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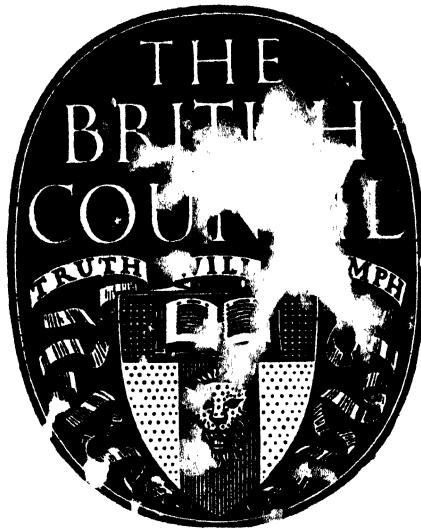
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THE LAMINAR BOUNDARY LAYER EQUATIONS

BY

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PREFACE

THE concept of the boundary layer, introduced by Ludwig Prandtl in 1904, has been a particularly fruitful one. Research on this topic has now reached the stage where there is a certain body of fundamental definitive information which is unlikely to be superseded to any great extent. This relates mainly to the steady incompressible laminar boundary layer in two dimensions. Work proceeds, however, and a considerable number of papers are being published, on unsteady boundary layers, on three-dimensional boundary layers in incompressible flow, and upon such topics as boundary-layer stability. In compressible flow, too, where additional important parameters arise, much is being done and more remains.

This monograph is one of a series, each of which is being written by an author on the general field in which his own research interests lie. It is inevitable, therefore, that there will be a certain amount of bias in the choice of material. I have tried to make the book reasonably self-sufficient, though lack of space has led to the omission of a number of very interesting problems of boundary layers. The topics so axed include unsteady boundary layers and boundary-layer stability, boundary layers on porous walls with suction or blowing, boundary layers in three dimensions (including axi-symmetric flow) and boundary layers with vorticity in the mainstream. For a discussion of these topics reference may be made to more encyclopaedic works, such as *Modern Developments in Fluid Dynamics* (Oxford, edited by S. Goldstein), the companion volumes on *High Speed Flow* (edited by L. Howarth), *Laminar Boundary Layers* (Oxford, edited by L. Rosenhead), and Volumes III to V of the series *High Speed Aerodynamics and Jet Propulsion*.

The purpose of Chapter 1 of this book is to introduce the boundary-layer concept, and to show how the equations of viscous flow are simplified thereby. The standard boundary-layer parameters, and the usual integral forms of the boundary-layer equations, are discussed, the incompressible flow forms being introduced as special cases of the more general compressible forms. Chapters 2 to 6 deal with various aspects of solutions in incompressible flow, commencing with analytic solutions for the velocity field, these being solutions which may be expressed in terms of functions which satisfy ordinary differential

equations (Chapter 2). There follow discussions of high-accuracy numerical solutions for the velocity field (Chapter 3), practical methods of calculation (Chapter 4), and an analysis of the factors which might govern the choice of a method (Chapter 5). Various types of solution of the temperature equation in incompressible flow with small temperature differences are considered in Chapter 6. Chapters 7 to 9 deal with compressible laminar boundary layers, consideration being first given to flow with zero pressure gradient (Chapter 7), then to flow with zero heat transfer (Chapter 8), and then to flow in which both pressure gradient and heat transfer are present (Chapter 9). Finally there is a brief discussion (Chapter 10) of some aspects of the problem of the interaction between shock waves and laminar boundary layers.

It is my hope that this book will be of value to a wide variety of workers. In the first place I have tried to present the material in a sufficiently ordered and logical manner as to make it of value as an introduction to boundary-layer theory for young research workers who are new to the subject, or to undergraduates who are familiar with the elements of classical inviscid fluid dynamics. Secondly, the book should be of some value to research workers in this field, since one of the things which has governed my choice of material has been the question of whether a particular piece of work has been an end in itself or whether it has assisted in opening up the way for further advances. Finally, I have borne in mind the needs of practising engineers, and have tried where possible to indicate the limitations, the likely accuracy, and the practical complexity of the methods described for calculating the various properties of laminar boundary layers.

In conclusion, I have great pleasure in expressing my thanks to the many people who have helped me, directly or indirectly, in the writing of this book. To Dr. M. J. Lighthill, F.R.S., Director of the Royal Aircraft Establishment, whose student I was at Manchester University, for the wise counsel he gave me then in so many branches of fluid dynamics, and whose influence is, I hope, evident in this book. To my colleagues at the National Physical Laboratory, for the stimulating discussions I have had with them at various times, and most particularly Dr. J. T. Stuart and Dr. G. E. Gadd. These two colleagues have been good enough to offer useful comments on a first draft of this book, although the responsibility for its deficiencies remains entirely my own. To Mrs. M. E. M. Sayer, for her patient and careful typing of the manuscript, and her cheerful approach to the difficult task of reading my writing. To Professor G. Temple, F.R.S., editor of this

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Finally, to my wife and, though they know it not, to my children, for so ordering their lives as to make the task of writing this book much less difficult. To all these people go my sincere thanks for their much appreciated help.

N. C.

Hanworth, Middlesex
June 1961

CONTENTS

1. INTRODUCTION	1
1. The equations of viscous flow	2
2. Boundary layers	4
~~3.~~ The laminar boundary layer on a plane wall	5
~~4.~~ The laminar boundary layer on a curved wall	8
5. Conditions in the mainstream	9
~~6.~~ Some standard boundary-layer characteristics	11
7. The momentum integral equation	13
8. The kinetic-energy integral equation	13
9. The thermal-energy integral equation	15
10. Incompressible flow	15
11. Crocco's transformation	17
~~12.~~ Von Mises's transformation	17
2. ANALYTIC SOLUTIONS FOR INCOMPRESSIBLE FLOW	19
~~1.~~ Flow parallel to a semi-infinite flat plate	19
~~2.~~ Flow near the stagnation-point of a cylinder	22
~~3.~~ The Falkner-Skan similarity solutions	23
~~4.~~ Series solutions from a stagnation-point	24
5. Series solutions from a sharp leading edge	28
6. Görtler's modified series expansions	29
7. Meksyn's technique	30
3. NUMERICAL SOLUTIONS FOR INCOMPRESSIBLE FLOW	31
1. The solution of Howarth	32
2. The solutions of Tani	35
3. The solutions of Curle	35
4. The solutions of Hartree, Leigh, and Terrill	38
4. PRACTICAL METHODS OF CALCULATION FOR INCOMPRESSIBLE FLOW	41
~~1.~~ Pohlhausen's method	41
2. Timman's method	44
3. The methods of Howarth and Walz	45
~~4.~~ The method of Thwaites	45
~~5.~~ The method of Stratford	48
6. The method of Curle	51
7. Use of the energy integral equation. The methods of Tani and Truckenbrodt	53
8. The method of Head	57

5. COMPARISONS OF APPROXIMATE METHODS OF SOLUTION	59
1. Information provided by various methods	60
2. Comparison of accuracy	60
3. Ease of computation	62
4. Conclusions	63
6. SOLUTIONS OF THE TEMPERATURE EQUATION AT LOW SPEEDS	65
1. Forced convection from a flat plate	65
2. Temperature of plate thermometer in moving fluid	67
3. Heat transfer near a stagnation-point	68
4. The solutions of Fage and Falkner	70
5. Lighthill's method	71
6. The work of Liepmann and Curle	73
7. Spalding's method	77
8. The method of Davies and Bourne	79
9. The Meksyn-Merk method	80
10. Curle's analysis by Stratford's method	81
11. Squire's method	84
12. Free convection from a heated vertical plate	86
7. THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH ZERO PRESSURE GRADIENT	88
1. Values for viscosity and Prandtl number	88
2. The solutions of Busemann and Kármán	90
3. The solutions of Kármán and Tsien	91
4. The solutions of Emmons and Brainerd	93
5. The calculations of Crocco	94
6. Summary of results for uniform wall temperature	100
7. The solutions of Chapman and Rubesin	102
8. Lighthill's analysis	105
8. THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH ZERO HEAT TRANSFER	108
1. Howarth's method	108
2. Young's method	110
3. The Stewartson-illingworth transformation	110
4. Rott's method	113
5. The method of Oswatitsch and Weighardt	114
6. The work of Cope and Hartree	114
7. The work of Illingworth, Frankl, and Gruschwitz	116
8. The investigations of Gadd	118

9. THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH PRESSURE GRADIENT AND HEAT TRANSFER	120
1. Accurate numerical solutions for special cases	120
2. Kalikhman's method	123
3. The method of Cohen and Reshotko	125
4. Monaghan's method	127
5. Curle's method	129
6. The method of Luxton and Young	131
7. The method of Poots	133
8. The methods of Lilley and Illingworth	135
9. Curle's method for calculating heat transfer	138
10. INTERACTIONS BETWEEN SHOCK WAVES AND BOUNDARY LAYERS	142
1. Principal results of experimental investigations	143
2. Summary of early theoretical investigations	145
3. Gadd's analyses for interactions causing separation	146
4. The analysis of Hakkinen, Greber, Trilling, and Abarbanel	150
5. The work of Gadd and Greber	152
6. The method of Curle	154
REFERENCES	157
INDEX	161

I

INTRODUCTION

THIS book is about solutions of the laminar-boundary-layer equations. The concept of the boundary layer, one of the corner-stones of modern fluid dynamics, was introduced by Prandtl (1904) in an attempt to account for the sometimes considerable discrepancies between the predictions of classical inviscid incompressible fluid dynamics and the results of experimental observations. As an example, we may remark that according to inviscid theory any body moving uniformly through an unbounded homogeneous fluid will experience zero drag!

Now the classical inviscid theories assume that the viscous forces in a fluid may be neglected in comparison with the inertia forces. This, indeed, would seem a reasonable approximation, since the viscosity of many fluids (and of air in particular) is extremely small. However, in certain regions of flow, fortunately often limited, the viscous forces can still be locally important, as Prandtl observed. The reason for this is that a typical viscous stress is of magnitude $\mu(\partial u/\partial y)$, where μ is the viscosity, u is the velocity measured in a direction parallel to that of the stress, and y is distance measured normal thereto, so that when the velocity gradient (or shear) $\partial u/\partial y$ is large the viscous stress can become important even though μ itself is small. It was Prandtl who remarked that in flow past a streamlined body, the region in which viscous forces are important is often confined to a thin layer adjacent to the body, and to a thin wake behind it. This thin layer is referred to as the boundary layer. When this condition holds the equations governing the motion of the fluid within the boundary layer take a form considerably simpler than the full viscous-flow equations, though less simple than the inviscid equations, and it is the solution of these equations with which we shall be presently concerned.

An alternative method of looking at this concept is as follows. The inviscid-flow equations are of lower order than the viscous-flow equations, so fewer boundary conditions can be satisfied in a mathematical solution of a given problem. Thus an inviscid-flow solution allows a finite velocity of slip at a solid boundary, whereas the solution of the viscous-flow equations does not allow such slip. In other words, the inviscid-flow solution assumes the existence of an appropriate vortex

sheet at the solid boundaries. Now in reality this vorticity will diffuse outwards from the boundary (in much the same way that heat would diffuse from a heated body) and will be convected with the stream. Thus, considering now the flow past a flat plate, the time t in which fluid travels a distance x parallel to the plate will be of order x/U , where U is a typical velocity, and in this time the vorticity will have diffused outwards through a distance of order $(\nu t)^{\frac{1}{2}} = (\nu x/U)^{\frac{1}{2}}$, where $\nu = \mu/\rho$ is the kinematic viscosity. This is an indication of the boundary-layer thickness.

Before turning to a more quantitative discussion, mention must be made of the phenomenon of boundary-layer separation. When the fluid is proceeding into a region of rising pressure, it is slowed down by this retarding force. In the outer part of the boundary layer, where the kinetic energy is large, this results only in a relatively small slowing down of the fluid, but the effect on the slower-moving fluid nearer to the wall can be considerable, and if the pressure rise is sufficient it can be brought to rest, and, farther downstream, a slow back-flow be set up. In such circumstances the forward flow must leave the surface to by-pass this region, and boundary-layer separation is said to have taken place. If the region of separated flow is extensive, the separation can have a back-reaction on the external flow, which is then quite different from what it would have been in the absence of the boundary layer. If the separated region is limited, on the other hand, the external flow may not be significantly affected, and the flow field may be calculated by calculating firstly the external flow (on the assumption of no boundary layer) and then calculating the boundary layer appropriate to this external flow.

The above qualitative analysis is restricted to low-speed flows, but serves to indicate the nature of the boundary layer in a simple way. At supersonic speeds, for example, interactions of the boundary layer with the external stream become more important, and lead to considerable theoretical difficulties which will not be discussed at this stage. In what follows a quantitative analysis will be given of how the boundary-layer equations may be deduced from the exact equations of viscous flow, and detailed discussions will be given of some of the points touched upon briefly above.

1. The equations of viscous flow

We take cartesian coordinates (x, y) , with associated components of fluid velocity (u, v) . The fluid is assumed to have pressure p , density ρ ,

and absolute temperature T , and these are functions of x and y only, in view of the approximation (made throughout this book) that the flow is two-dimensional and steady. The equations of motion then express the basic physical ideas that for a given element of fluid there is conservation of mass, momentum (excepting in so far as the element is acted upon by various forces), and energy (excepting in so far as work is done by these same forces). For a general derivation of these equations, reference may be made to volume I of *Modern Developments in Fluid Dynamics, High Speed Flow* (Howarth, 1953). In steady two-dimensional flow the equation of conservation of mass takes the form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \quad (1)$$

This equation is usually referred to simply as the equation of continuity. The equations of conservation of momentum in the x and y directions (the momentum equations) become

$$\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = & -\frac{\partial p}{\partial x} + \frac{1}{3}\mu \frac{\partial \Delta}{\partial x} + \rho X + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \\ & - \frac{2}{3}\Delta \frac{\partial \mu}{\partial x} + 2 \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \end{aligned} \quad (2)$$

and

$$\begin{aligned} \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = & -\frac{\partial p}{\partial y} + \frac{1}{3}\mu \frac{\partial \Delta}{\partial y} + \rho Y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \\ & - \frac{2}{3}\Delta \frac{\partial \mu}{\partial y} + 2 \frac{\partial v}{\partial y} \frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \end{aligned} \quad (3)$$

where

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad (4)$$

and X , Y are the external forces per unit mass of fluid. Finally the equation of conservation of energy (the thermal energy equation) takes the form

$$\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \mu \Phi, \quad (5)$$

$$\text{where} \quad \Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2. \quad (6)$$

There are thus four equations, (1), (2), (3), and (5), for the five unknowns u , v , p , ρ , T , and these are soluble, in principle at any rate, when an equation of state is defined, relating p , ρ , and T . For a perfect gas this takes the form

$$p = \rho \mathcal{R} T, \quad (7)$$

where $\mathcal{R}$ is the gas constant.

In the above equations c_p is the specific heat at constant pressure, usually taken to be constant, and k is the thermal conductivity, related to the thermometric conductivity κ by the relationship

$$\kappa = k/\rho c_p.$$

For a physical interpretation of the quantities κ , k , μ , and $\nu = \mu/\rho$, the reader is referred to the book *Modern Developments in Fluid Dynamics* (Goldstein, 1938). It will suffice for the present to remark that ν is a parameter determining the rate at which vorticity is diffused, whilst κ determines the rate at which heat is diffused. The ratio

$$\sigma = \nu/\kappa = \mu c_p/k$$

accordingly determines the relative rates of these two types of diffusion, and is called the Prandtl number. It is usual to assume that the Prandtl number is constant, and this holds true for air over quite a wide range of conditions, the value being about 0.72. It will be seen later that considerable simplifications are often possible if it is assumed that the Prandtl number is unity, an approximation that is not without value for air.

2. Boundary layers

It will be noted that the viscosity μ appears in equations (2) and (3) only as a multiplicative factor of velocity gradients, or of powers or products of velocity gradients. Accordingly, if the viscosity is small, classical fluid dynamics theory, which neglects viscosity, will be valid except in regions where velocity gradients are large. Similarly, provided the Prandtl number σ is not too small, small viscosity implies small thermal conductivity, so that the terms involving k in (5) are important only where the temperature gradients are large.

Now it is often found in practice that the regions of high velocity and temperature gradients are confined to a narrow region near to solid walls, known as the boundary layer, and to a thin wake behind streamlined bodies. In such a domain, considerable simplifications of the equations of motion are possible, even though all the terms involving μ and k cannot be neglected, as was first shown by Prandtl (1904) in a paper of fundamental importance. In low-speed flow it is usually possible to consider the development of the boundary layer as a separate problem from that of the substantially inviscid external flow, the exception being when there is a substantial region of separated flow. When the external flow is supersonic, however, there is an interaction between the boundary layer and the external stream which must be

taken into account. Crudely we may say that though an artificially introduced disturbance cannot be propagated upstream in a wholly supersonic flow, the presence of a boundary layer, in which the flow sufficiently near to the walls will be subsonic, provides a mechanism for such upstream influence. Accordingly the external stream does not approximate to that obtained in the absence of the boundary layer, and cannot be independently prescribed.

In spite of this difficulty it is still useful to begin by considering separately the boundary-layer approximation and the inviscid approximation, as use can be made of these results even in certain problems in supersonic flow where the interaction between boundary layer and external stream is particularly important, as, for example, when a shock wave interacts with a boundary layer.

3. The laminar boundary layer on a plane wall

We begin by deriving the boundary-layer equations for flow over a plane wall. The x -axis is taken along the wall and the y -axis perpendicular to it. We assume that the thickness δ_r of the layer in which the velocity gradient $\partial u/\partial y$ is large, where the velocity u rises rapidly from 0 at the wall to a value u_1 , is much smaller than a typical length l in the flow field as a whole. Equally well we assume that δ_t , the thickness of the layer in which the temperature gradient $\partial T/\partial y$ is large, and in which the temperature of the fluid changes from the temperature of the wall T_w to a value T_1 , is also much less than l . We shall assume at this stage that δ_r and δ_t are the same order of magnitude, each being of order $\delta \ll l$. This restriction can later be removed, provided δ_r and δ_t are both $\ll l$. We let u_0, ρ_0, T_0 be typical values of velocity u , density ρ , temperature T , and may then deduce the order of v from (1). This equation shows that

$$\frac{\partial}{\partial y}(\rho v) = -\frac{\partial}{\partial x}(\rho u),$$

which is of order $\rho_0 u_0/l$. Hence, upon integrating across the layer from $y = 0$ to $y = \delta$, we find that ρv is of order $(\rho_0 u_0/l)\delta$, so that

$$v = O\left(\frac{u_0 \delta}{l}\right).$$

Thus v is small compared with u , within and at the edge of the boundary layer.

We can now examine the magnitudes of the various terms in (2). In doing so we remember that the derivative $\partial F/\partial x$ of a function F

will be much smaller than the derivative $\partial F/\partial y$. In fact

$$\frac{\partial F}{\partial x} = O\left(\frac{F}{l}\right), \quad \frac{\partial F}{\partial y} = O\left(\frac{F}{\delta}\right).$$

Thus

$$\rho u \frac{\partial u}{\partial x}, \quad \rho v \frac{\partial u}{\partial y}, \quad \frac{\partial p}{\partial x} \quad \text{are} \quad O(\rho_0 u_0^2/l), \quad (8)$$

$$\mu \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial \mu}{\partial y} \frac{\partial u}{\partial y} \quad \text{are} \quad O(\mu_0 u_0/\delta^2), \quad (9)$$

and

$$\frac{1}{3}\mu \frac{\partial \Delta}{\partial x}, \quad \mu \frac{\partial^2 u}{\partial x^2}, \quad \frac{2}{3}\Delta \frac{\partial \mu}{\partial x}, \quad 2 \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x}, \quad \frac{\partial \mu}{\partial y} \frac{\partial v}{\partial x} \quad \text{are} \quad O(\mu_0 u_0/l^2). \quad (10)$$

Ideal fluid-dynamics theory rejects all the terms involving viscosity, namely those in (9) and (10), but boundary-layer theory retains the former, since δ^2 is small as well as μ_0 . We see that these terms are the same order of magnitude as (8) when

$$\delta_v^2 = O\left(\frac{\mu_0 l}{\rho_0 u_0}\right) \quad \text{or} \quad \frac{\delta_v}{l} = O(R^{-1/2}), \quad (11)$$

where R is the Reynolds number

$$R = \frac{u_0 l \rho_0}{\mu_0} = \frac{u_0 l}{\nu_0}.$$

We have chosen δ_v as the relevant value of δ since we are considering a momentum equation. Upon rejecting the terms in (10), then (2) simplifies to

$$\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= \rho X - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial \mu}{\partial y} \frac{\partial u}{\partial y} \\ &= \rho X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right). \end{aligned} \quad (12)$$

We note that the rejected terms are of order $(\delta/l)^2$ times those retained.

We now deal in a similar manner with equation (3). We can see that

$$\rho u \frac{\partial v}{\partial x}, \quad \rho v \frac{\partial v}{\partial y} \quad \text{are} \quad O\left(\frac{\rho_0 u_0^2 \delta}{l^2}\right) = O\left(\frac{\rho_0 u_0^2}{l} \frac{\delta}{l}\right) \quad (13)$$

$$\mu \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial \mu}{\partial x} \frac{\partial u}{\partial y}, \quad \frac{1}{3}\mu \frac{\partial \Delta}{\partial y}, \quad \frac{2}{3}\Delta \frac{\partial \mu}{\partial y}, \quad 2 \frac{\partial v}{\partial y} \frac{\partial \mu}{\partial y} \quad \text{are} \quad O\left(\frac{\mu_0 u_0}{\delta^2} \frac{\delta}{l}\right) \quad (14)$$

and

$$\mu \frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial \mu}{\partial x} \frac{\partial v}{\partial x} \quad \text{are} \quad O\left(\frac{\mu_0 u_0}{l^2} \frac{\delta}{l}\right). \quad (15)$$

Accordingly, the terms in (15) are of order δ/l times those rejected in

(2), and those in (13) and (14) are of order δ/l times those retained. It follows that

$$\rho Y - \frac{\partial p}{\partial y} = O\left(\frac{\rho_0 u_0^2 \delta}{l}\right). \quad (16)$$

Now Y will usually be zero. For example body forces can be neglected in problems of forced convection, and in problems of free convection the body force (gravity) will act in the x direction. Assuming, then, that $Y = 0$, it follows from (16) that the pressure gradient $\partial p/\partial y$ is small, and the pressure change across the boundary layer is very small, being $O(\rho_0 u_0^2 \delta^2/l^2)$, which is neglected. Thus (3) reduces simply to

$$p(x, y) = p(x). \quad (17)$$

We turn now to the thermal energy equation (5). By identical reasoning to that given above it follows that the two terms on the left-hand side are of equal order of magnitude. The term $v(\partial p/\partial y)$ vanishes by (17), and the term $\partial(k \partial T/\partial x)/\partial x$ is of order $(\delta/l)^2$ times $\partial(k \partial T/\partial y)/\partial y$. Finally, in Φ , the term $(\partial u/\partial y)^2$ is $O(u_0^2/\delta^2)$, which is at least $(l/\delta)^2$ times the other terms; these are therefore neglected. It follows that (5) becomes

$$\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2. \quad (18)$$

When body forces are negligible, an alternative form of (18) is obtained by adding u times (12) to (18). This yields

$$\begin{aligned} \rho u \frac{\partial}{\partial x}(c_p T + \tfrac{1}{2}u^2) + \rho v \frac{\partial}{\partial y}(c_p T + \tfrac{1}{2}u^2) &= \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 + u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial y}\left\{\mu \frac{\partial}{\partial y}\left(\frac{c_p T}{\sigma} + \tfrac{1}{2}u^2\right)\right\}. \end{aligned} \quad (19)$$

We note in passing that the terms $\rho u \partial(c_p T)/\partial x$ and $\partial(k \partial T/\partial y)/\partial y$ in (18) will be the same order of magnitude if

$$\delta_l^2 = O\left(\frac{k_0 l}{u_0 \rho_0 c_p}\right) = O\left(\frac{\kappa_0 l}{u_0}\right),$$

or

$$\delta_l/l = O(Pe^{-\frac{1}{2}}), \quad (20)$$

where (Pe) is the Peclet number

$$Pe = \frac{u_0 l}{\kappa_0}.$$

We deduce from (11) and (20) that

$$\frac{\delta_v}{\delta_t} = O\left(\frac{\nu_0}{\kappa_0}\right)^{\frac{1}{2}} = O(\sigma^{\frac{1}{2}}),$$

so that the assumption of similar velocity and thermal boundary-layer thicknesses is equivalent to the assumption that the Prandtl number is not too different from unity.

The equations (1), (7), (12), (17), and (19) are the boundary-layer equations for the five unknowns u , v , p , ρ , T , which must be solved subject to the following boundary conditions. At the wall $y = 0$ we have $u = v = 0$, since there is no slip or normal velocity at a fixed solid wall. Further there is a boundary condition on T at $y = 0$, which is usually that either T or $\partial T/\partial y$ is prescribed there (or determinable from a subsidiary equation). At the edge of the boundary layer we may assume that u , T , p are functions only of x since variations in the quantities with y outside the boundary layer are small compared with those within the boundary layer. Further, the boundary layer tends into the mainstream asymptotically, so that the edge of the boundary layer is $y \rightarrow \infty$, and the boundary conditions at the edge are $u \rightarrow u_1(x)$, $T \rightarrow T_1(x)$, $\rho \rightarrow \rho_1(x)$. From the practical point of view we may regard the edge of the boundary layer as the position where $u/u_1 = 0.99$ say, though mathematically there is no edge as such.

The solution of (17) is trivial, for if we let $y \rightarrow \infty$ in the momentum equation (12), then provided there are no body forces we find that

$$\rho_1 u_1 \frac{du_1}{dx} = -\frac{dp}{dx}, \quad (21)$$

so that the pressure is given in terms of the mainstream velocity and density.

4. The laminar boundary layer on a curved wall

The equations as derived above are valid only for flow past a plane wall. Physically one is often interested in flow past a curved surface, such as a cylindrical obstacle say. To consider this flow it is necessary to take a more general form of the Navier-Stokes equations than (1) to (6), by introducing general orthogonal coordinates x , y , with x measured parallel to and y normal to the wall. Since these coordinates are non-planar a considerable number of additional terms (involving the curvature κ_1 of the wall) appear in the equations. The details are here omitted, and the reader referred to Goldstein's book (§ 45), *Modern Developments in Fluid Dynamics*, vol. I, where the detailed analysis is

given for incompressible flow; the analysis for compressible flow follows the same lines. It is sufficient for us to remark that when the boundary-layer approximations are made exactly as in section 3, subject to the assumption that $\kappa_1 l$ is of order unity, then the equations reduce to the forms (1), (12), (18), and

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = \kappa_1 u^2.$$

Accordingly the only difference between the equations for plane and curved walls lies here, since a normal pressure gradient is required to balance the centrifugal force. Nevertheless, the total change of pressure across the boundary layer is still small, since the boundary layer is thin, so the pressure may still be taken to be a function of x only. Thus the equations for plane and curved boundaries are identical.

The practical significance of this fact is that in calculating laminar boundary-layer development, all curvature effects may be neglected, excepting in so far as these affect the flow outside and at the edge of the boundary layer.

5. Conditions in the mainstream

Certain relationships between quantities in the external stream will be constantly referred to in what follows. It is useful, therefore, to derive these relationships here briefly, so that they may be used when required, and further details will be found in any standard textbook on Gas Dynamics, such as *Modern Developments in Fluid Dynamics, High Speed Flow*, vol. I (Howarth, 1953).

It is continually necessary to introduce the local velocity of sound, for which the usual symbol is a , and in a moving stream this is the velocity with which sound waves are propagated relative to the stream. It is given by

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_S, \quad (22)$$

where the derivative is taken at constant entropy S . Now in the external stream, where viscosity and conductivity are negligible, the energy equation may be used to show that the entropy does not vary along a streamline, except in passing through a shock. Hence, outside the boundary layer, the derivative in (22) is taken along the streamlines. It also follows that if the entropy is uniform at infinity it will be the same everywhere (upstream of any shocks which may be present). Such a flow was formerly called isentropic, though Howarth (1953) uses the term homentropic and this term will be used here. The condition

of homentropic flow may be used, along with the equation of state for a perfect gas, (7), to show that

$$p \propto \rho^\gamma \quad (23)$$

in the mainstream, where γ is the ratio of the specific heats of the gas c_p/c_v , and is equal to 1.4 for a perfect gas. It is worthy of note that air behaves like a perfect gas over a considerable range of conditions. With the notation used in developing the boundary-layer theory earlier, (23) is written as

$$p/p_0 = (\rho_1/\rho_0)^\gamma, \quad (24)$$

and it then follows from (22) that a_1 , the local velocity of sound outside the boundary layer, is given by

$$a_1^2 = \gamma p/\rho_1 = \gamma \mathcal{R} T_1, \quad (25)$$

the second equality being deduced from (7). By making use of the fact that, for a perfect gas

$$\mathcal{R} = c_p - c_v,$$

we may deduce from (25) that

$$a_1^2 = \frac{\gamma p}{\rho_1} = \gamma \mathcal{R} T_1 = (\gamma - 1) c_p T_1. \quad (26)$$

It may also be shown from (7) and (24) that

$$\rho_1/\rho_0 = (T_1/T_0)^{1/(\gamma-1)} \quad (27)$$

and

$$p/p_0 = (T_1/T_0)^{\gamma/(\gamma-1)}. \quad (28)$$

From the Navier-Stokes equations, (1) to (6), it is not difficult to show, when viscosity and conductivity are neglected and the external forces are assumed to be conservative and derived from a potential Ω , that

$$c_p T + \frac{1}{2}(u^2 + v^2) + \Omega = \text{constant}$$

along a streamline. This result may be obtained by adding together (5), u times (2) and v times (3). Upon neglecting external forces, it follows that at the edge of the boundary layer (where $v^2 \ll u^2$)

$$c_p T_1 + \frac{1}{2} u_1^2 = \text{constant}, \quad (29)$$

which result agrees with that obtained by letting $y \rightarrow \infty$ in (19). Upon making use of (26) it follows that (29) may be written as

$$T_{H1} = T_1 + \frac{u_1^2}{2c_p} = T_1 \left\{ 1 + \frac{\gamma-1}{2} M_1^2 \right\} = \text{constant}, \quad (30)$$

where T_{H1} is the total temperature, or mainstream stagnation temperature, and M_1 is the local Mach number u_1/a_1 at the edge of the boundary layer. When M_1 is prescribed as a function of x , T_1 follows at once from (30). If, however, the local velocity u_1 at the edge of

the boundary layer is prescribed, the following relationships will be useful, namely

$$T_1 = T_0 \left(1 + \frac{\gamma-1}{2} M_0^2 \left(1 - \frac{u_1^2}{u_0^2} \right) \right), \quad (31)$$

and

$$M_1^2 = \frac{M_0^2 u_1^2 / u_0^2}{1 + \frac{1}{2}(\gamma-1)M_0^2 \{1 - (u_1^2/u_0^2)\}}. \quad (32)$$

6. Some standard boundary-layer characteristics

In practice we are often not interested in the full details of the velocity and temperature profiles, but only in some overall measure of boundary-layer thickness or in some quantity as calculated at the wall, such as the heat transferred from the fluid to the wall or the frictional drag exerted by the fluid on the wall. We will now discuss some of the more interesting (and important) of these characteristics.

Imagine inviscid flow past a body, the velocity of slip at the surface being $u_1(x)$. In a real fluid the effect of viscosity is to slow down the fluid near to the surface, bringing the fluid at the surface to rest and causing the streamlines to be displaced outwards by an amount which may be calculated as follows. Consider a streamline which is a distance $h(x)$ from the wall and, but for viscosity, would have been only $h(x) - \delta_1(x)$ from the wall. Then the total mass of fluid flowing in unit time between $y = 0$ and $y = h$ is that which would be flowing between $y = 0$ and $y = h - \delta_1$ if the velocity were everywhere $u = u_1(x)$. Thus

$$\rho_1 u_1 (h - \delta_1) = \int_0^h \rho u \, dy,$$

or

$$\rho_1 u_1 \delta_1 = \int_0^h (\rho_1 u_1 - \rho u) \, dy.$$

Thus streamlines far away from the wall have been displaced by an amount

$$\delta_1 = \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1} \right) dy, \quad (33)$$

which distance is known as the displacement thickness.

In a similar way the difference between the actual rate at which momentum is convected between $y = 0$ and $y = h$, that is $\int_0^h \rho u^2 \, dy$, and the rate at which the actual momentum ρu would be convected by a flow with zero viscosity, that is $\int_0^h \rho u u_1 \, dy$, may be compared with

inviscid flow momentum being convected in a region between $y = 0$ and $y = \delta_2$. Thus

$$\rho_1 u_1^2 \delta_2 = \int_0^h (\rho u u_1 - \rho u^2) dy,$$

or
$$\delta_2 = \int_0^h \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u}{u_1}\right) dy.$$

As $h \rightarrow \infty$ this yields

$$\delta_2 = \int_0^\infty \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u}{u_1}\right) dy, \quad (34)$$

which is known as the momentum thickness.

Two other measures of boundary-layer thickness which are of some interest are the kinetic-energy thickness, which is a measure of the defect of kinetic-energy flow within the boundary layer, and the enthalpy thickness which is a measure of the defect of heat flow within the boundary layer. The kinetic-energy thickness is defined as

$$\delta_3 = \int_0^\infty \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u^2}{u_1^2}\right) dy, \quad (35)$$

and the enthalpy thickness as

$$\delta_4 = \int_0^\infty \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{T}{T_1}\right) dy. \quad (36)$$

The skin friction at the wall, τ_w , is defined as the viscous force per unit area acting at the surface, and is given by

$$\tau_w = \lim_{y \rightarrow 0} \left(\mu \frac{\partial u}{\partial y} \right). \quad (37)$$

A position of zero skin friction is one for which $\partial u / \partial y$ changes sign near the wall, and hence, since $u = 0$ at $y = 0$, the local velocity must be of opposite sign on either side of such a point. It follows that when the boundary layer is of finite thickness the forward flow must 'separate' from the wall in order to by-pass the backward flow. For this reason a position of zero skin friction is referred to as a position of boundary-layer separation. The serious effects of boundary-layer separation have already been referred to.

The rate of heat transfer to unit area of wall, $y = 0$, is

$$Q_w(x) = \lim_{y \rightarrow 0} \left(k \frac{\partial T}{\partial y} \right). \quad (38)$$

Often either Q_w or T_w is given, and it is desired to calculate the other. However, at high speeds a relationship between Q_w and T_w may be used, expressing the balance between heat transfer and radiation.

7. The momentum integral equation

Equation (12) expresses local conservation of momentum in the x direction at points within the boundary layer. When integrated across the boundary layer from $y = 0$ to $y = \infty$ the resulting equation expresses conservation of momentum in the boundary layer as a whole. Upon neglecting the body force X , substituting for $\partial p / \partial x$ from (21), and integrating from $y = 0$ to $y = \delta$ we obtain

$$\begin{aligned} \int_0^\delta \rho u \frac{\partial u}{\partial x} dy + \int_0^\delta \rho v \frac{\partial u}{\partial y} dy &= \int_0^\delta \rho_1 u_1 \frac{du_1}{dx} dy - \left(\mu \frac{\partial u}{\partial y} \right)_w \\ &= \int_0^\delta \rho_1 u_1 \frac{du_1}{dx} dy - \tau_w. \end{aligned} \quad (39)$$

Now
$$\int_0^\delta \rho v \frac{\partial u}{\partial y} dy = [\rho uv]_0^\delta - \int_0^\delta u \frac{\partial}{\partial y} (\rho v) dy,$$

and by (1) this equals

$$\begin{aligned} \int_0^\delta u \frac{\partial}{\partial x} (\rho u) dy - u_1 \int_0^\delta \frac{\partial}{\partial x} (\rho u) dy &= \int_0^\delta (u - u_1) \frac{\partial}{\partial x} (\rho u) dy \\ &= \frac{d}{dx} \int_0^\delta \rho u (u - u_1) dy - \int_0^\delta \rho u \frac{\partial}{\partial x} (u - u_1) dy. \end{aligned}$$

Hence (39) yields

$$\begin{aligned} \tau_w &= \int_0^\delta (\rho_1 u_1 - \rho u) \frac{du_1}{dx} dy - \frac{d}{dx} \int_0^\delta \rho u (u - u_1) dy \\ &= \rho_1 u_1 \frac{du_1}{dx} \delta_1 + \frac{d}{dx} (\rho_1 u_1^2 \delta_2), \end{aligned} \quad (40)$$

upon formally letting $\delta \rightarrow \infty$. This equation is called the momentum integral equation, and was first derived by Kármán (1921) by more physical arguments. Afterwards, K. Pohlhausen (1921) derived the incompressible form by the procedure adopted above.

8. The kinetic-energy integral equation

If equation (12) is first multiplied by the local velocity u , and then integrated from $y = 0$ to $y = \delta$, an alternative integral equation is

obtained. Upon doing this we find

$$\int_0^{\delta} \rho u \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) dy + \int_0^{\delta} \rho v \frac{\partial}{\partial y} \left(\frac{1}{2} u^2 \right) dy = \rho_1 u_1 \frac{du_1}{dx} \int_0^{\delta} u dy + \int_0^{\delta} u \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) dy. \quad (41)$$

The second terms on both sides of (41) are integrated by parts. Thus

$$\int_0^{\delta} u \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) dy = - \int_0^{\delta} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy, \quad (42)$$

and

$$\begin{aligned} \int_0^{\delta} \rho v \frac{\partial}{\partial y} \left(\frac{1}{2} u^2 \right) dy &= -\frac{1}{2} u_1^2 \int_0^{\delta} \frac{\partial}{\partial x} (\rho u) dy + \int_0^{\delta} \frac{1}{2} u^2 \frac{\partial}{\partial x} (\rho u) dy \\ &= \frac{1}{2} \int_0^{\delta} (u^2 - u_1^2) \frac{\partial}{\partial x} (\rho u) dy. \end{aligned} \quad (43)$$

Also

$$\int_0^{\delta} \rho u \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) dy = \frac{1}{2} \int_0^{\delta} \rho u \frac{\partial}{\partial x} (u^2 - u_1^2) dy + u_1 \frac{du_1}{dx} \int_0^{\delta} \rho u dy. \quad (44)$$

Upon substitution into (41) from (42) to (44) we find that

$$\frac{1}{2} \frac{d}{dx} \int_0^{\delta} \rho u (u^2 - u_1^2) dy + u_1 \frac{du_1}{dx} \int_0^{\delta} (\rho - \rho_1) u dy = - \int_0^{\delta} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy,$$

or

$$\begin{aligned} \frac{d}{dx} \left(\rho_1 u_1^3 \int_0^{\infty} \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u^2}{u_1^2} \right) dy \right) &= 2 \int_0^{\infty} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy + 2u_1 \frac{du_1}{dx} \int_0^{\infty} (\rho - \rho_1) u dy \\ &= 2 \int_0^{\infty} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy + 2\rho_1 u_1^2 \frac{du_1}{dx} \int_0^{\infty} \left(\frac{\rho}{\rho_1} - 1 \right) \frac{u}{u_1} dy. \end{aligned} \quad (45)$$

Since, for a perfect gas, the equation of state is (7), it follows that

$$p = \rho \mathcal{R}T = \rho_1 \mathcal{R}T_1,$$

so that (45) becomes

$$\begin{aligned} \frac{d}{dx} \left(\rho_1 u_1^3 \int_0^{\infty} \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u^2}{u_1^2} \right) dy \right) &= 2 \int_0^{\infty} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy + 2\rho_1 u_1^2 \frac{du_1}{dx} \int_0^{\infty} \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{T}{T_1} \right) dy. \end{aligned} \quad (46)$$

This may be rewritten in terms of the energy thickness δ_3 and the enthalpy thickness δ_4 as

$$\frac{d}{dx}(\rho_1 u_1^3 \delta_3) = 2 \int_0^\infty \mu \left(\frac{\partial u}{\partial y} \right)^2 dy + 2\rho_1 u_1^2 \frac{du_1}{dx} \delta_4. \quad (47)$$

The incompressible form of this equation was first derived by Liebenson (1935).

9. The thermal-energy integral equation

This equation, as its name implies, is obtained by integrating the thermal energy equation (19) across the boundary layer. Now for any function F , the integral across the boundary layer of the expression $\{\rho u(\partial F/\partial x) + \rho v(\partial F/\partial y)\}$ becomes

$$\int_0^\delta \rho u \frac{\partial F}{\partial x} dy + \int_0^\delta \rho v dF = \int_0^\delta \rho u \frac{\partial F}{\partial x} dy + [\rho v F]_0^\delta - \int_0^\delta F \frac{\partial}{\partial y}(\rho v) dy.$$

Upon making use of (1) this becomes

$$\begin{aligned} \int_0^\delta \left(\rho u \frac{\partial F}{\partial x} + \rho v \frac{\partial F}{\partial y} \right) dy &= \int_0^\delta \frac{\partial}{\partial x}(\rho u F) dy - F_1 \int_0^\delta \frac{\partial}{\partial x}(\rho u) dy \\ &= \frac{d}{dx} \int_0^\delta \rho u (F - F_1) dy. \end{aligned}$$

$$\text{Choosing} \quad F = c_p T_H = c_p T + \frac{1}{2} u^2, \quad (48)$$

where T_H is the total temperature, (19) becomes

$$c_p \frac{d}{dx} \int_0^\infty \rho u (T_H - T_{H1}) dy = \int_0^\infty \frac{\partial}{\partial y} \left\{ \mu \frac{\partial}{\partial y} \left(\frac{c_p T}{\sigma} + \frac{1}{2} u^2 \right) \right\} dy = -\frac{\mu c_p}{\sigma} \left(\frac{\partial T}{\partial y} \right)_w,$$

$$\text{or} \quad \frac{d}{dx} \int_0^\infty \rho u (c_p T_H - c_p T_{H1}) dy = -k \left(\frac{\partial T}{\partial y} \right)_w. \quad (49)$$

This equation expresses the fact that the rate of change of flux of total temperature must be balanced by the heat being transferred across the boundary.

10. Incompressible flow

For flow of liquids, or of gases at sufficiently low speeds, the density ρ will be sensibly constant unless the temperature differences are great.

When this approximation is made the boundary-layer equations take a much simpler form. The continuity equation (1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (50)$$

Similarly the momentum equation (12) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} = u_1 \frac{du_1}{dx} + \nu \frac{\partial^2 u}{\partial y^2}. \quad (51)$$

Here we have neglected the body forces, and have taken the viscosity as constant, since μ is a function of the temperature, assumed not to vary much. Finally, the thermal energy equation, (19), remains. If typical temperature changes are small like the square of typical velocities (in non-dimensional form, of course), then (19) is trivial and vanishes. When, however, we consider the case of 'warm walls', with

$$\frac{\Delta T}{T} = O(U/a),$$

where U/a is the relevant non-dimensional velocity, then (19) reduces to

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\sigma} \frac{\partial^2 T}{\partial y^2}. \quad (52)$$

We notice that equations (50) and (51) are two equations for the velocity field (u, v) , which are uncoupled from the equation (52) for T . The relevant boundary conditions are

$$\begin{aligned} u &= v = 0 \quad \text{at } y = 0, \\ u &\rightarrow u_1(x) \quad \text{as } y \rightarrow \infty, \\ T &\rightarrow T_1 = \text{constant} \quad \text{as } y \rightarrow \infty, \\ T &= T_w(x), \quad \text{prescribed, at } y = 0. \end{aligned}$$

In a similar way the integral equations simplify. We find that (42) becomes

$$\frac{\tau_w}{\rho} = u_1 \frac{du_1}{dx} \delta_1 + \frac{d}{dx} (u_1^2 \delta_2),$$

$$\text{or} \quad \frac{\tau_w}{\rho u_1^2} = \frac{u'_1}{u_1} (\delta_1 + 2\delta_2) + \frac{d\delta_2}{dx}. \quad (53)$$

In the same way (47) becomes

$$\rho \frac{d}{dx} (u_1^3 \delta_3) = 2 \int_0^\infty \mu \left(\frac{\partial u}{\partial y} \right)^2 dy, \quad (54)$$

and (49) becomes

$$\rho c_p \frac{d}{dx} \int_0^\infty u(T - T_1) dy = -k \left(\frac{\partial T}{\partial y} \right)_w. \quad (55)$$

11. Crocco's transformation

Various attempts have been made to transform the laminar-boundary-layer equations into a form more amenable to calculation. These have proved most useful in restricted fields, though none has been found of universal value. We begin by considering a transformation due to Crocco (1939), in which x and u are taken as new independent variables, with the viscous stress $\tau = \mu(\partial u/\partial y)$ and the enthalpy $I = c_p T$ as dependent variables.

We may write the continuity, momentum, and energy equations, (1), (12), and (18), as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0,$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y},$$

and

$$\rho u \frac{\partial I}{\partial x} + \rho v \frac{\partial I}{\partial y} = u \frac{\partial p}{\partial x} + \frac{1}{\sigma} \frac{\partial}{\partial y} \left(\mu \frac{\partial I}{\partial y} \right) + \frac{\tau^2}{\mu}.$$

These equations are then transformed to new independent variables, x , u , and after some algebra (for which the reader is referred to Crocco's original paper) they become

$$\rho v = \rho u \frac{\partial y}{\partial x} - \frac{\mu}{\tau} \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial u}, \quad (56)$$

$$u \frac{\partial}{\partial x} \left(\frac{\mu \rho}{\tau} \right) + \frac{\partial^2 \tau}{\partial u^2} - \frac{\partial p}{\partial x} \frac{\partial}{\partial u} \left(\frac{\mu}{\tau} \right) = 0, \quad (57)$$

$$\text{and} \quad (1-\sigma) \frac{\partial I}{\partial u} \tau \frac{\partial \tau}{\partial u} + \tau^2 \left(\frac{\partial^2 I}{\partial u^2} + \sigma \right) - \sigma \mu \rho u \frac{\partial I}{\partial x} + \mu \sigma \frac{\partial p}{\partial x} \left(\frac{\partial I}{\partial u} + u \right) = 0. \quad (58)$$

Equations (57) and (58) are simultaneous equations for I and τ in terms of u and x . After these have been integrated, it is a straightforward matter to calculate

$$y(x, u) = \int_0^u \frac{\mu(I) du}{\tau(x, u)},$$

after which v follows from (56).

This form of the equations will be used in discussing the compressible laminar boundary layer on a flat plate in Chapter 7.

12. von Mises's transformation

A related transformation, in which x and $\psi = \int_0^y \rho u dy$ are taken as independent variables, was given by von Mises (1927) for incom-

pressible flow. Upon transforming, the equations become

$$u \frac{\partial u}{\partial x} - \frac{\rho_1 u_1}{\rho} \frac{du_1}{dx} = u \frac{\partial}{\partial \psi} \left(\mu \rho u \frac{\partial u}{\partial \psi} \right), \quad (59)$$

and
$$c_p \frac{\partial I}{\partial x} + \frac{\rho_1 u_1}{\rho} \frac{du_1}{dx} = \frac{\partial}{\partial \psi} \left(k \rho u \frac{\partial T}{\partial \psi} \right) + \mu \rho u \left(\frac{\partial u}{\partial \psi} \right)^2. \quad (60)$$

By adding these equations together we obtain, after some algebra,

$$\frac{\partial}{\partial x} \left(I + \frac{1}{2} u^2 \right) = \frac{\partial}{\partial \psi} \left\{ \mu \rho u \frac{\partial}{\partial \psi} \left(\frac{I}{\sigma} + \frac{1}{2} u^2 \right) \right\}.$$

The equations in this form have proved useful in considering the heat transfer through a laminar boundary layer in incompressible flow (Chapter 6), also the laminar boundary layer on a flat plate in compressible flow (Chapter 7).

ANALYTIC SOLUTIONS FOR INCOMPRESSIBLE FLOW

It has already been pointed out that in incompressible flow the momentum and energy equations become uncoupled, so that the momentum equation is independent of the temperature field. When this equation has been solved to yield the velocity distribution, the thermal energy equation, which is linear in temperature T , must be solved. In Chapters 2 to 5 various aspects of the first half of this problem will be considered, and solution of the thermal energy equation discussed in Chapter 6.

In this chapter we consider solutions of the incompressible momentum equation which reduce to the integration of one or more ordinary differential equations which can, in principle, be solved to any prescribed accuracy. Naturally such a situation only arises when the physical picture is extremely simple, such as in flow parallel to a flat plate, or when only a restricted domain is under consideration, such as the flow near to the forward stagnation point of a bluff body. Mathematically the solution will depend upon ordinary differential equations only if the velocity depends not upon x and y separately, but upon some suitable non-dimensional combination of these and of the basic fluid properties μ , ρ , ν . The obvious choice is y/δ , where δ is the boundary-layer thickness, and since this is typically of order $(\nu x/U)^{\frac{1}{2}}$ the non-dimensional parameter is $y(U/\nu x)^{\frac{1}{2}}$, where U is a representative velocity.

In spite of the simplicity and obvious limitations of these solutions they are none the less useful as exact solutions of the boundary-layer equations and, further, may serve as a valuable starting-point for a numerical investigation of more complex situations.

1. Flow parallel to a semi-infinite flat plate

We consider firstly the case of flow parallel to a semi-infinite flat plate, the flow field being otherwise unbounded and the plate assumed to have zero thickness. Then the everywhere uniform flow will be disturbed only within the thin boundary layer near to the plate, and the velocity outside this layer will accordingly be uniform, say equal to u_0 . We take axes x , parallel to the plate, and y normal to it, with

the origin at the leading edge. Then the boundary-layer continuity equation (50) and the momentum equation (51) become

$$\left. \begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2} \end{aligned} \right\}, \quad (61)$$

with boundary conditions $u = v = 0$ at $y = 0$, $u \rightarrow u_0$ as $y \rightarrow \infty$. These equations may be reduced to one ordinary differential equation by introducing a stream function ψ , such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad (62)$$

and writing $\psi = (u_0 \nu x)^{\frac{1}{2}} f(\eta)$, $\eta = \frac{1}{2} \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}} y$.

Upon substitution into (62) this yields

$$\left. \begin{aligned} u &= \frac{1}{2} u_0 f'(\eta) \\ v &= \frac{1}{2} \left(\frac{u_0 \nu}{x} \right)^{\frac{1}{2}} (\eta f' - f) \end{aligned} \right\}, \quad (63)$$

where dashes denote derivatives with respect to η , and when these values are substituted into (61) one obtains, after some algebra,

$$f''' + ff'' = 0. \quad (64)$$

The boundary conditions become $f = f' = 0$ at $\eta = 0$, $f' \rightarrow 2$ as $\eta \rightarrow \infty$. The first accurate solution of this equation was given by Blasius (1908), by a method which required a series expansion for small η and an asymptotic expansion for large η , sufficient terms being derived to yield a region of overlap. A better method of integration was later given by Töpfer (1912), and this has been used by Goldstein (1930) and Howarth (1938) to yield extremely accurate solutions. By expanding a solution of (64) about the origin it is found that

$$f = \sum_{n=0}^{\infty} a_n \alpha^{n+1} \eta^{3n+2}, \quad (65)$$

where

$$a_0 = \frac{1}{2!}, \quad a_1 = -\frac{1}{5!}, \quad a_2 = \frac{11}{8!}, \quad a_3 = -\frac{375}{11!},$$

and the constant α is to be determined from the condition that $f' \rightarrow 2$ as $\eta \rightarrow \infty$. Now (65) may be expressed in the form

$$f = \alpha^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n (\alpha^{\frac{1}{3}} \eta)^{3n+2} = \alpha^{\frac{1}{3}} F(\alpha^{\frac{1}{3}} \eta), \quad (66)$$

where $F(\eta)$ is the solution when $\alpha = 1$. By numerically integrating (64), using starting values derived from the series (66) with $\alpha = 1$, the function $F(\eta)$ may be obtained to any desired accuracy. Then the boundary condition at $\eta = \infty$ yields

$$\lim_{\eta \rightarrow \infty} f'(\eta) = \alpha^{\frac{3}{2}} \lim_{\eta \rightarrow \infty} F'(\alpha^{\frac{1}{2}}\eta) = \alpha^{\frac{3}{2}} \lim_{\eta \rightarrow \infty} F'(\eta),$$

so that

$$\alpha = \{\tfrac{1}{2}F'(\infty)\}^{-\frac{2}{3}}.$$

The value of α has by this means been determined as

$$\alpha = 1.32824,$$

whereupon numerical integration of (64) yields the function $f(\eta)$. The velocity distribution through the boundary layer, which by (63) is proportional to $f'(\eta)$, is tabulated below.

TABLE 1
The Blasius velocity profile

η	u/u_0	η	u/u_0	η	u/u_0
0	0	1.1	0.6813	2.1	0.9670
0.1	0.0664	1.2	0.7290	2.2	0.9759
0.2	0.1328	1.3	0.7725	2.3	0.9827
0.3	0.1989	1.4	0.8115	2.4	0.9878
0.4	0.2647	1.5	0.8460	2.5	0.9915
0.5	0.3298	1.6	0.8761	2.6	0.9942
0.6	0.3938	1.7	0.9018	2.7	0.9962
0.7	0.4563	1.8	0.9233	2.8	0.9975
0.8	0.5168	1.9	0.9411	2.9	0.9984
0.9	0.5748	2.0	0.9555	3.0	0.9990
1.0	0.6298				

The skin friction at the wall is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w = \tfrac{1}{2} \mu u_0 \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}} f''(0) = 0.33206 \mu u_0 \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}}. \quad (67)$$

The displacement thickness is

$$\begin{aligned} \delta_1 &= \int_0^{\infty} \left(1 - \frac{u}{u_0} \right) dy = \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}} \int_0^{\infty} (2 - f') d\eta = \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}} \lim_{\eta \rightarrow \infty} \{2\eta - f\} \\ &= 1.72077 \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}}, \end{aligned}$$

and the momentum thickness is

$$\delta_2 = \int_0^{\infty} \left(\frac{u}{u_0} - \frac{u^2}{u_0^2} \right) dy = \tfrac{1}{2} \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}} \int_0^{\infty} (2f' - f'^2) d\eta,$$

and upon making use of (64) this may be easily shown to yield

$$\delta_2 = \frac{1}{2} \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}} f''(0) = 0.66412 \left(\frac{\nu x}{u_0} \right)^{\frac{1}{2}}.$$

This result could have been very readily deduced from the momentum integral equation (53). When the external velocity u_1 is constant, so that $u'_1 = 0$, we see that δ_2 is directly proportional to the integral of the skin friction τ_w , given by (67).

2. Flow near the stagnation-point of a cylinder

Sufficiently near to the forward stagnation-point in two-dimensional flow past a cylinder the external velocity u_1 may be written as

$$u_1 = \beta_1 x/c, \quad (68)$$

where x is measured along the cylinder and y perpendicular to it. Then the continuity and momentum equations become

$$\left. \begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \beta_1^2 x/c^2 + \nu \frac{\partial^2 u}{\partial y^2} \end{aligned} \right\}, \quad (69)$$

with boundary conditions $u = v = 0$ at $y = 0$, $u \rightarrow u_1$ as $y \rightarrow \infty$. We introduce a stream function ψ , such that u, v are given by (62), and write

$$\psi = \left(\frac{\nu \beta_1}{c} \right)^{\frac{1}{2}} x f(\eta), \quad \eta = \left(\frac{\beta_1}{\nu c} \right)^{\frac{1}{2}} y.$$

Then

$$\left. \begin{aligned} u &= \beta_1 \frac{x}{c} f'(\eta) \\ v &= - \left(\frac{\nu \beta_1}{c} \right)^{\frac{1}{2}} f(\eta) \end{aligned} \right\}, \quad (70)$$

and upon substitution into (69) we find

$$f''' + ff'' - f'^2 + 1 = 0,$$

the boundary conditions being

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

This equation was derived by Blasius (1908), and improved numerical solutions have been obtained by various workers, including Heimenz (1911), Howarth (1934), and Tifford (1954). The first derivative f' is tabulated as f'_1 in Table 2. The skin friction and the momentum and

displacement thicknesses are easily shown to be

$$\begin{aligned}\tau_w &= \mu \left(\frac{\partial u}{\partial y} \right)_w = 1.23259 \mu \beta_1 \left(\frac{\beta_1}{\nu c} \right)^{\frac{1}{2}} \frac{x}{c}, \\ \delta_1 &= 0.64790 \left(\frac{\nu c}{\beta_1} \right)^{\frac{1}{2}}, \\ \delta_2 &= 0.29234 \left(\frac{\nu c}{\beta_1} \right)^{\frac{1}{2}}.\end{aligned}$$

It can be seen from (70) that $\partial^2 u / \partial x^2$ is identically equal to zero, so that the solution is an exact solution of the complete Navier-Stokes equations at points sufficiently close to the forward stagnation-point, where (68) holds. As an indication of the region over which this approximation holds, we may remark that for flow past a circular cylinder the development of the boundary layer follows this solution over roughly the front quadrant.

3. The Falkner-Skan similarity solutions

It was shown by Falkner and Skan (1930) that the two solutions given above are special cases of a general class of solutions characterized by the mainstream velocity

$$u_1 = \beta_m (x/c)^m.$$

Upon introducing a stream-function

$$\psi = (u_1 \nu x)^{\frac{1}{2}} f(\eta), \quad \eta = \left(\frac{u_1}{\nu x} \right)^{\frac{1}{2}} y,$$

then the momentum equation (51) may be shown to reduce to

$$f''' + \frac{1}{2}(m+1)ff'' + m(1-f'^2) = 0. \quad (71)$$

Solutions were given by Falkner and Skan for a range of values of m . Later Hartree (1937) studied (71) in greater detail, first making the linear transformation

$$Y = \left(\frac{m+1}{2} \right)^{\frac{1}{2}} \eta, \quad F = \left(\frac{m+1}{2} \right)^{\frac{1}{2}} f,$$

so that it becomes

$$\frac{d^3 F}{dY^3} + F \frac{d^2 F}{dY^2} + \frac{2m}{m+1} \left\{ 1 - \left(\frac{dF}{dY} \right)^2 \right\} = 0, \quad (72)$$

with boundary conditions

$$F = \frac{dF}{dY} = 0 \quad \text{at} \quad Y = 0, \quad \frac{dF}{dY} \rightarrow 1 \quad \text{as} \quad Y \rightarrow \infty.$$

For positive values of m , (72) has a unique solution, but more than one

solution exists for negative values of m . Hartree gave a family of solutions for positive values of m , and for negative values of m from zero decreasing to -0.0904 . For this last value d^2F/dY^2 becomes zero at $Y = 0$, so that

$$\frac{\partial u}{\partial y} = 0 \quad \text{when } y = 0 \quad \text{for all } x.$$

In other words the external velocity is such that the boundary layer is everywhere on the point of separation. For the numerical values of $F(Y)$ for various values of m see Hartree (1937).

A second family of solutions of (72), the so-called 'lower-branch' solutions, was given later by Stewartson (1954), again for negative values of m in the range $-0.0904 \leq m \leq 0$. These solutions all have the property that $(d^2F/dY^2)_w \leq 0$, so that

$$\frac{\partial u}{\partial y} \leq 0 \quad \text{when } y = 0 \quad \text{for all } x.$$

Accordingly these solutions can only correspond physically to flow in a laminar boundary layer beyond separation. These solutions, involving the possible singularity at separation (Goldstein, 1948; Stewartson, 1958), will not be further discussed here.

4. Series solutions from a stagnation-point

We now generalize the solution given in section 2, which was valid at points sufficiently close to the stagnation-point in flow past a round-nosed cylinder. Consider the case of flow past a symmetrically placed cylinder, for which the mainstream velocity is assumed to take the form

$$u_1(x) = \sum_0 \beta_{2n+1} \xi^{2n+1}, \quad \xi = \frac{x}{c}. \quad (73)$$

It was first noted by Blasius (1908) that a solution of the momentum equation (49) could be obtained by assuming an expansion for the stream-function

$$\psi = (\beta_1 \nu c)^{\frac{1}{2}} \sum_0 \xi^{2n+1} F_{2n+1}(\eta), \quad \eta = \left(\frac{\beta_1}{\nu c}\right)^{\frac{1}{2}} y, \quad (74)$$

where the functions F_{2n+1} are seen to depend only upon the distance y , normal to the wall. It follows that the local velocity is

$$u = \beta_1 \sum_0 \xi^{2n+1} F'_{2n+1}(\eta), \quad (75)$$

and the skin friction at the wall is similarly derived from

$$\left(\frac{\partial u}{\partial y}\right)_w = \left(\frac{\beta_1^3}{\nu c}\right)^{\frac{1}{2}} \sum_0 \xi^{2n+1} F''_{2n+1}(0). \quad (76)$$

Accordingly the skin friction is obtained as a simple power series in ξ . Upon substituting from (75) into the momentum equation (51) and equating powers of ξ , a series of equations for F_{2n+1} are obtained, and these may be solved in turn. Heimenz (1911) developed the work of Blasius and put F_1 and F_3 in universal forms, independent of the constants β_{2n+1} , but did not succeed in dealing similarly with F_5 .

The major progress to date was made by Howarth (1934) who showed that all the F_{2n+1} could be reduced to sums of universal functions. This is done as follows:

$$\left. \begin{aligned} F_1 &= f_1 \\ F_3 &= 4 \frac{\beta_3}{\beta_1} f_3 \\ F_5 &= 6 \left\{ \frac{\beta_5}{\beta_1} g_5 + \frac{\beta_3^2}{\beta_1^2} h_5 \right\} \\ F_7 &= 8 \left\{ \frac{\beta_7}{\beta_1} g_7 + \frac{\beta_3 \beta_5}{\beta_1^2} h_7 + \frac{\beta_3^3}{\beta_1^3} k_7 \right\} \\ F_9 &= 10 \left\{ \frac{\beta_9}{\beta_1} g_9 + \frac{\beta_3 \beta_7}{\beta_1^2} h_9 + \frac{\beta_5^2}{\beta_1^2} k_9 + \frac{\beta_3^2 \beta_5}{\beta_1^3} j_9 + \frac{\beta_3^4}{\beta_1^4} q_9 \right\} \\ F_{11} &= 12 \left\{ \frac{\beta_{11}}{\beta_1} g_{11} + \frac{\beta_3 \beta_9}{\beta_1^2} h_{11} + \frac{\beta_5 \beta_7}{\beta_1^2} k_{11} + \frac{\beta_3^2 \beta_7}{\beta_1^3} j_{11} + \frac{\beta_3 \beta_5^2}{\beta_1^3} q_{11} + \right. \\ &\quad \left. + \frac{\beta_3^3 \beta_5}{\beta_1^4} m_{11} + \frac{\beta_3^5}{\beta_1^5} n_{11} \right\} \end{aligned} \right\}. \quad (77)$$

The equations and boundary conditions for the universal functions will be found in Howarth's paper, together with tabulated values of the functions f_1 , f_3 , g_5 , h_5 , and k_7 . The functions f_3 depends upon f_1 , and h_5 in turn depends on both f_1 and f_3 . It follows that the accuracy obtainable in successive functions decreases rapidly, so that although Howarth's values of f_1 are given to four figures, those of f_3 are given only to three figures, g_5 and h_5 to two figures, and h_7 to one figure. Howarth's values were later improved, and further functions calculated, by Frössling (1940) and Ulrich (1949). More recently Tifford (1954), using high-speed computing apparatus, obtained values of all the universal functions shown in (77) to higher accuracy than obtained hitherto.

The universal functions being known, the first six terms in the series (75) for the velocity follow at once, as do the first six terms in the series (76) for the skin friction. Other boundary-layer characteristics, such as displacement and momentum thickness, are given as follows.

TABLE 2
Universal functions

η	f'_1	f'_3	g'_5	h'_5	g'_7	h'_7	k'_7	g'_9	h'_9
0.0	0.000000	0.00000	0.00000	0.00000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.226612	0.12511	0.10722	0.01413	0.0962	0.0173	0.0016	0.0884	+0.0112
0.4	0.414456	0.21288	0.17781	+0.01170	0.1563	+0.0030	0.0044	0.1413	-0.0079
0.6	0.566281	0.26881	0.21838	-0.00106	0.1879	-0.0286	0.0096	0.1669	0.0417
0.8	0.685937	0.29972	0.23665	0.01767	0.1994	0.0637	0.0174	0.1740	0.0760
1.0	0.777865	0.31250	0.23990	0.03306	0.1980	0.0925	0.0271	0.1700	0.1019
1.2	0.846671	0.31327	0.23415	0.04419	0.1896	0.1102	0.0369	0.1604	0.1157
1.4	0.896809	0.30702	0.22390	0.04989	0.1782	0.1159	0.0452	0.1489	0.1176
1.6	0.932348	0.29746	0.21226	0.05040	0.1665	0.1114	0.0506	0.1375	0.1101
1.8	0.956834	0.28712	0.20115	0.04682	0.1558	0.0997	0.0525	0.1276	0.0965
2.0	0.973217	0.27755	0.19158	0.04061	0.1469	0.0839	0.0510	0.1195	0.0798
2.2	0.983853	0.26950	0.18392	0.03317	0.1400	0.0669	0.0466	0.1132	0.0628
2.4	0.990549	0.26322	0.17814	0.02568	0.1349	0.0507	0.0402	0.1087	0.0470
2.6	0.994634	0.25859	0.17401	0.01891	0.1313	0.0367	0.0330	0.1055	0.0337
2.8	0.997046	0.25536	0.17119	0.01327	0.1288	0.0254	0.0257	0.1033	0.0231
3.0	0.998424	0.25322	0.16935	0.00890	0.1273	0.0168	0.0191	0.1020	0.0152
3.2	0.999186	0.25186	0.16820	0.00571	0.1263	0.0107	0.0135	0.1011	0.0096
3.4	0.999593	0.25103	0.16751	0.00351	0.1257	0.0065	0.0091	0.1006	0.0058
3.6	0.999803	0.25055	0.16711	0.00206	0.1254	0.0038	0.0059	0.1003	0.0034
3.8	0.999908	0.25028	0.16690	0.00116	0.1252	0.0021	0.0036	0.1002	0.0019
4.0	0.999959	0.25014	0.16678	0.00063	0.1251	0.0011	0.0021	0.1001	0.0010
4.2	0.999982	0.25007	0.16672	0.00033	0.1250	0.0006	0.0012	0.1000	0.0005
4.4	0.999993	0.25003	0.16669	0.00016	0.1250	0.0003	0.0006	0.1000	0.0003
4.6	0.999997	0.25001	0.16668	0.00008	0.1250	0.0001	0.0003	0.1000	0.0001
4.8	0.999999	0.25001	0.16667	0.00003	0.1250	0.0001	0.0002	0.1000	0.0001
5.0	1.000000	0.25000	0.16667	0.00001	0.1250	0.0000	0.0001	0.1000	0.0000
5.2	1.000000	0.25000	0.16667	0.00000	0.1250	0.0000	0.0000	0.1000	0.0000

η	k'_9	j'_9	q'_9	g'_{11}	h'_{11}	k'_{11}	j'_{11}	q'_{11}	m'_{11}	n'_{11}
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	+0.0019	0.0125	-0.0062	0.0824	+0.0074	-0.0041	0.0165	0.0237	-0.0359	0.0103
0.4	-0.0112	0.0288	0.0124	0.1299	-0.0145	0.0368	0.0371	0.0514	0.0721	0.0206
0.6	0.0311	0.0525	0.0190	0.1511	0.0489	0.0799	0.0651	0.0863	0.1095	0.0307
0.8	0.0501	0.0833	0.0262	0.1553	0.0816	0.1181	0.0992	0.1267	0.1489	0.0406
1.0	0.0639	0.1171	0.0341	0.1498	0.1046	0.1428	0.1342	0.1666	0.1895	0.0503
1.2	0.0704	0.1480	0.0423	0.1397	0.1152	0.1520	0.1641	0.1995	0.2288	0.0595
1.4	0.0703	0.1710	0.0502	0.1283	0.1146	0.1474	0.1841	0.2202	0.2627	0.0678
1.6	0.0649	0.1829	0.0567	0.1175	0.1055	0.1330	0.1919	0.2266	0.2874	0.0747
1.8	0.0562	0.1827	0.0610	0.1083	0.0911	0.1131	0.1876	0.2190	0.2996	0.0796
2.0	0.0460	0.1718	0.0625	0.1008	0.0746	0.0912	0.1731	0.2001	0.2982	0.0822
2.2	0.0359	0.1528	0.0610	0.0951	0.0581	0.0701	0.1515	0.1737	0.2834	0.0820
2.4	0.0267	0.1290	0.0568	0.0910	0.0432	0.0516	0.1263	0.1436	0.2575	0.0790
2.6	0.0190	0.1037	0.0504	0.0882	0.0308	0.0364	0.1003	0.1133	0.2237	0.0733
2.8	0.0129	0.0795	0.0426	0.0863	0.0210	0.0246	0.0761	0.0854	0.1858	0.0655
3.0	0.0085	0.0581	0.0344	0.0851	0.0138	0.0160	0.0552	0.0616	0.1476	0.0563
3.2	0.0053	0.0406	0.0265	0.0843	0.0087	0.0100	0.0383	0.0425	0.1121	0.0463
3.4	0.0032	0.0271	0.0195	0.0839	0.0052	0.0060	0.0255	0.0281	0.0814	0.0365
3.6	0.0019	0.0173	0.0137	0.0836	0.0030	0.0035	0.0162	0.0178	0.0565	0.0275
3.8	0.0010	0.0106	0.0092	0.0835	0.0017	0.0019	0.0099	0.0108	0.0375	0.0199
4.0	0.0006	0.0062	0.0059	0.0834	0.0009	0.0010	0.0057	0.0063	0.0238	0.0137
4.2	0.0003	0.0035	0.0036	0.0834	0.0005	0.0005	0.0032	0.0035	0.0144	0.0090
4.4	0.0001	0.0018	0.0021	0.0833	0.0002	0.0003	0.0017	0.0018	0.0083	0.0056
4.6	0.0001	0.0009	0.0011	0.0833	0.0001	0.0001	0.0008	0.0009	0.0045	0.0033
4.8	0.0000	0.0004	0.0006	0.0833	0.0000	0.0001	0.0004	0.0004	0.0022	0.0018
5.0	0.0000	0.0001	0.0002	0.0833	0.0000	0.0000	0.0001	0.0001	0.0008	0.0008
5.2	0.0000	0.0000	0.0000	0.0833	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

The displacement thickness is

$$\delta_1 = \int_0^{\infty} \left(1 - \frac{u}{u_1}\right) dy,$$

so that

$$\begin{aligned} u_1 \left(\frac{\beta_1}{\nu c}\right)^{\frac{1}{2}} \delta_1 &= \int_0^{\infty} (u_1 - u) d\eta = \int_0^{\infty} \left\{ u_1 - \beta_1 \sum_0^{\infty} \xi^{2n+1} F'_{2n+1}(\eta) \right\} d\eta \\ &= \lim_{\eta \rightarrow \infty} \left\{ u_1 \eta - \beta_1 \sum_0^{\infty} \xi^{2n+1} F_{2n+1}(\eta) \right\} \\ &= \beta_1 \lim_{\eta \rightarrow \infty} \sum_0^{\infty} \xi^{2n+1} \left\{ \frac{\beta_{2n+1}}{\beta_1} \eta - F_{2n+1} \right\}. \end{aligned}$$

The constant coefficients in the bracket are easily obtained from a knowledge of the asymptotic forms of $F_{2n+1}(\eta)$ as η tends to infinity. No similar simple expression for δ_2 has been obtained.

Some of the more important functions are tabulated here. In Table 2 are shown the first derivatives of all the universal functions in (77), for use in computing velocity profiles. Table 3 gives the values of the second derivatives at the wall, for computing the skin-friction. Table 4 gives the asymptotic values at infinity, for computing the displacement thickness.

TABLE 3

Second derivatives of universal functions

$f_1''(0) = 1.23259$	$g_9''(0) = 0.5399$	$g_{11}''(0) = 0.5100$
$f_3''(0) = 0.72445$	$h_9''(0) = 0.1520$	$h_{11}''(0) = 0.1323$
$g_5''(0) = 0.63470$	$k_9''(0) = 0.0572$	$k_{11}''(0) = 0.0742$
$h_7''(0) = 0.11918$	$j_9''(0) = 0.0607$	$j_{11}''(0) = 0.0806$
$q_7''(0) = 0.57920$	$q_9''(0) = -0.0308$	$q_{11}''(0) = 0.1164$
$h_7''(0) = 0.1829$		$m_{11}''(0) = -0.1796$
$k_7''(0) = 0.0076$		$n_{11}''(0) = 0.0516$

TABLE 4

Limiting forms of universal functions for large η

$f_1 \sim \eta - 0.64790$	$g_9 \sim \frac{1}{10} \eta + 0.0857$	$g_{11} \sim \frac{1}{10} \eta + 0.0856$
$f_3 \sim \frac{1}{2} \eta + 0.02847$	$h_9 = -0.1876$	$h_{11} = -0.1856$
$g_5 \sim \frac{1}{8} \eta + 0.07002$	$k_9 = -0.1153$	$k_{11} = -0.2479$
$h_7 = -0.07390$	$j_9 = 0.3567$	$j_{11} = 0.3731$
$q_7 \sim \frac{1}{8} \eta + 0.0824$	$q_9 = -0.1397$	$q_{11} = 0.4424$
$h_7 = -0.1809$		$m_{11} = -0.6750$
$k_7 = 0.0995$		$n_{11} = 0.2075$

The functions already computed are sufficient to give six terms of the series in any problem. In general, however, the convergence will

be such that more terms are required as separation is approached, and accordingly, since these could only be obtained at the cost of excessive labour, alternative methods must be used to continue the method to separation. These will be discussed in Chapter 3.

A similar procedure can be adopted for flow past a cylinder placed non-symmetrically, when the mainstream velocity takes the form

$$u_1(x) = \sum_1 \beta_n \xi^n$$

and the stream-function is expanded in a series

$$\psi = (\beta_1 \nu c)^{\frac{1}{2}} \sum_1 \xi^n F_n(\eta).$$

For details of this case the reader is referred to Howarth's paper (1934).

5. Series solutions from a sharp leading edge

We consider now the case of flow past a body which has a sharp leading edge, such as a flat plate placed parallel to the stream. The mainstream velocity is written in the form

$$u_1(x) = \sum_0 \beta_n \xi^n, \quad \xi = \frac{x}{c}, \quad (78)$$

and a solution sought by expanding the stream-function in a series

$$\left. \begin{aligned} \psi &= (\beta_0 \nu x)^{\frac{1}{2}} \sum_0 \xi^n F_n(\eta) \\ \eta &= \frac{1}{2} \left(\frac{\beta_0}{\nu x} \right)^{\frac{1}{2}} y \end{aligned} \right\}, \quad (79)$$

the local velocity thus being

$$u = \frac{1}{2} \beta_0 \sum_0 \xi^n F'_n(\eta), \quad (80)$$

and the skin friction being derived from

$$\left(\frac{\partial u}{\partial y} \right)_w = \frac{1}{4} \left(\frac{\beta_0^3}{\nu x} \right)^{\frac{1}{2}} \sum_0 \xi^n F''_n(0). \quad (81)$$

As for the case of expansion about a stagnation-point, substitution from (79) into the momentum equation (51) leads to a series of equations for F_n which may be solved in turn. It appears to have been noted first by Tifford (1954) that the solution for the general case may be

expressed in terms of universal functions by writing

$$\left. \begin{aligned} F_0 &= f_0 \\ F_1 &= 8 \frac{\beta_1}{\beta_0} f_1 \\ F_2 &= 8^2 \left\{ \frac{\beta_2}{\beta_0} g_2 + \frac{\beta_2^2}{\beta_0^2} h_2 \right\} \\ F_3 &= 8^3 \left\{ \frac{\beta_3}{\beta_0} g_3 + \frac{\beta_1 \beta_2}{\beta_0^2} h_3 + \frac{\beta_1^3}{\beta_0^3} k_3 \right\} \\ F_4 &= 8^4 \left\{ \frac{\beta_4}{\beta_0} g_4 + \frac{\beta_1 \beta_3}{\beta_0^2} h_4 + \frac{\beta_2^2}{\beta_0^2} k_4 + \frac{\beta_1 \beta_2}{\beta_0^3} j_4 + \frac{\beta_1^4}{\beta_0^4} q_4 \right\} \end{aligned} \right\}. \quad (82)$$

Not all the universal functions given above have been calculated, but f_0, f_1, h_2, k_3 , and q_4 have been tabulated by Howarth (1938) and g_2, g_4 , and k_4 have been calculated but not tabulated by Tani (1949). Accordingly in the general case, only the first three terms can be obtained from the functions at present calculated. However, if the four functions g_3, h_3, h_4 , and j_4 could be calculated, the first five terms would be known for arbitrary mainstream.

6. Görtler's modified series expansions

Görtler (1957*a*) has given an alternative series method which may be applied to cases with any shape of leading edge. He introduces new dimensionless independent variables

$$\xi = \frac{1}{\nu} \int_0^x u_1(x) dx, \quad \eta = \frac{y u_1(x)}{\left(2\nu \int_0^x u_1(x) dx \right)^{\frac{1}{2}}},$$

and writes the stream-function as

$$\psi(x, y) = \nu(2\xi)^{\frac{1}{2}} F(\xi, \eta).$$

Substitution into the momentum equation shows that $F(\xi, \eta)$ must satisfy the equation

$$\frac{\partial^3 F}{\partial \eta^3} + F \frac{\partial^2 F}{\partial \eta^2} + \beta(\xi) \left\{ 1 - \left(\frac{\partial F}{\partial \eta} \right)^2 \right\} = 2\xi \left\{ \frac{\partial F}{\partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} - \frac{\partial F}{\partial \xi} \frac{\partial^2 F}{\partial \eta^2} \right\},$$

where

$$\beta(\xi) = \frac{2u_1'(x) \int_0^x u_1(x) dx}{\{u_1(x)\}^2} = \frac{\xi(x)\xi''(x)}{\{\xi'(x)\}^2}.$$

It follows that the only explicit entry of the mainstream velocity is via the function $\beta(\xi)$.

Görtler indicates that when the mainstream velocity takes the form (78), then $\beta(\xi)$ takes the form

$$\beta(\xi) = \sum_1 \beta_n \xi^n,$$

where these β_n are different from those of section 5, but can easily be deduced from them. A solution for F in terms of universal functions may be obtained by writing

$$F(\xi, \eta) = \sum_0 \xi^n F_n(\eta),$$

where

$$F_0 = f_0,$$

$$F_1 = \beta_1 f_1,$$

$$F_2 = \beta_2 f_2 + \beta_1^2 f_{11}$$

$$F_3 = \beta_3 f_3 + \beta_1 \beta_2 f_{12} + \beta_1^3 f_{111},$$

and similar expressions for F_4 and F_5 .

Görtler (1957 *b*) has calculated these universal functions, and pointed out that the series expansion appears to converge rather better than the usual Blasius series, discussed in section 5. For the case of flow past a round-nosed cylinder, the Görtler series does not converge more quickly than the Blasius series, discussed in section 4. For details see Görtler (1957 *a*).

7. Meksyn's technique

It is perhaps valuable at this point to make brief mention of an interesting technique developed by Meksyn (1950, 1956, 1960, 1961), and applied by him to the calculation of laminar boundary layers. The method was later extended by Merk (1959) and applied with some success to the calculation of heat transfer through a laminar boundary layer. A discussion of the method, with particular reference to the heat transfer problem, will be presented in Chapter 6, section 9.

It may be remarked that the method is capable, in principle, of any required accuracy; in practice fair accuracy is attained with a fairly crude approximation, but considerable ingenuity is needed to improve on this. For full details reference may be made to the book by Meksyn (1961), *New Methods in Laminar Boundary Layer Theory*.

3

NUMERICAL SOLUTIONS FOR INCOMPRESSIBLE FLOW

APART from those special cases, considered in Chapter 2, where the boundary-layer equations may be reduced to one or a number of ordinary differential equations, one is faced in general with the problem of solving non-linear partial differential equations, and this is an exceedingly tricky numerical procedure. For this reason very few precise numerical solutions have been obtained. The importance of these few may perhaps best be illustrated by reference to the approximate method of Pohlhausen, to be considered in Chapter 4, section 1. For many years this method was regarded as being perfectly satisfactory for calculating the development of a laminar boundary layer in the presence of an arbitrary pressure gradient. Then, following the classical experimental work of Schubauer (1935), it was found that when Pohlhausen's method was applied to Schubauer's experimentally observed pressure distribution, it failed to predict separation by a considerable margin, whereas separation clearly took place in the experiments. It was then apparent that the method was inadequate in regions of rising pressure. In consequence comparison with at least one accurate solution has since been regarded as essential before reliance could be placed upon any approximate method.

It was to answer this need that Howarth (1938) considered a precise numerical solution for the flow in which the velocity at the edge of the boundary layer is given as

$$u_1 = \beta_0 \left(1 - \frac{x}{c}\right) = \beta_0(1 - \xi). \quad (83)$$

This solution will be considered in section 1 of this chapter.

Later Tani (1949) considered the three cases for external velocities

$$u_1 = \beta_0(1 - \xi^n),$$

with $n = 2, 4$, and 8 . The method used was precisely that of Howarth, and, although the accuracy is probably not quite so great (particularly for $n = 8$), there is no doubt that it is far more accurate than one could hope for by approximate methods.

In the meantime Hartree (1939 *a, b*) had re-solved Howarth's problem, using a differential analyser. Having thus confirmed the accuracy both of Howarth's calculations and of his own procedure, he then applied it to a numerically prescribed velocity distribution, derived from Schubauer's experimental results. This work gives a precise numerical solution for a flow containing a stagnation-point and separation, whereas the solutions of Howarth and Tani did not have a stagnation-point. The only drawback to the solution was that arising from the uncertainties of twice numerically differentiating a tabulated function, namely the experimental velocity distribution.

More recently, additional solutions have been obtained for flows with stagnation-point and separation for analytically prescribed velocity distributions. Curle (1958 *a*) has considered the case

$$u_1 = \beta_1(\xi - \xi^3)$$

by a semi-analytic method similar to but a little simpler than that used by Howarth above, and Terrill (1960) has used an electronic computer to obtain a solution for the case

$$u_1 = \beta_1 \sin \xi.$$

We shall now consider some of the details of these various solutions.

1. The solution of Howarth

As remarked earlier, Howarth (1938) considered the case of an external velocity

$$u_1(x) = \beta_0(1 - \xi).$$

By reference to Chapter 2, section 5, we see that this is the special case for which

$$\beta_1 = -\beta_0, \quad \beta_2 = \beta_3 = \beta_4 = \dots = 0.$$

Accordingly the stream function,

$$\psi = \int_0^y u \, dy,$$

may be expanded in a series

$$\psi = (\beta_0 \nu x)^{\frac{1}{2}} \sum_0 \xi^n F_n(\eta),$$

$$\eta = \frac{1}{2} \left(\frac{\beta_0}{\nu x} \right)^{\frac{1}{2}} y,$$

where

$$\begin{aligned} F_0 &= f_0, \\ F_1 &= -8f_1, \\ F_2 &= 8^2h_2, \\ F_3 &= -8^3k_3, \\ \text{and} \quad F_4 &= 8^4q_4. \end{aligned}$$

Howarth derived the equations satisfied by these five functions (and by further functions in the expansion) and integrated several of them. The first three equations take the form

$$f_0''' + f_0 f'' = 0,$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + 3f_0'' f_1 = -1,$$

and

$$h_2''' + f_0 h_2'' - 4f_0' h_2' + 5f_0'' h_2 = 2f_1'^2 - 3f_1 f_1'' - \frac{1}{8},$$

with boundary conditions

$$f_0 = f_1 = h_2 = 0, \quad f_0' = f_1' = h_2' = 0, \quad \text{when } \eta = 0,$$

$$f_0' \rightarrow 2, \quad f_1' \rightarrow \frac{1}{4}, \quad h_2' \rightarrow 0, \quad \text{as } \eta \rightarrow \infty.$$

By integrating these equations in turn, together with the next four, and by roughly integrating a further two, the detailed velocity profile was obtained for a considerable distance from the leading edge. In particular the skin friction at the wall, which is given by (81) as

$$\tau_w = \frac{1}{4}\mu \left(\frac{\beta_0^3}{\nu x}\right)^{\frac{1}{2}} \sum_0 \xi^n F_n''(0),$$

was found to equal

$$\frac{1}{4}\mu \left(\frac{\beta_0^3}{\nu x}\right)^{\frac{1}{2}} \{f_0''(0) - 8\xi f_1''(0) + (8\xi)^2 h_2''(0) - (8\xi)^3 k_3''(0) \dots\}. \quad (84)$$

The values of the second derivatives at the wall are perhaps best indicated by remarking that when $8\xi = 1$ the series in brackets in (84) took the form

$$1.328 - 1.021 - 0.069 - 0.056 - 0.037 - 0.027 - 0.021 - 0.017 - 0.015 \dots,$$

so that the series is converging rather slowly in this neighbourhood.

It is clear, however, that subsequent terms form only a relatively small correction to this series. For this reason Howarth suggested representing the terms after say the seventh in an approximate manner. Thus he approximated

$$u = \frac{1}{2}\beta_0 \sum_0^\infty \xi^n F_n'(\eta)$$

by

$$u = \frac{1}{2}\beta_0 \left\{ \sum_0^6 \xi^n F_n'(\eta) + A(\xi)B(\eta) \right\}. \quad (85)$$

In other words, it was assumed that the shapes of the subsequent terms $F'_n(\eta)$ ($n > 6$), could be adequately represented by a universal form $B(\eta)$, the various multiples of ξ^n being summed to give $A(\xi)$. By examining the functions $F'_5(\eta)$, $F'_6(\eta)$, and, as far as he had calculated them, $F'_7(\eta)$ and $F'_8(\eta)$, Howarth found that these were all multiples of the same functions, to a considerable accuracy. It remained, then, to determine $A(\xi)$, and this was done in each of the following two ways.

Firstly, Howarth substituted from (85) into the momentum integral equation, and upon assuming the shape $B(\eta)$, this led to a first order non-linear ordinary differential equation for $A(\xi)$, which took the form

$$\frac{dA}{d\xi} = \frac{P+QA+RA^2}{S+TA}, \quad (86)$$

where P , Q , R , S , T are polynomials which are determined by the known coefficients in (85). This equation was integrated by standard numerical techniques, and it turned out that $S+TA$ became zero and $dA/d\xi$ infinite at a position where the skin friction had decreased almost, but not quite, to zero. This happening is clearly bound up with the question of the singularity at separation, discussed earlier. Howarth then completed this calculation of the skin friction by an alternative procedure, and concluded that separation occurred when $\xi = 0.120$.

The function $A(\xi)$ was then calculated by the following alternative method. By considering the various boundary conditions at the wall (implied by the boundary layer equations), it is easily seen that the simplest one which is not automatically satisfied by (85) is that obtained by twice differentiating the boundary-layer momentum equation with respect to y , then putting $y = 0$. This yields

$$\left(\frac{\partial u}{\partial y}\right)_w \frac{d}{dx} \left\{ \left(\frac{\partial u}{\partial y}\right)_w \right\} = \nu \left(\frac{\partial^4 u}{\partial y^4}\right)_w.$$

By substituting into this relationship, an alternative equation for $A(\xi)$ is obtained. By essentially this process, Howarth obtained almost identical values for $A(\xi)$, the separation position again being $\xi = 0.120$.

It should be noted that the two methods used to determine the function $A(\xi)$ are quite independent, and the close agreement between the two sets of results is valuable confirmation of their accuracy. This has been further confirmed, beyond all doubt, by the subsequent work of Hartree (1939*a*) and Leigh (1955), who solved the same problem on a differential analyser and a modern high-speed computing machine, respectively.

2. The solutions of Tani

In an attempt to provide further accurate solutions of the laminar boundary-layer equations, for cases in which the second streamwise derivative of the external velocity, d^2u_1/dx^2 , is non-zero, Tani (1949) later considered the cases

$$u_1(x) = \beta_0(1 - \xi^n) \quad (n = 2, 4, 8).$$

As in Howarth's example, these are special cases of the family considered in Chapter 2, section 5. When $n = 2$, so that

$$u_1(x) = \beta_0(1 - \xi^2),$$

we have $\beta_1 = \beta_3 = \beta_4 = \dots = 0$, $\beta_2 = -\beta_0$.

Thus, by (82), we may write

$$\psi = (\beta_0 \nu x)^{\frac{1}{2}} \sum \xi^n F_n(\eta),$$

$$\eta = \frac{1}{2} \left(\frac{\beta_0}{\nu x} \right)^{\frac{1}{2}} y,$$

with

$$F_0 = f_0,$$

$$F_1 = 0,$$

$$F_2 = -8^2 g_2,$$

$$F_3 = 0,$$

and

$$F_4 = 8^4 k_4.$$

Tani calculated the first seven non-zero functions $F_0, F_2, \dots, F_{12}$, and then continued the solution by assuming a universal shape for the subsequent terms and determining the amplitude function $A(\xi)$ from the momentum integral equation. Similarly, for the cases $n = 4, 8$, he calculated six and five functions respectively before the approximate continuation. In each case he obtained an equation of the form (86), finding that $S + TA$ became zero and $dA/d\xi$ infinite just before separation was reached. It was easy, however, to extrapolate the extremely small distance to separation, and the separation-points in these three cases were determined as

$$\xi = 0.271 \quad (n = 2), \quad \xi = 0.462 \quad (n = 4), \quad \xi = 0.640 \quad (n = 8).$$

3. The solutions of Curle

As has already been mentioned briefly, Curle (1958*a*) has developed a method for calculating solutions of the laminar boundary-layer equations when the external velocity is that appropriate to flow past a cylinder symmetrically placed in a uniform stream. The external

velocity then takes the form of equation (73), with ψ , u , and $(\partial u/\partial y)_w$ given by (74), (75), and (76) respectively. In any chosen case the first six terms in these series expansions are known, since the relevant universal functions have been tabulated by Tifford, and in all the cases for which calculations have been made the series seem to converge well until just upstream of separation. For example, in calculating the solution for the case

$$u_1(x) = \beta_1(\xi - \xi^3), \quad (87)$$

Curle estimated by reference to approximate solutions that separation would occur in the vicinity of $\xi = 0.66$, and showed that at this position the series (76) for the skin friction takes the form

$$\left(\frac{\partial u}{\partial y}\right)_w \propto 0.8135 - 0.8331 + 0.0896 - 0.0033 - 0.0073 - 0.0064 \dots$$

We note that the fourth to sixth terms are of order 1 per cent. of the first two, so that even if the subsequent (unknown) terms continue to decrease rather slowly they will presumably be only a relatively small correction to the first six.

Following the idea used by Howarth (1938), Curle assumed that the shape of the subsequent functions F_{2n+1} ($n \geq 6$) could be approximated by F_{11} . The accuracy of this approximation can only be tested by comparing say $F'_7(\eta)/F'_7(\infty)$, $F'_9(\eta)/F'_9(\infty)$, $F'_{11}(\eta)/F'_{11}(\infty)$, to see whether there is any indication of a tendency towards universality in shape as n increases. In the examples he calculated, this tendency appeared to be present to a certain degree. On the basis of this approximation we write the velocity as

$$\frac{u}{\beta_1} = \sum_0^4 \xi^{2n+1} F'_{2n+1}(\eta) + A(\xi) F'_{11}(\eta), \quad (88)$$

and the skin friction as

$$\left(\frac{\nu c}{\beta_1^3}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y}\right)_w = \sum_0^4 \xi^{2n+1} F''_{2n+1}(0) + A(\xi) F''_{11}(0), \quad (89)$$

where $A(\xi)$ is to be determined. To do this, Curle suggested using the second of the two methods used by Howarth, that is by satisfying the first boundary condition at the wall which is not identically satisfied by a form (88). This, as in Howarth's case, is

$$\left(\frac{\partial u}{\partial y}\right)_w \frac{d}{dx} \left\{ \left(\frac{\partial u}{\partial y}\right)_w \right\} = \nu \left(\frac{\partial^4 u}{\partial y^4}\right)_w. \quad (90)$$

From (88), by four-fold differentiation with respect to $y = (\nu c/\beta_1)^{\frac{1}{2}}\eta$, we find

$$\frac{\nu^2 c^2}{\beta_1^3} \left(\frac{\partial^4 u}{\partial y^4} \right)_w = \sum_0^4 \xi^{2n+1} F_{2n+1}^{\nu}(0) + A(\xi) F_{11}^{\nu}(0). \quad (91)$$

Noting that, in (89) and (91), $A(\xi)$ is multiplied only by constants, we can easily eliminate $A(\xi)$ to give $(\partial^4 u/\partial y^4)_w$ in terms of $(\partial u/\partial y)_w$, and this yields

$$\nu \left(\frac{\partial^4 u}{\partial y^4} \right)_w = \frac{\beta_1^3}{\nu c^2} \left\{ \sum_0^4 b_{2n+1} \xi^{2n+1} + \frac{F_{11}^{\nu}(0)}{F_{11}''(0)} \left(\frac{\nu c}{\beta_1^3} \right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y} \right)_w \right\}, \quad (92)$$

where
$$b_{2n+1} = F_{2n+1}^{\nu}(0) - \frac{F_{11}^{\nu}(0)}{F_{11}''(0)} F_{2n+1}''(0).$$

Upon substituting from (92) into (90), an equation for $(\partial u/\partial y)_w$ results. This equation may be put into non-dimensional form by writing

$$T = \left(\frac{\nu c}{\beta_1^3} \right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y} \right)_w,$$

and defining
$$P(\xi) = \sum_0^4 b_{2n+1} \xi^{2n+1},$$

whence it becomes
$$T \frac{dT}{d\xi} = P + \frac{F_{11}^{\nu}(0)}{F_{11}''(0)} T.$$

Upon formally integrating with respect to ξ , this yields

$$T^2 = Q + \frac{2F_{11}^{\nu}(0)}{F_{11}''(0)} \int_0^{\xi} T d\xi, \quad (93)$$

where
$$Q = 2 \int_0^{\xi} P d\xi = \sum_0^4 \frac{b_{2n+1}}{n+1} \xi^{2n+2}.$$

The method of solution is therefore as follows. Given the external velocity $u_1(x)$, the coefficients β_{2n+1} are known, so that the first six terms in the series (89) for T , the function $Q(\xi)$, and the constant $2F_{11}^{\nu}(0)/F_{11}''(0)$ in (93) are easily obtained. The series may be used to obtain the values of T when ξ is small, and the solution is continued by numerical integration. A simple, and convenient, procedure for doing this is to replace the integral $\int_0^{\xi} T d\xi$ by its Simpson's rule equivalent, whereupon if $T(\xi)$ and $T(\xi+h)$ are known, $T(\xi+2h)$ is given by solution of a quadratic equation. This method appears to work particularly well, provided the constant $2F_{11}^{\nu}(0)/F_{11}''(0)$ is not so great that the error in Simpson's rule is multiplied by too large a factor. When $T(\xi)$ is thus determined, the details of the velocity profile easily

follow. The function $A(\xi)$ is given from (89), and then u from (88). An *a posteriori* check on the accuracy of the solution may be made, if desired, by using the momentum integral equation.

Curle used this method to calculate an accurate solution for the case (87). The values of $T(\xi)$ at $\xi = 0.250, 0.275, 0.300$ were given by the known terms of the series to six significant figures. With these starting values, the equation (93) was integrated with steps of $\Delta\xi = 0.025$ and $\Delta\xi = 0.050$, Richardson's h^2 -extrapolation being used to obtain improved values. Using smaller intervals as separation was approached, the integration proceeded without difficulty, and separation was predicted at $\xi = 0.655_1$.

Curle also gave solutions for two cases of the family of velocity distributions

$$u_1(x) = \beta_1(\xi - \xi^3 + \alpha\xi^5),$$

for which the value of the constant $F_{11}^Y(0)$ was zero. He showed that

$$F_{11}^Y(0) = 1.02966 - 4.58707\alpha - 107.422\alpha^2$$

is zero when $\alpha = 0.07885$ and $\alpha = -0.12156$,

and that in each case (87) takes the form

$$T^2 = Q(\xi).$$

Solutions were obtained, therefore, without numerical integration, and separation found at

$$\xi = 0.6647 \quad \text{when} \quad \alpha = 0.07885,$$

and $\xi = 0.6245$ when $\alpha = -0.12156$.

4. The solutions of Hartree, Leigh, and Terrill

The solutions discussed above were all obtained by semi-analytic means. We now turn to a number of precise numerical solutions obtained by predominantly non-analytic means, either by numerical analysis or by means of an automatic computing machine. We begin by considering a solution by Hartree (1939*a*) for the case of a linearly decreasing velocity, which had earlier been considered by Howarth. The main idea of Hartree's approach was that of approximating the partial differential equation (51) by an equivalent ordinary differential equation, derivatives with respect to x being replaced by finite differences and the integration being carried out with respect to y , that is across the boundary layer. In other words the distribution of velocity at one station is used to calculate the distribution at a downstream station.

The integration was started at $8\xi = 0.40$, Howarth's series having been used to give the velocity profile at that position. The profile at $8\xi = 0.80$ was derived in one and two steps, after which Richardson's h^2 -extrapolation formula was used and the results smoothed. Similar corrections for the finite size of the steps of integration were applied at $8\xi = 0.88, 0.94, 0.956$, and 0.958 . Now according to Howarth (1938), separation occurs when $8\xi = 0.960$, and Hartree encountered considerable difficulties as this position was approached. These difficulties pointed very strongly to the existence of a singularity at the position of separation, and indeed it was because of them that Goldstein's investigations into flow near to a position of boundary-layer separation were carried out. Hartree determined the position of separation by two independent methods, and concluded that it is close to $8\xi = 0.959$, in excellent agreement with Howarth's result.

This same problem was later considered with even greater care by Leigh (1955), using the E.D.S.A.C. machine of the University of Cambridge. Various special tricks were successfully used to get over some of the difficulties which Hartree had met, and it was established that separation took place where $8\xi = 0.95854$. The main object of Leigh's work was to improve on the accuracy of the earlier solutions, by making use of the greater storage capacity of the machine, so as to enable more information to be obtained regarding flow in the vicinity of separation. For details of the techniques see Leigh (1955).

These precise numerical solutions are extremely valuable in that they confirm (by independent methods) the considerable accuracy of Howarth's procedure, as well as throwing some light on the singularity at separation.

We now comment on a further solution obtained by Hartree (1939*b*), on a differential analyser, in which the external velocity (or pressure) distribution was that found experimentally by Schubauer (1935) in some experiments with an elliptic cylinder. The procedure used was essentially that developed for the case of a linearly retarded velocity save that in this case the integration was done mechanically rather than numerically. One practical difficulty arose, however, in that the assumed external velocity, being given from experimental data, had to be smoothed and later differentiated numerically. There is a certain latitude of interpretation in doing this, which may not normally be critical but appeared to be so in this particular case. For example, Hartree smoothed and differentiated the values of $u_1(x)$ given by Schubauer, and then calculated the development of the boundary layer

by numerical integration. With the minor axis of the ellipse c as the representative length, Schubauer's experiments indicated that the boundary layer separated at roughly $x/c = \xi = 2.0$, whereas in Hartree's calculations no separation was predicted. It was found that an extremely small change in the assumed values of $u_1(x)$ in the vicinity of $\xi = 2.0$ sufficed to cause a theoretical prediction of separation at the correct position. Accordingly we may regard Hartree's work as providing an accurate numerical solution for a case with a numerically prescribed external velocity, but it should be treated with some caution, in view of the uncertainties discussed above.

A final, and important, solution on an automatic computing machine is that due to Terrill (1960), who considered the case of an external velocity

$$u_1 = \beta_1 \sin \xi.$$

The method used involved the application of Görtler's transformation (1957*a*) followed by the technique so successfully used by Hartree above, in which derivatives with respect to x are replaced by finite differences. The size of step used was rapidly decreased as separation was approached, and this was predicted at

$$\xi = 1.823,$$

Terrill also carried out some calculations which throw some light on the nature of the singularity at the separation position. For details see Terrill (1960).

PRACTICAL METHODS OF CALCULATION FOR INCOMPRESSIBLE FLOW

IN practice one may not be interested in a solution of the boundary-layer equations as accurate as would be obtained by using the methods discussed in the preceding chapters, and certainly one would not normally wish to spend the considerable time required to obtain such solutions. Accordingly it may be sufficient to use methods which are quicker to apply, but yield less accurate results. Some of these will now be considered.

1. Pohlhausen's method

The method developed by K. Pohlhausen (1921) is not now used much in its original form, having been improved upon and superseded by other methods. It is included here, however, because it was historically the first general method to be developed, and because the idea used by Pohlhausen has found so many applications in various branches of fluid dynamics.

In this method the boundary-layer equations are not solved everywhere, but are satisfied at the wall, at the edge of the boundary layer, and on an average by satisfying the momentum integral equation. The velocity u is assumed to take the form

$$\frac{u}{u_1} = f(\eta), \quad \eta = y/\delta,$$

where δ may usefully be regarded at this stage as the effective boundary-layer thickness, but in the final formulae is no more than a convenient parameter. Then the skin-friction τ_w , the displacement thickness δ_1 , and the momentum thickness δ_2 are equal to

$$\left. \begin{aligned} \frac{\tau_w}{\rho u_1^2} &= \frac{\nu}{u_1 \delta} f'(0) \\ \frac{\delta_1}{\delta} &= \int_0^1 (1-f) d\eta \\ \frac{\delta_2}{\delta} &= \int_0^1 f(1-f) d\eta \end{aligned} \right\}. \quad (94)$$

Now the boundary conditions to be satisfied by the true local velocity u are

$$u = 0, \quad \mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} = -\rho u_1 u'_1, \quad \frac{\partial^3 u}{\partial y^3} = 0, \quad \dots \quad \text{at } y = 0 \quad (95)$$

$$\text{and} \quad u - u_1, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^3 u}{\partial y^3}, \quad \dots \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (96)$$

The idea used by Pohlhausen is to choose $f(\eta)$ to be a polynomial satisfying some of the boundary conditions, (95) and (96), leaving δ to be determined so that the resulting approximate u satisfies the momentum integral equation. If we write

$$\Lambda = \frac{\delta^2}{\nu} u'_1, \quad (97)$$

it can easily be shown that the above conditions lead to the equation

$$\frac{d\Lambda}{dx} = g(\Lambda) \frac{u'_1}{u_1} + \{\Lambda + \Lambda^2 h(\Lambda)\} \frac{u''_1}{u'_1}, \quad (98)$$

where $g(\Lambda)$ and $h(\Lambda)$ are universal functions which depend only upon which of the boundary conditions (94) and (95) are satisfied, and are independent of the data of the problem.

There are two criticisms which can immediately be levelled at Pohlhausen's method as presented above. The first is the arbitrary choice as to which boundary conditions shall be satisfied and the fact that the outer conditions are satisfied at $y = \delta$. Pohlhausen chose a quartic for u which satisfied the first two conditions of (95) at the wall, and the first three conditions of (96) at the edge of the boundary layer. This leads to a method which, when applied to the cases for which accurate solutions are known, yields accurate results in regions where the pressure is decreasing but becomes less accurate as separation is approached, so that the predicted distances to separation are typically 30 per cent. too high. Curle (1957) suggested that better results would be obtained in the region near to separation if an additional boundary condition at the wall were satisfied by choosing a quintic profile. Such a procedure leads to improved results throughout the region downstream of the pressure minimum, so that a typical predicted separation distance is say 6 per cent. too high only. However, the method then breaks down entirely near to a stagnation-point. Earlier Schlichting and Ulrich (1940), in connexion with some work on boundary-layer stability, had chosen to represent u by a sextic, satisfying still one condition more at the edge of the boundary layer. This procedure also leads to better results near to separation, say 15 per cent. error in the

separation distance, but again leads to a breakdown of the method near to a stagnation-point.

A second criticism of the Pohlhausen approach is that the second derivative of u_1 , that is u_1'' , appears explicitly in the formulation. Now if u_1 has been obtained from experimental data it will in many cases be extremely difficult to obtain even rough estimates of u_1'' . Fortunately the method can be recast, by a procedure due to Holstein and Bohlen (1940), so that u_1'' does not appear, and it is necessary only to derive u_1' by numerical differentiation. Holstein and Bohlen introduce a parameter λ , analogous to Λ , defined by

$$\lambda = \frac{\delta_2^2}{\nu} u_1' = \left(\frac{\delta_2}{\delta} \right)^2 \Lambda. \quad (99)$$

The momentum integral equation is written in the form

$$\frac{\tau_w \delta_2}{\mu u_1} = \frac{u_1 \delta_2}{\nu} \frac{d\delta_2}{dx} + \frac{\delta_2^2}{\nu} u_1' \left(2 + \frac{\delta_1}{\delta_2} \right),$$

which we write as
$$l = \frac{u_1}{2\nu} \frac{d}{dx} (\delta_2^2) + \lambda(H+2), \quad (100)$$

where
$$l = \frac{\tau_w \delta_2}{\mu u_1} \quad \text{and} \quad H = \frac{\delta_1}{\delta_2}. \quad (101)$$

Thus

$$\begin{aligned} u_1 \frac{d}{dx} \left(\frac{\delta_2^2}{\nu} \right) &= 2\{l - \lambda(H+2)\} \\ &= L. \end{aligned} \quad (102)$$

Now from (94) and (99) we know τ_w , δ_1 , δ_2 , and λ as functions of Λ . Accordingly by (101) and (102) we can deduce l , H , and L as functions of λ , and (102) reduces to a first-order equation for λ , as can be seen by writing it in the form

$$\frac{d}{dx} \left(\frac{\lambda}{u_1'} \right) = \frac{L(\lambda)}{u_1}. \quad (103)$$

The functions $g(\Lambda)$, $h(\Lambda)$, for use with the method in its original form, may be found in Pohlhausen's original paper or in either of the books by Goldstein (1938) and Schlichting (1955), whilst the functions $H(\lambda)$, $l(\lambda)$, $L(\lambda)$, can be found in Schlichting's book. We may mention here that Tani (1941) indicated that the function $L(\lambda)$ is approximately linear, so that (103) integrates analytically. This considerable simplification has been used by several other authors, notably Thwaites (1949).

2. Timman's method

Timman (1949) attempted to obtain an improved procedure for calculating laminar boundary-layer development by assuming a velocity profile of the form

$$\frac{u}{u_1} = f(\eta) = 1 - \int_{\eta}^{\infty} e^{-\eta^2(a+c\eta^2\dots)} d\eta - e^{-\eta^2(b+d\eta^2\dots)}, \quad (104)$$

where $\eta = y/\delta$. This form of profile is an improvement over the polynomials assumed in Section 1 in that u/u_1 tends to unity asymptotically. Accordingly only the boundary conditions at the wall need to be further considered, and these serve to determine a , b , c , d .

From (95), substituting from (104), these yield in turn

$$\left. \begin{aligned} 1 - b - \frac{1}{2}\sqrt{\pi}(a + \frac{1}{2}c\dots) &= 0 \\ 2(b-d) &= -\frac{u'_1 \delta^2}{\nu} = -\Lambda \\ 2(c-a) &= 0 \\ 12(2d-b) &= \frac{u_1 \delta^2}{\nu} a \left(\frac{da}{dx} + a \left(\frac{u'_1}{u_1} - \frac{1}{\delta} \frac{d\delta}{dx} \right) \right) \end{aligned} \right\}. \quad (105)$$

Initially Timman considers only the first three conditions of (105), and these determine a , b , c as functions of Λ when d is set equal to zero. The method then proceeds much as in Pohlhausen's method. From (104) we can obtain δ_1/δ , δ_2/δ , and $\tau_w \delta/\mu u_1$ as functions of Λ and also λ as a function of Λ . Thus we obtain l , H , and L , defined in (101) and (102), as functions of λ , with the momentum integral equation reducing to the form (103).

Timman found that this solution yielded accurate representations of the flow in a Blasius boundary layer and of the flow near to a stagnation point. In regions of retarded flow, however, the method was much less satisfactory, as application to the case $u_1 = u_0\{1 - (x/c)\}$ showed. Accordingly he suggested that in regions of adverse pressure gradient the condition $d = 0$ should be replaced by

$$2d - b = 0.$$

This condition was chosen to ensure that the complicated fourth condition in (105) should be satisfied at the separation position, and incidentally led to values of a , b , c , d , which were continuous at the join $\Lambda = 0$. This procedure leads to results which agree quite well with the accurate calculated results discussed in Chapter 3. The values of the relevant functions, required for integrating (103), are to be found in Timman's paper.

3. The methods of Howarth and Walz

In the methods of Pohlhausen and Timman the velocity profile at any station is chosen as one of a singly-infinite family of velocity profiles whose shape is characterized in either case by the parameter Λ and whose scale is characterized by $\delta = (\nu\Lambda/u_1')^{\frac{1}{2}}$. The choice as to which profile is to apply at each station is made by satisfying the momentum integral equation. Similarly the accurate solution of the boundary-layer equations for the case $u_1 = u_0\{1-(x/c)\}$, derived by Howarth (1938) and discussed in Chapter 3, section 1, yields a singly-infinite family of velocity profiles, whose shape may be characterized by the parameter λ , and with scale represented by say the momentum thickness $\delta_2 = (\nu\lambda/u_1')^{\frac{1}{2}}$. A method on these lines was given by Howarth (1938), by regarding a general velocity distribution as being replaced by a series of linear portions. His general result can easily be reduced to the form (103), the functions $H(\lambda)$, $l(\lambda)$, $L(\lambda)$ being obtained from Howarth's tabulated results. These values, of course, will all be for negative λ , since the velocity is everywhere retarded, but by considering the associated accelerated flow $u_1 = u_0\{1+(x/c)\}$ it would be possible, as Walz (1941) pointed out, to extend the functions to cover a range extending from the stagnation point to separation.

Walz (1941) also suggested a method in which the velocity profiles are chosen to be those of the Falkner-Skan similarity solutions. As distinct from Howarth's method, where each value of x corresponds to unique values of H , l , L , and λ , each complete similarity solution for a given value of the velocity exponent yields single values of H , l , L , λ for all x . However, the complete family of similarity solutions yields values for H , l , L , which are continuous functions of λ .

4. The method of Thwaites

It was pointed out by Thwaites (1949) that if one wishes to calculate only the boundary-layer thicknesses and the skin-friction distribution it is not necessary to introduce explicit assumptions concerning detailed velocity profiles, as the methods previously discussed all do. Provided suitable correlations are defined between the overall boundary-layer characteristics H , l , L , and the shape parameter λ , one can easily obtain λ as a function of x by numerical integration of (103), after which δ_2 follows from (99) and then δ_1 and τ_w from (101).

Each accurate solution of the laminar boundary-layer equations will correspond to a different set of functions $H(\lambda)$, $l(\lambda)$, $L(\lambda)$. Similarly any approximate method using a single-parameter family of profiles will

correspond to a definite set of functions. It follows from this that an approximate method will yield an accurate answer for a given problem if and only if the values of H , l , and L for the accurate and approximate solutions agree closely enough. Now the idea used by Thwaites was to examine and compare the functions H , l , and L , corresponding to all known exact and approximate solutions of the laminar boundary-layer equations. By considering the accuracy to which these various solutions are known, and how typical each is of a practical boundary layer, Thwaites defined what are virtually the optimum values of the functions for general use.

Thwaites found that as regards $H(\lambda)$ and $l(\lambda)$ the solutions lay very closely together for positive λ , that is for regions of increasing velocity—upstream of the pressure minimum in other words. Downstream of the pressure minimum, for negative λ , the curves deviated considerably, so that values of λ corresponding to boundary-layer separation, that is values for which $l(\lambda) = 0$, were obtained ranging from $\lambda = -0.068$ to $\lambda = -0.157$. Fortunately the two most important solutions of those available to him, namely Howarth's solution and Hartree's analysis of Schubauer's experimental results, lay quite close to each other, and Thwaites's choices of H and l lay close to both of these as well as being a rough average of the approximate solutions.

The values of $L(\lambda)$, derived from (102), for the various solutions were much closer than those of either $H(\lambda)$ or $l(\lambda)$, both for positive and negative λ , and in addition the function $L(\lambda)$ appeared to be linear to a high degree of accuracy. Thwaites accordingly suggested defining $L(\lambda)$ to be

$$L(\lambda) = 0.45 - 6\lambda, \quad (106)$$

the coefficients being chosen to give the best overall agreement with the available solutions. With $L(\lambda)$ given by (106) it is possible to integrate (103) in terms of a single quadrature, to yield

$$\delta_2^2 = \frac{0.45\nu}{u_1^6} \int_0^x u_1^5 dx, \quad (107)$$

$$\lambda = \frac{u_1' \delta_2^2}{\nu}. \quad (108)$$

Having thus obtained δ_2 , (101) then yields

$$\delta_1 = \delta_2 H(\lambda)$$

and

$$\tau_w = \frac{\mu u_1}{\delta_2} l(\lambda).$$

The idea of expressing $L(\lambda)$ as a linear function had been suggested earlier by both Walz and Tani, but Thwaites's coefficients are, if anything, more acceptable, since they are not based upon a single solution, and yield the computational simplification of requiring only integral powers of u_1 .

Some improvements to Thwaites's functions were suggested by Curle and Skan (1957). Their idea may be illustrated thus. Thwaites determined his $H(\lambda)$ by considering the ratios of exact δ_1 to exact δ_2 . Since the predicted δ_1 is obtained by multiplication of H and the approximate δ_2 , Curle and Skan suggested that more acceptable values for $H(\lambda)$ would be derived from the ratios of exact δ_1 to approximate δ_2 . This argument would be complicated further if one accounted for the fact that an approximate value of λ , derived from (107) and (108), is used in practice. In this way, and by similar arguments applied to $l(\lambda)$, Curle and Skan suggested improvements to Thwaites's functions in the region near separation where the functions for individual solutions differ most. These modified functions are given in Table 5.

TABLE 5
Universal functions for Thwaites's method

m	$l(m)$	$H(m)$	m	$l(m)$	$H(m)$
-0.25	0.500	2.00	0.040	0.153	2.81
-0.20	0.463	2.07	0.048	0.138	2.87
-0.14	0.404	2.18	0.056	0.122	2.94
-0.12	0.382	2.23	0.060	0.113	2.99
-0.10	0.359	2.28	0.064	0.104	3.04
-0.080	0.333	2.34	0.068	0.095	3.09
-0.064	0.313	2.39	0.072	0.085	3.15
-0.048	0.291	2.44	0.076	0.072	3.22
-0.032	0.268	2.49	0.080	0.056	3.30
-0.016	0.244	2.55	0.084	0.038	3.39
0	0.220	2.61	0.086	0.027	3.44
+0.016	0.195	2.67	0.088	0.015	3.49
0.032	0.168	2.75	0.090	0	3.55

The modifications of Curle and Skan also made use of the accurate solutions given by Tani (1949), which had not been available to Thwaites. Subsequently Curle (1958 *a*) has given the additional accurate solution discussed in Chapter 3, section 3. The greater number of accurate solutions now available makes it much easier to avoid the possibility of being unduly biased by the few accurate solutions available to Thwaites. Curle and Skan (unpublished) have examined the functions H and l which would lead to exact prediction of δ_1 and τ_w for each of the accurate solutions discussed in Chapter 3, and have

indicated that with the exception of the modifications near separation, the original values given by Thwaites probably reflect these as accurately as any single set of functions can do.

5. The method of Stratford

The method developed by Stratford (1954) is based upon the idea of dividing the boundary layer into outer and inner portions, for each of which a solution is obtained which joins smoothly onto the other. This idea was first used by Kármán and Millikan (1934).

Stratford begins by considering the outer part of the boundary layer, where the flow is nearly inviscid and so, by Bernoulli's equation, the total head is almost constant along streamlines. Thus if the pressure is constant between $x = 0$ and $x = x_0$, with a pressure rise downstream of $x = x_0$, then we have

$$(\tfrac{1}{2}\rho u^2)_{x,\psi} = (\tfrac{1}{2}\rho u^2)_{x_0,\psi} - (p - p_0) + \Delta H, \quad (109)$$

where the stream-function ψ is given by

$$\psi = \int_0^y u \, dy, \quad (110)$$

and ΔH is the small change in total head due to viscosity. Now the shape of the outer part of the boundary-layer velocity profile is not greatly changed by the pressure rise, so the viscous forces are given approximately as though there were no pressure rise. Hence

$$\Delta H = (\tfrac{1}{2}\rho u_B^2)_{x,\psi} - (\tfrac{1}{2}\rho u_B^2)_{x_0,\psi}, \quad (111)$$

where u_B is the Blasius solution for a boundary layer with no pressure gradient. From (111) and (109), remembering that $u \equiv u_B$ when $x = x_0$, we have

$$(\tfrac{1}{2}\rho u^2)_{x,\psi} = (\tfrac{1}{2}\rho u_B^2)_{x,\psi} - (p - p_0). \quad (112)$$

Stratford also indicates how this result may be deduced algebraically by writing the boundary-layer equation (51) in the form

$$\frac{\partial}{\partial s}(p + \tfrac{1}{2}\rho u^2) = \mu \frac{\partial^2 u}{\partial y^2}, \quad (113)$$

where $\partial/\partial s$ denotes differentiation along a streamline, and considering the Taylor expansion

$$(p + \tfrac{1}{2}\rho u^2)_{x,\psi} = (p + \tfrac{1}{2}\rho u^2)_{x_0,\psi} + \left[\frac{\partial}{\partial s}(p + \tfrac{1}{2}\rho u^2) \right]_{x_0,\psi} (x - x_0) + \dots \quad (114)$$

The first term on the right-hand side is independent of the pressure rise, since the effects have diffused outwards only downstream of $x = x_0$. Similarly the coefficient of $(x - x_0)$ is equal, by (113), to $\{\mu(\partial^2 u/\partial y^2)\}_{x_0,\psi}$,

and so again is independent of the pressure rise. Hence, if terms of order $(x-x_0)^2$ are neglected, the left-hand side of (114) is independent of the pressure rise, so that

$$(p + \frac{1}{2}\rho u^2)_{x,\psi} = (p_0 + \frac{1}{2}\rho u_B^2)_{x,\psi}, \quad (115)$$

which is (112) rewritten.

By similar arguments, or by differentiation with respect to ψ , one can also deduce from (112) or (115) that

$$\left(\frac{\partial u}{\partial y}\right)_{x,\psi} = \left(\frac{\partial u_B}{\partial y}\right)_{x,\psi} \quad (116)$$

$$\text{and} \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{x,\psi} = \left(\frac{u}{u_B} \frac{\partial^2 u_B}{\partial y^2}\right)_{x,\psi} \quad (117)$$

in the outer part of the boundary layer.

Sufficiently near to the wall, in contrast, viscous forces become very important, so that the resultant viscous force on an element of fluid at the wall must exactly balance the pressure-gradient force. Thus, by setting $y = 0$ in the boundary-layer equation (51) we find

$$\mu \left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{dp}{dx}. \quad (118)$$

By further considering the boundary conditions at the wall, Stratford indicates that in general

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_w = 0, \quad (119)$$

$$\text{but} \quad \left(\frac{\partial^4 u}{\partial y^4}\right)_w \neq 0. \quad (120)$$

By satisfying (118) and (119), Stratford suggests that at separation the velocity distribution near to the wall may be approximately obtained by writing

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + h y^4, \quad (121)$$

where h is to be determined.

With an inner profile given by (121), and an outer profile given by (115) to (117), it is postulated that there should be continuity of ψ , u , $\partial u/\partial y$, $\partial^2 u/\partial y^2$ at the position of the join, $\psi = \psi_j$, $u = u_j$. These four conditions serve to determine the values of u_j , ψ_j , h_j , and x at the position of separation. In general the solution of the four equations is difficult, since such functions as u_B , $\partial u_B/\partial y$, and $\partial^2 u_B/\partial y^2$ cannot be expressed in simple analytic terms. A considerable simplification is effected, however, if the assumption is made that the inner layer is sufficiently thin that the Blasius velocity profile $u_B(x, \psi)$ is sensibly

linear throughout $\psi \leq \psi_j$. This condition, which restricts the application of the method in a manner which we shall discuss later, leads to the simple result

$$\left. \begin{aligned} u_B(x, \psi) &= \left(\frac{2\tau_B}{\mu} \psi \right)^{\frac{1}{2}} \\ \left(\frac{\partial u_B}{\partial y} \right)_{x, \psi} &= \frac{\tau_B}{\mu} \\ \left(\frac{\partial^2 u_B}{\partial y^2} \right)_{x, \psi} &= 0 \end{aligned} \right\}. \quad (122)$$

By assuming that these values hold at the join, Stratford derives the simple result

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = 0.0065 \quad \text{at separation}, \quad (123)$$

where C_p is the pressure coefficient, defined as

$$C_p = \frac{p - p_0}{\frac{1}{2} \rho u_0^2}.$$

The condition on the inner layer is such that the method should strictly only apply when C_p is less than about 0.04, with an absolute upper limit of about 0.11. It has been found, however, that the method yields useful results over a larger range of pressure coefficients.

Stratford also gave an asymptotically exact solution for the case when the adverse pressure gradient downstream of $x = x_0$ is constant but asymptotically large, and the distance $x_s - x_0$ to separation tends asymptotically to zero, as does the thickness of the inner layer. For details of this solution reference may be made to Stratford's paper. The principal result is that

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = 0.0076 \quad \text{at separation}. \quad (124)$$

The numerical coefficient is not very different from that obtained by the more physical approach discussed earlier. Stratford suggested that in general (124) should lead to good predictions of separation.

Stratford also considered a second, very special, type of flow, in which the pressure gradient is zero upstream of $x = x_0$ and is such that the skin-friction is identically zero downstream of $x = x_0$. He suggests that a more appropriate form than (121) for the inner layer would be

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + jy^6.$$

An analysis along the same lines as that for the earlier case leads to a similar result to (123) or (124). The numerical coefficient becomes

0.0049, and a rigorous mathematical analysis for an asymptotically large adverse pressure gradient of the appropriate form leads to a value 0.0059. By carefully interpolating between these special cases and empirically adding further terms, Stratford also derived a considerably lengthier formula than those quoted above, which should yield exceedingly accurate predictions of separation. Later, however, Curle and Skan (1957) indicated that an equally accurate, but considerably simpler, empirical improvement of (124) could be obtained by replacing the numerical coefficient by 0.0104; separation is then predicted to occur where

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = 0.0104.$$

It should be pointed out that the method is based upon a consideration of the downstream deformation of a known Blasius velocity profile. Accordingly the method is not, strictly speaking, applicable to flows containing a forward stagnation-point. It was pointed out by Stratford, however, that in such a flow the velocity profile at the pressure minimum will be reasonably similar to a Blasius profile of suitably chosen scale. Accordingly the method may be applied downstream of the pressure minimum if we replace x by $x - x_L$ say, where x_L is chosen so that the true momentum thickness at the pressure minimum $x = x_m$ is equal to the momentum thickness of the equivalent Blasius layer at $x = x_m - x_L$. Since the momentum thickness δ_2 is usually given accurately by Thwaites's formula

$$\delta_2^2 = 0.45 \nu u_1^{-6} \int_0^x u_1^5 dx,$$

this condition is easily seen to yield

$$x_L = x_m - \int_0^{x_m} \left(\frac{u_1}{u_m} \right)^5 dx, \quad (125)$$

where u_m is the value of u_1 at the pressure minimum.

The method as developed by Stratford was designed merely for predicting separation. No estimate of the boundary-layer thicknesses or of the detailed distribution of skin-friction between the leading edge and separation is given. This restriction has been partly removed by the method which will now be discussed.

6. The method of Curle

This method is a generalization of Stratford's method, and leads to a simple method for predicting the detailed distribution of skin-

friction. The inner layer is assumed to be of the form

$$\mu u = \tau_w y + \frac{1}{2} \frac{dp}{dx} y^2 + ay^n, \quad (126)$$

where τ_w is the skin-friction $\mu(\partial u/\partial y)_{|w}$, and $a(x)$ and the constant n are to be determined. This profile generalizes Stratford's form in two ways, namely that the skin-friction is not assumed to be zero and that the value of n is not fixed *a priori*. The outer velocity profile is given, as in Stratford's method, by (112), (116), (117), and (122). The four joining conditions, continuity of ψ , u , $\partial u/\partial y$, $\partial^2 u/\partial y^2$, are sufficient to determine τ_w , a , y_j , ψ_j as functions of x , with n as an arbitrary parameter. After the relevant algebra has been performed the following results are obtained:

$$\frac{dp}{dx} y_j = \frac{n-1}{n-2} (\tau_B - \tau_w), \quad (127)$$

and

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = \frac{0.00405(n-1)^3(n^2+3)}{n^2(n+1)(n-2)^2} \left(1 - \frac{\tau_w}{\tau_B} \right)^3 \left\{ 1 + \frac{3(n^2-1)}{n^2+3} \frac{\tau_w}{\tau_B} \right\}, \quad (128)$$

where $\tau_B(x)$ is the skin-friction at station x of a Blasius boundary layer.

The position of separation, where $\tau_w = 0$, is obtained from (128) as

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = \frac{0.00405(n-1)^3(n^2+3)}{n^2(n+1)(n-2)^2}, \quad (129)$$

so that the predicted separation position depends upon the choice of the parameter n . When $n = 6$ the right-hand side becomes 0.0049, and when $n = 4$ it becomes 0.0065, the values obtained by Stratford. In view of the fact that, as Curle and Skan have shown, the best overall predictions of separation are obtained when the right-hand side of (129) is set equal to 0.0104, it would seem most natural to choose n accordingly, so that

$$n = 3.043.$$

With this value of n , the distribution of skin-friction is derived from (128), which now becomes

$$x^2 C_p \left(\frac{dC_p}{dx} \right)^2 = 0.0104 \left(1 - \frac{\tau_w}{\tau_B} \right)^3 \left(1 + 2.02 \frac{\tau_w}{\tau_B} \right). \quad (130)$$

It follows that when $x^2 C_p (dC_p/dx)^2$ has been obtained at any station x , the value of τ_w/τ_B follows by solution of a simple quartic algebraic equation.

For flows with an initial favourable pressure gradient (130) may be used to determine the distribution of skin-friction downstream of the

pressure minimum $x = x_m$, provided τ_B is interpreted as the skin-friction of a Blasius boundary layer at a station $x - x_L$, where x_L is given by (125).

The method was applied by Curle (1960) to consider some of the cases for which exact solutions are known, and good agreement was obtained, both as regards the distribution of skin-friction and, in particular, the position of separation.

7. Use of the energy integral equation. The methods of Tani and Truckenbrodt

The energy integral equation, due to Leibenson (1935), is

$$\frac{d}{dx} (u_1^3 \delta_3) = \frac{2D}{\rho}, \quad (131)$$

where

$$\delta_3 = \int_0^\infty \frac{u}{u_1} \left(1 - \frac{u^2}{u_1^2} \right) dy$$

and

$$D = \int_0^\infty \mu \left(\frac{\partial u}{\partial y} \right)^2 dy.$$

Upon writing

$$\alpha = \delta_3 / \delta_2, \quad (132)$$

and

$$\beta = \frac{D \delta_2}{\mu u_1^2}, \quad (133)$$

(131) becomes

$$\delta_2 \frac{d}{dx} (\alpha u_1^3 \delta_2) = 2\beta \nu u_1^2. \quad (134)$$

Upon multiplication by $2\alpha u_1^3$, this becomes

$$\frac{d}{dx} (\alpha^2 u_1^6 \delta_2^2) = 4\alpha\beta \nu u_1^5, \quad (135)$$

which integrates to give

$$\alpha^2 u_1^6 \delta_2^2 = 4\nu \int_0^x \alpha\beta u_1^5 dx,$$

plus a constant of integration which is zero. Thus

$$\delta_2^2 = \frac{4\nu}{\alpha^2} u_1^{-6} \int_0^x \alpha\beta u_1^5 dx. \quad (136)$$

Now Truckenbrodt (1952) suggested that in many cases it should be sufficiently accurate to regard α and β as constants. It is fairly easy to see that α should not vary very much, as the following argument

indicates. Now α , by (132), is the ratio of the energy thickness to the momentum thickness, each of which are integrals with integrands zero at the wall. Accordingly little contribution arises from the region very close to the wall. But it is just in this region that the greatest changes of profile shape take place, and the outer part of the boundary layer retains approximately the same general shape regardless of pressure gradient. Hence $\delta_3/\delta_2 = \alpha$ does not vary very much. This argument is borne out by the fact that for a stagnation-point $\alpha = 1.63$, for a Blasius boundary layer $\alpha = 1.57$, and for a typical separation profile $\alpha = 1.52$, so that α varies by a maximum of about 3 per cent. about its mean value. Similarly the value of β does not vary as much as might be expected. The reason for this is presumably that the decrease in the contribution to the dissipation integral as the skin-friction decreases to zero at separation is partially balanced by an increase in momentum thickness and a decrease in the local mainstream velocity. The values of β are 0.209 for a stagnation-point, 0.173 for a Blasius layer, and 0.157 for a typical separation profile, so that β varies by no more than 14 per cent. about its mean value.

Since α and β do not vary excessively it seems reasonable to replace them by their mean values, which will be close to the values appropriate to a Blasius layer. At positions downstream of the pressure minimum the most important contributions to the integral in (136) will arise from the region close to the pressure minimum, and the constant values chosen for α and β will be particularly accurate in this region. With $\alpha = 1.57$, $\beta = 0.173$, (136) becomes

$$\delta_2^2 = 0.441\nu u_1^{-6} \int_0^x u_1^5 dx, \quad (137)$$

which is identical to Thwaites's formula (107) save for a 2 per cent. change in the numerical factor, corresponding to a change of 1 per cent. in the resulting value of δ_2 .

Tani (1954) has developed a method of calculating laminar boundary layers which uses both the momentum and kinetic-energy integral equations. With the notation already used we may write the momentum equation in the form

$$\frac{u_1}{2\nu} \frac{d}{dx} (\delta_2^2) = l - (2+H)\lambda, \quad (138)$$

and the kinetic-energy integral equation in the form (135) may easily be rewritten as

$$\frac{u_1}{2\nu} \frac{d}{dx} (\delta_2^2) = \frac{2\beta}{\alpha} - \lambda \left(3 + \frac{\alpha' u_1}{\alpha u_1'} \right).$$

Hence, upon eliminating $(d/dx)(\delta_2^2)$, we find that

$$l = (H-1)\lambda + \frac{2\beta}{\alpha} - \lambda \frac{\alpha' u_1}{\alpha u_1'} \quad (139)$$

We may regard (136) and (139) as new forms of the two integral relationships. Tani integrates (136) in the approximate form (137) to yield δ_2 (and λ) as a function of x . He then assumes that l , H , α , β are related to each other by the same relationship as holds for Pohlhausen's family of profiles, but determines the dependence upon λ from (139). Thus, as a first approximation he ignores the last term and by replacing H , α , β with the relevant functions of l , a relationship between l and λ is obtained.† Having thus related l , H , α , β , and λ , these values can be used to estimate the neglected term, and a second approximation calculated. When this procedure has been repeated until sufficient accuracy is obtained, the relationship between α , β , and λ may be substituted into (136) to obtain an improved relationship between λ and x , but Tani remarks this will not in practice be necessary. The universal relationships between l , H , α , β are shown in Table 6.

TABLE 6
Universal relationships for Tani's method

H	l	α	β
3.50	0	1.53	0.157
3.33	0.029	1.53	0.157
3.18	0.059	1.54	0.158
3.04	0.089	1.54	0.159
2.92	0.120	1.54	0.161
2.81	0.150	1.55	0.164
2.72	0.180	1.56	0.167
2.63	0.210	1.56	0.170
2.55	0.235	1.57	0.174
2.49	0.260	1.58	0.179
2.43	0.283	1.59	0.184
2.33	0.322	1.60	0.193
2.27	0.347	1.61	0.200
2.25	0.356	1.62	0.203

Truckenbrodt (1952) introduces an alternative shape parameter to $H = \delta_1/\delta_2$ and $\alpha = \delta_3/\delta_2$. He writes (139) in the form

$$\frac{\alpha'}{\alpha(H-1)} = \frac{u_1'}{u_1} + \frac{2\beta/\alpha - l}{(H-1)\lambda} \frac{u_1'}{u_1}, \quad (140)$$

† It is interesting to note that according to this first approximation, in which H , l , α , β are functions of λ alone, separation occurs when $\lambda = -0.082$, which is much closer to the value appropriate to the known exact solutions than is Pohlhausen's value, $\lambda = -0.157$.

and introduces a shape parameter S defined by

$$\frac{dS}{dx} = \frac{d\alpha/dx}{\alpha(H-1)}$$

and $S = 0$ when $\lambda = 0$. He also defines $T(S)$ as

$$(H-1)T(S) = l - \frac{2\beta}{\alpha},$$

so that from (139), putting $\lambda = 0$, $T(0) = 0$. Then (140) becomes

$$\frac{dS}{dx} = \frac{u'_1}{u_1} - T \frac{u'_1}{u_1 \lambda},$$

$$\text{or} \quad \frac{dS}{dx} + \frac{u'_1}{\lambda u_1} T(S) = \frac{u'_1}{u_1}. \quad (141)$$

He then assumes that the relationship between T , S , H , l , α , β is exactly that which holds for the Falkner-Skan similarity solutions, but determines the dependence upon λ and x from the integral relationships as expressed in equations (137) and (141). The universal relationships are shown in Table 7.

TABLE 7

Universal relationships for Truckenbrodt's method

S	H	l	α	β
-0.018	4.04	0	1.52	0.157
-0.015	3.20	0.095	1.53	0.159
-0.010	2.88	0.149	1.55	0.163
0	2.59	0.220	1.57	0.173
+0.010	2.43	0.277	1.59	0.185
0.020	2.32	0.320	1.62	0.197
0.030	2.24	0.357	1.63	0.208
0.040	2.16	0.390	1.66	0.219

By examination of the similar solutions, Truckenbrodt found that the function $T(S)$ could be written as

$$T = cS, \quad c = 2.87 \quad \text{when} \quad S > 0, \\ c = 3.53 \quad \text{when} \quad S < 0,$$

$$\text{so that (141) becomes} \quad \frac{dS}{dx} + \frac{cu'_1}{\lambda u_1} S = \frac{u'_1}{u_1},$$

which may be written

$$\frac{d}{dx} \left(S \exp \int_{x_1}^x \frac{cu'_1}{\lambda u_1} dx \right) = \frac{u'_1}{u_1} \exp \left(\int_{x_1}^x \frac{cu'_1}{\lambda u_1} dx \right).$$

Now it may easily be shown that, with λ defined by (137),

$$\exp\left\{\int_{x_1}^x \frac{cu'_1}{\lambda u_1} dx\right\} = \frac{\xi(x)}{\xi(x_1)},$$

where

$$\xi(x) = \left\{\int_0^x u_1^5 dx\right\}^d$$

and

$$d = \left. \begin{aligned} & \frac{c}{0.441} = 8.0 \quad \text{when } S < 0 \\ & = 6.5 \quad \text{when } S > 0 \end{aligned} \right\}. \quad (142)$$

It then follows, after some algebra, that

$$\xi S(\xi) = \xi_1 \left\{ S(\xi_1) + \log \frac{u_1(\xi)}{u_1(\xi_1)} \right\} - \int_{\xi_1}^{\xi} \log \frac{u_1(z)}{u_1(\xi)} dz. \quad (143)$$

The integration starts from the leading edge, where $\xi = 0$, and if this is taken as the datum position, (143) yields

$$\xi S(\xi) = - \int_0^{\xi} \log \frac{u_1(z)}{u_1(\xi)} dz.$$

If $u_1(\xi)$ is an increasing function of ξ , then S will be positive, whereas S will be negative if $u_1(\xi)$ is a decreasing function of ξ . The value of d is determined accordingly from (142), and the integration proceeds easily until a point is reached at which S changes sign. At this point ξ changes discontinuously (because of the changed value of d), and the integration then proceeds by putting ξ_1 in equation (143) as the downstream value of ξ at this point.

It should be remarked that the two methods described above, due to Tani and Truckenbrodt, arose essentially as simplifications to a rather complex two-parameter method due to Weighardt (1946, 1948). A further method based upon that of Weighardt, designed in this case to improve rather than simplify it, is due to Head (1957*a*), and we shall now consider this method.

8. The method of Head

This method was designed to be considerably more general than any of the preceding methods, in that velocity profiles are predicted with considerable accuracy even when distributed suction is applied. Head prescribed a two-parameter family of velocity profiles dependent upon

the parameters l and λ . From the assumed profiles, chosen carefully to give the correct form in a number of representative cases, the various integral properties, α , β , H , can be deduced as functions of l and λ , and Head gives charts of these relationships. He also makes use of both the momentum and kinetic-energy integral equations, writing these as

$$\frac{d}{dx} (\delta_2^2/\nu) = \frac{2\{l - \lambda(H+2)\}}{u_1}, \quad (144)$$

$$\frac{d}{dx} (\alpha) = \frac{2\beta + \alpha\{\lambda(H-1) - l\}}{u_1 \delta_2^2/\nu}, \quad (145)$$

which are similar to (138) and (140) respectively. It is assumed that starting values of l and λ are known. These will be $l = 0.360$, $\lambda = 0.085$ at a stagnation-point, and $l = 0.221$, $\lambda = 0$ at a sharp leading edge. From the known values of l and λ , values of H , α , β , are read from the charts, whence (144) and (145) yield $d(\delta_2^2/\nu)/dx$ and $d(\alpha)/dx$. It follows that, corresponding to a small increment in x , the increments in δ_2 and α are found. Once the new value of δ_2 is known, λ follows. Then, by reference to the chart for $\alpha(\lambda, l)$, the values of α and λ enable the new value of l to be read off. The values of H and β follow and the solution at the new point is complete. By repeating these steps the development of the boundary layer may be calculated by a step-by-step procedure.

Head applied his method to several of the cases, considered in Chapter 3, for which accurate solutions are known, and obtained good agreement. He also did some calculations (Head, 1957*b*) for flows with distributed suction, and estimated the effects of distributed suction upon boundary-layer stability. These points, however, are outside the scope of this book, and will not be considered further.

COMPARISONS OF APPROXIMATE METHODS OF SOLUTION

IN the preceding chapter no less than ten different methods have been given for calculating the development of a two-dimensional incompressible laminar boundary layer. In view of the fact that little more remains to be done in this particular field, and there would appear to be little room for any improved methods, it seems desirable to make a critical comparison of the various methods, so that readers who are interested mainly in obtaining numerical results will be able to consider more readily which of the methods best suits their requirements.

There would appear to be three things which might determine the choice of a method. Firstly, the question arises as to how much information is required. For example, is it desired merely to calculate the position at which the boundary layer separates, or the momentum or displacement thickness at various stations? Or, on the other hand, is it required to obtain an accurate estimate of the detailed manner in which the boundary-layer velocity profile develops? Secondly, the question of accuracy arises. Is a rough estimate, say within 15 per cent., adequate, or is it essential to calculate some property or other to within say ± 5 per cent. at most? Thirdly, when those methods have been rejected which do not match up to the requirements on these two counts, the choice between the remainder can be determined by questions of speed and simplicity. We will now consider these three points in turn, and will first group the methods in the following way.

A. Methods based on the momentum integral equation:

Pohlhausen	(section 1)
Timman	(section 2)
Howarth	(section 3)
Walz	(section 3)
Thwaites	(section 4)

B. Methods based on outer and inner solutions:

Stratford and Curle	(sections 5 and 6)
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C. Methods based on the kinetic-energy integral equation:

Tani	(section 7)
Truckenbrodt	(section 7)
Head	(section 8)

1. Information provided by various methods

We begin by remarking that if only the position of boundary-layer separation is required, any of the methods is able to provide an estimate. If, further, the detailed distribution of skin-friction is sought, again any of the methods is satisfactory, with the qualification that for flow which starts from a stagnation-point rather than a sharp leading edge, the Stratford–Curle method is only used downstream of the pressure minimum.

If any further information is required the Stratford–Curle method must be rejected, but any of the remaining methods are able to give predictions of the boundary-layer momentum and displacement thicknesses. If, alternatively or additionally, the energy thickness is required the methods in group A fall down, strictly speaking. However, it would be such a simple process to make the relevant extensions to each of these methods that we may perhaps accept them from the practical point of view.

The only remaining possibility (of any great likelihood) is the detailed development of the velocity distribution, and the methods of Thwaites, Tani, and Truckenbrodt fail to answer this requirement.

2. Comparison of accuracy

Before assessing the accuracy of the various methods it is necessary to consider carefully just what accuracy is desired in any given problem. On the one hand it should be pointed out that the position at which boundary-layer separation takes place is somewhat difficult to determine experimentally. In this sense it may not be necessary, in certain circumstances, to have very great accuracy. On the other hand, if one is interested in a theoretical prediction of boundary-layer stability, extremely accurate detailed velocity profiles must be obtained. It follows that the ensuing remarks regarding accuracy must be interpreted in the light of the particular problem under consideration.

A natural starting-point is to compare the methods of type A above, which are all reducible to a form in which three functions, l , H , and L , are assumed given as functions of λ only. Of course, l , H , and L are not universal functions of λ . In fact, for any given pressure distribution the exact solution of the laminar boundary-layer equations would yield

a set of functions. In a sense, therefore, the question of accuracy is one of luck, whether the assumed functions happen to be sufficiently close to those relevant to the case in question.

An immediate consequence of this is that no one method of this type can be described absolutely as the most accurate or least accurate. For example, Pohlhausen's method would probably be considered by almost everyone to be the least accurate of the five methods of type A, yet there is no doubt that pressure distributions could be devised for which Pohlhausen's method would yield more accurate solutions than any of the other methods. In practice, however, there will be limits to the range of pressure distributions which might arise, and accordingly in the range of variation of l , H , and L as functions of λ . It was on this basis that Thwaites calculated these functions for all known exact and approximate solutions, and then suggested values for general use which are 'average' in the sense of being far removed from the extreme limits of variation.

One of the important results obtained by Thwaites was that the functions obtained from the various solutions agree well for favourable pressure gradients ($\lambda > 0$), but differ as separation is approached in the presence of unfavourable pressure gradients ($\lambda < 0$). Thus it matters little which of the methods is used from the forward stagnation point to the pressure minimum, but predictions of boundary-layer development downstream of this point will depend vitally upon the method used. Since the various curves do not differ much in their overall general shape, it is roughly true that the method which yields the best prediction of the position of separation will also tend to yield the best overall picture of skin-friction distribution. It is therefore natural to compare the predicted positions of separation for cases in which the position has been accurately determined. This has been done, not only for the methods of type A but also for the other methods discussed. The results, expressed non-dimensionally by quoting the pressure coefficients at separation, are given in Table 8. It appears from this Table that the most accurate methods (at least as regards predictions of separation) are those of Stratford, Tani, and Head, and that there is little to choose between these. It is also apparent that of the methods of type A, those due to Timman and Thwaites are the most accurate.

With regard to predictions of momentum thickness it would probably matter relatively little which of the methods is used. In methods of type A the momentum thickness is determined mainly by assumptions

TABLE 8
Pressure coefficients at separation

$u_1(\xi)/u_0$	$1-\xi$	$1-\xi^2$	$1-\xi^4$	$1-\xi^8$	$\xi-\xi^3$	Schubauer's ellipse
<i>Method</i>						
Pohlhausen	0.305	0.192	0.118	0.067	..	..
Limman	0.230	0.140	0.080	0.043	..	0.072
Valz	0.204	0.119	0.067	0.035	..	0.049
Thwaites	0.231	0.138	0.080	0.047	0.047	0.069
Stratford	0.227	0.141	0.088	0.055	0.062	0.088
Tani	0.225	0.142	0.088	0.058	..	..
Truckenbrodt	0.215	..	..	..	..	..
Head	0.226	..	..	..	0.075	0.089
Accurate	0.226	0.142	0.089	0.056	0.056	0.082

nade concerning the function $L(\lambda)$ in equation (102), and, as Thwaites showed, this function does not vary much between various exact and approximate solutions. Thwaites's method predicts the momentum thickness to within 1 or 2 per cent. at all stations in the case of the inearily retarded velocity, but is in error by about 6 per cent. near to stagnation-point.

Predictions of displacement thickness are given from the momentum thickness upon multiplication by the shape factor H . Accuracy in predicting displacement thickness is thus dependent upon accuracy in the value of H . Methods of type A assume that H depends only on λ , and are accordingly of uncertain accuracy. The best is likely, *a priori*, to be that of Thwaites, since it is 'geared' to the known accurate solutions. Comparisons between Thwaites's method and the exact solutions indicate that displacement thickness is usually predicted fairly accurately except near to separation where errors of up to about 10 per cent. can often arise. On the other hand, such comparisons as are available indicate that Head's method predicts displacement thickness to an accuracy of say 2 per cent. or thereabouts. Although detailed comparisons are not at present available, one would expect that methods like those of Tani and Truckenbrodt would be intermediate in accuracy, since some attempt is made to improve upon Thwaites's assumption that H depends upon λ alone.

With regard to predictions of velocity profile, the only method which makes any serious attempt to consider this is that of Head, and we content ourselves here with the comment that the accuracy achieved is considerable.

. Ease of computation

By far the simplest method of those considered is that due to Stratford and Curle. To determine the skin-friction at a given position it is neces-

sary only to calculate the value of $x^2 C_p (dC_p/dx)^2$, after which the ratio of skin-friction to Blasius skin-friction follows by solution of (130), a simple quartic equation.

The methods of type A, particularly that of Thwaites, are also fairly simple to apply. For this method, given the velocity $u_1(x)$ at a number of stations, it is required that u_1^5 , u_1^6 , and du_1/dx be calculated, and a simple quadrature (107) then yields δ_2^2 . The remaining characteristics then follow from prescribed universal functions. Other than Thwaites's method, the methods of type A are rather longer, in that either non-integral powers of u_1 are required or the quadrature is replaced by numerical solution of a differential equation.

The methods of Tani and Truckenbrodt are also rather longer than that of Thwaites. Thus, although δ_2 is calculated as a function of x exactly as in Thwaites's method, the relationships between H and λ , l and λ , etc., must be estimated by approximate solution of the kinetic-energy integral equation.

Head's method is the longest of all, as the simple method above for calculating δ_2 is rejected, and the method requires the simultaneous step by step integration of two first-order ordinary differential equations.

4. Conclusions

In summing up what has been discussed above, we begin by remarking that for calculations of skin-friction alone the obvious choice is the method of Stratford and Curle, since this method is both as accurate as any other and considerably simpler than any other. By this method the skin-friction is given by equation (130).

If, further, either the momentum or displacement thickness is required it is suggested that these be calculated by Thwaites's method, provided that errors of up to about 5 per cent. in δ_2 and 10 per cent. in δ_1 are acceptable. If similar errors in skin-friction are accepted the whole calculation can be done by Thwaites's method.

If the above errors are rather too great, they may be somewhat reduced by using the slightly longer methods of either Tani or Truckenbrodt. The probability is that Tani's method is slightly more accurate and marginally simpler.

Finally, if greater accuracy than this is required, or if velocity profiles are called for, it is suggested that the whole calculation be done by Head's method.

One factor which has not been discussed above, which is fundamentally important and very relevant to any possible extensions of a method,

is the extent to which the approximations used correspond to physical reality. In this respect it should be remarked that the approximations of Stratford's method are physically realistic, and that Thwaites's quadrature for momentum thickness rests on a firm physical basis by virtue of the work of Leibenson and Truckenbrodt.

SOLUTIONS OF THE TEMPERATURE EQUATION AT LOW SPEEDS

IN low-speed flow, provided the difference between the temperature of the stream and that of the wall is not too great (so that the density is sensibly constant), the velocity and temperature equations may be solved separately, as was previously shown in Chapter 1, section 10. The equations governing the flow then become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \quad (146)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (147)$$

and

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = K \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p} \left(\frac{\partial u}{\partial y} \right)^2, \quad (148)$$

where the last term in (148), representing frictional heating, may usually be neglected. Then (146) and (147) may be solved to obtain the velocity field (u, v) , the methods discussed earlier being available for this purpose, and the linear equation (148) for T remains. We now consider some solutions of these equations.

1. Forced convection from a flat plate

We begin with uniform flow parallel to a heated flat plate where the flow has zero pressure gradient, and the velocity u_1 at the edge of the boundary layer is constant, u_0 . Then the solution of (146) and (147), discussed in Chapter 2, section 1, is

$$\left. \begin{aligned} u &= \frac{1}{2} u_0 f'(\eta) \\ v &= \frac{1}{2} \left(\frac{u_0 \nu}{x} \right)^{\frac{1}{2}} (\eta f' - f) \\ \eta &= \frac{1}{2} \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}} y \end{aligned} \right\}, \quad (149)$$

where $f(\eta)$ is to be regarded as a known function.

We write

$$\frac{T - T_w}{T_0 - T_w} = \theta, \quad (150)$$

where suffix w refers to values at the wall and suffix zero to (constant) values in the mainstream, and look for a solution in which θ is a function of η alone. Then, upon substituting for u , v , T from (149) and (150) into (148), we find (when the wall is at constant temperature)

$$\theta'' + \sigma f \theta' = 0, \quad (151)$$

after neglecting the frictional heating term. The boundary conditions, $T = T_w$ when $\eta = 0$, $T \rightarrow T_0$ as $\eta \rightarrow \infty$, when substituted into (150), become

$$\theta(0) = 0, \quad \theta \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty. \quad (152)$$

The solution of (151), subject to the boundary conditions (152), was given by E. Pohlhausen (1921), and is

$$\theta(\eta) = \alpha_0(\sigma) \int_0^\eta \exp\left\{-\sigma \int_0^\eta f d\eta\right\} d\eta \quad (153)$$

where

$$\alpha_0(\sigma) = \left\{ \int_0^\infty \exp\left(-\sigma \int_0^\eta f d\eta\right) d\eta \right\}^{-1}$$

Since $f(\eta)$ satisfies (64) we may write

$$f = -f'''/f'',$$

whence (153) becomes

$$\theta(\eta) = \alpha_0(\sigma) \int_0^\eta \left\{ \frac{f''(\eta)}{f''(0)} \right\}^\sigma d\eta. \quad (154)$$

Pohlhausen calculated $\alpha_0(\sigma)$ for a range of values of σ , which are shown in Table 9, with the approximate formula

$$\alpha_0(\sigma) \simeq 0.664\sigma^{\frac{1}{2}}. \quad (155)$$

TABLE 9
The function $\alpha_0(\sigma)$

σ	$\alpha_0(\sigma)$	$0.664\sigma^{\frac{1}{2}}$
0.6	0.552	0.560
0.7	0.585	0.589
0.8	0.614	0.616
0.9	0.640	0.641
1.0	0.664	0.664
1.1	0.687	0.685
7.0	1.29	1.26
10.0	1.46	1.43
15.0	1.67	1.64

The local heat-transfer rate to the plate,

$$Q_w(x) = k \left(\frac{\partial T}{\partial y} \right)_w,$$

is

$$\begin{aligned} Q_w(x) &= \frac{1}{2} \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}} k (T_0 - T_w) \left(\frac{\partial \theta}{\partial \eta} \right)_w \\ &= \frac{1}{2} \left(\frac{u_0}{\nu x} \right)^{\frac{1}{2}} k (T_0 - T_w) \alpha_0(\sigma). \end{aligned} \quad (156)$$

2. Temperature of plate thermometer in moving fluid

The problem here considered is that of determining the temperature which a flat-plate thermometer will read when no heat is being transferred to it from the fluid. Viscous heating is not neglected, as it is the only reason why the wall temperature rises above that of the stream. Since T_w is not prescribed it is convenient to define an alternative non-dimensional temperature, so we write

$$T = T_0 + \frac{u_0^2}{8c_p} \theta(\eta). \quad (157)$$

Then upon substituting into (148), including the frictional heating term, we find that

$$\theta'' + \sigma f \theta' = -2\sigma f'^2,$$

the boundary conditions being

$$\theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty,$$

and

$$\theta' = 0 \quad \text{when} \quad \eta = 0.$$

The solution, first given by E. Pohlhausen (1921), is

$$\theta(\eta) = 2\sigma \int_{\eta}^{\infty} (f'')^{\sigma} \left[\int_0^{\eta} (f'')^{2-\sigma} d\eta \right] d\eta, \quad (158)$$

so that the temperature of the wall is

$$\begin{aligned} T_w &= T_0 + \frac{u_0^2}{8c_p} \theta(0) \\ &= T_0 + \frac{u_0^2 \sigma}{4c_p} \int_0^{\infty} (f'')^{\sigma} \left[\int_0^{\eta} (f'')^{2-\sigma} d\eta \right] d\eta. \end{aligned} \quad (159)$$

The function $\theta(0)$ has been calculated by Pohlhausen, and is tabulated in Table 10, with the approximate interpolation formula

$$\theta(0) = 4\sigma^{\frac{1}{2}},$$

with which approximation (159) becomes

$$T_w = T_0 + \frac{u_0^2}{2c_p} \sigma^{\frac{1}{2}}.$$

TABLE 10
The function $\theta(0)$

σ	$\theta(0)$	$4\sigma^{\frac{1}{2}}$
0.6	3.08	3.09
0.7	3.34	3.34
0.8	3.58	3.58
0.9	3.80	3.79
1.0	4.00	4.00
1.1	4.20	4.19
7.0	10.06	10.6
10.0	11.86	12.6
15.0	14.14	15.5

Before passing on, two points are worthy of mention. Firstly, when $\sigma = 1$, (158) takes a particularly simple form, namely

$$\theta(\eta) = 4 - f'^2$$

so that the temperature distribution is given by (157) as

$$T = T_0 + \frac{1}{2c_p} (u_0^2 - u^2),$$

or
$$T + \frac{u^2}{2c_p} = \text{constant.} \quad (160)$$

This result, in fact, holds when $\sigma = 1$ for flow at arbitrary Mach number. Secondly, since the thermal energy equation is linear, solutions can be superimposed, so that for flow past a flat plate with heat transfer and frictional heating, the solution is a suitable linear combination of the solutions given in this section and section 1 of this chapter.

3. Heat transfer near a stagnation-point

Near to the forward stagnation-point of a cylinder the velocity distribution, as was shown in Chapter 2, section 2, may be written as

$$\left. \begin{aligned} u &= u_1 f'(\eta) = \beta_1 \frac{x}{c} f'(\eta) \\ v &= -\left(\frac{\nu \beta_1}{c}\right)^{\frac{1}{2}} f(\eta) \\ \eta &= \left(\frac{\beta_1}{\nu c}\right)^{\frac{1}{2}} y \end{aligned} \right\}, \quad (161)$$

and a solution sought by writing

$$\frac{T - T_w}{T_1 - T_w} = \theta(\eta), \quad (162)$$

where T_1 is the temperature in the mainstream, constant in low-speed flow. Upon neglecting frictional heating, substitution from (161) and (162) into (148) leads to the equation

$$\theta'' + \sigma f \theta' = 0,$$

the boundary conditions being

$$\theta = 0 \quad \text{when} \quad \eta = 0,$$

$$\text{and} \quad \theta \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty.$$

The solution is

$$\theta(\eta) = \alpha_1(\sigma) \int_0^\eta \left[\exp \left(-\sigma \int_0^\eta f d\eta \right) \right] d\eta, \quad (163)$$

$$\text{where} \quad \alpha_1(\sigma) = \left\{ \int_0^\infty \left[\exp \left(-\sigma \int_0^\eta f d\eta \right) \right] d\eta \right\}^{-1}.$$

This solution, due to Squire, is reported in the book by Goldstein (1938), together with values of $\alpha_1(\sigma)$ for a range of values of σ . These results are shown in Table 11, with the approximate form

$$\alpha_1(\sigma) \simeq 0.570\sigma^{0.4}.$$

TABLE 11

The function $\alpha_1(\sigma)$

σ	$\alpha_1(\sigma)$	$0.570\sigma^{0.4}$
0.6	0.466	0.465
0.7	0.495	0.495
0.8	0.521	0.521
0.9	0.546	0.546
1.0	0.570	0.570
1.1	0.592	0.592
7.0	1.18	1.24
10.0	1.34	1.43
15.0	1.54	1.68

The local heat-transfer rate to the cylinder in the region of the stagnation-point is

$$\begin{aligned} Q_w(x) &= k \left(\frac{\partial T}{\partial y} \right)_w \\ &= k(T_1 - T_w) \left(\frac{\beta_1}{\nu c} \right)^{\frac{1}{2}} \theta'(0) \\ &= k(T_1 - T_w) \left(\frac{\beta_1}{\nu c} \right)^{\frac{1}{2}} \alpha_1(\sigma). \end{aligned} \quad (164)$$

4. The solutions of Fage and Falkner

The solutions given in sections 1 and 3 of this chapter are two special cases of a more general class of solutions, considered by Fage and Falkner (1931). As before we write

$$\frac{T - T_w}{T_1 - T_w} = \theta,$$

where the wall temperature T_w may be a function of x . Then upon substitution into the temperature equation (148), and again neglecting frictional heating, we find

$$u \frac{\partial \theta}{\partial x} + u \frac{\theta - 1}{T_w - T_1} \frac{dT_w}{dx} + v \frac{\partial \theta}{\partial y} = \kappa \frac{\partial^2 \theta}{\partial y^2}. \quad (165)$$

Fage and Falkner consider the cases in which the mainstream velocity is given by

$$u_1 = \beta_m (x/c)^m,$$

so that, as in Chapter 2, section 3, the velocity distribution is given in terms of a stream-function, and

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

where

$$\psi = (u_1 \nu x)^{\frac{1}{2}} f(\eta), \quad \eta = \left(\frac{u_1}{\nu x} \right)^{\frac{1}{2}} y,$$

and $f(\eta)$ is the solution of (71). Fage and Falkner also assume that the wall temperature takes the form

$$T_w = T_1 + \epsilon (x/c)^n. \quad (166)$$

Then upon substituting into (165), and looking for a solution in which θ is a function of η alone, we find that

$$\theta'' + \sigma \left[\frac{1}{2}(m+1) f \theta' - n f' (\theta - 1) \right] = 0 \quad (167)$$

with $\theta(0) = 0$, $\theta(\infty) = 1$. Fage and Falkner integrated (167) for a range of values of m and n , for a Prandtl number $\sigma = 0.77$, and making the approximation that

$$f(\eta) \sim \frac{1}{2} \alpha \eta^2, \quad f'(\eta) \sim \alpha \eta,$$

that is, replacing the curve of u against y by its tangent at the wall. The value of α is, of course, proportional to the skin-friction.

For the case of a wall at uniform temperature, $n = 0$, and (167) is integrable for arbitrary m without further approximation, as Squire has pointed out, the result being shown in equation (175).

5. Lighthill's method

Lighthill (1950*a*) generalized the work of Fage and Falkner, subject to their approximation to the velocity profile, to the case of arbitrary mainstream velocity and arbitrary wall temperature. Lighthill begins by making a transformation of the von Mises form, with new independent variables (x, ψ) , where

$$\psi = \int_0^y \rho u \, dy \quad (168)$$

is the mass flow between a given point (x, y) and the wall $(x, 0)$. Then the energy equation (148), with frictional heating neglected, takes the form

$$\frac{\partial T}{\partial x} = \frac{\mu \rho}{\sigma} \frac{\partial}{\partial \psi} \left(u \frac{\partial T}{\partial \psi} \right). \quad (169)$$

Fage and Falkner's approximation to the velocity,

$$u = \frac{\tau_w(x)}{\mu} y, \quad (170)$$

is written by Lighthill in the form

$$u = \left(\frac{2\tau_w}{\mu \rho} \psi \right)^{\frac{1}{2}}, \quad (171)$$

so that (169) becomes

$$\frac{\partial T}{\partial x} = \frac{1}{\sigma} (2\mu \rho \tau_w)^{\frac{1}{2}} \frac{\partial}{\partial \psi} \left(\psi^{\frac{1}{2}} \frac{\partial T}{\partial \psi} \right). \quad (172)$$

This equation was integrated by Lighthill, using the Heaviside operational technique, and his value for the local heat transfer to the wall is

$$Q_w(x) = -k \left(\frac{\sigma \rho}{9\mu^2} \right)^{\frac{1}{2}} \frac{\{\tau_w(x)\}^{\frac{1}{2}}}{(\frac{1}{3})!} \int_0^x \left(\int_{x_1}^x \{\tau_w(z)\}^{\frac{1}{2}} dz \right)^{-\frac{1}{3}} dT_\epsilon(x_1), \quad (173)$$

where T_ϵ is defined as the difference between the wall temperature $T_w(x)$ and the mainstream temperature T_1 (constant at low Mach number), that is

$$T_\epsilon(x) = T_w(x) - T_1. \quad (174)$$

The integral in (173) is a Stieltjes integral, which may be regarded as a shorthand notation for

$$\int_0^x f(t, x) d\{g(t)\} = f(0, x)g(0) + \int_0^x f(t, x)g'(t) dt,$$

when the only discontinuity in $g(t)$ occurs at $t = 0$.

Lighthill points out that the approximation (171) is essentially a high Prandtl number approximation. As the Prandtl number increases the thermal boundary layer becomes thinner, relative to the velocity boundary layer, so that for the purpose of calculating the thermal boundary layer it becomes increasingly accurate to replace the velocity curve by its tangent at the wall. Accordingly (173) is expected to be asymptotically exact as the Prandtl number tends to infinity.

Lighthill tests the accuracy of (173) at a Prandtl number $\sigma = 0.7$, by comparison with the similarity solutions. With a mainstream velocity

$$u_1 = \beta_m(x/c)^m$$

and a uniform wall temperature T_w , it can easily be deduced from (167) that

$$\begin{aligned} \alpha_m(\sigma) &= -\frac{1}{T_w - T_1} \left(\frac{\nu x}{u_1} \right)^{\frac{1}{2}} \left(\frac{\partial T}{\partial y} \right)_w \\ &= \left\{ \int_0^\infty \exp \left[-\frac{m+1}{2} \sigma \int_0^\eta f d\eta \right] d\eta \right\}^{-1}, \end{aligned} \quad (175)$$

a function only of m and σ . Exact results for $\sigma = 0.7$ and various values of m are compared in Table 12 with values deduced by Lighthill from (173).

TABLE 12
The function $\alpha_m(0.7)$

m	α_m (accurate)	α_m (approx.)
-0.0904	0.199	0
-0.0654	0.253	0.232
0	0.292	0.300
$+\frac{1}{9}$	0.331	0.360
$\frac{2}{3}$	0.384	0.435
1	0.495	0.587
4	0.813	0.995

It is immediately clear that agreement is excellent, about 3 per cent. error, when $m = 0$ (zero pressure gradient), and is reasonable in the vicinity of a stagnation point, $m = 1$, where the error is about 19 per cent. On the other hand the agreement is poor near a position of separation, where $m = -0.0904$. Lighthill indicates the reason why the error varies with m in this manner. When $m = 0$ the velocity boundary-layer profile is essentially linear over a considerable portion of the boundary layer. For other values of m , where there is a curvature in the velocity profile at the wall, there is a greater deviation from the

tangent at the wall in the outer parts of the boundary layer. Thus when $m > 0$ the assumed velocity is greater than the true value, so the heat transfer is overestimated. For $m < 0$ the assumed velocity is less than the true value, and the heat transfer is underestimated. As separation is approached this effect is excessively accentuated, since Lighthill's assumption is equivalent to putting the velocity equal to zero everywhere.

The aerodynamically desirable range is $0 \leq m \leq 1$, since negative values of m lead to the possibility of boundary-layer separation, and further have velocity profiles with a point of inflexion and a resultant possibility of instability. Now in this range the error in the predicted heat transfer varies between +3 per cent. and +19 per cent. Lighthill points out that if the predicted heat transfer be multiplied by a factor 0.904, so that the numerical factor

$$9^{-1}\{(\frac{1}{3})!\}^{-1} = 0.538$$

in (173) is replaced by 0.487, the error will lie between ± 7 per cent. Accordingly the accuracy is good when this empirical correction is made.

6. The work of Liepmann and Curle

Liepmann (1958) has derived an expression for the heat transfer which is identical in form to Lighthill's result (173), and is obtained by use of the thermal energy integral equation, (55). This equation may be written in the form

$$\frac{d}{dx} \int_0^{\infty} u(T - T_1) dy = -\frac{Q_w(x)}{\rho c_p}, \quad (176)$$

where $Q_w(x)$ is the heat transfer to the wall

$$Q_w(x) = k \left(\frac{\partial T}{\partial y} \right)_w.$$

Liepmann, in place of Lighthill's approximation (170), writes the velocity near to the wall as

$$u = \frac{\tau_w(x) c_p (T - T_w)}{\sigma Q_w(x)},$$

and by writing

$$dy = \frac{k}{Q} dT,$$

(176) may be transformed into the form

$$Q_w(x) = -\frac{\sigma \rho}{\mu^2} k^3 \frac{d}{dx} \left(\frac{\tau_w}{Q_w^2} (T_w - T_1)^3 \int_0^1 \frac{\theta(1-\theta)}{Q/Q_w} d\theta \right), \quad (177)$$

where θ is the usual non-dimensional temperature as defined in equation (162). Liepmann then considers the special case in which the wall temperature is equal to its zero heat-transfer value T_1 when $x \leq x_0$, say, and increases discontinuously to a new constant value downstream of this position. In such a special case Q/Q_w is approximately a universal function of θ alone, so that (177) becomes

$$Q_w(x) = -a \frac{\sigma \rho k^3}{\mu^2} (T_w - T_1)^3 \frac{d}{dx} \left(\frac{\tau_w}{Q_w^2} \right), \quad (178)$$

where

$$a = \int_0^1 \frac{\theta(1-\theta)}{Q/Q_w} d\theta \quad (179)$$

is a constant. When τ_w is known as a function of x , (178) may easily be integrated to yield $Q_w(x)$. Liepmann expressed this integral analytically, and then derived the value of $Q_w(x)$ for an arbitrary wall temperature by integrating the contributions from a distribution of elementary changes similar to the above, and so obtained a result identical to (173), with the constant $9^{-1}\{(\frac{1}{3})!\}^{-1} = 0.538$ replaced by $(\frac{2}{3}a)^{\frac{1}{3}}$. We note that the value $a = 0.215$ is obtained by setting $Q/Q_w = (1-\theta^2)^{\frac{1}{2}}$ in (179), and this leads to a value $(\frac{2}{3}a)^{\frac{1}{3}} = 0.523$, which is within 3 per cent. of Lighthill's value. Liepmann also shows how the same approach may be used to derive a formula which is valid in the vicinity of separation.

Curle (1961*a*) has shown how, by an approach modelled on that of Liepmann, Lighthill's method may be considerably improved in accuracy. The thermal-energy integral equation (176) is used, and the velocity u approximated by

$$\mu u = \tau_w y + \frac{1}{2} \frac{dp}{dx} y^2. \quad (180)$$

This leads to the result

$$\frac{d}{dx} \left\{ \tau_w \int_0^\infty y(T - T_1) dy + \frac{1}{2} \frac{dp}{dx} \int_0^\infty y^2(T - T_1) dy \right\} = -\frac{\mu^2}{\rho k \sigma} Q_w(x). \quad (181)$$

In the special case of a region of zero heat transfer with $T_w = T_1$, followed by a discontinuity in wall temperature to a new constant value, the temperature profile is represented by a universal shape, characterized by a boundary-layer thickness δ which varies with

position. It follows that

$$\int_0^{\infty} y(T-T_1) dy = O\{(T_w-T_1)\delta^2\},$$

$$\int_0^{\infty} y^2(T-T_1) dy = O\{(T_w-T_1)\delta^3\},$$

and
$$Q_w = O\left\{\frac{k(T_w-T_1)}{\delta}\right\}.$$

Thus
$$\int_0^{\infty} y(T-T_1) dy = \frac{ak^2(T_w-T_1)^3}{Q_w^2}, \quad (182)$$

and
$$\int_0^{\infty} y^2(T-T_1) dy = -\frac{bk^3(T_w-T_1)^4}{Q_w^3}, \quad (183)$$

where a and b are positive constants. Substitution from (182) and (183) into (181) yields

$$\frac{d}{dx}\left\{\frac{ak^2\tau_w(T_w-T_1)^3}{Q_w^2} - \frac{b(dp/dx)k^3(T_w-T_1)^4}{2Q_w^3}\right\} = -\frac{\mu^2}{\rho k\sigma} Q_w,$$

which is to be compared with Liepmann's equation (178). An alternative way of writing this equation is to introduce a representative velocity U_{∞} and a representative length l , a local Nusselt number

$$Nu = -\frac{l}{T_w-T_1}\left(\frac{\partial T}{\partial y}\right)_w = -\frac{lQ_w}{k(T_w-T_1)},$$

and a Reynolds number

$$R = \frac{U_{\infty}l}{\nu}.$$

Then the value of $K = NuR^{-\frac{1}{2}}$

is given by solution of the equation

$$K = l \frac{d}{dx}\left\{\frac{A}{K^2} - \frac{B}{K^3}\right\}, \quad (184)$$

or
$$\int_0^x K d\left(\frac{x}{l}\right) = \frac{A}{K^2} - \frac{B}{K^3}, \quad (185)$$

where
$$A = a\sigma \frac{\tau_w}{\mu} \left(\frac{\nu l}{U_{\infty}^3}\right)^{\frac{1}{2}},$$

and
$$B = -\frac{1}{2}b\sigma \frac{l}{\rho U_{\infty}^2} \frac{dp}{dx}.$$

The constants a , b are determined by reference to the similarity solutions. The values

$$a = 0.2226, \quad b = 0.1046$$

are such that agreement is within about ± 1 per cent. over the whole range of pressure gradients from stagnation-point to separation, and with values of the Prandtl number of order unity (typical of gases) and order ten (typical of liquids). The numerical accuracy is, accordingly, an improvement on that given by Lighthill's method, although the presence of two terms on the right-hand side of (184) or (185) precludes analytic integration. A simple method of numerical integration, due to Thwaites, is to replace the integral in (185) by its Simpson's rule value, so that when $K(x)$ and $K(x+h)$ are known, $K(x+2h)$ follows by solution of a quartic algebraic equation.

It has been remarked earlier that the above approaches are essentially high Prandtl number approximations. Curle (1961*a*) has also examined the possibility of deriving acceptable solutions at Prandtl numbers of order unity by empirical correction of a low Prandtl number solution. At low values of the Prandtl number the temperature boundary layer is thicker than the velocity boundary layer, and Curle replaces the local velocity u in (176) by βu_1 , where β will tend to unity as the Prandtl number tends to zero. He also writes

$$\int_0^\infty (T - T_1) dy = -\frac{a_0 k (T_w - T_1)^2}{Q_w},$$

with a_0 constant, in the special case of a single discontinuous jump in wall temperature. Then substitution into (176) yields

$$\frac{d}{dx} \left(\frac{u_1}{Q_w} \right) = \frac{\nu Q_w}{\beta a_0 k^2 \sigma (T_w - T_1)^2},$$

which may be integrated and transformed to yield

$$Q_w(x) = -k \left(\frac{\beta a_0 \sigma}{2\nu} \right)^{\frac{1}{2}} u_1(x) \left(\int_0^x u_1 dx \right)^{-\frac{1}{2}} (T_w - T_1).$$

The generalization to the case of arbitrary distribution of wall temperature is exactly as described by Liepmann, and yields the Steiltjes integral

$$Q_w(x) = -k \left(\frac{\beta a_0 \sigma}{2\nu} \right)^{\frac{1}{2}} u_1(x) \int_0^x \left\{ \int_\xi^x u_1 dx \right\}^{-\frac{1}{2}} dT_\epsilon(\xi), \quad (186)$$

where T_ϵ is defined as $T_w - T_1$, as in (174). The form of this equation

is in agreement with that obtained by Morgan *et al.* (1958), who showed that the asymptotically exact value of $(\frac{1}{2}\beta a_0)^{\frac{1}{2}}$ as $\sigma \rightarrow 0$ is

$$(\frac{1}{2}\beta a_0)^{\frac{1}{2}} = \pi^{-\frac{1}{2}} = 0.5642.$$

Curle remarks that when this constant is changed to 0.313, the formula (186) yields values of the heat transfer agreeing with the similar solutions to ± 25 per cent. Provided that the velocity boundary layer is not too thin in comparison with the thermal boundary layer Curle shows that a considerable improvement may be effected by writing

$$(\frac{1}{2}\beta a_0)^{\frac{1}{2}} = 0.5642 - 0.265\sigma^{\frac{1}{2}}(1 - 5\lambda),$$

where λ is the pressure gradient parameter, defined by (108), which may be calculated by means of equations (107) and (108).

Curle has applied these two methods to the case of flow past a heated cylinder, and compared his results with the experiments of Schmidt and Wenner (1941). The agreement is good in each case, the solution (186) (derived from the low σ approximation) giving particularly good agreement.

7. Spalding's method

An alternative method of improving Lighthill's method was given by Spalding (1958). His method begins with the equation (180) for u ,

$$\mu u = \tau_w y + \frac{1}{2} \frac{dp}{dx} y^2,$$

and he remarks that the correction to Lighthill's method will depend upon the relative importance of the second term on the right-hand side. Now at a position which is representative of the thickness of the thermal boundary layer, say

$$y = -\frac{k(T_w - T_1)}{Q_w},$$

the ratio of the second term to the first is

$$-\frac{1}{2}\chi = -\frac{k(T_w - T_1)(dp/dx)}{2Q_w \tau_w} \quad (187)$$

Accordingly he considers the case of a single step in wall temperature, writes (178) as

$$-\frac{\sigma \rho k^3}{\mu^2} \frac{(T_w - T_1)^3}{\tau_w^{\frac{1}{2}}} \frac{d}{dx} \left(\frac{\tau_w^{\frac{1}{2}}}{Q_w^3} \right) = \text{constant},$$

and suggests that the right-hand side of this equation be replaced by a function of the parameter χ defined in (187), so that it becomes

$$-\frac{\sigma \rho k^3}{\mu^2} \frac{(T_w - T_1)^3}{\tau_w^{\frac{1}{2}}} \frac{d}{dx} \left(\frac{\tau_w^{\frac{1}{2}}}{Q_w^3} \right) = F(\chi), \quad (188)$$

where χ is the expression $(\Delta_4 \delta_4 / \nu)(du_1/dx)$ of Spalding's paper. The function F is determined by reference to the available exact similar solutions. It is shown as a graph in Spalding's paper, and is tabulated here in Table 13. Spalding suggests the solution of (188) by iteration,

TABLE 13
The function $F(\chi)$

χ	$F(\chi)$
-4	3.5
-3	3.8
-2	4.3
-1	5.1
0	6.4
+1	8.5
2	11.6
3	15.8

so that a first approximation to Q_w , obtained by setting $F = 6.4$, is used to determine a better approximation to F , and so on. For the case in which $T_w = T_1$ when $x \leq \xi$, and $T_w = T_1 + \Delta T_\epsilon(\xi)$ when $x \geq \xi$, this yields the result

$$Q_w(x) = -k \left(\frac{\sigma \rho}{\mu^2} \right)^{\frac{1}{3}} \tau_w^{\frac{1}{3}} \left(\int_{\xi}^x \tau_w^{\frac{1}{3}} F(\chi_1) dx \right)^{-\frac{1}{3}} \Delta T_\epsilon(\xi),$$

where χ_1 is the value of χ obtained by using the previous approximation to Q_w . The solution for arbitrary wall temperature is obtained by adding the solutions for the relevant distribution of the above elementary increments in wall temperature.

It should be remarked at this stage that, although the method will work effectively (and yield an improvement in accuracy as compared with Lighthill's method) provided that F does not differ too much from 6.4, it will still break down as separation approaches and the parameter $\chi \rightarrow \infty$.

Spalding goes on to show that if only the total heat-transfer rate is required, it is not necessary to calculate (and then integrate) the local values. A quicker method is to set up an equivalent relationship to (188), which will allow this to be done directly. Spalding takes the thermal energy integral equation (55), and integrates it with respect to x to obtain

$$\begin{aligned} \int_0^x Q_w(x) dx &= -\rho c_p \int_0^\infty u(T - T_1) dy \\ &= -\rho c_p u_1(T_w - T_1) \Delta_2, \end{aligned}$$

where Δ_2 is a measure of boundary-layer thickness. Considering, as usual, the single-step distribution of wall temperature, Spalding shows that (173), which is Lighthill's approximation, may be written as

$$\frac{\sigma\rho}{(\mu\tau_w)^{\frac{1}{2}}} \frac{d}{dx} \{(u_1\Delta_2)^{\frac{3}{2}}\} = 0.725,$$

and that the appropriate generalization may be expressed as

$$\frac{\sigma\rho}{(\mu\tau_w)^{\frac{1}{2}}} \frac{d}{dx} \{(u_1\Delta_2)^{\frac{3}{2}}\} = 0.7 - 0.2 \left(\frac{\mu\rho^2 u_1^3}{\tau_w^3} \Delta_2 \right)^{\frac{1}{2}} u'_1. \quad (189)$$

The first step in an iterative solution of this equation is

$$u_1\Delta_2 = \left\{ 0.7 \frac{\nu}{\sigma} \int_{\xi}^x \left(\frac{\tau_w}{\mu} \right)^{\frac{1}{2}} dx - \frac{0.2}{\sigma} \int_{\xi}^x (u_1\Delta_2)^{\frac{1}{2}} \frac{\mu u_1 u'_1}{\tau_w} dx \right\}^{\frac{2}{3}}, \quad (190)$$

where again a breakdown occurs as separation is approached.

8. The method of Davies and Bourne

We consider now briefly one further method which has been suggested for improving the accuracy of Lighthill's method, due to Davies and Bourne (1956). Their method is to assume that the velocity profile $u(x, \psi)$ is known, and may be approximated by the expression

$$u(x, \psi) \simeq u_1(x) \cdot b_1 \psi^{\beta}, \quad (191)$$

so that

$$u(x, y) \propto y^{\beta/(1-\beta)}. \quad (192)$$

The values of b_1 and β are determined by the conditions of the particular problem under consideration, so that the exact and approximate velocity profiles are in good agreement over the greater part of the thermal boundary layer. As $\sigma \rightarrow \infty$ we have $\beta = \frac{1}{2}$, as in Lighthill's approximation. When considering boundary-layer flow with σ of order unity, Davies and Bourne chose b_1 and β to give good agreement over most of the velocity boundary layer, when applying their method to the case of the similar solutions. The temperature equation in the von Mises form (169), after substituting for u from (191), yields

$$\frac{\partial T}{\partial x} = \frac{\mu\rho}{\sigma} b_1 u_1 \frac{\partial}{\partial \psi} \left(\psi^{\beta} \frac{\partial T}{\partial \psi} \right),$$

which is a form amenable to analytic treatment. Davies and Bourne's results indicate that the percentage error in the predicted heat-transfer rate is roughly equal to a typical percentage error in the assumed velocity profile.

The method is neat, but does not appear to be of general use, since it relies upon a knowledge of the detailed velocity profile, which cannot

easily be estimated with any accuracy. Further difficulties arise if insufficient evidence is available concerning the relative extents of viscous and thermal boundary layers.

9. The Meksyn-Merk method

We now turn our attention to a method which was devised by Meksyn, applied originally to a number of viscous boundary-layer problems, and subsequently adapted by Merk (1959) for calculating thermal boundary layers. The applications to viscous boundary layers will not be discussed here, and the reader is referred to the book by Meksyn (1961) for details and for further references to the method.

The method as used to calculate thermal boundary layers may be illustrated by reference to the problem of section 1 of this chapter, namely forced convection from a flat plate. Consider the function $\alpha_0(\sigma)$, given by (153) as

$$\frac{1}{\alpha_0(\sigma)} = \int_0^\infty \exp\left(-\sigma \int_0^\eta f d\eta\right) d\eta.$$

We introduce the function $F(\eta)$ defined as

$$F(\eta) = \int_0^\eta f d\eta,$$

$$\text{so that} \quad \frac{1}{\alpha_0(\sigma)} = \int_0^\infty \exp(-\sigma F) d\eta. \quad (193)$$

Now by (65) we have

$$f(\eta) = \frac{1}{2!} \alpha \eta^2 - \frac{1}{5!} \alpha^2 \eta^5 + \frac{11}{8!} \alpha^3 \eta^8 \dots, \quad (194)$$

$$\text{so that} \quad F(\eta) = \frac{1}{3!} \alpha \eta^3 - \frac{1}{6!} \alpha^2 \eta^6 + \frac{11}{9!} \alpha^3 \eta^9 \dots, \quad (195)$$

where $\alpha = 1.32824$ and the series both converge for a finite range of η . The series (195) may be inverted to give

$$\eta = \left(\frac{6F}{\alpha}\right)^{\frac{1}{3}} \left\{1 + \frac{1}{60}F - \frac{1}{1260}F^2 \dots\right\}.$$

Then, upon substitution into (193) we have

$$\frac{1}{\alpha_0(\sigma)} = \left(\frac{6}{\alpha}\right)^{\frac{1}{3}} \int_0^\infty \exp(-\sigma F) \left\{\frac{1}{3}F^{-\frac{2}{3}} + \frac{1}{45}F^{\frac{1}{3}} - \frac{1}{540}F^{\frac{4}{3}} \dots\right\} dF.$$

This series may be integrated formally term by term, and when the reciprocal of the resulting series is calculated it is found that

$$\begin{aligned}\alpha_0(\sigma) &= \left(\frac{\alpha\sigma}{6}\right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{4}{3}\right) \right\}^{-1} \left\{ 1 - \frac{1}{45}\sigma^{-1} + \frac{2}{675}\sigma^{-2} \dots \right\} \\ &= 0.6775\sigma^{\frac{1}{3}} \left\{ 1 - \frac{1}{45}\sigma^{-1} + \frac{2}{675}\sigma^{-2} \dots \right\}. \quad (196)\end{aligned}$$

The limited range of convergence of the series (194) means that (196) is in the nature of an asymptotic expansion. However, even the three terms shown here are sufficient to give excellent results when $\sigma \geq \frac{1}{2}$. When $\sigma = \frac{1}{2}$, these three terms yield $\alpha_0(0.5) = 0.520$, compared with the exact value 0.518, and when $\sigma = 0.8$ the three terms yield three correct decimals in $\alpha_0(\sigma)$.

The above indicates in general terms the principles on which the method is based. For more general cases Meksyn and Merk make first a transformation of the variables which is virtually identical to that used by Görtler (1957*a*), so that the equations for the velocity and temperature functions take a form in which the above type of asymptotic expansion can be applied.

By developments along these lines, it has been possible to consider the boundary layer for Schubauer's experimental pressure distribution (Meksyn, 1956), and for retarded flow along a semi-infinite plane (Meksyn, 1950), the exact solutions to these problems being given by Hartree (1939*b*) and Howarth (1938) respectively. For the thermal boundary-layer problem Merk (1959) has derived solutions corresponding to the similar solutions of E. Pohlhausen and Squire, considered in sections 1 and 3 of this chapter. He has also considered more general cases of flow past circular and elliptic cylinders. For the case of a circular cylinder, the predicted heat transfer over the front 70° of the cylinder agrees exceedingly well with experiment, but the method breaks down as separation is approached at about 80°. Meksyn (1960) has also shown how accurate solutions for the plate-thermometer problem (section 2) may be obtained for large values of σ (> 100) where straightforward calculations of the integral (159) are inaccurate.

10. Curle's analysis by Stratford's method

It has been remarked earlier that one of the simplest and most accurate methods devised for calculating the skin-friction in an incompressible laminar boundary layer is that due to Stratford (1954), the method being based on the idea of dividing the boundary layer into inner and outer portions. Curle (1961*b*) has shown how a similar technique may be used for calculating the corresponding property for

the thermal boundary layer, namely the heat-transfer rate, when the wall is at uniform temperature.

Following the procedure adopted in the case of the viscous boundary layer, the thermal boundary layer is divided into an inner and an outer region. In the outer region the thermal energy equation is written as

$$u \frac{\partial T}{\partial s} = \kappa \frac{\partial^2 T}{\partial y^2}, \quad (197)$$

where $(\partial/\partial s)$ denotes differentiation at constant ψ , defined by (110). We also expand $T(x, \psi)$ in the form

$$T(x, \psi) = T(x_0, \psi) + (x - x_0) \left(\frac{\partial T}{\partial s} \right)_{x_0, \psi} + O(x - x_0)^2. \quad (198)$$

We consider the special case in which there is zero pressure gradient upstream of a position $x = x_0$. The downstream velocity profile is distorted by the pressure gradient, but only near to the wall initially. Accordingly we deduce from (197) that the value of $(\partial T/\partial s)_{x, \psi}$ is unaffected by the pressure gradient, except in the region near to the wall, so (198) shows that when $\psi > \psi'_j$, say, $T(x, \psi)$ is exactly as it would have been in the absence of pressure gradient, provided terms of order $(x - x_0)^2$ are neglected. Now it may be readily deduced from Pohlhausen's solution (154) that in the absence of pressure gradient

$$T_B = T_w + (T_1 - T_w)\theta(\eta),$$

and (anticipating the approximation below) this may be written

$$T_B(x, \psi) = T_w + \sigma^{\frac{1}{2}}(T_1 - T_w) \left(\frac{2\tau_B \psi}{\mu u_0^2} \right)^{\frac{1}{2}} \quad (199)$$

for small values of ψ , where $\sigma^{\frac{1}{2}}$ arises from the approximation (155). There is no *a priori* reason why ψ_j (determining the join in the inner and outer velocity profiles) and ψ'_j (determining the join of the temperature profiles) should be equal. It is assumed, however, that both are sufficiently small that the Blasius velocity profile remains sensibly linear throughout $\psi \leq \psi_j$ and $\psi \leq \psi'_j$. It follows from (199) that the value of $T(x, \psi'_j)$ given from the outer profile is

$$T(x, \psi'_j) = T_w + \sigma^{\frac{1}{2}}(T_1 - T_w) \left(\frac{2\tau_B \psi'_j}{\mu u_0^2} \right)^{\frac{1}{2}}, \quad (200)$$

and upon differentiating with respect to ψ

$$\left(\frac{\partial T}{\partial y} \right)_{x, \psi'_j} = \sigma^{\frac{1}{2}}(T_1 - T_w) u'_j \left(\frac{\tau_B}{2\mu u_0^2 \psi'_j} \right)^{\frac{1}{2}}. \quad (201)$$

Turning now to the inner region, $\psi \leq \psi'_j$, we assume a form

$$T = T_w + \frac{Q_w}{k} y, \quad (202)$$

which should be a good approximation, since both $\partial^2 T / \partial y^2$ and $\partial^3 T / \partial y^3$ are identically zero at $y = 0$. The values of T and $\partial T / \partial y$ at the join, as given by (202), may be equated to the values (200) and (201), yielding

$$\frac{Q_w}{k} y'_j = \sigma^{\frac{1}{2}} (T_1 - T_w) \left(\frac{2\tau_B}{\mu u_0^2} \psi'_j \right)^{\frac{1}{2}} \quad (203)$$

and
$$\frac{Q_w}{k} = \sigma^{\frac{1}{2}} (T_1 - T_w) u'_j \left(\frac{\tau_B}{2\mu u_0^2 \psi'_j} \right)^{\frac{1}{2}}. \quad (204)$$

These two equations connecting Q_w , ψ'_j , u'_j , y'_j must be supplemented by relationships giving ψ'_j and u'_j in terms of y'_j . To do this it is assumed for simplicity that the expression (126) is adequate throughout $\psi \leq \psi'_j$. This means, in reality, that $\psi'_j \leq \psi_j$ but since the inner and outer solutions for u have continuous values of u , $\partial u / \partial y$, and $\partial^2 u / \partial y^2$, it is possible that (126) holds for values of ψ rather greater than ψ_j . Then

$$\mu u'_j = \tau_w y'_j + \frac{1}{2} \frac{dp}{dx} y_j'^2 + a y_j'^n, \quad (205)$$

and
$$\mu \psi'_j = \frac{1}{2} \tau_w y_j'^2 + \frac{1}{6} \frac{dp}{dx} y_j'^3 + \frac{a}{n+1} y_j'^{n+1}, \quad (206)$$

with τ_w and a known from the solution for the velocity profile. The value of Q_w may be determined from (203) to (206), and is

$$Q_w = \sigma^{\frac{1}{2}} \frac{k(T_1 - T_w)}{\mu u_0} \tau_B \left\{ 0.6642 + 0.3358 \frac{\tau_w}{\tau_B} \right\}^{\frac{1}{2}}. \quad (207)$$

Accordingly, once the skin-friction τ_w has been determined, the heat-transfer rate is given by this explicit formula. This, of course, is by far the simplest method of calculating heat transfer. The limitations of the method are that it can only be used when the wall temperature is uniform and the pressure gradients are adverse.

To estimate the accuracy, we remark that the formula (207) is exact when there is zero pressure gradient. For small pressure gradients the accuracy remains good. Thus, if the mainstream velocity is

$$u_1 = u_0 \left(1 - \frac{x}{c} \right),$$

the heat-transfer rate is predicted from (207) and (130) to be

$$Q_w = \frac{k(T_1 - T_w)}{\mu u_0} \tau_B \left\{ 1 - 2.13 \frac{x}{c} \dots \right\},$$

whereas the exact solution replaces the coefficient of x/c by 1.86. We should expect the accuracy to decrease as we proceed farther from the accurately known starting conditions, that is as separation is approached. At separation (207) yields

$$Q_w = 0.815\sigma^{\frac{1}{2}} \frac{k(T_1 - T_w)}{\mu u_0} \tau_B, \quad (208)$$

the numerical coefficient being equal to 0.72 when $\sigma = 0.7$. For the similarity solution which corresponds to a separating boundary layer the numerical coefficient is $0.600(u_1/u_0)^{\frac{1}{2}}$, which might typically equal about 0.57. The agreement is still fairly good.

11. Squire's method

We shall now give a brief discussion of a method due to Squire (1942), modelled on the integral methods first used by K. Pohlhausen (1921) for the viscous layer. Squire's method rests initially upon the idea that the 'shape' of the temperature profile in a laminar boundary layer does not vary much over a wide range of pressure gradients. Since the temperature profile is the same as the Blasius velocity profile in the case of zero pressure gradient, we may suppose therefore that

$$\frac{T - T_1}{T_w - T_1} \simeq 1 - \frac{1}{2} f' \left(\frac{0.8604y}{\Delta_1} \right), \quad (209)$$

where

$$\frac{u}{u_1} = \frac{1}{2} f' \left(\frac{0.8604y}{\delta_1} \right) \quad (210)$$

is the Blasius velocity profile, $\frac{1}{2}f'(\eta)$ being tabulated in Table 1. We then assume that the velocity profile is always given by (210), with the displacement thickness δ_1 suitably chosen. Squire chooses

$$\delta_1^2 = \frac{2.960\nu}{u_1^6} \int_0^x u_1^5 dx, \quad (211)$$

which is the value given by an approximate method due to Young and Winterbottom (1942). Then, substituting from (209) and (210) into the thermal energy integral equation (55), we have

$$\rho c_p \frac{d}{dx} \int_0^\infty \frac{1}{2} u_1 f' \left(\frac{0.8604y}{\delta_1} \right) \left(1 - \frac{1}{2} f' \left(\frac{0.8604y}{\Delta_1} \right) \right) dy = -\frac{1}{2} k \frac{0.8604}{\Delta_1} f''(0),$$

which may be written as

$$\frac{d}{dx} \left\{ u_1 \frac{\Delta_1}{0.8604} \int_0^\infty f' \left(\frac{\Delta_1}{\delta_1} \eta \right) \{1 - \frac{1}{2} f'(\eta)\} d\eta \right\} = \frac{1.143\kappa}{\Delta_1},$$

or
$$\frac{d}{dx} \{2u_1 \Delta_1 \phi\} = \frac{1.143\kappa}{\Delta_1}, \quad (212)$$

where $\phi(\Delta_1/\delta_1)$, which was calculated by Squire, is defined as

$$2\phi = \frac{1}{0.8604} \int_0^\infty f' \left(\frac{\Delta_1}{\delta_1} \eta \right) \{1 - \frac{1}{2} f'(\eta)\} d\eta.$$

Upon integrating (212) we find that

$$u_1^2 \Delta_1^2 \phi^2 = 1.143\kappa \int_0^x u_1 \phi \, dx,$$

and when divided by (211) this yields

$$\frac{\Delta_1^2}{\delta_1^2} \phi \left(\frac{\Delta_1}{\delta_1} \right) = \frac{0.3861}{\sigma} \frac{u_1^4 \int_0^x u_1 \phi \, dx}{\phi \int_0^x u_1^5 \, dx}, \quad (213)$$

which is an equation for Δ_1/δ_1 .

Squire suggests that a good first approximation is to take ϕ as a constant on the right-hand side of (213), which may then easily be calculated as a function of x . Then the values of Δ_1/δ_1 may be determined from a table of the function $(\Delta_1^2/\delta_1^2)\phi(\Delta_1/\delta_1)$, which is given here as Table 14. A second approximation, if required, is obtained by using the resulting values of ϕ to recalculate the right-hand side of (213).

TABLE 14

The function $\eta^2\phi(\eta)$

η	$\phi(\eta)$	$\eta^2\phi(\eta)$
$\frac{1}{2}$	0.208	0.052
$\frac{5}{8}$	0.257	0.100
$\frac{3}{4}$	0.272	0.121
$\frac{5}{6}$	0.332	0.230
1	0.386	0.386
$\frac{5}{4}$	0.456	0.713
$\frac{3}{2}$	0.499	1.018
$\frac{5}{3}$	0.548	1.522
$\frac{2}{1}$	0.575	1.901
2	0.599	2.398

Squire compares the results of his method with some of the exact 'similar' solutions over a range of values of σ . He finds agreement

better than 1 per cent. as regards heat transfer in the absence of pressure gradient, but the predicted heat transfer is about 5 per cent. too low near a stagnation-point. A similar comparison for the case of the separation profile indicates that the predicted heat transfer is 30 per cent. too high. Squire has also used his method to calculate the heat transfer in flow past a circular cylinder, obtaining reasonable agreement with experiment.

12. Free convection from a heated vertical plate

To conclude this chapter we consider the problem of free convection when a heated plate is placed in still air. It is assumed that the temperature difference between the plate and the air is small, so that the fluid properties may be taken as constant, but that the (small) motion of the air is caused by a buoyancy force due to density variations. Then, with x measured upwards from the lower edge of the plate, y normal to it, u , v , the associated velocity components, T_w , T_0 , the temperature of the plate and the air respectively, the equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (214)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g \left(\frac{T}{T_0} - 1 \right), \quad (215)$$

and

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \frac{\partial^2 T}{\partial y^2}. \quad (216)$$

Upon introducing

$$\theta = \frac{T - T_0}{T_w - T_0},$$

(215) and (216) become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g \frac{T_w - T_0}{T_0} \theta \quad (217)$$

and

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \kappa \frac{\partial^2 \theta}{\partial y^2}. \quad (218)$$

These partial differential equations may be reduced to ordinary differential equations, as was first shown by E. Pohlhausen (1921). We write

$$\eta = \left\{ \frac{g(T_w - T_0)}{4\nu^2 T_0} \right\}^{\frac{1}{4}} \frac{y}{x^{\frac{1}{4}}} = C \frac{y}{x^{\frac{1}{4}}},$$

$$\psi(x, y) = 4\nu C x^{\frac{3}{4}} f(\eta),$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

$$\theta(x, y) = g(\eta),$$

so that $u = 4\nu C^2 x^{\frac{1}{2}} f'(\eta), \quad v = \nu C x^{-\frac{1}{2}} (\eta f' - 3f).$ (219)

Then (217) and (218) become

$$\left. \begin{aligned} f''' + 2ff'' - 2f'^2 + g &= 0 \\ g'' + 3\sigma fg' &= 0 \end{aligned} \right\}, \quad (220)$$

with boundary conditions

$$\left. \begin{aligned} f = f' &= 0, & g &= 1, & \eta &= 0 \\ f' = g &= 0, & \eta &\rightarrow \infty \end{aligned} \right\}. \quad (221)$$

These equations were integrated numerically by Pohlhausen, with the Prandtl number $\sigma = 0.733$. From his results we deduce that

$$\frac{1}{T_w - T_0} \left(\frac{\partial T}{\partial y} \right)_w = -0.508 \left(\frac{g(T_w - T_0)}{4\nu^2 T_0 x} \right)^{\frac{1}{4}}, \quad (222)$$

giving the heat-transfer rate at the plate.

THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH ZERO PRESSURE GRADIENT

THE essential difficulty in dealing with compressible laminar boundary layers is that the momentum and thermal energy equations are coupled, since the temperature field influences the density, which in turn influences the velocity field. Mathematically we are faced with the immense task of solving two simultaneous non-linear partial differential equations. For this reason the only flow which has been considered in detail over a wide range of conditions is that when there is zero pressure gradient. For such a flow the velocity u_1 , the temperature T_1 , and the density ρ_1 outside the boundary layer are constant. Each of these quantities is thus equal to its value at the leading edge, $x = 0$, and we shall accordingly denote them by suffix 0. The boundary-layer equations, (1), (12), and (19), then become, upon neglecting body forces

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (223)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right), \quad (224)$$

and

$$\rho u \frac{\partial}{\partial x} (c_p T + \tfrac{1}{2} u^2) + \rho v \frac{\partial}{\partial y} (c_p T + \tfrac{1}{2} u^2) = \frac{\partial}{\partial y} \left(\mu \frac{\partial}{\partial y} \left(\frac{c_p T}{\sigma} + \tfrac{1}{2} u^2 \right) \right). \quad (225)$$

Before proceeding farther it is necessary to discuss what assumptions are to be made concerning the values of μ and σ , and their dependence upon the various other properties of the fluid.

1. Values for viscosity and Prandtl number

An accurate representation of the variation of viscosity with absolute temperature is given by Sutherland's formula

$$\mu \propto \frac{T^{\frac{3}{2}}}{T + c}, \quad (226)$$

where c is a constant, taken to equal 114°K for air. The complexity of the compressible laminar boundary-layer equations is such that it is often necessary to use a simpler formula such as

$$\mu \propto T^\omega. \quad (227)$$

By suitable choice of ω and of the constant of proportionality it is possible to make (227) agree well with the accurate value (226) over a fairly wide range of temperatures. For example, $\omega = 0.76$ has often been used, and Cope and Hartree (1948) pointed out that good agreement in the range $90^\circ \text{K} \leq T \leq 300^\circ \text{K}$ could be obtained by taking $\omega = 8/9$. Considerable simplifications in the boundary-layer equations can be effected by taking $\omega = 1$, and this fact has been used to good effect in many theoretical investigations.

The value of the Prandtl number, σ , does not vary much with temperature for air, and a constant value of about 0.72 or 0.73 has usually been assumed. It must be pointed out, however, that considerable simplifications result from the choice $\sigma = 1$, and theories have often been developed on this basis.

We can easily see from equation (225) that when $\sigma = 1$ the thermal energy equation always has a solution

$$c_p T + \frac{1}{2} u^2 = \text{constant}. \quad (228)$$

This solution corresponds to the case of a thermally insulated wall, whose temperature is constant when $\sigma = 1$. For the case of zero pressure gradient we see further, by comparison between (224) and (225), that (225) always has a solution

$$c_p T + \frac{1}{2} u^2 = (\text{constant})u,$$

provided the wall is at uniform temperature. The general solution is then

$$c_p T + \frac{1}{2} u^2 = A + Bu,$$

and since $T = T_w$, $u = 0$, when $y = 0$, and $T = T_0$, $u = u_0$, when $y = \infty$, this becomes

$$T + \frac{u^2}{2c_p} = T_w + \left(T_0 + \frac{u_0^2}{2c_p} - T_w \right) \frac{u}{u_0},$$

$$\text{or} \quad T + \frac{u^2}{2c_p} = T_w + (T_z - T_w) \frac{u}{u_0}, \quad (229)$$

introducing the temperature T_z at which there would be zero heat transfer at the wall.

We note that when $\sigma = 1$, regardless of the viscosity-temperature relationship, the solution of (225) is given by (229) for the case of zero pressure gradient, and it remains only to solve (224) for u . We now consider various attempts to do this, with different assumptions regarding viscosity, as well as some solutions with values of σ other than unity.

2. The solutions of Busemann and Kármán

Busemann (1935) assumed $\sigma = 1$, and so deduced as above that T is a function only of u . He then reduced the equation (224) for u to an ordinary differential equation with $\eta = \frac{1}{2}y(u_0/\nu_w x)^{\frac{1}{2}}$ as independent variable, the form being similar to that of the Blasius equation. This equation was then integrated numerically for the case of Mach number $M_0 = 8.8$ and zero heat transfer at the wall.

As a result of these calculations he found that the velocity varies almost linearly with distance from the wall at this Mach number. Since a linear velocity profile is a fair approximation even in incompressible flow von Kármán (1935) considered the consequences of assuming a linear velocity profile at all Mach numbers, and calculated approximate solutions by the Kármán-Pohlhausen integral method. Zero heat transfer was taken, of course, and the Prandtl number was set equal to unity, so that the temperature profile was given by (229), with $T_w = T_z$, namely,

$$T = T_w - \frac{u^2}{2c_p}$$

or
$$T = T_0 \left\{ 1 + \frac{\gamma-1}{2} M_0^2 \left(1 - \frac{u_1^2}{u_0^2} \right) \right\}.$$

Now when there is no pressure gradient, the momentum integral equation (40) may be written as

$$\mu_w \left(\frac{\partial u}{\partial y} \right)_w = \frac{d}{dx} \left(\int_0^\infty \rho u (u_0 - u) dy \right).$$

Upon substituting for ρ ,

$$\rho = \rho_0 \left\{ 1 + \frac{\gamma-1}{2} M_0^2 \left(1 - \frac{u_1^2}{u_0^2} \right) \right\}^{-1},$$

setting

$$u = u_0 y / \delta,$$

$$\lambda^2 = \frac{\frac{1}{2}(\gamma-1)M_0^2}{1 + \frac{1}{2}(\gamma-1)M_0^2},$$

and

$$f(\lambda) = \int_0^1 \frac{z(1-z)}{1-\lambda^2 z^2} dz,$$

we deduce that
$$\mu_w \frac{u_0}{\delta} = \rho_0 u_0^2 \frac{f(\lambda)}{1 + \frac{1}{2}(\gamma-1)M_0^2} \frac{d\delta}{dx}. \quad (230)$$

If we let the viscosity law be $\mu \propto T^\omega$, then μ_w is related to the mainstream value μ_0 by

$$\mu_w = \mu_0 \left(\frac{T_w}{T_0} \right)^\omega = \mu_0 \left(1 + \frac{1}{2}(\gamma-1)M_0^2 \right)^\omega,$$

so that (230) integrates to give

$$\delta = \left\{ \frac{2\nu_0 x}{u_0 f(\lambda)} \right\}^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}(\gamma - 1) M_0^2 \right\}^{\frac{1}{2}(1+\omega)},$$

from which we deduce that the skin friction is

$$\tau_w = \left(\mu \frac{\partial u}{\partial y} \right)_w = \frac{\mu_w u_0}{\delta} = \left\{ \frac{\mu_0 \rho_0 u_0^3 f(\lambda)}{2x} \right\}^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}(\gamma - 1) M_0^2 \right\}^{\frac{1}{2}(\omega-1)}. \quad (231)$$

This is effectively the form in which Kármán presented his results. We note, however, that it may alternatively be written as

$$\tau_w = \left\{ \frac{\mu_w \rho_w u_0^3 f(\lambda)}{2x} \right\}^{\frac{1}{2}}. \quad (232)$$

We note that $\left\{ \frac{1}{2} f(\lambda) \right\}^{\frac{1}{2}}$ increases from 0.290 when $\lambda = 0$, i.e. $M_0 = 0$, to 0.393 when $\lambda = 1$, i.e. $M_0 \rightarrow \infty$. The accurate values are 0.332 and 0.400 respectively, so the accuracy is considerable at high Mach numbers, decreasing from 13 per cent. at zero Mach number to less than 2 per cent. as the Mach number tends to infinity.

3. The solutions of Kármán and Tsien

In view of the relatively large error at smaller Mach numbers, Kármán and Tsien (1938) later attempted a more accurate solution of the equations of motion. They began with the boundary-layer equations in the von Mises form (59), (60), and in the absence of pressure gradient these become

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi} \left(\mu \rho u \frac{\partial u}{\partial \psi} \right), \quad (233)$$

$$\text{and} \quad \frac{\partial}{\partial x} (c_p T + \frac{1}{2} u^2) = \frac{\partial}{\partial \psi} \left(\mu \rho u \frac{\partial}{\partial \psi} \left(\frac{c_p T}{\sigma} + \frac{1}{2} u^2 \right) \right). \quad (234)$$

Upon taking $\sigma = 1$, the solution of (234) is given by (229) as

$$T + \frac{u^2}{2c_p} = T_w + (T_\infty - T_w) \frac{u}{u_0}, \quad (235)$$

so that, once the wall temperature T_w has been prescribed, the temperature distribution is given explicitly in terms of u , which may then be determined from (233). To solve this equation, Kármán and Tsien introduce non-dimensional quantities,

$$u^* = \frac{u}{u_0}, \quad x^* = \frac{x}{c}, \quad \mu^* = \frac{\mu}{\mu_0}, \quad \rho^* = \frac{\rho}{\rho_0}, \quad T^* = \frac{T}{T_0},$$

and

$$\psi^* = \frac{\psi}{(\mu_0 \rho_0 u_0 x)^{\frac{1}{2}}}$$

where suffix zero refers to (uniform) values in the mainstream, and c is a representative length. Then (233) becomes

$$-\frac{1}{2}\psi^* \frac{du^*}{d\psi^*} = \frac{d}{d\psi^*} \left(\mu^* \rho^* u^* \frac{du^*}{d\psi^*} \right), \quad (236)$$

which equation may be solved by iteration. As a first approximation the function $u^*(\psi^*)$ appropriate to the incompressible Blasius solution is taken, with T^* deduced from (235) and μ^* , ρ^* , from the relationships

$$\mu^* = (T^*)^{0.76}, \quad (237)$$

and

$$\rho^* = 1/T^*, \quad (238)$$

this last result following immediately from the equation of state (7), since p is constant. Hence the product

$$\mu^* \rho^* u^* = f(\psi^*)$$

is known. Equation (236) then yields

$$u^* = C \int_0^{\psi^*} \frac{F}{f} d\psi^*, \quad (239)$$

where

$$F = \exp \left\{ - \int_0^{\psi^*} \frac{\psi^* d\psi^*}{2f} \right\},$$

and

$$C^{-1} = \int_0^{\infty} \frac{F}{f} d\psi^*.$$

This improved value of $u^*(\psi^*)$ may then be used as a starting-point for the next approximation. When $u^*(\psi^*)$ has been determined to sufficient accuracy, the value of the true coordinate y normal to the wall is calculated as

$$y = \int_0^{\psi} \frac{d\psi}{\rho u} = \left(\frac{\mu_0 x}{\rho_0 u_0} \right)^{\frac{1}{2}} \int_0^{\psi^*} \frac{d\psi^*}{\rho^* u^*}. \quad (240)$$

Kármán and Tsien calculated solutions for flow with zero heat transfer over a range of Mach numbers from 0 to 10. The values of the skin-friction coefficient are shown in Fig. 1, together with the results of other workers. They also carried out some calculations for cases in which the wall temperature was kept constant at one-quarter of the mainstream temperature. For details of these results reference may be made to the original paper.

4. The solutions of Emmons and Brainerd

Some solutions for zero heat transfer, with various viscosity-temperature laws and at several values of the Prandtl number, were given by Emmons and Brainerd (1941, 1942). These authors introduced a number of non-dimensional functions, defined as

$$u^* = \frac{u}{u_0}, \quad v^* = v \left(\frac{x}{u_0 \nu_0} \right)^{\frac{1}{2}}, \quad T^* = \frac{T}{T_0},$$

$$\mu^* = \frac{\mu}{\mu_0} = \frac{k}{k_0}, \quad \eta = y \left(\frac{u_0}{\nu_0 x} \right)^{\frac{1}{2}},$$

and showed that the equations of motion (223) to (225) could be reduced to three ordinary differential equations for u^* , T^* , and

$$\xi = \eta u^* - 2v^*$$

as functions of η . These equations take the form

$$T^* \left\{ \frac{d\xi}{d\eta} - u^* \right\} = \xi \frac{dT^*}{d\eta}, \quad (241)$$

$$\xi \frac{du^*}{d\eta} + 2T^* \frac{d}{d\eta} \left(\mu^* \frac{du^*}{d\eta} \right) = 0, \quad (242)$$

$$\text{and} \quad \frac{d\xi}{d\eta} - u^* + \frac{2}{\sigma} \frac{d}{d\eta} \left(\mu^* \frac{dT^*}{d\eta} \right) + 2(\gamma - 1) M_0^2 T^* \left(\frac{du^*}{d\eta} \right)^2 = 0. \quad (243)$$

These equations were then integrated on a differential analyser under the following assumptions.

Firstly, a range of solutions was obtained for $\mu^* = 1$, that is for constant viscosity and conductivity, equivalent to taking $\omega = 0$ in the viscosity-temperature relationship (227). These solutions were for values of the Prandtl number $\sigma = 0, 0.25, 0.733, 1.00, 1.20$, respectively, and at Mach numbers $M_0 = 0, 0.5, 1, 1.5, 2$, and $\sqrt{10}$. Some of the results for the variation of skin-friction with Mach number are shown in Fig. 1. An interesting conclusion which may be drawn from the results of these calculations is that when there is zero heat transfer the relationship between wall temperature T_w and mainstream temperature T_0 , is given with considerable accuracy by

$$T_w = T_0 \left\{ 1 + \frac{1}{2}(\gamma - 1) M_0^2 \sigma^{\frac{1}{2}} \right\}, \quad (244)$$

which is in agreement with the result obtained by E. Pohlhausen for low-speed flow.

In their second paper, Emmons and Brainerd initially took $\sigma = 0.733$ throughout, and examined various possible viscosity-temperature laws.

They concluded that the choice $\omega = 0.768$, close to the value (237) used by Kármán and Tsien, was the most satisfactory, and subsequently used only this relationship. They then calculated velocity and temperature distributions at various Mach numbers, and with the same Prandtl numbers as they used in their first paper. Some of these further results have also been included in the skin-friction coefficients in Fig. 1.

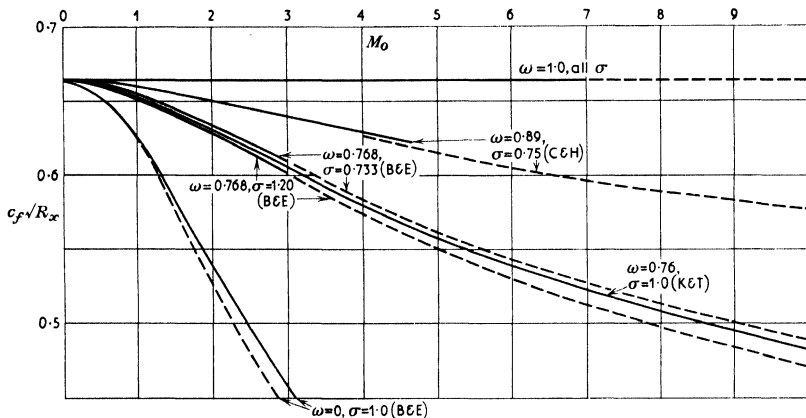


FIG. 1. Skin friction on a flat plate with no heat transfer.

— Calculated by various workers; - - according to equation (272). C & H, Cope and Hartree; K & T, von Kármán and Tsien; B & E, Brainerd and Emmons.

5. The calculations of Crocco

An exhaustive investigation of the laminar boundary layer with zero pressure gradient has been carried out by Crocco (1946). Upon setting the pressure gradient equal to zero and applying the Crocco transformation (Chapter 1, section 11), the equations (57), (58) become

$$u \frac{\partial}{\partial x} \left(\frac{\mu \rho}{\tau} \right) + \frac{\partial^2 \tau}{\partial u^2} = 0, \quad (245)$$

$$\text{and} \quad (1 - \sigma) \frac{\partial I}{\partial u} \tau \frac{\partial \tau}{\partial u} + \left(\frac{\partial^2 I}{\partial u^2} + \sigma \right) \tau^2 - \sigma \mu \rho u \frac{\partial I}{\partial x} = 0, \quad (246)$$

the dependent variables being τ and I , the independent variables x and u . Crocco looks for a solution in which the velocity and temperature profiles are each similar for different values of x (though not similar to each other). Thus I is a function of u alone, and τ may be expressed as

$$\tau = \phi(x)g(u).$$

After some straightforward analysis, Crocco deduces that $\phi \propto x^{-\frac{1}{2}}$, and accordingly writes

$$\tau = \frac{g(u)}{(2x)^{\frac{1}{2}}}, \quad (247)$$

whence substitution into (245) and (246) leads to

$$gg'' + \mu \rho u = 0, \quad (248)$$

and

$$(I'' + \sigma)g + (1 - \sigma)I'g' = 0, \quad (249)$$

where dashes denote derivatives with respect to u . Non-dimensional quantities, similar to those used in previously discussed methods, are introduced, so that

$$\left. \begin{aligned} u^* &= \frac{u}{u_0}, & I^* &= \frac{I}{I_0}, & \rho^* &= \frac{\rho}{\rho_0}, & \mu^* &= \frac{\mu}{\mu_0} = \frac{k}{k_0} \end{aligned} \right\}. \quad (250)$$

and

$$G(u^*) = \left(\frac{2}{\mu_0 \rho_0 u_0^3} \right)^{\frac{1}{2}} g(u)$$

Equations (248) and (249) then take the form

$$GG'' + 2\mu^* \rho^* u^* = 0, \quad (251)$$

and

$$\left(I^{*''} + \sigma \frac{u_0^2}{I_0} \right) G + (1 - \sigma) I^{*'} G' = 0, \quad (252)$$

primes now denoting derivatives with respect to u^* . The boundary conditions are

$$G' = 0, \quad I^* = \frac{I_w}{I_0} \quad \text{at} \quad u^* = 0,$$

and

$$G = 0, \quad I^* = 1 \quad \text{at} \quad u^* = 1.$$

We note that

$$\rho^* = (I^*)^{-1}$$

and

$$\mu^* = (I^*)^\omega$$

are functions of I^* alone.

Crocco considered first solutions with $\sigma = 1$. When this holds (252) becomes simply

$$I^{*''} + \frac{u_0^2}{I_0} = 0,$$

so that the relationship between I^* and u^* is quadratic, as has previously been indicated by other methods. It then follows that (251) takes the form

$$GG'' + h(u^*) = 0 \quad (253)$$

where h is a known function of u^* . This equation was solved by Crocco by a similar iterative method to that used by Kármán and Tsien when

they solved (236). Thus we have

$$G(u^*) = \int_{u^*}^1 \left\{ \int_0^{u^*} \frac{h}{G} du^* \right\} du^*, \quad (254)$$

and a first approximation G_1 , substituted into the right-hand side of this equation, leads to a second approximation G_2 . Crocco remarked that this procedure would not, in general, converge but that it would do so rapidly if either $\frac{1}{2}(G_1 + G_2)$ or $(G_1 G_2)^{\frac{1}{2}}$ were taken as the starting value for the next iteration.

Having thus considered the case of $\sigma = 1$, Crocco turned to the case $\omega = 1$, $\sigma \neq 1$. When this holds equation (251) takes the form

$$GG'' + 2u^* = 0, \quad (255)$$

so that G is independent of I^* and σ . It is fairly straightforward to transform this equation so that it becomes identical with the Blasius equation for incompressible flow. The required transformation is

$$G = \frac{1}{2} \frac{d^2 f}{d\eta^2}, \quad u^* = \frac{1}{2} \frac{df}{d\eta}, \quad (256)$$

whence (255) may be shown to become

$$f''' + ff'' = 0. \quad (257)$$

The boundary conditions are easily shown to be such that the relevant solution of (257) is that given in Chapter 2 for the Blasius profile. It follows from (255) that G as a function of u^* is simply the known relationship for $\frac{1}{2}f''$ as a function of $\frac{1}{2}f'$.

In particular, the skin-friction is given by (247) and (250) as

$$\tau_w = \frac{g(0)}{(2x)^{\frac{1}{2}}} = \left(\frac{\mu_0 \rho_0 u_0^3}{4x} \right)^{\frac{1}{2}} G(0),$$

and by (256), substituting the value of $f''(0)$ from (67), this yields

$$\tau_w = 0.33206 \left(\frac{\mu_0 \rho_0 u_0^3}{x} \right)^{\frac{1}{2}} \quad (258)$$

which may alternatively be written as

$$\left(\frac{\partial u}{\partial y} \right)_w = 0.33206 \left(\frac{u_0^3}{\nu_w x} \right)^{\frac{1}{2}}, \quad (259)$$

since $\mu \propto T$.

Once this relationship is known, it remains to solve (252) for I^* . Since this is a linear first-order equation for $I^{*'}$, it is a straightforward

procedure to obtain I^* by integration, and amongst the various forms in which the result can be expressed is

$$I^*(u^*) = 1 - I^{*'}(0)\Delta_1(\sigma, u^*) + \sigma \frac{u_0^2}{J_0} \Delta_2(\sigma, u^*), \quad (260)$$

where Δ_1 , Δ_2 are two functions, calculated by Crocco, and shown in Table 15 for various values of σ . They are defined as

$$\Delta_1(\sigma, u^*) = \int_{u^*}^1 \left[\frac{G(z)}{G(0)} \right]^{\sigma-1} dz,$$

$$\text{and} \quad \Delta_2(\sigma, u^*) = \int_{u^*}^1 \left\{ \left[\frac{G(z_1)}{G(0)} \right]^{\sigma-1} \int_0^{z_1} \left[\frac{G(z_2)}{G(0)} \right]^{1-\sigma} dz_2 \right\} dz_1.$$

In view of (256) these may alternatively be expressed in terms of the Blasius functions. Crocco showed, in fact, that $8\sigma\Delta_2$ is the function $\theta(\eta)$ defined by equation (158) and calculated by E. Pohlhausen (1921). An alternative method of expressing (260), which is more immediately useful when $I^{*'}(0)$ is unknown, is

$$I^*(u^*) = I^*(0) - \{I^*(0) - 1\}\Delta_3(\sigma, u^*) + \frac{u_0^2}{J_0} \Delta_4(\sigma, u^*), \quad (261)$$

$$\text{where} \quad \Delta_3 = 1 - \frac{\Delta_1(\sigma, u^*)}{\Delta_1(\sigma, 0)} \quad (262)$$

$$\text{and} \quad \Delta_4 = \sigma \left\{ \Delta_2(\sigma, u^*) - \frac{\Delta_2(\sigma, 0)}{\Delta_1(\sigma, 0)} \Delta_1(\sigma, u^*) \right\}. \quad (263)$$

For the more general case, when $\omega \neq 1$, $\sigma \neq 1$, it is necessary to integrate (251) and (252) simultaneously, and this was done by Crocco for a large number of cases, over a range of Mach numbers from 0 to 5, with $\omega = 0.5, 0.75, 1$, and 1.25 , and also in a number of cases in which the viscosity-temperature law was given by a parametric form of Sutherland's law (226). An important conclusion from these calculations is that the variation of enthalpy with velocity, that is the function $I^*(u^*)$, is virtually independent of which viscosity-temperature law is assumed. For this reason Crocco suggested that the relationship (260) or (261) could be assumed to hold in general, even when $\omega \neq 1$, so that the solution of (251) alone remains to be determined. This does depend, though, upon the viscosity-temperature law, and the accurate value should be used. The solution of the equation is formally exactly as in the case $\sigma = 1$, since $\mu^*\rho^*$ is known as a function of I^* , and hence of u^* and σ , and (251) takes the form

$$GG'' + h(\sigma, u^*) = 0.$$

This is similar to (253), and may be solved by an analogous procedure, so that we write

$$G(\sigma, u^*) = \int_{u^*}^1 \left\{ \int_0^{u^*} \frac{h}{G} du^* \right\} du^*,$$

and substitute an approximation G_1 in the right-hand side to obtain a second approximation G_2 . Then either $\frac{1}{2}(G_1 + G_2)$ or $(G_1 G_2)^{\frac{1}{2}}$ is the starting-point for the next iteration.

This considerable simplification of the simultaneous equations, (251) and (252), was used by van Driest (1952), who carried out extensive calculations with $\sigma = 0.75$, Mach numbers from 0 to 20 and wall-to-free-stream temperature ratios, $I^*(0)$, from 0.25 to 6.

We may deduce from Crocco's work, and in particular from (260), that the relationship between heat transfer and wall temperature, that is between $I^*(0)$ and $I^{*'}(0)$, is

$$I^*(0) = 1 - I^{*'}(0)\Delta_1(\sigma, 0) + \sigma \frac{u_0^2}{I_0} \Delta_2(\sigma, 0),$$

$$\text{or} \quad I^{*'}(0) = \frac{1 - I^*(0) + \sigma(u_0^2/I_0)\Delta_2(\sigma, 0)}{\Delta_1(\sigma, 0)}. \quad (264)$$

Now

$$\begin{aligned} Q_w &= \left(k \frac{\partial T}{\partial y} \right)_w = \frac{k_w}{c_p} \left(\frac{\partial I}{\partial u} \right)_w \left(\frac{\partial u}{\partial y} \right)_w = \frac{I_0}{\sigma u_0} \tau_w I^{*'}(0) \\ &= \frac{I_0}{u_0} \tau_w \left\{ \left(1 - \frac{T_w}{T_0} \right) \frac{1}{\sigma \Delta_w} + \frac{u_0^2}{I_0} \frac{\Delta_{20}}{\Delta_{10}} \right\}, \end{aligned} \quad (265)$$

where suffix zero on Δ_1 or Δ_2 denotes values at $u^* = 0$. Accordingly, when there is zero heat transfer the wall temperature $T_w = T_z$ is given by

$$0 = \left(1 - \frac{T_z}{T_0} \right) \frac{1}{\sigma} + \frac{u_0^2}{I_0} \Delta_{20},$$

or

$$\begin{aligned} T_z &= T_0 \{ 1 + (\gamma - 1) M_0^2 \sigma \Delta_{20} \} \\ &\simeq T_0 \{ 1 + \frac{1}{2} (\gamma - 1) M_0^2 \sigma^{\frac{1}{2}} \}, \end{aligned} \quad (266)$$

since it may be deduced from the results of Crocco, van Driest, and Brainerd and Emmons, and from the results of E. Pohlhausen for low speeds, that

$$\Delta_2(\sigma, 0) \simeq \frac{1}{2} \sigma^{\frac{1}{2}}.$$

A similar method to that of Crocco was used by Hantzsche and Wendt (1940, 1942) to consider cases with and without heat transfer. Their results are mainly for either $\omega = 1$ or $\sigma = 1$, and, as they are less comprehensive than those of Crocco, they will not be considered further here.

We will, however, refer briefly to an approximate analysis by Monaghan (1949) for the relationship between enthalpy and velocity at arbitrary σ close to unity. Monaghan makes the major assumption that enthalpy and velocity are related exactly as if each depended only upon local conditions, and deduces that

$$\frac{T}{T_0} \simeq \frac{T_w}{T_0} + \frac{T_z - T_w}{T_0} \sigma^{\frac{1}{2}} \frac{u}{u_0} - \frac{1}{2} \sigma (\gamma - 1) M_0^2 \left(\frac{u}{u_0} \right)^2. \quad (267)$$

Now it is not difficult to express (260) or (261) in this form. We write (261) as

$$\frac{T}{T_0} = \frac{T_w}{T_0} + \left(1 - \frac{T_w}{T_0} \right) \Delta_3 + (\gamma - 1) M_0^2 \Delta_4,$$

and add and subtract the quantity

$$\left(\frac{T_z}{T_0} - 1 \right) \Delta_3 = (\gamma - 1) M_0^2 \sigma \Delta_{20} \Delta_3,$$

whereupon we have

$$\frac{T}{T_0} = \frac{T_w}{T_0} + \frac{T_z - T_w}{T_0} \Delta_3 + (\gamma - 1) M_0^2 (\Delta_4 - \sigma \Delta_{20} \Delta_3).$$

By making use of (262) and (263) this may be simplified to

$$\frac{T}{T_0} = \frac{T_w}{T_0} + \frac{T_z - T_w}{T_0} \Delta_3 + (\gamma - 1) M_0^2 \sigma (\Delta_2 - \Delta_{20}). \quad (268)$$

Comparison with (267) shows that Monaghan's solution replaces

$$\Delta_3 \text{ by } \sigma^{\frac{1}{2}} u/u_0 \quad \text{and} \quad \Delta_2 - \Delta_{20} \text{ by } \frac{1}{2} (u/u_0)^2.$$

Monaghan compares these values for two cases, $\sigma = 0.725$ and $\sigma = 1.25$. In each case he finds agreement within 1 per cent. for values of u/u_0 less than 0.5, the error increasing to about 3 or 4 per cent. at $u/u_0 = 0.8$, and the order of 10 per cent. at $u/u_0 = 1.0$. The formula given by Monaghan is exact when $\sigma = 1$, and in general we can remark that for values of σ close to 1 the approximation is extremely good except perhaps at the very edge of the boundary layer.

6. Summary of results for uniform wall temperature

All the solutions considered so far in this chapter are for flows in which the wall is maintained at a uniform temperature. Taken together they provide a comprehensive set of results for values of σ between 0.5 and 2 and values of ω between 0 and 1.25, both with and without heat transfer. Young (1949) has carried out a detailed analysis of these results, and has succeeded in developing simple formulae for skin-friction and heat transfer which agree with the accurately calculated

values to within about 1 per cent. over the physically important range of the parameters considered.

Young expresses his results as a formula for the product $c_f\sqrt{R_x}$, where

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_0 u_0^2}$$

and

$$R_x = \frac{u_0 x}{\nu_0}$$

are the skin-friction coefficient and Reynolds number, respectively. He deduces from (258) that

$$c_f\sqrt{R_x} = 0.664 \quad \text{when} \quad \omega = 1, \quad (269)$$

and shows from the results of Hantzsche and Wendt that

$$c_f\sqrt{R_x} \simeq 0.664\{1 + 0.3(\gamma - 1)M_0^2 \sigma^{\frac{1}{2}}\}^{\frac{1}{2}(\omega - 1)}, \quad (270)$$

where the coefficient 0.3 and the factor $\sigma^{\frac{1}{2}}$ are approximate. These two formulae apply when there is zero heat transfer. For the case of heat transfer, Young obtains a rough solution of (251), which indicates that when $\sigma = 1$

$$c_f\sqrt{R_x} \simeq 0.664\{0.5 + 0.5(T_w/T_0) + 0.1(\gamma - 1)M_0^2\}^{\frac{1}{2}(\omega - 1)}, \quad (271)$$

where the numerical coefficients within the brackets are approximate. By examination of (269) to (271) we conclude that for arbitrary ω , σ , and T_w , the value of $c_f\sqrt{R_x}$ may be expressed as

$$c_f\sqrt{R_x} \simeq 0.664\{a + b(T_w/T_0) + c(\gamma - 1)M_0^2 \sigma^{\frac{1}{2}}\}^{\frac{1}{2}(\omega - 1)},$$

where a , b , c are roughly equal to

$$a = 0.5, \quad b = 0.5, \quad c = 0.1.$$

Young adjusts the values of these constants to get the best overall agreement with all of the available results, and thus finds that

$$c_f\sqrt{R_x} = 0.664\{0.45 + 0.55(T_w/T_0) + 0.09(\gamma - 1)M_0^2 \sigma^{\frac{1}{2}}\}^{\frac{1}{2}(\omega - 1)}. \quad (272)$$

The accuracy of this equation is considerable, since it may be shown to agree with the exact values to within about 1 per cent. over the range of the parameters likely to arise in practice with air as the working fluid, as is illustrated in Fig. 1.

It is now easy to derive a similar formula for the heat transfer rate. From (265) and (266) we may deduce that

$$Q_w = \frac{k_w \tau_w}{\mu_w u_0} (T_z - T_w) \frac{1}{\Delta_{10}},$$

and, since Δ_{10} may be shown to be approximately $\sigma^{-\frac{1}{2}}$, this yields

$$\frac{\mu_w Q_w}{k_w \tau_w} = \frac{(\partial T / \partial y)_w}{(\partial u / \partial y)_w} \simeq \frac{T_z - T_w}{u_0} \sigma^{\frac{1}{2}}. \quad (273)$$

To conclude the work on the compressible laminar boundary layer on a flat plate with uniform wall temperature we should make mention of a paper by Mack (1958) who has provided a considerable catalogue of results for the case of zero heat transfer over a range of Mach numbers from 0.4 to 5, with viscosity and thermal conductivity prescribed accurately as functions of temperature from experimental data, and the Prandtl number therefore not absolutely constant.

7. The solutions of Chapman and Rubesin

All the preceding solutions apply only when the wall is at a uniform temperature. The problem becomes considerably more difficult when T_w is allowed to vary with x in a prescribed manner. The first serious attempt to consider the high-speed boundary layer on a flat plate with non-uniform wall temperature was made by Chapman and Rubesin (1949), though some solutions for low-speed flow were given earlier by Fage and Falkner (1930).

Chapman and Rubesin begin by writing the boundary-layer equations in the von Mises form, (59) and (60). Upon approximating

$$\mu \rho = \mu_0 \rho_0 C,$$

which is equivalent to taking $\mu \propto T$ with the constant of proportionality left arbitrary, and then setting the pressure gradient equal to zero, these equations become

$$\frac{\partial u}{\partial x} = \mu_0 \rho_0 C \frac{\partial}{\partial \psi} \left(u \frac{\partial u}{\partial \psi} \right) \quad (274)$$

$$\text{and} \quad \frac{\partial I}{\partial x} = \frac{1}{\sigma} \mu_0 \rho_0 C \frac{\partial}{\partial \psi} \left(u \frac{\partial I}{\partial \psi} \right) + \mu_0 \rho_0 C u \left(\frac{\partial u}{\partial \psi} \right)^2. \quad (275)$$

They then introduce non-dimensional quantities

$$u^* = \frac{u}{u_0}, \quad T^* = \frac{T}{T_0} = \frac{I}{I_0}, \quad \mu^* = \frac{\mu}{\mu_0}, \quad \rho^* = \frac{\rho}{\rho_0},$$

$$x^* = \frac{x}{L}, \quad \text{and} \quad \psi^* = \frac{\psi}{(\mu_0 \rho_0 u_0 L C)^{\frac{1}{2}}},$$

and upon substituting into (274) and (275) this leads to

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial}{\partial \psi^*} \left(u^* \frac{\partial u^*}{\partial \psi^*} \right) \quad (276)$$

and
$$\frac{\partial T^*}{\partial x^*} = \frac{1}{\sigma} \frac{\partial}{\partial \psi^*} \left(u^* \frac{\partial T^*}{\partial \psi^*} \right) + (\gamma - 1) M_0^2 u^* \left(\frac{\partial u^*}{\partial \psi^*} \right)^2. \quad (277)$$

We note that the momentum equation (276) is now uncoupled from the thermal-energy equation, (277), so the solution of the former is simply the Blasius solution,

$$u^* = \frac{1}{2} f'(\eta),$$

where

$$\eta = \psi^* (x^*)^{-\frac{1}{2}}.$$

This suggests a further change of independent variable from (x^*, ψ^*) to (x^*, η) , whence (277) becomes

$$\frac{\partial^2 T^*}{\partial \eta^2} + \sigma f \frac{\partial T^*}{\partial \eta} - 2\sigma f' x^* \frac{\partial T^*}{\partial x^*} = -\frac{1}{4} \sigma (\gamma - 1) M_0^2 (f'')^2. \quad (278)$$

Now the general solution of this equation is the sum of a particular integral and the complementary function. For a particular integral we may choose a function of η alone,

$$T^* = N(\eta),$$

where

$$N'' + \sigma f N' = -\frac{1}{4} \sigma (\gamma - 1) M_0^2 (f'')^2,$$

which is very similar to Pohlhausen's equation. A solution is

$$N(\eta) = 1 + \frac{1}{8} (\gamma - 1) M_0^2 \theta(\eta),$$

where $\theta(\eta)$ is given by (159) and

$$\begin{aligned} N(0) &= 1 + \frac{1}{8} (\gamma - 1) M_0^2 \theta(0) \\ &\doteq 1 + \frac{1}{2} (\gamma - 1) M_0^2 \sigma^{\frac{1}{2}} = \frac{T_z}{T_0}. \end{aligned} \quad (279)$$

The complementary function is a solution of

$$\frac{\partial^2 T^*}{\partial \eta^2} + \sigma f \frac{\partial T^*}{\partial \eta} - 2\sigma f' x^* \frac{\partial T^*}{\partial x^*} = 0,$$

and Chapman and Rubesin write this solution as

$$T^* = Y,$$

where

$$Y = \sum_0^{\infty} a_n x^{*n} Y_n(\eta),$$

and Y_n is the solution of the equation

$$Y_n'' + \sigma f Y_n' - 2\sigma f' n Y_n = 0 \quad (280)$$

with boundary conditions

$$Y_n(0) = 1, \quad Y_n(\infty) = 0.$$

It follows that

$$Y(0) = \sum_0^{\infty} a_n x^{*n}. \quad (281)$$

Hence, when the wall temperature is given by the sum of (279) and (281), that is

$$\frac{T_w}{T_0} = \frac{T_z}{T_0} + \sum_0^{\infty} a_n x^{*n}, \quad (282)$$

the temperature profile is given by

$$\frac{T}{T_0} = 1 + \frac{1}{8}(\gamma - 1)M_0^2 \theta(\eta) + \sum_0^{\infty} a_n x^{*n} Y_n(\eta). \quad (283)$$

In particular, the heat-transfer rate is given as

$$\begin{aligned} Q_w &= \left(k \frac{\partial T}{\partial y} \right)_w = k_w T_0 \left(\frac{\partial \eta}{\partial y} \right)_w \sum_0^{\infty} a_n x^{*n} Y'_n(0) \\ &= \frac{1}{2} k_0 T_0 \left(\frac{C u_0}{\nu_0 x} \right)^{\frac{1}{2}} \sum_0^{\infty} a_n x^{*n} Y'_n(0). \end{aligned} \quad (284)$$

It should be remarked that the above work holds for non-integral values of n . Taking $\sigma = 0.72$, Chapman and Rubesin integrated equations (280) for several values of n , giving curves of the functions $Y_n(\eta)$. In particular they gave the values of $Y'_n(0)$ which are shown in Table 16. These values may be interpolated for intermediate values of n , to yield a solution for any wall temperature distributions T_w^* which can be adequately approximated as a sum of arbitrary powers of x^* .

TABLE 16†
Exact and approximate values of $\frac{1}{2}Y'_n(0)$

n	$-\frac{1}{2}Y'_n(0)$ (exact)	$-\frac{1}{2}Y'_n(0)$ (approx.)
0	0.296	0.304
1	0.489	0.490
2	0.597	0.594
3	0.678	0.671
4	0.744	0.734
5	0.799	0.787
10	1.006	0.987

† Reprinted from the *Journal of the Aeronautical Sciences*, vol. 16, p. 547, by permission of the Institute of the Aero/Space Sciences.

Amongst the conclusions which Chapman and Rubesin draw from their results is that when the wall temperature is not uniform, heat-transfer rates cannot be estimated from uniform-temperature results. In this context, reference should be made to a range of solutions obtained by Baxter and Flügge-Lotz (1957, 1958) including flows with zero pressure gradient and non-uniform wall temperature. These were obtained by means of a step-by-step solution on a digital computer. Amongst the cases considered are a number in which the wall tempera-

ture is uniform, $T_w = \frac{1}{2}T_z$ or $3T_z$ upstream of a position $x = l$, with dT_w/dx constant (but not zero) downstream of this position. These authors have also considered cases in which the wall temperature is uniform except in the region $1 \leq x/l \leq 1.0256$, where it increases continuously by a maximum of 2 per cent. and then decreases symmetrically to its upstream value.

8. Lighthill's analysis

A method for calculating heat transfer in low-speed flow, due to Lighthill (1950), was described in Chapter 6, section 5. Now Lighthill has indicated that in equation (169), or equation (172), μ and ρ only appear in the product $\mu\rho$. Accordingly, if this product is constant, the low-speed solution holds at all Mach numbers provided the appropriate value of the constant is taken. Thus we have, rewriting (173)

$$Q_w(x) = -0.538 \frac{k}{\mu} (\sigma\mu\rho)^{\frac{1}{2}} \{\tau_w(x)\}^{\frac{1}{2}} \int_0^x \left\{ \int_{x_1}^x [\tau_w(z)]^{\frac{1}{2}} dz \right\}^{-\frac{1}{2}} dT_\epsilon(x_1). \quad (285)$$

For the case of a flat plate we may write

$$\tau_w(x) = 0.332 \left(\frac{\mu\rho u_0^3}{x} \right)^{\frac{1}{2}},$$

so that (285) becomes

$$Q_w(x) = -0.339 \frac{k}{\mu} \sigma^{\frac{1}{2}} (\mu\rho u_0)^{\frac{1}{2}} x^{-\frac{1}{2}} \int_0^x (x^{\frac{1}{2}} - x_1^{\frac{1}{2}})^{-\frac{1}{2}} dT_\epsilon(x_1). \quad (286)$$

This is, strictly speaking, relevant to the case when $T = 0$ in the mainstream and $T = T_\epsilon$ at the wall. Lighthill shows that at high speeds it is also the value of the heat transfer for the case $T = T_1$ in the mainstream and $T = T_w$ at the wall provided T_ϵ is defined as

$$T_\epsilon = T_w(x) - T_z,$$

which, of course, reduces to the form (174) when the Mach number tends to zero. Accordingly, upon interpreting the Steiltjes integral in (286) we have

$$Q_w(x) = -0.339 \frac{k}{\mu} \sigma^{\frac{1}{2}} \left(\frac{\mu\rho u_0}{x} \right)^{\frac{1}{2}} \left(T_w(0) - T_z + x^{\frac{1}{2}} \int_0^x \frac{T'_w(x_1) dx_1}{(x^{\frac{1}{2}} - x_1^{\frac{1}{2}})^{\frac{1}{2}}} \right). \quad (287)$$

Lighthill has compared his results with those given by Chapman and Rubesin when the wall temperature is (282). Upon setting

$$T_w = T_z + T_0 \sum_{n=0}^{\infty} a_n x^{*n}$$

(287) yields

$$Q_w(x) = -\frac{k(\mu\rho u_0)^{\frac{1}{2}}}{\mu} T_0 \sum_0^{\infty} a_n x^{*n} \left[0.339\sigma^{\frac{1}{4}} \frac{(-\frac{1}{3})! (\frac{4}{3}n)!}{(\frac{4}{3}n - \frac{1}{3})!} \right], \quad (288)$$

which may readily be shown to be equal to (284) with the expression in square brackets replacing $-\frac{1}{2}V'_n(0)$. A comparison of results is given in Table 16 for the values of n considered by Chapman and Rubesin. We note that the error is nowhere as great as 3 per cent., indicating the considerable accuracy of Lighthill's method for flow with zero pressure gradient.

Having thus established the accuracy of his method, Lighthill uses it to calculate the distribution of temperature which might arise in the high-speed flight of a projectile, when heat transfer to the body is balanced mainly by radiation from it. Now the rate at which heat is radiated at the surface of the body is given by

$$Q_w(x) = \alpha \{T_w(x)\}^4, \quad (289)$$

where α is the product of the Stefan-Boltzmann constant and the emissivity of the wall. Upon equating (287) and (289) we have

$$T_w^4 = -0.339 \frac{k}{\alpha\mu} \sigma^{\frac{1}{4}} \left(\frac{\mu\rho u_0}{x} \right)^{\frac{1}{2}} \left(T_w(0) - T_z + x^{\frac{1}{2}} \int_0^x \frac{T'_w(x_1) dx_1}{(x^{\frac{1}{2}} - x_1^{\frac{1}{2}})^{\frac{1}{2}}} \right). \quad (290)$$

We note at this point that the only possible value for $T_w(0)$ is T_z . If it had any other finite value then it would be necessary for the integral to be of order $x^{-\frac{1}{2}}$ as $x \rightarrow 0$, implying that $T'_w(x_1) = O(x_1^{-1})$, which contradicts the assumption that $T_w(0)$ is finite. Physically the boundary layer is extremely thin near $x = 0$, so that heat transfer is extremely effective in forcing the wall to take up its equilibrium temperature T_z very quickly. If we neglect the term $T_w(0) - T_z$, the remainder of the equation may be simplified by writing

$$\frac{T_w}{T_z} = F(z), \quad z = \left(\frac{T_z}{T^*} \right)^3 \left(\frac{x}{l} \right)^{\frac{1}{2}},$$

and

$$T^{*3} = 0.678\sigma^{\frac{1}{4}} \frac{k}{\alpha\mu} \left(\frac{\mu\rho u_0}{l} \right)^{\frac{1}{2}},$$

whence it becomes

$$\{F(z)\}^4 = -\frac{1}{2z^{\frac{1}{2}}} \int_0^z \frac{F'(z_1)}{(z^{\frac{1}{2}} - z_1^{\frac{1}{2}})^{\frac{1}{2}}} dz_1, \quad (291)$$

with

$$F(0) = 1.$$

Lighthill showed that for small values of z a series solution

$$F(z) = 1 - 1.461z + 7.252z^2 - 46.46z^3 + 332.9z^4 \dots \quad (292)$$

could be calculated, and that for large z there is an asymptotic expansion

$$F(z) \sim 0.8409z^{-\frac{1}{2}} - 0.1524z^{-\frac{3}{2}} - 0.0195z^{-\frac{5}{2}} \dots \quad (293)$$

He also showed how the convergence of the series (292) could be extended reasonably accurately, and so was able to link the expansions successfully.

The principal conclusion to be drawn from Lighthill's analysis is that the temperature decreases extremely rapidly with increasing x near to the leading edge, with a more moderate rate of decrease further back. The actual temperature gradient predicted at $x = 0$ is infinite, because of the assumption of zero boundary thickness. In practice this is countered by conduction of heat within the solid and (though to a considerably lesser extent) by the fact that the boundary-layer approximation is inadequate there and the boundary-layer thickness should be regarded as small but non-zero. Even so one expects the wall temperature to fall from a maximum at the nose, so that melting may occur at high enough speeds with solidification further back. There is some evidence that this happens in the case of some stony meteorites.

Before concluding this chapter, mention should perhaps be made of a much less exact but considerably simpler solution of the above problem by the author (Curle, 1958*b*), based upon a method of calculating compressible laminar boundary layers which will be considered later (Chapter 9, section 5). By this method the heat-transfer rate through a laminar boundary layer is given as

$$Q_w(x) = 0.332 \frac{k}{\mu} \left(\frac{\mu \rho u_0}{x} \right)^{\frac{1}{2}} \{T_z + T_w(0) - 2T_w(x)\}, \quad (294)$$

the Prandtl number being taken as unity. This replaces the more complex expression (290), involving an integral, and when equated to (289) leads after the appropriate algebra to the following simple algebraic equation for F , namely

$$ZF^4 = 0.980(1 - F). \quad (295)$$

The solution of this equation is, of course, simple, and it follows fairly straightforwardly that the error in the predicted wall temperature increases from zero at the leading edge to a maximum of 18 per cent. far downstream, where the constant 0.8409 in (293) is approximated as 0.9950.

THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH ZERO HEAT TRANSFER

It has already been seen that the problem of the compressible laminar boundary layer with zero pressure gradient becomes extremely difficult unless suitable approximations are made concerning the value of the Prandtl number and the variation of viscosity with temperature. It is to be expected, therefore, that the same will be true when there is zero heat transfer but an arbitrary pressure gradient. Accordingly we consider first an approach to this problem in which the Prandtl number is taken to be unity, and the viscosity is assumed proportional to the absolute temperature.

1. Howarth's method

The equations of motion and continuity are given by (1) and (12), and with body forces neglected these become

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (296)$$

$$\text{and} \quad \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \quad (297)$$

Equation (297) may be satisfied by introducing a stream-function ψ such that

$$\rho u = \rho_s \frac{\partial \psi}{\partial y}, \quad \rho v = -\rho_s \frac{\partial \psi}{\partial x},$$

where suffix s denotes some standard datum position. Howarth (1948 *a*) assumes that the viscosity and temperature are related by

$$\mu = \mu_s T/T_s,$$

and then attempts partially to reduce (296) to an incompressible form by making the transformations

$$Y = \int_0^y \left(\frac{\nu_s}{\nu} \right)^{\frac{1}{2}} dy = \left(\frac{p}{p_s} \right)^{\frac{1}{2}} \int_0^y \frac{T_s}{T} dy, \quad (298)$$

$$\text{and} \quad \psi(x, y) = \left(\frac{p}{p_s} \right)^{\frac{1}{2}} \chi(x, Y). \quad (299)$$

He shows that
$$u = \frac{\partial \chi}{\partial Y}$$

and that the momentum equation (296) takes the form

$$\frac{\partial^2 \chi}{\partial x \partial Y} \frac{\partial \chi}{\partial Y} - \frac{\partial^2 \chi}{\partial Y^2} \frac{\partial \chi}{\partial x} = u_1 \frac{du_1}{dx} \left\{ \frac{T}{T_1} - \frac{\gamma}{2a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2} \right\} + \nu_s \frac{\partial^3 \chi}{\partial Y^3}, \quad (300)$$

with boundary conditions

$$\frac{\partial \chi}{\partial x} = \frac{\partial \chi}{\partial Y} = 0 \quad \text{at} \quad Y = 0; \quad \frac{\partial \chi}{\partial Y} \rightarrow u_1 \quad \text{as} \quad Y \rightarrow \infty.$$

This equation differs from the incompressible form only in the factor

$$G = \frac{T}{T_1} - \frac{\gamma}{2a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2} \quad (301)$$

which multiplies the pressure gradient term, and in the altered scale normal to the wall.

To proceed farther requires knowledge of the temperature field $T(x, y)$, given by the thermal energy equation. When the Prandtl number is taken to be unity it follows from (19) that

$$c_p T + \frac{1}{2} u^2 = \text{constant},$$

and Howarth writes this equation in the form

$$\frac{T}{T_1} = 1 + \frac{1}{2} (\gamma - 1) M_1^2 \left(1 - \frac{u^2}{u_1^2} \right). \quad (302)$$

It follows that (301) becomes

$$G = 1 + \frac{\gamma - 1}{2a_1^2} \left\{ u_1^2 - \left(\frac{\partial \chi}{\partial Y} \right)^2 \right\} - \frac{\gamma}{2a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2}, \quad (303)$$

so that (300) becomes an equation for $\chi(x, Y)$ alone.

Howarth then develops a momentum integral equation in the transformed variables, by integrating (300) from $Y = 0$ to $Y = \infty$. This yields

$$u_1^2 \frac{d\delta'_2}{dx} + u_1 \frac{du_1}{dx} \left\{ \left(2 - \frac{1}{2} M_1^2 \right) \delta'_2 + \left(1 + \frac{1}{2} (\gamma - 1) M_1^2 \right) \delta'_1 \right\} = \nu_s \left(\frac{\partial u}{\partial Y} \right)_w, \quad (304)$$

where

$$\left. \begin{aligned} \delta'_1 &= \int_0^\infty \left(1 - \frac{u}{u_1} \right) dY \\ \delta'_2 &= \int_0^\infty \frac{u}{u_1} \left(1 - \frac{u}{u_1} \right) dY \end{aligned} \right\} \quad (305)$$

and

are boundary-layer thicknesses measured in the transformed coordinate, Y , normal to the wall. He then seeks to develop a general method of

solution along the lines followed by Pohlhausen for incompressible flow. Thus u/u_1 is expressed as a quartic polynomial in Y/δ' , where δ' is the boundary-layer thickness in terms of the Y -coordinate, and the coefficients of the quartic are chosen to satisfy the corresponding boundary conditions to those satisfied in incompressible flow (Chapter 4, section 1). Howarth then finds that the relevant parameter is

$$\Lambda = \frac{\delta'^2}{\nu_s} \frac{du_1}{dx} \{1 - \frac{1}{2}(\gamma - 1)M_1^2\},$$

which may be compared with the incompressible form (97). Corresponding to (98), the momentum integral equation (304) yields an ordinary differential equation for Λ , which takes the form

$$\frac{d\Lambda}{dx} = \{g(1 + \frac{1}{2}(\gamma - 1)M_1^2) + (3\gamma - 2)M_1^2[\Lambda + \Lambda^2 h]\} \frac{u_1'}{u_1} + \{\Lambda + \Lambda^2 h\} \frac{u_1''}{u_1}, \quad (306)$$

where $g(\Lambda)$, $h(\Lambda)$ are the same universal functions as in incompressible flow. The inaccuracies inherent in the incompressible form of Pohlhausen's method are, of course, perpetuated in the compressible form.

Howarth used his method to consider flow with a linearly retarded external velocity

$$u_1 = u_0 \left(1 - \frac{x}{c}\right) \quad (307)$$

for various values of the Mach number $M_0 = u_0/a_0$. He finds that the position of separation moves towards the leading edge as the Mach number is increased, a result later obtained by the more acceptable method of Stewartson (1949).

2. Young's method

An alternative method of generalizing Pohlhausen's method was suggested by Young (1949). This method will not be discussed here, as it has since been improved and further generalized to flow with heat transfer at the wall (Luxton and Young, 1960). It is sufficient at this stage to point out that the method is valid for arbitrary σ and ω , where ω is the temperature-viscosity relationship index, and it is discussed in Chapter 9, section 6.

3. The Stewartson–Illingworth transformation

The crux of Howarth's method, which partially reduces the compressible boundary-layer equations to an incompressible form, is that the external velocity remains unchanged. By relaxing this condition Stewartson (1949) and Illingworth (1949) have independently shown

that it is possible to effect a complete transformation between a compressible boundary layer and an incompressible boundary layer with a different mainstream. The analysis given here follows Stewartson's approach.

Stewartson assumes that $\sigma = 1$, so that for the case of zero heat transfer the temperature field is given by (302). He then transforms the y -coordinate by writing

$$Y = \frac{a_1}{a_0 \nu_0^{\frac{1}{2}}} \int_0^y \frac{\rho}{\rho_0} dy, \quad (308)$$

where suffix 0 refers to any standard reference position, say at the leading edge in the mainstream. He also introduces a stream function ψ , such that

$$\rho u = \rho_0 \nu_0^{\frac{1}{2}} \frac{\partial \psi}{\partial y}, \quad \rho v = -\rho_0 \nu_0^{\frac{1}{2}} \frac{\partial \psi}{\partial x}, \quad (309)$$

which satisfies (297) automatically. Then upon substituting into (296), transforming from coordinates (x, y) to (X, Y) , making use of the assumption that $\omega = 1$, i.e. $\mu \propto T$, and the temperature profile (307), Stewartson finds that

$$\frac{\partial^3 \psi}{\partial Y^3} - \frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial X \partial Y} + \frac{\partial \psi}{\partial X} \frac{\partial^2 \psi}{\partial Y^2} = -U_1 \frac{dU_1}{dX}, \quad (310)$$

where
$$X = \int_0^x \left(\frac{a_1}{a_0} \right)^{(3\gamma-1)/(\gamma-1)} dx, \quad (311)$$

and
$$U_1 = a_0 M_1 = \frac{a_0}{a_1} u_1. \quad (312)$$

Since the relevant boundary conditions were shown by Stewartson to be

$$\psi = \frac{\partial \psi}{\partial Y} = 0 \quad \text{when} \quad Y = 0; \quad \frac{\partial \psi}{\partial Y} \rightarrow U_1 \quad \text{as} \quad Y \rightarrow \infty,$$

it follows that an exact correlation has been established between the compressible boundary layer in the (x, y) plane with external velocity $u_1(x)$, and an associated incompressible boundary layer in the (X, Y) plane with external velocity $U_1(X)$. It should be remarked that in deriving this same correlation Illingworth (1949) worked with the von Mises form of the boundary-layer equations.

Since many methods are available for calculating the development of the incompressible laminar boundary layer, the use of this transformation enables compressible laminar boundary layers to be calculated, subject to the restrictions made in the transformation, namely zero heat transfer, $\sigma = 1$, $\mu \propto T$. Stewartson himself used the trans-

formation simultaneously with Howarth's method (Chapter 4, section 3) to calculate the effect of compressibility on the separation of the laminar boundary layer with linearly retarded mainstream. His results are in qualitative agreement with the less accurate results obtained by Howarth, using the method of Chapter 8, section 1, and with the results given by Young's method in Chapter 8, section 2. Later, Rott and Crabtree (1952) suggested that for general calculations of compressible laminar boundary layers the transformation should be used with Thwaites's method, and this is done by many people. In view of the simplicity and overall accuracy of the latter method, this combination probably leads to good results in general, though it is possible that errors in the incompressible solution may be exaggerated at high Mach number, for example.

This possibility is borne out by an accurate solution obtained by the Mathematics Division, National Physical Laboratory, for a particular external velocity distribution,

$$u_1 = u_0(1-x/c),$$

the velocity u_0 at the leading edge being such that the Mach number is $M_0 = u_0/a_0 = 4$. It is known that when $M_0 = 0$, boundary-layer separation occurs at $x/c = 0.120$ (Howarth, 1938), and according to the precise numerical solution of N.P.L., separation occurs at $x/c = 0.04(5)$ when $M_0 = 4$. The corresponding solutions by Thwaites's method (as amended by Curle and Skan) and the Stewartson-illingworth transformation are $x/c = 0.123$ when $M_0 = 0$ and $x/c = 0.06(7)$ when $M_0 = 4$. We see that the error increases somewhat with Mach number in this particular case.

Stewartson has also shown that, by applying the transformation, a considerable simplification can be effected even when there is heat transfer at the wall. In such a case, he writes the temperature as

$$\frac{T}{T_1} = 1 + \frac{\gamma-1}{2a_1^2} (u_1^2 - u^2) + \frac{T_0}{T_1} \left\{ 1 + \frac{1}{2}(\gamma-1)M_0^2 \right\} S, \quad (313)$$

where $S \rightarrow 0$ as $Y \rightarrow \infty$, but the boundary condition on S at $Y = 0$ depends upon the temperature of the wall, and is $S(X, 0) = 0$ only when there is zero heat transfer. It may then be shown fairly easily that

$$\left. \begin{aligned} \frac{\partial^3 \psi}{\partial Y^3} - \frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial X \partial Y} + \frac{\partial \psi}{\partial X} \frac{\partial^2 \psi}{\partial Y^2} &= -U_1 \frac{dU_1}{dX} (1+S) \\ \text{and} \quad \frac{\partial \psi}{\partial Y} \frac{\partial S}{\partial X} - \frac{\partial \psi}{\partial X} \frac{\partial S}{\partial Y} &= \frac{\partial^2 S}{\partial Y^2} \end{aligned} \right\}, \quad (314)$$

and these equations must be solved simultaneously for S and ψ . Some solutions of these equations were given by Cohen and Reshotko (1956 *a*), and were made the basis of a general method for calculating compressible laminar boundary layers with pressure gradient and heat transfer, which will be discussed in Chapter 9, sections 1 and 3.

4. Rott's method

Stewartson's transformation holds rigorously only when $\sigma = 1$ and $\mu \propto T$. Rott (1953) showed how it may be generalized approximately to the case of arbitrary σ and general temperature-viscosity law. The three approximations upon which Rott's method is based are as follows. Firstly the assumption is made that the wall temperature for which there will be zero heat transfer is given by

$$\frac{T_w}{T_1} = 1 + \frac{1}{2}(\gamma - 1)r(\sigma)M_1^2, \quad (315)$$

where the recovery factor, $r(\sigma)$, is given to a good accuracy as

$$r(\sigma) \simeq \sigma^{\frac{1}{2}}.$$

This formula is exact when $\sigma = 1$, and has been found to be of good numerical accuracy for other values of σ close to unity. Secondly, the expression

$$\frac{T}{T_1} = 1 + \frac{1}{2}(\gamma - 1)r(\sigma)M_1^2 \left(1 - \frac{u^2}{u_1^2}\right)$$

is accepted as an adequate approximate solution for the temperature profile. This satisfies the boundary conditions at the wall, (315), and at the edge of the boundary layer (where $T \rightarrow T_1$), and is exact when $\sigma = 1$. Thirdly it is assumed that the viscosity-temperature relationship is

$$\frac{\mu(x, y)}{\mu_w(x)} = \frac{T(x, y)}{T_w(x)},$$

so that μ is proportional to T at constant x , but the variation with x is chosen so that the viscosity at the wall is given accurately, by Sutherland's law or by a carefully chosen power law. With these three approximations, the Stewartson transformation can be generalized, and the equations (308), (311), and (312) become respectively

$$Y = \left(\frac{a_1}{a_0}\right)^r \nu_0^{-\frac{1}{2}} \int_0^y \frac{\rho}{\rho_0} dy,$$

$$X = \int_0^x \left(\frac{a_1}{a_0}\right)^{\frac{1}{2}(\gamma-1)+r} \frac{T_0}{T_w} \frac{\mu_w}{\mu_0} dx,$$

and

$$U_1 = \left(\frac{a_0}{a_1}\right)^r u_1.$$

This transformation may then be used, in conjunction with say Thwaites's method, to provide a convenient approximate method for calculating the development of compressible laminar boundary layers with zero heat transfer, arbitrary σ , and general viscosity-temperature law.

5. The method of Oswatitsch and Weighardt

One of the earliest methods developed for calculating compressible laminar boundary layers was that due to Oswatitsch and Weighardt (1943), their method being a plausible generalization of the method of Walz (1941), using the 'similar' solutions of the boundary-layer equations. They write the momentum integral equation (40) in the alternative form

$$u_1 \frac{d}{dx} \left(\frac{\rho_1 \delta_2^2}{\mu_w} \right) + \frac{\rho_1 \delta_2^2}{\mu_w} u_1' \left(\frac{2\delta_1}{\delta_2} + 4 - M_1^2 \right) = \frac{2\delta_2}{u_1} \left(\frac{\partial u}{\partial y} \right)_w, \quad (316)$$

and further assume that the velocity profiles appropriate to specified values of the function

$$\lambda^* = \frac{\rho_1 \delta_2^2}{\mu_w} u_1'$$

are those given by the low-speed solutions of Falkner and Skan (1930), and used by Walz. It is then fairly straightforward to obtain δ_1/δ_2 and $\delta_2/u_1(\partial u/\partial y)_w$ as functions of λ^* and M_1 , the dependence upon M_1 arising because of the variation of density with Mach number. Then (316) becomes

$$u_1 \frac{d}{dx} \left(\frac{\lambda^*}{u_1'} \right) = f(\lambda^*, M_1),$$

and this is easily integrated by a step-by-step procedure.

As no numerical examples were calculated by Oswatitsch and Weighardt it is difficult to estimate what accuracy is to be expected from the method. However, the assumptions made are somewhat arbitrary, so it is unlikely that the method is of particularly wide application.

6. The work of Cope and Hartree

A considerable attack on the problem of the compressible laminar boundary layer with pressure gradient and zero heat transfer was made by Cope and Hartree (1948), involving the use of a high-speed computing machine. These authors make first the following transformation

of variables

$$X = \frac{x}{\rho_0 u_0^2} \frac{dp}{dx},$$

$$Y = \frac{1}{2} \left(\frac{u_0}{v_0 x} \right)^{\frac{1}{2}} \int_0^y \frac{\mu_0}{\mu} dy,$$

$$u^* = u/u_0,$$

$$r = \left(\frac{\rho_0}{\rho} - 1 \right) M_0^{-2},$$

and

$$W = \frac{1}{\rho_0 u_0^2} \frac{dp}{dx} \left(\frac{x}{\mu_0 \rho_0 u_0} \right)^{\frac{1}{2}} \frac{\partial \psi}{\partial X},$$

with ψ defined as in (168). They then derive the equations which u^* , r , W must satisfy, and seek solutions of the form

$$2u^* = \sum_0^{\infty} (-8X)^n h_n(Y),$$

$$r = \sum_0^{\infty} (-8X)^n r_n(Y),$$

$$2W = \sum_0^{\infty} (-8X)^n (2n+1) f_n(Y),$$

with two further expansions,

$$\frac{2\rho u}{\rho_0 u_0} = \sum_0^{\infty} (-8X)^n \chi_n(Y)$$

and

$$\frac{\mu}{\mu_0} = \sum_0^{\infty} (-8X)^n \phi_n(Y),$$

introduced for analytic convenience.

It is found that the zero-order terms, h_0 , r_0 , f_0 , χ_0 , ϕ_0 , satisfy non-linear equations, and these have been integrated by Cope and Hartree for various values of M_0 , assuming $\sigma = 0.715$ and $\mu \propto T^{8/9}$. Not surprisingly the solutions are barely dependent upon M_0 , since they would be identical under the similar assumption that $\mu \propto T$. Some of the first-order functions were also calculated approximately, but these exhibit a considerable dependence upon M_0 . The n th-order equations ($n \geq 1$) are all linear, and depend upon the functions of lower order. In principle, therefore, they may be calculated successively in order to obtain a complete solution.

Cope and Hartree have also given some consideration to the possibility of devising a solution along the lines of Pohlhausen's method for incompressible flow. They assume that the Prandtl number is close

to unity, so that an adequate solution of the thermal energy equation is given by (302) when there is zero heat transfer. The momentum integral equation (40) is first written in the alternative form

$$\frac{d\delta_2}{dx} + \frac{u'_1}{u_1}[\delta_1 + (2 - M_1^2)\delta_2] = \frac{\tau_w}{\rho_1 u_1^2},$$

and the velocity profile approximated as

$$\frac{u}{u_1} = f(\eta), \quad \eta = y/\delta,$$

where δ is a measure of the boundary-layer thickness, and the function f is chosen to satisfy a number of the boundary conditions on the velocity u and its derivatives. In the absence of a pressure gradient Cope and Hartree take

$$f(\eta) = \sin \frac{1}{2}\pi\eta,$$

and obtain results for the skin-friction which agree well with the calculations of Brainerd and Emmons.

When there is a pressure gradient the algebra becomes considerably more complicated, and Cope and Hartree concluded that the Pohlhausen method did not offer much promise of progress, a conclusion which was perhaps a little pessimistic.

7. The work of Illingworth, Frankl, and Gruschwitz

We might perhaps mention at this point some of the other investigations of the Pohlhausen-type which have been made by various authors. Illingworth (1946) has considered the case of zero heat transfer and arbitrary pressure gradient, with the assumption that viscosity μ and thermal conductivity k are constant. The velocity was defined by

$$\frac{u}{u_1} = f(\eta),$$

and the temperature (or density) by

$$\frac{\rho}{\rho_1} = \frac{\rho_w}{\rho_1} + \left(1 - \frac{\rho_w}{\rho_1}\right)g(\eta/\chi),$$

where $\chi = \delta'/\delta$ is the ratio of thermal boundary-layer thickness to velocity boundary-layer thickness. The functions f and g are chosen to satisfy the usual boundary conditions, derived from the differential forms of the equations. The momentum integral equation and the thermal energy integral equation then yield two relationships for the three unknowns ρ_w/ρ_1 , χ , and $\delta^2/u_1(\partial^2 u/\partial y^2)_w$, a third relationship being given by the boundary condition which follows from setting $y = 0$ in the thermal energy equation (18).

Illingworth has carried out detailed calculations for flow with zero pressure gradient, using a number of possible forms of the functions f and g . In particular he considered cases when f and g were of the same order in η , namely quadratic, cubic, and quartic, and one case in which f was quintic and g sextic. Further, he also considered trigonometric forms, in which f, g are

$$f = \sin \frac{1}{2}\pi\eta, \quad g = \left(\sin \frac{\pi\eta}{2\chi}\right)^2.$$

Upon comparing his results with the accurate calculations of Emmons and Brainerd (1942), Illingworth concludes that the trigonometric and cubic forms appear to give the most accurate values. He notes, however, that the particular form of the function g is critical and that the predictions of wall temperature leave much to be desired in all the cases considered.

Illingworth's work may be regarded as a generalization of the earlier work of Frankl (1934), who made the further approximation that the viscous and thermal boundary layers have equal thickness ($\chi = 1$), although the Prandtl number was not assumed to be unity. Various possible forms of the functions $f(\eta), g(\eta)$ were considered, and in spite of the apparent additional assumption (as compared with the later work of Illingworth) Frankl's results for zero pressure gradient, with f and g both quartics, were better than any of those obtained by Illingworth. This is perhaps indicative of the uncertainty which besets attempts to make direct extensions of Pohlhausen's method without first making a suitable transformation of coordinates and/or variables.

A slightly different approach was adopted by Gruschwitz (1950) who introduced a new boundary-layer thickness

$$\Delta = \int_0^{\delta(x)} \frac{\rho}{\rho_1} dy,$$

and expressed u/u_1 as a quartic in

$$\eta = \frac{1}{\Delta} \int_0^y \frac{\rho}{\rho_1} dy.$$

Further, instead of expressing ρ/ρ_1 in terms of η , he chose to write the product $\{1 - (u/u_1)\}(\rho_1/\rho)$ as a quintic in η , satisfying appropriate boundary conditions. Substitution into the momentum integral equation and the kinetic-energy integral equation leads, after some considerable calculation, to a first-order non-linear ordinary differential

equation for the momentum thickness δ_2 in terms of the known u_1 and M_1 and the unknown T_w/T_1 , and an explicit expression for T_w/T_1 in terms of M_1 and δ_2 . These two equations involve universal functions, analogous to those arising in Pohlhausen's method, which are dependent only upon the Prandtl number σ and the shape parameter

$$\lambda = \frac{\delta_2^2}{\nu_1} u_1' \frac{T_w}{T_1}.$$

It should be remarked that the method does not make use of the thermal-energy integral equation, and accordingly is suspect unless $\sigma = 1$, when the kinetic-energy integral equation is identical with it.

8. The investigations of Gadd

We conclude this chapter with a brief discussion of some work of Gadd (1953*a*), which investigates approximately the effects on the separation position of variations in Prandtl number σ and the index ω in the viscosity-temperature law (227).

He begins by considering an artificial case in which $\sigma = 0$. In such circumstances it is clear, by inspection of (19), that temperature gradients can remain finite only if

$$\frac{\partial}{\partial y} \left(\mu \frac{\partial T}{\partial y} \right) = 0,$$

so that

$$\mu \frac{\partial T}{\partial y} = 0$$

when the heat transfer at the wall is zero, and thus

$$T(x, y) = T_1(x).$$

With this solution of the thermal-energy equation, Gadd shows how the momentum equation may be transformed into an incompressible form by means of a transformation analogous to that of Stewartson and Illingworth. This equation may then be solved approximately by any convenient method, and Gadd gives some solutions (at various Mach numbers) by Thwaites's method for the case when $\omega = 1$ and the external velocity is of the form (307). Comparisons with solutions for the case $\omega = \sigma = 1$ suggest that the distance to separation increases with decreasing σ , and the solutions for $\sigma = 0$ provide an upper bound to this variation.

Gadd then considers approximate solutions for cases in which either ω or σ is equal to unity and the other does not differ much therefrom. In each case, he writes the equations in the Crocco form, with inde-

pendent variables x , u , and then further transforms to variables x and $f = u/u_1$.

When $\sigma = 1$, $\omega \neq 1$, the thermal energy equation has the simple solution (228) for temperature in terms of velocity. The solution of the momentum equation was then obtained by an approximate method. When $\omega = 1$, $\sigma \neq 1$, the process is slightly more difficult, as it is necessary to obtain first an approximate solution for the temperature distribution, and Gadd did this by means of a small perturbation about the solution (228) valid for small values of $|1 - \sigma|$.

The conclusions to be drawn from Gadd's calculations may be illustrated as follows. When $\omega = 1$, the proportionate increase in the distance to separation when $\sigma = 0.72$ (as compared with $\sigma = 1$) is about 8 per cent. when the Mach number at the leading edge is $M_0 = 4$, and about 25 per cent. when $M_0 = 10$, these results being for the case of a linearly retarded external velocity. Further, when $\sigma = 1$, the proportionate increase in the distance to separation when $\omega = \frac{8}{9}$ (as compared with $\omega = 1$) is about 2 per cent. when $M_0 = 4$ and about 6 per cent. when $M_0 = 10$. Gadd has also derived results by direct application of the approximate method of Young. At the lower Mach number, $M_0 = 4$, the results are almost identical with those quoted above, but at $M_0 = 10$ Young's method predicts proportionate changes in separation distance which are in each case approximately half those derived by Gadd's method.

We deduce from these results that, when there is zero heat transfer at any rate, the errors induced by assuming $\omega = \sigma = 1$ will be no more than about 10 per cent. when $M_0 = 4$ and 30 per cent. when $M_0 = 10$, 15 per cent. if Young's values are to be preferred, in practical applications with air as the constituent gas. These conclusions are broadly similar to those which would be reached if the method of Rott were used in the calculations.

THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH PRESSURE GRADIENT AND HEAT TRANSFER

As far as the formal boundary-layer problem is concerned, the case with both pressure gradient and heat transfer is by far the most difficult to deal with theoretically. The momentum and energy equations are simultaneous non-linear partial differential equations for which no simple approach is known. The problem becomes even more difficult if the wall temperature is not uniform (or very nearly so), a situation which could arise in practice at very high speeds. The mathematical and numerical difficulties encountered in solving these equations are such that, even with the assistance of modern high-speed computing machines, very few precise numerical solutions for special cases have been attempted, and some of these are of doubtful accuracy.

We shall begin this chapter by discussing briefly these numerical solutions, continuing with some of the approximate methods developed for this very general problem. In incompressible flow, where the pressure gradient alone is sufficient to prescribe the problem, there are now about eight really accurate solutions for particular cases, half of which are for flows in which the pressure gradient is everywhere adverse. These eight solutions are only just sufficient to enable us to make a reliable assessment of the accuracy and value of various approximate methods. In compressible flow things are further complicated by the presence of additional parameters such as Mach number and wall temperature. Clearly at least six or eight accurate solutions are required for several values of these parameters before we can assess the usefulness of the approximate methods with any certainty, and this point will accordingly not be considered as thoroughly as in the incompressible case.

1. Accurate numerical solutions for special cases

A range of accurate solutions are, of course, available for the case of zero pressure gradient, covering various values of the Prandtl number σ , several possible viscosity-temperature laws, and a few values of the

wall-temperature. These were discussed and summarized in Chapter 7, and will not be further considered here.

For flows with pressure gradient, the earliest accurate solutions (as far as the author is aware) were obtained by the Mathematics Division, National Physical Laboratory, and are first referred to in a paper by Gadd (1952). Three particular cases were considered, the assumptions that $\sigma = 1$ and $\mu \propto T$ being made in each case. All were such that the wall temperature was uniform and the pressure gradient everywhere adverse, so that flow near to the leading edge was like that on a flat plate with zero pressure gradient. Further details of the three cases are as follows: (i) In the first case a uniform adverse velocity gradient was assumed, so that the external velocity was given by (307), the Mach number at the leading edge was $M_0 = 4$ and zero heat transfer was considered. (ii) The data for this case differed from case (i) only in that the wall was taken at a temperature T_0 , which is the external stream temperature at the leading edge. Since T_z/T_0 is equal, by (244) to $1 + \frac{1}{2}(\gamma - 1)M_0^2$, upon putting $\gamma = 1.4$ we see that the ratio of wall temperature $T_w (= T_0)$ to its zero-heat-transfer value T_z is $\frac{5}{21}$. In other words the wall has been considerably cooled. (iii) In the third case, a uniform adverse pressure gradient was assumed, so that

$$p = p_0 \left(1 + \frac{x}{c} \right).$$

The Mach number at the leading edge was $M_0 = 2$, and the wall temperature was T_0 (defined as above), equal to five-ninths of its zero-heat-transfer value; the cooling at this lower Mach number is less severe.

For each of these cases detailed calculations were made of both the viscous and thermal boundary layers in the region between the leading edge and the position of zero skin-friction, and some of these results have been reported in a paper by Curle (1959). For further details, reference may be made to the Mathematics Division, National Physical Laboratory.

Some time later a series of 'similar' solutions were derived by Cohen and Reshotko (1956*a*), analogous to those found for incompressible flow by Falkner and Skan and by Squire. Following Li and Nagamatsu (1953), Cohen and Reshotko look for solutions of (314) in which the external velocity in the transformed plane is equal to

$$U_1(X) = CX^m,$$

and introduce new variables f, η , where

$$f(\eta) = \left(\frac{m+1}{2U_1 X} \right)^{\frac{1}{2}} \psi$$

and

$$\eta = \left(\frac{m+1}{2} \frac{U_1}{X} \right)^{\frac{1}{2}} Y.$$

Then equations (314) reduce to the form

$$\left. \begin{aligned} f''' + ff'' &= \frac{2m}{m+1} (f'^2 - 1 - S) \\ S'' + \sigma f S' &= 0 \end{aligned} \right\}. \quad (317)$$

Here primes denote derivatives with respect to η , and the boundary conditions are

$$\left. \begin{aligned} f(0) = f'(0) &= 0, \quad S(0) = S_w = \frac{T_w}{T_z} - 1 \\ f' &\rightarrow 1, \quad S \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \end{aligned} \right\}. \quad (318)$$

Equations (317) and (318) were integrated numerically for a series of values of the parameters m and S_w , corresponding respectively to different pressure gradients and wall temperatures, and the results are tabulated and plotted in their paper. Although the same reservations must be observed in applying these results to non-similar flows as were relevant to the Falkner-Skan solutions, there is no doubt that the solutions of Cohen and Reshotko are extremely valuable, none the less.

Subsequent to the above solutions Baxter and Flügge-Lotz (1957, 1958) have reported a series of some sixty examples which were calculated on a digital computer by means of a step-by-step solution, starting from Crocco's form of the equations, and have given detailed results for the skin-friction and heat transfer. As an illustration of the type of example considered by these authors, we mention four cases in which the external velocity is constant, equal to u_0 , when $x \leq l$, and varies when $x \geq l$ according to the law

$$\frac{dp}{dx} = \frac{5\rho_0 u_0^2}{l} \left(\frac{x}{l} - 1 \right).$$

The four cases differ only in that $T_w = \frac{1}{2}T_z$ in two of them, with $T_w = 3T_z$ in the other two, and $M_0 = \frac{1}{2}$ in two cases, with $M_0 = 3$ in the other two. In all the examples considered the Prandtl number was $\sigma = 0.72$, and Sutherland's law was used for the viscosity-temperature relationship. For details of the method used and for the numerical results, reference may be made to the two papers by these authors.

2. Kalikhman's method

Turning now to approximate methods for calculating the compressible laminar boundary layer with heat transfer and pressure gradient, we begin with a generalization of the Pohlhausen method, due to Kalikhman (1946). The method is valid for arbitrary constant Prandtl number, and assumes that the viscosity-temperature relationship is

$$\frac{\mu}{\mu_s} = \left(\frac{T}{T_s} \right)^\omega.$$

Kalikhman introduces a new independent variable η in place of y , where

$$\eta = \int_0^y \frac{\rho}{\rho_s} dy,$$

and suffix s denotes a standard reference condition, which he takes to be the free-stream stagnation value. He also introduces new dependent variables $\bar{v}$, E^* , defined by

$$\bar{v} = u \frac{\partial \eta}{\partial x} + \frac{\rho}{\rho_s} v,$$

$$T^* = T_H - T_w,$$

where T_H is the total temperature, as defined in equation (48). The continuity, momentum, and thermal energy equations (1), (12), and (19) then take the form

$$\frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial \eta} = 0, \quad (319)$$

$$u \frac{\partial u}{\partial x} + \bar{v} \frac{\partial u}{\partial \eta} = \frac{\rho_1}{\rho} u_1 \frac{du_1}{dx} + \nu_0 \frac{\partial}{\partial \eta} \left(\frac{\rho}{\rho_s} \left(\frac{T}{T_s} \right)^\omega \frac{\partial u}{\partial \eta} \right), \quad (320)$$

$$\text{and } u \frac{\partial T^*}{\partial x} + \bar{v} \frac{\partial T^*}{\partial \eta} = u_1 \frac{dT_1^*}{dx} + \nu_s \frac{\partial}{\partial \eta} \left(\frac{\rho}{\rho_s} \left(\frac{T}{T_s} \right)^\omega \frac{\partial}{\partial \eta} \left[T^* + \left(\frac{1}{\sigma} - 1 \right) T \right] \right). \quad (321)$$

It will be observed that the various transformations have gone some way towards reducing these equations to an incompressible form, in much the same way as the related transformation of Howarth (1948*a*). Kalikhman then integrates these equations with respect to η , and obtains the results

$$\frac{d\delta_2^*}{dx} + \frac{u_1'}{u_1} \{ \delta_1^* + 2\delta_2^* + \frac{1}{2}(\gamma-1)M_1^2(\delta_1^* + \delta_2^*) \} +$$

$$+ \frac{u_1'}{u_1} (1 + \frac{1}{2}(\gamma-1)M_1^2) \left(\frac{T_w}{T_s} - 1 \right) \Delta^* = \frac{\tau_w}{\rho_s u_1^2}, \quad (322)$$

$$\text{and } \frac{d}{dx} \{ u_1 T_1^* \delta_4^* \} = \frac{Q_w}{\rho_s c_p}, \quad (323)$$

where

$$\delta_1^* = \int_0^\infty \left(1 - \frac{u}{u_1}\right) d\eta,$$

$$\delta_2^* = \int_0^\infty \frac{u}{u_1} \left(1 - \frac{u}{u_1}\right) d\eta,$$

$$\Delta^* = \int_0^\infty \left(1 - \frac{T^*}{T_1^*}\right) d\eta,$$

$$\delta_4^* = \int_0^\infty \frac{u}{u_1} \left(1 - \frac{T^*}{T_1^*}\right) d\eta,$$

and δ_1^* , δ_2^* , and δ_4^* are analogous to the displacement thickness, momentum thickness, and enthalpy thickness of the boundary layer as previously defined. Following the idea of Pohlhausen, he then makes the approximations

$$\frac{u}{u_1} = \sum_0^4 A_n \left(\frac{\eta}{\delta}\right)^n, \quad (324)$$

$$\frac{T^*}{T_1^*} = \sum_0^4 B_n \left(\frac{\eta}{\Delta}\right)^n, \quad (325)$$

where the ten coefficients A_n , B_n are chosen to satisfy the usual boundary conditions, namely

$$u(0) = T^*(0) = 0,$$

and

$$\frac{u}{u_1} = \frac{T^*}{T_1^*} = 1, \quad \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial T^*}{\partial \eta} = \frac{\partial^2 T^*}{\partial \eta^2} = 0 \quad \text{at } \eta = 1,$$

together with the two relationships obtained by setting $\eta = 0$ in the momentum and energy equations, (320) and (321). Then (322) and (323) yield two simultaneous ordinary differential equations relating the boundary-layer thicknesses, δ and Δ , which appear in the parameters

$$\Lambda = \frac{u'_1 \delta^2 (T_w)}{\nu_s (T_s)}^{2-\omega} \{1 + \frac{1}{2}(\gamma-1) M_{11}^2\}^{(2\gamma-1)/(\gamma-1)},$$

and

$$A_1 = \frac{12 + \Lambda}{6 - \{3 - (9 - 12\alpha)^{\frac{1}{2}}\}(\delta/\Delta)},$$

where

$$\alpha = (1 - \omega) \left(1 - \frac{T_s}{T_w}\right).$$

No numerical results were given by Kalikhman, so the accuracy of the method is uncertain. It may be remarked, however, that the partial

transformation of the equations to incompressible form is a point in favour of the method, although the arbitrary choice of the order of the polynomials in (324) and (325) is unfortunately somewhat restrictive; an examination of which order would yield best results (by comparison with accurate solutions) would be extremely lengthy.

3. The method of Cohen and Reshotko

Mention has already been made of some 'similar' solutions for compressible laminar boundary layers with pressure gradient and heat transfer. These solutions were calculated by Cohen and Reshotko (1956 *a*) and were subsequently used (1956 *b*) as the basis of an approximate method for general use.

These authors begin by applying the Stewartson–Illingworth transformation, in a form which neglects the factor $\nu_0^{\frac{1}{2}}$ appearing in the stream function ψ as defined by Stewartson. Thus they write

$$\rho u = \rho_0 \frac{\partial \psi}{\partial y}, \quad \rho v = -\rho_0 \frac{\partial \psi}{\partial x},$$

and introduce the new independent variables,

$$X = \int_0^x \left(\frac{a_1}{a_0} \right)^{(3\gamma-1)/(\gamma-1)} dx, \quad Y = \frac{a_1}{a_0} \int_0^y \frac{\rho}{\rho_0} dy,$$

which differ from (308) and (311) only as regards the factor $\nu_0^{\frac{1}{2}}$. Then with

$$U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X},$$

the continuity, momentum, and thermal-energy equations may be written as

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (326)$$

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = (1+S)U_1 U_1' + \nu_0 \frac{\partial^2 U}{\partial Y^2}, \quad (327)$$

$$\text{and} \quad U \frac{\partial S}{\partial X} + V \frac{\partial S}{\partial Y} = \nu_0 \frac{\partial^2 S}{\partial Y^2}, \quad (328)$$

where S is defined by equation (313). The boundary conditions are

$$\left. \begin{aligned} U(X, 0) = V(X, 0) = 0, \quad S_w = S(X, 0) = \frac{T_w}{T_z} - 1 \\ U \rightarrow U_1, \quad S \rightarrow 0, \quad \text{as } Y \rightarrow \infty \end{aligned} \right\}. \quad (329)$$

Upon defining boundary-layer thicknesses, measured in the Y -coordinate, as follows,

$$\delta_1^* = \int_0^\infty \left(1 - \frac{U}{U_1} + S\right) dY,$$

$$\delta_2^* = \int_0^\infty \frac{U}{U_1} \left(1 - \frac{U}{U_1}\right) dY,$$

and
$$\delta_4^* = \int_0^\infty S \frac{U}{U_1} dY,$$

equations (326) to (328) may be integrated with respect to Y from $Y = 0$ to $Y = \infty$ to yield

$$\frac{d\delta_2^*}{dX} + \frac{U_1'}{U_1} (\delta_1^* + 2\delta_2^*) = \frac{\nu_0}{U_1^2} \left(\frac{\partial U}{\partial Y} \right)_w, \quad (330)$$

and
$$\frac{d\delta_4^*}{dX} + \frac{U_1'}{U_1} \delta_4^* = -\frac{\nu_0}{U_1} \left(\frac{\partial S}{\partial Y} \right)_w. \quad (331)$$

Cohen and Reshotko now introduce a number of non-dimensional quantities, analogous to those introduced by Thwaites (1949) for incompressible flow, namely

$$l = \frac{\delta_2^*}{U_1} \left(\frac{\partial U}{\partial Y} \right)_w, \quad (332)$$

$$n = \frac{\delta_2^{*2}}{U_1(1+S_w)} \left(\frac{\partial^2 U}{\partial Y^2} \right)_w = -\frac{U_1'}{\nu_0} \delta_2^{*2}, \quad (333)$$

$$r = \frac{\delta_2^{*3}}{U_1} \left(\frac{\partial^3 U}{\partial Y^3} \right)_w = n \delta_2^* \left(\frac{\partial S}{\partial Y} \right)_w, \quad (334)$$

and
$$H^* = \delta_1^*/\delta_2^*. \quad (335)$$

Upon substitution into (330) it is found that the following result may be obtained, namely

$$\begin{aligned} \frac{U_1}{\nu_0} \frac{d}{dX} (\delta_2^{*2}) &= 2\{n(H^*+2)+l\} \\ &= L, \end{aligned} \quad (336)$$

which is analogous to the low-speed form (103).

Cohen and Reshotko make the assumption at this stage that L , H^* , and l are all functions of n and S_w only, these functions being derived from the 'similar' solutions which they had calculated. This assumption is analogous to that made by Walz (1941). One consequence of the assumption is that equations (330) and (331) cannot be satisfied

simultaneously, and Cohen and Reshotko choose to satisfy (330) whilst rejecting (331). One expects intuitively, therefore, that in certain circumstances the method may give the temperature field with only low accuracy, whilst predicting the velocity field more successfully. When $L(n, S_w)$ is thus prescribed, equation (336) may be integrated numerically as in the incompressible case. When the wall temperature is uniform it is often possible to approximate L as a linear function of n , and an analytic solution of (336) then becomes possible. Having calculated δ_2^* , δ_1^* follows from (335), $(\partial U / \partial Y)_w$ from (332) and $(\partial S / \partial Y)_w$ from (334). It is then a simple matter to derive from these results the true boundary-layer thicknesses, the skin-friction, and the heat-transfer as measured in the physical plane. The relevant formulae are

$$\delta_2 = \delta_2^* \left(\frac{T_z}{T_1} \right)^{(\gamma+1)/2(\gamma-1)},$$

$$\delta_1 = \delta_2 \{ H^* + \frac{1}{2}(\gamma-1) M_1^2 (H^* + 1) \},$$

$$\left(\frac{\partial u}{\partial y} \right)_w = \frac{u_1}{\delta_2} \frac{T_1}{T_w} l,$$

and
$$\left(\frac{\partial T}{\partial y} \right)_w = \frac{T_1}{\delta_2} \frac{T_z}{T_w} \frac{r}{n}.$$

The functions l , H^* , L , and r are plotted in Cohen and Reshotko's paper as functions of n for various constant values of S_w .

Comparisons have been made with some of the accurate solutions discussed in section 1, and the agreement is quite good. For example, for the case of a uniform adverse external velocity gradient at a Mach number of 4, with the wall cooled to a temperature $T_w = \frac{5}{21} T_z$, separation is predicted when the external velocity has decreased by about $17\frac{1}{2}$ per cent., as compared with the accurate value of $22\frac{1}{2}$ per cent. This comparison indicates fairly well the overall accuracy of the method for this particular example.

4. Monaghan's method

Important simplifications and improvements to the method of Cohen and Reshotko have been suggested by Monaghan (1960). In the course of a more general study of the compressible laminar boundary layer, Monaghan found that by careful choice of parameters some of Cohen and Reshotko's curves for the function l , at different values of S_w , may be collapsed onto a single curve. In particular it appears that in regions

of favourable pressure gradient the function $l(m, S_w)$, where

$$m = n \frac{T_w}{T_z} = n(1 + S_w),$$

is almost independent of S_w , whilst in regions of adverse pressure the functions $l\{(m/m_{\text{sep}}), S_w\}$ fall on a single curve, m_{sep} being the value of m at which $l = 0$. A similar collapse onto a single curve is found for a function which is related to the r of Cohen and Reshotko's method. Monaghan further generalizes the transformation of coordinates used by Cohen and Reshotko, and writes

$$X = \frac{\int_0^x u_1^{q_1} T_1^{q_2} dx}{u_1^{q_1} T_1^{q_2}}.$$

His final equations take the form

$$\delta_2 = 0.664 \left(\frac{\nu_1 X}{u_1} \right)^{\frac{1}{2}},$$

$$\tau_w = 0.332 \left(\frac{\mu_1 \rho_1 u_1^3}{X} \right)^{\frac{1}{2}} \frac{l(m)}{0.22},$$

and

$$Q_w = 0.332 c_p (T_z - T_w) \left(\frac{\mu_1 \rho_1 u_1}{X} \right)^{\frac{1}{2}} \frac{h(m)}{0.22},$$

where

$$m = -0.441 \frac{T_w}{T_z} \frac{X}{M_1} \frac{dM_1}{dx},$$

and the functions $l(m)$, $h(m)$ are given to a good approximation as

$$\frac{l}{0.22} = 1 - 7.45m, \quad \frac{h}{0.22} = 1 + 2.7m \quad (m < 0),$$

and, when $m > 0$,

$$h = 0.225$$

and l is given as a function of m/m_{sep} by means of a graph in Monaghan's paper.

It remains to define g_1 , g_2 , and m_{sep} . To a first approximation g_1 and g_2 are given by

$$g_1 = 3 + 2 \frac{T_w}{T_z} \quad \text{and} \quad g_2 = \frac{5}{2} - \frac{T_w}{T_z}.$$

The value of m_{sep} is determined by means of a graph relating m_{sep}/m_0 to T_w/T_z , where m_0 is theoretically equal to 0.0681. However, Monaghan suggests an empirical improvement in which m_0 is taken to equal the value of m at which the same external velocity distribution would lead to separation in incompressible flow. If we may judge by experience

in incompressible laminar boundary-layer theory, it should be taken to equal about 0.090.

Comparisons with the exact solution for the case considered earlier ($M_0 = 4$, $T_w/T_z = \frac{5}{21}$, uniform adverse velocity gradient) indicate that, with all the empirical improvements noted above, Monaghan's method is remarkably accurate.

5. Curle's method

The two methods above, namely those of Cohen and Reshotko and of Monaghan, may be regarded as possible generalizations of the method of Thwaites (1949), or more precisely that of Walz (1941). An alternative method of doing this was presented by Curle (1959) in a paper giving a generalization of Thwaites's method starting from the Howarth transformation, as defined by equations (298) and (299). Upon formally applying this transformation we find (following Howarth's analysis) that

$$\frac{\partial^2 \chi}{\partial x \partial Y} \frac{\partial \chi}{\partial Y} - \frac{\partial^2 \chi}{\partial Y^2} \frac{\partial \chi}{\partial x} = u_1 u_1' \left\{ \frac{T}{T_1} - \frac{\gamma}{2a_1^2} \chi \frac{\partial^2 \chi}{\partial Y^2} \right\} + \nu_s \frac{\partial^3 \chi}{\partial Y^3},$$

and this may be formally integrated with respect to Y , from $Y = 0$ to $Y = \infty$, to yield

$$u_1^2 \frac{d\delta'_2}{dx} + u_1 u_1' \left\{ \delta'_1 + (2 - \frac{1}{2}\gamma M_1^2) \delta'_2 + \int_0^\infty \left(\frac{T}{T_1} - 1 \right) dY \right\} = \nu_s \left(\frac{\partial u}{\partial Y} \right)_w, \quad (337)$$

where δ'_1 and δ'_2 are defined by equation (305). Further progress depends upon a knowledge of the temperature profile.

Now it is well known that when $\sigma = 1$ the total temperature is constant if there is zero heat transfer, and is a linear function of the velocity u if there is zero pressure gradient. For the purpose of obtaining a straightforward solution of (337) for the velocity boundary layer (but not for calculating the heat transfer, for example) Curle suggests writing

$$T_H = T_w + (T_z - T_w) \frac{u}{u_1} + K T_1 \left(\frac{u}{u_1} - \frac{u^2}{u_1^2} \right), \quad (338)$$

where the last term should be zero when either the pressure gradient or the heat transfer is zero. The value

$$K T_1 = \frac{T_z - T_w}{1 + \theta(\xi - 1)} \xi,$$

where

$$\xi = 1 - \frac{u_1^2}{u_0^2}$$

and

$$\theta = \frac{T_w/T_0 - 1 + \frac{1}{2}(\gamma - 1)M_0^2}{T_w/T_0 + 1 + \frac{1}{2}(\gamma - 1)M_0^2},$$

is suggested for use in regions of retarded flow. The temperature T_0 , velocity u_0 , and Mach number M_0 are values at the leading edge if the velocity there is non-zero, otherwise they are values at the pressure minimum. The analysis cannot be directly applied to regions of accelerated flow, where Curle indicates that Cohen and Reshotko's method should give good accuracy.

With the approximation (338) it is easy to show that

$$\int_0^{\infty} \left(\frac{T}{T_1} - 1 \right) dY = \left(\frac{T_w}{T_1} - 1 \right) \delta'_1 + (K + \frac{1}{2}(\gamma - 1)M_1^2) \delta'_2,$$

whence (337) becomes

$$u_1^2 \frac{d\delta'_2}{dx} + u_1 u'_1 \left(\frac{T_w}{T_1} \delta'_1 + (2 + K - \frac{1}{2}M_1^2) \delta'_2 \right) = \nu_s \left(\frac{\partial u}{\partial Y} \right)_w. \quad (339)$$

Curle now introduces non-dimensional parameters

$$\begin{aligned} H' &= \delta'_1 / \delta'_2, \\ l' &= \frac{\delta'_2}{u_1} \left(\frac{\partial u}{\partial Y} \right)_w, \\ \lambda' &= \frac{\delta_2'^2}{\nu_s} u_1' \frac{T_w}{T_1}, \end{aligned}$$

whereupon (339) becomes

$$u_1 \frac{d}{dx} \left(\frac{\delta_2'^2}{\nu_s} \right) = L' + 4\lambda' - 2\lambda' \frac{T_1}{T_w} (2 + K - \frac{1}{2}M_1^2), \quad (340)$$

where

$$L' = 2l' - 2\lambda'(H' + 2).$$

By considering the consequences of determining relationships between L' , H' , l' , and λ' by means of velocity profiles prescribed in the Y -plane, Curle concludes that the functions $L'(\lambda')$, $H'(\lambda')$, $l'(\lambda')$ are largely independent of wall temperature and Mach number, and are given approximately by their incompressible values; the values appropriate to Thwaites's method are accordingly recommended. Since we can now use the approximation to $L'(\lambda')$ analogous to the Thwaites value (106) we have

$$L' \simeq 0.45 - 6\lambda',$$

and (340) is integrable in the form of a simple quadrature. Upon transforming from the Y -plane into the physical plane, the final equations of the method are

$$\delta_2^2 = \frac{0.45\nu_1}{G_1} \int_0^x \frac{G_1}{u_1} dx, \quad (341)$$

where

$$G_1 = \exp \left\{ 2 \int_0^x \left(\frac{T_w}{T_1} + 2 + K - \frac{1}{2} M_1^2 \right) \frac{u'_1}{u_1} dx \right\}, \quad (342)$$

$$\lambda' = \frac{\delta_2^2}{\nu_1} u'_1 \frac{T_w}{T_1}, \quad (343)$$

$$\left(\frac{\partial u}{\partial y} \right)_w = \frac{u_1 (\nu_1)}{\delta_2 (\nu_w)}^{\frac{1}{2}} l'(\lambda'), \quad (344)$$

$$\delta_1 = \delta_2 \left\{ \frac{T_w}{T_1} H'(\lambda') + K + \frac{1}{2} (\gamma - 1) M_1^2 \right\}. \quad (345)$$

The method is perfectly straightforward to apply, and comparisons with the three accurate solutions calculated by the Mathematics Division, National Physical Laboratory, indicate that the accuracy bears reasonable relationship to the work involved.

The method as discussed above is valid for $\sigma = 1$, $\mu \propto T$, and uniform T_w . In two later papers Curle (1958*b*, 1961*b*) has removed these restrictions. For full details reference may be made to these papers. Briefly, the removal of these restrictions leaves the general procedure unaltered, but amends some of the details, so that (for example) the coefficients of T_w/T_1 and M_1^2 in (342) and (345) are prescribed functions of σ , and an additional factor to account approximately for the generalized viscosity-temperature relationship appears in (344) and in the definition of the function G_1 .

6. The method of Luxton and Young

We now consider a method which is essentially a generalization of the Pohlhausen procedure, due to Luxton and Young (1960). It was originally developed by Young (1949) for the case of zero heat transfer, but was later generalized and improved by these two authors for use when there is heat transfer, and is applicable for arbitrary values of the Prandtl number σ and with the viscosity-temperature relationship given by (227) for arbitrary ω .

The method assumes that the velocity profile is of the form

$$\frac{u}{u_1} = a \frac{Y}{\delta} + b \left(\frac{Y}{\delta} \right)^2 + c \left(\frac{Y}{\delta} \right)^3 + d \left(\frac{Y}{\delta} \right)^4,$$

where

$$Y = \int_0^y \frac{\mu_1}{\mu} dy,$$

and a, b, c, d are chosen to satisfy the usual boundary conditions. Thus

$$a = 2 + \frac{1}{6}\Lambda, \quad b = \frac{1}{2}\Lambda, \quad c = 2 - \frac{1}{2}\Lambda, \quad d = 1 - \frac{1}{6}\Lambda,$$

and Λ is given by
$$\Lambda = \frac{\delta^2}{\nu_1} u_1' \frac{\mu_w}{\mu_1}. \quad (346)$$

It follows that
$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{(12 + \Lambda)\nu_1}{6u_1 \delta}. \quad (347)$$

The form parameter
$$H = \delta_1/\delta_2$$

is not chosen from the assumed velocity profile, but is taken to be

$$H = \left(2.59 \frac{T_w}{T_z} + k_2 \Lambda\right) \frac{T_z}{T_1} + \frac{1}{2}(\gamma - 1)M_1^2, \quad (348)$$

which is approximately correct in the case of zero pressure gradient (when σ is close to unity), and agrees fairly well with values derived from Cohen and Reshotko's solutions provided k_2 is a suitably chosen function of T_w/T_z . Further the ratio of total boundary-layer thickness to momentum thickness is not derived from the velocity profile, but is given by

$$\frac{\delta}{\delta_2} = f = 9.072 \left[0.45 + 0.55 \frac{T_w}{T_1} + 0.09(\gamma - 1)M_1^2 \sigma^{\frac{1}{2}} \right]^{1-\omega} (1 + k_1 \Lambda), \quad (349)$$

which again is correct in the case of zero pressure gradient, and gives approximately the correct effect of pressure gradient when compared with Cohen and Reshotko's values provided k_1 is correctly chosen as a function of T_w/T_z .

Upon substituting from (346) to (349) into the momentum integral equation, written first in the form

$$\frac{d\delta_2}{dx} + \delta_2 \left\{ (H + 2) \frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1} \right\} = \frac{\tau_w}{\rho_1 u_1^2},$$

it is found that

$$\frac{d}{dx} (\rho_1^2 \delta_2^2) + 2\rho_1^2 \delta_2^2 \frac{u_1'}{u_1} \left[H + 2 - \frac{1}{6} f \frac{\mu_w}{\mu_1} \right] = \frac{4\mu_1 \rho_1}{u_1 f}, \quad (350)$$

which may be formally integrated to give δ_2 as a function of x . Luxton and Young suggest a numerical method of integration, but upon introducing a preliminary quadrature to give

$$G_2 = \exp \left\{ 2 \int^x \left(H + 2 - \frac{1}{6} f \frac{\mu_w}{\mu_1} \right) \frac{u_1'}{u_1} dx \right\},$$

it may be integrated as

$$\delta_2^2 = \frac{4}{\rho_1^2 G_2} \int_0^x \frac{\mu_1 \rho_1 G_2}{u_1 f} dx.$$

Having obtained δ_2 , δ_1 follows from (348) and then τ_w from (347), making use of (349).

It will be recognized that this method contains elements of both Pohlhausen's method and of the similar solutions. At first sight the relative contributions might appear to be rather arbitrary. On the other hand, Pohlhausen's method, applied formally in incompressible flow, usually yields values of the skin-friction which are too high, with too late a separation, the value of the pressure gradient parameter at separation being $\lambda = -0.157$. Methods based on the similar solutions, such as that due to Walz, usually err in the opposite direction, with $\lambda = -0.068$ at separation. It is interesting to note that the method of Luxton and Young yields $\lambda = -0.090$ at separation in incompressible flow, which was shown by Curle and Skan (1957) to be the optimum value. Comparisons with some of the accurate solutions of section 1 suggest that at a Mach number of 4 the method still yields results of good accuracy.

7. The method of Poots

A method has recently been given by Poots (1960) which is theoretically more acceptable than any of the methods given in this chapter, but becomes much more complicated and lengthy in practice. The method begins with the Stewartson transformation, so that upon further introducing transformed velocities

$$U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X},$$

equations (314) become

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = U_1 \frac{dU_1}{dX} (1+S) + \frac{\partial^2 U}{\partial Y^2}, \quad (351)$$

$$\text{and} \quad U \frac{\partial S}{\partial X} + V \frac{\partial S}{\partial Y} = \frac{\partial^2 S}{\partial Y^2}, \quad (352)$$

where X, Y are defined by (308) and (311), and the boundary conditions are given by (329). By formally integrating (351) and (352) with respect to Y from $Y = 0$ to $Y = \infty$, and by multiplying (351) by U and then integrating, three equations are obtained, analogous to the momentum,

kinetic-energy, and thermal-energy integral equations. These may be written as

$$U_1 \frac{d\delta_2^{*2}}{dX} + 2U_1' \delta_2^{*2} \left(2 + \frac{\delta_1^* + \Delta^*}{\delta_2^*} \right) = \frac{2\delta_2^*}{U_1} \left(\frac{\partial U}{\partial Y} \right)_w, \quad (353)$$

$$U_1 \frac{d\delta_3^{*2}}{dX} + 2U_1' \delta_3^{*2} \left(3 + \frac{2\delta_4^*}{\delta_2^*} \right) = \frac{4\delta_3^*}{U_1^2} \int_0^\infty \left(\frac{\partial U}{\partial Y} \right)^2 dY, \quad (354)$$

and
$$U_1 \frac{d\delta_4^{*2}}{dX} + 2U_1' \delta_4^{*2} = -2\delta_4^* \left(\frac{\partial S}{\partial Y} \right)_w, \quad (355)$$

where

$$\delta_1^* = \int_0^\infty \left(1 - \frac{U}{U_1} \right) dY,$$

$$\delta_2^* = \int_0^\infty \frac{U}{U_1} \left(1 - \frac{U}{U_1} \right) dY,$$

$$\delta_3^* = \int_0^\infty \frac{U}{U_1} \left(1 - \frac{U^2}{U_1^2} \right) dY,$$

$$\delta_4^* = \int_0^\infty S \frac{U}{U_1} dY,$$

and
$$\Delta^* = \int_0^\infty S dY,$$

the definition of δ_1^* being different from that used by Cohen and Reshotko.

Poots now expresses U/U_1 and S as quartics in Y/δ^* , where δ^* is a boundary-layer thickness in the transformed plane. These quartics are chosen to satisfy the conditions

$$U = 0, \quad S = S_w \quad \text{at} \quad Y = 0,$$

and

$$\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} = S = \frac{\partial S}{\partial Y} = \frac{\partial^2 S}{\partial Y^2} = 0, \quad \frac{U}{U_1} = 1, \quad \text{at} \quad Y = \delta^*$$

but, following the method devised by Tani (1954) for incompressible flow, the conditions obtained by setting $Y = 0$ in (351) and (352) are

not satisfied, a free coefficient being left in each of the quartics. Thus, for the case when $S_w = 1$,

$$\frac{U}{U_1} = \frac{Y^2}{\delta^{*2}} \left(6 - 8 \frac{Y}{\delta^*} + 3 \frac{Y^2}{\delta^{*2}} \right) + a \frac{Y}{\delta^*} \left(1 - \frac{Y}{\delta^*} \right)^3$$

and
$$S = 1 - \frac{Y^2}{\delta^{*2}} \left(6 - 8 \frac{Y}{\delta^*} + 3 \frac{Y^2}{\delta^{*2}} \right) - b \frac{Y}{\delta^*} \left(1 - \frac{Y}{\delta^*} \right)^3.$$

Equations (353) to (355) must be solved simultaneously for δ^* , a , b . The problem has thus been reduced to that of solving three simultaneous ordinary differential equations.

Poots has expressed the various factors in (353) to (355) in terms of δ^* , a , and b (for the case when $S_w = 1$), and has gone on to show how the simultaneous integration might be done, taking as an example the case when the external velocity in the transformed plane is

$$U_1 = U_0(1 - X),$$

the wall temperature, of course, being $T_w = 2T_z$. He finds that separation of the boundary layer occurs at

$$X_s = 0.075,$$

and this value agrees with that obtained by expanding a power series solution of the partial differential equations, valid at small values of X , and continuing the solution by a numerical step-by-step method.

This method provides simultaneously solutions for skin-friction, heat transfer, and indeed any desired boundary-layer property. It is, therefore, of wider application than the methods previously considered, where the thermal energy equation is sacrificed in an attempt to obtain a quicker approximate solution for the viscous boundary layer. If the computing facilities are available the method should yield good accuracy, even in cases of non-uniform wall-temperature.

8. The methods of Lilley and Illingworth

Two rather similar methods of calculating the skin-friction and the heat-transfer rate in a compressible laminar boundary layer have been given by Illingworth (1954) and Lilley (1959). In what follows a brief summary of Lilley's approach will be given, together with some discussion of the numerical examples considered by both these authors.

The method adopted by these two authors is designed to extend the analysis of Lighthill (1950*a*) to the case of compressible flow with both heat transfer and pressure gradient. Unlike Illingworth, Lilley applies first the Stewartson-Illingworth transformation and then, following

Lighthill, transforms the equations further to the von Mises form, in which the independent variables are X and ψ , and the dependent variables are S and Z , where

$$Z = U_1^2(X) - U^2(X, \psi),$$

and S , X , ψ , are defined respectively by (313), (311), and (309). The momentum and energy equations then take the form

$$\frac{\partial Z}{\partial X} = \nu_0 U \frac{\partial^2 Z}{\partial \psi^2} - S U_1 U_1' \quad (356)$$

and
$$\frac{\partial S}{\partial X} = \nu_0 \frac{\partial}{\partial \psi} \left(U \frac{\partial S}{\partial \psi} \right), \quad (357)$$

and the assumptions $\sigma = 1$, $\mu \propto T$ (not made in Lilley's analysis) are made here for simplicity. Lilley rewrites (356) in the form

$$\frac{\partial}{\partial X} \left(Z + \int_0^X S(\eta, \psi) dU_1^2(\eta) \right) = \nu_0 U \frac{\partial^2}{\partial \psi^2} \left(Z + \int_0^X S(\eta, 0) dU_1^2(\eta) \right),$$

and notes that at both $\psi = 0$ and $\psi = \infty$ this may be written as

$$\frac{\partial G}{\partial X} = \nu_0 U \frac{\partial^2 G}{\partial \psi^2}, \quad (358)$$

where
$$G(X, \psi) = Z + \int_0^X S(\eta, \psi) dU_1^2(\eta). \quad (359)$$

Lilley assumes that (358) is an adequate approximation right across the boundary layer, and solves it subject to the boundary conditions

$$G(X, \infty) = 0, \quad G(0, \psi) = 0, \quad (360)$$

and for small ψ

$$G(X, \psi) = U_1^2(X) - U^2(X, \psi) + \int_0^X S_w(\eta) dU_1^2(\eta), \quad (361)$$

and since, by (313),
$$S_w = \frac{T_w}{T_z} - 1,$$

this further simplifies to

$$G(X, \psi) = U_1^2(+0) - U^2(X, \psi) + \int_0^X \frac{T_w(\eta)}{T_z} dU_1^2(\eta). \quad (362)$$

Lilley indicates that equation (358) may be integrated, subject to the boundary conditions (360) and (362), by a method analogous to that used by Lighthill, the term $U^2(X, \psi)$ in (362) being first replaced by

$$U^2(X, \psi) = \frac{2\tau_w(X)}{\mu_0} \psi + O(\psi^3),$$

where μ_0 is the viscosity calculated at the reference conditions. His solution is

$$U_1^2(X) + \int_0^X \frac{T_w(\eta)}{T_x} dU_1^2(\eta) = \frac{3^{\frac{1}{2}} 2}{(-\frac{2}{3})! (\mu_0 \rho_0)^{\frac{1}{2}}} \int_0^X \{\tau_w(\eta)\}^{\frac{1}{2}} \left\{ \int_{\eta}^X [\tau_w(z)]^{\frac{1}{2}} dz \right\}^{-\frac{1}{2}} d\eta. \quad (363)$$

Lilley further transforms back into the physical plane (for which see his paper), and also shows how (363) may be approximately inverted to yield τ_w as an explicit function of $U_1(X)$ and $T_w(X)$; again no details will be given here. It is easily seen that (363) reduces identically to the corresponding expression in Lighthill's paper when incompressible flow is considered.

Lilley deals with equation (357) in a similar manner, the boundary conditions being

$$S(X, \infty) = S(0, \psi) = 0,$$

and

$$S = \frac{T_w}{T_z} - 1 + \frac{Q_w^*}{c_p T_z (\mu_0 \tau_w)} \left(\frac{2}{\psi} \right)^{\frac{1}{2}} \psi^{\frac{1}{2}},$$

where

$$Q_w^* = k_0 \left(\frac{\partial T}{\partial Y} \right)_w.$$

The solution follows Lighthill's solution for the heat transfer, and is

$$Q_w(X) = - \left(\frac{\mu_0 \rho_0}{9\sigma^2} \right)^{\frac{1}{2}} \frac{c_p \tau_w^{\frac{1}{2}}}{(\frac{1}{3})!} \left\{ \frac{T_w(+0) - T_z}{\left[\int_0^X \{\tau_w(z)\}^{\frac{1}{2}} dz \right]^{\frac{1}{2}}} + \int_0^X \left[\int_{X_1}^X \{\tau_w(z)\} dz \right]^{-\frac{1}{2}} d[T_w(X_1) - T_z] \right\}. \quad (364)$$

Lilley indicates how the solutions given above may be improved by taking $\sigma \neq 1$, by introducing a factor $C(X)$, following Rott (1953), to account approximately for the viscosity-temperature law, and by adapting the technique of Spalding (1958) so as to extend the range of conditions over which the theory is accurate.

Both Lilley and Illingworth (by his basically similar method) have considered in detail the numerical results for a number of special cases, and obtain fair agreement with the accurate solutions. In particular, Lilley has considered the three cases considered by the Mathematics Division, National Physical Laboratory, and obtains an accuracy which, for the velocity field at any rate, is comparable with that given by the other methods of this chapter. No comparisons were made for heat-transfer rates. Illingworth has not made any calculations for flows

with both heat transfer and pressure gradient at high Mach numbers. He has, however, made some comparisons with the low-speed 'similar' solutions, and finds that the skin-friction is satisfactory in accelerated flow but not in retarded flow, especially in the vicinity of the true separation position. On the other hand, the heat-transfer rate as calculated using the erroneous skin-friction is exceedingly accurate owing to a fortuitous cancellation of errors. It is unlikely that this will happen in general. Illingworth has also calculated the separation positions for a uniform adverse external velocity gradient with zero heat transfer for several values of the Mach number. His solution at zero Mach number differs from the exact value by less than 8 per cent. and at $M_0 = 4$ appears to agree with the solution calculated by the Mathematics Division, National Physical Laboratory, to within about 1 per cent. It is difficult to estimate what accuracy might be expected in a general case, in view of the respective failure and success of the method in predicting separation in the two examples cited.

9. Curle's method for calculating heat transfer

In view of the predominance of methods which aim at calculating skin-friction as accurately as possible, whilst sacrificing accuracy of prediction of heat transfer, it is valuable to conclude this chapter with a method which aims at calculating the heat-transfer rate when the skin-friction is assumed known. The method, due to Curle (1962 *a*) is an extension to high-speed flow of the method described in section 6 of Chapter 6.

The method begins with the Stewartson transformation, and for convenience the factor $\nu_0^{\frac{1}{2}}$ is neglected in the definition of the stream function, so that the transformed velocity and temperature fields are given by (326) to (329). If it is assumed that the velocity field is adequately known by some other method, it remains to calculate the temperature field, which is given by the equation

$$U \frac{\partial S}{\partial X} + V \frac{\partial S}{\partial Y} = \nu_0 \frac{\partial^2 S}{\partial Y^2}.$$

Following the technique used at low speeds (Curle, 1961 *a*), based on the ideas of Lighthill and Liepmann, this equation is integrated to yield

$$\frac{d}{dX} \int_0^\infty US dY = -\nu_0 \left(\frac{\partial S}{\partial Y} \right)_w. \quad (365)$$

Now Lighthill indicated that for Prandtl numbers of order unity and

above the velocity could be taken in an approximate form valid near to the wall, since the thermal boundary layer is usually thinner than the viscous layer. Thus we write

$$U(X, Y) \simeq Y \left(\frac{\partial U}{\partial Y} \right)_w + \frac{1}{2} Y^2 \left(\frac{\partial^2 U}{\partial Y^2} \right)_w + \frac{1}{6} Y^3 \left(\frac{\partial^3 U}{\partial Y^3} \right)_w \dots, \quad (366)$$

where the first term only was retained by Lighthill, and the third term is negligible in the low-speed case provided temperature differences are small. In the case under consideration we assume that $(\partial U / \partial Y)_w$ is known, $(\partial^2 U / \partial Y^2)_w$ is obtained by setting $Y = 0$ in (327), and is

$$\left(\frac{\partial^2 U}{\partial Y^2} \right)_w = -(1 + S_w) U_1 U_1' \nu_0^{-1},$$

whilst $(\partial^3 U / \partial Y^3)_w$, obtained by differentiating (327) with respect to Y before setting $Y = 0$, is equal to

$$\left(\frac{\partial^3 U}{\partial Y^3} \right)_w = - \left(\frac{\partial S}{\partial Y} \right)_w U_1 U_1' \nu_0^{-1}.$$

Thus, writing $(\partial S / \partial Y)_w$ as S'_w , (366) becomes

$$U(X, Y) \simeq Y \left(\frac{\partial U}{\partial Y} \right)_w - \frac{U_1 U_1'}{\nu_0} \left\{ \frac{1}{2} (1 + S_w) Y^2 + \frac{1}{6} S'_w Y^3 \dots \right\},$$

and hence (365) may be written

$$\begin{aligned} -\nu_0 S'_w &\simeq \frac{d}{dX} \left(\left(\frac{\partial U}{\partial Y} \right)_w \int_0^\infty Y S dY - \right. \\ &\quad \left. - \frac{U_1 U_1'}{\nu_0} \left[\frac{1}{2} (1 + S_w) \int_0^\infty Y^2 S dY + \frac{1}{6} S'_w \int_0^\infty Y^3 S dY \right] \right). \end{aligned} \quad (367)$$

Following Liepmann, we write

$$\int_0^\infty Y^n S dY \propto \frac{S_w^{n+2}}{(S'_w)^{n+1}}, \quad (368)$$

for the case of a wall-temperature distribution which is either uniform or possibly uniform with zero heat transfer upstream of say $X = X_1$, with a step to a new uniform value downstream of this position. Then (367), after formal integration with respect to X , and substitution from (368), takes the form

$$\begin{aligned} -\nu_0 \int_0^X S'_w dX &= a \left(\frac{\partial U}{\partial Y} \right)_w \frac{S_w^3}{(S'_w)^2} + \frac{U_1 U_1'}{\nu_0} \left[\frac{1}{2} b (1 + S_w) \frac{S_w^4}{(S'_w)^3} - \frac{1}{6} c \frac{S_w^5}{(S'_w)^3} \right] \\ &= a \left(\frac{\partial U}{\partial Y} \right)_w \frac{S_w^3}{(S'_w)^2} + \frac{U_1 U_1'}{2\nu_0} \frac{S_w^4}{(S'_w)^3} [b(1 + S_w) - \frac{1}{3} c S_w], \end{aligned} \quad (369)$$

where a, b, c are positive constants, which will be determined at a later stage. Finally, if we write

$$F = \frac{LS'_w}{S_w} = \frac{L}{T_w - T_z} \left(\frac{\partial T}{\partial Y} \right)_w,$$

which is a Nusselt number in the transformed plane, (369) becomes

$$-\int_0^X F d\left(\frac{X}{L}\right) = a \frac{L^2}{\nu_0} \left(\frac{\partial U}{\partial Y} \right)_w \frac{1}{F^2} + \frac{L^3 U_1 U''}{2\nu_0^2} [b(1+S_w) - \frac{1}{3}cS_w] \frac{1}{F^3}. \quad (370)$$

The three terms whose coefficients include a, b, c are of decreasing orders of importance. Thus Lighthill found that upon retaining only the first term on the right-hand side of (370) (thus setting b and c equal to zero) the equation could be integrated analytically, and a formula obtained for F which agreed well with accurate solutions except near to separation, where $(\partial U / \partial Y)_w$ is zero. Similarly Curle (1961*a*) found that for the case of small temperature differences (i.e. S_w very nearly zero) the retention of the second term, involving the coefficient b , enabled the accuracy to be maintained right up to separation. The values suggested by Curle for the two coefficients are

$$a = 0.2226, \quad b = 0.1046. \quad (371)$$

It might be expected that these two terms alone would be equally satisfactory in more general cases, with S_w not almost zero, unless both almost vanish simultaneously. This could happen near to separation when S_w is close to -1 , that is for very low wall temperatures. Now examination of the Cohen and Reshotko similar solutions bears out this point, and Curle suggests retention of the third term to cover this relatively unlikely occurrence. It will be noted that the simplicity of equation (369) is not at all reduced by the retention of this term. The value of c is chosen to obtain good agreement with the accurately known similar solutions, and the term in square brackets in (369) becomes

$$b + (b - \frac{1}{3}c)S_w,$$

where b is given by (371) and

$$b - \frac{1}{3}c = 0.0833. \quad (372)$$

The application of the method is limited strictly to the case $\sigma = 1$ (although a factor $\sigma^{\frac{1}{2}}$ might well adequately convert the values of F into those appropriate to alternative values of σ close to unity), and to the case of uniform wall temperature, unless the assumption be made that the skin-friction is little dependent upon wall temperature, when

the heat-transfer rates appropriate to each of a number of elementary steps in wall temperature may be added. Within these limits the accuracy is considerable, and for the similar solutions heat-transfer rates are given to within about ± 2 per cent. for the considerable range of wall temperatures and pressure gradients considered.

INTERACTIONS BETWEEN SHOCK WAVES AND BOUNDARY LAYERS

It has already been remarked, in the first chapter of this book, that interactions between the boundary layer and the external stream are much more important in supersonic flow than in subsonic flow. Thus, although it is possible to treat the boundary layer and the external flow separately in many practical problems at low speeds, this is so only rarely in the supersonic case, since the presence of the boundary layer can make a tremendous difference to the external flow, particularly if the layer is a laminar one. Thus, although the inviscid theory of supersonic flow does not allow a disturbance at a point to have an upstream influence, the boundary layer provides the means whereby an upstream influence can exist.

One of the mechanisms whereby this can occur was suggested by Oswatitsch and Weighardt (1943), who remark that if a disturbance causes a positive pressure gradient the boundary layer will thicken, so that the external streamlines must begin to curve slightly upstream. Now this curvature will itself produce a pressure gradient in the same sense, and the boundary layer grows in equilibrium with the pressure gradient caused by its own growth. When the disturbance is strong enough to cause the boundary layer to separate the presence of a region of reversed flow (possibly extensive) makes a considerable difference to the mechanism, and, indeed, upstream spreading of a position of separation can occur in subsonic flow equally well. An important practical example of the interaction of a boundary layer with the external flow is the problem of the interaction between a shock wave and a boundary layer. In practice the nature of the interaction will depend vitally upon the state of the boundary layer. Three main possibilities exist, namely (i) when the boundary layer is laminar throughout the region of interaction; (ii) when the boundary layer goes turbulent downstream of the separation position, but is laminar upstream; (iii) when the boundary layer is turbulent throughout the interaction region. In this chapter it will only be possible to consider the case of boundary layers which are laminar throughout the region of interaction (and these

only very briefly). For details of the other cases, reference may be made to reviews by Holder and Gadd (1955) and by Gadd (1957*a*).

1. Principal results of experimental investigations

In practical cases of shock-wave and boundary-layer interaction, such as those occurring on an aerofoil surface at transonic speeds, it is sometimes difficult to separate the fundamental effects of the interaction

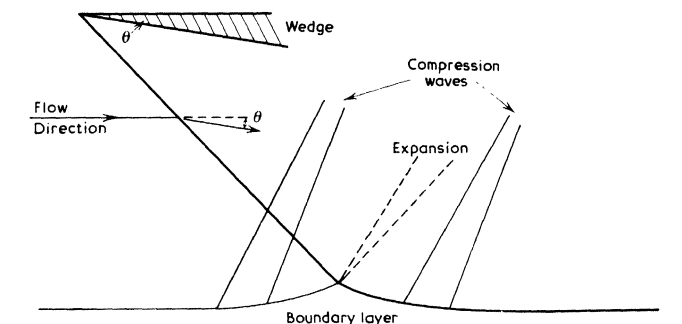


FIG. 2. Boundary layer and externally generated shock.

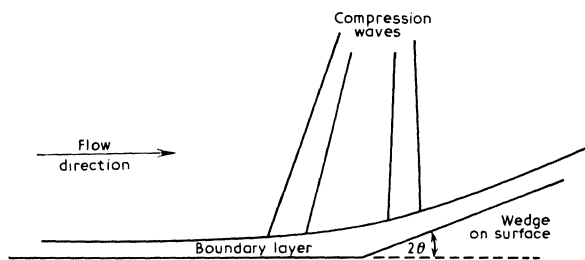


FIG. 3. Shock generated from within boundary layer by wedge on surface.

from those which are due to the limited extent of the supersonic region or to the pressure gradients associated with surface curvatures. For this reason the majority of the basic experimental investigations have been carried out with a boundary layer on a flat surface, so that the pressure gradients would be entirely absent if there were no shock waves. The shock is produced either by placing a wedge in the supersonic stream above the plate (Fig. 2) or by attaching a wedge (Fig. 3) or a step to the surface of the plate. If θ is the angle of deflexion through the external shock, then the streamlines at the edge of the boundary layer are turned through an angle 2θ in passing through the shock and

the reflected expansion. A wedge of angle 2θ on the plate will give roughly the same pressure rise, and in this case the external streamlines change direction continuously but are deflected discontinuously through an angle 2θ relative to the wall.

It is found experimentally that the pressure distribution is roughly symmetrical about either the apex of the wedge on the surface or the position where the shock strikes the boundary layer. When separation occurs the pressure distribution has three points of inflexion, which arise for the following reasons. The pressure gradient rises from zero well upstream of the interaction to a maximum value at separation. Downstream of separation the pressure gradient decreases with distance due to the thickening of the dead-air region, and then increases again when the dead-air region becomes thinner downstream of where the shock strikes the layer. It increases to a maximum value at the position of reattachment and then gradually decreases to zero as the pressure tends to its limiting downstream value.

In the case of zero heat transfer a considerable body of experimental evidence has been accumulated concerning the dependence of this pressure distribution upon such parameters as Mach number, Reynolds number, and shock strength. Chapman, Kuehn, and Larson (1958) have correlated some of these results. In particular they find that for a range of upstream Mach numbers $1.1 < M_0 < 3.6$, and with a Reynolds number (based on external conditions in the interaction region) $10^4 < R < 10^6$, the pressure coefficient C_p at separation takes the value

$$C_p = 0.93\{(M_0^2 - 1)R\}^{-1}, \quad (373)$$

the pressure coefficient being defined as

$$C_p = \frac{2}{\gamma M_0^2} \left(\frac{p}{p_0} - 1 \right).$$

The scatter in the experimental pressure coefficients is of order 10 per cent.† Various theoretical investigations give formulae like (373) with the numerical coefficient taking values ranging from 0.83 to 1.15.

More recently attention has been turned towards the effects of heat transfer upon the pressure distribution. Early experiments (Gadd 1957*b*) suggested that the pressure distribution was barely affected by heating or cooling the wall. This conclusion did not agree with the theoretical treatments then available, and further experiments carried

† A recent paper by Sterrett and Emery (1960) indicates that the numerical coefficient appropriate to their experimental observations decreases as M_0 increases from 4 to 6.

out by Gadd and Attridge (1961) with considerably greater rates of heat transfer have indicated that, provided the boundary layer remains laminar throughout, the length of the interaction region increases with wall temperature, with a corresponding decrease in the pressure gradients. Agreement between these experimental results and the latest theoretical investigations is promising.

2. Summary of early theoretical investigations

One of the first theoretical attempts to throw some light on the interaction of a shock wave with a laminar boundary layer was made by Howarth (1948*b*). Howarth's model consisted of two semi-infinite streams moving parallel to one another, one at a supersonic speed and the other at a subsonic speed. A small steady disturbance originates in the supersonic region and is propagated along the appropriate Mach line. At the boundary with the subsonic region it is partly reflected and partly transmitted. The portion transmitted into the subsonic region affects the whole of that part of the flow, including the interface, and causes further disturbances in the supersonic region which are propagated along the other family of Mach lines. For certain particular forms of incident wave, Howarth was able to calculate the details of the complete flow.

Howarth's results were somewhat inconclusive, because of the assumption of an infinite depth of subsonic flow, as compared with the thin subsonic region in a practical boundary layer. Accordingly Tsien and Finston (1949) improved on Howarth's model by making the subsonic region of only finite thickness and bounded on the other side by a plane wall, but again no conclusive results could be obtained as no indication was available as to what Mach number or thickness would be appropriate for the subsonic region. Lighthill (1950*b*) therefore further refined the analysis by replacing the subsonic layer by a region in which the Mach number falls continuously from M_1 in the mainstream to zero at the wall. Unfortunately these assumptions led to predictions of the upstream influence of the disturbance which were an order of magnitude smaller than experimental values.

Now the source of the error in Lighthill's work, as he himself pointed out in later papers (Lighthill 1953*a, b*), was one of inconsistent approximation. All three contributions discussed above assume that viscosity may be neglected except in the actual setting up of the basic steady flow, and that squares of disturbances may be neglected. Now in a boundary layer the pressure gradient at the wall is exactly equal to

the lateral derivative of the viscous shear stress, so the neglect of viscosity will force the pressure at the wall to remain constant unless a non-zero Mach number at the wall, M_2 , is postulated, allowing the pressure gradient to be balanced by inertial forces. Thus the continuous variation in Mach number from $M_1 > 1$ to $M_2 < 1$ is an improvement on the Tsien and Finston model, but the value of M_2 still remains to be found.

Lighthill then makes the crucial point that viscous forces have not been entirely neglected, but only disturbances to the viscous forces. Now, following the ideas of Stratford (1954), we know that the viscous forces in the outer part of the boundary layer are almost exactly what they would be in the absence of pressure gradient; in other words, the disturbances to the viscous forces are non-negligible only in an inner portion of the boundary layer. Lighthill assumes (and verifies afterwards) that this inner layer is sufficiently thin for the approximation of incompressibility to be made (because velocities very near to the wall are low). He assumes also that the velocity may be assumed to be linear within this region, which is probably adequate provided that the disturbance is not great enough to cause the boundary layer to be near to separation, and finds that the value of the effective Mach number at the wall for use in the inviscid theory is

$$M_2 = 0.78M'(0)L,$$

where $M'(0)$ is the lateral Mach number gradient at the wall in the undisturbed flow, which is proportional to the skin-friction, and L is a length defined in terms of the kinematic viscosity and skin-friction at the wall, and the wave number of the disturbance.

Comparisons of the magnitude of the upstream influence with those obtained experimentally show reasonable agreement. As an example, the upstream disturbance falls by a factor e^{-1} in about 5 or 6 boundary-layer thicknesses when the Reynolds number is of order 10^5 to 10^6 and the free-stream Mach number is about 2 to 3. Accordingly it will have fallen to 5 per cent. in about 3 times this distance, say 15 to 20 boundary-layer thicknesses.

3. Gadd's analyses for interactions causing separation

Provided that the incident shock is strong enough the boundary layer will separate and the assumptions of the preceding analysis will break down. Alternative methods have therefore been developed for considering shock-wave and boundary-layer interaction in cases where separation occurs well upstream of the agency which causes it. This

situation is of some considerable practical importance and can arise in a variety of ways, such as when a boundary layer encounters a step or a wedge on the surface, or when separation occurs upstream of a sufficiently strong externally generated shock. In such circumstances the pressure at the wall and the development of the boundary layer are governed by the equilibrium between the pressure gradient, the resulting boundary-layer thickening and the deflexion of the external streamlines. The relationship between pressure and the deflexion of the external streamlines is derived from the theory of supersonic flow and, provided the deflexion angle is not too great, that given by linear theory will be adequate, namely

$$\frac{p}{p_0} - 1 = \frac{\gamma M_0^2}{(M_0^2 - 1)^{\frac{1}{2}}} \alpha, \quad (374)$$

where suffix zero refers to conditions in the undisturbed stream, about which the linear perturbation is taken, and α is the angle through which the external streamlines are deflected. In the shock-wave and boundary-layer interaction problem it is usual to equate the deflexion of the external streamlines (caused by boundary-layer thickening) to the rate of growth of the displacement thickness, that is

$$\alpha = \frac{d}{dx} \{ \delta_1(x) - \delta_{10}(x) \}, \quad (375)$$

where $d\{\delta_{10}(x)\}/dx$ represents the rate of growth of displacement thickness in the absence of pressure gradient, which is often neglected if the interaction takes place far enough from the leading edge of the boundary layer. Combining (374) and (375) we have

$$\frac{p}{p_0} - 1 = \frac{\gamma M_0^2}{(M_0^2 - 1)^{\frac{1}{2}}} \frac{d}{dx} \{ \delta_1(x) - \delta_{10}(x) \}. \quad (376)$$

A second relationship connecting p and δ_1 is given from the basic theory of laminar boundary-layer development in the presence of a pressure gradient, and three papers by Gadd differ mainly in the method used to treat this part of the problem.

In the earliest paper (Gadd, 1953*b*) it is assumed that in the region upstream of the interaction, where there is no pressure gradient, the velocity within the boundary layer is given by

$$\frac{u}{u_0} = \sin\left(\frac{\pi y}{2\delta}\right),$$

where the boundary-layer thickness δ is chosen so that the displacement thickness is in agreement with that given by Crocco's calculations.

If we assume that there is no heat transfer at the wall, and that $\sigma = 1$ and $\mu \propto T$, this yields

$$\delta = \frac{1.721(1+0.277M_0^2)}{\{1-(1.422/M_0)\tan^{-1}(0.4472M_0)\}} \left(\frac{\nu_0 x}{u_0}\right)^{\frac{1}{2}}.$$

In the region where the boundary layer thickens under the influence of the shock it is assumed that

$$\frac{u}{u_1} = r_1 = \sin\left\{\frac{1}{2}\pi\left(\frac{p}{p_0}\right)^{(\gamma+1)/2\gamma}\left(\frac{y}{\delta}-\lambda\right)\right\},$$

when
$$n\lambda < \frac{y}{\delta} < \lambda + \left(\frac{p_0}{p}\right)^{(\gamma+1)/2\gamma},$$

and
$$\frac{u}{u_1} = r_1 + r_2, \quad \text{when} \quad 0 < \frac{y}{\delta} < n\lambda,$$

where
$$r_2 = \left(1 - \frac{y}{n\lambda\delta}\right)^3 \sin\left\{\frac{1}{2}\pi\left(\frac{p}{p_0}\right)^{(\gamma+1)/2\gamma}\lambda\right\}.$$

The form of r_1 is determined by the condition that in the outer part of the boundary layer changes in the profile caused by the pressure gradient may be determined by essentially inviscid considerations, following the ideas used by Stratford (1954). The form of r_2 ensures continuity of u , $\partial u/\partial y$, and $\partial^2 u/\partial y^2$ at the join, $y = n\lambda$, with the inner profile, and one of the two parameters n , λ , is determined by the balance between pressure gradient and viscous stresses at the wall. The condition on mass flow (corresponding to Stratford's condition that ψ is continuous at the join) is also approximately satisfied provided that $(p/p_0)-1$ and λ both remain small. The other relationship between n and λ , which is required to complete the solution, is given by (376).

Gadd made a number of comparisons between his theory and experiment. In particular the predicted value of the pressure coefficient at separation is of order 10 per cent. greater than the mean experimental value (373), which is extremely satisfactory.

In a second paper Gadd (1956) considers again the case when there is zero heat transfer and the Prandtl number is equal to unity, and assumes that the velocity profiles are given either by

$$\frac{\rho u}{\rho_1 u_1} = \frac{\tanh m + \tanh\{(y/l)-m\}}{1 + \tanh m},$$

or by a similar form which will not be considered here. The virtues of this family of profiles are firstly that it is simple, and secondly that for large values of m there is an extensive region of low-velocity flow near

to the wall, rather like that occurring in a well-separated laminar boundary layer. The main drawback is that no region of negative velocity occurs, so that the position of separation cannot be predicted.

The value of l is determined from the displacement thickness in the region upstream of the interaction, and is assumed to be constant. Thus

$$\delta_1 = \frac{l \log(1 + e^{2m})}{1 + \tanh m}, \quad (377)$$

so that just upstream of the interaction, where $m = 0$, we have

$$\delta_1(x_0) = l \log 2.$$

The parameter m is determined from the condition that the pressure gradient at the surface is given by

$$\frac{dp}{dx} = \left(\mu \frac{\partial^2 u}{\partial y^2} \right)_w,$$

which follows from the momentum equation (12) upon setting $(\partial \mu / \partial y)_w$ equal to zero for the case of zero heat transfer, and a second relationship between m and p is given by (376) with δ_1 given by (377). Simultaneous solution of these equations yields p as a function of x . Gadd remarks that in the case of an externally generated shock (strong enough to deflect the external streamlines through an angle θ) the streamlines are deflected instantaneously through an angle 2θ at the position where the shock strikes the boundary layer, and this corresponds to the discontinuous change in the slope of the wall when the interaction is due to a wedge on the surface.

Gadd's results may be expressed as

$$C_p \{ (M_0^2 - 1) R \}^{\frac{1}{2}} = 0.90 \frac{1 + 0.20 M_0^2}{(1 + 0.267 M_0^2)^{\frac{1}{2}}} \frac{dQ}{ds} + \frac{4\epsilon \theta R^{\frac{1}{2}}}{(M_0^2 - 1)^{\frac{1}{2}}},$$

in which dQ/ds is a numerically defined universal function of $s - \bar{s}$, s is a non-dimensional value of x , and $\bar{s}$ is the value of s corresponding to the upstream end of the interaction region. A comparison of the predictions of the upstream influence with the experimentally observed values shows fair agreement.

In a third paper Gadd (1957c) presents a more complete treatment of the problem (including the effects of heat transfer) by a similar approach to those discussed above, but involving fewer approximations. For the outer part of the boundary layer, following more closely the analysis used by Stratford (1954) in low-speed flow, it is assumed that the distortion of the upstream zero-pressure-gradient profile may again be determined by largely inviscid considerations, and in the inner part

of the boundary layer (close to the wall) the profile is assumed given by a form which is essentially equivalent to

$$\frac{\rho u}{\rho_0 u_0} = \frac{T_0}{T_w} \left(a \frac{y}{y_j} + b \left(\frac{y}{y_j} \right)^2 + c \left(\frac{y}{y_j} \right)^3 \right).$$

The three coefficients a , b , c , and the position of the join y_j , are then determined by the conditions that $\psi \left(= \int_0^y \rho u \, dy \right)$, ρu , $\partial(\rho u)/\partial y$, and $\partial^2(\rho u)/\partial y^2$ are all continuous at the join. Thus the complete profile is determined in principle at any rate. In practice two further approximations are made, namely that the inner profile is thin (so that the usual simplifications may be made to the joining conditions) and that the changes in displacement thickness under the action of the pressure gradient are mainly due to changes in the shape of the inner profile. Subject to these approximations it is indicated that the pressure at separation is given by an expression similar to the empirical form (373), but with the coefficient 0.93 replaced by 1.13. It is also predicted that the wall temperature T_w has no influence on the pressure coefficient at separation, but that the streamwise extent of the region of interaction varies with T_w as $T_w^{\frac{1}{2}}$. As has already been remarked, the earliest experiments on the effects of wall temperature suggested that the pressure distribution is barely dependent upon wall temperature. More recent experiments, however, with greater rates of heat transfer, have shown that the streamwise extent of the interaction region does indeed increase with T_w , though probably not as quickly as $T_w^{\frac{1}{2}}$.

4. The analysis of Hakkinen, Greber, Trilling, and Abarbanel

An interesting analysis, based on similar ideas to the above, has been given by Hakkinen *et al.* (1959). The approximations made, though numerous, are all physically sound, and the analysis deserves mention because of its essential simplicity.

It is assumed first that the boundary layer consists of an outer portion, differing little in 'shape' from the upstream undisturbed layer, and an inner sub-layer of small momentum. The join between the layers occurs at $y = \Delta_j$ say, and the apparent displacement of the undisturbed layer is Δ , with $\Delta \leq \Delta_j$. Two consequences of this assumption are that

$$\mu u(\Delta_j) \simeq \tau_0(\Delta_j - \Delta)$$

and

$$\tau(\Delta_j) \simeq \tau_0,$$

where τ is the local viscous stress and τ_0 the skin-friction immediately upstream of the interaction. Since the momentum therein is small, it

may be assumed that pressure and viscous forces balance throughout the inner layer. Thus

$$\frac{\partial \tau}{\partial y} \simeq \frac{dp}{dx} \quad \text{when } y \leq \Delta_j$$

and hence

$$\tau(\Delta_j) \simeq \tau_w + \Delta_j \frac{dp}{dx},$$

where τ_w is the local skin-friction and, more generally,

$$\mu u = \tau_w y + \frac{1}{2} \frac{dp}{dx} y^2 \quad \text{when } y \leq \Delta_j.$$

The conditions of continuity in u and τ at the join, $y = \Delta_j$, may then be shown to yield

$$\Delta = \frac{(\tau_0 - \tau_w)^2}{2\tau_0(dp/dx)}. \quad (378)$$

If it is assumed that the pressure rise due to the thickening of the boundary layer is mainly due to the growth in Δ , then

$$C_p \simeq \frac{2}{(M_0^2 - 1)^{\frac{1}{2}}} \frac{d\Delta}{dx}, \quad (379)$$

and elimination of Δ between (378) and (379) yields

$$\tau_0(M_0^2 - 1)^{\frac{1}{2}} \frac{dp}{dx} \int_{-\infty}^x C_p dx = (\tau_0 - \tau_w)^2,$$

or

$$\frac{1}{2} \rho_0 u_0^2 \tau_0 (M_0^2 - 1)^{\frac{1}{2}} \frac{dC_p}{dx} \int_{-\infty}^x C_p dx = (\tau_0 - \tau_w)^2. \quad (380)$$

Since it is found experimentally that dC_p/dx is very nearly constant, except in the region where C_p is very small, we may write approximately

$$\frac{dC_p}{dx} \int_{-\infty}^x C_p dx \simeq \frac{1}{2} C_p^2, \quad (381)$$

so that (380) becomes

$$\frac{1}{2} C_p [\rho_0 u_0^2 (M_0^2 - 1)^{\frac{1}{2}} \tau_0]^{\frac{1}{2}} \simeq \tau_0 - \tau_w,$$

or

$$C_p (M_0^2 - 1)^{\frac{1}{2}} (\frac{1}{2} c_{f0})^{\frac{1}{2}} \simeq c_{f0} - c_{fw}, \quad (382)$$

where

$$c_{f0} = \frac{\tau_0}{\frac{1}{2} \rho_0 u_0^2} \quad \text{and} \quad c_{fw} = \frac{\tau_w}{\frac{1}{2} \rho_0 u_0^2}$$

are skin-friction coefficients. Thus, at separation, (382) yields

$$C_{ps} (M_0^2 - 1)^{\frac{1}{2}} = (2c_{f0})^{\frac{1}{2}},$$

and, if $\mu \propto T$, it follows from (270) that this becomes

$$C_{ps} \{(M_0^2 - 1)R\}^{\frac{1}{2}} = (1.328)^{\frac{1}{2}} = 1.15, \quad (383)$$

where R is the local Reynolds number based on conditions at the edge of the boundary layer. This value of the pressure coefficient at separation is about 20 per cent. higher than the mean experimental value, so the agreement is fair. This is probably partly fortuitous, since the approximation (381) alone could introduce an error of order 20 per cent. in C_p , so some cancellation of the various sources of error has presumably been present.

These same authors have also carried out a similar analysis, making direct use of the momentum integral equation, and Greber (1960) has shown how to take account of convex wall curvature. This analysis has been somewhat improved upon by Gadd (Bray, Gadd, and Woodger, 1960), and will now be discussed.

5. The work of Gadd and Greber

The description given here follows that of Gadd, and will be restricted to the case of solid walls, although Gadd in fact developed the method in a form which makes it applicable when suction or injection takes place. Gadd writes the displacement thickness δ_1 as

$$\begin{aligned}\delta_1 &= \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy = \int_0^\infty \frac{\rho}{\rho_1} \left(\frac{T}{T_1} - \frac{u}{u_1}\right) dy \\ &= \left\{ \left(1 + \frac{1}{2}(\gamma - 1)M_1^2\right)r_1 + \frac{1}{2}(\gamma - 1)M_1^2 \right\} \delta_2,\end{aligned}\quad (384)$$

where

$$r_1 = \frac{1}{\delta_2} \int_0^\infty \left(\frac{T_H}{T_{H1}} - \frac{u}{u_1}\right) \frac{\rho}{\rho_1} dy,$$

and T_H is the total temperature. If, for simplicity, it is assumed that to a good approximation we may write

$$\frac{T_H}{T_1} = 1 + \frac{1}{2}(\gamma - 1)M_1^2 \frac{u}{u_1} + \left(\frac{T_w}{T_1} - 1\right)\left(1 - \frac{u}{u_1}\right), \quad (385)$$

which holds when there is zero heat transfer and when there is zero pressure gradient (provided $\sigma = 1$), then (384) becomes

$$\delta_1 = \left\{ \frac{T_w}{T_1} r_2 + \frac{1}{2}(\gamma - 1)M_1^2 \right\} \delta_2, \quad (386)$$

where

$$r_2 = \frac{1}{\delta_2} \int_0^\infty \frac{\rho}{\rho_1} \left(1 - \frac{u}{u_1}\right) dy.$$

This form will be accepted in what follows.

Now in an interaction which takes place over a distance which is small compared with the distance from the leading edge, changes in

displacement thickness are likely to be large compared with changes in momentum thickness. This is borne out by the experimental results and was surmised by Gadd by inspection of the momentum integral equation. Accordingly it may be assumed that δ_2 is approximately constant, as is M_1 . Then it follows from (386) that

$$\delta_1 \asymp \left\{ \frac{T_w}{T_0} r_2 + \frac{1}{2}(\gamma - 1) M_0^2 \right\} \delta_{20},$$

so that

$$\delta_1 - \delta_{10} \asymp \frac{T_w}{T_0} (r_2 - r_{20}) \delta_{20}, \quad (387)$$

where suffix zero refers to (constant) values in the region upstream of the interaction.

Gadd further deduces from the momentum equation (12) that

$$\frac{dp}{dx} = \frac{\mu_1 u_1}{\delta_2^2} \frac{T_1}{T_w} g, \quad (388)$$

where

$$g = \left[\frac{\partial^2}{\partial x^2} \left(\frac{u}{u_1} \right) \right]_w$$

and

$$z = \frac{1}{\delta_2} \int_0^y \frac{\rho}{\rho_1} dy.$$

He then assumes that g is a function only of r_2 , being independent of such parameters as the ratio of wall temperature to mainstream temperature, and deduces from (376) and (387) that

$$\frac{dp}{dx} = \frac{\gamma M_0^2 p_0}{(M_0^2 - 1)^{\frac{1}{2}}} \frac{d^2}{dx^2} (\delta_1 - \delta_{10}) = \frac{\gamma M_0^2 p_0}{(M_0^2 - 1)^{\frac{1}{2}}} \frac{T_w}{T_0} \delta_{20} \frac{d^2 r_2}{dx^2}. \quad (389)$$

By equating this value of dp/dx to that given by (388) it follows that

$$\frac{d^2 r_2}{dx^2} = \frac{(M_0^2 - 1)^{\frac{1}{2}}}{\gamma M_0^2 p_0} \frac{\mu_0 u_0}{\delta_{20}^3} \frac{T_0^2}{T_w^2} g(r_2).$$

Multiplication by $2(dr_2/dx)$ and formal integration with respect to x yield

$$\frac{dr_2}{dx} = \frac{T_0}{T_w} \left\{ \frac{\mu_0 u_0 (M_0^2 - 1)^{\frac{1}{2}}}{\gamma M_0^2 p_0 \delta_{20}^3} \int_0^{r_2} g(r) dr \right\}^{\frac{1}{2}}, \quad (390)$$

so that dr_2/dx is known as a function of r_2 . This relationship is universal except for the scaling factor, which is such that the extent of the region of interaction is proportional to T_w . This is in better agreement with experiment than the earlier predictions, in which the extent of the interaction was found to be proportional to $T_w^{\frac{1}{2}}$.

Gadd assumes that the universal relationship between g and r_2 is precisely that appropriate to the similar solutions with zero heat transfer, and deduces from (390) the relationship between r_2 and dr_2/dx , after which the value of r_2 (or p) as a function of x easily follows. He finds, in particular, that the value of the pressure coefficient at separation is equal to

$$C_{ps} = 0.94\{(M_0^2 - 1)R\}^{-1}, \quad (391)$$

which differs from the mean experimental value by only 1 per cent., in the case of zero heat transfer at least, over a range of Mach numbers up to about 4.

Gadd has also given a second method which does not involve the use of the approximate temperature relationship (385), but rather assumes that the function g is dependent upon r_1 and T_w , the dependence being given by the similar solutions with heat transfer. The results are broadly the same as those given by the method discussed above but indicate slightly less dependence upon T_w in the case of the extent of the interaction region, and a slight dependence of the pressure coefficient at separation upon T_w .

In the same paper Bray, Gadd, and Woodger have also given a simplified outline of a method originally due to Crocco and Lees (1952), and have discussed solutions of the shock-wave and boundary-layer interaction problem obtained by that method, making use of the National Physical Laboratory DEUCE computer. This work will not be discussed here, partly for reasons of space, and partly because it is the author's view that, for investigations of this type of interaction, the method is less convenient than others discussed in this book.

6. The method of Curle

We conclude this book with a brief description of a method due to Curle (1961*b*), which bears considerable similarity to that discussed in the previous section and, indeed, was developed almost simultaneously. The method begins with the approximate equation (341) for the momentum thickness of a compressible laminar boundary layer, from which it is deduced that if the pressure rise is a short sharp one (as has already been seen to be often true in interactions between shock waves and laminar boundary layers) then it automatically follows that approximately

$$\delta_2 \approx \text{constant}.$$

Under the same conditions it follows from (345) that

$$\delta_1 - \delta_{10} = \delta_{20} \frac{T_w}{T_0} \{H'(\lambda') - H'(0)\}, \quad (392)$$

where λ' is defined by (343) and $H'(\lambda')$ is a universal function. From this definition of λ' it follows that when the pressure (and velocity) changes are small we have

$$\lambda' \simeq \frac{\delta_{20}^2 T_w}{\nu_0 T_0} u_1' \simeq -\frac{u_0 \delta_{20}^2 T_w}{\gamma M_0^2 p_0 \nu_0 T_0} \frac{dp}{dx}. \quad (393)$$

Then elimination of δ_1 and λ' between equations (376), (392), and (393) leads to a single equation for p or u_1 . This equation may be expressed non-dimensionally by writing

$$F = 0.4096(M_0^2 - 1)^{-1} R^{\frac{1}{2}} \left(1 - \frac{u_1^2}{u_0^2} \right), \quad (394)$$

$$X = 1.820(M_0^2 - 1)^{-1} R^{\frac{1}{2}} \frac{T_0}{T_w} \left(\frac{x}{x_s} - 1 \right), \quad (395)$$

$$\text{and} \quad \Lambda = 1.638\theta R^{\frac{1}{2}} (M_0^2 - 1)^{-1}, \quad (396)$$

where x_s represents the position of separation (or any other convenient position within the region of interaction), the Reynolds number R is

$$R = \frac{u_0 x_s}{\nu_0},$$

and 2θ is the angle of deflexion of the streamlines relative to the wall caused by either the shock (Fig. 2) or the wedge (Fig. 3). We then find that

$$\int_{-\infty}^x F dX = H'(F') - H'(0) + \Lambda \epsilon (X - X_1), \quad (397)$$

where $\epsilon = 0$ upstream of the position $X = X_1$ where the shock strikes the boundary layer, and $\epsilon = 1$ downstream of this position. Solution of (397) yields F and F' as functions of X . This is done by multiplying the equation by F'' and formally integrating with respect to X from $-\infty$ to X , using integration by parts where necessary. This yields

$$(F - \Lambda \epsilon)^2 = 2F'H'(F') - 2 \int_0^{F'} H'(z) dz = P(F'), \quad (398)$$

where $P(F')$ may easily be determined once H' is known. Thus F is known as a function of F' , from which we easily deduce X as

$$X = \int_{F_s}^F \frac{dF}{F'}.$$

The condition that F is continuous at $X = X_1$ implies, by virtue of (398), that

$$F(X_1) = \frac{1}{2}\Lambda.$$

It is then easy to see that in taking the square root of equation (398) we must write

$$F = +\{P(F')\}^{\frac{1}{2}}, \quad X < X_1$$

and

$$F = \Lambda - \{P(F')\}^{\frac{1}{2}}, \quad X > X_1 \quad (399)$$

It will be noted that the pressure distribution is symmetrical about $X = X_1$. The results are shown in Table 17, where the relationship between F , F' , and X is given for $X < X_1$.

TABLE 17

Universal functions for shock-wave and boundary-layer interactions

F'	$\{P(F')\}^{\frac{1}{2}}$	X
0.0103	0.02	-7.03
0.0237	0.05	-5.12
0.0351	0.08	-4.09
0.0479	0.12	-3.14
0.0612	0.17	-2.21
0.0736	0.23	-1.32
0.0832	0.29	-0.55
0.0900	0.338	0
0.0885	0.40	+0.70
0.0828	0.50	1.86
0.0645	0.60	3.21
0.0465	0.70	5.03
0.0323	0.80	7.61
0.0174	0.90	11.75
0.0101	0.95	15.52
0.0042	1.00	23.33
0	1.03	∞

At separation $F' = \lambda' = 0.090$, and from Table 17 we note that

$$F'_s = 0.338,$$

and we may then deduce from (394) that

$$C_{ps}\{(M_0^2 - 1)R\}^{\frac{1}{2}} = 0.825. \quad (400)$$

This value is in good agreement with the mean experimental value (373) for zero heat transfer. That C_{ps} is independent of wall temperature is also predicted by the method described in the preceding section, due to Gadd.

It follows from (395) that the length of the region of interaction is directly proportional to T_w , which result is also predicted by Gadd's method, and is in at least reasonable accord with the most recent experimental results.

A similar approach has also been used (Curle, 1962*b*) to consider the interaction when a laminar boundary layer meets an expansive corner followed by a compressive agency, such as a step.

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INDEX

This index combines subjects and authors, the references being to page numbers. Where a given author or subject is referred to on consecutive pages only the first page number is given, unless the references are independent.

- Abarbanel, S. S., 150.
 Attridge, J. L., 145.
- Baxter, D. C., 104, 122.
 Bernoulli, D., 48.
 Blasius, H., 20, 22, 25, 48, 50, 82, 92.
 Böhlen, T., 43.
 Bourne, D. E., 79.
 Brainerd, J. G., 93, 116.
 Bray, K. N. O., 152.
 Busemann, A., 90.
- Chapman, D. R., 102, 105, 144.
 Cohen, C. B., 121, 125.
 Conservation of energy, 3.
 — mass, 3.
 — momentum, 3.
 Cope, W. F., 89, 115.
 Crabtree, L. F., 112.
 Crocco, L., 17, 94, 154.
 Crocco transformation, 17, 94.
 Curle, N., 32, 35, 38, 42, 47, 52, 60, 63, 74,
 81, 107, 112, 121, 129, 138, 154.
- Davies, D. R., 79.
 Displacement thickness, 11, 21, 27, 41, 60,
 84, 124, 127, 131.
 Driest, E. R. van, 99.
- Emery, J. C., 144.
 Emmons, H. W., 92, 116.
 Enthalpy thickness, 12, 124.
 Entropy, 9.
- Fage, A., 70, 102.
 Falkner, V. M., 23, 56, 70, 102.
 Finston, M., 145.
 Flugge-Lotz, I., 104, 122.
 Frankl, F. I., 117.
 Frictional heating, 65, 67.
 Frössling, N., 25.
- Gadd, G. E., 118, 121, 143, 145, 147, 152.
 Goldstein, S., 4, 20, 24, 43, 69.
 Görtler, H., 29, 30, 40, 81.
 Greber, I., 150, 152.
 Gruschwitz, E., 117.
- Hakkinen, J., 150.
 Hantzsche, W., 99, 101.
- Hartree, D. R., 23, 32, 34, 38, 39, 81, 89,
 115.
 Head, M. R., 58, 62.
 Heat transfer rate, 12, 67, 69, 71, 78, 83,
 87, 99, 101, 104, 105, 127, 135, 141.
 Heimenz, K., 22, 25.
 Holder, D. W., 143.
 Holstein, H., 43.
 Howarth, L., 3, 20, 22, 25, 28, 29, 32, 36,
 39, 45, 81, 108, 145.
 Howarth transformation, 108, 123, 129.
- Illingworth, C. R., 110, 116, 135.
- Kalikhman, L. E., 123.
 Kármán, T. von, 48, 90, 91.
 Kinetic energy integral equation, 13, 54,
 63, 117, 134.
 Kinetic energy thickness, 12, 53, 58.
 Kuehn, D. M., 144.
- Larson, H. K., 144.
 Lees, L., 154.
 Leigh, D. C. H., 34, 39.
 Liebson, L. S., 15, 53, 64.
 Liepmann, H. W., 73, 76, 138.
 Lighthill, M. J., 71, 76, 105, 135, 138, 145.
 Lilley, G. M., 135.
 Li, T. Y., 121.
 Luxton, R. E., 110, 131.
- Mach number, 10.
 Mack, L. M., 102.
 Meksyn, D., 30, 80.
 Merk, H. J., 30, 80.
 Millikan, C. B., 48.
 Mises, R. von, 17, 71.
 Mises transformation, 17, 71, 102, 136.
 Momentum integral equation, 13, 43, 54,
 58, 90, 109, 114, 117, 132, 134.
 Momentum thickness, 12, 21, 41, 51, 54,
 124, 127, 130.
 Monaghan, R. J., 100, 127.
 Morgan, G. W., 77.
- Nagamatsu, H. T., 121.
- Oswatitsch, K., 114, 142.

- Plate thermometer, 67, 81.
 Pohlhausen, E., 66, 67, 82, 86, 97, 103.
 Pohlhausen, K., 31, 41, 55, 61, 110.
 Poots, G., 133.
 Prandtl, L., 1, 4.
 Prandtl number, 4, 89.
 Pressure coefficient, 50.

 Radiation, 106.
 Reshotko, E., 121, 125.
 Reynolds number, 6.
 Rott, N., 113, 137.
 Rubesin, M. W., 102, 105.

 Schlichting, H., 42, 43.
 Schmidt, E., 77.
 Schubauer G. B., 31, 39, 81.
 Separation, 2, 34, 35, 49, 60, 73, 84.
 Series solutions, 24, 28, 35.
 Shock wave/Boundary layer interactions:
 Deflexion of streamlines, 143.
 Heat transfer effects, 144, 149, 154, 156.
 Pressure coefficient at separation, 144, 148, 151, 154, 156.
 Pressure distribution, 144, 156.
 Thickening of boundary layer, 147, 151.
 Upstream influence, 146.
 Skan, Miss S. W., 23, 47, 56, 112.
 Skin friction, 12, 27, 34, 36, 41, 52, 61, 63, 83, 91, 94, 101, 127, 131, 135.
 Spalding, D. B., 77, 137.
 Specific heats, 4, 10.
 Squire, H. B., 69, 81, 84.
 Stagnation-point, 22, 24, 68.
 Stagnation temperature, 10.
 State—equation of, 3, 14.

 Sterrett, J. R., 144.
 Stewartson, K., 24, 110, 112.
 Stewartson—Illingworth transformation, 110, 113, 125, 133, 138.
 Stratford, B. S., 63, 81, 146.

 Tani, I., 29, 31, 35, 47, 54, 62, 134.
 Terrill, R. M., 32, 38, 40.
 Thermal conductivity, 4.
 Thermal energy integral equation, 73, 78, 134.
 Thermometric conductivity, 4.
 Thwaites, B., 45, 62, 112, 126.
 Tifford, A. N., 22, 25, 28, 36.
 Timman, R., 44.
 Töpfer, C., 20.
 Total temperature, 10, 15.
 Trilling, L., 150.
 Truckenbrodt, E., 53, 55, 60, 62.
 Tsien, H. S., 91, 145.

 Ulrich, A., 25, 42.
 Universal functions, 25, 35.

 Velocity profile, 36, 42, 44, 48, 63, 72, 74, 82, 90.
 Viscosity, 88, 97, 113.
 Viscous stress, 1.

 Walz, A., 45, 47, 114, 126.
 Weighardt, K., 57, 114, 142.
 Wendt, K., 99, 101.
 Wenner, K., 77.
 Winterbottom, N. E., 84.
 Woodger, M., 152.

 Young, A. D., 84, 100, 110, 119, 131.

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