the case of direct light, the decrease in intensity is measured by the extinction term, that is

$$\frac{I}{I_0} = \exp\left(-\frac{k_2 V^2 l}{\lambda^4}\right) = \exp\left(-\frac{k_2 \pi^2 N d^6 l}{36 \lambda^4}\right)$$

Thus we see that within the limitations of Rayleigh's assumptions, as d increases the ratio of residual to initial intensity, I/I_0 diminishes, as it also does when l and N increase, although not at as great a rate. The influence of any variation in λ is also evident.

Cheveneau and Audubert (1920) made extensive studies of Rayleigh's formula and have given a modification which they state is more general, namely

$$I = I_0 \cdot \exp\left(-kNld/\lambda^n\right) \qquad \text{Eq (9-20)}$$

where k is a constant, n is a variable exponent, l is the transverse dis-

tance, and N is the number of particles per unit-volume. This corresponds closely to the equation developed by Simon *et al* (1931)

$$I = I_0 \cdot \exp(-K'Nld) \qquad \text{Eq (9-21)}$$

In this equation the wave length has been lumped into the constant K'. Using this equation for spherical particles, Drinker and Hatch (1936) have shown that a concentration of 350 particles per cc having a uniform diameter of 2 μ would require a distance, l, of 47 meters to reduce the intensity of the light 5 percent, that is, from 100 to 95 percent.

Cheveneau and Audubert's equation, and that attributed to Simon *et al*, were developed by Werner (1923) on the following basis: Let there be a suspension of N^3 particles per unit-volume; consider a unit-surface of these particles, N^2 in number, and let there be Nl such surfaces through which the incident beam passes. If the particles are circular, such that the cross section of each is $\pi d^2/4$, and if the area of the beam is A, then the latter covers an area $N^2A\pi d^2/4$ of particulate material. Hence

$$dI = I(AN^2\pi d^2/4)dl$$

and integrating dI between the limits I_0 and I, and l from 0 to Nl we obtain

$$I = I_0 \cdot \exp(-N^3A\pi d^2l/4)$$

so that if N_0 denotes the number of particles per unit-volume and A is taken as unity*

$$I = I_0 \cdot \exp(-N_0\pi d^2l/4)$$

With increasing particle-size, the intensity of scattered light obeys a new law. Stutz (1930) studied the transmission of monochromatic light of different wave lengths through suspensions of zinc oxide These suspensions were carefully sized and covered a range of 0.1 to more than 1.0 μ. It was found that the intensity of the scattered beam increases rapidly to a maximum as the particle-size approaches the wave length of the incident beam and then decreases with further increase in particle-size. From these data Stutz concluded: (*a*) That the maximum turbidity is independent of concentration, (*b*) that the particle-size at which the maximum turbidity occurs varies with wave length, and (*c*) that suspensions comprised of particles over 1 μ in size transmit all wave lengths with equal facility.

Large Particles—The intensity of light reflected from particles

* This equation is often referred to as the Beer-Lambert law. Since d^2 is proportional to particle-surface, we may substitute for this the specific surface $1/d$.

larger than the wave length of the incident beam has been shown by Tolman *et al* (1919*a*, *b*, *c*) to be given approximately by the equation

$$I_r = \frac{KC}{d} \qquad \text{Eq (9-22)}$$

where C is the concentration (mass) of dust per unit-volume, and d the diameter of the particles. Note that for a given concentration the intensity of reflected light varies inversely as the particle-diameter, and that for a constant particle-diameter the intensity of reflected light varies with the concentration. If we write $C = \rho Nd^3$, where ρ is the density of the particles, and N their number per unit-volume, we have, substituting in Eq (9-22)

$$I_r = K'Nd^2 \qquad \text{Eq (9-23)}$$

where K' includes a number of optical constants. Thus, we see that the reflected light is a function of particle-surface. This fact has been used in developing certain apparatus for measuring particulate surface.

Absorption—The light absorbed by a suspension is, of course, the difference between the light transmitted with and without particles present. Thus, using the Cheveneau and Audubert equation Eq (9-20), the absorption is

$$1 - \frac{I}{I_0} = \exp\,(-KNdl/\lambda^4) \qquad \text{Eq (9-24)}$$

With reference to Eqs (9-22) and (9-23) we cannot write the absorption equation since much depends upon the surface characteristics of the particles. Their color, opaqueness, and other optical characteristics have an important bearing on absorption so that no general equation can be given.

Mixtures of Particles—The development thus far concerns a medium composed of particles of uniform size. When a mixture of different sizes is used, the following equation due to Cheveneau and Audubert may be used:

$$I = I_1 \cdot \exp\,(-a/m^\beta) \qquad \text{Eq (9-25)}$$

where m represents the total mass of the particles Nd^3, and a and β are two constants depending only on the nature of the granules in suspension and the wave length of light used. This formula has some interesting applications. Thus for a given medium, determination of the ratio I/I_0 permits us to obtain the total mass of its constituent particles. This measurement is possible whenever we have a suspension which does not settle too rapidly. It is only necessary to calculate the constants a and β of the exponential formula from two determinations of

known concentrations. We can likewise obtain the mass m without calculating it by formula. In fact, all that is required is construction of a curve for the values of I/I_0 for known concentrations of the given medium. Then, to determine the concentration of another suspension of the same kind we merely measure I/I_0, and relate it to the standard curve. In essence this is what is done with the Tyndall meter, described in another section of this chapter. In the present instance the value of the Cheveneau and Audubert equation lies in the fact that it permits determination of the physical constants of the suspension.

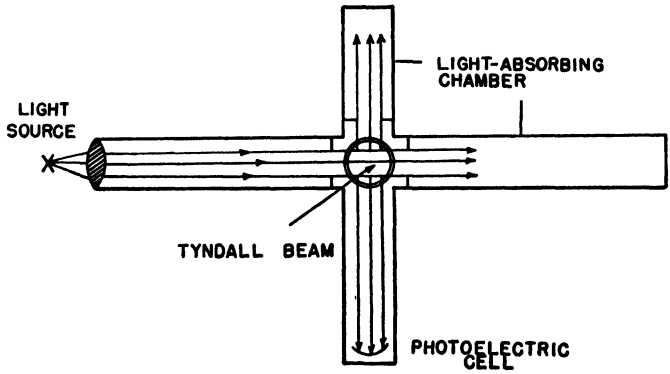


FIGURE 54. ILLUSTRATING PRINCIPLE OF TYNDALL METER.

Tyndall Meter—The Tyndall meter is a device developed by Tolman and Vliet (1919) for measuring the intensity of a Tyndall beam. The apparatus is shown in Figure 54 and consists of a central cell which contains the suspension to be measured. This is illuminated by a light beam whose intensity has been standardized. This beam of light passes through the cell. The intensity of the resulting Tyndall beam in the cell is measured by means of a photoelectric meter, or an illuminometer. Several variations of the Tyndall meter are possible, as follows:

1. Instead of a cell containing the suspension in either gas or liquid medium, the cell may be connected at right angles to the plane of the paper to a suitable gas-chamber, or by means of a glass cylinder to any liquid suspension which is made to flow through it. Thus, it is continuously possible to observe the change in concentration which takes place in a given medium.
2. The photoelectric cell may be connected to a recorder and a complete permanent record of change conditions in any suspension medium determined. The theory of the device has already been given. (See Eqs 9-19, 9-20, and 9-23).

From a consideration of these equations it may be seen that for a given concentration of particles larger than 1 μ, the reflected light varies inversely as the diameter of the particles, while for particles less than 1 μ the intensity is as the cube.

Hatch and Choate (1929), in studying the relationship between the strength of a Tyndall beam and parameters of particle-size, obtained the relation

$$\log T_{50} = \log 85.0 - \log d_g - 5.757 \log^2 \sigma_g \qquad \text{Eq (9-26)}$$

where T_{50} indicates the value of Tyndall readings for suspensions in distilled water of 50 mg per liter, d_g is the geometric-mean diameter of the particles, and σ_g the geometric-standard deviation. These constants are explained in Chapter 3. Comparison of Eq (9-26) with Eq (3-46) shows that the Tyndall reading is a measure of the specific surface. The use of the Tyndall beam as a means of measuring particle-surface is described in Chapter 16. It must be remembered that Eq (9-26) is not a general equation, but has been found to apply to dusts such as quartz, granite, and calcite. Particles with other surface characteristics will alter the first term to the right of the sign of equality.

Polarization

In addition to transmitting, absorbing, and reflecting, small particles in suspension polarize light. The degree of polarization varies with the angle at which it is observed with Nicol prisms. Usually the polarization is a maximum at right angles to the light beam passing the particle-suspension. In addition to angle, the degree of polarization depends upon particle-size and concentration.

Light scattered by small particles will be completely polarized if they are optically isotropic and distributed entirely at random. This polarization will take place in any plane normal to the scattered rays. In practice complete polarization by a suspension of particles is never realized, and consequently suitable corrections must be applied to Rayleigh's equation to account for the shape of the particles.

Gans (1912, 1920) and Lange (1928) developed some interesting relationships concerning polarization of light by suspensions of particles. If I_p and I_p' are the intensities recorded for the natural and polarized components, respectively, and if

$$\theta = \frac{I_p'}{I_p + I_p'} \qquad \text{Eq (9-27)}$$

then, θ is proportional to the concentration as shown by Gans

$$\frac{\theta - \theta_0}{1 + \frac{5}{3}\theta} = 2aC \qquad \text{Eq (9-28)}$$

where a is a constant depending upon the shape and size of the particles and C is proportional to the concentration. The value of θ_0 is defined by Eq (9-27) as the "depolarization" at zero concentration, that is, for a single particle. Lange has given the equation for determining θ_0 in terms of θ_1 and θ_2 for two corresponding concentrations of suspensions, C_1 and C_2. Thus

$$\theta_0 = \frac{\frac{C_1}{C_2}\theta_2\left(1 + \frac{5}{3}\theta_1\right) - \theta_1\left(1 + \frac{5}{3}\theta_2\right)}{\frac{C_1}{C_2}\left(1 + \frac{5}{3}\theta_1\right) - \left(1 + \frac{5}{3}\theta_2\right)} \qquad \text{Eq (9-29)}$$

where it is assumed that the concentrations C_1 and C_2 are known. For a given wave length and for particles of uniform size and known concentration the relation between the depolarization and particle-size a is easily determined from Eq (9-28). Stutz (1930) obtained fairly good agreement between values of θ_0 and particle-size derived experimentally and calculated from Eq (9-28).

Hatch and Choate (1930) derived an expression connecting particle-size, polarization, Tyndall reading, and other constants of the suspension. The expression obtained was as follows:

$$\log d_g = \frac{3}{2m}\left[\log\left(\frac{b}{10 - b}\right) - \log K\right] + \frac{1}{3}\left(\log T_c - \log C\right) \qquad \text{Eq (9-30)}$$

where d_g is the geometric-mean size of the suspension, T_c the Tyndall reading, b the Tyndall reading given by nonpolarized light when the maximum light reading had a value of 10, and m and C constants depending upon concentration, optical properties of the material, and characteristics of the apparatus. The constants C, K, and m are determined by the physical properties of the material, and also by the characteristics of the apparatus. Values of these constants for three different materials are given in Table 34.

TABLE 34—VALUE OF CONSTANTS IN EQ (9-30)

Material	Constant		
	C^a	k	m
Silica	74.1	0.893	0.627
Granite	74.1	0.893	0.627
Calcite	113.8	1.860	0.291

[a] For suspensions of 50 mg per liter of water.

In measuring the Tyndall beam and polarization, Hatch and Choate used the Tyndall meter previously described. Observations were made with water suspensions. A Nicol prism was placed at right angles to the beam and so mounted that it could be rotated about its axis. Two readings were obtained, one of which gave the strength of the full light beam and the second the strength of that portion of the beam which was composed entirely of nonpolarized light. The latter was obtained by turning the prism 90 deg from its position of maximum reading. Hatch and Choate obtained good agreement between observed diameters and those calculated by means of Eq (9-30). The average size of material used ranged from 1 to 50 μ in diameter.

Photospheric Particles

It is well known that small particles are repelled by light rays with an intensity which is proportional to the surface exposed to light. Thus, if the radiation pressure equals p, we have

$$p = kd^2$$

where d is the diameter of the particle and k is a constant. If we consider a particle having a diameter d at the sun's surface, we have a pressure exerted which is equivalent to 2.75 dynes per sq cm. The force of gravitation is obviously proportional to the cube of the diameter, that is

$$g = k'd^3$$

where k' is a constant. Hence, it is possible that the repulsion exerted on the particle may be greater or less than the pull of gravity, depending upon its diameter. It is clear that as the diameter of the particle diminishes, the force of repulsion becomes greater than the force of gravity. If the particle is capable of absorbing all radiation (black body) and is of unit-density, the repulsion will equal the gravitational force when d equals approximately 1.5 μ. Figure 55 shows the relations between repulsive and gravitational forces, and diameter.

The reasoning given in the preceding paragraph has given rise to the theory that finely charged dust or smoke may reach the earth from the sun. This theory was suggested by Arrhenius (1900) The particles may be charged both positively and negatively, although some are undoubtedly neutral due to recombination of particles with opposite charge. The particles are charged by the ultraviolet radiation at the sun's surface and by the high temperature Not all the particles reach the surface of the earth, since some are drawn back into the sun and others remain in interstellar space. Gibbs (1924) cites Fleming as having computed ". . . . that particles of unit-density and of particles, say equal to 0.16, 0.50,

and 1.0 μ projected from the sun at the velocity of 200 km per sec, would reach the earth in 22 hrs, 42 hrs, and 76 hrs, respectively, their final velocities being 1700, 780, and 350 km per sec."

EFFECT OF SOUND

The familiar test for antinodes due to standing sound waves in a closed horizontal cylinder is well known to students of elementary physics. Fine dust initially scattered through the cylinder gathers at

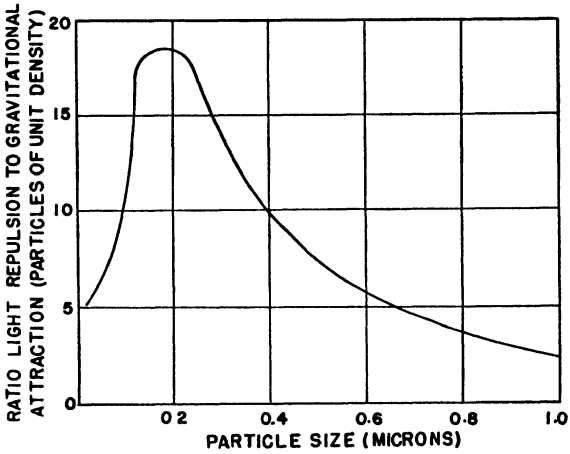


FIGURE 55. VARIATION OF RATIO OF LIGHT PRESSURE TO GRAVITATION FOR PARTICLES OF UNIT DENSITY AND VARIOUS DIAMETERS.

certain fixed points when sound waves are set up. The points at which the dust gathers are so spaced that their distance apart measures the wave length of the sound produced.

In addition to this phenomenon, sound waves of very high frequency and amplitude have a faculty of coagulating small particles. Thus a cloud of fine dust or mist introduced into a vertical column with an oscillator for producing sound waves on one end will coagulate or force the particles to coalesce. This causes the suspension to settle rapidly. Devices which accomplish this are called "sonic flocculators." The vibratory element consists of a thin plate actuated by a ferromagnetic metal or alloy rod connected to it. This rod is placed in a high-frequency magnetic field causing the element to be deformed. For example, a nickel rod placed in a field of 100 gausses decreases in length approximately 0.00004 cm per cm length. Hollow tubes are more effective for the production of sound waves since the so-called magneto-

strictive effects do not penetrate deeply. In almost every particular the sound field produced in the cylinder is essentially the same as that obtained in the usual dust-tube experiment above referred to. Flocculation at the antinodes can be seen in a cloud of ammonium chloride when the cylinder is held vertically.

Briefly the action taking place is as follows: When the column is made to resonate by adjustment of the reflector on the end opposite to the sound-producing mechanism, the radiated and reflecting waves interfere in such a manner as to produce a standing wave. The nodes and antinodes correspond to minimum and maximum intensities of sound.

Theories of Sonic Flocculation—Theories pertaining to the action of sonic flocculations have been advanced by Andrade (1931–1932*a*, *b*), Brandt and Hiedemann (1936), and St. Clair (1938, 1940). Particles in a sound field being caused to vibrate with the gas in which they are suspended set up attractive and repulsive forces, similar to those of two spheres moving together in a fluid.*

The particles in a sound field, however, behave in a peculiar manner which is not quite so simple. St. Clair points out that at a frequency of 5000 cycles, particles of 0.5 μ or smaller (density = 1.5) have an amplitude and velocity the same as that of the surrounding gas, while particles of 10 μ will scarcely vibrate at all. The intermediate sizes will pulsate out-of-phase with the pulsations of the gas stream. Thus particles 2 μ will have an amplitude of 0.87 that of the gas and will be about 30 deg out-of-phase. It has been suggested by Brandt and Hiedemann that the flocculating effect is due to the increased number of collisions between the particles due to the kinetic energy imparted to them. However, as St. Clair points out, this cannot be the sole factor since flocculation is observed at a few hundred cycles for which the suspended particles remain stationary.

If we consider two spheres in the vibratory field of a gas stream it may be shown that the forces acting on the spheres (in the xz-plane), provided the motion is streamline, are (König, 1891),

$$F_x = -\frac{3\pi\rho_0 d_1^3 d_2^3 v_0^2}{2l^4}\sin\theta(1-5\cos^2\theta)$$

$$F_z = -\frac{3\rho_0 d_1^3 d_2^3 v_0^2}{2l^4}\cos\theta\,(3-5\cos^2\theta)$$

where d_1 and d_2 are the diameters of the particles, v_0 the maximum velocity of vibration, ρ_0 the density of the fluid, θ the angle between the line of centers and the z-axis, and l the distance between them (the

* See, for example, Ramsey (1920), pp. 199–207.

center of one particle being taken as the intersection of the yz-axis). We see, therefore, that the forces are proportional to the square of the velocity of the gas *relative to the particles*. Hence for rapid flocculation, as St. Clair points out, it is necessary that the particles do not participate to too great an extent in the motion of the gas. There is, therefore, a minimum frequency, depending on the size and density of the particles and the viscosity of the gas, below which these forces will not be effective.

St. Clair states that the lowest frequency which is found effective for small particles is about 4000 cycles. About 7000 cycles are required for tobacco smoke. With regard to rate of flocculation, St. Clair (1940) found that the rate of flocculation was proportional to the power input and almost proportional to the concentration of suspended material. By *rate of flocculation* is meant the relation between the final and initial average particle-size determined by optical methods similar to those implied in Eq (9-30).

On the basis of what is the most extensive work thus far done, St Clair concluded that flocculation by a sound field may be attributed to two effects· (1) To the increased effective diameter of each particle as a result of the hydrodynamic forces set up between them, and (2) the *radiation* pressure of the sound field causing the particles to concentrate at the antinodes.

Problems

1. A particle 2 μ in diameter having a dielectric constant 3.1 is placed in an electric field for 0.02 sec. The constants of the field are as follows: voltage gradient 900 volts per cm, current density 9×10^{-9} amp per sq cm. Calculate the charge on the particle in terms of the number of unit-charges. *Note:* 1 unit-charge $= 4.8 \times 10^{-10}$ esu and 1 amp $= 3 \times 10^{9}$ esu.

2. If the gas stream, in which the particle in Problem 1 is suspended, moves through the cylinder at 1 meter per sec, how far has the particle traveled through the cylinder before it is 90 percent charged?

3. For a given concentration and size of suspended particulate material, plot the variation of intensity with distance traversed by a beam of light. (Use Eq 9-21)

4. Using Eq (9-21) how much more light will pass a given distance through a dust concentration of 10 million particles per cu ft having an average diameter of 1 μ than through the same distance if the concentration of 60 million particles per cu ft has an average diameter of 3 μ ? Assume the optical properties of the two dusts to be the same.

5. Compare the result obtained in Problem 4 with the result that would be obtained if Eq (9-23) were used. Discuss the limitations of both equations.

6. Discuss the method for determining the constants m, K, and C in Eq (9-30) Note the ratio $b/(10 - b)$ varies logarithmically with size, as follows:

$$\log \left(\frac{b}{10 - b} \right) = \log K + m \log d_{av}$$

But by Eq (3-25)

$$\log d_{av} = \log d_g + 1.151 \log^2 \sigma_g$$

And for the generalized Eq (9-26) we have

$$\log T_e = \log C - \log d_g - 5.757 \log^2 \sigma_g.$$

7. For the "Senate-House Problems and Riders (Mathematical Tripos)" for the year 1875, P. G. Tait proposed the following for solution: "Show that a cloud of small particles or of fine dust if only deep enough, however far the particles may be separated in comparison with their diameters, can give a brightness equal to half that of a slab of the same material similarly illuminated by a distant source of light."

Note: Let I_0 be the amount of incident light, I_a the amount absorbed, I_t the amount transmitted, and I_r the amount reflected, then

$$I_0 = I_a + I_t + I_r$$

$$1 = \frac{I_a + I_t}{I_0} + \frac{I_r}{I_0} = \theta + \theta_r$$

where θ represents the fraction of light absorbed and transmitted and θ_r the fraction of light reflected. If we imagine the cloud to be divided into layers, $(1 - \theta)$ represents the amount of incident light absorbed and transmitted and $\theta\theta_r$ the amount returned. Arranging our analysis layer by layer, we obtain

Layer	*Amount received*	*Amount returned*
1	1	$\theta\theta_r$
2	$1 - \theta$	$(1 - \theta)^2\theta\theta_r$
3	$(1 - \theta)^2$	$(1 - \theta)^4\theta\theta_r$
	etc.	etc.

the factors $(1 - \theta)$, $(1 - \theta)^2$, etc , being squared because of the loss on return. Hence the total amount of light returned is the sum of the terms of the last column which if the layers are thin and infinite in number is

$$\theta\theta_r [1 + (1 - \theta)^2 + (1 - \theta)^4 + \text{etc.} = \frac{\theta\theta_r}{1 - (1 - \theta)^2} = \frac{\theta_r}{1 - \theta} = \frac{\theta_r}{2}$$

when θ is very small. Hence, under the most favorable conditions, the amount of light reflected from a cloud of particles cannot exceed half the amount reflected from a slab composed of the same particles.

CHAPTER 10

THERMODYNAMICS OF PARTICLES

BEHAVIOR OF PARTICLES IN A HEAT FIELD

It is well known that in the immediate vicinity of a hot body the air is relatively free of small particles. Thus, a hot wire which is placed in a cloud of dust will have a well-defined area which is free of any particles. This effect is not due to radiant energy of a hot body since these particles are in a state of continual motion and have surface temperatures equal to that of the hot wire. It is believed highly probable that conduction of heat from solid bodies to gases affords the most reasonable explanation. Cawood (1936) advanced this theory and set forth its mathematical basis. His analysis is based on the concept of a small cylinder of gas of a length about twice the mean free path of the gas in a temperature gradient above a hot plane. The cylinder is further assumed to have a very small radius so that convection currents cannot affect the movements of molecules parallel to the axis of the cylinder. The perpendicular velocity gradient of a convection current is conceived as discontinuous and varying in steps having the order of magnitude of the new free path of the gas.* Hence the velocity on both sides of such a cylinder will be approximately equal. Now if a smoke particle in the middle of the cylinder is considered, it will be bombarded by the gas molecules in all directions. However, we are considering only the collisions caused by molecules moving perpendicular to the hot plate. If the force per unit-area parallel to the plate is taken as F, then the total force on one side of the spherical particle (of diameter d) being considered is

$$\frac{\pi d^2}{2} F \int_0^{\pi/2} \sin\theta \cos\theta d\theta = \frac{\pi d^2}{4} F$$

The force on the spherical particle due to the bombardment of the gas molecules may be computed by a method due to A. Einstein (1924).

* Cawood cites Knudsen for justification of the assumption that if there were no convection currents then in any small surface in the temperature gradient $N_1 v_1 = N_2 v_2$. As a consequence of this assumption which applies only at low pressures, $N_1 v_1^2 / N_2 v_2^2 = v_1/v_2$, $P_1/P_2 = (T_1/T_2)^{1/2}$, and $N_1/N_2 = (T_2/T_1)^{1/2}$, where v_1 and v_2 are average velocities corresponding to a difference of temperature $T_1 - T_2$.

Let n_1 and n_2 be the number of molecules on each side of an imaginary surface and let v_1 and v_2 be their corresponding average velocities. Then the number of molecules colliding with the particle on one side of the boundary will be $^1/_{24}\pi n_1 v_1 d^2$ and on the other side $^1/_{24}\pi n_2 v_2 d^2$. The differences in velocities of the molecules account for the heat flow. Assuming that the particle causes no interference, the flow across the imaginary boundary will be

$$\frac{1}{4} f\pi d^2 = \frac{1}{24}\pi n_1 v_1 d^2 \cdot \frac{m v_1^2}{2} - \frac{1}{24}\pi n_2 v_2 d^2 \cdot \frac{m v_2^2}{2}$$

$$= \frac{1}{24}\pi n_1 v_1 d^2 \left(\frac{m v_1^2}{2} - \frac{m v_2^2}{2}\right) \qquad \text{Eq (10-1)}$$

$$(n_1 v_1 = n_2 v_2 \textit{ approximately})$$

where f is the heat or energy flow per unit-area and m is the mass of the gas molecules. From physical chemistry it may be shown that the velocity of a molecule is given by the equation

$$\frac{1}{2} m v^2 = \frac{3}{2} BT$$

and the change in energy by

$$\frac{1}{2} m(v_1^2 - v_2^2) = -\frac{3}{2} B \cdot \frac{dT}{dx} \cdot 2\lambda$$

Hence

$$f = -\frac{1}{2} n_1 v_1 B \cdot \frac{dT}{dx} \cdot \lambda \qquad \text{Eq (10-2)}$$

where λ is the mean free path of the gas, T the absolute temperature, B the gas constant, and the ratio dT/dx a temperature gradient. The excess of momentum ΔM from the hot plate is obtainable again from the Einstein concept. This excess is

$$\Delta M = \frac{1}{24}\pi n_1 v_1 d^2 \cdot m v_1 - \frac{1}{24}\pi n_2 v_2 d^2 \cdot m v_2 \qquad \text{Eq (10-3)}$$

$$= \frac{1}{24}\pi n_1 v_1 d^2 (m v_1 - m v_2)$$

so that if impulse due to recoil is neglected, ΔM is the force on the particle. Hence

$$\Delta M = \frac{\pi d^2}{4v} = -\frac{1}{8}\pi d^2 n_1 B \cdot \frac{dT}{dx} \cdot \lambda = -\frac{1}{8}\pi d^2 P \cdot \frac{dT}{dx} \cdot \frac{\lambda}{T}$$

$$\left(\text{if } v = \frac{v_1 + v_2}{2} \text{ and } v \text{ is nearly } = v_1\right)$$

where P is the pressure equal to nTB. From this point Cawood assumes that the excess momentum is equal to the resistance encountered by a particle moving according to Stokes' law (with Cunningham's correction), that is,

$$\Delta M = \frac{3\pi\mu v_0 d}{1 + \frac{2A\lambda}{d}}$$

where v_0 is the velocity, μ the viscosity of the gas, and A a constant. From these considerations we obtain

$$\frac{P\lambda\pi d^2}{8T} \cdot \frac{dT}{dx} = \frac{3\pi\mu v_0 d}{1 + \frac{2A\lambda}{d}}$$

and

$$v_0 = \frac{PLd^2}{24\mu T}\left(1 + \frac{2A\lambda}{d}\right)\cdot\frac{dT}{dx} \qquad \text{Eq (10-4)}$$

In the final step, in order to estimate the accuracy of this equation, Cawood assumed that dT/dx is constant. If T_g is the gas temperature, then we have, approximately,

$$\frac{T - T_g}{x} = \frac{dT}{dx}$$

From Eq (10-4), $v_0 = dx/dt$, so that the time required for a particle to move from the plate to a point x is

$$t = \frac{24\mu\left[\left(\frac{dT}{dx}\right)\frac{x^2}{2} + T_g x\right]}{P\lambda d\left(\frac{dT}{dx}\right)\left(1 + \frac{2A\lambda}{d}\right)} \qquad \text{Eq (10-5)}$$

The time of movement of a particle from a given point near the plate to the edge of the dust-free space may be observed experimentally. The temperature of the gas is known and the temperature gradient is easily determined from the gas temperature and the observed thickness of the dust-free space. Cawood found that Eq (10-5) gives results of the correct order of magnitude but somewhat greater than observed experimentally. The difference between the results obtained by Eq (10-5) and those obtained experimentally may be attributed to the fact that the temperature gradient is not constant.

Experimental—Extensive studies of the nature of the dust-free space about hot bodies were made by Watson (1936) who established the relation between the thickness of the dust-free space and the tempera-

ture difference between the hot body and the walls of the containing vessels. If δ is the thickness of the dust-free space and ΔT the temperature difference, the expression derived is of the form

$$\delta = K(\Delta T)^b \qquad \text{Eq (10-6)}$$

where for bodies of different shapes the values of K and b are as given in Table 35. Watson then proceeded to obtain a single formula for all bodies and attempted to determine the manner in which K varied with shape and size. The rate of convective heat loss from a hot body is given by

$$H_c = c(\Delta T)^{1.25} \qquad \text{Eq (10-7)}$$

where H_c is the convective heat loss per unit-time from a unit-area of a hot surface, and c is a constant which has been experimentally deter-

TABLE 35—VALUES OF K AND b AND c IN EQS (10-6) AND (10-7)

Type of body	Value of K	Value of b	Value of c
Vertical plane surface, 6 × 3 cm	0.0061	0.51	0.000103
Vertical plane surface, 3.1 × 3 cm	0.0054	0.55	0.000153
Horizontal circular rod, 0.9 cm diam	0.0035	0.49	0.000093
Horizontal circular rod, 0.46 cm diam	0.0027	0.55	0.000125
Horizontal circular rod, 0.0254 cm diam	0.0015	0.52	0.000587
Mean value of b		0.52	

mined for horizontal rods and vertical planes. The value of c is independent of the length of the rod and extent of vertical plane. The values of c obtained by Fishenden and Saunders (1932) are shown in Table 35. The best relation between K and c as obtained by Watson is

$$K = K'c^{-0.38} \qquad \text{Eq (10-8)}$$

where K' is a constant. Eq (10-6) reduces to

$$\delta = K'TH_c^{-0.38} \qquad \text{Eq (10-9)}$$

The mean value of K' for vertical planes is approximately 15.6×10^{-5} and for horizontal rods is 7.6×10^{-5}. Thus, for horizontal rods $\delta = 7.6 \times 10^{-5}\ \Delta T \cdot H_c^{-0.38}$ and for vertical plane surfaces $\delta = 15.6 \times 10^{-5}\ \Delta T \cdot H_c^{-0.38}$, where δ is expressed in cm, ΔT in deg C, and H_c in calories per sq cm per sec. Watson reports that Iles extended the work discussed above to a study of the effect of air pressure on the dust-free surface for a vertical-plane surface 3.1 cm high and 3.0 cm wide. Iels found that at any one temperature the thickness of the dust-free space

opposite the vertical-plane surface 3.1 cm high varied inversely as the 0.61 power of the pressure. By combining the variation of δ with ΔT and p we obtain the equation for the vertical plane, as follows:

$$\delta = K''(\Delta T)^{0.52}p^{-0.61} \qquad \text{Eq (10-10)}$$

where K'' is a particular constant for the type of body in question.

THEORY OF HEAT CONDUCTION IN PACKINGS

Within a packing there are two paths by which heat may flow through the particles and two paths by which it may flow through the pore-spaces. These paths are made clear by reference to Chapter 6. In the first case heat passes through the solid structure, and in the second by conveyance through the gas phase. Kistler and Caldwell (1934) give the following equation for conductivity in the latter case

$$C_g = k\left(\frac{0.0075}{p} - \frac{d}{2}\right)\log\frac{0.0037}{p} \qquad \text{Eq (10-11)}$$

where C_g is the conductivity at 34 deg C referred to 4 deg C, k a constant, p the pressure in mm Hg, and d the average diameter of the particles in cm. For a solid with gas film, the conductivity of the two is

$$\frac{1}{C} = \frac{1}{C_s} + \frac{1}{C_g} \qquad \text{Eq (10-12)}$$

The value of C_s (the conductivity through the solid structure) will be independent of pressure. For air the constant k has a value of 1.75 Hence, the conductivity can be determined for any pressure by means of Eqs (10-11) and (10-12).

If the material is porous and similar to silica gel, the two conditions referring to the packing as a whole apply to each granule. Kistler and Caldwell obtained the following relation for heat flow through such material from the kinetic theory of gases

$$C_{ig} = 5.8 \times 10^{-5}\frac{\lambda p}{10 + \lambda p} \qquad \text{Eq (10-13)}$$

where C_{ig} is the conductivity at 34 deg C referred to 2 deg C for the granule, and λ is the mean free path of the gas molecules in cm within the porous body when the pressure is so low that all the molecules travel in straight lines until they strike some structural element. The contact conductivity of the granules may be assigned a value, say C_{is}. The value of C_{is} for silica gel is $0.8C_s$. The ratio of C_{is}/C_s should not de-

part greatly from this value for other materials Thus the over-all conductivity is

$$\frac{1}{C} = \frac{1}{C_s + C_g} + \frac{1}{C_{ts} + C_{tg}} \qquad \text{Eq (10-14)}$$

Taking $k = 1.75$ and $\lambda = 4 \times 10^{-6}$ cm for air, and d the average diameter 4/10-mesh silica gel, Kistler and Caldwell evaluated the change in conductivity of the substances for pressures ranging from 0.00011 to 760 mm Hg. Corresponding values of C determined by the above equations were 2.7 and 5.1 cal per sec per deg C per cm $\times 10^5$ and were in excellent agreement with experimental observations. Values of conductivity for a number of materials are given in Table 36.

TABLE 36—CONDUCTIVITIES OF VARIOUS PARTICULATE SUBSTANCES

Substance	Temp range (deg C)	Conductivity (cal/sq cm/sec/cm per deg change in temperature gradient)
Sand	...	0.00074–0.0026
Quartz sand	18–98	0.0006
Carborundum sand	18–98	0.0005
Calcined magnesia	20–100	0.00034
Infusorial earth	17–98	0.00013–0.00038
Clinker, small grains	0–700	0.0011
Ordinary brick dust	0–100	0.00039
Powdered charcoal	0–100	0.00022
Pumice	...	0.0006

Specific Heat—The specific heat of particles is determined by heating a given weight of a sample to nearly 100 deg C in a steam bath, and quickly stirring the hot particles into the water in a calorimeter. The temperature rise may thus be determined and the specific heat obtained from this observation and the known masses involved. Most soils, whether loams, sands, or clays, have specific heats ranging from 0.15 to 0.20 cal per g per deg C.

FLOW OF HEAT IN GRANULAR MEDIA

The flow of heat in a granular medium may be approximated by the classical heat-flow equations of mathematical physics. Assume a packing of particles contained in a rectangular box, one end of which is kept at a constant temperature. If T is the temperature at a distance x from the heat source, the flow of heat across an element of area A parallel to the heated surface will be $-CA(\partial T/\partial x)$, where C is the conductivity of the packing which is numerically equal to the quantity of heat flowing,

per unit-time, through a unit-area of a plate with a unit-difference of temperature between its faces. At a distance $x + dx$, the heat leaving the same area will be

$$- CA\left[\frac{\partial}{\partial x}\left(T - \frac{\partial T}{\partial x}dx\right)\right]$$

so that the net amount of heat retained within the space dx will be

$$-CA\left(\frac{\partial T}{\partial x}\right) + CA\left[\frac{\partial}{\partial x}\left(T + \left(\frac{\partial T}{\partial x}\right)dx\right)\right] = CA\frac{\partial^2 T}{\partial x^2}\,dx$$

If there are no heat losses, the quantity of heat added to cause a rise in temperature ∂T will be $AK_p(\partial T/\partial t)dx$, where K_p is the thermal capacity

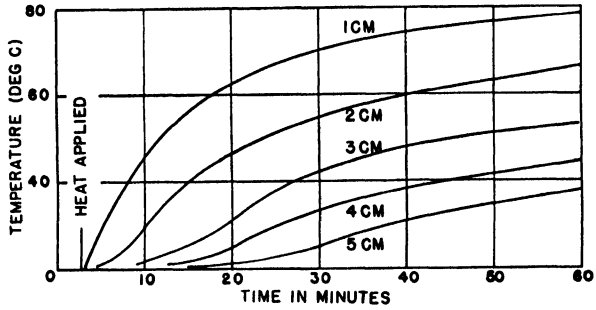


FIGURE 56. TEMPERATURE-TIME CURVES FOR QUARTZ PARTICLES HAVING A SIZE-DISTRIBUTION AS SHOWN IN TABLE 37.

of the substance (effective specific heat divided by apparent specific volume). Hence

$$\frac{C}{K_p}\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad \text{Eq (10-15)}$$

The term C/K_p is called the diffusivity. Eq (10-15) may be solved by Fourier series for any boundary conditions. If the temperature of the packing at the start is T_0 (room temperature), and that of one end 100 deg C, then $T = T_0$ when $t = 0$, and $T = 100$ when $x = 0$. Since the box is of finite length, L, the end limit may be defined by writing $T = T_0$, when $x = L$. The complete solution is

$$T - T_0 = 100\left[\frac{L - x}{L} - \frac{2}{\pi}\sum_{m=1}^{m=\infty}\frac{1}{m}\cdot\exp\left(-m^2a^2\pi^2t/L^2\right)\sin\frac{m\pi x}{L}\right] \qquad \text{Eq (10-16)}$$

where m is an integer and $a = C/K_p$.

Eq (10-16) is known to conform closely with experimental observations. The chief difficulties in applying Eq (10-16) are due to determination of the constants C and K_p. These are functions of many variables including temperature, packing voids, and particle size.

Experimental Work—The nature of heat flow through a packing was studied by Patten (1909), using different materials with varying moisture content. The critical constants C and K_p contained in Eq (10-16) were carefully evaluated. Patten used a water bath of 100 deg C at one end of a box containing the packing, the porosity of the latter being adjusted in a uniform manner. Temperatures were taken along the center of the box. The changes of temperature with time for various distances along the center line are shown in Figure 56. The data are for coarse quartz particles having a size-distribution as shown in Table 37. The distribution of temperatures along the center line of the test box after a period of 20 min is plotted in Figure 57. It may be seen from this figure that the temperature is less than 50 deg C at a distance of 2 cm from the heated face, thus showing that heat is conducted very slowly.

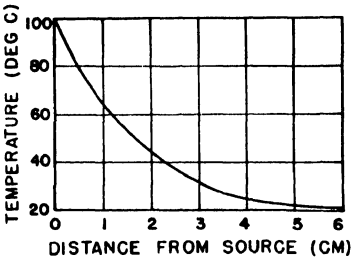


FIGURE 57 TEMPERATURE GRADIENT FOR AIR-DRY QUARTZ PARTICLES ALONG CENTER LINE OF CONTAINER, WHEN RATE OF TEMPERATURE RISE AT EVERY POINT IS CONSTANT.

TABLE 37—SIZE-DISTRIBUTIONS OF COARSE AND FINE QUARTZ PARTICLES USED BY PATTEN (1908)

Particle-diameter (mm)	Percent by weight in each size	
	Coarse quartz	Fine quartz
1–0.5	14.74	1.01
0.5–0.25	58.22	0.09
0.25–0.1	26.11	0.16
0.1–0.05	0.61	8.00
0.05–0.005	...	80.47
<0.005	...	10.84

Patten studied the change on heat conductivity C, diffusivity C/K_p, and specific volume (the reciprocal of the apparent density) of various packings with varying moisture contents. Figure 58 shows some of the data obtained by Patten on coarse and fine quartz. The curves, with the

exception of conductivity, are characteristic of most packings. With increasing moisture content the heat conductivity increases rapidly and approaches that of water. On the other hand, the diffusivity rises to a maximum and apparently is a function of particle-size. The diffusivities are calculated graphically from data using increment forms of the equation

$$\text{Diffusivity} = \frac{C}{K_p} = \frac{\partial T/\partial t}{\partial^2 T/\partial x^2} \qquad \text{Eq (10-17)}$$

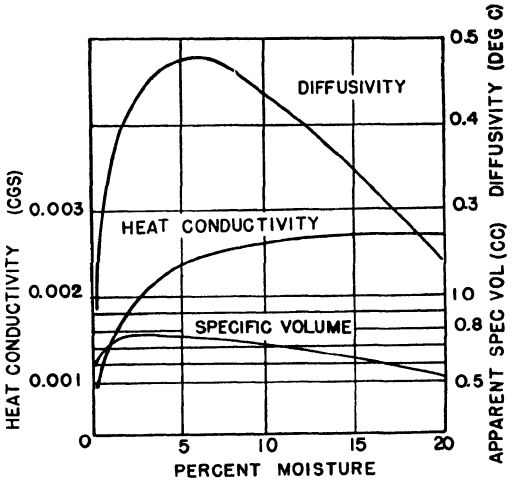


FIGURE 58. EFFECT OF MOISTURE ON HEAT CONDUCTIVITY, APPARENT SPECIFIC VOLUME, AND DIFFUSIVITY OF QUARTZ WHOSE SIZE-DISTRIBUTION IS GIVEN IN TABLE 37.

The determinations are based on values of T at a point 5 cm from the heat source. Numerically, these values are the "diffusivity rate of rise of temperature at 5 cm from the heat source per degree change in temperature gradient."

The heat-conductivity data obtained by Patten were by no means uniform in character. This may be attributed to a number of factors, but particularly to the nature of the packings used. Conductivity when plotted against moisture content as illustrated in Figure 58 was linear, concave upward or concave downward, depending upon the material used.

Relation between Particle-Size and Thermal Properties—The only data available giving the relation between particle-size and thermal properties of packings are those obtained by Patten (1909) on carborundum

of different degrees of fineness. His data are presented in Table 38. It may be seen from the table that diffusivity and heat conductivity follow the same pattern and have two maxima. The specific volume decreases rapidly up to 0.09 mm average diameter, and then is constant. The second maximum of the upper curves occurs while the specific volume remains constant. Since the material was relatively free from moisture, it must be assumed that the differences are attributable not so much to particle-size as to the possibility that the samples tested were not from the same source, and therefore inherently different.

TABLE 38—EFFECT OF PARTICLE-SIZE ON THERMAL PROPERTIES OF CARBORUNDUM

Average diameter of particle (mm)	Apparent specific volume (cc/g)	Period required for rise of 6.3 deg C at 5 cm from heat source (min)	Conductivity (cal per sq cm per sec per deg change in temperature gradient)	Diffusivity (rate of rise of temperature at 5 cm from heat source per deg change in temperature gradient)
0.60	0.584	24.5	0.00214	0.175
0.45	0.548	24.7	0.00225[a]	0.280[a]
0.30	0.559	27.9	0.00192	0.243
0.175	0.659	24.6	0.00197	0.213
0.125	0.601	24.3	0.00219[a]	0.217[a]
0.090	0.596	26.3	0.00216	0.256
0.075	0.625	29.7	0.00175	0.220
0.015	0.945	29.9	0.00112	0.185
0.006	1.066	36.0	0.000763	0.163

[a] Peak values.

HEAT TRANSFER

If a hot fluid is made to flow through a packing at lower temperature, heat will be transferred to the packing. If the latter is contained in a well-lagged pipe so that losses to the surroundings were kept at a minimum and thus rendered negligible, we may develop a differential equation whose solution gives the temperature of the fluid or packing at any point or time.

Let us suppose that the pipe containing the packing is uniform in cross section, and that its axis lies wholly in the direction x. If A denotes the area of the pipe and we consider a section between x and $x + dx$, then at times t and $t + dt$ the heat content of the packing is given by the expression

$$K_p T_p A \vartheta dx - K_p\left(T_p + \frac{\partial T_p}{\partial t}\,dt\right)A\vartheta dx = K_p A \vartheta \frac{\partial T_p}{\partial t}\,dt dx$$

where K_p is the specific heat of the packing, ϑ the fractional voids, and T_p the temperature of the packing.

At the same time t, we have for the heat content of the fluid $K_p'T_f A\vartheta dx$, and at time $t + dt$

$$K_p'\left(T_f + \frac{\partial T_f}{\partial t}\,dt\right)A\vartheta dx$$

where K_p' is the specific heat of the fluid and T_f its temperature. But with the fluid crossing the plane at x, the amount of heat transferred in time dt is $K'_pT_fA\vartheta vdt$, v being the velocity through the packing. At the same time the heat leaving at $x + dx$ is

$$K_p'\left(T_f + \frac{\partial T_f}{\partial x}\,dx\right)A\vartheta vdt$$

The amount of heat conveyed to the packing is thus

$$-K_p'\left(\frac{\partial T_f}{\partial t} + v\,\frac{\partial T_f}{\partial x}\right)A\vartheta dtdx$$

From Newton's law of heat transfer, the heat transferred into the element Adx in time dt is

$$C(T_f - T_p)A\vartheta dtdx$$

where C is the coefficient of heat conductivity. Hence, we obtain the simultaneous differential equations which must be solved:

$$\left.\begin{aligned} K_p\frac{\partial T_p}{\partial t} &= C(T_f - T_p) \\ K_p'\left(\frac{\partial T_f}{\partial t} + \frac{\partial T_f}{\partial x}\right) &= C(T_p - T_f) \end{aligned}\right\} \qquad \text{Eq (10-18)}$$

and these are independent of the porosity of the packing.

By writing $a = C/K_p'v$ and $b = C/K_pv$ and introducing the new variables $m = b(vt - x)$, $n = ax$, Eq (10–18) becomes

$$\left.\begin{aligned} \frac{\partial T_p}{\partial m} &= T_f - T_p \\ \frac{\partial T_f}{\partial n} &= T_p - T_f \end{aligned}\right\} \qquad \text{Eq (10-19)}$$

Solutions of this equation are given by Bateman (1944) and Furnas (1930).

The transfer of heat from a fluid stream to a column of irregular particles was studied by Furnas (1930*a*, *b*). A closely allied problem concerning the transfer of heat from the container surface to gas flowing through the packing was studied by Colburn (1931).

It is well known that the transfer of heat from the walls of a cylinder to a fluid flowing within it varies as the 0.8 power of the mass velocity

The general equation obtained by Furnas for the coefficient of heat transfer from a gas to the particles of a packing through which it flows is

$$C = \frac{mq^{0.7}T^{0.3}10^{(1.68\vartheta - 3.56\vartheta^2)}}{d^{0.9}} \qquad \text{Eq (10-20)}$$

where

C = coefficient of heat conductivity measured in heat units per unit time per unit of packing volume per deg temperature difference; usual units are cal per sec per deg C per cc

m = a constant; for limestone packings, m = 0.0073; for iron ore, 0.0105; anthracite and other coals, 0.0050; blast-furnace charge, 0.0072. The values are in metric units.

q = volume of gas flow per unit-time per square unit of cross-sectional packing area, expressed as liters per sec per sq cm

T = temperature, deg abs, C

ϑ = fractional voids (dimensionless)

d = diameter of particles, cm

Furnas also derived equations for conditions simulating blast-furnace practice. These equations permit the calculation of the steady-state temperature at any point within a column through which the packing and a hot gas move counter to each other. Furnas assumed that the solids entering the column were at zero deg C. His equations giving the temperature at any point in the column x-distant from the bottom are:

$$\ln\left(\frac{T' + bT_{sx}}{T'}\right) = \frac{bC}{K_{ps}q_s}(L - x) \qquad \text{Eq (10-21)}$$

$$T_s = T_g\left[\frac{1 - \exp(-bCL/K_{ps}q_s)}{\theta' - \exp(-bCL/K_{ps}q_s)}\right] \qquad \text{Eq (10-22)}$$

The latter equation gives the temperature of the solid at the bottom of the furnace when the solid fed at the top of the column of height L has a temperature of zero. The meanings of the terms included in these equations are as follows:

$$T' = T_g - \frac{K_{ps}q_s}{K_{pg}q_g}T_s = T_g - \theta' T_s; \quad \theta' = \frac{K_{ps}q_s}{K_{pg}q_g}$$

$$b = \frac{K_{ps}q_s}{K_{pg}q_g} - 1 = \theta' - 1$$

T_{sx} = temperature of solid at any point x

T_s = temperature of solid at bottom of column (exit)

T_g = temperature of gas at bottom of column (entrance)

C = value calculated by means of Eq (10-20)

q_s = rate of solid flow in volume of solid material per unit-cross-sectional-area of packing per unit-time

q_0 = rate of gas flow, standard volume per unit-cross-sectional-area of packing per unit-time
K_{ps} = heat capacity of solid per unit-volume per deg
K_{pg} = heat capacity of gas per unit-standard-volume per deg

Thus, when T_s is known, T' and b may be calculated and the temperature of the solid at any point determined by means of Eq (10–21). Corresponding equations were also developed by Furnas giving the temperature at any point in the column, thus

$$T_{gx} = T_g - \theta' (T_s - T_{sx}) \qquad \text{Eq (10-23)}$$

$$\ln \left(\frac{T' + bT_{gx}}{T' + bT_g} \right) = - \frac{bCx}{K_{ps} q_s} \qquad \text{Eq (10-24)}$$

where T_{gx} is the temperature at any point x in the column. Therefore we see that Eqs (10-20) to (10-24) define completely the thermal conditions existing in a packed column with a counterflow of solids and gas.

Colburn's studies were made with 3-in. and $1^1/_4$-in. diameter cylinders provided with steam jacket. The tubes were approximately 4.5 ft in length. The large tube was packed to a depth of 3.8 ft and the smaller to 1.5 ft. The heat transfer area in each case was 3.06 and 0.539 sq ft respectively. The packing material consisted of various substances including pebbles, porcelain balls, and zinc balls, ranging from approximately 0.2 to 1 in. in diameter. Colburn derived the following general equation for heat transfer for turbulent flow

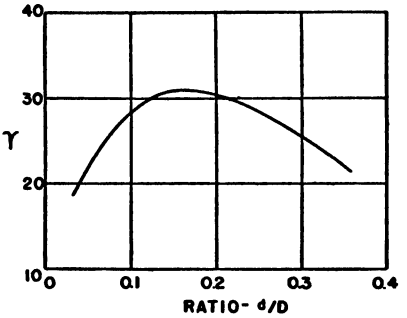


FIGURE 59. RELATION OF WALL FACTOR TO DIAMETER OF PARTICLES AND CONTAINER.

$$H_t = K'' \gamma K_p \mu^{0.2} G^{0.8} \qquad \text{Eq (10-25)}$$

where γ is a constant which is a function of the ratio of particle-diameter d to the container-diameter D, K_p the specific heat of the gas, μ the viscosity, and G the mass velocity. The value of K'' depends on the material contained in the packing. Values of the factor γ, which is essentially a wall factor, are shown in Figure 59. The significance of this wall factor has been discussed in Chapter 6. The curve of Figure 59 is to correct for the nonuniformity of packing next to the tube wall. Eq (10-20) was tested for air, using mass velocities ranging from 0.2 to 2.6 lb per sq ft per sec. Experimental results obtained were about eight times as high as would be expected for the same rate through the empty

container. Colburn obtained a value of $K'' = 8.0$ for most packings, when G is expressed in lbs per sq ft of empty tube cross-sectional area per sec and μ in centipoises.

The work of Furnas above described and those of others were re-examined by Lovell and Karnofsky (1943). Their mathematical studies were confined to packings composed of spheres of uniform diameter and made allowance for resistance to heat transfer by conduction within the solids. The equation obtained by Lovell and Karnofsky, without correcting for resistance, is of the form encountered in the design of continuous heat exchangers,

$$\ln\left[\frac{T_{gx} - T_{so}}{T_g - T_{so}}(1 - \theta) + \theta\right] = \frac{6C_t t}{K_{ps}\rho d} \cdot \frac{1 - \theta}{\theta} \quad \text{Eq (10-26)}$$

where T_{so} is the temperature of the spheres entering the exchanger, $\theta = K_{pg}q_g/K_{ps}q_s$, ρ the density of the packing in the exchanger, t the time measured from the entrance of a sphere to the section x, C_t the coefficient of heat transfer in heat units per unit time per unit area per deg temperature difference, K_{pg} and K_{ps} the specific heat of the gas and sphere respectively measured in heat units per unit weight per deg temperature difference, and other terms are as previously defined. From this equation we may solve for T_{gx} and obtain an approximate solution for the temperature of the gas in a packing at any section x-distant from the point at which the spheres enter. To correct for resistance to heat transfer by conduction in the spheres, it is best to solve Eq (10-26) for

$$\frac{T_{gx} - T_{so}}{T_g - T_{so}}$$

and to subtract from this ratio the product

$$\frac{5tC_s}{K_{ps}\rho d^2} \cdot \frac{T_{gx} - T_{so}}{T_g - T_{so}}$$

where C_s is the conductivity of the solid in heat units per sq ft per hr per (deg temperature difference per length).

Values of T_{gx} computed by this procedure will conform closely to the theoretical values set forth by Lovell and Karnofsky.

Problems

1. Limestone having an average diameter of 3 cm is packed in a 100-cm cylindrical column. The voids are calculated to be 34.5 percent. Air having a temperature of 320 deg C is made to flow through the column at a rate of 500 liters per sec. Determine the coefficient of heat conductivity.

CHAPTER 11

SURFACE PROPERTIES OF PARTICLES

We shall now discuss a subject of great importance to the technology of fine particles—surface energy and the relation it bears to heat produced by particles on adsorption, wetting of particles, and other diverse phenomena which have found practical applications, such as flotation of ores and minerals. Information on this subject is still limited, but sufficient is known to permit explanation of the behavior of particulate matter.

If we concern ourselves with a liquid we become aware of the tautness (or state of tension) on its surface by noting that dust sprinkled lightly on the surface remains there even though its density is greater than that of the liquid. This tension is referred to as surface tension and can be shown to be characteristic of solids and gases as well as liquids. Surface tension, σ, is defined as the force with which the surface on one side of a line one cm long pulls against that on the other side of the line.

SURFACE ENERGY

The term "surface energy" means simply the following, or its equivalent: If a strip of surface one cm in width is stretched one cm longer, the work done is measured by the contractile force or surface tension, σ, multiplied by the distance it is drawn out. The increase of surface by one sq cm is work done, or surface energy expended. This energy is stored in the surface and is released when the surface contracts. Therefore, we conclude that the molecules of a liquid near the surface possess more energy than those in the interior. Furthermore, from what has been said, the increase in energy in *ergs per sq cm* of surface is numerically equal to the surface tension in *dynes per linear cm.*

Surface tension of a liquid or solid is markedly affected by temperature or cleanliness so that values observed must take these variables into account.

Contact Angle—Suppose a clean glass plate is held vertically in water. The water will rise on both sides of the glass plate, and the free surface of the water is increased in spite of the contractile force—surface tension. The expanding force or *negative* tension is due to the attraction between the water and the glass. The condition described is shown for one side

of the glass plate in Figure 60. Let σ_{WG} represent the negative tension between the water and the glass plate and σ_{WA} the surface tension of the water-air interface. Then for equilibrium we must have

$$\sigma_{WG} = \sigma_{WA} \cos \theta \qquad \text{Eq (11-1)}$$

where θ is the angle at which equilibrium takes place and is usually referred to as the *contact angle* for the two substances involved.

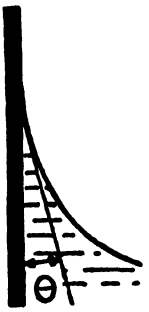


FIGURE 60. ADHESION OF LIQUID TO SOLID.

If kerosene is used instead of water the negative tension is much greater and hence equilibrium cannot be achieved. Thus, the contact angle is 0 deg and the oil spreads over the whole surface of the glass. On the other hand, if the glass plate is held vertically in a pool of mercury, the mercury curves downward instead of upward at the line of contact. The contact angle in this case is about 140 deg so we conclude that the surface between mercury and glass has a positive or contractile tension.

Thus far we have merely touched on the various forces entering into play with regard to surface phenomena. Suppose, as in Figure 61, the glass plate is held in a horizontal position and that a liquid denoted by L makes a contact angle θ with it. Let A denote the air and S the glass plate (solid), and write the following symbols to denote the interfacial tensions:

σ_{SA}	surface tension solid-air
σ_{LA}	surface tension liquid-air
σ_{SL}	surface tension solid-liquid

If now by any suitable means we attempt to separate the solid from the liquid in a direction perpendicular to each other, then work w_{SL} will be done. The amount of this work (work of adhesion) is

$$w_{SL} = \sigma_{SA} + \sigma_{LA} - \sigma_{SL} \qquad \text{Eq (11-2)}$$

There is no known method for determining directly the surface tension of solids against gases. However, we can determine the difference $\sigma_{SA} - \sigma_{SL}$ which is called the *adhesion tension* in Eq (11-2) by resolving the surface tension parallel to the solid surface. Thus, for equilibrium

$$\sigma_{SA} = \sigma_{SL} + \sigma_{LA} \cos \theta \qquad \text{Eq (11-3)}$$

and

$$w_{SL} = \sigma_{LA} (1 + \cos \theta) \qquad \text{Eq (11-4)}$$

This last equation is susceptible of interesting interpretations: (*a*) It shows that θ is determined by the relative strengths of the liquid to the solid and to the air. (*b*) If $\theta = 0$, $w_{SL} = 2\sigma_{LA}$, so that the liquid attracts the solid as much as it does itself. Hence the attraction of the liquid for itself is $2\sigma_{LA}$. (*c*) If $\theta = \pi/2$, the attraction of the liquid for the solid is half that for itself. And, (*d*) if $\theta = \pi$, there is no adhesion between the liquid and the solid.

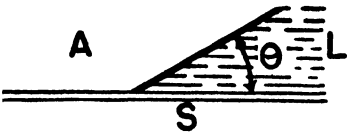


FIGURE 61. LIQUID-SOLID-AIR INTERFACES.

Therefore, we see that the contact angle is an important quantity. Yet unfortunately this is difficult to determine since it is not a constant quantity even for a given liquid and solid. Its value is a function depending upon whether the liquid advances on a dry surface or recedes from a previously wetted one. Adam (1941) attempted to formulate the concept of frictional force involved in an effort to obtain an explicit relationship. If $\mathfrak{R}$ denotes this frictional force and it is assumed to operate with equal intensity when advancing or receding motions are just prevented, we have for the former

$$\sigma_{SA} - \sigma_{SL} = \sigma_{LA} \cos \theta_A + \mathfrak{R}$$

and for receding motion

$$\sigma_{SA} - \sigma_{SL} = \sigma_{LA} \cos \theta_R - \mathfrak{R}$$

where θ_A and θ_R indicate contact angles for advancing and receding motions, respectively. From these equations we obtain a value of the equilibrium contact angle θ

$$\cos \theta = \frac{\cos \theta_R + \cos \theta_A}{2}$$

and

$$2\mathfrak{R} = \sigma_{LA}(\cos \theta_R - \cos \theta_A)$$

whence $\mathfrak{R}$ is determined.

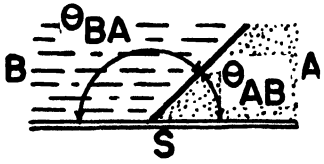


FIGURE 62. LIQUID-LIQUID SOLID INTERFACES.

Contact Angle of Solid at Interface of Two Liquids—One other situation of practical importance should be considered, namely the contact angle of two liquids and a solid. Let one of the liquids be designated by *A* and the other by *B*, and the contact angle between them by θ_{AB} (Figure 62). Resolving our surface tensions, as above, we obtain, as in Eq (11-3), the adhesion tensions

$$\sigma_S - \sigma_{SA} = \sigma_A \cos \theta_{SA} \qquad \text{Eq (11-5)}$$

$$\sigma_S - \sigma_{SB} = \sigma_B \cos \theta_{SB} \qquad \text{Eq (11-6)}$$

and at equilibrium

$$\sigma_{SB} - \sigma_{SA} = \sigma_{AB} \cos \theta_{AB}$$

The work of adhesion between the two liquids and the solid may be written

$$w_{SA} = \sigma_A + \sigma_S - \sigma_{SA}$$

$$w_{SB} = \sigma_B + \sigma_S - \sigma_{SB}$$

where the subscripts A, B, and S apply to the surface energy of the component referred to. Thus,

$$w_{SA} - w_{SB} = \sigma_A - \sigma_B + \sigma_{AB} \cos \theta_{AB} \qquad \text{Eq (11-7)}$$

If the fluid B is air, σ_B and w_{SB} are both zero and $\sigma_{AB} = \sigma_A$. Thus, Eq (11-7) reduces to Eq (11-3) under such conditions.

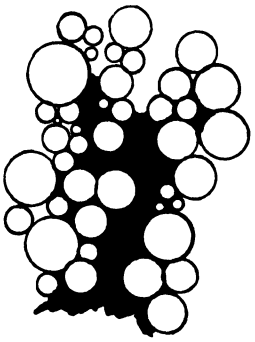


FIGURE 63. PARTICULATE CAPILLARITY.

It is well known that under certain conditions one liquid will displace another from a particulate mass (in a tube, for example). This will occur when the contact angle made by the liquid-liquid interface with the solid is acute, the direction of displacement being toward the concavity. This is shown in Figure 63, illustrating a capillary formed of particles. The direction of movement will be as indicated as long as the angle of contact is small. The conditions for this situation are set forth by Eq (11-5), and θ_{AB} is acute as long as

$$w_{SA} - \sigma_A > w_{SB} - \sigma_B$$

The discussion just given again emphasizes the important role played by contact angle and adhesion tension. However, while the theory involved is simple and direct, we must have at our disposal prior knowledge of the contact angle values. These must be obtained experimentally—and often, as in the case of granular materials, we must be content with approximate determinations. As a matter of fact there is no completely satisfactory way of determining the contact angle of granular material. We shall here present an ingenious method due to Bartell and Osterhof (1927) which is one of the best, though not without limitations. Referring to Figure 64, we have a short cylinder 3 in. in length and 1 in. in diameter, capable of withstanding high pressures. The particles wetted with the liquid to be displaced are put in the end marked A and packed to a pressure of 150

atm. The second layer, B, consists of the same particles wetted with the displacing liquid or gas and compressed to the same pressure. The cylinder is then sealed and end B connected by a capillary to a manometer capable of measuring the amount of displacement. The other end, A, is connected to a manometer for measuring the equilibrium pressure, this condition being reached shortly after connections to the manometers are made. In essence we have this: If the contact

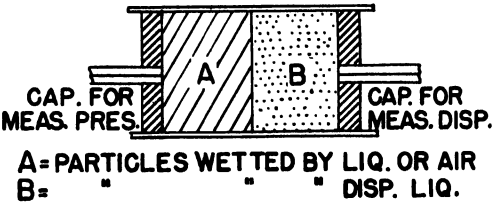


FIGURE 64. METHOD OF MEASURING WETTABILITY OF PARTICLES (BARTELL AND OSTERHOF, 1927).

angle is less than $\pi/2$, liquid enters through the capillary at the B-end, and a pressure p will be built up at the A-end, which will be proportional to the surface tension of the liquid and the cosine of the angle, a situation analogous to that expressed by Eq (11-1).

Now for any capillary of diameter D_c (and in a packing we have many such capillaries) we shall show in Chapter 15 (Eq 15-2) that

$$\cos\theta = \frac{D_c\rho_0 gh}{4\sigma} = \frac{981pD_c}{4\sigma} \qquad \text{Eq (11-8)}$$

In our particular problem, using liquids A and B we have $\theta = \theta_{AB}$ and $\sigma = \sigma_{AB}$. The pressure p is measured in g per sq cm and D_c in cm.

The adhesion tensions of the particles, at the interface of two liquids A and B are easily written down and are precisely those of Eq (11-5). Let Σ denote the adhesion tension, then these equations become

$$\Sigma_{SA} = \sigma_A \cos\theta_{SA},\ \theta_{SA} > 0$$

$$\Sigma_{SB} = \sigma_B \cos\theta_{SB},\ \theta_{SB} > 0$$

Subtracting

$$\Sigma_{SA} - \Sigma_{SB} = \sigma_{SB} - \sigma_{SA}$$

and from our equation of equilibrium, Eq (11-6)

$$\sigma_{AB}\cos\theta_{AB} = \sigma_{SB} - \sigma_{SA}$$

so that

$$\Sigma_{SA} - \Sigma_{SB} = \sigma_{AB}\cos\theta_{AB} \qquad \text{Eq (11-9)}$$

These equations together with our capillary equation Eq (11-7) suffice to compute several important constants, as will now be shown.

Carbon tetrachloride (CCl_4) is known to displace water from carbon. Using data of Bartell and Osterhof, let A denote the water phase and B the CCl_4, and let the carbon be identified as S so that the above equations may be used without difficulty. The following data are known, being determined by other methods: $\sigma_{SA} = 0$, $\sigma_{SB} = 43° 35'$, $D_c = 18.6 \times 10^{-6}$ cm, $\sigma_{AB} = 44.54$ dynes per cm and p (from experiment) = 6935 g per sq cm.

TABLE 39 --DISPLACEMENT PRESSURES FOR ORGANIC LIQUID-WATER-SOLID INTERFACE SYSTEMS. (BARTELL AND OSTERHOF, 1927)

Organic liquid	Surface tension (dynes per cm)	Interfacial tension—Organic liquid–water (dynes per cm)	Displacement pressure (g per sq cm)
Organic liquid displaces water from carbon black[a]			
Aniline	43 40	6 00	1266
Carbon tetrachloride	26.12	44.54	6935
Hexane	17 82	50.25	3330
Decaline	31.00	26.68	4745
Benzene	28 25	34.65	5775
Water displaces organic liquid from silica[b]			
Aniline	43.40	6.00	8
Carbon tetrachloride	26.12	44.54	409
Hexane	17.82	50.25	395
Benzene	28 25	34.65	295

[a] Diameter of packing pores = 18.6×10^{-6} cm.
[b] Diameter of packing pores = 4.2×10^{-4} cm.

Hence, using Eq (11-8)

$$\cos \theta_{AB} = \frac{6935 \times 981 \times 18.6 \times 10^{-6}}{4 \times 44.54} = 0.7103$$

and

$$\Sigma_{SA} - \Sigma_{SB} = \sigma_{AB} \cos \theta_{AB} = 0.7103 \times 44.54 = 31.63 \text{ dynes per cm}$$

If $\sigma_B = 72.08$ (from tables) we have, since $\Sigma_{SB} = \sigma_B \cos \theta_{SB}$, that

$$\Sigma_{SB} = \cos 40° 35' \times 72.08 = 54.74 \text{ dynes per cm}$$

Also

$$\Sigma_{SA} = \Sigma_{SB} + \sigma_{AB} \cos \theta_{AB}$$
$$= 54.64 + 31.63 = 86.37 \text{ dynes per cm}$$

Certain experimental data obtained by Bartell and Osterhof are presented in Tables 39 and 40. These apply to finely divided carbon black

and silica and illustrate the surface forces affecting the displacement of liquids from solids. In Table 39 it will be seen that the organic liquids displace water from carbon black, whereas in the case of finely divided silica the opposite is true. Table 40 gives the adhesion tensions computed as indicated above.

Remarks on Adhesion Tension—It is clear that adhesion tension (or the work of adhesion) measures the degree of wetting of a solid by a liquid. A high interfacial tension in a solid-liquid system indicates a low adhesion tension and conversely a low interfacial tension is associated with a high adhesion tension.

Among other things, a suspension of particles will be most stable in a liquid having the highest adhesion tension for the solid. Referring to Table 40 we see that the adhesion tension for silica in water and in ani-

TABLE 40—SOLID-LIQUID ADHESION TENSIONS. (BARTELL AND OSTERHOF, 1927)

Liquid	Silica		Carbon Black	
	Displacement pressure (g per sq cm)	Adhesion tension (dynes per cm)	Displacement pressure (g per sq cm)	Adhesion tension (dynes per cm)
Water		82.82		54.74
Aniline	8	82.00	1266	60.51
Carbon tetrachloride	409	40.69	6935	89.45
Hexane	395	42.13	3330	69.93
Decaline			4745	76.38
Benzene	295	52.43	5775	81.08

line is high. These suspensions are stable and will give little evidence of flocking. Again, a suspension in a system of two immiscible liquids will tend to go to that liquid giving the higher adhesion tension with the solid.

In a tightly packed particulate system with two liquids present, that liquid giving the higher adhesion tension with the particles will tend to displace the other liquid.

ADSORPTION BY PARTICLES

Any change in the surface of a particle by liquid or gas entails the release of energy. For, as we have shown, the alteration of surface tensions involves work. Thus, we may expect that a mass of fine particles will release energy when wetted either by a liquid, a vapor, or a gas. This release of energy manifests itself by an increase in temperature of the wetted mass of particles—that is, in the formation of heat. The magnitude of these effects will be discussed later. At this point we shall present the theory of adsorption with which heats of wetting are allied.

Since the last quarter of the eighteenth Century, it has been known that finely divided matter is capable of adsorbing vapors and gases. This phenomenon has engaged, and continues to engage, the foremost students of physical chemistry. The fundamental relationships concerned are far-reaching, involving not only particle-size but the nature of the particle-surface, surface-energy, heats of adsorption, and many other phenomena. The concepts involved furnish the laws governing adsorption and evaporation for a given material and gas or vapor. The intricate and complex nature of adsorption and the theories pertaining to its mechanism are beyond the scope of this text. We shall consider only the more obvious and simpler phases of the subject, especially regarding particle-size.

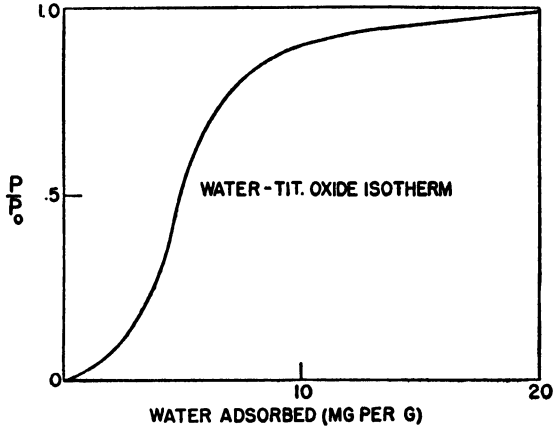


FIGURE 65. ADSORPTION ISOTHERM. WATER VAPOR ON TITANIUM OXIDE (25 DEG C).

Theory of Adsorption—If at a given temperature we obtain by experiment the amounts of vapor adsorbed by a unit weight of small particles at various vapor pressures (up to the maximum vapor pressure p_0) and if we plot the data so obtained we shall have a sigmoid curve similar to that shown in Figure 65. This curve indicates the volume of water vapor adsorbed by a grain of titanium oxide at 25 deg C at various ratios of p and p_0. If a temperature other than 25 deg C had been used, another curve would have been obtained. Because such adsorption curves as shown in Figure 65 were investigated by van der Waals, they are appropriately called van der Waals' adsorption isotherms. From such isotherms, as will be shown subsequently, may be determined particle surface and certain surface energy constants of practical value.

A possible theory concerning adsorption was developed by Langmuir

(1915), and recently extended by Brunauer *et al* (1938). The extent of adsorption is dependent upon the rate of evaporation and reactions taking place at the interface. (See section on hygroscopicity, Chapter 14). Thus, if the rate of adsorption is low and the surface is chemically inactive, the molecules will remain on the surface; while if the rate of evaporation is high, the molecules will leave the surface quickly. Langmuir's theory is that adsorption is due to the difference of time required for condensation and evaporation of the molecules on a surface (assumed chemically inactive), and that actually not more than one layer of gas molecules is adsorbed. For this reason it is commonly referred to as the monomolecular theory in contradistinction to the polymolecular theory of Brunauer and his associates. However, the latter theory does not necessarily assume that all adsorbed gases on different solids are polymolecular or that every layer is necessarily complete. It is a matter of general agreement that most nonporous solids adsorb more than one layer of molecules.

The area s_m occupied by a molecule of a gas in closest packing on a surface is

$$4 \times 0.866 \left(\frac{M}{4\rho_0 \sqrt{2} N_A} \right)^{2/3} \qquad \text{Eq (11-10)}$$

where M is the molecular weight of the gas adsorbed, ρ_0 its density and N_A Avogadro's number. The value of s_m is, of course, expressed in sq cm. Some values of s_m computed from this equation are presented in Table 41. Hence, if we know the amount of any vapor listed in

TABLE 41—AREA OCCUPIED BY MOLECULES OF VARIOUS GASES AND VAPORS IN CLOSE PACKING

Vapor of	Temperature (deg C)	Area occupied by single molecule (sq. A°)
Liquid nitrogen	−195 8	16 24
Solid nitrogen	−252 5	13 8
Liquid butane	0	32 1
Liquid *N*-heptane	25	45 0
Water	25	10 0

Table 41 which has been adsorbed at the temperature stipulated, and if we assume that the layer adsorbed is one molecule thick, we are in a position to compute the area of an irregular solid.

Brunauer and his associates have developed from their theory of the adsorption isotherm the volume of a gas necessary to form a monomolecular layer on a mass of particles. Their equation follows:

$$Q = \frac{Q_m C p}{(p_0 - p)[1 + (C - 1)p/p_0]} \qquad \text{Eq (11-11)}$$

where Q is the volume of gas or vapor adsorbed at the pressure p, Q_m the volume of gas or vapor necessary to form a monomolecular layer, and p_0 the vapor pressure of the gas or vapor at the temperature of the isotherm. The quantity C is given by the expression

$$C = \exp\,[(E_1 - E_L)/BT] \qquad \text{Eq (11-12)}$$

where E_1 is the energy of vaporization of the first layer of molecules, E_L the energy of vaporization of the liquid, T the absolute temperature at which adsorption takes place, and B the gas constant in such units as to render the ratio $(E_1 - E_L)/BT$ dimensionless. In Chapter 16 we shall indicate how Eq (11-10) may be arranged so that C and Q_m are directly determinable.

When Q_m is determined from Avogadro's number and the area occupied by a single molecule (Eq 11-10), we have a means of computing the area of a mass of irregular particles. Thus, if Q_m' is the cu cm of gas or vapor adsorbed per g of particles, N_A the number of molecules per cc of gas adsorbed, and s_m the area in sq cm occupied by each molecule, the total surface S of the particles must be

$$S_m = Q_m' N_A s_m \qquad \text{Eq (11-13)}$$

The basic theorem of adsorption may be stated as follows: Adsorption on a solid by a vapor continues as long as the process can occur with a decrease in the surface energy—free energy—of the solid surface. If π denotes this decrease in free energy per unit area, we have symbolically

$$\pi = \sigma_{SO} - \sigma_{SF} \qquad \text{Eq (11-14)}$$

where σ_{SO} denotes the free surface energy of a clean solid or system of particles, and σ_{SF} the same quantity when the surface is covered by an adsorbed film. (Free surface energy is the same quantity as surface tension, discussed earlier. It cannot, of course, be measured directly.) The decrease in free surface energy π can be computed by a method outlined by Jura and Harkins (1937) using the Gibbs adsorption equation

$$\pi = \sigma_{SO} - \sigma_{SF} = \frac{BT}{Q'S}\int_0^P \frac{Q}{p}\,dp \qquad \text{Eq (11-15)}$$

where S is the surface of the particles per g, Q the volume of the gas adsorbed per g of particles at pressure p and absolute temperature T, Q' the molar volume of the gas, and B the gas constant, its units being such as to render π in ergs per cm^2. The above equation may be integrated graphically from adsorption isotherm data by plotting Q/p against p. Thus, π for any vapor solid system for any value of p/p_0 may be obtained.

Once π has been computed for a saturated vapor of a liquid in equilibrium with a given solid, we can determine the work of adhesion between the liquid and solid as

$$w_{SL} = \pi_a + \sigma_L (1 + \cos \theta) \qquad \text{(Eq 11-16)}$$

where π_a is the reduction in free surface energy of the solid in equilibrium with the saturated vapor of the liquid. If $\theta = 0$, that is, if the surface is completely wetted,

$$w_{SL} = \pi_a + 2\sigma_L = \sigma_S - \sigma_{SL} + \sigma_L \qquad \text{Eq (11-17)}$$

Also, we may write for p corresponding to π_a

$$p_a = \pi_a + \sigma_L \cos \theta \qquad \text{Eq (11-18)}$$

and for $\theta = 0$

$$p_a = \pi_a + \sigma_L = \sigma_S - \sigma_{SL} \qquad \text{Eq (11-19)}$$

The heat of *emersion* per unit area (H_e') of a solid in equilibrium with a vapor can be shown to be*

$$H_e' = \sigma_{SF} - \sigma_{SL} - T\left(\frac{\partial \sigma_{SF}}{\partial T} - \frac{\partial \sigma_{SL}}{\partial T}\right) \qquad \text{Eq (11-20)}$$

where T is the absolute temperature and the partial derivatives are taken at constant pressure and solid area. The adhesion tension

$$\sigma_{SF} - \sigma_L = \sigma_L \cos \theta \qquad \text{Eq (11-21)}$$

may be used to simplify the equation. Substituting σ_{SF} from this equation into Eq (11-20)

$$H_e' = \left(\sigma_L - T\frac{\partial \sigma_L}{\partial T}\right) \cos \theta + T\sigma_L \sin \theta \frac{\partial \theta}{\partial T}$$

Thus, when $\theta = 0$

$$H_e' = \sigma_L - T\frac{\partial \sigma_L}{\partial T} = H_L$$

which is the "total heat" of the liquid surface and is obtainable for most liquids from tables of σ_L and its variation with absolute temperature.

If the adsorbed film is in equilibrium with the vapor, it will be about evenly distributed over the solid surface. If the actual solid surface is denoted by S_m sq cm per g and that with the adsorbed layer by S_m' sq

* See, for example, Müller-Pouillet, *Lehrbuch der Physik*. Band III Pt. 1, Zweite Aufl, Braunschweig, 1926.

cm per g, it may be shown (Harkins and Jura, 1944*b*) that the latter is given by the relation

$$S_m' = 4.185 \times 10^7 H_e/H_L \qquad \text{Eq (11-22)}$$

where H_e is the heat of emersion of the film-covered solid (particles) in cal per g and H_L the total energy of the surface of the liquid. The difference between S_m and S_m' is slight and correction may be made by a procedure outlined by Harkins and Jura (1944*b*). An average thickness of adsorbed film is about 20 Å. Eqs (11-13) and (11-22) give results which are identical within the limits of experiment. The determination of H_e unfortunately requires an elaborate calorimeter and other equipment beyond the means of most laboratories.

Mention should be made of an important relation first brought out by Jura and Harkins (1944). It is simply that the adsorption isotherm is closely represented by the equation

$$\ln \frac{p}{p_0} = B - \frac{A}{Q^2} \qquad \text{Eq (11-23)}$$

where A and B are constants. Hence $\ln p/p_0$ against Q plots as a straight line, from which the constants A and B may be determined. The surface of a particulate adsorbent is further related to A as follows:

$$S_m = k/\sqrt{A} \qquad \text{Eq (11-24)}$$

where S_m is the surface per g of solid material and k a constant. Values of k when S_m is in sq m per g and Q in cu cm are given by the above investigators for the following gases: For nitrogen at -195.8 deg C, $k = 4.06$, for water at 25 deg C, $k = 3.83$, for *n*-butane at 0 deg, $k = 13.6$, and for *n*-heptane at 25 deg C $k = 16.9$. The usefulness of the procedure just outlined should not be underestimated. However, it cannot be used if the adsorption isotherm does not plot as a straight line in accordance with the development given.

It is of interest to note that since the number of molecules adsorbed per g of solid may be determined by the method of Brunauer and his associates, and the surface of the same solid by the technique just discussed, then the area occupied by a molecule is at once determined. For

$$N = Q_m' N_A$$
$$s_m = S_m/N \qquad \text{Eq (11-25)}$$

where N is the number of molecules adsorbed per g of solid, Q_m' the cu cm of gas adsorbed per g of solid, and N_A is Avogadro's number.

Heat of Adsorption—Some data on the heat of adsorption on zinc oxide by different gases and vapors are presented in Table 42. This

table should be compared with data on heats of wetting given in Table 45. It will be seen that the heat of adsorption of water is greater than the heat of wetting. There is, however, no relation between the heat of wetting and the heat of adsorption.

TABLE 42—HEAT OF ADSORPTION OF ZINC OXIDE USING VARIOUS LIQUIDS. (AVERAGE DIAMETER OF PARTICLES 0.607 μ, SPECIFIC SURFACE 1.8 SQ METER/G)

Gas or vapor	Pressure of gas or vapor (mm Hg)	Heat of adsorption (cal/g)	Heat of adsorption cal/sq m
Carbon dioxide	49	0.74	0.4
Ammonia	49	0.94	0.5
Sulfur dioxide	90	1.40	0.8
Benzene	95	0.10	0.5
Xylene	7	0.78	0.4
Pyridine	20	0.80	0.4
Water	23	2.40	1.3

So far as the adsorption of nitrogen by soils is concerned, Makower *et al* (1937) present data showing that from 0.36 to 0.98 cal are produced per g of soil. The soils tested had surface-areas ranging from 14.2 to 34.9 m per g. The amount was found to be directly proportional to the surface-area of the particles.

Determination of Heat of Adsorption—The heat of adsorption may be determined by a method due to Ewing (1931) whose work on the measurement of the heat of wetting has already been cited. The determination is made by a calorimeter in which the sample is placed and the adsorbent material admitted. If the adsorbent is a vapor, it is admitted through the device *a* of Figure 66, and if a gas by means of the device *b*. The iron rod which is raised by an external permanent magnet is used to break the tips of the tubes containing the adsorbent material.

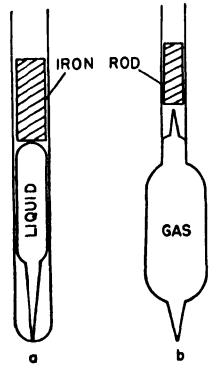


FIGURE 66. EWING'S DEVICE FOR INTRODUCING GASES AND VAPORS INTO A CALORIMETER FOR DETERMINING HEAT OF ADSORPTION

Adsorption by Silica Gel—The adsorptive qualities of silica gel are well known. This material is highly porous and possesses a vast amount of internal surface. When the structure becomes saturated with moisture, the gel may be reactivated by heating. Not all gels are alike in their capacity to adsorb vapors and gases, since it has been found that small amounts of impurities may greatly affect this quality.

Adsorption capacity for water vapor appears to be little affected at ordinary room temperatures and only slightly by velocity. It would be expected that the latter is important inasmuch as high rates of gas flow bring more adsorbable gas into contact with the gel. At the same time, owing to the greater heat capacity of the larger volume of gas, the heat of adsorption is more readily removed from the gel. Particle-size is perhaps the most important single variable since the surface exposed in a given amount of material is greater as the particle-size decreases. Also, smaller particles permit a more rapid diffusion of heat. However, there is an optimum size which is determined by the volume of gas made to pass through the gel without disturbing the packing, the resistance to flow encountered, and the frequency of reactivation required. It has been found that for air the amount of water vapor adsorbed in a given time may be represented by the empirical formula

$$M = 0.065\zeta L\sqrt{t}\,[(1.9 + 6.1d)(4.2 - 0.004v)]^{-1/2} \quad \text{Eq (11-26)}$$

where M is the water vapor adsorbed expressed in lb per cu ft of silica gel, ζ the relative humidity in percent, L the thickness of the bed in inches, t the time in min, d the average screen diameter in inches, and v the approach velocity in ft per min. The equation cannot be applied beyond the point at which saturation begins. It gives reasonably accurate results at ordinary temperatures and pressures for adsorbing periods up to 2 hrs. The data on which Eq (11-26) is based were obtained for humidities ranging from 15 percent to 70 percent, particle-diameters from 0.08 to 0.31 in., and air velocities from 40 to 400 ft per min.

Volume Adsorbed by Particles—The volume of gas adsorbed by particles is a function of the surface-area. Most soils at ordinary temperatures and pressures adsorb from 5 to 8 cc of nitrogen per g. Colloid fractions of these soils averaging less than 0.05 μ adsorb more than three times these amounts. It is interesting to point out that approximately 0.8 of the nitrogen is adsorbed in forming a monomolecular layer. With regard to activated charcoal and silica gel, the amounts of nitrogen adsorbed may be more than 20 times those mentioned above for soils.

Thickness of Adsorbed Film—Briggs (1905) determined the thickness of water vapor adsorbed by quartz particles. The surface was calculated from the mechanical analysis. By assuming that the water vapor was uniformly distributed over all the particles to the same thickness, Briggs obtained a maximum thickness of 2.66×10^{-6} cm in an atmosphere within one percent of saturation at 30 deg C. Patten and Gallagher (1908) repeated Briggs' experiments for a number of soils. Their results for Norfolk sand, quartz flour, and a special soil sepa-

rate are given in Table 43. The adsorption data for Norfolk sand and quartz flour are the maximum possible under the conditions of the experiment, but for the soil separate the data are for adsorption taking place over a period of eight days. The data in Table 43 assume that the film is evenly distributed over the particles, that the densities of the vapors adsorbed are those of the liquids themselves, that the soil grains are spheres, and that the separations are clean-cut.

More recent investigations by Makower *et al* (1937), cited above, show that the thickness of nitrogen films adsorbed by soils were of the order of 3.3×10^{-5} cm. On the other hand Harkins and Jura (1944*a*) report film thicknesses for water, nitrogen, and butane on titanium oxide of 15, 36, and 64 Å, respectively. The volume of gas adsorbed and the surface measurements made in these tests are the most accurate thus far obtained.

TABLE 43—ADSORPTION CAPACITIES OF DIFFERENT SOILS AND VAPORS AND THE THICKNESS OF ADSORBED LAYER. (TEMPERATURE 28 DEG C)

Soil	Area per g (sq cm)	Vapor	Vapor pressure (mm Hg)	Thickness of layer (cm $\times 10^{-5}$)
Norfolk sand	300	Water	28.1	1.50
		Toluene	33.5	0.94
		Ether	575.0	2.83
Quartz flour	1260	Water	28.1	0.64
		Toluene	33.5	0.36
Soil separate	320	Water	28.1	2.00
		Toluene	33.5	1.42
		Ether	575.0	3.25

The thickness of an adsorbed film may be determined by weighing the solid before and after adsorption. If we assume the density of the adsorbed film to be that of the liquid, and if the surface of the solid is known, the thickness δ is easily computed. Thickness will, of course, depend upon the amount of vapor or gas adsorbed as well as on the size of the vapor or gas molecule.

HEAT OF WETTING

It has long been known that heat is generated when fine particles are wetted. This is a surface phenomenon which does not necessarily depend upon chemical activity. The latter is important and markedly affects observations; but entirely apart from chemical action the creation of a film requires a certain amount of work which is expended in the form of heat.

The heat of wetting has been studied for many materials, with different liquids. Not all the data are accurate, and few even attempt to estimate the relation between the heat generated and particle-size. Studies made in the field of soil physics have provided many interesting results and some of the more important of these are presented here. Table 44

TABLE 44—HEAT OF WETTING OF VARIOUS SOILS, WETTED WITH WATER

Soil	Surface area per g (sq cm)	Heat evolved (cal/g)	Heat evolved (cal/sq cm)
Coarse quartz	97	0 150	0.00155
Fine quartz	740	0.386	0.000522
Norfolk sand	300	0.347	0.00116
Hudson River sand	345	0.179	0.000519
Galveston clay	3280	3.79	0.00116

shows the heat of wetting obtained by Patten (1909) for various soils, using water. The data show that there is little concordance between the various soils These soils (initially dry) are composed of many substances, each of which produces heat to a different degree and which may also be chemically activated by moisture. Some question also arises with regard to surface measurements which at best are only approximate.

TABLE 45—HEAT OF WETTING RESULTING FROM MIXING DIFFERENT AMOUNTS OF DRY AND WET SOIL

Wet soil (g)	Dry soil (g)	Temperature rise (deg C)	Heat evolved (cal)	Moisture content (percent)
35	5	0.65	10.9	20.1
30	10	1.75	28.0	16.6
25	15	2.24	34.2	13.6
15	25	2.55	35.7	7.8
10	30	4.05	47.0	5.0
5	35	4.20	52.9	2.3

When dry and wet soils are mixed, heat is produced. Wadsworth (1938) studied the effect of this mixing on Hawaiian soil, but failed to state whether such heat was free from chemical activity. His tests consisted in mixing various weights of dry and wet soils in an insulated container and then measuring temperature rise. Forty grams of soil were used in each test, and his results are presented in Table 45 Other tests based on soils of different moisture content when wetted with 100 cc of

distilled water indicate that the heat produced varies inversely with the initial moisture content. Wadsworth concluded that the heat produced may be represented by the equation

$$H_w = H_0^{-bM} \qquad \text{Eq (11-27)}$$

where H_w is the heat of wetting in cal per g, H_0 is the heat produced with initially dry soil, b another constant, and M the percentage of moisture in the soil on a dry basis.

Of significance are the data by Ewing (1931) dealing with the heat of wetting in zinc oxides. Ewing's experiments were made with purified substances. Vapors, especially water, adsorbed by pigments affect their properties. Ewing's results in Table 46 show that the heat of wetting varies according to the liquid used. The surface measurements indicated in the table were obtained by photographic measurement of about 500 zinc-oxide particles. The average diameter was determined by Eq (3-10), that is, $d_{ss} = \Sigma nd^3/\Sigma nd^2$ and the specific surface by Eq (3-6), the density of the oxide being taken as 5.47 g per cc.

Detailed studies on the effect of surface-area and heat of wetting are not available. However, it is known that the heat of wetting of silica gel, which is a highly porous material, is approximately 30 cal per g (approximately 200 times that of coarse quartz as shown in Table 46).

TABLE 46—HEAT OF WETTING OF ZINC OXIDE USING VARIOUS LIQUIDS (AVERAGE DIAMETER OF PARTICLES 0.607 μ, SPECIFIC SURFACE 1.8 SQ METER/G)

Liquid	Heat of wetting (cal/g)
Nujol	0.20
Benzene	0 56
Xylene	0.64
Pyridine	0 82
Linseed oil	2 00
1.5 Percent zinc oleate in benzene	1.28
5 Percent zinc oleate in benzene	0.90
Water	1.12

Determination of Heat of Wetting—The heat of wetting is determined by a calorimeter, although direct temperature measurements of a bulk of material are often used. Ewing described a method which permits determination of both heat of wetting and heat of adsorption by a simple technique. While intended primarily for zinc oxide, the method may be applied to other particulate material. This method is as follows: A 5-g sample is heated to 500 deg C in an electric furnace and evacuated at that temperature for several hours. The evacuation is made with a

mercury pump to about 5×10^{-5} mm Hg. The evacuated sample is sealed off and mounted as shown in Fig. 67. This device is then introduced into a calorimeter containing the appropriate wetting fluid and the tip is broken. In order to compute the heat of wetting, the temperature of the sample must be known.

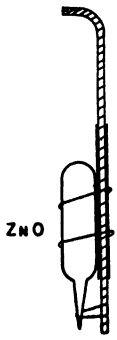


FIGURE 67. EWING'S DEVICE FOR BRINGING LIQUID INTO CONTACT WITH ZINC OXIDE IN CALORIMETER.

CONDENSATION PROCESSES

The vapor pressure of a liquid is affected by the curvature of the surface to which it adheres. Liquid on the convex surface of a particle with extremely small radius of curvature exerts a vapor pressure greater than that on a plane surface. Conversely, the vapor pressure due to a thin film on a concave surface is less than that for a film on a plane surface. Hence, vapor will condense more readily within a concave surface than upon a convex one. For this reason an irregular particle will condense moisture better than the surface of a smooth sphere with approximately the same diameter. The relation between the vapor pressures and the density and surface tension of a liquid was obtained by Lord Kelvin and is given in texts on physical chemistry. If the vapor pressure due to a plane liquid surface is denoted by p_0, and the vapor pressure above a curved surface whose radius of curvature is R denoted by p then,

$$\rho_0 BT \ln \frac{p}{p_0} = \frac{2\sigma}{R} \qquad \text{Eq (11-28)}$$

where σ is the surface tension of the liquid, ρ_0 the density of the liquid, B the gas constant, and T the absolute temperature. At a plane water surface at 0 deg C the vapor pressure is approximately 6000 dynes per sq cm. It is clear from Eq (11-28) that the logarithm of the pressure ratio varies inversely as the diameter. It is to be remembered that when the surface is convex $p > p_0$, and that for a concave surface $p < p_0$, so that p/p_0 will possess values less than unity and the curve will be inverted. Since we are particularly concerned with small particles approximating spheres, that is, convex surfaces, the smaller the particle the greater the evaporation. Hence, it is impossible to condense moisture on small particles except at a very high vapor density.

The value of surface tension is assumed to be constant for all radii, although when the particles are extremely small the value of surface ten-

sion may differ from that ordinarily given in tables of physical constants.

It has been shown that the tendency of droplets (with small particles as nuclei) is to evaporate rapidly due to the increased vapor pressure. In order to produce condensation it is necessary to modify the ratio of p/p_0. This may be done in several ways:

1. By introducing particles having plane surfaces, the vapor pressures of such surfaces being equal to p.
2. By introducing particles having porous surfaces such as activated charcoal and silica gel whose surfaces are internal and extremely small. Hence, having thus introduced a large amount of concave surface the pressure of the vapor becomes less than that of a plane surface. This constitutes the basic theory for the adsorption obtained by porous material such as activated charcoal. As the water vapor condenses within such porous material the radii of the concave surfaces are made smaller; hence their capacity to condense moisture increases until the porous material becomes saturated.
3. By the introduction of small particles carrying an electrical charge. The addition of a charge to the surface of a particle diminishes the vapor pressure. J. J. Thomson (1903) deduced the following formula for charging liquid surfaces:

$$\rho_0 BT \ln \frac{p}{p_0} = \frac{2\sigma}{R} - \frac{\mathcal{Q}_e^2}{8\pi R^4} \qquad \text{Eq (11-29)}$$

where $\mathcal{Q}_e$ is the quantity of charge on the particle. It is readily seen from this equation that the application of a charge greatly modifies the condensation. Thus, a critical examination of Eq (11-29) shows that when p/p_0 is plotted against R it will have a maximum. The radius of the particle when this maximum is obtained is 0.63 $\mu\mu$. Hence condensation is impossible on a charged particle if the radius is less than this amount. Condensation is possible only when the vapor pressure decreases as R increases, that is, when $R < 0.63$ $\mu\mu$. Unfortunately the phenomena of condensation and evaporation are modified under some conditions, in a manner not accounted for by Eq (11-29).

4. Condensation may be induced by substances having a strong chemical affinity for the vapor. Hygroscopic substances such as calcium chloride, sulfur trioxide, etc., rapidly combine with the water vapor and form droplets which have lower vapor pressures upon which more moisture readily condenses. Because they are essentially new substances these droplets possess a permanently lowered vapor pressure.

EVAPORATION

The evaporation of small droplets was studied by Langmuir (1913) who demonstrated that the rate of loss in weight is proportional to the diameter of the particle plus the water film, and not to the surface. Langmuir showed that the rate of loss of weight due to evaporation is given by the expression

$$-\frac{dw}{dt} = \frac{2\pi d \mathfrak{D} M p}{BT} \qquad \text{Eq (11-30)}$$

where $-dw/dt$ is the rate of loss in weight due to evaporation, d the diameter of the particle plus the diameter of the water film, M the molecular weight, p the pressure of the gas or vapor, B the gas constant, T the absolute temperature, and $\mathfrak{D}$ the diffusion coefficient.

Houghton (1933) has given a simple formula for the evaporation of small water drops. If d_0 denotes the initial drop diameter and d the diameter after t-sec, then

$$d^2 = d^2_0 - 8k(\rho_v' - \rho_v)t \qquad \text{Eq (11-31)}$$

where ρ_v' is the vapor density close to the drop (saturation) and ρ_v that of the surrounding air, and k is a constant depending on drop size and vapor density difference $(\rho_v' - \rho_v)$ given in Table 47.

TABLE 47 VALUES OF k IN EQS (11-31) AND (11-32). (HOUGHTON, 1933)

Vapor-density difference (g per cc) $\times 10^{-6}$	Size in microns							
	100	200	400	600	800	1000	1600	2000
1	0 26	0 27	0 27	0 27	0 27	0 27	0 27	0 27
5	0 165	0 17	0 175	0.18	0 185	0 19	0 21	0 22
3	0 20	0 205	0 21	0 215	0 22	0 225	0 23	0 235
10	0 15	0 16	0 17	0.175	0 18	0 185	0 19	0 20

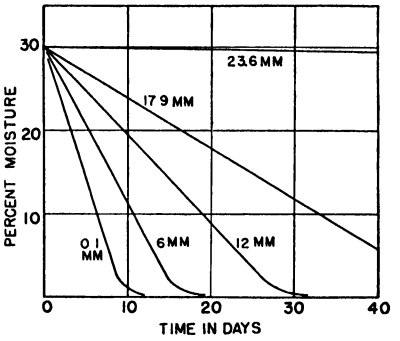


FIGURE 68. CURVES FOR FINE QUARTZ SHOWING THE EFFECT OF HUMIDITY UPON RATE OF EVAPORATION.

If evaporation takes place in t seconds

$$d^2 = 8k(\rho_v' - \rho_v)(t_0 - t)$$

or if we desire the total time for the evaporation of the complete drop

$$t_0 = d_0^2/8k(\rho_v - \rho_v') \qquad \text{Eq (11-32)}$$

These equations can only be considered approximate for thin films surrounding particles.

Patten and Gallagher (1908) presented data showing that the rate of evaporation from soils is linear until the moisture content approaches 1 percent. Some of their results for fine quartz exposed to atmospheres of different humidities are given in Figure 68.

Evaporation from land and water surfaces has been studied extensively by Thornthwaite and Holzman (1942) who applied the general theory of turbulence outlined in Chapter 8. Using the proper modification of Eq

(8-20), if E denotes evaporation and M_1 and M_2 the moisture concentrations at heights z_1 and z_2, respectively, then

$$E = \frac{\rho_0 k_0^2 (M_1 - M_2)(v_{x2} - v_{x1})}{\ln^2 (z_2/z_1)} \qquad \text{Eq (11-33)}$$

where ρ_0 is the density of the air, k_0 is von Kármán's turbulence constant, and the v's denote the mean wind velocities in the x-direction at z_1 and z_2. In ordinary terms, E being expressed in inches depth per hr,

$$E = \frac{833 k_0^2 (h_1 - h_2)(v_2 - v_1)}{\ln^2 (z_2/z_1)} \qquad \text{(Eq 11-34)}$$

and h_1 and h_2 are the vapor pressures in mm Hg at heights z_1 and z_2, respectively, and v_1 and v_2 are corresponding wind velocities in miles per hr.

Problems

1. Using Eq (11-11) calculate the amount of nitrogen adsorbed by a gram of fine particles, given $P_0 = 76$ cm, $Q_m = 10.3$ cu cm per g, and $C = 3.0$.

2. Calculate the surface of a g of titanium oxide powder, given that heat of emersion of the powder with water vapor is 0.409 cal per g and the enthalpy of the water surface 118.5 erg per sq cm.

3. Calculate the molecular area of water, assuming spherical molecules in "closest" packing.

4. Calculate the amount of water vapor adsorbed by silica gel which has passed 14- and is retained on 20-mesh Tyler sieves in 20 min, assuming a 3-in. thickness of the substance in a frame 3 ft × 3 ft in area, the approaching air velocity being 100 lfm and the relative humidity 60 percent.

CHAPTER 12

CHEMICAL PROPERTIES OF SMALL PARTICLES

THE rate of any chemical reaction taking place between solid and liquid or gas phases is markedly influenced by surface. As has been pointed out repeatedly, small particles in the aggregate present unusually large amounts of surface. Hence, we conclude that small particles will greatly affect chemical activity. This activity is further enhanced by the Brownian motion of the finest particles, thus constantly exposing new surface and transferring the reaction throughout the liquid or gas phase in which the particles are contained.

It is characteristic of many metals which ordinarily are relatively inactive in air to react with great vigor when divided into a fine state and scattered in air. This is true of such substances as iron, zinc, magnesium, and even lead. Under certain conditions organic substances ordinarily noninflammable in air, such as sugars, starches, and coal will explode with great violence when finely divided and dispersed.

The properties of small particles affecting chemical activity are discussed in this chapter. We begin by outlining those factors which influence solubility. Only those studies having a bearing on particulate matter will be covered. It cannot be hoped that equations of such generality will be obtained that all types of reactions are encompassed. This cannot be done, and we are limited to consideration of only the most elementary items. Rate of crystal growth and granulation will form another section. Regarding the first of these, no simple satisfactory approach exists for its evaluation, as practicing chemical engineers well know. So many variables are encountered that at best we can interpret results only by empirical expressions. The subject of granulation is relatively new, but considering the commercial aspects of packaging, bagging, and ease of application, a knowledge of granulation and its control assumes great importance.

The remainder of the chapter concerns oxidation and explosibility of fine matter. The former includes a discussion of the properties of fine fuels and is based on recent experiments conducted at the Massachusetts

Institute of Technology. The subject of explosibility of certain dusts brings together some widely scattered data.

SOLUBILITY OF SMALL PARTICLES

Ostwald (1900), using the thermodynamic relationship between vapor pressure of spherical drops and the curvature of surface, derived an equation giving the relation between solubility and particle size. Ostwald's equation was later modified by Freundlich (1909), and is as follows:

$$\frac{BT}{M} \ln \frac{S_2}{S_1} = \frac{4\sigma}{\rho}\left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

where B is the gas constant and M the molecular weight of the liquid, T the absolute temperature, S_1 and S_2 the solubilities of the particles with radii d_1 and d_2, respectively, σ the surface energy, and ρ the true density of the particles. This equation was subsequently modified by Dundon and Mack (1923) in calculating the surface energy of calcium sulfate. The equation is as follows:

$$(1 - \alpha + n\alpha) \frac{BT}{M} \ln \frac{S_2}{S_1} = \frac{4\sigma}{\rho}\left(\frac{1}{d_2} - \frac{1}{d_1}\right) \qquad \text{Eq (12-1)}$$

where α is the degree of dissociation and n is the number of ions from one molecule. If d_2 is regarded as the diameter of a small particle and d_1 that of another particle so large that the solubility is that of a plane surface, Eq (12-1) becomes

$$(1 - \alpha + n\alpha) \frac{BT}{M} \ln \frac{S}{S_\infty} = \frac{4\sigma}{\rho d} \qquad \text{Eq (12-2)}$$

This equation may be used to calculate either the relative solubility S/S_∞ or the surface energy, whichever is desired.

Rate of Solubility—The rate of solubility of small particles depends on a great number of variables. Eq (12-2) takes into account free surface energy (σ) and particle surface ($1/d$). These are purely surface considerations, and are scarcely complete in themselves. The "shape" of the surface and its physical state must also be specified, that is, its relative freedom from contamination which might influence the speed of reaction. The effect of packing density and the extent of agitation imparted to the particles are also important, particularly with regard to exposure of fresh surfaces and formation of possible gas pockets. The liquid and liquid-solid phases jointly are additional important considerations. The volume of the liquid, its temperature, and the amount of dissolved solid already in solution must enter into all calculations. Nor can we ignore the *chemical* nature of the substances involved in the

reactions and the products formed. The manner by which these variables affect the *rate of solution* must be known, otherwise a complete understanding of the problem from the mathematical standpoint is impossible.

Many theories have been proposed regarding the rate of solubility. Fick (1855) proposed a diffusion law which bears his name. Fick's expression for rate of solubility is as follows:

$$\frac{dw}{dt} = - \mathfrak{D}A \frac{dC}{dx} \qquad \text{Eq (12-3)}$$

This equation states that the quantity of solute, dw, which diffuses through an area A at right angles to the surface being dissolved in an element of time dt, is proportional to the rate of change of concentration dC in a distance dx from the surface. (Eq 12-3 is identical with Eq 8-2.) The constant of proportionality, $\mathfrak{D}$, is called the diffusion constant and is equal to the amount of solute crossing an area of 1 sq cm in a unit-time, if the change of concentration per cm in a direction perpendicular to this cross section is unity. Fick's law applies only for *linear* diffusion, that is, the diffusion takes place in one direction only. It is based on the assumption that all motions of the medium and particle are excluded and that molecular agitation is the only variable of consequence. The law has been tested by various investigators. It was found that the diffusion constant $\mathfrak{D}$ varies with the concentration of the solution, and that it is markedly affected by the slightest agitation. This led to the theory that at the boundary surfaces of the particles there is a thin adhering layer of saturated solution, which affects the rate of diffusion. Differences in the diffusion constant are attributed to the rate of chemical reaction and surface characteristics which alter the state of the adhering layer. Some investigators have emphasized that the chemical reaction rate is of primary importance and diffusion processes are secondary. At the present time the choice depends upon attendant conditions, as little has been done to develop a generally accepted theory.

Noyes and Whitney (1897) proposed an equation which makes no assumptions in regard to the mechanism of the solution process, and which is substantiated by experiment. Their equation is

$$\frac{dw}{dt} = - K'S(C_s - C) \qquad \text{Eq (12-4)}$$

This is in every respect similar to Fick's law, with the following interpretation: The time rate of change of concentration (loss in weight dw of a particle per given time) at any instant is proportional to the surface S of the particle, the difference in concentration of a saturated solution C_s

(at the particle-surface), and the concentration existing in solution at that instant, C. The constant K' is a function of surface condition and of other variables discussed above. The equation applies for a given set of conditions and changes when the conditions change. For example, the constant K' will depend upon the degree of agitation provided and, as will be shown later, is a good index of its measure. Eq (12-4) has been solved by various investigators based on the following assumptions: (1) That the rate of dissolution takes place normal to the surface, and that the surfaces of the particles are exposed to the liquid phase in a uniform manner; (2) that the shape-factor of the particles is constant; and (3) that the particles are agitated in such a manner that no adhering layer is permitted to form. While the solutions of Eq (12-4) given below are limited to a single particle, they have general application to a number of particles. Let w be the weight of a particle at time t, and w_0 its original weight. Also let w_s be the amount (weight) of the particle required to saturate the solution, S the surface of the particle at the time t, and V the volume of the solution. Then, $(w_0 - w)$ is the amount of the particle dissolved in the time t and the concentration is this divided by V. Similarly, $w_s/V = C_s$, the concentration of dissolved material at the point of saturation. Hence, from Eq (9-4)

$$V\frac{dw}{dt} = -K'S(w_s - w_0 + w) \qquad \text{Eq (12-5)}$$

Now the value of S changes with time. This variable may be related to the weight of the particle at any instant by means of the shape-factors discussed in Chapter 3. Thus $\rho\alpha_v d^3 = w$ and $\alpha_s d^2 =$ particle-surface where ρ is the density of the particle, and α_v and α_s the volume and surface-shape factors, respectively. Combining these expressions as in Eq (3-47), we obtain

$$S = \frac{\alpha_{sv}}{\rho^{2/3}} w^{2/3} = aw^{2/3} \qquad \text{Eq (12-6)}$$

where α_{sv} is the specific volume-shape factor and a is the constant $\alpha_{sv}/\rho^{2/3}$. We shall now introduce a new variable, w_d, which is the amount of dissolved matter present in the solution. Since $w = w_0 - w_d$, we have $dw/dt = -dw_d/dt$ and substituting in Eq (12-5), we obtain

$$V\frac{dw_d}{dt} = K'S(w_s - w_0 + w_0 - w_d) = K'S(w_s - w_d) \qquad \text{Eq (12-7)}$$

Dividing through by w_s and writing $w_d/w_s = \theta$

$$V\frac{d\theta}{dt} = K'S(1 - \theta) \qquad \text{Eq (12-8)}$$

and since

$$\left(\frac{w}{w_s}\right)^{2/3} = \left(\frac{w_0}{w_s} - \frac{w_d}{w_s}\right)^{2/3}$$

$$S = aw_s^{2/3} \cdot (\theta_0 - \theta)^{2/3} \qquad \text{Eq (12-9)}$$

where $\theta_0 = w_0/w_s$. Substituting in Eq (12-8)

$$\frac{d\theta}{dt} = \frac{K'aw_s^{2/3}}{V}(\theta_0 - \theta)^{2/3} \cdot (1 - \theta) \qquad \text{Eq (12-10)}$$

The solution of this equation between the limits of $\theta = 0$ and $\theta = 1$ can be obtained by graphical methods and is given by:

$$\frac{1}{\theta_0}\left(\frac{\theta}{1 - \theta}\right) = \frac{K'at}{V} \cdot w_s^{2/3} \qquad \text{Eq (12-11)}$$

Eq (12-11) is another solution of Eq (12-5) previously obtained by Hixson and Crowell (1931*a*, *b*), and by Wilhelm, Conklin, and Sauer (1941). To introduce a particulate parameter, let N be the number of particles under consideration. Then the weight of a single particle is w/N. Hence the area of a particle is $\alpha_{sv}(w/N\rho)^{2/3}$ and the total area $\alpha_{sv}N(w/N\rho)^{2/3}$ or $\alpha_{sv}N^{1/3}(w/\rho)^{2/3}$. Hence the right-hand term of Eq (12-11) is simply multiplied by $N^{1/3}$, that is,

$$\frac{1}{\theta_0}\left(\frac{\theta}{1 - \theta}\right) = \frac{K'aN^{1/3}t}{V} \cdot w_s^{2/3} \qquad \text{Eq (12-12)}$$

This equation agrees closely with the integral solution of Wilhelm *et al*, which corrects for changes in initial volume of solution as dissolution continues. In general the equation fits most conditions, provided the number of particles and their shape-factors are determinable.

Wilhelm *et al* found that their equation holds for crystals of rock salt ranging from 2 to 14 U. S.-mesh size. The average diameters calculated, determined from the weight of a known number of particles (assumed spheres) ranged from 1.0 to 7.7 mm. The constant K', calculated by Eq (12-12), varied with particle-size according to the equation $K' = 0.37d^{0.28}$, and also with changes in the amount of salt dissolved (expressed by the function $\theta_0 = w_0/w_s$), namely $K' = 0.57\theta_0^{-0.07}$. However, for a given set of particles K' is *constant*, although during dissolution the diameter of the particles decreases. Actually K' is proportional to both d and θ_0 simultaneously and these counteract each other; that is:

$$K' = f\left(\frac{d^{0.28}}{\theta_0^{0.07}}\right) = f\left(\frac{d^{0.28}}{w^{0.07}}\right) = f\left(\frac{d^3}{w}\right)^{0.08} \text{(approximately)}$$

We see therefore that since w varies as d^3, the ratio d^3/w is invariant.

We next consider some special cases of Eq (12-12). These cases were first treated by Hixson and Crowell.

Case 1. For this case take the initial weight as that required for saturation, $w_s = w_0$. Hence, substituting in Eq (12-5)

$$V\frac{dw}{dt} = K'Sw = K'aw^{2/3}$$

which upon integrating gives

$$\frac{K'at}{V} = w^{-1/3} - w_0^{-1/3} \qquad \text{Eq (12-13)}$$

Case 2. Assume the amount dissolved is negligible so that the concentration does not change. Using Eq (12-4) and putting $(C_s - C) = constant$, make the rate of dissolution proportional to the surface, so that

$$-\frac{dw}{dt} = K''S = K''aw^{2/3}$$

and on integrating between limits w_0 and w

$$K''at = w_0^{1/3} - w^{1/3} \qquad \text{Eq (12-14)}$$

The various equations given above have been tested and found satisfactory.

CRYSTAL GROWTH AND GRANULATION

Crystal Growth—In many lines of chemical research it is useful to be able to control the size of crystal. This is important not only from the standpoint of the ultimate product desired for some particular purpose, but also that crystallization equipment may be designed properly. Only the most elementary treatment has been given the general problem of calculating the ultimate size of crystals, when given the original size of the seeds and assuming, of course, that no new seed nuclei are formed. The problem itself is not mathematically difficult when all factors are known and can be evaluated. The difficulty lies in the fact that the variables affecting crystal growth are many, and often beyond control. Each crystallization problem requires separate experimental study until all variables have been accounted for. Thereafter, given a size-distribution of seed and assuming (other things being equal) that each crystal grows at the same rate, the size-distribution of the final product can be obtained.

Solution of the problem for spherical growth of a single particle is simple. Let the rate of heat loss for a sphere be given by its steady-state conduction. If the diameter of the surface is d_2 and that of the seed d_1, and the temperatures T_2 and T_1, respectively, then

$$dH = 2\pi C\frac{T_1 - T_2}{d_2 - d_1}d_1d_2dt \qquad \text{Eq (12-15)}$$

where C is the conductivity. This loss will crystallize a layer of the seed of thickness $d(d_1)$. If the latent heat of fusion is denoted by H_L, the heat loss must be equal to $\alpha H_L m = \alpha H_L \cdot \pi d_1^2 d(d_1)$, where α may be termed the "accommodation" coefficient and m the mass of the layer crystallized on the seed. Hence

$$2\pi C \frac{T_1 - T_2}{d_2 - d_1} d_1 d_2 dt = \alpha H_L \cdot \pi d_1^2 d(d_1)$$

from which the time required to lay down an amount $(d_2 - d_1)/2$ in the thickness may be determined. Thus

$$t = \frac{\alpha H_L}{2Cd_2(T_1 - T_2)} \int_0^{d_1/2} d_1(d_2 - d_1) d(d_1)$$

$$t = \frac{\alpha H_L}{2Cd_2(T_1 - T_2)} \left[\frac{d_2 d_1^2}{2} - \frac{d_1^3}{3} \right]_0^{d_1/2} \qquad \text{Eq (12-16)}$$

The factor α is difficult to evaluate and depends upon a number of conditions. It may be considered the percentage of supersaturated liquid which crystallizes. Eq (12-16) must be considered approximate, in view of the assumptions made, but it should be useful in determining the time required to crystallize a product of required size from a known size-distribution.

Since every crystal grows at the same rate, the increase in diameter of every seed at the end of a given time will be the same. Thus, if this increase is known for a single seed, it is the same for all other seeds and we can predict the size-distribution of the product, provided, of course, that the size-distribution of the seeds is known.

Of some interest is the expression derived by Montillon and Badger (1927) who studied the growth of large crystals of Glauber's and Epsom salts under carefully controlled conditions. Their quantitative results were expressed by the following empirical equation

$$\frac{\Delta w}{\Delta S} = 10^3 a \cdot \exp(bt) \qquad \text{Eq (12-17)}$$

where Δw and ΔS are, respectively, the increase in weight and surface of the crystals which develop per unit-time, t the time expressed in minutes, and a and b are constants. The constant b appears to be the same for both salts used, but a is definitely a function of temperature. When w is the increased weight expressed in g per hr and S the increased surface in sq cm per hr, b has a value of 17.8; a varies with the kind of salt and with temperature, having values ranging from 12.57 at 27.1 deg C to 16.26 at 30.9 deg C in the case of Glauber's salts, and from 18.52 at 29.25 deg C to 22.13 at 31.6 deg C in the case of Epsom salts. While

Eq (12-17) is admittedly empirical, it serves to indicate the relationship of the factors involved.

Granulation—Granulation is that property of certain substances which, due to cohesive forces, surface tension, etc., acting between the particles, causes the material to form itself into grains of various sizes. Granulation is an important property often desired in some products, particularly fertilizers and certain foods. The property of granulation differs from what is ordinarily attributed to the word. In its industrial usage it signifies the cohesive character of small grains to combine into larger ones. The factors affecting granulation vary widely, depending on the material under treatment. In general, granulation is a function of moisture content, temperature, and frequently the ratio of various components. These same factors not only affect the kind of granulation, but also its strength and size-distribution.

In the specific instance of fertilizer mixtures, Hardesty and Ross (1938) indicated the importance of initial fineness of the material to be granulated. Thus, in the case of a fertilizer mixture containing 26 percent of material coarser than 20-mesh and 53 percent coarser than 40-mesh, the highest proportion of 5/20-mesh granules was obtained with an initial moisture content of only 5 percent. The proportion of granules larger than 5-mesh increased as the moisture was increased, and conversely a decrease in moisture below this percentage increased the proportion of granules smaller than 30-mesh. When this same mixture was ground so as to pass a 40-mesh screen, a water content of at least 9 percent was required for satisfactory granulation.

Considering a rotary drum (which is generally used to granulate material), the following forces are involved: When sufficient moisture is present the particles of a mixture increase in size as the rolling process continues. This is effected by two or more particles combining to form a particle of larger size. The process continues until the cohesive forces are offset by friction and crushing which take place and tend to reduce the size of the granules.

Hardesty and Ross made careful studies of the pressure to which fertilizer granules could be subjected without degradation. They described a simple device for determining the force necessary to crush individual particles consisting of a rack and pinion similar to that used in adjusting the elevation of a microscope. Below the rack, and resting on a sensitive scale or balance, is a small anvil. The particle is placed on a container in the anvil directly below the rack, the latter being turned down so as to rest on the particle. The pinion is turned until the particle breaks, at which time the maximum deflection of the scale pointer is observed. In this type of experiment it is important that the particles

be of the same shape and size so that significant comparisons may be made with respect to moisture, temperature, and other variables.

OXIDATION PHENOMENA

Oxidation of Coal Particles—It is important in combustion engineering to have available data on the oxidation of coal particles. The relationships between oxidation-rate and temperature and particle-size here described, while specifically limited to carbonaceous materials,

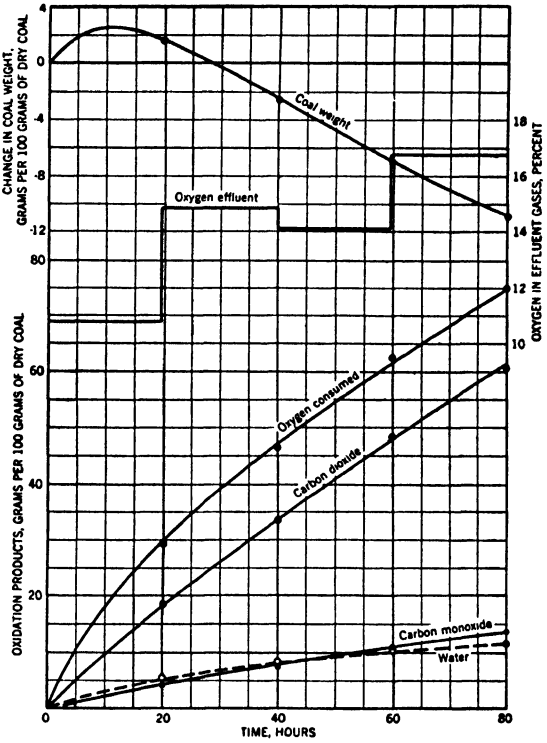


FIGURE 69. LOW-TEMPERATURE-OXIDATION TEST OF ANTHRACITE COAL 48/100-MESH, TEMPERATURE 300 DEG C.

have general applications to other substances. When coal particles are heated a certain amount of oxygen is consumed. The amount of oxygen thus used is a function of particle-surface and temperature. In many respects the action taking place is similar to the problem of dissolution discussed earlier. However, a precise treatment of the problem is impossible for a number of reasons, but especially because

the condition of the particle-surface is altered chemically, and usually increases. The general effect of passing air through a packing of heated carbon particles is illustrated in Figure 69. The data shown in the figure are from the investigations of Scott and Jones (1941) on the oxidation of anthracite coal. The oxidation products are plotted cumulatively while

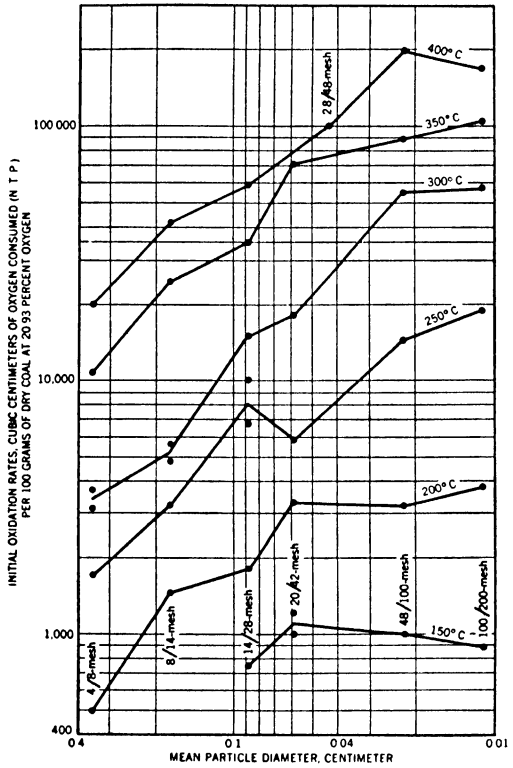


FIGURE 70. INITIAL OXIDATION RATES OF ANTHRACITE COAL.

the curves of coal weight and oxygen in the effluent air indicate conditions existing at the times indicated. Curves similar to those shown in Figure 69 were obtained by Scott and Jones for different temperatures and sizes of materials. From these curves were determined the initial oxidation rates for a given weight of material; in turn these data were plotted against mean particle-diameter. The results are shown in Figure 70. These are descriptive of critical conditions only. Similar results might be obtained for rates at any subsequent time or oxygen demand. The data presented in Figure 70 indicate the following kind of relationship between the variables

$$q_0 = ad^b \cdot f(T)$$

where q_0 is the initial oxidation rate, d the diameter of coal particles, T the temperature, and a and b are constants. In the range of particle-diameters and temperatures studied (excluding the 150 deg C curve) the data of Figure 70 are represented to a fair degree of accuracy by the expression

$$q_0 = 1.2 \times 10^{-8} \cdot T^5 d^{-0.8} \qquad \text{Eq (12-18)}$$

In this equation q_0 is the *initial* oxidation rate in cu cm of oxygen consumed (N.T.P.) per 100 g of dry anthracite coal at 20.93 percent oxygen, T the temperature in deg C, and d the diameter of the particles in mm. Eq (12-18) is empirical. If the oxidation were precisely proportional to the surface, the exponent of d would be -1. It is interesting to note the importance of temperature. Small variations in temperature will produce large changes in the value of q_0. Scott and Jones stressed the importance of constant temperature during tests.

Combustion Rate of Carbon—Eq (12-18) can hardly be expected to hold at ignition temperatures. Indeed it is doubtful whether it will hold at temperatures much above 400 deg C even in the case of anthracite coal. Above this point the carbon ignites and a completely new set of conditions pertains to the problem. Studies on the combustion rate of carbon particles have been studied in considerable detail by several investigators. The general method used was described by Parker and Hottel (1936). The device used consists of a furnace in which is suspended a cylindrical rod of carbon 2.5 cm in diameter with a hemispherical end. The rod is mounted on a porcelain tube and suspended from one arm of a balance so that the hemispherical end is downward in the furnace. In this way loss of weight is easily determined. The surface of the specimen is also capable of measurement. Known volumes of air heated to the required temperatures are then made to flow past the carbon, and by means of a small quartz sampling-tube (which can be adjusted at any distance from the specimen) samples are withdrawn for analysis. In this way samples of air may be analyzed for the amount of carbon dioxide and oxygen present at any distance from the heated particle-surface.

A general equation giving the specific combustion rate was developed by Tu *et al* (1934), and by Parker and Hottel. The former presented an excellent discussion of the general problem as well as a review of available literature. The equation for specific combustion rate may be written as follows:

$$K = \frac{P_{ox}}{\text{diffusional resistance} + \text{chemical resistance}} \qquad \text{Eq (12-19)}$$

where K is the specific reaction or combustion rate expressed in units of weight per sec $\times$ area, and P_{ox} is the partial pressure of oxygen in the ambient air stream, usually expressed in atmospheres. For a small sphere of carbon in still air, the diffusional resistance is given by the equation

$$\text{diffusional resistance (still air)} = \frac{10^5 d}{1.09\sqrt{T}} \qquad \text{Eq (12-20)}$$

where d is the diameter of the particle in cm, and T the absolute temperature (deg K) of the carbon. The diffusional resistance is expressed in (sec $\times$ sq cm per g) $\times$ atm. If the sphere is in an ambient air stream or is in motion relative to the surrounding gas, the term d in Eq(12-20)must be corrected, since the resistance to diffusion is reduced—that is, the value of d in the equation must be replaced by $1/((1/d) + (1/2\delta))$, where δ is the thickness of the film.

The complete general equation for diffusional resistance obtained by Tu *et al* is

$$\text{diffusional resistance} = 10^5 \left[0.537\, P_{ox} \sqrt{T} \left(\frac{117}{d \cdot T^{0.185}} (\rho_0 v d)^{0.37} + \frac{2}{d} \right) \right]^{-1} \qquad \text{Eq (12-21)}$$

where v is the velocity of the air stream or particle, and ρ_0 the density of the gas.

The chemical resistance of carbon was found by the investigators mentioned to be represented by the equation

$$\text{chemical resistance} = 1.05 \times 10^{-7} \sqrt{T} \cdot \exp\,(44{,}000/BT) \qquad \text{Eq (12-22)}$$

where B is a gas constant in this equation expressed as cc $\times$ atm per deg C. For combustion in air, B = 82.07 cc atm per deg C. Note that the chemical resistance is independent of particle-diameter. At high temperatures the chemical resistance becomes extremely small and may be neglected. Thus, for still air at very high temperatures

$$K = 1.09 \times 10^{-5}\, P_{ox} \sqrt{T}/d \qquad \text{Eq (12-23)}$$

the units of K being g per (sec $\times$ sq cm). It must be remembered that the chemical resistance is not constant, but varies with different kinds of carbons. Hence the constant must be evaluated for each combustible material. Diffusional resistance, on the other hand, should be relatively independent of the nature of the carbon. The constants used in the above equations apply to "brush" carbon.

The time required for burning a carbon particle may be calculated by means of Eq (12-23). The rate of combustion of a sphere of density ρ is its rate of change of volume times its density, and this must be equal

to the surface of the particle multiplied by the specific combustion rate K

$$\frac{\rho\pi d^2}{2}\frac{d(d)}{dt} = 1.09 \times 10^{-5} \times \pi d^2 \cdot P_{ox}\sqrt{T}/d$$

Integrating from an initial diameter d_0 at time 0 to diameter d at time t

$$\tfrac{1}{2}(d_0^2 - d^2) = 2.18 \times 10^{-5} P_{ox} t\sqrt{T}/\rho$$

$$t = \frac{\rho(d_0^2 - d^2) \times 10^5}{4.36 P_{ox}\sqrt{T}} \qquad \text{Eq (12-24)}$$

or under normal conditions of combustion with an infinite supply of air, $P_{ox} = 0.2$, so that

$$t = \frac{1.1\rho(d_0^2 - d^2)}{\sqrt{T}} \times 10^5 \qquad \text{Eq (12-25)}$$

If the supply of air is not infinite, but limited, say, to $1 + \epsilon$ (ϵ = fractional excess air) times that theoretically necessary for combustion, Parker and Hottel obtain

$$t = \frac{7.3 \times 10^4 \rho(d_0^2 - d^2)(1+\epsilon)}{\epsilon^{1/3}\sqrt{T}}\left[\frac{1}{2}\ln\frac{\epsilon^{2/3} - \epsilon^{1/3} + 1}{(\epsilon^{1/3}+1)^2} + \sqrt{3}\left(\tan^{-1}\frac{2\epsilon^{-1/3} - 1}{\sqrt{3}} + \frac{\pi}{6}\right)\right] \qquad \text{Eq (12-26)}$$

DUST EXPLOSIONS

Fine particles of various substances when suspended in air behave like mixtures of inflammable gas and air, that is, under certain conditions they are explosive. And just as in the case of gas-air mixtures, two conditions are necessary to cause an explosion, namely the proper concentration of particles and heat of sufficient intensity to ignite the mixture. Following ignition, there is a propagation due to the ignition of particles in the immediate vicinity and this will continue throughout the region where the concentration of particles is within the explosive range. Naturally, the rate of propagation is dependent on the concentration, and on the amount of surface exposed—in other words, particle-size. But in the case of dust, the manner in which the dust is ignited and certain other specific characteristics also must be considered.

Dusts vary in their inflammability, depending chiefly on chemical composition, concentration, and size. While tests to determine inflammability of different substances are relatively easy, few data are available giving all the conditions under which a particular dust will ignite. The inflammability of dusts is more difficult to rationalize than mixtures of gases.

Effect of Particle-Size—Given a known weight of dust dispersed in a special apparatus and ignited in a fixed manner, the dust will generate pressure on explosion. Similar weights of the same dust will generate different pressures if there is any difference in particle-size distribution; also similar weights of different substances will generate different pressures. The pressure developed is termed "relative inflammability."

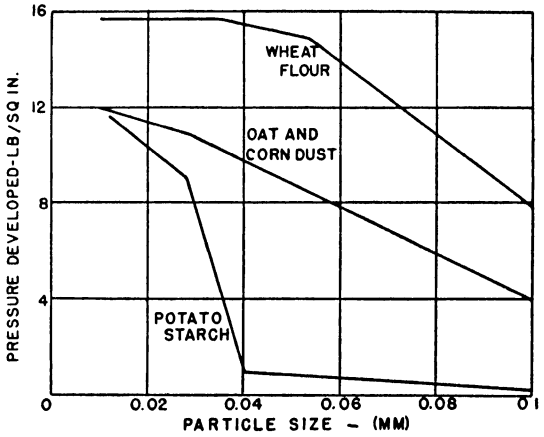


FIGURE 71. RELATIVE INFLAMMABILITY OF THREE KINDS OF AGRICULTURAL DUSTS, ACCORDING TO PARTICLE-SIZE.

The relative inflammability of a few common explosive dusts, according to particle-size, is shown in Figure 71. These data are from studies by Price and Brown (1922) and show the decrease in relative inflammability with increasing particle-size. It is necessary to point out that the samples tested are not clean-cut size-distributions. They include unknown fractions of particles less than stated size, so that relationships to surface cannot be determined. However they are indicative of the effect of increasing size on relative inflammability.

Wheeler (1936) uses a different method of determining inflammability. Having determined whether a given dust is explosive, he next proceeds to add known percentages of inert dust such as fuller's earth (200 I.M.M.-mesh) until the dust is no longer explosive. This critical point is called the "index of inflammability." If the amount of fuller's earth used is denoted by f (in percent), the index of inflammability I may be written

$$I = \frac{100}{100 - f} \qquad \text{Eq (12-27)}$$

Thus, an index of inflammability of 1.00 is a safe dust in any circumstance; the higher the value of I, the more dangerous the dust.

Since the fineness of a given dust has an important bearing upon its inflammability, Wheeler arbitrarily defined a measure of surface by use of a ratio he called the "fineness-factor." This factor is merely a ratio between the opening of a 200 I.M.M.-mesh sieve and the mean diameter of the particle as determined by the average of two successive sieve-sizes. The fineness-factor for a set of I.M.M. sieves is given in Table 48. The fineness-factor for material passing a 200-mesh sieve is taken as 2.000. The procedure in using the fineness-factor is as follows: Given a percentage sieve-size distribution, each value is multiplied by the corresponding fineness-factor given in Table 48 and divided by 100

TABLE 48—FINENESS-FACTORS FOR VARIOUS I.M.M -SERIES SIEVES

Mesh (1)	Width of mesh (in.) (2)	Mean diameter successive meshes (in.) (3)	Fineness-factor 0.00375 ÷ (3)
3	0.1667	....	. .
5	0.1000	0.1333	0.028
8	0.0625	0.0813	0.046
20	0.0250	0.0438	0.086
50	0.0100	0.0175	0.214
100	0.0050	0.0075	0.50
200	0.0025	0.00375	1.00
>200	....	.....	2.00

The fineness-factor for the total analysis is the sum of the individual fineness-factors of the component size-fractions. The example given in Table 49 shows the method of determining the factor.

Wheeler found that when the index of inflammability was plotted against the fineness-factor (for a given dust) a straight line was obtained.

Effect of Concentration—As in the case of gas mixtures, there are lower and upper limits of dust inflammability. There is also a concentration at which the relative inflammability is a maximum, as well as an optimum concentration for the propagation of an explosion. With regard to the latter, little information is available. Data pertaining to the lower limits of inflammability of a number of dusts are given in Table 50. It must be realized that fixed limits cannot always be stated since moisture content, percentage of volatile matter, and other variables affect the inflammability of a dust. The optimum amount would, of course, be that which is just enough to utilize all the oxygen in a given volume.

Pressure and Velocity of Flame Propagation– Dusts, in general, do not develop pressures as high as those obtained with inflammable mixtures of gases. Nor does the flame propagate with the same speed. Price and Brown state that on large-scale tests with coal dusts, pressures as high as 270 lb per sq in. have been obtained. They cite some French tests in which it was estimated that pressures exceeding 500 lb per sq in. were produced. Pressures of 300 to 400 lb per sq in. are considered reasonable. These pressures do not apply for all dusts, although some dusts composed of sugar or starch may exceed these pressures.

TABLE 49—DETERMINATION OF FINENESS-FACTOR FOR A SAMPLE OF INFLAMMABLE COAL DUST

Sieve-size (I.M.M.) (1)	Amount (percent) (2)	Sieve fineness-factor (Table 48) (3)	Fineness-factor components (2) × (3) ÷ 100
3/5	8	0.028	0.002
5/8	17	0.046	0.008
8/20	13	0.086	0.011
20/50	10	0.214	0.021
50/100	4	0.500	0.020
100/200	10	1.000	0.10
>200	38	2.000	0 76
	100		0.92 = Fineness-factor of sample

TABLE 50—LOWER LIMITS OF INFLAMMABILITY OF CERTAIN DUSTS

Dust	Concentration (mg/liter)	Authority
Pittsburgh coal	32–80	Rice and Greenwald (1931)
Aluminum	40–50	Mason and Taylor (1937)
Magnesium	>40	Brown (1941)
Zinc	100	Edwards and Leinbach (1935)

The velocities of propagation also depend upon the character of the dust. As a rule, these velocities are less than 2000 ft per sec. Rice, Greenwald, and Howarth (1932), while testing rock barriers in an experimental coal mine, showed that velocity of flame propagation was generally less than 1000 ft per sec, and that for the particular experiments made the explosion-wave pressures did not exceed 30 lb per sq in. and were generally less than 20 lb per sq in.

Problems

1. Quartz particles ($\rho = 2.65$) are immersed in a solution of acid at 37 deg C. If $B = 8.3 \times 10^7$ ergs per gram-molecule, $M = 96$, and $\sigma = 10^3$ dynes per cm, plot the change of S_2/S_∞ with particle-size in the expression

$$\frac{BT}{M} \ln \frac{S_2}{S_\infty} = \frac{4\sigma}{\rho d}$$

2. An amount of uniformly sized rock salt (3500 g) is added to 9.8 liters of water. After 1 min 15 sec the solution is analyzed and found to contain 200 g of salt per liter. If the saturation concentration is 350 g per liter at 15 deg C, and there are 4100 particles per 100 g of salt ($\rho = 2$ 17), calculate the diffusion rate constant K' in Eq (12-12) expressed as g dissolved per min × sq cm × g per cu cm. Assume a shape-factor 6.1.

3. A naphthalene ball having an area of approximately 5 sq cm is placed in 95 percent alcohol. The weight of the ball is 3 88 g and the alcohol 59 g. The following results were obtained:

Time (*min*)	*Weight of ball* (*g*)
0	3.88
10	3.60
30	3.15
50	2.77
70	2.46
100	2.01

Calculate values of the constant K' in Eq (12-13). Take the shape-factor of the particle as 6.1.

4. Calculate the time required for a crystal of ice (hail) to increase from a diameter of 1 cm to a diameter of 2 cm, due to a change of temperature of 20 deg C. Take the latent heat as 80 cal per g and C the conductivity of ice as 0.0044 cal per cm per sec. Use an accommodation coefficient of 12.

5. Calculate the initial oxidation rate of 1000 lb anthracite coal at 20.93 percent oxygen, the average diameter of the particles being 0.2 cm and the temperature of the surrounding air 350 deg C.

6. Calculate the time required to burn 40 percent of the mass of a sphere of pure carbon at 1500 deg C, if the initial diameter of the sphere is 1 in. (Assume density of carbon to be 2.3.)

CHAPTER 13

FLOW OF FLUIDS THROUGH PACKINGS

THE laws governing flow of fluids through packings have many important applications. The operation of water and chemical filters, absorbing towers, heat exchangers, and other industrial equipment depends upon a knowledge of the factors affecting flow. A great deal of work has been done on this subject and as a result much has been learned regarding the characteristics of packings. In fact, a better description of the over-all characteristics of a packing are derivable from its hydrodynamic behavior than from any other physical factor. While ample data have been obtained which are applicable to all kinds of conditions, the basic formulas pertaining to the flow of fluids through packings still remain empirical. No completely satisfactory theory of this subject exists at the present time.

DIFFUSION THROUGH PACKINGS

Buckingham (1904) showed that when two or more gases diffuse through a packing the diffusion constant is a function of the porosity of the packing. The diffusion constant was defined as the number of cc of each of two gases which in 1 sec would pass in opposite directions through 1 sq cm of cross section of a packing 1 cm thick, if the partial pressure of one gas were 1 mm Hg greater on one side than on the other, and if the partial pressure of the second gas on the other side were the same amount greater than on the first side, the total pressure being the same on both sides. Buckingham's experiments concerned several soils with varying degrees of porosity.

The following example given by Buckingham illustrates how the diffusion constant is determined for CO_2 and air. Streams of carbon dioxide and air are made to pass a layer of soil from opposite sides, the pressures being the same. Suppose the packing used is 100 sq cm in area and 5 cm thick and that 0.057 cu cm of CO_2 passes through to the air side while 0.083 cu cm of air passes through to the CO_2 side, the concentration of each gas differing by 10 percent on the two sides of the packing The average flow is $(0.057 + 0.083)/2$ or 0.070 cu cm per sec. The mean

flow per sq cm per sec is 0.070/100 = 0.0007, and since the difference of partial pressure is 10 percent of 760 mm Hg, or 76 mm Hg, the gradient is therefore 76/5 or 15.2 mm Hg per cm. For a packing 1 cm in thickness and a partial pressure difference of 1 mm Hg, the flow is 0.0007/15.2, or 4.61×10^{-5} cu cm per cm per sec. Hence the diffusion constant is $\mathfrak{D} = 4.61 \times 10^{-5}$.

Results on the speed of diffusion of air and carbon dioxide into each other through soils is approximately represented by the equation

$$\mathfrak{D} = 2.6\vartheta^2 \times 10^{-4} \qquad \text{Eq (13-1)}$$

which means that the speed of diffusion $\mathfrak{D}$ through a packing is proportional to the square of the fractional free space in the packing ϑ. Texture, structure, and moisture-content of the soil appear to affect the degree of diffusion but slightly. The value of $\mathfrak{D}$ if no packing were present, say $\mathfrak{D}_0$, is approximately 2.20×10^{-4}. For comparative purposes these constants are reduced to normal temperature and pressure by proper correction factors.

In general, the value of $\mathfrak{D}_0$ is derivable from Stefan's law which states that

$$\mathfrak{D}_0 \propto \frac{T^2}{P} \cdot \frac{1}{\sqrt{\rho_1 \rho_2}}$$

where T is the absolute temperature, P the total pressure, and ρ_1, ρ_2 the densities of the two gases diffusing into each other. This equation is convenient for determining the diffusion constant for any set of conditions, provided it has been established for a given temperature and pressure.

An important special case of diffusion is that of two gases diffusing through a third gas saturating a packing. In this case, the saturating gas will always move toward the region occupied by the lighter gas. The general theory involved is beyond the scope of this text, and the reader is referred to advanced treatises on the kinetic theory of gases.

FLOW THROUGH PACKINGS

We present below an equation giving the flow of gas at low pressures through an orifice. If the orifice is of length L and diameter D (in cm) through which a gas is flowing, the number of molecules n passing in one sec due to a pressure difference across L of $P_2 - P_1$ measured in bars (1 mm Hg = 1325 bars, one bar being the force of one dyne over one sq cm) is

$$n = 2.8 \times 10^{20} \frac{D^3}{\sqrt{MT}} \cdot \frac{P_2 - P_1}{L}$$

where M is the molecular weight of the gas passing the tube and T its absolute temperature in deg K. To obtain the flow in cu cm we merely divide this equation by Avogadro's number, $N_A = 6.06 \times 10^{23}$, so that

$$q = 4.6 \times 10^{-4} \frac{D^3}{\sqrt{MT}} \cdot \frac{P_2 - P_1}{L}$$

DARCY'S LAW

Early investigations of the laws governing flow of fluids through packings at low pressure indicate that the rate of flow is very nearly proportional to the pressure-drop per unit-length of packing.

$$q = \frac{Q}{t} = \mathcal{P}_0 A \frac{\Delta P}{L} \qquad \text{Eq (13-2)}$$

where q is the rate of flow through sectional area A, ΔP the pressure-drop through the packing of depth L, and $\mathcal{P}_0$ a constant depending on the fluid, temperature, and packing characteristics. For a given packing, and using air as the fluid, $\mathcal{P}_0$ is inversely proportional to the 0.7 power of the absolute temperature.

$$\mathcal{P}_0 \propto T^{-0.7}$$

where T is the absolute temperature. Hence, if $\mathcal{P}_0$ is known for one set of conditions, it is determinable for any other set of conditions.

Buckingham has shown that the constant $\mathcal{P}_0$ is also a function of the porosity or fractional free space of the packing. If the particles are not wetted,

$$\mathcal{P}_0 \propto \vartheta^7 \text{ (approximately)}$$

Hence the complete expression where ϑ is the porosity may be written

$$q = k_0 \vartheta^7 \frac{\Delta P}{L} \qquad \text{Eq (13-3)}$$

The interrelationships of the factors entering into the constant form the subject matter of the paragraphs which follow.

Eq (13-2) is known as Darcy's law (1856). The equation is usually written

$$v = \frac{\mathcal{P}}{\mu} \frac{dP}{dx} \qquad \text{Eq (13-4)}$$

where v is the velocity of the fluid along the x-axis and μ its viscosity. The constant $\mathcal{P}$ is termed the permeability. It is the volume of a fluid of unit-viscosity passing through a unit cross-section of a packing under

a unit-pressure gradient in unit-time. The unit of permeability in cgs units has been termed the *darcy* by Fancher, Lewis, and Barnes (1933). According to their definition, a given packing has a permeability of 1 darcy if, in a section 1 sq cm in area perpendicular to the flow, 1 cc fluid possessing unit viscosity travels 1 cm per sec through the packing under a pressure-drop of 1 atm. The following order of magnitudes has been suggested: 1 darcy = 10 decidarcys = 100 centidarcys = 1000 millidarcys = 1,000,000 microdarcys. Eq (13-4) forms the foundation for the subject of flow of fluids through porous media. The dynamic quantities v and P must be regarded as statistical averages—applying to the body of the packing as a whole rather than to its smallest elements. The velocity v is in fact not a velocity in the sense commonly used, but rather a quantity of flow per unit of area of the medium. If q is the volume of fluid flowing per unit-time through the cross-sectional area A of a packing L units in length, the permeability as defined by Eq (13-4) becomes

$$\mathcal{P} = \frac{\mu q L}{A(P_1 - P_2)} \qquad \text{Eq (13-5)}$$

where $(P_1 - P_2)$ is the pressure difference between the extremes of the section L. This equation assumes constancy of the term dP/dx throughout the length of the packed column.

Darcy's law holds for all fluids, but certain adjustments must be made for gases. The velocity v is not constant, but increases toward the outlet end (at point of lowest pressure) due to expansion. Likewise the term dP/dx changes in such a way that its ratio to the velocity is constant. On the other hand, the mass velocity ($v\rho_0$) and the gradient in squares of the pressure remain constant at every point along the length of the packing. This means

$$\frac{dP^2}{dx} = 2P\frac{dP}{dx} = \textit{constant}$$

$$\frac{P_1^2 - P_2^2}{L} = 2\mathbf{P}\left(\frac{P_1 - P_2}{L}\right) = \textit{constant}$$

where $\mathbf{P}$ is the mean pressure $(P_1 + P_2)/2$. If now the velocity v is chosen as referring to the mean pressure $\mathbf{P}$, and if $\mathbf{q}$ denotes the total fluid flowing through the packing reduced to $\mathbf{P}$, from Eq (13-4) we have

$$\mathcal{P} = \frac{\mu v P L}{\mathbf{P}(P_1 - P_2)} = \frac{\mu \mathbf{q} L}{A(P_1 - P_2)} \qquad \text{Eq (13-6)}$$
$$(\mathbf{P} = \mathbf{P})$$

Hence $\mathcal{P}$ may be calculated from gas-flow experiments except that the

volume rate of outflow must be reduced to the mean pressure across the packing.

Units of Permeability—Permeability as defined by Darcy's law (Eq 13-4) has the units $[\mathcal{P}] = [L^2]$. Expressed in terms of the definition given above, using cgs units

$$\mathcal{P} = \frac{\text{poises} \times \text{cm}^3 \times \text{cm}}{\text{sec} \times \text{cm}^2 \times \text{atms}} = \frac{\text{poises} \times \text{cm}^4}{\text{sec} \times \text{dynes}} = [L]^2$$

As an example of the calculations involved in computing permeability, let us assume that a volume of 750 cc of air is moved through a packing 10 sq cm in area and 30 cm long in 50 sec, and that the pressure at entrance is 5 cm H_2O. The outlet pressure is taken as zero, the air temperature 0 deg C, and the barometric pressure 755 mm Hg. We then have

1. Volume per unit-area per unit-time per sec $= \dfrac{750}{10 \times 50} = 1.5$ cc per sq cm
2. Total inflow pressure = 755 mm Hg + 5 cm H_2O = 75.5 × 13.6 + 5 = 1030 cm H_2O
3. Outflow pressure = 1025 cm H_2O
4. One atmosphere of pressure = 760 × 13.6 = 1033 cm H_2O
5. P_1 = (2) ÷ (4) = 1030 ÷ 1033 = 0.9971 atm
6. P_2 = (3) ÷ (4) = 1025 ÷ 1033 = 0.9922 atm
7. $(P_1 + P_2)/2 = 0.9947$
8. Corrected volume = q/A = (1) ÷ (7) = 1.5 ÷ 0.9947 = 1.51 cc per sq cm per sec
9. $P_1 - P_2/L$ = 0.0049 ÷ 30 = 0.000163 atm per cm
10. From Eq (13-6) ($\mu = 178 \times 10^{-6}$ poises at 20 deg C)

$$\mathcal{P} = \frac{178 \times 10^{-6} \times 1.51}{1.63 \times 10^{-4}} = 1.64 \text{ darcys}^*$$

The permeability calculated from Darcy's law applies to a uniform packing. However, the law may be extended to packings composed of layers having different permeabilities. We shall indicate how the permeability of such packings is calculated when the flow is perpendicular to the layers, and when parallel to them.

Case 1. Flow perpendicular to layers—Let us write Darcy's law ($\mu = 1$) as $q = \mathcal{P}A(P_1 - P_n)/L$, where A is the cross-sectional area of the packing composed of n layers, having permeabilities $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3 \ldots \mathcal{P}_n$. As written, the equation expresses the permeability $\mathcal{P}$ across n layers of thicknesses L_1, L_2, $L_3, \ldots L_n$, so that $L = L_1 + L_2 + L_3 + \ldots + L_n$. If the cross-sectional area is constant, q is also a constant

* In the petroleum industry it is usual to define the permeability in darcys for a fluid of 1 centipoise viscosity. In this case the result is 164 darcys.

$$q = \frac{\mathcal{P}_1 A(P_1 - P_2)}{L_1} = \frac{\mathcal{P}_2 A(P_2 - P_3)}{L_2} = \ldots . = \frac{\mathcal{P}_n A(P_{n-1} - P_n)}{L_n}$$

Since

$$P_1 - P_n = (P_1 - P_2) + (P_2 - P_3) + \ldots . + (P_{n-1} - P_n)$$

$$\frac{q_1 L_1}{\mathcal{P}_1 A} + \frac{q_2 L_2}{\mathcal{P}_2 A} + \ldots . + \frac{q_n L_n}{\mathcal{P}_n A} = \frac{qL}{\mathcal{P} A}$$

$$\mathcal{P} = \frac{\sum_{n=1}^{n=n} L_n}{\sum_{n=1}^{n=n} \frac{L_n}{\mathcal{P}_n}} \qquad \text{Eq (13-7)}$$

Case 2. Flow parallel to layers—For this case, since the thickness of the layers is different, we cannot assume the areas to be constant. However, the thickness of each layer may be taken as constant, $L = L_1 = L_2 = L_3 = \ldots = L_n$. Let the total area through which the flow takes place be defined as $A = A_1 + A_2 + A_3 + \ldots + A_n$; the total flow is $q = q_1 + q_2 + q_3 + \ldots . + q_n$, where the q's are the volumes flowing through each layer, and since the pressure-drop across each layer must also be constant,

$$\frac{(P_1 - P_2)}{L_1} = \frac{(P_1 - P_2)}{L_2} = \ldots . = \frac{(P_1 - P_2)}{L_n}$$

Solving for q

$$\mathcal{P}' A = \mathcal{P}_1 A_1 + \mathcal{P}_2 A_2 + \ldots . + \mathcal{P}_n A_n$$

$$\mathcal{P}' = \frac{\mathcal{P}_1 A_1 + \mathcal{P}_2 A_2 + \ldots . + \mathcal{P}_n A_n}{A_1 + A_2 + \ldots . + A_n} \qquad \text{Eq (13-8)}$$

If in Eqs (13-7) and (13-8) we put $L_1 = L_2 = \ldots . = L_n$ and $A_1 = A_2 = \ldots = A_n$, it may be shown that

$$\frac{\mathcal{P}'}{\mathcal{P}} = \frac{\sum_{n=1}^{n=n} \mathcal{P}_n \sum_{n=1}^{n=n} \frac{1}{\mathcal{P}_n}}{n^2} \qquad \text{Eq (13-9)}$$

Applications of Darcy's Law—It is clear that the permeability constant is a function of many variables, of which particle-size and porosity are obviously the most important. Darcy's law, however, has important applications to ground-water hydrology where methods of classical

hydrodynamics thus far have been in vain, to problems of irrigation and water filtration, and especially to the production of gas and oil from underground reservoirs. It is the basic formula of fluid flow through porous media. This chapter deals with the most elementary uses of Darcy's formula. For more complete information the reader is referred to Muskat's (1937*a*) important contributions in the field of fluid flow through porous media. The general equations here developed show the relationships of the variables involved. As might be expected, certain general assumptions must be made. Well pressures can be measured and also the depth of the water- or oil-bearing stratum; but the extent of the stratum or its permeability (whichever is to be determined) depends upon the accuracy with which either is estimated. Let us consider first the problem of radial flow. Assume a well just penetrates a water-bearing stratum of thickness δ; then if the stratum is everywhere homogeneous, the permeability is given by the equation

$$\mathcal{P} = \frac{\mu q \ln (R_1/R_2)}{2\pi\delta(P_1 - P_2)} \qquad \text{Eq (13-10)}$$

where μ is the viscosity, q the rate of flow, R_1 the radius of the reservoir, R_2 the radius of the well, P_1 the reservoir pressure at R_1, and P_2 the bottom-hole pressure. When the units are as previously discussed, the permeability is expressed in darcys. Eq (13-10) applies when the fluid is a liquid; for gases, $\mathbf{q}$ (the rate of flow reduced to mean pressure $(P_1 + P_2)/2$) is substituted for q. To determine permeability, the measurements which must be taken are the closed-in bottom-hole pressure, which is essentially the effective reservoir pressure for the well, and the bottom-hole pressure for a measured rate of flow. The value of R_1 can only be estimated, but it is to be noted that $\ln (R_1/R_2)$ changes very slowly.

The following problem is of special interest. Assume a water-bearing stratum on an impermeable base with the well penetrating into it. When water is pumped, its normal level is reduced. If the pump is stopped for a sufficient length of time, the water level will return to its normal level. Let the height of the normal water level from the impervious layer be h_1, and h_2 the level after pumping a volume qt, where q is the volume of flow per unit-time, and t the time of pumping. Then

$$\mathcal{P} = \frac{\mu q \ln (R_1/R_2)}{\pi\rho_0 g(h_1^2 - h_2^2)} \qquad \text{Eq (13-11)}$$

If $\mathcal{P}$ is to be expressed in darcys, μ is the viscosity in poises, q the volume of water in cc per sec, R_1 and R_2 respectively the radii of reservoir and well (any units), ρ_0 the density of the fluid, g the acceleration of gravity

(0.00097 atm for unit-density), h_1 the normal water level in cm, and h_2 the reduced water level, also in cm. This equation can only apply when $h_1 \leqq$ the thickness of the stratum δ. If an artesian drive exists so that $h_1 > \delta$,

$$\mathcal{P} = \frac{\gamma\mu q \ln (R_1/R_2)}{\pi\rho_0 g(2\delta h_1 - \delta^2 - h_2^2)} \qquad \text{Eq (13-12)}$$

When the well does not completely penetrate the stratum to the impermeable bed, the permeability must be multiplied by a correction factor, γ. Values of this correction factor computed by Wyckoff *et al* (1934) are presented in Table 51.

TABLE 51—CORRECTION-FACTOR γ FOR WELLS PARTIALLY PENETRATING A PERVIOUS STRATUM

Depth of stratum	Per cent penetration							
	5	10	15	20	30	50	70	100
50	0.35	0.45	0.51	0.57	0.64	0.75	0.82	1.0
75	0.40	0.50	0.56	0.61	0.69	0.79	0.85	1.0
125	0.45	0.56	0.62	0.67	0.72	0.81	0.89	1.0
200	0.51	0.61	0.68	0.71	0.78	0.83	0.90	1.0

POISEUILLE'S LAW

Poiseuille's equation is in some respects closely analogous to Darcy's.

$$q = \frac{\pi D_c^4 P}{128\mu L} = \frac{\mathcal{P}A}{\mu} \cdot \frac{P}{L} \qquad \text{Eq (13-13)}$$

where the terms are as previously defined. This equation is derivable from classical hydrodynamics and gives the flow in terms of tube diameter and other constants. If now the diameter D_c can be considered as that of an average pore through the packing (see Chapter 6), then theoretically we should obtain results comparable to those given by Darcy's law. As a matter of historical interest, Poiseuille announced his results several years prior to Darcy, and it may seem as though the latter could have been anticipated on the basis of his law. However, each was developed for entirely different conditions and there is no reason to presume that they are alike. In fact, such an idea of their identicalness, in so far as packings are concerned, is purely intuitive and no rigorous proof can be given.

Slichter's Equation—In Chapter 6 we have given a rather extensive account of Slichter's (1899) analysis of the packing of spheres, and we discussed the average pore-area in a unit-cell and the mean length of path. The former was expressed in terms of the diameter of the spheres,

assumed to be uniform throughout the packing. After making certain corrections and adjustments to suit all conditions, Slichter derived the modified Poiseuille relationship

$$q = \frac{10.2Ad^2h}{K_s\mu L} \qquad \text{Eq (13-14)}$$

where q is expressed in cc H_2O per sec, A the cross-sectional area of the packing in sq cm, h the driving pressure in cm H_2O, d the diameter of the spheres in cm, K_s a packing constant depending upon porosity, μ the viscosity in poises, and L the thickness along the direction of flow in cm. Thus we see that for Darcy's permeability $\mathcal{P}$ Slichter substituted

$$\mathcal{P} = \frac{10.2d^2}{K_s} \qquad \text{Eq (13-15)}$$

A few words should be said about modification of the Slichter equation when applied to irregular particles. In this case, the diameter d is replaced by a diameter d_s which may be defined as a diameter of the particles such that if all of them were of this diameter the packing would have the same permeability as it actually has at a given temperature and porosity. This diameter has resulted in much confusion and misunderstanding. For closest packing $K_s = 84.4$ (porosity 26 percent), and for porosities of 30, 35, 40, and 45 percent, K_s has the values 52.5, 31.6, 20.3, and 13.7, respectively. The importance of Slichter's formula was recognized by King (1899), who applied it in calculating a particular average diameter of irregular particles. Mavis and Wilsey (1936) give the following equation for computing values of K_s

$$\frac{1}{K_s} = 0.05\left(\frac{\vartheta}{40}\right)^{3.3} \qquad \text{Eq (13-16)}$$

where ϑ is the porosity (in percent). While Slichter's formula fails in many respects (for reasons that will soon be apparent), his analysis of packing behavior is not only the first of its kind, but the clearest and most scholarly presentation of the subject yet written. Professor King's companion article also must be highly regarded and deserves serious attention, not only for the vast amount of valuable information it contains but also because his monograph touches upon the experimental aspects of studies in porous media that have been duplicated unwittingly by scores of workers during recent years.

A comprehensive formula based upon an analysis similar to Slichter's is that of W. O. Smith (1932), whose analysis is as follows: The value for the mean velocity through a triangular capillary (see Chapter 6) formed by a rhombohedral packing is (Slichter, 1899)

$$v_{av} = 0.0289A\,\Delta P/\mu L \qquad \text{Eq (13-17)}$$

where v_{av} is the mean velocity, A_{av} the mean area of the triangular pore, and ΔP the pressure-drop across a length of packing. Now A_{av} has been shown to be

$$A_{av} = 0.360d^2\left[\frac{0.985}{(1-\vartheta)^{2/3}} - 1\right] \qquad \text{Eq (13-18)}$$

by Eq (6-5) where d is some average diameter of the particles, and ϑ the voids. Thus, from Eqs (13-17) and (13-18) we obtain

$$v_{av} = 0.008\left[\frac{1}{(1-\vartheta)^{2/3}} - 1\right]d^2\Delta P/\mu L \qquad \text{Eq (13-19)}$$

The quantity of fluid moved per second through each capillary is $v_{av}A_{av}$ and the number of capillaries per unit-area is $1/A'$, where A' has been taken by Smith to be the area of the triangle through the centers of the pores. The amount of fluid moved through a section of packed column of area A'' in a unit-time is therefore

$$q = v_{av}A_{av}A''/A'$$

Using the procedure given in Chapter 6, we have

$$A' = \frac{\sqrt{3}}{4}(d+\delta)^2 = \frac{0.360d^2}{(1-\vartheta)^{2/3}} \qquad \text{Eq (13-20)}$$

Hence, from Eqs (13-18) to (13-20) we obtain

$$q = 0.008(1-\vartheta)^{2/3}\left[\frac{0.985}{(1-\vartheta)^{2/3}} - 1\right]^2\frac{d^2\Delta PA''}{\mu L} \qquad \text{Eq (13-21)}$$

giving the quantity of flow per sec through a packed column L and section A''.

We shall now introduce the permeability $\mathcal{P}$,

$$\mathcal{P} = qL\mu/A''\Delta P \qquad \text{Eq (13-22)}$$

Since $\Delta P = gh$ in the case of downward flow, and $g = 980$ cm per sec^2, from Eqs (13-21) and (13-22) we obtain

$$\mathcal{P} = 7.9(1-\vartheta)^{2/3}\left[\frac{1}{(1-\vartheta)^{2/3}} - 1\right]^2 d^2 \qquad \text{Eq (13-23)}$$

Hence, the quantity $\mathcal{P}/d^2$ is a function of the voids only.

Using data obtained by Green and Ampt (1911, 1912), Eq (13-23) was tested by Smith who found he could accurately predict the value of $\mathcal{P}$ and $\mathcal{P}/d^2$ for glass beads having diameters ranging from 0.09 to 0.025 cm, and voids from 0.36 to 0.40. Smith also obtained good agreement with sand packings (diameter of particles ranging from 0.08 to 0.019 cm, and voids between 0.34 and 0.40). He secured better agreement,

however, when instead of the average section A_{av} he used the minimum pore section A_{min} (area of a triangular pore).

$$A_{min} = \sqrt{3}\,(d + \delta)^2/4 - \pi d^2/8 = 0.390\left[\frac{0.912}{(1 - \vartheta)^{2/3}} - 1\right]d^2 \quad \text{Eq (13-24)}$$

The flow through the packing, now assumed as taking place through the minimum areas of the pore sections, is thus

$$q = 0.01(1 - \vartheta)^{2/3}\left[\frac{0.912}{(1 - \vartheta)^{2/3}} - 1\right]^2 \frac{d^2 \Delta P A'}{\mu L} \quad \text{Eq (13-25)}$$

and

$$\mathcal{P} = 9.5(1 - \vartheta)^{2/3}\left[\frac{0.912}{(1 - \vartheta)^{2/3}} - 1\right]^2 d^2$$

This equation gives better results for sand. It is to be noted that $\mathcal{P}/d^2$ is dimensionless. Plots of $\mathcal{P}/d^2$ versus ϑ show a straight-line relationship.

Hazen's Equation—Anticipating Slichter by some seven years, Hazen (1892, 1893) proposed the following formula based on studies of sand filters

$$q = K_H d_H^2 \frac{h}{L} \quad \text{Eq (13-26)}$$

where d_H is Hazen's "effective" diameter of sand particles having uniformity coefficients of less than 5.0 (see Chapter 3), and h/L the pressure-drop through the filter per unit-depth. Hazen also supplied a temperature correction-factor by which the right-hand term must be multiplied, but this may be neglected in view of the wide range of values assignable to K_H. If q is expressed as cu ft of water per day per sq ft of filter area, d_H in mm, h the loss of head in feet of water, and L in feet, then values of K_H range from 1300 for dirty sand to above 4000 for clean sands. The usual limits of K_H for ordinary sands lie between 2300 and 3300.

The above equation applies to values of d_H between 0.1 and 3 mm and contains no term for porosity. His formula has found wide acceptance among sanitary engineers ever since its publication. It is of historical interest because it represents the first attempt to recognize the importance of particle-size and to give a reasonably satisfactory method for its representation.

Mavis and Wilsey (*loc cit*) recently retested Hazen's formula, and on the basis of extensive experiments on Iowa River sands give the following equations

$$q = 2300 d_H^2 \frac{h}{L}\left(\frac{T + 10}{60}\right)$$

$$q = 980 d_{34}^2 \left(\frac{\vartheta}{40}\right)^6 \left(\frac{T+10}{60}\right) \qquad \text{Eq (13-27)}$$

where T is the temperature of the water in deg F, d_{34} the 34 percent size in mm, and other terms are the same as in the previous equation. Note that the porosity in the second equation enters as the sixth power, while in Eq (13-3) the flow is as the seventh power of the porosity.

Kozeny's Equation—The equations thus far presented show that the flow is proportional to the square of the particle-diameter and to some power of the porosity. The shape of the particles is also of some consequence, but this factor is contained in the proportionality constant. The most important of these variables is the porosity since, as we have shown, the flow may vary from the 3.3-power (Slichter) to the 7.0-power (Buckingham) of this factor. Next in importance is a concept of the size-distribution. We have shown in Chapter 6 that some relation exists between pore-space and particle-size; but obviously the methods used heretofore for expressing particle-size make small allowance for size-distribution and inadequately take cognizance of the shapes of the particles themselves. The general problem of the relationship of these variables was first made empirically by Krüger (1918–1920), but its exact mathematical formulation is due to Kozeny (1927). His equation may be stated as follows:

$$q = \frac{kA\vartheta_w^3}{\mu S_w^2} \cdot \frac{\Delta P}{L} \qquad \text{Eq (13-28)}$$

where S_w is the specific surface of the material whose fractional voids per unit-weight are ϑ_w. If the size and shape of the particles are kept constant, the surface per unit-volume, S_v, is also constant and is related to S_w by the equation

$$S_w = (1 - \vartheta_w) S_v \qquad \text{Eq (13-29)}$$

Hence Eq (13-28) becomes

$$q = \frac{kA\vartheta_w^3}{\mu(1-\vartheta_w)^2 S_v^2} \cdot \frac{\Delta P}{L} \qquad \text{Eq (13-30)}$$

If the fluid flows through a layer of particulate matter under its own head, h

$$\frac{h}{L} = \frac{1}{kg}\left(\frac{\mu}{\rho_0}\right)\left(\frac{q}{A}\right) S_v^2 \frac{(1-\vartheta_w)^2}{\vartheta_w^3} \qquad \text{Eq (13-31)}$$

where ρ_0 is the density of the fluid and g the gravitational acceleration constant. This equation may be written as follows:

$$q = \frac{\mathcal{P}_c A}{\mu} \cdot \frac{h}{L}$$

where

$$\mathcal{P}_c = \frac{kg\rho_0}{S_v^2} \cdot \frac{\vartheta_w^3}{(1-\vartheta_w)^2} \qquad \text{Eq (13-32)}$$

Carman (1938) defined permeability as follows:

$$\mathcal{P}_c' = \frac{g}{k'\nu S_v^2} \cdot \frac{\vartheta_w^3}{(1-\vartheta_w)^2} \qquad \text{Eq (13-33)}$$

where $k' = 1/k$ and $\nu = \mu/\rho_0$, and by means of this equation determined the surface of an aggregate of particles (see Chapter 16). The value of k' was found to be 5.0 for most materials. Note that $\mathcal{P}_c'$ is defined by the modified Darcy equation (Eq 13-31)

$$q = \mathcal{P}_c' A \frac{h}{L}$$

so that if q and h are measured, and A and L are constants in a given permeability measuring device, $\mathcal{P}_c'$ is easily determined. Carman found that the Kozeny equation was reliable over a wide range of conditions.

The basic Kozeny equation (Eq 13-28) may be recast to include particle-diameter. It is clear from the discussion given in Chapter 3 that $S_w \propto 1/d$. Hence

$$q = \frac{k''A\vartheta_w^3}{\mu(1-\vartheta_w)^2} \cdot d^2 \cdot \frac{\Delta P}{L} \qquad \text{Eq (13-34)}$$

where k'' is another constant.

The Kozeny form of equation was also obtained by Fair and Hatch (1933). These investigators were unaware of Kozeny's contribution and arrived at the porosity relationship by a simple and ingenious device. Because the ultimate equation developed applies to both streamline- and turbulent-flow conditions, we shall review their contribution briefly.

It may be shown by the methods of dimensional analysis that the flow of fluids through granular material is given by the general equation

$$\frac{h}{L} = \frac{k}{g}\left(\frac{\mu}{\rho_0}\right)^{2-n} v^n D_c^{n-3} \qquad \text{Eq (13-35)}$$

where h is the head of fluid (assumed to be flowing under influence of gravity), D_c the diameter of channel, and n an exponent ranging in value from 1.0 for streamline flow to 2.0 for turbulent flow. The other constants and variables are as previously defined. The conduit or channel diameter D_c is replaced by the hydraulic radius, a general term used in hydraulics and defined subsequently. The following transformations are made:

$$\text{hydraulic radius} = \frac{\text{cross-sectional area of conduit}}{\text{wetted perimeter of conduit}}$$

$$= \frac{\text{cross-sectional area} \times \text{length}}{\text{wetted perimeter} \times \text{length}}$$

$$= \frac{\text{volume of voids}}{\text{surface-area of solids}} = \frac{\text{volume of solids}}{\text{surface-area of solids}} \cdot \frac{\vartheta}{1 - \vartheta}$$

Now the velocity v through the packing conduits is clearly v'/ϑ where v' is the approach velocity. If S_v is the surface represented by a unit-volume of the particles, then Eq (13-35) becomes

$$\frac{h}{L} = \frac{k}{g}\left(\frac{\mu}{\rho_0}\right)^{2-n}\left(\frac{v'}{\vartheta}\right)^{n}\left(\frac{S_v(1 - \vartheta)}{\vartheta}\right)^{3-n}$$

Thus, for streamline motion, putting $n = 1$,*

$$\frac{h}{L} = \frac{k}{g}\left(\frac{\mu}{\rho_0}\right) v' S_v^{\,2} \frac{(1 - \vartheta)^2}{\vartheta^3} \qquad \text{Eq (13-36)}$$

The value of S_v may be determined from Eq (3-46), which by the methods discussed in Chapter 3 is equivalent to

$$S_v = \frac{\alpha_s}{100\alpha_v} \sum \frac{w}{d}$$

where w is the weight percentage of particles of uniform sieve-diameter d. Values of the ratio α_s/α_v have been shown to lie between 6.0 and 7.7.

If the beds are stratified and graded so that fine particles are at the top and coarse ones at the bottom Fair and Hatch assume that for all practical purposes the porosity factor of the various layers is constant, and that the depth of layer created by each sieve-separated portion is proportional to its weight. Since the area-volume shape-factor, α_s/α_v, is greatest at the top and lowest at the bottom of the stratified column, the pressure-drop will vary accordingly—in other words, approximately inversely as the particle-size. Since the sizes are graded, the whole thickness of the packing must be considered. Fair and Hatch suggest an average value of S_v, weighted with respect to flow resistance, and based upon the fact that for individual layers the percentage of total weight w is also a percentage of the total depth; thus

$$S_v^{\,2} = \frac{\alpha_s^{\,2}}{100\alpha_v^{\,2}} \sum \frac{w}{d^2}$$

* Note that in this equation k is dimensionless whereas in Eq (13-31) it has dimensions $[ML^{-3}]$. Obviously $\vartheta_w = \vartheta/\rho$, where ρ is the density of the particles Also $(1 - \vartheta_w) = (1 - \vartheta)/\rho$, so that the ratio $\vartheta_w^3/(1 - \vartheta_w)^2 = \vartheta^3/(1 - \vartheta)^2$. We have retained the designations given in the Kozeny equation since it is consistent with the original development. This procedure is not followed in Chapter 16 where ϑ_w is replaced by ϑ.

or its logarithmic equivalent based on size-measurements

$$\log S_v^2 = \log\left(\frac{\alpha_s}{\alpha_v}\right)^2 - \log d_g^2 + 4.605 \log^2 \sigma_g$$

Relative Permeability—The relative permeability may be defined as the variation of flow with respect to the porosity factor. The flow at any other porosity is determined by any of the following equations:

Buckingham $q \propto \vartheta^{7.0}$ Eq (13-3)

Slichter $q \propto \vartheta^{3.3}$ Eq (13-16)

Smith $q \propto (1 - \vartheta)^{2/3}\left[\frac{1}{(1 - \vartheta)^{2/3}} - 1\right]$ Eq (13-23)

Mavis and Wilsey $q \propto \vartheta^{6.0}$ Eq (13-27)

Kozeny-Hatch $q \propto \frac{\vartheta^3}{(1 - \vartheta)^2}$ Eq (13-29)

The effect of porosity, as accounted for in the various equations given, is illustrated in Figure 72.

OTHER EQUATIONS OF FLOW

Chilton and Colburn Equation A correlation of existing data on flow through packings under all types of conditions was undertaken by Chilton and Colburn (1931). These data were plotted as a friction factor (Eq 2-13) against Reynolds number. Their equations cover both streamline and turbulent conditions of flow. The equation applying to the streamline range is

$$\frac{\Delta P}{L} = \frac{0.97\gamma\mu G}{\rho_0 d_{as}^2} \quad \text{Eq (13-37)}$$

and for the turbulent range

$$\frac{\Delta P}{L} = \frac{2.27\gamma\mu^{0.15}G^{1.85}}{\rho_0 d_{as}^{1.15}} \quad \text{Eq (13-38)}$$

where ΔP is the pressure-drop in inches of water, γ a wall-effect factor (see Chapter 6) depending upon the ratio of particle-diameter

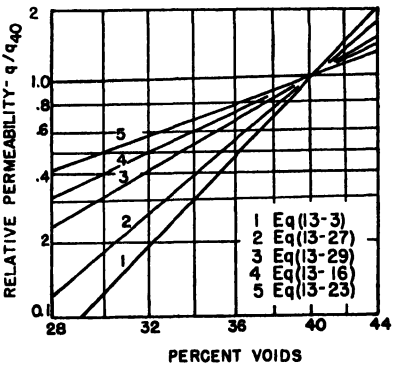


FIGURE 72. EFFECT OF POROSITY ON FLOW THROUGH SAND ACCORDING TO THE EQUATIONS OF VARIOUS INVESTIGATORS (REFERRED TO POROSITY 40 PERCENT = 100).

to container diameter and upon the type of flow (presented in Table 52), μ the viscosity in centipoises, L the length of packing in ft, $G = \rho_0 v$, the mass velocity based on cross-sectional area of empty tube in lb per sq ft per sec, ρ_0 the average density of the fluid, and d_{av} the average diameter of the particles in inches. The critical region between streamline and turbulent flow is defined by the relation

$$\frac{d_{av}G}{\mu} = 0.32 \text{ to } 0.64 \qquad \text{Eq (13-39)}$$

This equation gives the critical velocity which separates the two regions. Considering the large amount of data analyzed by Chilton and Colburn and the critical study of their limitations, there can be no question that with regard to the flow of gas through packings Eqs (13-37) and (13-38) are the best available at the present time. The data apply to particles ranging from 0.1 to 1.0 in. in diameter.

TABLE 52—VALUES OF WALL-FACTOR γ AS DETERMINED BY FURNAS (1930) FOR FRACTIONAL VOIDS 0 50

Ratio particle-diameter to container-diameter	Values of	
	Streamline flow	Turbulent flow
0.0	1.00	1.00
0.05	0.90	0.84
0.10	0.83	0.71
0.15	0.78	0.65
0.20	0.74	0.60
0.25	0.72	0.58

Furnas' Equation—Probably the most complete analysis of gas flow through beds of broken solids ever undertaken was made by Furnas (1930*a*, *b*). A full outline of Furnas' technique cannot be presented in these pages, but his basic equation is as follows:

$$\frac{\Delta P}{L} = aq^B$$

where a and B are constants, and his final equation for a container of large size in cgs units is

$$\frac{\Delta P}{L} = \frac{82.8}{P d_p^{1.68-\vartheta}} 10^{B_{air}(25) + 3.91 - 3.44(\alpha/\vartheta_n) + 4.2\vartheta + 0.62(\alpha\vartheta_n)^2 - 8.9\vartheta^2} \times \left(\frac{T}{298}\right)^{1.28 - 0.57 B_{air}(25)} \times \left(\frac{\nu_{air}}{\nu_{gas}}\right)^{\frac{3.51 B_{air}(T') - 1.75}{\log 1/\nu_{gas} + 2.70}} \times q^{B_{gas}(T')} \qquad \text{Eq (13-40)}$$

As a first approximation

$$\frac{\Delta P}{L} = \frac{1390 T^{0.15} q^{B_{air}(T')}}{P d_p^{1.2}} \qquad \text{Eq (13-41)}$$

The significance of the various constants is as follows:

ΔP = pressure-drop in cm of water
P = pressure in system in similar units
L = length of column in cm
T = absolute temperature, deg C
T' = air or gas temperature, deg C
d_n = nominal spherical diameter (see Chapter 3)
α = shape-factor—nominal spherical diameter divided by the average screen size
ϑ_n = normal voids (see Chapter 6)
ϑ = actual voids
ν = μ/ρ_0; μ = viscosity in poises; ρ_0 = density
q = rate of gas flow—liters per sec per sq cm

$$B_{\text{air (25)}} = \frac{0.006 + d_n}{0.036 + 0.51 d_n}$$

$$B_{\text{air }(T')} = 2.19 B_{\text{air (25)}} - 0.59 - 0.48\,(B_{\text{air (25)}} - 0.50) \log T$$

or as a first approximation,

$$B_{\text{air }(T')} = 3.78 - 0.72 \log T$$

$$B_{\text{gas }(T')} = 0.77 B_{\text{air (25)}} + 0.12 + \frac{B_{\text{air (25)}} - 0.50}{3.51} \log \frac{1}{\nu}$$

Furnas gives the average value of $\alpha/\vartheta_n = 1.65$. Eq (13-40) has been criticized by Chilton and Colburn (1931) on the grounds that the equation is based on flow in the turbulent and streamline region and is therefore open to question. It is well known that pressure-drop is markedly affected by the nature of the flow through the packing. In justice to Furnas, however, it is necessary to point out that his data cover a range of particle-size and temperature-conditions not achieved by any other equation yet proposed. For blast-furnace practice Furnas' equation is unquestionably more accurate than those given elsewhere in this text.

FLOODING VELOCITIES

Packings are often used as scrubbers in which gases to be absorbed or cleaned rise vertically against a countercurrent of liquid. In designing scrubber columns of this type, it is necessary that the flow of the fluids be such that there will be no liquid holdup at the top, so that the gases after passing through the packing must bubble through a layer of liquid. Such a holdup will take place if the gas velocities are high, and the efficiency of the column as an absorber is reduced.

The critical gas velocity can be determined by several methods. Direct observation will serve if the column is made of glass; or if pressure drop is plotted against gas velocity, a break in the curve will be noted.

In case of visual determination, the decision must be made whether to record the first sign of buildup (often indefinite) or to record splashing.

Several investigations of flooding velocities have been made and the results of most are in general agreement with one another. Relationships developed, however, are empirical and follow the pattern developed by Sherwood and his associates (1938). We reproduce here a simple relationship due to Elgin and Weiss (1939). If values of

$$\frac{v_L^2 S_v}{g\vartheta^2}$$

are plotted against

$$\left[\left(\frac{\rho_L}{\rho_G}\right)\left(\frac{G_G}{G_L}\right)\right]^{1/2}$$

when flooding occurs, all points will lie on a single curve. Thus, for a given packing, gas, and liquid, we are enabled to determine the gas velocity necessary to prevent flooding. In the above relationship v_L is the superficial velocity of the liquid, S_v the surface area of the packing per unit volume, ϑ the fractional voids after the packing has been wetted and drained, ρ_L and ρ_G the densities, G_L and G_G the corresponding mass velocities of the liquid and gas, respectively, and g the acceleration constant of gravitation. Since the relationships are dimensionless the units must be such as to reduce each part to unity.

MEASUREMENT OF PERMEABILITY

We have shown the importance of permeability in the solution of problems dealing with the flow of fluids through packings. Its determination by laboratory procedures is consequently of interest. Many permeability devices have been developed and when used with proper precautions each one gives accurate results. We shall here describe one of the simplest forms developed by Fishel and Stringfield (1937) of the Ground Water Division of the U. S. Geological Survey. The essential elements of the device are shown in Figure 73, and consist of a brass percolation cylinder A, having an inside diameter 3.5 cm and height 14.5 cm. This is connected with a graduated manometer tube B, with an inside diameter 0.89 cm. A screen is placed on the bottom of the manometer tube with a copper tube F. In making the permeability test the manometer tube is adjusted so that when the water overflows the percolation chamber, the meniscus in the manometer tube is exactly at the zero line. After this adjustment, the water is withdrawn from the apparatus until the water level stands about half an inch from the screen in the bottom of the percolation cylinder. The material to be tested

is then poured into the percolation cylinder, care being taken to keep the liquid in the manometer within sight. Water may be added slowly to the manometer tube to prevent the decline of the water level during the filling process. It is essential that great care be taken in filling the percolation cylinder since porosity greatly affects the permeability. Then the cylinder is completely filled, and water is added slowly to the manometer tube until the material in the test cylinder becomes saturated and the water level in the manometer is raised nearly to the top. The water level in the tube then declines as the water percolates upward through the material in the percolation cylinder. The passage of the meniscus at each marking of the manometer is then noted. The permeability may then be computed as follows. Using Darcy's equation in increment form

$$dQ = \frac{\mathcal{P}_c}{\mu} \cdot A \cdot \frac{h}{L} dt$$

where A and L are the area and length of the test cylinder, and h the height of liquid at the time t. If h_0 is the initial height of the liquid and A_m is the area of the manometer tube, then

$$h = h_0 - \frac{Q}{A_m}$$

$$dh = -\frac{1}{A_m} dQ$$

and

$$-\frac{A_m dh}{h} = \frac{\mathcal{P}_c}{\mu} \cdot \frac{A}{L} \cdot dt$$

Integrating between the limits $t = 0$ and $t = t$, and $h = h_0$ and $h = h$ and solving for $\mathcal{P}_c$

$$\mathcal{P}_c = -\frac{\mu L A_m}{At} \ln \frac{h}{h_0}$$

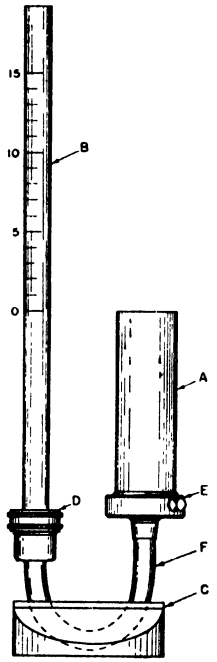


FIGURE 73. DIAGRAM OF PERMEABILITY APPARATUS FOR USE IN THE FIELD. (*A*) BRASS PERCOLATION CYLINDER WITH SCREEN IN BOTTOM; (*B*) GLASS MANOMETER TUBE GRADUATED IN CENTIMETERS; (*C*) LEAD BASE; (*D*) PACKING GLAND; (*E*) PIPE REDUCER; (*F*) COPPER TUBE.

If the diameters of the percolation cylinder and the manometer tube are D and D_m, respectively,

$$\mathcal{P}_c = -\frac{\mu L}{t} \left(\frac{D}{D_m}\right)^2 \ln \frac{h}{h_0}$$

or in terms of common logarithms

$$\mathcal{P}_c = -2.303\frac{\mu L}{t}\left(\frac{D}{D_m}\right)^2 \log\frac{h}{h_0} \qquad \text{Eq (13-42)}$$

Problems

1. Given a sand-filter bed 2 ft in depth and composed of particles having an average diameter of 0.2 mm. The porosity is estimated to be 34 percent. If a constant head of 5 ft of water at 77 deg F is maintained on the bed, calculate the rate of flow per sq ft through the bed. Assume the shape-factor of the particles to be 6.1.

2. Water passes through a sand bed 2.54 cm in depth at 2 cu cm per sec under a net pressure of 2.84 atm. If the temperature is 78 deg F and the viscosity 0.884 centipoise and the area of the bed 2.85 sq cm, compute the permeability.

3. If air flows through the sand bed in Problem 2, the rate is 55.6 cu cm per sec, the inlet pressure being 1 983 atm and the outlet pressure 0.966 atm. Take the viscosity of air at 78 deg F as 0.0182 centipoise. *Note:* Compute the mean pressure P

$$\mathbf{P} = \frac{1\ 983 - 0.966}{2} = 1\ 425 \text{ atm}$$

Hence,

$$\mathbf{q} = 55.6 \times \frac{0.966}{1.425} = 38\ 5 \text{ cu cm per sec}$$

4. In Eq (13-10), assume the following:

S—6 ft	P_1—1470 lb per sq in
R_1—500 ft	P_2—1270 lb per sq in
R_2—0.3 ft	q—2000 cu cm per sec

$\mu = 1.1$ centipoises at 60 deg C

Compute the permeability of the stratum.

5. A certain ore is sized to pass a square-mesh sieve opening of 2.67 cm and to be retained on a 1.89-cm sieve. The bed made from this material is 95 cm high and 25.9 cm in diameter. Gas at 800 deg C is made to flow through the packing at a rate of 0.2 liter per sq cm per sec and the outlet pressure is 76 cm. What is the pressure-drop through the cylinder? Assume voids to be 0.52, gas density 0.00024 g per cu cm, and viscosity 4000×10^{-7} poises.

6. Water flows upward through a bed of sand, causing an expansion of the sand Determine the condition for equilibrium and deduce a possible relation between velocity and porosity.

Note: Using the generalized equation above Eq (13-36) of the text, we have that h/L, which is the resistance to flow, must be equal to the weight of the particles in the fluid, that is

$$h/L = (\rho - \rho_0)/\rho(1 - \vartheta)$$

or

$$\vartheta^3/(1 - \vartheta)^{2-n} = [(k/g)(\mu/\rho_0)^{2-n}v'^n S_v^{3-n}][\rho/(\rho - \rho_0)]$$

L. Hatch (1943) found that

$$\vartheta^3/(1 - \vartheta) \propto v^1 \text{ to } v^{1.7}$$

depending on the size of the particles. Sand particles $\geqq$ 20/28 mesh varying as the 1.7 power of the velocity.

CHAPTER 14

INFILTRATION AND PARTICLE-MOISTURE RELATIONSHIPS

THIS chapter is a continuation of the preceding one. Herein we shall consider the moisture-distribution in packings under various natural conditions. The laws discussed have an important bearing on soil-moisture studies, particularly capillarity.

MOISTURE-DISTRIBUTION IN PACKINGS

Effect of Particle-Size—The effect of particle-size on the water-holding capacity of a packing was investigated by King (1899) in a series of carefully conducted experiments. King filled five galvanized-iron cylinders, 8 ft long and 5 in. in diameter, with sands of different degrees of fineness. These sands were screened and their effective size based on Slichter's formula, Eq (13-14), determined. Each column was filled with water from the bottom to expel the air, and percolation was started in all columns simultaneously. Evaporation was kept at a minimum by a small opening for replacing the water drained downward with air.

TABLE 53—SUMMARY TABLE OF MOISTURE-RETENTION STUDIES OF SANDS OF DIFFERENT SIZES (KING 1899)

Item	Type of sand				
	No. 20	No. 40	No. 60	No. 80	No. 100
Effective size (mm)	0.47	0.18	0.16	0 12	0 083
Voids	0.39	0.49	0 41	0 41	0 40
Total moisture retained after $2^1/_2$ years:					
grams	2121	2475	3515	4576	5832
percent	4.24	5.05	7.25	9.41	11.82
Percolation first 30 minutes:					
grams	3298	6427	1730	486	390
percent	3 01	3 44	3 00	0 86	0.56
Percolation first 9 days:					
grams	7499	7043	6234	4874	4154
percent	14.98	14.35	12.86	10.02	8.42
Percolation after first 9 days:					
grams	805	839	580	740	621
percent	1.61	1.71	1.20	1.73	1.26

Although King's experiments continued for two and one-half years, more than half of the water was drained from the coarsest sand within the first hour, while in the finest sand (No. 100) the amount drained in the same period was approximately 10 percent. Data pertaining to these experiments are given in Table 53. This table shows that the retentivity of a packing becomes less as the particle-size increases.

King's experiments, described above, showed a complicating feature which makes it difficult to determine the borderline between gravitational and capillary moisture. After the columns drained for two and one-half years, King determined the moisture-content of each column at 3 in. sections. These results are shown in Table 54. The table illus-

TABLE 54—PERCENTAGE DISTRIBUTION OF MOISTURE LEFT IN COLUMNS OF SAND AFTER PERCOLATION HAD CONTINUED FOR 2 AND ONE-HALF YEARS (KING 1899)

Height of section above ground (in.)	Type of sand				
	No. 20	No. 40	No. 60	No. 80	No. 100
96–93	0.27	0.17	0.22	1.26	3.44
90–87	0.23	0.16	0.29	1.34	3 84
81–78	0.29	0.19	1.07	2.32	4.19
72–69	1.18	1.16	1.80	3.12	4.94
60–57	1.83	1.86	2.26	4.22	6.77
51–48	2.02	1.92	2.36	5.42	9.42
42–39	2.06	2.18	3.08	7.47	12.58
30–27	2.63	3.14	6.36	12.95	17.20
21–18	4.26	6.76	23.57	19 08	20.49
12–9	16.08	21.31	22.46	22.69	22.68
3–0	21.58	24.61	23.54	25 07	24.08

trates the change in moisture-content along the length of the column, as well as the differences occurring between the types of sand. However, with the procedure just outlined it is impossible to determine the boundary between gravitational and capillary moisture. In the next chapter it will be shown that if the bottom of each column were placed in water (the sand originally being dry) and the water allowed to creep upward by capillary action, the moisture-content would vary with distance from the water level.

Moisture-Equivalent—Briggs and McLane (1907) made extensive physical studies on what has been termed "moisture-equivalent"—that is, "the percentage of water retained by a soil, when the moisture content is reduced by means of a constant centrifugal force until it is brought into a state of capillary equilibrium." While King's procedure yields specific information on the relative retentiveness of various soils only after considerable time, the method of Briggs and McLane is rapid,

easily standardized, and gives good and consistent results on relative retentiveness. The device used consists of a series of eight cups mounted on a heavy cylindrical head. This head is rotated at a speed of 5000 rpm producing a centrifugal force of 2940 ± 60 times gravity. The soil sample is placed in one of the cups (eight samples may be tested at one time) in a layer approximately 5 mm in depth. The moisture released passes through a filter paper backing to cotton or other absorbent material. Samples are centrifuged for 30 or 45 min, although if more than 2 g (approximately) are centrifuged the time required may be proportionately greater. The percentage of moisture remaining at the end of the test is called the *moisture-equivalent.**

When the moisture-equivalent is plotted against the reciprocal of the centrifugal force, resulting curves for most soils and particularly for loose sands and loams containing small amounts of clay, are straight lines. If M_e represents the moisture-equivalent for a given material and F the centrifugal force, then

$$M_e = M_{oe} + \frac{a}{F}$$

where M_{oe} is the intercept on the ordinate axis and a the slope of the curve. This relation holds true over only a limited range of velocities, but is sufficiently accurate for the usual test runs.

Briggs and McLane investigated the moisture-equivalent of 104 soils, and on the basis of composition and particle-size distributions obtained the following relation:

$$0.022A + 0.002B + 0.13C + 0.622D + 0.627E = M_e \pm 3.1 \qquad \text{Eq (14-1)}$$

where

M_e = the moisture-equivalent in percent by weight
A = percent of soil in size range 2–0.25 mm
B = percent of soil in size range 0.25–0.5 mm
C = percent of soil in size range 0.05–0.005 mm
D = percent of soil in size range 0.005 mm and less
E = organic matter

The probable error of a single determination was found to be ±3.1. Eq (14-1) shows that the finest particles and the organic matter on a weight-for-weight basis overbalance all the other constituents combined.

The moisture-equivalent for a given centrifugal force is useful, but it is not certain that this actually represents capillary moisture as believed

* For modern tests of moisture-equivalent see A.S.T.M. Standard D 425–39.

by many to be the case. The force necessary to overcome surface tension cannot be determined by centrifuging. Moisture-equivalent is merely a relative measurement of moisture retentiveness in various soils. It has proved a useful tool in agricultural studies and has contributed greatly to an understanding of soil moisture relationships. The chief drawbacks of moisture-equivalent determination are: (*a*) It makes no allowance for porosity or method of packing, and (*b*) it is not specific in so far as actual capillary moisture is concerned.

It must be pointed out that Eq (14-1) was developed with centrifugal speeds of 5000 rpm. Later, Briggs and Shantz (1912) defined the mois-

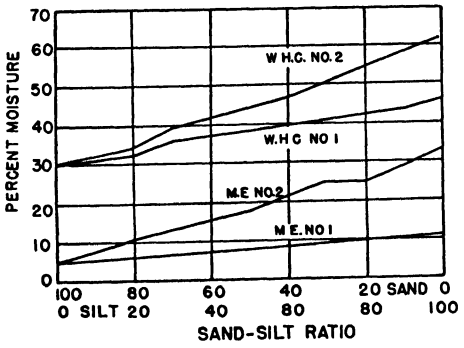


FIGURE 74. PERCENT MOISTURE-EQUIVALENT AND WATER-HOLDING CAPACITY IN SAND-SILT MIXTURES. (YUMA-MESA SANDY SOIL.) NUMBERS REFER TO ANALYSES GIVEN IN TABLE 55.

ture-equivalent of a soil as the percentage of water it can retain in opposition to a centrifugal force 1000 times that of gravity. This has been the accepted definition for over three decades, although from time to time Eq (14-1) above has been assumed to apply to it.

Moisture-Holding Capacity—King's experiments, described earlier, indicate that no part of a moist-sand column is in complete equilibrium, even when permitted to drain for many months. However, if a small quantity of soil is saturated with moisture and allowed to drain (without evaporation), a certain degree of equilibrium will be reached. The moisture remaining in the soil, expressed as a percentage of the dry weight of the soil, is called the "moisture-holding capacity." Theoretically, as Alway and Clarke (1916) point out, actual equilibrium is not obtainable until the moisture-content of the soil can be shown equal to that of the same soil in actual contact with the water.

Hilgard's technique (1906) for the determination of moisture-holding capacity is as follows: A layer of soil 1 cm thick is placed in a screen-bottom cup and saturated with water by capillarity. The sample is then drained by placing it in a saturated atmosphere for 24 hrs, after which the moisture is determined in the usual way.

Experimental Data—Figure 74 shows the relation between moisture-equivalent (1000 times gravity) and the moisture-holding capacity of sand and silt mixtures investigated by McGeorge (1941). The mechanical analysis of these components is given in Table 55. It is to be noted in Figure 74 that as the percentage of silt increases the moisture-holding capacity increases at a greater rate than the moisture-equivalent. Thus, ordinary sand may be made to hold larger amounts of moisture by adding silt.

TABLE 55—MECHANICAL ANALYSES OF SAND AND SILTS SHOWN IN FIGURE 74

Material	Percent in size range 0.05–1 (mm)	0.005–0.05 (mm)	0.005 (mm)
Sand	88	6	6
Silt No. 1	60	27	13
Silt No. 2	27	22	51

Hygroscopic Moisture—In addition to gravitation and capillary moistures, there is another type of moisture with which we are concerned in packing problems. This may be considered as a solid-solution, a sort of superficial colloidal characteristic, called hygroscopic moisture. The nature of this moisture is not very well understood, but it is considerable in view of the large amount of surface presented by the particles. Loughridge (cited by Briggs, 1897) determined hygroscopic moisture in a number of soils. His results ranged from 0.8 to 14.5 percent by weight, sandy soils showing the smallest amounts and clays the highest.

Briggs and Shantz defined an upper limit of hygroscopic moisture which is determinable by experiment. If a soil is placed in a water-saturated atmosphere, it will absorb water vapor until a condition of approximate equilibrium is reached. The moisture content of a soil under these conditions is called the "hygroscopic coefficient" of that soil. Determination of the hygroscopic coefficient is subject to error unless carried out with considerable care. It has been pointed out that hygroscopic moisture is tenacious and difficult to drive off even at elevated temperatures. For this reason, and also because the equilibrium

condition cannot be determined precisely, the hygroscopic coefficient is subject to certain inherent disadvantages.*

The problem of hygroscopic moisture was thoroughly examined by Bouyoucos (1917, 1921). He used a dilatometer and freezing-point method, and showed that a certain portion of the water contained in a given soil fails to freeze at a temperature of −1.5 deg C. Some of this water was capable of freezing at −4 deg C, while still other amounts could not be frozen at temperatures below −78 deg C. Bouyoucos' experiments were conducted by means of a Beckman apparatus, the depression of the freezing point being measured at different moisture-contents. In moist quartz sand, and some extreme types of sandy soils, depression of the freezing point was found to vary inversely as the water-content. Keen (1921–1922) points out that in *certain instances* the moisture obeys the law of dilute solutions, the freezing-point depression varying as the concentration. Bouyoucos had interpreted this to mean that in soils a certain amount of the moisture was rendered "unfree," in the sense that it did not take part in the freezing-point depression. In substance, the fundamental assumption is that the freezing-point depression is inversely proportional to the free water present. This, as has been stated, is not generally the case but applies fairly well in the case of moist sands or sandy soils. On the basis of this assumption, Keen was enabled to relate the free moisture† to the total moisture-content, obtaining the relation

$$M_f = cM_T^{\beta} \qquad \text{Eq (14-2)}$$

where M_f is the free-moisture content of a soil having a total moisture content M_T, and c and β are constants for the soil in question. Since $M_T = M_f + M_u$, where M_u is the "unfree" moisture, by substitution in Eq (14-2) we obtain the relation

$$M_u = \left(\frac{M_f}{c}\right)^{1/\beta} - M_f \qquad \text{Eq (14-3)}$$

which gives the "unfree" moisture in terms of the free-moisture content. If we differentiate this equation with respect to M_f, we have

$$\frac{dM_u}{dM_f} = \frac{M_f^{1/\beta - 1}}{\beta c^{1/\beta}} - 1$$

and putting this equation equal to zero, we obtain

$$M_f^{1/\beta - 1} = \beta c^{1/\beta} \qquad \text{Eq (14-4)}$$

* See Chapter 11 for a discussion of adsorption phenomena. A clean-cut separation between this and hygroscopicity does not seem to have been attempted. There is undoubtedly a difference but the difficulty apparently lies in the fact that the meaning of hygroscopicity has never been adequately defined.

† This free moisture at its maximum determines the moisture-holding capacity.

Again by substitution, this equation may be thrown into the form

$$M_u = M_T - \exp\left[\frac{\beta}{1-\beta}\left(\ln\beta + \frac{1}{\beta}\ln c\right)\right] \qquad \text{Eq (14-5)}$$

That is, the amount of unfree water has a maximum which depends on the total moisture and the constants of the soil. In commenting upon this result, Keen observed that it is difficult to conceive any physical reason why the unfree water should show such a maximum, although experimental data tend to confirm this result.

Relation of Soil-Constants—The relationship between the various constants discussed in the preceding paragraphs has been worked out by Briggs and Shantz (1912) and are as follows:

Moisture-equivalent = hygroscopic coefficient × 2.71 = (moisture-holding capacity − 21) × 0.635 = 0.02 percent sand + 0.22 percent silt + 1.05 percent clay Eq (14-6)

Moisture-holding capacity = (moisture-equivalent × 1.57) + 21 = (hygroscopic coefficient × 4.26) + 21 = 0.03 percent sand + 0.35 percent silt + 1.65 percent clay + 21 Eq (14-7)

Hygroscopic coefficient = moisture-equivalent × 0.37 = (moisture-holding capacity − 21) × 0.234 = 0.007 percent sand + 0.082 percent silt + 0.39 percent clay Eq (14-8)

In these equations the moisture-equivalents are based on a centrifugal force of 1000 times gravity. The particle-size of sand, silt, and clay (used in the experiments, and for which the equations apply) is:

Sand = 0.2–0.05 mm diameter
Silt = 0.05–0.005 mm diameter
Clay < 0.005 mm diameter

It is clear that the constants are very much affected by the clay portion of the soil. Since porosity appears to have had little effect on the constants, it seems safe to assume that points of particle-contact are of greater significance.

Physical Basis of the Briggs-Shantz Equations—In Eq (14-7) it was shown that the moisture-holding capacity is equal to 4.26 times the hygroscopic coefficient, plus 21. If the constants are expressed as percentages of dry soil, the value of 21 percent is the intercept of the ordinate axis, that is, for a zero-hygroscopic coefficient the moisture-holding capacity of a soil is 21 percent. This fact was the subject of an interesting investigation by Wilsdon (1924).

Consider two spheres in contact, and let these spheres have equal diameters. Assume further that at the point of contact there is an amount of moisture held by surface tension consistent with the configura-

tion. To a first degree of approximation the meniscus may be taken as the arc of a circle touching the two spheres, as indicated in Figure 75. The volume of a drop formed by revolution of the meniscus between the arcs of the spheres can be shown to be

$$V = \frac{\pi d^3 \sin^4 \theta}{\cos^2 \theta}\left[1 - \tan 2\theta\left(\frac{\pi}{2} - 2\theta\right)\right] \qquad \text{Eq (14-9)}$$

where d is the diameter of the spheres, and 2θ is the angle between the point of contact of the spheres and the point of tangency of the meniscus. With the aid of this expression we may obtain an equation for the free moisture as a fraction of the weight of a packing of spheres. The number of spheres per 100 g of the packing is

$$N = \frac{6}{\pi \rho d^3} \times 100$$

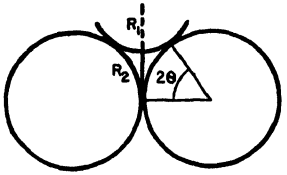


FIGURE 75. MENISCUS FORMED AT CONTACT OF TWO SPHERES.

where ρ is the real density of the spheres. Without loss of generality, we may assume that we are dealing with soil particles instead of spheres, and take $\rho = 2.4$, thus

$$N = 79.4/d^3 \qquad \text{Eq (14-10)}$$

Now each of these spheres touches a number of adjacent spheres, and the number of contacts depends upon the type of packing. Each sphere touches another sphere, so that each contact enters twice. Thus, if C is the number of contacts, the actual contacts are $CN/2$. The actual volume of the liquid in 100 g of soil is $VCN/2$, and this is the free moisture M_f which with the aid of Eqs (14-9) and (14-10) we may write

$$M_f = 125C \frac{\sin^4 \theta}{\cos^2 2\theta}\left[1 - \tan 2\theta\left(\frac{\pi}{2} - 2\theta\right)\right] \qquad \text{Eq (14-11)}$$

It now remains to determine θ, and this may be done as follows. The relation connecting pressure P and the surface tension σ is

$$P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are the radii of curvature (R_2 being negative). The condition of null pressure is obtained when $R_1 = R_2$, and if gravity is neglected this should represent the free moisture. From Figure 75 it is not difficult to show that

$$R_1 = \frac{d(1 - \cos 2\theta)}{2 \cos 2\theta} \text{ and } R_2 = \frac{d}{2 \cos 2\theta}(\sin 2\theta + \cos 2\theta - 1)$$

Equating these R's and solving for θ, we obtain

$$\theta = 26° 33' \text{ and } 2\theta = 53° 6'$$

Putting this value of θ in Eq (14-11) and using a packing of spheres in closest array (C = 12) we obtain a value of the free moisture $M_f = 23.46$ percent, a value remarkably close to the value of 21 percent given by Briggs and Shantz assuming, of course, that the latter truly represents the free moisture.

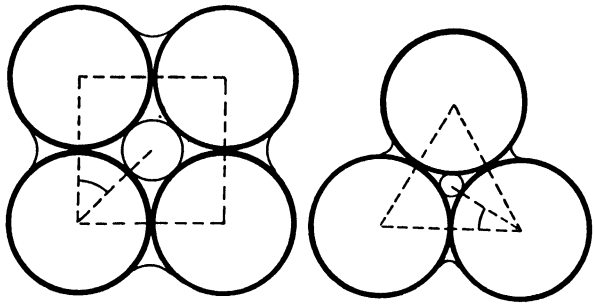


FIGURE 76. GEOMETRIC ARRANGEMENT OF SPHERES IN MOST-OPEN AND CLOSEST PACKINGS.

Actually the above analysis, given by Wilsdon, is in error. As Keen (1924) noted, he considered the equilibrium of a wedge between two spheres. In closest packing, however, we are concerned with a triangular pore-element, and in open packing with a rectangular one (Figure 76). Under these conditions, fusion begins when $\theta = 15°$ in the first case, and when $\theta = 22\frac{1}{2}°$ in the second instance. Using Eq (14-11) and substituting the above values (remembering that for closest packing the volume of a unit cell is $0.71d^3$ and d^3 for open packing) we obtain a total moisture of 6.3 percent and 8.7 percent by volume, or respectively, 24.3 percent and 18.2 percent of the pore-space.

Keen (1924) severely criticized Wilsdon's assumption that M_f is in any sense the true free moisture, as implied in the empirical equation of Briggs and Shantz.

It is simple to calculate the moisture-holding capacity for spheres in closest packing, since we know the voids amount to approximately 26 percent. If we take the gravity of the spheres to be 2.4, and assume the voids to be completely filled with water, the moisture-holding capacity M_c is

$$M_c = \frac{26 \times 100}{74 \times 2.4} = 14.6 \text{ percent}$$

This value is much less than obtained by Wilsdon and, as Keen points out, the discrepancy is much more serious inasmuch as the value of 14.6 percent is a maximum value, whereas the mathematical value of 23.46 percent assumes the pores to be only partly filled. In the case of cubical packings (voids = 47.5 percent $C = 6$) the moisture-holding capacity at saturation is 36.2 percent. Keen pointed out the fallacies of Wilsdon's procedure, but the question as to whether the additional constant in the Briggs and Shantz equation represents the "free" moisture remains unanswered. If it is not the free moisture because it is in excess of that given by actual computation in terms of voids for closest packings, then certainly the Briggs-Shantz equation does not hold for such packings. The Wilsdon theory seems to have some basis for packings with larger voids, even though derived from considerations of closest packing. Since we know nothing about the true state of soil-particle contacts, and less yet about the characteristics of the soils used by Briggs and Shantz, it seems academic to argue that Wilsdon's theory is very far from the truth.

Keen's own studies on hygroscopic moisture offer some useful extensions. If we assume that total moisture is taken at the moisture-holding capacity M_c, we have

$$M_f = cM_c^{\beta}$$

and

$$M_c = 4.3M_h + 21$$

where M_h is the hygroscopic coefficient. Hence, the relation between free moisture and hygroscopic moisture is given by

$$M_f = c(4.3\ M_h + 21)^{\beta} \qquad \text{Eq (14-12)}$$

LYSIMETRY

The moisture-relations of complex soils cannot be predetermined and we must resort to experimental means. For this purpose an apparatus called a lysimeter has been devised. This consists of a tray carefully mounted on a delicate scale, and provided with suitable drainage. As a general rule, the tray is made large and deep enough to contain a representative section of the soil profile to be studied. The lysimeter may be housed in a pit having its top flush with the surface of the ground, or it may be set up well above the ground. However, for best results, it is better to simulate conditions as they exist at the soil surface and to install the lysimeter in the ground.

The purpose of the lysimeter is to study the behavior of soil-moisture over long periods of time, under all types of weather conditions, and if

desired, under controlled conditions with plant growth. Kohnke, Dreibelbis, and Davison (1940) have shown how the data obtained can be used with a lysimeter connected to a sensitive balance.

WATER ADDITION	= WATER LOSS	+ WATER STORAGE
Precipitation	*Run-off* (direct	*Direct measurements by weight*
(direct measurement)	measurement)	
Rain	*Percolation*	
Snow	(direct measurement)	
Hail	*Evapo-transpiration**	Eq (14-13)
Sleet	Evaporation	
Fog	Transpiration	
Condensation		
Dew (plants)		
Condensation (soil)		
Adsorption (soil)		

PRECIPITATION + CONDENSATION = RUN-OFF + PERCOLATION + EVAPO-TRANSPIRATION + STORAGE† Eq (14-14)

PRECIPITATION − RUNOFF − PERCOLATION − STORAGE = EVAPO-TRANSPIRATION − CONDENSATION Eq (14-15)

In using these equations, it must be remembered that weight changes in a lysimeter may be caused by factors other than water. Thus, chemical and biochemical changes, soil erosion, growth, harvest of plants, etc., must be included in the calculations. In general, however, when these factors are noted the equations are applicable to any period of time, and all factors on the left-hand side of Eq (14-15) can be determined directly. With regard to the difference between evapo-transpiration and condensation, Kohnke *et al* state that a close separation can be obtained by plotting accumulated evapo-transpiration minus condensation against time, wherever accurate weights are recorded for 30 min intervals or less. Therefore it is seen that lysimeter studies may yield important correlated information for soils of different composition, under all kinds of conditions.

Noncapillary-Porosity—Noncapillary-porosity is an index of the actual free space in a given volume of soil under field conditions. The term was introduced in recent years, but appears to have given no remarkable results. Free *et al* (1940) studied the relation between three sets of noncapillary-porosity indices and infiltration. The indices used were as follows:

* Evaporation and transpiration cannot be separated accurately in a single lysimeter experiment.

† Storage may be positive or negative (loss of weight). If negative, the opposite sign should be used.

1. Volume of drained pores 24 hrs after natural percolation of water for 3 hrs.
2. Total porosity minus moisture-equivalent (volume).
3. Total porosity minus moisture-equivalent (volume) corrected for texture by a method attributed to Harding (1919). Harding's method is to correct the moisture-equivalent by a multiplier which takes into account texture of the soil.

Free *et al* found infiltration was more highly correlated with noncapillary-porosity than with total porosity for both surface and subsurface soils.

Percolation and Infiltration—We shall now consider the flow of water through soils at very low heads. The more general problem of packing behavior with high heads and pressures is discussed in Chapter 13.

Free *et al* (1940) and Musgrave and Free (1936) made a comprehensive study of the variables affecting infiltration. While clay, silt and clay, and aggregates (particles greater than 0.20 mm) were each correlated separately, no single parameter of the size-distribution was so correlated. However, the study referred to is outstanding for obtaining indices as to the relative importance of the variables studied. A description of the technique used by Free and his co-workers is given below.

Nine-inch diameter galvanized-steel tubes were pressed into the soil. These tubes varied in length depending upon the depth of the sub-soil which they penetrated. About 2 to 3 in. of the tube were left protruding above the surface of the ground. A head of water of approximately $^1/_4$-in. was maintained within the tube by a self-dispensing buret. At various intervals of time readings were taken of the amount of water supplied by the buret. These tests required (*a*) that the soil be disturbed as little as possible in the course of placing the tube, and (*b*) that sufficient replicas be taken to give representative infiltration rates of a given soil. At least six replicas were found necessary for certain soils, while others required as many as 22 replicas.

Free and his associates studied the infiltration rates of 68 soils in the manner described, and in addition studied the interrelation of some 12 determinable soil characteristics. Their results applied to "as found" soil conditions and "wet" soil conditions. The first of these conditions applied to soil as found, irrespective of previous rainfall, drought, or other climatic conditions which might conceivably affect the characteristics of the soil. This test was carried out as above described for 3 hrs. The "wet" soil results concern identical tests made 24 hrs later on the same tubes.

Table 56 gives the range of results obtained for surface soils in the

above-mentioned study, and in Table 57 are one-to-one correlation coefficients of the soil. The negative correlations mean that the variables are inversely related. Table 57 indicates significant correlations with filtration of the following physical characteristics:

Positive { voids, aggregation, organic matter, noncapillary-porosity }

Negative { volume weight, suspension, dispersion }

Note also how voids are highly correlated with aggregation (0.57), silt and clay (0.70), volume-weight (−0.99), and moisture-equivalent (0.82).

TABLE 56—RANGE OF RESULTS ON INFILTRATION RATES AND OTHER PHYSICAL CHARACTERISTICS OF 68 SURFACE SOILS

Physical characteristic	Type of soil: Silt loams	Clay and clay loams	Sandy loams	Gravelly silt loams	Loams
Rate of infiltration (surface inches per hour)[a]	0.02– 3.98	0 01– 1 78	0 00– 1.93	0.55– 4.96	0.00– 0.50
Total soil in aggregates (weight percent >0.20 mm)	8 4 –49.8	6.0 –72.5	1 4 –33.1	24 1 –61.5	8.9 –12.7
Silt and clay (weight percent <0.05 mm)	76 2 –97 4	33.6 –95 3	11.9 –68.0	57.1 –76 0	33.8 –62.5
Clay (weight percent <0.002 mm)	3.9 –25.4	7.9 –61.1	3.3 –24.4	8.5 –23.3	10.3 –18 0
Volume-weight ratio	1 05– 1.46	0.94– 1 54	1 26– 1.66	1.04– 1.36	1.16– 1.49
Voids	0 44– 0 61	0 41– 0.64	0.36– 0.52	0.46– 0.61	0.46– 0.55
Noncapillary-porosity index	0.13– 0.36	0.14– 0.37	0.12– 0.36	0.15– 0.34	0.17– 0.30
Organic matter (weight percent)	1.4 – 5.3	0.7 – 3.2	0.7 – 2.4	2.5 – 6.8	1.3 – 2.9
Moisture-equivalent	19 4 –44 3	14.1 –41.2	3.9 –19.7	20.4 –31.9	11.7 –23 6
Suspension (percent)	11 6 –38.6	8.7 –26.6	5.8 –41.5	9.4 –22.2	14 2 –33.4
Dispersion ratio[b]	20.0 –50.8	9.3 –50.8	20.4 –62.9	13.3 –34.2	29.2 –53.3
pH value	4 88– 6.79	5.14– 9.22	5.01– 8.72	5.53– 7.80	5.49– 7.34

[a] Rate measured during third hour of wet run, initial rates were higher.
[b] Suspension (per cent) ÷ silt and clay (weight per cent <0.05 mm).

The correlations given in the table are consistent with the principles of soil physics discussed in the present chapter and in Chapter 15 (Capillarity).

Rate of Infiltration—The rate of infiltration is invariably high at the start of the test and changes rapidly during the first hour. Thereafter, the rate (generally expressed in surface inches per hour) changes slightly so that after 3 hrs it is fairly constant. "Wet" runs showed initial infiltration rates very much less than those on soil "as found."

TABLE 57—CORRELATION COEFFICIENTS[a] OF CERTAIN CHARACTERISTICS OF SURFACE SOILS

Characteristics considered	Infiltration	Aggregation >0.20 mm	Silt and clay	Clay	Volume-weight	Total porosity	Non-capillary-porosity	Organic matter	Moisture-equivalent	Suspension	Dispersion	pH value
Infiltration	....	0.30	−0.11	−0.16	−0.24	0.24	0.36	0.50	0.02	−0.29	−0.29	0.16
Aggregation >0.20 mm	0.30	....	0.46	0.48	−0.56	0.57	0.30	0.55	0.64	−0.30	−0.74	0.28
Silt and clay	−0.11	0.46	....	0.35	−0.69	0.70	0.25	0.40	0.80	0.50	−0.41	0.06
Clay	−0.16	0.48	0.35	. ..	−0.36	0.36	0.10	0.12	0.50	−0.24	−0.55	0.43
Volume-weight	−0.24	−0.56	−0.69	−0.36	....	−0.99	−0.75	−0.63	−0.81	−0.11	0.53	−0.31
Total porosity	0.24	0.57	0.70	0.36	−0.99	....	0.76	0.62	0.82	−0.12	−0.53	0.26
Noncapillary-porosity	0.36	0.30	0.25	0.10	−0.75	0.76	...	0.38	0.30	−0.09	−0.21	0.20
Organic matter	0.50	0.55	0.40	0.12	−0.63	0.62	0.38	....	0.62	−0.06	−0.46	0.05
Moisture-equivalent	0.02	0.64	0.80	0.50	−0.81	0.82	0.30	0.62	....	0.18	−0.59	0.21
Suspension	−0.29	−0.30	0.50	−0.24	−0.11	−0.12	−0.09	−0.06	0.18	.. .	0.50	−0.32
Dispersion	−0.29	−0.74	−0.41	−0.55	0.53	−0.53	−0.21	−0.46	−0.59	0.50	....	−0.45
pH value	0.16	0.28	0 06	0.43	−0.31	0.26	0 20	0.05	0.21	−0.32	−0.45	....

[a] Value of coefficients at the 1 percent point (odds 99 to 1) is 0.31; at the 5 percent point (odds 19 to 1). 0.24.

Final infiltration rates on both types of runs, however, became appreciably the same. The general equation fitting the infiltration data obtained by Free *et al* was found by these investigators to be

$$I = bt^a \qquad \text{Eq (14-16)}$$

where I is the cumulative infiltration in inches, t the time in min after start of infiltration, and a and b are constants depending upon the soil under test. The coefficient b for soils "as found" ranged from unity to 0.0087, while the exponent a for the same conditions ranged from 0.04 to 0.82. It should be held in mind that the above equation does not provide for a constant rate of infiltration except when $a = 1$, and for large values of t.

Morosow (1935) has given formulas for calculating the time taken by a liquid to move from the upper limit of a packing of depth L_0 to any point L, under a constant head h_0

$$t = K\left[\frac{\vartheta}{100-\vartheta}(L - L_0) - \frac{\vartheta}{(100-\vartheta)^2}(100h_0 + L_0\vartheta) \times \ln\frac{h_0 + \frac{L_0\vartheta}{100} + \frac{L(100-\vartheta)}{100}}{h_0 + L_0}\right] \qquad \text{Eq (14-17)}$$

where ϑ is the porosity of the packing in percent and K a suitable constant.

For calculating the absorption time through a homogeneous packing of a head of liquid equal to h_0

$$t = Kh_0\left[\frac{100}{100-\vartheta} - \frac{100\vartheta}{(100-\vartheta)^2}\ln\left(\frac{100}{\vartheta}\right)\right] \qquad \text{Eq (14-18)}$$

This equation measures the progress of a liquid of height h_0 until it completely disappears into the packing.

Factors Affecting Rate of Percolation—The change in percolation rate was examined by Bodman (1937) who showed that removal of electrolytes caused the permeability to decrease. Initial permeabilities (that is, rate of flow per unit-time per unit-head per unit cross-section) decreased from 17 to 80 times over a period of 14 days of test with given soils. Bodman stated that small changes in electrolyte concentration made large changes in permeability, and that "of all the physical properties of the soil columns which were considered in relation to saturated permeability, decreases were found to be closely related only to silt content." No relation was found with respect to clay content, moisture-equivalent, or to small increases in apparent density which often occur from water saturation and flow. The change in permeability found by Bodman is not entirely explained by any change in the voids caused by

the dispersion and rearrangement of clay particles, thus reducing the pore size. When ordinary water was replaced by a solution having an electrical resistivity approximately equal to that in the original soil,* it was found that the permeability increased. However, if leaching with ordinary water was continued for too long, the permeability was permanently altered and normal amounts of the electrolyte were unable to affect it. This leads to the conclusion that in complex packings, such as soils, colloid phenomena may play an important part in affecting their physical characteristics.

PENETRATION OF PRESSURE-WAVES

Buckingham's investigations discussed in Chapter 13 were extended to determination of the quantity of air driven into or removed from soil by fluctuations in barometric pressure. The equations developed by Buckingham are based on the assumption that the soil is pervious to the movement of air as far as the water table or base rock. The following quotation from Buckingham's paper illustrates the significance of the assumptions: "If the soil be sufficiently open there is free communication of pressure between the soil air and the outside air, and as the outside pressure increases, that of the soil air must increase by the same amount. Hence the air in the soil is compressed to a smaller volume and more air enters from outside. It is evident that if the outside pressure increases from 760 mm to 761 mm, the air in the soil will be compressed to $^{760}/_{761}$ of its former volume, and therefore, since the total free space in the soil is unchanged, that there must enter the soil from the outside a volume of air equal to the seven-hundred-and-sixty-first part of what was there already." Thus, if the depth of the impervious layer and the barometric fluctuation are known, the depth of penetration can be determined. With an increase in pressure from 760 mm to 761 mm Hg, the outside air will advance $^{1}/_{761}$ of the total depth of the pervious soil. If the soil is L mm in depth, the total penetration is $L/761$ mm. We shall now develop Buckingham's equations for penetration of barometric waves in the soil.

Consider an infinitesimal volume element $dxdydz$ of a soil layer infinite in extent and let the flow be parallel to the axis of x which is measured vertically downward through the surface of the soil. The flow downward for an element of time dt through the upper surface of the element will be, by Eq (13-2),

* Bodman used Hoaglund's solution consisting of $^{1}/_{200}$ molar of calcium nitrate, potassium nitrate, magnesium phosphate, and potassium dihydrogen phosphate. Iron tartrate was omitted.

$$- \mathcal{P}_0 dydz \frac{\partial P}{\partial x} dt$$

Through the lower face of the element, $x + dx$ below the flow will be

$$- \mathcal{P}_0 dydz \left[\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial P}{\partial x} \right) dx \right] dt$$

Hence the total inflow will be the difference between these flows, or

$$\mathcal{P}_0 dxdy\, dz \frac{\partial^2 P}{\partial x^2} dt$$

If dQ is the volume of air flowing into the element, writing dV for $dxdydz$ we have

$$\frac{1}{dV} \times \frac{\partial Q}{\partial t} = \mathcal{P}_0 \frac{\partial^2 P}{\partial x^2}$$

But the term on the left is the time rate of change of the volume V' of gas present in the volume element, that is

$$\frac{\partial V'}{\partial t} = \mathcal{P}_0 \frac{\partial^2 P}{\partial x^2}$$

If ϑ is the fractional pore-space, then in accordance with the discussion above

$$\frac{V'}{\vartheta} = \frac{P}{760}$$

so that

$$\frac{\partial P}{\partial t} = \frac{760 \mathcal{P}_0}{\vartheta} \frac{\partial^2 P}{\partial x^2}$$

or writing $760\mathcal{P}_0/\vartheta = a^2$

$$\frac{\partial P}{\partial t} = a^2 \frac{\partial^2 P}{\partial x^2} \qquad \text{Eq (14-19)}$$

Buckingham's next step was to consider small oscillations about the mean pressure of 760 mm Hg, that is, $P = 760 + \theta$, and

$$\frac{\partial P}{\partial t} = \frac{\partial \theta}{\partial t}; \quad \frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 \theta}{\partial x^2}$$

Thus Eq (14-19) becomes

$$\frac{\partial \theta}{\partial t} = a^2 \frac{\partial^2 \theta}{\partial x^2} \qquad \text{Eq (14-20)}$$

This equation may be solved by Fourier series, for given initial and boundary conditions. These conditions may be determined as follows:

Assume initially that the excess pressure θ is determinable by some function $\theta = f(x)$ for $t = 0$ at any depth to the impervious stratum. Let the depth of this stratum be $L/2$. If the excess of pressure at the surface is simply harmonic, $\theta = \theta_0 \sin \omega t$ for $x = 0$ and any value of t, we have the upper surface boundary condition. The value of θ_0 is of course the maximum oscillation from the mean value 760 mm Hg. At $L/2$ no flow can take place. This boundary can be accomplished by the following mathematical artifice. Instead of assuming the impervious layer to exist at $L/2$, assume that it exists at a depth L. At the depth L, let there be a forced oscillation of the same magnitude as that at the surface, so that the region then encompassed will be symmetrical about $L/2$, and no flow will take place across a plane parallel to the surface at this point. The boundary conditions to be satisfied by Eq (14-20) will then be:

$$\theta = 0, \text{ for } t = 0$$
$$\theta = \theta_0 \sin \omega t, \text{ for } x = 0$$
$$\theta = \theta_0 \sin \omega t, \text{ for } x = L$$

and Eq (14-20) may be solved by the usual methods (Byerly, 1893). The complete solution is

$$\frac{\theta}{\theta_0} = \sin \omega t \left[1 - \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sin n\pi x}{n} \right] + 4a_1^2 \pi \sum_{n=1,3,5,\ldots}^{\infty} \tau n \sin \pi x \quad \text{Eq (14-21)}$$

where

$$\tau = \frac{1}{(a_1 n\pi)^4 + \omega^2} [(a_1 n\pi)^2 \sin \omega t - \omega \cos \omega t]$$

and

$$a_1^2 = \frac{760\mathcal{P}_0}{L^2\vartheta}$$

where the former value of a^2 has been corrected so that the unit of length is taken equal to L in order to simplify the original equation.

To get the average value of the pressure in the soil at any time we must obtain

$$\int_0^1 \frac{\theta}{\theta_0} dx = \bar{\theta}$$

carrying through the integration of Eq (14-21)

$$\frac{1}{8a_1^2} \frac{\bar{\theta}}{\theta_0} = -A\omega \cos \omega t + Ba_1^2\pi^2 \sin \omega t \quad \text{Eq (14-22)}$$

where

$$A = \sum_{n=1,\,3,\,5,}^{\infty} \frac{1}{(a_1 n\pi)^4 + \omega^2}$$

$$B = \sum_{n=1,\,3,\,5,}^{\infty} \frac{n^2}{(a_1 n\pi)^4 + \omega^2}$$

The mean value of θ, $\bar{\theta}$, is the deviation of the average pressure in the soil at any instant from its mean value of 760 mm. It is a periodic function of time with the same period as the pressure above the stratum considered. It has a smaller amplitude, however, and a small time lag. Let this amplitude be denoted by $\bar{\theta}_0$, and let us investigate the *greatest* divergence of the mean pressure in the stratum from the surface to the impervious stratum $L/2$ below. In other words $\bar{\theta}_0/\theta_0$ is what we desire, and to obtain it we must find the time t for which $\bar{\theta}/\theta_0$ is a maximum or minimum and then insert this time in Eq (14-22). Performing these operations we obtain

$$\frac{1}{8a_1^2} \cdot \frac{\bar{\theta}_0}{\theta_0} = A\omega \cos \omega t + Ba_1^2\pi^2 \sin \omega t \qquad \text{Eq (14-23)}$$

where A and B are as above, and where t satisfies the condition

$$\tan \omega t = \frac{a_1^2\pi^2 B}{\omega A} \qquad \text{Eq (14-24)}$$

The values of A and B may be arranged so as to make them rapidly convergent.

After computing A and B from data on hand and inserting them in Eq (14-24) we may determine values of sin ωt and cos ωt. These in turn may be inserted in Eq (14-23), thus obtaining the value of $\bar{\theta}_0/\theta_0$ which gives the ratio of amount of oscillation of the mean pressure in the stratum deficient in its range to the range of oscillation of the barometric pressure above the stratum. It is the error that would be committed by regarding the stratum as so porous as to permit free communication of pressure throughout its volume.

The value of tan ωt also permits the computation of the phase relationship.

Eqs (14-21) and (14-23) are important inasmuch as they show how changes in barometric pressure affect the volume of air added or subtracted within a soil. The harmonic variation imposed for the sake of illustration can be approximated for any locality where data or barometric changes are available. Buckingham computed the mean pressure

in a soil and the maximum variation of the outside pressure for a number of special cases. Thus, for a pervious stratum of 15 ft and a soil porosity of 30 percent, the ratio of the mean pressure $\bar{\theta}_0$ to the maximum variation of pressure outside θ_0 was $\theta_0/\bar{\theta}_0 = 1.00$ for a variation within a period of one day, and $\theta_0/\bar{\theta}_0 = 0.86$ for a variation within a period of one hour. The "transpiration" constant $\mathcal{P}_0$ was taken as unit. Buckingham has shown that depth of soil is not as important as the period of the barometric variation. Variations of 12 hours or more are most significant.

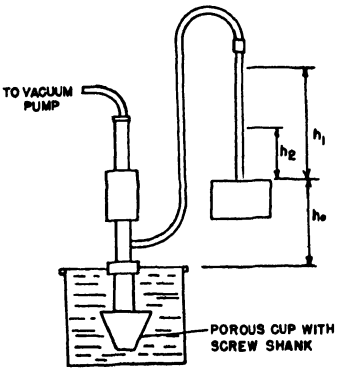


FIGURE 77. SCHEMATIC DIAGRAM ILLUSTRATING PRINCIPLE OF TENSIOMETER.

While in general the ratio of $\theta_0/\bar{\theta}_0$ is not great, it is obvious that changes in barometric pressure do exert an effect on soil behavior. The addition or depletion of air and other gases from the soil, coupled with gas diffusion as well as seasonal temperature changes, makes the soil a living entity with a respiratory system.

TENSIOMETRY

A tensiometer consists of a small porous cup which is inserted into a soil or packing and connected to a suitable mercury manometer. When the porous cup is partly filled with water and the soil in which it is inserted is dry, water will leave the porous cup and cause a slight vacuum. For a given type of porous cup, the amount of decrease in pressure is indicative of the moisture conditions in the soil. Figure 77 shows the essential features of a tensiometer.

Several features concerning a tensiometer must be considered carefully. Whether the manometer gives a true indication of capillary tension depends upon many factors. Among these factors, discussed by L. A. Richards *et al* (1937), we may mention (1) amount of air in the tensiometer, (2) rate of change of temperature, (3) sensitivity of the gauge, (4) accuracy of the gauge, and (5) cup-to-soil water-transfer conductance. Regarding the first of these factors, good design requires that air be removed from the cup since the presence of air causes a sluggish response. The factors of sensitivity and conductance in the design of a tensiometer are discussed below.

When a tensiometer is in equilibrium with the soil in which it is placed (the temperature being constant) any abrupt change in moisture volume

of the device will change the reading of the manometer by an amount Δh, since the rate of passage of water q through the porous cup will generally be proportional to the difference in tension between cup-water and soil-water. Thus

$$q = \frac{dQ}{dt} = C\Delta h \qquad \text{Eq (14-25)}$$

where Q is the volume of water in the cup at a time t, Δh the change in manometer reading, and C a proportionality constant expressed in cu cm per cm per sec. It is equivalent to the cup-to-soil water conductance and depends upon the permeability, thickness, and area of the porous cup wall, and cup-to-soil contact.

If ζ is now taken as the gauge sensitivity (= rate of change of manometer reading with respect to water transfer through cup), then

$$\zeta = \frac{\partial h}{\partial Q} = \frac{\partial h}{\partial t} \cdot \frac{\partial t}{\partial Q}$$

and from Eq (14-25)

$$q = \frac{1}{\zeta} \frac{\partial h}{\partial t} = C\Delta h$$

So that

$$C = \frac{1}{\Delta h} \cdot \frac{1}{\zeta} \cdot \frac{\partial h}{\partial t} \qquad \text{Eq (14-26)}$$

Thus, it is possible to compute C from a set of tension-time curves. It is desirable to make C as large as possible. L. A. Richards *et al* defined cup conductance (C) as the volume of water passing through the cup wall in unit-time with a unit-difference of pressure, the walls of the cup being in contact with free water.

The effect of soil temperature on tension was studied by Richards *et al* (1937), who showed the tension to be inversely related to the temperature. A drop of approximately 20 deg C caused a tension increase of 10 cm Hg. The full significance of tensiometer measurements has yet to be revealed. Haines (1927) pointed out the serious limitations of such measurements since they give no indication as to whether the moisture is on the rising or falling capillarity cycles. (See Chapter 15.)

Problems

1. A soil is tested at two different times. The cumulative infiltration after 6 hrs was found to be 3 in. When tested again for 2 hrs the cumulative infiltration was 1 in. From these data compute the value of a in Eq (14-16).

2. A head of water of 1 ft is absorbed in a cylindrical soil column, the porosity being 34 percent. How much longer will it take for a head of 2 ft to be absorbed by the soil if the porosity is increased by compacting to 31.5 percent?

CHAPTER 15

CAPILLARITY

THE precise nature of the behavior of moisture in packings of particulate substances has been the subject of numerous investigations for more than fifty years. The importance of soil moisture in agricultural science warrants even greater interest than has thus far been given it.

Chief among the various phenomena contributing to the behavior of moisture in packings is capillarity. Investigations on this subject thus far may be divided into three parts: (*a*) Those based upon extensions of Poiseuille's capillary law, (*b*) those based on Buckingham's theory of capillary potential, and (*c*) Haines' capillary-quantum theory. Other investigations have developed equations of capillary flow with some degree of success, and some of these will be discussed. However, from the larger viewpoint, the three major divisions mentioned are the most significant.

The reasons for the difficulties encountered in the study of capillary phenomena in packings are not hard to find. We may cite the following.

1. Inability to describe packing characteristics as a simple mathematical function.
2. Inadequate knowledge of particle-moisture relationships.

These are difficulties which occur repeatedly whenever it is necessary to describe the behavior of a packing in terms of certain characteristic constants. Fundamentally, the nature of the packing should be related to particle-size and distribution. However, so little has been done with regard to the physical characteristics of the materials dealt with that it is small wonder so little information is available. Investigators have generalized when actually no generalizations were possible without knowledge of the media concerned.

When a column of particulate matter is flooded with water and then allowed to drain, several important points must be considered. The water in excess of the amount the particles are able to retain is called gravitation water. The water retained is called capillary water. The amount of water retained depends upon the effect of gravitational forces and the surface tension. Briggs (1897) has given what is probably the best description of gravitation and capillary waters, and the following is quoted from his investigation of the subject:

"There is no sharply drawn line between these two quantities of water. The relative proportion depends, among other factors, upon the texture and structure of the soil, the surface tension of the soil water, the temperature, and the length of the column of soil considered. The importance of this last factor can be shown from the following considerations: Suppose we have 100 cubic inches of soil packed into 100 cubical boxes without bottom or top, each containing 1 cubic inch. Suppose the soil in each box to be saturated with water. There will be a free water surface at the top and at the bottom of each box. By means of forces existing in these surfaces the water in each cube is enabled to overcome the attraction of gravity, so that each cube is able to retain an amount of water equal to that necessary to produce saturation. In this case, therefore, there is no gravitation water. Suppose now that these cubical boxes are built up in a vertical column 100 inches high. The water surfaces previously existing at the top and bottom, respectively, of two cubes now disappear when one cube is placed on top of the other. Instead of having 200 surfaces as before, we now have only two surfaces, and they are called upon to support a column of water one hundred times as high as before. This they are unable to do, and water begins to drip from the lower surface. This water, which was previously what we have termed capillary, now becomes gravitational in its nature, due simply to a change in the length of the column. If the water in the soil was held in vertical capillary tubes running throughout the length of the column the water in each tube would simply fall until the two surfaces were able to support the weight of the liquid. In the soil, however, we have a different condition. As the water begins to leave the upper part of the column new surfaces are developed within the soil. As the water continues to drain away, these surfaces become more efficient . . . and finally there comes a time when the opposing force exerted by these surfaces is sufficient to balance the weight of the liquid and drainage ceases."

*SURFACE TENSION**

As has been mentioned the height of capillary rise is dependent upon surface tension. This is distinct from the force exerted on a liquid through gravity, which varies as the mass of the liquid. In a capillary tube a liquid will rise to a height such that the gravitational force will just balance the force of surface tension.

The pressure due to surface tension is given in standard physics texts, and the derivation of the general equation need not be given here in detail. Some phases of the general theory of capillary pressure, however, are worth mentioning. Assume a liquid droplet in contact with a plane solid surface and assume further that the surface tensions of three surfaces which separate the solid and liquid, liquid and gas, and solid and gas, are known.

* See Chapter 11 for a detailed discussion of this subject.

At any contact point of the three substances there exist three forces equal in magnitude to the tensions at the surface of contact, and these forces are directed along lines tangent to the surfaces at that point. Since the system is in complete equilibrium, any of these forces is the resultant of the other two. Equilibrium can exist only when subject to this condition. In general, when the tension of the solid-liquid surface is greater than the sum of the other two tensions, the liquid will assume the appearance of a drop. On the other hand, if the solid-gas surface has a tension greater than the vector sum of the other two tensions, the liquid will spread over the solid. Again, if the tension of the liquid-gas surface is greater than the vector difference of the tensions of the other surfaces, then the liquid will assume a configuration of equilibrium. In other words, whenever the tensions are unbalanced, the liquid will spread over the surface of the solid; when the tensions are balanced, the configuration at the points of common contact are completely determined by the triangle of forces.

The pressure exerted by a surface is dependent on the form of the film. It may be shown that the pressure, P, exerted by a surface tension, σ, at any point in the surface of the fluid can be expressed by the equation

$$P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad \text{Eq (15-1)}$$

where R_1 and R_2 are the radii of curvature of the sections formed by passing two planes at right angles through the point in question.

In the special case of a fluid rising in a capillary tube we have the following situation. The upward pull due to surface tension must balance a column of liquid with height, say h, and density ρ_0. Since the tension is exerted along the contact of the liquid with circumference of the tube, then the total upward pull due to this tension is $\pi D_c\sigma$ where D_c is the diameter of the capillary and σ the surface tension. This upward pull must equal the downward gravitational pull on the mass of liquid which is obviously $\frac{\pi D_c^2}{4} h\rho_0 g$, so that

$$\pi D_c\sigma \cos\theta = \frac{\pi}{4} D_c^2 h\rho_0 g$$

$$h = \frac{4\sigma \cos\theta}{D_c\rho_0 g} \qquad \text{Eq (15-2)}$$

where g is the gravitational constant and θ is the angle of contact. Thus, we see that the height of rise for a given liquid is proportional to the surface tension, and inversely proportional to the diameter of the capillary; that is, the height of rise will increase with decreasing di-

ameter of the capillary. The significance of the result in its application to the capillary movement of moisture in packed columns is apparent.

Turning now to the configuration of the top surface of the capillary we have the fact, as previously discussed, that this surface will be determined by the vector triangle of the tensions involved at the common line of contact. The pressure at any point on this surface will be given by Eq (15-1).

POISEUILLE'S EQUATION

The flow of fluids through capillaries is given by the equation

$$q = \frac{\pi D_c^4 \Delta P}{128 \mu L} \qquad \text{Eq (15-3)}$$

in which q is the flow per unit-time, D_c the diameter of the capillary, ΔP the pressure difference between the ends of the capillary, μ the viscosity of the fluid, and L the length of the capillary. This is known as Poiseuille's equation, having been derived experimentally by Poiseuille (1846). It is derivable from purely theoretical considerations, and this derivation is to be found in most texts on hydrodynamics. The velocity of the fluid in the capillary is obviously

$$v = \frac{q}{A_c} = \frac{A_c \Delta P}{8 \pi \mu L} \qquad \text{Eq (15-4)}$$

where A_c is the area of the capillary. Eqs (15-3) and (15-4) are fundamental to the flow of fluids through packings. Although the Poiseuille equation is remarkably accurate in describing the motion of fluids in capillaries of uniform and nonuniform cross sections (subject to slight modifications), it has not proved successful with regard to capillary motion in packings. Although, as will be shown subsequently, the equation has been modified in various ways to account for the nature of the packing dealt with, no notable success has been attained by its use. The reason for the failure of the Poiseuille equation when applied to capillary flow in packings is difficult to explain. This failure is not due to any single circumstance, but rather to a number of complicated causes. The first and most important of these is the fact that it has thus far been impossible to give complete representation of a packing by a simple function. The reader is referred to Chapters 6 and 13 for details concerning packing characteristics and applications of the Poiseuille equations.

A second important reason for the failure of the Poiseuille equation is the nature of the contacts established between particles. This, coupled with surface-tension phenomena which depend upon the curva-

ture of the fluid film, does not make for conditions similar to those in tubes. Actually, in a packing there is a cyclic alteration of the capillary fluid surface - a spurting and then a creeping motion, as if the fluid were passing through narrow constrictions, filling the voids, and then spurting again through the constrictions.

Limitations of the Poiseuille Equation—It has been mentioned that the Poiseuille equation does work in some instances when it is modified slightly in regard to the capillary diameter used. The reason for a certain amount of success is explained on the basis of mathematical investigations by Boussinesq (1868) and Greenhill (1881). Boussinesq was the first to demonstrate that for small changes in the form of capillary cross section, the average velocity v is substantially constant, as long as the cross-sectional area of the capillary is substantially constant. Greenhill extended Boussinesq's investigations and showed the methods for computing flow for a variety of complicated cross sections. In general, the investigations of Boussinesq and Greenhill indicate that the approximate velocity of flow through the pores of an assembly of spheres may be considered uniform, if the space through which the flow takes place is taken as a triangular or circular cylinder of equal cross-sectional area. This is precisely what was done by Slichter (1899) in his classical discussion of fluid flow through soil, as explained in Chapter 13.

As far as application of the above principles is concerned, one fact is obvious, namely that the Boussinesq and Greenhill theorem applies to a *single* capillary. Actually, in an assemblage of spheres or particles the capillaries are interconnected and it is doubtful that the theorem has any validity in such circumstances; thus, the fact that the Poiseuille equation works may be merely fortuitous.

CAPILLARY MOVEMENT

Capillary movement may take place in any direction; upward, downward, or laterally. After moisture has been added to the surface of a packing, capillarity may help in draining it to lower levels; conversely, in surface drying, capillarity will draw water from the deeper areas. The rate of downward movement has already been discussed. The movement of moisture upward through a packing is difficult to predetermine, and forms another section of the present chapter. King (1899), in his remarkable memoir on the movements of ground water, states:

"Where the movement is vertically upward through a distance of 1 foot it has been found by experiment that the rate for a fine sand was 2.37 pounds per square foot per day of twenty-four hours; when the lift was increased to 2 feet the movement became 2.07 pounds; at 3 feet it was 1.23 pounds, and at 4 feet only 0.91 pound per square foot. A

similar trial with medium clay loam gave a movement of 2.05 pounds for a lift of 1 foot; 1.62 pounds for 2 feet; 1 pound for 3 feet, and but 0.9 pound where the lift was 4 feet.

"There are as yet no sufficiently exact data relating to field conditions to show through what distances vertical capillary movements of water take place. But the observations quoted show that it is quite rapid at 4 feet; so rapid indeed that were it maintained throughout the year it would deliver at the surface the equivalent of 63.85 inches of water and almost prevent any part of the rain from sinking deeply beneath the surface over much the larger part of the United States."

TABLE 58—EVAPORATION OF WATER FROM CYLINDERS FILLED WITH GRADED SAND (KING, 1899)

Depth of packing (in sand)	Total evaporation in 40 days (in water)	Mean evaporation per day (in water)
6	4.56	0.114
12	4.45	0.111
18	3.19	0.08
24	1.36	0.034
30	0.78	0.0195

TABLE 59—LOSS OF MOISTURE BY EVAPORATION FROM COLUMNS OF SOIL OF VARIOUS THICKNESSES AND POROSITIES, THE MOISTURE REACHING THE SURFACE BY CAPILLARY ACTION (BUCKINGHAM, 1907)

Soil	Initial moisture content (percent)	Depth of soil column (in.)	Duration of experiment (days)	Initial porosity	Rate of moisture loss (in. per yr)
Coarse sand	Air dry	2	4	0.45	4.3
Takoma lawn soil	10	12	53	0.44	0.12
Takoma lawn soil	10	12	53	0.50	0.15
Takoma lawn soil	10	12	53	0.52	0.17
Cecil clay	Air dry	12	441	0.46	0.60
Cecil clay	Air dry	..	441	0.66	0.80
Leonardtown loam	Air dry	1	140	0.54	2.71
Leonardtown loam	Air dry	2	140	0.51	1.60
Leonardtown loam	Air dry	4	140	0.49	0.95
Leonardtown loam	Air dry	6	140	0.51	0.69
Podunk fine sandy loam	Air dry	1	140	0.48	2.52
Podunk fine sandy loam	Air dry	2	140	0.46	1.59
Podunk fine sandy loam	Air dry	4	140	0.41	0.93
Podunk fine sandy loam	Air dry	6	140	0.46	0.67

King also conducted a number of interesting tests to indicate the effect of evaporation. A series of cylinders, each having a cross-sectional area of 0.1 sq ft, were cut to different heights and filled with graded sand. (Unfortunately, the size-distribution was not stated.) The

water levels in the various cylinders were maintained 6, 12, 18, 24, and 30 in. below the surface. Evaporation was observed for 40 days with results as shown in Table 58. The experiment was conducted in a laboratory having a mean temperature of 70 deg F. The relative humidity was said to be low. It is seen from Table 58 that the rate of evaporation is less as the depth of the bed above the water level increases. This is to be expected since the height reached by the fluid through capillarity is counteracted by the action of gravity.

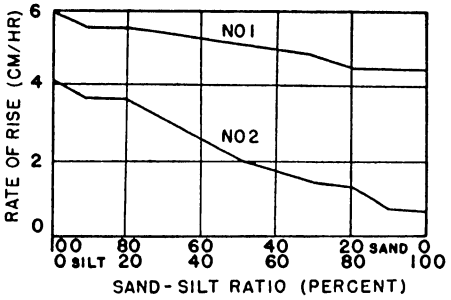


FIGURE 78. RATE OF RISE OF MOISTURE IN YUMA-MESA SOIL. NUMBERS REFER TO DATA IN TABLE 60.

Buckingham (1907) conducted further experiments on soil capillarity, but again gave no data on the particle-size of the materials used. For a given soil, however, it was found that the rate of evaporation was less as the porosity increased. Thus, a change in porosity from 47 to 55 percent changed the mean rate of drying from 5.24 to 4.39 in. per yr. Other results obtained by Buckingham are shown in Table 59.

Rate of Capillary Rise—McGeorge (1941) determined the capillary rise in different mixtures of sand and silt. This was done by packing a 50-cm column of soil in a 1 in. diameter glass tube and setting the bottom end in a container of water. The results of these tests are shown in Table 60, giving the total rise of moisture in 8 hrs, and in Figure 78 where the rate of rise per hr for the same period is plotted

TABLE 60—TOTAL CAPILLARY RISE OF WATER IN A COLUMN CONTAINING SAND-SILT MIXTURES[a] (MCGEORGE, 1941)

Mixture		Total capillary rise (cm per 8-hr period)	
Sand (percent)	Silt (percent)	Sand-silt No. 1	Sand-silt No. 2
100	0	49	47
80	20	45	41
60	40	42	27
40	60	40	1
20	80	36	11
0	100	35	7

[a] For the analysis of these sand-silt mixtures see Table 55.

The size-analysis of the sand-silt mixtures to which the data apply is contained in Table 55.

CAPILLARY-TUBE THEORY

The possibilities of the Poiseuille equation for explaining capillary phenomena in soils have had a certain amount of appeal to soil physicists. The theory of capillarity is so simple and so well established that if suitable soil parameters to express a "bundle" of capillaries could be found, the problem would be solved in the most direct fashion. Unfortunately, however, there has as yet been no completely satisfactory formula for the capillary "bundle" theory, although it has often given results of the proper magnitude in certain instanees. We shall here develop the Poiseuille equation for special capillary conditions and point out the fallacies of this approach.

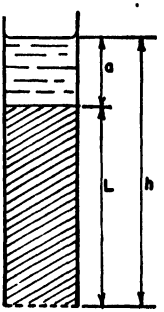


FIGURE 79. ILLUSTRATING UPWARD FLOW OF CAPILLARY MOISTURE.

Consider a vertical tube filled to a height L with small capillaries each having a diameter D_c. Let us assume that all movement of water takes place through the capillaries only. Let the tube be filled above the top of the capillaries to a height a, so that the total height above the open bottom of the capillaries is $a + L = h$, as shown in Figure 79. Under these conditions the Poiseuille equation becomes

$$q = \frac{Q}{t} = \frac{\pi}{128\mu} \cdot \frac{gh\rho_0}{L} \Sigma D_c^4 \qquad \text{Eq (15-5)}$$

Now if we assume that D_c represents some average measure of the pore-diameter through which moisture moves in a soil column we may, with only a slight change in Eq (15-5), make it more general. Since $Q = Ax$, where A is the area of the cylinder at a point x that the water front has moved in the time t

$$\frac{x}{t} = \frac{\pi}{128\mu} \cdot \frac{gh\rho_0}{AL} \Sigma D_c^4 \qquad \text{Eq (15-6)}$$

$$= C \frac{h}{L} \varphi$$

where $C = \pi g\rho_0/128\mu$ and $\mathcal{P} = \Sigma D_c^4/A$ has been termed by Green and Ampt (1911, 1912) the absolute permeability. Hence

$$C\mathcal{P} = \frac{x}{t} \cdot \frac{L}{h} = \mathcal{P}_c' \qquad \text{Eq (15-7)}$$

where $\mathcal{P}_c'$ is called the permeability, also defined as

$$\mathcal{P}_c' = \frac{Q}{t} \cdot \frac{L}{Ah} \qquad \text{Eq (15-8)}$$

In other words, the permeability is the volume of water passing in a unit-time through a unit cross section of soil under a pressure equal to the length of the column.

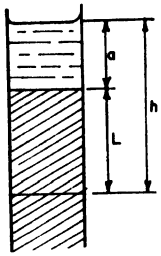


FIGURE 80. ILLUSTRATING DOWNWARD FLOW OF CAPILLARY MOISTURE

Downward Movement—In the case of downward movement, it is necessary to adjust the driving force, in Eq (15-5) to take care of the capillary forces. Assume, following Green and Ampt, that this added driving force is taken as an extra head, h_c, so that the total driving force is $g\rho(h + h_c)$ instead of $gh\rho$. Also, since the water occupies the pore-space ϑ of the soil, it follows that at any stage the velocity of the water front will be given by the equation

$$\frac{dQ}{dt} = A\vartheta \frac{dL}{dt} \qquad \text{Eq (15-9)}$$

where Q and L represent the volume of water and the length of the wetted soil (Figure 80). Hence, combining Eqs (15-5) and (15-9),

$$\frac{dL}{dt} = \frac{\pi}{128\mu} \cdot \frac{g\rho_0(h + h_c)}{\vartheta LA} \Sigma D_c^4$$

$$= \frac{C\mathcal{P}}{\vartheta} \cdot \frac{h + h_c}{L} = \frac{\mathcal{P}_c'}{\vartheta} \cdot \frac{h + h_c}{L} \qquad \text{Eq (15-10)}$$

Since $h = a + L$ and a, h_c, $\mathcal{P}_c'$, and ϑ may be held constant during a test

$$\frac{\mathcal{P}_c'}{\vartheta} t = \int_L^0 \frac{L dL}{L + a + h_c}$$

and since $L = 0$ when $t = 0$

$$\frac{\mathcal{P}_c'}{\vartheta} t = L - (a + h_c) \ln \left(1 + \frac{L}{a + h_c}\right) \qquad \text{Eq (15-11)}$$

where ln is written for logarithm to the base e. In this equation, h_c is determinable only by trial and error.

Green and Ampt tested Eq (15-11) as follows: A glass tube about 1 in. in diameter and 30 in. long was carefully and uniformly packed with air-dry loam to within 6 in. of the top rim. The lower end was closed with a copper-gauze disk. A layer of water was placed in the tube and, by means of a constant leveling device, maintained about 10 cm above the surface of the soil. Thus, any percolation taking place through the soil was *made good*. Observations were taken of the downward percolation of moisture at intervals during 60 hrs, at the end of which time water began dripping from the tube. Green and Ampt found fairly good agreement of their experimental data when h_c was taken to equal 90.

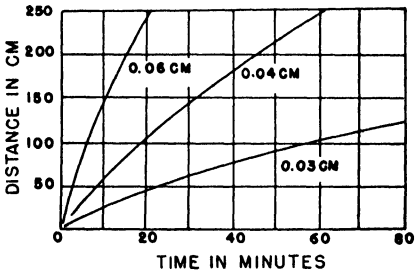


FIGURE 81 RATE OF PERCOLATION THROUGH COLUMNS OF SAND OF VARIOUS SIZES

Gardner and Widstoe (1921) made attempts to develop a general equation. They assumed an ideal soil to be one in which the capillary potential was a linear function of the reciprocal of the moisture-content (after Buckingham, 1907), and that the inherent moisture conductivity is independent of the moisture-content. These assumptions are necessary for mathematical reasons. For downward flow through sand, whose surface is kept saturated, the equation connecting the time t and distance from the surface L was as follows:

$$L = c_1 t + c_2[1 - \exp\ (-bt)] \qquad \text{Eq (15-12)}$$

where c_1, c_2, and b are constants. Results for three grades of sand are shown in Figure 81 and are from data given by Gardner and Widstoe.

It should be remembered that Eq (15-12) contains three constants and an exponential term, and will fit a wide range of results. For this reason, although derived from rational hypotheses, it must be regarded as empirical. Eq (15-12) has one distinct advantage over Eq (15-10), in that the initial value of dL/dt, the rate of advance of the water front, is not infinite but a finite positive quantity.

Vertical Rise—If the lower end of the soil column now touches a water surface, we have $h = L$ and $a = 0$, so that Eq (15-10) becomes

$$\frac{dL}{dt} = \frac{\mathcal{O}_c'}{\vartheta} \cdot \frac{h_c - L}{L} \qquad \text{Eq (15-13)}$$

which on integration gives

$$\frac{\mathcal{P}_c'}{\vartheta} t = h_c \ln \left(\frac{L}{h_c - L} \right) - L$$

Green and Ampt did not obtain good agreement in applying this equation to Loughridge's data given in Hilgard's "Soils" (1906), page 205.

Hackett (1921–1922), on the other hand, seems to have achieved better success. If we plot the velocity dL/dt against $1/L$ we should obtain a straight line (Eq 15-13), that is,

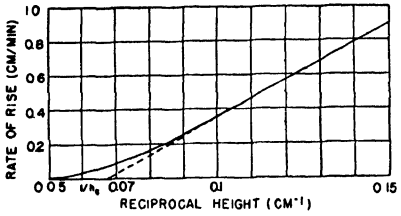


FIGURE 82. CURVE SHOWING RATE OF RISE DUE TO CAPILLARITY IN A PACKING.

$$\frac{dL}{dt} = \frac{\mathcal{P}_c'}{\vartheta} \cdot \frac{h_c - L}{L} \qquad \text{Eq (15-14)}$$

$$= C' h_c \left(\frac{1}{L} - \frac{1}{h_c} \right)$$

where C' is written for $\mathcal{P}_c'/\delta$. In this way Hackett determined the values of h_c and $C'h_c$ for the rise of paraffin oil through sand having a mean diameter of 0.0168 cm, obtaining the following values, $h_c = 15.6$, $C' = 10.6$. The data are shown in Table 61. The value of h_c may be obtained by plotting dL/dt against $1/L$. The intercept of the straight-line position of the curve on the abscissae is the reciprocal of h_c (see Figure 82).

TABLE 61—RAPID RISE OF PARAFFIN OIL THROUGH SAND. AVERAGE SCREEN SIZE—0.0168 CM

Distance (cm)	4.7	7.85	10.9	13.9	14.6	15.0	15.3
Observed time (min)	1.0	4.0	12.0	31.5	42.0	52.0	57.5
Calculated time (min)	1.3	4.3	11.5	31.6	42.0	52.0	67.5

Hackett's results apply to the initial stages of ascent through granular media. With increasing time, the rate of advance diminishes and fails to follow the linear relation of Eq (15-14). Hackett attempted to explain the slow rise by constant readjustment of his datum plane, but failed to give any consistent or general theory.

Washburn (1921) from purely theoretical considerations derived the following expression for horizontal capillarity:

$$L = \frac{1}{2} \sqrt{\frac{\sigma D_c t \cos \theta}{\mu}} \qquad \text{Eq (15-15)}$$

where σ is the surface tension, θ the angle of contact, μ the viscosity of the liquid, and D_c the diameter of the capillary. If we regard a packing as a bundle of capillaries, then D_c is merely a measure of capillary size and L will be proportional to the volume moving at any given instant. For a given packing and any liquid, the volume penetrating all tubes is

$$Q = K\sqrt{\frac{\sigma t}{\mu}} \qquad \text{Eq (15-16)}$$

and the rate of flow at any moment

$$q = K\sqrt{\frac{\sigma}{\mu t}} \qquad \text{Eq (15-17)}$$

where K is a constant for a given packing, and is independent of the liquid.

Referring to Eq (15-2) for capillary rise in a tube of diameter D_c we had

$$h = \frac{4\sigma}{g\rho_0 D_c}$$

where σ is the surface tension, ρ_0 the density of the fluid, and g the gravitational acceleration constant. For a given tube and liquid at constant temperature

$$hD_c = \frac{4\sigma}{g\rho_0} = K_c\frac{\sigma}{g\rho_0} \qquad \text{Eq (15-18)}$$

where K_c may be called the capillary constant. This relation holds for a packing where D_c as an order of approximation is taken as the diameter of the particles, d. The value of h may be considered the maximum height to which the liquid will rise in rapid ascent and is equal to h_c in Eq (15-14). Hence, the value of K_c for a packing may be determined. Hackett and Keen (1918–1919) succeeded in showing that hd is a sensibly constant quantity for a given packing of irregular particles in close array. Hackett experimented with various sands having mean diameters ranging from 0.0085 to 0.042 cm.* Following Slichter (1899) and from sand experiments, the above investigators computed the constants for different kinds of capillaries shown in Table 62.

The values of K_c for sand are less than those for open packing, although the voids are undoubtedly much less. This shows the importance of points of contact as well as surface characteristics which may greatly influence the character of the capillary. Note that K_c is dimensionless.

* The mean size was the average of two successive I.M.M. screen sizes.

TABLE 62—VALUES OF CAPILLARY CONSTANT K_c AFTER KEEN (1918–1919) AND HACKETT (1921–1922)

Type of capillary	Value of Kc
Capillary tube of diameter D_c	4
Cubical (open) packing	14.6
Rhombohedral (close) packing (Keen)	19.8
Rhombohedral (close) packing (Hackett)	39.0
Experimental sand	7.84

Horizontal Capillarity—When the capillary motion is horizontal, it is due entirely to the capillary phenomenon itself, so that from Eq (15-10)

$$\frac{dL}{dt} = \frac{\mathcal{P}_c'}{\vartheta} \cdot \frac{h_c}{L}$$

$$\frac{\mathcal{P}_c'}{\vartheta} h_c t = \frac{1}{2} L^2 \qquad \text{Eq (15-19)}$$

or for a given liquid and packing

$$\frac{L^2}{T} = constant$$

Green and Ampt (1911) tested this equation for a soil and found the L^2/t to be constant only after several hours. They found better agreement for the same data by using the expression

$$M^2/t = constant \qquad \text{Eq (15-20)}$$

where M is the total moisture absorbed by the column in the time t.

CAPILLARY POTENTIAL

Work of Buckingham—In an attempt to explain the general theory of soil capillarity, Buckingham (1907) formulated the concept of capillary potential. Buckingham conceived capillary movement to be analogous to the flow of heat and electricity. Let M be the mass of water flowing through a cross section perpendicular to the direction of flow, λ the capillary conductivity or facility offered the flow of water, and ψ the capillary potential. Then, using the analogy of heat flow

$$M = -\lambda \frac{\partial \psi}{\partial x} \qquad \text{Eq (15-21)}$$

This analogy, as Buckingham pointed out, is only formal. In the flow of heat and electricity, conductivity is a constant quantity. In capillary theory, however, λ will vary with distance and time. The capillary

potential up to a certain point is a function of the moisture-content.

Buckingham showed that the relation between ψ and the moisture-content M_c could be ascertained from measurement of the equilibrium moisture distribution (which is very slowly attained and never certain) in a vertical column of soil, whose lower end was in water and upper end protected from evaporation. Consider the work required to move a small mass of water from x to $x + dx$, the capillary potentials being ψ and $(\psi + (\partial\psi/\partial x)dx)$, respectively. The total work done against capillary potential is therefore

$$\psi dM_c - \left(\psi + \frac{\partial\psi}{\partial x}\right) dM_c = \frac{\partial\psi}{\partial x} dx dM_c$$

But in this process the mass, M_c, has been raised a distance dx against gravity, and this work is $gdxdM_c$, where g is the gravitational constant. Since the system is in equilibrium

$$\frac{\partial\psi}{\partial x} dx dM_c = g dx dM_c$$

$$\frac{\partial\psi}{\partial x} = g$$

The dependence of capillary potential on moisture-content, as we have seen in King's (1899) experiments, is a function of x, so that

$$\frac{\partial\psi}{\partial M_c} = \frac{\partial\psi}{\partial x} \cdot \frac{\partial x}{\partial M_c}$$

and in combination with the above equations

$$\frac{\partial\psi}{\partial M} = \frac{g}{\frac{dM_c}{dx}} \qquad \text{Eq (15-22)}$$

Thus, a change of capillary potential with moisture-content is equal to a constant divided by the rate of moisture change with height. Integrating, we obtain

$$\psi = gx \qquad \text{Eq (15-23)}$$

or that the capillary potential varies directly as the height.

Buckingham was able to prove his concept only qualitatively and was unable to arrive at any relationship between λ and the moisture-content. From a general consideration of the distribution of water over soil grains, he concluded that at high moisture-contents the films would be thick and continuous, and the conductivity λ at its maximum; that is, conductivity would increase with the moisture-content.

Some of Buckingham's results are shown in Figure 83. The data shown in this figure apply to data on three soil columns whose bottoms rested in water and whose surfaces were protected from evaporation. The particle-size of the soils studied for their porosities under the given conditions was not stated. The curves show that the general level of capillary potential is lower for coarser particles (Curve A).

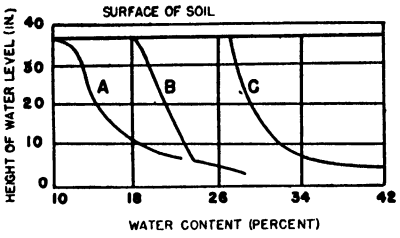


FIGURE 83 RELATION BETWEEN HEIGHT OF WATER LEVEL AND MOISTURE CONTENT. (*A*) NORFOLK FINE SANDY LOAM; (*B*) LEONARDTOWN LOAM, (*C*) CECIL CLAY.

Work of Gardner—Gardner (1919*a*, *b*, 1920*a*, *b*) reëxamined the general theory of capillary potential and attempted to develop suitable equations for the variation of moisture-content with height and conductivity. We shall here develop Gardner's method. Let us consider the water wedge shown in Figure 75. When $R_1 = R_2 = 0$, we have a condition of zero moisture-content; when we have a condition of maximum moisture possible, R_1 will approach the radius of the sphere (approximately) and R_2 will approach infinity. To comply with these limits, Gardner formed the function

$$R_1 = \frac{aR_2}{bR_2 + c} \qquad \text{Eq (15-24)}$$

where a, b, and c are constants. In this expression $R_1 = 0$ when $R_2 = 0$, and when R_2 is infinity, R_1 is a finite quantity a/b. The volume of the wedge was next assumed by Gardner to be proportional to the cube of the diameter of the sphere, that is,

$$V = 8K_1R_1^3 = K_1d^3 \qquad \text{(nearly)} \qquad \text{Eq (15-25)}$$

where d is the diameter of the sphere and approximately equal to $2R_1$, V is the volume of the water per unit-volume of soil (the particles being assumed to be spheres of equal size), and K_1 is a constant. Now using Eq (15-1) relating capillary pressure P and surface tension σ, we may eliminate R_1 and R_2 from the above equations, obtaining

$$P = -\frac{K_2}{V^{1/3}} \qquad \text{Eq (15-26)}$$

and differentiating with respect to x, the distance along the capillary columns

$$\frac{dP}{dx} = \frac{K_2'}{V^{1/3}} \cdot \frac{dV}{dx} \qquad \text{Eq (15-27)}$$

where K_2 and K_2' involve the constants K_1, σ, a, b, and c. Gardners' next step was to assume some relation for dP/dx, the rate of change of film pressure with distance. For this he suggested that the movement of moisture from one datum plane to another under the influence of this pressure is analogous to the motion of a body through a viscous fluid. Hence, using Stokes' law, and writing v for the velocity of a unit-volume of fluid and μ for the viscosity,

$$v = \frac{K_3}{\mu} \cdot \frac{dP}{dx} \qquad \text{Eq (15-28)}$$

where K_3 is a proper constant and dP/dx is the same as g. Eliminating dP/dx from Eqs (15-27) and (15-28) gives

$$Vv = \frac{K_3 K_2'}{\mu V^{1/3}} \cdot \frac{dV}{dx} \qquad \text{Eq (15-29)}$$

or

$$(Vv)\, V^{1/3} \Big/ \frac{dV}{dx} = a\ constant = \lambda_G \qquad \text{Eq (15-30)}$$

The dimensions of this constant are clearly $[L^3\ T^{-1}]$ = rate of flow, $q/t = \lambda_G$. Gardner regards this constant as equivalent to the capillary conductivity on the basis of heat and electrical analogies. This result is remarkable considering the assumptions involved. Eq (15-30) indicates that λ_G, the capillary conductivity, may be determined directly from measurement of the volume moisture-content V, the actual volume of flow per unit cross section Vv, and the gradient moisture-content dV/dx at the point x. These quantities are all experimentally observable.

Capillary conductivity constants are difficult to determine since slight disturbances of the soil structure have a marked effect upon its value. Gardner's results for Greenville soil are given in Table 63. The first three values indicate that there is an optimum porosity for capillary flow.

To obtain a relation between the volume moisture-content and the distance x for a given point, we may write the equation of continuity for one-dimensional flow as follows:

$$\frac{\partial V}{\partial t} = -\frac{\partial}{\partial x}(Vv)$$

so that from Eq (12-30), we obtain

$$\frac{\partial V}{\partial t} = -\frac{K_3}{\mu} \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} \cdot \frac{1}{V^{1/3}}\right) \qquad \text{Eq (15-31)}$$

as the differential equation for capillary motion. For a steady state

$$V\frac{\partial^2 V}{\partial x^2} = \frac{1}{3}\left(\frac{\partial V}{\partial x}\right)^2$$

A solution of the equation is

$$V = (ax + b)^{4/3} \qquad \text{Eq (15-32)}$$

Gardner claimed a certain amount of success with Eq (15-30). As an example, let us calculate the amount of water available at the surface of a plot of Greenville soil, due to a water table 100 cm below it. Take $\lambda_G = 6 \times 10^{-3}$, V (average) as 0.25 cu cm per cu cm of soil (0.5/2), and assume V/x is constant through the distance concerned. We are interested, therefore, in computing the value of Vv. Using Eq (15-30)

$$Vv = \frac{6.0 \times 10^{-3}}{(0.25)^{1/3}} \cdot \frac{0.5}{100} = 4.6 \times 10^{-5} \text{ cu cm per sec}$$

This amounts to 50 in. of water per month, which is quite high. However, considering the factors involved in the derivation of the formula, the order of approximation is noteworthy.

TABLE 63—VALUES OF CAPILLARY CONDUCTIVITY FOR VARIOUS SOIL CONDITIONS

Condition of soil	Value of λ_G (cu cm/sec) × cm
Loosely packed	1.8×10^{-3}
Well packed and dry	7.4
Further packed, moistened and dried	5.4
Undisturbed field condition (downward flow)	8.7
Average	5.8

Work of Richards—L. A. Richards (1931) gave an excellent theoretical presentation of the factors involved in a study of capillary constants. He succeeded in establishing a general capillary equation and gave a method for determining conductivity. The conductivity was defined as the constant contained in Darcy's equation (Eq 13-2)

$$q = -\mathcal{P}_0 A\frac{\Delta P}{L} = -\mathcal{P}_0 A \nabla \Phi$$

where q is the volume of water passing in a unit-time perpendicular to the flow, $\mathcal{P}_0$ is the permeability, and Φ the pressure gradient producing the flow. By a suitable laboratory procedure in which the soil sample could be placed between two porous plates, Richards measured the rate of fluid flow and the pressure gradient. If Q/t represents the flow

per unit-time, A the area of the sample cross section, ΔP the pressure difference, and L the thickness of the sample

$$q = \frac{Q}{t} = -\mathcal{P}_0 A \nabla \Phi \qquad \text{Eq (15-33)}$$

and under gravity and pressure

$$\nabla \Phi = g + \Delta P/L$$

so that, if we write $\mathcal{P}_0 = \lambda_R$ = conductivity according to Richards,

$$\lambda_R = \frac{Q}{t} \cdot \frac{1}{A(g + \Delta P/L)} \qquad \text{Eq (15-34)}$$

For conditions of steady flow, Richards developed an equation relating the conductivity λ_R and capillary potential ψ

$$\lambda_R \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial \lambda_R}{\partial z} \cdot \frac{\partial \psi}{\partial z} + g \frac{\partial \lambda_R}{\partial z} = 0 \qquad \text{Eq (15-35)}$$

where z, of course, is any point on the vertical soil column. Thus, if ψ and λ_R have some functional relationship the equation is easily simplified. As a matter of fact, experimental data are available indicating that such a relationship may exist as a linear function

$$\lambda_R = a\psi + b \qquad \text{Eq (15-36)}$$

where a and b are constants determinable from the data. Substituting this in Eq (15-35), and integrating

$$\frac{\partial \psi}{\partial z} = \frac{1 - gac_1(a\psi + b)}{ac_1(a\psi + b)}$$

Integrating again, we obtain the relation between z and λ_R

$$\frac{1}{g^2 a^2 c_1} [1 - gac_1(a\psi + b) - \ln (1 - gac_1(a\psi + b))] = z + c_2 \qquad \text{Eq (15-37)}$$

where c_1 and c_2 are constants of integration and ln denotes logarithm to the base e. Now the equation of flow is precisely Eq (15-33), that is,

$$\begin{aligned} q &= -\lambda_R \nabla \Phi = \lambda_R[(-\partial\psi/\partial z) - g] \\ &= (a\psi + b)\left[\frac{-1 + gac_1(a\psi + b)}{ac_1(a\psi + b)} - g\right] = -\frac{1}{ac_1} \end{aligned} \qquad \text{Eq (15-38)}$$

Richards has given some experimental data relating to the conductivity and capillary potential. These are shown plotted on semi-

log grid in Figure 84 and indicate that a suitable relation of these variables is

$$\lambda_R = a' \cdot \exp\ (b'\psi)$$

Substitution of this relation in Eq (15-35) results in a slight simplification in so far as the computation of $\partial\psi/\partial z$ and ψ are concerned.

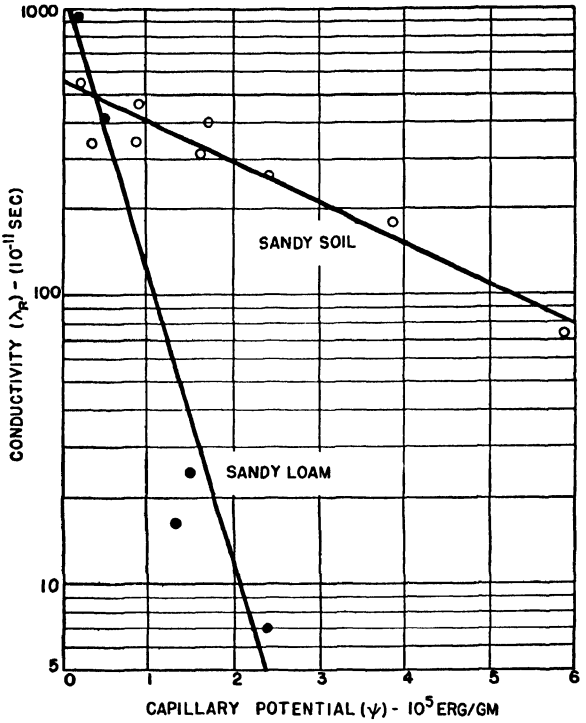


FIGURE 84. RELATIONSHIP OF SOIL CONDUCTIVITY TO CAPILLARY POTENTIAL FOR TWO TYPES OF SOILS. DATA FROM L. A. RICHARDS (1931).

Example—Using Greenville soil, Richards obtained a linear relation between λ_R and ψ in accordance with Eq (15-36). The constants a and b in this equation were as follows: $a = 5.35 \times 10^{-15}$ and $b = 3.85 \times 10^{-9}$. It is desired to calculate the flow at the surface of the soil due to a water table 100 cm below, assuming a capillary potential at the surface of -6×10^5 ergs per g. From these data it is possible to calculate values of c_1 and c_2 in Eq (15-37), ($\psi = 0$ when $z = 0$, and $\psi = -6 \times 10^5$ when $z = 100$ being the boundary conditions). These constants can be shown to have the following values: $c_1 = -1.66 \times 10^{19}$ and $c_2 =$

-2288. Since we are interested only in the computation of q in accordance with Eq (15-38)

$$q = -\frac{1}{ac_1} = \frac{1}{1.66 \times 10^{19}} \times \frac{10^{15}}{5.35} = 1.13 \times 10^{-5} \text{ cu cm/sec}$$

or q = approximately 1 cu cm per day or about 12 in. per month.

HAINES' THEORY

The various methods for computing capillary rise given in preceding paragraphs are based upon analogies with known solutions for special problems on the flow of liquids or heat. In none of the various formulas given was the configuration of the pore-space dealt with. It remained for Haines (1925*a*, *b*, 1927, 1930) to develop the true nature of capil-

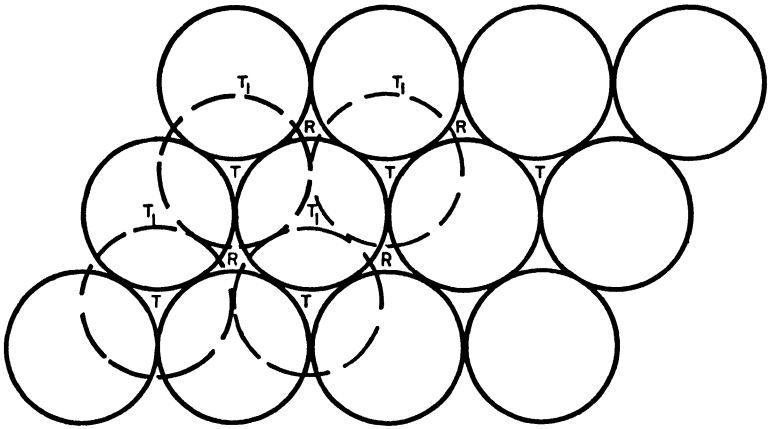


FIGURE 85. ARRANGEMENT OF SPHERES IN CLOSEST PACKING. SOLID LINES DENOTE FIRST LAYER OF SPHERES AND DOTTED LINES THE SPHERES LYING IN THE HOLLOWS FORMED BY THE FIRST LAYER. T, T_1 ARE FORMS OF TETRAHEDRAL CELLS; R DENOTES RHOMBOHEDRAL CELLS.

lary flow in packings and the pressures involved. Haines' approach and experimental techniques were so simple it is surprising they had escaped the many earlier investigators. The results of Haines' work have not as yet been utilized completely but they should have a great bearing upon the theory of drying of particulate substances and perhaps even in defining configuration of the pore-space in a packing. In the following discussion we shall constantly refer to Chapter 6 on the characteristics of packings.

Liquid Rising—Let us consider an ideal soil made up of spheres in closest packing, and direct our attention to just two rows of such a packing as shown in Figure 85. Haines has shown that there are two

types of cells, one of tetrahedral form and the other rhombohedral. The former lies at the center of four spheres packed in the form of an equilateral pyramid. The rhombohedral cell is enclosed by a group of six spheres. The relative positions of both kinds of cells can be seen from the figure, which shows in full line the plan of one layer of spheres in close packing. The next layer above is shown in dotted lines. Each of the spheres marked T forms a depression or socket into which one of the dotted spheres fits, so that each dotted sphere and each full-line sphere below it enclose a tetrahedral cell. The depressions marked R are not so enclosed. These are the rhombohedral cells formed by three full-line spheres and three dotted-line spheres. In addition there is a second series of tetrahedral cells marked T_1 which are identical with the first set of tetrahedral cells marked T. Therefore, there are two tetrahedral cells to each rhombohedral cell. Each of the corners or apices of an R cell connects with an apex of a T cell and each apex of the latter is connected with an apex of an R cell.

We shall attempt to ascertain how the water is distributed within the cellular structure at different moisture-contents. Restricting ourselves for the moment to a wedge formed by two spheres as shown in Figure 75, we have that the suction at the waist of the spheres is determined by the relation

$$P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are radii of curvature of the wedge. Also $R_1 = r$ (sec 2θ − 1) and $R_2 = -r$ (tan 2θ − sec 2θ + 1) where r is the radius of the spheres. Hence

$$P = \frac{\sigma}{r}\,\frac{(2 + \tan 2\theta + \sec 2\theta)}{(1 + \tan 2\theta - \sec 2\theta)(\sec 2\theta - 1)}$$

We see therefore that as the volume of the wedge decreases θ decreases and approaches zero. At this point P approaches infinity. On the other hand, as the moisture increases, θ increases but even for closest packings cannot exceed 30°, as indicated in Figure 76. Hence, up to the point when the wedges begin to coalesce $\theta = 30°$, and

$$P = 4.1\,\frac{\sigma}{r}$$

Haines termed this the "pendular stage."

Now as the limit approaches and coalescence starts, let us focus our attention on the void shown in Figure 28. The water wedges are now in contact along the lines S so that a continuous meshwork is formed,

and we can pass throughout the liquid film without leaving the liquid phase. As the liquid rises it enters a rhombohedral cell R (Figure 85), and a new action is created: The liquid proceeds upward to the waist of three spheres, to the triangular capillary. The rhombohedral cell is gradually encompassed with water wedges and the air bubble is forced to escape through the triangular capillary. Haines assumes the radius of this bubble to be 0.29 r. As the liquid rises, the suction created by the bubble is

$$P = \frac{\sigma}{r}\left(\frac{1}{0.29} + \frac{1}{0.29}\right) = 6.9\,\frac{\sigma}{r}$$

This stage is termed the "funicular" stage. It is to be noted that the pendular stage broke when the suction was 4.1 σ/r. Then a new episode took place, and a suction of 6.9 σ/r was achieved. This was done entirely at the expense of the configuration dealt with, and the results are therefore not incompatible as might be presumed. As indicated there are two states of equilibrium and they are maintained as long as the cells at any given level are in their respective states.

As more liquid is added, the capillary pores (OO' in Figure 26) eventually fill with liquid, and this condition continues at the expense of the funicular stage. The suction gradually diminishes as the menisci and other pores disappear. This is called the "capillary" stage, and the cycle repeats in each successive layer of spheres.

Liquid Falling—As the liquid drops from the top surface of the spherical packing just discussed, menisci are formed at the surface pores and advance downward. The suction will be maximum at the throat of the pore when the radius is 0.155 r, that is

$$P = \frac{\sigma}{r}\left(\frac{1}{0.155} + \frac{1}{0.155}\right) = 12.9\,\frac{\sigma}{r}$$

After passing the throat the menisci start entering the next cell. This entrance is abrupt because the liquid attempts to redistribute itself so as to attain equilibrium. Again we can move throughout the surface of the liquid without difficulty. The suction decreases gradually but can be given no fixed value since not all the waists need to be opened. Thus, the funicular stage during decreasing moisture is indefinite.

As the liquid continues decreasing we revert to the pendular state. This state is perfectly reversible for increasing or decreasing moisture conditions and, as before, is 4.1 σ/r. This value is the one obtained when the water wedge is the maximum which can be maintained. If we consider Eq (15-1), we see that as θ decreases R_1 and R_2 become equal, and P approaches infinity. Thus, the pendular stage during the decreasing

moisture cycle can have any value greater than 4.1 σ/r. This fact permits us to calculate the probable moisture-content of the packing at any point during the pendular stage. The volume of the wedge when at its maximum ($P = 4.1\ \sigma/r$) has already been found and is 24.3 percent of the total pore-space (see Chapter 14). The sequences discussed may be extended to any packing, but it must be remembered that a given concentration of moisture does not necessarily have a single type of distribution. Within certain values of moisture-content the three stages outlined may exist simultaneously, but we are restricted in calculating any but the pendular stage. The reason for the existence of all three stages is not difficult to explain since the stages are within the suction range 0–12.9 σ/r. The sequence of possibilities is briefly as follows:

Stage	*Range of suction* ($\times\ \sigma/r$)	*Range of concentration moisture (percent pore-space)*
Pendular	4.1	0–24.3
Funicular	6.9–12.9	6–undetermined but greater than 50
Capillary	0–12.9	100

It is seen from this arrangement that the funicular stage may persist down to a moisture concentration of 6 percent, and thus overlap an identical condition encountered during the pendular stage.

Experimental Verification—Haines verified the critical suction values of the three stages described for an array of spheres in closest packing. A number of layers of metal balls were arrayed in this system

TABLE 64—RESULTS OF HAINES' EXPERIMENTS ON SUCTION OBTAINABLE FOR AN ARRAY OF SPHERES IN CLOSEST PACKING

Stage	Observed suction	Theoretical suction
Capillary (complete saturation)		
First layer of cells entered	11.3	12.9
Funicular decreasing suction, air cell collapsed at	6.7	6.9
Pendular		
Closure of waists	4.3	4.1

on a horizontal plate, fixed on one arm of an adjustable U-tube. The balls were large enough to permit systematic packing, but not too large for observing capillary phenomena. Haines filled the apparatus with a light oil and observed the behavior of the advancing oil level, which could be adjusted by moving the opposite arm of the U-tube. The suction created by the capillary actions showed close agreement with theoretical results. Haines' results are shown in Table 64.

The above experiment was repeated with small spheres, more or less uniform and having a diameter of 0.038 cm. These spheres were placed in a Büchner funnel to a depth of 2 cm and the funnel was connected to a buret manometer by means of a U-tube as shown in Figure 86. The equipment was filled with water. The reference level was taken at the top of the spheres, and the moisture-content could be followed under changes of suction without difficulty.

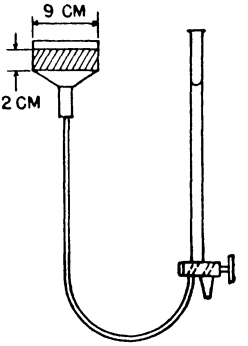


FIGURE 86. HAINES' APPARATUS FOR MEASURING CHANGES IN CAPILLARY FORCES.

The results of this experiment and a later one using sand are shown in Figure 87. It is seen in this figure that the curves for increasing and decreasing moisture-content are not the same. Within the loop formed by these cycles, which Haines has termed capillary "hysteresis," we may form similar cycles. Thus, for a complete cycle of falling and rising moisture for spheres in Figure 87, we have a loop *OADEK*. But if at a point *C* we began increasing the moisture, the path traced would be from *C* to *C'* on the main hysteresis curve for increasing moisture. If again at the point *C'* the moisture-content is decreased, then we follow the path *C'C"C*.

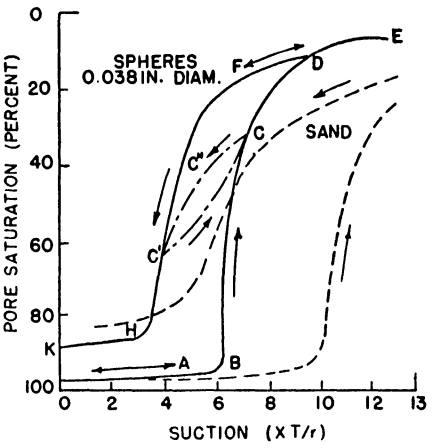


FIGURE 87. CAPILLARY HYSTERESIS DETERMINED FOR TWO MATERIALS BY HAINES' APPARATUS.

Several features concerning the loop are worthy of mention. Consider first the falling moisture curve. Between the points *OAB* the curve is reversible. At the entry value of *A*, 4.1 σ/r, Haines observed air to enter the pores at isolated spots, and at *B*, 6.1 σ/r, the entry of air was general, leading to the conclusion that most pores in the surface layers have diameters about equal to rhombohedral cells in closest packing. About 70 percent of the moisture is lost at values slightly higher than 6.1 σ/r, the process constituting an evacuation of the cells and a

reduction of the funicular water. At the point E the pendular state is reached, suction is 11.2 σ/r, and saturation is 8 percent.

On the curve for rising moisture, some difficulty was experienced with the pendular stage, because the film conductivity was broken. Haines therefore started at the point D, where continuity was established. The stage DF was found to be reversible. At F the smaller cells began to close, until at G the closure was general. The process was completed at about H.

The hysteresis loop for sand shown in Figure 87 (the difference between the rising and falling curves) is enhanced, due to irregularities of the packings and other characteristics. Haines (1927) also experimented on the entry values (decreasing) of moisture-content with a wide variety of spherical and quasispherical materials having voids of approximately 0.37. Using water and benzene as liquids, his entry values ranged from 4 $\times$ to 6.9 $\times$ σ/r. Particle-diameters were as low as 0.0026 cm in the case of potato starch, and as high as 0.105 for lead shot. It was found that samples with poor uniformity gave entry values considerably higher than those recorded here.

Problems

1. Paraffin oil rises through a sand column by capillarity. Calculate the rate of rise at a distance 10 cm upward from the paraffin-oil surface in which the column is placed. Take $h_c = 15.6$ and $C'h_c = 10.6$ in Eq (15-14).

2. Determine the rate of moisture movement upward through a soil from a water table 50 ft below the surface. Assume a porosity of 34 percent and a conductivity $\lambda_G = 4.4 \times 10^{-3}$.

3. Explain how Haines' capillary theorem can be applied to explain drying of particulate matter by natural means.

CHAPTER 16

DETERMINATION OF PARTICLE SURFACE

MEASUREMENT of the surface of an aggregate of irregular particles is of great importance in chemical engineering. Catalysis, solubility, and adsorption are all phenomena dependent on surface. Similarly, the power required for crushing rock and other substances is a function of the surface produced. While a knowledge of particulate-surface measurement is readily admitted as necessary to the design of certain equipment, no absolutely sure method exists for accurately determining surface. In general, there are only two approaches, one of which may be termed statistical, and the other experimental. Both approaches are probably equally reliable if only relative comparisons of surfaces for particle groups of the same kind are considered. It is only when absolute surface measurements are required that experimental methods possess certain advantages.

Before we discuss the measurement of particle surface, it must be pointed out that this process is variously defined. We may consider a surface in a physical sense as a continuous array of perfectly smooth, infinitesimal surfaces without discontinuities, or as a chemical surface composed of an infinite number of discontinuous surfaces which are molecules of its constituent material. The surface measured in the first instance will always be less than in the second. The methods of surface measurement described in the present chapter include both types of surface measurement. Fortunately, the differences between the methods presented are not great, but it must be kept in mind that surface as ordinarily understood applies only to indirect statistical measures, as outlined in Chapter 3. In experimental methods of surface measurement, the kind of surface determined must be understood distinctly.

Some estimate of the relation between particle-size and surface may be obtained by systematic splitting of a cube whose sides are each initially 1 cm. At first the total surface of this particle will be 6 sq cm. If this cube is divided into 8 equal cubes, the total surface becomes 12 sq cm, and if this process be repeated until the original cube has been subdivided into 1000 cubes each side 1 mm in length, the total surface will be 60 sq cm. Thus, we see that continued subdivision of the cube

into progressively smaller sizes creates greater amounts of surface. There are several important considerations with regard to the constant subdivision into small particles. Among these are (1) the increase in bulk (which is further elaborated upon in Chapter 6); (2) increased chemical activity due to surface exposed; and (3) increased physical activity such, for example, as is associated with the condensation and evaporation of moisture, electrical activities, etc. One other feature which is more closely linked with particle-size may be mentioned, namely that the dynamic behavior of particles is affected by their degree of fineness.

STATISTICAL METHODS

Surface in Terms of Statistical Diameters—A few aspects of the statistical method have been discussed briefly. Reference to Chapter 3 indicates that there are several formulas for computing approximate particle surface. They are based on the assumption that the particles are spheres having particular average diameters. If it is desired to compute the specific surface of an aggregate (surface per unit-weight), Eq (3–6) may be used, that is,

$$S_w = \frac{6}{\rho d} \qquad \text{Eq (16-1)}$$

where S_w is the specific surface, ρ the density of the material, and d a suitable average particle-size as defined in Chapter 3. The specific surface on a volume basis (surface per unit-volume) is independent of density, and is simply

$$S_v = \frac{6}{d} \qquad \text{Eq (16-2)}$$

Eq (16-1) may be cast into another form by using Eq (3-45). If N defines the number of particles per g, we obtain

$$SN = S_w = \frac{1}{\rho} \cdot \frac{\alpha_s}{\alpha_v} \cdot \frac{d_s^2}{d_v^3} \qquad \text{Eq (16-3)}$$

By methods explained in Chapter 3, this equation for a distribution having a known geometric mean d_g and logarithmic standard deviation σ_g (Eq 16-3) is

$$\log S_w = \log \frac{6}{\rho} - \log d_g - 5.757 \log^2 \sigma_g$$

where for α_s (the surface-shape factor), we have written its value for spheres, π, and for α_v (the volume-shape factor), $\pi/6$. This is the Hatch-

Choate equation. The corresponding Roller equation is given by Eq (3-35).

Surface Obtained by Sieve Methods—When sieves are used for determining surface the various transformation equations, Eqs (5-10)–(5-14), may be used in the above equation. However, for crushing studies, where it is necessary to determine energy requirements on the basis of surface produced, it is more convenient to use an estimation method according to Coghill (1928). The basic principle of this method was outlined in Chapter 5, but a few modifications extending the method to surface measurement require explanation. Coghill uses a sequence of Tyler sieves, 3, 6, 10, 20, etc., whose dimensions decrease in the ratio 1:2. A geometric progression of weights 1, 2, 4, etc., is assigned to each of the sieves starting with the largest, and the moments of the weight frequency obtained. Table 65 illustrates the method for the feed and dis-

TABLE 65—COGHILL'S METHOD OF MEASURING SURFACE PRODUCED BY CRUSHING
OUTPUT OF CRUSHER $^1/_2$ TON PER HORSEPOWER-HOUR PER HOUR

		Feed		Discharge	
Mesh	Weight assigned (*a*)	Weight of material (percent) (*b*)	Moment (*a*) × (*b*)	Weight of material (percent) (*c*)	Moment (*a*) × (*c*)
26.7 mm	1	1.1	1.1	..	....
13.3 mm	2	21.2	42.4	..	....
3	4	36.7	146.8	..	....
6	8	16.1	128.8	8.1	64.8
10	16	8.5	136.0	35.6	569.4
20	32	4.8	153.6	9.1	291.2
35	64	3.9	249.6	10.3	659.2
65	128	3.3	422.4	7.0	896.0
150	256	4.4	1126.4	19.9	3994.4
		100.0	2407.1		6475.0

Units of surface produced = 64.8 − 24.1 = 40.7.
Surface-tons produced per horsepower-hour per hour = 0.5 × 40.7 = 20.4

charged product of a ball mill. From the sum of the moments of feed and discharge the units of surface of each per unit-weight may be determined. The difference between these units of surface is the surface produced per unit-weight of material. If the power input and time of grinding and the amount of material ground are known, then the output of a given mill can be expressed in terms of surface-tons per horsepower-hour per hour. This method of expressing results is valuable for comparing different equipment grinding the same product. Coghill's method assumes that the surface-characteristics (surface-shape factors) of the feed and discharge are the same and that the absolute surface is

not necessary for comparative tests of grinding equipment and materials. Basically, the method differs little from the usual formula explained above for determining specific surface, although for routine laboratory work it is of great convenience and deserves wide acceptance.

EXPERIMENTAL METHODS

Determination of Surface-Shape Factor—The shape-factor is one which, when multiplied by the square of the diameter of a hypothetical particle having an average surface, gives the true surface. The shape-factor is given by Eq (3-41)

$$S = \alpha_s d_s^2 = \alpha_s \left(\sqrt{\frac{\Sigma n d^2}{\Sigma n}} \right)^2$$

where S is the average surface, α_s the surface-shape factor, and d_s the statistical surface diameter. The expression means simply that the surface of an average irregular particle can be represented by a sphere of the same average diameter, but in the second instance the surface equivalent to that of the particle will be equal to $\alpha_s d_s^2$ rather than πd_s^2.

The general equation for determining specific surface involving the use of shape factors is given by Eq (16-3) above.

The computation of α_v, the volume-shape factor, is relatively simple and values of this constant for certain materials are given in Table 66. The determination of α_s is more involved, and depends on a method of solubilities first used by Martin *et al* (1925–1926*a*, *b*) for quartz particles, and later improved by Gross and Zimmerley (1928*a*, *b*, *c*). Martin's method is briefly as follows: Quartz particles whose average diameter is known are placed in a definite concentration of hydrofluoric acid (0.5 1.0 g in 5 *N* solution). The particles dissolve slowly, the reaction reducing the strength of the acid. The number of particles and the amount of hydrofluoric acid must be such that their rate of solution is a linear function. If the strength of the acid decreases rapidly, the rate of solution diminishes, and the method fails. The proper quantity of solution for a given material is best determined experimentally.

The length of time the particles are in the solution is another important variable. This should not be so long that particles of extremely fine sizes are dissolved completely, nor so long that the diameters of the other particles are so reduced that they are no longer representative of the material studied. The best time for a given amount of material to be given solution should be one which will reduce the surface an infinitesimal amount, say a molecular layer. Now let the reduction in concentration of the hydrofluoric-acid solution be denoted by ΔC_x and

assume that for the length of time that particles have been immersed ΔC_x is proportional to the surface of the particles S_x. The next step is to duplicate the experiment with a known surface of quartz. Preferably this quartz should be of the same kind as the quartz being tested. Martin used a parallelopiped of quartz which he carefully measured with a

TABLE 66—VALUES OF VOLUME- AND SURFACE-SHAPE FACTORS OF VARIOUS MINERALS

Material	Average size (cm)	Shape factor	Authority
	Volume-shape factor, α_v		
Crushed quartz	0 19	0 17	Goldman and DallaValle (1939)
Crushed quartz	0 089	0.15	Goldman and DallaValle (1939)
Crushed quartz	0.022	0.28	Martin *et al* (1927–1928*c*)
Crushed quartz	0 049	0.27	Martin *et al* (1927–1928*c*)
Crushed quartz	.	0 14	Hatch and Choate (1929)
Calcite	.	0 135	Hatch and Choate (1929)
Hornblende	0 005	0.02	Goldman and DallaValle (1939)
Feldspar	0 004	0.26	Goldman and DallaValle (1939)
Dolomite	0 004	0 19	Goldman and DallaValle (1939)
	Surface-shape factor, α_s		
White sand (smooth)	0 29	2 1	DallaValle (1938)
White sand (smooth)	0 08	2.7	DallaValle (1938)
White sand (smooth)	0.04	2.6	DallaValle (1938)
Filter sand (smooth)	0 06	2 7	DallaValle (1938)
Filter sand (smooth)	0.05	2.9	DallaValle (1938)
Crushed quartz	0.003	2 1	Martin *et al* (1927–1928*c*)
Crushed quartz	0.011	2.2	Martin *et al* (1927–1928*c*)
Crushed quartz	0.049	2.5	Martin *et al* (1927–1928*c*)
Crushed quartz	0 083	2.5	Martin *et al* (1927–1928*c*)

micrometer gauge. If the reduction in acid concentration is denoted by ΔC_0 for the known surface, and again if it is assumed to be proportional to the surface S_0, then

$$S_x = S_0 \left(\frac{\Delta C_x}{\Delta C_0} \right)$$

from which we may determine S_x. On a weight basis (specific surface)

$$S_w' = \frac{S_0}{w} \left(\frac{\Delta C_x}{\Delta C_0} \right)$$

where w is the weight of the sample used in the experiment. If loss-in-weight method is used, then $\Delta C_x / \Delta C_0$ is replaced by $\Delta w_x / \Delta w_0$. The

value of α_s is then easily determined if we assume that the average diameter of the particles is known, and that their number is N, thus

$$\frac{S_w'}{S_w} = \frac{N\alpha_s d_s^2}{N\pi d^2} = \frac{\alpha_s}{\pi}$$

where S_w is the surface obtained on the assumption that the particles are spheres. Hence for a single particle of average diameter, the surface is

$$S = \alpha_s d_s^2$$

In his experiments, Martin used the average arithmetic diameter d_{av} as a measure of surface. Using this diameter he obtained values of α_s for crushed quartz ranging from 2.0–2.5. These values of α_s are considerably less than the value of $\pi = 3.1416$ taken for spheres.

The investigations of Gross and Zimmerley were also concerned with surface generated by crushing and grinding. No attempt was made to determine the surface-shape factor, but the modifications of the Martin method merit discussion.

Gross and Zimmerley used approximately a gram sample for fine materials, and upward to 16 g for larger materials. The standard concentration of hydrofluoric-acid solution was 3.66 normal, and the volume varied from 25 to 250 cc. The smaller volumes were found easier to manipulate. The sample placed in the solution was agitated in a thermostatically controlled water bath. After agitation for the desired length of time the contents were rapidly filtered and washed. The filter paper containing the quartz particles was placed in a weighed platinum crucible, ignited, and weighed after cooling. The length of time that the sample was subjected to hydrofluoric-acid treatment varied from $^1/_2$ to 2 hrs. From the experimental data at various times two curves were drawn: (*a*) The cumulative curve as a percentage of silica dissolved plotted against time and (*b*) the rate curve as a percentage of silica dissolved per hr plotted against time. By extrapolation of the rate curve to the zero time ordinate the initial rate (I.R.) or the index of the surface was obtained. The extrapolation must be done carefully. Gross and Zimmerley obtained the following expressions for their data:

$$c = \frac{w_0 t + at^2}{1 + kt}$$

$$b = \frac{w_0 + at}{1 + kt}$$

where c is the percentage of silica dissolved, b the percentage of silica dissolved per hr, w_0 the initial rate, t the time in hrs, and a and k are constants. The general equation in which $kc + b$ is a straight line is $kc + b$

$= w_0 + at$, in which at is the distance from the intercept w_0 to the straight line. The initial rate of solution (I.R.) was calculated by means of the above equations.

The surface of irregular crushed quartz particles is shown in Table 67. The theoretical surface given in this table was obtained by Eq (16-1)

TABLE 67—DATA ON SIEVE-SIZES OF CRUSHED QUARTZ

Mesh	Average size of particle (mm)	Theoretical surface per g of quartz cubes or spheres (sq cm)	Initial rate, I.R.	Measured surface per g, (sq cm)[a] I.R. × 170	Ratio of measured to theoretical surface
3/4	5.690	4.0	0 201	34.2	8.55
4/6	4.013	5.65	0.258	43.9	7.77
6/8	2.845	8.0	0.307	52.2	6.53
8/10	2.007	11.3	0.351	59.7	5.28
10/14	1.410	16.0	0.425	72.3	4.52
14/20	1.001	22.5	0.525	89.3	3.97
20/28	0.711	31.8	0.632	107.4	3.38
28/35	0.503	45.0	0.823	139.9	3.11
35/48	0.356	63.5	1.038	176.5	2.78
48/65	0.252	90.0	1.393	236.8	2.63
65/100	0.178	127.0	1.851	314.7	2.48
100/150	0.126	180.0	2.504	425.7	2.37
150/200	0.089	254.0	3.219	547.2	2.15
200/270	0.063	260.0	4.287	728.8	2.02

[a] Calibration factor.

TABLE 68—DATA ON SIEVE-SIZES OF OTTAWA SAND

Mesh	Average size of particle (mm)	Theoretical surface per g of quartz spheres (sq cm)	Initial rate, I.R.	Measured surface per g, (sq cm)[a] I.R. × 170	Ratio of measured to theoretical surface
20/38	0.711	31.8	0.255	43.4	1.37
28/35	0.503	45.0	0.358	60.9	1.35
35/48	0.356	63.5	0.497	84.5	1.33
48/65	0.252	90.0	0.732	124.4	1.38
65/100	0.178	127.0	1.020	173.4	1.37
100/150	0.126	180.0	1.386	235.6	1.31
150/200	0.089	254.0	1.923	326.9	1.29

[a] Calibration factor.

where the diameter was taken as the mean of the mesh sizes given in the first column. The ratio of measure to theoretical surface given in the last column shows that surface of irregular particles as measured by solubility methods is from two to more than eight times those obtained by Eq (16-1). Further tests by Gross and Zimmerley with smooth Ottawa sand are given in Table 68. The last column of this

table indicates that the ratio of measured to theoretical surface is constant. This implies that material which is smooth and approximates spheres has a more or less constant shape-factor.

The limitations of the solubility method for surface measurement are readily apparent. Mineral particles are usually conglomerate, or composed of crystals often loosely bound. A strong acid such as hydrofluoric acid tends to split up the particles or penetrate so far that the true surface is exaggerated. Another difficulty concerns the assumption that all portions of a given particle-surface dissolve uniformly; this is not the case. The solubility of indentations or discontinuities is not uniform, especially since such points have occluded gases which are not affected. In most instances these factors are probably averaged out, and excellent approximations of the true surface may be made; but instances will undoubtedly arise where they are of great importance.

PERMEABILITY METHODS

Carman's Method—This method is based on the Kozeny equation, Eq (13-30). In terms of permeability the Kozeny equation is easily put into the form

$$S_v = \sqrt{\frac{g}{k\mathcal{O}_c'\nu}\frac{\vartheta^3}{(1-\vartheta)^2}} \qquad \text{Eq (16-4)}$$

where S_v is the specific volume surface (surface per unit-volume), g the gravitational constant, $\mathcal{O}_c'$ the permeability, k a constant = 5.0 for streamline flow, ν the kinematic viscosity = μ/ρ_0, and ϑ the fractional voids or porosity. If the equation is limited to streamline motion, Eq (16–4) becomes, in cgs units,

$$S_v = 14\sqrt{\frac{1}{\mathcal{O}_c'\nu}\frac{\vartheta^3}{(1-\vartheta)^2}}$$

If the density of the material is ρ, then the specific surface (weight basis) is

$$S_w = \frac{14}{\rho}\sqrt{\frac{1}{\mathcal{O}_c'\nu}\frac{\vartheta^3}{(1-\vartheta)^2}} \qquad \text{Eq (16-5)}$$

Eq (16-5) was tested by Carman (1938) for spheres of various materials and sizes for both air and liquids. Excellent agreement was obtained with irregular materials whose surface could be determined. From these experiments it was assumed that the equation has general application to particles as small as 2 μ in diameter.

Blaine (1941) used the Carman method, modified by Lea and Nurse (1939), to determine the specific surface of cement and developed a con-

venient apparatus using air as the fluid medium. The apparatus is shown schematically in Figure 88. The essential part of the device consists of a small cylinder (B of figure) in which is placed a Norton RA225 filter. The cement to be tested is placed in this cylinder and tamped with a special plunger so that it occupies a set volume. The weight of the cement is then determined by difference between the weight of the

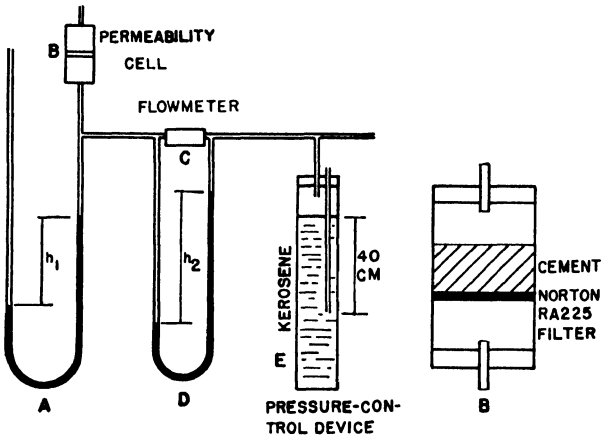


FIGURE 88. BLAINE'S APPARATUS FOR DETERMINING THE SURFACE OF CEMENT.

unit B and its final weight, with the cement in place. Hence, knowing the density of the cement, the porosity ϑ is quickly determined. The manometer A measures the pressure-drop through the cement while the assembly D is a flow-meter manometer. From the definition of permeability (unit-volume through unit-length per unit-time per unit-pressure gradient, we have

$$\mathcal{P}_c' = q/\left[A \times \frac{h_1\rho_1}{L\rho_a}\right]$$

where q is the volume passing through the cement per unit-time, A the cross-sectional area of the bed, L the depth of the bed, h_1 the manometer pressure difference across the cement, ρ_1 the density of the manometer fluid, and ρ_a the density of the air. From Poiseuille's equation, Eq (13-13), we have

$$q = Kh_2\rho_2/\mu$$

where K is the constant of the flowmeter, h_2 the manometer deflection,

ρ_2 the density of fluid used in the manometer, and μ the viscosity of the air. Hence

$$\mathcal{P}_c' = \frac{Kh_2\rho_2}{\mu A} \cdot \frac{\rho_a L}{h_1\rho_1}$$

Since the density of the manometer fluids are the same, that is, $\rho_1 = \rho_2$, then

$$\mathcal{P} = \frac{Kh_2\rho_a L}{\mu h_1 A}$$

Substituting in Eq (16-5), we obtain the complete expression for determining surface in terms of the constants of the apparatus*

$$S_w = \frac{14}{\rho(1 - \vartheta)} \sqrt{\frac{\vartheta^3 h_1 A}{KLh_2}} \qquad \text{(Eq 16-6)}$$

where ρ is the density of the cement or any other material used. Note that by the assumption of the Poiseuille relation, the expression is independent of the kinematic viscosity of the fluid (as it should be) and the surface is related only to the relative permeability $\vartheta^3/(1 - \vartheta)^2$. Experimental values obtained by Blaine, using cement and Potters' flint, indicated that values obtained by the device described range from 1.5 to almost 2 times those obtained by means of the Wagner (1933) turbidimeter, described in another section of this chapter.

Dalla Valle's Method—The author used an equation due to Burke and Plummer (1928) for determining the pressure-drop through packed columns. For streamline flow of air through packings of various sands in a 3 in. diameter column, the modified Burke-Plummer equation takes the form

$$S_w = \frac{1050}{\rho} \sqrt{\frac{\vartheta^3 \Delta P}{Lv}} \qquad \text{Eq (16-7)}$$

where ΔP is the pressure-drop through the packing in cm of water, L the depth of the packing, v the average velocity in cm per sec, and other terms are as previously defined. The value of v is the volume of air passed divided by the area of the column. Correction for fractional voids is included in the derivation of the equation. Eq (16-7) is not as accurate as the Carman equation since the relative permeability ϑ^3 is not as sensitive as the Kozeny permeability $\vartheta^3/(1 - \vartheta)^2$. For closest packings, however, Eq (16-7) gives consistent and fairly accurate results.

* With regard to the standardization of the technique used in applying this equation, see "Report on Air-Permeability Method for Determining Fineness of Cement" *A.S.T.M. Bulletin*, No. 118, October 1942, pp. 31-36

As has been indicated, Carman obtained excellent agreement with his equation using measurable surfaces. Nevertheless, there are a number of difficulties inherent in permeability methods. These concern the evaluation of the voids ϑ and the shape of the particulate matter for which the surface is to be determined. The method cannot be used for disk-like or flat material which packs in peculiar fashion since only external surface is measured. In this respect it differs from the solubility method described earlier and the adsorption method discussed in the next paragraph. Theoretically, permeability methods should give higher surface values than statistical determinations, but lower values than the two mentioned.

ADSORPTION METHOD

Emmett-Brunauer Method—This method of surface determination consists in measuring the volume of adsorbed gas forming a unimolecular layer over the surface of the particles. It is particularly adapted to

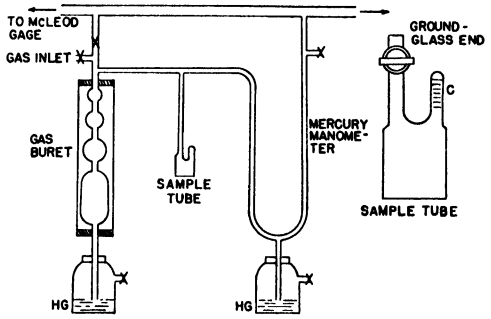


FIGURE 89. MODIFICATION OF THE EMMETT-BRUNAUER METHOD FOR DETERMINING THE SURFACE OF SMALL PARTICLES

extremely fine particles, provided the volume of the sample is such that the total surface is in excess of 10,000 sq cm. In fact, the method may be used conveniently for measurement of colloid diameters. Results thus far obtained indicate that the method is probably the best available for fine particles and is widely used for the study of pigments and other commercial materials such as activated charcoal (W. R. Smith *et al*, 1941).

Measurement of surface-area by the adsorption method is accomplished by the van der Waals adsorption isotherm of a suitable gas (usually nitrogen because of its chemical inactivity) at a temperature

close to the boiling point. In the following paragraphs we shall describe a modification of the Emmett-Brunauer method according to Makower *et al* (1937). The arrangement of the apparatus used is shown in Figure 89. A sample of the material to be measured is placed in the sample tube through a nipple C and evacuated at a low pressure with a mercury pump to remove moisture and other adsorbed substances. Approximately four days are required (unless temperatures of 200 deg C are used) until the pressure is $< 10^{-5}$ mm Hg. The free space is then determined with helium while the sample bottle is immersed in a liquid-nitrogen bath. The density of the material is then determined by the relation

$$\text{density} = \rho = \frac{\text{weight of sample}}{\text{volume of empty bottle} - \text{volume of free space}}$$

The helium is then pumped off and, while the tube is still immersed in the liquid-nitrogen bath, nitrogen gas is admitted. Equilibrium pressure is attained for low pressures (10–100 mm Hg) in approximately 10 min. The adsorbed gas is then measured from the temperature and volume of gas remaining in the free space. Readings may be taken at higher pressures by forcing mercury into the gas buret. When expressed graphically, the results constitute the so-called van der Waals isotherm. These are S-shaped curves, as shown in Figure 90. The basis of the adsorption method is that the intercept of the extrapolated linear portion of the isotherm on the volume axis represents within 15 percent the volume of nitrogen required to form a monomolecular layer of adsorbed molecules. The multiplication of the number of molecules in the layer by the average cross-sectional area of each molecule gives the surface-area of the particles in the sample. The nitrogen adsorbed may, of course, be expressed on a unit-weight basis. The expression found to fit the isotherm is as follows:

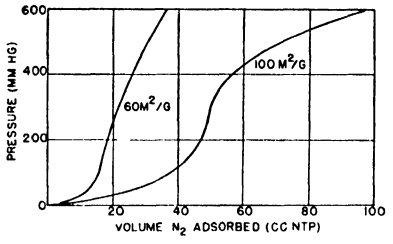


FIGURE 90. TYPICAL VAN DER WAALS ISOTHERMS FOR FINELY GROUND ACTIVATED CARBONS.

$$\frac{P}{Q(P_0 - P)} = \frac{1}{Q_m C} + \frac{(C-1)}{Q_m C}\cdot\frac{P}{P_0} \qquad \text{Eq (16-8)}$$

where Q is the volume of nitrogen (or other gas) adsorbed at the pressure P, P_0 the vapor pressure of the liquid nitrogen at bath temperature

(−195.8 deg C), Q_m the volume of gas at normal temperature and pressure required to form a monomolecular layer over the entire surface of the particles, and C a constant related to the difference between the heat of vaporization of a monomolecular layer and the heat of vaporization of liquid nitrogen.*

Eq (16-8) is in linear form, so that by plotting $P/[Q(P_0 - P)]$ against P/P_0 a straight line is obtained. The intercept of the line on the ordinate axis gives the value of $1/Q_mC$ and the slope $(C - 1)/Q_mC$. Hence, the values of Q_m and C are determined graphically.

The next basic assumption of the Emmett-Brunauer technique is that the average cross-sectional area of the adsorbed molecules is the same as that corresponding to the plane of closest packing in the solidified gas. It may be shown that when nitrogen is used as an adsorbate at −195.8 deg C, each cu cm adsorbed represents 4.38 sq m of surface.

The limitations of the adsorption method are those implied in the basic assumptions, namely the existence of a monomolecular layer as above determined, and the arrangement of closest packing assumed. However, results thus far obtained appear to warrant the assumptions made. There can be no question but that the method gives the correct order of surface measured. As in the solubility method previously described, the surface measured by the Emmett-Brunauer technique measures the surface at all indentations and infinitesimal cracks present. It possesses a marked advantage over other methods since it can be used with sponge-like particles possessing infinite labyrinth-like structures and is independent of chemical activity.

Method of Jura and Harkins—This adsorption method was discussed in Chapter 11 (Eqs 11-23 and 11-24). The method involves use of the same procedure as the Brunauer technique—that is, the determination of the isotherm. Thereafter, the linear plot of data in accordance with Eq (11-23) yields the constant necessary to determine the surface of a mass of particles. This method is comparable to that of Brunauer and his associates just described, and may be used as a check.

OPTICAL METHODS

Optical methods for measuring particle surface are widely used. They are based largely on the obscuring power of particles held in suspension, or the degree of polarization exhibited. The first-mentioned method depends upon Eq (9 23) where it was shown that the reflection of light by a Tyndall beam is a function of particle surface. The polarization of light by small particles is a function of particle-diameter

* See Chapter 11

but may be shown to be related to surface by convenient expressions. Optical methods give rise to many errors, among which the most important are due to optical properties (color, adsorption, birefringence, refraction, reflectivity, etc.) which are involved in an indeterminate manner. Only one method is discussed here since other methods are variations of the one presented.

Wagner Turbidimeter—Wagner (1933), utilizing the principles outlined in Chapter 9, developed a simple and rapid method for estimating the relative surface of fine particles. The device consists of a glass sedimentation tube 8 in. in depth and $1^1/_2$ in. square. The suspension of particles whose surface is to be measured is placed in the sedimentation tube and stirred. At a fixed distance from the tube is placed a standard lamp which passes a beam of light through the suspension. The light absorbed is indicated on the other side of the tube by means of a photoelectric cell. The standard lamp (6 candle power) is usually adjusted to 100 microamperes and is provided with a condenser to remove radiant heat. Wagner used Stokes' law to determine the size-distribution of particles at any level in the settling tube (see Chapter 4). The general equation fitting the observed data was

$$S_d = c(\log I_0 - \log I_d) = c\ (2 - \log I_d)$$

where S_d is the surface-area of all particles in the sample smaller than d, c a constant for the material observed, I_0 the strength of the incident light (100 microamperes), and I_d the intensity of the light in microamperes transmitted through the suspension at the time interval required for a particle $d\ \mu$ in diameter to settle from the surface of the suspension to the center line of the light beam.

In use the light beam and photoelectric cell may be moved up or down along the axis of the sedimentation tube. Thus, the time required to obtain a sedimentation curve of the suspension may be shortened considerably. By adjustment of the lamp and photoelectric cell to any level it is not necessary to wait for small particles to settle to a given level. If a level has been chosen to which particles less than d units in size have settled at a given time, then

$$\frac{S_d\ (\text{unknown sample})}{S_d\ (\text{known sample})} = \frac{2 - \log I_d\ (\text{unknown})}{2 - \log I_d\ (\text{known})}$$

where the constant c is assumed to be the same for both the known and unknown materials. It is obvious that after a standard curve has been derived, only a single reading is required for determination of the surface of particles less than a stated size. Integration over-all sizes may be accomplished easily.

At best the Wagner method can be considered only a means of obtaining a relative surface. The surface measured depends upon the standard used and the accuracy with which the surface was originally determined. The method is in no sense absolute, nor can it be applied safely to different materials unless suitable standards are available for each. Moreover, the accuracy of the method is affected by color and other optical properties of the materials suspended. Schweyer (1937) studied the effect of light of different wave lengths upon the turbidity-concentration relation of white and colored mineral powders in aqueous suspension, using Wagner's turbidimeter. He showed that suspensions tend to absorb the shorter wave lengths, thereby directly affecting the output of the photoelectric cell and indirectly influencing the surface as measured by this method.

Problems

1. Determine the specific surface of 14/20-mesh Tyler sieved sand ($\rho = 2.65$). For 200/325-mesh.

2. Compute the specific surface per unit-volume for the sands given in Problem 1.

3. The following size-distribution (count basis) was obtained for a product whose density is 2.4.

Average group size (microns)	*Percent less than stated size*
2	33
4	54
6	81.3
8	87.0
10	94.1
12	96.3
14	97
16	99

Calculate the specific surface, assuming the particles are spheres.

4. Air is made to pass through a column of sand having a porosity of 33.7 percent. The approach velocity of the air is 30 cm per sec and the drop is found to be 0 008 cm H_2O per cm of packing. The diameter of the particles is 0 38 cm and the air temperature 68 deg F ($\mu = 0.0182$ centipoise). Compute the surface per unit-volume.

5. The surface of a certain soil is measured by means of the Emmett-Brunauer method using nitrogen. The amount of nitrogen adsorbed at STP is 6 cu cm per g. Calculate the surface-area of the particles per cu cm and their average diameter (assumed spheres).

6. Two particles of radii r_1 and r_2 are each enveloped by a fluid whose surface tension is σ. If the external pressure on each is p_a and the internal pressure p_1 and p_2, respectively, show that when the two particles coalesce, there is a decrease in the total surface.

The relations between the surface tension and the pressures are

$$p_1 - p_a = \frac{2\sigma}{r_1} \text{ and } p_2 - p_a = \frac{2\sigma}{r_2}$$

and for the two after coalescence, assuming the configuration to be spherical and to have a radius r and an internal pressure p

$$p - p_a = \frac{2\sigma}{r}.$$

By Boyle's law, we have

$$p_1 r_1^3 + p_2 r_2^3 = p r^3$$

and substituting and combining terms

$$2\sigma(r^2 - r_1^2 - r_2^2) + p_a(r^3 - r_1^3 - r_2^3) = 0.$$

The terms in parentheses denote the change in surface and volume which may be denoted by dS and dV, respectively. Hence, multiplying through by 4/3, we obtain

$$\frac{dS}{dV} = -\frac{3}{2}\frac{p_a}{\sigma}$$

We see, therefore, that there is a diminution of surface and hence of surface energy the loss being accounted for as heat.

CHAPTER 17

MUDS AND SLURRIES

DENSE suspensions of fine particles, commonly called muds or slurries, are to be found in almost every industry. They occur in concentrating and thickening processes where moisture is to be removed from solids; they are used in metallurgy to separate substances of different gravities, and in the petroleum industry for removing coarse material formed during drilling operations. This chapter discusses the chief characteristics of muds and slurries, as well as methods for determining their important properties. The subject is complicated by many factors and no generalizations can be made. So much depends on the specific properties of every substance dealt with that separate treatment and study are necessary. In this chapter we cannot hope to include the whole field of behavior of suspensions, and shall limit the discussion to their most important properties and the methods used for their determination.

PHYSICAL PROPERTIES

Dispersion of Particles—When lumpy material is mixed with a liquid, some measure of the rate at which the larger masses are disintegrated is necessary. What factors affect its breaking up into elemental units? The size-distribution of the lumps is one factor, the solubility of the salts holding the particles together is another; moisture-content may be a third. We shall discuss only those lumps easily disintegrated, and not those which are impermeable or so composed chemically and physically that the individual particles remain bound indefinitely.

Naturally, it is often desirable to accelerate dispersion of a mass of particles and, for comparative purposes, to have an acceptable method for their determination. The following procedure which gives a so-called "dispersion-factor" was developed by Puri and Keen (1925). Their method consists in taking a known weight of material (about 5 g) and adding water so as to obtain 500 cu cm of suspension. This is placed in a 500-cu cm flask and rotated end over end at 40 rpm. After shaking, the suspension is allowed to settle for 24 hrs, at the end of which time the top 100 cu cm are removed, evaporated to dryness at 100 deg C, and weighed. This weight is called the "dispersion-factor." Puri and

Keen state that satisfactory replicas can be obtained with a given material, and that the dispersion coefficient is given reasonably well by the formula

$$\xi = a + k \log t \qquad \text{Eq (17-1)}$$

where ξ is the dispersion-factor, t the time of shaking, and a and k are constants. It is necessary to point out that Eq (17-1) applies best to soils. Puri and Keen found that the dispersion-factor decreases with decrease in initial moisture-content; therefore, when materials are compared, care should be taken to have their moisture-contents correspond if possible.

Gravimetric Properties—The important equations pertaining to suspensions have been developed in Chapter 4 in conjunction with the determination of particle-size. The density of the suspension is readily apparent in the characteristic suspension equation developed there, but will be repeated for the sake of completeness. Let W be the total weight of the particles and ρ their density; if the volume of the suspension is designated as V, and if the suspending fluid is water, then the specific gravity φ_0 of the suspension at time $t = 0$ (before settling takes place) is

$$\varphi_0 = \frac{\text{weight of liquid} + \text{weight of particles}}{\text{volume of liquid} + \text{volume of particles}}$$

$$= \frac{(V - W/\rho) + W}{V} = 1 + \frac{W}{V}\left(1 - \frac{1}{\rho}\right) \qquad \text{Eq (17-2)}$$

Hence, the density of the suspension is greater than water by an amount $W/V(1 - 1/\rho)$. From this equation the weight of solids per unit-volume of suspension W may be determined:

$$W = \frac{\rho V(\varphi_0 - 1)}{\rho - 1}$$

If now W_s is the weight of a unit-volume of suspension, then $V = W_s/\varphi_0$, or

$$W = \frac{\rho W_s(\varphi_0 - 1)}{\varphi_0(\rho - 1)} \qquad \text{Eq (17-3)}$$

The weight of solids per unit-volume of suspension, W, entails no difficulty in its determination; φ_0 is readily determined by a hydrometer, and since the volume of the suspension is known, W_s may be calculated. The density of the particles is obtainable by pycnometric methods, or simply by weighing a given amount and displacement in a calibrated graduate.

CONSISTENCY

Viscosity and Plasticity--Viscosity and plasticity are closely related. Viscosity may be defined as the force required to move a unit-area of plane surface with unit-speed relative to another parallel plane surface, from which it is separated by a layer of the liquid of unit-thickness. Other definitions have been applied to viscosity, an equivalent one being the ratio of shearing stress to rate of shear. When a mud or slurry is moved in a pipe in more or less plastic condition the viscosity is not the same for all rates of shear, as in the case of ordinary fluids. A material may be called "plastic" if the apparent viscosity varies with the rate of shear. The physical behavior of muds and slurries is markedly affected by viscosity. However, consistency of muds and slurries is not necessarily the same as viscosity but is dependent upon a number of factors, many of which are not yet clearly understood. The viscosity of a plastic material cannot be measured in the manner used for liquids. The usual instrument consists of a cup in which the plastic material is placed and rotated at constant speed, causing the deflection of a torsional pendulum whose bob is immersed in the liquid. The Stormer viscosimeter, for example, consists of a fixed outer cylinder and an inner cylinder which is revolved by means of a weight or weights.

It is also possible to determine the viscosity by extrusion methods. This is analogous to the determination of viscosity with fluids, except that the material is extruded under pressure. For a fluid, the slightest pressure P will produce motion, and the volume of flow will be proportional to the pressure applied. Thus for fluids, we have the simple form of Maxwell's equation

$$-\frac{dv}{dx} = \frac{P}{\mu} \qquad \text{Eq (17-4)}$$

where v is the velocity at any point x perpendicular to the direction of fluid flow. When this equation is applied to the flow in a capillary tube, we obtain Poiseuille's equation. The relation between flow and pressure for a liquid is shown in Figure 91 by the line OO'. However, when we deal with a plastic material the total force may be regarded as made up of two portions, one equal to OA' which overcomes solid frictional resistance P, and the remainder $OA = p$. For this type of flow Bingham (1922) proposed the equation

$$-\frac{dv}{dx} = \frac{P - p}{\mu'} \qquad \text{Eq (17-5)}$$

where μ' is analogous to μ for fluids. Using Eq (17-5) Bingham was

led to the modified Poiseuille relation expressing flow of plastic material under gravity

$$\mu' = \frac{\pi g D_c^4 t \varphi_0 (h - h_0)}{128 Q L} \qquad \text{Eq (17-6)}$$

where g is the gravitational acceleration constant, D_c and L the diameter and length of the capillary, respectively, Q/t the rate of flow of material through it, whose density is φ_0, and h and h_0 are the heads of plastic material corresponding to P and p, respectively. Eq (17-6) is a first approximation. Actually the curve for plastic flow may be extremely complex so that no over-all general theory is possible. Henry Green (1920), by microscopic observations of the flow of paint in capillary tubes, found that when the pressures were low the material moved as a single plug. When higher pressures were used the paint moved in "telescopic" layers. The inner layers moved at a higher velocity than the outer layers. Buckingham (1921*b*) was led to modify (Eq 17-6) in accordance with Green's observations

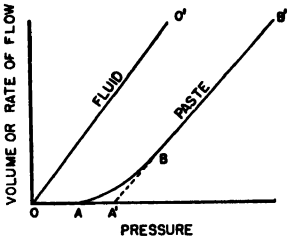


FIGURE 91. ILLUSTRATING LAWS OF FLUID AND PLASTIC FLOW.

$$\mu' = \frac{\pi g D_c^4 t \varphi_0}{128 L Q}\left(h - \frac{4}{3}h_0 + \frac{h_0^4}{3h^3}\right) + \frac{\pi D_c^3 \delta \mu'' h}{16 L} \qquad \text{Eq (17-7)}$$

The last term takes account of slippage at the boundaries. Buckingham assumes that there is a very thin lubricating layer (of thickness δ and fluidity μ'') between the plastic material and the wall. Examination of Buckingham's equation indicates that flow at pressures less than h increases directly as the pressure. At pressure h_0 shearing begins and the relation is shown by a curved line AB' in Figure 91.

Farrow *et al* (1923) used the modified Maxwell equation $dv/dx = P^n/\mu'$. This equation has a certain amount of generality. Using this basic equation for plastic flow in a capillary tube, the following equation was derived,

$$q = \frac{\pi D_c^3}{8}\left(\frac{PD_c}{4L}\right)^n \Big/ (n - 3)\mu' \qquad \text{Eq (17-8)}$$

It is important to note from this equation that μ' has dimensions which depend on the dimensions of n, and is not expressible in simple units.

The above equations represent a few of the many efforts to arrive at a

satisfactory explanation of plasticity. The behavior of a given material can be expressed empirically, but no generalizations are possible. For example, Traxler *et al* (1937) investigated the relation between the logarithm of viscosity for viscous mixtures and solids, in terms of the percent of solid present. Four mixtures of filler were prepared and then were mixed with asphalt in certain definite proportions. When their data are plotted on semi-log grid a linear relation is found between the logarithm of the viscosity and the volume percent of solid present. The relation may be expressed as follows:

$$\log \mu' = \log K + a\theta \qquad \text{Eq (17-9)}$$

where μ' is the viscosity in poises, θ the volume of the solid in percent, and a and K are constants which may be determined from plots of the data. Thus, we are left with two constants which depend on unknown factors of the suspension, and have little information regarding the true nature of plasticity.

Stability Index—For systems in which Eq (17-9) holds, the percentage increase in viscosity caused by an increase in the volume percent of solid is given by the following equation:

$$\text{percent change} = 100\left(\frac{\mu_2' - \mu_1'}{\mu_1'}\right)$$

where μ_1' and μ_2' are the viscosities of two similar liquid-solid mixtures. If θ_2 and θ_1 are corresponding volume percentages of the solid materials ($\theta_2 > \theta_1$), we have from Eq (17-9) that

$$a = \frac{\log \mu_2' - \log \mu_1'}{\theta_2 - \theta_1} = \frac{\log \dfrac{\mu_2'}{\mu_1'}}{\theta_2 - \theta_1}$$

Hence

$$\log \frac{\mu_2'}{\mu_1'} = a(\theta_2 - \theta_1)$$

and

$$\text{percent change} = 100[10^{a(\theta_2 - \theta_1)} - 1]$$

This equation states that the percentage increase in viscosity of a suspensoid is constant for a constant increase in the volume percent of the solid. If this increase is taken to equal unity ($\theta_2 - \theta_1 = 1$), an expression is obtained which evaluates the percentage change in viscosity for a one percent increase in the volume percent of the solid. Traxler *et al* state that this expression is a quantitative index of the stabilizing ability of a solid; that is

$$\text{stability index} = \text{S.I.} = 100\,(10^a - 1) \qquad \text{Eq (17-10)}$$

The stability indices of a number of particulate substances are given in Table 69. The value of the stability index lies in the fact that it is

TABLE 69—STABILITY INDEX OF VARIOUS PARTICULATE SUBSTANCES

Substance	Stability index	Substance	Stability index
Limestone	4.5	Black slate	6.6
Graphite	4.5	Talc	7.1
Loess (Nebraska)	4.8	Gray-green mica	9.8
Silica (quartz)	5.3	Diatomaceous earth	15.0
Portland cement	5.7	Wood flour	16.0
Clay	6.6	Asbestos	21.0

independent of the nature and viscosity of the liquid, provided no reaction takes place between liquid and dispersed solvent. Traxler *et al*, in extending their studies, were led to the conclusion that the viscosities of mixtures of a given liquid and solid are inversely proportional to the average void diameter of dispersed solid and the percentage of voids represented by the volume percent of liquid present in the mixture. Thus, if μ_m' is the viscosity of the mixture in poises, D_v the average void

TABLE 70—RELATION BETWEEN VARIOUS CONSTANTS OF A SUSPENSION AS THEY APPLY TO EQ (17-11)

Dispersion media	Substance	Percentage of solid	Viscosity of mixture (poises $\times 10^{-6}$)	Average pore-diameter, D_v	Value of $\mu'_m D_v$
California asphalt	Red slate	10	5.14	4.31	22.1
		15	6.35	3.46	22.0
		20	7.85	2.78	21.8
		25	9.70	2.24	21.7
		30	12.0	1.18	21.6
	Trap rock	10	4.95	12.8	63.3
		20	7.69	8.27	63.6
		30	12.0	5.34	64.1
Venezuelan asphalt	Limestone	5	4.05	16.5	66.8
		10	5.05	13.2	66.7
		15	6.31	10.1	63.7
		20	7.88	8.55	67.4
	Silica	10	5.25	9.91	52.0
		15	6.80	7.96	54.1
		19	8.36	6.40	53.5
		23.8	10.7	5.42	58.0

diameter of the powders present in the mixture, and K a constant for the particular liquid and solid considered, we have

$$\mu'_m = \frac{K}{D_v} \qquad \text{Eq (17-11)}$$

The data on which this relationship is based are shown in Table 70. The average void diameter was obtained from the equation $\log D_v = m\vartheta + b$, where ϑ = *percent voids*, and m and b are constants. (See Eq 6-28.)

Traxler and Baum (1936) showed that the value for the slope m ($m = 0.019$) holds for a large number of mineral powders. For irregular particles it is necessary to determine the average void diameter experimentally. This subject was discussed in Chapter 6. Note that in Table 70 the value of $C = \mu_m' D_v$ is practically constant for a given mixture, regardless of the amount of solids present.

Einstein Equation for Viscosity of Suspensions—Einstein derived a general equation for computation of the viscosity of a suspensoid in terms of viscosity of the medium, and the ratio of aggregate volume of solid particles to total volume of suspensoid. Einstein's equation is

$$\mu' = \mu(1 + f\theta) \qquad \text{Eq (17-12)}$$

where μ' = viscosity of suspensoid, μ = viscosity of medium, θ = ratio of aggregate volume of solids to total volume of suspensoid, and f = *constant.* The value of f varies between 2.5 and 2.9. Einstein's equation may be considered as giving an approximate value of the viscosity of the suspensoid when the concentrations of suspended material do not exceed 2 percent. In general, the consistency of suspensoids determined by viscosity must be considered a matter of experiment rather than of mathematical computation.

Broughton and Windebank (1938) re-examined the basis of the Einstein relation and found that it held for spheres of relatively large size (20 to 200 μ) provided the concentrations did not exceed 2 percent. These investigators found that for small spheres no agreement was obtained with the Einstein equation. No completely satisfactory reason for breakdown of the equation for particles smaller than 50 μ was given, but it was pointed out that Brownian movement decreases as particle-size increases, thus reducing the rate of agglomeration. It may be mentioned that Einstein based his equation on the assumption that the particles were spherical in shape, uniform in size, random in orientation, and that coagulation and agglomeration did not take place. While in theory it was intended that the equation should apply to colloids, it is remarkable that it is better suited to particles of fairly large size. Eq (17-12) does not hold if any charges exist on the particles. In such cases a more general expression due to von Smoluchowski (1917) must be used.

Measurement of Consistency—As a rule the determination of viscosity of fine-particle suspensions, such as most pigments and clays, involves little difficulty. Many types of viscosimeters for plastic materials are available and have been discussed by Bingham (1922). However, when rather large particles are in suspension, say in making "heavy liquids" for float-and-sink separation, vigorous agitation is required to maintain the suspension so that correct determinations can be made. For this reason, many standard viscosimeters cannot be used for particles of rather large diameters or of high densities.

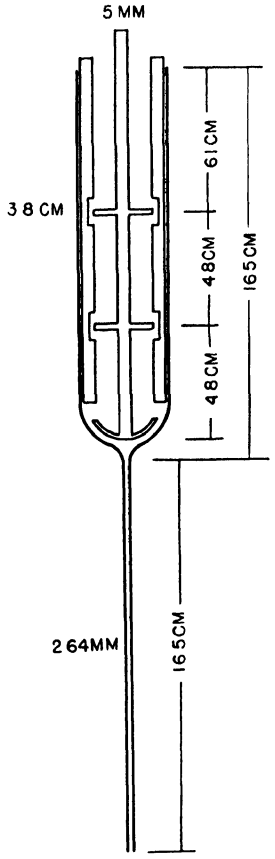


FIGURE 92. CONSISTOMETER CHAMBER AND DISCHARGE TUBE.

DeVaney and Shelton (1940) developed a consistometer (viscosimeter) which is easily made in the laboratory and has proved fairly accurate for a wide variety of suspensions, both large and fine. Because of its simplicity and its range of usefulness a description is presented here. The device is shown in Figure 92 and consists of a graduate cylinder 3.8 cm in diameter and 16.5 cm in length, capable of holding 200 cu cm of suspended material. A glass tube 16.5 cm in length and having an inside diameter of 0.264 cm is attached to the cylinder. Within the cylinder is placed an impeller actuated by an electric mixer (kitchen-type) at a speed of 890 rpm, or such speed as will keep the particles in suspension. Baffle plates are fixed on the walls of the cylinder to reduce vorticity. To measure consistency, the suspension is first brought to the temperature at which the observation is to be made. It is then quickly stirred and placed in the cylinder, the material being retained by a stopper at the end of the discharge tube. The motor is started and the contents stirred. A graduated flask is then placed at the end of the discharge tube, the stopper removed, and the time required for 100 cu cm of the suspension to flow into a graduate or volumetric flask is measured. The viscosity is then directly determined from Bingham's approximate equa-

tion, Eq (17-6)

$$\mu' = \frac{\pi g D_c^4}{128QL} \cdot \varphi_0 ht$$

φ_0 being the density of the suspension and h the head in cm of the suspension. If φ_0 approximates the density of the liquid and h is kept constant, and if t is the time required for 100 cu cm of suspension to move through the tube then

$$\mu' = Kt$$

where K is a suitable constant. DeVaney and Shelton found good agreement between calculated and observed results by using standard sucrose solutions. Their results are shown in Figure 93.

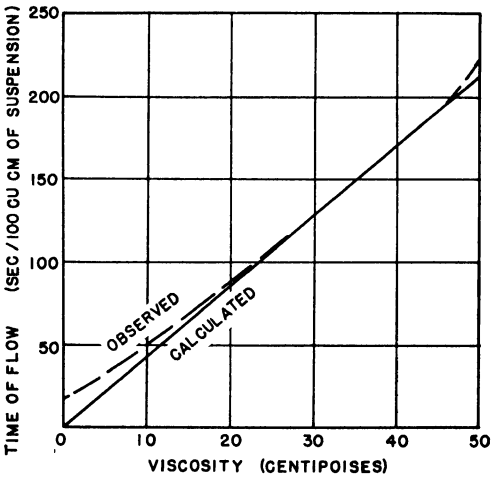


FIGURE 93. COMPARISON OF OBSERVED AND CALCULATED RATES OF FLOW FOR SUCROSE SOLUTIONS IN CONSISTOMETER.

FACTORS AFFECTING CONSISTENCY

Effect of Specific Gravity—The effect of specific gravity on consistency was studied by DeVaney and Shelton (1940) using the consistometer described above. Suspensions were made of fine substances ranging from a density of 2.65 for quartz to 11.3 for metallic lead. Each of the materials was screened so as to pass a 200-mesh sieve and be retained on a 325-mesh. The materials were carefully washed so no slime

or dirt adhered to their surfaces. Results of the tests are shown in Figure 94. Each curve shows that as material is added to the suspension the consistency increases gradually and almost in proportion to the

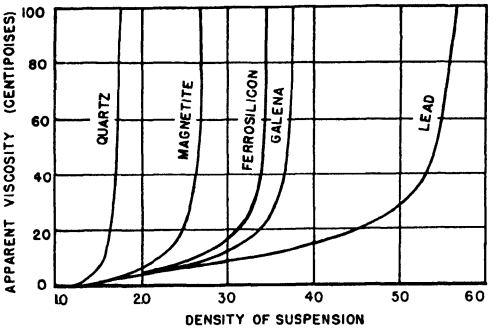


FIGURE 94. EFFECT OF SPECIFIC GRAVITY OF PULP ON CONSISTENCY OF SEVERAL SUSPENSOIDS—200/325-MESH.

density of the pulp or suspension, until a certain point is reached. Thereafter consistency increases enormously, while density of the suspension scarcely changes. The critical points are given in Table 71.

TABLE 71—CRITICAL POINTS OF 200/325-MESH SUSPENSIONS SHOWN IN FIGURE 94

Material	Specific gravity	Maximum gravity of suspension obtainable (estimated)	End of linear relation	
			Specific gravity of suspension	Volume of solids (percent)
Quartz	2.65	1.8	1.38	23.1
Magnetite	5.18	2.7	2.08	25.8
15 percent ferrosilicon	6.80	3.5	2.30	22.4
Galena	7.50	3.8	2.50	23.0
Lead	11.30	6.0	3.90	28.0

It is a curious fact that for all substances studied, the proportionate relationship ends at a point where the solids approximate 25 percent by volume of the suspension.

Effect of Particle-Size and Shape—The effect of particle-size on consistency is clearly shown in Figures 95 and 96, taken from the investigations of Ward and Kammermeyer (1940) and DeVaney and Shelton, respectively. Note that as the size of the average particle decreases, the consistency of the pulp increases. In other words, pulps composed

of fine materials are much less fluid than those made of coarse grains.

The shape and surface characteristics of particles naturally affect consistency. Studies by DeVaney and Shelton verify this fact. Thus, one sample consisting of round grains of sand was compared with freshly ground angular quartz particles in the same size range (100/150-mesh). At a consistency of 10 centipoises the pulp of round particles had a specific gravity of 1.54 as compared with 1.59 for supensions of angular particles. This effect was even more striking when new and abraded ferrosilicon were compared. There can be no question but that the ease with which particles roll and slip past each other must influence the consistency of suspensions made from them. In making heavy liquids for float-and-sink operations these considerations take on a certain significance.

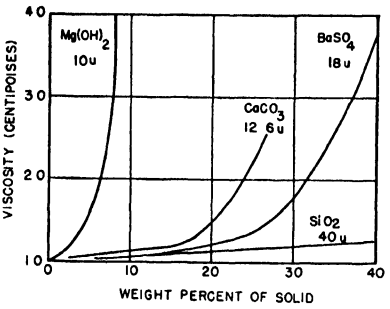


FIGURE 95 EFFECT OF PARTICLE-SIZE ON CONSISTENCY OF VARIOUS SLURRIES.

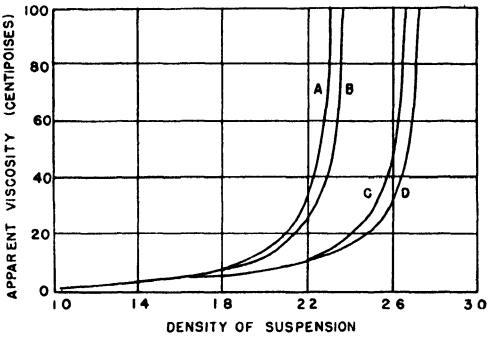


FIGURE 96. EFFECT OF PARTICLE-SIZE ON CONSISTENCY OF MAGNETITE. AVERAGE DIAMETER IN MICRONS: *A*, 15.8; *B*, 25 7; *C*, 37.6; *D*, 51.7.

THICK MUDS AND CLAYS

Haines' Extrusion Apparatus—Haines (1925*a*), in studying the physical properties of soils, developed an extrusion apparatus. This

consists of a plunger fitting into a cylinder in which is placed a sample of the soil made into a paste. The soil is extruded through an opening. The plunger is forced downward by the screw arrangement and the soil is made to press against a cylinder containing oil. The oil cylinder is connected to a suitable pressure gauge so that pressure readings are readily indicated. In operation the pressure was found to increase considerably before extrusion commenced. Once extrusion started Haines found that the rate of extrusion was constant as long as it was slow. Haines carefully studied the effect of varying moisture on the extrusion pressure. His general equation was as follows:

$$M = e - aP - \exp - (bP + c) \qquad \text{Eq (17-13)}$$

where M is the percent moisture, P the extrusion pressure, and a, b, c, and e are constants. With regard to the general properties of a plastic material Haines (1925*a*) made the following statement: "Determinations of the rate of flow for different stresses have indeed been made, but their interpretation is quite obscured by the fact that the conditions of flow change from one determination to another. Practically every property of a fluid which is dependent upon the various methods of measuring viscosity, fails to be fitted in the case of plastic substances like soils and clays. In the first place a fluid can suffer deformation to an indefinite extent without physical change, but usually a plastic substance begins to crack and break under comparatively small deformations."

Atterberg (1911–1912) developed an apparatus shown in Figure 97 for the determination of soil cohesion. This apparatus depends on the force required to make a steel wedge penetrate a specially prepared block of soil. It is important that the wedge be sharp and that the force be applied gradually. The force required to make the soil yield is indicated by the pointer and is taken as a measure of soil cohesion. Atterberg's wedge is approximately 15 deg. As might be expected there is always some difficulty in determining when the yield point has been reached.

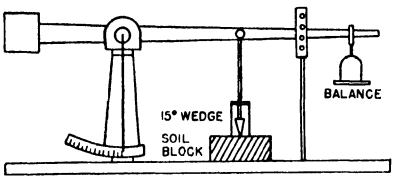


FIGURE 97. ATTERBERG'S APPARATUS FOR DETERMINATION OF SOIL COHESION.

The general relationship between cohesion E and moisture-content M, as determined from this equation, is as follows:

$$E = aM^{-c} \qquad \text{Eq (17-14)}$$

where a and c are constants determined by the condition of the soil. The force necessary to overcome cohesion by the Atterberg apparatus is given by the following

$$F' = \frac{Fl_w}{l_h + 2l_w\tau} \qquad \text{Eq (17-15)}$$

where F' is the force tending to split the block, F the force driving the wedge, l_w and l_h the width and height of the wedge, respectively, and τ the coefficient friction, usually taken as 0.3.

Some further light may be cast on the subject of cohesive forces by the following analysis. Suppose we have a mixture of particles and water which exhibits a cohesive character. The complex will be held together by free surface energies in accordance with Dupre's equation. The work of adhesion w_{SL} is that necessary to disrupt the latent cohesive energies; thus, using negative signs to indicate tearing apart rather than formation,

$$-w_{SL} = \sigma_{SL} - \sigma_{SA} - \sigma_{LA} \qquad \text{Eq (17-16)}$$

where σ_{SL} is the free energy (surface tension) produced by the distribution of the particle-water interface, σ_{SA} that used up in the formation of the particle-air interface, and σ_{LA} that used up in the formation of the water-air interface.

We have noted on page 238 that fine particles when wetted release heat. This heat may be termed the total surface energy and can be determined calorimetrically. Let H_e represent the heat of emersion of fine particles in water; then what is measured is given by the following equation·

$$H_e = \sigma_{SA}' + \lambda_{SA} - \sigma_{SL}' - \lambda_{SL} \qquad \text{Eq (17-17)}$$

where the λ's are the latent heats of surface formation, and the σ's are expressed in heat units.

The least amount of water necessary to cause cohesion is readily extrapolated from a series of experiments with a given material, as may also the corresponding value of H_e. The values so obtained may be regarded as the critical cohesive limit of the material with water. A similar procedure may of course be applied to any other solid-liquid-gas phases.*

COAGULATION

In all suspensions we must contend with constant motion and adherence of small particles to each other. This effect constantly reduces

* The procedures outlined here are discussed in Chapter 11.

their number—an effect which is further augmented by accelerated settling action due to the increased mass of some particles. Some estimate of the nature of the action taking place may be obtained by referring to Figure 98. This figure shows the change in particle-size of zinc-oxide fume at various instants of time. While the data presented apply to an aerosol, the phenomenon is not greatly different for particles suspended in fluids.

The theory of coagulation was developed by von Smoluchowski (1917) and found to be correct as long as particles are not attracted or repulsed by electric charges, or go into solution. Von Smoluchowski's basic assumptions are as follows: (1) There must be a sphere of attraction around every particle such that every other particle coming within this sphere adheres to it; if d is the diameter of the particle, this sphere of attraction is taken to have a diameter of $2d$; (2) the probability that one particle (motionless) in a medium containing n particles per cu cm gets within the sphere of attraction of another particle is $4\pi\mathfrak{D}d$, where $\mathfrak{D}$ is the mean displacement (diffusion constant) as defined in the case of Brownian motion by Eq (8-2) *et seq.*

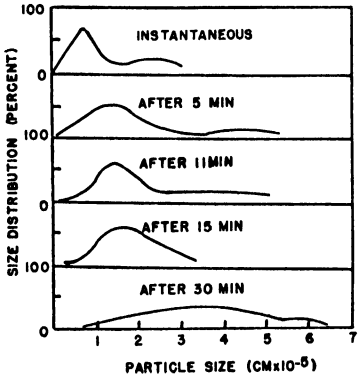


FIGURE 98. PARTICLE-SIZE DISTRIBUTION IN A SMOKE AFTER VARIOUS INTERVALS OF TIME. (WELLS AND GERKE, 1919.)

$$\mathfrak{D} = \frac{BT}{N} \cdot \frac{1}{3\pi\mu d} \qquad \text{Eq (17-18)}$$

where B is the gas constant, T the absolute temperature, and N Avogadro's number. Thus, starting with n_0 single particles, n_1 after time t_1, n_2 after time t_2, etc., we have

$$n_1 = n_0 \cdot \exp\ [-4\pi\mathfrak{D}dn_0 t]$$

Hence their decrease is

$$-\frac{dn_1}{n_1} = 4\pi\mathfrak{D}dn_0 \cdot dt = K_s n_0 dt \qquad \text{Eq (17-19)}$$

where K_s is called the coagulation constant. When only a small part has come together so that approximately $n_0 = n_1$ we have on integrating

$$n_1 = \frac{n_0}{1 - 4\pi\mathfrak{D}dn_0t} = \frac{n_0}{1 + \frac{t}{t_c}}$$

where t_c equals $1/(4\pi\mathfrak{D}dn_0)$ is called the time of coagulation. Now, double particles exist, and when these meet they become quadruple particles; if they meet single particles they form triple ones, etc. The procedure of calculating the new numbers is the same, with the following assumptions: That the displacement of i-single particles composing one particle, $\mathfrak{D}_i$, and one composed of j-single particles, $\mathfrak{D}_j$ on coming together possess a mean displacement $\mathfrak{D}_{ij} = \mathfrak{D}_i + \mathfrak{D}_j$ and that the sphere of attraction of the combination is $d_{ij} = (d_i + d_j)/2$. The following series of equations developed by von Smoluchowski gives the manner of formation for single, double, triple, etc., particles

$$n_1 = \frac{n_0}{\left(1 + \frac{t}{t_c}\right)^2} = \frac{n_0}{(1 + \chi n_0 t)^2}$$

$$n_2 = n_0 \frac{\chi n_0 t}{(1 + \chi n_0 t)^3}$$

. . . . . .

$$n_j = n_0 \frac{(\chi n_0 t)^{j-1}}{(1 + \chi n_0 t)^{j+1}}$$

The total number of particles of all kinds at any time t is given by the equation

$$\Sigma n = \frac{n_0}{1 + \frac{t}{t_c}} \qquad \text{Eq (17-20)}$$

From this equation we see that when $t = t_c$ the total number of particles is just halved. The significance of t_c is therefore apparent. Note that n_1 and Σn start at the same point ($t = 0$), since originally the suspension consisted only of single particles.

Whytlaw-Gray *et al* (1936), while considering only disperse systems in gases, found that experimental data fitted the Smoluchowski equation in the form:

$$\frac{1}{n_2} - \frac{1}{n_1} = K_s(t_2 - t_1) \qquad \text{Eq (17-21)}$$

where n_2 and n_1 are the number of particles per unit-volume at times t_2 and t_1, respectively, and K_s is the coagulation constant as defined by Eq (17-19).

Whytlaw-Gray *et al*, citing Cawood, state that the coagulation process is to some extent independent of the nature of the particles, although the constant for any one system of particles may vary as follows:

1. With the average size of the particle, the smaller the size, the faster the rate of coagulation.
2. With size-distribution, when the size-range is small (that is, if the particles approach the homogeneous), coagulation is smaller than is the case with widely heterogeneous systems.
3. With the form and shape of the particles and the aggregates formed.

Attraction of Two Spheres—An expression for the attraction between two or more particles, based on collision theory rather than attraction due to the motion of spheres, may be developed by the methods of dimensional analysis. Let the force of attraction between two particles of diameters d_1 and d_2 be F and assume that when the particles are close enough so that the gaseous film enveloping each particle coalesces over a region about the point of contact, then the whole attraction is due to a free surface energy σ. If the *average* surface of contact of the particles is denoted by S_c then

$$F = \psi(S_c, \sigma) \qquad \text{Eq (17-22)}$$

Hence, by the methods of dimensional analysis,

$$[\mathrm{MLT}^{-2}] = [\mathrm{L}]^{2a} \cdot [\mathrm{MT}^{-2}]^b$$

and, $a = {}^1/_2$, $b = 1$, so that

$$F = k\sigma\sqrt{S_c}$$

where k is a constant of proportionality.

The value of S_c is difficult to determine unless we assume that the surface of a given volume of particles is proportional to the harmonic mean. If this assumption is valid, then such contacts as we have described are also proportional to this mean (see Chapter 3). Let us now write

$$F = \psi(d_h^{-1}, \sigma)$$

where d_h denotes the harmonic mean of the particle diameters. Proceeding as before, we find for this case that

$$F = k' d_h^{-1} \sigma$$

For two particles $2/d_h = 1/d_1 + 1/d_2$, so that

$$F = k' \frac{d_1 d_2}{d_1 + d_2} \cdot \sigma \qquad \text{Eq (17-23)}$$

This is identical with Bradley's (1930) expression where k' was found to be equal to 2π. However, Bradley used a general law of attraction to derive Eq (17-23), and his derivation rests on a firmer foundation than the one we have derived by dimensional analysis. Bradley was able to show for quartz spheres that F was in fact proportional to $d_1d_2/(d_1 + d_2)$ as in the above expression.

The question now arises as to the significance of Eq (17-23) in a vibrating or turbulent field of small particles. The particular expression we have developed assumes two particles which have come into contact and are held together by their mutual surface energy. Bradley has examined this problem for smoke particles, and has found (*a*) that particles of equal size will cohere, even at high temperatures; (*b*) that a very small particle will cohere with a very large particle even if it will not do so with one of its own size; and (*c*) that linear chains and aggregates yield results similar to (*a*) and (*b*). All this provided that the particles are very small and of the order of 10^{-8} cm.

The situation here described applies equally well to a vibrating field. For, consider a wide range of particle sizes. Obviously, they will have different amplitudes of vibration, thus leading to a large number of collisions and eventual coagulation of many pairs, so that the phenomenon of sonic precipitation of aerosols is readily explained. In addition we must consider the effect of the constantly changing size distribution, but in this connection we should appreciate the following fact: It is clear that the net effect of numerous collisions is to reduce the concentration, and hence the frequency of collisions (for a fixed frequency). Therefore, we must alter the frequency of oscillation in order to maintain efficient coagulation up to the point at which the particles are large enough to settle rapidly under gravity, remembering at the same time that amplitude and phase exert a definite influence. Whether direct contact leads to coagulation in every instance (when particles are vibrated) cannot be stated.

SETTLING OF SUSPENSIONS

It must be realized from the foregoing that if settling is assumed to follow Stokes' law, the viscosity is a variable and a function of the concentration. Moreover, when particles are in great concentration a certain amount of disturbance is set up in the fluid, which may be sufficient to destroy the usefulness of this law. As particles settle toward the bottom of the container their concentration increases, and with time a kind of packing action sets in. In the design of settling basins and in gauging the ultimate size of a plant, some knowledge of the laws which govern settling of great concentrations of particles is necessary and valuable.

Most of the available information on settling concerns small laboratory units. In practice, settling basins and thickeners are of large size and are kept in continuous operation. Comings (1940) presented an excellent practical discussion of these and other factors affecting their operation. However, it remained for Kammermeyer (1941) to evaluate the importance of these factors quantitatively.

Ultimate Settling Height—Ward and Kammermeyer (1940) found that initial and final heights of the suspension (without stirring) are related to the weight percent of suspended solids by the expression

$$\frac{H_u}{H_0} = aC^b \qquad \text{Eq (17-24)}$$

where H_u and H_0 are the ultimate and initial heights of the suspension, C is the weight percent of suspended solids, and a and b are constants. The constant a is a function of particle-size, and b is characteristic of a given suspension. According to Ward and Kammermeyer, Eq (17-24) may be used to determine average particle-size. Thus, values of H_u/H_0 for weight concentration of 30 percent quartz in water, of 0.302, 0.383, and 0.432 have average particle-diameters corresponding to 80, 32, and 10 μ, respectively.

Effect of Container Diameter—The rate of settling, as well as ultimate compacting of the particles, is affected by the diameter of the container in which settling takes place. No data are available on particle-size, but in general, for laboratory experimentation the diameter of the settling tube should not be less than 40 mm for substances such as calcium carbonate, barium sulfate, and silica (Kammermeyer, *loc cit*).

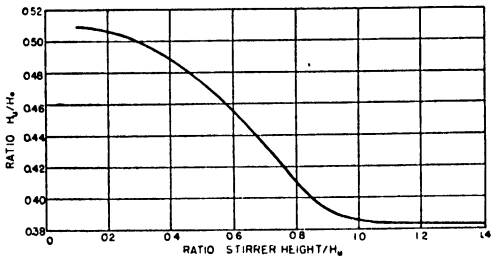


FIGURE 99. EFFECT OF STIRRER HEIGHT ON RATIO H_u/H_0.

Effect of Stirrer Height—Figure 99, taken from data by Kammermeyer (1941), indicates the effect of stirrer height on the ratio H_u/H_0. The data apply to a 19 percent suspension of calcium carbonate. The stirrer speed was adjusted to 2 rpm. From the curve shown in the

figure it is seen that the ratio H_u/H_0 becomes constant when the ratio of stirrer height to ultimate height of suspension approaches one.

The effect of stirring is interesting. For most substances the curve of H_u/H_0 versus concentration is a straight line, almost parallel to the curve for normal settling without stirring. This is true up to a point where the mud or slurry thickens. At that point there is a sharp rise in the slope of the curve which may be explained in terms of compressibility of the material, and possibly an accelerating action due to a decrease in voids. This effect is not so noticeable in the case of hard, firm particles which do not compact easily. It is necessary to point out that the break in the curve is sharp. This break may signify a characteristic of packings of some importance, probably in their bulking properties. Work on this subject merits further attention.

Rate of Settling—At the present time two methods are widely used for computing settling rates of liquid suspensions. Rollason, cited by Egolf and McCabe (1937), proposed the use of Stokes' equation in the form

$$\frac{dH}{dt} = K\,\frac{d^2(\rho - \rho_0)}{\mu}$$

We have pointed out that the difficulty of this equation lies in the fact that μ cannot be considered constant, but is a function of the concentration at any given time. The value of K must be determined experimentally.

Egolf and McCabe propose the following equation to determine the rate of settling:

$$\frac{dH}{dt} = \phi\left[\frac{d^2(\rho - \rho_0)}{\mu} \cdot \frac{H_0}{H_u} \cdot (0.415 - \theta)^{1.4}\right] \qquad \text{Eq (17-25)}$$

where dH/dt is the rate of settling, which is assumed linear up to the time the settled particles are compressed by their own weight (hence, $dH/dt = H/t$), ϕ symbolizes "function of," and θ is the volume concentration (cu cm/cu cm) of suspension at ultimate height. The other terms have been defined previously. With regard to this equation, Egolf and McCabe state substantially as follows: The equation is for the most part empirical, but does have some rational basis. The initial rate increases and the concentration decreases, and the ratio H_0/H_u is equivalent to the reciprocal of the concentration since the ultimate height is practically constant. The quantity $(0.415 - \theta)$ is a measure of the intensity of the flocculation since it has been observed that the more strongly the particles flocculate the less dense the final sediment, and hence the greater the value of $(0.415 - \theta)$. Eq (17-24) does not hold

for values of θ equal to or greater than 0.415. Ward and Kammermeyer state that for quartz θ becomes equal to 0.415 at an average particle-diameter of 50 μ.

The compression part of the curve as given by Egolf and McCabe is

$$\log (H/H_u) = K' \log at \qquad \text{Eq (17-26)}$$

where H is the height of the suspension at time t, and a is a compression constant. Thus K' is determined from the slope of the curve. The value of K' was found to be fairly constant for any given material; determination of a entails no difficulty. Plots of concentration of suspension versus aH_0 (or H_u/H_0) were found to give a straight line.

MIXING OF SUSPENSIONS

It is often useful to determine the concentration of a suspension after a given amount of diluent has been added. The procedure involved in calculating the resulting mixture is simple. Suppose we have a volume Q_0 of a suspension whose initial concentration is C_0 and that a volume Q_1 of diluent is added to this suspension. Let $Q = \theta Q_0$ where θ represents the fractional part of Q_0 added. If the diluent possesses a concentration C_1 of the component in Q_0 being diluted, the final concentration of the suspension C is then simply the weighted average of the components

$$C = \frac{C_0Q_0 + C_1Q_1}{Q_0 + Q_1} = \frac{C_0 + \theta C_1}{1 + \theta} \qquad \text{Eq (17-27)}$$

When $C_1 = 0$, the condition for a pure diluent,

$$C = \frac{C_0}{1 + \theta} \qquad \text{Eq (17-28)}$$

If we now consider a situation where there is an influx of diluent and an efflux of desired concentration C, we may arrive at a general solution for the case of two liquids for which the two equations above are special cases. This may be done provided that the liquids diffuse instantaneously (a condition which may be accomplished mechanically) and that no chemical reaction takes place. Let C_0 and C_1, as before, represent the initial concentration of suspension to be diluted and the concentration of the same suspension in the diluent. If ξ denotes the ratio of efflux to influx into the container whose volume is Q_0, then

$$dC = \frac{C_1Q_0d\theta}{Q_0[1 + \theta(1 - \xi)]} - \frac{CQ_0d\theta}{Q_0[1 + \theta\,(1 - \xi)]}$$

The first term on the right gives the fraction of increment $Q_0d\theta$ mixed with the original amount of Q_0 of the fluid, and the second term gives the

fraction forced out by the admission of $Q_0 d\theta$. Hence, on reducing and integrating between limits $C = C_0$, $C = C$, and $\theta = 0$, $\theta = \theta$, we obtain

$$\frac{\ln [1 + \theta(1 - \xi)]}{1 - \xi} = - \ln \frac{C_1 - C}{C_1 - C_0}$$

or

$$\theta = \frac{1}{1 - \xi}\left[\left(\frac{C_0 - C_1}{C - C_1}\right)^{1-\xi} - 1\right] \qquad \text{Eq (17-29)}$$

Thus, if the initial volume Q_0 of the suspension to be diluted is known, we may calculate the volume $Q = \theta Q_0$ of the diluent required to give a concentration C for any ratio of efflux to influx. Note that when $\xi = 0$ (no efflux), we obtain Eq (17-27) and that when $\xi = 0$ and $C_1 = 0$ (fluid without suspension), we obtain Eq (17-28). Intermediate cases for $C_1 = 0$ with efflux and influx are directly obtainable for Eq (17-29). Note further that Eq (17-29) is indeterminate when $\xi = 1$. Also, if $C_1 < C$, $C_0 > C > C_1$, and if $C_1 > C$, then $C_1 > C > C_0$. The latter case represents an increase in the concentration of the suspension in the container.

Problems

1. Calculate the density of a suspension of particles, given the following information: Volume 500 cu cm, total weight of particles 50 g, density of particles 2.2.

2. From the data of Problem 1, calculate the weight of a unit-volume of the suspension.

3. Using Eq (17-12) calculate the viscosity of the suspension given in Problem 6, Chapter 4.

4. A suspension of zinc oxide was measured at 2 P.M. and the concentration estimated to be 4 particles per cu cm; at 4 P.M. the concentration was estimated to be 1 particle per cu cm. Determine the coagulation constant.

5. (A) Suppose a volume V of a fluid is present in a container and V' is the volume of another fluid equal to V which is added to give a concentration C of a fluid. If the initial concentration of the fluid is C_0, determine the quantity of fluid V' required to yield a concentration to a value C.

Note: The change in concentration dC produced by the admission of a small increment VdV of the added fluid is proportional to the difference between this increment and that contained in the mixture forced out of the container. Hence

$$dC = VdV' - VCdV'/V$$

whence

$$V' = - \ln (1 - C) = -2.303 \log (1 - C)$$

(B) If the concentration of the fluid being added is C_1, determine the value of V' in part A of this problem to give a concentration C.

(C) Given that the fluid introduced contains the fractional part C_1 of the component whose concentration is to be changed, determine the volume of the fluid which

must be added to give a concentration C, assuming that the initial concentration in the receiver is C_0, and that there exists a free efflux of fluid mixture.

Note: Here we have

$$dC = (C_1 V dV' - CV dV')/V = (C_1 - C)\, dV'.$$

CHAPTER 18

TRANSPORT OF PARTICLES

THIS chapter will deal with the movement of particles in liquids and gases. The subject matter will be considered in three major sections: (*a*) Silt transportation in open channels, (*b*) movement of sand and clays in pipes, and (*c*) pneumatic transport. In each type of transport there are different laws of behavior—in fact, many of them. The whole field dealing with transport of particles has been in a state of flux for many years. Each investigation seems to have produced a separate concept of the importance of the variables affecting transport, and formulas developed are equally numerous. In this chapter we can only indicate the significant factors involved. For more complete discussions the reader is referred to the bibliography which contains a fairly representative list of contributors to the subject.

As a matter of general interest we shall show that the force required to just move a particle varies as the sixth power of the velocity if the motion is turbulent. As will be shown later, this law has been amply verified, but its use thus far seems to be limited almost exclusively to problems of silt transportation. Referring to Eq (2-10)

$$d = \frac{v^2}{K_T^2 \left(\frac{\rho - \rho_0}{\rho_0} \right)} \propto v^2$$

Since the mass of the particle is as the cube of the diameter, we have *moving force* $\propto d^3 \propto v^6$.

SILT TRANSPORTATION

Land contours have been shaped largely through the movement of fine grains by streams and rivulets. The accumulation of particles has created fertile fields in some places and important deposits of minerals in others. All this has been going on for countless ages and still continues. We have learned to appreciate the significance of the earth's shifting surface because the forces of erosion have also deprived us of untold agricultural wealth. And equally important is the effect of river silt on reservoir basins used for power and irrigation; heavy silt-

loads quickly reduce capacities. Some concept of the amount of silt moved by a river can best be shown by data obtained for the Missouri. Thus, for the 2-yr period July 1, 1929, to June 30, 1931, the silt-load at Williston, N. D. (1659 miles above the mouth), was 64 million tons; at Sioux City, Iowa (769 miles above the mouth), the silt increased to 141 million tons; and at Howard Bend (36.5 miles above the mouth) the total load was 270 million tons. All this is suspended silt. Between the stations, some silt was deposited and more picked up.

When a river enlarges or comes to the still waters of a reservoir, its velocity is reduced and the silt-load deposited forms a delta. The material in the delta is therefore sorted as to size, the coarser particles settling first and the finer ones being carried farther on.

Relations of Silt-Load to Flow and Velocity—It is difficult to obtain a relation between silt-load and flow and velocity, since the amount of silt present depends on unpredictable factors. In connection with studies of silt transportation in the Missouri River by the Corps of Engineers (1935*b*), the following empirical relations were obtained at Kansas City

$$G = 30\ 4 \times 10^{-11} Q^{2.16}$$

$$G = 0.00862\ v^{4.8}$$

where G is the total amount of silt transported in tons per sec, Q the flow in cu ft per sec, and v the mean velocity in ft per sec. If the first of these equations is written

$$G/Q = 30.4 \times 10^{-11} Q^{1.16}$$

then the amount of silt per unit-volume is nearly proportional to the flow. These equations are interesting, but it should be remembered that they cannot be expected to hold for all times. In the many hundreds of miles of rivers and tributaries, the greatest variety of local conditions may exist, such as cloudbursts, melting snows, droughts, etc., all affecting the main stream far below.

Distribution of Particles—The distribution of particles at various depths in a slowly moving stream may be determined as follows. From the fact that the rate of change in weight of silt at a depth L is proportional to the weight of silt per unit-volume of water vertically above a unit-area, we obtain the following expression

$$\frac{dw}{dt} = -k\frac{w}{L}$$

and the solution is

$$w = w_0 \cdot \exp\ (-kt/L) \qquad \text{Eq (18-1)}$$

where w_0 is the weight of silt originally in the water. The w's in the equation may be replaced by number of particles per unit-volume. However, the constant k involves certain difficulties in its determination. A more complete equation for distribution of suspended silt in a slow-moving stream was obtained by Christiansen (1935)

$$C = C_0 \cdot \exp\left(v_m \rho_0 \int_0^L \frac{dL}{\epsilon}\right) \qquad \text{Eq (18-2)}$$

where C is the number of particles per unit-volume at a point L measured downward from the surface, C_0 the initial concentration, v_m the settling velocity defined by Eq (2-9), ρ_0 the density of the fluid, and ϵ a coefficient defined by the equation

$$\epsilon = \frac{\rho_0 SL}{dv/dL}$$

where v is the velocity of the water, and S the slope of the channel. If ϵ is constant, then

$$C = C_0 \cdot \exp\left(v_m \rho_0 L/\epsilon\right) \qquad \text{Eq (18-3)}$$

which on reduction ($dv/dL = v/L$) gives the same result as Eq (18-1) The above equations do not hold at points near the bottom of the channel. Moreover, in computing the concentration at any point it is preferable to use Eq (18-2) since the ratio of $L/(dv/dL)$ is not necessarily constant. Values of dv/dL are derived from velocity traverses in the stream.

Sorting Phenomenon—It is a common observation that larger particles are dragged or rolled more rapidly than small ones along the bed of a stream. Thus, just as there is a gradation effect on particles in the formation of a delta (previously discussed), along a stream bed a similar gradation also takes place. However, the causes are different in each case. We explained the delta formation in reservoirs as due to settling phenomena, while in the present instance the effect is due to large changes in the stream velocity gradient near the bed surface. The effect is shown schematically in Figure 100. This figure shows two particles (assumed spherical), and the accompanying velocity distribution for the flume in which they lie. For purposes of illustration the curve is exaggerated, but it is obvious that the mean velocity pressure is much

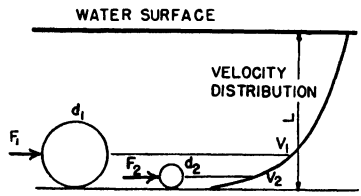


FIGURE 100. ILLUSTRATING ROLLING EFFECT ON STREAM BOTTOM.

greater over the upper portions of the larger particles. The effect is easily shown quantitatively: If v_b is the velocity of the bed stream, then the equilibrium forces on a particle, say of diameter d, are given by the equation

$$\frac{\pi d^2}{4} \cdot \frac{\rho v_b^2}{2g} = f \frac{\pi d^3}{6} (\rho - \rho_0)$$

where f is the coefficient of friction and the other terms are as defined previously. Hence

$$v_b = \left(\frac{2}{3} f \cdot 2g \cdot \frac{\rho - \rho_0}{\rho_0} \right)^{1/2} \sqrt{d} = K \sqrt{d}$$

Thus if $d_1 = 2d_2$, the velocity of the larger particle requires a bed velocity of $\sqrt{2}$ times that of the smaller. However, the velocity change between the points identified as centers of the two particles is generally more than doubled, so that the coarser particle is more easily moved.

Bed Velocity—It is customary to use the term "mean" velocity to define movement of streams. In silt transportation we are primarily interested in the velocities near the bed of transportable material. It is difficult to define precisely what is meant by bed velocity, but in general, we refer to the average velocity over a narrow region next tc the bed. The thickness of this region and the velocity distribution across it require certain fundamental assumptions beyond the scope of this text (see Rubey, 1938). The bed velocity v_b may be shown to be related to the mean velocity v_{av} as follows:

$$v_b^4 = \frac{g}{k_b} v_{av}^2 RS \qquad \text{Eq (18-4)}$$

where g is the gravitational constant, k_b the coefficient of bed resistance referred to the bed velocity, R the hydraulic radius (cross-sectional area of stream divided by the wetted perimeter), and S the slope of the stream. From the same equation

$$\sqrt{k_b}\, v_b^2 = \sqrt{g}\, v_{av} \sqrt{RS} \qquad \text{Eq (18-5)}$$

and for a particular bed velocity

$$\sqrt{k_b} \propto v_{av} \sqrt{RS}$$

Manning's Formula—From time to time we shall have occasion to refer to the coefficient of roughness contained in Manning's formula for flumes (in fps units)

$$v = \frac{1.486}{n} R^{2/3} S^{1/} \qquad \text{Eq (18-6)}$$

where R is the hydraulic radius, S the slope, and n the coefficient of roughness. The coefficient of roughness depends upon the nature of the bottom and sides of the flume, and frequently is difficult to determine. There has also been some discussion regarding the dimensions of n, those generally accepted being $[L]^{1/6}$. Chang (1939) cites Strickler as having proposed an equation for n in terms of the diameter of particles near the surface layer of the bed

$$n = 0.015 d_{m_e}^{1/6}$$

where d_{m_e} is the median diameter in mm. Chang himself verified this formula, obtaining in his experiments

$$n = 0.0166 d_{av}^{1/6}$$

where d_{av} denotes the mean diameters of the particles also expressed in mm. Both Strickler's and Chang's equations must be used cautiously

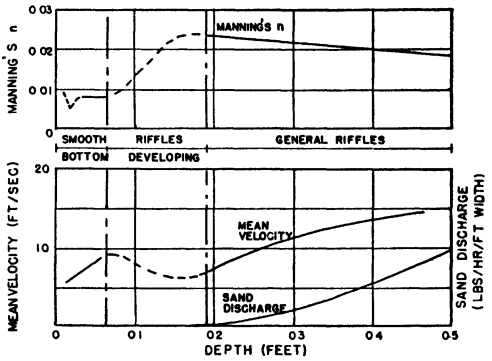


FIGURE 101. VARIATION OF SAND DISCHARGE, MEAN VELOCITY, AND MANNING'S n FOR AN EXPERIMENTAL FLUME. AVERAGE DIAMETER OF SAND 0.347 MM, KRAMER'S MODULUS 0.643, AND SLOPE 0.0015.

although they give values of the right order of magnitude. Difficulty in applying them generally lies in the fact that n is not constant, but fluctuates with the conditions of flow. Some estimate of the variation of n may be seen by inspection of Figure 101 taken from data obtained by the Corps of Engineers (1935*a*), at the U. S. Waterways Station, Vicksburg, Miss. Note the change in values of n with the development of riffles.

THEORIES REGARDING TRANSPORT

Several theories have been proposed to explain the laws governing transport of particles in streams. Each theory seems to have points in

its favor, but until the movement of water on stream beds is better understood, it is doubtful whether a completely satisfactory theory can be developed. In general, there are three theories regarding silt transport. One of these is implied in the sixth-power law, discussed in the opening paragraphs of this chapter, although this law has not been used in the form given. As indicated elsewhere, the velocity in question is that near the bed of the stream, and it is practically impossible to measure this velocity. The bed velocity must be expressed as the mean stream velocity, but such alteration involves approximations often leading to errors of considerable magnitude. In spite of these limitations, the velocity theory finds an important place in silt problems.

The second theory is called the "tractive-force theory" and utilizes the concept of current drag or "Geschiebe" required to move particles of a given size. Later we shall develop the underlying basis of tractive force, but for the moment we merely note that it is distinguishable from the velocity theory in that the movement of a particle by a stream varies as the depth and slope of the stream.

Finally, the third theory states that the dislodging of particles from a stream bed does not depend upon velocity or tractive force, but upon the lift induced by the velocity gradient or the rate of shear between adjacent fluid filaments. There can be no question but that "lift" plays an important role in silt transport. However, the author believes it is ancillary to the two preceding theories and hardly to be considered self-sufficient or capable of explaining the movement of particles. At least until such time as more is known regarding the nature of flow at boundary layers, we cannot hope to use the lift theory alone in predicting what may be expected from a given stream, even though we do know its depth, slope, and velocity, as well as the size-distribution of the silt composing the bed.

VELOCITY THEORY IN SILTING

It was shown that large particles are more easily transported by streams than smaller ones. This phenomenon becomes more pronounced with small particles since the cohesive forces existing between the particles increase as the latter decrease in size. Chatley, cited by Tchikoff (1937), gives the formula

$$v_c = 0.02/d$$

where v_c is the critical or eroding velocity (average velocity of stream) in cm per sec, and d the diameter of the particles in cm. The method of determining d was not given, but we infer that it applies to classes of

particles as given in Table 72. The formula is limited, but offers an idea as to the importance of small particles in scouring.

TABLE 72—ERODING VELOCITIES FOR STREAM BEDS COMPOSED OF CERTAIN PARTICLES

Class of particles	Diameter of particles (mm)	Critical velocity (velocity of erosion) (cm/sec)
Fine gravel to coarse sand	1.00	0.2
Fine sand to very fine sand	0.10	2.0
Very fine sand to silt	0.05	4.0
Silt to clay	0.005	40 0
Clay	0.001	200.0
Colloids	0.0001	...

Kennedy's Formula—Kennedy (1894–1895) proposed an empirical formula of the type

$$v_c = aL^b$$

where v_c as before is the velocity of the stream required to cause movement of silt, L the distance between the surface of the bed and the stream, and a and b are constants. For the Upper Bari Doab Canal, Punjab, India, Kennedy obtained a value of $a = 0.84$ and $b = 0.64$. Subsequent investigations of other canals by Kennedy and later workers on the subject obtained values of a ranging from 0.39 to 0.95, and b from 0.52 to 0.57. Kennedy assumed that vertical eddies alone were responsible for silt transportation and that width of the channel was unimportant. It is now known that other dimensional factors in addition to depth are necessary. Lacey (1930) re-examined existing data utilizing the Kennedy formula, and proposed a formula of the type

$$v_c = \phi(R, \xi) = \xi R^c$$

where R is the hydraulic radius, c a constant, and ξ a silt-factor, derived in a manner to be described later. Kennedy's equation for the Upper Bari Doab Canal, $v_c = 0.84\,L^{0.64}$, was recast in the form $v_c = 1.17\sqrt{R}$. Other data, when similarly treated, assumed the form $v_c = \alpha\sqrt{R}$, where α is a specific constant applying to the stream.

Now to introduce the Lacey silt-factor we write a generalized formula such as

$$v_c = 1.17\sqrt{\xi R} \qquad \text{Eq (18-7)}$$

For the original Kennedy formula $\xi = 1$, an arbitrary arrangement. If v_{c1} is the critical velocity for a flume, such that $v_{c1} = 1.17\sqrt{\xi_1 R}$, where ξ_1 is the specific silt-factor for the flume, then $v_{c1}/v_c = \sqrt{\xi_1}$ and we note that the silt-factor measures the ease with which a stream bed is moved.

Lacey next developed an important concept of the silt-factor-area relationship of a channel. Following Kennedy's formula and defining a "regime channel" as a channel whose bed-silt and also self-silted side berms are nearly stable, self-derived, and long continued, then for two channels having the same mean velocities

$$\xi R = \xi_1 R_1$$

If the wetted perimeters are considered to follow the same law as the depth or hydraulic radius, then

$$\xi \pi = \xi_1 \pi_1$$

where the π's are the wetted perimeters. Multiplying these equations together and generalizing for all regime channels

$$A\xi^2 = A_1\xi_1^2 = A_2\xi_2^2 = \text{etc.}$$

where A, A_1, A_2, etc., are the respective cross-sectional areas of the regime channels (equals πR). By plotting the available data, $A\xi^2$ against v_c, Lacey derived the relationships

$$A\xi^2 = 3.8v_c^5 \qquad \text{Eq (18-8)}$$

and

$$Q\xi^2 = 3.8v_c^6 \qquad \text{Eq (18-9)}$$

Therefore the silt-transporting power is as the sixth power of the velocity, as it should be.

Gilbert's Experimental Results—Gilbert (1914) carried out extensive flume experiments which are regarded as models of excellence. Gilbert was interested primarily in stream capacity, but the data collected have been used often to calculate the ability of a stream to move particles on the bed surface. Following are the developments due to Rubey (1938) who derived formulas of critical bed velocities. By plotting a roughness ratio, $d_{av}/2R$, logarithmically against the inverse measure of bed velocity and resistance coefficient, $1/(v_{av}\sqrt{RS})$, and drawing a line of best fit between region marking no movement of particles and that indicating movement, Rubey obtained the following equations:

For particles of sand 0.250–0.370 cm (average = 0.304 cm)

$$\frac{1}{v_{av}\sqrt{RS}} = 0.016 \log \frac{2R}{d_{av}} + 0.048$$

For particles of sand 0.175–0.250 cm (average = 0.209 cm)

$$\frac{1}{v_{av}\sqrt{RS}} = 0{\cdot}025 \log \frac{2R}{d_{av}} + 0.074$$

For particles of sand 0.045–0.105 cm (average = 0.069 cm)

$$\frac{1}{v_{av}\sqrt{RS}} = 0.11 \log \frac{2R}{d_{av}} + 0.32$$

Using Eq (18-5) *et seq*, an approximate general equation for bed velocity was derived:

$$v_b^2 = \left(1.05 \log \frac{2R}{d_{av}} + 3.11\right)\sqrt{g}v_{av}\sqrt{RS} \qquad \text{Eq (18-10)}$$

where the term in parentheses corresponds to $1/\sqrt{k_b}$ in Eq (18-5). Rubey also derived the following equations giving the maximum particle-diameter capable of being moved by a stream

$$d_{max} = 0.44\left(\log \frac{2R}{d_{av}} + 2.96\right)\frac{\rho_0}{\rho - \rho_0} \cdot \frac{v_{av}\sqrt{RS}}{\sqrt{g}} \qquad \text{Eq (18-11)}$$

where ρ_0 is the density of the water, ρ that of the particle, the other terms being as previously defined.

As a measure of the amount of silt moved by a stream, Gilbert obtained the approximate expression

$$G = \frac{a}{d_{av}^{0.58}} S^{1.59} Q^{1.02} - b$$

where G is the rate of silt movement, expressed as weight per unit-time, Q the volume of flow per unit-time, and a and b are constants.

TRACTIVE-FORCE THEORY

In the velocity theory above developed it is apparent that while fundamentally sound, the chief difficulty in practice concerns the measurement of bed velocities. This has been overcome in part by Rubey's analysis of the subject, and by the general theory of Kennedy and Lacey. However, in recent years studies of silt movement have utilized DuBoys' (1879) expression of "tractive force." This expression is simple and convenient and involves the basic elements of channel hydrology: depth and slope. Tractive force means the force activity on the bed causing movement of the particulate material. The force required to impart initial motion to the bed material is called the "critical tractive force." General movement is defined as the condition where particles up to and including the largest composing the bed are in motion.

DuBoys' expression for the tractive force may be derived in the following manner: Consider a channel with a uniform flow of water of depth L: if no resistance is offered to the flow the kinetic energy of a

prism of unit cross-sectional area and depth L in a time dt is very nearly

$$dE = \frac{m}{2}\left[(v + dv)^2 - v^2\right] = mvdv$$

where E is the energy, m the mass of fluid, and v its velocity. Since $dv/dt = gS$, and $dE = mgSvdt$, and assuming that there is no increase in kinetic energy, there must be a force F opposing it, and since this force acts in the distance vdt, we have

$$F \cdot vdt = dE$$
$$F = mgS$$

If the density of the fluid is taken as ρ_0, then $m = \rho_0 L/g$, so that

$$F = \rho_0 LS \qquad \text{Eq (18-12)}$$

This is DuBoys' equation for tractive force. The equation makes no allowance for internal friction and turbulence, bed friction, or movement of bed load. However, in spite of these omissions, it has been found that for a given bed material, the movement is proportional to the tractive force. DuBoys' equation for the rate of movement of silt was of the form

$$G = \psi F(F - F_c) \qquad \text{Eq (18-13)}$$

where G is the amount of sand moved per unit-time, ψ a factor which is a function of particle-size, condition of bed, etc., and F_c is the critical tractive force.

Kramer's Equation—Kramer (1935) made a series of measurements of flume traction in which the important variables (including particle-size) were carefully controlled. Kramer's equation of the critical tractive force was as follows:

$$F_c = \frac{100}{6} \cdot \frac{d_{me}\,(\rho - \rho_0)}{K} \qquad \text{Eq (18-14)}$$

in which F_c is the critical tractive force in g per sq m, d_{me} is the median diameter of the particles composing the bed in mm, ρ and ρ_0 are the density of particles and fluid, respectively, and K is Kramer's size-factor or uniformity modulus, discussed in Chapter 3. It has a value of 1.0 for uniform particle-size, and a value of $^1/_3$ for a uniform distribution. The addition of fine or coarse particles to a given mixture tends to reduce its value. Kramer used the following sizes of sand:

Sand No.	*Size-range of particles (mm)*	*Median diameter (mm)*
I	0.0 –5.00	0.53
II	0.0 –1.77	0.51
III	0.385–5.00	0.55

Therefore, Eq (18-14) may be said to hold for a range of sand sizes from 0.0 to 5.00 mm.

Work of U. S. Waterways Experiment Station—The most comprehensive study of silt movement in recent years is that by the Corps of Engineers (1935*a*), published by the U. S. Waterways Experiment Station. To some extent this study followed the pattern used by Kramer, but included more different kinds of material. The mean sizes for eight of the sands composing the beds ranged from 0.205 to 0.586 mm; for one other sand the mean size was 4.08 mm. The sand, having a mean diameter of 0.586 mm, had a uniformity modulus of 0.280; all the others had moduli of approximately 0.5. The critical tractive-force equation obtained was

$$F_c = 29\sqrt{\frac{d_{av}(\rho - \rho_0)}{K}} \qquad \text{Eq (18-15)}$$

where the units are the same as in Kramer's Eq (18-14). In addition to the experimental data obtained in this study, the data of various other investigators (including Kramer) were found to fit Eq (18-15).

An equation was also obtained for the rate movement of silt; that is, the amount of silt moving per unit-width of channel per unit-time. The following empirical formula was derived

$$G = \frac{1}{n}\left(\frac{LS - L_cS_c}{k}\right)^m \qquad \text{Eq (18-16)}$$

where G is the rate of movement, n Manning's coefficient of roughness as given in Eq (18-6), L and S, respectively, the depth and slope of the channel, L_cS_c the value of the depth-slope product for the given sand mixture at the time movement begins (determined by the linear plot of Gn against LS), and k and m are functions of the physical characteristics of the bed-load material. Table 73 gives values of the various quantities included in Eq (18-16). The values of G are expressed in lb per ft width per hour.

Work of Chang—A great number of formulas on flume traction are available. We cannot give details on all of them, nor discuss their relative merits, but shall confine the discussion to those of most recent origin and general interest.

Chang (1939) conducted numerous experiments and correlated the data of several earlier investigators. Unlike Kramer and his co-workers at the U. S. Waterways Experiment Station, Chang found that when values of F_c were plotted against the product of the variables $[(\rho - \rho_0)/\rho_0]d_{av}'\theta^{1/3}$, the various data plotted as two straight intersecting lines on double log grid. The intersection was found to occur at a value of the

TABLE 73—VALUE OF CONSTANTS IN EXPRESSION $G = \frac{1}{n}\left(\frac{LS - L_eS_e}{k}\right)^m$

Sand No.	d_{av} (mm)	K	Slope	Range of values of n	L_eS_e	k	m
1	0.586	0.280	0.0010	0.0112 to 0.0118	0.000100	0.00063	1.5
			0.0015	0.0116 to 0.0124			
			0.0020	0.0118 to 0.0128			
2	0.541	0.439	0.0010	0.0120 to 0.0132	0.000095	0.00061	1.6
			0.0015	0.0124 to 0.0143			
			0.0020	0.0132 to 0.0144			
3	0.525	0.539	0.0010	0.0146 to 0.0193	0.000270	0.00046	1.6
			0.0015	0.0165 to 0.0232			
			0.0020	0.0132 to 0.0188			
4	0.506	0.406	0.0010	0.0152 to 0.0188	0.000300	0.00049	1.7
			0.0015	0.0184 to 0.0248			
			0.0020	0.0140 to 0.0260			
5	0.483	0.438	0.0010	0.0166 to 0.0194	0.000350	0.00058	1.5
			0.0015	0.0170 to 0.0260			
			0.0020	0.0178 to 0.0260			
6	0.347	0.643	0.0010	0.0195 to 0.0212	0.000280	0.00100	1.8
			0.0015	0.0190 to 0.0242			
			0.0020	0.0184 to 0.0250			
7	0.310	0.525	0.0010	0.0214 to 0.0238	0.000250	0.00110	1.7
			0.0015	0.0218 to 0.0252			
			0.0020	0.0206 to 0.0272			
8	0.205	0.560	0.0010	0.0176 to 0.0296	0.000260	0.00066	1 6
			0.0015	0.0236 to 0.0276			
			0.0020	0.0180 to 0.0292			
9	4.077	0.566	0.0030	0.0164 to 0.0172	0.000940	0.00060	1.6
			0.0040	0.0162 to 0.0165			
			0.0045	0.0164 to 0.0193			

product of approximately 1.0. The general equation obtained by Chang is

$$F_c = 0.0175\left(\frac{\rho - \rho_0}{\rho_0}\cdot d_{av}'\theta^{1/3}\right)^{\beta} \qquad \text{Eq (18-17)}$$

where d_{av}' is the average of the shortest and longest dimensions of some 150 particles, θ is the ratio of the longest to the shortest diameter of a sand particle, and the ρ's have their usual significance. For values of $[(\rho - \rho_0)/\rho_0]d_{av}'\theta^{1/3}$ greater than 1, $\beta = 1$, and for values of this product less than 1, $\beta = {}^1/_2$. The discontinuity obtained by Chang is assumed due to a critical particle-diameter composing the bed. It will be recalled that in our discussions regarding movement of particles, large particles are moved more easily than smaller ones. According to Chang's

thesis, which apparently is widely accepted, there is a size of particle at which the larger size no longer moves first, and above which finer particles are placed in motion first. In the work of the U. S. Waterways Experiment Station this critical diameter was placed at 0.6 mm. Chang's value is surprisingly close to this, being in the neighborhood of 0.5 mm. Chang's use of a shape-factor in terms of the ratio of long to short dimensions of particles ($\theta = 1.75$ for sand) involves a great deal of labor. Undoubtedly any of the various shape-factors discussed in Chapter 3 would have proved equally suitable and could have been derived with less work.

Chang's results for the rate of movement G, of various materials, expressed in lb of silt per sec are:

For sand ($\rho = 2.645$, $d_{av}' = 0.226$ mm, $\theta = 1.4$)

$$G = 100{,}000\ F(F - F_c) \qquad \text{Eq (18-18}a\text{)}$$

For pumice ($\rho = 2.053$, $d_{av}' = 0.289$ mm, $\theta = 1.52$)

$$G = 210{,}000\ F(F - F_c) \qquad \text{Eq (18-18}b\text{)}$$

For emery ($\rho = 3.89$, $d_{av} = 0.210$ mm, $\theta = 1.33$)

$$G = 50{,}000\ F(F - F_c) \qquad \text{Eq (18-18}c\text{)}$$

The tractive force is, of course, expressed in lb per sq ft.

OTHER FORMULAS FOR RATE OF SILT MOVEMENT

Below are a few of the more common formulas used for estimating the rate of silt movement. The rate of movement, G, is expressed in lb per sec.

(*A*) *Schoklitsch's Equation*—Shulits (1935) gives an equation due to Schoklitsch which is based on Gilbert's experiments. The equation is

$$G = \frac{436.5}{\sqrt{d_{av}}} S^{3/2} B(q - q_c) \qquad \text{Eq (18-19)}$$

where d_{av} is the average size of the particles in mm, whose range is confined to narrow limits, S is the hydraulic slope of the channel, B the surface width of the channel in ft, q the water discharge per unit-width of the channel in cu ft per sec, and q_c the critical discharge to move the sand or silt per unit-width of channel, expressed in cu ft per sec. The value of q_c for sand is given to a fair degree of approximation by the equation

$$q_c = 0.00021\ LS^{-4/3} \qquad \text{Eq (18-20)}$$

where L is the depth of flow in ft.

(*B*) *Fabre's Equation*—Chang states that Fabre proposed the following general equation

$$G = 2.16\left(\frac{\rho_0}{\rho - \rho_0}\right)^{1/3} S^{1/3} Q - 20.7\left(\frac{\rho - \rho_0}{\rho_0}\right)^{7/9} d_{av} \quad \text{Eq (18-21)}$$

where Q is the total discharge in cu ft per sec. The equivalent critical discharge is

$$Q_c = k\left(\frac{\rho - \rho_0}{\rho_0}\right)^{7/9} \frac{d_{av}}{S^{3/2}} \quad \text{Eq (18-22)}$$

Fabre's results apply to materials having specific gravities slightly greater than 1 and to materials having specific gravities greater than 4.

(*C*) *Meyer-Peter's Equation*—Meyer-Peter *et al* (1934) obtained an equation of the form

$$G = (1.32\, S^{3/2}\, Q - 0.34\, d_{av})^{3/2} \quad \text{Eq (18-23)}$$

This equation was found to apply to most sands. The equivalent critical discharge is given by the expression

$$Q_c = c\left(\frac{d_{av}}{S}\right)^{3/2} \quad \text{Eq (18-24)}$$

where c is a constant. Note the difference between this equation and the corresponding equation developed by Fabre.

MOVEMENT OF SANDS AND CLAYS IN PIPES

Much less is known about the movement of sands and clays in pipes than of silt in regular channels. In spite of the industrial importance of the subject, the amount of research done is extremely limited. While it may seem that fundamental information could easily be obtained from various installations, this is not generally the case. Wherever sands, clays, or other particulate matter are moved in pipes, little attempt is made to control the concentration of the mixture moved. When such material is moved from a vat or thickener, the most concentrated matrial is removed first. In other words, since the concentration of the material moved seems a most important variable at first, and since it usually cannot be controlled, there has been little incentive to experimentation.

In addition to concentration of the material moving in a pipe, we must consider particle-size. Obviously, material of relatively large particle-size offers fewer interfacial contacts than smaller particles. The latter will behave differently—more like a viscous fluid. This is amply borne out by many investigations. In general, it may be said that the energy

required to move a mass of particles in suspension is dependent upon (*a*) accelerating the particles (that is, imparting kinetic energy); (*b*) providing kinetic energy for moving the fluid; (*c*) overcoming interparticulate and fluid friction; and (*d*) (for vertical pipes) energy to lift the particles. In the following paragraphs, only horizontal movement is discussed and there will be no attempt to distinguish between the different energy components, since this is frequently impossible.

TRANSPORTATION OF SAND IN WATER

In considering transportation of sand, we deal with particles of relatively large size, so that settling rate becomes an important variable. Velocities must be so maintained that no settling takes place. It is a curious fact, in so far as sand and substances similar to it are concerned, that at comparatively low water velocities, say 5 to 10 ft per sec, the resistance to flow is quite high (most sands begin to move at the lower velocity); but at higher water velocities, the lines of resistance for water and water and sand (up to 25 percent at least) approach each other and become parallel. The higher initial resistance is undoubtedly due to constriction effects (settling) and to the lower velocity of sand movement with respect to the water. Figure 102 shows data compiled by Dent (1939) from various experiments made by Howard (1939) on the transportation of sand and gravel in a 4-in. pipe.

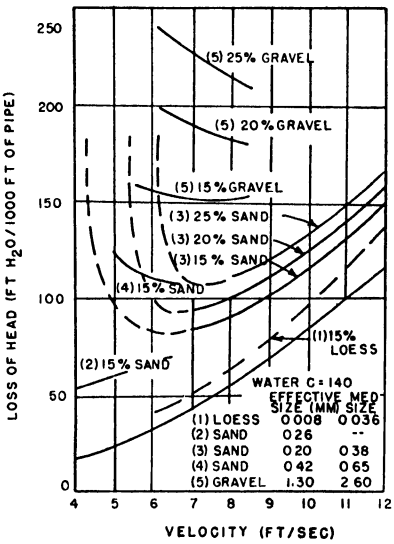


FIGURE 102. LOSS OF HEAD DUE TO TRANSPORTATION OF VARIOUS MATERIALS IN A 4-INCH PIPE.

Transportation Equations for Sand—Dent (1915) gave the following practical equation for use in dredge design,

$$h = h_w + h_s = h_w + C\chi \qquad \text{Eq (18-25)}$$

where h is the total loss in head due to flow of a mixture of sand and water (per 1000 ft of pipe line), h_w the loss due to the flow of water at the same velocity, h_s the excess loss due to the presence of the sand, χ a factor depending upon the character of the sand and diameter of the pipe,

and C the weight percentage of sand present in the mixture. The mean value of χ obtained by Dent (1939) to fit the data of the above figure is 2.14 for a 4-in. pipe. For 3-in. and 1-in. pipes, Dent estimates

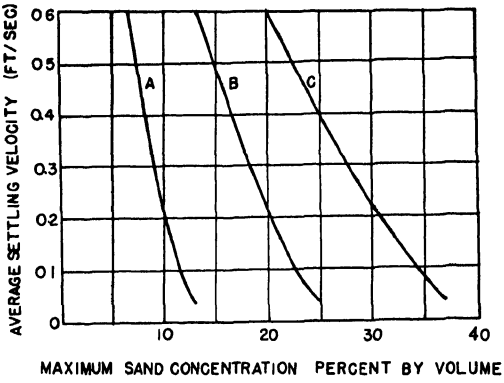


FIGURE 103. RESULTS OF O'BRIEN AND FOLSOM (1937) ON THE VELOCITY OF IMPENDING CLOGGING AT VARIOUS SAND CONCENTRATIONS FOR 2-INCH AND 3-INCH PIPES.

values of χ at 3.04 and 10.10, respectively; while for pipes exceeding 12 in., he estimates values between 1.0 and 1.5.

Experiments by O'Brien and Folsom (1937), with three different sands having the following median sizes: 0.17, 0.33, and 1.3 mm showed (1) that above the critical velocity the head loss in any given pipe is the same as would occur with clear water at the same mean velocity, and (2) that the pressure-drop increases in proportion to the specific gravity of the mixture. These data correspond to Dent's findings, as well as those of Howard whose experiments will be discussed later. We have shown that with increasing velocities the resistance offered to flow by particulate material does approach that of clear water. However, it is hard to conceive why resistance at rather heavy concentrations should be the same as that for water unless we consider that the particle-size distributions used in Howard's experiments differed considerably from those of O'Brien and Folsom. As the latter have pointed out, the importance of settling velocities cannot be ignored, and since this is a function of particle-size it may be a possible explanation of the differences. Thus, both investigators may be entirely correct. Reproduced in Figure 103 is an important set of curves by O'Brien and Folsom, based on experiments on 2-in. and 3-in. pipes. It is to be noted in these curves that the average settling velocities and the velocity of impending clogging are closely related.

We have already cited the work of Howard (1939) whose experiments were carried out with 4-in. pipe. In addition to the losses given in the figure, Howard also computed the Fanning friction factor from the equation

$$h_f = \frac{fLv^2}{2gD} \qquad \text{Eq (18-26)}$$

where h_f is the friction loss per unit-length of pipe, v the velocity of the fluid, L and D respectively the length and diameter of the pipe, g the gravitational constant, and f the Fanning friction factor. Factors were computed from experimental observations for different water velocities and concentrations of mixtures. Values of f obtained by Howard are presented in Table 74 and may be compared with a similar compila-

TABLE 74—VARIATIONS IN FRICTION FACTOR, f[a], FOR CHANGES IN SOLIDS CONCENTRATION IN A 4-IN. PIPE

Velocity (ft per sec)	Values of f for the following percentages of sand in the mixture							Values of f for the following percentages of gravel		
	10	15	20	25	30	35	40	15	20	25
5.5	...	...	...	...	...	...	...	0.089	...	...
6.0	...	...	...	...	...	...	...	0 084	0.102	0.118
6.5	0.035	...	...	...	...	...	...	0.069	0.084	0 096
7 0	0.030	0 031	0 033	...	...	...	..	0.058	0 070	0 082
7 5	0 027	0 028	0 030	.	0.034	...	..	0.050	0.060	0 068
8 0	0.024	0 026	0 027	0 029	0 030		0 038	0 044	0.051	0 058
8 5	0 022	0 024	0 026	0 026	0 027	0 029	0.036	0 040	0 045	0 051
9 0	0 019	0 023	0 024	0 025	0 025	0 029	0 033	...	...	..
9.5	0.020	0 022	0 023	0 023	0 024	0 028	0 031	...	...	...
10.0	0.020	0 021	0 021	0 022	0 023	0 026	0.029	...	...	...
10 5	0 020	0 020	0 020	0 021	0 022	0 024	0 027	...	...	...
11 0	0 020	0 020	0 020	0 021	0 021	0 023	0 025	...	...	...
11 5	0 019	0 019	0 020	0 020	0 020	0 022	...	...	...	..
12 0	0.019	0 019	0 019	0 019	0 019	...	...	...	...	...
12 5	0 018	0 018	0 017	...	...	...	...	...	...	...
13 0	0.017	..	. .	..	...	..	...	...	...	...

[a] All values of f are computed from a head that is in ft of mixture rather than in ft of water

tion (also given by Howard) for data obtained on a 1-in. pipe by Miss Blatch (1906) as well as for 3-in. and 4-in. pipes by Hazen and Hardy (1906) given in Table 75. Inspection of these tables shows decreasing values of f for increasing velocities, as well as increases in f for increasing concentrations of solids transported.

For sand having a median diameter of 0.38 mm (geometric standard deviation 1.64) Howard derived the following formula to fit his data

$$h_f = 0.0044\, C^{1.74} v^{6.4\, C^{-0.60}}$$

$$\log h_f = -2.36 + 1.74 \log C + \frac{6.4}{C^{0.60}} \log v \qquad \text{Eq (18-27)}$$

TABLE 75—VARIATIONS IN FRICTION FACTOR, f^a, FOR CHANGES IN SOLIDS CONCENTRATION IN PIPES OF VARIOUS SIZES

Velocity (ft per sec)	Values of f for the following percentages of solid matter								
	1-in. pipe			3-in. pipe			4-in. pipe		
	10	20	30	10	20	30	10	20	30
4.0	0.060	0.084	0.098	0.070	0.081	0.098	0.076	0.100	0.123
5.0	0.046	0.054	0.066	0.048	0.059	0.069	0.055	0.071	0.085
6.0	0.038	0.041	0.050	0.039	0.047	0.053	0.044	0.055	0.064
7.0	0.036	0.037	0.041	0 034	0.039	0.043	0.037	0.045	. .
8.0	0.034	0.034	0.038	0.030	0.034	...	0.033	0.038	...
9.0	...	...	...	0.028	0.030	...	0.029	0.034	...

[a] All values of f are computed from a head that is in ft of mixture rather than in ft of water.

where h_f is the loss of head in ft of mixture per ft of pipe, C the concentration of solids in percent, and v the velocity in ft per sec. Eq (18-27) ap-

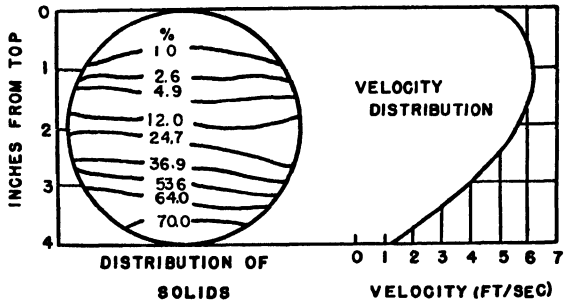


FIGURE 104. DISTRIBUTION OF SOLIDS IN A 4-INCH PIPE.

plies only for the sand in question and only for a 4-in. pipe. It is presented above merely to indicate a possible relationship between the variables considered.

Distribution of Solids—The distribution of solids and the velocity traverse of the 4-in. pipe above referred to are shown in Figure 104. It is seen from this figure that the material is highly concentrated at the bottom of the pipe. The amount of material discharged section by section is shown in Figure 105, the data for which were obtained by multiplying the material passing a given section by the area and velocity at that section.

FLOW OF MUDS AND SLURRIES IN PIPES

When material of extremely small size such as muds and slurries is moved, our problem becomes exceedingly complicated. So much depends upon the nature of the material moved that generalizations

are almost impossible. As is well known, the shearing stress of a plastic or semi-formable material varies with the velocity at which the material is moving. Therefore, it is always necessary to define the limits of application of a formula.

A form of equation applicable to a large number of industrial muds and slurries was advanced by Babbitt and Caldwell (1939, 1940),

$$\frac{h_w}{L} = \frac{16\tau}{3\varphi_0 D} + \frac{\mu' v}{\varphi_0 D^2} \qquad \text{Eq (18-28)}$$

This equation is based on fundamental considerations pertaining to the flow of plastic materials. The terms have the following significance in fps units: h_w = difference in static head between two points of a pipe in ft of water, L = distance between the points in ft, τ the shearing stress at the yield point of a plastic material in lb per sq ft, μ' the viscosity of the material in lb per ft sec, v the velocity of the material in ft per sec, φ_0 the density of the flowing substance in lb per cu ft, and D the diameter of the pipe in ft. To determine the loss of head between two points, it is only necessary to determine φ_0, μ', and τ for the suspension in question; φ is easily determined gravimetrically, and μ' and τ may be obtained with a Stormer viscosimeter. The values of μ' and τ may be obtained as follows: Plots are made of the driving force W (in g) against the revolutions of the drum. The intercept of the line on the W-axis is a measure of the shearing stress, and its slope S is a measure of viscosity. Converting to pipe-flow constants in the units above given

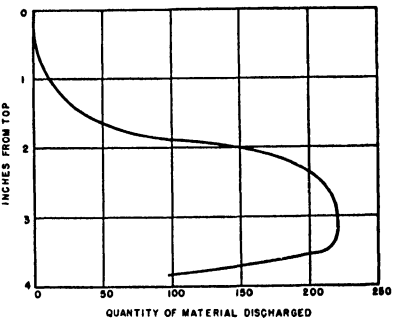


FIGURE 105. AMOUNT OF MATERIAL DISCHARGED THROUGH VARIOUS SECTIONS OF A 4-INCH PIPE.

$$\mu' = 0.0035S$$

$$\tau = 0.0020W$$

Eq (18-28) was tested by Babbitt and Caldwell and found to apply to streamline-flow conditions. Between this type of flow and turbulent flow there is an indeterminate region. The lower and upper limits of this region are given by the following equations:

$$v_l = \frac{1000\mu' + 103\sqrt{94(\mu')^2 + D^2\tau\varphi_0}}{D\varphi_0} \qquad \text{Eq (18-29)}$$

$$v_u = \frac{1500\,\mu' + 127\,\sqrt{140\,(\mu')^2 + D^2\tau\varphi_0}}{D\varphi_0} \qquad \text{Eq (18-30)}$$

where v_l and v_u refer to the lower and upper limits of velocity within the critical region of flow.

Wilhelm and his co-workers (1939) made a study of cement-rock suspensions, 92 percent of which passed a 200-mesh sieve. These investigators obtained the following general equation, based on experiments with $^3/_4$-, $1^1/_2$-, and 3-in. pipes:

$$\frac{\Delta P}{L} = k\,\frac{v^{0.2}}{D} \qquad \text{Eq (18-31)}$$

where ΔP is the pressure-drop through length of pipe L, D the diameter of the pipe, and k a constant which is a function of the nature and concentration of the material transported. Note that the pressure-drop varies as the one-fifth power of the velocity, and inversely as the diameter. Eq (18-31) was determined with two mixtures of rock-cement which were 54 and 62 percent by weight; and the range of velocities studied was from 0.3 to 4 ft per sec. The Fanning friction factors (Eq 18-26) were given by the following equations: For the 54 percent suspension, $f = 0.8v^{-2}$; and for the 62 per cent suspension, $f = 1.5v^{-2}$. In these equations the velocity is expressed in ft per sec; the equations do not apply for velocities above 4 ft per sec, when turbulent motion starts.

Problems involving head loss for thick suspensions of fine materials at velocities greater than calculated by Eq (18-30)—that is, for flow in the turbulent region—can be solved by the usual friction charts or formulas developed for the flow of water, after making due corrections for the density of the suspension.

PNEUMATIC TRANSPORTATION

In Chapter 2, Eqs (2-9) to (2-11) giving the terminal velocity of particles may be used to calculate the velocities required to move such particles through vertical pipes. The velocities calculated by these formulas are those required to support the particles in a vertical fluid stream. Motion in the direction of the moving stream will be imparted when the velocities exceed the calculated amounts. Little is known with regard to velocities for horizontal transportation of particles. In general, velocities capable of moving particles vertically will be more than sufficient for horizontal transportation.

In this discussion we confine ourselves to two types of transportation: (*a*) Transportation of low concentrations, as given in Eqs (2-9) to (2-11)

and Eq (17-28); and (*b*) transportation of high concentrations of materials, commonly referred to as pneumatic conveying. The former entails little difficulty in design of equipment, but the latter is complicated by many factors and is not well understood

TRANSPORTATION OF LOW CONCENTRATIONS

While the equations cited above may be used with considerable certainty of success, we must remember that the usual particles encountered in industry are by no means regular in shape, nor do they move in straight lines. The continual rebound of particles in pipes, which often brings them to rest, requires that more accurate equations be obtained. The author (1932, 1942) developed the following equations for transporting materials having specific gravities less than 3.0. For horizontal transportation

$$v = 6000 \left(\frac{\rho}{\rho + 1}\right) d^{3/4} \quad \text{Eq (18-32)}$$

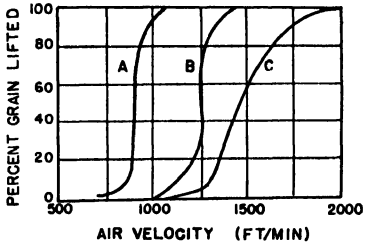


FIGURE 106. VELOCITIES REQUIRED TO LIFT VARIOUS GRAINS. (*A*) OATS; (*B*) WHEAT; (*C*) CORN

and for vertical transportation

$$v = 13{,}300 \left(\frac{\rho}{\rho + 1}\right) d^{3/4} \quad \text{Eq (18-33)}$$

where v is the velocity in ft per min, ρ the specific gravity of the particle, and d the diameter in inches of the largest particle to be transported.

Lifting Velocities for Grain—H. R. Brown and J. O. Reed (1926) made tests to determine the air velocity required to lift grains of oats, wheat, and corn. One hundred kernels of the various grains were uniformly distributed in a single layer on a screen placed at the intake of a vertical air duct. By controlling the air volume they were able to determine the percentage of material moved at various velocities. The data obtained in these studies are shown in Figure 106.

It will be seen from the data presented in this figure that lift begins at the following velocities: for oats, 685 ft per min; for wheat, 986 ft per min; and for corn, 1070 ft per min. All grains are moved when the velocities for the same grains are 1050, 1300, and 2000 ft per min, respectively.

TRANSPORTATION OF HEAVY CONCENTRATIONS

The movement of large quantities of fine material, especially grains, was studied by Cramp (1925). The analysis given here closely follows that developed by him. While theoretical developments were given by M. Jennings (1940) and Gasterstädt (1924), no practical comprehensive work is available other than that of Cramp. We begin with the generalized energy equation

$$F = \alpha (v_a - v)^2$$

where α is a constant having the dimensions $[ML^{-1}]$, and $(v_a - v)$ is the relative velocity of the air with respect to that of the particle. The value of α depends on the shape, density, and other characteristics of the particle. If $F = w$ = the weight of a single particle,

$$v_a - v = \sqrt{\frac{w}{\alpha}} \qquad \text{Eq (18-34)}$$

and this is the terminal velocity as discussed in Chapter 2. We see from this equation that $\sqrt{w/\alpha}$ measures the vertical speed at which the

TABLE 76—VALUES OF v_a, v, AND $(v_a - v)$ FOR MANITOBA WHEAT MOVING IN A VERTICAL PIPE. (VELOCITIES EXPRESSED IN M PER SEC)

Air velocity in pipe (v_a)	Particle velocity in pipe (v)	Value of ($v_a - v$)
10	1.5	8.5
11	2.5	8.5
13	3.2	9.8
14	5.2	8.8
14	6.1	7.9
14	6.0	8.0
15	5.3	9.7
16	6.7	9.3
16	8.7	7.3
18	9.0	9.0
18	9.0	9.0
19	10.0	9.0
19	9.4	9.6
20	10.2	9.8

air must slip past the particles in order to support them. Values of v_a and v given by Cramp are presented in Table 76. It will be seen that although the velocity of the air increases, the velocity of the particle does not increase in the same proportion Experimentally, Cramp

found the value of α for Manitoba wheat having a bulk density of 52 lb per cu ft was 0.00052 in cgs units.

Regarding the horizontal velocity of particles in pipes, the following experiment was made: Wheat was introduced into the positive side of the duct connecting the fan, and the material was blown from the open end. The distance at which the grain struck the floor was measured from this end. Tests with dry wheat showed that the extreme values of the ratio v/v_a were 0.48 and 0.41, but the bulk of the material gave a value of 0.47. If v_h denotes the horizontal velocity it may be represented with a fair degree of accuracy by the equation

$$v_h = v_a\left(\frac{\alpha - \sqrt{\alpha\psi}}{\alpha - \psi}\right) \qquad \text{Eq (18-35)}$$

where ψ is a constant to be determined by experiment. For Manitoba wheat $\psi = 0.0005$. Note that if l is the distance from the discharge end of the pipe to the point where the grain touches the floor, and if H is the height of outlet above the floor, then $v_h = l/\sqrt{2H/g}$.

Pressure Loss—The equation obtained by Cramp, giving the relation between the various factors thus far discussed is

$$\Delta h = \frac{20.7\,W}{A}\left[\frac{v}{g} + \frac{L}{kv}\left(1 + \frac{mv^2}{\sqrt{W}}\right)\right] + \frac{0\ 0011\ v_a^2 L}{D} + 0.00092\ v_a^2 \qquad \text{Eq (18-36)}$$

This equation is composed of the following forces, with the following units:

1. Δh = pressure-drop from end of pipe D cm in diameter to any point along the pipe L meters distant expressed in cm Hg per sq cm of pipe area.
2. $20.7\ WL/kv$ = force required to support the weight of material in the vertical section of the pipe, where W is the weight of material conveyed in tons per hr, k is a constant to allow for acceleration period of grain in the pipe, and v is the velocity of the material in m per sec given by the equation

$$v = \frac{v_a - \sqrt{\frac{w}{\alpha}\left[\frac{m}{\sqrt{W}}\left(v_a^2 - \frac{w}{\alpha}\right) + 1\right]}}{1 + \frac{m}{\sqrt{W}}\cdot\frac{w}{\alpha}} \qquad \text{Eq (18-37)}$$

where v_a is the velocity of the air in m per sec, w is the average weight of a single grain of wheat, and m is a pipe friction constant = 0.018 for wheat.

3. $20.7\ Wv/g$ = force required to accelerate the material in the pipe, and g is the gravitational constant = 9.81 m per sec.

4. $20.7\sqrt{W \cdot L\, m\, v/k}$ = force required to overcome the friction on the walls of the pipe

5. $0\ 0011\ v_a^2 L/D$ = force required to overcome air friction in the pipe.

6. $0.00092\ v_a^2$ = force required to accelerate the air in the pipe.

From the above we see that the first force is equal to the sum of forces (2) to (6), inclusive. The constant m may be set equal to zero whenever it cannot be determined and an approximate solution for v obtained. Of course the term A is the area of the pipe.

Design of Nozzle—Cramp has given an equation for the design of a nozzle for raising material into the pipe system. For each type of nozzle there is the relation

$$\zeta = \frac{W}{A_n\, v_n}$$

where ζ is a nozzle constant, A_n the area of the nozzle in sq cm, and v_n the velocity of the air through the nozzle in m per sec. Since

$$v_n = \frac{A}{A_n} \cdot \frac{h}{h_n} \cdot v_a \quad \text{(approximately)}$$

where h_n is the pressure at the nozzle, and h the pressure at any section A of the pipe, we obtain

$$\frac{W}{A} = \zeta \left(1 - \frac{h_n - h}{h_n}\right) v_a \qquad \text{Eq (18-38)}$$

Hence for maximum effectiveness it is necessary to use a nozzle with a value of ζ as large as possible. Cramp found that ζ for various nozzles ranged from 0.017 to 0.05, and was generally lowest for a vertical circular nozzle. He also recommended a value of v_n just small enough to lift the material. It must therefore exceed $\sqrt{w/\alpha}$, and preferably should be about $1.3\sqrt{w/\alpha}$.

Problems

1. Calculate the critical tractive force on a flume bed of sand whose average diameter is 0.033 in. and Kramer's modulus 0.44. Take the specific gravity of sand as 2.65.

2. Given a flume with a sandy bed whose particles are 0.59 mm and Kramer's modulus 0.28. Using Eq (18-16) and taking $n = 0.0118$, $L_c S_c = 0.00011$, $k = 0.00063$, and $m = 1.6$, compute the rate of bed-load movement in lb per ft width of flume per hr.

3. The flow of water in a sandy-bottom flume depends upon many factors. Assume that the following factors are involved: The average velocity over the cross section v, the hydraulic radius R, the acceleration of gravity g, the slope of the flume S, the density and viscosity of the fluid ρ and μ, respectively, and a

roughness coefficient of the bed material n, which may be taken as proportional to the diameter of the sand grains. From the function

$$f(v, R, g, S, \rho, \mu, n) = 0$$

indicate by use of Buckingham's Π-theorem a possible relationship of the variables involved. *Note:* The function may be replaced by

$$\phi(\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

as indicated in the text, where the π's are dimensionless quantities. Obviously, for π_1 we may write Froude's number $\pi_1 = v^2/Rg$ and for π_2, Reynolds number $\pi_2 = VR\rho/\mu$. Thus, we have included all variables except n and S, the coefficient of roughness and the slope. If the former is taken as proportional to the average diameter of the sand particles composing the bed, then a possible dimensionless ratio for π_3 is d/R. Since the slope is dimensionless, we may write

$$\phi(\mathrm{F}, \mathrm{R}, \mathrm{n}, \mathrm{S})$$

Hence, derive a possible expression for n.

Determine the dimensions of n in Manning's equation

$$v = \frac{KR^{1/3}\,S^{1/2}}{n}$$

Note: In the above discussion, assuming F to be the most important factor, write

$$S = \frac{k'\,v^2}{gR}$$

where $k' = \varphi(R, n)$, a dimensionless quantity. Substitute this value of S in the above equation.

4. Using the data of Problem 1 and assuming the particles to be spherical, compute the critical tractive force by means of Chang's equation.

5. What is the velocity required to transport vertically a mixture of particles the largest of which is 1 mm in diameter, and whose density is 2 1?

CHAPTER 19

DUST CLOUDS

This chapter deals with the motion of masses of particles over extended terrain.* Thus far we have considered what may be termed "transport" in channels and pipes (Chapter 18). The more general problem of dust cloud movement and formation forms the discussion of the present chapter.

The dissemination of particles from point, line, or areal sources by winds has both economic and climatic aspects. Thus, the smoke from industrial chimneys not only causes wastage by its very uncleanliness, but in many cities it contributes to formation of heavy fogs. Such conditions are trifling compared to the magnitude of dust and sand storms. These are large-scale phenomena affecting terrain and habitation for thousands of square miles. What causes sand storms and what motivates them? We cannot give all the answers, of course, but the behavior of such storms properly comes within the scope of micromeritics. The laws governing them are those we have sketched in preceding paragraphs and are natural consequences of them.

Because smoke clouds may be regarded as arising from point or line sources, we shall begin this chapter with a treatment of them. Dust and sand storms are more complicated phenomena arising from areal sources. The former are, in a sense, limiting cases of the latter, although we cannot generalize as yet. There are certain important physical differences, which will be indicated as the subject matter develops.

SMOKE CLOUDS

Diffusion Equation—We shall now proceed to discuss the general equations applying to a few special cases. Consider first a line source of particle emission in the yz-plane, at a height z above the xy-plane; then any occurrence in the xz-plane through the line source will be the same in any other xz-plane. In other words, the problem is reduced to a two-dimensional analysis. If the particles are subject to a fluid velocity in the x-direction equal to v_x, and if the particles are small enough that

* The reader is referred to Chapter 8 for fundamental theory relating to diffusion of particles by air turbulence.

this velocity tends to make the concentration gradient in the x-direction small as compared to that in the z-direction, we may by a simple procedure establish the necessary relation to compute the concentration everywhere in the space involved. Consider a small rectangle whose coordinates are (x,z), $(x + dx, z)$, $(x, z + dz)$, and $(x + dx, z + dz)$. Considering only the diffusion through the sections represented by the lines parallel to the xy-plane at a distance dz apart, we have through the lower boundary a gradient $-\mathfrak{D}_e \dfrac{\partial C}{\partial z} dx$, and through the upper one $\mathfrak{D}_e\left[\dfrac{\partial C}{\partial z} + \dfrac{\partial}{\partial z}\left(\dfrac{\partial C}{\partial z}\right)\right] dx$, so that by adding the loss through the element $\mathfrak{D}_e \dfrac{\partial}{\partial z}\left(\dfrac{\partial C}{\partial z}\right) dx$ the rate of increase in concentration in the element is clearly $(\partial C/\partial t)dx$ and this must equal the loss. Hence

$$\frac{\partial C}{\partial t} = \mathfrak{D}_e \frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right)$$

But $\mathfrak{D}$ is clearly a function of z, in accordance with Eq (8-33)

$$\frac{\partial C}{\partial t} = k_1 v_x \frac{\partial}{\partial z}\left(z\frac{\partial C}{\partial z}\right)$$

Also, since $v_x = dx/dt$

$$\frac{\partial C}{\partial x} = k_1 \frac{\partial}{\partial z}\left(z\frac{\partial C}{\partial z}\right) \qquad \text{Eq (19-1)}$$

Diffusion Pattern from a Continuous Point Source—The distribution of particles from a point source in a moving fluid can be determined provided we assume that the concentration gradients in the direction of fluid motion are small compared to those at right angles to it. If C_s is defined as the concentration of particles over a unit area of a plane horizontal surface downstream and to one side of the mean path of the diffusing stream from a point source, then the equation of diffusion at any point x downstream and at a distance y from the mean path is

$$\frac{\partial C_s}{\partial x} = \frac{\mathfrak{D}_e \partial^2 C_s}{v_x \partial y^2} \qquad \text{Eq (19-2)}$$

where v_x is the fluid velocity and $\mathfrak{D}_e$ the eddy diffusion coefficient. At $y = 0$, along the mean path, the concentration C_s must be a maximum and must diminish as we recede in the y-direction on either side of it. If C_t defines the concentration of particles issuing from the source in

unit time, then C_t/v_x must be constant since it is the concentration per unit length of diffusing particles and $[C_t/v_x x] = [C_s]$, dimensionally. Subject to these conditions and noting that $\mathfrak{D}_e = kv_x\xi$, Eq (8-33), and $\bar{\xi} = kx$, Eq (8-35), we obtain as a solution of the above equation

$$C_s = \frac{C_t}{\sqrt{2\pi}kv_x x} \cdot \exp\left(-\frac{y^2}{2k^2x^2}\right) \qquad \text{Eq (19-3)}$$

Thus the distribution curve is Gaussian, which conforms to observed results. (See, for example, Kalinski and Pien, 1944.)

Distribution from a Point Source—Eq (19-1) has been integrated by Bosanquet and Pearson (1936) for both point and line sources. The solution for the latter, when the source is at ground level, yields

$$C_s = \frac{C_t}{k_1 v_x x} \cdot \exp\left(-\frac{z}{k_1 x}\right) \qquad \text{Eq (19-4)}$$

where C_s is the concentration at z expressed in weight of particles above a unit area and C_x the amount of material issuing from a unit length of source in unit time. These units may appear confusing but it is obvious that if v_x is in mi per hr, and z and x are in miles, C_t would be in weight units per hour and C_s in corresponding weight units per sq mile.

In the case of smoke we are chiefly concerned with determining the concentrations C_{os} at ground level. Bosanquet and Pearson obtained a solution for this case for a line source of height H,

$$C_{os} = \frac{C_t}{k_1 v_x x} \cdot \exp\left(-\frac{H}{k_1 x}\right) \qquad \text{Eq (19-5)}$$

where H, of course, must correspond to the units of x. This equation gives the ground-level concentration in weight per unit area.

Both Eq (19-4) and Eq (19-5) give the concentration along the mean centerline of travel of the cloud in the x-direction, subject to the condition that $\int_0^\infty C dz = C_t/v_x$. From the second of these equations it is clear that C_{os} is a maximum when $x = H/k_1$. Thus if k_1 is taken as 0.05, C_{os} is maximum at about 20 chimney heights downstream. For large values of x

$$C_{os} = \frac{C_t}{k_1 v_x x} = \frac{C_t}{\mathfrak{D}_e} \qquad \text{Eq (19-6)}$$

so that the concentration falls off inversely as the distance.

If now we concern ourselves with the concentration at ground level on either side of the cloud axis a distance y, we may substitute the value of $C_s v_x$ in Eq (19-3) for C_t in Eq (19-5), obtaining

$$C_0 = \frac{C_t}{\sqrt{2\pi}kk_1 v_x x^2} \cdot \exp\left(-\frac{H}{k_1 x} - \frac{y^2}{2k^2x^2}\right) \qquad \text{Eq (19-7)}$$

where C_{os} now becomes the weight concentration of particles per unit volume, C_0. Along the mean axis of cloud movement, the concentration per unit volume is therefore

$$C_0 = \frac{C_t}{\sqrt{2\pi k k_1 v_x x^2}} \cdot \exp\left(-\frac{H}{k_1 x}\right) \qquad \text{Eq (19-8)}$$

Figure 107 is taken from work by Bosanquet and Pearson to whom we are indebted for much of the mathematical treatment of the smoke problem we have thus far presented. The lower curve is a plot of Eq (19-8), where the following values apply: $k = 0.08$, $k_1 = 0.05$, $H = 100$ m, $v_x = 5$ m per sec; C_0 is in mg per cu m and C_t in metric tons per day. In the figure values of the ratio, C_0/C_t are plotted against distance.

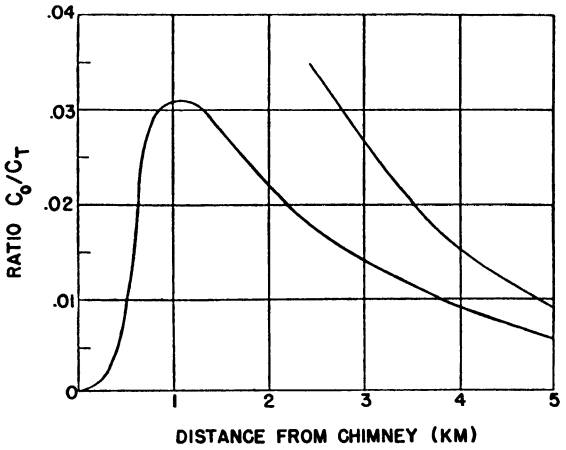


FIGURE 107. CONCENTRATION OF SETTLED SMOKE ISSUING FROM A TALL STACK. NOTE POSITION OF MAXIMUM CONCENTRATION DOWNWIND. DATA FROM BOSANQUET AND PEARSON (1936).

The upper curve in the figure applies to a source at ground level, that is, $H = 0$. Note that in the first case the concentrations are low within $^1/_4$ km of the stack and that a maximum occurs at about 1 km. However, at great distances the difference between the two types of sources disappears. Thus the effect of stack height is to reduce the concentrations of smoke in the immediate region of the stack.

The variation of concentration over long periods of time due to changes in wind direction can be obtained from Eq (19-8). For, let the fraction of time (say a year) during which the wind direction falls

within an arc θ be $a\theta$, and the mean velocity be $\bar{v}$. Then the average value of C_0 for the year will be

$$C_0 = \frac{aC_t}{\sqrt{2\pi k k_1 v x^2}} \cdot \exp\left(-\frac{H}{k_1 x}\right) \qquad \text{Eq (19-9)}$$

where the mean value of a is $^1/_{2\pi}$ and the variation with direction may be obtained from weather bureau observations for the locality in question.

Line Source—The equation governing concentrations from a line source, such as, for example, a city whose length is very great as compared to its width, is the same as Eq (19-5), where now C_t is defined as the weight of particles emitted from the line per unit length per unit time when v_x is the velocity of the wind at all points along the line.

Distribution Allowing for Settling—The equations given above apply generally for particles below 20 μ in size (approximately). For larger particles certain corrections must be applied to the diffusion equations to allow for settling. If v_m is the terminal velocity of a particle of given diameter, then Eq (19-1) becomes

$$\frac{\partial C}{\partial x} = k_1 \frac{\partial}{\partial z}\left(z \frac{\partial C}{\partial x}\right) + \frac{v_m}{v_x} \cdot \frac{\partial C}{\partial z} \qquad \text{Eq (19-10)}$$

subject to the condition

$$\frac{d}{dx}\int_0^\infty C dz = -C_{os}\frac{v_m}{v_x}$$

Eq (19-10) cannot be integrated in convenient form, but a solution has been given by Bosanquet and Pearson for small values of $v_m/k_1 v_x$. Since C_t is expressed as weight per unit length per unit time, we have for a line source

$$C_{os} = \frac{C_t(1.78H/k_1)^{v_m/k_1 v_x}}{k_1 v_x x^{(1+v_m/k_1 v_x)}} \cdot \exp\left(-\frac{H}{k_1 x}\right) \qquad \text{Eq (19-11)}$$

For a point source, taking account of the distribution on either side of the mean direction of motion

$$C_0 = \frac{C_t(1.78\ H/k_1)^{v_m/k_1 v_x}}{\sqrt{2\pi} k_1 k v_x x^{(2+v_m/k_1 v_x)}} \cdot \exp\left(-\frac{H}{k_1 x} - \frac{y^2}{2k^2 x^2}\right) \text{Eq (19-12)}$$

Note of course that as before C_{os} is an areal ground-level concentration along the mean axis of cloud motion, and that C_0 is the ground level concentration at any point xy expressed in weight per unit volume. When $v_m = 0$, Eqs (19-12) and (19-7) are identical.

EFFECT OF FLUID MOVEMENT ON SETTLED PARTICLES

Superficial Motion of Particles If loose soil or sand is disturbed by wind, we may readily hypothesize two possibilities regarding the subsequent motion of particles composing them, (*a*) a diffusion of fines as discussed in Chapter 8, and (*b*) a superficial motion of the particles. The latter may further be considered as composed of two kinds of motion, the first consisting of particles which rebound on settling and are carried on in this manner, and the second consisting of a creeping motion of the soil or sand layer caused by the drag of the fluid and by partial impact. A singular characteristic of superficial motion with which we shall be concerned is the large size of particles which are dislodged and carried along by the fluid. This effect is in part explained by the extra-surface projected by such particles so that they receive a greater amount of drag (Chapter 18), but as we shall show later there are other forces at work which help contribute to the motion of these particles.

For example, small particles moving downwind possess considerable energy. These particles have well-defined trajectories and when they reach their respective termini, this energy may be imparted to a larger particle which the smaller ones strike. This process may be repeated often enough so that the large particle is dislodged and set in motion. The bounding motion of particles near a surface immersed in a moving fluid is called "saltation."

In the following paragraphs we shall devote much attention to the effect of fluid motion over a bed composed of particles. We shall have occasion to discuss not only the theory regarding superficial motion of particles, but also the principles underlying true suspension of particles in a moving fluid.

Dynamics of Saltation—The concept of "susceptibility" of particle motion has been introduced by Bagnold (1943). It is the ratio of two forces, that of the wind on the particle to that of gravity. When the two forces are equal, the particle moves at constant velocity. If the force of gravity is greater, the particle motion is accelerated, and conversely if it is less.

The resistance encountered by a particle in a fluid in a direction opposite to the motion has been shown in Chapter 2 (Eq 2-12) to be

$$\mathcal{R} = \frac{1}{2} C_R \rho_0 A v^2$$

where C_R is the coefficient of resistance, ρ_0 the density of the fluid, A the projected area of the particle, and v the velocity of the fluid. The force of gravity is obviously $mg = \pi d^3 \rho g/6$ where m is the mass of the particle having a diameter d and density ρ and g is the acceleration constant of

gravity. If **B** is termed the susceptibility (Bagnold's number) we have for a spherical particle

$$\mathbf{B} = \frac{3}{4} C_R \frac{\rho_0 v^2}{d\rho g} \qquad \text{Eq (19-13)}$$

Since **B** varies inversely with particle diameter, we see that with small particles the effect of the wind is greater than that of gravity. Hence we conclude that such particles conform nearly to the motion of the fluid stream in which they are carried. Let us examine Eq (19-13) more closely. The constant C_R is obviously a function of Reynolds number which for streamline motion is equal to 24/**R** and for turbulent motion has a constant value of 0.4. Hence for Bagnold's number we may write:

$$\text{for streamline motion—}\mathbf{B} = \frac{18\mu v}{\rho d^2 g}$$

$$\text{for turbulent motion—}\mathbf{B} = \frac{3}{10} \cdot \frac{\rho_0 v^2}{\rho d g}$$

where μ is the coefficient of viscosity which enters the Reynolds relation $\mathbf{R} = \rho_0 dv/\mu$.

We see that for streamline motion the susceptibility varies inversely as the square of the diameter and directly as the viscosity and velocity of the fluid. Hence on a small particle the effect of gravity is offset by the viscosity of the fluid, while the converse is true for large particles. When the motion is turbulent the effect noted above is further enhanced by the fact that the value of **B** varies as the square of the velocity so that small particles conform even more closely to the motion of the fluid than was at first suspected.

Angle of Particle Fall—After a particle has reached the top of its flight path it begins to fall. If we assume the particle to be large enough so that the terminal velocity of fall is quickly reached and that near the end of its motion it has attained the velocity of the fluid, then the particle will hit the ground at an angle with the horizontal such that

$$\tan \alpha = \frac{\text{terminal velocity of particle}}{\text{velocity of fluid}} = \frac{v_m}{v}$$

where α is the angle made with the horizontal. The expression is approximate since it is never known when the terminal velocity of a particle is reached, unless, of course, it reaches the top of its flight curve at a considerable distance from the ground. On the other hand, if the particle does not reach a sufficient height so as to attain its terminal velocity before hitting the ground, neither will it have attained the horizontal speed of the fluid. Hence, as Bagnold points out, the angle α remains

fairly constant over a wide range of conditions. For desert sands it is estimated that α is between 10 and 16 deg.

Impact and Fluid Threshold—Two important concepts have been introduced by Bagnold regarding the behavior of saltating particles which we shall explain briefly. Bagnold found in controlled experiments when particles were permitted to fall from the roof of his wind-tunnel on a layer of mobile sand that the former would saltate for a certain distance downwind. The energy of these particles is dissipated by forming small craters in the sand surface. As the wind intensity was increased in the tunnel, but not sufficiently so as to cause a movement of the mobile sand layer, the saltation of the falling particles was extended downwind. However, the saltation ceased the moment the flow of particles from the tunnel roof was stopped. This experiment indicated that at wind speeds below that necessary to disturb the sand surface, the saltation of particles is not perpetuated by the direct action of the wind but by the elastic rebound of the descending grains.

The upper limit of wind intensity at which saltation is perpetuated without the wind affecting the mobile sand surface has been termed the *impact threshold.* It is readily determined for a given sand by wind-tunnel experiments, by direct observation of the moving particles, or by the formation of a fine crater-like roughness on the once-smoothed sand surface throughout the length of the tunnel. The term impact threshold applies only when the saltation as above described extends indefinitely downwind.

When the wind intensity in the same tunnel is increased (after the fall of the particles has stopped) above the impact threshold, another critical limit is reached. This limit is difficult to observe because it is not well defined. It is that wind intensity at which a general movement of sand particles begins. At this limit, called the *fluid threshold,* the grains of sand gather speed in their movement downwind, and begin to saltate and give the appearance of a sand cloud.

When the fluid threshold has been reached, the sand movement is general. As we have stated, we then have in addition to saltation a surface "creep." This action is due to the impact of the falling particles. It is not, as might be supposed, induced by the drag of the wind, but is almost entirely an impact effect. Bagnold's experiments indicate that for desert sands having diameters ranging from 0.18 to 0.3 mm about $^1/_4$ of all the sand moved by a wind above the fluid threshold is accounted for by creeping.

Both saltation and creeping are characteristic motions of large particles. Finer particles may be swept upward by a turbulent fluid and kept in motion by it so that the effect of gravity is nullified. Such par-

ticles are said to be in suspension. They travel at the velocity of the fluid and hence offer no appreciable resistance to its motion. "Susceptibility" as we have defined it in an earlier paragraph might well be used as a modulus of suspension, although application of such a modulus, especially as regards its critical value, has yet to be undertaken on experimental data and observations.

WIND SPEED AND SURFACE ROUGHNESS

It is now well established that wind velocity varies logarithmically with height—at least to those heights of interest to us. It is also well estab-

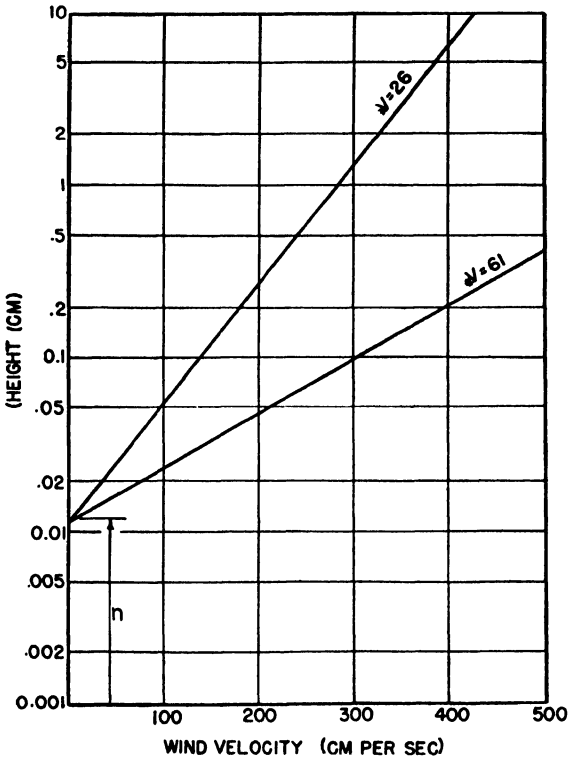


FIGURE 108. WIND VELOCITY DISTRIBUTION PLOTTED ON LOGARITHMIC HEIGHT SCALE.

lished that near any surface across which a fluid moves, there is a certain height within which the movement of the fluid is streamline. In other words, turbulence does not extend to the surface itself but to a height designated as the source point at which the transition from streamline to

turbulent motion theoretically takes place. The source point is readily determined if we plot the velocities corresponding to various heights above the surface on the semi-log grid. By extending the velocity-height line to zero-velocity, we see that it intersects the ordinate at a fixed height as shown in Figure 108. For a given surface, it will be found that all wind intensities when plotted as above intersect at the point shown in the figure.

The intersection is determined by the condition of the surface over which the wind (or any fluid we desire) is moving. It is a function of the surface. For particles, it has been found that the position of the point O is

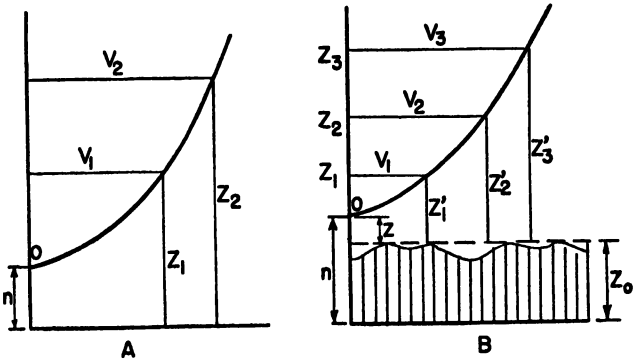


FIGURE 109. COORDINATE SYSTEMS REQUIRED FOR COMPUTATION OF n AND z_0.

about $^1/_{30}$ the scale of surface roughness. Thus if on a flat surface there is a layer composed of sand particles of diameter d, the point O is located $d/30$ above the flat surface.

The theoretical relation between velocity and end height has already been developed in Chapter 8 and we have exhibited a special form of this relationship in the first part of this chapter. Let us, however, return to the fundamental equation, Eq. (8-17)

$$v_2 - v_1 = \frac{1}{k_0}\sqrt{\frac{\tau}{\rho_0}} \ln \frac{z_2}{z_1}$$

in which motion in a specified direction is here assumed and the various terms are as already defined. If we set $v_1 = 0$, then z_1 becomes the distance from the surface to the point O of Figure 109, and this distance we shall denote by n, so that the above equation becomes for any given point

$$\bar{v} = \frac{1}{k_0}\sqrt{\frac{\tau}{\rho_0}} \ln \frac{z}{n}$$

or

$$\bar{v} = \frac{1}{k_0} {}_*v \ln \frac{z}{n} \qquad \text{Eq (19-14)}$$

where ${}_*v = \sqrt{\tau/\rho_0}$ is defined as the drag velocity, The term k_0 is von Kármán's turbulence constant. The value of k_0 is generally accepted as being between 0.2 and 0.4, although values outside of this range are not infrequent.

Note that ${}_*v$ which we have defined as the drag velocity $\sqrt{\tau/\rho_0}$ is also a measure of the rate of increase of velocity with *ln*-height, or is, in other words, a velocity gradient. It is the variation of velocity with height which causes surface drag as we have shown in Chapter 8. We may also state with reference to ${}_*v$ that the greater the strength of the wind (or fluid), the greater must be the inclination of the velocity-height line from the vertical as shown in Figure 108.

Variation of n with Surface Character—The position of the point O in Figure 109 which characterizes the roughness of the surface is most important. It is therefore necessary to go into some detail relative to the computation of its position; for it is readily apparent that if this can be done in terms of velocity-height data much can then be said regarding the diffusion of particles or other diffusible phenomena, as, for example, temperature and humidity. The position of n is easily computed from two wind velocity observations at z_1 and z_2. Since the velocity is known to vary logarithmically with height (Figure 109*A*)

$$z_1 = n \cdot \exp\,(bv_1)$$

$$z_2 = n \cdot \exp\,(bv_2)$$

or

$$\ln z_1 = \ln n + bv_1$$

$$\ln z_2 = \ln n + bv_2$$

from which we obtain

$$b = \frac{\ln z_1 - \ln z_2}{v_1 - v_2}$$

and

$$\ln n = \ln z_1 - \frac{\ln z_1 - \ln z_2}{v_1 - v_2} \cdot v_1$$

Thus if n is measured directly from the ground surface

$$n = z_1 \left(\frac{z_1}{z_2}\right)^{v_1/(v_2 - v_1)} \qquad \text{Eq (19-15)}$$

Often it is desirable to obtain some estimate of the height of the obstruction on which n depends. We have stated that if the surface is composed of particles of diameter d that the position of n is $d/30$ above the surface. Obviously this does not permit generalization and leaves much to be desired. Roughness depends upon so many obstacles forming the surface that the source point O in Figures 108 and 109*A* must be considered as an intergrated average of all the obstacles. Let us now refer to Figure 109*B* which shows the source point above an irregular surface whose height is z_0 above the ground. Our problem is to determine z_0 in terms of the velocity-height relationship. This may be done by making three velocity-height observations and eliminating the measure of roughness n. From the figure, we see that

$$n = z_0 + z$$
$$z_1' = z_1 - z_0$$
$$z_2' = z_2 - z_0$$
$$z_3' = z_3 - z_0$$

From these three points it is possible, from our equation involving the natural logarithm, height, and velocity to eliminate n and compute z_0. Thus using z_0 as our reference point

$$\ln (z_1 - z_0) = \ln n + bv_1$$
$$\ln (z_2 - z_0) = \ln n + bv_2$$
$$\ln (z_3 - z_0) = \ln n + bv_3$$

Subtracting

$$\ln (z_1 - z_0) - \ln (z_3 - z_0) = b(v_1 - v_3)$$
$$\ln (z_2 - z_0) - \ln (z_3 - z_0) = b(v_2 - v_3)$$

Dividing, these equations after changing signs lead to

$$\frac{v_3 - v_1}{v_2 - v_3} = \frac{\ln \dfrac{z_3 - z_0}{z_1 - z_0}}{\ln \dfrac{z_3 - z_0}{z_2 - z_0}} \qquad \text{Eq (19-16)}$$

from which z_0 may be computed by trial and error.

Effect of Roughness Change—If a wind is moving over a terrain with obstacles determined by the constant z_0 and if the wind suddenly encounters another terrain with obstacles determined by the constant z_0', there will be an immediate effect on the wind velocity near the source point. Eventually the effect will be imparted to succeeding layers upward, but a certain amount of time will be required before the full effect of the change will be transmitted to the whole mass of wind. In other words, the wind will have traveled a considerable distance before the ground change will have been fully developed within it.

The locus of a fully developed velocity change with height from the new point source as the wind moves downstream is called the *boundary layer*. Above this layer the effect of the change in roughness has not been completed. It appears from experiments that the boundary layer increases rather slowly in thickness as the wind moves downstream from the new obstacle. Some recent data show that the thickness at any given distance is proportional to the change in surface roughness and is independent of velocity.

Eventually the effect of a change in surface roughness after causing an immediate shift in the source point, will be to shift the velocity-height line on the semi-log grid until it is parallel to the original.

Further consideration of the general theory of surface roughness here given leads us to the following corollary: If the roughness *increases* suddenly, the velocity gradient over the surface at the point of change is greater than at any other point, simply because near that point it is checked while as we move upward the effect of the change has not been transmitted. Hence, near the source point the drag is greatest momentarily, and gradually decreases as the wind moves downstream. At the moment an obstruction is encountered the drag may be reduced by altering the tail profile of the obstacle downwind. By curving the rear of the obstruction gradually downward, the drag may be kept constant. The tendency of the wind to preserve a constant drag as it moves over an obstruction is in part responsible for the type of ripple profile found in desert regions and in stream bottoms.

Effect of Particle Movement on Drag Since the velocity-height relationship is altered by surface roughness, it must follow that once the movement of sand begins —that is, once the fluid threshold has been reached—the relationship is similarly altered. However, in this case we are confronted with a surface moving at some velocity relative to the fluid.

In reference to Figure 110, we have shown by dashed lines the velocity-height relationship for winds of different intensity moving over a surface of mobile sand particles. The impact threshold— that is, the velocity at which saltation of grains can be sustained downwind indefinitely—is indicated by the line marked ${}_{*}v = 19$.*

As the intensity of the wind is increased up to the fluid threshold ${}_{*}v = 22$ and above, the movement of sand becomes general At the fluid threshold and above the velocity-height relationship undergoes a marked change, as shown by the heavy lines in the upper part of the figure.

* The source point in this case, as Bagnold points out, is not a function of the sand diameter, but of the height of the craters produced by the bombardment of saltating particles.

Within 2 cm of the surface, the relationship is not strictly linear, but at greater heights the lines become straight. This is explained by the fact that the concentration of particles in motion is not uniformly distributed with respect to height.

The theoretical source point lies some distance above that for the

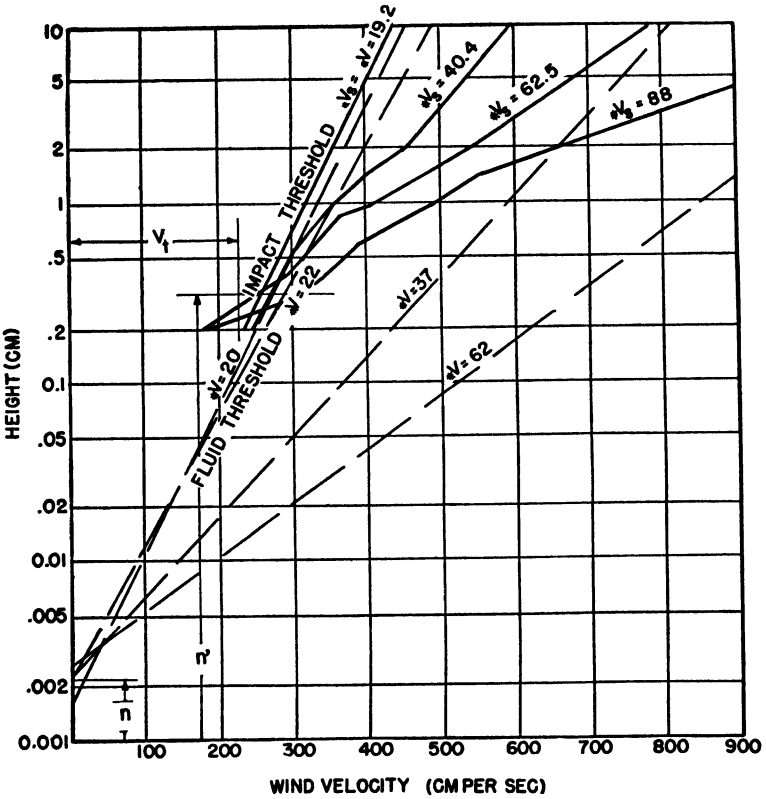


FIGURE 110. VARIATION OF WIND VELOCITY WITH HEIGHT. DASHED LINES ARE FOR CONSTANT DRAG VELOCITIES OVER A FIXED SAND SURFACE. SOLID LINES FOR A MOBILE SAND SURFACE, BAGNOLD (1943).

stationary bed, but is not constant for all wind velocities. Actually a point exists about 0.2 cm above the surface *on the impact threshold line* at which the velocity-height lines intersect, provided, of course, that the wind intensity exceeds this threshold. Note that in the case of general sand movement, increasing the wind intensity above the impact threshold actually decreases the velocity near the surface. Thus at 1 cm above a "fixed" immobile sand surface the velocity for a wind intensity

shown by the dashed line $_*v = 62$ is 9 cm per sec. For a mobile surface we have at the same height for the same wind intensity a velocity of 4.4 cm per sec. As we approach nearer and nearer the surface the difference becomes greater.

If for a moment we disregard the kinks in the velocity-height relationships for general sand movement shown in Figure 111 and assume the lines to be straight, we then have as the relationship applying to these lines

$$v = \frac{1}{k_0} {}_*v_s \ln \frac{z}{n_s} + v_x \qquad \text{Eq (19-17)}$$

where $_*v_s$ and n_s are the drag velocity and roughness coefficient applying to the altered conditions caused by the sand movement and v_t is the threshold velocity applying at the new source point. Thus $v - v_t$ is the relative velocity of the wind with respect to the sand.

The extension of the intensity lines from the focus so as to intersect the zero-ordinate permit us by means of Eqs (19-15) and (19-16) to compute the characteristics of a static "fixed" surface necessary to produce the result given by Eq (19-17).

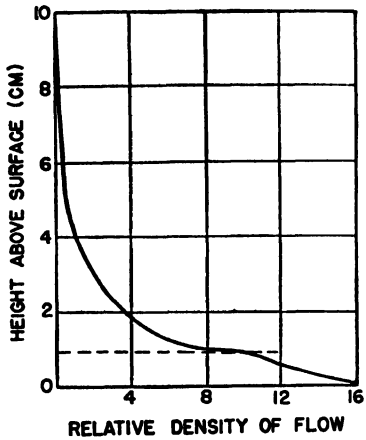


FIGURE 111. RELATIVE CONCENTRATION OF SAND PARTICLES ACCORDING TO HEIGHT ABOVE THE SURFACE. BAGNOLD (1943).

Effective Saltation—On the basis of the departure from the straight line relationships near the surface when general sand movement takes place (solid lines, Figure 110), Bagnold assumes that it is possible to replace the paths of all the particles by a *characteristic* grain path. This concept is similar to the computation of effective diameter by sedimentation techniques. However it must be remarked that the analogy is not perfect for in one case we are concerned with mass and shape while in the other with mass, resilience, angle of terminal contact (of particles in flight), and wind velocity. The usefulness of the concept lies in the fact that it can be applied to the computation of distance between ripples formed when wind blows over a mobile sand surface.

The kinks in the height-velocity relationship shown in Figure 110 are due to the variation in sand concentration with height. If such a concentration were plotted in the manner shown in Figure 111, it would be

found that more than 50 percent of the sand mass in motion lies within one cm or so of the surface. Moreover, as reference to the figure shows, this mean height corresponds in magnitude with the kinks in the velocity-height relationship shown in Figure 110. Bagnold has calculated the possible paths of a particle by assuming that the height of the kink or its equivalent position on the height-concentration curve, is a manifestation of the average height to which the saltating grains rise. His computations are based on the assumption that a particle after striking the surface rebounds vertically upward. By further assuming different values of the upward velocity of rebound he was enabled to extrapolate the distance traveled upward and the horizontal length of flight. These he found would correspond to the kinks in Figure 110 and to the distance between ripples observed in his wind-tunnel experiments. He concludes that ripples are manifestations of the travel of average sand grains downwind.

The results noted above apply to more or less uniform sand. Neither the kinks in the velocity-height curves of Figure 110 nor the spacing between ripples are clearly defined with mixed sands. This may be attributed to the different effective paths of various fractions.

Computation of Impact and Fluid Thresholds—The velocity necessary to sustain a particle in a fluid must be the same as that required to keep it in suspension, or, in other words, the terminal velocity v_m. Thus whether the forces acting on a stationary particle causing it to move are vertical or horizontal, they are encompassed in the equations developed in Chapter 2, Eqs (2-9) to (2-11). Since we are generally interested in conditions corresponding to turbulent motion we may write

$$*v_0 \propto v_m \propto \left(\frac{\rho - \rho_0}{\rho_0} \cdot d\right)^{1/2} = K \left(\frac{\rho - \rho_0}{\rho_0} \cdot d\right)^{1/2}$$

which is identical to Eq (2-10). The constant K is experimentally determined and for irregular particles it has a value of 2.5 at the impact threshold and 3.1 at the fluid threshold. These values differ very much from the value of $K_T = 50$ given in Table 2 for irregular particles, but we must remember that the moments of horizontal forces acting on a particle may first cause the particle to roll. If we place the above expression for $*v$ in Eq (19-14), we obtain an expression for the impact and fluid thresholds in the form

$$v_0 = \frac{K}{k_0}\left(\frac{\rho - \rho_0}{\rho_0} \cdot d\right)^{1/2} \ln \frac{z}{n} \qquad \text{Eq (19-18)}$$

so that we have a relation involving the densities of the particle and fluid

and the particle diameter. The value of K for water has been found to be about twice that for air.*

The value of v_t in Eq (19-17) is derivable from Eq (19-18) if in the latter we place $z = n_s$, so that

$$v_t = \frac{K}{k_0}\left(\frac{\rho - \rho_0}{\rho_0} \cdot d\right)^{1/3} \ln \frac{n_s}{n} \qquad \text{Eq (19-19)}$$

We have assumed that v is determined by conditions of turbulence. However, this is not always the case and it is an observed phenomenon that fine particles are not necessarily set in motion with the same fluid intensities needed to move the larger particles. Greater intensities are often necessary for fine than for large particles. This is due to the fact that when the surface is composed of small particles the drag is distributed over the whole surface. A large "exposed" particle, on the other hand, may carry not only the drag on the area it occupies but also for an area about it as much as twenty times its projected area.

These differences in the state of flow due to particle size cause the value of v_0 in Eq (19-18) to vary as shown in Figure 112. For large

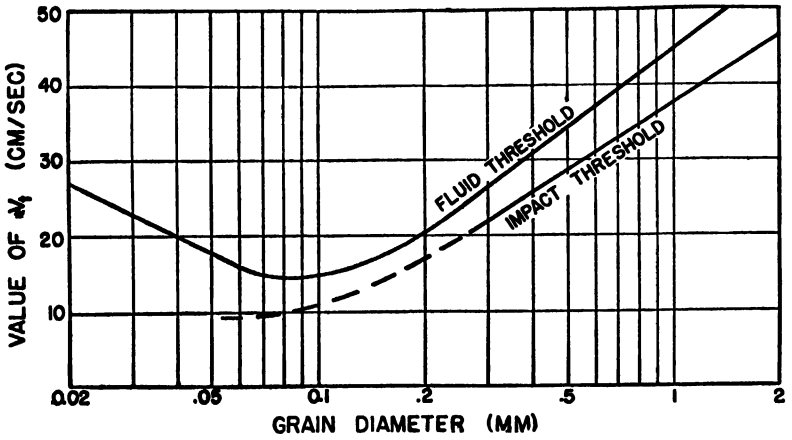


FIGURE 112. VARIATION OF FLUID AND IMPACT THRESHOLDS WITH GRAIN DIAMETER, BAGNOLD (1943).

grains the coefficient K is seen to be constant, but there is a certain particle diameter, about 0.2 mm, at which this is no longer true. (For particles in water this occurs at a diameter of 0.6 mm.) A critical diameter

* A detailed account of the reasons for the different behavior of particles in air and water is given later in this chapter. The equations thus far developed are general and apply to all fluids. There are real differences between fluids which can generally be attributed to the density factor $(\rho - \rho_0)/\rho_0$.

is reached in the case of air when the particles are 0.08 mm in size. Thereafter the value of ${}_*v_0$ increases. The variation of threshold velocity gradient with grain size shows the same similarity when the fluid is water, although the critical diameter is larger, being about 0.2 mm.

Bagnold has determined the Reynolds number which fixes either the critical diameter or the threshold velocity gradient. He has found that when

$$\frac{{}_*v_0 z_0}{\nu} > 3.5$$

the particle behaves as though it were isolated and eddies are thrown off at the lee face. The term z_0 is, of course, some function of the particle diameter d. When Reynolds number is less than 3.5, that is, when either the grain diameter or the threshold velocity gradient is smaller, the surface is smooth and the flow streamline.

When the wind moves over a bed composed of particles of mixed sizes, those most exposed will be moved first. Much therefore depends on what the bed-surface is like. If the sand is spread out so that particles of all sizes are equally exposed, then those particles will move first for which v_0 is a minimum. The threshold velocity gradient as we have employed the term, refers to that value of v_0 required to move the particles of predominant diameter. As the grading action continues, larger and larger particles remain on the bed-surface, and hence for these the threshold velocity gradient must increase.

SAND FLOW

The equations governing the amount of sand transported superficially are easily derived. Let v_1 be the velocity with which a particle rises after impact by another saltating particle and v_2 its final velocity after it has traveled a distance x downstream. Since v_2 is very large as compared with v_1, the amount of momentum lost in saltation per unit length of path by a mass of sand q of unit width is

$$q\frac{v_2 - v_1}{x} = \frac{qv_2}{x}$$

This movement of sand is resisted by the drag. Denoting this drag by $\tau_s = \rho_0 {}_*v_s^2$, we obtain

$$\frac{qv_2}{x} = \tau_s = \rho_0 {}_*v_s^2$$

Now x, the distance moved by the particle, is proportional to the velocity

gradient, so that we may write the quantity of sand moved by saltation as

$$q = \zeta' \rho_{0*} v_s^3 \qquad \text{Eq (19-20)}$$

Hence the amount of sand moved varies as the cube of the velocity gradient. For uniform sand having a diameter of 0.25 mm, the constant ζ' has a value of 8.1×10^{-4} in cgs units. The term q represents the amount of sand moving by saltation in a lane of unit width past a given point in one second. If the total sand moved were experimentally measured, a certain portion of the total would be that due to creep. The quantity of sand moved by creeping, as we have stated earlier, is about one-fourth that moving by saltation. Hence using this factor we may predict from Eq (19-20) the total amount of sand moving.

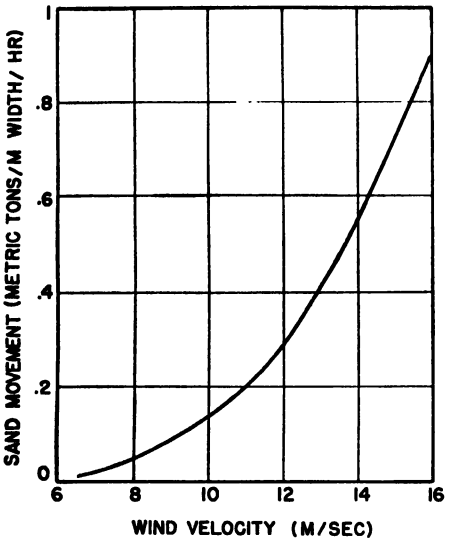


FIGURE 113. EFFECT OF WIND VELOCITY MEASURED AT A HEIGHT OF ONE M ON THE FLOW OF SAND. SAND DIAMETER 0.25 MM. DATA FROM BAGNOLD (1943).

It has been found from experiment that the amount of sand moved also varies as the square root of the particle diameter d, or

$$q = \zeta d^{1/2} \rho_{0*} v_0^3 \qquad \text{Eq (19-21)}$$

where ζ is another constant having the following values when d is measured in cms:

0.97×10^{-2} for nearly uniform sand
1.1×10^{-2} for dune sand (naturally graded)
1.8×10^{-2} for sand with wide range of particle size

If we wish to convert Eq (19-21) so as to obtain the sand flow from a wind measurement at a given height, we may use Eq (19-17). If we take 1 m as the standard height of measurement and the roughness coefficient $n_s = 1$ cm, then using logarithms to the base 10, $\log z/n_s = 2$, we obtain

$$q = 6.85 \zeta d^{1/2} \rho_0 (v - v_t)^3 \times 10^{-4} \qquad \text{Eq (19-22)}$$

The curve shown in Figure 113 gives the sand movement in metric tons per hr per m width for various wind velocities. This curve is for sand 0.025 cm in diameter and having a threshold velocity $v_t = 400$ cm per sec.

SUSPENSION OF PARTICLES

In a wind of strong intensity fine particles do not saltate but move with the fluid and may be said to be in a state of suspension. Under such conditions Bagnold's number is very large although unfortunately, as we have already noted, the criterion for distinguishing suspension from saltation in terms of this number is lacking at present. However, there is evidence that when the settling velocity v_m is less than seven times the drag velocity $*v$ suspension begins to set in.

White (1939) has proposed a formula which gives the concentration of particles at any depth in a flume or stream which is of the following form

$$\log \frac{C}{C_{1/2}} = 5.75 \frac{v_m}{*v} \log\left(\frac{L}{y} - 1\right)$$

where C is the concentration of particles at any height y in a stream of depth L and $C_{1/2}$ is the concentration at midstream depth, $y = L/2$. In the case of a wind blowing over extended terrain, the midstream depth is infinite. By taking the concentration $C_{1/2}$ as the concentration at the bed (mass of sand per unit volume of stationary bed), Bagnold obtains a suspension equation of the form

$$q' = \frac{v^3 d}{2.5 v_m{}^2}\left(0.076 + \frac{1}{6.6\theta}\right) \qquad \text{Eq (19-23)}$$

$$\theta = 2.5 v_m / *v$$

where q' is the mass of particles in suspension moving past a fixed point in a lane of unit width in a unit time. This equation applies only when $v_m < 7 *v$.

We have set forth in Eq (8-17) a possible expression for determining the "pick-up" of fine material by a turbulent surface wind,

$$C_{os} = \frac{\rho_0 k_0{}^2 (C_1 - C_2)(\bar{v}_2 - \bar{v}_1)}{\ln^2 \frac{z_2}{z_1}} \qquad \text{Eq (19-24)}$$

In this equation ρ_0 is the density of the air and $\bar{v}_1$ and $\bar{v}_2$ the mean wind velocities in a specified direction at two corresponding heights z_1 and z_2, and C_1 and C_2 the corresponding concentrations as weight per weight of air. The constant k_0 is von Kármán's turbulence constant which, as we have elsewhere noted, has a value of approximately 0.4 and is independent of the fluid. The value of C_{os} is simply the mass of material raised from a unit surface in a unit time.

The derivation of Eq (19-24) assumes that the coefficient of eddy viscosity, μ_e, for momentum and matter are the same (see Chapter 8). It further assumes that the distribution of vertical velocity and concentration varies logarithmically with height from the ground surface. Actually this is not the case, because near the ground surface turbulence does not operate, and the variation begins from the source point as we have shown earlier in the present chapter.

If now in Eq (19-24) we refer one of our observations to the source point, for which $v_1 = 0$ when $z_1 = n$, and if further we note with respect to units that $[C_{os}/v] = [\rho_0(C - C_2)]$, we obtain in conjunction with Eq (19-14)

$$\frac{C}{C_0} = 1 - \frac{\xi}{k_0} \ln \frac{z}{n} \qquad \text{Eq (19-25)}$$

or

$$\frac{C}{C_0} = 1 - \xi \frac{v}{{}_*v} \qquad \text{Eq (19-26)}$$

where C is the weight concentration of particles per unit volume at height z (or at corresponding velocity v), C_0 the concentration at the source height, and ξ a constant which is a function of the height to which the particles rise due to the wind gradient. The units of C and C_0 correspond to C_0/v. The above equations do not allow for settling nor do they state the suspension threshold to which they apply.

Eqs (19-24) to (19-26) apply only so long as the wind moves over an area where particle pick-up is possible. Beyond the limit of this area downstream, we may treat the problem as a line source, utilizing Eq (19-5) or (19-6). If, for example, we assume that the mean height of a dust cloud is 500 m and that it issues from the edge of a dust bowl at the rate of 10^7 grams per km per hr, then with a wind rate of 10 km per hr with $k_0 = 0.05$, we have at a distance of 100 km

$$C_{os} = \frac{10^7}{0.05 \times 10 \times 100} \cdot \exp\left(-\frac{0.5}{0.05 \times 100}\right)$$
$$= 2 \times 10^5 \cdot \exp(-0.1) = 1.8 \times 10^5 \text{ g/sq km}$$

EFFECT OF FLUID DENSITY ON PARTICLE BEHAVIOR

It is readily apparent that the behavior of particles in a wind is not similar to the behavior in a denser fluid such as a stream of water. There are several reasons for such differences as occur. Aside from the fact that observations in flowing water are difficult to make, especially if the particles are fine, there are two factors of significance which we shall

proceed to discuss. The first of these is the density factor for air and water which affects buoyancy. Thus, the density ratio for air and quartz is about $^1/_{1400}$ and for water and quartz only $^1/_{1.64}$. For the same speed of travel of a grain of quartz in both fluids, it is obvious that the energy of the grain suspended in air is 1400 times that of an equivalent volume of air, whereas in water it is only 1.64 times that of the same volume of water. It follows, therefore, that the energy of motion of a particle, taken at the expense of the fluid, is greater in the case of air.

When a wind moves over a surface of sand, the reduction in velocity as we approach the surface is so great that the unevenness due to ripples and small craters is relatively unimportant as compared with the effect produced by the saltating particles. In fact when saltation sets in, the vertical velocity profile is almost independent of the actual surface contour. On the other hand, in the case of water the reverse is true, and since saltation is negligible the velocity profile is almost entirely accounted for by the stationary surface contour. Thus while saltation is a dominant factor in the case of air which controls sand movement, in the case of water it appears as though the movement of particles depends on the probability of the occurrence at the surface of an eddy of sufficient intensity to move an exposed particle. Furthermore, since in water the drag of particles already in motion is not limited as in the case of air moving over saltating particles, higher velocities increase the probability of grain dislodgement and the concentration of particles moving with the fluid is increased. The differences in ripple formation in sand beds in flowing water and air can be attributed to the differences in mode of particle pick-up.

RIPPLES, RIDGES, AND DUNES

The formation of land surfaces by wind and water is not properly subject matter for a text such as this. However, there are basic principles underlying the formation of such surfaces that we should consider with some discussion at this point. More detailed discussion will be found in Bagnold's (1943) book with ample illustrations.

We shall begin our treatment of the subject by a discussion of the instability of a flat surface and the effect of pebbles. In the first instance it is readily apparent that the finite character of grains of sand or soil excludes the possibility of having an absolutely plain surface. If, then, a wind moves over as smooth a surface as it is possible to have, there is a chance that a few particles will be picked up. These will saltate in the manner we have already described, and will set in motion other particles until a condition is reached where the whole surface is in motion.

The saltation in turn initiates a creep and a surface will be set up such that the advance profile of a small section will be humped. The windward side of this hump receives the full impact of the saltating grains which strike it at a small angle to the horizontal, while the lee side receives few impacts and is not built up. This procedure continues until a more or less equilibrium condition exists. If the wind blows with great intensity the uneven surface will be leveled and a "grading" action will set in. Hence, we conclude that a flat surface is unstable since any small deformation sets up a chain of circumstances affecting the whole surface.

The effect of pebbles, too large to be moved or carried by the strongest wind, is most interesting. If these are scattered about the surface of a sand bed, they tend to keep the bed smooth and prevent the formation of ripples and ridges. The action of these pebbles is somewhat as follows: As the wind blows over the surface it tends to move the particles most exposed. Thereafter as the pebbles are bared, a new roughness condition sets in and the source point (the value of n) changes position. Between the pebbles are particles which are sheltered. As the wind intensity increases, more particles will be picked up between the pebbles, and no opportunity is given for a ripple or ridge to form. For, suppose a few pebbles are in close proximity to each other; the wind will tend to clear the particles about the pebbles, leaving them to rest on a sort of elevation. In time, the pebbles will roll into the lower area about them and the elevation of finer particles on which they rested will be smoothed over. The pebble agglomeration is then distributed over a wider area so that whatever wind condition may arise no further opportunity is afforded for sand accumulation.

On the other hand, if the wind intensity decreases, particles will begin to fill the space between the pebbles, but only to a definite height. Lesser wind intensities will eventually cover the pebbles. We may conclude that, in general, sands with a wide range of particle size tend to maintain smooth surfaces in a wind.

Ripples—The regular pattern of ripples running crosswise to the wind can probably be attributed to the average length of path traveled by the saltating sand. The ripples begin at an obstructing hump. The particles strike the windward side of this hump, bound upward and are carried downwind. Relatively few particles strike because of the small angle at which the particles terminate their fall. Thus the average distance traveled by the saltating particles causes them to fall within a narrow strip which gradually builds up into a ripple. Particles saltating from the first ripple then create the second and the process is repeated many times. As Bagnold has pointed out the spacing between the ripples

occurs with quasi-mathematical regularity which may be readily computed in terms of particle mass and wind velocity.

As might be expected, various types of ripples are formed depending upon the availability of transportable sands upwind. Nor can we overlook the fact that fluctuations in wind intensity can alter from moment to moment the character of the ripple formed. The end result is not necessarily something developed in an incrementally constant manner. There may be periods when the ripple is more extensive and higher, or composed of particles of much different size range than those found after the wind has ceased blowing.

Ridges—The term ripple is applied to a repetition of wave form whose distances between crests depend upon the average distance traveled by the saltating particles, or, in other words, ripples possess a wave length which depends upon the wind intensity. Other forms of sand waves whose wave lengths increase indefinitely with time are called "ridges." They differ fundamentally from ripples in that they occur whenever the grading of surface particles becomes coarser. Ripples occur when the surface grading is uniform or nearly so, and their amplitude decreases if deposition is heavy. The converse is the case with ridges.

The oncoming saltation on a ripple tends to build up the top portion so that fewer and fewer particles reach the lee slopes. As the wind intensity increases larger particles are deposited on the ripple. With this accumulation the form is preserved and continues to increase in size. These large particles resistant to the effects of strong winds are a chief characteristic of ridges. The increase in height of a ridge tends to increase the distance between it and the next one formed downwind. Ripples of uniform sand 0.25 mm in diameter disappear at intensities of about $_{*}v_{s} = 65$ or three times the threshold value, while ridges are sustained and even built up at these wind intensities.

While ridges are ordinarily formed in a direction transverse to the wind, they may also be formed in longitudinal strips parallel to the wind. The latter type occurs when the wind moves over an existing ridge which is parallel to the direction of its motion. During a wind of high intensity, the velocity at the top of such a ridge is less than that on either side of it so that a horizontal swirling effect takes place on either side of the ridge and this tends to build up two other ridges parallel to the first. This is followed by a repetition of form as the wind continues to blow. On the other hand, when the wind is gentle, it moves with greater speed along the top of the existing ridge than on either side of it. Ridges formed in the direction of wind motion are called sand strips, and the existing ridge giving rise to their formation is called a sand-pebble border.

Dunes—Dunes are formed in much the same way as ripples. They represent an accretion of sand which under favorable circumstances of sand supply and wind direction often attain extensive proportions.

Bagnold classifies dunes into two forms, namely, the barchan or crescentic dune and the seif or swordlike dune. The former are perhaps the more basic form since they may be shown to degenerate into the seif type. Dunes are characterized by the fact that they are capable of moving over terrain often without change of form. While the contour of a dune affects the flow pattern of the wind (which in turn affects it), it is possible by a simple analysis to obtain a relationship of other pertinent variables. Let us refer to Figure 114 which illustrates the cross

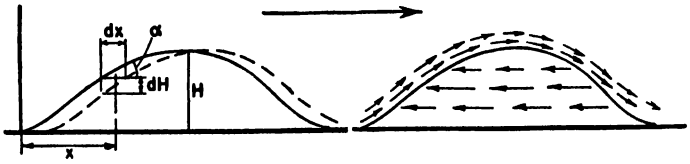


FIGURE 114. SCHEMATIC ILLUSTRATION OF DUNE TRAVEL.

section of a barchan type dune and consider a volume element of sand dq at a height H above the base of the dune. If all elements of the dune move a distance c in a unit time and the apparent density of the dune is ρ_a, then for an infinitesimal difference in level dH, we have

$$\frac{dq}{\rho_a} = c \cdot dH \qquad \text{Eq (19-27)}$$

and since $dH = dx \tan \alpha$

$$\frac{dq}{dx} = \rho_a c \cdot \tan \alpha \qquad \text{Eq (19-28)}$$

where α is the angle made by the dune surface with the horizontal at the point considered. If we assume for an ideal situation that all the sand removed from the windward side of the dune is deposited on the lee side, we readily see that as the dune continues to travel downwind, the sand once taken from the windward side and deposited over the brink on the lee side again returns to the windward side. This effect is shown schematically in Figure 114 and illustrates the circulatory motion within a moving dune.

Let us now return to Eq (19-28). We see that the rate of removal per unit area at any point on the surface of the dune is proportional to the tangent of the angle of inclination at that point. At the summit of the dune $dq/dx = 0$, so that there can be neither removal nor deposition.

As the sand moves over the summit dq/dx is negative and deposition is indicated. This deposition does not take place uniformly over the lee side of the dune, but just over the brink. This deposition continues until the angle assumed by the deposited sand with the horizontal approaches the angle of repose (about 34 deg for sand) and then the deposit slides down. Thus the lee slope, or slip-face as it is called, never exceeds the angle of repose.

The equations we have developed thus far are oversimplifications, but as Bagnold has pointed out they give remarkably good agreement with such observations of dune movement as are available. If we write $\bar{v}$ for c in Eq (19-27), since c is actually the rate of dune movement

$$\bar{v} = \frac{q}{\rho_a H} \qquad \text{Eq (19-29)}$$

Thus the higher the dune, the less is its rate of movement, subject of course to the supposition that all the sand moved from the windward side has been deposited on the slip-face. As a corollary we also conclude that, if the dune decreases in size for any reason, its rate of travel increases. As an example of the magnitude of the quantities involved in the present discussion, suppose that with a wind intensity of 14 m per sec, $q = 0.46$ ton per meter-width per hr. If $\rho_a = 1.7$ and $H = 15$ m, we then have from Eq (19-29) that

$$\bar{v} = \frac{0.46}{1.7 \times 15} = 1.8 \text{ cm per hr}$$

Using the development here given, Bagnold has computed from measurements made of an Egyptian barchan belt that if the rate of travel of the barchans was at all times proportional to their height, the time taken for them to move from their suspected place of origin to their present position is of the order of 7000 yrs. He suggests from this computation a measure of confirmation regarding the migration from the Libyan Desert to the Nile Valley of pre-Dynastic peoples. These peoples are known to have commenced their migration about 5000 B.C.

Problems

1. Approximately 1 ton per day of fly ash issues from a stack 250 ft in height. Using values of k and k_1 equal to 0.06, determine the concentration distribution at ground level along the direction of a wind blowing 10 mi per hr.

2. In Problem 1, determine the concentration distribution at ground level perpendicular to the wind direction 0.5 mi downstream.

3. The following data were recorded during an investigation of a dust storm:

Height (meters)	Concentration (M P P.C F.)*	Wind velocity (mi per hr)
1	2000	20.0
10	600	17.4

* Million particles per cu ft

Using a value of von Kármán's constant of 0.4 and assuming the density of the particles to be 2.4, compute the number of particles lifted from the ground surface by the wind.

CHAPTER 20

ATMOSPHERIC AND INDUSTRIAL DUST

ATMOSPHERIC POLLUTION

With the exception of dust storms, most of the dust found in municipal air comes from chimneys and stacks. With the advent of powdered fuel and ultimate discharge of fine ash through stacks, the degree of dustiness in most cities increased enormously. However, in recent years with the application of electric precipitators, cyclones, and other types of arresters, the amount of fly-ash discharged industrially has been reduced. Indeed, industry has given so much attention to the efficient combustion of coal and elimination of air pollution that the amount of atmospheric pollution is no longer in direct proportion to the number of industries in any given locality. There have been many studies to indicate this, as well as the fact that small houses and apartments with poor fuel regulations are the chief cause of atmospheric pollution.

In addition to the method of firing, atmospheric pollution also depends upon the frequency of rainfall and direction of prevailing winds. It has been estimated that if all the pollution existing at a given time in an industrial city actually settled within its limits, it would amount to approximately 5 tons per year per acre. Some areas in highly industrialized cities without the benefit of smoke-control programs have recorded dust settling to an amount of over 300 tons per year per acre. This is an enormous quantity of dust which, if it were not for the natural elements, would require much more frequent cleaning of streets and rooftops.

Economic Aspects—The economic aspects, from the standpoint of efficient use of fuel, have been touched upon but briefly. It is clear that black smoke is the result of ineffective coal combustion. As a matter of fact, effective combustion of fuel may actually represent a saving and has been the most pertinent argument for the elimination of black smoke. However, the use of powdered coal with complete combustion and the production of a white smoke or fly-ash is equally detrimental. When we consider that approximately 15 to 20 percent of coal is ash, and if this is poured out continuously from an industrial chimney day after

day it represents a sizable quantity of material. This fly-ash is relatively free of carbon but possesses features which are equally bad; since it is discharged in such high quantities it quickly covers a vast area, penetrating into homes, offices, and stores; it crumbles easily and smudges, not as badly as soot but noticeably nevertheless. Naturally this fly-ash is combined with all other fine matter present in the air, and the total amount of air contamination determines the frequency of laundering, house-painting, window-cleaning, loss of goods in department stores, and many other similar items. But this is not all. In addition to dust *per se*, we must consider its acidity when wetted. Atmospheric dust generally runs high in acidity, so that following a rain it gradually attacks limestone, mortar, and many other structural materials. To a certain degree it is also responsible for the destruction of plant life. Everything considered, if atmospheric pollution is very high, it represents a serious monetary loss to both citizens and community.

Results of Studies—The concentration of dust in city air depends upon many factors such as time of day, degree of activity, prevailing winds, etc. While clean country air is estimated to contain 0.2 mg of dust per cu m, a careful study of dust concentration in some 14 American cities undertaken by the U. S. Public Health Service (1936) showed the following results:

1. The average amount of suspended matter in the air during the winter months in the 14 cities in which the studies were made was found to be 5.1 mg per 10 cu m of air, of which 65 percent consisted of carbonaceous matter, 35 percent of ash, 12 percent of silica, and 2 percent of iron oxide. An average amount of 0.73 mg of sulfur was found to be present in 10 cu m of air during the winter months. The amount of sulfur in the air was found to be closely correlated with the amount of carbonaceous matter present. The modal size of the dust particles in the air was found to be about half a micron. The median size of the dust particles was found to be 0.58 μ. Only a small percentage of the particles were more than $1^1/_2$ μ in diameter. The variation in particle-size from city to city was found to be very small. The average number of particles per cu cm in winter was 815.
2. As was to be expected, atmospheric pollution was found to be greatest in the winter, reaching a maximum in December. The occurrence of the maximum in December seemed to be associated with unusually cold Decembers in the two winters studied, and it is possible that the peak would occur normally in January. The winter months showed about twice as much pollution as the summer months. This excess was particularly evident for carbonaceous matter and sulfur.
3. There was found to be a great variation in atmospheric pollution at different times of day. A minimum was found at about 3:30 A.M., a maximum at about 7:30 A.M., a dip in the middle of the day, and changes in the slope of the curve at about 8 P.M. and again at midnight

The pollution at the time of the morning maximum was about twice as great as that in the middle of the day. These variations in pollution with the hour of the day were clearly marked for all seasons of the year, but the morning peak occurred about two hours earlier in the summer than in the winter. The minimum in the middle of the day was much more marked in the summer.

4. Atmospheric pollution was less on Sundays than on week days, indicating the importance of the pollution due to industry. However, the nonindustrial pollution in the winter, resulting from the heating of residences, apartment houses, hotels, and other buildings, appeared to be even more important.

5. The velocity of the wind was found to have a marked effect on the degree of atmospheric pollution, increase in velocity being associated with a decrease in pollution. In the winter, on cloudy days, the pollution, as measured with the Owens automatic air filter, had an average shade of about 1.9 for a velocity of 5 miles per hr, 1.2 for a velocity of about 10 miles, and 0.8 for a velocity of about 20 miles, or when the velocity of the wind doubled, the pollution decreased to about six-tenths of its original value, and when the velocity quadrupled, the pollution decreased to about four-tenths. Wind direction also affected the degree of atmospheric pollution, but this factor depended on local conditions, such as the position of industrial areas, large bodies of water, etc.

6. Aside from the morning maximum, which appears to be due primarily to the starting up of industrial and domestic fires, the density of the pollution during the day appears to depend largely on the vertical air currents. This is indicated, among other things, by the fact that on clear days there was a marked minimum in the pollution in the middle of the day, which was not present on cloudy days.

7. An examination of the records of the automatic air filters iailed to show any definite decrease fn atmospheric pollution either during or after the fall of rain. This result is curious in view of the generally accepted belief that "rain clears the air."

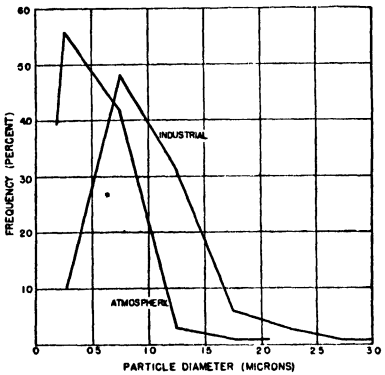


FIGURE 115. SIZE-FREQUENCY DISTRIBUTION OF ATMOSPHERIC AND INDUSTRIAL DUSTS.

Size-Distribution—The size-frequency distribution of atmospheric dust is shown in Figure 115.

In this figure it will be seen that the average size of atmospheric dust particles is below 1 μ. This is much lower than that obtained for industrial dust shown in the same figure. A cumulative plot of the curves for the two dusts indicates that more than 50 percent of in-

dustrial dust is greater than 1 μ in size, while for atmospheric dust the median size is approximately 0.5 μ.

INDUSTRIAL DUST

Industrial dust is of general interest because it is associated with certain diseases of the lungs, commonly referred to as the pneumoconioses. It has long been known that workers in dusty trades become disabled after a few years' exposure. While the etiology of the disease is known to be due to dusts inhaled in high concentrations, in sizes generally below 10 μ, the exact nature of the action on the lung tissue itself remains a mystery. Not all dusts react alike and some, such as quartz-containing dusts, are very active. It is probable that any dust when breathed in sufficiently high concentrations for long periods of time is injurious.

In addition to those dusts producing lung injury are dusts which are systemic in their effects. Common examples of these dusts are the lead compounds, manganese dioxide, cadmium oxide, etc. On inhalation, the effect of these dusts is much more severe than if they were ingested. They are easily absorbed into the blood stream and act directly on many vital organs.

Finally, there is the third group, such as pollens and certain organic substances, whose action produces allergy. As a rule the severity of their effect is due to the individual susceptibility of the person exposed; but there are instances, notably exposure to castor-bean dust, where susceptibility does not play an important role. However, as in the classes previously discussed, degree and length of exposure are the most important considerations. In this chapter the discussion is limited to three topics of interest, namely: (1) Particle-size range and concentration; (2) the methods used to determine safe limits of exposure; and (3) a brief outline of control methods.

PARTICLE-SIZE OF INDUSTRIAL DUST

There has been much discussion on the significant particle-sizes giving rise to pneumoconioses. It is the contention of one school of investigators that particles of decreasing fineness have greater significance than the larger particles. This question cannot be answered until more exact data are available concerning the action inducing fibrosis in the lung tissue. Many investigators have recovered particles from lung tissue, and it has been shown that such particles rarely exceed 10 μ in longest dimension, most of them falling below 3 μ in size. In obtaining inhaled particles from autopsy material it must be remembered that the lung tissue is subjected to rigorous chemical treatment, which may

naterially affect the amount of small particles present. As a general ·ule particles recovered from lung tissue are rarely noted when less than).5 μ. It is curious that in ordinary microscopy the particle-size distribution of dust found in lung tissue closely follows that found in normal ndustrial air. The size-range (0.5 to 3 μ) is that which would be sustained in the air long enough to be inhaled. Extremely fine particles submicroscopic in size are not noted, because they go beyond the resolution of most microscopic systems now in use. T. Hatch and Pool (1934) using dark-field technique for counting showed that concentrations obtained with light field were greatly underestimated. This leads to the conclusion that the finest particles have greatest physiological significance. At the same time it must be indicated that the work of C. E. Brown (1931) shows that fine particles are not retained by the lungs; the efficiency of retention decreases as the particle-size decreases. It is conceivable that we may approach particle-sizes of the order of tobacco smoke (0.05 μ) for which retention efficiency by the lungs is nil. Particles of this size may well be produced industrially. Whether they should be considered of physiological significance still remains a moot question. The particle-size distribution of outdoor and industrial dust (a composite of several industries) is shown in Figure 115.

With regard to concentration, several important variables must be considered. These are size and number of the dust-producing operations, space in which they are located, material being worked, degree of ventilation provided, amount of activity, housekeeping provided, etc. Rock drilling in tunnels may produce concentrations exceeding 2 billion particles per cu ft of air, foundry shakeout operations more than 500 million particles. No set figure can be given for any single operation or process, but it may be stated that, without control, concentrations generally exceed 10 million particles per cu ft which is the accepted lower limit for safe exposure to most dusts.

METHODS OF SAMPLING INDUSTRIAL DUST

Much of the confusion arising in the interpretation of dust counts may be traced to an inadequate appreciation of their actual significance. Many investigators have ignored the limitations of the technique used and this has caused severe criticism of such counts in evaluating a dusty environment, particularly in attempts to comprehend the meaning of wide variations so often obtained in results. Therefore, what are the criteria which make dust counting definitely valuable?

Dust counts have a threefold application: (1) They are an index of the cleanliness of a plant; (2) they help to determine the effectiveness of dust-removal equipment; and (3) they can be used in conjunction with

medical data to determine the threshold or safe limit of exposure to a specific dust. In practice, the first and second applications are more common than the third. It is important to check upon the performance of control measures in order to protect workers from excessive dust exposures. Dust counts are a quantitative measure of environment, which, coupled with a knowledge of the safe or threshold limit, tell whether the conditions found are inimical to the health of workers. The third application of dust counts, namely, their use in establishing threshold limits, is a research function. In order to understand the causes of the difficulties encountered by so many investigators in making dust counts, a brief discussion is given of the characteristics of instruments and techniques generally used for dust sampling, the environmental conditions affecting the sample, and the methods of averaging and weighting of counts in order to arrive at an index of dustiness which accurately defines the environment of the worker. These factors have a direct bearing on the applications of dust counts mentioned above.

There are two general types of sampling, which may be designated as continuous or integrating, and grab or instantaneous. The first is represented by the impinger device as used by the U. S. Public Health Service (1935*b*). This instrument samples at a constant rate for any period of time and thus "integrates" variations in dust concentrations. The second mode of sampling includes devices similar to the Owens jet dust sampler and the Kotze konimeter, which take small and relatively instantaneous samples. Both methods of sampling are valuable and widely used. As with all instruments, they have certain applications for which they are best fitted. These must be appreciated if the results obtained are to be reliable indices of the environment.

There is no need for absolute sampling devices—that is, equipment which will capture *all* dust suspended in air. Such a device would be of little value. The only requirements for any method are that it secure a representative sample of dust present in the air, and that such tests can be duplicated by other investigators.

There is an exaggerated tendency among many technicians making dust surveys toward a "simple" dust-counting instrument. Generally, it is desired to make daily or weekly inspections, but the actual need is for a rapid method of making dust counts. While it is admittedly useful to have an instrument both rapid and reliable, it must be stressed that frequent dust surveys are not necessary. The pneumoconioses develop only after many years of exposure to dust. Consequently, only two, or at most three, annual surveys are necessary to evaluate exposure in any dusty trade. Control equipment, once tested to assure effective removal of dust, will continue to perform so long as it is maintained in

good condition. Finally, in making repeated checks of a given activity, it is important to have designated sampling points accurately spotted, since dust concentrations vary considerably with distance from the source of production.

The Greenburg-Smith Impinger—The Greenburg-Smith impinger is the device most commonly used for dust sampling in this country. Its advantages over other instruments have been discussed in U. S. Public Health Bulletin 217. The impinger is adapted to taking samples of airborne dust over short or long periods of time. It is not efficient for very small particles ($<0.5\ \mu$), but the fact that samples can be duplicated, plus its reliability, are distinct advantages. The lack of efficiency of the impinger device for small particles need cause no alarm, because it is doubtful whether very small particles ($<0.5\ \mu$) are sufficiently injurious to warrant an accurate estimation of them. The use of dark-field methods adds little to the reliability of the count since, as has been mentioned, the impinger itself has a vanishing efficiency as the size of the particles sampled diminishes.

Volume of sample—There are several points of view as to the amoun or volume of impinger sample which should be taken. Many believe that numerous short-time samples have greater significance than a few long-time samples. There can be no question but that a large number of samples taken at various times depict the environment better than single samples, even if taken over a period of several hours. The reason for this is not far to seek. No activity produces a constant quantity of dust. The human element, as well as variations in dust-producing processes, change from day to day. Samples taken for too long a time are subject to two serious criticisms: (1) Long-time samples cause many particles to go into solution, and (2) if a process tends to produce dust floods, there is a tendency to overlook important causes of dust production. Moreover, long-time samples lead to results which do not correctly picture the dust generated by the activity itself. This phase of the dust-sampling problem will be discussed more fully in another section.

Dilution of sample—The dilution of concentrated samples may lead to large errors in counting. Concentrated samples flocculate and settle at higher rates than dilute samples. Consequently, the portions transferred by pipette are not always representative. However, this in itself may not be a serious drawback except when the samples are taken for long periods of time. The dilution of samples should proceed to a point where the average count per one-fourth field is greater than 20 and less than 40 particles. Control samples may be relatively free from particles, but as a rule they average from 4 to 10 per one-fourth field.

Therefore, if the count is much less than 20 per one-fourth field, it is probable that some fields may be constituted almost entirely of dust contained in the control sample itself.

Microscopy—Mention has already been made of the limitations of the microscopic system in counting impinger dust samples. In addition, it may be stated that accuracy in counting depends to a great extent on experience. The technician skilled in recognizing dust particles employs just enough illumination so that all particles stand out distinctly. Less experienced technicians frequently use too much illumination which may obscure highly refractive particles as well as tire the eye. Application of color adapters to the illumination system on microscopes often helps in making certain particles, such as quartz, stand out clearly.

Grab-Sample Devices—The chief criticisms of grab-sample devices are the smallness of the sample taken and the lack of any reliable data as to the significance of the results obtained. These devices should not be compared with continuous-sampling devices such as the impinger. There are no points of similarity to warrant comparisons; the impinger device integrates the dust sampled over a period of time, while the grab sampler represents conditions during a brief instant. The devices are to be compared only when the dust concentration is known to be constant. Even then care should be exercised, since the selectivity and other characteristics of the instruments may vary with dust concentration.

Field of Application—It must not be concluded from the foregoing that grab samplers are not to be used. For general air samples and routine control work these devices are quite effective. Similarly, repeated samples taken at a station may often reveal the variations in dust concentration with time. Such studies help to locate sources of activities producing excessive amounts of dust. It is in the field of dust control that grab-sample devices find their most valuable application. Once a criterion or threshold limit has been decided upon for the device selected, it is merely necessary to determine from time to time whether this criterion is being met. Hatch and Thompson (1934) discussed a novel arrangement with one form of grab-sampling device which is both rapid and simple to use. In the hands of a trained technician and used at fixed sampling stations, routine surveys of dust conditions in a plant can be expedited, and all failures in control equipment noted and remedied. Exact quantification is not necessary; only a visual comparison of the sample taken with the standard is required.

The conditions producing a dusty atmosphere are the all-important considerations limiting the quantitative aspects of dust-counting methods. No greater degree of accuracy is needed in sampling than warranted by the nature of the dust-producing activities. Dust counts will

average either high or low depending upon whether large or small quantities of dust are produced by the activity in question. This will be the case regardless of refinements of technique in sampling.

No dust sampling should be undertaken until a careful study of the occupations and activities involved in producing dust has been made. Dust samples alone should never be taken to represent the exposure of a group of activities since such samples do not indicate the relative amounts of dust to which all the workers are exposed. In its many field investigations, the United States Public Health Service (1935, 1938, 1940) has generally given its attention to determination of the exposure of specific occupational groups. This has been done because similar occupations function in identical ways and have corresponding dust exposures. The occupation can also be analyzed on a time basis. The need for this is apparent when we consider that it is possible to have dust produced only during a part of normal operations associated with an occupation. An occupation is composed of many activities, and the worker is exposed to intermittent concentrations of dust. Hence, samples must cover every phase of the workers' activities.

It must be remembered that the dust counts are used to obtain an *index of dustiness*. Some of the data presented undoubtedly show wide variations which might make the averages based upon them open to serious criticism. But if we bear in mind that only an index—or relative degree of dustiness—is sought, no difficulties should exist. Statistical methods for the treatment of raw-dust-count data are not always warranted since variations in processes, natural ventilation, and other factors affecting dust concentration change from time to time. It is one of the advantages of this method that the effect of such factors is accounted for without incurring serious discrepancies in large series of dust counts.

AVERAGING AND WEIGHTING OF DUST EXPOSURES

In small plants, workers are frequently engaged in several different activities. In order to reduce the problem of determining the exposure of such workers to simple terms, it is essential that a study be made of the time spent in each activity. Table 77 illustrates the first steps in the technique used in estimating the exposure of 39 workers in two plants who were engaged in grinding, drying, and bagging mica. The two plants differed as noted in the column headed "activity." In plant *A*, the ground-mica sludge was filtered, dried in rotary kilns, and bagged by mechanical equipment of modern construction; plant *B* utilized ordinary oven-type kilns to dry the sludge and used manual methods of bagging. In the last column of the table is given the time spent by the

various workers in each activity during the course of an 8-hr day. The other columns give the individual dust counts and their arithmetical averages.

Determination of Average Exposure—Table 78 shows the method of arriving at a weighted average of the workers' exposure. This consists

TABLE 77—AVERAGE DUST COUNTS AND TIME ESTIMATES MADE IN TWO MICA-GRINDING PLANTS WHERE WORKERS PARTAKE IN ALL THE ACTIVITIES

Activity	Results of individual samples (million particles per cu ft)	Arithmetic average (million particles per cu ft)	Time estimated as spent in activity per day (hrs)
Wet grinding and mica settling (both plants)	1.8 5.0 2.6	3.1	2
Filter-cake drying (plant *A*)	24.2 39.6	32.0	3
Kiln drying (plant *B*)	123 193	158.0	3
Screening and machine bagging (plant *A*)	34.2 64.6	49.0	3
Screening and hand bagging (plant *B*)	133 123	128.0	3

TABLE 78—METHOD OF DETERMINING EXPOSURE OF WORKERS IN A WET MICA-GRINDING PLANT

Activity	Time estimated as spent in each activity (hrs) (*a*)	Average dust concentration (million particles per cu ft) (*b*)	Million-particle hrs exposure, plant *A* (*a* × *b*)	Million-particle hrs exposure plant *B*, (*a* × *b*)
Wet grinding and mica settling (both plants)	2	3.1	6.2	6.2
Filter-cake drying (plant *A*)	3	32.0	96.0	...
Kiln drying (plant *B*)	3	158.0	...	474.0
Screening and machine bagging (plant *A*)	3	49.0	147.0	...
Screening and hand bagging (plant *B*)	3	128.0	...	384.0
Total	8	...	249.2	924.2

Weighted average: Plant *A* $\frac{248 \text{ million-particle hrs per cu ft}}{8 \text{ hrs}}$ = 31 million particles per cu ft

Plant *B* $\frac{924 \text{ million-particle hrs per cu ft}}{8 \text{ hrs}}$ = 116 million particles per cu ft

in multiplying the average dust concentration in each activity by the time spent in it. The sum of these products for each plant, divided by the number of hours (8) spent at work, gives the weighted average exposure. Table 78 shows that the workers in plant *A* were exposed to a weighted average concentration of 31 million particles per cu ft, while those in plant *B* were exposed to 116 million particles per cu ft. If straight arithmetical averages of all the samples for each plant had been used, the estimates would have been 24.6 and 83.0 million particles per cu ft, respectively. Again, if different times had been assigned to the activities in question, it is evident that very wide discrepancies would exist between the weighted and arithmetical averages. The chief differences between the two methods of expressing the dust exposure associated with a set of activities are that in the case of weighted average emphasis is placed upon duration of exposure, while in the arithmetic average dependence is placed upon the number and magnitude of in-

TABLE 79—METHOD OF CALCULATING THE EXPOSURE OF FELDSPAR DRILLERS AND HELPERS WORKING IN AN OPEN-PIT MINE

Activity	Number of samples	Time estimated as spent in each activity (hrs) (*a*)	Average dust concentration (million particles per cu ft) (*b*)	Million-particle hrs per cu ft exposure (*a* × *b*)	Remarks
Drilling	2	4	65.0	260.0	Dry drilling
Mucking, sorting, and cobbing	2	4	2.1	8.4	No drilling
Total	4	8	...	268.4	

Weighted average: $\frac{\text{268.4 million-particle hrs per cu ft}}{\text{8 hrs}}$ = 33.6 million particles per cu ft

dividual samples. Here again, however, it must be stressed that there is no certainty in the dust exposures arrived at by weighting. It is a convenient way of obtaining an index of dustiness which accounts for all the activities associated with a given job.

Another example is furnished in estimating the exposure of drillers and helpers in an open-pit feldspar mine. In this case, the drillers were engaged in drilling operations for only four hours each day. The remaining four hours were spent in mucking and cobbing the feldspar blasted from the working face. The data and computations are shown in Table 79. The weighted average is seen to be 33.6 million particles per cu ft. If no time studies had been made, and reliance placed merely

on the designation implied in the term "driller," the exposure would have been assumed to be 65 million particles per cu ft.

Method of Averaging a Widely Fluctuating Dust Exposure—The examples given above are especially adapted to the weighting of occupations. However, with slight modifications the technique can also be applied to the problem of determining the effectiveness of a given piece of dust-control equipment. The procedure consists in taking dust samples at regular intervals over a period of time sufficient to represent all variations in the operation to which it is applied. Figure 116 shows

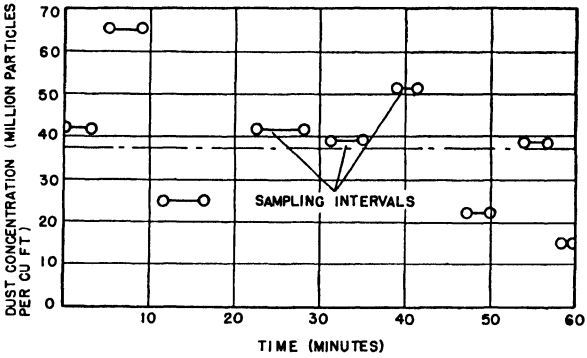


FIGURE 116. VARIATION OF DUST CONCENTRATION AS INDICATED BY IMPINGER SAMPLE FOR A FELDSPAR-CRUSHING MILL.

the variations in dust production generated by a hammer-type crusher. The dust peaks occur at the times the crusher is loaded. The crusher in question is small and is exhausted near the opening at which the rock is fed.

To calculate the average dust concentration generated by the crusher it is necessary to determine the area under a curve of the points plotted in Figure 116 and divide by the period of time over which the samples were taken, in this case 1 hr. The average concentration is then seen to be 42 million particles per cu ft. This value should then be compared with the criterion of permissibility to determine whether the amount of dust generated is hazardous to health.

Weighting of Occupational Histories—If a worker has been employed in several occupations over a period of years, some measure of his total dust exposure becomes necessary. This is particularly true when medical findings must be correlated with the amount of the workers' exposure. Pneumoconiosis is the result of prolonged exposure to dust, and

if the worker has been employed in several occupations, all involving different dust exposures, a method of integrating them is essential. The tendency to utilize the exposure in the present occupation in correlating medical with engineering findings may lead to many difficulties. The solution of this problem follows the procedure developed in the previous paragraphs, subject to the following restrictions: (1) The dust associated with the various occupations must not have changed very much for a number of years, and (2) the dust breathed must not have changed in composition with change of occupation.

Table 80 gives the occupational history of a worker 45 years of age who has been working since the age of 15. Note that the first occupation, namely that of farmer, was for a period of 10 years and involves no exposure to dust. The succeeding years were spent in feldspar mining and feldspar milling and crushing. The exposure due to these occupations, together with the method of calculating the total and average exposure of the worker, is given in Table 81.

TABLE 80—OCCUPATIONAL HISTORY OF A FELDSPAR WORKER, 45 YEARS OLD, WHO BEGAN WORK AT THE AGE OF 15

Occupation	Years in nondusty occupation	Years in dusty occupation	Remarks
Farmer	10	..	
Mucker and sorter	..	2½	Underground mine, full-time work
Driller	..	3	⅓ time drilling
Trammer[a]	..	5	½ time sorting and mucking, feldspar plant
Farmer	1	..	
Mill operator	..	7	½ time sorting and mucking, feldspar plant
Idle	1½	..	Estimated
Total	12½	17½	

[a] Handling wheelbarrow.

The total given in the last column of Table 81 is a measure of the lifetime exposure of the worker to feldspar dust. It is an index often sufficiently absolute to indicate whether or not pneumoconiosis is present; that is, when the million-particle years of exposure exceed a particular value it may be expected that some lung pathology will be found. The weighted average is equally valuable for statistical purposes when it is desired to group workers with the same degree of exposure by years of employment.

In view of the number of variables involved, it would seem preferable to classify dust counts in broad categories. Thus, counts could simply

be expressed as greater or less than a maximum tolerance, say the threshold limit, without reference to the actual counts themselves. This term is usually taken to mean the maximum permissible limit of dust concentration. The threshold limit may be arbitrary, being based on engineering practices alone, or it may be derived from a correlation of dust concentrations with medical findings. It is likewise apparent from a consideration of the foregoing that a standard may be considered a maximum irrespective of the activities engaged in, or as a weighted average based on a careful time analysis of all activities. The former

TABLE 81—WEIGHTING OF EXPOSURE OF WORKER WHOSE OCCUPATIONAL HISTORY IS GIVEN IN TABLE 80

Activity	Time spent in stated activity (yrs) (*a*)	Average dust concentration associated with occupation (million particles per cu ft) (*b*)	Million-particle years exposure ($a \times b$)
Mucking and sorting	2½	239	597
Drilling	3	500[a]	1500
Tramming	5	20	100
Mill operating	7	270	1890
Total	17½	..	4087

$$\text{Weighted average exposure} = \frac{4087 \text{ million-particle yrs per cu ft}}{17\ 5 \text{ yrs}} = 234 \text{ million particles per cu ft}$$

[a] Weighted average to take account of time spent mucking and sorting.

interpretation appears in some state regulations. A threshold limit having a medico-engineering basis requires that all factors entering into the occupational environment, including time studies, be considered. By this latter method, an upper permissible limit can be exceeded for an interval of time although the weighted average may well be less than the limit. A threshold limit, therefore, should state whether it is a maximum to be complied with at all times during a given activity, or whether it is a weighted average based on a consideration of the time engaged in various phases of the work.

CONTROL OF INDUSTRIAL DUST

Isolation of Source—Dust can best be controlled at the source of generation by enclosures, wet methods, or exhaust ventilation. The first of these methods, although obviously the simplest, cannot be used where the process requires attention by an operator. In theory this control method utilizes segregation of the dust-producing source. It is

usually better adapted to large-scale operations, so as to protect workers in nearby nondust-producing operations. This system cannot be applied successfully unless means are provided to prevent accumulated dust from interfering with the process.

Wet Method—The second control method, wet method, is generally used in rock drilling, blasting, and mucking in tunnels, and in these operations it has proved effective. Wet drilling consists in introducing approximately $^1/_2$ gal of water per min through the drill steel.

The effect of water introduced through the drill steel during rock drilling is shown in Figure 117. It is seen that amounts of water as low as 0.2 gal per min reduce the dust concentration below 10 million particles per cu ft. Higher rates of flow will further reduce the amount of dust produced, so that with approximately $^1/_2$ gal per min the reduction is sufficient to maintain concentrations below the generally accepted limits. It must be remembered that in tunnels, where a number of drills are used simultaneously, the dust count may be greater than indicated in Figure 117, so that wet methods of control should be supplemented with ventilation.

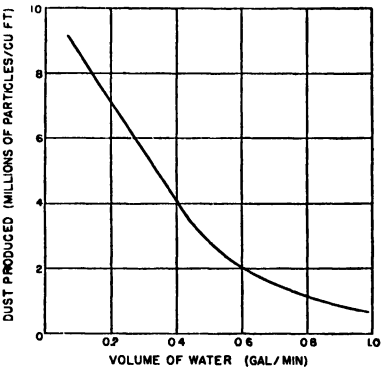


FIGURE 117 EFFECT OF RATE OF WATER FLOW THROUGH DRILL STEEL ON DUST COUNT.

Dust produced in blasting is controlled by a fine water spray placed near the face. The spray is made to form a curtain and is so fine that the dust formed is immediately wetted. Mucking is accomplished by wetting down the blasted rock. The effectiveness of wet methods in tunnel operations is shown in Table 82.

In using wet methods due allowance should be made for freezing weather and possible accident hazards. The use of wetting agents may in many cases aid in controlling dust and also cut down on the volume of water required. Except in the case of drilling, no fixed amounts of water can be specified since this is dependent upon the size of the operation and other variable factors.

Local Exhaust Ventilation—Exhaust ventilation at the source of dust generation is perhaps the most widely used method of dust control. This method of control requires that an air movement be provided at the source sufficient to convey the dust into the hood and duct system

and thence to a suitable point of disposal. Maximum effectiveness is obtained when the hood encloses the dust source as much as practicable. Complete enclosure is not advisable since the basis of control by exhaust ventilation requires air movement to carry the dust into the duct system. The duct system should be sized so that the velocity is great enough to move the largest particles.

Since exhaust ventilation depends upon the provision of air movements, near the source of dust production, sufficiently great to convey the dust into the hood we must have information available on the nature of the dust source and the air velocities generated by the hood. The former entails difficulties because processes vary greatly. Some opera-

TABLE 82—VARIATION IN CONCENTRATION OF THE DUST IN THE AIR IN PASSING FROM THE FIRST TO THE SECOND SAMPLING PLACES, 70 TO 100 FT FARTHER FROM THE FACE

Test heading	Sampling time (min after last hole blasted) Location		Dust concentrations, million particles per cu ft (light-field technique) Test without spray Location		Test with spray Location	
	1[a]	2[b]	1[a]	2[b]	1[a]	2[b]
1	10–15	25–30	452.0	71.6	1.6	1.3
1	20–25	35–40	80.9	41.5	0.4	0.5
1	35–45	50–60	57.5	37.8	0.5	1.0
2	12–17	27–32	77.4	129.5	0.3	0.6
2	22–27	37–42	52.4	97.9	0.1	0.2
2	38–48	53–63	49.6	91.6	3 1	0.2
3	10–15	18–23	141.7	403.4	19.1	114.3
3	20–25	28–33	146.1	199 4	9.2	104.8
3	35–45	43–53	77.2	83.3	20.8	55.8

[a] 70 ft from working face.
[b] 100 ft from working face.

tions, such as grinding, throw off dust at high velocities in a well-defined direction; others, such as sandblasting, generate dust over relatively wide areas. Nor can the influence of the operator be overlooked since his methods in manipulating a particular job may influence both the rate and amount of dust produced. In general, the control by exhaust ventilation means taking advantage of directional effects and enclosing the source so that the dust is confined as much as possible.

The problem of predicting the velocities necessary to control a given hazard cannot be given in a form applicable to all operations. In fact, little information is available on this subject. As a general rule, the air velocities may be as high as 1200 ft per min for grinding-wheel dusts and as low as 100 ft per min for sandblasting in a cabinet or room open at one end and exhausted at the other. Some data on the air volumes or

velocities necessary to control common industrial operations are given in Table 83. Difficulty is also experienced in predetermining the velocity characteristics of hoods under suction. These characteristics are largely determined by the shape of the hood opening and the obstructions forward of the opening. For hoods under suction which are free

TABLE 83—AIR VELOCITIES REQUIRED TO CONTROL DUST IN CERTAIN OPERATIONS

Process	Required air velocity or volume (ft per min) At point of dust origin	At hood opening
Granite		
Hand pneumatic tool	200	..
Surface machine	1500	..
Swing grinder	.	100
Crusher (rock, ore, etc.)	..	200
Flat-deck screen (enclosed)	..	200[a]
Cylindrical screen (enclosed)	..	600[b]
Mixers	..	200
Bagging (powdered sand)	..	400
Bucket elevator	..	200[c]
Conveyor belt (transfer points)	..	200[d]
Metal spraying		
Lead	200	..
Zinc	125	..
Welding	100	..

[a] Through inspection openings or 50 cu ft per min per sq ft of screen.
[b] Cu ft per min per ft of screen diameter.
[c] Through openings or 100 cu ft per min per sq ft of cross-sectional area of enclosure.
[d] Or 350 cu ft per min per ft of belt width.

of any obstructions forward of the openings, the author (1939*a*) has obtained the following formulas for determining air velocities outward along the center line of the opening: For rectangular openings

$$\frac{Y}{100 - Y} = \left[\frac{0.83}{1 + 0.26\left(\frac{1 - \beta}{\beta}\right)^{1.1}}\right]\frac{A}{x^{1.9}} \qquad \text{Eq (20-1)}$$

and for circular openings

$$\frac{Y}{100 - Y} = \frac{0.65D^{2.1}}{x^{1.9}} \qquad \text{Eq (20-2)}$$

where Y is the percent average velocity at the opening at the point x, A the area of the opening, β the ratio of the short to the long side of a rectangular opening, and D the diameter of a round opening. The units

of A, x, and D must be consistent; that is, if A is in sq ft, x and D must be in ft. A simple general formula adapted to most types of openings is

$$v = \frac{0.1Q}{x^2 + 0.1A} \qquad \text{Eq (20-3)}$$

where v is the velocity at the point x in ft per min and Q the volume of air flow in cu ft per min.

Problems

1. Calculate the value of C_z/q for various values of x (in miles) for a stack 250 ft in height, assuming a wind velocity of 10 ft per sec. Plot the results and from them determine where the concentration will be a maximum.
2. What are the important variables to be considered in a study of atmospheric pollution?
3. Show why dust suspended in air is generally less than 3 μ in size
4. Discuss the procedure used for making a survey of dust conditions in an industrial plant. What is meant by "index of dustiness?"
5. What methods may be used to control industrial dust?
6. Calculate the velocity 6 in. in front of a rectangular opening 5 in. by 10 in., if the velocity at the opening is 1000 ft per min

CHAPTER 21

COLLECTION AND SEPARATION OF PARTICULATE MATTER FROM AIR

IN THIS chapter various methods for the collection and separation of small materials will be discussed. One of the important aspects of this problem has already been touched upon in Chapter 17, where it was shown how particles maintained in suspension by agitation alter the specific gravity of the medium as a whole. Thus, given a mixture of materials the components may be separated by sinking or floating the particular fractions desired. This method merely requires prior knowledge of the densities of the various components, and that the spread between them is sufficient for a sharp separation. Since the basic theory has been amply covered in Chapter 17 no further discussion is believed necessary. In filtration the character of the filtrate is dependent upon so many variables that a prediction of performance in terms of particle-size or other properties of the material cannot be given. The degree of compaction of individual particles in the filtrate mass influences the behavior of a filter to such a degree that estimates for the most part can be determined only by experimentation. For the reasons cited, only the collection or separation of particles from an air or gas stream will be given. The discussion is limited to dry filters, cyclones, electric precipitators, and electrostatic separators.

DRY FILTERS

Dry filters considered here are of the cloth type, although in general the treatment discussed is applicable to all types possessing a fair degree of retention. The theory of filtration given involves the application of the Kozeny equation presented in Chapter 13 (see Eq 13-30). The use of this equation in this particular problem is due to Williams *et al* (1940). Remembering that we are dealing with a gaseous medium, the pressure-drop through the filter is

$$h = \frac{k\mu v\delta}{g\rho_0} \cdot S_v^2 \cdot \frac{(1 - \vartheta)^2}{\vartheta^3} \qquad \text{Eq (21-1)}$$

where h is the pressure-drop measured in units of air column, k a constant theoretically equal to 5 when cgs units are used, g the gravitational

constant, μ and ρ_0 respectively the viscosity and density of the air, v the velocity of the air, δ the thickness of the deposit, S_v the surface per unit-volume of particles collected, and ϑ the fractional voids. Considering a unit-area of the filter, we have

$$\rho(\delta \times 1 \times 1)(1 - \vartheta) = w \qquad \text{Eq (21-2)}$$

where ρ is the true density of the material collected and w is the weight of the material collected on the filter. Also, if C is the weight of the particles per unit-volume of air,

$$w = C \cdot vt$$

where t is the time during which the resistance of the filter increased by an amount h. Hence combining the last two equations above with Eq (21-1)

$$h = \left[\frac{k\mu}{g\rho_0} \cdot S_v^2 \cdot \frac{1 - \vartheta}{\vartheta^3}\right]\frac{v^2Ct}{\rho} \qquad \text{Eq (21-3)}$$

since $S_v = k'/d$ where k' is a constant (Eq 16-2) and d is some average particle diameter. Thus Eq (21-3) becomes

$$h = \left[\frac{k\mu}{g\rho_0} \cdot \frac{k'}{\rho} \cdot \frac{1 - \vartheta}{d^2}\right]\frac{v^2Ct}{d^2}$$

$$= \frac{KCv^2t}{d^2} = \frac{Kwvt}{d^2} \qquad \text{Eq (21-4)}$$

where K has been written for the expression in brackets. As a rule the value of ϑ cannot be determined and it is customary to lump the constants in the manner shown. If h is expressed in inches of water, then K becomes another constant, K', and

$$h_w = \frac{K'wvt}{d^2} = \frac{K'Cv^2t}{d^2} \qquad \text{Eq (21-5)}$$

Expressing w in lbs per sq ft, t in min, and v in ft per min, Williams *et al* determined the constant K' for a number of dusts of different diameters. These are given in Table 84. The constant K' may be termed the specific resistance and is expressed as in. water gauge per lb dust per sq ft filter area per lineal ft per min filtering velocity. The data contained in this table show that the resistance coefficient (1) increases with decreasing particle-size, and (2) does not vary greatly among several materials when compared at equivalent sizes. These facts are quite clear from Eq (21-5) since h varies inversely as d^2.

Whether K' is constant for all values of the time t is not known pre-

cisely. It would seem that with increasing time the pores through which the clean air passes become smaller and fewer in number, thus affecting both ϑ and the constant itself. On the other hand, for short intervals of time (the usual practice in industry) K' may be regarded as substantially constant.

TABLE 84—FILTER-RESISTANCE COEFFICIENTS, K', FOR CERTAIN INDUSTRIAL DUSTS. (INDUSTRIAL CLOTH-TYPE AIR FILTERS)

Dust	Particle-size (microns)[a]					
	840	105	90	45	20	2
Granite	1.58	2.20	..	..	19.8	..
Foundry	0.62	1.58	..	3.78	..	..
Gypsum	..	..	..	6.30	18.9	..
Feldspar	..	..	..	6.30	27.3	..
Stone	0.95	..	6.30	..	..	..
Lampblack	..	..	..	..	..	47.2
Zinc oxide	..	..	..	..	..	15.7[b]
Wood	..	.	6.30	..	..	..
Resin	..	0.62	..	..	25.2	.
Oats	1.58	..	9.60	..	..	..
Corn	0.62	..	3.78	1.58	..	..

[a] Theoretical size.

[b] Flocculated material. Size actually larger than indicated.

Actually K in Eq (21-4) is a permeability constant. If we assume that it varies linearly with time such that

$$K = K_0 + K_1 t$$

and this is substituted in Eq (21-4), we will obtain an expression in t^2. The term $K_0 t$ will generally be small in comparison with $K_1 t^2$ for most filters and thus may be neglected. In such cases

$$h = \frac{K_1 C v^2 t^2}{d^2}$$

Now vt is the rate of flow through a unit area, q, and we see that a possible law of resistance for a given dust concentration is one proportional to the square of the flow and inversely proportional to the square of the average particle diameter,

$$h = K_1 C \left(\frac{q}{d}\right)^2 \qquad \text{Eq (21-6)}$$

CYCLONE SEPARATORS

The cyclone or centrifugal separator is a device utilizing radial acceleration for separating particles suspended in a gas stream. It con-

sists of an outer cylindrical-shell and cone attachment, and is so arranged that dust-laden gas enters tangentially. This causes a vortex which descends into the cone, and then ascends to an outlet concentric with the outer cylinder. The principal details of the usual commercial type cyclone are shown in Figure 118. The circular motion thus imparted to the air stream causes the suspended matter to move toward the outer shell, then fall into a collecting bin.

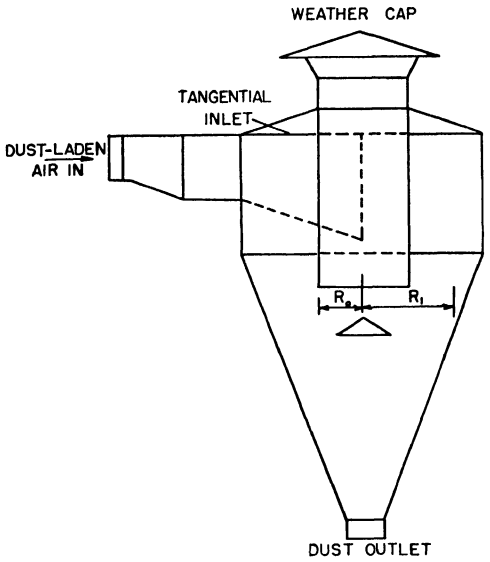


FIGURE 118. CROSS SECTION OF INDUSTRIAL-TYPE CYCLONE SEPARATOR.

In spite of a cyclone's simplicity and remarkable effectiveness in separating fine particles, few data are available on the combination of structural features most conducive to maximum efficiency. It is clear that the outer shell and cone may be either long or short, that the outlet may be of large or small diameter, and may be cut off in either cylinder or cone. The best combinations for a unit of given size can be determined only by experiment. As a general rule the outer shell should be from two to three entrance-pipe diameters in length, and the cone three times the length of the shell. The outlet diameter should have the same area as the entrance duct and extend about $^1/_2$ diameter into the cone. An inverted cone placed $^3/_4$ outlet diameter below the exit, and made somewhat smaller (30 percent) than the outlet, has been found to improve efficiency although no theoretical basis for this deflector can

be found. Many other combinations of dimensions and designs are possible. Those described above apply to a type widely used in this country.

The effectiveness of a cyclone separator depends upon its size and the size of particles to be removed. The smaller the outer shell the more effective the separation; conversely, the smaller the particle the greater the separating force required to move it from the air stream to the outer wall. The importance of cyclone size is evident from a consideration of the separation factor discussed in Chapter 2, Eq (2-46). If ξ denotes the separation factor we have

$$\xi = \frac{v_t^2}{Rg} \qquad \text{Eq (21-7)}$$

where v_t is the tangential velocity of the stream circulating about the outer shell, R the radius of the outer shell, and g the gravitational constant. From this equation we note that the separation factor (or capacity to cast particles out of the air stream) decreases with increasing size of the outer shell. To determine the effect of the inner vortex, we may assume that its radius of rotation is half that of the outer one; hence the tangential velocity of this vortex is twice that of the outer one, and we see that

$$\xi = \frac{2(2v_t)^2}{Rg} = \frac{8v_t^2}{Rg}$$

This means that the separating ability of the inner vortex is 8 times that of the outer one.

As to the distance traveled by the particles themselves, it is obvious that this varies directly as the diameter. Since the separation factor varies inversely as the diameter, we see that large cyclones are less effective in handling fine particles than would be expected from consideration of the separating factors alone.

Pressure Loss through Cyclone—Lissman (1930) has shown how pressure-drop through a cyclone may be determined. The total drop in pressure consists of the following losses: (*a*) Entrance losses including effect of enlargement; (*b*) cyclone friction loss; and (*c*) exit losses. The first and third of these losses entail no difficulty since suitable formulas are available in most texts on ventilation. We are concerned only with the pressure-drop or loss within the cyclone from the point of entrance to the point of exit.

At any two radial points, R and R_1, by the principle of conservation of angular momentum we know that $vR = v_1R_1$, where the v's denote tangential velocities at distances R and R_1, from the center. If the pres-

sures at R and R_1 are P and P_1, respectively, and the corresponding densities of the gas are ρ_0 and ρ_0' then the following equations hold

$$v = \frac{v_1 R_1}{R}$$

$$\rho_0 = \frac{\rho_0' P}{P_1}$$

The distance rate of pressure change for a unit-volume of gas at a radial distance R is

$$\frac{dP}{dR} = \frac{\rho_0' P}{P_1} \cdot \frac{v^2}{gR} = \frac{\rho_0' P}{P_1} \cdot \frac{v_1 R_1^2}{R^3}$$

which, on integrating between limits R_0 and R_1, and P_0 and P_1 gives:

$$\ln \frac{P_0}{P_2} = -\frac{1}{2}\left(\frac{\rho_0' v_1^2 R_1^2}{P_1 g}\right)\left(\frac{1}{R_0^2} - \frac{1}{R_1^2}\right)$$

If R_0 is the radial distance of the inner vortex, and R_1 that of the outer one, then according to Lissman we may put $R_0 = {}^1/_2 R_1$ (approx.) so that

$$\ln \frac{P_0}{P_1} = -\frac{3}{2}\left(\frac{\rho_0' v_1^2}{P_1 g}\right)$$

$$\frac{P_0}{P_1} = \exp\left[-\frac{3}{2}\left(\frac{\rho_0' v_1^2}{P_1 g}\right)\right] \qquad \text{Eq (21-8)}$$

Since P_1 is very close to P_0, this equation may be written as $\exp(-x) = 1-x$. Hence

$$\frac{P_0}{P_1} = 1 - \frac{3}{2}\left(\frac{\rho_0' v_1^2}{P_1 g}\right)$$

$$P_0 - P_1 = \frac{3}{2}\left(\frac{\rho_0' v_1^2}{g}\right) \qquad \text{Eq (21-9)}$$

This equation gives the draft loss and shows that pressure-drop is independent of the dimensions of the cyclone. Expressed in inches of water gauge, Eq (21-9) may be written

$$h_w = 0.29\frac{\rho_0 v_1^2}{g} \qquad \text{Eq (21-10)}$$

($\rho_0' = \rho_0$ approximately)

where ρ_0 is the density of air in lbs per sq ft, v_1 the velocity in ft per sec, and g the gravitational acceleration constant in ft per sec per sec. The total head to be supplied must also include the velocity head of the outer vortex. If this is denoted by h_t inches water gauge then the total

drop through a cyclone from the point of entrance to the point of exit is $h_w + h_t$.

Shepherd and Lapple (1939) made an experimental study of the losses above discussed. By altering the dimensions of the test cyclone and studying the velocity distribution within the cyclone, they arrived at the following empirical relationship:

$$\beta = 7.5\frac{ab}{D_0^2} \qquad \text{Eq (21-11)}$$

where β is the cyclone friction loss expressed as the number of cyclone inlet velocity heads, a the height of cyclone entrance, b the width of the cyclone entrance, and D_0 the diameter of the exit duct. Shepherd and Lapple criticized Lissman's equation, pointing out that it assumes a variation of tangential velocity along a radial line varying inversely as the radius, whereas it should vary as $1/R^n$, where n is a suitable exponent. Proceeding on this basis they found the theoretical value of β to be

$$\beta = \frac{R_e}{\sqrt{R_0 R_t}} \qquad \text{Eq (21-12)}$$

where R_e = radius at which the outer vortex velocity is equal to the entrance velocity, R_0 the radius to the boundary of the inner and outer vortex, and R_t is the inside radius of the inner vortex.

Lapple and Shepherd (1940) have given the general equations for motion of particles in a cyclone. These equations cannot be solved except by the method of approximations. If the tangential and radial accelerations are neglected, the radial velocity of the particle is given by the equation

$$v^2 = \frac{2mv_t^2(\rho - \rho_0)}{\rho_0 \rho C_R A R}$$

where m is the mass of the particle, v_t the tangential velocity of the air stream at entrance, C_R the resistance coefficient as explained in Chapter 2, R the radial distance traveled, and A the projected area of the particle. For spherical particles in streamline motion

$$v = \frac{v_t^2(\rho - \rho_0)d^2}{18\mu R}$$

The above equations hold when v is small and the tangential velocity of the particle is close to that of the air stream.

As regards the effectiveness of removal of particles Rosin, Rammler,

and Intelmann (1932) give the theoretical equation for the smallest particle removed by a cyclone under a given set of conditions

$$d_{min}^2 = \frac{9\mu D_0}{2\pi\alpha v_c(\rho - \rho_0)(4R_a/D_0)^n} \qquad \text{Eq (21-13)}$$

where α is the equivalent number of turns of the gas stream in the inner spiral of the cyclone, R_a the radius at which the spiral velocity is equal to the cyclone entrance velocity v_c, and n an exponent. Other terms are as previously defined. Values of α are constant for all cyclones of similar proportion and range from 0.5 to 3.0 For n Lapple and Shepherd give values ranging 0.5 to 0.7.

ELECTRICAL PRECIPITATION

Electrical precipitators are widely used for the collection of suspended matter. They are constructed in several forms, but basically they may be described as being of the electrode-in-cylinder or the electrode-and-plate type. The former consists of a suspended wire electrode and a concentric cylinder which is grounded. The plate type consists of a series of discharge electrodes and parallel grounded plates. Between the electrode and the cylinder or plates a high voltage is applied. In some cases dust-laden air is made to pass through a series of electrodes and plates at a relatively low sub-corona voltage for precharging, and then made to pass another series of electrodes and plates of higher voltage difference. Whatever the method of collection used, it is necessary that the electrodes, cylinders, or plates be spaced accurately and symmetrically arranged; otherwise the electrical field is altered and the collection efficiency reduced.

Motion of Particles in an Electric Field—The motion of a spherical particle obeying Stokes' law in an electric field of strength E is easily determined. When the velocity is steady, the resistance it encounters just balances the electric force on the particle. Let the diameter of the particle be d, and the viscosity of the gas in which it moves, μ. Then when the velocity becomes constant,

$$3\pi\mu v d = E\mathcal{Q} = En'\epsilon \qquad \text{Eq (21-14)}$$

when $\mathcal{Q}$ is the charge on the particle which is determinable from Eq (9-7). The total number of unit charges ϵ which we shall call n' can be obtained from the same equation by dividing $\mathcal{Q}$ by ϵ. From Eq (21-14) we calculate the velocity of the particle

$$v = \frac{E\mathcal{Q}}{3\pi\mu d} = \frac{En'\epsilon}{3\pi\mu d} \qquad \text{Eq (21-15)}$$

The value of n' is not constant, but is a function of time, Eq (9-7). The particle is gradually accelerated and must move some distance before it is fully charged, and even more before it reaches its terminal velocity. We may by use of Eq (9-7) and Eq (21-15) determine this velocity in air for a 10 μ particle having a dielectric constant $k = 2$, assuming that the particle is 90 percent charged in 10^{-1} sec, and that the field it traverses is uniform. Then if $E = 10$ esu, $u = 600$ esu per cm per sec, $\epsilon = 4.8 \times 10^{-10}$ esu and $n = 10^8$, then $un\epsilon t = 2.9$. Hence, substituting the given data in Eq (9-7), we obtain $\mathfrak{Q} = 27 \times 10^{-7}$ esu. Placing this value of $\mathfrak{Q}$ in Eq (21-15) and assuming $\mu = 178 \times 10^{-6}$ poises (20 deg C), we obtain a value of v of about 16.2 cm per sec. This is the rate at which the particle will move after it reaches its terminal velocity, assuming it receives no further charge, though it is obvious that a further increase in charge will only slightly alter this velocity. Note that the total number of unit charges received by the 10 μ particle amounts to $n' = \mathfrak{Q}/\epsilon = 27 \times 10^{-7}/4.8 \times 10^{-10} = 6520$.

If air or a gas containing particles moves between two charged plates or through coaxial cylinders, the motion of some of the particles may be complicated by the fact that the motion in the direction of the field (assuming the particles are charged) is streamline, while that along the air stream is turbulent. As a general rule the former is streamline for almost all sizes normally found in air. Since the collection of suspended particles is accomplished with air in turbulent motion, we must resort to another method to calculate the motion of the particle. If we assume that because of air turbulence the motion in the direction of the field is also turbulent, the terminal velocity is then given by applying Newton's expression of resistance to motion, Eq (2-2), so that

$$v = \frac{4}{d}\sqrt{\frac{E\mathfrak{Q}}{\pi\rho_0}} \qquad \text{Eq (21-16)}$$

where ρ_0 ($= 1.14 \times 10^{-3}$ g per cm at 20 deg C) is the density of the air and the other terms are as previously defined. If we now calculate v for the problem from this equation, we obtain $V = 170$ cm per sec, a velocity in accordance with experience.

The path followed by a charged particle in an electric field between two parallel plates is easily obtained. Assume that the plates are perpendicular to the x-axis, and that the flow is upward through them in the z-direction. Using the above equation and writing V/x for E where V is the voltage applied to the plates and $v = dx/dt$, we obtain for any two points $x = x_0$ and $x = x$ at corresponding times $t = 0$ and $t = t$

$$\frac{1}{3}(x^{3/2} - x_0^{3/2}) = \frac{2t}{d}\sqrt{\frac{V\mathfrak{Q}}{\rho_0\pi}}$$

or since x_0 is arbitrary and may be taken as the zero point of observation

$$x^3 = \frac{36t^2}{d^2} \cdot \frac{V\mathcal{Q}}{\rho_0 \pi} \qquad \text{Eq (21-17)}$$

Since the distance traveled in the time t along the z-axis is $z = v_g t$ where v_g is the air or gas velocity, then

$$z = \frac{dv_g}{6} \cdot \left(\frac{\pi \rho_0}{V\mathcal{Q}}\right)^{1/2} \cdot x^{3/2} \qquad \text{Eq (21-18)}$$

which is the equation of a cubic parabola.

If the particle motion had been streamline along both directions we have considered, we would have obtained

$$z = \frac{3}{2} \cdot \frac{\pi \mu v_g d}{V\mathcal{Q}} \cdot x^2 \qquad \text{Eq (21-19)}$$

which is the equation of an ordinary parabola. We note in passing that the effect of gravitation on the particles is neglected, which is permissible for short intervals of time for fine particles of the kind usually suspended in air or industrial gases.

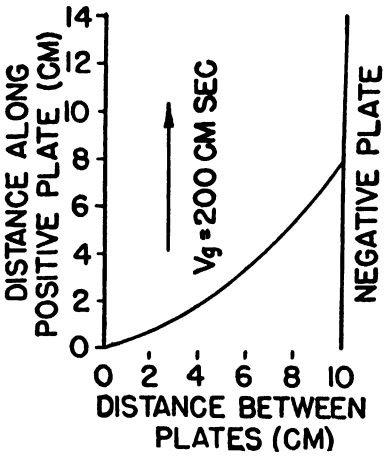


FIGURE 119. PATH OF 10 μ PARTICLE IN AN ELECTROSTATIC FIELD OF 30 KV, v_g = 200 CM PER SEC.

As an example, let us apply the data given in connection with Eqs (9-7), and (21-18) above, to determine the path of the 10 μ particle. Assume that it is initially close to the positive plate, just at the point where it and the air entering meet the uniform field. However, let the applied voltage be 30,000 ordinary volts (100 esu), the distance between the plates 10 cm, and v_g = 200 cm per sec. Substituting these and other values known or previously determined in Eq (21-18) we have

$$z^2 = 0.059\, x^3, \qquad 0 \leqq x \leqq 10 \text{ cm}$$

Plotting the data as in Figure 119 we obtain the path of the particle indicated. Hence, for the conditions stated we see that the particle is completely precipitated on the positive plate in a distance of 7.7 cm after it has received a charge of 90 percent of the total possible. The time required to precipitate the particle is 0.039 sec, or about $^4/_{10}$ the

time required to charge it to that amount. For the same particle, therefore, subject to the conditions of the problem the maximum distance of travel along the plate is about 28 cm.

Deutsch (1922) has given the following expression for determining the efficiency of collection: For cylinders

$$\frac{C_0 - C}{C_0} = 1 - \exp\left(-\frac{2vL}{Rv_g}\right) \qquad \text{Eq (21-20)}$$

and for a wire-and-plate system

$$\frac{C_0 - C}{C_0} = 1 - \exp\left(-\frac{vL}{Rv_g}\right) \qquad \text{Eq (21-21)}$$

where C_0 is the concentration of suspended matter entering the system, C the concentration leaving it, v the drift velocity of the particle as given by Eq (21-15), L the length of the collector, R the distance between electrodes, and v_g the gas velocity.

Calculation of v, the drift velocity, depends on the voltage gradient. For cylinders we may use Eq (9-6) or its equivalent $E = \sqrt{2i/u}$ in esu, where i denotes the current per cm of precipitator length. If a value of 0.01 ma is taken for i ($= 3 \times 10^4$ esu per cm) which is a reasonable value, and u as before is 600 esu per cm per sec, we obtain a value of $E = 10$ esu/cm or 3000 ordinary volts per cm. The expression to be used for a wire-and-plate system is complicated, and the procedure described above for a cylinder may be used as an approximation.

ELECTROSTATIC SEPARATION

Electrostatic separation is widely used in the mineral and food industries for separating two or more components from a mixture. The equipment is generally applied to granular material, although in recent years it has been used for separating powders. Separation by electrostatic methods is dependent only on the charge and electrostatic properties of the materials dealt with, being independent of specific gravity and shape of the particles.

The difference in charges or field strength which makes electrostatic separation possible may be due to several causes. These charges may be described according to origin as contact, conductance, pyroelectric, or photoelectric. They are generated by friction and contact with dissimilar substances as explained in Chapter 9 or by heat and light. Application of this separation method will be discussed later. Conductance charges applied to separation require special sources of electrical potential in order to condition the material. These charges arise from contact of a mixture with a charged plate or cylinder. By using an electrode of opposite polarity, certain particles are attracted and thus separated.

Nonconducting particles of the mixture are acted upon by induction so that they acquire equal and opposite charges, the total net charge being zero. On the other hand, conducting particles present in the mixture permit the flow of electricity, with the result that they assume a charge of the same polarity as the plate or cylinder. An electrode of polarity opposite that of the plate or cylinder will therefore attract the conducting particles.

Pyroelectric charges are generated in certain materials by heating or cooling them, or by subjecting them to pressure. The stresses set up

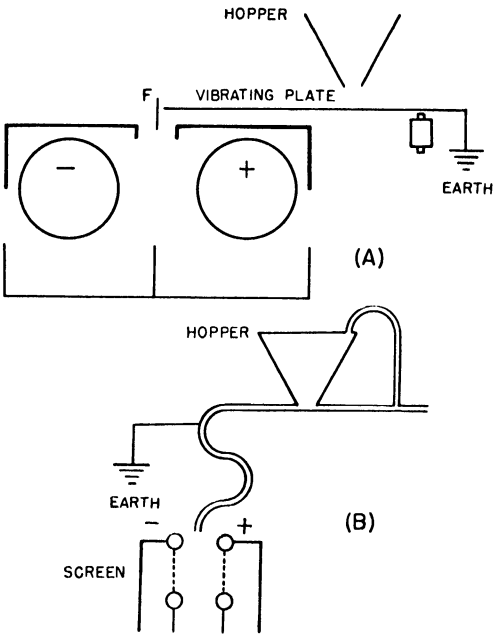


FIGURE 120. ILLUSTRATING TWO TYPES OF ELECTROSTATIC SEPARATORS.

in the crystals produce opposite charges at their extremities. When one of these charges is neutralized by the charged surface of a plane or cylinder, the particles are left with a charge corresponding to the contact surface and therefore subject to the attractive force of an oppositely charged electrode. Photoelectric charges are produced only on substances which react to light radiations of the proper frequency. Up to the present time they have not been widely used in electrostatic work.

Contact Separators—Two simple designs for contact separators, described by Fraas and Ralston (1940), are shown in Figure 120. Sepa-

rators depending on contact charges require a movement of the constituent materials. Thus, sketch *A* of the figure shows a vibrating plate on which is placed a thin layer of a mixture of material from a hopper. These vibrations cause the particles to move toward a baffle plate *F*, where they are dropped vertically through an electric field and separated according to the polarity of their charges. It is known that vibrating surfaces enhance the charges on particles. The vibrating plate is electrically conductive and held at a potential about midway between the contact potentials of the materials to be separated. Thus the plate will be positive toward one constituent and negative toward the other

Another form of contact separator is shown in *B* of Figure 120. This type of device is suitable for material of small particle-diameter ($<40\ \mu$). Particles from the hopper are passed into a stream of dry gas, then through a sinuous passageway where they become electrically charged.

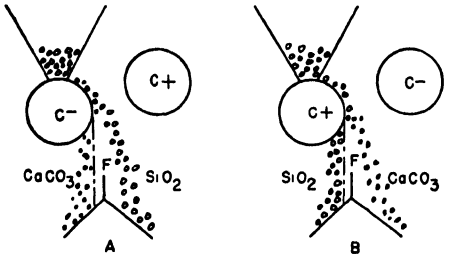


FIGURE 121. ILLUSTRATING ELECTROSTATIC-CONDUCTANCE SEPARATOR.

On leaving the end of the passageway, the particles move through two parallel grids of opposite polarity and are separated according to the polarity of their charge.

Conductance Separator—The conductance separator is an extremely simple device, as may be seen by reference to Figure 121. A mixture flows from a hopper over a charged revolving cylinder $C-$, the particles for the most part being one layer in depth. The cylinder moves so slowly that centrifugal forces do not come into play. Cylinder $C-$ may be designated as the material-conveying electrode. The revolving electrode $C+$ is oppositely charged so that an electric field is set up between the cylinders. The $C-$ electrode is usually grounded while $C+$ is charged with static electricity from a high-voltage vacuum-tube rectifier.

Equipment in general use has the $C-$ cylinder in a fixed position. The strength of the field is varied either by moving $C+$ closer to $C-$, or by increasing the voltage on $C+$, or both. The dividing gauge *F* is ad-

justable so that material attracted by $C+$ falls to the right of the gauge, and that which is unaffected falls directly, due to gravity.

It is obvious that in the contact or other method of separation much depends upon the surfaces of the particles. These surfaces may be dry or wet, or treated in various ways to alter their electrical properties.

Not all substances exhibit the same degree of electrostatic behavior. However, it is possible to arrange dielectric substances so that ability to separate is exhibited in a remarkable manner. For example, it is well known that when two different dielectric substances are rubbed together one acquires a positive charge and the other a negative charge. If these various substances are studied as to properties exhibited between any two, it will be found that they may be arranged so that certain ones will predominate in acquiring a positive charge. The relative characteristics of a selected group of dielectric substances are shown in Table 85. The substances are so arranged that when any two are

TABLE 85—POSITIVE-NEGATIVE SERIES OF COMMON MATERIALS[a] (BULLOCK, 1941

1. Fur	8. Paraffin wax
2. Flannel	9. Ebonite
3. Ivory	10. The hand
4. Glass	11. Metals
5. Cotton	12. Sulfur
6. Paper	13. Celluloid
7. Silk	14. Rubber tubing

[a] The list is so arranged that if any two materials are placed in rubbing contact and separated, the first on the list will become positively charged while the other will be negative.

rubbed together, the one higher in the list receives a positive charge and the lower one a negative charge. Thus, it is seen that there is a difference in electrostatic intensity between the components of a mixture of particles such that separation of these components in a suitable electrostatic machine is facilitated.

Bullock (1941) has carefully examined the applications of electrostatic separation and points out that while such separation methods are sensitive the electric forces generated in any machine are very small. Hence, the applications of electrostatic principles to all phases of the separation field are not as universal as might be supposed. Certain substances must be dried or chemically treated, and nearly all must be ground to an economical size for handling in machines. This latter step entails a certain loss in fines. A 6-mesh grind is recommended though 60/120 mesh material can also be separated. The relative costs of grinding and sizing are important considerations in the selection of an electro-

static technique if economics of separation are considered. On the other hand, electrostatic separation is the only method which can be used in many instances, and when successful, can handle large quantities of material at low cost. The life of an electrostatic unit probably exceeds that of any other method of separation.

Pyroelectric Separator—The pyroelectric effect is used mainly for the separation of quartz from feldspar. The mixture is heated in the hopper feeder by means of steam. On passing to a cold rotating cylinder below, the material causes pyroelectric polarization to appear on the quartz. This mineral adheres to the cylinder, while the feldspar is not affected.

Problems

1. Show how the use of sawdust mixed in an air stream containing dust may be used to prolong the use of a filter.
2. Compute the draft loss through a cyclone separator in inches of water gauge, given the entrance velocity as 3600 ft per min. Take the density of air as 0.075 lb per cu ft.
3. The dust load entering a 10 in. diameter precipitator is 8 grains per cu ft. If the length of the precipitator is 10 ft and the gas velocity 500 ft per min, compute the concentration of dust leaving the device.
4. Discuss the general theory and limitations of electrical precipitators.
5. Show that one inlet velocity head of any gas in inches H_2O is equal to 0.003 $\rho_0 (Q/A)^2$ when ρ_0 is the density of the fluid in lb per cu ft, Q the volume of flow in cu ft per sec, and A the area in sq ft of the duct through which the gas flows.

CHAPTER 22

THEORY OF FINE GRINDING

THE purpose of grinding is to subdivide a given substance in order to achieve certain results. Thus, in concentration of ore by flotation methods the degree of fineness in grinding operation is determined by the size of crystals containing a maximum amount of the mineral desired. Again, other grinding processes are for the purpose of grinding new surface either for roasting and oxidation, or for chemical treatment. Finally, there are certain materials whose physical properties are of value only when they exist in a fine state of subdivision.

FRIABILITY

The tendency of certain substances, especially coals, to break into smaller sizes in the course of handling is termed friability. Measurement of this characteristic has broad economic implications; for instance, in handling coals it is important to know what may be expected to happen in the course of treatment from the mine, through the washing and screening processes, in shipment, and in other stages of handling until reaching the consumer. Measurement of this property gives an *a priori* estimate of the severity of treatment to which a given substance may be subjected.

Friability depends upon a number of factors, among which are age, degree of weathering, temperature, and moisture content. In geological formations, local faulting and earth pressures may seriously affect this property. It must be remembered that friability and grindability are not identical properties; in the first instance we are concerned only with fragility in normal handling, while in the second case we must consider the extent of forces necessary to produce a given size-distribution.

While friability is a property that may be attributed to crystals and many chemical compounds, tests devised thus far pertain only to coals; however, the same tests may be used on other materials having the same relative degree of friability. Several methods have been used to determine friability, and are based on two procedures: (*a*) Measurement of work done in reducing coal of a given size to a smaller size, or (*b*) subjecting the coal to a definite amount of work and measuring the reduction in coal size due to such work.

The first procedure, developed in England by Stopes and Wheeler (1923), consists in grinding a certain weight of coal in a ball mill, run under fixed conditions and for a stated length of time. The coal is then removed from the mill and the weight of the coal passing a 200-mesh used as a measure of friability. In this country the second procedure has been more widely used, resulting in two more or less widely accepted tests.

The first of these tests was developed by the University of Illinois Engineering Experiment Station, and forms one of the tentative A.S.T.M. standards (Tentative method of drop shatter test for coal—A.S.T.M. Designation D 440 – 37 T). The drop shatter test, as developed by the University of Illinois, consists in dropping 60 pieces of 3- to $2^1/_2$-in. coal on a concrete floor from a height of 10 ft. The shattered product is carefully collected, screened according to a fixed procedure, the material on each screen counted and weighed, and the size and weight of each fraction reported.

The second test, known as the "tumbler-test" method, also forms one of the tentative A.S.T.M. standards (A.S.T.M. Designation D 441 – 37 T). The apparatus for this test consists of a $7^1/_4$-in. diameter pebble mill in which a 1000-g sample is rotated at a speed of 40 rpm. No pebbles or balls are used in the jars. Three lifting shelves 120 deg apart, mounted on a removable circular frame, are provided; the frame is fastened to the jar during the test. The jar (four jars containing the same sample are usually run simultaneously on one machine) is turned for one hour at the speed indicated.

In conducting the tumbler test for friability, great care must be taken in selecting and preparing the sample. Since the procedure for friability determinations requires the use of 1.5- to 1.05-in. coal, it is necessary that samples be selected from this size range. It is also necessary that each lump be free from cracks or any indications of faults which might bias the results. The lumps selected are carefully screened through a 1.5-in. screen and those sizes retained on the 1.05-in. screen to the amount of 1000 g are placed in the pebble mill and turned as described above.

After being subjected to 2400 revolutions in the course of an hour the material is screened on a series of square-meshed sieves, 8 in. in diameter, with the following size openings: 1.05 in., 0.742 in., 0.525 in., 0.371 in., 0.0469 in., and 0.0117 in. The particle-size of the material remaining on a given screen is assumed to be the average of size openings in two successive screens—the one through which the material passed, and the one on which it is retained. In the A.S.T.M. procedure, the average size of two successive screens is divided by the average size of the coal before the test, *e.g.*, $(1.50 + 1.05)/2 = 1.275$. This gives a factor which is then

multiplied by the weight of coal in its size range. The result of each multiplication performed in this way is termed the "effective size stability" and is usually expressed as a percentage. In turn the sum of these effective size stabilities for all seven screens used is called the "total size stability." Friability is then defined quantitatively as follows:

$$\text{Friability (percent)} = (100 - \text{total size stability})$$

A typical tabulation of a friability test is shown in Table 86.

TABLE 86—RESULTS OF FRIABILITY TESTS ON A CERTAIN COAL

Retained on sample 1.05 in.	Passing sieve size 1.5	Weight (percent) (1) 100.0	Average of screen openings In. (2) 1 275	Factor (3) 1.0	Effective stability (1) × (3) 100
Tumbled coal					
1 05	1 5	35 3	1.275	1.0	35.30
0.742	1.05	19.1	0 896	0.7	13.37
0.525	0 742	9.2	0.634	0 5	4.60
0 371	0 525	8 0	0.448	0.35	2.80
0.469	0.371	5.2	0.209	0.15	0.780
0.117	0.469	3 1	0 029	0 025	0.078
0.0	0 0117	20.1	0 006	0 005	0.100
Total size stability					57.028

Friability = (100 − 57.03) = 42.97 percent.

The percent of fines passing the 0.0117-in. screen is considered breakage due to attrition or abrasion rather than shattering, and is reported as "dust index." In the above example this is 20.1 percent. Note that the higher the friability value, the more easily the material under test degradates.

GRINDABILITY

The relative grindability of a material may be defined as the work necessary to produce a given amount of new surface. So far a general method for determining grindability is lacking. The A.S.T.M. recognizes two methods for determining grindability of coals. These are: Test for Grindability of Coal by the Ball-Mill Method (Tentative) D 408 – 37 T, and Test for Grindability of Coal by the Hardgrove-Machine Method (Tentative) D 409 – 37 T. These methods are based on the principle of performing a constant amount of work. We shall briefly discuss the details of the A.S.T.M. ball-mill method only, which appears

to have received a greater acceptance than the Hardgrove method.

The ball-mill method consists of a porcelain or steel jar with three lifters spaced at 120 deg. The mill is charged with 100 polished steel balls, each 1 in. in diameter, and a 500-g sample of coal which has been carefully screened on a U. S. standard No. 12 sieve, 10 in. in diameter (aperture openings 0.0661 in.). The mill is rotated at 40 rpm for 50 revolutions; the contents are then emptied into a 4-mesh riddle screen to separate the steel balls from the coal. The coal is carefully screened through a 200-U. S. sieve, 10 in. in diameter, until not more than $^1/_2$ g passes in 1 min of sieving. The weight of the undersize is determined and the number of revolutions required to increase its weight to 50 g or 10 percent of the sample is calculated by direct proportion. The total number of revolutions required for grinding 10 per cent of the sample fine enough to pass 200-mesh is considered one cycle. The oversize and undersize are returned to the mill and another cycle completed. The mill is again emptied, the contents screened as before, the undersize discarded, and the oversize returned to the mill for another cycle. These cycles are repeated with due regard to maintaining correct number of revolutions per cycle until at least 80 percent (or 400 g) of the original sample has passed the 200-mesh sieve. The grindability criterion is based on the number of revolutions required to grind exactly 80 percent of the sample so as to pass a 200-mesh sieve. Grindability is defined as 50,000 divided by the number of revolutions required to grind 80 percent of the charge fine enough to pass a 200-mesh sieve. Thus, a coal sample requiring 500 revolutions to produce this result would have a grindability index of 100 percent. The factor 50,000 is arbitrary, and founded on accumulated experience that no coal can have a grindability criterion as high as 100 percent. Thus, by the technique outlined, the higher the grindability value the easier the substance is to grind.

The technique outlined above gives reproducible results and for this reason is widely used. No such simple method has been devised for the harder rocks and minerals. There are some data in the literature on the efficacy of certain mills and the grinding of various substances, but so much depends upon such factors as design of mills, speed of rotation, size of load, grinding media, packing of fines, etc., that comparison of the grindability for two different materials seems out of the question. For a given material, however, the difficulty is lessened and a standardized procedure is possible as in the case of coal above described. Where it is desired to determine grindability of a given rock, mineral, or other substance under various operating conditions it should be borne in mind that the basic unit of comparison is the amount of new surface created per unit-weight per hp, per given unit of time.

TYPES OF MILLS

Many different forms of equipment are used for fine grinding. One of the oldest types, and still the most successful, is the ball mill: A cylinder mounted horizontally with separate heads or trunnions attached to the ends and resting on suitable bearings. The trunnions are hollow for the introduction and discharge of material; the cylinder is partially filled with steel balls and a charge of the material to be ground. The individual balls may vary from $^1/_2$ to 5 in. in diameter and may occupy from one-third to one-half the mill volume of the shell. The material is fed to the ball mill through one of the hollow trunnions by means of a chute; the material and balls are then rotated. The power required for driving a ball mill depends on the peripheral speed of the mill, the weight of the total charge, the contour of the cylinder lining, and the load level within the mill. Horsepower requirements for most commercial types are expressed in terms of the length of the cylinder. In general, the amount of power varies from approximately 2 hp per ft of cylinder length for a 3-ft diameter mill, up to 30 hp per ft of length for a mill 8 ft in diameter.

Another type of mill commonly used for fine grinding is known as the "rod mill." In general construction the rod mill is similar to a cylindrical ball mill, except that the cylinder length is approximately twice the diameter. The grinding medium consists of steel rods from 2 to 5 in. in diameter and a few inches shorter than the length of the mill cylinder. Rod mills are operated at speeds varying from 25 rpm for a 4-ft diameter mill to 13 rpm for an 8-ft mill. The power requirements range from 5 to 20 hp per ft of length, depending upon the diameter of the mill. The rod mill produces a more uniformly sized product than the ball mill. It is considered a good granulator when grinding friable ores or minerals, and when grinding 10- or 20-mesh material it produces a minimum quantity of fines.

Classification of Grinding Operations—Grinding may be accomplished either wet or dry, and in what are called "open" or "closed" circuits. When grinding wet, sufficient water must be added to give the mixture a cream-like consistency. In dry grinding the moisture must be maintained at a minimum so the ground material will not stick. Open-circuit grinding may be used with either wet or dry methods; it consists in feeding material at one end of the mill and discharging the ground product at the other. This is a simple procedure and is generally employed for relatively coarse products; its efficiency for fine grinding is considered low. As the name implies, closed-circuit grinding consists in conveying the finished material to some type of sizing or collecting device and returning the oversize to the mill for regrinding. With constant re-

moval of fine materials in this fashion the mill has increased capacity. Closed-circuit grinding can also be applied to either wet or dry grinding. The fines in closed-circuit grinding may be removed continuously and passed on to screens, the oversize being returned to the mill; or an air stream may be fed through the mill at such velocity that particles smaller than a desired size are carried off to a separator or classifier. Thus, if extremely fine particles are desired the oversize may be returned to the mill for further crushing.

CRITICAL SPEED OF BALL-MILL GRINDING

The speed of grinding operations is determined by the diameter of the mill, the size of the material entering the mill, the size of the product desired, and whether the material is being ground wet or dry. It is customary to express the speed of a ball-mill grinder in terms of a "critical speed," at which the centrifugal and centripetal forces, acting on a small particle traveling on and with the breast of the mill, are equal. Under this condition the particle remains in position and becomes a part of the grinder cylinder. The critical speed (rpm) is given by the formula:

$$\omega_{cs} = \frac{54.2}{\sqrt{R}} \qquad \text{Eq (22-1)}$$

where R is the radius of the mill in ft. Critical speed is independent of the density of the material. Some criticism has been expressed regarding the use of this formula; many practical engineers feel that the theoretical speed can rarely be applied in practice because of the variables involved in every case. Nevertheless, in testing equipment and evaluating power requirements, critical speed is a valuable index for determining the performance of a given machine. In fact, there is no other means of safely comparing grinders of different size.

Ball Loads—Davis (1919) developed a method for selecting the proper types of balls to attain a given grind. He took into account the rate of wear of different sizes of balls and derived a load balanced against such wear when the renewal or rejection of various sizes was known. Davis' formula made no allowance for the nature of the material being ground. Coghill and DeVaney (1937) developed a so-called "rationed" load based on a consideration of particle-size of the original material to be ground and particle-size of the final product.

Davis assumed that the rate of wear for the different sizes of balls in a mill was proportional to the weight of each ball. The formula developed by Davis has wide application, and has been used by operators to determine the quantity of new balls which must be put in a mill to replace those which are worn. The formula requires that the largest ball size

and the size to be rejected be determined, and after that the other sizes are set. Some differences of opinion exist regarding the Davis formula, particularly in grinding material to an extremely fine size. Tests by various experimenters resulted in methods for determining optimum size of balls for grinding a material with given size characteristics. Figure 122 indicates what may be achieved with grinding media of different sizes. All elements of the tests, from which the curves shown in the figure were obtained, were held constant. It will be seen from the figure that the largest balls failed in the coarsest sizes due to the few points of contact with the grains. On the other hand, when they did contact a particle the larger balls imparted such a strong crushing impulse that some sub-sieve

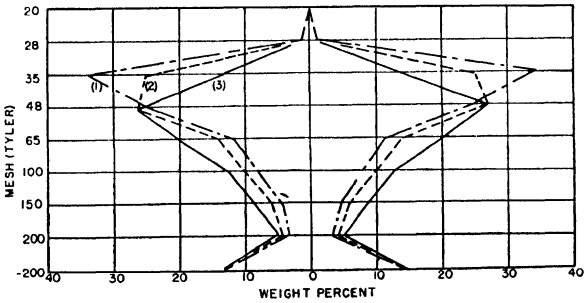


FIGURE 122 SIZING DIAGRAM OF BALL-MILL PRODUCTS FROM (1) OVERSIZE, (2) UNDERSIZE, AND (3) OPTIMUM SIZE BALL. FEED 28/35-MESH CHERT.

sizes resulted. This was not true of the smallest balls although they had many more points of contact. However, a few of the particles were reduced in size and again comminuted to sub-sieve sizes. It is clear that between the two extremes there is one size of ball which gives a uniform gradation through all sizes of material. The optimum ball size, as determined by Coghill and DeVaney for closely sized ore, may be expressed by the following equation:

$$D^2 = Kd \qquad \text{Eq (22-2)}$$

where D is the diameter of the ball, d the diameter of the particle to be ground, and K a constant depending on the character of the material to be ground. When D and d are expressed in inches, the value of K for chert is 55, and for dolomite 35. Starke (1935) had previously obtained an expression similar to the foregoing equation.

It must not be assumed that the power requirements will remain con-

stant for a given size ball. Figure 123 taken from data obtained by Coghill and DeVaney illustrates the variation in net horsepower with time, when different balls are used. The data are for a 14/20-mesh dolomite. Results with 2.75-in. balls show that less horsepower is required than with any other grinding medium, but for the particular material used the balls were too big, and as grinding continued the material tended to stick to the balls, with attendant reduction in power consumption. The final material showed very poor particle-size distribution. While this effect with large balls is noteworthy, it must be remembered that certain materials frequently cause the same results even with what might be considered optimum ball size. The reason for this is the formation of a tenacious coating which acts as a buffer and tends to reduce grinding efficiency. This buffer may be removed in many instances by a suitable wetting agent, although care must be used with such agents so that the physical properties of the final product will not be harmed.

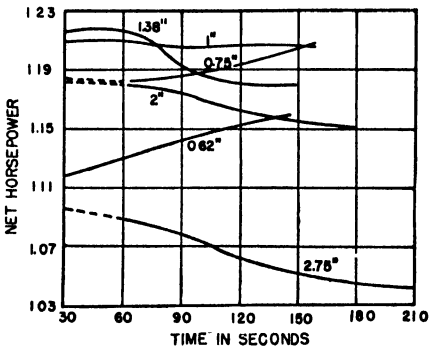


FIGURE 123. POWER CHANGE DURING GRINDING PERIOD FOR BATCH GRINDING WITH VARIOUS BALL SIZES. CHARGE, 50 LB 14/20-MESH DOLOMITE. PULP CONSISTENCY 60 PERCENT, CRITICAL SPEED 50 PERCENT.

In a series of tests on grading cement clinker with various types of mills, Starke arrived at the following major conclusions:

1. During the early periods of grinding in batch mills, the specific surface of the charge increases approximately as the time, but during later periods the rate of increase of specific surface tends to diminish.
2. Mill efficiency is influenced by the rate at which material is fed in a continuous-discharge mill.
3. A mill is most efficient in producing surface when the ratio initial diameter of the particle/(ball diameter)^2 is approximately 600 $\times$ 10^{-8}, the diameters being expressed in microns. The optimum value of this ratio depends upon the characteristics of the material ground.

With regard to the first of these conclusions, namely that the surface of a product is increased in proportion to the length of period of grinding, Starke found a straight-line relation for particles of large mesh. For smaller-mesh sizes it was found that as time increased the lines had a

tendency to curve slightly downward. Starke also found that the grinding effectiveness of a given size of balls is markedly affected by the initial size of the particles being ground. While the initial surfaces of the finer meshes were very high after prolonged grinding, the specific surfaces for this size varied appreciably, in comparison with the product of the larger screens.

With reference to development of the proper size of ball diameter to be used, Starke found that the relation given by Eq (22-2) holds over a wide range of conditions. The curve shown in Figure 124 indicates the relation

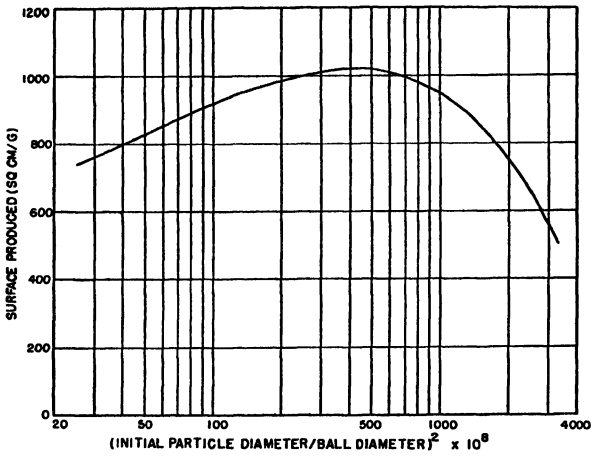


FIGURE 124. STARKE'S CURVE OF THE RELATION BETWEEN SURFACE PRODUCED AND RATIO OF PARTICLE TO BALL DIAMETER. CURVES ARE FOR BATCH GRINDING OF CEMENT CLINKER. MILL DIAMETER 3 FT, LENGTH 3 FT, RPM 33, TOTAL REVOLUTIONS 800. BALL SIZES $^1/_8$, 1, $1^1/_4$, $1^1/_2$ INCHES.

obtained by plotting the surface produced against the Starke ratio law. Regarding the analysis of particle-size distribution, it was found that grinding at Starke ratios greater than the optimum results in products with very small weights of particles between one-half and one-tenth the original size. In other words, for large values of the Starke ratios the product is composed essentially of the original-size materials and very fine particles. On the other hand, in grinding at small ratios, values of 100×10^{-8} or less yielded products with appreciable weights of particle-sizes smaller than the original size. In every case, as may be seen by reference to Figure 124, the specific surface of the total product is essentially greatest when the optimum Starke ratio is used. Materials of equal specific surfaces produced at different grinding ratios will not neces-

sarily have similar distributions of weight to particle-size, but materials ground to equal specific surfaces at equal Starke ratios will probably have similar distributions of weight to particle-size. The products of the smaller balls at the smaller ratios differ widely from those produced by larger balls at the same Starke ratios.

RELATIONSHIPS BETWEEN GRINDING-MILL VARIABLES

Relation between Grinding Medium and Material Ground—In a ball mill operating at a given speed it is natural to assume that the results obtained will depend on the kind of ball material used. As the mill revolves the grinding medium is set in motion and assumes an oval position at the ends. One part of the load is pulled under the mass, and another part travels on the outside of the batch. The part rising to the top drops or cascades and is folded into the mass. Thus, reduction in particle-size is attained by a combination of rubbing, attrition, and impact, set up by contact of the balls with themselves, the material, and the outer cylinder of the mill. In general, smaller balls grind faster because there are many more contacts.

Coghill and DeVaney (1937) investigated the effect of density of grinding media on power requirements, and size-distribution of material produced through use of pipes stuffed with different materials so that specific gravities varying from 2.89 to 7.30 could be obtained. These pipes measured $1^3/_8$ by 35 in. and were placed in a mill 19 by 35 in. The charge was Tri-State chert, and the grinding was done wet at a fixed consistency (60 percent solids). The investigators found that for a given speed the ratio of total weight of ore plus water, plus grinding medium, was constant, leading to the conclusion that the power varied as the total weight of the contents of the mill. The relation between total weight of charge per horsepower and the critical speed may be shown to be given by the following equation

$$\frac{W}{E} = \frac{K}{\omega_{cs}}$$

where W is the total weight of the charge, E the power required, ω_{cs} the percent critical speed, and K a constant. When the same set of conditions was analyzed as to particle-size for a 50 percent critical speed, it was found that the media having a lighter density were not as effective as the heavier media. Results of the study were not as striking as might be expected, but did show that for chert the heavier grinding media (all other factors being the same) yielded a finer product. This may not be the case for softer materials and the results should not be generalized.

When tests were made to compare large balls and small rods (both having the same volume) it was found that although the rods required more power they gave a much better (more uniformly fine) product. This is in accordance with results generally obtained in rod mills. Individually the rods were heavier than the balls, the rods weighing 7 lb and the balls 5 oz each; on the other hand, the diameter of the rods was smaller, the rods measuring 35 in. in length. Finally, we must consider various types of media, nonferrous and ferrous. Nonferrous media include pebbles, generally consisting of quartzite. The pebbles are irregular in shape but are considered effective and economical in many operations. Pebbles have a relatively low density, and while not as effective as denser media the power requirements are proportionately lower since, as has been shown, the power varies as the weight of the medium, plus the charge, plus water (if used). The hardness of various media affects fineness of grind and power requirements. Coghill and DeVaney showed that for chert and dolomite approximately 10 percent less power is required for annealed balls having a Brinell hardness of 150 than for nickel balls having a Brinell hardness of 548. With regard to fineness, the nickel balls gave slightly better grinding.

Charging rates of ball and mix are important inasmuch as grinding is a function of the number of ball contacts. If the charge does not exceed the volume of voids in the ball mass, optimum grinding conditions are obtained. Grinding conditions in excess of this volume may markedly decrease the efficiency of the mill. On the other hand, a charge smaller than the volume of the balls has only slight advantage. The total charge of balls or pebbles and mix in relation to volume of the mill is important chiefly as it affects the motion of the charge. A mill completely filled does no grinding. The height from which the balls may fall or cascade affects the optimum grinding conditions. Ball charges consisting of mixtures with two or more sizes have a purely additive effect. The rate of grinding is proportional to the number of ball-surface contacts and is greatest with very small balls. When two or more sizes are used there is also a tendency for the balls to classify.

As might be expected, the wear of the balls is in proportion to the number of contacts and to the nature of the material being ground. When grinding pigments in a viscous medium best results are obtained: (1) When the balls are permitted to cascade freely; (2) when the charge approximates the volume of the ball space of the grinding medium; (3) when the mill is operated at approximately 50 percent critical speed; and (4) when the grinding operation is started with a charge and vehicle of slightly higher viscosity than considered desirable for maximum ball mobility.

In general, the viscosity of a ground pigment and vehicle increases in proportion to the increasing degree of dispersion. In this connection Fischer (1941) states that the change in viscosity may be considered a measure of the new surface exposed at the higher pigment interface.

Effect of Grinding Speed—Too much emphasis cannot be placed on the fact that the peak of a power curve comes at a lower speed than that which centrifugalizes the outermost layer of balls. It may be shown that the power peak comes when the outer circle of the balls reaches the breast

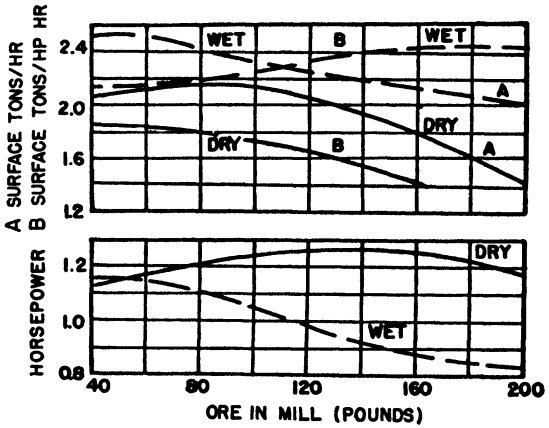


FIGURE 125. COMPARISON OF WET AND DRY BATCH GRINDING OF CHERT AT 50 PERCENT CRITICAL SPEED. WET GRINDING AT 60 PERCENT PULP CONSISTENCY.

of the mill, in what may be called the eight o'clock position. The power curve drops when the balls are higher or lower than this position. Tests have shown that the maximum power is consumed at a much lower speed than necessary to centrifugalize the balls. In general, however, ball-mill efficiencies do not vary greatly between speeds ranging from 30 to 80 percent critical speed, although in the case of rod mills the efficiency drops off markedly at high speeds.

Wet and Dry Grinding—On the basis of experiments made with chert, Coghill and DeVaney concluded that for best capacity and efficiency dry grinding should be done with small ore charges and at high speeds. The same is true with wet grinding in regard to capacity, but the efficiency is best with a large ore load at low speed. However, the values for capacity are more significant than those for efficiency since

efficiency values do not indicate the nature of the grind. Figure 125 gives the characteristics of wet and dry grinding for a set of conditions indicated. The lower figure shows that in wet grinding the power decreases with increased amount of ore charge, whereas in dry grinding the power increases. In wet grinding also the capacity is best with a moderate ore charge, but a large ore charge is required for good efficiency. In Figure 126 in which the speed varied, the capacities increased with speed and the efficiencies changed but slightly. As indicated previously the efficiency in dry grinding is slightly better at a high speed. The power curves in Figure 126 show that there is a break in the dry-grinding curve. The relatively high powers in dry grinding at from 40 to 60 percent critical speeds show that the balls are thrown high and reach the eight o'clock position at a lower speed than in wet grinding. The eight o'clock position in dry grinding is attained at 80 percent critical speed. The effect of grinding the same material in wet and dry conditions with varying load for a fixed critical speed is shown in Table 87.

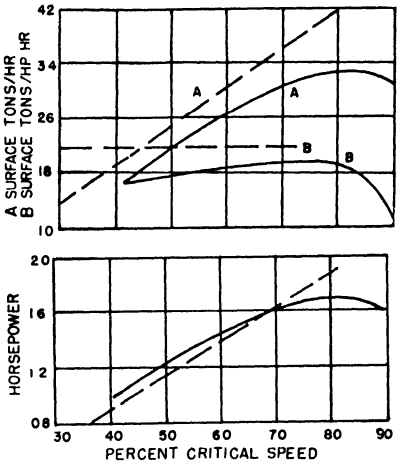


FIGURE 126. COMPARISON OF WET AND DRY BATCH GRINDING AT VARIOUS CRITICAL SPEEDS. LOAD 75 LB CHERT. WET GRINDING (BROKEN LINES) AT 60 PERCENT PULP CONSISTENCY.

With regard to this table, Coghill and DeVaney stated as follows:

". . . . The conclusions to be drawn are not in harmony with some of those in the literature where experimental evidence was too scanty to justify the broad statements that were made. Such statements have prompted the compilation (of the table) now to be discussed. . . Throughout the table it holds true that with the heavy ore charge the coarse particles were reduced more in wet than in dry grinding. The reverse is true when the ore charge is light. Midway, that is, at about 75 pounds of ore, the degree of selective grinding was about the same. With that charge closed-circuit work would be expected to develop the same type of circulating load, whether wet or dry.

"In wet grinding the heavy ore charge in the mill would give a closed-circuit product with the minimum amount of coarse particles, but in dry grinding the advantage would be in favor of the small ore charge. Hence, it is seen that the writer who said that 'the median diameter of the circulating load is larger when grinding wet than when grinding dry'

TABLE 87—TYPE OF GRIND WITH DIFFERENT AMOUNTS OF ORE IN WET AND DRY BALL-MILLING OF CHERT (TOP TABLE), AND DOLOMITE (BOTTOM TABLE)

Data Are for a Cylindrical Mill 19 × 36 In., Batch Circuit 50 Percent Critical Speed; Ball Load 28 Percent 2½ In., 40 Percent 2 In., 18 Percent 1½ In., 7 Percent 1¼ In., 7 Percent 1 In. Consistency 60 Percent Solids; Load 796 Lb; Volume 45 Percent

Size, mesh	Feed, weight percent	Products, weight percent: 200 lb Wet	200 lb Dry	150 lb Wet	150 lb Dry	125 lb Wet	125 lb Dry	100 lb Wet	100 lb Dry	75 lb Wet	75 lb Dry	50 lb Wet	50 lb Dry	35 lb Wet	35 lb Dry
10	25.2	0.4	1.7	0.3	1.4	0.4	1.8	0.5	1.3	1.6	1.2	3.7	1.4	4.5	2.3
14	55.0	3.3	9.3	3.2	8 1	3.1	8.4	3.2	6.6	6.0	6 0	10.4	6.2	12.4	8.7
20	76.7	14.0	24.3	13.1	21.1	12.8	22.1	12.2	18.0	16.0	17.2	20.7	16.5	21.5	21.0
28	87.2	30.0	39.8	28.4	36.4	28.3	37.6	27.8	32.8	30.9	32.1	33.6	30.7	34.5	34.2
35	93.4	48.8	52.6	46.7	49.8	46.9	51.3	46.9	47.1	48.2	47.1	49.4	45.4	49.6	48.7
48	96.2	60.5	62.8	58.7	60.6	59.0	62.4	59.8	59.1	61.0	59.9	61.1	58 2	60.5	61.1
65	97.9	70.7	70.6	68.8	69.0	69.2	70.7	70.5	68.2	70.9	69.5	71.0	67.9	69.9	70.5
100	98.9	77.7	77.6	76.3	76.4	76.6	78.0	77.8	76.2	78.1	77.9	78.2	76.4	77.2	78.5
150	99.4	83.1	82.0	82.0	81.2	82.2	82.6	83.2	81.4	83.3	83.1	83.5	81.8	82.5	83.5
200	99.6	86.4	85.5	85.5	84.9	85.9	86.1	86.6	85.3	86.5	87.0	86.9	85.8	85.9	87.1
−200	0.4	13.6	14.5	14.5	15.1	14.1	13.9	13.4	14.7	13.5	13.0	13.1	14.2	14.1	12.9
10	15.3	0.6	2.9	0.6	2.2	0.6	2.1	0.5	2.0	1.8	2.1	3.6	1.8	..	2.1
14	50.8	6.4	15.6	5.8	12.5	5.9	12.5	5.8	11.8	10.7	11.4	16.2	10.3	..	11.2
20	79.5	20.9	35.5	20.0	31 2	19.8	31.1	19.5	29.6	27.0	29.1	32.5	27.1	..	27.9
28	90.2	37.7	51.1	36.8	47.0	36.0	47.5	36.7	46.1	42.9	45.4	46.4	43.1	..	44.1
35	95.4	53.7	62.0	53.3	58.9	52.5	59.9	53.8	58.4	58.1	57.7	59.1	56.0	..	56.8
48	97.3	62 8	69.6	62.5	67.3	61.8	68.7	63 5	67.2	66.6	66.9	66.5	65.4	..	66.4
65	98.2	69.5	74.5	69.2	73.0	68.6	74.3	70.6	73.2	73.0	73.1	72.2	71.7	..	72.5
100	98.6	74.9	78.8	74.6	78.1	74.2	79.2	76.0	78.4	77.8	78.3	76.9	77.2	..	78.0
150	98.8	79.2	81.8	78.8	81.7	78.7	82.6	80.1	82.0	81.6	82.1	80.8	81.1	..	81.7
200	99.0	82.6	84.6	82.0	84.9	82.4	85.6	83.3	85.3	84.5	85.3	83.8	84.7	..	85.0
−200	1.0	17.4	15.4	18.0	15.1	17.6	14.4	16.7	14.7	15.5	14.7	16.2	15.3	..	15.0

did not realize that the median diameter did not depend on the amount of ore in the mill."

As to an outright comparison of wet and dry open-circuit ball milling, wet grinding is much more efficient and capable of greater capacity than dry grinding.

Effect of Classifier—Since in many kinds of grinding it is found that fines impede the efficiency of grinding, it is desirable to have some method for their continuous removal. This is done in dry grinding with an air classifier or water classifier. The former is more widely used and is described here. The method consists in forcing an air stream through the mill in such a volume that the fines are conveyed out of it to a separator device. The volume is so adjusted that only particles below a given size are removed.

The operation of an air classifier in a closed circuit may take place in either of two ways: (1) The product of the mill may be conveyed to the classifier as produced; or (2) the air classifier may be placed in circuit with the mill in such a way that the fine particles are carried off as soon as produced. The latter procedure is preferred in many cases. By operating in either way, a product can be turned out by closed-circuit grinding with an air classifier that has a variety of characteristics controllable at will, though the fineness of the particles remains constant. It is possible to obtain an extremely fine-grained product by operating the mill and classifier in one way, while a slight change can produce granular material with an absence of fine particles. If it is assumed that a certain grain size and a uniform gradation of fineness are desired so that the voids may be filled, the closed-circuit grinding system is fed with such a volume of air that more material is taken out than is actually ground. Thus, the grinding medium is made more efficient and the product taken from the mill (containing a quantity of oversize) is classified, the larger particles being returned for further grinding. On the other hand, if a granular product is required to pass a specified screen test and the amount of fines has to be kept at a minimum, the grinding medium is of large size and the mill is operated at high speeds. The air is moved through the mill at a very high speed to remove the particles desired as soon as they are formed. In this way fines are kept at a minimum and the material passing through the classifier may be as closely sized as desired.

Effect of Grinding on Density of Materials—When quartz is subjected to continuous grinding so that particles are constantly reduced to finer and finer sizes, the density of the product is reduced. As a matter of experimental observation there is a tendency for the density of quartz

to increase slightly during the first hour and then gradually to decrease. The general trend of the change is indicated in Figure 127. Martin (1926–1927*c*) gives the following explanation for initial increase in density by assuming that cracks, gas-holes, and other pores of microscopic and submicroscopic dimensions are eliminated in the first stages of grinding. As the grinding continues an amorphous form of silica is produced. The amount of amorphous silica produced with prolonged grinding may be considerable. Ray (1923) states that a sample of quartz after 15 hrs' grinding contained 25.7 percent amorphous silica. Martin's results were previously borne out by Dale (1923–1924).

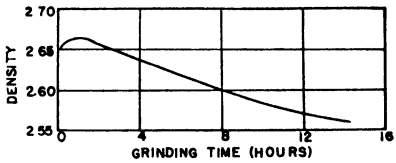


FIGURE 127. VARIATION OF DENSITY OF QUARTZ WITH DURATION OF GRINDING.

Martin's work with quartz, which was carried out in considerable detail with a vacuum pycnometer, led to the following conclusions:

1. The densities of the 17 carefully graded sands studied were the same or slightly higher than that of unground sand.
2. Grinding of a sand, whose particle-size initially averaged 0.75 mm, for a period of 25 to 50 min caused an increase in density of approximately 0.1 percent (that is, the gravity changed from 2.650 to 2.653).
3. Reduction in the particle-size of quartz from 0.75 to 0.3 mm did not affect its density.
4. Reduction in particle-size from 0.3 to 0.03 mm caused an increase in density of approximately 0.2 percent.
5. Further reduction of particle-size from 0.03 mm downward caused the density to decrease slightly by an amount equivalent to 0.07 percent. Martin estimated that grinding a standard sand for a period ranging from 25 to 50 min caused production of 0.46 percent amorphous silica and that this form of silica was confined to particles below 0.03 mm in size.
6. Prolonged grinding of quartz caused a further reduction in density, amounting to as much as 4 percent after 15 hrs in one series of experiments.

THEORY OF CRUSHING

The random fracture of brittle solids was studied by Griffith (1920), Sunatuni (1922), Hirata (1931), Terada (1931), Lienau (1936), and others. The problem is complicated by many factors and a satisfactory theory is lacking, especially with regard to ultimate distribution of sizes. Lienau, while devoting considerable attention to the nature of the force

distribution and properties of a solid which determine fracture and ultimate shapes, fails to prove generally what size distribution may be expected on random fracture. To state that the distribution is according to the Poissonian law requires no analysis of the physics of fracture but purely statistical assumptions of random distribution whose choice is optional as long as it is consistent with observations. As we know today, a logarithmic distribution which is the consequence of Poisson's law is but one form of possible distribution. There are others, for materials are heterogeneous in the elements which compose them, not only as to size but as to composition as well. In general, a homogeneous material of one component will fracture in accordance with Poisson's law, and undoubtedly for some forms of heterogeneous materials which are not too finely crushed the law will hold.

Nevertheless, Lienau's analysis constitutes an important contribution and his theorem, that the mean fragment of a homogeneous crushed material tends toward a configuration having maximum surface per unit volume, is fundamental to crushing theory.

Mechanics of Fracture—We shall here proceed to discuss the conditions under which fracture takes place. We shall assume a solid made up of small irregular elements and we shall define a "point" as being some average of these elements. The linear dimension of this point is Δ, its surface Δ^2, and its volume Δ^3. We need not consider shape factor as the shape of the element cannot be specified. The elements of the solid form a closely bound aggregate and considering the macroscopic scale, the composition of the solid is uniform and homogeneous. In every respect, for purposes of discussion, the behavior of the solid under stress is considered to be governed by elastic parameters, β, the bulk modulus having dimensions of force intensity $[ML^{-1}T^{-2}]$ and E_u the ultimate tension reckoned as positive for outward displacement of an element ready for fracture, and reckoned on the same units as β.

If now a solid of volume V_0 with no elastic stresses within it initially is subjected to an amount of work W_0 prior to fracture of the solid, we may by dimensional analysis reckon the average density of elastic energy. Let W_0/V_0 represent this average; then, assuming that this energy is a function of β and E_u above, we have

$$W_0/V_0 = f(\beta, E_u)$$

whence

$$\frac{W_0}{V_0} = k\frac{E_u^2}{\beta}$$

Since we are dealing with an average over the duration of the stress, $k = {}^1/_2$ and

$$\frac{W_0}{V_0} = \frac{1}{2}\frac{E_u^2}{\beta} \qquad \text{Eq (22-3)}$$

If S_0 is the surface of the solid, and if the boundary is displaced an amount δ by an *average* pressure p_{av}

$$W_0 = p_{av}\delta S_0 \qquad \text{Eq (22-4)}$$

Hence

$$p_{av} = \frac{1}{2}\frac{E_u^2 V_0}{\beta\delta S_0} \qquad \text{Eq (22-5)}$$

The measure of δ is easily derived from definitions of the parameters E_u and β if we assume that neither the surface S_0 nor the volume V_0 is changed up to the point of fracture; then

$$\frac{E_u}{\beta} = \frac{S_0}{V_0}\delta \qquad \text{Eq (22-6)}$$

Combining Eqs (22-5) and (22-6), we obtain as the average tension

$$p_{av} = \frac{E_u}{2} \qquad \text{Eq (22-7)}$$

and for a "point" Δ, as defined above

$$p' = \text{constant} = \frac{E_u\Delta^2}{2} \qquad \text{Eq (22-8)}$$

We have here written p' for p_{av} since the forces acting on the point source are several in number. These are oriented in every direction so that, in so far as the point is concerned, these forces exist as though it were a sphere. If the diameter of this representative solid is taken as d, then

$$V_0 = \frac{\pi d^3}{6},\ S_0 = \pi d^2, \text{ and } \frac{V_0}{S_0} = \frac{d}{6} \qquad \text{Eq (22-9)}$$

Now it can be shown that at least four forces are necessary to act on a sphere before deformation or rupture can take place, that is, if n is the number of forces

$$n \geqq 4 \qquad \text{Eq (22-10)}$$

Let us now consider the units of probable fragmentation. If the physical point is Δ, then the number of fragments on the average is of the order $d^3/\Delta^3 = \xi$. Whatever deformation takes place in the solid

must be at a rate less than the velocity of propagation of the forces. For a homogeneous solid, the transverse velocity of force propagation which is less than the longitudinal is given by the expression $\sqrt{\mu/\rho}$ where μ is the shear modulus and ρ the density of the solid Hence, for one point

$$\frac{d\varphi}{dt} = \frac{1}{\xi}\sqrt{\frac{\mu}{\rho}} \qquad \text{Eq (22-11)}$$

If τ represents the duration of n forces of intensity p', we also have

$$\frac{W_0}{\tau} = \frac{d\varphi}{dt}p'n = \frac{1}{\xi}p'n\sqrt{\frac{\mu}{\rho}} \qquad \text{Eq (22-12)}$$

or since

$$\tau\frac{d\varphi}{dt} = \delta,\ W_0 = \delta p'n \qquad \text{Eq (22-13)}$$

Thus, from Eqs (22-4), (22-7), (22-8), and (22-12) we obtain

$$n = \pi\xi^{3/2} \qquad \text{Eq (22-14)}$$

and since $n \geqq 4$

$$\xi \geqq \left(\frac{n}{\pi}\right)^{3/2} \geqq \left(\frac{4}{\pi}\right)^{3/2} \qquad \text{Eq (22-15)}$$

We are now in position to obtain equations of the lower limits of magnitude of p', p_{av}, δ, τ, and W_0, and to discuss the significance of some of these quantities. From Eqs (22-4), (22-9), and (22-15), we have

$$p' = \frac{E_u d^2}{2\xi^{3/2}} \leqq \frac{E_u S_0}{8} \qquad \text{Eq (22-16)}$$

Also, the velocity of normal displacement from Eqs (22-11) and (22-15) is

$$\frac{d\delta}{dt} \leqq \sqrt{\frac{\pi^3\mu}{27\rho}} \qquad \text{Eq (22-17)}$$

and the average prefracture period τ becomes from Eqs (22-6), (22-17), and (22-9)

$$\tau = \frac{\delta}{(d\delta/dt)} \geqq \frac{E_u V_0}{\beta S_0} = \frac{E_u d}{6\beta}\sqrt{\frac{27\rho}{\pi^3\mu}} \qquad \text{Eq (22-18)}$$

Efficiency of Crushing—It is recognized that the creation of new surfaces involves certain explicit energy relationships. These have been investigated by Lienau (1936). Let s_{av} denote the surface of an average grain of crushed material, whose total surface is S. Then if n_a denotes the number of these average elemental surfaces

$$dS = s_{av}dn_a \qquad \text{Eq (22-19)}$$

Since the boundary surface of any interior particle is in contact with an equivalent surface contributed by other particles, the work done in separating the elemental surface is

$$dW = \delta u \cdot dS/2 \qquad \text{Eq (22-20)}$$

where the terms are as defined above. The term dW represents the work directly attributable to fracture of the element. If dW' denotes the work done at the boundaries of the element and dW'' the random strain energies which do not contribute to new surface formation, then the energy-balance equation is

$$dW = dW' + dW''$$

At the instant fracture occurs, the boundaries separate and within a time very short as compared with the prefracture period the new surface has been formed. Thus, equilibrium takes place under the following conditions:

$$t = \tau$$
$$dW = dW' = dW'' = 0$$
$$dS = s_{av} dn_s = 0$$
$$dn_s = 0$$

Integrating over the time τ which represents the interval from the time of application of the fracturing pressures to the time of fracture, we get

$$S = s_{av} n_s \qquad \text{Eq (22-21)}$$

$$W = \delta u \cdot S/2 \qquad \text{Eq (22-22)}$$

$$W' = W + W'' \qquad \text{Eq (22-23)}$$

Combining, we obtain the absolute amount of new surface

$$S = \frac{2(W' - W'')}{\delta u}$$

The random non-surface producing energy is comparable in magnitude at the time of fracture with the energy actually required to produce new surface. Moreover, since crushing efficiency is defined as

$$\epsilon = \frac{W}{W'} = \frac{W' - W''}{W'} \qquad \text{Eq (22-24)}$$

it is not difficult to comprehend the low efficiency in producing new surface by mechanical methods. Now consider n particles wholly within the mass at the moment of crushing, and let the diameter of any random particle be d. Since we are concerned with rupture surface, we may

write the geometrical probability, P, of n particles ($n = n_s$) within the volume V_0 as $n\alpha_s d^2 \cdot d / V_0$. From Eqs (22-21), (22-22), and (22-24), this becomes

$$P = \frac{\alpha_s \epsilon}{2\delta u} \frac{W' d^2}{s_{av} V_0} \cdot d$$

where α_s is the surface shape factor. But $s_{av} = \alpha_s d^2$ and we obtain

$$P = \frac{1}{2} \frac{\epsilon W'}{V_0} \cdot \frac{d}{\delta u} \qquad \text{Eq (22-25)}$$

Since P is dimensionless it is not difficult to show that this expression reduces to

$$\frac{1}{2} \frac{\epsilon W'}{V_0} \cdot \frac{d}{\delta u} = \frac{d}{d_{av}}$$

where we may identify $2V_0 S u / \epsilon W'$ as equivalent to an average diameter of particle d_{av}. Hence, we obtain explicitly

$$d_{av} = \frac{2V_0 \delta u}{\epsilon W'} \qquad \text{Eq (22-26)}$$

Lienau's Theorem--If d is taken as the random mean diameter of a fractured element, the mean volume of an equivalent sphere is

$$V = \alpha_v d_{av}^3 / 6 = V_0 / n_0$$

where α_v is the volume shape factor and N_0 the number of elements making up the original volume V_0. The mean elemental surface is therefore

$$\bar{s} = \frac{S + S_0}{n_0}$$

Since the production of new surface is improbable once fracture has taken place and there is a relaxation of strains, we have on the scale of observation for a small element that

$$dS \geqq 0 \qquad \text{Eq (22-27)}$$

It may be shown by the methods of geometric probabilities that a pair of opposite faces of a fragment are parallel on the average, but the chance that these faces are sensibly out of parallel is more than certain in 98 out of 100 cases. The chance that these faces do not intersect is negligible, although, as in the previous case, it may be shown that on the average they are orthogonal.

The chance that $(n + 3)$-planes intersect at a definite point is greater for heterogeneous material, and it can be shown that the probabilities are a maximum for $n = 1$. The probabilities of $(n + 3)$-edge vertices di-

minish as n increases. The general tendencies of particle shape are toward a plane, convex, equidimensional polygon. The vertices are largely three-edged, with opposite faces parallel and adjacent faces orthogonal—or simply, a cube.* The condition of high specific surface is also attained for 4-, 5-, 6- and 7-sided hedra, any of which can be produced by a single plane—cut through the cube.† These display 3 three-edge vertices. The faces are 3-, 4-, and 5-sided polygons, and these are the most probable fragments of a sheet. The hexagon and octagon are unlikely because they depend on improbable vertex conditions.

If the surface per unit volume of the mean fragment is $\bar{s}/\bar{V}$, we have from Eq (22-27) that

$$d\left(\frac{\bar{s}}{\bar{V}}\right) = \frac{dS}{V_0} \geqq 0$$

Consequently, the mean element tends toward a configuration having a maximum surface per unit volume, and since the crushed element tends to be equidimensional, the average element is an equidimensional irregular polyhedron whose form approaches that of a cube. This is Lienau's theorem.

PARTICLE-SIZE DISTRIBUTION OF GROUND MATERIAL

Theory of Particle-Size Distribution in Grinding—Martin (1923) explained the basis for the type of distribution obtained as follows: If we have given a set of particles of mean diameter d_1, and these particles are ground to a series of smaller particles in grades whose mean diameters are d_2, d_3, d_4, etc., statistically speaking, we may state that each particle crushed generates a family of smaller particles of the same number as another particle the same size. Thus, if we start with one particle and it is crushed into two particles, then each of the two particles in turn is crushed into two more so that at the end of the second stage of crushing we have four particles, and eight particles at the third stage.

* The sides were counted on 396 pieces of crushed granite, $1^1/_2/^1/_2$ mesh. Edges and sides were sharp and well defined Following is the frequency distribution of the sides of the various pieces counted:

4 sides	88
5 sides	122
6 sides	174
7 sides	12

† It may be shown that the probability of a section of a random plane cutting a cube is a hexagon given by the expression

$$1/4\pi(\sqrt{3}\cot^{-1}\sqrt{3} - \sqrt{2}\cot^{-1}\sqrt{2}) = 0.046$$

or that the chances are approximately 5 in every 100.

This is entirely a statistical assumption, but is a means of explaining the type of distribution that has been obtained.

Particle-Size Distribution—The size-frequency distribution of ground materials differs from the usual type of frequency-distribution in chance sampling. As a general rule the number of particles increases with decreasing particle-diameter. Martin (1923) has shown that frequency-distribution follows the law of compound interest, namely

$$nd = N \cdot \exp(-ad)$$

where nd is the number of particles of diameter d, and N and a are constants. Actually N is the number of particles when d becomes infinitely small.*

The problem of size-frequency distribution has been the object of numerous investigations. Particle-size determinations of ground material are important because they give the manner of subdivision and because it is possible to obtain a measure of the energy requirements from them. Figure 128 indicates a typical size-distribution obtained in the course of crushing a given material. The curve is a plot of number of particles versus size. The method for making size-distributions of this kind was explained in Chapter 3. However, in practice we must resort to screen sizes for particle-size determinations and in these instances it is customary to express the percentage of total material screened remaining on each of the screens of the series used. The type of distribution obtained by gravimetric methods does not differ from the frequency-distribution shown in Figure 128. The reason for the parallel nature of the distributions was explained in Chapters 3 and 5. When weights and counts are made of the same material, size by size, the slopes of the resultant distributions are the same.

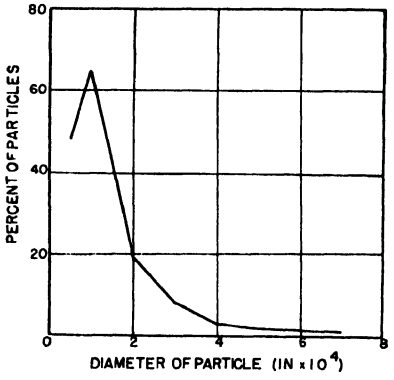


FIGURE 128 TYPE OF FREQUENCY-DISTRIBUTION OBTAINED IN GRINDING

While it is generally accepted that the size-distributions of

* However, much depends upon the initial size-distribution of the material being ground and the time of grinding If at the start of grinding the sizes are more or less uniform, the relation given generally follows With varying conditions such as time of grinding, ball sizes used, nature of the material, etc., almost any type of frequency-distribution may be obtained

ground material follow an exponential law, this is not always precisely the case. Generally, as particle-size decreases the distribution passes through a maximum and decreases to zero. After screening, there is always a residue which constitutes a percentage less than the maximum. Martin (1923) discussed the significance of this type of distribution. He raised the question whether the maximum value corresponds to a true physical reality, or whether it is due to the fact that when the size of particles falls below a certain magnitude the microscope does not detect them (assuming the particles are measured by means of a microscope). These points were tested by using different microscope magnifications and making counts on the same sample. On the basis of these experiments Martin concluded that the maximum point shifts to smaller and smaller diameters as the magnifying power of the microscope increases.

When summation curves are made of the particle-size distribution obeying the exponential law, it is found that the curve follows an exponential relationship similar to that obtained when the frequency distribution is plotted.*

Gaudin (1926) reported details on jaw crushers, rolls, rod mills, and ball mills using quartz, galena, and granite, in terms of graded sieve openings and percentages included in the sieve intervals. When percentages of a sieve interval and sieve openings were plotted on a logarithmic scale he found a constant paraboloid loop in the coarse range, and a straight-line relation in the fine range. He noted a constant exception to the straight-line relation when the materials were not homogeneous, and in such cases assumed that breakages take place preponderantly at crystal size. Gaudin checked these observations by measuring the crystal size in the original rock.

The statistical studies of Lienau cited above have amply supported the work of Martin and Gaudin. In general the probability of obtaining n_1 particles of diameter d_1, n_2 particles of diameter d_2, etc., is given by the binomial law, but when the possible number of particles of any diameter is large and their probability P small, as is the case in crushing, the binomial distribution reduces to Poisson's law

$$P(n) = \frac{1}{n!}\left(\frac{d}{d_{av}}\right)^n \cdot \exp\left(-\frac{d}{d_{av}}\right)$$

Subject to this law the chance that no diameter less than d will be found is

$$P(0) = \exp -\left(\frac{d}{d_{av}}\right)$$

* This follows from a consideration of the integral of the equation: Number of particles greater than size $d = N\int \exp(-ad)d(d) = \frac{N}{a}.\exp(-ad)$

and for a finite but large number of elements N, we have that the number n of elements greater than d is simply

$$n = N \cdot \exp\left(-\frac{d}{d_{av}}\right)$$

which is Martin's equation.

SURFACE-AREA AND ENERGY

The energy necessary to grind a given material depends upon the ease with which the material may be broken, the fineness of the product desired, and the design characteristics of the equipment. For a number of years there were two schools of thought as to the energy required for production of particles with a given size-distribution. Rittinger (1867) stated that the energy necessary for size reduction of particles is directly proportional to the increase in surface. Martin (1925–1926*a*) cites Kick, who stated that the energy required for producing identical changes of configuration in geometrically similar grains of the same composition varies as reduction in the volume or weight of those grains.*

As a matter of fact it is common opinion that Rittinger's law applies very closely to the actual performance of ball mills and other mills used to grind very fine material. Kick's law seems to measure the performance in coarse grinding. In recent years Martin (1925–1926*a*) and Gross and Zimmerley (1928*a*, *b*, *c*) re-examined the relation of energy input to surface produced. These investigators made careful measurements of the surface of crushed material and utilized different methods for measuring the energy required to produce known amounts of surfaces. The results indicate that, as far as the crushing of quartz is concerned, the new surface produced is in direct proportion to the work input, in accordance with Rittinger's law of crushing. However, there still remains some question whether the work done by Martin and by Gross and Zimmerley is correct. Some exception has been taken to the method by which particle-surface was measured. It was pointed out (1) that quartz becomes amorphous when subjected to crushing action, and (2) that chemical methods of producing surface are subject to error when the rate of solution of the amorphous forms of quartz are not known. In spite of these criticisms, however, it would seem that the careful work

* I have been unable to verify the reference given to Kick's paper, cited by Martin and many others, in any of the libraries in New York City. In view of recent investigations indicating that the work involved in crushing varies as the amount of surface produced, Kick's law is tenable only in the first stages of the crushing process, that is, when the number of new particles produced is very low.

done by Gross and Zimmerley on surface measurement deserves general acceptance.

The surface measurements made by Martin and by Gross and Zimmerley are absolute measurements. In practice it is necessary to resort to the use of parameters based on particle-size distribution, as explained in Chapters 3 and 16. At the present time there is no standard for absolute surface measurement and consequently no uniformity in the expression of grinding efficiencies and power requirements.

Martin's Experiments—In a series of comprehensive tests with quartz, Martin (1925–1926*a*, *b*) arrived at the following conclusions with regard to the relation between grinding and surface produced: (1) In an ordinary tube mill the surface produced is directly proportional to the work done; (2) approximately 61 ft-lb of work are required to increase the surface of sand 1 sq ft when 1-in. balls are used for grinding in an 18-in. × 18-in. mill; (3) with prolonged grinding, the fine particles tend to cushion the grinding somewhat and alter the proportionality between surface produced and work done. In ordinary tube-mill grinding it was found that only a small amount of power saving could be effected by removing the dust with an air stream, but this was not true in prolonged grinding.

If S_2 is the final surface of the material ground in sq ft, and S_1 the original surface, then on the basis of the foregoing the work done in foot-pounds equals $a(S_2 - S_1)$, where a is a constant which is peculiar both to the grinding apparatus and material used. The constant is actually the work required to increase the material surface by 1 sq ft. An example serves to illustrate the use of this simple formula. Assume it is desired to calculate the power needed to grind 1 ton of quartz per hr from a fineness equivalent to 25 sq ft per lb to a fineness represented by 1000 sq ft per lb. Assume further that the grindability constant $a = 61$ and that the material is ground in an 18-in. × 18-in. mill, using steel balls. The calculations are as follows: Since 1 ton = 2000 lb, the final surface is $2000 \times 1000 = S_2 = 2{,}000{,}000$ sq ft. The initial surface is $2000 \times 25 = S_1 = 50{,}000$ sq ft. Therefore, work $= 61 \times (2{,}000{,}000 - 50{,}000) = 119{,}000{,}000$ ft-lb. Since one hp = 33,000 ft-lb per min = 1,980,000 ft-lb per hr, the theoretical hp required will be 119,000,000/1,980,000 = 50.5 hp.

Experiments of Gross and Zimmerley—Gross and Zimmerley (1928*c*) made very careful measurements of the work required to crush quartz. These investigators determined the energy needed to crush particles by dropping weights on pieces of quartz from different heights. Since the weight rebounded immediately after impact (representing a loss of work done). it was necessary to determine the work done in terms of the de-

formation of aluminum wires placed beneath the anvil holding the quartz. Aluminum was found to be relatively inelastic, and suitable for this purpose. After the surface of the crushed quartz was determined (the initial surface having been measured previously), the work necessary to crush a given surface was calculated. Gross and Zimmerley found that the work done was proportional to the surface produced, but that the actual amount of work needed to crush quartz was 3.83 ft-lb per sq ft of surface produced. This amounts to 6.3 percent of the value of 61 ft-lb per sq ft determined by Martin in his ball-mill experiments.

TABLE 88—EXPLOSIVE SHATTERING EXPERIMENTS ON 3/18-MESH DOLOMITE (MADE ON CONTINUOUS MACHINE ON 2000 G IN 200-G CHARGES)

Steaming time (sec)	Steam condensed (lb/ton of charge)	Btu of steam condensed per ton of charge	Lb produced from 1 ton of charge 8/14	14/48	48/100	100/200	< 200	Condition of experiment
180	864	991,900	395	368	244	163	112	
120	748	858,700	368	352	226	150	100	Wet, cold
60	546	626,800	442	354	232	154	106	
15	423	485,600	504	376	252	170	118	
15	291	334,100	504	372	276	186	128	
8	251	288,100	464	336	242	164	114	Dry, cold
4	226	259,400	454	338	214	144	100	
2	206	236,500	468	364	216	144	100	
15	377	432,800	482	370	246	166	114	Wet, heated
8	337	386,900	474	358	254	172	120	(90 deg C)
15	142	163,000	410	312	234	154	106	
8	102	117,100	430	306	224	150	104	Dry, heated
4	77	88,400	458	354	218	140	94	(181 deg C)
2	57	65,400	410	314	190	124	84	

When we consider efficiency of grinding, there is no general agreement as to the true magnitude. Gross and Zimmerley computed efficiency of grinding in terms of surface energy. Their method of crushing was found to have an efficiency of 3 percent, and Martin's ball-mill method 0.06 percent. The efficiency is seen to be very low. Gross and Zimmerley question these efficiencies since the surface energies of solids on which they are based are at present unsatisfactory. When it is considered that more than 97 percent of energy supplied is dissipated in the form of heat, it is natural to inquire into other more suitable methods. This was done by Dean and Gross (1932, 1933) who in a series of remarkable papers outlined a wholly new attack on the problem of producing fines.

EXPLOSIVE SHATTERING

Dean and Gross, and subsequently Gross (1934), described a method for crushing particles by means of steam. This method of producing fines may be considered similar to that of ball milling except that the shattering force is furnished by the sudden expansion of water films between the ore particles rather than by balls falling under gravity. When a dry material is subjected to the action of saturated steam under pressure, a film of water is deposited on the particles. Within a very short time the film becomes overheated and when the pressure is released the resulting sudden expansion of the overheated water throws the particles together so violently as to produce shattering. Gross and Wood (1935) stated that the efficiency of such a system would be theoretically practical with sufficiently large charges. Expansion is so rapid as to be adiabatic, so that the efficiency would be substantially the same as that for a steam engine. Some data taken from Gross and Wood's report are given in Table 88. It will be seen that the costs for crushing by this method are very low. Gross (1934) arrived at the following conclusions with regard to explosive shattering of minerals:

1. Minerals are now considered far from solid and impermeable substances. They may not only have cracks, cleavage planes, pores, and cavities, but according to our present concepts, are built up of small unit "blocks" somewhat like a brick wall.
2. Subjection of minerals to an explosive wave will break up the mineral along this secondary structure.
3. The cost of accomplishing such rupture is theoretically so low that the possibility of economic application is very promising.
4. All minerals are by no means equally shatterable, due to different degrees of permeability and variable tenacity between the individual blocks. The probability that the individual blocks constituting a mineral are in themselves impermeable leads to the conclusion that extremely fine material will not be produced in the explosive method to the same extent as in ordinary crushing or grinding.
5. Rupture conceivably takes place by detonation of an explosive within or on the mineral, or by sudden release of pressure on a superheated liquid.
6. No liquid or solution has been found more efficient than water for producing an explosion by suitable release of pressure after superheating.
7. Only two conditions have been found to affect shattering: (*a*) Increased pressure, and (*b*) increased quantity of water present at the time of the explosion. Prolonged soaking of the material was found to have no beneficial effect. No preliminary treatment before wetting, or wetting with substances other than plain water, had any tendency to produce better shattering.

Problems

1. Discuss the essential differences between friability and grinding
2 What are the essential differences in the product ground by ball mills and tube mills?
3. Calculate the critical speed of a mill 28 in. in diameter. What is the significance of critical speed in grinding operations?
4. What is the best ball size for grinding clinker initially averaging 0.2 in. in diameter?
5. Explain the conditions which determine whether wet or dry methods should be used for grinding a given material
6 What is the effect of a classifier?
7. Discuss the limitations of Martin's general theory regarding size-distribution obtained by grinding.

CHAPTER 23

SAMPLING

ALMOST every problem of micromeritics concerns itself with samples. Conclusions are drawn from a study of these samples and inferences made as regards the properties or behavior of the material in question. It is clear that much depends upon the degree to which one sample, or many samples, actually represents the bulk of material under investigation. A generalization is warranted only if we know that the composition of the sample is identical with the bulk of the material and that the properties and behavior revealed apply irrespective of the amount considered, or at least within specified limits. Thus, the sample is the first and most important element of micromeritics since the accuracy of the results, however obtained, depends upon the goodness of the sample and how closely it represents the bulk of the material of which it is a part.

A sample is an aliquot approximating the attributes of the whole. If it is known that a mixture of fine particles is uniform throughout, one sample is generally sufficient to yield the necessary information. If the mixture is not homogeneous, then several samples must be taken. These samples should be drawn at suitable places within a given bulk of material so that an average of the analyses obtained corresponds closely to an analysis of the whole bulk, assuming this were possible.

It is one of the major problems of micromeritics to ascertain what sampling procedures should be used to obtain a true representation of a given material. To do this in the absence of any information about the material we must resolve the following questions: (1) What sampling method should be used? (2) What method of analysis should be used?

In the following sections we shall not discuss personal errors regarding these items, but only those which may be characterized as inherent in the methods themselves.

SAMPLING PROCEDURE

The selection of the proper technique of sampling is one of judgment and experience; the test of the selected technique lies in its reliability for estimating a given attribute for the universe under consideration. Thus, the attribute measured from the sample should approximate the

true contents of the universe,* or at least estimate within known limits what it is likely to be. We may *a priori* assume the following: (*a*) That the more heterogeneous the material, the more samples are necessary to measure a given attribute; and (*b*) that as the number of samples is increased, the amount of added information gained about a measured attribute diminishes. It is the function of statistics to determine when a sufficient number of samples has been taken, so that our results represent, with some stated degree of accuracy, the contents of the universe. Furthermore, it is axiomatic of all sampling techniques that we must presume nothing about the universe until samples have been drawn and the attributes determined from them.

Random and Orderly Sampling—While the mathematics of statistics is based on the random selection of samples—that is, that the likelihood of drawing a representative aliquot from one place in the universe is no different from another—in practice samples are usually drawn according to an orderly fixed scheme. As a general rule the final results using either method should approximate each other closely. Randomness is rarely achieved for we are influenced by the previous sample as to where we shall take the next. Orderly prearranged sampling, provided it covers representative portions of the universe and is not biased as to quantities taken, is preferable, all things considered. The significant difference in either method lies in the fact that random procedures ignore the volume or mass of the universe, whereas orderly sampling must take these items into account.

Having obtained samples (the number of samples necessary will be discussed elsewhere) we must next determine what should be done with them. The following procedures may be used: (*a*) Each sample may be analyzed separately, or (*b*) the samples may be combined and analyzed, or (*c*) the combined samples may be mixed thoroughly and an aliquot taken for analysis, or (*d*) each sample may be thoroughly mixed and an aliquot of each analyzed. Any of these methods may be used, provided we note that in the case of (*c*) and (*d*) the *amounts* combined should be of the same weight or volume. Otherwise, if the universe is not homogeneous, the amounts of the individual samples will influence the results of analysis.

As to the quantity of the sample which should be taken, no fixed rule can be given. We may, if we choose, select individual particles at random (assuming this is possible), or we may take as much as a pound or more from each of the places where the single particles were taken. In

* The term universe as here used means the whole mass of material from which the sample is drawn. An attribute is regarded as some component, as, for example, particle size or chemical analysis.

the latter case the final results are more reliable in so far as the calculated chance of any one sample falling within specified limits is concerned; the same degree of reliability with single particles can be obtained by increasing the number of particles selected for the sample.

Errors of Sampling—Whatever device or method is used, sampling requires a complete understanding of all its characteristics. Consider, for example, the pipette method of particle size measurement as compared with the microscopic procedure using a pinch of the same particulate material. In the first instance the location of the pipette tip at a given point in a graduate of suspended material does not yield accurate information relative to the particle size to be expected above that point.* The pipette tip when the pipette itself is being filled causes a flow of particles from regions above and below the level at which it is placed. Usually, the particles drawn into the tip are those from a spherical volume approximating that drawn into the pipette. Thus, pipette volume influences the particle-size determination.

In the case of microscopy, our chief concern is with the representativeness of the sample itself, and hence the uppermost consideration is with its preparation. The amount of sample measured affects only the *accuracy* of the final average, whereas the pipette volume actually alters the true level from which computations are made.

The above are mere distinctions of procedure. The necessity for uniformity of procedure has long been recognized, so that except for comparative purposes the above discussion may appear to be only of academic interest. However, method of procedure is important and must be carefully stated whenever specified techniques are under discussion. In the cases under discussion either technique yields a particle-size distribution. It is when we compare their relative merits that knowledge of every step of procedure must be had, since the persons who are to judge these merits must be informed of every detail. Obviously there will be differences in the results obtained by either technique, but if procedures are standardized we are more certain that these differences are peculiar to the techniques and may be expected to occur consistently. If these differences are then admitted and their reasons understood, we may then ask what determines the choice of either sampling procedure in a given instance.

Let us review briefly the meaning of the two procedures. The pipette method furnishes a means for measuring the diameter of particles settling at the same rate in accordance with Stokes' law. The diameter is calcu-

* We assume for the sake of discussion that all the particles have the same shape and density, but are graded as to size.

lated as though the particles are spheres. Hence, the method disregards the shape of the particles and in every sense is a *gravimetric* measure. The microscopic method measures the length of the projections of superficial particle termini along any random line. It assumes that all orientations are represented, and that particle-depth need not be accounted for. This method is best described as *statistical*. If the density of the particles is known and all are of the same density, this method permits the evaluation of a volume shape factor, thus rendering it comparable to the pipette technique, theoretically. However, the procedure is tedious and constitutes a serious defect of the microscopic technique. On the other hand, in so far as the pipette method is concerned, we must appreciate that particle shape has a bearing on the settling velocity so that in either case particle shape is not satisfactorily accounted for, although it *can* be measured by the microscopic method as we have indicated. However, we may conclude that for particles other than extreme shapes such as mica or needle-like particles, the pipette method yields at once a direct gravimetric measure. The microscopic method in such cases is likewise unsatisfactory.

Our analysis thus far reveals significant differences in attributes of the sampling techniques discussed.* As regards applications which often determine the procedure to be followed, we must consider (*a*) whether the particles are soluble, in which case if the pipette is used we must find a liquid other than water; (*b*) whether shape factors are important, thus ruling out the pipette on heterogeneous material; (*c*) whether gravimetric or volumetric distribution by size is desired, thus ruling out microscopy; and (*d*) the maximum and minimum sizes to be covered. As a general rule, the pipette method is more satisfactory for fine sizes ($< 5\ \mu$) than the microscopic, and conversely for the larger sizes.

We have dealt in detail regarding the inherent errors of two sampling procedures. The discussion, however, applies generally to all kinds of sampling and is intended to indicate three things of importance, namely, a knowledge of the fundamental theory underlying various methods, limitations in their practical use, and the ultimate purpose to which the data obtained are intended.

Mechanical Aspects of Sampling—We have thus far discussed sampling methods in a general way. There are certain situations in which it is desired to predetermine exactly the influence of sampling devices themselves on the universe under consideration. For example, it is relatively easy to deal with particles in a fluid medium so long as the fluid is still,

* For an excellent critical analysis of the differences in various particle-size measuring methods, reference is made to Schweyer's studies (1942*a*), discussed briefly in Chapter 4.

but quite difficult if the fluid is in motion. To understand what happens in the latter case, we must know precisely what happens at the tip of a pipette when a sample is drawn from a static medium.

The theory of sampling of particles in a moving suspension especially when the fluid is moving at high speeds has never been treated in a satisfactory manner. The problem is complicated not only by the fact that in horizontal flow we have stratification and saltation of particles, but also by the fact that not all the particles move with the speed of the fluid, which in itself may not be uniform. The extent of errors involved in determining the concentration of particles in a moving fluid can only be approximated, especially if a probe device is used. For we must not overlook the possibility that the probe itself often affects the nature of the flow and may favor the collection of only those particles whose momentum is sufficient to thrust themselves into the probe opening. Thus, there may arise situations in which the probe exhibits a certain degree of selectivity.

Static Suspension—Let us analyze the effect of drawing a sample from a suspension which was originally homogeneous throughout. It does not matter what property we are to measure since, as shown in Chapter 4, the density of the suspension and particle-size are related to time and position. Let us assume that we are considering a suspension contained in a glass cylinder and that at a stated point x and time t after the suspension was rendered homogeneous we insert a pipette and withdraw an amount V in Δt seconds. We wish to find the effect of the pipette in selecting particles not at the position of the tip but over a volume of influence created in drawing the sample. What is intended here is that the act of sucking a volume V in the time Δt draws fluid and suspended material not alone from the plane of the pipette tip but from a certain region above and below it as well.

First of all it is clear that by defining both x and t_0 we can eliminate particles which obey Newton's law, and thus confine ourselves only to such particles as obey Stokes' law. In the first case (Newton's law) we observe that

$$F(d) = -\frac{t}{d_e'} \cdot \frac{1}{\varphi_0 - \rho_0} \cdot \frac{d\varphi}{dt}$$

and in the case of Stokes' law we again have Eq (4-8)

$$F(d) = -\frac{2t}{d_e'} \cdot \frac{1}{\varphi_0 - \rho_0} \cdot \frac{d\varphi}{dt}$$

where the terms, except d_e', Newton's effective diameter, are the same as those used in Chapter 4. Both of these equations apply to a fixed point x_0. The question we must answer is as follows: What is the pre-

cise region sampled so that we can correct for either of the laws of settling we may apply?

The simplest procedure is to regard the time of sampling as negligible in comparison to the distance traveled by the largest particle from the point x_1, through x_0 and to x_2, $|x_1 - x_2|$ defining the limits of the zone of influence. The latter may be approximated by drawing an imaginary sphere of volume Q about the center of the tip. This then represents approximately the volume of the fluid sampled.

From the standpoint of applying Stokes' law this procedure is sufficiently accurate since the rate of settling is actually small enough to result in only a slight error. On the other hand, if we are concerned with particles falling according to Newton's law, both Q and the time Δt are of great importance. Let us now determine what really happens about the tip of the pipette.

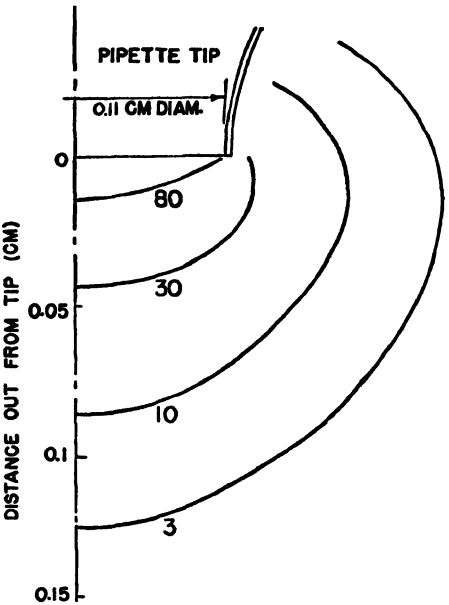


FIGURE 129. VELOCITY CONTOURS ABOUT TIP OF PIPETTE DURING SAMPLING. VELOCITY AT TIP = 500 CM PER SEC; OPENING AREA = 0.01 SQ CM. VOLUME OF LIQUID MOVED IS CONTAINED WITHIN CONTOUR REPRESENTING 0.5 PERCENT OF THE VELOCITY AT THE OPENING. CONTOURS IN FIGURE EXPRESSED A PERCENT OF VELOCITY AT THE OPENING.

The distribution of flow about a circular opening under suction has been investigated by the author (1939) and the flow pattern is as shown in Figure 129. The various contours are expressed as percentages of the velocity at the tip. Furthermore, when the coordinates are expressed in terms of the tip diameter, the contours for all openings are identical and their absolute values are functions of the velocity at the opening. Thus, what is important to consider is the ratio $Q/A \cdot \Delta t$, where A is the area of the pipette opening. Let us assume for the sake of a concrete example that $Q = 10$ cu cm, $A = 0.01$ sq cm, and $\Delta t = 2$ sec. Then the flow at the pipette tip is 500 cu cm per sec. It may be shown that this volume is confined within the 0.5 percent velocity contour. This conforms with a velocity contour surface of $0.005 \times 500 = 2.5$ cm per sec. Hence all particles

below the tip of the pipette which can be lifted at that velocity will be drawn up to the pipette but will not actually enter it. No particle, wherever it may be within the dotted contour, will enter the pipette tip if it is settling at a rate greater than the velocity contour at which it resides at the moment of sampling. Furthermore, particles at the sides of the contours need not even move at the speed of the contour indicated, but because the effect of the contour is a lateral one, they will continue to accelerate downward and may never be brought to the tip during the time of sampling. All these are important considerations in sampling rapidly-settling particles, and exhibit the selectivity of the pipette method in such instances.

Returning now to the above example, we note the following: (1) In Figure 129 all particles are displaced toward the tip. (2) Particles forward from the tip, except those whose settling rates at any specified contour are greater than the value expressed by the contour, will be brought into the pipette. (3) Particles at the sides of the contour surfaces will be displaced, and many may continue to accelerate downward and pass completely below the dotted contour, even though these particles have settling rates less than the contours at which they find themselves during the interval Δt. (4) In general, the space sampled (10 cu cm) lies about 1 cm above the tip and about 3 cm below it. (5) More large particles will come from below the tip than laterally. (6) If the space inside the 0.5 percent contour is divided into two equal areas, the mean depth sampled below the surface is actually 0.8 cm.

We may remark in passing that the contours shown in Figure 129 are applicable to any circular opening and that the procedure outlined here may be used for conditions other than those given in illustration.

Suspension in a Fluid Stream—While diffusional and eddy processes tend to keep a suspension more or less homogeneous, there are many occasions when in spite of these processes conveyed material tends to stratify or distribute itself in a non-uniform manner. This difficulty is not present when particles are conveyed vertically, and yet we must appreciate the fact that the movement of a fluid in any duct creates a velocity gradient which is a maximum along the axis and decreases to very low values next to the duct surface. Thus, all the particles do not move at a uniform rate and in the case of vertical motion the largest particles may move only along the duct axis. Those near the duct surface may, in fact, tend to fall unless projected by eddies and particle impacts toward the duct axis. In any case, it is generally easier to sample a suspension flowing in a vertical duct than in a horizontal duct.

Perhaps it is best to begin our analysis with the effect of the sampling

probe on the flow pattern. We shall regard the flow as uniform, and the probe the only obstruction encountered.

Let us first assume that the probe is a solid plate of thickness $2a$ and so as to reduce our problem to simplest terms let it be of infinite length. Then our problem is two-dimensional. As the fluid moves past such a probe, it breaks away from the edges somewhat as shown in Figure 130. The flow along the plate does not become uniform until a distance of some $6a$ downstream. A particle P at one of the leading edges having a mass greater than the fluid, and hence more momentum, departs from the main stream line and follows the course shown by the dotted curve.

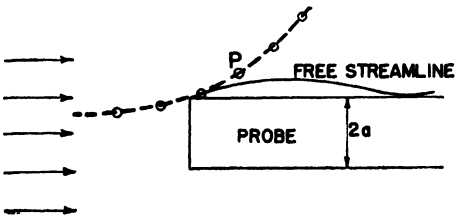


FIGURE 130 ACTION OF SAMPLING PROBE IN DEFLECTING PARTICLE TO BE SAMPLED

Suppose now that the end facing the stream is open and maintained under suction. The condition of flow will then be somewhat as sketched in Figure 131, which shows the reinforcing effect of a fluid velocity directed toward the opening of 500 cm per sec when superimposed on a flow into the probe opening of 2500 cm per sec. If the probe were under suction in a still fluid the velocity contour would be as shown by the dashed line in the figure. Thus, the effect of the probe under suction in a moving air stream increases the intensity of the velocity contours forward from the opening and decreases them in the rear. The effect noted in Figure 131 is diminished, so that in general, sampling is improved, especially as regards large particles. The development given actually permits determination of the path of a particle to the point where it either comes into the zone of probe influence or else misses it and goes on unaffected.

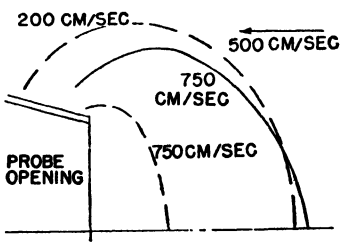


FIGURE 131. ILLUSTRATING THE EFFECT OF SUPERIMPOSING A FLUID VELOCITY OF 500 CM PER SEC IN DIRECTION OF OPENING OF SAMPLING PROBE. FLUID VELOCITY AT OPENING OF SAMPLING PROBE IS INITIALLY 2500 CM PER SEC.

DISTRIBUTIONS AND THEIR MEASURES

If we examine a number of particles which are not uniform as to size, and if these particles are measured by any of the methods described in Chapters 4 and 5, one size will be found to occur with more frequency

than the others. The frequency of sizes above and below that of maximum frequency will diminish by varying degrees. In general, if we plot frequency against size, the resulting curve will resemble the outline of a bell, though it will not always be as symmetrical. The shape of the distribution—that is, the curve fitting the data—can be ascertained mathematically from the frequency data themselves. We shall show that only two constants need be determined in order to describe most distributions completely. These are (1) some mean or average of the size measured, and (2) some method of calculating how the various sizes are dispersed. Needless to say, while we shall consider size distributions in the following paragraphs, it must be understood that for "size" we may substitute any other attribute in which we are interested.

Let us assume that we have at our disposal a sample containing N-particles to be measured. Let p denote the fraction (expressed decimally) of particles falling below a stipulated size and q the fraction exceeding this size; then Np denotes the number of particles less than the stipulated size and Nq those which are greater. Let this process be repeated n-times. Then since each event is independent of the previous one, the frequency of 0, 1, 2, . . . particles being less than the stated size must be given by the binomial expansion

$$N(p + q)^n = N\left[p^n + np^{n-1}q + \frac{n(n-1)}{2!}p^{n-2}q^2 + \ldots\right] \qquad \text{Eq (23-1)}$$

Symmetrical Distribution—Since $p + q = 1$, we see that the above series is weighted to the left or right depending upon whether p is greater or less than q. In other words, if the terms of the series are plotted we obtain different bell-shaped curves which for values of $p > q > {}^1/_2$ are peaked to the left and for $p < q < {}^1/_2$ are mirror images of the first set but peaked to the right. When $p = q = {}^1/_2$, we have a symmetrical curve. For this case the above series becomes

$$N({}^1/_2 + {}^1/_2)^n = N({}^1/_2)^n\left[1 + n + \frac{n(n-1)}{2!} + \ldots\right] \qquad \text{Eq (23-2)}$$

so that the frequency m, that particles less than a stated size will occur, is given by the general term

$$n({}^1/_2)^n \cdot \frac{n!}{m!(n-m)!}$$

Since the curve for $p = q = {}^1/_2$ is symmetrical, we may put $n = 2k$ (for simplicity) and derive the midterm or maximum of the series. If we denote this midterm by N_0, we obtain

$$N_0 = N({}^1/_2)^{2k} \cdot \frac{(2k)!}{k!k!} \qquad \text{Eq (23-3)}$$

The general binomial expansion, Eq (23-1), can be fitted to most size distributions encountered in micromeritics. The special expansion Eq (23-2) can be simplified somewhat. Provided neither p nor q is small, it may be shown that when n becomes very great the frequency of particles less than a stipulated diameter d is

$$F(d) = N_0 \cdot \exp(-d^2/k) \qquad \text{Eq (23-4)}$$

It may further be shown by means of Sterling's formula that the value of N_0 in Eq (23-3) is equivalent to

$$N_0 = \frac{N}{(\pi k)^{1/2}} \qquad \text{Eq (23-5)}$$

The significance of k is easily derived. Let m-particles be measured and let the departure of each from the average be denoted by x. Then, since each measurement is independent of others, the frequency of this system is

$$F(d) = \frac{N^m}{(\pi k)^{m/2}} \cdot \exp\left(-\frac{x_1^2 + x_2^2 + \cdots + x_m^2}{k}\right)$$

The most probable value of k is that which renders k a maximum. Let us write $1/k = h^2$, then

$$\frac{dF(d)}{dh} = \frac{N^m h^{m-1}}{\pi^{m/2}} \cdot \exp\left(-h^2 \sum_1^m x_r^2\right) \cdot \left(m - 2h^2 \sum_1^m x_r^2\right) = 0$$

whence

$$h^2 = \frac{m}{2\sum_1^m x_r^2}$$

or

$$k = 2\sum_1^m x_r^2/m$$

Now the summation

$$\frac{\sum_1^m x_r^2}{m} = \frac{\sum_1^m (d_r - d_{av})^2}{m} \qquad \text{Eq (23-6)}$$

and if there is a stated frequency n' for every d, we may write

$$\frac{\sum_1^m x_r^2}{m} = \frac{\sum[n'(d - d_{av})^2]}{\sum n'}$$

and this is called the standard deviation squared $(S.D.)^2$, and is usually denoted by the symbol σ^2. The S.D. is a measure of the dispersion of our data as we shall show later.

Having thus determined the significance of k, we may write Eq (23-4) so that the maximum frequency is taken at the zero of the abscissa as follows, remembering that now $\Sigma n' = N$,

$$F(d) = \frac{N}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{d^2}{2\sigma^2}\right) \qquad \text{Eq (23-7)}$$

If we shift our maximum to the right or left of our zero abscissa, then

$$F(d) = \frac{N}{\sigma\sqrt{2\pi}} \cdot \exp\left[-\frac{(d - d_{av})^2}{2\sigma^2}\right] \qquad \text{Eq (23-8)}$$

Area under Curve—It is obvious that the integral of either Eq (23-7) or (23-8) gives the area under the curve. Since all eventualities are included within the abscissa and the curve, the area must be equal to unity. Similarly the integral to any value of the abscissa yields the frequency of particles up to that diameter (if we remain concerned with particle measurement).

In Eqs (23-7) and (23-8) we note that the distribution represented applies to a size range which is infinite. Generally speaking, particle-size measurements (provided, of course, that the measurements yield a symmetrical curve) are confined to particles having well-defined limits of size, and the distribution given by the above equations does not fit extremes very well. In other words, our ideal curve tapers off asymptotically at both extremes, whereas our experimental observations are limited. Thus, we have a significant departure,* but not one which results in serious errors (for example, see Yule, 1927, pp. 304–305).

If we write t for $|d_{av} - d|/\sigma\sqrt{2}$ in Eq (23-8) we then have the probability integral

$$\frac{2N}{\sqrt{\pi}}\int_0^t \exp(-t^2)dt \qquad \text{Eq (23-9)}$$

which represents the area under the curve to the upper limit. If $t = \infty$ the value of the integral is unity, so that for any value of t the integral merely gives the chance of an observation being greater (or less) than t.

* As a matter of fact, the kind of distribution of interest to us is actually represented by a Type III Pearsonian curve (see Elderton, 1938). We have, however, chosen to omit discussion of Pearson's distribution since it is simpler to deal with the forms of equations we developed. There are few situations where rigorous adherence to distribution types is required in micromeritics. In our developments the ease of obtaining parameters has governed our selection of formulas (See Chapter 4.)

Tables of $|d_{av} - d|/\sigma = \sqrt{2}t$ are to be found with most mathematical tables. In Appendix VII we reproduce such a table, where x is the difference $|d_{av} - d|$. Note that only one-half of the area under the curve is given in the table. Since the curve is symmetrical, twice the values listed gives the total area. In particular, we should remember that x/σ is the distance from the mean to any particular ordinate of the curve in terms of standard deviations. We see from the table cited that

when $x = \sigma$, 68.26 percent ($2 \times 0.3412 \times 100$) are contained within the limits
when $x = 2\sigma$, 95.46 percent are contained within the limits
when $x = 3\sigma$, 99.73 percent are contained within the limits

Thus, the table forms a convenient reference to check observed data—that is, to test what would be expected if the normal law of errors applied. Note, too, how σ measures the dispersion of our data.

Asymmetrical Distributions—These are included in our binomial expansion Eq (23-1). However, it is more convenient to use another equation similar to Eq (23-8) for such distributions by merely changing variables. Many frequency distribution data which plot asymmetrically on arithmetic grid become symmetric if the independent variable is plotted logarithmically. When a normal distribution results by this method we may apply Eqs (23-8) and (23-6) by taking the logarithms of the variables, thus*

$$F(d) = \frac{N}{\ln \sigma_g \sqrt{2\pi}} \cdot \exp\left[-\frac{(\ln d - \ln d_g)^2}{2 \ln^2 \sigma_g}\right] \quad \text{Eq (23-10)}$$

$$\log \sigma_g = \sqrt{\frac{\sum[n'(\log d - \log d_g)^2]}{\sum n'}} \quad \text{Eq (23-11)}$$

where d_g now refers to the geometric mean defined as in Eq (3-4) and σ_g is the *geometric* S.D.

Poisson Distribution—When either p or q is small but n very large, the binomial expression

$$N(p + q)^n$$

may be written thus

$$N(p + q)^n = N(1 - q + q)^n$$

since $(p + q) = 1$. Thus, if we write $nq = \lambda$

$$N(p + q)^n = N(1 - q)^{\lambda/q} \cdot \left(1 + \frac{q}{1 - q}\right)^{\lambda/q}$$

As q becomes indefinitely small, we have that

$$N(1 - q)^{\lambda/q} = N \cdot \exp(-\lambda)$$

*See Chapter 3.

and the second bracket to the right can be expanded and it may be shown that it becomes

$$1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^n}{n!}$$

Hence, for the special case under discussion

$$N(p + q)^n \equiv N \cdot \exp(-\lambda) \cdot \left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^n}{n!}\right)$$

(for $q \to 0$, $n \to \infty$, but nq finite) and this equation is called Poisson's distribution.

If p denotes success and q failure, we see from this equation (writing $P(m)$ for the probability of m-successes in $m + m'$ trials, $m'/n \to 0$) that

$$P(m) = \exp(-\lambda) \cdot \left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^{m'}}{m'!}\right) \quad \text{Eq (23-12)}$$

If $m' = 0$ the chance that the event succeeds every time is $\exp(-\lambda)$. If $m' = 1$ the chance that the event cannot fail more than once is $\exp(-\lambda) \cdot (1 + \lambda)$, so that $\lambda \cdot \exp(-\lambda)$ is the chance of exactly one failure. The last term in parentheses therefore gives us the frequencies of 0, 1, 2, 3, etc., failures.

The Poisson distribution has been used several times in this text in connection with problems involving small independent probabilities. For example, suppose we desire to find the distribution of ν random particles on an N-square uniform grid, N being large and each square equally accessible; the distribution will then be given by the expression

$$N\left(\frac{1}{N} + \frac{N-1}{N}\right)^{\nu}$$

The expansion of this equation reduces in the limit to Eq (23-12). Thus, we are enabled to predict the frequency distribution of 1, 2, 3, etc., particles in the squares constituting the grid.

It may be shown that the mean of the Poisson distribution is λ and the standard deviation $\sqrt{\lambda}$. Hence, in any set of observations conforming to the distribution, *mean* = (S.D.)^2.

Relationship between Various Means—In Chapter 3 we discussed various averages or means commonly used. There are a few simple relationships between these means which though approximate are often useful. If M denotes the arithmetic mean, G the geometric mean, and H the harmonic mean of any set of data for which the deviations x are small in comparison with the arithmetic mean ($x^2/M^2 << x/M$), then

$$G = M\left(1 - \frac{1}{2}\frac{\sigma^2}{M^2}\right)$$

$$II = M\left(1 - \frac{\sigma^2}{M^2}\right)$$

where σ is the arithmetic S.D. To a comparable degree of approximation

$$M^2 - G^2 = \sigma^2$$

so that

$$G^2 = MII$$

NUMBER OF SAMPLES REQUIRED

Earlier in this chapter a general description of sampling procedure was discussed. It was there recognized that the number of samples taken markedly affected the accuracy of the result and hence the reliability of the result as a measure of extent of a particular attribute in the universe. We shall here consider two cases, the first of which applies to samples which represent a sizable portion of the universe. In the second instance we shall deal extensively with the "random law" of sampling and show how the standard deviation of the universe is derivable from the standard deviation of the samples and the number of samples.

Limited Universe—If in the case of a finite universe n represents the number of samples taken from a universe of N samples, and if p represents decimally an attribute observed or measured, and q the decimal in which the attribute is lacking ($p + q = 1$), then the standard deviation (as a decimal) of the samples is expressed by the relationship

$$\sigma^2 = \frac{pq}{n} \cdot \frac{N - n}{N - 1}$$

Hence the number of samples needed to assure a standard deviation of a stipulated amount σ (expressed as a decimal) is

$$n = \frac{Npq}{\sigma^2(N - 1) + pq} \qquad \text{Eq (23-13)}$$

If we select σ, which we are at liberty to do, and if p can be approximated as to order of magnitude, then the number of samples required to obtain the degree of dispersion measured by σ is readily obtained from this expression. For example, suppose N is a universe of 1000 g and we desire to determine the number of 1-g random samples needed to yield a standard deviation of 10 percent. Assume further that the attribute p measures roughly 5 percent of the sample. Then $p = 0.05$, $q = 0.95$, and $\sigma = 0.1$, and we obtain by substitution in Eq (23-13) that five 1-g samples are required.

If we know nothing about the universe, and the probabilities of an

attribute being present or absent can be assumed equal, then $p = q = 0.5$ and $pq = 0.25$. By substituting in Eq (23-13) any one of three variables may be computed, provided, of course, the other two are known. Note that for a given N and σ, n is a maximum for the condition discussed.

The Random Law—We now arrive at a discussion of one of a series of cases developed by Landry (1944) which are of great importance to the samlping of particulate matter. Suppose we have a sample of N-pieces, each of weight w and let σ_w^2 be the standard deviation squared (S.D.)^2 of the curve of variability of any attribute, as, for example, diameter or a chemical constituent, taking each piece singly. Then it may be shown that the (S.D.)^2 of the gross sample ($\sigma_{N(w)}^2$) is given by the relation

$$\sigma_{N(w)}^2 = \frac{\sigma_w^2}{N} \qquad \text{Eq (23-14)}$$

Thus, given N and σ_w^2, we may at once determine $\sigma_{N(w)}^2$, or if we desire to set a tolerance, that is, a stated deviation in the attributes from the average in a given percentage of the sample, we may determine the number of pieces, N, having an observed σ_w^2 necessary to meet it. The determination of σ_w^2 is given in the section on distribution curves, Eq (23-6).

There is no direct way of computing $\sigma_{N(w)}^2$ when N is not known. However, we may overcome this difficulty by using the symmetric (normal) probability curve, assuming that the kind of distribution we are dealing with is random. Suppose, as an example, that we desire 99 percent of gross samples to be within ± 1 percent of the average of a determinable attribute whose (S.D.)^2 is 20, what will be the value of $\sigma_{N(w)}$ and N? In this case, using the probability table in Appendix VII, we obtain or $x/\sigma_{N(w)}$,

$$\frac{0.99}{2} = 0.495$$

and

$$\frac{1}{\sigma_{N(w)}} = 2.56$$

Since $x = 1$, we have $\sigma_{N(w)} = 0.39$, and $\sigma_{N(w)}^2 = 0.15$. Hence

$$N = \frac{\sigma_w^2}{\sigma_{N(w)}^2} = \frac{20}{0.15} = 144$$

Thus, in order that 99 percent of the gross samples will be within ± 1 percent of the average of the determinable attribute we must have a total of 144 random samples.

Problems

1. Small samples are to be taken from 100 lots of fine material. These samples are to be analyzed into two fractions, A and B. How many samples must be taken to give an S. D. of 0.5 if the value desired of the fraction A occurs five times out of ten?

2. The center of a glass slide has 100 ruled squares. Over the area occupied by these squares was placed a small amount of xylene suspension containing particles. After drying the distribution over the rulings was examined. The following count distribution was obtained:

Number of particles	Number of squares having this count
1	2
2	10
3	21
4	52
5	63
6	77
7	62
8	46
9	41
10	11
11	8
12	5
13	2

Show that the distribution follows the law of large numbers (Poisson distribution)

Note: The Poisson distribution may be derived as follows (see chapter 23): Let the number of ruled squares be N and let n be the number of particles. Then the probability of a particle entering a square is $1/N$ and the probability of its not entering $(1 - 1/N)$. The probability of any r particles entering is therefore

$$P_r = \frac{n!}{(n-r)!r!}\left(1 - \frac{1}{N}\right)^{n-r}\left(\frac{1}{N}\right)^r$$

This is the $(r + 1)$th term of the binomial

$$\left[\left(1 - \frac{1}{N}\right) + \frac{1}{N}\right]^n$$

Let m be the mean number of particles per square and write $n = m/(1 - 1/N)$. Now if $(1 - 1/N)$ is small and m not large the terms of the binomial above may be expanded as follows:

$$\left(1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots + \frac{m^n}{n!}\right) \cdot \exp(-m)$$

The first term gives the probability of 0 particles, the second the probability of 1 particle, the third the probability of 2 particles, etc. The standard deviation of this distribution is given by $\sigma = \sqrt{m}$. The above expansion is known as Poisson's series

In solving the problem given compute the probabilities for each number of particles in a given square and multiply by the total number of squares under observation.

3. A sample of uniform particles is drawn from a universe and is to be analyzed as to properties A and B If m particles have the attribute A and n the attribute B, determine the probability that the ratio of the numbers having each attribute lies between specified limits.

Note: Suppose the particles are arranged in a line OO', those having the attribute A being to the left of a point X (whose position is unknown) and all those having the attribute B to the right. Then, of $m + n$ particles taken at random in OO', n will be found to fall on OX and m on XO'. If the distance X measured from O is denoted by x, the number of cases for X between x and $x + dx$ is measured by

$$\frac{(m + n)!}{m!\ n!}\ x^m\ (1 - x)^n\ dx$$

Hence the probability that the ratio of particles having the attribute A to the whole number of particles lies between given limits α and β is

$$p = \frac{\int_\alpha^\beta x^m\ (1 - x)^n\ dx}{\int^1 x^m\ (1 - x)^n\ dx}$$

Here α and β are fractions of the whole number of particles

Suppose, for example, of 10 particles selected for analysis 7 had the attribute A and 3 the attribute B, we are to determine the chance that the proportion of those having attribute A is between 0.6 and 0 8 of the whole, that is, that it differs by $\pm$ 0.1 of its expected value 0 7 Thus

$$p = \frac{\int_{0\,6}^{0\,8} x^7\ (1 - x)^3\ dx}{\int_0^1 x^7\ (1 - x)^3\ dx}$$

The frequency distribution for the point x is, of course,

$$F(x) = x^m\ (1 - x)^n.$$

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APPENDIX

Useful Constants and Conversion Factors of Frequent Occurrence

I. Constants

Densities (g per cu cm)	
Water (3.98 deg C)	1.000
Air (20 deg C)	0 00114
Nitrogen (−196 deg C)	0 803
Viscosity (poises)	
Water (10 deg C)	0 01308
(20 deg C)	0.01005
(30 deg C)	0 00801
Air (0 deg C)	171×10^{-6}
(10 deg C)	172×10^{-6}
(20 deg C)	178×10^{-6}
(30 deg C)	183.6×10^{-6}

II. Conversion Factors

Dimension	To convert	To	Multiply by
Length, [L]	ft	cm	30.48
	in.	cm	2.540
	cm	ft	3.28×10^{-2}
	cm	in.	0 3937
Area, [L^2]	sq ft	sq cm	929 0
	sq in.	sq cm	6.452
	sq cm	sq ft	1.076×10^{-3}
	sq cm	sq in.	1.055
Volume, [L^3]	cu ft	cu cm	2.823×10^4
	cu in.	cu cm	16.39
	cu cm	cu ft	3.531×10^{-5}
	cu cm	cu in.	6.102×10^{-2}
Velocity, [LT^{-1}]	ft per sec	cm per sec	30.48
	ft per min	cm per sec	0.508
	cm per sec	ft per min	1.969
Acceleration, [LT^{-2}]	ft per sec per sec	cm per sec per sec	30.48
	cm per sec per sec	ft per sec per sec	3.281×10^{-2}

Mass, [M]	lb	grams	453.6
	lb	grains	7000.0
	grains	grams	6.481×10^{-2}
	grams	lb	2.205×10^{-3}
	grains	lb	1.429×10^{-4}
	grams	grains	15.43
Force, [MLT^{-2}]	dynes	grams	1.020×10^{-3}
	dynes	lb	2.248×10^{-6}
	grams	dynes	980.7
	lb	dynes	4.448×10^{5}
Pressure, [$ML^{-1}T^{-2}$]	dynes per sq cm	lb per sq in.	1.450×10^{-5}
	dynes per sq cm	lb per sq ft	2.089×10^{-3}
	dynes per sq cm	atm	9.869×10^{-7}
	dynes per sq cm	cm Hg	7.50×10^{-5}
	lb per sq in.	dynes per sq cm	6.895×10^{4}
	lb per sq in.	atm	0.068
	lb per sq in.	cm Hg	5.17
	cm Hg	dynes per sq cm	13.33×10^{2}
	cm Hg	atm	1.32×10^{-2}
	cm Hg	lb per sq in.	1.934×10^{-1}
	cm Hg	in. H_2O	5.354
	in. H_2O	cm Hg	0.187
Energy, [$ML^{2}T^{-2}$]	Btu	ft-lb	777.97
	Btu	joules (abs)	1054.8
	Btu	gram cal	251.98
	Btu	erg	1.055×10^{10}
	ft-lb	Btu	1.285×10^{-3}
	ft-lb	gram cal	0.3239
	ft-lb	erg	1.356×10^{7}
	joules (abs)	Btu	9.48×10^{-4}
	joules (abs)	gram cal	0.239
	joules (abs)	erg	10^{7}
	joules (abs)	ft-lb	0.738
	gram cal	Btu	3.969×10^{-3}
	gram cal	ft-lb	3.087
	erg	Btu	9.481×10^{-11}
	erg	gram cal	2.389×10^{-8}
	erg	ft-lb	7.376×10^{-8}
Viscosity, [$ML^{-1}T^{-1}$]	poise	fps-unit	0.0672
	fps-unit	poise	14.88

III. Temperature Conversion Equations

$$\text{Deg C} = \frac{5}{9}(\text{deg F} - 32)$$

$$\text{Deg F} = \frac{9}{5}\,\text{deg C} + 32$$

$$\text{Deg K} = \text{deg C} + 273.18 = \text{deg F} + 459.4$$

1 Btu = 453.59 gram calories

IV. Electrical Conversion Units

$$1 \text{ volt} = \frac{1}{3} \times 10^{-2} \text{ volt (esu)}$$

$$1 \text{ ampere} = 3 \times 10^{9} \text{ amperes (esu)}$$

$$1 \text{ coulomb} = 3 \times 10^{9} \text{ units of charge (esu)}$$

$$1 \text{ electron charge} = 4.8 \times 10^{-10} \text{ esu}$$

V. Constants for Hydrometer Computations, Chapter 4 (Bauer, 1937)

Density and Viscosity of Water

Densities and viscosities are taken from *Smithsonian Physical Tables*, 8th edition.

Temperature (deg C)	Density (g./cc.) (ρ_0)	Viscosity poises (μ)
16	0 998 9705	0.011 111
17	8029	0.010 828
18	6244	559
19	4347	299
20	2343	050
21	0233	0 009 810
22	0.997 8019	579
23	5702	358
24	3286	142
25	0770	0.008 937
26	0.996 8158	737
27	5451	545
28	2652	360
29	0.995 9761	180
30	6780	007

Values of $\sqrt{\frac{30}{980}\frac{\mu}{\rho - \rho_0}}$ for various values of temperature and specific gravity of particles

Temperature (deg C)	Specific gravity of soil particles, ρ											
	2 35	2 40	2 45	2.50	2 55	2 60	2 65	2.70	2.75	2 80	2.85	2 90
16	01587	01558	01531	01505	01481	.01457	.10435	01414	.01394	.01374	.01356	.01338
17	01566	01538	.01511	01486	.01462	01439	.01417	01396	.01376	.01356	.01338	.01320
18	01546	01519	01492	01467	01443	01421	.01399	01378	.01359	.01339	.01321	.01303
19	01527	01499	01474	.01449	01425	.01403	.01382	.01362	01342	.01323	.01305	.01287
20	.01509	01481	.01456	01431	.01408	.01386	.01365	.01344	.01325	.01307	.01289	.01272
21	01490	01463	01438	01414	.01391	01369	.01348	01328	.01309	.01201	.01273	.01257
22	01473	.01446	01421	.01397	.01374	.01353	.01332	.01312	.01294	.01276	.01258	.01242
23	01455	01429	01404	.01381	.01358	.01337	.01317	.01297	.01279	.01261	.01233	.01227
24	01438	01412	01388	01365	01342	01321	01301	.01282	.01264	.01246	.01229	.01213
25	01422	01396	01372	.01349	01327	.01306	.01286	01267	.01249	.01232	.01215	.01199
26	01406	01381	01357	.01334	.01312	.01291	.01272	.01253	.01235	.01218	.01201	.01185
27	01390	.01365	.01342	.01319	.01297	.01277	.01258	01239	.01221	.01204	.01188	.01172
28	.01375	01350	01327	.01304	.01283	01264	.01244	01225	.01208	.01191	.01175	.01159
29	.01360	.01335	.01312	.01290	.01269	01249	.01230	01212	.01195	.01178	.01162	.01146
30	01345	01321	01298	.01276	01256	.01236	.01217	.01199	.01182	.01165	.01149	.01134

VALUES OF $\frac{\rho}{\rho - \rho_0}$ FOR VARIOUS VALUES OF SPECIFIC GRAVITY, ρ, OF PARTICLES

Table is computed for values of ρ_0 at 20 deg C. The effect of temperature on values of $\frac{\rho}{\rho - \rho_0}$ is exceedingly small and may be neglected.

Specific gravity (ρ)	$\frac{\rho}{\rho - \rho_0}$	Specific gravity (ρ)	$\frac{\rho}{\rho - \rho_0}$
2.35	1 738	2 65	1.604
2 40	1 712	2.70	1.587
2 45	1 688	2.75	1.570
2.50	1 665	2 80	1.554
2.55	1.643	2 85	1.539
2.60	1 623	2.90	1.525

VI. ELECTROSTATIC AND ELECTROMAGNETIC UNITS -DIMENSIONS

Unit	Dimensions: Electrostatic	Dimensions: Electromagnetic	Ratio of electromagnetic to electrostatic unit	Practical unit: Name	Practical unit: Ratio of size to that of electromagnetic unit
Charge	$M^{1/2}L^{3/2}T^{-1}k_0^{1/2}$	$M^{1/2}L^{1/2}\mu_0^{1/2}$	c	coulomb	10^{-1}
Electromotive force	$M^{1/2}L^{1/2}T^{-1}k_0^{-1/2}$	$M^{1/2}L^{3/2}T^{-2}\mu_0^{1/2}$	c^{-1}	volt	10^{8}
Electric intensity	$M^{1/2}L^{-1/2}T^{-1}k_0^{-1/2}$	$M^{1/2}L^{1/2}T^{-2}\mu_0^{1/2}$	c^{-1}		
Electric displacement	$M^{1/2}L^{-1/2}T^{-1}k_0^{1/2}$	$M^{1/2}L^{-3/2}\mu_0^{-1/2}$	c		
Electric current	$M^{1/2}L^{3/2}T^{-2}k_0^{1/2}$	$M^{1/2}L^{1/2}T^{-1}\mu_0^{1/2}$	c	ampere	10^{-1}
Electric resistance	$L^{-1}Tk_0^{-1}$	$LT^{-1}\mu_0$	c^{-2}	ohm	10^{9}
Capacity	Lk_0	$L^{-1}T^2\mu_0^{-1}$	c^2	farad	10^{-9}
Inductance	$L^{-1}T^2k_0^{-1}$	$L\mu_0$	c^2	henry	10^{9}
Magnetic field	$M^{1/2}L^{1/2}T^{-2}k_0^{1/2}$	$M^{1/2}L^{-1/2}T^{-1}\mu_0^{-1/2}$	c	gauss or oersted	1
Magnetic induction	$M^{1/2}L^{-3/2}k_0^{-1/2}$	$M^{1/2}L^{-3/2}T^{-1}\mu_0^{1/2}$	c^{-1}		
Magnetic flux	$M^{1/2}L^{1/2}k_0^{-1/2}$	$M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$	c^{-1}	maxwell	1
Magnetic moment	$M^{1/2}L^{3/2}k_0^{-1/2}$	$M^{1/2}L^{5/2}T^{-1}\mu_0^{1/2}$	c^{-1}		
Intensity of magnetization	$M^{1/2}L^{3/2}k_0^{-1/2}$	$M^{1/2}L^{-1/2}T^{-1}\mu_0^{1/2}$	c^{-1}		
Magnetic pole	$M^{1/2}L^{1/2}k_0^{-1/2}$	$M^{1/2}L^{1/2}T^{-1}\mu_0^{1/2}$	c^{-1}		
Dielectric constant	k_0	$L^{-2}T^2\mu_r$	c^{-2}		
Magnetic permeability	$L^{-2}T^2k_0$	μ_0	c^2		
Energy	ML^2T^{-2}	ML^2T^{-2}		joule	10^{7}
Rate of working	ML^2T^{-3}	ML^2T^{-3}		watt	10^{7}

VII. Normal Probability Curve Area Table

x/σ	.00	01	02	03	04	05	.06	07	08	09
0 0	0000	0040	.0080	0120	.0159	.0199	0239	0279	0319	.0359
0 1	0398	0438	0478	0517	0557	0596	0636	0675	0714	0753
0 2	0793	0832	0871	0910	0948	0987	1026	.1046	1103	1141
0 3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
0 4	1554	1591	1628	1664	1700	1736	1772	1808	1844	1879
0 5	1915	1950	1985	2019	2054	.2088	2123	2157	.2190	2224
0 6	.2257	2291	2324	2357	2389	2422	2454	2486	2518	2549
0 7	2580	2612	2642	2673	.2704	2734	2764	2794	2823	2852
0 8	2881	.2910	2939	2967	2995	3023	3051	3078	3106	3133
0 9	3159	3186	3212	3238	.3264	3289	3315	3340	3365	3389
1 0	.3413	3438	3461	3485	3508	.3531	.3554	3577	3599	3621
1.1	.3643	3665	3686	3718	3729	3749	3770	3790	3810	3830
1 2	.3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4083	4099	4115	4131	.4147	4162	4177
1 4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1 5	4332	4345	4357	4370	4382	4394	4106	4418	4430	4441
1 6	4452	4463	4474	4485	4495	4505	4515	4525	4535	1545
1 7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1 8	4641	4649	4656	4664	.4671	4678	4686	4693	4699	4706
1 9	4713	4719	4726	4732	4738	4744	4750	4758	4762	4767
2 0	4773	4778	4783	4788	4793	4798	4803	4808	4812	4817
2 1	4821	4826	4830	1834	4838	4842	4846	4850	4854	4857
2 2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2 3	4893	4896	4898	4901	4904	4906	4909	4911	.4913	4916
2 4	4918	4920	4922	4925	4927	4929	4931	4932	.4934	4936
2 5	4938	4940	4941	4943	4945	1946	4948	4949	4951	4952
2 6	.4953	4955	4956	4957	4959	4960	4961	4962	.4963	4964
2 7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	.4974	.4975	.4976	4977	4977	4978	4979	4980	4980	4981
2 9	4981	4982	.4983	.4984	4984	4984	4985	4985	4986	4986
3 0	.4987	4987	.4987	.4988	4988	4988	4989	.4989	.4989	.4990
3.1	4990	.4991	.4991	.4991	.4992	4992	.4992	.4992	.4993	.4993

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