

GUSTAV DOETSCH

GUIDE TO THE APPLICATIONS
OF LAPLACE TRANSFORMS

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DOETSCH

Guide to the Applications of Laplace Transforms

GUIDE TO THE APPLICATIONS OF LAPLACE TRANSFORMS

by

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With 45 Figures
and a Table of Laplace Transforms

by

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Preface

When applying the Laplace transformation method to practical problems it is necessary to have some knowledge of its properties. Many of these properties are difficult to prove, and some of the proofs are rather lengthy. Because of the great increase in the amount of mathematical knowledge required by engineers, it is impossible for them to study all the proofs in detail. Frequently they must leave it to the professional mathematician to guarantee that proofs are rigorous and to provide clearly-worded statements of the results—the latter is important, because inaccurate wording, which suppresses part of the validity conditions, can be the cause of errors in applications.

This has led me to write this book, and it is for this reason, in Chapter 2, that I have given a clear and precise list of the results which are most frequently required and used in calculations. These are called 'Rules', and are stated without proofs. Aids to further study of the theory are presented as precisely-worded 'theorems', printed in special type for easy identification. A 'danger sign' is used to emphasize those validity conditions which experience has shown to be often overlooked in practice, and to cause errors.

An engineer who aspires to greater heights and who is interested in the proofs because he wishes to use the concepts in other investigations, can study the complete proofs by consulting the references to a recent book by the present author.

Nevertheless there are numerous proofs in this book. These are of results which would be unintelligible without proof, or for which proofs are not found elsewhere.

By separating much of the descriptive material from the theoretical discussion, the methods of applying the Laplace transformation have been clearly set forth and applied to many numerical examples. In these examples I have tried always to reveal the underlying ideas. If a technical problem is formulated mathematically, perhaps as a system of differential equations, then the Laplace transformation affords a complete and unique method of solution, and requires no further reference to the physical significance of the quantities involved. Because of this, practical problems which appear to be completely different can sometimes be solved by the same method because they are equivalent mathematically. When engineers write about the Laplace transformation they provide as many examples as possible, and express them separately in physical terms, although mathematically the examples deal with the same problem. By contrast, in my numerical examples, which have all been taken from practice, I have rather tended to keep the physical description in the background, in order to emphasize more clearly the technique of the calculations.

I have taken this opportunity to discuss rigorously some topics which have been incorrectly treated in technical literature. In particular, I should like to draw attention to the discussion in §§ 15 and 19 of systems of simultaneous differential equations. I discuss the property, often thought to be paradoxical, that the initial values of the solution are not zero for a system which can be described by differential equations and which initially is in a passive state. There have been many recent attempts to solve this problem by the use of

inappropriate methods. To clarify the situation, I consider a problem in automatic control systems (§ 15); I also discuss a particularly enlightening problem in network theory (§ 19), and, at the end of that paragraph, I describe the correct interpretation for a completely general network.

The relation between the Laplace transformation and the spectral description used by engineers is given particular consideration. The spectral description of a function is established rigorously and discussed fully in § 1. The Laplace transformation includes the spectral description but goes far beyond it. Its real significance is that it transforms a time function into an analytic function, which can be thought of as an amplification of the spectral density. However, the properties of the time function are reflected more completely in the Laplace transform than in the spectral density, which in practice does not exist at all for many functions.

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Contents

Symbols	10
Chapter 1. <i>Definition of the Laplace Transformation</i>	
§ 1. Spectral Representation of a Function by Fourier Series and Fourier Integral	11
§ 2. The Laplace Integral and its Physical Meaning	24
§ 3. Some Properties of Functions Described by the Laplace Integral and Examples of Laplace Transforms	27
§ 4. The Laplace Integral as a Transformation	31
Chapter 2. <i>Rules for the Calculus of the Laplace Transformation</i>	
§ 5. The Mapping of Operations	34
§ 6. Linear Substitutions	35
§ 7. Differentiation	37
§ 8. Integration	39
§ 9. Multiplication and Convolution	40
Chapter 3. <i>Ordinary Differential Equations</i>	
§ 10. The Differential Equation of First Order	44
§ 11. The Differential Equation of Second Order	47
§ 12. The Inhomogeneous Differential Equation of n th Order with Vanishing Initial Values	53
§ 13. The Response to Special Excitations	60
1. The Step Response	60
2. The Impulse Response	63
3. Frequency Response	68
§ 14. The Homogeneous Differential Equation of n th Order with Arbitrary Initial Values. The Characteristic Oscillations	72
§ 15. Solution of a Particular System of Simultaneous Differential Equations	76
§ 16. A General System of Simultaneous Differential Equations	83
§ 17. A System of Differential Equations with Piecewise Different Constants and Excitations	85
§ 18. The Equation System of an Electrical Network	88
§ 19. The Initial Values with Systems of Differential Equations	94
§ 20. Nonlinear Differential Equations	105

Chapter 4. *Difference Equations and Sampled-Data Systems*

§ 21. The Homogeneous Difference Equation with Arbitrary Initial Values . . .	109
§ 22. The Inhomogeneous Difference Equation with Vanishing Initial Values . . .	116
§ 23. The Boundary Value Problem for the Difference Equation of Second Order . . .	119
§ 24. A System of Simultaneous Difference Equations with Initial or Boundary Conditions (Electric Ladder Network)	120
§ 25. Solution of Difference Equations by Means of the Discrete Laplace Transformation	127
§ 26. Solution of Difference Equations by Means of the Laurent or $\mathfrak{B}$ -transformation	133
§ 27. The Pulse-former and its Description by $\mathfrak{L}$ -, $\mathfrak{D}$ -, and $\mathfrak{B}$ -transformation	138
§ 28. Networks Containing Pulse Generators	148
1. Impulse Generators	148
2. Pulse Generators	153
3. Pulses Formed by a Modulating Function	156

Chapter 5. *Partial Differential Equations*

§ 29. General Principles in the Application of the Laplace Transformation to Partial Differential Equations	161
§ 30. The Heat Conduction Equation	166
1. Vanishing Initial Temperature, Arbitrary Boundary Temperatures	168
2. Arbitrary Initial Temperature, Vanishing Boundary Temperatures	171
§ 31. The Equations for a Twin Conductor Line with Distributed Constants	173

Chapter 6. *Integral Equations and Integral Relations*

§ 32. Integral Equations of the Convolution Type	181
§ 33. Integral Relations	184

Chapter 7. *Calculation of the Original Function from the Image Function*

§ 34. The Complex Inversion Formula	187
§ 35. Series Expansions	191
1. Expansion in Power Series	191
2. Series Expansion with Respect to Exponential Functions	193
3. Expansion in a Series of Arbitrary Functions	198
§ 36. Numerical Calculation of the Original Function	201
§ 37. Determination of the Maximum of the Original Function from the Image Function	203

Chapter 8. *The Asymptotic Behaviour of Functions and the Question of Stability*

§ 38. Some Limit Theorems	206
§ 39. General Definitions of Asymptotic Representation and Asymptotic Expansion of Functions	207
§ 40. Asymptotic Expansion of the Image Function	210
§ 41. Asymptotic Expansion of the Original Function	212
§ 42. Investigation of Stability	216
Subject Index	221

Appendix. *Table of Laplace Transforms*

1. Operations	227
2. Rational Functions	230
3. Irrational Functions	238
4. Transcendental Functions	240
5. Original Functions, Defined Piecewise	244
Index of Functions	255

Symbols

j = imaginary unity

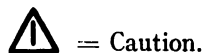
$$z = x + jy, \quad x = \Re z, \quad y = \Im z;$$

$$z = r e^{j\varphi}, \quad r = |z|, \quad \varphi = \arg z;$$

$$z = x - jy = r e^{-j\varphi} \text{ (complex conjugate of } z\text{)}.$$

Proofs of the theorems quoted in the present book are found in the author's work "Einführung in Theorie und Anwendung der Laplace-Transformation", Birkhäuser Verlag, Basel und Stuttgart 1958, 301 p. This book is referred to by EINF. and number of page.

The Appendix "Table of Laplace Transforms" is referred to by TAB. and number of correspondence.



CHAPTER 1

Definition of the Laplace Transformation

§ 1. Spectral Representation of a Function by Fourier Series and Fourier Integral

Engineers and physicists have a preference for mathematical concepts that can be clearly visualized. With the so-called Laplace integral

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

this is in fact the case.

To obtain a physical meaning for this integral, we start by expanding the function $F(t)$, of period 2π , in the well-known Fourier series

$$(1.1) \quad F(t) = a_0 + 2 \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where the coefficients a_n and b_n are determined by the formulas

$$(1.2) \quad a_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(t) \cos nt dt, \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(t) \sin nt dt \quad (n = 0, 1, 2, \dots).$$

When the function $F(t)$ satisfies certain conditions, for example, when we can decompose $F(t)$ into a finite number of continuous and monotonic

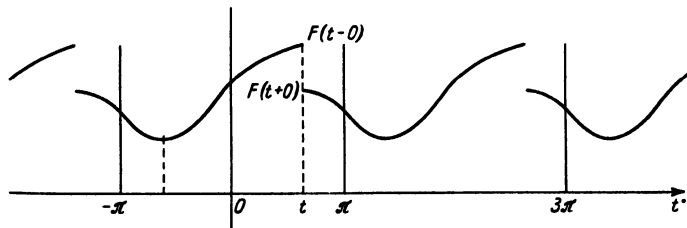


Figure 1.1 Periodic function, consisting of more than one continuous and monotonic pieces.

pieces, then the Fourier series converges and is equal to $F(t)$ ¹⁾. Substituting

$$(1.3) \quad a_n = r_n \cos \varphi_n, \quad b_n = r_n \sin \varphi_n,$$

where

$$r_n = (a_n^2 + b_n^2)^{1/2}, \quad \tan \varphi_n = \frac{b_n}{a_n},$$

into the series (1.1), we obtain

$$(1.4) \quad F(t) = r_0 + 2 \sum_{n=1}^{\infty} r_n \cos(n t - \varphi_n).$$

When t represents time, this formula can be interpreted as the resolution of a periodic occurrence $F(t)$, which, for example, describes the position of a point moving along a line, into a *sum of harmonic oscillations*, that is, oscillations whose frequencies are integral multiples of the fundamental frequency. The n th harmonic has amplitude $2r_n$, and phase lag $-\varphi_n$.

In physics, the functions that occur in many cases satisfy differential equations, so that their derivatives are required. Series (1.1) and (1.4) have the disadvantage that differentiation converts the cosine function into the sine function and conversely. We can avoid this difficulty by writing the *Fourier series in the complex form*

$$(1.5) \quad F(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j n t},$$

where the coefficients c_n are determined by the formula²⁾

$$(1.6) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(t) e^{-j n t} dt \quad (n = 0, \pm 1, \pm 2, \dots).$$

The two representations (1.1) and (1.5) are mathematically completely equivalent. When $F(t)$ is real, c_0 is real and $c_{-n} = \overline{c_n}$, and therefore

$$F(t) = c_0 + \sum_{n=1}^{\infty} (c_n e^{j n t} + \overline{c_n} e^{-j n t}) = c_0 + 2 \sum_{n=1}^{\infty} \Re(c_n e^{j n t}).$$

¹⁾ Where two continuous monotonic pieces are adjacent, the function can have a discontinuity (see Fig. 1.1), a case which occurs frequently in applications. There the series represents the mean-value between the limiting values from left and right, i.e., $(F(t-) + F(t+))/2$.

²⁾ That the summation index in the series (1.5), in contrast to that in (1.1), extends from $-\infty$ to $+\infty$ is connected with the fact that the functions $\cos nt$, $\sin nt$, with positive integral n ($n = 0, 1, 2, \dots$), form a complete orthogonal system in the interval $(-\pi, +\pi)$, whereas the orthogonal system $e^{j n t}$ is complete only when the n extends also over the negative integers.

If we set

$$(1.7) \quad c_0 = a_0, \quad c_n = (a_n - j b_n) \quad (n = 1, 2, \dots),$$

it follows that

$$F(t) = a_0 + 2 \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

and, from (1.6), a_n and b_n are given by

$$a_n = \Re c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(t) \cos nt \, dt, \quad b_n = -\Im c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(t) \sin nt \, dt.$$

With the definition (1.7) formulas (1.5) and (1.6) are equivalent to (1.1) and (1.2) respectively.

We can also write the complex Fourier series (1.5) in a form analogous to (1.4). By comparing (1.7) with (1.3), we obtain

$$c_0 = r_0, \quad \varphi_0 = 0; \quad c_n = r_n (\cos \varphi_n - j \sin \varphi_n) = r_n e^{-j\varphi_n} \quad (n = 1, 2, \dots).$$

Further

$$c_{-n} = \overline{c_n} = r_n e^{j\varphi_n},$$

and when we set

$$r_{-n} = r_n, \quad \varphi_{-n} = -\varphi_n,$$

we obtain

$$c_{-n} = r_{-n} e^{-j\varphi_{-n}},$$

and therefore (Fig. 1.2)

$$(1.8) \quad F(t) = \sum_{n=-\infty}^{+\infty} r_n e^{j(n t - \varphi_n)}.$$

The series (1.5) and (1.8) are convenient for differentiation. They are, however, not so vivid as (1.1) and (1.4) because in them $F(t)$ is described as the sum of complex functions: thus it is not immediately obvious that $F(t)$

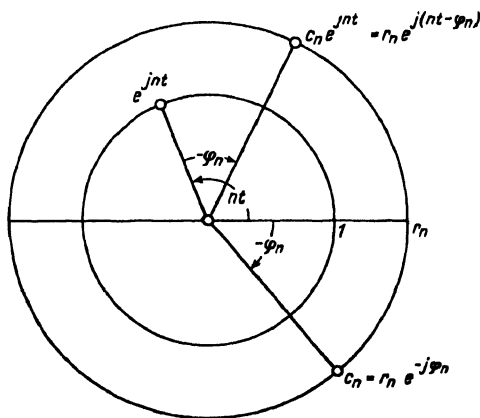


Figure 1.2 Representation of $c_n e^{jnt}$ in the complex plane.

is constructed from functions which represent real harmonic oscillations. To describe the term $c_n e^{jnt}$ graphically, it is convenient to use the *complex plane*: $c_n e^{jnt} = r_n e^{j(nt - \varphi_n)}$ determines a vector extended from the origin, which has length r_n and makes an angle $(nt - \varphi_n)$ with the positive real axis. When t increases, the end point of this vector rotates on a circle of radius r_n , in a positive direction of rotation (counter-clockwise) when n is positive, and in a negative direction of rotation (clockwise) when n is negative. $F(t)$ is obtained by adding together all these different vectors.

We can return now to the real oscillations, $2r_n \cos(nt - \varphi_n)$, if we always combine the two mirror-image vectors $c_n e^{jnt}$ and $c_{-n} e^{-jnt}$, as has been done already in the proof of the equivalence of (1.1) and (1.5):

$$c_n e^{jnt} + c_{-n} e^{-jnt} = 2 \Re(c_n e^{jnt}) = 2 \Re(r_n e^{j(nt - \varphi_n)}) = 2 r_n \cos(nt - \varphi_n).$$

This term represents a projection of the vectors onto the real axis. The end points of the two mirror-image vectors rotate round the circle of radius r_n in opposite directions of rotation and the point of intersection of the real axis and the line connecting the end-points of the two vectors oscillates back and forth between the points $2r_n$ and $-2r_n$, according to the formula $2r_n \cos(nt - \varphi_n)$. (Figure 1.3 illustrates this motion.)

By combining always the two vectors with indices $+n$ and $-n$ and not adding them together in an arbitrary manner, we return to the concept of real oscillations, and we can even say that a better representation of the occurrence has been obtained through the linking of the point oscillating

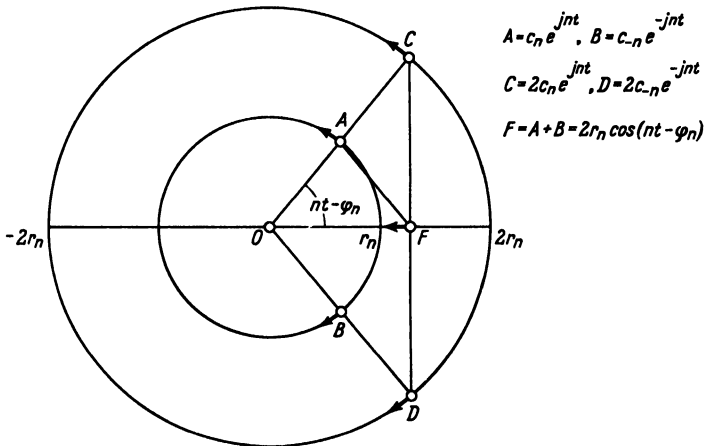


Figure 1.3 Combination of the mirror-image vectors OA and OB to form the real vector OF .

back and forth on the real axis with the points rotating on the circle. Although we cannot represent the movement on the real axis as a function of the time, we can clearly visualize the variable $(nt - \varphi_n)$ as an angle subtended at the centre or as an arc on the circle. The points on the circle rotate with constant angular velocity, while the point on the axis has variable velocity depending on the inclination of the circular arc to the horizontal.

Even if we do not combine the complex terms in this particular way which takes us back to the original representation of the oscillations, we can still give the representation (1.5) a *vivid meaning*, but this certainly requires more power of imagination. To do this, we interpret (1.5) in the following new way: we embed the one-dimensional line on which the point oscillates in a two-dimensional plane and use as the components from which $F(t)$ is composed, not the linear oscillations $\cos nt$ and $\sin nt$, but the functions e^{int} , ($n = 0, \pm 1, \pm 2, \dots$), which describe motions in the plane, viz., *rotations on the circumference of the unit circle* with frequencies $n = 0, \pm 1, \pm 2, \dots$; that is, the point rotates around the circle n times in 2π units of time. A positive frequency means that the rotation sense is positive, and a negative frequency means that the rotation sense is negative.

We can now call a rotation on a circle a *complex oscillation* with a positive or negative frequency according to whether the rotation is in the positive or negative sense. This has been done in electrical engineering theory for many years. (Although negative frequency has no meaning in linear oscillations, it does have a very definite meaning in complex oscillations.) The "amplitude" of a complex oscillation is the radius of the circle in the complex plane, and the phase is the angle defining the position of the radius to the point at $t = 0$. The term $c_n e^{int}$ with $c_n = r_n e^{-j\varphi_n}$ represents a *complex oscillation of frequency n , amplitude r_n , and phase $-\varphi_n$* ; briefly:

$$(1.9) \quad \text{Amplitude} = |c_n|, \quad \text{Phase} = \arg c_n.$$

For this reason we can call the sequence c_n the *spectrum* of the function $F(t)$. The values c_n indicate the oscillations that actually occur (when $c_n = 0$, oscillation of frequency n does not occur) and determine the amplitude and phase of these oscillations. Here $F(t)$ need not be real, but can have complex values.

In what follows, we shall always use the *complex* oscillations with positive and negative frequencies, since they are easier to manipulate than the real oscillations. The above discussion can be condensed as follows:

We obtain the spectrum c_n of the function $F(t)$, defined in the interval $(-\pi, +\pi)$ and periodically continued, by means of the integral (1.6). The knowledge of the spectrum can be substituted for the knowledge of the function

$F(t)$, because we can always reconstruct the function $F(t)$ from the spectrum with the help of the formula (1.5). The decomposition into oscillations gives a clear concept of the character of the physical occurrence $F(t)$.

If the physical event is not periodic and the function $F(t)$ is given in an arbitrary manner for all t , ($-\infty < t < +\infty$), then with suitable assumptions it can be described, as is well known, by a *Fourier integral* instead of by a Fourier series:

$$(1.10) \quad F(t) = \int_{-\infty}^{+\infty} f(y) e^{jyt} dy,$$

where the function $f(y)$ can be determined from $F(t)$ by

$$(1.11) \quad f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(t) e^{-jyt} dt.$$

This is valid if $\int_{-\infty}^{+\infty} |F(t)| dt$ converges and if $F(t)$ can be decomposed in every finite interval into a finite number of continuous and monotonic pieces. At the jump points of $F(t)$ the integral (1.10) represents the value $[F(t-0) + F(t+0)]/2$ as does the Fourier series. Formulas (1.10) and (1.11) are similar to (1.5) and (1.6): $f(y)$ corresponds to the coefficients c_n , except that now, instead of integral indices n , the continuous variable y appears (and also the integral instead of the sum). This means that $F(t)$ is not constructed from oscillations with integral frequencies, but that oscillations of all frequencies are required.

In this case, $f(y)$ cannot be interpreted immediately as the "coefficient" of e^{jyt} which indicates the amplitude and the phase of the oscillation e^{jyt} , that is, the spectrum. The individual oscillation here has no finite amplitude, but is multiplied by the infinitesimal amount $f(y) dy$. Perhaps this is difficult to imagine, but it is easier to understand when we compare it with an analogous concept in mechanics, which is easier to visualize. If we think of the y axis as being covered with a certain mass that has *density* $f(y)$, then an individual point has no finite mass. The line element dy is covered with the infinitesimal mass $f(y) dy$; a finite mass can correspond only to a complete interval and a mass function can be defined only in relation to an interval. We can, for example, introduce the mass of the interval $(-\infty, y)$ as a function of y :

$$(1.12) \quad \int_{-\infty}^y f(\eta) d\eta = g(y).$$

This is called in mathematics the *distribution function* of the mass covering³⁾. It follows that

$$(1.13) \quad \frac{dg(y)}{dy} = f(y) \quad \text{or} \quad f(y) dy = dg(y).$$

The distribution function as the mass of an interval has a clear meaning; the density, which appears at a single point, can only be conceived as a limiting concept, viz., as the limit of the quotient of mass increment and increment of y .

When we consider the spectrum of $F(t)$ in the form of a mass distribution then we call the function defined by (1.12) the *spectral distribution*. It is the "sum" of the infinitesimal spectral quantities between the frequencies $-\infty$ and y . The function $f(y)$ is called the *density of the spectral distribution at the frequency y* , or briefly, the *spectral density*. The spectral representation of $F(t)$ can be written with the help of the spectral density in the form (1.10), or with the help of the spectral distribution in the form

$$(1.14) \quad F(t) = \int_{-\infty}^{+\infty} e^{iyt} dg(y).$$

When we write the complex quantity $f(y)$ in the form

$$f(y) = r(y) e^{-i\varphi(y)},$$

where

$$r(y) = |f(y)|, \quad -\varphi(y) = \arg f(y),$$

the formula

$$(1.15) \quad F(t) = \int_{-\infty}^{+\infty} r(y) dy \cdot e^{iyt - \varphi(y)},$$

shows that the oscillation of frequency y has the infinitesimal amplitude $r(y) dy = |f(y)| dy$ and the phase $-\varphi(y) = \arg f(y)$. Consequently the quantity $r(y) = |f(y)|$ is called the "amplitude density":

$$(1.16) \quad \text{Amplitude density} = |f(y)|, \quad \text{Phase} = \arg f(y).$$

³⁾ For example the integral $\int_{-\infty}^{+\infty} y^n f(y) dy$ is called the " n th moment" of the function f (in relation to the point 0). Underlying this is also the idea that the y -axis is covered with a mass of density $f(y)$. In the theory of the moment problem the function $g(y)$ is called, as before, a distribution function. In the same manner in probability theory the probability that a quantity, which can assume values between $-\infty$ and $+\infty$, lies in the interval (y_1, y_2) is represented by an integral $\int_{y_1}^{y_2} f(y) dy$. $f(y)$ is called the probability density and $g(y)$ the probability distribution.

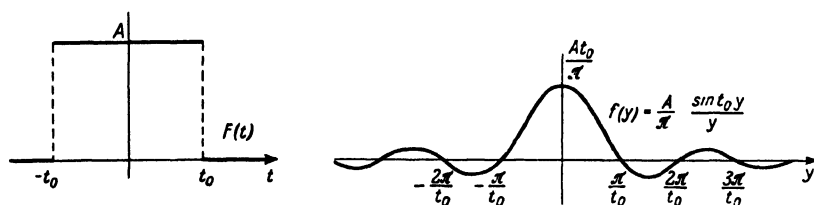


Figure 1.4 Rectangular time function and the corresponding spectral density.

Examples

Example 1. We consider the rectangular time function

$$(1.17) \quad F(t) = \begin{cases} A & \text{for } |t| \leq t_0 \\ 0 & \text{for } |t| > t_0 \end{cases}$$

Its spectral density is

$$(1.18) \quad f(y) = \frac{1}{2\pi} \int_{-t_0}^{+t_0} A e^{-jyt} dt = \frac{A}{2\pi} \frac{e^{jyt_0} - e^{-jyt_0}}{jy} = \frac{A}{\pi} \frac{\sin t_0 y}{y}.$$

This is real, but can be positive or negative (see Fig. 1.4). The amplitude density is its absolute value; the phase is 0 for $f(y) > 0$ and π for $f(y) < 0$. The spectral distribution is given by

$$(1.19) \quad g(y) = \frac{A}{\pi} \int_{-\infty}^y \frac{\sin t_0 \eta}{\eta} d\eta = \frac{A}{\pi} \int_{-\infty}^{t_0 y} \frac{\sin u}{u} du = \frac{A}{\pi} (\pi + \text{si } t_0 y) \\ = \frac{A}{\pi} \left(\frac{\pi}{2} + \text{Si } t_0 y \right),$$

where, as is customary, we define the functions “integral sine”, si and Si, by

$$(1.20) \quad \text{si } x = - \int_x^{\infty} \frac{\sin u}{u} du, \quad \text{Si } x = \int_x^{\infty} \frac{\sin u}{u} du = \text{si } x + \frac{\pi}{2}.$$

Example 2. We consider next the time function

$$F(t) = e^{-|t|}.$$

Its spectral density is (see Fig. 1.5)

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-|t|} e^{-jyt} dt = \frac{1}{\pi} \int_0^{\infty} e^{-t} \cos yt dt = \frac{1}{\pi(1+y^2)}.$$

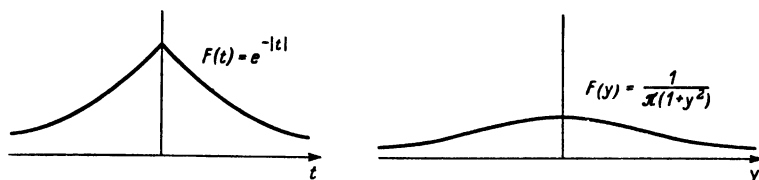


Figure 1.5 Exponentially increasing and decreasing time function and the corresponding spectral density.

It is positive and therefore equal to the amplitude density. The phase of all oscillations is 0. The spectral distribution is

$$g(y) = \frac{1}{\pi} \int_{-\infty}^y \frac{d\eta}{1+\eta^2} = \frac{1}{\pi} \arctan y + \frac{1}{2} \quad \left(\text{since } \arctan(-\infty) = -\frac{\pi}{2} \right).$$

In these examples $F(t)$ decreases so rapidly for $t \rightarrow \pm \infty$ that the integral for $f(y)$ converges and consequently the spectral density exists. However, if we had not defined two different functions $F(t)$ for $t < 0$ and $t > 0$, viz., e^{+t} for $t < 0$ and e^{-t} for $t > 0$, but used always e^{-t} , then the integral would not converge, because $e^{-t} \rightarrow \infty$ for $t \rightarrow -\infty$. The same difficulty occurs in practice in the particularly frequent case when $F(t)$ is an *oscillation of frequency* ω , that is, equal to $e^{j\omega t}$, because

$$(1.21) \quad f(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-jyt} e^{j\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j(\omega-y)t} dt$$

converges for no y . Because this case is treated throughout the technical literature in an obscure and unmathematical way, we shall describe here in detail how such a time function can also be represented by a spectral integral.

It is clear that the spectral decomposition of the function $e^{j\omega t}$ is exceptionally simple: here there is only a single spectral line under consideration, that is, a single oscillation with frequency ω and amplitude 1; all oscillations with other frequencies have amplitude 0, that is, they do not appear. It is evident, however, that such a "spectral decomposition" cannot be given by the Fourier integral (1.10), because in this integral every frequency corresponds to an infinitesimal "mass" $f(y) dy$, whereas in the preceding example the finite mass 1 belonged to the frequency ω . Also the concept of spectral density $f(y)$ fails here and this explains why the integral for $f(y)$ diverges. On the other hand, the concept of spectral distribution $g(y)$ retains its meaning: $g(y)$ is the total mass of the interval $(-\infty, y)$. As y increases from $-\infty$ and approaches the point ω from the left, the mass $g(y)$ is identically 0, because these frequencies correspond to no mass. However, at the moment that y reaches the point ω , the mass $g(y)$ jumps from 0 to the value 1, because

the finite mass 1 is concentrated at the point ω . As y continues to increase, $g(y)$ remains constant and equal to 1, because no mass is added. Hence

$$(1.22) \quad g(y) = \begin{cases} 0 & \text{for } y < \omega \\ 1 & \text{for } y \geq \omega. \end{cases}$$

$g(y)$ has the form shown in Fig. 1.6.

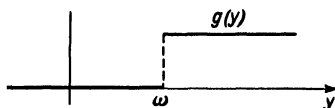


Figure 1.6 Spectral distribution for the time function $e^{j\omega t}$.

The density $f(y)$, which is the derivative of $g(y)$ (see (1.13)), apparently has no meaning at the point $y = \omega$, since $g(y)$ is not differentiable there. The most that we can say is that the (left side) derivative of $g(y)$ at ω

$$\lim_{h \rightarrow 0} \frac{g(\omega) - g(\omega - h)}{h} = \lim_{h \rightarrow 0} \frac{1}{h},$$

is equal to ∞ , and this corresponds to the physical representation that an infinitely great density is produced when a mass 1 is concentrated at a single point.

In technical literature this situation is usually handled in the following way: from (1.21)

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 1 \, dt = +\infty,$$

whereas for $y \neq \omega$ the function

$$e^{j(\omega - y)t} = \cos(\omega - y)t + j \sin(\omega - y)t,$$

is 0 when integrated over $(-\infty, +\infty)$ because the arcs bounded by the cosine and sine curves are alternatively positive and negative and therefore cancel each other.

However, when this non-rigorous description, which is inconsistent with the mathematical idea of convergence of an integral, is permitted, the density obtained

$$(1.23) \quad f(y) = \begin{cases} \infty & \text{for } y = \omega \\ 0 & \text{for } y \neq \omega \end{cases}$$

cannot be used in the spectral description (1.10) of $F(t)$, because this function is not a function in the mathematical sense and cannot be substituted in the integral for $F(t)$.

For cases of this kind, which occur frequently in applications, physicists employ a "pseudo function"⁴⁾, the so-called *impulse function* or *Dirac δ -function*. This function $\delta(y)$ vanishes for $y \neq 0$, but is infinitely great⁵⁾ at $y = 0$, so that it has the property of "sifting out" the value of the other integrand at zero,

$$(1.24) \quad \int_{-\infty}^{+\infty} h(y) \delta(y) dy = h(0).$$

If we set $f(y) = \delta(y - \omega)$ in (1.10) we obtain

$$(1.25) \quad F(t) = \int_{-\infty}^{+\infty} e^{j\omega t} \delta(y - \omega) dy = \int_{-\infty}^{+\infty} e^{j(\omega + \omega)t} \delta(y) dy = e^{j\omega t},$$

and hence the description of $e^{j\omega t}$ is supplied by a Fourier integral.

When we employ this artifice to make the spectral decomposition of $e^{j\omega t}$ perceptible and physically illuminating, we must not forget that in using the pseudo function we abandon the basis of our mathematics and continue the analysis of the integral (1.25) in a very uncertain domain where the usual mathematical rules (for example, integration by parts, substitution of a new variable, etc.) are no longer required to be valid⁶⁾.

Fortunately it is also possible to accomplish the description of the spectral decomposition of $e^{j\omega t}$ by means of the Fourier integral using rigorous mathematical methods. Thus we can build a foundation supplementary to the inexact physical idea of the impulse function, so that this function can be used without contradiction, provided that we use the necessary caution. The ideas that we shall introduce here will also be of use to us later.

For this purpose we remember that in the preceding case the spectral density $f(y)$ did not exist but the spectral distribution $g(y)$ was a well-defined function. We shall therefore make use of the description (1.14) of $F(t)$ with the help of $g(y)$, in place of the description (1.10) which has no meaning. In its original meaning as an abridgment for

$$\int_{-\infty}^{+\infty} e^{j\omega t} \frac{dg(t)}{dt} dt$$

⁴⁾ We call it a pseudo function because it is not a true function from a mathematical standpoint.

⁵⁾ In this way we interpret an "impulse" in mechanics or a "current impulse" or a "voltage impulse" in electrical engineering, and thence the name "impulse function" (which originated from Heaviside).

⁶⁾ In the modern extension of classical analysis to "distribution theory" the δ -function and even stranger functions such as, for example, the derivatives δ' , δ'' , ... are given mathematical rigour. A real knowledge of this theory can be acquired only through extensive study and therefore it cannot be assumed that engineers possess it.

(1.14) has as little meaning as (1.10). We can however confer meaning to it by considering it as a Stieltjes integral. The concept of a Stieltjes integral shall prove later to be indispensable and because the engineer in general is not familiar with it, we shall give a brief outline of its properties.

The *Riemann integral* of a function $\varphi(x)$, in a finite interval $a \leq x \leq b$, is defined as follows, as is well known: in the interval (a, b) a finite number of points are taken such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, and in each subinterval an intermediate point is chosen: $x_{v-1} \leq \xi_v \leq x_v$. The sum

$$\sum_{v=1}^n \varphi(\xi_v) (x_v - x_{v-1})$$

is formed and its limit (when it exists) for $\max (x_v - x_{v-1}) \rightarrow 0$ is defined as $\int_a^b \varphi(x) dx$. When we consider the x -axis to be covered with a mass of density 1, the difference $x_v - x_{v-1}$ has the significance of the mass existing in the interval (x_{v-1}, x_v) . If a more general mass distribution is substituted for this special one, by introducing an arbitrary distribution function $\psi(x)$ in the sense explained above ($\psi(x)$ is the mass of the interval from a fixed point, for example $x = a$, to a variable point x), then the mass of the interval (x_{v-1}, x_v) is equal to $\psi(x_v) - \psi(x_{v-1})$. When we substitute this for the special mass $x_v - x_{v-1}$, we obtain the sum

$$\sum_{v=1}^n \varphi(\xi_v) [\psi(x_v) - \psi(x_{v-1})].$$

Its limit (when it exists) for $\max (x_v - x_{v-1}) \rightarrow 0$ is called the *Stieltjes integral* of $\varphi(x)$ ⁷⁾ relative to the function $\psi(x)$ and is written in the form

$$\int_a^b \varphi(x) d\psi(x).$$

As with the *Riemann integral*, we can define also a *Stieltjes integral* with infinite limits by passing to the limits $a \rightarrow -\infty$ and $b \rightarrow \infty$. When $\psi(x)$ is differentiable, the Stieltjes integral is equal to the Riemann integral

$$\int_a^b \varphi(x) \frac{d\psi(x)}{dx} dx = \int_a^b \varphi(x) \psi'(x) dx.$$

We wish now to interpret the integral (1.14) as a Stieltjes integral and to substitute the distribution (1.22) for $g(y)$. We return to the definition of the

⁷⁾ A compilation of the most important rules for the Stieltjes integral together with proofs are found in the author's "*Handbuch der Laplace-Transformation*," Band 1 (Birkhäuser Verlag, Basel 1950), pp. 61—67.

Stieltjes integral based on the division of the integration interval by division points, which we now denote by y_r . We see that only the interval with $y_{r-1} < \omega$, $y_r \geq \omega$ contributes to the value of the integral, namely,

$$e^{j\eta_r t} [g(y_r) - g(y_{r-1})] = e^{j\eta_r t} \cdot 1,$$

and the contributions from all other intervals vanish because $g(y)$ is constant in them. When the interval lengths tend to zero then $\eta_r \rightarrow \omega$ and the Stieltjes integral gives

$$(1.26) \quad \int_{-\infty}^{+\infty} e^{j\omega t} dg(y) = e^{j\omega t}.$$

Thus the oscillation $e^{j\omega t}$ is described in a perfect mathematical way, not through the Fourier integral in the previous sense, but through the so-called "*Fourier-Stieltjes integral*", in which the spectral distribution is used in the description instead of the spectral density.

We can now interpret the description (1.25) with the help of the impulse function

$$e^{j\omega t} = \int_{-\infty}^{+\infty} e^{j\omega t} \delta(y - \omega) dy$$

as an intuitive transcription of the Stieltjes integral (1.26). If, when deriving proofs or when operating with the integral (1.25), we replace this integral by the Stieltjes integral and employ rules that are valid for that integral, we can be certain that the mathematics is rigorous and that we will not make a mistake nor give a fallacious proof.

For the sake of completeness, we mention that the spectral distribution can be calculated from the time function $F(t)$ by the formula

$$(1.27) \quad g(y) - g(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1 - e^{-j\omega t}}{jt} F(t) dt.$$

This can be obtained formally by integrating formula (1.11) from 0 to y . This formula is associated with the Fourier-Stieltjes integral (1.14) in the same way as formula (1.11) is with the Fourier integral (1.10).

Summarizing, we are in the position to give a spectral description of time functions $F(t)$ defined in the interval $(-\infty, +\infty)$, and with a convergent integral

$$\int_{-\infty}^{+\infty} |F(t)| dt$$

as well as of the oscillation $e^{j\omega t}$. However, for example, the damped oscillations $e^{(\alpha + j\omega)t}$ can be described neither through the ordinary nor the Stieltjes-

Fourier integral because the respective integrals (1.11) for $f(y)$ and (1.27) for $g(y)$ diverge with positive α for $t \rightarrow +\infty$ and with negative α for $t \rightarrow -\infty$.

The following section shows how these difficulties can be eliminated by the introduction of the Laplace integral.

§ 2. The Laplace Integral and its Physical Meaning

We assumed tacitly in the previous section that time varied in the interval $(-\infty, +\infty)$. Occurrences that can be observed from $t = -\infty$ to $t = +\infty$ seldom happen in practice. In general, the occurrence starts at a definite time point, which can be taken as the zero point, and is observed from this point for a long time, theoretically until $t = +\infty$; therefore the time interval is $0 \leq t \leq \infty$ and is only one-sided infinite. This case can be included in the preceding case if we define $F(t)$ to be 0 for $t < 0$. Then the integral (1.11) for the spectral density extends only from 0:

$$(2.1) \quad f(y) = \frac{1}{2\pi} \int_0^{\infty} e^{-iyt} F(t) dt,$$

and the spectral representation (1.10) for $F(t)$ is now as follows⁸⁾

$$(2.2) \quad \int_{-\infty}^{+\infty} e^{ity} f(y) dy = \begin{cases} F(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

It is possible to avoid the difficulty that the integral (2.1) may not converge. We consider, along with the time function $F(t)$, the "damped" function $e^{-\alpha t} F(t)$, with parameter $\alpha > 0$, and form its spectral density, which naturally now depends on α and therefore is denoted by $f_{\alpha}(y)$:

$$(2.3) \quad f_{\alpha}(y) = \frac{1}{2\pi} \int_0^{\infty} e^{-iyt} [e^{-\alpha t} F(t)] dt.$$

The integral converges due to the strong decrease⁹⁾ of the factor $e^{-\alpha t}$ as $t \rightarrow +\infty$ for all bounded $F(t)$, and even for functions $F(t)$ that increase to infinity like an exponential function $e^{\alpha t}$ ($\alpha \geq 0$) when we choose $\alpha > \alpha$.

⁸⁾ As in the case of Fourier series the Fourier integral (1.10) represents the mean value of the limits from right and left at jump points; in the present case, therefore, for $t = 0$ this is the mean value of the limit 0 from the left and the limit $F(0+)$ from the right, i.e., $F(0+)/2$.

⁹⁾ In the case of the original interval $-\infty < t < +\infty$ the multiplication by $e^{-\alpha t}$ would not achieve this purpose because this function increases rapidly for $t \rightarrow -\infty$ and therefore the convergence for $t \rightarrow -\infty$ is impaired.

Therefore, all functions that occur in practice can be included. The spectral representation is now as follows:

$$(2.4) \quad \int_{-\infty}^{+\infty} e^{jty} f_x(y) dy = \begin{cases} e^{-xt} F(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

When we write formulas (2.3), (2.4) in the form

$$(2.5) \quad \frac{1}{2\pi} \int_0^{\infty} e^{-(x+jy)t} F(t) dt = f_x(y),$$

$$(2.6) \quad \int_{-\infty}^{+\infty} e^{(x+jy)t} f_x(y) dy = \begin{cases} F(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases}$$

we see that the parameter x and the frequency y appear only in the combination $x + j y$. A complex variable has appeared naturally in the formulas: for this variable we can introduce a single letter. In mathematics it is customary to use the letter s for this purpose¹⁰:

$$s = x + j y.$$

Since $f_x(y)$ depends only upon this complex variable it is convenient to write

$$f_x(y) = f(x + j y) = f(s).$$

Of course, we shall introduce the variable s also into (2.6) instead of the integration variable y . Because y varies from $-\infty$ to $+\infty$, while x is constant, $s = x + j y$ varies from $x - j \infty$ to $x + j \infty$, and therefore along the vertical line in the complex plane that has abscissa x (Fig. 2.1). We also set $ds = j dy$.

To realize the customary mathematical form, we substitute $2\pi F(t)$ for $F(t)$, and formulas (2.5) and (2.6) assume the final form:

$$(2.7) \quad \int_0^{\infty} e^{-st} F(t) dt = f(s),$$

¹⁰ In technical literature the letter p is often used instead of the letter s . This originates from Heaviside's calculus which, for differential equations, leads to a similar formalism to the Laplace Transformation. The letter p originally did not signify a variable, but the differential operator D . Heaviside wrote p instead of D to prevent confusion with the letter D which is used for the dielectric displacement. Since it is usual when two variables appear simultaneously to use letters of the alphabet which are next to one another (as x , y or u , v), it is natural to use with the letter t in F , the letter s in f and not the letter p which otherwise is never used as a symbol for a variable, but generally for a positive constant.

$$(2.8) \quad \frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} e^{ts} f(s) ds = \begin{cases} F(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

Formula (2.7) is nothing but the Laplace integral; formula (2.8), when we disregard its meaning as a spectral representation, is called the “inversion” of the Laplace integral.

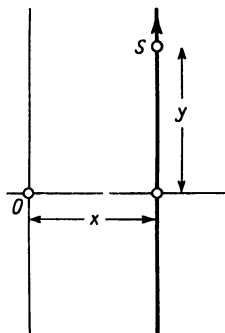


Figure 2.1 The complex variable $s = x + jy$.

According to the above derivation, formulas (2.7) and (2.8) can be interpreted as follows:

When, in the function $f(s)$ defined by the Laplace integral (2.7), we consider the variable s to be complex, $s = x + jy$, then $f(x + jy)$ with constant x describes the spectral density of the damped time function $e^{-xt} F(t)$ with frequency variable y ¹¹). The time function $F(t)$ can be constructed from the spectral density $f(s)$ by formula (2.8), which, however, when interpreted as a spectral representation, must be written in the form

$$(2.9) \quad \int_{-\infty}^{+\infty} e^{jyt} f(x + jy) dy = \begin{cases} 2\pi e^{-xt} F(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0 \end{cases}$$

that is, as a spectral representation of the function $2\pi e^{-xt} F(t)$ (completed for negative time by 0).

Transferring the factor e^{xt} to the left-hand side, we obtain

$$(2.10) \quad \int_{-\infty}^{+\infty} [e^{xt} e^{jyt}] f(x + jy) dy = 2\pi F(t),$$

¹¹) In technical literature s is sometimes called the “complex frequency” but this has no physical meaning and obscures the real importance of s . It is as always a question of real frequencies and these are given by the values $\Im s = y$. $\Re s = x$ is the parameter which determines the damping e^{-xt} of $F(t)$.

and thus the representation of $2\pi F(t)$ can be thought of as the superposition of exponentially *increasing* oscillations $e^{xt} e^{j\omega t}$; that, however, is less vivid¹²).

§ 3. Some Properties of Functions Described by the Laplace Integral and Examples of Laplace Transforms

If the spectral density of damped function $e^{-x_0 t} F(t)$ exists for a value x_0 of the parameter x , it is plausible that it exists for a stronger damping $x > x_0$. Mathematically this means that when the integral (2.7) converges for $s = x_0 + jy$, it converges also for $s = x + jy$ with $x > x_0$. Therefore we can conclude (EINF. p. 26) that the exact region of convergence of the integral (2.7) is a half-plane $\Re s > \beta$, the so-called *convergence half-plane*. Thus $f(s) = f(x + jy)$ describes infinitely many spectral densities, viz., those of the infinitely many functions $e^{-x t} F(t)$ with $x > \beta$.

In physical applications $F(t)$ is often real (but not always, see the case $F(t) = e^{j\omega t}$). Then

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt = \overline{\int_0^{\infty} e^{-s^* t} F(t) dt} = \overline{f(s^*)},$$

that is, to values of s which are reflected images with respect to the x -axis there correspond similar mirror-image function values.

When $f(s)$ is regarded as a function and is detached from its physical meaning as a spectral density, it exists (if at all) in a half-plane of the complex plane. It is of the greatest importance that in that region it describes an analytic function which is differentiable in the complex sense arbitrarily many times (EINF. p. 35). We obtain derivatives by differentiation under the integral sign:

$$(3.1) \quad f^{(n)}(s) = (-1)^n \int_0^{\infty} e^{-st} t^n F(t) dt.$$

Hence it is possible to use the very extensive methods of complex function theory. From the results obtained with $f(s)$ we can draw interesting conclusions about the time function $F(t)$, defined rather arbitrarily only for real t , for which no general method exists.

Before we carry these ideas further, we shall determine the corresponding functions $f(s)$ for a few time functions $F(t)$.

¹² In books on electrical engineering the assertion can be found that $F(t)$ is represented by (2.8) or (2.10) as a superposition of dying away oscillations. This would be correct only when x is negative. Then we would not require to introduce the Laplace integral because the original integral (1.11) for $f(y)$ would converge.

We stress that $F(t)$ is required to be defined only for $t > 0$. Whether and how $F(t)$ is defined for $t < 0$ does not matter for the Laplace integral. When on occasion the interval $t < 0$ is considered, as, for example, with the inversion integral, it is to be noted that $F(t)$ is always zero in the sense of § 2.

1. We consider the time function $F(t) = 1$. This function occurs frequently and in principle it is practical to complete the function by 0 for $t < 0$. We then call it the *unit step function* $U(t)$, defined by

$$(3.2) \quad U(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

We leave open the definition at the point of discontinuity $t = 0$, because the value of $U(t)$ at $t = 0$ is irrelevant to the calculation of the Laplace integral. In many cases it is advantageous to set $U(0) = 0$; in other cases to set $U(0) = 1/2$ or $U(0) = 1$.

The formula

$$(3.3) \quad f(s) = \int_0^{\infty} e^{-st} U(t) dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{1}{s}$$

is valid if and only if $e^{-st} \rightarrow 0$ for $t \rightarrow \infty$ that is, $\Re s > 0$. Hence we confirm the previous result (as also in the following examples) that the convergence domain is a half-plane. The time function $2\pi e^{-xt} U(t)$ has therefore the spectral density $1/s = 1/(x + jy)$ and the amplitude density $1/(x^2 + y^2)^{1/2}$. We have the inversion formula

$$(3.4) \quad \frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} e^{ts} \frac{1}{s} ds = U(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases}$$

or, written as a spectral representation,

$$(3.5) \quad \int_{-\infty}^{+\infty} e^{jyt} \frac{1}{x + jy} dy = 2\pi e^{-xt} U(t).$$

$U(t)$ itself possesses no spectral density and consequently no spectral representation in the sense of (1.10)¹³. On the other hand, we could think of $U(t)$ as a complex oscillation $e^{j\omega t}$ with the frequency $\omega = 0$ and describe it with the help of the spectral distribution $g(t)$ by a Fourier-Stieltjes integral (see (1.26)). If we set $x = 0$ in (3.4), we would obtain the integral

$$\int_{-\infty}^{+\infty} e^{jyt} \frac{1}{jy} dy,$$

which diverges because $1/y$ is not integrable in the neighbourhood $y = 0$.

¹³) This especially must be stressed since sometimes the contrary is asserted in technical literature.

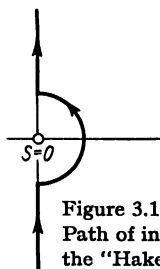


Figure 3.1
Path of integration for
the "Hakenintegral".

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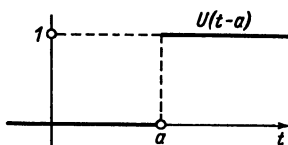


Figure 3.2
Unit step function with jump
point at $t = a$.

We shall mention a formula which is often used for $U(t)$. Setting $x = 0$ in formula (3.5) means that the path of integration in (3.4) is along the imaginary axis. In order to avoid the divergence of the integral, which arises from the singularity at $s = 0$, we go round the zero point along a semicircle to the right whereby the path of integration $\mathfrak{C}$ sketched in Fig. 3.1 is produced. With this path the valid formula is (denoted in German by "Hakenintegral"):

$$(3.6) \quad \frac{1}{2\pi j} \int_{\mathfrak{C}} \frac{e^{ts}}{s} ds = U(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

This is, however, not a spectral representation, because on the semicircle the function e^{ts} is not of the form of an oscillation $e^{j\omega t}$ nor of the form of a damped oscillation $e^{\alpha t} e^{j\omega t}$ with constant α .

2. The jump of the unit step function may not be at time $t = 0$ but at time $t = a > 0$; then it is described by $U(t - a)$ (Fig. 3.2). We have

$$(3.7) \quad f(s) = \int_0^{\infty} e^{-st} U(t - a) dt = \int_a^{\infty} e^{-st} dt = \frac{e^{-as}}{s} \quad \text{for } \Re s > 0.$$

The "delay" of the time function by a span of time a corresponds, therefore, to multiplying the spectral density by the factor e^{-as} . We shall find later that this relation is valid in general.

3. The time function may be equal to $e^{\alpha t}$ (α arbitrary complex) or, more exactly, equal to the function that is defined for $t > 0$ by $e^{\alpha t}$ and for $t < 0$ by 0. We can denote it by

$$U(t) e^{\alpha t}.$$

When $\alpha = \sigma + j\omega$, with $\omega \neq 0$, this function represents a complex oscillation for $\sigma = 0$, a damped oscillation for $\sigma < 0$, and an excited oscillation for $\sigma > 0$; for $\omega = 0$ it represents an aperiodic decreasing or increasing function. $f(s)$ is

$$(3.8) \quad \int_0^{\infty} e^{-st} e^{\alpha t} dt = \int_0^{\infty} e^{-(s-\alpha)t} dt = \frac{1}{s-\alpha} \quad \text{for } \Re s > \Re \alpha.$$

4. The time function may be the real oscillation $\cos \omega t$, more precisely, $U(t) \cos \omega t$. We consider this time function as the sum of two complex oscillations in the sense of § 1 and employ the previous formula:

$$(3.9) \quad \int_0^{\infty} e^{-st} \cos \omega t \, dt = \frac{1}{2} \int_0^{\infty} e^{-st} (e^{j\omega t} + e^{-j\omega t}) \, dt \\ = \frac{1}{2} \left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega} \right) = \frac{s}{s^2 + \omega^2}.$$

The integral converges if $\Re s > \Re(j\omega)$ as well as $\Re s > \Re(-j\omega)$, that is, for $\Re s > 0$.

5. Analogously the time function $\sin \omega t$ gives

$$(3.10) \quad \int_0^{\infty} e^{-st} \sin \omega t \, dt = \frac{1}{2j} \int_0^{\infty} e^{-st} (e^{j\omega t} - e^{-j\omega t}) \, dt \\ = \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2} \quad \text{for } \Re s > 0.$$

6. We consider next the time function t^a (a real). In order that the Laplace integral exists at $t = 0$, we require $a > -1$. $f(s)$ is

$$\int_0^{\infty} e^{-st} t^a \, dt = \frac{1}{s^{a+1}} \int_0^{\infty} e^{-\tau} \tau^a \, d\tau.$$

We have here put $st = \tau$. For τ to be real and for the integral to equal the Euler gamma function, we require that s must be real and greater than 0. Then the formula yields

$$(3.11) \quad \int_0^{\infty} e^{-st} t^a \, dt = \frac{\Gamma(a+1)}{s^{a+1}}.$$

This Laplace integral, however, is an analytic function; therefore the expression is valid also for complex s with $\Re s > 0$.

The special case t^n ($n = 0, 1, 2, \dots$) gives

$$(3.12) \quad \int_0^{\infty} e^{-st} t^n \, dt = \frac{n!}{s^{n+1}} \quad \text{for } \Re s > 0.$$

7. The time function may be $t^n e^{\alpha t}$ ($n = 0, 1, \dots$; α arbitrary complex). Then $f(s)$ is

$$(3.13) \quad \int_0^{\infty} e^{-st} t^n e^{\alpha t} \, dt = \int_0^{\infty} e^{-(s-\alpha)t} t^n \, dt = \frac{n!}{(s-\alpha)^{n+1}} \quad \text{for } \Re s > \Re \alpha,$$

a result which can be obtained from formula (3.12), when $s - \alpha$ is substituted for s .

§ 4. The Laplace Integral as a Transformation

We have seen that for every time function defined for $t > 0$, for which the integral $\int_0^{\infty} e^{-st} F(t) dt$ converges for sufficiently large x_0 , there exists a spectral density $f(s) = f(x + jy)$, which is defined for $x > x_0$. Therefore every such $F(t)$ is related to a $f(s)$ by the relation

$$(4.1) \quad f(s) = \int_0^{\infty} e^{-st} F(t) dt.$$

This relation can be interpreted as a transformation, called the *Laplace transformation*, by which the function $F(t)$ is converted or transformed into the function $f(s)$. This is a typical mathematical concept that adds a new interpretation to the former physical meaning of the Laplace integral. This will prove to be extremely useful. A closely related concept is the representation of the transformation as a mapping: thus as a photographic camera produces an image from the original, the Laplace transformation of an *original function* $F(t)$ produces an *image function* $f(s)$. This is an especially striking terminology which we shall use henceforth. The totality of all $F(t)$ is called "*original space*" and the totality of all $f(s)$ is called "*image space*" (in imitation of the usual topological concept of function sets as abstract spaces).

We shall, as far as is possible, denote the original function by a capital letter and the image function by the corresponding letter in lower case, for example, $F(t)$ and $f(s)$, $Y(t)$ and $y(s)$. Many authors use exactly the opposite notation, that is $f(t)$ and $F(s)$, or they denote the pair of functions by $f(t)$ and $\bar{f}(s)$ (this can cause confusion with the complex conjugate) or by $f(t)$ and $\tilde{f}(s)$. The reader should avoid the custom which is sometimes used in technical literature of denoting original and image functions by the same letter and of distinguishing them only by their arguments t and s ; frequently these variables must be replaced by others and confusion can arise.

The concept of a transformation or a mapping is remarkably similar to the concept of a function which expresses the association between two variables, for example, z and w . Because with ordinary functions we do not write " w is a function of z ", but instead employ a functional symbol such as $w = \varphi(z)$, we introduce for the relation established by the Laplace transformation the symbol $\mathfrak{L}$ and write

$$f(s) = \mathfrak{L}\{F(t)\},$$

spoken: $f(s)$ is the *Laplace transform* of $F(t)$. In what follows we shall use the abbreviations $\mathfrak{L}$ -transformation, $\mathfrak{L}$ -transform, $\mathfrak{L}$ -integral.

Formula (2.8) which describes $F(t)$ with the help of its spectral density and which we have already called the "inversion" of (4.1), can likewise be regarded as a transformation, in which $f(s)$ is converted into $F(t)$:

$$(4.2) \quad F(t) = \frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} e^{ts} f(s) ds \quad \text{for } t > 0.$$

We call this formula the *inverse transformation* of the $\mathfrak{L}$ -transformation and denote it by the symbol $\mathfrak{L}^{-1}$:

$$F(t) = \mathfrak{L}^{-1}\{f(s)\}.$$

The relation between F and f is also called a *correspondence* and is denoted by the sign $\circ\bullet$:

$$F(t) \circ\bullet f(s) \quad \text{or} \quad f(s) \bullet\circ F(t).$$

The transformations $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ have different *uniqueness* properties. It is clear that by (4.1) to every F there corresponds only one f . However, if we consider all $f(s)$ produced by formula (4.1), we see that every $f(s)$ is produced by infinitely many functions $F(t)$. For example, when we vary the definition of $F(t)$ at a finite number of points, $f(s)$ does not vary, because the integral (4.1) is unchanged by such variations. Nevertheless, it is easy to examine the totality of functions $F(t)$ that correspond to a fixed $f(s)$: they differ only by so-called null functions (EINF. p. 30), that is, by functions $N(t)$ with the property

$$\int_0^t N(\tau) d\tau = 0 \quad \text{for all } t \geq 0.$$

When one of the original functions $F(t)$ is continuous, no such other exist (EINF. p. 34). This is sufficient in most cases to determine the corresponding original function uniquely from the image function because the former is often taken to be differentiable and therefore continuous. When we consider all original functions that differ only by null functions to be the same (as is usual in mathematics), the inversion of the $\mathfrak{L}$ -transformation is unique.

Through the interpretation of the $\mathfrak{L}$ -integral as a transformation, we have arrived at a meaning of $f(s) = f(x + jy)$ which is different from the previous one of spectral density of $e^{-st} F(t)$; that meaning will prove to be much more important in applications. We have considered the spectral representation in detail in order to show that the $\mathfrak{L}$ -integral is not merely an abstract mathematical idea, but can also be combined with an intuitive representation. In this way the $\mathfrak{L}$ -transformation absorbs the engineer's customary way of spectral thinking (in particular, the spectral representation of the time function with the help of the spectral density, if it exists, is contained

in the inversion integral (4.2) for $x = 0$) and it is often employed with advantage, for example, in filter theory. However, it can be extended to functions for which the spectral description is not convenient and it goes far beyond the restricted question of the representation of a function in one particular form. Regarding this we make the following preliminary observations.

The functions with which the engineer is concerned appear always as solutions of functional equations such as differential, difference and integral equations; therefore certain operations, such as differentiation, difference building, and integration, must be undertaken with them. *The true significance of the $\mathfrak{L}$ -transformation lies now in its mapping properties*, namely, it replaces the functions and the operations in the original space, by their images in the image space, which have a much simpler appearance. Consequently the images of the functional equations have a simpler form in the image space and are easier to solve than the original equations in the original space. In the next chapter we discuss the mapping of the operations in question.

The $\mathfrak{L}$ -transformation has a further important property, which has already been mentioned in § 3: the image functions produced are analytic functions, and therefore we can apply the very powerful methods of complex function theory. Thus we can, for example, consider formula (4.2), which expresses $F(t)$ through $f(s)$, as an integral of the analytic function $e^{ts} f(s)$ along a line in the complex plane. When we use, with the aid of Cauchy's theorem, a different path of integration we can obtain interesting properties of the function $F(t)$, such as its asymptotic behaviour.

Generally speaking, it will prove again and again to be advantageous that the variable s varies in the complex plane. To cite a simple example, we remember that the solutions of a linear homogenous differential equation with constant coefficients are of the form $t^n e^{\alpha t}$, where α is, in general, complex. The image functions of these are, from example 7 in § 3, the rational functions $1/(s - \alpha)^{n+1}$. The true character of such a function can only be understood properly when it is considered in the complex plane as an analytic function with pole at $s = \alpha$ of multiplicity $(n + 1)$.

Rules for the Calculus of the Laplace Transformation

§ 5. The Mapping of Operations

The practical application of the $\mathfrak{L}$ -transformation is accomplished by working with the image functions instead of the original functions; similarly, instead of multiplying numbers directly, we calculate with logarithms, because this leads to the simpler operation of addition.

The procedure of mapping can be likened to the *translation* of one language into another: as one word translates another word, so, with $\mathfrak{L}$ -transformations, every function has its corresponding function. The role of a *dictionary* in the translation is played by a *table of corresponding original and image functions* in the transformation, some of which have been considered in § 3, and which is developed to a greater extent in the appendix of this book. However, to be able to translate an entire sentence, it is not sufficient to know only the translation of single words, but we must know also how the *grammatical formations* of the one language (for instance the inflections of words or the relations between several words in a more complex structure) are rendered in the other language. In the case of the $\mathfrak{L}$ -transformation this means that if we perform an *operation* on an original function (e. g. differentiation or integration), then this operation must correspond to a certain operation on the corresponding image function; or when several functions are combined with one another, as in multiplication, then a certain combination of the corresponding image functions must correspond to this combination.

Therefore we must know not only the mapping or translation of single functions, but also those of the operations performed on them. It is in this sense that the following "grammatical rules" are to be understood. Here only the most important are presented, these being the ones which are most frequently required. Further rules are to be found in the first section of the appendix¹⁴).

¹⁴) It is recommended that the reader verifies a few of the rules so that he will have more insight into the mechanism; for example,

$$\mathfrak{L}\{F(at)\} = \int_0^{\infty} e^{-st} F(at) dt = \int_0^{\infty} e^{-s \frac{\tau}{a}} F(\tau) \frac{d\tau}{a} = \frac{1}{a} \int_0^{\infty} e^{-\frac{s}{a} \tau} F(\tau) d\tau = \frac{1}{a} f\left(\frac{s}{a}\right).$$

Further, it is advised that the reader who wants to learn only the "technique" of the $\mathfrak{L}$ -transformation, first takes cognizance only of the framed formulas which state the rules, and reads the explanations added when the rules are applied in Chapters 3—6.

In Chapters 3 to 6 we will show how these rules can be used to solve certain mathematical problems.

§ 6. Linear Substitutions

In this paragraph the operations performed on the functions will consist of linear substitutions of the variable either in the original or in the image function.

Rule I (*Similarity theorem*)

$$\text{or } \left[\begin{array}{l} F(at) \circ \bullet \frac{1}{a} f\left(\frac{s}{a}\right) \\ f(as) \circ \bullet \frac{1}{a} F\left(\frac{t}{a}\right) \end{array} \right] \quad (a > 0)$$

The graphs of the functions are contracted or expanded in a certain ratio of similitude (EINF. p. 40).

Rule II (*First translation theorem*)

$$\left[F(t-a) \circ \bullet e^{-as} f(s) \right] \quad \begin{array}{l} (a > 0) \\ F(t < 0) = 0 \end{array}$$

The convention that $F(t)$ is to be set equal to 0 for negative t is unnecessary in many cases, but here it is particularly important (EINF. p. 41). For $t < a$, $t-a$ is negative and therefore $F(t-a) = 0$.

The graph of $F(t-a)$ is formed by translating the graph of $F(t)$ through the distance a to the right and completing it in the interval between 0 and a by the t -axis (Fig. 6.1a).

This rule is particularly significant if the start of an occurrence is delayed (example: regulation with dead time).

When we read Rule II from right to left (that is, when we go from the image function to the original function), we must not forget to state explicitly that $F(t-a)$ has the value 0 for $t < a$. Had the calculation of an

image function produced $e^{-s} \frac{1}{s^2+1}$, for example, then the original function is $\sin(t-1)$ for $t \geq 1$, and 0 for $0 \leq t < 1$.



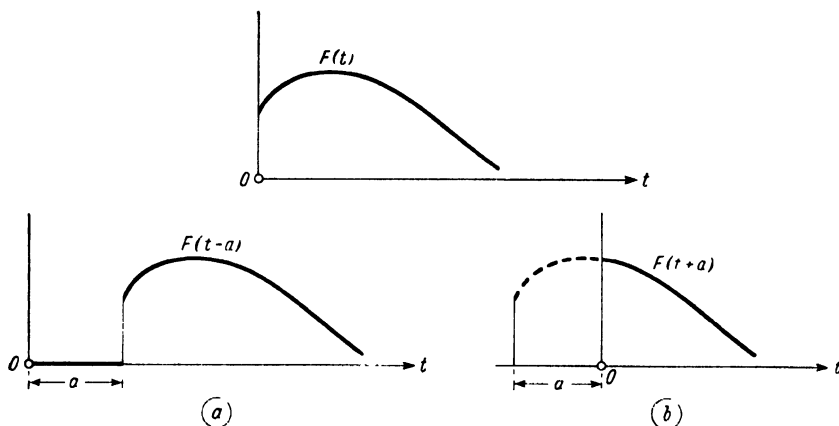


Figure 6.1 Translation of a function (a) to the right, (b) to the left.

Rule III (*Second translation theorem*)

$$F(t+a) \circ \bullet e^{as} \left(f(s) - \int_0^a e^{-st} F(t) dt \right) \quad (a > 0)$$

Here we have the relation opposite to Rule II. The graph of $F(t)$ is translated through the distance a to the left, and thus the initial part from 0 to a is lost, because the variable t of the new function is considered only for $t \geq 0$ (Fig. 6.1b). It is clear that $f(s)$ cannot be connected immediately with this truncated function, hence the appearance of the "finite" Laplace integral on the right hand side, in which the value of $F(t)$ for $0 \leq t < a$ appears (EINF. p. 44).

This rule is important for handling difference equations, in which in addition to $F(t)$ the values $F(t+a)$, $F(t+2a)$, ... also appear.

Rule IV (*Damping theorem*)

$$e^{-\alpha t} F(t) \circ \bullet f(s + \alpha) \quad (\alpha \text{ arbitrary complex})$$

Of course, an actual damping appears in the original function only when α is real and positive (EINF. p. 45).

§ 7. Differentiation

In the preceding paragraphs we have treated the simplest elementary operations. In the following paragraphs we go on to consider other operations, such as differentiation and integration.

Rule V (*Differentiation theorem for the original function*)

When F' possesses an image function, so also does F . The converse is not always true: thus, for example, $\mathfrak{L}\{\log t\}$ exists, but $\mathfrak{L}\{1/t\}$ does not because $1/t$ is not integrable in the neighbourhood of $t = 0$. With the following rule it is to be presupposed always that the highest derivative appearing exists for $t > 0$ and possesses an image function; hence, it follows automatically that the lower derivatives, including the function itself, have image functions (EINF. pp. 48—51).

$$F'(t) \circ \bullet s f(s) = F(0+)$$

$$F''(t) \circ \bullet s^2 f(s) = F(0+)s - F'(0+)$$

$$F^{(n)}(t) \circ \bullet s^n f(s) = F(0+)s^{n-1} - F'(0+)s^{n-2} - \dots - F^{(n-2)}(0+)s - F^{(n-1)}(0+)$$

This rule is the most important for practical applications. It expresses the remarkable fact that differentiation, which is a transcendental process in original space, is described in the language of image space by the completely elementary process of multiplication by a power of the variable. This is excluding the addition of a polynomial in s , whose coefficients are the “initial values” of the original function. The appearance of these initial values proves to be particularly valuable in the application of Rule V to differential equations.

Remarks

1. The “initial values” are denoted not by $F(0)$, $F'(0)$, ..., but by $F(0+)$, $F'(0+)$, ...; this expresses the fact that it is not the *values of the functions at the point 0* which are involved, but the *limits of the functions as t tends towards 0 from the right*. Conversely we can say that these are the values from which the function starts with continuous connection to the right. Thus, for example, the unit step function $U(t)$ of § 3, No. 1 can have any value $U(0)$ according to the situation at hand. The limit $U(0+)$, however,



is uniquely equal to 1. Because $U'(t) = 0$ for $t > 0$, $\mathfrak{L}\{U'\} = 0$ and since $\mathfrak{L}\{U\} = 1/s$, Rule V yields:

$$0 = s \frac{1}{s} - U(+0).$$

This equation is indeed correct since $U(0+) = 1$. With a different value $U(0)$ in place of $U(0+)$, it would not be true.

The distinction between "value at a point" and "limit as the variable tends towards this point" is particularly important in the solution of partial differential equations.



2. With Rule V it is presupposed that the highest derivative appearing, $F^{(n)}$, exists at every point $t > 0$ ¹⁵. We will show by an example that if this condition is not satisfied, Rule V is not necessarily correct. We consider the unit step function $U(t-a)$ with $a > 0$, which was considered in § 3, No. 2; its $\mathfrak{L}$ -transform is $\mathfrak{L}\{U(t-a)\} = \frac{e^{-as}}{s}$, and its limit when t tends to 0 from the right is 0. Therefore, Rule V seems to yield

$$(7.1) \quad \mathfrak{L}\{U'(t-a)\} = e^{-as}.$$

However, when we calculate this transformation directly, by forming first

$$(7.2) \quad U'(t-a) = \begin{cases} 0 & \text{for } t \neq a \\ \text{not defined} & \text{for } t = a, \end{cases}$$

we obtain something entirely different, namely

$$(7.3) \quad \mathfrak{L}\{U'(t-a)\} = 0.$$

This is because a function which is 0 everywhere with the exception of a single point gives the same integral as a function which is actually equal to 0 everywhere.

The discrepancy between (7.1) and (7.3) is explained by the fact that $U(t-a)$ is not differentiable for all t ; this function is not continuous at $t = a$, nor differentiable there. Rule V cannot therefore be applied.

For a different interpretation of (7.2) and an explanation of the formula (7.1), see p. 65.

¹⁵ It is not required that the n th derivative exists for $t = 0$ and this case appears fairly frequently. An example with $n = 1$ is $F(t) = t^{1/2}$, $F'(t) = \frac{1}{2}t^{-1/2}$. Here

$$\mathfrak{L}\{F\} = \frac{\Gamma(3/2)}{s^{3/2}}, \quad \mathfrak{L}\{F'\} = \frac{1}{2} \frac{\Gamma(1/2)}{s^{1/2}}, \quad F(0+) = 0.$$

Rule V supplies the correct equation

$$\frac{1}{2} \frac{\Gamma(1/2)}{s^{1/2}} = s \frac{\Gamma(3/2)}{s^{3/2}} \quad \text{because } \Gamma(3/2) = \frac{1}{2} \Gamma(1/2).$$

Rule VI (*Differentiation theorem for the image function*)

As was mentioned in § 3, the image function is always analytic and therefore possesses derivatives of all orders. These can be obtained by differentiation under the integral sign (EINF. p. 35), and this leads to the following rule:

$$\left| \begin{array}{l} -t F(t) \circ \bullet f'(s) \\ t^2 F(t) \circ \bullet f''(s) \\ \dots\dots\dots \\ (-1)^n t^n F(t) \circ \bullet f^{(n)}(s) \end{array} \right|$$

This rule shows that, as for original functions, the complicated operation of differentiation for image functions reflects into the other space as an elementary operation, viz., multiplication by the negative of the variable.

§ 8. Integration**Rule VII** (*Integration theorem for the original function*)

The differentiation of the original function corresponds (apart from the addition of a constant) to the multiplication of the image function by s . The integration of the original function from 0 to a variable point t corresponds to the division of the image function by s :

$$\left| \int_0^t F(\tau) d\tau \circ \bullet \frac{1}{s} f(s) \right|$$

It has been assumed that $F(t)$ possesses an image function. The integral has then an image function automatically (EINF. p. 46).

Rule VIII (*Integration theorem for the image function*)

The *differentiation* of the image function corresponds to the *multiplication* of the original function by $-t$, the *integration to division* by $-t$, if $s = \infty$ is chosen as the lower limit for the integration. However we replace $\int_s^\infty$ by $-\int_\infty^s$:

$$\frac{F(t)}{t} \circ \bullet \int_s^{\infty} f(\sigma) d\sigma$$

For this it has been assumed that $\frac{F(t)}{t}$ possesses an image function, if so, $F(t)$ certainly has one. The path of integration in $\int_s^{\infty}$ can be a ray radiating from s , and making an acute angle with the real axis.

This rule, in contrast to the others, is seldom employed and we mention it here only for completeness.

§ 9. Multiplication and Convolution

Up to now we have established how a certain operation executed on *one* function is reflected upon its corresponding function. We consider now combinations of *several* functions and seek the corresponding combination in the other space.

It is self-evident that the *sum* of a finite number of original functions corresponds to the sum of the corresponding image functions, so we do not formulate this as a special rule¹⁶). The next simplest combination is the *product* and in this case we begin with the product of image functions.

Rule IX (*Convolution theorem*)

To the product $f_1(s) \cdot f_2(s)$ there corresponds a peculiar integral combination of the functions $F_1(t)$ and $F_2(t)$, which appears very often in physics, namely¹⁷)

$$(9.1) \quad \int_0^t F_1(\tau) F_2(t - \tau) d\tau.$$

¹⁶) For the sum of an infinite number of terms this naturally is not true in general. See also § 35.

¹⁷) When, for example, certain causes $F_1(\tau)$ have effect during the time from $\tau = 0$ to $\tau = t$, the total effect is described in the simplest case by $\int_0^t F_1(\tau) d\tau$. If, however, every cause has a weighting factor F_2 , which depends upon the time difference between the time t of its observation and the time τ of its occurrence, i.e., $t - \tau$, then the total effect is given by integral (9.1).

It is called the *convolution* (German: Faltung) of F_1 and F_2 and is symbolically denoted by $F_1 * F_2$:

$$F_1 * F_2 = \int_0^t F_1(\tau) F_2(t - \tau) d\tau.$$

This symbol looks like a product and indeed the convolution behaves completely like a product: it is *commutative*, that is,

$$(9.2) \quad F_1 * F_2 = F_2 * F_1 \quad \text{or} \quad \int_0^t F_1(\tau) F_2(t - \tau) d\tau = \int_0^t F_2(\tau) F_1(t - \tau) d\tau,$$

and *associative*,

$$(9.3) \quad (F_1 * F_2) * F_3 = F_1 * (F_2 * F_3),$$

so that the convolution of several functions, $F_1 * F_2 * \dots * F_n$ always leads to the same result, independent of the order in which the convolution is calculated (EINF. p. 56).

This correspondence between convolution and product can be formulated as a rule:

$$(9.4) \quad \left[F_1 * F_2 \circ \bullet f_1 \cdot f_2 \right]$$

For this rule it is assumed the $\mathfrak{L}$ -integrals of F_1 and F_2 exist and that at least one converges absolutely¹⁸⁾; then the $\mathfrak{L}$ -integral of $F_1 * F_2$ converges automatically (EINF. p. 59).

After Rule V this rule is the most important in applications.

Rule X (Complex convolution theorem)

To the product of two original functions there corresponds the "complex convolution" of the image functions. With the assumption that for two real values x_1, x_2 the integrals

$$(9.5) \quad \int_0^\infty e^{-x_1 t} |F_1(t)| dt, \quad \int_0^\infty e^{-2x_2 t} |F_2(t)|^2 dt \quad (\nu = 1, 2)$$

¹⁸⁾ Absolute convergence of $\int_0^\infty e^{-st} F(t) dt$ means that

$$\int_0^\infty |e^{-st} F(t)| dt = \int_0^\infty e^{-\Re s \cdot t} |F(t)| dt$$

converges. This is fulfilled for all functions $F(t)$ for which $|F(t)| < A e^{at}$, when $\Re s > a$.

converge, the following rule is valid for all s with $\Re s \geq x_1 + x_2$ (EINF. p. 210):

$$(9.6) \quad F_1(t) \cdot F_2(t) \circ \bullet \begin{cases} \frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} f_1(\sigma) f_2(s-\sigma) d\sigma & \text{with } x_1 \leq x \leq \Re s - x_2 \\ \text{or} \\ \frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} f_1(s-\sigma) f_2(\sigma) d\sigma & \text{with } x_2 \leq x \leq \Re s - x_1 \end{cases}$$

When the abscissae x of the integration lines are chosen as stated, then the variables in f_1 and f_2 move in the half-planes of absolute convergence of the $\mathfrak{L}$ -integrals $\mathfrak{L}\{F_1\}$ and $\mathfrak{L}\{F_2\}$.

The assumption (9.5) holds when F_2 is replaced by $\overline{F_2}$. The corresponding image function is:

$$\mathfrak{L}\{\overline{F_2}\} = \int_0^\infty e^{-st} \overline{F_2(t)} dt = \int_0^\infty e^{-st} \overline{F_2(t)} dt = \overline{f_2(s)}.$$

When we replace F_2 by $\overline{F_2}$ and $f_2(s)$ by $\overline{f_2(s)}$ in the upper row of (9.6) and choose $s = x_1 + x_2$, $x = x_1$, we obtain, on replacing the correspondence sign by the explicit $\mathfrak{L}$ -integral,

$$\int_0^\infty e^{-(x_1+x_2)t} F_1(t) \overline{F_2(t)} dt = \frac{1}{2\pi j} \int_{x_1-j\infty}^{x_1+j\infty} f_1(\sigma) \overline{f_2(x_1+x_2-\sigma)} d\sigma$$

or, with $\sigma = x_1 + jy$, $\sigma = x_1 - jy$:

$$(9.7) \quad \int_0^\infty e^{-(x_1+x_2)t} F_1(t) \overline{F_2(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_1(x_1 + jy) \overline{f_2(x_2 + jy)} dy$$

This relation is called the *generalized Parseval equation* for the $\mathfrak{L}$ -transformation. It is valid under the assumption (9.5).

For $F_1 = F_2 = F$, $x_1 = x_2 = x$, it reduces to the *Parseval equation*

$$(9.8) \quad \int_0^\infty e^{-2xt} |F(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(x + jy)|^2 dy$$

This is valid under the assumption that the integrals

$$(9.9) \quad \int_0^{\infty} e^{-x t} |F(t)| dt, \quad \int_0^{\infty} e^{-2x t} |F(t)|^2 dt$$

converge.

If we can set $x = 0$, that is, when the integrals

$$(9.10) \quad \int_0^{\infty} |F(t)| dt, \quad \int_0^{\infty} |F(t)|^2 dt$$

converge, the Parseval equation assumes the form:

$$(9.11) \quad \int_0^{\infty} |F(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(jy)|^2 dy$$

This equation is of manifold importance in engineering. When $F(t)$ is the output of an electrical network, equation (9.11) represents the total energy dissipated in a resistance of one ohm, expressed through the amplitude density $|f(jy)|$. If $F(t)$ is the controlled variable of an automatic control system, then $\int_0^{\infty} |F(t)|^2 dt$ represents the so-called "quadratic control area".

One of the methods of optimum adjustment in the field of automatic control consists of making this area a minimum. In this case $F(t)$ is real; therefore $\overline{f(jy)} = f(-jy)$ and (9.11) takes the form:

$$(9.12) \quad \int_0^{\infty} F(t)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(jy) f(-jy) dy = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} f(\sigma) f(-\sigma) d\sigma.$$

If $f(s)$ is a rational function, this complex integral can be calculated by the calculus of residues. The result depends on the finite number of constants in the rational function and the integral can be minimized by the standard methods of calculus.

* * *

We shall see in the next chapters, how certain types of functional equations can be solved in a simple way by means of the $\mathfrak{L}$ -transformation.

Ordinary Differential Equations

§ 10. The Differential Equation of First Order

We consider the differential equation of first order with constant coefficients but with arbitrary "excitation or perturbation function" $F(t)$:

$$(10.1) \quad Y' + c_0 Y = F(t).$$

We have implied by use of capital letters that the equation is given in original space. The basic idea consists of employing the $\mathfrak{L}$ -transformation, which we have so far used only with single functions, directly on the equation (10.1). This can be expressed by saying that we desire the *image equation* of the *original equation*. Of course this is done by multiplying both sides by e^{-st} and integrating from 0 to ∞ , but we adapt ourselves to the routine of the calculation most quickly if we do not write out the explicit $\mathfrak{L}$ -integrals but only employ the symbol $\mathfrak{L}$:

$$(10.2) \quad \mathfrak{L}\{Y'\} + c_0 \mathfrak{L}\{Y\} = \mathfrak{L}\{F\}.$$

We come now to the second important step which consists of employing Rule V and thus expressing $\mathfrak{L}\{Y'\}$ in terms of $\mathfrak{L}\{Y\}$. At the same time we introduce the lower case letters and obtain:

$$(10.3) \quad s y(s) - Y(0+) + c_0 y(s) = f(s).$$

After a little practice we can omit equation (10.2) and write down equation (10.3) immediately under (10.1) and say "We write down the image equation of the original equation" or "We transform the equation into image space".

We observe two important properties of the image equation: 1. It is a linear *algebraic equation* in $y(s)$, and therefore incomparably more simple than the original differential equation. 2. It contains the value $Y(0+)$. This proves to be very useful since to determine a definite solution from the infinitely many solutions of the differential equation, we must be given the value of $Y(t)$ at one point and in applications this is generally at the "initial point" $t = 0$; therefore the name "initial value" for $Y(0+)$.

The expression "initial value" is to be understood as that $Y(t)$ shall be continuously connected to it or, in other words, that this value shall be the limiting value of the solution when t tends to 0 from the right. It is exactly

the value $Y(0+)$ that enters also into Rule V, cf. also remark 1, p. 37. This prescribed initial value enters into the image equation and thus is considered automatically. This is an important advantage over the classical method, in which we first find the "general solution" which depends upon an arbitrary constant and then subsequently determine this constant so that $Y(t)$ assumes the prescribed value at $t = 0$. (We can say: a constant also appears in the method of the $\mathfrak{L}$ -transformation but this constant has a known relation to the solution; it is equal to $Y(0+)$.)

The solution of the image equation (10.3) can be written down immediately

$$(10.4) \quad y(s) = f(s) \frac{1}{s + c_0} + Y(0+) \frac{1}{s + c_0}.$$

Hence the image function $y(s)$ of the wanted function $Y(t)$ has been found already and we require only to determine its *corresponding original function*. For this purpose we could use the complex inversion formula (2.8). However, we wish to avoid this as much as possible and so we proceed in the same way as when we meet an integral in the process of a calculation. We do not evaluate the integral from its definition as the limit of a sum but we consult a table of integrals; when we do not find the required result immediately, we decompose and reshape the integrand so that we reach a known integral. Analogously, in our case we consult the attached *Table of Correspondences* and see whether the original function of the image function under consideration is written down or whether perhaps by using our "grammatical rules" of Chapter 2 the original function can be built up from the functions in the table.

In the case of the image function (10.4) this is very simple. According to TAB. 35 to the image function $1/(s + c_0)$ there corresponds the original function $e^{-c_0 t}$, and thus the original function of the second term in $y(s)$ is already established. The first term is the product of two image functions and by Rule IX the convolution of the original functions corresponds to it. Hence by combining these results it follows that

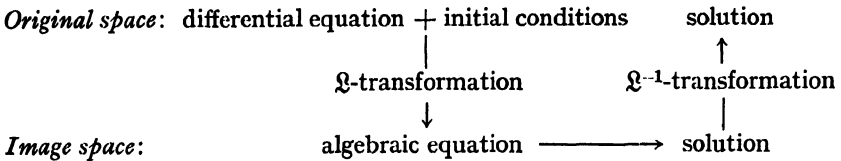
$$(10.5) \quad \left\{ \begin{aligned} Y(t) &= F(t) * e^{-c_0 t} + Y(0+) e^{-c_0 t} \\ &= e^{-c_0 t} \int_0^t F(\tau) e^{c_0 \tau} d\tau + Y(0+) e^{-c_0 t}, \end{aligned} \right.$$

and from this the solution of the differential equation (10.1) with given initial value is obtained.

It must be emphasized that we have obtained here immediately the solution of the inhomogeneous equation (with excitation function F), while by the classical method we find first the solution of the homogeneous equation ($F \equiv 0$) and then obtain the solution in the inhomogeneous case by the method of variation of parameters.

We can exhibit the method of solution used here in a clearly arranged scheme, to which we always return when we employ the $\mathfrak{L}$ -transformation on functional equations.

Scheme



Explanation: Instead of solving the given differential equation with given initial conditions directly, we make a detour across into the image space. We go from the original equation to the image equation (an algebraic equation) by the $\mathfrak{L}$ -transformation, we solve this and then translate the solution back to original space with the help of the inversion of the $\mathfrak{L}$ -transformation.

We observe that the image function $f(s)$ of the perturbation function $F(t)$ is not needed for the explicit expression of the solution $Y(t)$ but is used only in the derivation of the result. Nevertheless, it is practical in many cases to determine $f(s)$ and to translate the complete expression $f(s)/(s + c_0)$ back into original space instead of calculating the convolution integral in (10.5). For example, let $F(t) = U(t)$; then $f(s) = 1/s$; thus

$$(10.6) \quad \frac{f(s)}{s + c_0} = \frac{1}{s(s + c_0)},$$

and for this function we find immediately from TAB. 40 the original function

$$\frac{1}{c_0} (1 - e^{-c_0 t}).$$

Similarly, when $F(t)$ is a power or an exponential function. We wish to find the solution when $F(t)$ is the sinusoidal oscillation $e^{j\omega t}$. Then

$$f(s) \frac{1}{s + c_0} = \frac{1}{(s - j\omega)(s + c_0)}.$$

From TAB. 44 the corresponding original function is

$$\frac{1}{c_0 + j\omega} (e^{j\omega t} - e^{-c_0 t}).$$

To obtain the solution for $F(t) = \cos \omega t$ or $\sin \omega t$, we require to write this expression in the form

$$\frac{c_0 - j\omega}{c_0^2 + \omega^2} (\cos \omega t + j \sin \omega t - e^{-c_0 t})$$

and to split it into its

$$\text{real part} = \frac{1}{c_0^2 + \omega^2} (\omega \sin \omega t + c_0 (\cos \omega t - e^{-c_0 t})),$$

$$\text{and its imaginary part} = \frac{1}{c_0^2 + \omega^2} (c_0 \sin \omega t - \omega (\cos \omega t - e^{-c_0 t})).$$

In the complete solution the term $Y(0+)e^{-c_0 t}$ still appears. With c_0 positive the parts containing $e^{-c_0 t}$ tend to 0 for increasing t and there are left only pure oscillations with the same frequency as the excitation function, but with different amplitude and phase.

§ 11. The Differential Equation of Second Order

For the differential equation of second order

$$(11.1) \quad Y'' + c_1 Y' + c_0 Y = F(t),$$

whose coefficients are taken to be real¹⁹⁾, we will be more brief. Its image equation by Rule V is

$$[s^2 y - Y(0+)s - Y'(0+)] + c_1 [s y - Y(0+)] + c_0 y = f(s).$$

Two initial values appear in it, namely those of the function Y and its first derivative and these two values must be given for the solution of a differential equation of second order to be determined uniquely. The image equation is a linear algebraic equation with solution

$$(11.2) \quad y(s) = f(s) \frac{1}{s^2 + c_1 s + c_0} + Y(0+) \frac{s + c_1}{s^2 + c_1 s + c_0} + Y'(0+) \frac{1}{s^2 + c_1 s + c_0}.$$

The corresponding original functions of the individual terms of this sum can be found from the table of correspondences. However, we wish to show how the original functions can be found, because this is a good preparatory exercise to the solution of a differential equation of arbitrary order. The fractions appearing in (11.2) are rational functions in which the degree of the numerator is less than the degree of the denominator. Such functions, as is well known, can be decomposed by the so-called partial fraction expansion which is used also in the integral calculus for the integration of rational functions. We first determine the zeros α_1, α_2 , of the polynomial in the denominator and then decompose it into linear factors:

$$p(s) = s^2 + c_1 s + c_0 = (s - \alpha_1)(s - \alpha_2).$$

The treatment is different now according as $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2$.

¹⁹⁾ We have divided by the coefficient of Y'' so that the highest coefficient is equal to 1. This simplifies the calculations which follow.

Case 1: $\alpha_1 \neq \alpha_2$

The rational function can be expanded into fractions in which the denominators contain only one linear factor,

$$\frac{d_1}{s - \alpha_1} + \frac{d_2}{s - \alpha_2}.$$

We apply this subsequently to the two different fractions appearing in (11.2). We can determine the coefficients d_1, d_2 in the equation

$$(11.3) \quad \frac{1}{p(s)} = \frac{d_1}{s - \alpha_1} + \frac{d_2}{s - \alpha_2}$$

by a simple procedure²⁰), in which all the terms on the right-hand side are brought again over one denominator and the numerators are then compared. We obtain from

$$\frac{1}{p(s)} = \frac{(d_1 + d_2)s - (d_1\alpha_2 + d_2\alpha_1)}{(s - \alpha_1)(s - \alpha_2)}$$

the equations

$$d_1 + d_2 = 0, \quad \alpha_2 d_1 + \alpha_1 d_2 = -1,$$

and hence the coefficients have the values

$$d_1 = \frac{1}{\alpha_1 - \alpha_2}, \quad d_2 = -\frac{1}{\alpha_1 - \alpha_2}.$$

Because the original function of a single partial fraction is known, the original function of (11.3) can be found:

$$(11.4) \quad q(s) = \frac{1}{p(s)} \Leftrightarrow \frac{1}{\alpha_1 - \alpha_2} (e^{\alpha_1 t} - e^{\alpha_2 t}) = Q(t).$$

Explicitly we have

$$\alpha_1 = -\frac{c_1}{2} + \sqrt{-D}, \quad \alpha_2 = -\frac{c_1}{2} - \sqrt{-D}$$

$$\text{with} \quad D = c_0 - \frac{c_1^2}{4} \quad (\text{discriminant}).$$

For $D < 0$, that is $\frac{c_1^2}{4} > c_0$,

$k = \sqrt{-D}$ is real and

$$(11.5) \quad Q(t) = \frac{1}{2k} e^{-c_1 t/2} (e^{kt} - e^{-kt}) = \frac{1}{k} e^{-c_1 t/2} \sinh kt \quad (\text{aperiodic}).$$

For $D > 0$, that is $\frac{c_1^2}{4} < c_0$,

²⁰) In § 12 we shall give a more elegant method.

α_1 and α_2 are complex conjugates and $Q(t)$ appears in complex form; it is in fact real. For if we set $\sqrt{-D} = \omega j$, then

$$(11.6) \quad Q(t) = \frac{1}{2\omega j} e^{-c_1 t/2} (e^{j\omega t} - e^{-j\omega t}) = \frac{1}{\omega} e^{-c_1 t/2} \sin \omega t.$$

We can establish this real form immediately if we express $q(s) = 1/p(s)$ in the following way:

$$q(s) = \frac{1}{(s + c_1/2)^2 + (c_0 - c_1^2/4)} = \frac{1}{(s + c_1/2)^2 + D} = \frac{1}{(s + c_1/2)^2 + \omega^2}.$$

From the correspondence (see § 3, No. 5)

$$\frac{1}{s^2 + \omega^2} \longleftrightarrow \frac{1}{\omega} \sin \omega t$$

it follows, with the help of Rule IV (damping theorem), that

$$q(s) \longleftrightarrow \frac{1}{\omega} e^{-c_1 t/2} \sin \omega t = Q(t).$$

$Q(t)$ is therefore a periodic oscillation which is damped or amplified by the factor $e^{-c_1 t/2}$.

When $c_1^2/4 < c_0$, it is helpful to write the coefficients first in the form

$$(11.7) \quad c_1 = 2\delta, \quad c_0 = \delta^2 + \omega^2,$$

and then the differential equation is of the form:

$$(11.8) \quad Y'' + 2\delta Y' + (\delta^2 + \omega^2) Y = F(t).$$

Also $D = \omega^2$, and

$$(11.9) \quad q(s) = \frac{1}{p(s)} \longleftrightarrow \frac{1}{\omega} e^{-\delta t} \sin \omega t.$$

We can conclude immediately from the differential equation (11.8) that the function $Q(t)$ has frequency ω and damping δ . Moreover, the ratio between the amplitudes of two successive maxima of the damped oscillation is given by the constant value $e^{-\pi\delta/\omega}$. The so-called damping ratio δ/ω gives a good estimate of the rate at which the oscillations decrease.

For the other fraction in (11.2) we obtain by partial fraction expansion

$$\frac{s + c_1}{p(s)} = \frac{d_1}{s - \alpha_1} + \frac{d_2}{s - \alpha_2} = \frac{(d_1 + d_2)s - (d_1\alpha_2 + d_2\alpha_1)}{p(s)}$$

and the coefficients are determined by the equations

$$d_1 + d_2 = 1, \quad \alpha_2 d_1 + \alpha_1 d_2 = -c_1.$$

Hence, since $-c_1 = \alpha_1 + \alpha_2$, we obtain the values

$$d_1 = \frac{-\alpha_1 - c_1}{\alpha_2 - \alpha_1} = \frac{\alpha_2}{\alpha_2 - \alpha_1}, \quad d_2 = \frac{\alpha_2 + c_1}{\alpha_2 - \alpha_1} = -\frac{\alpha_1}{\alpha_2 - \alpha_1}.$$

Therefore

$$(11.10) \quad \frac{s + c_1}{p(s)} \bullet \bullet \frac{1}{\alpha_2 - \alpha_1} (\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}) = Q_1(t).$$

When we use the form (11.7) in the case when $D > 0$, the translation into original space can be achieved particularly simply (see § 3, Nos. 4, 5 and Rule IV):

$$(11.11) \quad \frac{s + 2\delta}{s^2 + 2\delta s + \delta^2 + \omega^2} = \frac{s + \delta}{(s + \delta)^2 + \omega^2} + \frac{\delta}{\omega} \frac{\omega}{(s + \delta)^2 + \omega^2} \\ \bullet \bullet e^{-\delta t} \cos \omega t + \frac{\delta}{\omega} e^{-\delta t} \sin \omega t.$$

Case 2: $\alpha_1 = \alpha_2$ ($D = 0$)

In this case $\alpha_1 = \alpha_2 = -\frac{c_1}{2}$ and (cf. § 3, No. 7)

$$(11.12) \quad \frac{1}{p(s)} = \frac{1}{(s - \alpha_1)^2} \bullet \bullet t e^{\alpha_1 t} = Q(t),$$

$$(11.13) \quad \frac{s + c_1}{p(s)} = \frac{s + c_1}{(s + c_1/2)^2} = \frac{1}{s + c_1/2} + \frac{c_1/2}{(s + c_1/2)^2} \bullet \bullet e^{-c_1 t/2} + \frac{c_1}{2} t e^{-c_1 t/2} = Q_1(t).$$

After the original functions of the rational functions in (11.2) are found, the solution of the differential equation is determined by the application of Rule IX (Convolution theorem):

$$(11.14) \quad Y(t) = F(t) * Q(t) + Y(0+)Q_1(t) + Y'(0+)Q(t).$$

In practice we do not use this formula for the general solution, but execute the above steps afresh each time.

By doing this it is often possible to avoid the inconvenient computation of the convolution integral $F * Q$ by determining $f(s)$ explicitly and then translating the image function $f(s)/p(s)$ into original space. It happens frequently in practice that $F(t)$ is one of the functions $U(t)$, $e^{\alpha t}$, $t^n e^{\alpha t}$, $\cos \omega t$ or $\sin \omega t$ so that $f(s)$ is a rational function and likewise $f(s)/p(s)$. We can then find the original function by partial fraction expansion. The following example illustrates this procedure.

Numerical Example

The differential equation

$$Y'' + 10 Y' + 74 Y = 28 \sin 4t$$

with the initial conditions

$$Y(0+) = 0, \quad Y'(0+) = 2.$$

The image equation is

$$s^2 y - 2 + 10 s y + 74 y = \frac{112}{s^2 + 16}.$$

Its solution is

$$y = \frac{112}{(s^2 + 16)(s^2 + 10s + 74)} + \frac{2}{s^2 + 10s + 74}.$$

Consider first the second term and determine the zeros of

$$s^2 + 10s + 74 = 0.$$

They are

$$\alpha_1 = -5 + 7j, \quad \alpha_2 = -5 - 7j.$$

We set

$$\frac{1}{s^2 + 10s + 74} = \frac{d_1}{s + 5 - 7j} + \frac{d_2}{s + 5 + 7j},$$

and by comparing coefficients we have

$$\begin{aligned} d_1 + d_2 &= 0 \\ (5 + 7j) d_1 + (5 - 7j) d_2 &= 1. \end{aligned}$$

From these equations we obtain

$$d_1 = \frac{1}{14j}, \quad d_2 = -\frac{1}{14j}.$$

Hence the original function of

$$\frac{1}{s^2 + 10s + 74} = \frac{1/14j}{s + 5 - 7j} - \frac{1/14j}{s + 5 + 7j},$$

is

$$Q(t) = \frac{1}{14j} (e^{(-5+7j)t} - e^{(-5-7j)t}) = \frac{1}{7} e^{-5t} \sin 7t.$$

We could have obtained this result more skilfully by writing

$$\frac{1}{s^2 + 10s + 74} = \frac{1}{7} \frac{7}{(s+5)^2 + 49},$$

since from the correspondence

$$\frac{7}{s^2 + 49} \longleftrightarrow \sin 7t$$

and the damping theorem $Q(t)$ can then be found immediately.

We could calculate the original function of the first term in y immediately by using the convolution theorem. However, when the explicit expression for function $f(s)$ is inserted, this term is a rational function and it is simpler to decompose this function into partial fractions. The denominator has four different zeros

$$4j, -4j, -5+7j, -5-7j,$$

and thus we have four different partial fractions:

$$\frac{112}{(s^2+16)(s^2+10s+74)} = \frac{d_1}{s-4j} + \frac{d_2}{s+4j} + \frac{d_3}{s+5-7j} + \frac{d_4}{s+5+7j}.$$

When the right-hand side is put over one denominator, the numerator is

$$\begin{aligned} & d_1(s+4j)(s^2+10s+74) + d_2(s-4j)(s^2+10s+74) \\ & + d_3(s^2+16)(s+5+7j) + d_4(s^2+16)(s+5-7j) \\ = & s^3(d_1+d_2+d_3+d_4) + s^2[(10+4j)d_1 + (10-4j)d_2 + (5+7j)d_3 + (5-7j)d_4] \\ & + s[(74+40j)d_1 + (74-40j)d_2 + 16d_3 + 16d_4] \\ & + [296jd_1 - 296jd_2 + (80+112j)d_3 + (80-112j)d_4]. \end{aligned}$$

Therefore d_1 to d_4 must satisfy the linear equations:

$$\begin{aligned} d_1 + d_2 + d_3 + d_4 &= 0 \\ (10+4j)d_1 + (10-4j)d_2 + (5+7j)d_3 + (5-7j)d_4 &= 0 \\ (74+40j)d_1 + (74-40j)d_2 + 16d_3 + 16d_4 &= 0 \\ 296jd_1 - 296jd_2 + (80+112j)d_3 + (80-112j)d_4 &= 112. \end{aligned}$$

Solving these equations we find:

$$\begin{aligned} d_1 &= -\frac{7}{1241}(20+29j), & d_2 &= -\frac{7}{1241}(20-29j), \\ d_3 &= \frac{4}{1241}(35+4j), & d_4 &= \frac{4}{1241}(35-4j). \end{aligned}$$

Hence, for the first term in y we obtain the original function

$$\begin{aligned} & -\frac{7}{1241}(20+29j)e^{4jt} - \frac{7}{1241}(20-29j)e^{-4jt} \\ & + \frac{4}{1241}(35+4j)e^{(-5+7j)t} + \frac{4}{1241}(35-4j)e^{(-5-7j)t} \\ = & -\frac{7}{1241}[20(e^{4jt} + e^{-4jt}) + 29j(e^{4jt} - e^{-4jt})] \\ & + \frac{4}{1241}e^{-5t}[35(e^{7jt} + e^{-7jt}) + 4j(e^{7jt} - e^{-7jt})] \\ = & -\frac{14}{1241}(20\cos 4t - 29\sin 4t) + \frac{8}{1241}e^{-5t}(35\cos 7t - 4\sin 7t). \end{aligned}$$

Thus the problem is completely solved.

We have undertaken the expansion into partial fractions in the most primitive way by placing the fractions over one denominator and setting up a system of linear equations for $d_1, d_2, \dots$ by comparing coefficients. The solution of such a system for a large number of unknowns is very troublesome, and for this reason we will show how to achieve partial fraction expansions in a much simpler way. For this purpose we consider next the differential equation of arbitrary order.

§ 12. The Inhomogeneous Differential Equation of n th Order with Vanishing Initial Values

As in the previous examples we make the coefficient of the highest derivative equal to 1, because it simplifies the calculation and guards against errors. The differential equation of n th order then has the form

$$(12.1) \quad Y^{(n)} + c_{n-1} Y^{(n-1)} + \dots + c_1 Y' + c_0 Y = F(t).$$

For the solution to be determined uniquely, n initial values must be given. These are the values

$$Y(0+), \quad Y'(0+), \quad Y''(0+), \quad \dots, \quad Y^{(n-1)}(0+).$$

To simplify the notation in what follows, we denote these by $Y(0)$, $Y'(0)$, etc. This is therefore the so-called "initial value problem" for the differential equation (12.1). To make the calculations simpler, we divide the problem into two parts: first we assume that the equation is inhomogeneous, that is, the perturbation function $F(t)$ does not vanish identically, but that the initial values all vanish (this is the most frequent case in practice). Then in § 14 we consider the homogeneous equation, that is, $F(t) = 0$, with arbitrary initial values. These solutions when added together give the general solution.

The differential equation (12.1) is now considered with the initial conditions

$$(12.2) \quad Y(0) = Y'(0) = \dots = Y^{(n-1)}(0) = 0.$$

According to Rule V it has the image equation

$$(12.3) \quad s^n y + c_{n-1} s^{n-1} y + \dots + c_1 s y + c_0 y = f(s).$$

Using the abbreviation

$$(12.4) \quad s^n + c_{n-1} s^{n-1} + \dots + c_1 s + c_0 = p(s)$$

the solution in image space is

$$(12.5) \quad y(s) = \frac{1}{p(s)} f(s).$$

We can write down immediately the solution of the differential equation with the help of Rule IX (convolution theorem) when we have determined the original function of

$$q(s) = \frac{1}{p(s)}.$$

This we accomplish, as in the previous special cases, by partial fraction expansion. For this purpose we must first determine the zeros $\alpha_1, \dots, \alpha_n$ ²¹⁾ after which $p(s)$ can be represented as a product of linear factors:

$$(12.6) \quad p(s) = (s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_n).$$

The determination of the zeros is the most tedious part of the entire calculation when $n \geq 4$. The real zeros are easily found by calculating a few function values of $p(s)$ from which we can sketch the curve for $p(s)$ and thus determine approximately the points of intersection of this curve with the real axis; then they can be found more and more exactly by known iteration processes. The calculation of the complex zeros, which occur as complex conjugate pairs when the coefficients $c_{n-1}, \dots, c_0$ are real, is, on the other hand, an intricate procedure (cf. also the processes given in § 35 and 36 for the determination of the original function).

It has been shown already in the case $n = 2$ that the appearance of equal zeros requires special handling. For that reason we make here also the corresponding distinction.

Case 1: The zeros α_ν are all different

The partial fraction expansion of $q(s)$ has the form

$$(12.7) \quad q(s) = \frac{1}{p(s)} = \frac{d_1}{s - \alpha_1} + \frac{d_2}{s - \alpha_2} + \dots + \frac{d_n}{s - \alpha_n}.$$

This corresponds to the fact that $q(s)$ possesses the simple poles $\alpha_1, \dots, \alpha_n$, (this is made clear by the partial fractions). The coefficients $d_1, \dots, d_n$ (the residues at the poles) can be determined by the following method which is simpler than that used in the previous special cases. We multiply equation (12.7) by $s - \alpha_1$; then the first term on the right-hand side consists only of d_1 , while all other terms contain $(s - \alpha_1)$. When we let s tend to α_1 , then all terms on the right vanish except the first; hence d_1 is isolated. On the left-hand side, when we remember that $p(\alpha_1) = 0$, we obtain by this limiting process

$$\lim_{s \rightarrow \alpha_1} \frac{s - \alpha_1}{p(s)} = \lim_{s \rightarrow \alpha_1} \frac{1}{\frac{p(s) - p(\alpha_1)}{s - \alpha_1}} = \frac{1}{p'(\alpha_1)}.$$

²¹⁾ This is also the first step in the classical method of solution of differential equations. There $p(s) = 0$ is called the characteristic equation.

Therefore the result is

$$d_1 = \frac{1}{p'(\alpha_1)}$$

and by analogy

$$d_2 = \frac{1}{p'(\alpha_2)}, \quad \dots, \quad d_n = \frac{1}{p'(\alpha_n)}.$$

The partial fraction expansion of $q(s)$ reads

$$(12.8) \quad q(s) = \frac{1}{p(s)} = \sum_{v=1}^n \frac{1}{p'(\alpha_v)} \frac{1}{s - \alpha_v}.$$

For this expression we require to form only the derivative of the polynomial $p(s)$ and to calculate the values $p'(\alpha_v)$: this is much easier than the earlier method of solving a system of n linear equations with n unknowns.

Moreover we can use the following observation: from the rule for the differentiation of a product, we have

$$p'(s) = (s - \alpha_2)(s - \alpha_3) \dots (s - \alpha_n) + (s - \alpha_1)(s - \alpha_3) \dots (s - \alpha_n) + \dots,$$

and therefore

$$p'(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \dots (\alpha_1 - \alpha_n),$$

$$p'(\alpha_2) = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3) \dots (\alpha_2 - \alpha_n), \quad \text{etc.}$$

When we apply this to the numerical example on p. 52 with

$$p(s) = (s^2 + 16)(s^2 + 10s + 74)$$

we obtain

$$p'(\alpha_1) = 8j(5 - 3j)(5 + 11j) = -320 + 464j,$$

$$\frac{1}{p'(\alpha_1)} = \frac{1}{-320 + 464j} = \frac{-320 - 464j}{320^2 + 464^2} = \frac{-20 - 29j}{19856}.$$

In the example $1/p(s)$ was multiplied by 112 and when that is done here we obtain the same value for d_1 as on p. 52, but with much less calculation.

We return now to (12.8). To $q(s)$ there corresponds the original function:

$$(12.9) \quad Q(t) = \sum_{v=1}^n \frac{1}{p'(\alpha_v)} e^{\alpha_v t}$$

Using this we can write the original function of (12.5), that is, the solution of the differential equation (12.1) with the initial conditions (12.2), in the simple form

$$(12.10) \quad Y(t) = Q(t) * F(t)$$

In practice the following steps have to be carried out: 1. Calculation of the zeros of $p(s)$. 2. Determination of $Q(t)$ by formula (12.9). 3. Calculation of the convolution integral (12.10). However, as has been shown already in the example on p. 52, we can often avoid the convolution integral by forming the image function $f(s)$ and translating the solution $f(s)/p(s)$ in closed form in image space back to original space. We shall give a further example on pp. 62, 63.

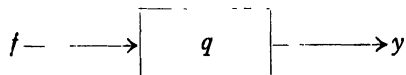
In electrical engineering certain names are used for the functions appearing here. When the differential equation describes an electric network the function $F(t)$ may mean the voltage at the input terminal and $Y(t)$ the current at the output terminal. For this reason $F(t)$ is generally called the "input" and $Y(t)$ the "output". Sometimes $F(t)$ is called the "excitation" and $Y(t)$ the "response". These expressions are carried over into the image space and $f(s)$ is also called input or excitation and $y(s)$ output or response. In image space the connection between the input and the output is particularly simple:

$$(12.11) \quad y(s) = q(s) f(s)$$

Therefore we work as much as possible in the image space and only go over to the original space when we require numerical results for the function $Y(t)$.

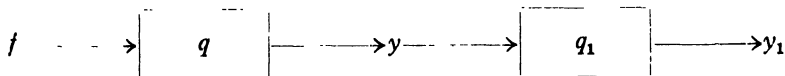
The function $q(s)$, which depends upon the constants $c_{n-1}, \dots, c_0$, and therefore on the inner structure of the physical system, and which establishes the connection between $f(s)$ and $y(s)$ by (12.11) is called the "system function". The corresponding original function $Q(t)$ is known in mathematics as the "Green's function" of the problem and in electrical engineering as the "Weight function" (cf. footnote 17 p. 40).

The connection between f , y and q is most clearly represented by the following figure



The physical system is represented by a "block", which is labelled by the letter q . This quantity is the only one that we require to know for the system: it is completely characterized through q . f enters into the block and y comes out of it.

The block symbol is especially useful when several physical systems are coupled together, as, for example, when the output of the first system with the system function q is the input of the second system with the system function q_1 :



Then

$$y = qf, \quad y_1 = q_1 y,$$

and therefore

$$y_1 = q q_1 f.$$

Hence the two systems can be substituted by a single system with the system function $q q_1$:

$$f \longrightarrow \boxed{q q_1} \longrightarrow y_1$$

Particularly in feedback, as it occurs in servomechanism, the analysis in image space by means of the block symbol leads to very clear relations (EINF. p. 89).

Case 2: Some of the zeros α_v are equal

These zeros can be regarded as "multiple". When we denote the numerically different zeros by $\alpha_1, \alpha_2, \dots, \alpha_m$, then we must ascribe to every zero α_v a multiplicity k_v , and the decomposition of $p(s)$ into linear factors has the form:

$$(12.12) \quad p(s) = (s - \alpha_1)^{k_1} (s - \alpha_2)^{k_2} \dots (s - \alpha_m)^{k_m}.$$

In the partial fraction expansion of $1/p(s)$ there now corresponds to a zero α_v not merely one fraction with $s - \alpha_v$ in the denominator, but k_v fractions with $(s - \alpha_v), (s - \alpha_v)^2, \dots, (s - \alpha_v)^{k_v}$ as denominators:

$$(12.13) \quad q(s) = \frac{1}{p(s)} = \frac{d_1^{(1)}}{s - \alpha_1} + \frac{d_1^{(2)}}{(s - \alpha_1)^2} + \dots + \frac{d_1^{(k_1)}}{(s - \alpha_1)^{k_1}} \\ + \dots + \frac{d_m^{(1)}}{s - \alpha_m} + \frac{d_m^{(2)}}{(s - \alpha_m)^2} + \dots + \frac{d_m^{(k_m)}}{(s - \alpha_m)^{k_m}}.$$

To this corresponds the original function (cf. § 3, No. 7)

$$(12.14) \quad Q(t) = e^{\alpha_1 t} \left(d_1^{(1)} + d_1^{(2)} \frac{t}{1!} + \dots + d_1^{(k_1)} \frac{t^{k_1-1}}{(k_1-1)!} \right) \\ + \dots + e^{\alpha_m t} \left(d_m^{(1)} + d_m^{(2)} \frac{t}{1!} + \dots + d_m^{(k_m)} \frac{t^{k_m-1}}{(k_m-1)!} \right).$$

With this expression for $Q(t)$ the differential equation has again solution of the form (12.10).

In this case also there exists a general formula for the determination of the coefficients $d_v^{(\lambda)}$ (EINF. p. 76). We remove the factor $(s - \alpha_v)^{h_v}$ in the representation (12.12) of $p(s)$ and thus obtain a function which we can denote by

$$r_v(s) = \frac{p(s)}{(s - \alpha_v)^{h_v}}.$$

Then

$$(12.15) \quad d_v^{(\lambda)} = \frac{1}{(h_v - \lambda)!} \left\{ \frac{d^{h_v - \lambda}}{ds^{h_v - \lambda}} \left[\frac{1}{r_v(s)} \right] \right\}_{s = \alpha_v}.$$

In practice it is often simpler to proceed as in the following example.

Numerical Example

After determination of the zeros, let $p(s)$ have the form:

$$p(s) = (s - 2)^3 (s + 5) (s + 7).$$

That the multiplicity of $(s - 2)$ is 3, and those of $(s + 5)$ and of $(s + 7)$ are 1, is also shown by

$$\begin{aligned} p(2) &= p'(2) = p''(2) = 0, & p'''(2) &\neq 0; \\ p(-5) &= 0, & p'(-5) &\neq 0; \\ p(-7) &= 0, & p'(-7) &\neq 0. \end{aligned}$$

We set

$$\frac{1}{(s + 2)^3 (s + 5) (s + 7)} = \frac{g_3}{(s - 2)^3} + \frac{g_2}{(s - 2)^2} + \frac{g_1}{s - 2} + \frac{h}{s + 5} + \frac{k}{s + 7}.$$

Now multiply by $(s - 2)^3$ to isolate g_3 ,

$$\frac{1}{(s + 5) (s + 7)} = g_3 + g_2 (s - 2) + g_1 (s - 2)^2 + \frac{h (s - 2)^3}{s + 5} + \frac{k (s - 2)^3}{s + 7},$$

and then set $s = 2$. This gives

$$\frac{1}{7 \cdot 9} = g_3.$$

We bring this known term $\frac{g_3}{(s - 2)^3}$ over to the left-hand side, which is then of the form

$$\frac{1}{(s - 2)^3 (s + 5) (s + 7)} - \frac{1}{63 (s - 2)^3} = \frac{63 - (s + 5) (s + 7)}{63 (s - 2)^3 (s + 5) (s + 7)}.$$

The numerator equals

$$-s^2 - 12s + 28 = -(s-2)(s+14),$$

and thus contains the factor $(s-2)$, which can be cancelled out from both numerator and denominator. The left-hand side is now equal to

$$\frac{-(s+14)}{63(s-2)^2(s+5)(s+7)}.$$

When we equate this to the remaining part of the right-hand side, we obtain

$$\frac{-(s+14)}{63(s-2)^2(s+5)(s+7)} = \frac{g_2}{(s-2)^2} + \frac{g_1}{(s-2)} + \frac{h}{s+5} + \frac{k}{s+7}.$$

(It is now clear why the numerator on the left-hand side contained the factor $(s-2)$. On the right-hand side $(s-2)$ appears in the denominator raised to the second power at most, therefore the same must be true on the left-hand side. Thus one factor out of $(s-2)^3$ must cancel $(s-2)$ in the numerator.) We can now isolate g_2 by multiplying by $(s-2)^2$:

$$\frac{-(s+14)}{63(s+5)(s+7)} = g_2 + g_1(s-2) + \frac{h(s-2)^2}{s+5} + \frac{k(s+2)^2}{s+7}$$

and again setting $s = 2$:

$$-\frac{16}{63 \cdot 7 \cdot 9} = g_2.$$

This term $\frac{g_2}{(s-2)^2}$ is now brought over to the left-hand side and we obtain

$$\frac{-(s+14)}{63(s-2)^2(s+5)(s+7)} + \frac{16}{63^2(s-2)^2} = \frac{-63(s+14) + 16(s+5)(s+7)}{63^2(s-2)^2(s+5)(s+7)}.$$

The numerator equals

$$16s^2 + 129s - 322 = (s-2)(16s + 161),$$

and we can again cancel the factor $(s-2)$. Equating this to the right-hand side gives

$$\frac{16s + 161}{63^2(s+5)(s+7)} = \frac{g_1}{s-2} + \frac{h}{s+5} + \frac{k}{s+7}.$$

After multiplication by $(s-2)$ the equation, when $s = 2$, gives

$$\frac{193}{63^2 \cdot 7 \cdot 9} = g_1.$$

$\frac{g_1}{(s-2)}$ is now brought over to the left-hand side and this is then equal to

$$\frac{16s + 161}{63^3(s-2)(s+5)(s+7)} - \frac{193}{63^3(s-2)} = \frac{63(16s + 161) - 193(s+5)(s+7)}{63^3(s-2)(s+5)(s+7)}.$$

The numerator equals

$$-193s^2 - 1308s + 3388 = -(s-2)(193s + 1694),$$

and after cancelling $(s-2)$ in numerator and denominator we obtain

$$-\frac{193s + 1694}{63^3(s+5)(s+7)} = \frac{h}{s+5} + \frac{k}{s+7}.$$

We multiply now by $(s+5)$, set $s = -5$ and find

$$-\frac{1}{7^3 \cdot 2} = h$$

and therefore

$$-\frac{193s + 1694}{63^3(s+5)(s+7)} = -\frac{729}{63^3 \cdot 2(s+5)} + \frac{k}{s+7}.$$

To evaluate k it is sufficient to set $s = 0$. Then

$$\frac{1}{9^3 \cdot 2} = k.$$

The partial fraction expansion now reads:

$$\begin{aligned} \frac{1}{(s-2)^3(s+5)(s+7)} &= \frac{1}{7 \cdot 9(s-2)^3} - \frac{16}{7^2 \cdot 9^2(s-2)^2} - \frac{193}{7^3 \cdot 9^3(s-2)} \\ &\quad - \frac{1}{7^3 \cdot 2(s+5)} + \frac{1}{9^3 \cdot 2(s+7)}. \end{aligned}$$

§ 13. The Response to Special Excitations

The behaviour of a physical system can be well estimated when we know its response to certain special excitations (test functions), which can easily be realized technically. We choose for this purpose the unit step function $U(t)$, the impulse function $\delta(t)$ and the sinusoidal oscillation $e^{i\omega t}$.

As in the previous paragraphs we presume that the initial values vanish.

1. The Step Response

The solution of the differential equation (12.1) which corresponds to the excitation $F(t) \equiv U(t)$ is called the "step response" $Y_U(t)$. By (12.10), we have

$$(13.1) \quad Y_U(t) = Q(t) * 1 = \int_0^t Q(\tau) d\tau \quad \text{or} \quad Q(t) = \frac{dY_U(t)}{dt}.$$

Thus the step response and the weight function are connected in a simple way.

The connection in image space is even simpler, for since $\mathfrak{L}\{U\} = 1/s$, we have by (12.11)

$$(13.2) \quad y_U(s) = \frac{1}{s} q(s) = \frac{1}{s p(s)}.$$

Here $1/p(s)$ is a rational function and, therefore, so is $1/sp(s)$. For this reason we shall not calculate $Y_U(t)$ from formula (13.1) by first finding $Q(t)$ and then integrating, but we shall expand $1/sp(s)$ into partial fractions and translate these to original space. A particularly simple formula is obtained when the zeros of $p(s)$ are all different and none is zero (the last requirement means $c_0 \neq 0$). Then the zeros $0, \alpha_1, \dots, \alpha_n$ of $sp(s)$ are all different and the partial fraction expansion is

$$\frac{1}{s p(s)} = \frac{d_0}{s} + \frac{d_1}{s - \alpha_1} + \dots + \frac{d_n}{s - \alpha_n}.$$

From

$$\frac{1}{p(s)} = d_0 + \sum_{v=1}^n \frac{d_v}{s - \alpha_v}$$

it follows that

$$d_0 = \frac{1}{p(0)}.$$

Further, since

$$\begin{aligned} \frac{s - \alpha_v}{s p(s)} &= \frac{1}{s \frac{p(s) - p(\alpha_v)}{s - \alpha_v}} \\ &= \frac{d_0}{s} (s - \alpha_v) + \frac{d_1}{s - \alpha_1} (s - \alpha_v) + \dots + d_v + \dots + \frac{d_n}{s - \alpha_n} (s - \alpha_v), \end{aligned}$$

we have, for $s \rightarrow \alpha_v$:

$$\frac{1}{\alpha_v p'(\alpha_v)} = d_v,$$

and so we obtain

$$y_U(s) = \frac{1}{p(0)} \frac{1}{s} + \sum_{v=1}^n \frac{1}{\alpha_v p'(\alpha_v)} \frac{1}{s - \alpha_v}$$

and

$$(13.3) \quad \left[Y_U(t) = \frac{1}{p(0)} + \sum_{v=1}^n \frac{e^{\alpha_v t}}{\alpha_v p'(\alpha_v)} \right]$$

This formula is known in electrical engineering as “*Heaviside's expansion theorem*”. When all the zeros α_r have negative real parts, $Y_U(t)$ tends towards the constant $1/p(0)$ for increasing t ; that is, a system which by the discontinuous excitation $U(t)$ is put in a fairly irregular condition (an aggregate of “characteristic oscillations”, see p. 74), approaches a new equilibrium state $1/p(0) = 1/c_0$. We can see this also from the differential equation; when Y becomes smooth, the derivatives $Y', \dots, Y^{(n)}$ tend to 0, and we have $c_0 Y \approx 1$.



Formula (13.3) is employed frequently in engineering. However, we wish to point out explicitly that it is correct only when the initial values vanish and when the zeros are different from one another, none being equal to 0. In numerical calculations we notice this automatically. For, if a multiple zero α_r appears, then $p'(\alpha_r) = 0$, and when an α_r is equal to 0, then $p(0) = 0$. In both cases formula (13.3) becomes meaningless.

When the step response has been determined either mathematically or experimentally, we can determine the response $Y(t)$ for an arbitrary excitation $F(t)$. For by (12.10), $Y(t) = Q * F$ and by (13.1) $Q = Y'_U$; therefore

$$(13.4) \quad \boxed{Y(t) = Y'_U * F}$$

This is known as *Duhamel's formula*. We can also write (cf. Theorem 32.1)

$$(13.5) \quad \boxed{Y(t) = \frac{d}{dt} [Y_U * F]}$$

Numerical Example

We consider again the differential equation on p. 51, but with the right-hand side equal to $U(t)$ and with vanishing initial values; that is, we wish to find the step response of the system that is described by the differential equation

$$Y'' + 10 Y' + 74 Y = U(t), \quad Y(0) = Y'(0) = 0.$$

Here

$$p(s) = s^2 + 10s + 74, \quad p'(s) = 2s + 10;$$

$$\alpha_1 = -5 + 7j, \quad \alpha_2 = -5 - 7j,$$

and therefore

$$p(0) = 74, \quad p'(\alpha_1) = 14j, \quad p'(\alpha_2) = -14j.$$

From formula (13.3), the system function is given by

$$Y_U(t) = \frac{1}{74} + \frac{1}{(-5 + 7j) 14j} e^{(-5+7j)t} - \frac{1}{(-5-7j)(-14j)} e^{(-5-7j)t}.$$

The second and third terms are complex conjugates of each other and therefore their sum is equal to twice the real part of either of them:

$$\begin{aligned} 2 \Re \frac{1}{(-5 + 7j) 14j} e^{(-5+7j)t} &= \frac{2}{14} e^{-5t} \Re \frac{(-5-7j)(-j)}{(25+49)} e^{7jt} \\ &= \frac{1}{7 \cdot 74} e^{-5t} \Re (-7 + 5j) (\cos 7t + j \sin 7t) \\ &= -\frac{1}{74 \cdot 7} e^{-5t} (7 \cos 7t + 5 \sin 7t). \end{aligned}$$

Thus we obtain

$$Y_U(t) = \frac{1}{74} \left\{ 1 - \frac{1}{7} e^{-5t} (7 \cos 7t + 5 \sin 7t) \right\}.$$

Moreover

$$Y'_U(t) = \frac{1}{7} e^{-5t} \sin 7t, \quad Y''_U(t) = \frac{1}{7} e^{-5t} (7 \cos 7t - 5 \sin 7t),$$

from which we see that $Y_U(t)$ satisfies the differential equation and has vanishing initial values.

For $t \rightarrow \infty$, Y'_U and Y''_U tend to zero and Y_U tends to $\frac{1}{74}$, as it must by the differential equation.

2. The Impulse Response

The impulse function $\delta(t)$ has been introduced already in § 1. It represents an excitation which vanishes for all time $t \neq 0$, while at $t = 0$ it takes an infinitely great value in such a way that the value $\varphi(0)$ is "sifted" out of the integral $\int_a^b \varphi(t) \delta(t) dt$ when the interval $a \leq t \leq b$ contains the zero point:

$$(13.6) \quad \int_a^b \varphi(t) \delta(t) dt = \begin{cases} \varphi(0) & \text{for } a \leq 0 < b \\ 0 & \text{in any other case.} \end{cases}$$

Correspondingly

$$(13.7) \quad \int_a^b \varphi(t) \delta(t - t_0) dt = \begin{cases} \varphi(t_0) & \text{for } a \leq t_0 < b \\ 0 & \text{otherwise.} \end{cases}$$

Because a function (in the sense of traditional analysis) cannot have this property, $\delta(t)$ was called a "pseudo function". We can think of it as the limit of a family of functions $S_\varepsilon(t)$, which are very great in the neighbourhood of $t = 0$ and otherwise zero; for example,

$$S_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } 0 \leq t < \varepsilon \\ 0 & \text{for } t < 0 \text{ and } t \geq \varepsilon. \end{cases}$$

If $\varphi(t)$ is continuous in the neighbourhood of $t = 0$, the mean value theorem of integral calculus gives

$$\int_a^b \varphi(t) S_\varepsilon(t) dt = \frac{1}{\varepsilon} \int_0^\varepsilon \varphi(t) dt = \varphi(\vartheta \varepsilon) \quad (0 \leq \vartheta \leq 1).$$

For $\varepsilon \rightarrow 0$, $S_\varepsilon(t)$ tends to $\delta(t)$ and $\int_a^b \varphi(t) S_\varepsilon(t) dt = \varphi(\vartheta \varepsilon)$ tends to $\varphi(0)$.

From the conclusions on p. 23, we can place "pseudo analysis" (that is, analysis that uses the pseudo function) on a more rigorous basis when we introduce the jump function²²)

$$(13.8) \quad U(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}$$

and replace integrals (13.6), (13.7) by Stieltjes integrals relative to the function $U(t)$. The integrals then give the same results in a legitimate way if, following the pattern of p. 23, we go back to the definition of these integrals:

$$(13.9) \quad \int_a^b \varphi(t) dU(t) = \begin{cases} \varphi(0) & \text{for } a \leq 0 < b \\ 0 & \text{otherwise,} \end{cases}$$

$$(13.10) \quad \int_a^b \varphi(t) dU(t - t_0) = \begin{cases} \varphi(t_0) & \text{for } a \leq t_0 < b \\ 0 & \text{otherwise.} \end{cases}$$

From these Stieltjes integrals we obtain formally the above integrals with δ by writing, for example, (13.9) in the form

$$\int_a^b \varphi(t) dU(t) = \int_a^b \varphi(t) \cdot \frac{dU(t)}{dt} dt$$

and setting

$$(13.11) \quad \frac{dU(t)}{dt} = \delta(t).$$

In this way we consider $\delta(t)$ to be the derivative of $U(t)$, and this is intuitively justified, since $U'(t)$ equals zero for $t \neq 0$ and the right-sided derivative of $U(t)$ at $t = 0$ has the value

$$\lim_{t \rightarrow 0+} \frac{U(t) - U(0)}{t} = \lim_{t \rightarrow 0+} \frac{1 - 0}{t} = \infty.$$

²²) We notice that $U(t)$ is now defined to be 0 at $t = 0$, while on p. 28 the value $U(0)$ was indefinite and on p. 20 was set equal to 1 (when $\omega = 0$) because this corresponded best to the physical meaning.

Because equations (13.6, 7) are in some respect only shorthand for the legitimate equations (13.9, 10), we can calculate with them without hesitation and this is the normal practice of engineers. When doubts arise we can always refer back to the Stieltjes integrals.

What interests us most are the $\mathfrak{L}$ -transformations of the pseudo functions $\delta(t)$ and $\delta(t - t_0)$. From formulas (13.6, 7), we have

$$(13.12) \quad \mathfrak{L}\{\delta(t)\} = \int_0^{\infty} e^{-st} \delta(t) dt = 1,$$

$$(13.13) \quad \mathfrak{L}\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st} \delta(t - t_0) dt = e^{-st_0} \quad \text{for } t_0 > 0.$$

Remark: It was pointed out on p. 38 that Rule V (Differentiation theorem) is not applicable to the function $U(t - t_0)$ because it is not differentiable at $t = t_0$. This is correct if we define $U'(t - t_0)$ in the sense of classical analysis as was done in (7.2). If however, we set

$$(13.14) \quad U'(t - t_0) = \delta(t - t_0),$$

Rule V becomes applicable, since

$$\mathfrak{L}\{U'(t - t_0)\} = s \mathfrak{L}\{U(t - t_0)\} - U(-t_0) \quad (t_0 > 0)$$

means, by (13.13) and § 3, No. 2,

$$e^{-st_0} = s \frac{e^{-st_0}}{s} - 0.$$

We can explain this by the fact that the function $U(t - t_0)$ by the definition (13.14) now possesses a derivative *everywhere*, as it is required to by Rule V. This rule holds also for $t_0 = 0$, if we replace $U(0+) = 1$ by $U(0) = 0$ ²³⁾; therefore, thinking of $U(t)$ as $\lim_{t_0 \rightarrow 0+} U(t - t_0)$:

$$\mathfrak{L}\{U'(t)\} = s \mathfrak{L}\{U(t)\} - U(0), \quad \text{i.e. } 1 = s \frac{1}{s} - 0.$$

After these general discussions about $\delta(t)$ we now consider $\delta(t)$ as an excitation or input and denote the response or output by $Y_\delta(t)$ and its image function by $y_\delta(s)$. Because $\mathfrak{L}\{\delta\} = 1$, we have by (12.11)

$$(13.15) \quad y_\delta(s) = q(s) = \frac{1}{p(s)},$$

²³⁾ If we rewrite $\int_0^{\infty} e^{-st} dU(t)$ according to the rule of integration by parts for Stieltjes integrals (for this see Footnote 7), we see indeed that here $U(0)$ occurs and not $U(0+)$.

and therefore

(13.16)

$$Y_{\delta}(t) = Q(t)$$

Here we meet a new aspect of the function $Q(t)$. In § 12 it was interpreted as a Green's function or as a weight function, but now it appears as the response to an impulse excitation, therefore as an "*impulse response*". Thus it is possible to determine the weight function experimentally when the differential equation describes an electric network. An impulse $\delta(t)$ is applied to the input (this can be approximately realized in practice by a large surge during a very small time interval) and the output is measured with an oscillograph.

By (13.1) there is a connection between the step and the impulse responses

$$(13.17) \quad Y_U(t) = \int_0^t Y_{\delta}(\tau) d\tau \quad \text{or} \quad Y_{\delta}(t) = \frac{dY_U(t)}{dt}.$$

If the impulse occurs at the time $t = t_0$, its image function is equal to $e^{-t_0 s}$, and that of the output, according to (12.11), is equal to $e^{-t_0 s} q(s)$. The output itself is, by Rule II (first translation theorem), equal to

$$(13.18) \quad Q(t - t_0) = Y_{\delta}(t - t_0),$$

where $Q(t) = 0$ for $t \leq 0$.

The response (12.10) for an arbitrary excitation $F(t)$ is given by

$$Y(t) = \int_0^t Q(t - \tau) F(\tau) d\tau$$

and this can now be explained as a superposition of the responses to single impulses of intensity $F(\tau) d\tau$, which occur at the time points τ ($0 \leq \tau \leq t$). From (13.18) one such impulse supplies the response $F(\tau) d\tau \cdot Q(t - \tau)$, and the superposition of these values yields the integral (12.10) for $Y(t)$.

Technical literature is usually content with the above derivation of (13.16). On closer examination it turns out that the result still requires a correction. The differential equation in the preceding case reads

$$(13.19) \quad Y^{(n)} + c_{n-1} Y^{(n-1)} + \dots + c_0 Y = \delta(t),$$

with the given initial conditions

$$(13.20) \quad Y(0+) = Y'(0+) = \dots = Y^{(n-1)}(0+) = 0.$$

By (13.16) the solution allegedly is equal to $Q(t)$, an ordinary function. This cannot be so, because it follows that the left-hand side of (13.19) for

an ordinary function gives an ordinary function and therefore, certainly not the pseudo function $\delta(t)$. Indeed the function $Q(t)$ satisfies a differential equation different from (13.19) and in addition does not have the initial values (13.20). As will be shown in § 14,

$$(13.21) \quad Q^{(n)}(t) + c_{n-1}Q^{(n-1)}(t) + \cdots + c_0Q(t) = 0$$

and

$$(13.22) \quad Q(0+) = Q'(0+) = \cdots = Q^{(n-2)}(0+) = 0, \quad Q^{(n-1)}(0+) = 1.$$

How can we show that this agrees with the assertion that $Y_\delta(t) = Q(t)$ satisfies equation (13.19,20)? For this purpose we complete the function $Q(t)$ for $t \leq 0$ by the value 0, as already in (13.18). We define, therefore,

$$(13.23) \quad Y_\delta(t) = \begin{cases} Q(t) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Then all left-sided derivatives of $Y_\delta(t)$ are equal to zero at $t = 0$, and consequently, by (13.22), the functions $Y_\delta(t)$, $Y'_\delta(t)$, ..., $Y^{(n-2)}_\delta(t)$ are continuous at $t = 0$, while $Y^{(n-1)}_\delta(t)$ has the jump 1 at $t = 0$ (as has the func-

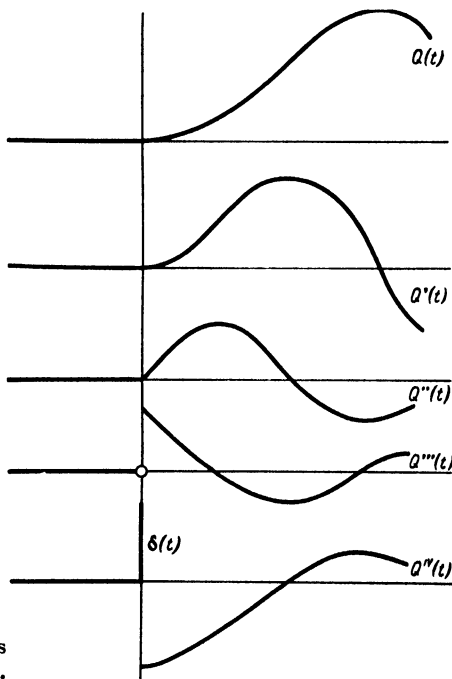


Figure 13.1
The impulse response and its
derivatives in the case $n = 4$.

tion U), cf. Fig. 13.1. We can express this in the simplest way when we set

$$(13.24) \quad Y_{\delta}^{(n-1)}(t) = Q^{(n-1)}(t) U(t).$$

Then we see that the derivative of this function, that is $Y_{\delta}^{(n)}(t)$, is no ordinary function, but of the same character as the derivative of $U(t)$ which is described by $\delta(t)$. This is obtained also when formula (13.24) is differentiated formally using the product rule:

$$(13.25) \quad \begin{aligned} Y^{(n)}(t) &= Q^{(n)}(t) U(t) + Q^{(n-1)}(t) \delta(t) \\ &= Q^{(n)}(t) U(t) + \delta(t). \end{aligned}$$

If we insert $Y_{\delta}(t)$ as defined by (13.23) in the left-hand side of (13.19) and use expression (13.25) for $Y^{(n)}(t)$, then $\delta(t)$, and not 0, appears. Further, concerning the initial values, we have seen already on p. 65 that Rule V for differentiation, which we have used to establish the image equation of (13.19), is correct only when we apply it to $U(t)$ and its derivative $\delta(t)$ if the limit value $U(0+) = 1$ is replaced by the function value $U(0) = 0$. The expression (13.25) for $Y^{(n)}(t)$ therefore requires that the condition $Y_{\delta}^{(n-1)}(0+) = 0$ is replaced by $Y_{\delta}^{(n-1)}(0) = 0$, and this is fulfilled by the function (13.23).

We cannot say that the impulse response is given by $Q(t)$ for $t \geq 0$, instead we must use the corrected expression represented by (13.23).

Example. A mechanical oscillator (for example, a pendulum suspended by a long string) satisfies the known oscillation equation of order $n = 2$. If the oscillator is initially at rest, then $Y(0) = Y'(0) = 0$. If the pendulum receives an impulse then the velocity $Y'(t)$ jumps suddenly from 0 to a positive value, while the displacement $Y(t)$ varies continuously from 0.

3. Frequency Response

The special excitation $F(t) = e^{i\omega t}$ is of particular interest in practice. We use this excitation when we wish to know how the physical system reacts to oscillations of different frequencies.

In alternating current theory this case is treated as follows: although in fact we are considering a "transient" beginning from $t = 0$ with definite initial conditions, we handle it as if it were an occurrence in the interval from $t = -\infty$ to $+\infty$. Because the excitation is an oscillation, we assume that the response is also an oscillation with the same frequency, but with different amplitude and phase; therefore it has the form

$$Y(t) = G(\omega) e^{i\omega t}$$

where $G(\omega)$ is a complex number depending on ω . If we insert this $Y(t)$ into the differential equation, we obtain

$$G(\omega)(j\omega)^n e^{j\omega t} + c_{n-1} G(\omega)(j\omega)^{n-1} e^{j\omega t} + \dots + c_1 G(\omega)(j\omega) e^{j\omega t} + c_0 G(\omega) e^{j\omega t} = e^{j\omega t}$$

or, with our earlier notation,

$$G(\omega) e^{j\omega t} p(j\omega) = e^{j\omega t},$$

$$G(\omega) = \frac{1}{p(j\omega)} = q(j\omega).$$

Therefore

$$(13.26) \quad Y(t) = q(j\omega) e^{j\omega t}.$$

The function $q(j\omega)$, which for the excitation $e^{j\omega t}$ represents the quotient

$$\frac{\text{response}}{\text{excitation}} = \frac{Y(t)}{e^{j\omega t}}$$

is called the “*frequency response*” of the physical system. If we set

$$q(j\omega) = |q(j\omega)| e^{j\psi(\omega)},$$

then $|q(j\omega)|$ is called the “*amplitude characteristic*” and $\psi(\omega)$ the “*phase characteristic*”.

As we see, the frequency response is nothing but the *value of the system function* $q(s)$ (see p. 56) *on the imaginary axis*²⁴.

The above derivation, which arises more from experience and intuition and which is not mathematically rigorous, can be made exact by means of the $\mathfrak{L}$ -transformation. Because, as always in this paragraph, we have vanishing initial conditions, the output $Y_\omega(t)$ for an input $e^{j\omega t}$ is given from (12.10) by

$$(13.27) \quad Y_\omega(t) = Q(t) * e^{j\omega t} = e^{j\omega t} \int_0^t e^{-j\omega\tau} Q(\tau) d\tau.$$

We now make the essential assumption that $q(s) = \mathfrak{L}\{Q\}$ still converges on the imaginary axis and this is equivalent to saying that the poles α_i of the rational function $q(s)$ lie to the left of the imaginary axis. Then

$$q(j\omega) = \int_0^\infty e^{-j\omega\tau} Q(\tau) d\tau,$$

²⁴ In electrical engineering literature $q(s)$ also is sometimes called the frequency response; this, however, is to be rejected as improper. The importance of $q(s)$ as system function far surpasses the importance of $q(j\omega)$ as frequency response.

and we can write expression (13.27) in the form

$$(13.28) \quad Y_{\omega}(t) = e^{j\omega t} \left\{ \int_0^{\infty} e^{-j\omega\tau} Q(\tau) d\tau - \int_t^{\infty} e^{-j\omega\tau} Q(\tau) d\tau \right\} \\ = q(j\omega) e^{j\omega t} - e^{j\omega t} \int_t^{\infty} e^{-j\omega\tau} Q(\tau) d\tau.$$

Since the second term obviously tends to 0 for $t \rightarrow \infty$, we see that the first term, which is identical to (13.26), gives the “steady state” of the system, which appears after a sufficiently long time. We denote it by $\tilde{Y}_{\omega}(t)$:

$$(13.29) \quad \boxed{\tilde{Y}_{\omega}(t) = q(j\omega) e^{j\omega t}}$$

When we are not interested in the value of $Y_{\omega}(t)$ for small t , that is for transient phenomena, then the solution (13.29) is sufficient for practical purposes even although it does not represent the complete solution, but only the “asymptotic” behaviour.

This latter result is of particular value, because it can be obtained completely from image space without translation into the original space.



It is to be observed, however, that we can use (13.29) only if the poles of $q(s)$ have negative real parts. This means that the “characteristic oscillations” (see p. 74) die away with increasing t ; that is, the physical system is “passive” (possesses no internal energy sources) and cannot by itself amplify its oscillations. We add the remark that (13.29) also describes the steady state in the case of non-vanishing initial values. This is because the initial values only add to the solution an aggregate of characteristic oscillations which die away to zero with increasing t (see § 14).

We can relate the frequency response to the impulse response if we go back to the concept of spectral density introduced in § 1. By (1.11) and (13.6) the impulse function $\delta(t)$ has the spectral density

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = \frac{1}{2\pi} \cdot 1.$$

(Therefore all frequencies appear in $\delta(t)$ with equal amplitude density $1/2\pi$ and without phase displacement²⁵.) As an excitation the impulse function

²⁵) The spectral density of $\delta(t)$ exists, but $\delta(t)$ cannot be represented by it as a Fourier integral using formula (1.10) because $\int_{-\infty}^{+\infty} 1 e^{j\omega t} d\omega$ diverges for every t . Nevertheless this meaningless expression is sometimes written in the technical literature for $\delta(t)$, in which case the same interpretation is used as is given on p. 20 (see before formula (1.23)).

corresponds, by (13.23), to the response $Y_\delta(t) = Q(t)$ for $t > 0$, and $Y_\delta(t) = 0$ for $t \leq 0$. Its spectral density is

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} Y_\delta(t) e^{-j\omega t} dt = \frac{1}{2\pi} \int_0^{\infty} Q(t) e^{-j\omega t} dt = \frac{1}{2\pi} q(j\omega).$$

For the impulse function as excitation, the quotient

$$(13.30) \quad \frac{\text{spectral density of response}}{\text{spectral density of excitation}} = q(j\omega).$$

This equation supplies a *new interpretation of the frequency response* $q(j\omega)$. The difference from the earlier definition is the following: the response $Y_\delta(t)$ to the excitation $\delta(t)$ is found first. Then a frequency analysis is executed for both functions; that is, for every component oscillation $e^{j\omega t}$ the corresponding spectral density is found and the quotient (13.30) formed. On the other hand, in the earlier definition the frequency analysis was executed by selecting a single component oscillation $e^{j\omega t}$, and then the ratio response: excitation was formed.

Connection between frequency response and step response

Technically the frequency response is, in general, more exactly and more easily measurable than the step response. For this reason it is advantageous that the latter can be calculated from the frequency response, if it exists, that is, when the poles of the rational function $q(s)$ have negative real parts. If we express $q(j\omega)$ by modulus and phase

$$q(j\omega) = |q(\omega)| e^{j\psi(\omega)},$$

then the formula

$$(13.31) \quad Y_U(t) = \frac{q(0)}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{|q(j\omega)|}{\omega} \sin[t\omega + \psi(\omega)] d\omega \quad (t > 0)$$

is valid (EINF. p. 96). If, on the other hand, we decompose $q(j\omega)$ into real and imaginary parts

$$q(j\omega) = u(\omega) + jv(\omega),$$

then both formulas

$$(13.32) \quad \begin{aligned} Y_U(t) &= \frac{2}{\pi} \int_0^{\infty} \frac{u(\omega)}{\omega} \sin t\omega d\omega \\ Y_U(t) &= \frac{2}{\pi} \int_0^{\infty} \frac{v(\omega)}{\omega} \cos t\omega d\omega \end{aligned} \quad (t > 0)$$

In the rational functions occurring here, the degree of the numerator is less than the degree of the denominator and therefore we can expand the solution into partial fractions and transform into original space. For higher orders, however, we can use the following process with advantage.

If the special initial values

$$Y(0) = Y'(0) = \dots = Y^{(n-2)}(0) = 0, \quad Y^{(n-1)}(0) = 1$$

are given, the solution of the image equation is simply

$$y(s) = \frac{1}{p(s)} = q(s),$$

and therefore that of the original equation is

$$Y(t) = Q(t).$$

This can be expressed as follows.

The *weight function* $Q(t)$ has the following important and much used properties: $Q(t)$ satisfies the *homogeneous differential equation*

$$(14.3) \quad Q^{(n)}(t) + c_{n-1}Q^{(n-1)}(t) + \dots + c_1Q'(t) + c_0Q(t) = 0$$

and has the *initial values*

$$(14.4) \quad Q(0+) = Q'(0+) = \dots = Q^{(n-2)}(0+) = 0, \quad Q^{(n-1)}(0+) = 1.$$

From this, by Rule V, we have

$$(14.5) \quad \left\{ \begin{array}{lll} \frac{1}{p(s)} & = & q(s) \quad \bullet \circ Q(t) \\ \frac{s}{p(s)} & = & s q(s) \quad \bullet \circ Q'(t) \\ \dots & & \dots \\ \frac{s^{n-1}}{p(s)} & = & s^{n-1} q(s) \quad \bullet \circ Q^{(n-1)}(t) \\ \frac{s^n}{p(s)} - 1 & = & s^n q(s) - 1 \quad \bullet \circ Q^{(n)}(t). \end{array} \right.$$

We can therefore construct very simply from the derivatives of $Q(t)$, which can be calculated easily, the original functions of the multipliers of $Y(0)$, $Y'(0)$, ... in (14.2), and obtain

$$(14.6) \quad Y(t) = \left\{ \begin{array}{l} Y(0) [Q^{(n-1)}(t) + c_{n-1}Q^{(n-2)}(t) + \dots + c_2Q'(t) + c_1Q(t)] \\ + Y'(0) [Q^{(n-2)}(t) + c_{n-1}Q^{(n-3)}(t) + \dots + c_2Q(t)] \\ \dots \\ + Y^{(n-2)}(0) [Q'(t) + c_{n-1}Q(t)] \\ + Y^{(n-1)}(0) Q(t). \end{array} \right.$$

$Q(t)$ itself is obtained using the method in § 12 by a partial fraction expansion of $q(s)$ and is an aggregate of functions $e^{\alpha_i t}$, which, if α_i has multiplicity k_i , are multiplied by $t, t^2, \dots, t^{k_i-1}$. The derivatives of $Q(t)$ have the same form. Therefore $Y(t)$ is a linear combination of these functions with coefficients that are formed from the initial values and the c_i . If no outside excitation occurs, no functions appear other than

$$e^{\alpha_1 t}, t e^{\alpha_1 t}, \dots, t^{k_1-1} e^{\alpha_1 t}; \quad e^{\alpha_2 t}, t e^{\alpha_2 t}, \dots$$

which are called the *characteristic oscillations* of the system for this reason. With arbitrary initial values, the system when left to itself executes oscillations which are linear combinations of these characteristic oscillations.

Numerical Example

On pp. 52 and 55 we have considered the case

$$\begin{aligned} p(s) &= (s^2 + 16)(s^2 + 10s + 74) \\ &= s^4 + 10s^3 + 90s^2 + 160s + 1184. \end{aligned}$$

This corresponds to the differential equation of the fourth order

$$Y'''' + 10Y''' + 90Y'' + 160Y' + 1184Y = F(t).$$

We found (notice that there $\frac{1}{p(s)}$ was multiplied by 112) that

$$Q(t) = -\frac{1}{9928}(20 \cos 4t - 29 \sin 4t) + \frac{1}{17374}e^{-5t}(35 \cos 7t - 4 \sin 7t).$$

Therefore

$$Q'(t) = \frac{1}{2482}(20 \sin 4t + 29 \cos 4t) - \frac{1}{17374}e^{-5t}(203 \cos 7t + 225 \sin 7t),$$

$$Q''(t) = \frac{1}{1241}(40 \cos 4t - 58 \sin 4t) - \frac{1}{8687}e^{-5t}(280 \cos 7t - 1273 \sin 7t),$$

$$\begin{aligned} Q'''(t) &= -\frac{1}{1241}(160 \sin 4t + 232 \cos 4t) \\ &\quad + \frac{1}{8687}e^{-5t}(10311 \cos 7t - 4405 \sin 7t). \end{aligned}$$

With these functions we can combine the factors by which the initial values are multiplied:

$$Y(+0): Q'''(t) + c_3 Q''(t) + c_2 Q'(t) + c_1 Q(t) = Q'''(t) + 10 Q''(t) + 90 Q'(t) + 160 Q(t),$$

$$Y'(+0): Q''(t) + c_3 Q'(t) + c_2 Q(t) = Q''(t) + 10 Q'(t) + 90 Q(t),$$

$$Y''(+0): Q'(t) + c_3 Q(t) = Q'(t) + 10 Q(t),$$

$$Y'''+(+0): Q(t) = Q(t).$$

The reader is advised to verify that

$$Q(0) = Q'(0) = Q''(0) = 0 \quad \text{and} \quad Q'''(0) = 1.$$

Because of vagueness in the technical literature it is again pointed out that the initial values entering in the image equation are the numbers $Y(0+)$, $Y'(0+)$, ... It cannot be otherwise because the differential equation is considered only for $t \geq 0$ and $Y(t)$ with its derivatives must be connected continuously to the initial values from the right. A different question is, where do these values originate in practice? With a physical system they come naturally from the past history of the system. (Thus, for example, the position and velocity of a linear oscillator at the time $t = 0$ is a necessary consequence of its motion in the time before.) The physical system reaches the null point from the left with certain values $Y(0-)$, $Y'(0-)$, ..., and it is now required that the system begins the time section $t \geq 0$ with the same values. This demand can always be satisfied, because we can verify that the solution obtained by $\mathfrak{L}$ -transformation always has the prescribed initial values (EINF. pp. 78, 80), however these initial values may be given and an excitation may be chosen which can be totally different from the excitation acting before $t = 0$. This surprises us only when we find that with systems of differential equations quite different things may occur. (Compare § 19.)



We state once again

the Advantages of the $\mathfrak{L}$ -Transformation Method.

1. In the classical method we establish first a "general" solution in which the constants must be made to fit the initial conditions. This necessitates the solution of an additional system of linear equations with n unknowns (for $n \geq 3$ this is fairly tedious). By contrast, the $\mathfrak{L}$ -transformation considers these initial values from the beginning and introduces them automatically into the solution. Because of this their influence is clear from the start. Therefore the method is particularly suitable for the initial value problem. The frequent case of vanishing initial values, which does not simplify the classical method and which still requires the solution of the previously mentioned system of linear equations, is solved very simply by the $\mathfrak{L}$ -transformation.

2. While in the classical method we solve first the homogeneous equation and then the inhomogeneous equation by variation of parameters, in using the $\mathfrak{L}$ -transformation we can immediately solve the inhomogeneous equation, which in practice is the more important.

§ 15. Solution of a Particular System of Simultaneous Differential Equations

As has already been shown, the calculations involved in the solution of single differential equations with order greater than three are much simpler by the $\mathfrak{L}$ -transformation method than by the classical method. However the method shows its full power in the solution of a system of several differential equations where it leads to much greater insight and much less calculation than the classical method, which in reality is not practicable at all.

We consider next a special example from engineering; this arises from the mathematical treatment of a certain automatic control system²⁶). To obtain the most realistic relations we have chosen it so that one of the equations appearing is not a differential equation but an algebraic equation, a case that appears frequently in technical problems. All together three unknown functions Y_1, Y_2, Y_3 appear (in the automatic control system Y_1 is the controlled variable [the voltage of a generator], Y_2 is the adjustable variable, and Y_3 is the current in the field of the generator); the right-hand sides F_1, F_2, F_3 of the equations are combinations of known functions (disturbance and control values).

The equations are

$$(15.1) \quad \begin{cases} Y_1 & - 8 Y_3 & = F_1 \\ & - Y_2 & + 2 Y_3 + 10 Y_3' = F_2 \\ Y_1 + \frac{1}{2} Y_2 + Y_2' & & = F_3. \end{cases}$$

The most interesting function is Y_1 (the controlled variable), of which, as we see, no derivative is present in the equations.

By the usual method we proceed to eliminate Y_2 and Y_3 and set up a single differential equation for Y_1 . In (15.1) this is a very simple matter because in the first equation Y_3 and in the second Y_2 appear without derivatives. We obtain from the first equation

$$Y_3 = \frac{1}{8} (Y_1 - F_1).$$

Substituting this in the second gives

$$Y_2 = \frac{1}{4} Y_1 + \frac{5}{4} Y_1' - \frac{1}{4} F_1 - \frac{5}{4} F_1' - F_2.$$

²⁶) The description of this automatic control system and the deduction of the equations can be found in: *Die Laplace-Transformation und ihre Anwendung in der Regelungstechnik*. Oldenbourg Verlag, München 1955, pp. 9—15: W. OPFELT, Ableitung der Gleichungen eines Regelvorganges aus dem technischen Aufbau einer Regelanlage.

Using this value in the third equation, we obtain

$$(15.2) \quad \frac{5}{4} Y_1'' + \frac{7}{8} Y_1' + \frac{9}{8} Y_1 = \frac{1}{8} F_1 + \frac{7}{8} F_1' + \frac{5}{4} F_1'' + \frac{1}{2} F_2 + F_2' + F_3.$$

If the initial values $Y_1(0+)$, $Y_1'(0+)$ are known, we can solve this equation uniquely.

In the present case we see immediately that *these initial values cannot be given arbitrarily*. Since the derivative of Y_1 does not appear in the original equations, we do not expect to prescribe the initial values for Y_1 . In fact they are determined by the initial values $Y_2(0+)$, $Y_3(0+)$ for Y_2 and Y_3 (each appears differentiated once). It follows from the first equation that the initial value of Y_1 is

$$(15.3) \quad Y_1(0+) = 8 Y_3(0+) + F_1(0+),$$

and also from the first equation differentiated

$$Y_1'(0+) = 8 Y_3'(0+) + F_1'(0+)$$

and the second equation

$$Y_3'(0+) = \frac{1}{10} Y_2(0+) - \frac{1}{5} Y_3(0+) + \frac{1}{10} F_2(0+)$$

that the initial value of Y_1' is

$$(15.4) \quad Y_1'(0+) = \frac{4}{5} Y_2(0+) - \frac{8}{5} Y_3(0+) + \frac{4}{5} F_2(0+) + F_1'(0+).$$

There are, however, the following considerations: 1. In more general cases the reduction of a system of differential equations to one differential equation (of higher order) for a single function is not as simple as in the preceding case, where only two unknowns appear in each equation. 2. The question of which initial values can be prescribed is, in general, not easily answered by inspection (see for this question § 19). 3. The gravest difficulty is that the above operations assume that the functions $F_1, \dots$ are differentiable and together with certain derivatives have limits for $t \rightarrow 0+$. This is precisely what is not true in many practical cases; for example, for $F_1 = U(t-1)$ (this is not differentiable at $t = 1$) or for $F_1 = t^{1/2}$ ($F_1' = 1/2 t^{-1/2}$ has no limit for $t \rightarrow 0+$).

None of these difficulties arises when we use the $\mathcal{L}$ -transformation method and set up the image equations of the given equations without previous elimination. In our example this leads to the following system of algebraic equations

$$(15.5) \quad \begin{cases} y_1 & - 8 y_3 & = f_1 \\ - y_2 & + 2 y_3 + 10 [s y_3 - Y_3(0+)] & = f_2 \\ y_1 + \frac{1}{2} y_2 + [s y_2 - Y_2(0+)] & & = f_3. \end{cases}$$

In doing this, no assumptions are made about the differentiability of the input functions $F_1, \dots$, and the initial values required are obvious.

We can calculate the unknowns y_1, y_2, y_3 by Cramer's rule with the help of determinants or by elimination, which can be carried out in the same order as above for the differential equation (15.1). Only now everything is much simpler because here only the three unknowns themselves appear and not, as above, together with their derivatives. (Even in the most complicated cases we do not need to consider how to pick out the unknowns.) We determine y_3 from the first equation:

$$y_3 = \frac{1}{8} (y_1 - f_1),$$

substitute this into the second equation and obtain

$$y_2 = \left(\frac{1}{4} + \frac{5}{4}s \right) y_1 - \left(\frac{1}{4} + \frac{5}{4}s \right) f_1 - f_2 - 10 Y_3(0+).$$

We insert this value into the third equation:

$$(15.6) \quad \left(\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8} \right) y_1 = \left(\frac{1}{8} + \frac{7}{8}s + \frac{5}{4}s^2 \right) f_1 + \left(\frac{1}{2} + s \right) f_2 + f_3 \\ + Y_2(0+) + 5 Y_3(0+) (2s + 1).$$

When, as is frequently the case, we describe the excitation functions by unit step functions $U(t)$, then $f_1 = f_2 = f_3 = 1/s$, and we obtain for y_1 a sum of rational functions, in which the numerators are of lower degree (at most 2) than the denominators (equal to 3). Such functions can be expanded by the method of § 12 into partial fractions and then transformed easily into original space. Here, however, we wish to assume the general case when F_1 , at least, is an arbitrary function, because from this we can learn something new.

On dividing by the multiplier of y_1 we obtain as first term on the right-hand side

$$\frac{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{1}{8}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} f_1(s).$$

If the first factor were a $\mathfrak{L}$ -transform, we could use the convolution theorem. It is, however, definitely not a $\mathfrak{L}$ -transform, since a $\mathfrak{L}$ -transform tends to zero as s increases (cf. Theorem. 38.1) whereas this expression tends to 1. Nevertheless we can expand a rational function in which the numerator and denominator have the same degree into a constant and a function in which the numerator is of lower degree than the denominator.

If as previously (§ 11) we use the method of undetermined coefficients, we find that

$$\frac{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{1}{8}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} = 1 - \frac{1}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}}.$$

When this is multiplied by f_1 the first term is f_1 in isolated form and in the second place f_1 is multiplied by a $\mathfrak{L}$ -transform, so that transformation back into original space is possible. To

$$\frac{1}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} = \frac{\frac{4}{5}}{s^2 + \frac{7}{10}s + \frac{9}{10}} = \frac{0.8}{(s + 0.35)^2 + (0.882)^2}$$

there corresponds, from the TAB. 38 and Rule IV, the original function

$$\frac{0.8}{0.882} e^{-0.35t} \sin 0.822t.$$

Consequently, corresponding to the first term in the solution (15.6) for y_1 the original function is

$$F_1(t) - \frac{0.8}{0.882} (e^{-0.35t} \sin 0.882t) * F_1(t).$$

The transformation of the other terms does not require any new methods and is left to the reader as an exercise. Products of exponential and trigonometrical functions and convolutions of such functions with F_2 and F_3 , appear (as in the first term). These are all continuous functions, even in the case when F_2 and F_3 have discontinuities. However, the first term, in which F_1 appears by itself, shows that at every jump point of F_1 the solution Y_1 exhibits the same jump as F_1 .

The reader is advised to apply the $\mathfrak{L}$ -transformation directly to the differential equations and not to derive first by elimination a single differential equation of higher order for one of the unknown functions. In engineering the latter procedure is used, because usually only one unknown is desired while the others are required only for the calculation. However, even in this case it is much simpler and safer to transform all the given differential equations first and then calculate that image function which is desired from the resultant system of linear algebraic equations.

Many engineers habitually prefer to set up one differential equation for one unknown, even if, during the elimination, functions must be differentiated which contain discontinuities and therefore are not differentiable. We shall therefore show how the $\mathfrak{L}$ -transformation must be



employed in such a case. For this purpose, we use again the above system of equations and assume that the input $F_2(t)$ is equal to 1 from $t = 0$ to $t = 3$, but then jumps to the value 0.3; this case occurs frequently in engineering, for example, when a step regulator causes an unsteady variation of the adjustable variable in certain intervals. We can write F_2 in the form (cf. Fig. 15.1)

$$(15.7) \quad F_2(t) = U(t) - 0.7 U(t-3).$$

In order to demonstrate only significant points we set

$$F_1(t) \equiv F_3(t) \equiv 0, \quad Y_2(0+) = Y_3(0+) = 0.$$

We proceed first in the safe way recommended above by transforming the given system of equations and solving for y_1 as already has been done in (15.6): since

$$f_2(s) = \frac{1}{s} - 0.7 \frac{e^{-3s}}{s}$$

we obtain

$$\left(\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8} \right) y_1 = \left(\frac{1}{2} + s \right) \left(\frac{1}{s} - 0.7 \frac{e^{-3s}}{s} \right)$$

and this equation gives

$$(15.8) \quad y_1(s) = \frac{1 + \frac{1}{2s}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} - 0.7 e^{-3s} \frac{1 + \frac{1}{2s}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}}.$$

We now proceed in the other way, by setting up a single differential equation for Y_1 . From (15.2), we have in this special case

$$(15.9) \quad \frac{5}{4} Y_1'' + \frac{7}{8} Y_1' + \frac{9}{8} Y_1 = \frac{1}{2} F_2 + F_2'.$$

The initial values by (15.3, 4), since $Y_2(0+) = Y_3(0+) \equiv F_1(0+) = F_3(0+) = 0$, are

$$(15.10) \quad Y_1(0+) = 0, \quad Y_1'(0+) = \frac{4}{5} F_2(0+).$$

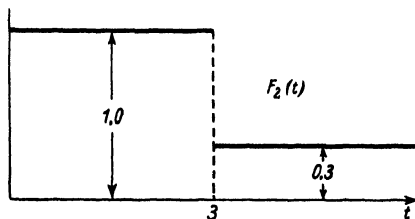


Figure 15.1
Excitation function $F_2(t)$ with jump at $t = 3$.

The image equation of (15.9) is (if the term $-\frac{5}{4}\frac{4}{5}F_2(0+)$ is brought to the right-hand side)

$$(15.11) \quad \left(\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}\right)y_1 = \frac{1}{2}\mathfrak{L}\{F_2\} + \mathfrak{L}\{F'_2\} + F_2(+0).$$

By (15.7), we have

$$\mathfrak{L}\{F_2\} = \frac{1}{s} - 0.7 \frac{e^{-3s}}{s}.$$

Before we form $\mathfrak{L}\{F'_2\}$, we remark that $U(t)$ is defined by (13.8) so that $F_2(t)$ exhibits discontinuities at the points $t = 0$ and $t = 3$. When we wish to speak of derivatives, we must introduce the δ -function and set

$$F'(t) = \delta(t) - 0.7 \delta(t-3).$$

Then by (13.12, 13), we have

$$\mathfrak{L}\{F'_2\} = 1 - 0.7 e^{-3s}.$$

However, with this definition of U' it is necessary, following the remark on p. 65, to replace $U(0+) = 1$ by $U(0) = 0$. Therefore $F_2(0+) = U(0+)$ in (15.11) must also be replaced by $F_2(0) = U(0) = 0$, so that we obtain finally for (15.11)

$$\left(\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}\right)y_1 = \frac{1}{2}\left(\frac{1}{s} - 0.7 \frac{e^{-3s}}{s}\right) + (1 - 0.7 e^{-3s}).$$

From this we obtain the same expression for y_1 as in (15.8). We see, however, how much more artificial this derivation is: we must employ the pseudo function and moreover take care that we replace $U(0+)$ by $U(0)$ ²⁷.

We wish to determine the original function of $y_1(s)$ as an exercise. Because the original function of

$$\frac{1}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}}$$

has been calculated already on p. 79, we require to establish only the original function of

$$\frac{\frac{1}{2s}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} = \frac{4}{s(10s^2 + 7s + 9)}.$$

²⁷) There is yet another possible way of converting equation (15.11) into the correct expression. In forming F'_2 we can give the function $U(t)$, in the usual way, the derivative $U'(t) = 0$, and then from remark 1 in § 7 the value $U(0+) = 1$ and therefore also $F_2(0+) = 1$ has to be retained. Then the right-hand side of (15.11) yields:

$$\frac{1}{2}\mathfrak{L}\{F_2\} + \mathfrak{L}\{F'_2\} + F_2(+0) = \frac{1}{2}\left(\frac{1}{s} - 0.7 \frac{e^{-3s}}{s}\right) + (0 - 0.7 e^{-3s}) + 1.$$

This is the same expression as above.

First we apply a partial fraction expansion

$$\frac{4}{s(10s^2 + 7s + 9)} = \frac{a}{s} + \frac{bs + c}{10s^2 + 7s + 9} = \frac{(10a + b)s^2 + (7a + c)s + 9a}{s(10s^2 + 7s + 9)}.$$

Comparison of coefficients gives

$$10a + b = 0, \quad 7a + c = 0, \quad 9a = 4,$$

or

$$a = \frac{4}{9}, \quad b = -\frac{40}{9}, \quad c = -\frac{28}{9}.$$

Hence, we obtain

$$\begin{aligned} \frac{\frac{1}{2s}}{\frac{5}{4}s^2 + \frac{7}{8}s + \frac{9}{8}} &= \frac{4}{9} \frac{1}{s} - \frac{4}{9} \frac{10s + 7}{10s^2 + 7s + 9} \\ &= \frac{4}{9} \frac{1}{s} - \frac{4}{9} \frac{(s + 0.35) + 0.35}{(s + 0.35)^2 + (0.882)^2} \\ &\rightarrow \frac{4}{9} - \frac{4}{9} \left[e^{-0.35t} \cos 0.882t + \frac{0.35}{0.882} e^{-0.35t} \sin 0.882t \right]. \end{aligned}$$

The second term in $y_1(s)$ is equal to the first multiplied by $0.7 e^{-3s}$. This means, by Rule II, that apart from a factor 0.7 the second term corresponds to the same function as the first, but translated by the amount 3 to the right and completed between $t = 0$ and $t = 3$ by the value 0. This is also obvious physically. Hence, combining the results we obtain

$$Y_1(t) = \begin{cases} \frac{4}{9} + e^{-0.35t} \left[\frac{0.8 - \frac{4}{9} 0.35}{0.882} \sin 0.882t - \frac{4}{9} \cos 0.882t \right] & \text{for } 0 \leq t \leq 3 \\ \text{the same function} - 0.7 \left\{ \frac{4}{9} + e^{-0.35(t-3)} \left[\frac{0.8 - \frac{4}{9} 0.35}{0.882} \sin 0.882(t-3) - \frac{4}{9} \cos 0.882(t-3) \right] \right\} & \text{for } t > 3. \end{cases}$$

We can easily verify that $Y_1(t)$ actually has the initial values $Y_1(0+) = 0$, $Y_1'(0+) = \frac{4}{5}$. We can also obtain this immediately from the image function described by (15.8), when we use Theorem 38.2. Because the limits of Y_1 and Y_1' for $t \rightarrow 0+$ exist, we have

$$Y_1(0+) = \lim_{s \rightarrow \infty} s y_1(s) = 0,$$

$$Y_1'(0+) = \lim_{s \rightarrow \infty} s \mathfrak{L}\{Y_1'\} = \lim_{s \rightarrow \infty} s [s y_1 - Y_1(0+)] = \lim_{s \rightarrow \infty} s^2 y_1 = \frac{4}{5}.$$

§ 16. A General System of Simultaneous Differential Equations

The essentials of the method used have been demonstrated in the example of the preceding paragraph. To make the formalism still more clear, we will consider a *system of more general character*. So that the equations are not too complicated, we take as a basis three differential equations of the first order. In these, we write down all terms that theoretically can appear, although usually in engineering a number of these terms are absent, so that their coefficients have then to be set equal to 0. The system to be considered is

$$(a_{11} Y_1' + b_{11} Y_1) + (a_{12} Y_2' + b_{12} Y_2) + (a_{13} Y_3' + b_{13} Y_3) = F_1(t)$$

$$(a_{21} Y_1' + b_{21} Y_1) + (a_{22} Y_2' + b_{22} Y_2) + (a_{23} Y_3' + b_{23} Y_3) = F_2(t)$$

$$(a_{31} Y_1' + b_{31} Y_1) + (a_{32} Y_2' + b_{32} Y_2) + (a_{33} Y_3' + b_{33} Y_3) = F_3(t).$$

The transformation into image space yields²⁸⁾

$$a_{11} [s y_1 - Y_1(0)] + b_{11} y_1 + a_{12} [s y_2 - Y_2(0)] + b_{12} y_2 + a_{13} [s y_3 - Y_3(0)] + b_{13} y_3 = f_1(s)$$

$$a_{21} [s y_1 - Y_1(0)] + b_{21} y_1 + a_{22} [s y_2 - Y_2(0)] + b_{22} y_2 + a_{23} [s y_3 - Y_3(0)] + b_{23} y_3 = f_2(s)$$

$$a_{31} [s y_1 - Y_1(0)] + b_{31} y_1 + a_{32} [s y_2 - Y_2(0)] + b_{32} y_2 + a_{33} [s y_3 - Y_3(0)] + b_{33} y_3 = f_3(s).$$

With the abbreviation

$$a_{ik}s + b_{ik} = p_{ik}(s)$$

we can write these equations in the less recondite form

$$p_{11} y_1 + p_{12} y_2 + p_{13} y_3 = f_1 + a_{11} Y_1(0) + a_{12} Y_2(0) + a_{13} Y_3(0)$$

$$p_{21} y_1 + p_{22} y_2 + p_{23} y_3 = f_2 + a_{21} Y_1(0) + a_{22} Y_2(0) + a_{23} Y_3(0)$$

$$p_{31} y_1 + p_{32} y_2 + p_{33} y_3 = f_3 + a_{31} Y_1(0) + a_{32} Y_2(0) + a_{33} Y_3(0).$$

If the differential equations were not of first but of second order, then the $p_{ik}(s)$ would be polynomials not of the first but of the second degree and the values $Y_1'(0), \dots$ would appear on the right-hand side in addition to the values of $Y_1(0), \dots$. In principle, however, the equations would have the same form; they are a system of linear algebraic equations for the unknowns y_1, y_2, y_3 . Theoretically such a system is solved most elegantly by Cramer's Rule with determinants, but in practice it is better to use the successive elimination of the unknowns or (with a great number of equations) one of the several different methods developed for solving such a system.

²⁸⁾ For ease of printing we write $Y(0)$ instead of $Y(0 +)$.

Because we wish in particular to set up a scheme for the formalism of the method, we make use of determinants. On the right-hand side of the equations there are the image functions $f_i(s)$ of the input functions, and numerical constants which depend on the initial values; these we combine as follows

$$a_{i1} Y_1(0) + a_{i2} Y_2(0) + a_{i3} Y_3(0) = r_i.$$

We introduce the letter $D(s)$ for the determinant of the system which is built from the $p_{ik}(s)$ and which is in general a polynomial of the third degree in s (for exceptions see § 19), and we obtain by Cramer's Rule

$$y_1 = \frac{1}{D} \begin{vmatrix} f_1 + r_1 & p_{12} & p_{13} \\ f_2 + r_2 & p_{22} & p_{23} \\ f_3 + r_3 & p_{32} & p_{33} \end{vmatrix}, \quad y_2 = \frac{1}{D} \begin{vmatrix} p_{11} & f_1 + r_1 & p_{13} \\ p_{21} & f_2 + r_2 & p_{23} \\ p_{31} & f_3 + r_3 & p_{33} \end{vmatrix}, \quad y_3 = \dots$$

We can show even more clearly how the solutions are influenced by the excitation functions and the initial values by inserting explicit expressions for the r_i and splitting up the determinants as follows:

$$\begin{aligned} D y_1 &= \begin{vmatrix} f_1 + a_{11} Y_1(0) + a_{12} Y_2(0) + a_{13} Y_3(0) & p_{12} & p_{13} \\ f_2 + a_{21} Y_1(0) + a_{22} Y_2(0) + a_{23} Y_3(0) & p_{22} & p_{23} \\ f_3 + a_{31} Y_1(0) + a_{32} Y_2(0) + a_{33} Y_3(0) & p_{32} & p_{33} \end{vmatrix} \\ &= \begin{vmatrix} f_1 & p_{12} & p_{13} \\ f_2 & p_{22} & p_{23} \\ f_3 & p_{32} & p_{33} \end{vmatrix} + Y_1(0) \begin{vmatrix} a_{11} & p_{12} & p_{13} \\ a_{21} & p_{22} & p_{23} \\ a_{31} & p_{32} & p_{33} \end{vmatrix} + Y_2(0) \begin{vmatrix} a_{12} & p_{12} & p_{13} \\ a_{22} & p_{22} & p_{23} \\ a_{32} & p_{32} & p_{33} \end{vmatrix} + Y_3(0) \begin{vmatrix} a_{13} & p_{12} & p_{13} \\ a_{23} & p_{22} & p_{23} \\ a_{33} & p_{32} & p_{33} \end{vmatrix}. \end{aligned}$$

In the terms containing $Y_1(0)$, ... the first columns are constants and the second and third columns are polynomials of the first degree; thus the determinant is a polynomial of the second degree in s . Dividing by $D(s)$ yields a rational function whose numerator is of lower degree than the denominator and which we can expand into partial fractions by the previous method and transform back into original space. The determinant containing the f_i 's can be expanded in the form

$$\begin{vmatrix} f_1 & p_{12} & p_{13} \\ f_2 & p_{22} & p_{23} \\ f_3 & p_{32} & p_{33} \end{vmatrix} = f_1 \begin{vmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{vmatrix} - f_2 \begin{vmatrix} p_{12} & p_{13} \\ p_{32} & p_{33} \end{vmatrix} + f_3 \begin{vmatrix} p_{12} & p_{13} \\ p_{22} & p_{23} \end{vmatrix}.$$

The two row determinants, after division by $D(s)$, again give rational functions which we can transform into original space, after which we have to use the convolution theorem on the products with f_i .

From this general description we observe the following

advantages of the $\mathfrak{L}$ -transformation method:

1. It requires only the solution of a *single* linear algebraic equation system, viz., the system for y_i .
2. The *initial values* appear in this system and are thence automatically considered, whereas in the classical method first the general solution is set up (which is a very complicated process), and then the initial conditions are fitted to it: this requires the repeated solution of further linear systems of equations. The frequent case in engineering of vanishing initial values produces particularly simple relations, whereas no such simplification occurs in the classical method.
3. A remarkable advantage exists in that *any unknown can be computed individually* without knowing the others, and this is not possible in general in the classical method when initial values are given. This is important if we are interested in only one unknown, while the others are required only for the calculations, but are not themselves wanted.

§ 17. A System of Differential Equations with Piecewise Different Constants and Excitations

The fact that the initial values can be considered so easily in applying the $\mathfrak{L}$ -transformation, is particularly advantageous when the constants of the system and the excitations are not fixed at the same values for all $t > 0$, but vary discontinuously as soon as one of the unknowns is over or under a certain value. This happens for example, when a regulation loop possesses a zone of insensitivity within which the controlled variable can vary without engagement of the servomechanism. We then perform the integration of the equations up to the point where the unknown in question reaches its critical value; at that point the value obtained for the unknown is read off, and using this value as a new initial value we continue the integration until we reach a critical value again and so on.

We illustrate this process by an example which represents the mathematical description of a control loop of the above kind²⁹⁾. It involves a system of two differential equations with two unknowns $Y_1(t)$, $Y_2(t)$. The first equation has the usual form in which the excitation function is equal to a multiple of the unit step function. The second equation, however, requires

²⁹⁾ The description of this automatic control system, the derivation of the equations and the carrying through of the calculation can be found in the book quoted in footnote 26, pp. 104—114: R. OLDENBOURG, Anwendung der Laplace-Transformation bei abschnittsweise linearen Regelvorgängen.

that $Y_2'(t)$ vanishes as long as $Y_1(t)$ lies between the limits $-\eta$ and $+\eta$, but on the other hand is equal to $-Y_1(t) \pm \eta$ as soon as $Y_1(t)$ lies outside these limits (the algebraic sign of η shall agree with that of $Y_1(t)$). Explicitly, the two equations are

$$Y_1' + 2\delta Y_1 - (\delta^2 + \omega^2) Y_2 = c U(t)$$

$$Y_2' = \begin{cases} 0 & \text{for } |Y_1| \leq \eta \\ -Y_1 + (\text{sign } Y_1) \eta & \text{for } |Y_1| > \eta. \end{cases}$$

The given initial values are

$$Y_1(0+) = 0, \quad Y_2(0+) = 0.$$

Because $Y_1(0+) = 0$, initially $|Y_1| \leq \eta$; therefore $Y_2'(t) = 0$ and consequently, since $Y_2(0+) = 0$, $Y_2(t) = 0$ for a certain time. Thus in an interval beginning from $t = 0$ the first equation reduces to

$$Y_1' + 2\delta Y_1 = c U(t).$$

We pretend that this is valid for all $t > 0$ and use the $\mathfrak{L}$ -transformation. Subsequently we are free to use the solutions only to a definite time point $t = T_1$. Since $Y_1(0+) = 0$, the image equation is

$$s y_1 + 2\delta y_1 = \frac{c}{s}.$$

To its solution

$$y_1 = \frac{c}{s(s+2\delta)} = \frac{c}{2\delta} \left(\frac{1}{s} - \frac{1}{s+2\delta} \right)$$

there corresponds the original function

$$Y_1(t) = \frac{c}{2\delta} (1 - e^{-2\delta t}).$$

For $\delta > 0$, this function is monotonic, increasing to the value $\frac{c}{2\delta}$. If $\eta \geq \frac{c}{2\delta}$, $Y_1(t) < \eta$ is always true and in the equation for Y_2' the first row always holds; that is, $Y_1(t)$ is equal to the function found and $Y_2(t) = 0$ for all $t > 0$. If however,

$$\eta < \frac{c}{2\delta}$$

then there is a time $t = T_1$ at which $Y_1(T_1) = \eta$. T_1 is determined from the equation

$$\frac{c}{2\delta} (1 - e^{-2\delta T_1}) = \eta$$

and so

$$T_1 = \frac{1}{2\delta} \log \frac{c}{c - 2\delta\eta}.$$

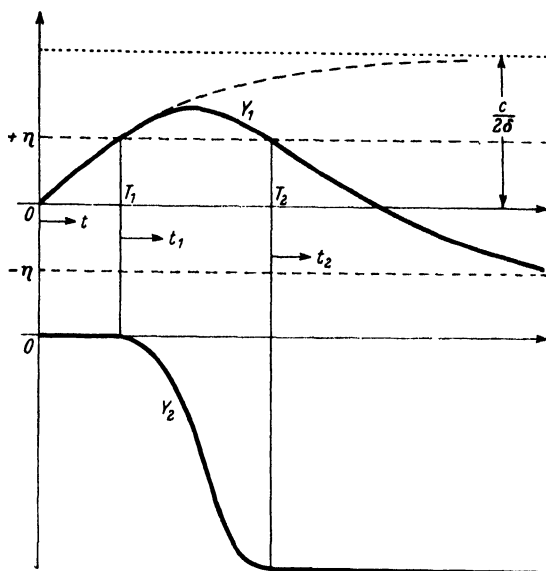


Fig. 17.1

Graph of the controlled and the controlling variables in a regulator with a zone of insensitivity.

We now make the time point T_1 the initial point of a new time scale t_1 by setting $t = t_1 + T_1$. Primes shall refer now to differentiation with respect to t_1 . Since from $t = T_1$, we have $Y_1 \geq \eta$, the second row in the second differential equation holds, so that the system has the form

$$\begin{aligned} Y_1' + 2\delta Y_1 - (\delta^2 + \omega^2) Y_2 &= c U(T_1 + t_1) \\ Y_1 + Y_2' &= \eta. \end{aligned}$$

The initial conditions are

$$Y_1(T_1 +) = \eta, \quad Y_2(T_1 +) = 0$$

(for until $t = T_1$, Y_2 was equal to 0). We pretend again that the system is valid for all $t_1 > 0$ and transform it, obtaining

$$\begin{aligned} s y_1 - \eta + 2\delta y_1 - (\delta^2 + \omega^2) y_2 &= \frac{c}{s} \\ y_1 + s y_2 &= \frac{\eta}{s}. \end{aligned}$$

This yields as solutions

$$\begin{aligned} y_1 &= \eta \frac{s + \frac{c}{\eta} + (\delta^2 + \omega^2) \frac{1}{s}}{s^2 + 2\delta s + \delta^2 + \omega^2}, \\ y_2 &= \frac{2\delta\eta - c}{s(s^2 + 2\delta s + \delta^2 + \omega^2)}. \end{aligned}$$

For the inverse transformation we require the formulas

$$\begin{aligned} \frac{1}{(s+\delta)^2 + \omega^2} &\longleftrightarrow \frac{1}{\omega} e^{-\delta t_1} \sin \omega t_1, \\ \frac{s}{(s+\delta)^2 + \omega^2} &\longleftrightarrow \frac{d}{dt_1} \left(\frac{1}{\omega} e^{-\delta t_1} \sin \omega t_1 \right) = \frac{1}{\omega} e^{-\delta t_1} (-\delta \sin \omega t_1 + \omega \cos \omega t_1), \\ \frac{\delta^2 + \omega^2}{s[(s+\delta)^2 + \omega^2]} &= \frac{1}{s} - \frac{s+\delta}{(s+\delta)^2 + \omega^2} - \frac{\delta}{(s+\delta)^2 + \omega^2} \\ &\longleftrightarrow 1 - e^{-\delta t_1} \cos \omega t_1 - \frac{\delta}{\omega} e^{-\delta t_1} \sin \omega t_1. \end{aligned}$$

Hence we obtain the formulas

$$\begin{aligned} Y_1(t_1 + T_1) &= \eta \left[1 + \frac{1}{\omega} \left(\frac{c}{\eta} - 2\delta \right) e^{-\delta t_1} \sin \omega t_1 \right], \\ Y_2(t_1 + T_1) &= \frac{2\delta\eta - c}{\delta^2 + \omega^2} \left[1 - e^{-\delta t_1} \left(\frac{\delta}{\omega} \sin \omega t_1 + \cos \omega t_1 \right) \right]. \end{aligned}$$

Y_1 starts from the value η and behaves as a damped sine function; first it increases and then it decreases. It reaches the value η at a point $t_1 = T_2$, which is determined from the equation $\sin \omega T_2 = 0$ to be

$$T_2 = \frac{\pi}{\omega}$$

We take this point T_2 as the initial point of a new time scale t_2 . From $t_1 = T_2$, that is from $t_2 = 0$, the first row in the second differential equation holds now because $Y_1 < \eta$. The new initial values are

$$Y_1(T_1 + T_2) = \eta, \quad Y_2(T_1 + T_2) = \frac{2\delta\eta - c}{\delta^2 + \omega^2} \left[1 + \frac{\delta}{\omega} e^{-\delta \frac{\pi}{\omega}} \right].$$

Because Y_2' is equal to 0 from now on, Y_2 retains this constant initial value, while Y_1 is determined by the equation

$$Y_1' + 2\delta Y_1 - (\delta^2 + \omega^2) Y_2(T_1 + T_2) = c U(T_1 + T_2 + t_2).$$

From the preceding analysis it is clear how the process is continued.

§ 18. The Equation System of an Electrical Network

When determining the currents and voltages in electrical networks, we are led to a system of functional equations which are very similar to the system of differential equations that was treated in § 16. In this system, however,

there is an important difference. Because of the great importance of network theory, we will demonstrate the solution of the system by the $\mathfrak{L}$ -transformation.

We consider first a *single closed current loop* with lumped parameters: inductance L , resistance R , and capacitance C . In the loop there is a generator which produces the variable voltage $E(t)$, by which a current flows, of strength $I(t)$. The potential drop across L is $L \frac{dI}{dt}$, across R , RI , and across C , $\frac{1}{C} \int_{-\infty}^t I(\tau) d\tau$. By Kirchhoff's current law, the sum of these potential drops is equal to the impressed EMF $E(t)$, so that we have the equation

$$(18.1) \quad L \frac{dI}{dt} + RI + \frac{1}{C} \int_{-\infty}^t I(\tau) d\tau = E(t).$$

This is an *integro-differential equation* for $I(t)$. We could make a differential equation of second order from it by introducing the charge $K(t) = \int_{-\infty}^t I(\tau) d\tau$ of the condenser as a *new variable*:

$$L \frac{d^2 K}{dt^2} + R \frac{dK}{dt} + \frac{1}{C} K = E(t),$$

or by differentiating the equation with respect t . The latter is only possible, however, when $E(t)$ is differentiable and this frequently is not the case; for example, when $E(t)$ is represented by a jump function. However, for the application of the $\mathfrak{L}$ -transformation it is not necessary to convert to a differential equation. We set

$$\int_{-\infty}^t I(\tau) d\tau = \int_0^t I(\tau) d\tau + \int_{-\infty}^0 I(\tau) d\tau = \int_0^t I(\tau) d\tau + A,$$

and this gives, by Rule VII,

$$\frac{1}{C} \int_{-\infty}^t I(\tau) d\tau \circ \bullet \frac{1}{s} i(s) + \frac{A}{s},$$

hence we can write down the image equation of (18.1) immediately.

For the purpose of obtaining a better picture, we assume in what follows that the current in the loop is at rest until $t = 0$; thus

$$I(t) = 0 \quad \text{for } t \leq 0,$$

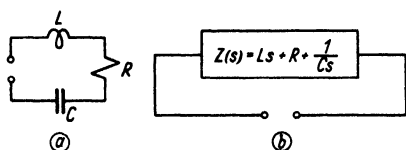


Fig. 18.1

LRC-loop and equivalent impedance.

The image equation of (18.1) is then

$$Ls i(s) + R i(s) + \frac{1}{Cs} i(s) = e(s)$$

or with

$$(18.2) \quad Ls + R + \frac{1}{Cs} = Z(s)$$

we obtain

$$(18.3) \quad Z(s) i(s) = e(s).$$

If, as in § 12, we carry over the names from original space to image space and call $i(s)$ “current” and $e(s)$ “voltage”, we have in image space simply Ohm’s law, if $Z(s)$ is called “resistance”. Instead $Z(s)$ is called the “*impedance*” and its reciprocal $Y(s) = 1/Z(s)$, the “conductance”, is called the “*admittance*”. The loop which possesses inductance, resistance and capacitance, and is governed by the integro-differential equation (18.1) can be treated more simply in image space as a *loop which possesses only the resistance Z and obeys Ohm’s law*. Therefore instead of the usual figures L, R, C in a sketch (see Fig. 18.1a) we require only to draw a block and to denote it by the functional expression $Z(s)$ (see Fig. 18.1b). This is similar to what happened previously with a system which was governed by a differential equation, where we denoted a block by the system function $q(s)$.

If two circuits with impedances Z_1 and Z_2 are connected in series, a circuit is formed with impedance $Z = Z_1 + Z_2$ (see Fig. 18.2a). Two circuits with admittances Y_1 and Y_2 when connected in parallel form a circuit with admittance $Y = Y_1 + Y_2$ (see Fig. 18.2b).

This description of the connections in the language of image space proves itself particularly useful when many loops (meshes) are connected together in a *network*, because the complicated differential and integral relations in original space are replaced by linear algebraic relations in image space. We shall illustrate this with the network represented by Fig. 18.3, which

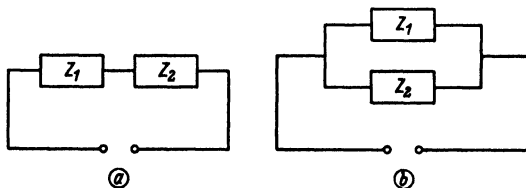


Fig. 18.2

Impedances in series and in parallel.

has two input and two output terminals and thus is a four-terminal network. A voltage E_1 is applied to the input and we require the voltage E_0 of the output.

We proceed by assigning a mesh current $I_\nu(t)$ to each mesh and by defining arbitrarily a current direction as positive (clockwise in the Figure). Then the current rule is applied in every mesh. It is to be noted that for our network in the mesh denoted by I_1 the current through C_1 is $I_1 - I_2$, that through L_2 in the loop denoted by I_2 is $I_2 - I_4$, and so forth. Therefore Kirchhoff's equations are

$$R_1 I_1 + \frac{1}{C_1} \int_0^t (I_1 - I_2) d\tau + L_1 \frac{d}{dt} I_1 = E_1$$

$$L_2 \frac{d}{dt} (I_2 - I_4) + \frac{1}{C_2} \int_0^t (I_2 - I_3) d\tau + R_2 I_2 + \frac{1}{C_1} \int_0^t (I_2 - I_1) d\tau = 0$$

$$R_3 I_3 + \frac{1}{C_2} \int_0^t (I_3 - I_2) d\tau = 0$$

$$R_4 I_4 + L_2 \frac{d}{dt} (I_4 - I_2) = 0$$

$$R_3 I_3 = E_0.$$

Using the $\mathfrak{L}$ -transformation and under the assumption made above that the network was passive until $t = 0$, that is, $I_\nu(t) = 0$ for $t \leq 0$, these equations transform into the following linear algebraic equations, ordered according to the unknowns i_ν and e_0 :

$$\begin{array}{rcccccl} \left(L_1 s + R_1 + \frac{1}{C_1 s} \right) i_1 & - \frac{1}{C_1 s} i_2 & & & & = e_1 \\ - \frac{1}{C_1 s} i_1 + \left(L_2 s + R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s} \right) i_2 & - \frac{1}{C_2 s} i_3 & - L_2 s i_4 & & & = 0 \\ & - \frac{1}{C_2 s} i_2 + \left(R_3 + \frac{1}{C_2 s} \right) i_3 & & & & = 0 \\ & - L_2 s i_2 & + \left(L_2 s + R_4 \right) i_4 & & & = 0 \\ & & & & R_3 i_3 & - e_0 = 0. \end{array}$$

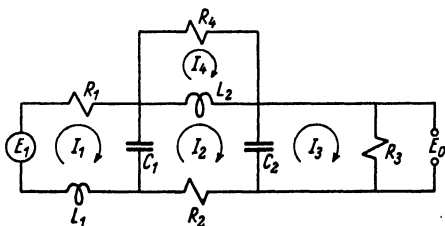


Fig. 18.3

A network with five meshes.

this and on expanding by this column, the determinant reduces to the product of $(-1)^{1+\nu}$, $e_1(s)$ and the minor

$$D_{1\nu}(s) = \begin{vmatrix} Z_{21} & \dots & Z_{2\nu-1} & Z_{2\nu+1} & \dots & Z_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ Z_{n1} & \dots & Z_{n\nu-1} & Z_{n\nu+1} & \dots & Z_{nn} \end{vmatrix},$$

which is obtained from $D(s)$ by crossing out the first row and the ν^{th} column. Then $i_\nu^{(1)}(s)$ is given by the quotient:

$$i_\nu^{(1)}(s) = (-1)^{1+\nu} \frac{D_{1\nu}(s)}{D(s)} e_1(s).$$

If we assume that an *arbitrary* $e_\mu(s)$ is not equal to zero but that all others are equal to zero, we obtain analogously

$$(18.6) \quad i_\nu^{(\mu)}(s) = (-1)^{\mu+\nu} \frac{D_{\mu\nu}(s)}{D(s)} e_\mu(s),$$

where $D_{\mu\nu}$ is obtained from D by crossing out the μ^{th} row and the ν^{th} column. Setting

$$(18.7) \quad (-1)^{\mu+\nu} \frac{D_{\mu\nu}(s)}{D(s)} = q_{\mu\nu}(s),$$

$i_\nu^{(\mu)}$ assumes a form analogous to equation (12.11):

$$(18.8) \quad i_\nu^{(\mu)}(s) = q_{\mu\nu}(s) e_\mu(s).$$

We call $q_{\mu\nu}(s)$ a *system function*; in contrast to § 12, we have here n^2 such functions.

After the determination of the special currents $i_\nu^{(\mu)}(s)$, we can write the currents for arbitrary voltages $e_1, e_2, \dots, e_n$ in the form

$$(18.9) \quad i_\nu(s) = \sum_{\mu=1}^n q_{\mu\nu}(s) e_\mu(s).$$

When in particular we choose as voltage (actual not transformed) in the μ^{th} mesh the impulse function

$$E_\mu(t) = \delta(t),$$

then the current $I_\nu^{(\mu)}(t)$ arising in the ν^{th} mesh is called an *impulse response*. In this case

$$e_\mu(s) = 1, \quad \text{and therefore} \quad i_\nu^{(\mu)}(s) = q_{\mu\nu}(s),$$

that is, the system function $q_{\mu\nu}(s)$ is the $\mathfrak{L}$ -transform of the response of the ν^{th} mesh to an impulse in the μ^{th} mesh.

It can be proved that if and only if the determinant of the coefficients of the highest derivatives does not vanish, i.e. if and only if

$$(19.1) \quad \begin{vmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{vmatrix} \neq 0,$$

the initial values $Y_1(0+)$, $Y_1'(0+)$; $Y_2(0+)$, $Y_2'(0+)$, ... can be given arbitrarily. They are taken with certainty from the solutions.

By this criterion, which is easy to use in practice, we can establish immediately whether we can rely upon the solutions to supply the given initial values. If the determinant vanishes there is a procedure for setting up the "compatibility relations" which must be satisfied so that the initial values are actually taken. Yet this is of little interest for the practical man, because he cannot select his initial conditions, but must calculate with the values which he is forced to accept according to the situation.

It is still an open question how we are to interpret the case where we apply the general method to a system and then find that there is a *discrepancy between the given initial values and those actually taken*.

For this purpose, we refer back to what has been said about the single differential equation on p. 75. The given values *result* from the *past* of the physical system described by the differential equations. These are the values with which the system arrives at the point $t = 0$ approaching through negative t values; that is, they are the values $Y_1(0-)$, $Y_1'(0-)$, ...; $Y_2(0-)$, $Y_2'(0-)$, ..., ... The *reason* why we do not always succeed in imparting these values to the solution as initial values $Y_1(0+)$, $Y_1'(0+)$, ..., and so make a continuous connection between the past and the future, is contained in the following. It is not the fault of the initial values which are necessarily prescribed by the system and the excitation operating in the past, but it is due to the excitation operating for $t \geq 0$ which in the process of switching on we choose in an arbitrary manner without respect to the past of the system. If the new excitations (for $t > 0$) do not "fit" the old excitations (for $t < 0$), that is, when jumps occur at $t = 0$, it is not surprising that the system values Y_1 , Y_1' , ...; Y_2 , Y_2' , ... also exhibit jumps; that is, it can be that

$$Y_1(0-) \neq Y_1(0+), \quad Y_1'(0-) \neq Y_1'(0+), \quad \dots; \quad Y_2(0-) \neq Y_2(0+), \quad \dots$$

It can be proved rigorously that the discontinuities which must appear are precisely those given in the solutions supplied by our method, that is, that these solutions are "physically right" (EINF. p. 121). We can therefore always use the method of the $\mathfrak{L}$ -transformation without hesitation, however the initial values are given.

Example

We consider the network of two meshes shown in figure 19.1 (LRC four-terminal network). Before time $t = 0$ its energy is zero; that is, there are no currents, no voltages and no charges. At time $t = 0$ the input voltage

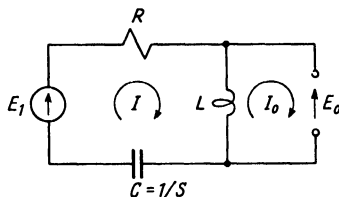


Figure 19.1 An LRC four-terminal network.

$E_1(t)$ is applied and we wish to determine the voltage $E_0(t)$ at the open output. We denote the currents in the meshes by I and I_0 so that, since $I(t) = I_0(t) = 0$ for $t \leq 0$, by § 18 the circuit equations are

$$\begin{aligned} L \frac{d}{dt}(I - I_0) + R I + S \int_0^t I(\tau) d\tau &= E_1(t) \\ L \frac{d}{dt}(I_0 - I) &= -E_0(t), \end{aligned}$$

where, to simplify the notation, we have put $1/C = S$ (elastance). Since the second mesh is open, $I_0(t) \equiv 0$. Thus there are two unknowns, I and E_0 , for which the following system of integro-differential equations is valid:

$$\begin{aligned} (19.2) \quad L I' + R I + S \int_0^t I(\tau) d\tau &= E_1(t) \\ L I' - E_0 &= 0. \end{aligned}$$

By § 18 we can apply the $\mathfrak{L}$ -transformation directly to these equations. However, because we wish to have an example of a system of differential equations, we transform system (19.2) by introducing the charge of the capacitor, $K(t) = \int_0^t I(\tau) d\tau$, as a new variable. Thus we obtain a system of differential equations of the second order:

$$(19.3) \quad L K'' + R K' + S K = E_1(t)$$

$$(19.4) \quad L K'' - E_0 = 0.$$

The determinant (19.1) of the coefficients of the highest order derivatives is

$$\begin{vmatrix} L & 0 \\ L & 0 \end{vmatrix} = 0,$$

and therefore not all the initial values of this second order system, namely $K(0+)$, $K'(0+)$, $E_0(0+)$, $E'_0(0+)$, can be chosen arbitrarily. We return to this point later, namely, the question of which initial values can be given from the mathematical point of view (see remark on p. 101). For the present we take the point of view of the engineer, that the initial values are determined from the past: thus we could say that, since there was no current in the network, $I(0-) = K'(0-) = 0$; since there were no charges present, $K(0-) = 0$; and since there was no voltage, $E_0(0-) = 0$. By analogy to the behaviour of a single differential equation the engineer expects, perhaps, that the solution takes these "left-sided" initial values as its "right-sided" initial values $K'(0+)$, $K(0+)$, $E_0(0+)$. The calculation will show whether or not this is the case.

We treat system (19.3,4) with the $\mathfrak{L}$ -transformation, using Rule V in which right-sided initial values occur. No initial values occur for E_0 , which does not appear differentiated. As we shall see later, when the values $E_0(0+)$, $E'_0(0+)$ are not determined, one can prescribe the values $K(0+)$, $K'(0+)$. On physical grounds, since the current in a loop with self-inductance cannot be discontinuous, we set $K'(0+) = I(0+)$ equal to $I(0-) = 0$.

From $K(t) = \int_0^t I(\tau) d\tau$ it is obvious that $K(0+) = 0$. Thus the image equations of (19.3,4) are

$$(19.5) \quad (L s^2 + R s + S) k(s) = e_1(s)$$

$$(19.6) \quad L s^2 k(s) - e_0(s) = 0.$$

Since only the voltage E_0 is of interest, we calculate first only e_0 (using Cramer's Rule so that we have the scheme for the calculation also in more complicated cases):

$$\begin{vmatrix} L s^2 + R s + S & 0 \\ L s^2 & -1 \end{vmatrix} e_0(s) = \begin{vmatrix} L s^2 + R s + S & e_1(s) \\ L s^2 & 0 \end{vmatrix}$$

or

$$(19.7) \quad (L s^2 + R s + S) e_0(s) = L s^2 e_1(s),$$

and therefore

$$(19.8) \quad e_0(s) = \frac{L s^2}{L s^2 + R s + S} e_1(s).$$

Here we have the same case as on p. 78. Since the numerator and the denominator of the rational function are of equal degree, (19.8) is not a $\mathfrak{L}$ -transform and thus the convolution theorem is not applicable. If, however, we divide out

$$(19.9) \quad \frac{L s^2}{L s^2 + R s + S} = 1 - \frac{R s + S}{L s^2 + R s + S} = 1 - a(s),$$

then $a(s)$ is an image function, whose original function $A(t)$ can be determined by the methods of § 12. From

$$e_0(s) = e_1(s) - a(s) e_1(s)$$

it follows that

$$(19.10) \quad E_0(t) = E_1(t) - A(t) * E_1(t).$$

In the solution of a single differential equation the excitation $E_1(t)$ occurs only in a convolution integral (see (12.10)), which represents a continuous function. Here, however, $E_1(t)$ appears by itself, so that the output voltage $E_0(t)$ reproduces all the jumps which occur in the input voltage. (Thus the four-terminal network is a "jump-capable" system.)

Which right-sided initial value has $E_0(t)$? On passing to the limit $t \rightarrow 0+$ in (19.10) we obtain

$$(19.11) \quad E_0(0+) = E_1(0+).$$

Thus $E_0(0+)$ is not equal to the left-sided initial value $E_0(0-) = 0$, but it is determined by the initial value of the input voltage (e.g. if the unit step function $U(t)$ is applied as input, to investigate the transfer function, then $E_0(0+) = 1$). This corresponds completely to physical reality: because the capacitor held no charge for $t \leq 0$, the input voltage $E_1(0+)$ appears also at the output as $E_0(0+)$.

We wish still to examine the right-sided initial values of K and I . From (19.5) it follows that

$$k(s) = \frac{1}{Ls^2 + Rs + S} e_1(s) = b(s) e_1(s)$$

and, since $k(s) = \frac{1}{s} i(s)$:

$$(19.12) \quad i(s) = \frac{s}{Ls^2 + Rs + S} e_1(s) = b_0(s) e_1(s),$$

where $b(s)$ and $b_0(s)$ are image functions, whose original functions $B(t)$ and $B_0(t)$ are easily calculated. With these we obtain

$$(19.13) \quad I(t) = B_0(t) * E_1(t) \quad \text{and} \quad K(t) = B(t) * E_1(t).$$

Hence it follows that, for $t \rightarrow 0+$,

$$(19.14) \quad I(0+) = K'(0+) = 0, \quad K(0+) = 0,$$

so that the left- and right-sided initial values of I and K are in agreement.

To sum up, the application of the $\mathfrak{L}$ -transformation gives exactly that solution of system (19.3,4) which was obviously expected on physical grounds.

In practice we can avoid the transition from integro-differential to differential equations, and this simplifies the method of solution. The only initial condition required is $I(0+) = I(0-) = 0$. Using this and transforming (19.2), we have

$$\left(Ls + R + S \frac{1}{s}\right) i(s) = e_1(s)$$

$$Ls i(s) - e_0(s) = 0.$$

From this it follows that

$$\left(Ls + R + S \frac{1}{s}\right) e_0(s) = Ls e_1(s)$$

and hence the same value of $e_0(s)$ as in (19.8).

This example shows that input into a system, which is completely at rest and therefore for which all left-sided initial values of current and voltage are zero, results in right-sided initial values which are not necessarily zero. Mathematically this is not surprising; it is a consequence of criterion (19.1) not being satisfied. However it has caused much discussion in the technical literature which sees therein a problem that is, in fact, not present. This problem arises solely because the network is treated in a way which is not mathematically suitable: in general it is customary—especially in the “system theory” of electrical engineering—to proceed exactly in the way against which one was warned on p. 79: when a system of simultaneous differential equations with several unknowns is considered, one derives a single differential equation (in general of higher order) for the single unknown which is of most interest. In the case of system (19.3,4) this is done as follows: we write it in the form

$$(19.15) \quad \left(L \frac{d^2}{dt^2} + R \frac{d}{dt} + S\right) K = E_1(t)$$

$$(19.16) \quad L \frac{d^2}{dt^2} K - E_0 = 0.$$

We treat the differential operators as coefficients, and solve the system for E_0 by Cramer's Rule:

$$\begin{vmatrix} L \frac{d^2}{dt^2} + R \frac{d}{dt} + S & 0 \\ L \frac{d^2}{dt^2} & -1 \end{vmatrix} E_0 = \begin{vmatrix} L \frac{d^2}{dt^2} + R \frac{d}{dt} + S & E_1(t) \\ L \frac{d^2}{dt^2} & 0 \end{vmatrix}$$

or

$$(19.17) \quad \left(L \frac{d^2}{dt^2} + R \frac{d}{dt} + S\right) E_0(t) = L \frac{d^2}{dt^2} E_1(t).$$

This is done in the same way as in the solution of the algebraic system (19.5,6), where the variable s corresponds to the operator d/dt here.

We must now apply the $\mathfrak{L}$ -transformation to the resulting differential equation (19.17). For this we require

- 1) the right-sided initial values $E_1(0+)$, $E_1'(0+)$ of $E_1(t)$; these we know since $E_1(t)$ is given;
- 2) the right-sided initial values $E_0(0+)$, $E_0'(0+)$ of $E_0(t)$; these are not known.

If one uses the left-sided initial values 0,0 for the latter, one obtains

$$(L s^2 + R s + S) e_0(s) = L [s^2 e_1(s) - E_1(0+) s - E_1'(0+)],$$

which, by (19.7), is *false*, except in the case $E_1(0+) = E_1'(0+) = 0$.

The electrical engineer, also without knowledge of (19.7), sees that the use of the values $E_0(0+) = E_0'(0+) = 0$ is not admissible, since they lead to a result which does not correspond to physical reality. In the literature there are two different attempts to overcome this difficulty.

- 1) One ignores completely the initial values of E_1 as well as those of E_0 by putting

$$\frac{d^2}{dt^2} E_0(t) \circ\!\!\!\bullet s^2 e_0(s), \quad \frac{d}{dt} E_0(t) \circ\!\!\!\bullet s e_0(s), \quad \frac{d^2}{dt^2} E_1(t) \circ\!\!\!\bullet s^2 e_1(s)$$

(if it occurs, $\frac{d}{dt} E_1(t) \circ\!\!\!\bullet s e_1(s)$), as is done in Heaviside calculus. Surprisingly, this obviously *false method* gives the *correct result* (19.7) — the reason for this we shall see later.

- 2) One determines the right-sided initial values of $E_0(t)$ *before* applying the $\mathfrak{L}$ -transformation: this can be done.

This determination is made most simply by returning to the original system (19.3,4), and is the same as the establishing of the mathematically well-known "compatibility conditions" (EINF. p. 116), which were mentioned previously on p. 95. To do this one eliminates the highest derivatives K'' , E_0'' from equations (19.3,4). This is possible (also in the general case) because it is a system in which the determinant (19.1) of the coefficients vanishes. In the present example, in which E_0'' does not appear, the elimination is accomplished simply by subtracting (19.3) from (19.4):

$$(19.18) \quad RK' + SK + E_0 = E_1.$$

On passing to the limit $t \rightarrow 0+$ we obtain

$$(19.19) \quad R K'(0+) + S K(0+) + E_0(0+) = E_1(0+).$$

If, as was assumed above, $K'(0+) = I(0+) = 0$ then (19.19) reduces to

$$(19.20) \quad E_0(0+) = E_1(0+),$$

which is the same result (19.11) as was found from the solution established above. To determine $E'_0(0+)$ as well, one first differentiates equation (19.18)

$$R K'' + S K' + E'_0 = E'_1$$

and then eliminates K' by multiplying equation (19.4) by R/L and subtracting:

$$(19.21) \quad S K' + E'_0 + \frac{R}{L} E_0 = E'_1.$$

On passing to the limit $t \rightarrow 0+$ we obtain

$$(19.22) \quad S K'(0+) + E'_0(0+) + \frac{R}{L} E_0(0+) = E'_1(0+)$$

or, since $K'(0+) = I(0+) = 0$,

$$(19.23) \quad E'_0(0+) = E'_1(0+) - \frac{R}{L} E_0(0+) = E'_1(0+) - \frac{R}{L} E_1(0+).$$

(The same result follows if one differentiates equation (19.10) using theorem 32.1, goes to the limit $t \rightarrow 0+$, and determines $A(0+)$ by theorem 38.2: $A(0+) = \lim_{s \rightarrow \infty} s a(s) = R/L$.)

If equation (19.17) is $\mathfrak{L}$ -transformed, using the initial values (19.20,23), the result is

$$\begin{aligned} L \left[s^2 e_0(s) - E_1(0+) s - \left(E'_1(0+) - \frac{R}{L} E_1(0+) \right) \right] + R \left[s e_0(s) - E_1(0+) \right] \\ + S e_0(s) = L [s^2 e_1(s) - E_1(0+) s - E'_1(0+)]. \end{aligned}$$

It is seen that all the terms containing the initial values cancel out, and the correct equation (19.7) remains. This explains why the procedure in which *all* the initial values are set equal to zero leads to the correct result, despite the obvious errors.

Remark: Equations (19.19,22) show that two linear relations exist among the four theoretically possible initial values $K(0+)$, $K'(0+)$, $E_0(0+)$ and $E'_0(0+)$. Thus two of them can be chosen arbitrarily, the other two then being determined.

Let us compare the two methods: in the method just described we first derive a single differential equation for E_0 , calculate the right-sided initial values and then apply the $\mathfrak{L}$ -transformation; in the previous method we apply the $\mathfrak{L}$ -transformation directly to the given system of differential equations. From the standpoint of economy in calculation, it is preferable

to proceed by the latter method. Moreover, from a mathematical standpoint, it is not at all permissible to derive a single equation for E_0 , because that requires that the excitation E_1 is differentiable (even more than once), and that is generally not the case in practice; for example, when E_1 is a jump-function. Moreover, when the derivatives do not exist, Rule V for the $\mathfrak{L}$ -transformation is not applicable, unless we introduce the δ -function and its derivatives as on pp. 80, 81: however that is both a clumsy and a dangerous procedure.

The derivation of a single differential equation for E_0 produces artificially the difficulty that the initial values required in applying the $\mathfrak{L}$ -transformation to this differential equation are not necessarily zero for a network which is at rest at $t = 0$. In technical literature this has sometimes been considered to be a contradiction and an argument against the application of the $\mathfrak{L}$ -transformation. However, in the correct mathematical treatment, this difficulty does not arise.

When we recommend the application of the $\mathfrak{L}$ -transformation directly to the given system of differential equations we do so in the interest of mathematical tidiness. Those who prefer to retain the customary, but generally incorrect, derivation of the single differential equation for E_0 have the consolation that, in spite of this error, they obtain the correct result, provided that they combine this first error with a second: that is, when applying the $\mathfrak{L}$ -transformation, they set the initial values of both the given functions and the required functions equal to zero. For our particular problem this method has given the correct result (p. 100): however, it is completely general and is valid for all networks. If a voltage $E_1(t)$ is introduced into only the first mesh of a network, if the last mesh is open ($I_n = 0$) and if the voltage $E_0(t)$ across its terminals is wanted, then, using the notation of § 18, the network equations are

$$\begin{aligned}
 & Z_{11} \left(\frac{d}{dt} \right) I_1(t) + \dots + Z_{1, n-1} \left(\frac{d}{dt} \right) I_{n-1}(t) &= E_1(t) \\
 (19.24) \quad & Z_{21} \left(\frac{d}{dt} \right) I_1(t) + \dots &= 0 \\
 & \dots &= 0 \\
 & Z_{n1} \left(\frac{d}{dt} \right) I_1(t) + \dots + Z_{n, n-1} \left(\frac{d}{dt} \right) I_{n-1}(t) - E_0(t) &= 0,
 \end{aligned}$$

where

$$Z_{\mu\nu} \left(\frac{d}{dt} \right) = L_{\mu\nu} \frac{d}{dt} + R_{\mu\nu} + S_{\mu\nu} \left(\frac{d}{dt} \right)^{-1} \text{ and } \left(\frac{d}{dt} \right)^{-1} = \int_0^t \dots d\tau.$$

If the network was at rest before $t = 0$ then all $I_r(t) = 0$ for $t \leq 0$. Since currents are continuous when inductances are present, we must set the initial values $I_r(0+)$ equal to zero. This is all that must be assumed. (If there is a branch which contains no inductance, then no initial value is required.) On application of the $\mathfrak{L}$ -transformation the following system of algebraic equations is obtained:

$$\begin{aligned} Z_{11}(s) i_1(s) + \dots + Z_{1, n-1}(s) i_{n-1}(s) &= e_1(s) \\ Z_{21}(s) i_1(s) + \dots &= 0 \\ \dots &= 0 \\ Z_{n1}(s) i_1(s) + \dots + Z_{n, n-1}(s) i_{n-1}(s) - e_0(s) &= 0. \end{aligned}$$

Using Cramer's Rule to solve for the unknown $e_0(s)$,

$$\begin{vmatrix} Z_{11}(s) & \dots & Z_{1, n-1}(s) & 0 \\ \dots & \dots & \dots & \dots \\ Z_{n-1, 1}(s) & \dots & Z_{n-1, n-1}(s) & 0 \\ Z_{n1}(s) & \dots & Z_{n, n-1}(s) & -1 \end{vmatrix} e_0(s) = \begin{vmatrix} Z_{11}(s) & \dots & Z_{1, n-1}(s) & e_1(s) \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots \\ Z_{n1}(s) & \dots & Z_{n, n-1}(s) & 0 \end{vmatrix}$$

or, when the determinants are expanded by the last column,

$$- \begin{vmatrix} Z_{11}(s) & \dots & Z_{1, n-1}(s) \\ \dots & \dots & \dots \\ Z_{n-1, 1}(s) & \dots & Z_{n-1, n-1}(s) \end{vmatrix} e_0(s) = (-1)^{n+1} \begin{vmatrix} Z_{21}(s) & \dots & Z_{2, n-1}(s) \\ \dots & \dots & \dots \\ Z_{n1}(s) & \dots & Z_{n, n-1}(s) \end{vmatrix} e_1(s).$$

In order to remove the fraction $1/s$ in the $Z_{\mu\nu}(s)$ we multiply each row of both left and right sides by s (thus multiplying both sides by s^{n-1}), and also write

$$s Z_{\mu\nu}(s) = L_{\mu\nu} s^2 + R_{\mu\nu} s + S_{\mu\nu} = V_{\mu\nu}(s).$$

This gives

$$\begin{vmatrix} V_{11}(s) & \dots & V_{1, n-1}(s) \\ \dots & \dots & \dots \\ V_{n-1, 1}(s) & \dots & V_{n-1, n-1}(s) \end{vmatrix} e_0(s) = (-1)^n \begin{vmatrix} V_{21}(s) & \dots & V_{2, n-1}(s) \\ \dots & \dots & \dots \\ V_{n1}(s) & \dots & V_{n, n-1}(s) \end{vmatrix} e_1(s).$$

On evaluating the determinants explicitly we obtain, in general, a polynomial of degree $2(n-1) = m$ on both sides, except when the degree is lowered because some $V_{\mu\nu}$ are of lower degree than 2, or because there is cancellation of higher powers. The result is therefore an equation of the form

$$(19.25) \quad (a_m s^m + \dots + a_1 s + a_0) e_0(s) = (b_m s^m + \dots + b_1 s + b_0) e_1(s),$$

from which it follows that

$$e_0(s) = \frac{b_m s^m + \dots + b_0}{a_m s^m + \dots + a_0} e_1(s) = \left[\frac{b_m}{a_m} + r(s) \right] e_1(s),$$

where $r(s)$ is a rational function whose numerator is of lower degree than its denominator. The inverse transformation gives

$$(19.26) \quad E_0(t) = \frac{b_m}{a_m} E_1(t) + R(t) * E_1(t),$$

from which it is seen that in general the network is a "jump-capable" system, and that the following equation holds:

$$(19.27) \quad E_0(0+) = \frac{b_m}{a_m} E_1(0+).$$

We have

$$a_m = \begin{vmatrix} L_{11} & \dots & L_{1,n-1} \\ \dots & \dots & \dots \\ L_{n-1,1} & \dots & L_{n-1,n-1} \end{vmatrix}, \quad b_m = (-1)^n \begin{vmatrix} L_{21} & \dots & L_{2,n-1} \\ \dots & \dots & \dots \\ L_{n1} & \dots & L_{n,n-1} \end{vmatrix}.$$

This derivation is free from errors. If we now apply the incorrect procedure and construct a single differential equation for E_0 from the system (19.24), then first all the equations must be differentiated to give the system

$$\begin{aligned} V_{11} \left(\frac{d}{dt} \right) I_1(t) + \dots + V_{1,n-1} \left(\frac{d}{dt} \right) I_{n-1}(t) &= E'_1(t) \\ \dots &= 0 \\ V_{n1} \left(\frac{d}{dt} \right) I_1(t) + \dots + V_{n,n-1} \left(\frac{d}{dt} \right) I_{n-1}(t) - E'_0(t) &= 0, \end{aligned}$$

in which

$$V_{\mu\nu} \left(\frac{d}{dt} \right) = L_{\mu\nu} \frac{d^2}{dt^2} + R_{\mu\nu} \frac{d}{dt} + S_{\mu\nu}.$$

On solving for $E'_0(t)$ by Cramer's Rule and expanding the determinants by their last columns, we obtain the differential equation

$$\begin{vmatrix} V_{11} \left(\frac{d}{dt} \right) & \dots & V_{1,n-1} \left(\frac{d}{dt} \right) \\ \dots & \dots & \dots \\ V_{n-1,1} \left(\frac{d}{dt} \right) & \dots & V_{n-1,n-1} \left(\frac{d}{dt} \right) \end{vmatrix} E'_0(t) = (-1)^n \begin{vmatrix} V_{21} \left(\frac{d}{dt} \right) & \dots & V_{2,n-1} \left(\frac{d}{dt} \right) \\ \dots & \dots & \dots \\ V_{n1} \left(\frac{d}{dt} \right) & \dots & V_{n,n-1} \left(\frac{d}{dt} \right) \end{vmatrix} E'_1(t),$$

which, when written out explicitly, is of the form

$$\left(a_m \frac{d^m}{dt^m} + \dots + a_1 \frac{d}{dt} + a_0 \right) E'_0(t) = \left(b_m \frac{d^m}{dt^m} + \dots + b_1 \frac{d}{dt} + b_0 \right) E'_1(t).$$

If we apply the $\mathfrak{L}$ -transformation to this equation and also make the second error of assuming that all the initial values of E_0 and E_1 are zero (i.e. replace all the differential operators by powers of s as in Heaviside's calculus), then we obtain the following equation

$$(a_m s^m + \dots + a_1 s + a_0) s e_0(s) = (b_m s^m + \dots + b_1 s + b_0) s e_1(s),$$

and, dividing by s , the correct equation (19.25).

When the initial values of the currents are not zero then, naturally, the only legitimate method which can be used is that described first.

The proof given above can be extended easily to general systems of differential equations for which vanishing initial values are given.

§ 20. Nonlinear Differential Equations

For nonlinear differential equations there are no general methods which are applicable to all cases and no solutions in closed form; in general the solutions cannot be written in terms of the classical transcendents. We must be satisfied with approximate methods. Although the $\mathfrak{L}$ -transformation is not applicable immediately in the previous way, since it is a linear transformation and particularly suited to solving linear problems, it can, nevertheless, be useful in approximate methods.

Usually the left-hand side of the differential equation consists of a linear part of the same form as in § 12 and a nonlinear part $\Phi(Y, Y', \dots, Y^{(m)})$, so that the equation has the form

$$(20.1) \quad \sum_{\nu=0}^n c_{\nu} Y^{(\nu)} + \Phi(Y, Y', \dots, Y^{(m)}) = F(t).$$

When a current loop is treated, an integro-differential expression of form (18.1) can replace the linear differential expression. After applying the $\mathfrak{L}$ -transformation and writing

$$\sum_{\nu=0}^n c_{\nu} s^{\nu} = p(s)$$

we obtain the image equation

$$(20.2) \quad p(s) y(s) - g(s, Y(0), Y'(0), \dots, Y^{(n-1)}(0)) + \mathfrak{L}\{\Phi(Y, \dots)\} = f(s),$$

where g is a polynomial in s , which depends upon the initial values (cf. § 14). With

$$\frac{1}{p(s)} = q(s)$$

we obtain

$$(20.3) \quad y(s) = q(s) h(s) - q(s) \mathfrak{L}\{\Phi(Y, \dots)\},$$

where we have used the abbreviation

$$f(s) + g(s, Y(0), Y'(0), \dots, Y^{(n-1)}(0)) = h(s).$$

In original space the function corresponding to $y(s)$ is

$$(20.4) \quad \begin{aligned} Y(t) &= Q(t) * H(t) - Q(t) * \Phi(Y, \dots) \\ &= Q(t) * H(t) - \int_0^t Q(t-\tau) \Phi(Y(\tau), \dots) d\tau, \end{aligned}$$

where $H(t)$ is the original function of the known image function $h(s)$.

(20.4) is a *nonlinear integral equation* for $Y(t)$, which we can solve in the usual way by successive approximations. Neglecting the nonlinear term Φ we obtain as first approximation the solution of the linear problem found in §§ 12 and 14

$$Y_0(t) = Q * H.$$

Inserting this function in the right-hand side of (20.4) we obtain the next approximation

$$Y_1(t) = Y_0(t) - \int_0^t Q(t-\tau) \Phi(Y_0(\tau), \dots) d\tau,$$

which is inserted again in the right-hand side of (20.4), and so on; in general

$$(20.5) \quad Y_n(t) = Y_0(t) - \int_0^t Q(t-\tau) \Phi(Y_{n-1}(\tau), \dots) d\tau.$$

If the function Φ is sufficiently regular, the solution $Y(t)$ is found as $\lim_{n \rightarrow \infty} Y_n(t)$.

In practice we calculate only a few approximations and then we see whether they are approaching a limit; this will happen only within a certain interval $0 \leq t \leq t_0$. The process therefore adapts itself to the calculation of transient phenomenon but, for example, we can conclude nothing about the stability of the system (asymptotic behaviour).

This approximation process can be simplified by the $\mathfrak{L}$ -transformation. Applying the $\mathfrak{L}$ -transformation to (20.5), we obtain by the convolution theorem

$$(20.6) \quad y_n(s) = y_0(s) - \mathfrak{L}\{Q\} \mathfrak{L}\{\Phi(Y_{n-1}, \dots)\}.$$

Transforming this equation back to original space without using the convolution theorem, we obtain

$$(20.7) \quad Y_n(t) = Y_0(t) - \mathfrak{L}^{-1}\{q(s) \mathfrak{L}\{\Phi(Y_{n-1}, \dots)\}\}.$$

If the original function of $q(s)$ $\mathfrak{L}\{\Phi(Y_{n-1}, \dots)\}$ can be determined easily, the calculation of Y_n by using (20.6, 7) is considerably simpler than by using (20.5) because the difficult integration does not arise.

We illustrate this procedure by an example.

Example

A problem in automatic control³⁰⁾ leads to the following nonlinear differential equation

$$Y'' + (2 + a Y^2) Y' + (1 + b Y^2) Y = 0,$$

which we write in the form corresponding to (20.1):

$$(Y'' + 2 Y' + Y) + (a Y^2 Y' + b Y^3) = 0.$$

Using the $\mathfrak{L}$ -transformation³¹⁾ it becomes

$$(s^2 + 2s + 1)y(s) - Y(0)s - Y'(0) - 2Y(0) + \mathfrak{L}\{a Y^2 Y' + b Y^3\} = 0,$$

from which it follows that

$$y(s) = \frac{Y(0)s + Y'(0) + 2Y(0)}{(s+1)^2} - \frac{1}{(s+1)^2} \mathfrak{L}\{a Y^2 Y' + b Y^3\}.$$

If we take the particular initial values

$$Y(0) = 0, \quad Y'(0) = c \neq 0,$$

then

$$y(s) = \frac{c}{(s+1)^2} - \frac{1}{(s+1)^2} \mathfrak{L}\{a Y^2 Y' + b Y^3\}.$$

Neglecting the second term on the right we have the first approximation

$$y_0(s) = \frac{c}{(s+1)^2}.$$

To it corresponds

$$Y_0(t) = ct e^{-t}.$$

³⁰⁾ For the description of this automatic control see H. MATUSCHKA: Nichtlinearitäten im Regler zur Verbesserung der Regelgüte. *Regelungstechnik, Moderne Theorien und ihre Verwendbarkeit* (Report of the Conference held at Heidelberg 1956), Oldenbourg Verlag, 1957, pp. 172—177. There the solution is calculated numerically by substituting a difference equation for the differential equation and is shown graphically.

³¹⁾ The treatment of the differential equation using the $\mathfrak{L}$ -transformation originates from P. J. NOWACKI: Die Behandlung von nichtlinearen Problemen in der Regelungstechnik. *Regelungstechnik* 8 (1960), pp. 47—50. There the first approximations for $0 \leq t \leq 1.5$ are represented by curves and compared with the solution found using the difference equation.

Now with

$$Y_0^2 = c^2 t^2 e^{-2t}, \quad Y_0^3 = c^3 t^3 e^{-3t}, \quad Y_0' = c(1-t)e^{-t}$$

we have to calculate

$$a Y_0^2 Y_0' + b Y_0^3 = c^3 (a t^3 + (b-a) t^3) e^{-3t}$$

and then to form its $\mathfrak{L}$ -transform:

$$\mathfrak{L}\{a Y_0^2 Y_0' + b Y_0^3\} = c^3 \left(\frac{2a}{(s+3)^3} + \frac{6(b-a)}{(s+3)^4} \right).$$

Hence we obtain in image space the approximation

$$y_1(s) = \frac{c}{(s+1)^2} - \frac{2ac^3}{(s+1)^2(s+3)^3} - \frac{6(b-a)c^3}{(s+1)^2(s+3)^4}.$$

The corresponding original function $Y_1(t)$ is found by a partial fraction expansion using the method described in § 12. To shorten the algebra we set $b = a$, and then the last term is suppressed. The partial fraction expansion of y_1 is

$$y_1(s) = \frac{c}{(s+1)^2} - 2ac^3 \left(\frac{-3/16}{s+1} + \frac{1/8}{(s+1)^2} + \frac{3/16}{s+3} + \frac{1/4}{(s+3)^2} + \frac{1/4}{(s+3)^3} \right).$$

To this corresponds

$$Y_1(t) = ct e^{-t} - 2ac^3 \left[\left(-\frac{3}{16} + \frac{1}{8}t \right) e^{-t} + \left(\frac{3}{16} + \frac{1}{4}t + \frac{1}{8}t^2 \right) e^{-3t} \right].$$

The continued application of this process obviously produces a sequence of exponential functions multiplied by polynomials³²).

³²) For further examples of the same method see L. A. PIPES: Application of integral equations to the solution of nonlinear electric circuit problems. *Trans. Amer. Inst. Electr. Engin.*, Sept. 1953, pp. 445—450. For a different method of approximation which likewise uses the $\mathfrak{L}$ -transformation see B. N. NAUMOV: Eine Näherungsmethode zur Berechnung der Übergangsprozesse in selbsttätigen Regelungssystemen mit nicht-linearen Elementen. *Regelungstechnik* (the conference report quoted in footnote 30), pp. 184—197. A short description of this method is found in the work of Nowacki quoted in footnote 31.

Difference Equations and Sampled-Data Systems

§ 21. The Homogeneous Difference Equation with Arbitrary Initial Values

The difference equation of the r^{th} order is taken to be of the form

$$(21.1) \quad Y_{n+r} + c_{r-1} Y_{n+r-1} + \cdots + c_1 Y_{n+1} + c_0 Y_n = F_n.$$

It is recommended here, as with differential equations, to set the coefficient c_r equal to 1. The sequence F_n is given and the sequence Y_n is to be determined ($n = 0, 1, \dots$). To obtain a definite solution, the "initial values" $Y_0, Y_1, \dots, Y_{r-1}$ must be given in advance. Then the value Y_r can be calculated from equation (21.1) with $n = 0$. From the values $Y_1, \dots, Y_r$ and from (21.1) with $n = 1$ the value Y_{r+1} can be found and so forth. In this way we can calculate recursively all values Y_n and, because of this, (21.1) is also called a "recurrence relation". What we are aiming for, however, is to obtain a general expression for Y_n .

We can split the problem in the same way as that involving differential equations: we determine first a solution $Y_n^{(1)}$ for an arbitrary F_n (inhomogeneous equation) with vanishing initial values and then a solution $Y_n^{(2)}$ for the particular $F_n \equiv 0$ (homogeneous equation) with arbitrary initial values $Y_0, \dots, Y_{r-1}$. Then, because of the linearity of the equation, the solution of the inhomogeneous equation (21.1) with the initial values $Y_0, \dots, Y_{r-1}$, is the sequence $Y_n = Y_n^{(1)} + Y_n^{(2)}$.

The homogeneous difference equation with arbitrary initial values is (in contrast to differential equations) the problem which arises most frequently, and for that reason we consider it first:

$$(21.2) \quad Y_{n+r} + c_{r-1} Y_{n+r-1} + \cdots + c_1 Y_{n+1} + c_0 Y_n = 0.$$

We wish to apply the $\mathfrak{L}$ -transformation to this equation. Because the transformation refers to functions and not to sequences, we first construct from the sequence Y_n a stair-case function defined for $t \geq 0$ by

$$(21.3) \quad Y(t) = Y_n \quad \text{for } n \leq t < n+1 \quad (n = 0, 1, \dots)$$

(see Fig. 21.1). $Y(t)$ obviously satisfies the difference equation "with continuous variable"³³⁾

$$(21.4) \quad Y(t+r) + c_{r-1} Y(t+r-1) + \cdots + c_1 Y(t+1) + c_0 Y(t) = 0.$$

The values of $Y(t)$ in the interval $0 \leq t < r$, are now the given "initial values", viz.,

$$(21.5) \quad Y(t) = Y_0 \text{ in } 0 \leq t < 1, \quad Y(t) = Y_1 \text{ in } 1 \leq t < 2, \quad \dots, \\ Y(t) = Y_{r-1} \text{ in } r-1 \leq t < r.$$

In applying the $\mathfrak{L}$ -transformation to equation (21.4) we use Rule III (the 2nd translation theorem) with each term and obtain

$$\mathfrak{L}\{Y(t+k)\} = e^{ks} \left[y(s) - \int_0^k e^{-st} Y(t) dt \right].$$

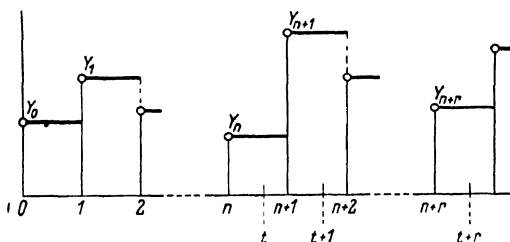


Fig. 21.1

The stair-case function constructed from a sequence.

This integral can be evaluated because $Y(t)$ is piecewise constant³⁴⁾:

$$\begin{aligned} \int_0^k e^{-st} Y(t) dt &= Y_0 \int_0^1 + Y_1 \int_1^2 + \cdots + Y_{k-1} \int_{k-1}^k e^{-st} dt \\ &= Y_0 \frac{1-e^{-s}}{s} + Y_1 \frac{e^{-s}-e^{-2s}}{s} + \cdots + Y_{k-1} \frac{e^{-(k-1)s}-e^{-ks}}{s} \\ &= \frac{1-e^{-s}}{s} (Y_0 + Y_1 e^{-s} + \cdots + Y_{k-1} e^{-(k-1)s}). \end{aligned}$$

³³⁾ Pure mathematics is concerned predominantly with such difference equations, but they are of less importance in practice. For the application of the $\mathfrak{L}$ -transformation it would be more suitable to use from the beginning a difference equation of the form (21.4). We do not do this because the calculation is much simpler for the step functions connected with sequences, cf. footnote 34.

³⁴⁾ Here we see the advantage which the step function has in contrast to an arbitrary function.

Thus the second translation theorem in the case of a step function takes the following form:

$$(21.6) \quad \mathfrak{L}\{Y(t+k)\} = e^{ks} \left[y(s) - \frac{1-e^{-s}}{s} \sum_{\nu=0}^{k-1} Y_{\nu} e^{-\nu s} \right]$$

The image equation of (21.4) is therefore

$$(21.7) \quad \sum_{k=0}^r c_k e^{ks} \left[y(s) - \frac{1-e^{-s}}{s} \sum_{\nu=0}^{k-1} Y_{\nu} e^{-\nu s} \right] = 0$$

or

$$y(s) \sum_{k=0}^r c_k e^{ks} = \frac{1-e^{-s}}{s} \sum_{k=0}^r c_k e^{ks} \sum_{\nu=0}^{k-1} Y_{\nu} e^{-\nu s},$$

where $c_r = 1$. If we introduce the polynomial

$$\sum_{k=0}^r c_k z^k = p(z),$$

then $y(s)$ is multiplied by $p(e^s)$. When the order of summation is interchanged³⁵⁾, we obtain for the solution in image space

$$(21.8) \quad y(s) = \frac{1-e^{-s}}{s} \frac{1}{p(e^s)} \sum_{\nu=0}^{r-1} Y_{\nu} \sum_{k=\nu+1}^r c_k e^{(k-\nu)s}.$$

$y(s)$ is a sum of functions of the form

$$(21.9) \quad \frac{1-e^{-s}}{s} \frac{1}{p(e^s)} e^{\mu s} \quad (\mu = 1, \dots, r).$$

The most elegant method of determining the original functions is the following: the solution of the difference equation (21.2) with the specific initial values

$$(21.10) \quad Y_0 = Y_1 = \dots = Y_{r-2} = 0, \quad Y_{r-1} = 1,$$

when completed to a stair-case function, has by (21.8) the image function

$$(21.11) \quad y(s) = \frac{1-e^{-s}}{s} \frac{1}{p(e^s)} e^s.$$

However, we already know the solution Y_n from our earlier results on differential equations. If we set, as in § 12,

$$\frac{1}{p(s)} = q(s),$$

³⁵⁾ If the outer series is summed with respect to k and the inner series is summed with respect to ν , then $0 \leq k \leq r$, $0 \leq \nu \leq k-1$. If the order of summation is interchanged, then we have to notice that ν runs through the values $0 \leq \nu \leq r-1$, while $\nu \leq k-1 \leq r-1$; therefore $\nu+1 \leq k \leq r$.

then by (14.3, 4) the corresponding original function $Q(t)$ (called there the weight function or impulse response) has the properties³⁶⁾

$$(21.12) \quad Q^{(r)}(t) + c_{r-1}Q^{(r-1)}(t) + \dots + c_1Q'(t) + c_0Q(t) = 0,$$

$$(21.13) \quad Q(0) = Q'(0) = \dots = Q^{(r-2)}(0) = 0, \quad Q^{(r-1)}(0) = 1.$$

If equation (21.12) is differentiated n times and if we set $t = 0$, we obtain

$$Q^{(n+r)}(0) + c_{r-1}Q^{(n+r-1)}(0) + \dots + c_1Q^{(n+1)}(0) + c_0Q^{(n)}(0) = 0.$$

This implies that if we consider $Q^{(n)}(0)$ as a sequence depending on the index n (that is to say, the upper index n is lowered), then this sequence satisfies the difference equation (21.2). However, the relations (21.13) show that it has the initial values (21.10). Because the solution of the initial value problem is unique (this follows from the recurrence calculation referred to at the beginning), the image function (21.11) corresponds to the step function determined by $Q^{(n)}(0)$, which we can denote by $Q^{(l)}(0)$ ³⁷⁾:

$$(21.14) \quad Q^{(l)}(0) \circ \bullet \frac{1 - e^{-s}}{s} \frac{1}{p(e^s)} e^s = \frac{e^s - 1}{s} q(e^s).$$

This original function has a strange form at first sight; however, it has the great advantage that it is completely general and independent of the multiplicity of zeros of $p(z)$; we will return later to that distinction.

We obtain now the original function of the more general function (21.9) in the following way. We denote the function (21.11) temporarily by $r(s)$; then we have

$$\frac{1 - e^{-s}}{s} \frac{1}{p(e^s)} e^{\mu s} = e^{(\mu-1)s} r(s) \quad (\mu - 1 = 0, 1, \dots, r-1).$$

By (21.6)

$$e^{(\mu-1)s} \left[r(s) - \frac{1 - e^{-s}}{s} \sum_{\nu=0}^{\mu-2} Q^{(\nu)}(0) e^{-\nu s} \right] \circ \bullet Q^{(l+\mu-1)}(0).$$

Because $Q^{(\nu)}(0) = 0$ ($\nu = 0, 1, \dots, r-2$), we obtain the correspondence

$$(21.15) \quad \boxed{\frac{1 - e^{-s}}{s} \frac{1}{p(e^s)} e^{\mu s} \circ \bullet Q^{(l+\mu-1)}(0) \quad (\mu = 1, \dots, r)}$$

³⁶⁾ Previously we denoted the order by n , but now we use r , because in difference equations the letter n usually means the index of the sequence.

³⁷⁾ $[t]$ indicates the greatest integer in t ; for example, $[5.7] = 5$, $[\pi] = 3$, $[2] = 2$. Using this symbol we can denote quite generally the step function corresponding to a sequence Y_n by $Y_{[t]}$.

With this we can transform the image function (21.8) into original space:

$$(21.16) \quad Y(t) = \sum_{\nu=0}^{r-1} Y_{\nu} \sum_{k=\nu+1}^r c_k Q^{(l+k-\nu-1)}(0).$$

Setting $t = n$, $Y(n) = Y_n$ in this, we obtain (with $k = l + \nu + 1$ in the inner sum) the solution of our original difference equation (21.2)³⁸⁾:

$$(21.17) \quad Y_n = \sum_{\nu=0}^{r-1} Y_{\nu} \sum_{l=0}^{r-\nu-1} c_{l+\nu+1} Q^{(n+l)}(0)$$

In practical applications of this formula only the function $Q(t)$ requires to be calculated using the method described in § 12. This is particularly simple when the zeros, $\alpha_1, \dots, \alpha_r$ of the characteristic polynomial are all different. Then we have by (12.9)

$$Q(t) = \sum_{\mu=1}^r \frac{1}{p'(\alpha_{\mu})} e^{\alpha_{\mu} t},$$

and therefore

$$Q^{(n+l)}(0) = \sum_{\mu=1}^r \frac{\alpha_{\mu}^{n+l}}{p'(\alpha_{\mu})}.$$

The solution of the difference equation can therefore be written in the form:

$$(21.18) \quad Y_n = \sum_{\nu=0}^{r-1} Y_{\nu} \sum_{l=0}^{r-\nu-1} c_{l+\nu+1} \sum_{\mu=1}^r \frac{\alpha_{\mu}^{n+l}}{p'(\alpha_{\mu})}$$

Y_n is a linear combination of the "characteristic solutions" α_{μ}^n , which correspond to the characteristic oscillations $e^{\alpha_{\mu} t}$ of the differential equation. Although for differential equations it is the real part of α_{μ} which determines the asymptotic behaviour of the characteristic oscillations as $t \rightarrow \infty$, with difference equations it is the absolute value of α_{μ} , since α_{μ}^n for $n \rightarrow \infty$ tends toward 0 or ∞ , or oscillates, according as $|\alpha_{\mu}| < 1$ or > 1 or $= 1$.

The Difference Equation of Second Order

Since the case $r = 2$ is by far the most frequent in engineering we will solve the equation once again and obtain the solution explicitly. At the same time those readers for whom the preceding explanations were too general may understand the principle of the method from this example.

³⁸⁾ Comparing this formula with (14.6) we see that the sequence $Q^{(n)}(0)$ has the same importance in the difference equation as the function $Q(t)$ in the corresponding differential equation. This will be obvious also in the solution of the inhomogeneous difference equation.

The equation is

$$Y_{n+2} + c_1 Y_{n+1} + c_0 Y_n = 0,$$

where the initial values Y_0, Y_1 are given. We construct from the sequence Y_n a step function $Y(t)$ as in Fig. 21.1, for which the difference equation with continuous variables

$$Y(t+2) + c_1 Y(t+1) + c_0 Y(t) = 0$$

together with the initial values

$$Y(t) = Y_0 \quad \text{in } 0 \leq t < 1, \quad Y(t) = Y_1 \quad \text{in } 1 \leq t < 2$$

is valid. After using the translation theorem (21.6) for step functions the image equation is

$$e^{2s} \left[y(s) - \frac{1-e^{-s}}{s} (Y_0 + Y_1 e^{-s}) \right] + c_1 e^s \left[y(s) - \frac{1-e^{-s}}{s} Y_0 \right] + c_0 y(s) = 0$$

or

$$(e^{2s} + c_1 e^s + c_0) y(s) = \frac{1-e^{-s}}{s} \left[Y_0 (e^{2s} + c_1 e^s) + Y_1 e^s \right].$$

Setting

$$z^2 + c_1 z + c_0 = p(z)$$

we have

$$(21.19) \quad y(s) = \frac{1-e^{-s}}{s} \frac{1}{p(e^s)} \left[Y_0 (e^{2s} + c_1 e^s) + Y_1 e^s \right].$$

We obtain the original functions of each term from the correspondence (21.15):

$$(21.20) \quad Y(t) = Y_0 [Q^{(t)+1}(0) + c_1 Q^{(t)}(0)] + Y_1 Q^{(t)}(0),$$

where $Q(t)$ is the original function of

$$q(s) = \frac{1}{s^2 + c_1 s + c_0}.$$

$Q(t)$ is calculated differently, depending on whether the zeros α_1, α_2 of $p(s)$ are equal or unequal.

Case 1: $\alpha_1 \neq \alpha_2$

From (11.4)

$$Q(t) = \frac{1}{\alpha_1 - \alpha_2} (e^{\alpha_1 t} - e^{\alpha_2 t}),$$

and therefore

$$Q^{(m)}(0) = \frac{\alpha_1^m - \alpha_2^m}{\alpha_1 - \alpha_2}.$$

We use this expression in (21.20) and return at the same time from the function $Y(t)$ to the sequence Y_n by setting $t = n$:

$$(21.21) \quad Y_n = Y_0 \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} + c_1 \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right] + Y_1 \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

Observing that $c_1 = -(\alpha_1 + \alpha_2)$ and $\alpha_1 \alpha_2 = c_0$, this formula simplifies to

$$(21.22) \quad Y_n = -Y_0 c_0 \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} + Y_1 \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$$

Case 2: $\alpha_1 = \alpha_2$

From (11.12)

$$Q(t) = t e^{\alpha_1 t},$$

and therefore

$$Q^{(m)}(0) = m \alpha_1^{m-1}.$$

Thus we obtain from (21.20)

$$(21.23) \quad Y_n = Y_0 [(n+1) \alpha_1^n + c_1 n \alpha_1^{n-1}] + Y_1 n \alpha_1^{n-1}$$

or, since $c_1 = -2\alpha_1$ and $\alpha_1^2 = c_0$:

$$(21.24) \quad Y_n = -Y_0 c_0 (n-1) \alpha_1^{n-2} + Y_1 n \alpha_1^{n-1}$$

Remarks

1. As we shall see in the example on p. 125, the coefficients of Y_0 and Y_1 in formulas (21.21, 22) can be expressed in a more convenient form for the purposes of numerical calculation.

2. The form of the solution can be obtained also by partial fraction expansion analogous to the solution of differential equations. For this purpose we write the image function (21.19) in the form

$$y(s) = \frac{e^s - 1}{s} \frac{Y_0(e^s + c_1) + Y_1}{e^{2s} + c_1 e^s + c_0}.$$

Setting $e^s = z$, the second factor becomes

$$Y_0 \frac{z + c_1}{(z - \alpha_1)(z - \alpha_2)} + Y_1 \frac{1}{(z - \alpha_1)(z - \alpha_2)}.$$

In the case $\alpha_1 \neq \alpha_2$, we can expand both fractions in partial fractions of the form

$$\frac{d_1}{z - \alpha_1} + \frac{d_2}{z - \alpha_2}.$$

Substituting e^s for z , $y(s)$ appears as the sum of terms of the form

$$\frac{e^s - 1}{s} \frac{1}{e^s - \alpha}$$

with coefficients independent of s ($\alpha = \alpha_1$ or α_2). To this there corresponds a very simple original function:

$$(21.25) \quad \boxed{\frac{e^s - 1}{s} \frac{1}{e^s - \alpha} \longleftrightarrow \alpha^{[t]}}$$

Using this correspondence for the transformation of the terms in $y(s)$, with a little more calculation we arrive at formula (21.21). Further useful correspondences of the type (21.25) are found in TAB. 244—247.

This method is naturally applicable to difference equations of higher order.

§ 22. The Inhomogeneous Difference Equation with Vanishing Initial Values

We consider now the difference equation

$$(22.1) \quad Y_{n+r} + c_{r-1} Y_{n+r-1} + \cdots + c_1 Y_{n+1} + c_0 Y_n = F_n,$$

where F_n is not equal to 0 for all n , and where the initial values prescribed are

$$(22.2) \quad Y_0 = Y_1 = \cdots = Y_{r-1} = 0.$$

If we construct the step functions $Y(t)$ and $F(t)$ from the sequences Y_n and F_n as in (21.3) then we obtain a difference equation with continuous variable

$$(22.3) \quad Y(t+r) + c_{r-1} Y(t+r-1) + \cdots + c_1 Y(t+1) + c_0 Y(t) = F(t),$$

with initial condition

$$(22.4) \quad Y(t) = 0 \quad \text{for } 0 \leq t < r.$$

Using the translation theorem (21.6) and by (22.2)

$$\mathfrak{L}\{Y(t+k)\} = e^{ks} y(s).$$

Thus the image equation

$$y(s) \sum_{k=0}^r c_k e^{ks} = f(s)$$

corresponds to (22.3) and with the abbreviations

$$\sum_{k=0}^r c_k z^k = p(z), \quad \frac{1}{p(z)} = q(z)$$

it follows that

$$(22.5) \quad y(s) = \frac{1}{p(e^s)} f(s) = q(e^s) f(s).$$

The convolution theorem is not applicable here because $1/p(e^s)$ is not an $\mathfrak{L}$ -transform³⁹⁾. Therefore we must proceed differently.

The original function $Q(t)$ of $q(s)$ is, by (12.9,14), a sum of exponential functions, which may be multiplied by polynomials; in every case it is an entire function which can be expanded for all t in a convergent Taylor-series

$$Q(t) = \sum_{\nu=0}^{\infty} \frac{Q^{(\nu)}(0)}{\nu!} t^{\nu}.$$

Its image function can be obtained from Theorem 35.1 by applying the $\mathfrak{L}$ -transformation term by term:

$$q(s) = \sum_{\nu=0}^{\infty} \frac{Q^{(\nu)}(0)}{s^{\nu+1}}.$$

(This series converges outside a circle, centre the origin, that includes all poles of $q(s)$.) Therefore we obtain

$$(22.6) \quad q(e^s) = \sum_{\nu=0}^{\infty} \frac{Q^{(\nu)}(0)}{e^{(\nu+1)s}} = \sum_{\nu=1}^{\infty} Q^{(\nu-1)}(0) e^{-\nu s}$$

and

$$(22.7) \quad y(s) = \sum_{\nu=1}^{\infty} Q^{(\nu-1)}(0) e^{-\nu s} f(s).$$

By Rule II (first translation theorem)

$$e^{-\nu s} f(s) \Leftrightarrow \begin{cases} F(t-\nu) & \text{for } t \geq \nu \\ 0 & \text{for } t < \nu. \end{cases}$$

³⁹⁾ This follows from the theorem that a $\mathfrak{L}$ -transform which is not identically 0 cannot be periodic (EIN§. p. 34). $1/p(e^s)$ has period $2\pi j$.

If the inversion of the $\mathcal{Q}$ -transformation can be applied term by term, we obtain

$$Y(t) = \sum_{\nu=1}^{\infty} Q^{(\nu-1)}(0) F(t-\nu).$$

This series is in fact finite because for fixed t the terms with $\nu > t$ are zero; hence

$$Y(t) = \sum_{\nu=1}^{[t]} Q^{(\nu-1)}(0) F(t-\nu).$$

The sum can be reduced further because by (21.13) $Q^{(\nu-1)}(0) = 0$ for $\nu = 1, \dots, r-1$. If we go back from the step function to the sequence we obtain as the solution of the difference equation (22.1) with the initial conditions (22.2)

$$(22.8) \quad Y_n = \sum_{\nu=r}^n Q^{(\nu-1)}(0) F_{n-\nu} \quad \text{for } n \geq r$$

(For $n < r$, Y_n does not need to be defined because it is then equal to 0, by the initial conditions.) We can verify that Y_n gives the solution of the problem independently of the assumptions made in the derivation.

As in § 21, in practical applications of the formula the function $Q(t)$ has to be calculated explicitly. If the zeros of the polynomial $p(z)$ are different, we obtain

$$(22.9) \quad Y_n = \sum_{\nu=r}^n F_{n-\nu} \sum_{\mu=1}^r \frac{\alpha_{\mu}^{\nu-1}}{p'(\alpha_{\mu})} \quad \text{for } n \geq r$$

The Difference Equation of Second Order

For the difference equation of second order we can take the value of $Q^{(\nu-1)}(0)$ from p. 114, 115.

Case 1: $\alpha_1 \neq \alpha_2$

$$(22.10) \quad Y_n = \sum_{\nu=2}^n F_{n-\nu} \frac{\alpha_1^{\nu-1} - \alpha_2^{\nu-1}}{\alpha_1 - \alpha_2} \quad \text{for } n \geq 2$$

Case 2: $\alpha_1 = \alpha_2$

$$(22.11) \quad Y_n = \sum_{\nu=2}^n F_{n-\nu} (\nu-1) \alpha_1^{\nu-2} \quad \text{for } n \geq 2$$

§ 23. The Boundary Value Problem for the Difference Equation of Second Order

It frequently happens in applications that the values of the unknown Y_n in the difference equation do not occur for all $n \geq 0$, but only for a finite number of indices $0 \leq n \leq N$. Then, in general, a definite solution is not determined by prescribing initial values. For the usual case, in which the order of the difference equation is equal to two, the two values Y_0 and Y_N , the "boundary values", are given and this is why we call it a "boundary value problem". However (for example in the theory of ladder networks) other possibilities can occur; for example, a linear relation may be given between Y_0 and Y_1 and another between Y_{N-1} and Y_N .

We consider here the following problem. The difference equation of second order

$$(23.1) \quad Y_{n+2} + c_1 Y_{n+1} + c_0 Y_n = F_n$$

is given together with the pair of boundary values Y_0, Y_N (naturally $N \geq 2$). The zeros α_1, α_2 of the characteristic polynomial

$$p(z) = z^2 + c_1 z + c_0$$

may be different. (The case of equal zeros is simpler and is omitted here.) We require no new theory since we can use the general solution of the initial value problem which arises from the superposition of the partial solutions (22.10) and (21.22):

$$(23.2) \quad Y_n = \sum_{\nu=2}^n F_{n-\nu} \frac{\alpha_1^{\nu-1} - \alpha_2^{\nu-1}}{\alpha_1 - \alpha_2} - Y_0 c_0 \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} + Y_1 \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \quad (n \geq 2).$$

In this formula we must introduce the given value Y_N instead of the unknown value Y_1 . For this purpose we set $n = N$ in (23.2) and obtain a relation between Y_0, Y_1 , and Y_N , from which we can calculate Y_1 as a function of Y_0 and Y_N :

$$Y_1 = \frac{1}{\alpha_1^N - \alpha_2^N} \left\{ Y_0 c_0 (\alpha_1^{N-1} - \alpha_2^{N-1}) + Y_N (\alpha_1 - \alpha_2) - \sum_{\nu=2}^N (\alpha_1^{\nu-1} - \alpha_2^{\nu-1}) F_{N-\nu} \right\}.$$

If we insert this into (23.2), we obtain

$$(23.3) \quad \boxed{Y_n = \frac{1}{\alpha_1 - \alpha_2} \sum_{\nu=2}^n (\alpha_1^{\nu-1} - \alpha_2^{\nu-1}) F_{n-\nu} - \frac{1}{\alpha_1 - \alpha_2} \frac{\alpha_1^n - \alpha_2^n}{\alpha_1^N - \alpha_2^N} \sum_{\nu=2}^N (\alpha_1^{\nu-1} - \alpha_2^{\nu-1}) F_{N-\nu} + \frac{1}{\alpha_1^N - \alpha_2^N} \{ Y_0 (\alpha_1^N \alpha_2^n - \alpha_1^n \alpha_2^N) + Y_N (\alpha_1^n - \alpha_2^n) \}}$$

This is valid for $2 \leq n \leq N$, but it gives the correct values also for $n = 0$ and 1, if we set the first sum equal to 0 for $n = 0$ and 1.

Of course the solution has meaning only if

$$\alpha_1^N - \alpha_2^N \neq 0.$$

If the given "interval length" N and the constants in the difference equation are such that $\alpha_1^N - \alpha_2^N = 0$, then the problem has, in general, no solution. In analogy to the well-known results with differential equations, there arise characteristic values and characteristic functions, which are of importance also in engineering; however, we cannot discuss this subject in greater detail here.

§ 24. A System of Simultaneous Difference Equations with Initial or Boundary Conditions (Electric Ladder Network)

We return to the treatment of the electric network of § 18; there we considered first a single mesh and set up the integro-differential equation (18.1) for the current $I(t)$ produced by a voltage $E(t)$ of an entirely arbitrary nature. The linear algebraic equation (18.3) connects the image functions $e(s)$ and $i(s)$. In complex alternating-current calculations one is restricted to the special case of an *alternating voltage* $E(t) = E e^{j\omega t}$ and assumes that an *alternating current* of the same frequency is produced; $I(t)$ is therefore of the form $I(t) = I e^{j\omega t}$. (E and I are complex constants which determine the amplitude and phase.) Strictly speaking this is only possible with a steady state ($-\infty < t < +\infty$). For the case of a transient ($0 \leq t < \infty$) such a steady state arises only after a sufficiently long time (in mathematical language, it is given by the asymptotic behaviour, cf. § 13 No. 3).

Differentiating equation (18.1) with respect to t ⁴⁰

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I(t) = E'(t)$$

and introducing $I(t) = I e^{j\omega t}$, $E(t) = E e^{j\omega t}$, we obtain

$$I \left(L(j\omega)^2 + Rj\omega + \frac{1}{C} \right) e^{j\omega t} = E j\omega e^{j\omega t}$$

or

$$(24.1) \quad \left(Lj\omega + R + \frac{1}{Cj\omega} \right) I = E.$$

⁴⁰) If $I e^{j\omega t}$ were inserted straight into equation (18.1) the integral $\int_{-\infty}^t I e^{j\omega \tau} d\tau$ would diverge.

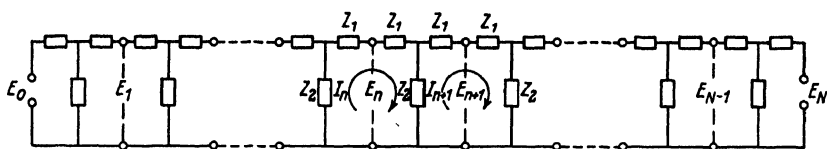


Figure 24.1 A ladder network consisting of T-structures.

This remark was made in advance because subsequently we wish to use alternating current methods to simplify calculations. We consider now a special network for which we can employ the theory developed previously for difference equations.

A *ladder network* is a special network in which all meshes (with the possible exception of the first and the last) have the same structure and are ordered in series so that every mesh has a branch in common with each branch preceding and following it. For the meshes of the ladder network shown in Fig. 24.1 there are two equal impedances Z_1 in every upper branch, there is no impedance in the lower branch and there is one impedance Z_2 in each transverse branch.

Because we wish to determine only the steady state of the ladder network, we can use the alternating current calculation. We could set up the mesh equations (24.4) for the network, but it proves simpler to think of the network as built from four-terminal networks of the type indicated in Fig. 24.2, which is called the T-structure (instead of from meshes). Also it is advantageous for calculation to attach a complete T-structure at the beginning and at the end, and this means that there is only a half-mesh present at these

the pure imaginary variable $j\omega$, without sufficient motives and without introducing the $\mathfrak{L}$ -transformation, and in order to make this unpalatable complex variable appetizing to the engineer they give it the well-sounding but meaningless name "complex frequency". — As always it requires time for new and better concepts to replace old and insufficient, however beloved, notions. Before the $\mathfrak{L}$ -transformation was introduced into electrical engineering we had to be satisfied with a network theory for $e^{j\omega t}$ -excitations and $e^{j\omega t}$ -responses. However, the $\mathfrak{L}$ -transformation allows entirely arbitrary excitations and responses from the beginning; it supplies the complete solution which includes the transient as well as the steady state. Above all, however, it enables us to discover deep properties of the often complicated solutions from the much simpler $\mathfrak{L}$ -transforms. Examples of this will be shown in Chapters 7 and 8. The complex alternating current calculus is contained in the theory of the $\mathfrak{L}$ -transformation and in it is also its foundation (cf. § 13 No. 3): if the excitations are sinusoidal and all of the same frequency, then this calculus furnishes a certain *part* of the *exact* solution supplied by the $\mathfrak{L}$ -transformation. This part describes the asymptotic behaviour of the solution; expressed technically it represents the steady state. This is, however, only under the assumption that the zeros of the characteristic polynomial (with systems, the determinant $D(s)$ in (18.5)) have negative real parts. If only alternating current and the steady state are of interest, then under this assumption we can safely use the complex alternating current calculation as a shorthand method; this we will do in the further development of this paragraph.

points. Let the number of T -structures equal N , and denote the variable index by n . Let the complex voltage at the output of the n^{th} T -structure and therefore at the input of the $(n+1)^{\text{th}}$ T -structure be E_n , and the corresponding complex current I_n . With the direction arrows as indicated in Fig. 24.1, the following loop equations hold for the left and the right halves of the $(n+1)^{\text{th}}$ T -structure:

$$(24.5) \quad \begin{aligned} (Z_1 + Z_2) I_n - Z_2 I_{n+1} &= E_n \\ Z_2 I_n - (Z_1 + Z_2) I_{n+1} &= E_{n+1} \end{aligned} \quad (n = 0, 1, \dots, N-1).$$

These are $2N$ linear equations for the $2N+2$ values $I_0, \dots, I_N, E_0, \dots, E_N$. When two of the values are given, the other $2N$ can be calculated.

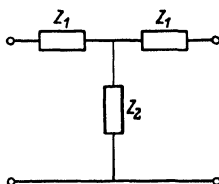


Figure 24.2 T -structure.

Instead of solving by using the usual rules with determinants, we consider only the general equation pair (24.5) and think of it as a system of two difference equations of the first order for the unknown sequences E_n and I_n . We distinguish two problems.

I. Initial Value Problem

Because the equations are of the first order, one initial value must be given for each unknown: these are E_0 and I_0 . The initial value problem corresponds to the practical problem in which the voltage and current are given at the input of the ladder network and are required at the output of an arbitrary member. As in § 21 the sequences E_n, I_n are used to construct the step functions $E(t), I(t)$ and then their image functions $e(s), i(s)$ are found. After using the translation theorem (21.6) the image equations are

$$\begin{aligned} (Z_1 + Z_2) i(s) - Z_2 e^s \left[i(s) - \frac{1 - e^{-s}}{s} I_0 \right] &= e(s) \\ Z_2 i(s) - (Z_1 + Z_2) e^s \left[i(s) - \frac{1 - e^{-s}}{s} I_0 \right] &= e^s \left[e(s) - \frac{1 - e^{-s}}{s} E_0 \right] \end{aligned}$$

or

$$(24.6) \quad e(s) - (Z_1 + Z_2 - Z_2 e^s) i(s) = I_0 Z_2 \frac{e^s - 1}{s}$$

$$e^s e(s) - (Z_2 - (Z_1 + Z_2) e^s) i(s) = E_0 \frac{e^s - 1}{s} + I_0 (Z_1 + Z_2) \frac{e^s - 1}{s}.$$

The determinant of this system of linear equations for $e(s)$ and $i(s)$ is

$$D(s) = Z_2 e^{2s} - 2(Z_1 + Z_2) e^s + Z_2.$$

This plays the same role here as the characteristic polynomial does for a single difference equation. In order to set the coefficient of the highest power equal to 1, when computing $e(s)$ and $i(s)$ we divide numerator and denominator by Z_2 and obtain

$$(24.7) \quad e(s) = \frac{e^s - 1}{s} \frac{E_0 \left[e^s - \left(\frac{Z_1}{Z_2} + 1 \right) \right] - I_0 Z_2 \left[\left(\frac{Z_1}{Z_2} + 1 \right)^2 - 1 \right]}{e^{2s} - 2 \left(\frac{Z_1}{Z_2} + 1 \right) e^s + 1}$$

$$i(s) = \frac{e^s - 1}{s} \frac{-E_0 \frac{1}{Z_2} + I_0 \left[e^s - \left(\frac{Z_1}{Z_2} + 1 \right) \right]}{e^{2s} - 2 \left(\frac{Z_1}{Z_2} + 1 \right) e^s + 1}.$$

As for a single difference equation, in image space these are rational functions of e^s . Following § 21 we set

$$p(z) = z^2 - 2 \left(\frac{Z_1}{Z_2} + 1 \right) z + 1, \quad \frac{1}{p(s)} = q(s) \bullet \bullet Q(t).$$

Then by (21.15) the correspondence

$$\frac{e^s - 1}{s} \frac{1}{p(e^s)} e^{\nu s} \bullet \bullet Q^{(t)+\nu}(0) \quad (\nu = 0, 1)$$

is true. On returning from the image functions (24.7) to the original functions and setting $t = n$ we obtain

$$(24.8) \quad E_n = E_0 \left[Q^{(n+1)}(0) - \left(\frac{Z_1}{Z_2} + 1 \right) Q^{(n)}(0) \right] - I_0 Z_2 \left[\left(\frac{Z_1}{Z_2} + 1 \right)^2 - 1 \right] Q^{(n)}(0)$$

$$I_n = -E_0 \frac{1}{Z_2} Q^{(n)}(0) + I_0 \left[Q^{(n+1)}(0) - \left(\frac{Z_1}{Z_2} + 1 \right) Q^{(n)}(0) \right].$$

We have still to calculate explicitly the values $Q^{(n)}(0)$ and for this, by (11.4), we require the zeros α_1, α_2 of $p(z)$. We can write the formulas in a

form more suitable for numerical calculation, when we notice that α_1, α_2 are connected to the coefficients by

$$\alpha_1 + \alpha_2 = 2 \left(\frac{Z_1}{Z_2} + 1 \right), \quad \alpha_1 \alpha_2 = 1.$$

This suggests setting

$$(24.9) \quad \alpha_1 = e^\tau, \quad \alpha_2 = e^{-\tau}.$$

Then we have

$$e^\tau + e^{-\tau} = 2 \left(\frac{Z_1}{Z_2} + 1 \right),$$

so that τ is defined by

$$(24.10) \quad \cosh \tau = \frac{Z_1}{Z_2} + 1.$$

Thus we find

$$(24.11) \quad Q^{(n)}(0) = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} = \frac{e^{n\tau} - e^{-n\tau}}{e^\tau - e^{-\tau}} = \frac{\sinh n\tau}{\sinh \tau}.$$

For the other coefficients which appear in (24.8) we obtain, since $\sinh(n+1)\tau = \sinh n\tau \cosh \tau + \cosh n\tau \sinh \tau$:

$$Q^{(n+1)}(0) - \left(\frac{Z_1}{Z_2} + 1 \right) Q^{(n)}(0) = \frac{\sinh(n+1)\tau}{\sinh \tau} - \cosh \tau \frac{\sinh n\tau}{\sinh \tau} = \cosh n\tau,$$

$$\left[\left(\frac{Z_1}{Z_2} + 1 \right)^2 - 1 \right] Q^{(n)}(0) = (\cosh^2 \tau - 1) Q^{(n)}(0) = \sinh^2 \tau Q^{(n)}(0) = \sinh \tau \sinh n\tau.$$

The solution (24.8) assumes the simple form

$$(24.12) \quad \begin{aligned} E_n &= E_0 \cosh n\tau - I_0 Z_2 \sinh \tau \sinh n\tau, \\ I_n &= -E_0 \frac{1}{Z_2 \sinh \tau} \sinh n\tau + I_0 \cosh n\tau, \end{aligned}$$

which cannot be found easily from the solution of linear equations (24.5) using determinants.

If we terminate the ladder network at the n^{th} T -structure, E_n and I_n represent the complex output voltage and current. Formulas (24.12) are especially suitable for numerical calculation. $Z_1 = Z_1(j\omega)$ and $Z_2 = Z_2(j\omega)$ are complex, and so also is τ . Extensive tables exist for hyperbolic functions with complex arguments.

We observe in addition that the determinant of the equations (24.12) is equal to 1. This is a well-known general property for a passive four-terminal network.

The above solution refers to the case when the input voltage and current are sinusoidal and only the steady state solution is wanted. If an arbitrary

input voltage and current are given and the complete solution is needed, we must substitute s in equations (24.12) for $j\omega$, and then determine the original functions of the image functions which arise.

II. Boundary Value Problem

We assume that the ladder network consists of N T -structures, where N is a fixed number. It may be that two initial values are not given, but that the initial value is given for one variable and that the end value is given for the same or the other variable; in other words, two boundary values are given, for example, E_0 and E_N , or E_0 and I_N . We restrict ourselves to the physically most important case when the end value is equal to 0.

1. Given: E_0 arbitrary, $E_N = 0$.

This means that the output of the ladder network is short-circuited. We use equations (24.12), in which we introduce the abbreviation

$$Z_2 \sinh \tau = W \quad (= \text{iterative impedance}).$$

In order to eliminate the unknown value I_0 , we set $n = N$ and $E_N = 0$ in the first equation:

$$0 = E_0 \cosh N \tau - I_0 W \sinh N \tau.$$

Inserting the solution

$$I_0 = E_0 \frac{\cosh N \tau}{W \sinh N \tau}$$

into equation (24.12), we obtain

$$E_n = E_0 \cosh n \tau - E_0 \frac{\cosh N \tau}{\sinh N \tau} \sinh n \tau = E_0 \frac{\sinh (N - n) \tau}{\sinh N \tau},$$

$$I_n = -E_0 \frac{1}{W} \sinh n \tau + E_0 \frac{\cosh N \tau}{W \sinh N \tau} \cosh n \tau = E_0 \frac{\cosh (N - n) \tau}{W \sinh N \tau}.$$

2. Given: E_0 arbitrary, $I_N = 0$.

This means that the output of the ladder network is open (no load). If we set $n = N$ and $I_N = 0$ in the second equation (24.12):

$$0 = -E_0 \frac{1}{W} \sinh N \tau + I_0 \cosh N \tau,$$

we obtain

$$I_0 = E_0 \frac{\sinh N \tau}{W \cosh N \tau}.$$

With this value of I_0 , equations (24.12) give

$$E_n = E_0 \cosh n\tau - E_0 \frac{\sinh N\tau}{\cosh N\tau} \sinh n\tau = E_0 \frac{\cosh (N-n)\tau}{\cosh N\tau},$$

$$I_n = -E_0 \frac{1}{W} \sinh n\tau + E_0 \frac{\sinh N\tau}{W \cosh N\tau} \cosh n\tau = E_0 \frac{\sinh (N-n)\tau}{W \cosh N\tau}.$$

§ 25. Solution of Difference Equations by Means of the Discrete Laplace Transformation

In order to be able to apply the $\mathfrak{L}$ -transformation to the sequences appearing in difference equations, we have constructed a stair-cose function $Y(t)$ from the sequence Y_n by definition (21.3). We have

$$(25.1) \quad \mathfrak{L}\{Y(t)\} = \sum_{n=0}^{\infty} \int_n^{n+1} e^{-st} Y_n dt = \sum_{n=0}^{\infty} Y_n \frac{e^{-ns} - e^{-(n+1)s}}{s}$$

$$= \frac{1 - e^{-s}}{s} \sum_{n=0}^{\infty} Y_n e^{-ns}.$$

The factor $(1 - e^{-s})/s$ appears also in the correspondences (21.15) used in the solution of difference equations⁴²). Therefore it is natural to omit it entirely. The remaining infinite series in (25.1) then depends not on the function $Y(t)$ but only on the sequence Y_n . We can therefore associate this infinite series with the sequence Y_n immediately, without introducing the function $Y(t)$, and consider this as a transformation applied to the sequence Y_n . It is called the "*Discrete Laplace Transformation*", and is denoted by the operator symbol $\mathfrak{D}$. It produces a function of s which we denote by $y^*(s)$ ⁴³):

$$(25.2) \quad \mathfrak{D}\{Y_n\} = \sum_{n=0}^{\infty} Y_n e^{-ns} = y^*(s).$$

We call it simply the $\mathfrak{D}$ -transformation and use the same notation as earlier; that is, *original sequence* Y_n and *image function* $y^*(s)$. Also we use the correspondence sign $Y_n \circ \bullet y^*(s)$.

⁴²) To make this evident, we have kept the factor in (21.15) and have not written the correspondence in the more agreeable form

$$\frac{e^s - 1}{s} \frac{e^{\mu s}}{p(e^s)} \circ \bullet Q^{(t) + \mu}(0).$$

Moreover, the factor is interpreted physically on p. 141.

⁴³) A series of the form $\sum c_n e^{-\lambda_n s}$ is called in mathematics a Dirichlet series. $y^*(s)$ corresponds to the special case $\lambda_n = n$.

The $\mathfrak{D}$ -transformation can be described as a $\mathfrak{L}$ -transformation applied to a certain pseudo function. We think of a sequence of impulses which act at the time points $t=n$, and which therefore is represented by $\sum_{n=0}^{\infty} \delta(t-n)$. We modulate this by a time function $Y(t)$; that is, we form

$$(25.3) \quad Y^*(t) = Y(t) \sum_{n=0}^{\infty} \delta(t-n).$$

If we apply the $\mathfrak{L}$ -transformation to this we have by (13.13)

$$\mathfrak{L}\{Y^*\} = \int_0^{\infty} e^{-st} Y(t) \sum_{n=0}^{\infty} \delta(t-n) dt = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-st} Y(t) \delta(t-n) dt = \sum_{n=0}^{\infty} e^{-sn} Y(n).$$

Therefore, the important relation

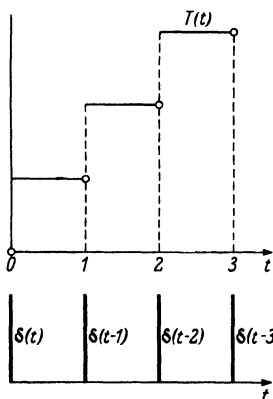
$$(25.4) \quad \boxed{y^*(s) = \mathfrak{L}\{Y^*(t)\} = \mathfrak{D}\{Y(n)\}}$$

is valid for every function $Y(t)$. In this way the $\mathfrak{D}$ -transformation is included in the $\mathfrak{L}$ -transformation.

It should be noted that all functions $Y(t)$ which have the same values $Y(n)$, give the same pseudo function $Y^*(t)$ and the same $\mathfrak{D}$ -transform $\mathfrak{D}\{Y(n)\} = y^*(s)$.

The operation with the pseudo function Y^* can be made rigorous by the Stieltjes integral as in § 13 No. 2. If we introduce a step function $T(t)$ which has the jump 1 at $t = n$

$$T(0) = 0, \quad T(t) = n + 1 \quad \text{for } n < t \leq n + 1 \quad (n = 0, 1, \dots)$$



(see Fig. 25.1), then we have

$$(25.5) \quad \int_0^{\infty} e^{-st} Y(t) dT(t) = \sum_{n=0}^{\infty} e^{-sn} Y(n) = \mathfrak{D}\{Y(n)\}.$$

Fig. 25.1 Stair-case function and the corresponding sequence of impulse functions.

Naturally there are sequences for which $\mathfrak{D}\{Y_n\}$ does not converge for any s . If, however, $\mathfrak{D}\{Y_n\}$ converges anywhere, it converges in a right half-plane.

$y^*(s)$ has period $2\pi j$ because $e^{-n(s+2\pi j)} = e^{-ns}$. Hence, it suffices to consider $y^*(s)$ in the strip $-\pi j < \Im s \leq +\pi j$ (Fig. 25.2).

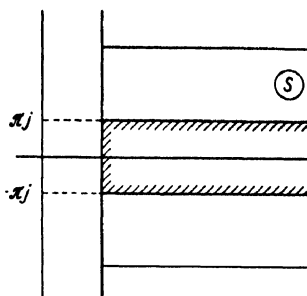


Figure 25.2 Basic region for definition of the periodic function $y^*(s)$.

As an example we calculate (τ a complex constant)

$$\mathfrak{D}\{e^{\tau n}\} = \sum_{n=0}^{\infty} e^{-(s-\tau)n} = \frac{1}{1-e^{-(s-\tau)}} = \frac{e^s}{e^s - e^{\tau}} \quad \text{for } \Re s > \Re \tau.$$

From this we obtain $\mathfrak{D}\{1\}$ by $\tau = 0$, $\mathfrak{D}\{\alpha^n\}$ by $\tau = \log \alpha$, and the $\mathfrak{D}$ -transformations of the hyperbolic and trigonometric functions by the same combinations as for the $\mathfrak{L}$ -transformation in § 3. The short Table 25.1 gives the correspondences which we require in the solution of difference equations.

Table 25.1. The most important correspondences of the $\mathfrak{D}$ -transformation.

	Y_n	$y^*(s) = \mathfrak{D}\{Y_n\}$
1	1	$\frac{e^s}{e^s - 1}$
2	$e^{\tau n}$	$\frac{e^s}{e^s - e^{\tau}}$
3	$\cosh \tau n$	$\frac{e^s (e^s - \cosh \tau)}{e^{2s} - 2e^s \cosh \tau + 1}$
4	$\sinh \tau n$	$\frac{e^s \sinh \tau}{e^{2s} - 2e^s \cosh \tau + 1}$
5	$\cos \tau n$	$\frac{e^s (e^s - \cos \tau)}{e^{2s} - 2e^s \cos \tau + 1}$
6	$\sin \tau n$	$\frac{e^s \sin \tau}{e^{2s} - 2e^s \cos \tau + 1}$
7	α^n	$\frac{e^s}{e^s - \alpha}$

The following basic "grammatical rules" (cf. chap. 2) of the $\mathfrak{D}$ -transformation are needed in applications⁴⁴):

First translation theorem:

$$Y_{n-k} \circ \bullet e^{-ks} y^*(s),$$

if we define $Y_n = 0$ for $n < 0$.

Second translation theorem:

$$Y_{n+k} \circ \bullet e^{ks} \left[y^*(s) - \sum_{r=0}^{k-1} Y_r e^{-rs} \right].$$

Damping theorem:

$$e^{-\lambda n} Y_n \circ \bullet y^*(s + \lambda).$$

Convolution theorem:

Let the original functions $Y_n^{(1)}$ and $Y_n^{(2)}$ correspond to $y_1^*(s)$ and $y_2^*(s)$. Then

$$y_1^*(s) y_2^*(s) \circ \bullet \sum_{r=0}^n Y_r^{(1)} Y_{n-r}^{(2)}.$$

The sum on the right-hand side is the discontinuous analogue to the convolution of two functions and is also called a convolution.

We mention in addition the *inversion formula*

$$(25.6) \quad Y_n = \frac{1}{2\pi j} \int_{x-j\pi}^{x+j\pi} e^{ns} y^*(s) ds \equiv \mathfrak{D}^{-1} \{y^*(s)\},$$

where x is an abscissa in the convergence half-plane of $\mathfrak{D}\{Y_n\}$. It is the analogue to the complex inversion formula (2.8) or (4.2) of the $\mathfrak{L}$ -transformation. For a simple proof see p. 136.

Application of the $\mathfrak{D}$ -transformation to Difference Equations

We consider first a difference equation of second order

$$Y_{n+2} + c_1 Y_{n+1} + c_0 Y_n = F_n$$

with the initial values Y_0, Y_1 . If we apply the $\mathfrak{D}$ -transformation to it, we obtain the image equation by using the second translation theorem:

$$e^{2s} [y^*(s) - Y_0 - Y_1 e^{-s}] + c_1 e^s [y^*(s) - Y_0] + c_0 y^*(s) = f^*(s),$$

⁴⁴) The $\mathfrak{D}$ -transformation is treated in detail in the book by J. S. ZYPKIN. *Differenzengleichungen der Impuls- und Regeltechnik*. VEB Verlag Technik, Berlin 1956, where proofs of the following theorems can be found.

which with

$$p(z) = z^2 + c_1 z + c_0$$

gives as solution

$$(25.7) \quad y^*(s) = \frac{1}{p(e^s)} f^*(s) + Y_0 \frac{e^s(e^s + c_1)}{p(e^s)} + Y_1 \frac{e^s}{p(e^s)}.$$

First we must determine the original sequence of $e^s/p(e^s)$. This is obviously the solution when $F_n = 0$, $Y_0 = 0$, $Y_1 = 1$. When the zeros α_1 , α_2 of $p(z)$ are different, we have

$$\frac{1}{p(z)} = \frac{1}{\alpha_1 - \alpha_2} \left(\frac{1}{z - \alpha_1} - \frac{1}{z - \alpha_2} \right),$$

and therefore

$$\frac{e^s}{p(e^s)} = \frac{1}{\alpha_1 - \alpha_2} \left(\frac{e^s}{e^s - \alpha_1} - \frac{e^s}{e^s - \alpha_2} \right).$$

From Table 25.1, No. 7, we obtain

$$\frac{e^s}{p(e^s)} \rightarrow \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \quad (n = 0, 1, \dots).$$

If we use the abbreviation

$$(25.8) \quad \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} = Q_n,$$

then

$$(25.9) \quad \frac{e^s}{p(e^s)} \rightarrow Q_n.$$

For the inversion of (25.7), we need additional correspondences.

a) Application of the second translation theorem to (25.9) gives

$$Q_{n+1} \rightarrow e^s \left[\frac{e^s}{p(e^s)} - Q_0 \right].$$

Because $Q_0 = 0$, this reduces to

$$(25.10) \quad e^s \frac{e^s}{p(e^s)} \rightarrow Q_{n+1} \quad (n = 0, 1, \dots).$$

b) Application of the first translation theorem to (25.9) gives

$$Q_{n-1} \rightarrow e^{-s} \frac{e^s}{p(e^s)} \quad (n = 0, 1, \dots),$$

if Q_n is put equal to 0 for $n < 0$, therefore $Q_{-1} = 0$. Since also $Q_0 = 0$, we have

$$(25.11) \quad \frac{1}{p(e^s)} \rightarrow \begin{cases} 0 & \text{for } n = 0, 1 \\ Q_{n-1} & \text{for } n \geq 2. \end{cases}$$

If we apply the convolution theorem to the first term in (25.7), we obtain finally

$$(25.12) \quad Y_n = \sum_{r=2}^n Q_{r-1} F_{n-r} + Y_0(Q_{n+1} + c_1 Q_n) + Y_1 Q_n \quad (n \geq 2).$$

The derivation of this solution is exactly parallel to that in § 21 and § 22 using the $\mathfrak{L}$ -transformation. The correspondences (25.9, 10) check with (21.15) when the factor $(1 - e^{-s})/s$ is omitted and $Q^{([j]+k)}(0) = Q^{(n+k)}(0)$ is replaced by Q_{n+k} . The first term in (25.12) can now be obtained in a simpler way, because the convolution theorem for sequences is applicable and this was not the case with functions.

From this special case we see how the method can be applied to a difference equation of the r^{th} order. The solution in image space, apart from the factor $(1 - e^{-s})/s$, looks exactly like the image functions (21.8) or (22.5) which were found using the $\mathfrak{L}$ -transformation. First $e^s/p(e^s)$ is expanded into partial fractions and, when $p(z)$ has multiple zeros, terms occur of the form

$$\frac{e^s}{e^s - \alpha}, \quad \frac{e^s}{(e^s - \alpha)^2}, \quad \dots$$

From the correspondence

$$\frac{e^s}{e^s - \alpha} \bullet \alpha^n$$

we can obtain the others by differentiation with respect to α :

$$(25.13) \quad \frac{e^s}{(e^s - \alpha)^2} \bullet n \alpha^{n-1}$$

$$(25.14) \quad 2 \frac{e^s}{(e^s - \alpha)^3} \bullet n(n-1) \alpha^{n-2}$$

etc. When the original sequence Q_n of $e^s/p(e^s)$ has been found, we obtain, by the translation theorem,

$$(25.15) \quad \frac{e^{ks}}{p(e^s)} \bullet Q_{n+k-1} \quad (k = 1, \dots, r)$$

$$(25.16) \quad \frac{1}{p(e^s)} \bullet \begin{cases} 0 & \text{for } n = 0, 1, \dots, r-1 \\ Q_{n-1} & \text{for } n \geq r. \end{cases}$$

From these correspondences and the convolution theorem we can determine the original function of the image function and obtain the same formulas as (21.17) and (22.8).

With the above difference equation of the second order we have transformed to the original space by a partial fraction expansion because this method is applicable to equations of every order r . However, a shorter method can be used for equations of the second order; this we illustrate

using the example of two simultaneous equations of first order considered in § 24. (For each unknown we could derive one equation of the second order from these.) When the $\mathfrak{D}$ -transformation is applied, image functions arise which differ from (24.7) only by the factor $(1 - e^s)/s$ and hence are of the form (it is sufficient to consider one of the two functions)

$$e^*(s) = \frac{E_0 e^s \left[e^s - \left(\frac{Z_1}{Z_2} + 1 \right) \right] - I_0 Z_2 e^s \left[\left(\frac{Z_1}{Z_2} + 1 \right)^2 - 1 \right]}{e^{2s} - 2 \left(\frac{Z_1}{Z_2} + 1 \right) e^s + 1}.$$

The individual terms have the form of the image functions in Table 25.1, Nos. 3, 4, if we set

$$\frac{Z_1}{Z_2} + 1 = \cosh \tau$$

and therefore

$$\left(\frac{Z_1}{Z_2} + 1 \right)^2 - 1 = \sinh^2 \tau.$$

We can now write down immediately the original sequences using Table 25.1 and obtain the same expressions as in (24.12).

In this example the constant term in the characteristic polynomial $p(z)$ was equal to 1. When this is not so, we use the correspondences which are obtained by applying the damping theorem to Nos. 3, 4 of Table 25.1:

$$(25.17) \quad e^{\lambda n} \cosh \tau n \circ \bullet \frac{e^s (e^s - e^\lambda \cosh \tau)}{e^{2s} - 2 e^s e^\lambda \cosh \tau + e^{2\lambda}},$$

$$(25.18) \quad e^{\lambda n} \sinh \tau n \circ \bullet \frac{e^\lambda e^s \sinh \tau}{e^{2s} - 2 e^s e^\lambda \cosh \tau + e^{2\lambda}}.$$

§ 26. Solution of Difference Equations by Means of the Laurent or $\mathfrak{Z}$ -Transformation

In the definition of $\mathfrak{D}$ -transformations as well as in their "grammatical rules" the variable s entered only in the combination e^s . Therefore we can set e^s equal to a new variable (for this see footnote 46, p. 137)

$$e^s = z.$$

Then the original sequence Y_n is associated with the image function $\sum_{n=-\infty}^{\infty} Y_n z^{-n}$ which we denote by $\tilde{y}(z)$. In mathematics a series of the form $\sum_{n=-\infty}^{+\infty} a_n z^n$ is called a Laurent series. $\tilde{y}(z)$ is therefore a Laurent series from which the powers with positive exponents are absent and $a_{-n} = Y_n$.

Hence we call this association of $\tilde{y}(z)$ with Y_n the “*Laurent transformation*”. Since this transformation has occurred in technical literature (from about 1950), it has been customary to call it the “*z-transformation*”. This is not in accordance with the general custom of associating transformations which appear frequently with the name of a mathematician. The custom of calling a transformation by its image variable has little to recommend it because this variable is not always denoted by the same letter (for example, with the Fourier transformation we sometimes use y , and sometimes ω). Since we cannot use the initial letter $\mathfrak{L}$ for the Laurent transformation, because it has been used already for the Laplace transformation, we make an exception to the usual practice and use the symbol $\mathfrak{L}$ for the Laurent transformation:

$$(26.1) \quad \tilde{y}(z) = \sum_{n=-\infty}^{\infty} Y_n z^{-n} \equiv \mathfrak{L}\{Y_n\},$$

and we use the abbreviation “ $\mathfrak{L}$ -transformation”.

We have

$$(26.2) \quad \tilde{y}(e^s) = y^*(s) \quad \text{or} \quad \tilde{y}(z) = y^*(\log z).$$

When we work exclusively with the $\mathfrak{L}$ -transformation, we can write $y(z)$ instead of $\tilde{y}(z)$ because no confusion can arise with the $\mathfrak{L}$ -transform $y(s)$. Since, however, in this book we are considering the $\mathfrak{L}$ -, $\mathfrak{D}$ -, and $\mathfrak{L}$ -transformations side by side, we must use the more exact notation $\tilde{y}(z)$.

$\mathfrak{D}\{Y_n\} = y^*(s)$ had period $2\pi j$ and therefore had to be considered only in that part of the strip $-\pi j < \Im s \leq +\pi j$ which lies to the right of the convergence line $\Re s = \beta$. Under the mapping $z = e^s$ this region becomes the outside of a circle with the radius $R = e^\beta$. $y(z)$ converges therefore, if anywhere, for $|z| > R$ (Fig. 26.1). This agrees with the well-known fact that the convergence domain of a series with descending powers is the outside of a circle.

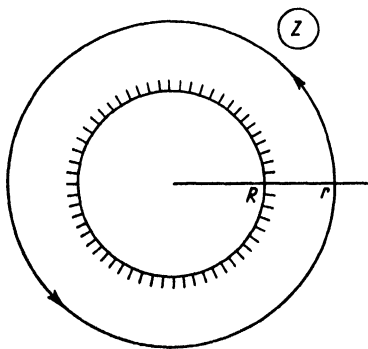


Figure 26.1 Domain of convergence for $\tilde{y}(z)$.

Using the substitution $e^s = z$ we transform the "grammatical rules" of the $\mathfrak{D}$ -transformation into rules for the $\mathfrak{Z}$ -transformation (which can easily be proved directly) and obtain also Table 26.1 of correspondences arising from Table 25.1.

First translation theorem:

$$Y_{n-k} \circ \bullet z^{-k} \tilde{y}(z),$$

when we define $Y_n = 0$ for $n < 0$.

Second translation theorem:

$$Y_{n+k} \circ \bullet z^k \left[\tilde{y}(z) - \sum_{\nu=0}^{k-1} Y_{\nu} z^{-\nu} \right].$$

Damping theorem:

$$\kappa^{-n} Y_n \circ \bullet \tilde{y}(\kappa z).$$

Convolution theorem:

$$\tilde{y}_1(z) \tilde{y}_2(z) \bullet \circ \sum_{\nu=0}^n Y_{\nu}^{(1)} Y_{n-\nu}^{(2)}.$$

Table 26.1. The most important correspondences of the $\mathfrak{Z}$ -transformation.

	Y_n	$\tilde{y}(z) = \mathfrak{Z}\{Y_n\}$
1	1	$\frac{z}{z-1}$
2	$e^{\tau n}$	$\frac{z}{z-e^{\tau}}$
3	$\cosh \tau n$	$\frac{z(z - \cosh \tau)}{z^2 - 2z \cosh \tau + 1}$
4	$\sinh \tau n$	$\frac{z \sinh \tau}{z^2 - 2z \cosh \tau + 1}$
5	$\cos \tau n$	$\frac{z(z - \cos \tau)}{z^2 - 2z \cos \tau + 1}$
6	$\sin \tau n$	$\frac{z \sin \tau}{z^2 - 2z \cos \tau + 1}$
7	a^n	$\frac{z}{z-a}$

The Inversion Formula for the $\mathfrak{Z}$ -transformation: By the Cauchy formula the coefficients in a Laurent series are given by

$$(26.3) \quad Y_n = \frac{1}{2\pi j} \oint z^{n-1} \tilde{y}(z) dz = \mathfrak{Z}^{-1}\{\tilde{y}(z)\},$$

where the contour for the complex integral is a circle with the radius $r > R$ (Fig. 26.1). By the substitution $z = r e^{j\varphi}$ we obtain an integral extending over the real interval $(-\pi, +\pi)$:

$$(26.4) \quad Y_n = \frac{r^n}{2\pi} \int_{-\pi}^{+\pi} e^{jn\varphi} \tilde{y}(r e^{j\varphi}) d\varphi.$$

Setting $z = e^s$ and $r = e^w$ in (26.3), we obtain the inversion formula (25.6) of the $\mathfrak{D}$ -transformation.

The inversion $\mathfrak{Z}^{-1}$ of the $\mathfrak{Z}$ -transformation is unique; that is to say, only one original sequence Y_n corresponds to a particular image function $y(z)$. This is mentioned, because in technical literature the contrary is sometimes maintained⁴⁵⁾.

Application of the $\mathfrak{Z}$ -transformation to Difference Equations

We require only to repeat the steps made on p. 130ff. and set $e^s = z$. From the difference equation

$$Y_{n+2} + c_1 Y_{n+1} + c_0 Y_n = F_n$$

with the initial values Y_0, Y_1 , by applying the $\mathfrak{Z}$ -transformation and using the second translation theorem we obtain the image equation

$$z^2 [\tilde{y}(z) - Y_0 - Y_1 z^{-1}] + c_1 z [\tilde{y}(z) - Y_0] + c_0 \tilde{y}(z) = \tilde{f}(z),$$

⁴⁵⁾ In the book by J. G. TRUXAL: *Automatic Feedback Control System Synthesis*, McGraw-Hill Book Co., New York, 1955, p. 512, it is stated that the image $z/(z-1)$ corresponds not only to the sequence $Y_n = 1$, but also to the impulse sequence $d(t) = \sum_{n=0}^{\infty} \delta(t-n)$. This erroneous statement has arisen by identifying two different transformations: $\mathfrak{Z}\{1\} = z/(z-1) = \tilde{y}(z)$ is equivalent to $\mathfrak{D}\{1\} = e^s/(e^s-1) = y^*(s)$. This $\mathfrak{D}$ -transform can be interpreted by (25.4) as a $\mathfrak{L}$ -transform of the impulse sequence $d(t)$ modulated by the function 1:

$$\mathfrak{L}\{d(t)\} = \sum_{n=0}^{\infty} e^{-ns} = \frac{1}{1-e^{-s}} = \frac{e^s}{e^s-1} = y(s).$$

We can then say only that the $\mathfrak{Z}$ -transform $\mathfrak{Z}\{1\} = z/(z-1)$ of the unit sequence can be interpreted when we substitute $z = e^s$ as a $\mathfrak{L}$ -transform $\mathfrak{L}\{d(t)\}$ of the impulse sequence. From this, however, no ambiguity arises in the $\mathfrak{Z}^{-1}$ -transformation: it signifies only that the image function in its two different forms $z/(z-1)$ and $e^s/(e^s-1)$ corresponds, in the domain of two different transformations, to two different original functions.

from which

$$(26.5) \quad \tilde{y}(z) = \frac{1}{p(z)} \tilde{f}(z) + Y_0 \frac{z(z+c_1)}{p(z)} + Y_1 \frac{z}{p(z)},$$

where $p(z) = z^2 + c_1 z + c_0$ ⁴⁶. The original sequence of $z/p(z)$ must be determined first. For $\alpha_1 \neq \alpha_2$ we have

$$\frac{z}{p(z)} = \frac{1}{\alpha_1 - \alpha_2} \left(\frac{z}{z - \alpha_1} - \frac{z}{z - \alpha_2} \right),$$

and correspondence No. 7, Table 26.1, gives

$$(26.6) \quad \frac{z}{p(z)} \bullet \circ \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} = Q_n.$$

Since $Q_0 = 0$, the second translation theorem gives

$$(26.7) \quad z \frac{z}{p(z)} \bullet \circ Q_{n+1},$$

and from the first translation theorem we have

$$\frac{1}{p(z)} = z^{-1} \frac{z}{p(z)} \bullet \circ Q_{n-1},$$

when $Q_{-1} = 0$; therefore

$$(26.8) \quad \frac{1}{p(z)} \bullet \circ \begin{cases} 0 & \text{for } n = 0, 1 \\ Q_{n-1} & \text{for } n \geq 2. \end{cases}$$

From these correspondences and the convolution theorem we obtain, by inverting (26.5), the same solution as found previously in (25.12).

The method of partial fraction expansion employed here is applicable also to difference equations of higher order. For equations of second order we can obtain the solution quicker by using correspondences Nos. 3 and 4 in Table 26.1. The reader may compare this with the corresponding calculation using the $\mathfrak{D}$ -transformation in § 25.

⁴⁶) It is clear at this point why we have set $z = e^s$ and not $z = e^{-s}$ in the $\mathfrak{D}$ -transformation. If we had used the latter we should have obtained the series $\sum_{n=0}^{\infty} Y_n z^n = \eta(z)$ which appears initially to be simpler. It has positive exponents and can be suitably called the "Taylor transformation" $\mathfrak{T}\{Y_n\}$. For this transformation the second translation theorem is

$$Y_{n+k} \bullet \circ z^{-k} \left[\eta(z) - \sum_{\nu=0}^{k-1} Y_{\nu} z^{\nu} \right],$$

which gives as the coefficient of $\eta(z)$ in the image equation of the difference equation

$$z^{-2} + c_1 z^{-1} + c_0 = p(z^{-1}).$$

This is not a characteristic polynomial, but a rational function: the calculus is more involved than that for the $\mathfrak{B}$ -transformation because z^{-1} must be substituted everywhere for z .

We see that the methods of the $\mathfrak{L}$ -, $\mathfrak{D}$ -, and $\mathfrak{Z}$ -transformations are equivalent, but that there is less writing when the method of the $\mathfrak{Z}$ -transformation is used: compared with the $\mathfrak{L}$ -transformation, in the $\mathfrak{D}$ -transformation we do not need to carry the factor $(1 - e^{-s})/s$ and in the $\mathfrak{Z}$ -transformation we write simply the letter z instead of the expression e^s which appears in the $\mathfrak{D}$ -transformation. The disadvantage of the $\mathfrak{D}$ - and $\mathfrak{Z}$ -transformations is that they can be used only to solve difference equations but not differential equations, whereas the $\mathfrak{L}$ -transformation can be used for both types of functional equations (and also integral equations, see § 32) so that no new "grammatical rules" and correspondence tables are required.

§ 27. The Pulse-former and its Description by $\mathfrak{L}$ -, $\mathfrak{D}$ -, and $\mathfrak{Z}$ -transformation

A pulse-former is a sampling device which converts a continuous input function $F(t)$ to a step function in the following way. The value of $F(t)$ is periodically sampled at discrete time points nT ($n = 0, 1, \dots$), multiplied by a constant k (amplified) and then held at that value during a time span ϑ ($0 < \vartheta \leq T$); in the remaining time intervals $F(t)$ is switched off so that the functional value is zero. The output function of the pulse-former is therefore the stair-case function (Fig. 27.1)

$$(27.1) \quad \bar{F}(t) = \begin{cases} k F(nT) & \text{for } nT \leq t < nT + \vartheta \\ 0 & \text{for } nT + \vartheta \leq t < (n+1)T \end{cases}$$

(for $\vartheta = T$ the second row does not exist). We call T the pulse period, ϑ the pulse length and k the amplifying factor of the pulse former.

Using the unit step function, $\bar{F}(t)$ can be written in the following form

$$(27.2) \quad \bar{F}(t) = k \sum_{n=0}^{\infty} F(nT) [U(t - nT) - U(t - nT - \vartheta)].$$

The $\mathfrak{L}$ -transform of $\bar{F}(t)$ is

$$\begin{aligned} \bar{f}(s) &= \sum_{n=0}^{\infty} \int_{nT}^{nT+\vartheta} e^{-st} \bar{F}(t) dt \\ &= k \sum_{n=0}^{\infty} F(nT) \int_{nT}^{nT+\vartheta} e^{-st} dt = k \sum_{n=0}^{\infty} F(nT) \frac{e^{-nTs} - e^{-(nT+\vartheta)s}}{s}, \end{aligned}$$

and therefore

$$(27.3) \quad \bar{f}(s) = k \frac{1 - e^{-\vartheta s}}{s} \sum_{n=0}^{\infty} F(nT) e^{-nTs}.$$

For $k = 1$, $T = 1$, $\vartheta = 1$, we have the stair-case function used with difference equations.

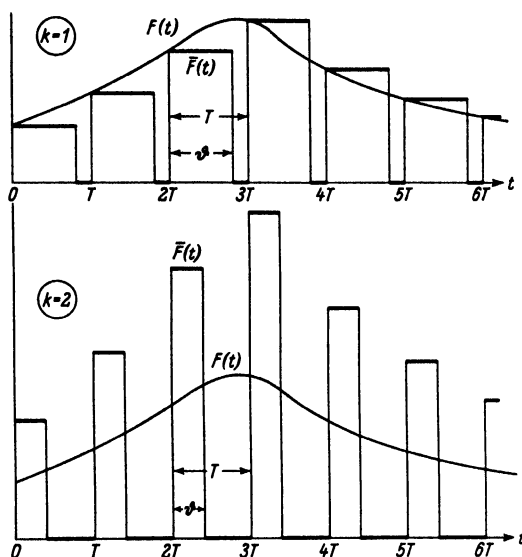


Figure 27.1 Conversion of a continuous function into a stair-case function by a periodic sampler.

The area between $\bar{F}(t)$ and the t -axis consists of a sequence of rectangles. If k and ϑ are chosen so that $k\vartheta = 1$, then we have that the area of a single rectangle is

$$k F(nT) \vartheta = F(nT),$$

so that the sampled value $F(nT)$ is represented also by the area of the rectangle. It is helpful for understanding the following to keep this interpretation in mind.

In practice ϑ is generally chosen small and k large. In order to make the mathematical treatment independent of the choice of ϑ and k , we idealize the preceding relations by passing over to the limit $\vartheta \rightarrow 0$, and therefore $k \rightarrow \infty$. This limiting form of a pulse is called an *impulse*. The limit function $\lim \bar{F}(t)$ is then equal to ∞ at the points nT and otherwise equal to 0, but the area in the neighbourhood of $t = nT$ keeps its previous value $F(nT)$; mathematically this states that the integral of the limit function shall increase by the value $F(nT)$ on passing through the point nT . We see that $\lim \bar{F}(t)$ is the same as the pseudo function $F^*(t)$ which has been introduced already in § 25 (called there Y^*), and which arises through modulation of the impulse sequence $\sum_{n=0}^{\infty} \delta(t - nT)$ with the function $F(t)$ (in § 25, T was equal to 1);

$$(27.4) \quad \lim_{\vartheta \rightarrow 0, k \rightarrow \infty} \bar{F}(t) = F^*(t) = F(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_{n=0}^{\infty} F(nT) \delta(t - nT).$$

This representation expresses that the "sampled-data" of the function $F(t)$ are taken at the points nT .

Recently, the case has been considered in which the functional value $F(nT)$ is not held constant during the time interval ϑ but modulated by a function $\Psi(t - nT)$ (Fig. 27.2).

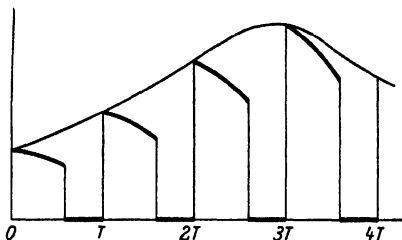


Figure 27.2 Pulse formed by a modulating function.

The $\mathfrak{L}$ -transform of the pseudo function $F^*(t)$ is, by (13.13),

$$(27.5) \quad \mathfrak{L}\{F^*(t)\} = f^*(s) = \sum_{n=0}^{\infty} F(nT) e^{-nTs},$$

and therefore the $\mathfrak{D}$ -transform of the sequence $F(nT)$, with the generalization that now the interval 1 is replaced by T , is

$$(27.6) \quad \boxed{f^*(s) = \mathfrak{L}\{F^*(t)\} = \mathfrak{D}\{F(nT)\}}$$

with

$$(27.7) \quad \mathfrak{D}\{F(nT)\} = \sum_{n=0}^{\infty} F(nT) e^{-nTs}.$$

The same result is obtained by passing over to the limit in (27.3). Since $k\vartheta = 1$ and

$$k \frac{1 - e^{-\vartheta s}}{s} = k\vartheta \frac{1 - e^{-\vartheta s}}{\vartheta s} \rightarrow 1 \quad \text{for } \vartheta \rightarrow 0$$

we have

$$(27.8) \quad \lim_{\vartheta \rightarrow 0, k \rightarrow \infty} \bar{f}(s) = \sum_{n=0}^{\infty} F(nT) e^{-nTs} = f^*(s).$$

The $\mathfrak{D}$ - (and consequently also the $\mathfrak{Z}$ -) transformation is therefore not only useful for the treatment of difference equations, but also suitable for pulse formers when the pulse length can be considered to be vanishingly small.

From $f^*(s)$ we can obtain $F(nT)$ again by the *inversion of the $\mathfrak{D}$ -transformation* (generalization of formula (25.6))

$$(27.9) \quad F(nT) = \frac{T}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} f^*(s) ds \equiv \mathfrak{D}^{-1}\{f^*(s)\},$$

where x is an abscissa in the convergence half-plane of $\mathfrak{D}\{F(nT)\}$.

The periodic sampling mechanism which produces the instantaneous impulse gives the output function $F^*(t)$; the mechanism that produces the rectangular pulse gives $\bar{F}(t)$. How are these two functions connected? In image space this connection is very simple; it is

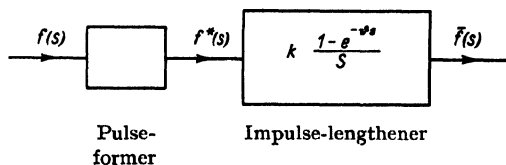
$$(27.10) \quad \bar{f}(s) = k \frac{1 - e^{-\theta s}}{s} f^*(s).$$

We can think of the factor of $f^*(s)$,

$$(27.11) \quad q_\theta(s) = k \frac{1 - e^{-\theta s}}{s},$$

as a system function $q_\theta(s)$ of a network which is inserted between the input $F^*(t)$ and the output $\bar{F}(t)$ (See the block description in Fig. 27.3). Because

Figure 27.3 Conversion of impulse into a pulse.



this network holds the functional value $F(nT)$ supplied by $F^*(t)$ during the interval θ , it is called a hold circuit or an *impulse lengthener*. The original function of $q_\theta(s)$ is the weight function $Q_\theta(t)$ of the network which is given by

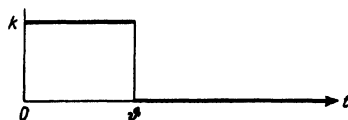
$$(27.12) \quad Q_\theta(t) = k [U(t) - U(t - \theta)]$$

and has the form shown in Fig. 27.4. From the convolution theorem, we obtain

$$(27.13) \quad \bar{F}(t) = Q_\theta(t) * F^*(t),$$

and thus the connection between $\bar{F}$ and F^* is established. If we calculate the convolution integral explicitly, we obtain the description (27.2) for $\bar{F}(t)$. Because $q(s)$ is not a rational function, the hold circuit can be realized only approximately by a network.

Figure 27.4 Weight function for the hold circuit.



Because $F^*(s)$ can be calculated from $F(t)$ (but not conversely), it must be possible to calculate $f^*(s)$ from $f(s)$. Before we do this, we go one step further and take the sampled-data from $F(t)$ not only at the time points nT , but also at equidistant points which lie between; at the points $nT + \tau$, where τ is fixed and $0 \leq \tau < T$. Every fixed τ gives a sequence $F(nT + \tau)$; the totality of these sequences with variable τ represent all functional values of $F(t)$. With the sequence $F(nT + \tau)$ we associate the generalized $\mathfrak{D}$ -transform

$$(27.14) \quad f^*(s, \tau) = \sum_{n=0}^{\infty} F(nT + \tau) e^{-(nT + \tau)s}.$$

The $\mathfrak{D}$ -transforms $f^*(s)$ and $f^*(s, \tau)$ can now be expressed by the $\mathfrak{L}$ -transform $f(s)$ as follows:

$$(27.15) \quad f^*(s) = \frac{F(0)}{2} + \frac{1}{T} \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right)$$

$$(27.16) \quad f^*(s, \tau) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right) e^{j m \frac{2\pi}{T} \tau} \quad \text{for } 0 < \tau < T$$

In the literature a number of different proofs are given for these formulas, but generally no conditions of validity are given and in part the proofs are mathematically meaningless. An especially simple proof which at the same time clarifies the essential features of these formulas is the following. From the inversion formula (2.8) of the $\mathfrak{L}$ -transformation, we have

$$\begin{aligned} (27.17) \quad F(nT) &= \frac{1}{2\pi j} \int_{s-j\infty}^{s+j\infty} e^{nTs} f(s) ds && \text{for } n \geq 1 \\ &= \frac{1}{2\pi j} \sum_{m=-\infty}^{+\infty} \int_{s+j(2m-1)\frac{\pi}{T}}^{s+j(2m+1)\frac{\pi}{T}} e^{nTs} f(s) ds && \left(s = \sigma + j m \frac{2\pi}{T}\right) \\ &= \frac{1}{2\pi j} \sum_{m=-\infty}^{+\infty} \int_{s-j\frac{\pi}{T}}^{s+j\frac{\pi}{T}} e^{nT\sigma} e^{nm2\pi j} f\left(\sigma + j m \frac{2\pi}{T}\right) d\sigma && (e^{nm2\pi j} = 1) \\ &= \frac{1}{2\pi j} \sum_{m=-\infty}^{+\infty} \int_{s-j\frac{\pi}{T}}^{s+j\frac{\pi}{T}} e^{nTs} f\left(s + j m \frac{2\pi}{T}\right) ds. \end{aligned}$$

If the series

$$(27.18) \quad \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right)$$

is uniformly convergent in the interval $\left(x - j \frac{\pi}{T}, x + j \frac{\pi}{T}\right)$, we can interchange the sum and the integral and obtain

$$F(nT) = \frac{1}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right) ds.$$

This is valid for $n \geq 1$. By footnote 8, p. 24 the inversion formula for $t = 0$ does not describe $F(0)$ but $F(0)/2$. We have therefore

$$(27.19) \quad \frac{1}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right) ds = \begin{cases} \frac{F(0)}{2} & \text{for } n = 0 \\ F(nT) & \text{for } n \geq 1. \end{cases}$$

Now ($s = x + jy$)

$$(27.20) \quad \frac{1}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} T \frac{F(0)}{2} ds = \frac{T}{2\pi} e^{nTx} \frac{F(0)}{2} \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} e^{jnTy} dy \\ = \begin{cases} \frac{F(0)}{2} & \text{for } n = 0 \\ 0 & \text{for } n \geq 1. \end{cases}$$

If we add equations (27.19, 20), the following equation, which is valid for all $n \geq 0$, is obtained

$$(27.21) \quad \frac{1}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} \left\{ T \frac{F(0)}{2} + \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right) \right\} ds = F(nT).$$

We compare this with the inversion formula (27.9):

$$\frac{T}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} f^*(s) ds = F(nT).$$

In both equations the left-hand side describes a Fourier coefficient; for example ($s = x + jy$),

$$(27.22) \quad \frac{T}{2\pi j} \int_{x-j\frac{\pi}{T}}^{x+j\frac{\pi}{T}} e^{nTs} f^*(s) ds = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} e^{jnTy} [e^{nTx} f^*(x + jy)] dy.$$

From the uniqueness theorem for Fourier series⁴⁷⁾ it follows that if the coefficients are equal the functions are equal; therefore we have

$$T f^*(s) = T \frac{F(0)}{2} + \sum_{m=-\infty}^{+\infty} f\left(s + j m \frac{2\pi}{T}\right).$$

This is equation (27.15). Equation (27.16) is obtained in the same way. However, with it there is no complication for $n = 0$ because the value of $F(0)$ does not appear⁴⁸⁾.

Because we can consider (27.17) to be a representation of $F(nT)$ as a Fourier Transform, this proof means that, for integral values of the argument, we can represent the Fourier Transform of a function as the Fourier coefficient of another function.

Reconstruction of a Time Function out of the Sampled-Data

The *sampling theorem* of SHANNON plays an important role in information theory:

If the spectrum of a signal (a time function) $F(t)$ contains only frequencies in a finite band $-b \leq \gamma \leq +b$, that is, the spectral density is⁴⁹⁾

$$(27.23) \quad f(\gamma) = \int_{-\infty}^{+\infty} e^{-j\gamma t} F(t) dt = 0 \quad \text{for } |\gamma| > b,$$

then it is sufficient to sample $F(t)$ with the pulse period

$$T = \pi/b$$

to obtain again the complete signal $F(t)$ from the sampled values $F(nT)$ ($n = 0, \pm 1, \pm 2, \dots$).

⁴⁷⁾ If two continuous functions have the same Fourier coefficients then they are identical. $f^*(s)$ is continuous as a power series in e^s . The series (27.18) is assumed to be uniformly convergent and, because the terms are continuous functions, it represents also a continuous function.

⁴⁸⁾ The above proof uses an assumption about the series (27.18) and therefore about $f(s)$. If the Poisson summation formula is employed in the proof, then the following assumptions must be made about $F(t)$: $F(t)$ has a bounded derivative in every finite interval, $\Re \{ |F'| \}$ exists. See G. DOERSCH: Der Zusammenhang zwischen den Laplace-Transformierten einer Funktion und der zugeordneten Treppenfunktion. *Regelungstechnik* 5 (1957), pp. 86—88.

⁴⁹⁾ Because of what follows we carry over the factor $1/2\pi$ from the integral for f to that for F (as in § 2 when passing from the Fourier to the Laplace integral).

This theorem can be proved formally (i. e. without conditions on the validity of the operations) in a very simple way. By (1.10) we have

$$(27.24) \quad F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{jtv} f(y) dy = \frac{1}{2\pi} \int_{-b}^{+b} e^{jtv} f(y) dy,$$

therefore

$$(27.25) \quad F(nT) = \frac{1}{2\pi} \int_{-b}^{+b} e^{jnTv} f(y) dy = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} e^{jnTv} f(y) dy.$$

$TF(nT)$ is the complex Fourier coefficient (cf. § 1) of $f(y)$ in the interval $(-\pi/T, +\pi/T)$, and therefore $f(y)$ is represented by the Fourier series

$$(27.26) \quad f(y) = T \sum_{n=-\infty}^{+\infty} F(nT) e^{-jnTv}.$$

Thus $F(T)$ is also known by (27.24):

$$\begin{aligned} F(t) &= \frac{T}{2\pi} \int_{-b}^{+b} e^{jtv} \sum_{n=-\infty}^{+\infty} F(nT) e^{-jnTv} dy \\ &= \frac{T}{2\pi} \sum_{n=-\infty}^{+\infty} F(nT) \int_{-b}^{+b} e^{j(t-nT)v} dy \\ (27.27) \quad &= \frac{T}{\pi} \sum_{n=-\infty}^{+\infty} F(nT) \frac{\sin b(t-nT)}{t-nT}. \end{aligned}$$

Therefore the complete function $F(t)$ can be constructed from the sampled-data $F(nT)$ by a so-called "cardinal series". We observe that the proof is correct only when t and n extend from $-\infty$ to $+\infty$.

In technical literature⁵⁰⁾ the proof of the sampling theorem is given sometimes as a consequence of formula (27.15). If we assume that the imaginary axis is included in the domain of convergence of $f(s)$ and that $F(0) = 0$, then with

$$\frac{2\pi}{T} = \omega = \text{circular frequency of sampling,}$$

we have

$$f^*(jy) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} f(j(y + m\omega)),$$

and we see that $f^*(jy)$ is represented as the superposition of the spectral densities $f(j(y + m\omega))/T$. (The present $f(jy)$ was denoted by $f(y)$ in the above proof.) When $f(jy)$ is different from 0 only for $-b \leq y \leq +b$, then

⁵⁰⁾ See for example, the book of TRUXAL quoted in footnote 45, p. 504, and H. KAUFMANN: *Dynamische Vorgänge in linearen Systemen der Nachrichten- und Regelungstechnik*. Oldenbourg Verlag, 1959, p. 160.

the functions $f(j(y + m\omega))$ are different from 0 only in $-m\omega - b \leq y \leq -m\omega + b$ (Fig. 27.5). If $\omega = 2b$ (Fig. b), the curves of these functions lie exactly side by side; for $\omega > 2b$ (Fig. a) they disengage from one another; for $\omega < 2b$ (Fig. c) they overlap on themselves and give, by superposition, the heavily drawn curve, in which it is no longer possible to recognize the true form of $f(jy)$.

It is then concluded that in the case $\omega = 2b$, that is, as above in the sampling theorem for $T = \pi/b$ (and *a fortiori* for $T < \pi/b$), the spectral

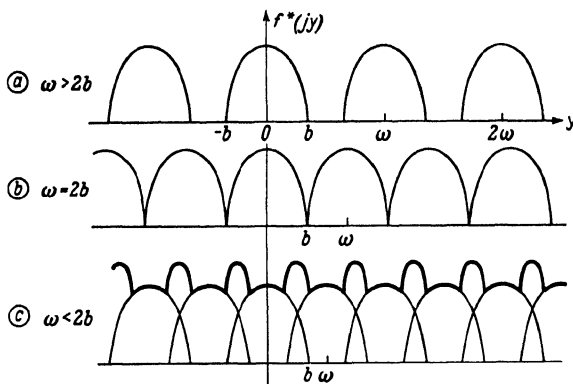


Figure 27.5 Representation of $f^*(jy)$ for a time function with finite frequency band $|y| \leq b$ when sampled at frequency ω .

density $f(j\omega)$ can be filtered out of the "spectral density" $f^*(s)$ by a low pass filter with bandwidth $b = \omega/2$. When $f(j\omega)$ is known, then $F(t)$ also is known.

This alleged proof for the sampling theorem is based upon several erroneous assumptions. 1) If a signal consists only of oscillations with frequencies $|y| \leq b$, and therefore is represented by the integral (27.24) with finite limits, then $F(t)$ is necessarily a function which is analytic for every t in $(-\infty, +\infty)$ (an entire function). Therefore it cannot vanish for $t < 0$ ⁵¹⁾ (and be arbitrary for $t \geq 0$) as was assumed above in the derivation of (27.15) where the $\mathfrak{L}$ -integral is formed for $t \geq 0$ and the data $F(nT)$ are

⁵¹⁾ This follows also from a theorem of N. WIENER: "A necessary and sufficient condition that a time function $F(t)$ which vanishes for $t < 0$ exists with a spectral density $f(jy)$ is that the integral

$$\int_{-\infty}^{+\infty} \frac{|\log |f(jy)||}{1+y^2} dy$$

converges." Therefore a function $f(jy)$ which vanishes in an interval, so that there $|\log |f(jy)|| = \infty$, cannot be the spectral density of a time function which vanishes for $t < 0$.

sampled only for $n \geq 0$. 2) It was tacitly assumed that $f^*(s)$ is a spectral density in the same sense as $f(s)$. However in formula (27.9), which for $\alpha = 0$ has the form

$$(27.28) \quad F(nT) = \frac{T}{2\pi} \int_{-\pi/T}^{+\pi/T} e^{jnTy} f^*(jy) dy,$$

$f^*(jy)$ does not enter as a factor of an oscillation e^{jty} of frequency y , but only as a factor of the discrete values $e^{j(nT)y}$. The other values of e^{jty} are not present, and therefore no oscillations e^{jty} can be filtered out. If, however, the spectrum of $F(t)$ contained only the frequencies $|y| \leq \pi/T$, then the above equation (27.24) as well as (27.25) would be true and the comparison of (27.28) with (27.25) would yield $Tf^*(jy) = f(jy)$ in the interval $|y| \leq \pi/T$. Thus $Tf^*(jy)$ would be identical with the spectral density $f(jy)$ in this interval and therefore could serve as its measurement. By 1), however, this case cannot occur.

It is not surprising that in the sampling theorem $F(t)$ can be calculated from the discrete values $F(nT)$ because then $F(t)$ is an *analytic* (entire) function. On the other hand, it is obvious that for an arbitrary function defined for $t \geq 0$, the values $F(nT)$ can supply no information about the intermediate values of $F(t)$.

Example

The time function may be

$$F(t) = \frac{\sin \alpha t}{\pi t},$$

which is the standard function from which the development (27.27) proceeded. If we consider $F(t)$ for all t ($-\infty < t < +\infty$) the spectral density is

$$f(y) = \int_{-\infty}^{+\infty} e^{-jyt} \frac{\sin \alpha t}{\pi t} dt = \begin{cases} 1 & \text{for } |y| \leq \alpha \\ 0 & \text{for } |y| > \alpha. \end{cases}$$

The entire transcendental function $F(t)$ has therefore the frequency band $|y| \leq \alpha$. If we sample with the pulse period $T = \pi/\alpha$, that is, with the circular frequency $\omega = 2\alpha$, we obtain the sampled-data

$$F(nT) = \frac{\sin \alpha nT}{\pi nT} = \begin{cases} \frac{\alpha}{\pi} & \text{for } n = 0 \\ 0 & \text{for } n = \pm 1, \pm 2, \dots \end{cases}$$

The development (27.27) supplies the correct expression

$$F(t) = \frac{T}{\pi} F(0) \frac{\sin \alpha t}{t} = \frac{\sin \alpha t}{\pi t}.$$

If on the other hand we consider $F(t)$ only for $t \geq 0$, that is, we set

$$F(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{\sin \alpha t}{\pi t} & \text{for } t \geq 0, \end{cases}$$

then we obtain

$$\begin{aligned} f(y) &= \int_0^{\infty} e^{-iyt} \frac{\sin \alpha t}{\pi t} dt = \int_0^{\infty} \cos yt \frac{\sin \alpha t}{\pi t} dt - j \int_0^{\infty} \sin yt \frac{\sin \alpha t}{\pi t} dt \\ &= \begin{cases} 1/2 & \text{for } |y| \leq \alpha \\ -j \frac{1}{2\pi} \log \left| \frac{y+\alpha}{y-\alpha} \right| & \text{for } |y| > \alpha. \end{cases} \end{aligned}$$

This time function therefore does not have a finite frequency band. We have

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt = \frac{1}{\pi} \operatorname{arc tg} \frac{\alpha}{s}$$

and with $T = \pi/\alpha$

$$f^*(s) = \sum_{n=0}^{\infty} F(nT) e^{-nTs} = F(0) = \frac{\alpha}{\pi}.$$

Therefore, $f(s)$ and $Tf^*(s) = 1$ are not equal in $|y| \leq \pi/T = \alpha$.

§ 28. Networks Containing Pulse Generators

We pass on now to the insertion of pulse generators into networks. It may be remembered that a network is characterized by its system function $q(s)$ or its weight function $Q(t)$ (symbolized in a figure by a block with the description $q(s)$), and that with vanishing initial values the output $y(s)$ or $Y(t)$ corresponding to an input $f(s)$ or $F(t)$ is given by

$$y(s) = q(s)f(s) \quad \text{or} \quad Y(t) = (f(t) * F(t)).$$

We shall consider the most fundamental of the many possible combinations of networks and pulse generators.

1. Impulse Generators

1. The impulse generator is inserted between the (continuous) input $F(t)$ and the network (Fig. 28.1).

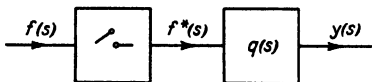


Figure 28.1 Impulse generator between input and network.

The network is fed with the output $f^*(s)$ of the impulse generator, and therefore

$$(28.1) \quad y(s) = q(s) f^*(s) \quad \text{or} \quad Y(t) = Q(t) * F^*(t)$$

Explicitly

$$Y(t) = Q(t) * \left[F(t) \sum_{n=0}^{\infty} \delta(t - nT) \right] = \int_0^t Q(t - \tau) F(\tau) \sum_{n=0}^{\infty} \delta(\tau - nT) d\tau,$$

and therefore

$$(28.2) \quad Y(t) = \sum_{n=0}^{\lfloor t/T \rfloor} Q(t - nT) F(nT)$$

This is the "convolution sum" of Q and F . The expression also comes from the fact that, by § 13.2, the response to an impulse of intensity $F(nT)$ at time $t = nT$ is $F(nT) Q(t - nT)$. Up to the time t , impulses occur at times $0, T, 2T, \dots, \lfloor t/T \rfloor T$. The output $Y(t)$ is also continuous at the points $t = nT$, because there $Q(t - nT) = Q(0) = 0$ (see (14.4)).

2. The impulse generator is inserted behind the network (Fig. 28.2)

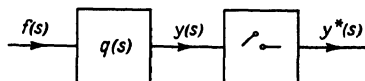


Figure 28.2 Impulse generator after network.

$y(s) = q(s) f(s)$ comes from the network and enters the impulse generator, from which comes $y^*(s) = [q(s) f(s)]^*$. This function can be expressed from (27.15) by the $\mathfrak{L}$ -transforms $q(s)$, $f(s)$, so that we obtain

$$(28.3) \quad y^*(s) = [q(s) f(s)]^* = \frac{1}{T} \sum_{m=-\infty}^{+\infty} q\left(s + j m \frac{2\pi}{T}\right) f\left(s + j m \frac{2\pi}{T}\right)$$

(We notice that the original function $Q(t) * F(t)$ of $q(s) f(s)$ vanishes for $t = 0$.)

The original function of $y^*(s)$ is the pseudo function

$$(28.4) \quad Y^*(t) = Q(t) * F(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_{n=0}^{\infty} \int_0^{nT} Q(nT - \tau) F(\tau) d\tau \cdot \delta(t - nT)$$

3. *Synchronized impulse generators are inserted before and behind the network* (Fig. 28.3).

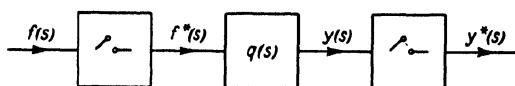


Figure 28.3 Synchronized impulse generators before and after network.

From 1), $y(s) = q(s) f^*(s)$ comes out of the network and enters into the second impulse generator; $y^*(s) = [q(s) f^*(s)]^*$ emerges from this. This function can be expressed by q^* and f^* . To $q f^*$ there corresponds by (28.2) the original function

$$Q * F^* = \sum_{\nu=0}^{\lfloor t/T \rfloor} Q(t - \nu T) F(\nu t),$$

which for the time points $t = nT$ has the value

$$\sum_{\nu=0}^n Q((n - \nu) T) F(\nu t).$$

Therefore by (27.5)

$$(28.5) \quad y^*(s) = \sum_{n=0}^{\infty} e^{-nTs} \sum_{\nu=0}^n Q((n - \nu) T) F(\nu t).$$

The Cauchy product of two power series is

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} z^n \sum_{\nu=0}^n a_{n-\nu} b_{\nu}.$$

If we set $a_n = Q(nT)$, $b_n = F(nT)$, $z = e^{-Ts}$, we see that

$$y^*(s) = \sum_{n=0}^{\infty} Q(nT) e^{-nTs} \sum_{n=0}^{\infty} F(nT) e^{-nTs}.$$

Therefore we have⁵³⁾

$$(28.6) \quad y^*(s) = q^*(s) f^*(s) \quad \text{or} \quad \mathfrak{D}\{Y\} = \mathfrak{D}\{Q\} \mathfrak{D}\{F\}$$

With $e^{-Ts} = z$ this relation, by (26.1), can be written also in the form

$$(28.7) \quad \tilde{y}(z) = \tilde{q}(z) \tilde{f}(z) \quad \text{or} \quad \mathfrak{B}\{Y\} = \mathfrak{B}\{Q\} \mathfrak{B}\{F\}$$

⁵³⁾ In literature this equation is generally derived by using formula (27.15), but this requires certain assumptions. These assumptions are not necessary with the above proof.

where

$$\tilde{y}(z) = \sum_{n=0}^{\infty} Y(nT) z^{-n}, \quad \tilde{q}(z) = \sum_{n=0}^{\infty} Q(nT) z^{-n}, \quad \tilde{f}(z) = \sum_{n=0}^{\infty} F(nT) z^{-n}.$$

For the treatment of a network in which only sampled values are sent in and taken out, the $\mathfrak{Z}$ -transformation gives the shortest description.

We wish to establish the above result in a general form. For two arbitrary $\mathfrak{Z}$ -transforms $f_1(s)$, $f_2(s)$ and their corresponding $\mathfrak{D}$ -transforms $f_1^*(s)$, $f_2^*(s)$ the following is true:

$$(28.8) \quad [f_1(s) f_2^*(s)]^* = f_1^*(s) f_2^*(s)$$

We can think of the application of an asterisk to a function as an operation performed on the function, which satisfies rule (28.8).

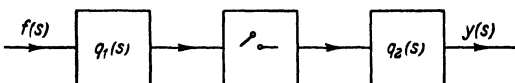
To obtain the sampled values $Y(nT)$ themselves, we require only to remember that to the product of two $\mathfrak{D}$ - or $\mathfrak{Z}$ -transforms there corresponds the "convolution sum" of the original functions:

$$(28.9) \quad Y(nT) = \sum_{\nu=0}^n Q((n-\nu)T) F(\nu T)$$

This can be seen also from (28.5).

4. An impulse generator is inserted between two networks in series (Fig. 28.4).

Figure 28.4 Impulse generator between two networks.



From (28.3) the output of the impulse generator is $[q_1 f]^*$. This enters into the second network which has the output $q_2 [q_1 f]^*$. We have therefore

$$(28.10) \quad y(s) = q_2(s) [q_1(s) f(s)]^*$$

By the convolution theorem the corresponding original function is

$$\begin{aligned} Y(t) &= \int_0^t Q_2(t-\tau) \left[Q_1 * F \sum_{n=0}^{\infty} \delta(t-nT) \right]_{t=\tau} d\tau \\ &= \sum_{n=0}^{\lfloor t/T \rfloor} Q_2(t-nT) [Q_1 * F]_{t=nT}, \end{aligned}$$

or

$$(28.11) \quad Y(t) = \sum_{n=0}^{[t/T]} Q_2(t-nT) \int_0^{nT} Q_1(nT-\tau) F(\tau) d\tau$$

Supplement. We can bring the representation of the output in Case 1 into the same form as that in Case 3. Setting $t = nT + \tau$ ($n = [t/T]$, $0 \leq \tau < T$) and calling the summation index r instead of n , we obtain

$$(28.12) \quad Y(nT + \tau) = \sum_{r=0}^n Q((n-r)T + \tau) F(rT).$$

The establishing of the values $Y(nT + \tau)$ is equivalent to attaching an impulse generator at the output of the network: this samples the function $Y(t)$ at the times $nT + \tau$. Therefore (28.12) is a generalization of (28.9) in which $\tau = 0$. By the convolution theorem for the $\mathfrak{D}$ -transformation it follows from (28.12) that

$$\sum_{n=0}^{\infty} Y(nT + \tau) e^{-nTs} = \sum_{n=0}^{\infty} Q(nT + \tau) e^{-nTs} \sum_{n=0}^{\infty} F(nT) e^{-nTs}.$$

If we set

$$\sum_{n=0}^{\infty} Y(nT + \tau) e^{-(nT+\tau)s} = y^*(s, \tau), \quad \sum_{n=0}^{\infty} Q(nT + \tau) e^{-(nT+\tau)s} = q^*(s, \tau)$$

(this generalized $\mathfrak{D}$ -transformation was introduced previously in (27.14)), we obtain

$$(28.13) \quad y^*(s, \tau) = q^*(s, \tau) f^*(s)$$

The sequences $Y(nT + \tau)$ constitute, in their totality for $0 \leq \tau < T$, all values of the function $Y(t)$, and therefore represent also a complete description of the function. Hence the pair of formulas (28.1, 2), which express the connection between input and output in the language of the $\mathfrak{L}$ -transformation, can be replaced by the pair (28.12, 13), which describe this relationship in the language of the $\mathfrak{D}$ -transformation. Translated into the language of the $\mathfrak{L}$ -transformation formula (28.13) is⁵³⁾

$$(28.14) \quad \tilde{y}(z, \tau) = \tilde{q}(z, \tau) \tilde{f}(z)$$

⁵³⁾ Some authors imagine that this representation demonstrates a result peculiar to the $\mathfrak{D}$ - and $\mathfrak{L}$ -transformations, and believe that impulse regulated systems can be treated only by these transformations. In fact the representations by the $\mathfrak{L}$ -transformation or $\mathfrak{D}$ - or $\mathfrak{L}$ -transformation are equally possible; they express one and the same thing in different languages. Which language is preferred is a question of what one is used to.

with

$$\tilde{y}(z, \tau) = \sum_{n=0}^{\infty} Y(nT + \tau) z^{-n}, \quad \tilde{q}(z, \tau) = \sum_{n=0}^{\infty} Q(nT + \tau) z^{-n}.$$

(It is convenient to omit the factor $e^{-\tau s}$ which appeared in the $\mathfrak{D}$ -transformation.)

2. Pulse Generators

1. Pulse generator between the input and the network (Fig. 28.1).

The output of the pulse generator which is fed into the network is the function $\bar{f}(s)$ determined by (27.3); the output of the network is $y(s) = q(s) \bar{f}(s)$ or, by (27.10),

$$(28.15) \quad y(s) = k \frac{1 - e^{-\vartheta s}}{s} q(s) \bar{f}^*(s)$$

$\frac{1}{s} q(s)$ is the image function of the network's step response, which we denoted by $Y_v(t)$ in § 13.1. To prevent confusion we denote it now by $H(t)$:

$$\frac{1}{s} q(s) \bullet \circ H(t).$$

Then by Rule II we have

$$(28.16) \quad \frac{1}{s} q(s) - e^{-\vartheta s} \frac{1}{s} q(s) \bullet \circ H(t) - H(t - \vartheta),$$

where H is to be set equal to zero for negative arguments. The convolution theorem gives

$$\begin{aligned} Y(t) &= k [H(t) - H(t - \vartheta)] * F^*(t) \\ &= k \int_0^t [H(t - \tau) - H(t - \tau - \vartheta)] F(\tau) \sum_{n=0}^{\infty} \delta(\tau - nT) d\tau \\ &= k \sum_{n=0}^{\lfloor t/T \rfloor} [H(t - nT) - H(t - nT - \vartheta)] F(nT). \end{aligned}$$

In the last term where $n = \lfloor t/T \rfloor$ we have

$$t - nT - \vartheta = t - \lfloor t/T \rfloor T - \vartheta = t - \lfloor t \rfloor - \vartheta \begin{cases} \geq 0 & \text{for } t - \lfloor t \rfloor \geq \vartheta \\ < 0 & \text{for } t - \lfloor t \rfloor < \vartheta. \end{cases}$$

Because H is equal to zero for negative arguments, the last term has a special form when $t - [t] < \theta$ so that we have

$$(28.17) \quad Y(t) = \begin{cases} k \sum_{n=0}^{[t/T]} [H(t-nT) - H(t-nT-\theta)] F(nT) & \text{for } t - [t] \geq \theta \\ k \sum_{n=0}^{[t/T]-1} [H(t-nT) - H(t-nT-\theta)] F(nT) + kH(t-[t]) F([t]) & \text{for } t - [t] < \theta \end{cases}$$

2. Pulse generator behind the network (Fig. 28.2)

$y(s) = q(s) f(s)$ comes from the network and that is converted by the pulse generator into

$$(28.18) \quad \bar{y}(s) = \bar{q}(s) \bar{f}(s) = k \frac{1 - e^{-\theta s}}{s} [q(s) f(s)]^*$$

The original function $\bar{Y}(t)$ is obtained most simply from formula (27.1). The values of $Y(t) = Q(t) * F(t)$ at the points $t = nT$ are fixed during the span θ and multiplied by k giving

$$(28.19) \quad \bar{Y}(t) = \begin{cases} k \int_0^{nT} Q(nT - \tau) F(\tau) d\tau & \text{for } nT \leq t < nT + \theta \\ 0 & \text{for } nT + \theta \leq t < (n+1)T. \end{cases}$$

3. Pulse generators before and after the network (Fig. 28.3).

By 1. $y(s) = q(s) \bar{f}(s)$, or according to (27.10, 11)

$$y(s) = q(s) q_\theta(s) f^*(s)$$

leaves the network. We put $q_\theta(s) q(s) = r(s)$. From (28.16) we obtain

$$(28.20) \quad r(s) \Leftrightarrow k [H(t) - H(t - \theta)] = R(t).$$

Then we have

$$y(s) = r(s) f^*(s).$$

This function enters the second pulse generator. If this generator provided impulses, then, by (28.8), the output of the generator would be

$$y^*(s) = [r(s) f^*(s)]^* = r^*(s) f^{**}(s).$$

But because the generator now supplies pulses, we must go over to $\bar{y}(s)$ from $y^*(s)$ by multiplying by $q_\theta(s)$:

$$q_\theta(s) y^*(s) = r^*(s) q_\theta(s) f^*(s)$$

or

(28.21)

$$\bar{y}(s) = r^*(s) \bar{f}(s)$$

This is the description of what happens in the language of the $\mathfrak{L}$ -transformation. $r^*(s)$ plays the role of the system function for the network if the input is changed into pulses before entering the network, and the output after leaving it (f into $\bar{f}$, y into $\bar{y}$).

Because $y(s)$ is determined by $y^*(s)$ up to the constant factor $q_\theta(s)$, we can represent the occurrence in the language of the $\mathfrak{D}$ -transformation by the equation

(28.22)

$$y^*(s) = r^*(s) f^*(s) \quad \text{or} \quad \mathfrak{D}\{Y\} = \mathfrak{D}\{R\} \mathfrak{D}\{F\}$$

or in the language of the $\mathfrak{Z}$ -transformation by

(28.23)

$$\bar{y}(z) = \bar{r}(z) \bar{f}(z) \quad \text{or} \quad \mathfrak{Z}\{Y\} = \mathfrak{Z}\{R\} \mathfrak{Z}\{F\}$$

We obtain the original function most easily by determining first $Y(nT)$ using the convolution theorem on (28.22) or (28.23):

$$Y(nT) = \sum_{\nu=0}^n R((n-\nu)T) F(\nu t)$$

and then by applying formula (27.1):

(28.24)

$$\bar{Y}(t) = \begin{cases} \sum_{\nu=0}^n R((n-\nu)T) F(\nu t) & \text{for } nT \leq t < nT + \theta \\ 0 & \text{for } nT + \theta \leq t < (n+1)T \end{cases}$$

4. Pulse generators between two networks (Fig. 28.4).

We have

$$y(s) = q_2(s) \overline{q_1(s) \bar{f}(s)} = q_\theta(s) q_2(s) [q_1(s) f(s)]^*$$

The general expression for $Y(t)$ is very cumbersome and is not given here. For particular functions f , q_1 , q_2 it can be calculated easily.

3. Pulses formed by a modulating function

In this case only insignificant variations occur in the formulas. Let $\Psi(t)$ be an arbitrary function, defined in $0 \leq t < T$. Then the time function $F(t)$ is transformed into a sequence of pulses (Ψ -pulses)

$$(28.25) \quad \bar{F}(t) = F(nT) \Psi(t - nT) \quad \text{for } nT \leq t < (n+1)T$$

(Fig. 27.2). Rectangular pulses correspond to the case

$$\Psi(t) = \begin{cases} k & \text{for } 0 \leq t < \theta \\ 0 & \text{for } \theta \leq t < T. \end{cases}$$

We obtain

$$\begin{aligned} \mathfrak{L}\{\bar{F}\} = \bar{f}(s) &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} F(nT) \Psi(t - nT) dt \\ &= \sum_{n=0}^{\infty} \int_0^T e^{-s(\tau + nT)} F(nT) \Psi(\tau) d\tau = \sum_{n=0}^{\infty} F(nT) e^{-nTs} \int_0^T e^{-st} \Psi(t) dt. \end{aligned}$$

The “finite” $\mathfrak{L}$ -transformation appears here: i.e.

$$\psi(s) = \int_0^T e^{-st} \Psi(t) dt.$$

From this we obtain

$$(28.26) \quad \bar{f}(s) = \psi(s) \sum_{n=0}^{\infty} F(nT) e^{-nTs} = \psi(s) f^*(s).$$

When comparison is made with the case of the rectangular pulses the only difference is that $q_\theta(s)$ is replaced by $\psi(s)$.

We consider the two most important of the linkages treated above for the case in which the generator forms Ψ -pulses.

1. Generator between input and network.

The output of the network is

$$y(s) = q(s) \bar{f}(s) = \psi(s) q(s) f^*(s).$$

We set

$$(28.27) \quad \psi(s) q(s) = r(s).$$

The corresponding original function is

$$R(t) = \Psi(t) * Q(t),$$

where $\Psi(t)$ is to be set equal to zero for $t \geq T$. Then

$$(28.28) \quad R(t) = \begin{cases} \int_0^t \Psi(\tau) Q(t-\tau) d\tau & \text{for } t < T \\ \int_0^T \Psi(\tau) Q(t-\tau) d\tau & \text{for } t \geq T. \end{cases}$$

From

$$(28.29)$$

$$y(s) = r(s) f^*(s)$$

it follows that

$$Y(t) = R(t) * \left[F(t) \sum_{\nu=0}^{\infty} \delta(t - \nu T) \right] = \int_0^t R(t-\tau) F(\tau) \sum_{\nu=0}^{\infty} \delta(\tau - \nu T) d\tau,$$

and therefore

$$(28.30) \quad Y(t) = \sum_{\nu=0}^{\lfloor t/T \rfloor} R(t - \nu T) F(\nu T) \quad \text{or with } t = nT + \tau:$$

$$(28.31) \quad Y(nT + \tau) = \sum_{\nu=0}^n R((n - \nu)T + \tau) F(\nu T)$$

3. Generators before and after the network.

The considerations are formally the same as in the case of a rectangular pulse, except that in the definition of $r(s)$, $q_\theta(s) q(s)$ is replaced by $\psi(s) q(s)$. Formulas (28.21–24) are the same but with the new meanings of $r(s)$ or $R(t)$.

We can repeat for Ψ -pulses the argument on p. 152 (Supplement) and bring the representation of the output in Case 1 into the same form as in Case 3. If we apply the convolution theorem of the $\mathfrak{D}$ -transformation to (28.31), we have

$$\sum_{n=0}^{\infty} Y(nT + \tau) e^{-nTs} = \sum_{n=0}^{\infty} R(nT + \tau) e^{-nTs} \sum_{n=0}^{\infty} F(nT) e^{-nTs}.$$

With

$$\sum_{n=0}^{\infty} Y(nT + \tau) e^{-(nT + \tau)s} = y^*(s, \tau), \quad \sum_{n=0}^{\infty} R(nT + \tau) e^{-(nT + \tau)s} = r^*(s, \tau)$$

the equation assumes the form

$$(28.32)$$

$$y^*(s, \tau) = r^*(s, \tau) f^*(s)$$

or in the language of the $\mathfrak{Z}$ -transformation

(28.33)

$$\tilde{y}(z, \tau) = \tilde{r}(z, \tau) \tilde{f}(z)$$

where

$$\tilde{y}(z, \tau) = \sum_{n=0}^{\infty} Y(nT + \tau) z^{-n}, \quad \tilde{r}(z, \tau) = \sum_{n=0}^{\infty} R(nT + \tau) z^{-n}.$$

Example of Case 1.

Let the network be an I -system (integrating system) with $q(s) = a/s$. Then we have

$$r(s) = a \frac{\psi(s)}{s}$$

and

$$R(t) = a \int_0^t \Psi(\tau) d\tau = \begin{cases} a \int_0^t \Psi(\tau) d\tau = a \Phi(t) & \text{for } t < T \\ a \int_0^T \Psi(\tau) d\tau = a \Phi(T) & \text{for } t \geq T. \end{cases}$$

From (28.31) we obtain

$$\begin{aligned} Y(nT + \tau) &= \sum_{\nu=0}^{n-1} R((n-\nu)T + \tau) F(\nu T) + R(\tau) F(nT) \\ &= a \Phi(T) \sum_{\nu=0}^{n-1} F(\nu T) + a \Phi(\tau) F(nT) \end{aligned}$$

(the sum is absent for $n = 0$, i.e., $0 \leq t < T$).

In particular, for $F(t) = F_0$ we obtain the "step response"

$$Y(nT + \tau) = a F_0 [n \Phi(T) + \Phi(\tau)].$$

If we choose $\Psi(t) = 1 - (t/T)$ as a modulating function (Fig. 28.5) we have

$$\Phi(\tau) = \tau - \frac{\tau^2}{2T}, \quad \Phi(T) = \frac{T}{2},$$

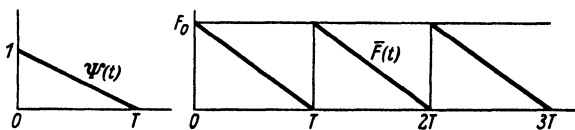
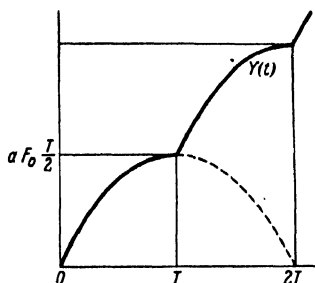


Figure 28.5 The pulses formed from the input $F(t) \equiv F_0$ by a modulating function $\Psi(t)$.

Figure 28.6 Output of an I -system when the input is the pulse shown in figure 28.5.



and therefore (Fig. 28.6)

$$Y(nT + \tau) = a F_0 \left(n \frac{T}{2} + \tau - \frac{\tau^2}{2T} \right).$$

In this special case the different transforms are

$$\psi(s) = \int_0^T e^{-st} \left(1 - \frac{t}{T} \right) dt = \frac{1}{s} - \frac{1}{T} \frac{1 - e^{-Ts}}{s^2}, \quad r(s) = a \left(\frac{1}{s^2} - \frac{1}{T} \frac{1 - e^{-Ts}}{s^3} \right),$$

$$f^*(s) = \sum_{n=0}^{\infty} F_0 e^{-nTs} = F_0 \frac{1}{1 - e^{-Ts}}, \quad \tilde{f}(z) = F_0 \frac{1}{1 - z^{-1}} = F_0 \frac{z}{z - 1},$$

$$\begin{aligned} \tilde{r}(z, \tau) &= R(\tau) + \sum_{n=1}^{\infty} R(nT + \tau) z^{-n} = a \left(\tau - \frac{\tau^2}{2T} + \sum_{n=1}^{\infty} \frac{T}{2} z^{-n} \right) \\ &= a \left(\tau - \frac{\tau^2}{2T} + \frac{T}{2} \frac{1}{z - 1} \right). \end{aligned}$$

The description of the occurrence in the image space of the $\mathfrak{L}$ -transformation is, by (28.29),

$$y(s) = a F_0 \frac{1}{s} \frac{\psi(s)}{1 - e^{-Ts}}$$

and in the image space of the $\mathfrak{Z}$ -transformation,

$$\tilde{y}(z, \tau) = a F_0 \left(\tau - \frac{\tau^2}{2T} + \frac{T}{2} \frac{1}{z - 1} \right) \frac{z}{z - 1}.$$

If we do not use the previous formulas (28.30, 31), but instead transform $y(s)$ or $\tilde{y}(z, \tau)$ into the original space, we use for $\tilde{y}$ the correspondences

$$\frac{z}{z - 1} \longleftrightarrow 1, \quad \frac{z}{(z - 1)^2} \longleftrightarrow n$$

and obtain the same expression as above. For the transformation of y we use the following theorem (EINF. p. 43), which is useful in all pulse problems for which $f^*(s) = 1/(1 - e^{-Ts})$, and therefore $F(t) \equiv 1$:

Theorem 28.1. *If the function $\Psi(t)$ defined in $0 \leq t < T$ is continued periodically with period T , there arises a function $\Psi_p(t)$ defined for $t \geq 0$. If we put*

$$\psi(s) = \int_0^T e^{-st} \Psi(t) dt, \quad \psi_p(s) = \int_0^\infty e^{-st} \Psi_p(t) dt,$$

then we have

$$\psi_p(s) = \frac{\psi(s)}{1 - e^{-Ts}}.$$

From this we obtain the original function of $y(s)$

$$Y(t) = a F_0 \int_0^t \Psi_p(\tau) d\tau$$

where $\Psi_p(t)$ is the periodic continuation of the function $1 - (t/T)$, and therefore is identical to the function $\bar{F}(t)$ (see Fig. 28.5). This is equivalent to the expression obtained above for $Y(nT + \tau) = Y(t)$, because

$$\int_0^T \left(1 - \frac{\tau}{T}\right) d\tau = \frac{T}{2}, \quad \int_0^\tau \left(1 - \frac{u}{T}\right) du = \tau - \frac{\tau^2}{2T} \quad (0 \leq \tau < T).$$

Partial Differential Equations

§ 29. General Principles in the Application of the Laplace Transformation to Partial Differential Equation⁵⁴⁾

In a partial differential equation the unknown is a function of several variables. We consider here the case of two variables which we denote by x and t ; the unknown function is denoted by $U(x, t)$. In a partial differential equation a certain domain in the xt -plane is given at the outset, and it is within this domain that the unknown is to be determined. For the

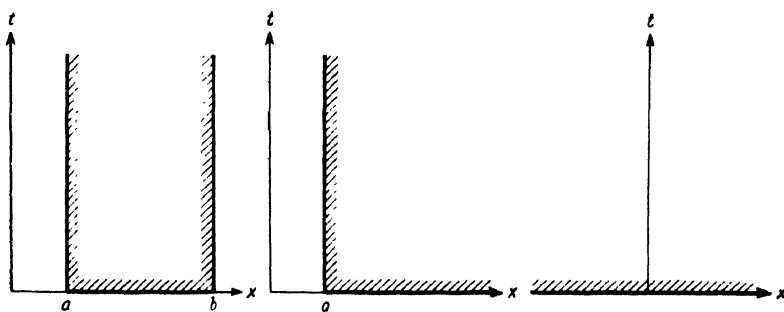


Figure 29.1 Some basic regions in the xt -plane.

equations considered here, we assume that t varies in the one-sided infinite interval $0 \leq t < \infty$, and x varies in a finite or infinite interval, so that the basic region of the xt -plane is a semi-infinite strip, a quadrant or a half-plane according as x varies in a finite, one-sided or two-sided infinite interval (Fig. 29.1).

To determine uniquely a definite function from the infinitely many which satisfy the differential equation in the domain, we must prescribe certain *conditions on the boundary*. In practice, conditions of the following kind occur: the values of the function itself or of certain of its partial derivatives

⁵⁴⁾ A reader to whom the subject is completely new is recommended to pay only brief attention to § 29 at first, and to read this paragraph more carefully after having studied the example in § 30.

are given, or some relations between these values are prescribed. In general nothing can be said about the number and kind of conditions. For the differential equations in which the engineer is interested, the conditions which can and must be prescribed in order that a unique solution exists can be found, generally, from the physical meaning of the problem under consideration.

When t is the time, the conditions referring to the horizontal part of the boundary (that is, the x -axis) are called the *initial conditions* and those referring to the vertical boundary part (when present) are called *boundary conditions*.

If we wish to solve partial differential equations by using the $\mathfrak{L}$ -transformation, we must apply the $\mathfrak{L}$ -transformation to the function $U(x, t)$ and to the derivatives which occur. Because the transformation represents an integration with respect to a *single* variable, we must undertake the $\mathfrak{L}$ -transformation relative to one of the variables in U , while the other is not involved. The variable to which the transformation is applied is t , which we have assumed from the first to vary between 0 and ∞ , because this is the interval over which the $\mathfrak{L}$ -transformation extends. The variable x is to be thought of as kept constant. For each fixed value of x we obtain a different transform which therefore depends not only as previously on s , but also on x ; it is therefore a function $u(x, s)$:

$$(29.1) \quad \mathfrak{L}\{U(x, t)\} = \int_0^{\infty} e^{-st} U(x, t) dt = u(x, s).$$

If we wish to transform *derivatives with respect to t* , we can employ Rule V, and here also we think of x as kept constant. Then we have, for example,

$$(29.2) \quad \mathfrak{L}\left\{\frac{\partial U(x, t)}{\partial t}\right\} = s u(x, s) - U(x, 0+),$$

$$(29.3) \quad \mathfrak{L}\left\{\frac{\partial^2 U(x, t)}{\partial t^2}\right\} = s^2 u(x, s) - U(x, 0+) s - U_t(x, 0+),$$

where $U_t = \frac{\partial U}{\partial t}$. To use our method for *derivatives with respect to x* it must be assumed that the order of the differentiation and the integration in the $\mathfrak{L}$ -integral can be interchanged; for example,

$$(29.4) \quad \mathfrak{L}\left\{\frac{\partial U(x, t)}{\partial x}\right\} = \frac{\partial}{\partial x} \mathfrak{L}\{U(x, t)\} = \frac{\partial u(x, s)}{\partial x},$$

$$(29.5) \quad \mathfrak{L}\left\{\frac{\partial^2 U(x, t)}{\partial x \partial t}\right\} = \frac{\partial}{\partial x} \mathfrak{L}\left\{\frac{\partial U(x, t)}{\partial t}\right\} = \frac{\partial}{\partial x} [s u(x, s) - U(x, 0+)].$$

We see that the values $U(x, 0+)$, $U_t(x, 0+)$, ... are required when derivatives are transformed. As *initial conditions*, therefore, a certain number of these "*initial values*" must be given. As in ordinary differential equations, there is the advantage that they occur in the image equation and are therefore considered automatically.

Because the derivatives with respect to t have been eliminated by the transformation, only the derivatives of U with respect to x remain in the equation and this means that *the image equation is an ordinary differential equation*. Since such an equation represents an incomparably simpler problem than a partial differential equation, the original problem is simplified considerably by using the $\mathfrak{L}$ -transformation. This explains why many problems, which cannot be treated at all by other methods, or at best, in a complicated way, can be handled by this method.

A few remarks must be made about the boundary conditions which may be present on the vertical boundary of the domain. It may be, for example, that $x = a$ is the left boundary of the x -interval and as a boundary condition the value of the function $U(x, t)$ may be given there, i.e., $U(a, t)$. Now, however, it is to be noted that this boundary value is to be understood, as we already know from initial values with ordinary differential equations, in the sense of a limiting value; i.e., if we denote the function given at $x = a$ (which is a function of t alone) by $A(t)$ then we require that

$$(29.6) \quad \lim_{x \rightarrow a+} U(x, t) = A(t) \text{ or more briefly } U(a+, t) = A(t).$$

This formulation is much more important in solving partial differential equations than in solving ordinary differential equations, because in most problems the solution for $x = a$ has no meaning at all so that we cannot form $U(a, t)$. The requirements that a limiting value exists for $x \rightarrow a$ (or, more exactly, from the right, i.e. $x \rightarrow a+$) and that it shall have a given value are the only conditions that can be realized.

If we assume that the limit $x \rightarrow a+$ is interchangeable with the $\mathfrak{L}$ -integral, we have

$$\lim_{x \rightarrow a+} u(x, s) = \lim_{x \rightarrow a+} \mathfrak{L}\{U(x, t)\} = \mathfrak{L}\left\{\lim_{x \rightarrow a+} U(x, t)\right\} = \mathfrak{L}\{A(t)\} = a(s),$$

and therefore

$$(29.7) \quad \lim_{x \rightarrow a+} u(x, s) = a(s).$$

This means that the transform of the boundary value $A(t)$ is taken as the *boundary value of the transform* $u(x, s)$ at $x = a$. This is shown most clearly by Fig. 29.2; the function $A(t)$ is transformed on the vertical with the particular value $x = a$ in the same way as $U(x, t)$ is transformed on the vertical with the abscissa x . The result is the boundary value of $u(x, s)$.

If the right boundary $x = b$ of the x -interval is present, then the corresponding result holds there. The boundary values of $u(x, s)$ are obtained by transformation of the boundary values of $U(x, t)$.

It is not surprising that we must make certain assumptions (namely that the $\mathfrak{L}$ -transformation shall be interchangeable with differentiation with respect to x and with the limit $x \rightarrow a$). No method can give a solution without assumptions having been made; frequently these are not quoted explicitly but are tacitly assumed, for example, that the solution has a

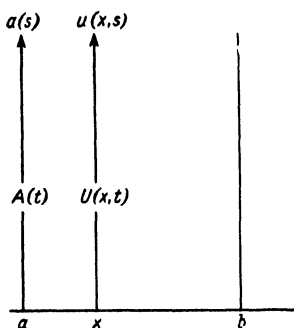


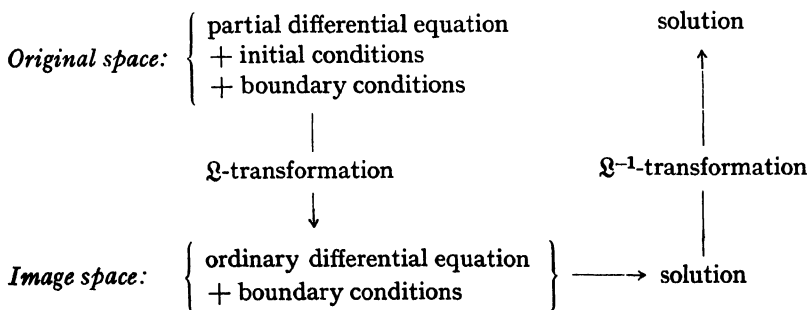
Fig. 29.2

The $\mathfrak{L}$ -transformation of $U(x, t)$ with x held constant.

certain form, that it may be developed into a series, etc. Therefore, strictly speaking, after having found the solution, we must verify that the function obtained actually satisfies the differential equation and the boundary and initial conditions⁵⁵).

Summing up the method we obtain the following scheme:

Scheme



⁵⁵) For the existence of solutions which do not satisfy the assumptions of the method see EINF. p. 269.

Explanation: Instead of solving the differential equation with initial and boundary conditions directly, we detour into the image space: using the $\mathfrak{L}$ -transformation the partial differential equation becomes an ordinary differential equation, in which the initial conditions are present and so are considered automatically; the boundary conditions transform into boundary conditions. When we have solved the ordinary differential equation we find the solution of the original problem by inverting the $\mathfrak{L}$ -transformation.

If the variable x moves in the interval $0 \leq x < \infty$, then only conditions at the left boundary $x = 0$ have to be considered; these have the character of initial conditions. Then the $\mathfrak{L}$ -transformation can be applied again, this time with respect to x , and an algebraic equation is found for the transform; this can be solved without difficulty. To obtain the solution of the original problem we have to apply the inverse $\mathfrak{L}$ -transformation twice⁵⁶⁾.

When the original differential equation refers to a function of three variables, $U(x, y, t)$, where t varies in the interval $0 \leq t < \infty$, and x and y vary in a region of the xy -plane, the application of the $\mathfrak{L}$ -transformation with respect to t results in a partial differential equation in two variables, x and y , because of the elimination of the derivatives with respect to t ; again this is a considerably simpler equation.

The most difficult part of the transformation method is normally the *determination of the original function* of the solution found in image space. If the present tables are not sufficient, then the methods given in Chapter 7 can be applied.

The partial differential equations appearing most frequently in engineering are of the second order. These are divided into the following three types, of which we have given the most important representative in each case:

elliptic type	potential equation	$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2} = 0,$
hyperbolic type	wave equation	$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0,$
parabolic type	heat conduction equation	$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = 0.$

Greater difficulties are encountered in the solution of the elliptic type when the $\mathfrak{L}$ -transformation is used and therefore this type is set aside. As a first example we consider the heat conduction equation, because the method works best with this type.

⁵⁶⁾ For the method of the "double $\mathfrak{L}$ -transformation", which we shall employ in an example on p. 173, the reader is referred to the book of D. VOELKER and G. DOETSCH: *Die zweidimensionale Laplace-Transformation. Eine Einführung in ihre Anwendung zur Lösung von Randwertproblemen nebst Tabellen von Korrespondenzen*. Birkhäuser Verlag, Basel 1950.

§ 30. The Heat Conduction Equation

The heat conduction equation

$$(30.1) \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

is of interest not only to the heat engineer, but also to the electrical engineer because heat production is often of importance in electrical engineering. (A transistor, for example, can only be operated up to a maximum temperature if the semiconductor is not to be changed or destroyed.) The equation is also known in electrical engineering under the name of the equation for the "Thomson Cable". At the time of the laying of the first submarine cable, the solution of the telegraph equation had not yet been obtained. On neglecting the inductance and leakage conductance the telegraph equation passes over into the heat conduction equation, the solution of which was well known and was used by Thomson as an approximate solution of the telegraph equation.

We treat the equation in the language of heat conduction theory. Then $U(x, t)$ is the temperature of a linear heat conductor (or of a heat conductor in a space where the temperature depends only upon one coordinate, and therefore is constant in a layer). Let x extend from $x = 0$ to $x = l$; t runs from $t = 0$ to $t = \infty$. The basic region in the xt -plane is therefore a semi-infinite strip when l is finite and a quadrant when $l = \infty$.

At the time $t = 0$ the conductor has a certain temperature, which can depend upon x ; we denote it by $U_0(x)$. It represents the "initial value" of $U(x, t)$ in the sense of § 29, i.e. $U(x, t)$ must satisfy the condition

$$(30.2) \quad U(x, 0+) = U_0(x).$$

As is seen from physical considerations and as is explained also in the course of the solution, a single initial value is sufficient in this problem and we do not require to use $U_i(x, 0+)$, ...; this, of course, comes from the fact that equation (30.1) is of first order with respect to t .

Both ends $x = 0$ and $x = l$ are kept at definite temperatures by heaters; these temperatures depend upon the time. Therefore we have

$$(30.3) \quad U(0+, t) = A_0(t), \quad U(l-, t) = A_1(t).$$

These are the "boundary conditions" in the sense of § 29. Naturally these are not the only possible ones, and with other physical assumptions other conditions are given. When, for example, heat can radiate from one end, a linear relation is given between U and $\partial U / \partial x$.

We must now set up the image equation of equation (30.1) under the initial conditions (30.2) and the boundary conditions (30.3). If we write $\mathfrak{L}\{U(x, t)\} = u(x, s)$ we have, by (30.2),

$$\mathfrak{L}\left\{\frac{\partial U}{\partial t}\right\} = s u(x, s) - U(x, 0+) = s u(x, s) - U_0(x).$$

If further, the second derivative with respect to x is interchangeable with the $\mathfrak{L}$ -transformation, we have

$$\mathfrak{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2} \mathfrak{L}\{U\} = \frac{\partial^2 u(x, s)}{\partial x^2}.$$

Because differentiation occurs now only with respect to x and with respect to no other variable, we can replace partial derivatives by ordinary derivatives and obtain the image equation

$$(30.4) \quad \frac{d^2 u}{dx^2} = s u - U_0(x).$$

In this equation s plays the role of a parameter upon which the solution u depends; this is already expressed by the notation $u(x, s)$.

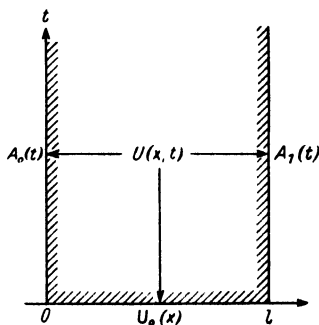


Figure 30.1 Boundary and initial values of $U(x, t)$.

If we also transform the boundary conditions (30.3)

$$\mathfrak{L}\{U(0+, t)\} = \mathfrak{L}\{A_0(t)\} = a_0(s), \quad \mathfrak{L}\{U(l-, t)\} = \mathfrak{L}\{A_1(t)\} = a_1(s),$$

we obtain, according to (29.7), the boundary conditions for the equation:

$$(30.5) \quad u(0+, s) = a_0(s), \quad u(l-, s) = a_1(s).$$

The given initial condition (30.2) is present in the image equation and is no longer a supplementary condition. It is considered automatically and this is an important advantage.

A differential equation such as (30.4) with boundary conditions is treated as follows. First the homogeneous equation is solved with arbitrary boundary values; the inhomogeneous term $U_0(x)$, that is the initial temperature, is assumed to vanish. Then the inhomogeneous equation is solved with vanishing boundary values $a_0(s)$, $a_1(s)$, and this is equivalent to saying that the boundary temperatures $A_0(t)$, $A_1(t)$ are assumed to vanish.

Then obviously the sum of these two solutions gives the solution of the general problem.

1. Vanishing initial temperature, arbitrary boundary temperatures.

For the solution of the homogeneous equation

$$(30.6) \quad \frac{d^2 u}{dx^2} = s u$$

we substitute, in the usual way, $u = e^{\alpha x}$ and obtain for α the equation $\alpha^2 = s$. Both solutions $\alpha = \pm \sqrt{s}$ give particular integrals, viz. $e^{x\sqrt{s}}$ and $e^{-x\sqrt{s}}$, and from these the general integral can be written in the form $c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}}$; c_1 and c_2 are now to be determined so that the boundary conditions (30.5) are fulfilled. It is best to set up first two particular solutions, one of which has the values 1 and 0 at the left and right boundary points, and the other of which has the values 0 and 1, i.e.

$$u_0(x, s) = \frac{e^{(l-x)\sqrt{s}} - e^{-(l-x)\sqrt{s}}}{e^{l\sqrt{s}} - e^{-l\sqrt{s}}} = \frac{\sinh(l-x)\sqrt{s}}{\sinh l\sqrt{s}},$$

$$u_1(x, s) = \frac{e^{x\sqrt{s}} - e^{-x\sqrt{s}}}{e^{l\sqrt{s}} - e^{-l\sqrt{s}}} = \frac{\sinh x\sqrt{s}}{\sinh l\sqrt{s}}.$$

Using these functions the desired solution of the image equation can be written in the form

$$u(x, s) = a_0(s) u_0(x, s) + a_1(s) u_1(x, s).$$

The original function of this has now to be determined. Next we wish to consider the limiting case $l = \infty$. $u_1(x, s)$ vanishes as $l \rightarrow \infty$ and for $u_0(x, s)$ we obtain

$$u_0(x, s) = \frac{e^{-x\sqrt{s}} - e^{-(2l-x)\sqrt{s}}}{1 - e^{-2l\sqrt{s}}} \rightarrow e^{-x\sqrt{s}}.$$

In image space the solution is therefore

$$(30.7) \quad u(x, s) = a_0(s) e^{-x\sqrt{s}}.$$

From the correspondence (TAB. 167)⁵⁷⁾

$$(30.8) \quad e^{-x\sqrt{s}} \leftrightarrow \frac{x}{2\sqrt{\pi} t^{3/2}} e^{-\frac{x^2}{4t}} = \psi(x, t) \quad (x > 0)$$

⁵⁷⁾ ψ is the usual notation for the so-called double source function of heat conduction and for this reason we do not denote it by an upper case letter although it occurs here as an original function. Similarly it is customary to denote by χ the "source function" appearing on p. 172.

and by using the convolution theorem (Rule IX) we obtain

$$(30.9) \quad U(x, t) = A_0(t) * \psi(x, t).$$

Writing this expression explicitly in the form

$$(30.10) \quad U(x, t) = \frac{x}{2\sqrt{\pi}} \int_0^t A_0(t-\tau) \frac{e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}} d\tau,$$

we see that we were justified in formulating the boundary condition as a limiting relation,

$$\lim_{x \rightarrow 0+} U(x, t) = A_0(t)$$

and not in the form $U(0, t) = A_0(t)$, because if we simply set $x = 0$, the denominator $\tau^{3/2}$ causes the integral to be divergent at $\tau = 0$. Even if this integral had a definite value, $U(0, t)$ would have the value 0 and not $A_0(t)$ because of the factor x . On the other hand it can be proved by a somewhat longer procedure that the function (30.10) has the limiting value $A_0(t)$ for $x \rightarrow 0$, if A_0 is continuous at the point t .—We see immediately that the initial condition $U(x, 0+) = 0$ is fulfilled.

In the case when the length l is finite, the original functions $U_0(x, t)$, $U_1(x, t)$ of $u_0(x, s)$, $u_1(x, s)$ can be found from TAB. 186, 185. If these correspondences were not available, a development in series could be tried to obtain the original functions (cf. § 35). We have

$$(30.11) \quad u_0(x, s) = \frac{e^{-x\sqrt{s}} - e^{-(2l-x)\sqrt{s}}}{1 - e^{-2l\sqrt{s}}} = (e^{-x\sqrt{s}} - e^{-(2l-x)\sqrt{s}}) \sum_{n=0}^{\infty} e^{-2nl\sqrt{s}} \\ = \sum_{n=0}^{\infty} e^{-(2nl+x)\sqrt{s}} - \sum_{n=0}^{\infty} e^{-(2(n+1)l-x)\sqrt{s}} = \sum_{n=0}^{\infty} e^{-(2nl+x)\sqrt{s}} - \sum_{n=1}^{\infty} e^{-(2nl-x)\sqrt{s}}.$$

For $0 < x < l$, $2nl + x > 0$ ($n = 0, 1, \dots$) and $2nl - x > 0$ ($n = 1, 2, \dots$), and therefore the correspondence (30.8) is applicable. Further, we can prove with the help of Theorem 35.4 that the interchanging of the sum and the $\mathfrak{L}$ -integral is justified here (see p. 200). Therefore we have

$$U_0(x, t) = \sum_{n=0}^{\infty} \psi(2nl + x, t) - \sum_{n=1}^{\infty} \psi(2nl - x, t)$$

or, since by (30.8)

$$-\psi(2nl - x, t) = \psi(-2nl + x, t),$$

we obtain

$$(30.12) \quad U_0(x, t) = \sum_{n=-\infty}^{+\infty} \psi(2nl + x, t).$$

In the case $l = \infty$ treated above only the term with $n = 0$ occurs. Analogously we obtain

$$(30.13) \quad U_1(x, t) = \sum_{n=-\infty}^{+\infty} \psi(2nl + l - x, t).$$

These series are particularly suitable for practical calculation when t is small because then the factors $e^{-(2nl+x)^2/4t}$ in the terms of the series are small. For large t the other form given in TAB. 186, 185 is more suitable.

The original function of $u(x, s)$ is determined from the functions U_0, U_1 by

$$(30.14) \quad U(x, t) = U_0(x, t) * A_0(t) + U_1(x, t) * A_1(t).$$

Special Boundary Values

We compile the special solutions which arise when $A_1(t) = 0$ at the right boundary and the temperature on the left boundary is described by certain simple functions which occur frequently.

1. We take $A_0(t) = \delta(t) =$ impulse function, and therefore $a_0(s) = 1$. Then we have for $l = \infty$

$$U(x, t) = \psi(x, t)$$

and for finite l

$$U(x, t) = U_0(x, t).$$

The boundary function $\delta(t)$ describes ideally a "heat explosion" at the point $x = 0$ at the time $t = 0$. The "Green's functions" ψ or U_0 can be interpreted as the temperature distribution which results from such an explosion.

2. We next take $A_0 = U(t)$, and therefore $a_0(s) = 1/s$. Then we have for $l = \infty$

$$u(x, s) = \frac{1}{s} e^{-x\sqrt{s}}.$$

This function corresponds to the original function (TAB. 168)

$$(30.15) \quad U(x, t) = \operatorname{erfc} \frac{x}{2\sqrt{t}} \quad (x \geq 0, t > 0).$$

With finite l we have, by (30.11),

$$u(x, s) = \frac{1}{s} u_0(x, s) = \sum_{n=0}^{\infty} \frac{1}{s} e^{-(2nl+x)\sqrt{s}} - \sum_{n=1}^{\infty} \frac{1}{s} e^{-(2nl-x)\sqrt{s}},$$

and therefore

$$(30.16) \quad U(x, t) = \operatorname{erfc} \frac{x}{2\sqrt{t}} + \sum_{n=1}^{\infty} \left[\operatorname{erfc} \frac{2nl+x}{2\sqrt{t}} - \operatorname{erfc} \frac{2nl-x}{2\sqrt{t}} \right].$$

3. Let the time function be $A_0(t) = e^{j\omega t}$. Then we have for $l = \infty$

$$(30.17) \quad U(x, t) = e^{j\omega t} * \psi(x, t) = e^{j\omega t} \int_0^t e^{-j\omega \tau} \psi(x, \tau) d\tau.$$

Analogously to § 13.3 we can determine the steady state of U as $t \rightarrow \infty$. Because $\mathfrak{L}\{\psi(x, t)\}$ converges for $s = j\omega$, we have

$$\begin{aligned} U(x, t) &= e^{j\omega t} \left\{ \int_0^\infty e^{-j\omega \tau} \psi(x, \tau) d\tau - \int_t^\infty e^{-j\omega \tau} \psi(x, \tau) d\tau \right\} \\ &= e^{j\omega t} \left\{ e^{-x(j\omega)^{1/2}} - \int_t^\infty e^{-j\omega \tau} \psi(x, \tau) d\tau \right\}. \end{aligned}$$

The second term inside the brackets tends to zero for $t \rightarrow \infty$ so that the steady state is given by

$$(30.18) \quad \tilde{U}(x, t) = e^{-x(j\omega)^{1/2}} e^{j\omega t}.$$

With $j = e^{j\pi/2}$, $j^{1/2} = e^{j\pi/4} = \cos \frac{\pi}{4} + j \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} (1 + j)$ we have

$$(30.19) \quad \tilde{U}(x, t) = e^{-\sqrt{\omega/2} x} e^{j(\omega t - \sqrt{\omega/2} x)}.$$

$\tilde{U}$ oscillates with the same frequency as the boundary function; its amplitude and phase depend on x and ω . As in § 13.3 the factor

$$q(j\omega) = e^{-x(j\omega)^{1/2}} = \mathfrak{L}\{\psi(x, t)\}_{s=j\omega}$$

can be called the frequency response of the temperature in an infinite conductor.

2. Arbitrary Initial Temperature, Vanishing Boundary Temperatures

We consider now the inhomogenous equation (30.4) with the boundary values

$$u(0+, s) = 0, \quad u(l-, s) = 0.$$

If we have no book on differential equations at our disposal from which we could find the solution of this boundary value problem, we could obtain the solution as follows. We use the method of the $\mathfrak{L}$ -transformation to solve the equation $u'' - su = -U_0(x)$ with given initial values $u(0) = 0$ and $u_x(0)$. In this solution we set $x = l$ and we obtain an equation connecting $u_x(0)$ and $u(l)$ from which we can express $u_x(0)$ in terms of $u(l)$. If we insert this value of $u_x(0)$ into the solution, we obtain

$$(30.20) \quad u(x, s) = \int_0^l \gamma(x, \xi; s) U_0(\xi) d\xi$$

with

$$(30.21) \quad \gamma(x, \xi; s) = \begin{cases} \frac{\sinh \xi \sqrt{s} \sinh (l-x) \sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & \text{for } 0 \leq \xi \leq x \\ \frac{\sinh x \sqrt{s} \sinh (l-\xi) \sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & \text{for } x \leq \xi \leq l. \end{cases}$$

Again we consider first the case $l = \infty$. Here γ reduces to

$$(30.22) \quad \gamma(x, \xi; s) = \begin{cases} \frac{1}{\sqrt{s}} e^{-x\sqrt{s}} \sinh \xi \sqrt{s} & \text{for } 0 \leq \xi \leq x \\ \frac{1}{\sqrt{s}} e^{-\xi\sqrt{s}} \sinh x \sqrt{s} & \text{for } x \leq \xi < \infty. \end{cases}$$

If we express $\sinh$ in terms of exponential functions and use the correspondence (TAB. 169)

$$(30.23) \quad \frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \longleftrightarrow \frac{1}{\sqrt{\pi} t} e^{-k^2/4t} = \chi(k, t) \quad (k \geq 0),$$

we obtain the following original function of γ ; in contrast to the image function, this is given by a single expression:

$$(30.24) \quad \Gamma(x, \xi; t) = \frac{1}{2} [\chi(\xi - x, t) - \chi(\xi + x, t)].$$

If the integral in (30.20) (whose upper limit is now ∞) is interchanged with the inversion of the $\mathcal{L}$ -transformation, we obtain

$$(30.25) \quad U(x, t) = \frac{1}{2} \int_0^\infty [\chi(\xi - x, t) - \chi(\xi + x, t)] U_0(\xi) d\xi.$$

When $U_0(x)$ possesses a limiting value for $x \rightarrow \infty$, this integral converges for $x > 0, t > 0$.

In the case of finite length l , we proceed as on p. 169. We substitute exponential functions for the $\sinh$ in $\gamma(x, \xi, s)$, develop it in a series and transform term by term. Then we obtain as the original function of γ :

$$(30.26) \quad \Gamma(x, \xi; t) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} [\chi(2nl + x - \xi, t) - \chi(2nl + x + \xi, t)]$$

and as the solution to the boundary value problem we have

$$(30.27) \quad U(x, t) = \int_0^l \Gamma(x, \xi; t) U_0(\xi) d\xi.$$

Form (30.26) of Γ is particularly suitable for small t ; for large t it is better to use the form given in TAB. 188. Moreover, the functions $\Gamma(x, \xi, t)$, $U_0(x, t)$, $U_1(x, t)$ can be written in terms of the theta function ϑ_3 , defined in the theory of elliptic functions.

§ 31. The Equations for a Twin Conductor Line with Distributed Constants

If an electrical conductor is so large that its constants (resistance, etc.) cannot be thought of as being concentrated in single places, then the voltage and current depend not only on t but also on the length coordinate x and are therefore functions $E(x, t)$ and $I(x, t)$. If the conductor consists of two parallel wires, then the currents $I(x, t)$ are equal at points opposite each other, but have opposite directions. $E(x, t)$ is the voltage between the two conductors at the point x .

R is the resistance, L the inductance, C the capacitance, and G is the leakage conductance of one conductor to the other (Fig. 31.1), all being

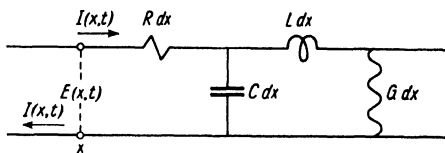


Figure 31.1 Twin conductor line.

measured per unit of length. Then we have the following system of partial differential equations of the first order:

$$(31.1) \quad \begin{aligned} \frac{\partial E}{\partial x} + L \frac{\partial I}{\partial t} + R I &= 0 \\ \frac{\partial I}{\partial x} + C \frac{\partial E}{\partial t} + G E &= 0. \end{aligned}$$

We can derive one partial differential equation of the second order for $E(x, t)$ alone by writing the equations in the form

$$\begin{aligned} -\frac{\partial E}{\partial x} &= \left(L \frac{\partial}{\partial t} + R \right) I \\ -\frac{\partial I}{\partial x} &= \left(C \frac{\partial}{\partial t} + G \right) E. \end{aligned}$$

On differentiating the first equation with respect to x

$$-\frac{\partial^2 E}{\partial x^2} = \left(L \frac{\partial}{\partial t} + R \right) \frac{\partial I}{\partial x}$$

and on inserting the value of $\partial I / \partial x$ from the second equation, we obtain

$$\frac{\partial^2 E}{\partial x^2} = \left(L \frac{\partial}{\partial t} + R \right) \left(C \frac{\partial}{\partial t} + G \right) E$$

or

$$(31.2) \quad \frac{\partial^2 E}{\partial x^2} = LC \frac{\partial^2 E}{\partial t^2} + (LG + RC) \frac{\partial E}{\partial t} + RGE.$$

The same equation, the so-called *telegraph equation*, holds also for $I(x, t)$. For $R = G = 0$, this equation reduces to the wave equation

$$\frac{\partial^2 E}{\partial x^2} = LC \frac{\partial^2 E}{\partial t^2};$$

for $L = G = 0$, it reduces to the heat conduction equation

$$\frac{\partial^2 E}{\partial x^2} = RC \frac{\partial E}{\partial t}$$

(Thomson's Cable Equation, cf. § 30).

However, we examine the full system (31.1) and assume that the twin conductor line was at rest at the time $t = 0$; i.e., that⁵⁸⁾

$$(31.3) \quad E(x, 0+) = I(x, 0+) = 0.$$

Then the image equations of (31.1) are

$$(31.4) \quad \begin{aligned} \frac{d e(x, s)}{dx} + L s i(x, s) + R i(x, s) &= 0 \\ \frac{d i(x, s)}{dx} + C s e(x, s) + G e(x, s) &= 0. \end{aligned}$$

This is a system of ordinary differential equations of the first order which we can solve by a further transformation with respect to x if we consider the initial values $e(0, s)$, $i(0, s)$ as given. This means that their original functions $E(0, t)$, $I(0, t)$, i.e., the values of the voltage and the current, are given at the initial point of the conductor. We introduce the following notation

$$\int_0^\infty e^{-yx} e(x, s) dx = e_1(y, s), \quad \int_0^\infty e^{-yx} i(x, s) dx = i_1(y, s).$$

This $\mathfrak{L}$ -transformation with respect to x transforms the system (31.4) into the following system of linear algebraic equations:

$$(31.5) \quad \begin{aligned} y e_1(y, s) - e(0, s) + (Ls + R) i_1(y, s) &= 0 \\ y i_1(y, s) - i(0, s) + (Cs + G) e_1(y, s) &= 0. \end{aligned}$$

From this we can easily calculate e_1 and i_1 and obtain

$$(31.6) \quad \begin{aligned} e_1(y, s) &= \frac{e(0, s) y - i(0, s) (Ls + R)}{y^2 - (Ls + R)(Cs + G)} \\ i_1(y, s) &= \frac{i(0, s) y - e(0, s) (Cs + G)}{y^2 - (Ls + R)(Cs + G)}. \end{aligned}$$

⁵⁸⁾ The complete solution for arbitrary initial conditions in the case of an infinitely long conductor is found in the book quoted in footnote 56, p. 74, and in G. DOETSCH: *Handbuch der Laplace-Transformation*. 3. Band, Birkhäuser-Verlag, Basel 1956, p. 49.

If we set

$$(31.7) \quad (Ls + R)(Cs + G) = h(s)^2$$

and go from the image functions with the variable y back to the original functions with the variable x , then from TAB. 49, 39 we obtain

$$(31.8) \quad e(x, s) = e(0, s) \cosh x h(s) - i(0, s) \frac{Ls + R}{h(s)} \sinh x h(s)$$

$$(31.9) \quad i(x, s) = i(0, s) \cosh x h(s) - e(0, s) \frac{Cs + G}{h(s)} \sinh x h(s).$$

These are the solutions of the system (31.4) when $e(0, s)$, $i(0, s)$, i.e., originally $E(0, t)$, $I(0, t)$, are given. If a conductor of length l is being considered, then normally, instead of one of these two functions, the value of E or I is given at the end $x = l$. We wish to consider the case when the values of the voltage are given at both ends, that is $E(0, t)$ and $E(l, t)$. For this we can confine ourselves to the case where $E(l, t) = 0$, that is, the conductor is short-circuited at the end $x = l$, since we can obtain the solution for $E(0, t) = 0$ and an arbitrary $E(l, t)$ by replacing x by $l - x$ and combining the two solutions to give the general solution.

To eliminate $i(0, s)$ and introduce $e(l, s)$, we set $x = l$ and $e(l, s) = 0$ in (31.8). Then

$$0 = e(0, s) \cosh l h(s) - i(0, s) \frac{Ls + R}{h(s)} \sinh l h(s),$$

from which we obtain

$$i(0, s) = \frac{h(s)}{Ls + R} \frac{\cosh l h(s)}{\sinh l h(s)} e(0, s).$$

If we insert this value into (31.8), and use the addition theorem for the $\sinh$ function we have

$$(31.10) \quad e(x, s) = e(0, s) \frac{\sinh(l - x) h(s)}{\sinh l h(s)}.$$

Analogously, if we use definition (31.7) of $h(s)$, we obtain from (31.9):

$$(31.11) \quad i(x, s) = e(0, s) \sqrt{\frac{Cs + G}{Ls + R}} \frac{\cosh(l - x) h(s)}{\sinh l h(s)}.$$

Before we transform the image functions $e(x, s)$, $i(x, s)$ into original space we shall consider a special case.

The Distortionless Conductor

The formulas are simplified considerably when $h(s)$ is a linear function. Since

$$\begin{aligned} h(s)^2 &= (Ls + R)(Cs + G) \\ &= \frac{1}{LC} \left\{ \left(LCs + \frac{LG + RC}{2} \right)^2 + \left[LCRG - \left(\frac{LG + RC}{2} \right)^2 \right] \right\} \end{aligned}$$

this is true if and only if

$$LCRG - \left(\frac{LG + RC}{2} \right)^2 = - \left(\frac{LG - RC}{2} \right)^2 = 0,$$

that is

$$(31.12) \quad LG = RC.$$

Then we have

$$\begin{aligned} h(s) &= (LC)^{-1/2} \left(LCs + \frac{LG + RC}{2} \right) = (LC)^{-1/2} (LCs + RC) \\ &= (LC)^{-1/2} (LCs + LG), \end{aligned}$$

Therefore

$$(31.13) \quad h(s) = \left(\frac{C}{L} \right)^{1/2} (Ls + R) = \left(\frac{L}{C} \right)^{1/2} (Cs + G),$$

and further

$$\frac{Cs + G}{Ls + R} = \frac{G}{R} = \frac{C}{L}.$$

If we use the abbreviation

$$(31.14) \quad \left(\frac{C}{L} \right)^{1/2} = a,$$

then, when the condition (31.12) holds, we have

$$(31.15) \quad e(x, s) = e(0, s) \frac{\sinh a(l - x)(Ls + R)}{\sinh al(Ls + R)} = e(0, s) v_0(x, s)$$

$$(31.16) \quad i(x, s) = e(0, s) a \frac{\cosh a(l - x)(Ls + R)}{\sinh al(Ls + R)}.$$

The same quotient of two sinh functions as now occurs in $v_0(x, s)$ occurred in the function $u_0(x, s)$ for the heat conduction equation; however, in place of the linear function $Ls + R$, $\sqrt{s}$ occurred there. That quotient was a $\mathfrak{L}$ -transform and therefore the convolution theorem could be used when going over into the original space. But $v_0(x, s)$ is not a $\mathfrak{L}$ -transform so that a different method must be applied. We rewrite $v_0(x, s)$ in the following way

$$(31.17) \quad v_0(x, s) = \frac{e^{a(l-x)(Ls+R)} - e^{-a(l-x)(Ls+R)}}{e^{aL(Ls+R)} - e^{-aL(Ls+R)}} = \frac{e^{-ax(Ls+R)} - e^{-a(2l-x)(Ls+R)}}{1 - e^{-2aL(Ls+R)}}$$

and consider first the case $l = \infty$. Then $v_0(x, s)$ reduces to

$$e^{-ax(Ls+R)}.$$

If this is put into (31.15) and Rule II is applied, we obtain, with the notation of (31.14),

$$(31.18) \quad E(x, t) = \begin{cases} e^{-R(CL)^{1/2}x} E(0, t - (CL)^{1/2}x) & \text{for } t \geq (CL)^{1/2}x \\ 0 & \text{for } 0 \leq t < (CL)^{1/2}x. \end{cases}$$

This solution signifies a propagation of the input voltage $E(0, t)$ in the direction of increasing x with velocity $(CL)^{-1/2}$. A certain value $E(0, t_0)$ reaches the point x at time t when

$$t - (CL)^{1/2}x = t_0.$$

It requires the time $t - t_0$ to pass along the interval between 0 and x , and therefore we have

$$\frac{x}{t - t_0} = (CL)^{-1/2}.$$

The time $t = (CL)^{1/2}x$ elapses before the first excitation occurs at $x(t_0 = 0)$, i.e., until then $E(x, t) = 0$. This corresponds to the second row in (31.18). As it propagates, the voltage is damped by the factor $e^{-R(CL)^{1/2}x}$.

At a fixed point only a single value of the input voltage arrives at a definite time t . There is, therefore, no superposition of values which have arrived previously and remain there as residues, as in the case of a conductor with arbitrary constants (see p. 179). For this reason a conductor which satisfies the condition (31.12) is called "distortionless".

If l is finite, we expand the function represented by (31.17) into a series

$$\begin{aligned} v_0(x, s) &= (e^{-ax(Ls+R)} - e^{-a(2l-x)(Ls+R)}) \sum_{n=0}^{\infty} e^{-2nla(Ls+R)} \\ &= \sum_{n_1=0}^{\infty} e^{-(2n_1l+x)a(Ls+R)} - \sum_{n_2=1}^{\infty} e^{-(2n_2l-x)a(Ls+R)}. \end{aligned}$$

Multiplying by $e(0, s)$ and transforming term by term we obtain the original function of (31.15):

$$(31.19) \quad E(x, t) = \sum_{n_1=0}^{\infty} e^{-R(CL)^{1/2}(2n_1l+x)} E(0, t - (CL)^{1/2}(2n_1l+x)) \\ - \sum_{n_2=1}^{\infty} e^{-R(CL)^{1/2}(2n_2l-x)} E(0, t - (CL)^{1/2}(2n_2l-x)),$$

where $E(0, t)$ is to be set equal to 0 for $t < 0$. For a fixed pair of values x, t , each sum contains only a finite number of terms. A certain value $E(0, t_0)$ of the input voltage reaches the points x, t , for which

$$t - (CL)^{1/2} (2n_1 l + x) = t_0, \text{ with positive sign,}$$

$$t - (CL)^{1/2} (2n_2 l - x) = t_0, \text{ with negative sign.}$$

These equations represent straight lines which in the strip $0 \leq x \leq l$ of the xt -plane form a zigzag line (Fig. 31.2). Thus the input voltage runs from $x = 0$ to $x = l$, is reflected at $x = l$ with inversion of sign (phase shift of π), then runs back to $x = 0$ and is reflected there with change of sign, etc. Also there is a constant damping which increases with the length of the path. At a point x, t all the values of the input voltage which are carried by the zigzag lines passing through x, t are superimposed; the sign is adjusted according to the number of reflections (Fig. 31.3).

The current $I(x, t)$ is calculated from (31.16) in a similar way.

The Conductor with Arbitrary Constants

Let $LG \neq RC$. We consider first the case $l = \infty$, when (31.10) reduces to

$$(31.20) \quad e(x, s) = e(0, s) e^{-x h(s)} = e(0, s) e^{-x \sqrt{(Ls+R)(Cs+G)}}.$$

To invert we use a formula which involves the modified Bessel function

$$I_1(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+1}}{n! (n+1)!},$$

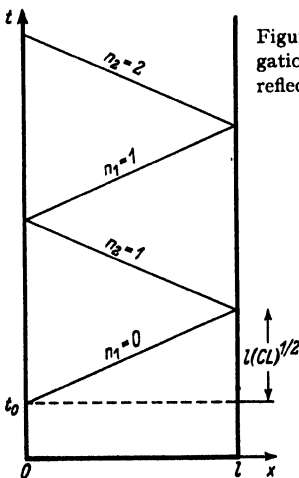


Figure 31.2 (left). Propagation of excitation with reflection at the endpoints.

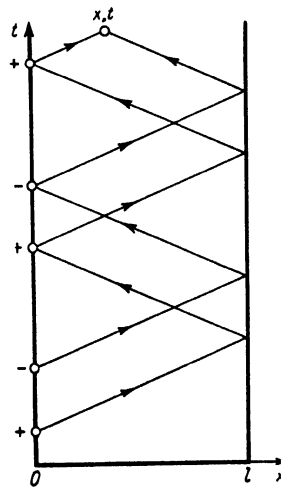


Figure 31.3 (right). Excitations arriving simultaneously at a point after reflections at the endpoints.

$$e^{-x\sqrt{as^2+bs+c}} = e^{-(b/2\sqrt{a})x} e^{-\sqrt{a}xs} + \sqrt{\frac{d}{a}} x \int_{\sqrt{a}x}^{\infty} e^{-st} e^{-(b/2a)t} \frac{I_1\left(\frac{\sqrt{d}}{a}\sqrt{t^2-ax^2}\right)}{\sqrt{t^2-ax^2}} dt$$

with

$$d = (b/2)^2 - ac.$$

We have to set

$$a = LC, \quad b = LG + RC, \quad c = RG, \quad d = (LG - RC)^2/4.$$

If we introduce the abbreviations

$$\begin{aligned} \sqrt{L\bar{C}} &= \alpha, & \frac{1}{2}\left(\frac{G}{\bar{C}} + \frac{R}{\bar{L}}\right) &= \beta_1, & \frac{1}{2}\left(\frac{G}{\bar{C}} - \frac{R}{\bar{L}}\right) &= \beta_2, \\ \frac{1}{2}\left(G\sqrt{\frac{\bar{L}}{\bar{C}}} + R\sqrt{\frac{\bar{C}}{\bar{L}}}\right) &= \gamma_1, & \frac{1}{2}\left(G\sqrt{\frac{\bar{L}}{\bar{C}}} - R\sqrt{\frac{\bar{C}}{\bar{L}}}\right) &= \gamma_2, \end{aligned}$$

then the formula takes the form

$$(31.21) \quad e^{-x\sqrt{(Ls+R)(Cs+G)}} = e^{-\gamma_1 x} e^{-\alpha xs} + \gamma_2 x \int_{\alpha x}^{\infty} e^{-st} e^{-\beta_1 t} \frac{I_1(\beta_2 \sqrt{t^2 - \alpha^2 x^2})}{\sqrt{t^2 - \alpha^2 x^2}} dt.$$

The first term, when multiplied by $e(0, s)$, corresponds in original space to a translation of $E(0, t)$. The second term represents the $\mathfrak{L}$ -transformation of the function

$$(31.22) \quad V(x, t) = \begin{cases} 0 & \text{for } 0 \leq t < \alpha x \\ \gamma_2 x e^{-\beta_1 t} \frac{I_1(\beta_2 \sqrt{t^2 - \alpha^2 x^2})}{\sqrt{t^2 - \alpha^2 x^2}} & \text{for } t \geq \alpha x. \end{cases}$$

Therefore the convolution theorem can be applied to the product of this term and $e(0, s)$. Taking these together the original function is

$$(31.23) \quad E(x, t) = \begin{cases} 0 & \text{for } 0 \leq t < \alpha x \\ e^{-\gamma_1 x} E(0, t - \alpha x) \\ + \gamma_2 x \int_{\alpha x}^t E(0, t - \tau) e^{-\beta_1 \tau} \frac{I_1(\beta_2 \sqrt{\tau^2 - \alpha^2 x^2})}{\sqrt{\tau^2 - \alpha^2 x^2}} d\tau & \text{for } t \geq \alpha x. \end{cases}$$

The first term in the second row corresponds to a propagation of the input voltage to the right with a damping which increases with x . The second term represents the sum of all input voltages $E(0, 0)$ to $E(0, t - \alpha x)$ which have arrived at the point x up to time t , and each voltage is multiplied by a certain weight function. These remaining residue voltages cause a distortion in the transmission.

If the conductor has finite length l , we can proceed exactly as in the case of the distortionless conductor: we expand the factor of $e(0, s)$ in (31.10) into a series and transform term by term. The single terms have the same form as (31.23), except that x is replaced by $2n_1 l + x$ or $2n_2 l - x$. This means, as on p.178, that the values of the input voltage run through the conductor and are reflected at the end points. Also every value leaves behind a residue at the point x and these residues are summed by integrals of the kind appearing in (31.23).

* *

*

The examples which have been considered are intended to show that the $\mathfrak{L}$ -transformation provides a clear and standard method of dealing with boundary and initial value problems in certain partial differential equations. It reduces those which contain two independent variables with constant coefficients to boundary value problems of ordinary differential equations. In general, these are easy to solve. Difficulties first appear in transforming the solution back to original space. The methods demonstrated in the examples, of using tables and of developing in series, can be applied to many other problems. Further methods are found in Chapter 7.

Integral Equations and Integral Relations

§ 32. Integral Equations of the Convolution Type

In physics integral equations of the following type frequently occur ($F(t)$ is the unknown function, $K(t)$ and $G(t)$ are given functions):

$$(32.1) \quad \int_0^t K(t-\tau) F(\tau) d\tau = G(t) \quad (1^{\text{st}} \text{ kind}),$$

$$(32.2) \quad F(t) = G(t) + \int_0^t K(t-\tau) F(\tau) d\tau \quad (2^{\text{nd}} \text{ kind}).$$

Since the integral which appears is a convolution, the integral equation is said to be of convolution type.

We consider first the *integral equation of the second kind*. This is the easier case to deal with. If $\mathfrak{L}\{K\}$ converges absolutely, by the convolution theorem (Rule IX) the $\mathfrak{L}$ -transformation converts the convolution $K * F$ into the simple product of the image functions which, as always, we denote by the corresponding lower case letters. Corresponding to (32.2) we have the image equation

$$f(s) = g(s) + k(s) f(s)$$

with solution

$$(32.3) \quad f(s) = \frac{g(s)}{1 - k(s)}.$$

In this form $f(s)$ cannot be transformed back immediately into original space. If, however, we write $f(s)$ in the form

$$(32.4) \quad f(s) = g(s) + \frac{k(s)}{1 - k(s)} g(s),$$

it can be shown (EINF. p. 286) that an original function $Q(t)$ always corresponds to

$$q(s) = \frac{k(s)}{1 - k(s)}$$

and thus (32.4) can be transformed into original space, giving

$$(32.5) \quad F(t) = G(t) + Q(t) * G(t).$$

This solution, when it is written in the form

$$G(t) = F(t) + (-Q) * G,$$

has the same form as the original integral equation except that the roles of F and G are now interchanged and in place of the "kernel" $K(t)$ we have the "reciprocal kernel" $-Q(t)$.

$Q(t)$ can be represented by an infinite series and also can be calculated in that way. From

$$q(s) = \sum_{n=1}^{\infty} [k(s)]^n$$

it follows that

$$(32.6) \quad Q(t) = \sum_{n=1}^{\infty} K(t)^{*n},$$

where

$$K^{*1} = K, \quad K(t)^{*n} = \underset{1}{K} * \underset{2}{K} * \dots * \underset{n}{K} \quad (n = 2, 3, \dots).$$

In the general theory of integral equations the series (32.6) is called the "Neumann Series".

Sometimes the original function of $q(s)$ can be found immediately. If, for example, the kernel $K(t)$ is a polynomial (or can be approximated to by a polynomial)

$$K(t) = a_0 + a_1 t + \dots + a_r t^r,$$

then

$$k(s) = \frac{a_0}{s} + \frac{1! a_1}{s^2} + \dots + \frac{r! a_r}{s^{r+1}},$$

and

$$q(s) = \frac{k(s)}{1 - k(s)} = \frac{a_0 s^r + 1! a_1 s^{r-1} + \dots + r! a_r}{s^{r+1} - a_0 s^r - 1! a_1 s^{r-1} - \dots - r! a_r}$$

is a rational function whose numerator is of lower degree than the denominator and therefore can be transformed by the methods of § 12.

The *integral equation of the first kind* cannot be solved by a general formula, except in certain cases. The image equation of (32.1) is

$$k(s) f(s) = g(s).$$

Its solution

$$(32.7) \quad f(s) = \frac{g(s)}{k(s)}$$

cannot be transformed back into original space by means of the convolution theorem because $\frac{1}{k(s)}$ is not a $\mathfrak{L}$ -transform. Sometimes the integral equation of the first kind can be converted into one of the second kind. To do this we require the following theorem (EINF. p. 62).

Theorem 32.1. *If $F_1(t)$ is differentiable for $t > 0$ and continuous for $t = 0$, and if $F'_1(t)$ and $F_2(t)$ are absolutely integrable in every interval $0 \leq t \leq T$ and bounded in every interval $0 < T_1 \leq t \leq T_2$, then at every point $t > 0$ where F_2 is continuous from the right (left), $F(t) = F_1 * F_2$ is differentiable from the right (left) and we have*

$$F'(t) = F_1(0) F_2(t) + F'_1 * F_2.$$

If $F_1(0) = 0$, then the assumption that F_2 is continuous is unnecessary.

(We require the theorem in this very general form, for example, in the theory of ordinary differential equations if the excitation function possesses jump points.)

If $K(t)$ and $G(t)$ are differentiable and $K(0) \neq 0$, by differentiation of (32.1) we obtain an integral equation of the second kind

$$K(0) F(t) + \int_0^t K'(t-\tau) F(\tau) d\tau = G'(t),$$

which we can handle by the above method. If

$$K(0) = K'(0) = \dots = K^{(n+1)}(0) = 0, K^{(n)}(0) \neq 0,$$

then by differentiating $(n+1)$ times we obtain

$$K^{(n)}(0) F(t) + \int_0^t K^{(n+1)}(t-\tau) F(\tau) d\tau = G^{(n+1)}(t).$$

This is also an integral equation of the second kind.

This method fails if $K(t)$ possesses no derivatives at $t = 0$, as for example, in the case $K(t) = t^{-\alpha}$, $0 < \alpha < 1$. Then the following method sometimes leads to the desired result: we set

$$\int_0^t F(\tau) d\tau = \Phi(t).$$

Since by Rule VII

$$\mathfrak{L}\{\Phi\} = \varphi(s) = \frac{1}{s} f(s)$$

(32.7) takes the form

$$\varphi(s) = \frac{1}{s k(s)} g(s).$$

If $k(s)^{-1}$ is not a $\mathfrak{L}$ -transform, $[s k(s)]^{-1}$ may be one and then the convolution theorem is applicable.

This is the case with the *Abel integral equation*

$$(32.8) \quad \int_0^t (t-\tau)^{-\alpha} F(\tau) d\tau = G(t) \quad (0 < \alpha < 1)$$

which is of importance in many branches of physics. If we set $F * 1 = \Phi(t)$, then, since

$$t^{-\alpha} \circ \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} = k(s) \quad (\alpha < 1)$$

we obtain the image equation for $\varphi(s)$

$$\frac{\Gamma(1-\alpha)}{s^{-\alpha}} \varphi(s) = g(s)$$

with solution

$$\varphi(s) = \frac{1}{\Gamma(1-\alpha) s^{\alpha}} g(s).$$

Since

$$\frac{1}{s^{\alpha}} \circ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad (\alpha > 0)$$

the original function corresponding to $\varphi(s)$ is

$$\Phi(t) = \frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} t^{\alpha-1} * G(t).$$

(For the direct transformation it is necessary to have $\alpha < 1$ and for the inverse transformation $\alpha > 0$, therefore both together give $0 < \alpha < 1$.)

From the well-known formula

$$\frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)} = \frac{\sin \alpha \pi}{\pi}$$

$\Phi(t)$ has the form

$$\Phi(t) = \frac{\sin \alpha \pi}{\pi} \int_0^t \tau^{\alpha-1} G(t-\tau) d\tau.$$

If $G(t)$ is differentiable and continuous for $t = 0$, we obtain by theorem 32.1

$$(32.9) \quad F(t) = \Phi'(t) = \frac{\sin \alpha \pi}{\pi} \left[G(0) t^{\alpha-1} + \int_0^t \tau^{\alpha-1} G'(t-\tau) d\tau \right].$$

§ 33. Integral Relations

In § 32 we went from an equation for an unknown function, in which a convolution integral appeared, to an algebraic equation by transforming it using the $\mathfrak{L}$ -transformation. Conversely, we now start from algebraic equations for known functions and by the inversion of the $\mathfrak{L}$ -transformation obtain relations in which convolution integrals appear. In this simple way we can set up important integral relations for functions appearing in physics. These are very difficult to prove by a direct approach. We shall illustrate this method by two examples.

For the *double source function* which appears in heat conduction (see (30.8))

$$\psi(x, t) = \frac{x}{2\sqrt{\pi t^{3/2}}} e^{-x^2/4t}$$

we have the correspondence

$$\psi(x, t) \circ \bullet e^{-x} \mathcal{V}_s^- \quad (x > 0).$$

From

$$e^{-x_1} \mathcal{V}_s^- \cdot e^{-x_2} \mathcal{V}_s^- = e^{-(x_1+x_2)} \mathcal{V}_s^- \quad (x_1 > 0, x_2 > 0)$$

the following relation is obtained for ψ :

$$(33.1) \quad \psi(x_1, t) * \psi(x_2, t) = \psi(x_1 + x_2, t).$$

When written explicitly this looks fairly complicated. When we attempt to confirm the relation by direct calculation, we find that it is a very tedious task.

For the *Bessel function*, which occurs frequently in physics,

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{t}{2}\right)^{2n}$$

we have the correspondence

$$J_0(t) \circ \bullet \frac{1}{(s^2 + 1)^{1/2}}.$$

If we factorize the image function in the following way

$$\frac{1}{(s^2 + 1)^{1/2}} = \frac{1}{(s + j)^{1/2}} \frac{1}{(s - j)^{1/2}}$$

and notice that

$$\frac{\Gamma(1/2)}{(s + \alpha)^{1/2}} \circ \bullet t^{-1/2} e^{-\alpha t} \quad (\Gamma(1/2) = \sqrt{\pi}),$$

we obtain by the convolution theorem

$$\begin{aligned} J_0(t) &= \frac{1}{\pi} (t^{-1/2} e^{-jt}) * (t^{-1/2} e^{jt}) = \frac{1}{\pi} \int_0^t \tau^{-1/2} e^{-j\tau} (t-\tau)^{-1/2} e^{j(t-\tau)} d\tau \\ &= \frac{1}{\pi} e^{jt} \int_0^t \tau^{-1/2} (t-\tau)^{-1/2} e^{-2j\tau} d\tau = \frac{1}{\pi} e^{jt} \int_0^1 u^{-1/2} (1-u)^{-1/2} e^{-2ju} du \\ &= \frac{1}{\pi} \int_0^1 e^{jt(1-2u)} [u(1-u)]^{-1/2} du. \end{aligned}$$

If we set

$$1 - 2u = v, \quad \text{and therefore} \quad u = \frac{1-v}{2}, \quad 1-u = \frac{1+v}{2},$$

we obtain the following formula

$$(33.2) \quad J_0(t) = \frac{1}{\pi} \int_{-1}^{+1} e^{jt v} (1-v^2)^{-1/2} dv,$$

which by the substitution

$$v = \cos \varphi, \quad 1-v^2 = \sin^2 \varphi$$

gives the Poisson integral for J_0 :

$$(33.3) \quad J_0(t) = \frac{1}{\pi} \int_0^\pi e^{jt \cos \varphi} d\varphi = \frac{2}{\pi} \int_0^{\pi/2} \cos(t \cos \varphi) d\varphi.$$

These examples are sufficient to show that deep properties of a function can often be found in a simple and clear way by means of the $\mathfrak{L}$ -transformation.

Calculation of the Original Function from the Image Function

§ 34. The Complex Inversion Formula

The solution of functional equations (differential, difference and integral) by means of the $\mathfrak{L}$ -transformation is always accomplished by transforming the equation in original space into image space and then by solving the image equation. The last and, in general, the most difficult step is the calculation of the original function corresponding to the image function found.

The most important auxiliary method for doing this is the use of the complex inversion formula (2.8)

$$(34.1) \quad F(t) = \lim_{Y \rightarrow \infty} \frac{1}{2\pi j} \int_{\alpha - jY}^{\alpha + jY} e^{ts} f(s) ds \quad (t > 0),$$

where α is an abscissa in the half-plane of absolute convergence of the Laplace integral $\mathfrak{L}\{F\}$. This formula is not well adapted for the immediate calculation of $F(t)$, because what is required is knowledge of $f(s)$ for the complex values $s = \alpha + jy$ ($-\infty < y < +\infty$). Its importance lies in that it is a contour integral of an analytic function so that the methods of complex function theory can be employed (such as deformation of the path of integration, calculus of residues etc.) and by this means the properties of the integral can be determined.

We illustrate this method by the following example. The heat conductor which was considered in § 30 extends from $x = 0$ to $x = \infty$ and has initial temperature 0. At the end $x = 0$ the boundary temperature $\cos \omega t$ is applied. From (30.9) the temperature U at the point x and time t is given by

$$U(x, t) = \cos \omega t * \psi(x, t).$$

This is a very elegant expression which, however, gives no information about the behaviour of $U(x, t)$. Therefore we go back to the image function (30.7) which, since

$$\cos \omega t \longleftrightarrow \frac{s}{s^2 + \omega^2}$$

has the form

$$u(x, s) = \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}}.$$

Using the complex inversion formula we have

$$(34.2) \quad U(x, t) = \lim_{Y \rightarrow \infty} \frac{1}{2\pi j} \int_{\alpha-jY}^{\alpha+jY} e^{ts} \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}} ds.$$

Because of the denominator $s^2 + \omega^2$, and the exponent $\sqrt{s}$, $u(x, s)$ has simple poles at $s = \pm j\omega$ and a branch point at $s = 0$. Any real positive number can be used as the abscissa α .

We consider now the closed curve $\mathcal{C}$ given in Fig. 34.1. It consists of the

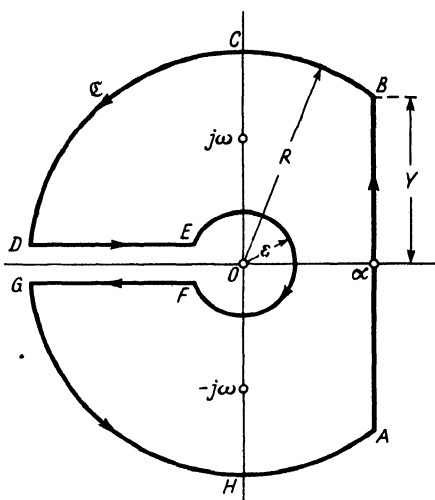


Figure 34.1 A curve, inside which the function $u(x, s)$ is single-valued and analytic except at poles.

vertical at $s = \alpha$, two circular arcs around the origin of radius ϵ and R and two horizontals, one above and one below the negative real axis. Inside this curve $u(x, s)$ is single-valued and analytic with the exception of poles $s = \pm j\omega$. The Cauchy residue theorem gives

$$\frac{1}{2\pi j} \int_{\mathcal{C}} e^{ts} u(x, s) ds = \text{sum of residues of } e^{ts} u(x, s) \text{ at } s = \pm j\omega.$$

The residues are calculated by the following theorem (EINF. p. 167).

Theorem 34.1. *If $f_1(s)$ and $f_2(s)$ are analytic in the neighbourhood of s_0 and if $f_1(s_0) \neq 0$, but $f_2(s)$ has a simple pole at s_0 , then the residue of $\frac{f_1(s)}{f_2(s)}$ at s_0 is equal to $\frac{f_1(s_0)}{f_2'(s_0)}$.*

If we write $e^{ts} u(x, s)$ in the form

$$\frac{e^{ts} s e^{-x\sqrt{s}}}{s^2 + \omega^2} = \frac{f_1(s)}{f_2(s)},$$

then $f_2'(s) = 2s$, and the residue

$$\text{at } s_0 = j\omega \quad \text{is equal to } \frac{e^{tj\omega} j\omega e^{-x\sqrt{j\omega}}}{2j\omega},$$

$$\text{and at } s_0 = -j\omega \text{ is equal to } \frac{e^{-tj\omega} (-j\omega) e^{-x\sqrt{-j\omega}}}{-2j\omega}.$$

Therefore we have

$$\frac{1}{2\pi j} \int_{\mathcal{C}} e^{ts} u(x, s) ds = \frac{1}{2} (e^{j\omega t - x\sqrt{j\omega}} + e^{-j\omega t - x\sqrt{-j\omega}}).$$

Because

$$\sqrt{j} = \cos \frac{\pi}{4} + j \sin \frac{\pi}{4} = \frac{1+j}{\sqrt{2}},$$

$$\sqrt{-j} = \cos \frac{\pi}{4} - j \sin \frac{\pi}{4} = \frac{1-j}{\sqrt{2}}$$

this simplifies to

$$(34.3) \quad \frac{1}{2\pi j} \int_{\mathcal{C}} e^{ts} u(x, s) ds = \frac{1}{2} e^{-x\sqrt{\omega/2}} (e^{j(\omega t - x\sqrt{\omega/2})} + e^{-j(\omega t - x\sqrt{\omega/2})}) \\ = e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2}).$$

We now wish to let ε tend to zero and R to ∞ . Then by (34.2) the integral along AB tends to $U(x, t)$. The behaviour of the integral along the two circular arcs CD and GH is found from a theorem which we can use very frequently in investigations of this type (EINF. p. 157).

Theorem 34.2. *If, for $r \rightarrow \infty$, $f(s) = f(re^{j\varphi})$ tends to zero uniformly in φ in the left half-plane $\Re s \leq 0$, i.e., $\frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2}$, and if $\mathcal{S}$ is a semicircle in the left half-plane of radius r around the origin, then*

$$\int_{\mathcal{S}} e^{ts} f(s) ds \rightarrow 0 \quad \text{for } r \rightarrow \infty,$$

when $t > 0$. The same is true when the full semicircles are replaced by circular arcs which subtend the same angle at the centre⁵⁹.

For s in the left half-plane $\sqrt{s}$ lies in the right half-plane; therefore $\Re \sqrt{s} \geq 0$ and

$$|e^{-x\sqrt{s}}| = e^{-x\Re \sqrt{s}} \leq 1.$$

⁵⁹) This theorem is by no means self-evident. For although e^{ts} is bounded for $t > 0$ in the left half-plane and therefore the integrand tends uniformly to 0, the length of the path of integration tends to infinity.

Further for large values of $|s|$ we have

$$\frac{s}{s^2 + \omega^2} \sim \frac{1}{s},$$

so that in the left half-plane when $|s| \rightarrow \infty$, $u(x, s)$ tends uniformly in φ to zero. From the theorem stated, the integrals along the arcs CD and GH tend to zero as $R \rightarrow \infty$ if $t > 0$, which is true in this case.

e^{ts} and $e^{-s\sqrt{r}}$ are bounded along the arcs BC and HA and the factor $\frac{s}{s^2 + \omega^2}$ causes the integrand to tend uniformly to zero so that these contributions to the integral also converge to zero because the length of the path of integration is bounded.

On the circular arc EF the whole integrand is bounded and the length of the path of integration converges to zero for $\varepsilon \rightarrow 0$: thus this contribution to the integral vanishes.

Therefore, there remain on the left-hand side of (34.3) only the integral over the vertical with abscissa α , which equals $U(x, t)$, and the integrals over the two horizontal rays, which are now replaced by the upper and lower borders of the negative real axis. From (34.3) we have

$$U(x, t) = e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2}) - \frac{1}{2\pi j} \int_{-\infty}^0 e^{ts} u(x, s) ds - \frac{1}{2\pi j} \int_0^{-\infty} e^{ts} u(x, s) ds.$$

In the last two integrals we put $s = re^{j\pi}$ for the upper border and $s = re^{-j\pi}$ for the lower border:

$$\begin{aligned} \int_{-\infty}^0 e^{ts} u(x, s) ds &= - \int_0^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} e^{-j\pi\sqrt{r}} dr, \\ \int_0^{-\infty} e^{ts} u(x, s) ds &= \int_0^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} e^{+j\pi\sqrt{r}} dr. \end{aligned}$$

From these we obtain finally

$$(34.4) \quad U(x, t) = e^{-x\sqrt{\omega/2}} \cos\left(\omega t - x\sqrt{\frac{\omega}{2}}\right) - \frac{1}{\pi} \int_0^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} \sin x\sqrt{r} dr.$$

This form of the solution which has been obtained from the complex inversion integral, gives us a deeper insight into the behaviour of $U(x, t)$ than the earlier form. The integral appearing is a Laplace integral, but t appears instead of s , and r instead of t . It tends to zero for $t \rightarrow \infty$ (see Theorem 38.1) and therefore represents a diminishing transient phenomenon, the behaviour of which can be described more exactly by an asymptotic expansion as will be shown in § 40. The important part of $U(x, t)$ is the first term. This shows that the temperature at a point $x > 0$ executes the same oscillation as was

applied on the boundary $x = 0$ except that it is damped by the factor $e^{-x\sqrt{\omega/2}}$ and has the phase shift $x\sqrt{\omega/2}$. (Cf. the result (30.19) where the transient phenomenon appears in a different form.)

In many problems we can obtain a useful expression for the original function in an exactly similar way.

§ 35. Series Expansions

An obvious method often used in practice for obtaining $F(t)$ from $f(s)$, i.e., to reverse the $\mathfrak{L}$ -transformation, consists of expanding $f(s)$ in a series,

$$f(s) = \sum_{n=0}^{\infty} f_n(s), \text{ in which the terms } f_n(s) \text{ are image functions; i.e. } f_n(s) = \mathfrak{L}\{F_n(t)\}, \text{ and then of transforming this series term by term: } F(t) = \sum_{n=0}^{\infty} F_n(t).$$

It is clear that this cannot always be correct because it amounts to the interchange of an infinite summation and an integration (even an improper one). Nevertheless there are certain types of series in which the transformation may be performed term by term without hesitation.

1. Expansion in Power Series

Theorem 35.1. *If $f(s)$ can be expanded into a series of decreasing powers of s ⁶⁰⁾*

$$(35.1) \quad f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}},$$

which converges for $|s| > R$, then the transformation may be undertaken term by term:

$$(35.2) \quad F(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n.$$

The series for $F(t)$ converges for all real and complex t .

This case refers to a very special class of functions because $f(s)$ is then an analytic and vanishing function at $s = \infty$, and $F(t)$ is an entire function of "exponential type" i.e. it can be bounded by $|F(t)| < C e^{a|t|}$ (EINF. p. 191).

⁶⁰⁾ The reader may notice that the series begins at least with $\frac{1}{s}$ and that there is no term independent of s . This results from the fact that every $\mathfrak{L}$ -transform tends to 0 for $s \rightarrow \infty$ (see Theorem 38.1).

An example is supplied by the Bessel function considered on p. 185. The image function can be expanded as

$$\frac{1}{(s^2+1)^{1/2}} = \frac{1}{s} (1+s^{-2})^{-1/2} = \frac{1}{s} \sum_{n=0}^{\infty} \binom{-1/2}{n} s^{-2n} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{1}{s^{2n+1}},$$

and therefore

$$J_0(t) = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{t^{2n}}{(2n)!}.$$

Since

$$\begin{aligned} \binom{-1/2}{n} &= \frac{(-1/2)(-3/2)\cdots(-n+1/2)}{n!} = \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{n! 2^n} \\ &= \frac{(-1)^n (2n)!}{n! 2^n 2 \cdot 4 \cdots 2n} = \frac{(-1)^n (2n)!}{n! 2^{2n} n!} \end{aligned}$$

we have

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{t}{2}\right)^{2n}.$$

The preceding theorem can be extended to power series with non-integral exponents (EINF. p. 188).

Theorem 35.2. When $f(s)$ can be expanded in an absolutely convergent series for $|s| > R$ of the form

$$(35.3) \quad f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{\lambda_n}}$$

where the λ_n form an arbitrary increasing sequence of numbers $0 < \lambda_0 < \lambda_1 < \dots \rightarrow \infty$, then the transformation can be executed term by term:

$$(35.4) \quad F(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\lambda_n-1}}{\Gamma(\lambda_n)}.$$

The series for $F(t)$ converges for all real and complex $t \neq 0$.

This theorem is, for example, applicable to the Bessel function $J_\nu(t)$ of arbitrary order $\nu > -1/2$. We have

$$\begin{aligned} \mathfrak{L}\{t^\nu J_\nu(t)\} &= \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1) 2^\nu} \frac{1}{(s^2+1)^{\nu+1/2}} = \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1) 2^\nu} \frac{1}{s^{2\nu+1}} (1+s^{-2})^{-\nu-1/2} \\ &= \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1) 2^\nu} \sum_{n=0}^{\infty} \binom{-\nu-1/2}{n} \frac{1}{s^{2n+2\nu+1}}. \end{aligned}$$

By transforming term by term we obtain from this

$$J_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+1)} \left(\frac{t}{2}\right)^{2n+\nu}.$$

2. Series Expansion with Respect to Exponential Functions

If $f(s)$ is a *rational function*

$$f(s) = \frac{r_1(s)}{r_2(s)},$$

in which the numerator r_1 is of lower degree than the denominator r_2 , then $f(s)$ can be expanded in a finite number of partial fractions. If the denominator has the simple zeros $\alpha_1, \dots, \alpha_n$,

$$r_2(s) = (s - \alpha_1) \cdots (s - \alpha_n),$$

then the coefficients which appear are the residues of $f(s)$ at the poles α_ν , which, by Theorem 34.1, are equal to $\frac{r_1(\alpha_\nu)}{r_2'(\alpha_\nu)}$. (If $r_1(\alpha_\nu) = 0$, then $r_1(s)$ also contains the factor $s - \alpha_\nu$, which cancels in $\frac{r_1(s)}{r_2(s)}$, and there is no pole at α_ν and no residue.) Therefore we have

$$(35.5) \quad f(s) = \sum_{\nu=1}^n \frac{r_1(\alpha_\nu)}{r_2'(\alpha_\nu)} \frac{1}{s - \alpha_\nu}.$$

Because there is only a finite number of terms, the transformation can be made term by term and the result is

$$(35.6) \quad F(t) = \sum_{\nu=1}^n \frac{r_1(\alpha_\nu)}{r_2'(\alpha_\nu)} e^{\alpha_\nu t}.$$

In the case of multiple zeros, the exponential functions are multiplied by polynomials, cf. (12.14). This representation appeared in the solutions of ordinary differential equations and of systems of such equations and was used in several examples (see §§ 11, 12).

In partial differential equations there frequently arise meromorphic functions instead of rational functions; these are functions which are analytic in the entire plane except at isolated poles. The number of these poles can be finite or infinite. For example, in solving the heat conduction equation (p. 168) the function $u_0(x, s)$ appeared which, when we set $l = \pi$, has the form

$$u_0(x, s) = \frac{e^{(\pi-x)\sqrt{s}} - e^{-(\pi-x)\sqrt{s}}}{e^{\pi\sqrt{s}} - e^{-\pi\sqrt{s}}}.$$

This function is apparently two-valued in the neighbourhood of $s = 0$ because of the square root $\sqrt{s}$. But after one cycle when $\sqrt{s}$ has become $-\sqrt{s}$, $u_0(x, s)$ has become

$$\frac{e^{-(\pi-x)\sqrt{s}} - e^{(\pi-x)\sqrt{s}}}{e^{-\pi\sqrt{s}} - e^{\pi\sqrt{s}}} = \frac{e^{(\pi-x)\sqrt{s}} - e^{-(\pi-x)\sqrt{s}}}{e^{\pi\sqrt{s}} - e^{-\pi\sqrt{s}}},$$

and therefore has returned to its original value.

The only singularities are the (simple) zeros of the denominator. From

$$e^{\pi\sqrt{s}} - e^{-\pi\sqrt{s}} = 0 \quad \text{or} \quad e^{2\pi\sqrt{s}} = 1$$

it follows that

$$2\pi\sqrt{s} = \nu \cdot 2\pi j \quad (\nu = 0, \pm 1, \pm 2, \dots),$$

i.e.

$$s = -\nu^2.$$

Because negative values of ν give the same results as positive, we can omit them. Further since the numerator of $u_0(x, s)$ vanishes at $s = 0$ there is no singularity at that point. Therefore we have to consider only the poles

$$\alpha_\nu = -\nu^2, \quad \nu = 1, 2, \dots$$

By theorem 34.1 the residue of $u_0(x, s)$ at α_ν is equal to

$$\begin{aligned} \frac{e^{(\pi-x)\sqrt{s}} - e^{-(\pi-x)\sqrt{s}}}{2\sqrt{s}(e^{\pi\sqrt{s}} + e^{-\pi\sqrt{s}})} \bigg|_{s=\alpha_\nu} &= \frac{e^{(\pi-x)\nu j} - e^{-(\pi-x)\nu j}}{2\nu j(e^{\pi\nu j} + e^{-\pi\nu j})} = \frac{2j \sin(\pi-x)\nu}{\nu j \cos \pi\nu} \\ &= \frac{2}{\pi} \nu \sin \nu x. \end{aligned}$$

The so-called principal part of $u_0(x, s)$ at the pole $\alpha = -\nu^2$ is therefore the partial fraction

$$\frac{2}{\pi} \nu \sin \nu x \frac{1}{s + \nu^2}.$$

From purely formal considerations one might imagine that the meromorphic function $u_0(x, s)$ could be constructed from these partial fractions like a rational function,

$$(35.7) \quad u_0(x, s) = \frac{2}{\pi} \sum_{\nu=1}^{\infty} \nu \sin \nu x \frac{1}{s + \nu^2},$$

and that the original function could then be obtained by term-by-term transformation

$$(35.8) \quad U_0(x, t) = \frac{2}{\pi} \sum_{\nu=1}^{\infty} \nu \sin \nu x e^{-\nu^2 t}.$$

This method, which is often used in technical publications, happens to be correct in the case being discussed, but it can lead to incorrect results. To begin with, the expansion (35.7) is doubtful, because we can say only that if we subtract all principal parts from $u_0(x, s)$ then there remains a function which has no singularities and is therefore an entire function. The determination of this entire function, which here happens to be equal to 0, is extremely difficult in general.

It can happen that this entire function is the most important part of the image function. If, for example, it contains a term which is a finite $\mathfrak{L}$ -transform as

$$\int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{s}$$

(this function can be defined analytically at $s = 0$), then this represents an entire function and consequently possesses no singularities in the finite plane. Such a part of the image function is never considered in the above procedure.

It is obvious that the partial fraction expansion does not exist when the meromorphic function has only a finite number of poles but is not a rational function; this can happen. If we simply write down the partial fraction expansion it would convey the erroneous impression that the function was rational. Moreover, as has already been stated at the beginning of this paragraph, the transition from (35.7) to (35.8) i.e. the term-by-term transformation, is also suspect.

With the help of the complex inversion formula we can place this method upon a solid foundation, in which the expansion of $u_0(x, s)$ into a series of partial fractions (used above only as an intermediate stage) no longer plays a role and we obtain immediately an expansion of the form (35.8). We use the Cauchy calculus of residues. For this purpose, $U(x, t)$ is written in the form

$$(35.9) \quad U(x, t) = \lim_{Y \rightarrow \infty} \frac{1}{2\pi j} \int_{\alpha-jY}^{\alpha+jY} e^{ts} u(x, s) ds.$$

To the left of the vertical which has abscissa α we add the supplementary curves $\mathfrak{C}_1, \mathfrak{C}_2, \dots$, which begin at the heights $Y_1, Y_2, \dots$, end at $-Y_1, -Y_2, \dots$ and include successively the poles $\alpha_1, \alpha_2, \dots$. These can be semi-circles, rectangles, or similar curves (see Fig. 35.1). Then we have

$$\begin{aligned} & \frac{1}{2\pi j} \int_{\alpha-jY_n}^{\alpha+jY_n} e^{ts} u(x, s) ds + \frac{1}{2\pi j} \int_{\mathfrak{C}_n} e^{ts} u(x, s) ds \\ &= \text{sum of the residues of } e^{ts} u(x, s) \text{ at } \alpha_1, \dots, \alpha_n. \end{aligned}$$

If $u(x, s)$ has a simple pole at α_v , then the principal part has the form

$$\frac{b_v}{s - \alpha_v}.$$

b_v is dependent on x , and the residue of $e^{ts} u(x, s)$ is equal to

$$b_v e^{\alpha_v t}.$$

If, for example, $u(x, s)$ has a pole of order 3 at α_v , then the principal part is of the form

$$\frac{b_v}{s - \alpha_v} + \frac{c_v}{(s - \alpha_v)^2} + \frac{d_v}{(s - \alpha_v)^3},$$

and the residue of $e^{ts} u(x, s)$ is equal to

$$\left(b_v + \frac{c_v}{1!} t + \frac{d_v}{2!} t^2 \right) e^{\alpha_v t}.$$

For the purpose of simplifying the formulas, we assume in what follows that all poles are simple. Then we obtain

$$(35.10) \quad \frac{1}{2\pi j} \int_{\alpha_{-j} Y_n}^{\alpha_{+j} Y_n} e^{ts} u(x, s) ds = \sum_{v=1}^n b_v(x) e^{\alpha_v t} - \frac{1}{2\pi j} \int_{\mathfrak{C}_n} e^{ts} u(x, s) ds.$$

If $Y_n \rightarrow \infty$ with increasing n and if

$$(35.11) \quad \frac{1}{2\pi j} \int_{\mathfrak{C}_n} e^{ts} u(x, s) ds \rightarrow 0,$$

then we obtain, from (35.9)

$$(35.12) \quad U(x, t) = \sum_{v=1}^{\infty} b_v(x) e^{\alpha_v t}.$$

Thus we have established the desired expansion. *The condition (35.11) is decisive for its validity.* For this, a lot depends upon how the curves $\mathfrak{C}_n$

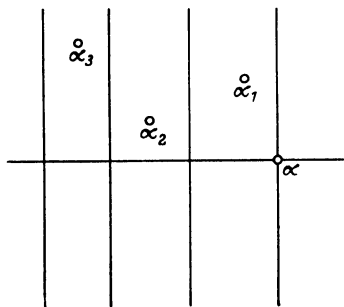
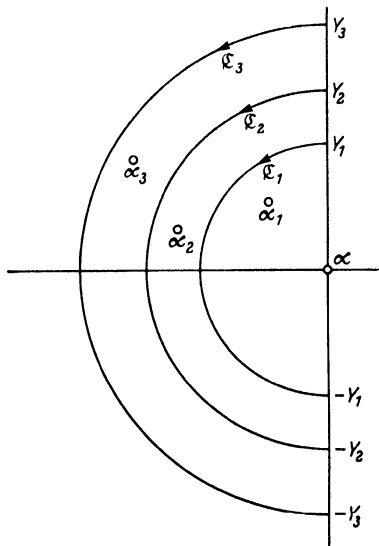


Figure 35.2 The replacement of the original straight line path of integration by a set of parallel vertical lines.

Figure 35.1 The supplementary curves $\mathfrak{C}_n$ which include poles $\alpha_1, \dots, \alpha_n$.

are chosen. They must suit the character of the function $u(x, s)$ so that the behaviour of the function along the curves can be determined easily. For example, the proof of (35.11) for the expansion (35.8) can be demonstrated most simply when the $\mathfrak{C}_n$ are chosen as parabolas⁶¹).—Naturally it is not always possible to establish (35.11). Then the validity of (35.12) must remain undecided.

In a *variation of the process* described, the line of integration ($x = \alpha$) is replaced by vertical lines which have different abscissae lying between the poles α_n (Fig. 35.2). This is possible if, in the strips between these lines and the original line of integration, $u(x, s)$ tends uniformly to zero as s runs towards the top or bottom of these strips. What is left is an integral whose path runs between the n^{th} and the $(n+1)^{\text{th}}$ pole. For the expansion (35.12) to be valid this integral must tend to zero for $n \rightarrow \infty$.

The method can be used only if $u(x, s)$ is a unique function because only then is the Cauchy residue theorem applicable. If the singularities of $u(x, s)$ are branch points, then we must deform the path of integration as on p. 188 or apply the method of § 41.

Due to lack of space we give no special example, but refer those readers who wish to know how the process is carried out in complicated cases to the literature quoted in footnote 61.

In technical papers the method is sometimes applied to *rational functions* where, however, it is unnecessary because then we can simply set up the partial fraction expansion and transform it term by term. The proof of (35.11) is generally very lengthy for *meromorphic functions*, and therefore it is seldom accomplished in technical literature. Most of the time only the initial steps, up to equation (35.10), of the process are performed, while the main point, namely, the proof of (35.11) is omitted. Then, however, the result (35.12) can be given only a heuristic meaning. Without the proof of (35.11), the method is no more exact than the simpler process described on p. 194 which sets up the image function as an infinite series of partial fractions (the sum of the principal parts) and transforms it term by term. If we cannot, or do not wish to, prove the validity of (35.11) then we can save ourselves the whole labour of the calculus of residues, because then the greater rigour is only simulated and is not really present.

Because it can happen that equations (35.11, 12) are not true at all, every author who does not give the proof of (35.11) should state explicitly that the result is true only if equation (35.11) can be proved.

⁶¹) See G. DOETSCH: *Handbuch der Laplace-Transformation*, 1. Band, Birkhäuser Verlag, Basel 1950, pp. 277—281. An example for which the $\mathfrak{C}_n$ can be chosen as circular arcs is given in EINF. p. 169. Further examples are found in H. S. CARSLAW and J. C. JAEGER: *Operational Methods in Applied Mathematics*, Oxford University Press 1941, and R. V. CHURCHILL: *Operational Mathematics*, McGraw-Hill Book Co., New York 1958.

3. Expansion in a Series of Arbitrary Functions

The preceding expansions use special functions, namely, power and exponential functions. Two other theorems may be quoted which are related to expansions with respect to arbitrary functions and have proved to be very useful in many practical cases (EINF. p. 186).

Theorem 35.3. *A function $f(s)$ may be described as an infinite series of $\mathfrak{L}$ -transforms in $\Re s \geq x_0$*

$$f(s) = \sum_{n=0}^{\infty} f_n(s), \quad f_n(s) \bullet \bullet F_n(t).$$

For this all integrals

$$\int_0^{\infty} e^{-st} F_n(t) dt = f_n(s) \quad (n = 0, 1, \dots)$$

must exist in a common half-plane $\Re s \geq x_0$. Two additional stipulations are also required:

a) *The integrals*

$$\int_0^{\infty} e^{-st} |F_n(t)| dt = \varphi_n(s) \quad (n = 0, 1, \dots)$$

shall exist in this half-plane $\Re s \geq x_0$.

b) *The series*

$$\sum_{n=0}^{\infty} \varphi_n(x_0)$$

shall converge.

Then $\sum_{n=0}^{\infty} F_n(t)$ converges absolutely to a function $F(t)$ for almost all $t \geq 0$, and $\mathfrak{L}\{F\} = f(s)$, i. e.,

$$\sum_{n=0}^{\infty} f_n(s) \bullet \bullet \sum_{n=0}^{\infty} F_n(t).$$

The expression "almost all" comes from Lebesgue integration theory and it means "for all t with the exception of a set of measure zero". The latter is a set whose points can be enclosed in denumerably infinite intervals of arbitrarily small total length. In practical cases the series generally converges for all t .

Instead of considering a single series, we wish to show the applicability of the theorem to an entire class of series.

An important class of series which are useful for solving differential and difference equations is obtained from the *factorial series*:

$$(35.13) \quad \sum_{n=0}^{\infty} \frac{a_n n!}{s(s+1) \cdots (s+n)}.$$

Every term is a $\mathfrak{L}$ -transform, since

$$\frac{n!}{s(s+1) \cdots (s+n)} \bullet \circ (1 - e^{-t})^n \text{ for } \Re s > 0.$$

Because the original function is positive, we obtain

$$\int_0^{\infty} e^{-st} |F_n(t)| dt = \int_0^{\infty} e^{-st} |a_n| (1 - e^{-t})^n dt = |a_n| \frac{n!}{s(s+1) \cdots (s+n)} = \varphi_n(s) \text{ for } \Re s > 0.$$

The stipulation b) of the preceding theorem demands therefore that the series

$$(35.14) \quad \sum_{n=0}^{\infty} \frac{|a_n| n!}{x_0(x_0+1) \cdots (x_0+n)}$$

converges for a certain $x_0 > 0$. This is nothing more than the series (35.13) with $s = x_0 > 0$, in which the absolute value of every term has been taken. It is true in general that the factorial series converges in a half-plane $\Re s > \lambda$ (when it has a convergence point at all), and that it converges absolutely at least for $\Re s > (\lambda + 1)$. If we now choose $x_0 > 0$ as well as $x_0 > \lambda + 1$, then the stipulation b) is satisfied for (35.14). Thus a factorial series (35.13) is always a $\mathfrak{L}$ -transform (if it converges anywhere), and almost everywhere its original function is equal to

$$(35.15) \quad \sum_{n=0}^{\infty} a_n (1 - e^{-t})^n.$$

But this is a power series in $1 - e^{-t}$. If a series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z_0 > 0$, then it certainly converges for $0 \leq z \leq z_0$. Because $0 \leq 1 - e^{-t} \leq 1 - e^{-t_0}$ for $0 \leq t \leq t_0$, it follows from the convergence of (35.15) for $t_0 > 0$ that it converges for $0 \leq t \leq t_0$. The series converges therefore for all t and not only almost all t .

Under very general assumptions a $\mathfrak{L}$ -transform can be expanded into a factorial series; the coefficients are determined as follows

$$sf(s) \rightarrow a_0, \quad \frac{1}{1!} s(s+1) \left[f(s) - \frac{a_0}{s} \right] \rightarrow a_1,$$

$$\frac{1}{2!} s(s+1)(s+2) \left[f(s) - \frac{a_0}{s} - \frac{a_1 1!}{s(s+1)} \right] \rightarrow a_2, \dots \text{ for } s \rightarrow \infty.$$

In cases when these values can be calculated easily, (35.15) supplies a convenient formula for the determination of the original function.

This method, which plays an important role in the theory⁶²), should be applied also in practice, though up till now it has been used rarely.

Theorem 35.4. *If $\mathfrak{L}\{F_1\} = f_1(s)$ and $\mathfrak{L}\{F_2\} = f_2(s)$ converge absolutely, if $\varphi(z_1, z_2)$ is an analytic function of two variables in the neighbourhood of $z_1 = z_2 = 0$ and if $\varphi(z_1, z_2)$ equals 0 at $z_1 = z_2 = 0$, i.e.,*

$$\varphi(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} a_{n_1, n_2} z_1^{n_1} z_2^{n_2} \quad \text{with } a_{00} = 0$$

converges in the two circles $|z_1| < r$, $|z_2| < r$, then $\varphi(f_1(s), f_2(s))$ is a $\mathfrak{L}$ -transform whose original function is given by the term-by-term transformation of

$$\sum_{n_1, n_2=0}^{\infty} a_{n_1, n_2} f_1(s)^{n_1} f_2(s)^{n_2}.$$

A corresponding theorem holds when φ depends on only one variable or on more than two variables.

With this theorem (EINF. p. 191) the expansion (30.12) for $U_0(x, t)$ can be made rigorous very simply. We had

$$u_0(x, s) = \frac{\sinh(l-x)\sqrt{s}}{\sinh l\sqrt{s}} = \frac{e^{(l-x)\sqrt{s}} - e^{-(l-x)\sqrt{s}}}{e^{l\sqrt{s}} - e^{-l\sqrt{s}}} = \frac{e^{-x\sqrt{s}} - e^{-(2l-x)\sqrt{s}}}{1 - e^{-2l\sqrt{s}}}.$$

With

$$f_1(s) = e^{-x\sqrt{s}}, \quad f_2(s) = e^{-(2l-x)\sqrt{s}}$$

we obtain

$$u_0(x, s) = \frac{f_1(s) - f_2(s)}{1 - f_1(s)f_2(s)}.$$

If we set

$$\begin{aligned} \varphi(z_1, z_2) &= \frac{z_1 - z_2}{1 - z_1 z_2} = (z_1 - z_2) \sum_{n=0}^{\infty} z_1^n z_2^n \quad (|z_1 z_2| < 1) \\ &= \sum_{n=0}^{\infty} (z_1^{n+1} z_2^n - z_1^n z_2^{n+1}), \end{aligned}$$

then this function satisfies the conditions of the theorem. Therefore $u(x, s)$ is a $\mathfrak{L}$ -transform whose series expansion

$$\begin{aligned} \sum_{n=0}^{\infty} [e^{-(n+1)x\sqrt{s}} e^{-n(2l-x)\sqrt{s}} - e^{-nx\sqrt{s}} e^{-(n+1)(2l-x)\sqrt{s}}] \\ = \sum_{n=0}^{\infty} [e^{-(2nl+x)\sqrt{s}} - e^{-(2(n+1)l-x)\sqrt{s}}] \end{aligned}$$

may be transformed term by term, and hence (30.12) is obtained.

⁶²) See G. DOETSCH: *Handbuch der Laplace-Transformation*, 2. Band, 11. Kap., Birkhäuser Verlag, Basel 1955.

§ 36. Numerical Calculation of the Original Function

The methods given in § 35 for determining the original function from the image function assume that the image function is known in so far as its singularities can be determined, or that it can be expanded into a series of a definite form. Then we obtain a representation of the original function in a general form which is useful for theoretical purposes: for example, the examination of the behaviour at small or large t . In many cases however, we wish to know the numerical values of the original function in certain intervals. Naturally the representations obtained in § 35 can also be used for the numerical calculation, but in practice that is a needlessly lengthy procedure. The ideal method would be one by which the *original function could be calculated immediately by a definite scheme from a number of values of the image function $f(s)$* . So many methods of this kind have been given that it is impossible to quote them all. We will confine ourselves to demonstrating one which uses only well-known methods and which gives a numerical expression for $F(t)$ rather quickly⁶³.

Two assumptions are made; these, however, do not limit the generality of the process. 1. $f(s)$ exists for $\Re s > 0$. This can always be achieved when $f(s+a)$ is taken with sufficiently large a in place of $f(s)$ and this causes the function $F(t)$ to be multiplied by e^{-at} . 2. $F(0+) = 0$. If this is not the case the value of $F(0+)$ is determined by Theorem 38.2, this value is subtracted from $F(t)$ and thus $f(s)$ is replaced by $f(s) - F(0+)/s$.

The process uses only the values of $f(s)$ at the equidistant points $s = (2n+1)\sigma$ (σ arbitrary > 0 , $n = 0, 1, \dots$)⁶⁴. For this purpose the $\mathfrak{L}$ -integral

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

is transformed by the substitution

$$(36.1) \quad e^{-\sigma t} = \cos \vartheta, \quad F(t) = F\left(-\frac{1}{\sigma} \log \cos \vartheta\right) = \Phi(\vartheta)$$

into

$$(36.2) \quad \sigma f(s) = \int_0^{\pi/2} (\cos \vartheta)^{s/\sigma-1} \sin \vartheta \Phi(\vartheta) d\vartheta.$$

⁶³) This process originates from A. PAPOULIS: A new method of inversion of the Laplace transform. *Quarterly of Applied Mathematics* 14 (1957), pp. 405–414. There two additional methods are given which are based on expansions in Legendre and Laguerre functions. For such expansions cf. *Handbuch* quoted in footnote 61, 1. Band, p. 301.

⁶⁴) A $\mathfrak{L}$ -transform is determined uniquely by its values at a sequence of equidistant points parallel to the real axis (EINF. p. 32).

Then we have

$$(36.3) \quad \sigma f((2n+1)\sigma) = \int_0^{\pi/2} \cos^{2n} \vartheta \sin \vartheta \Phi(\vartheta) d\vartheta.$$

We expand $\Phi(\vartheta)$ in a Fourier series with respect to the functions $\sin(2\nu+1)\vartheta$, which form a complete orthogonal system in the interval $(0, \pi/2)$:

$$(36.4) \quad \Phi(\vartheta) = \sum_{\nu=0}^{\infty} C_{\nu} \sin(2\nu+1)\vartheta.$$

This is suggested by the fact that the kernel in the integral (36.3) can be represented as a linear combination of the functions $\sin(2\nu+1)\vartheta$:

$$(36.5) \quad \cos^{2n} \vartheta \sin \vartheta = 2^{-2n} \sum_{k=0}^n \left[\binom{2n}{k} - \binom{2n}{k-1} \right] \sin[2(n-k)+1]\vartheta \quad \text{with} \quad \binom{2n}{-1} = 0.$$

If we insert (36.4, 5) into (36.3) then, since

$$\int_0^{\pi/2} \sin(2\mu+1)\vartheta \sin(2\nu+1)\vartheta d\vartheta = \begin{cases} 0 & \text{for } \mu \neq \nu \\ \pi/4 & \text{for } \mu = \nu \end{cases}$$

only the terms with $\nu = n-k$ ($k=0, \dots, n$) remain for fixed n :

$$\sigma f((2n+1)\sigma) = 2^{-2n} \frac{\pi}{4} \sum_{k=0}^n \left[\binom{2n}{k} - \binom{2n}{k-1} \right] C_{n-k}$$

or

$$(36.6) \quad \left[\binom{2n}{n} - \binom{2n}{n-1} \right] C_0 + \dots + \left[\binom{2n}{k} - \binom{2n}{k-1} \right] C_{n-k} + \dots + C_n = \frac{4^{n+1}}{\pi} \sigma f((2n+1)\sigma).$$

If we set in turn $n=0, 1, \dots$, then we obtain a linear recurrence relation for the C_{ν} :

$$C_0 = \frac{4}{\pi} \sigma f(\sigma)$$

$$C_0 + C_1 = \frac{4^2}{\pi} \sigma f(3\sigma)$$

$$2C_0 + 3C_1 + C_2 = \frac{4^3}{\pi} \sigma f(5\sigma)$$

.

The C_n can be calculated easily from these equations. The coefficients of $C_0, \dots, C_n$ for $n = 0$ to $n = 6$ are given in Table 36.1⁶⁵).

Table 36.1. The coefficient of C_n in equation (36.6)

n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
0	1						
1	1	1					
2	2	3	1				
3	5	9	5	1			
4	19	28	20	7	1		
5	42	90	75	35	9	1	
6	132	297	275	154	54	11	1

Using a finite number of coefficients C_n we obtain a partial sum of (36.4) and thus an approximation for $\Phi(\vartheta)$. If $F(t)$ is required for a fixed value of t then $\Phi(\vartheta)$ is to be calculated for $\vartheta = \arccos e^{-\sigma t}$. The value of σ is chosen according to the interval in which $F(t)$ has to be found; for small t , σ should be chosen to be large, and conversely for large t , σ should be chosen to be small, because $t \rightarrow 0$ corresponds to $s \rightarrow \infty$, and $t \rightarrow \infty$ corresponds to $s \rightarrow 0$ (see Theorems 38.2 and 3).

§ 37. Determination of the Maximum of the Original Function from the Image Function

In many cases we are not interested in the entire course of the time function, but only in its absolute maximum; for instance, when we desire to know whether or not a voltage exceeds a certain amount. It would be advantageous if the maximum could be obtained immediately from the image function without calculating the original function. However this problem has not yet been solved. This is not surprising because the project is an ambitious one. Even when an explicit expression is known for $F(t)$ the absolute maximum cannot be determined directly: we have to look for the zeros of $F'(t)$ at which points, according to the rules of the differential calculus, $F(t)$ possesses a relative extremum (maximum or minimum) or a point of inflection with horizontal tangent. Then the function values must be calculated at these points; the greatest supplies the absolute maximum.

⁶⁵) The table is taken from the work quoted in footnote 63. There the coefficients are given up to $n = 10$.

The more modest problem, that of determining the positions of the relative extrema, can, in fact, be solved by means of the $\mathfrak{L}$ -transformation, by using the following theorem⁶⁶):

Theorem 37.1. *If $F(t)$ is real and has n changes of sign for $t > 0$, then the derivatives $f^{(k)}(s)$ possess exactly n real zeros for all sufficiently large k 's. When $F(t)$ has a change of sign at t_0 then $f^{(k)}(s)$ possesses a zero s_k with*

$$\lim_{k \rightarrow \infty} \frac{k}{s_k} = t_0.$$

It is to be noticed that with a continuous function a point where a change of sign takes place is a zero, but a zero does not require a change of sign to take place.

For our purpose we apply the theorem to F' instead of to F . We have

$$\mathfrak{L}\{F'\} = s f(s) - F(0+) = f_1(s),$$

where $F(0+)$ can be determined by Theorem 38.2 from $f(s)$ as $\lim_{s \rightarrow \infty} s f(s)$.

Therefore $f_1(s)$ is known. A number of the derivatives $f_1^{(k)}(s)$ are calculated and for each derivative the zeros $s_k^{(1)}, \dots, s_k^{(n)}$ are found. The limiting values of $k/s_k^{(1)}, \dots, k/s_k^{(n)}$ give the points $t_1, \dots, t_n$, at which $F'(t)$ has a sign change and therefore also a zero. These are the points where $F(t)$ possesses a relative maximum or minimum. (A point of inflection with a horizontal tangent does not occur at these points.) Then the function values themselves must be calculated, perhaps by the method of § 36.

This process requires much calculation and in some cases converges very slowly. However, there are cases when it converges quickly. As an example we consider

$$F(t) = t e^{-t}.$$

The derivative $F'(t) = e^{-t}(1-t)$ shows that the only maximum lies at $t_0 = 1$. If $F(t)$ itself was not known, but only its $\mathfrak{L}$ -transform

$$f(s) = \frac{1}{(s+1)^2},$$

then we would first determine

$$F(0+) = \lim_{s \rightarrow \infty} \frac{s}{(s+1)^2} = 0$$

and then consider

$$f_1(s) = \frac{s}{(s+1)^2}.$$

⁶⁶) The proof of this theorem is found in D. V. WIDDER: The inversion of the Laplace integral and the related moment problem. *Trans. Amer. Math. Soc.* 36 (1934), pp. 107—200 (p. 156).

Using Leibniz Rule we have

$$f_1^{(k)} = s \frac{d^k}{ds^k} (s+1)^{-2} + k \frac{d^{k-1}}{ds^{k-1}} (s+1)^{-2}.$$

Since

$$\frac{d^k}{ds^k} (s+1)^{-2} = (-1)^k (k+1)! (s+1)^{-(k+2)}$$

we obtain

$$\begin{aligned} f_1^{(k)} &= s(-1)^k (k+1)! (s+1)^{-(k+2)} + k(-1)^{k-1} k! (s+1)^{-(k+1)} \\ &= (-1)^{k-1} k! \frac{k-s}{(s+1)^{k+2}}. \end{aligned}$$

Therefore every k^{th} derivative has the single zero $s_k = k$, and $k/s_k = 1$. In this case we have $k/s_k = t_0$ for every k .

The Asymptotic Behaviour of Functions and the Question of Stability

§ 38. Some Limit Theorems

Frequently the engineer is not interested in the explicit expression for the solution $Y(t)$ of a differential equation, i.e., in the *complete course* of an occurrence, but only in certain *properties* as, for example, the behaviour of $Y(t)$ in the neighbourhood of $t = 0$, i.e. immediately after "switching on", or in the neighbourhood of $t = \infty$, i.e., for large t , when it is of interest whether or not the solution is stable. It is desirable to be able to answer these types of questions purely from inspection of the *image function* without obtaining the original function explicitly. Conversely, sometimes we wish to calculate the behaviour of the image function from the original function. For example, in the evaluation of the complex inversion integral of § 34 by deformation of the path of integration in formula (34.4) a $\mathfrak{L}$ -integral is produced: the behaviour of this is of interest mainly for large values of the variable.

Expressed mathematically it is a question of finding from one function the "*asymptotic behaviour*" of its corresponding function, as the variable approaches a certain value such as 0 or ∞ . This chapter is concerned with problems of this type, and first we give a few particularly simple theorems which are often of use in practice.

Theorem 38.1. *It is a general property of all $\mathfrak{L}$ -transforms $f(s)$ that they converge to 0 when s tends through real values to $+\infty$. This is true even when s tends to ∞ along a ray in the complex plane which makes an angle $< \pi/2$ with the real axis (EINF. p. 138).*

This theorem represents a *necessary* condition that every image function must satisfy, and it often allows one to conclude immediately that a function cannot be a $\mathfrak{L}$ -transform. Thus, for example, it is clear that a constant $c \neq 0$ or a power s^α , with a positive α , are not image functions.

The condition $f(s) \rightarrow 0$ for $s \rightarrow +\infty$ is by no means sufficient for $f(s)$ to be an image function. Thus, for example, e^{-s} tends to 0 on every ray which makes an angle $< \pi/2$ with the positive real axis, but it does not represent an image function (EINF. p. 35).

Theorem 38.2. *If $\lim_{t \rightarrow 0} Y(t)$ exists then (EINF. p. 219)*

$$\lim_{t \rightarrow 0} Y(t) = \lim_{s \rightarrow \infty} s y(s).$$

This theorem can be used to determine $\lim_{t \rightarrow 0} Y(t) = Y(0+)$ simply from knowledge of $y(s)$ if we know that the value $Y(0+)$ exists (but without knowing its value).

There are cases where $\lim_{s \rightarrow \infty} s y(s)$ exists, but $\lim_{t \rightarrow 0} Y(t)$ does not exist; for example, 

$$Y(t) = \frac{1}{\sqrt{t}} \cos \frac{1}{t}, \quad y(s) = \sqrt{\frac{\pi}{s}} e^{-\sqrt{2}s} \cos \sqrt{2}s.$$

Here $\lim_{s \rightarrow \infty} s y(s) = 0$, but $\lim_{t \rightarrow 0} Y(t)$ does not exist.

Theorem 38.3. *If $\lim_{t \rightarrow \infty} Y(t)$ exists then*

$$\lim_{t \rightarrow \infty} Y(t) = \lim_{s \rightarrow 0} s y(s).$$

With the help of this theorem $\lim_{t \rightarrow \infty} Y(t) = Y(+\infty)$ can be determined from $y(s)$, if we know that $Y(+\infty)$ exists.

As with theorem 38.2 we must take care, as the following example shows: 

$$Y(t) = \sin t, \quad y(s) = \frac{1}{s^2 + 1}.$$

Here $\lim_{s \rightarrow 0} s y(s) = 0$, but $Y(+\infty)$ does not exist.

§ 39. General Definitions of Asymptotic Representation and Asymptotic Expansion of Functions

If a function $\varphi(z)$ tends toward the limit l as it approaches the point z_0 (which can be the point at infinity), then it is said that in the neighbourhood of z_0 it behaves as the constant l . Frequently we wish to describe the behaviour of $\varphi(z)$ more exactly. Thus $\sin z$ tends to 0 for $z \rightarrow 0$, but it is much more instructive that $\sin z$ behaves as z for $z \rightarrow 0$, and this means that

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1.$$

Although $\varphi(z)$ sometimes has no limit for $z \rightarrow z_0$, nevertheless its behaviour can be described by another (simpler) function. Thus, for example, the function

$$\frac{2z^2 + 3z + 4}{5z + 6}$$

has no limit for $z \rightarrow \infty$, but it behaves as $(2/5)z$ (becomes infinite as $(2/5)z$); i. e.,

$$\lim_{z \rightarrow \infty} \frac{2z^2 + 3z + 4}{5z + 6} : \frac{2}{5}z = 1.$$

Thus, in general, it is customary to make the following definition:

The function $\varphi(z)$ behaves for $z \rightarrow z_0$ as the comparison function $\psi(z)$ (or $\varphi(z)$ is represented asymptotically for $z \rightarrow z_0$ by $\psi(z)$), in symbols

$$\varphi(z) \sim \psi(z) \quad \text{for } z \rightarrow z_0,$$

if

$$\frac{\varphi(z)}{\psi(z)} \rightarrow 1 \quad \text{for } z \rightarrow z_0.$$

In some cases it is possible to give not one function only but a complete sequence of comparison functions, which have the form

$$\psi_0(z), \quad \psi_0(z) + \psi_1(z), \quad \psi_0(z) + \psi_1(z) + \psi_2(z), \quad \dots$$

and to obtain "better" representations of the function $\varphi(z)$ by increasing the number of terms. Expressed exactly this means the following.

If we take n terms of the sequence, i. e.,

$$\sum_{\nu=0}^{n-1} \psi_\nu(z),$$

then the difference between $\varphi(z)$ and this sum shall behave as the next term $\psi_n(z)$ (in sense of the previous definition):

$$(39.1) \quad \varphi(z) - \sum_{\nu=0}^{n-1} \psi_\nu(z) \sim \psi_n(z) \quad \text{for } z \rightarrow z_0.$$

This corresponds to the obvious tendency that when we have obtained a certain comparison function we consider the "remainder", i. e., the difference between the function and the comparison function, and seek a comparison function for it, which is then used as the next improvement.

Explicitly (39.1) means

$$\frac{\varphi(z) - \sum_{\nu=0}^{n-1} \psi_\nu(z)}{\psi_n(z)} \rightarrow 1 \quad \text{or} \quad \frac{\varphi(z) - \sum_{\nu=0}^{n-1} \psi_\nu(z)}{\psi_n(z)} - \frac{\psi_n(z)}{\psi_n(z)} \rightarrow 0,$$

i. e.

$$\frac{\varphi(z) - \sum_{\nu=0}^n \psi_\nu(z)}{\psi_n(z)} \rightarrow 0.$$

If $\frac{f_1(z)}{f_2(z)}$ tends to 0, then we write $f_1(z) = o(f_2(z))$; this means that $f_1(z)$ is of lower order of magnitude than $f_2(z)$. Using this symbol, (39.1) can be replaced by

$$(39.2) \quad \varphi(z) - \sum_{\nu=0}^n \psi_{\nu}(z) = o(\psi_n(z)) \quad \text{for } z \rightarrow z_0.$$

The "remainder" or "error" when φ is replaced by a sum of the ψ_{ν} is therefore of lower order of magnitude than the last term in the sum.

If we can find an arbitrarily large number of ψ_{ν} with property (39.1) or (39.2) then we say that for $z \rightarrow z_0$, $\varphi(z)$ has the asymptotic expansion

$$\sum_{\nu=0}^{\infty} \psi_{\nu}(z),$$

which we write in the form

$$\varphi(z) \approx \sum_{\nu=0}^{\infty} \psi_{\nu}(z) \quad \text{for } z \rightarrow z_0.$$

The asymptotic expansions which occur most frequently with finite z_0 are series of increasing powers of $z - z_0$ with arbitrary exponents which may be non-integral:

$$(39.3) \quad \varphi(z) \approx \sum_{\nu=0}^{\infty} c_{\nu}(z - z_0)^{\lambda_{\nu}}, \quad -N < \lambda_0 < \lambda_1 < \dots \rightarrow +\infty;$$

with $z_0 = \infty$ they are series of decreasing powers of z with arbitrary exponents:

$$(39.4) \quad \varphi(z) \approx \sum_{\nu=0}^{\infty} \frac{c_{\nu}}{z^{\lambda_{\nu}}}, \quad -N < \lambda_0 < \lambda_1 < \dots \rightarrow +\infty.$$

For either of these it is very clear which meaning to attach to the asymptotic expansion. Thus (39.4) means, for example,

$$\varphi(z) - \sum_{\nu=0}^n \frac{c_{\nu}}{z^{\lambda_{\nu}}} = o\left(\frac{1}{z^{\lambda_n}}\right)$$

or

$$z^{\lambda_n} \left[\varphi(z) - \sum_{\nu=0}^n \frac{c_{\nu}}{z^{\lambda_{\nu}}} \right] \rightarrow 0 \quad \text{for } z \rightarrow \infty,$$

i.e., the error $\varphi(z) - \sum_{\nu=0}^n$ not only tends to zero, but it tends so fast to zero that even its product with z^{λ_n} tends to zero.

If the series (39.3) or (39.4) *converge absolutely* within or outside a circle, then they are also *asymptotic* expansions.

Basic properties of the $\mathfrak{L}$ -transformation are that a certain asymptotic expansion of the image function for $s = \infty$ corresponds to an asymptotic expansion of the original function for $t = 0$, and conversely a certain asymptotic expansion of the original function for $t = \infty$ corresponds to an asymptotic expansion of the image function at a finite point s_0 ⁶⁷). For these results we require the theorems in the two following paragraphs. In practice the series from which we start is usually not only asymptotic, but also convergent. In the following theorems we will confine ourselves to this case.

§ 40. Asymptotic Expansion of the Image Function

We begin with the statement of an asymptotic expansion of the image function because this is particularly simple to obtain (EINF. p. 222).

Theorem 40.1. $\mathfrak{L}\{F\} = f(s)$ converges somewhere. If $F(t)$ can be expanded in the neighbourhood of $t = 0$ in an absolutely convergent series of the form

$$F(t) = \sum_{\nu=0}^{\infty} c_{\nu} t^{\lambda_{\nu}} \quad (-1 < \lambda_0 < \lambda_1 < \dots \rightarrow \infty),$$

then for $s \rightarrow \infty$, $f(s)$ possesses the asymptotic expansion

$$f(s) \approx \sum_{\nu=0}^{\infty} c_{\nu} \frac{\Gamma(\lambda_{\nu} + 1)}{s^{\lambda_{\nu} + 1}}.$$

The series for $f(s)$ is simply the term-by-term transformation of the series for $F(t)$. Therefore the theorem supplies an interesting contribution to the question of the possibility of *transforming a series term by term*. If the series is regarded only from the standpoint of convergence, then simple examples show that this term-by-term transformation is not always possible. Thus for $F(t) = e^{-t^2}$ the $\mathfrak{L}$ -integral exists in the entire s -plane and the power series

$$e^{-t^2} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{t^{2\nu}}{\nu!}$$

converges for all t . However, the series obtained by term-by-term transformation

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(2\nu!)}{\nu!} \frac{1}{s^{2\nu+1}}$$

⁶⁷) The theorems 38.2 and 38.3 represent the simplest cases of this connection. With our present terminology we can state them in the form: if $Y(t) \sim l$ for $t \rightarrow 0$ ($t \rightarrow \infty$) it follows that $y(s) \sim \frac{l}{s}$ for $s \rightarrow \infty$ ($s \rightarrow 0$).

converges for no s , because for every s the terms increase in absolute magnitude after a certain index. However, the above theorem now shows that the series is not completely meaningless, but that it gives the asymptotic expansion of $f(s)$ for $s \rightarrow \infty$. We mention that

$$f(s) = e^{(s/2)^2} \int_{s/2}^{\infty} e^{-x^2} dx.$$

When evaluating the complex inversion formula by deforming the path of integration we met an example where the asymptotic expansion of the image function was required. In formula (34.4) we had expressed the temperature in a heat conductor as the sum of a steady state oscillation (of the same frequency as the cosine oscillation at the boundary) and a transient phenomenon, which was represented by a $\mathfrak{L}$ -integral: in the usual notation it has the form

$$(40.1) \quad f(s) = -\frac{1}{\pi} \int_0^{\infty} e^{-st} \frac{t}{t^2 + \omega^2} \sin x \sqrt{t} dt.$$

From Theorem 38.1 we know that $f(s)$ tends to zero for $s \rightarrow \infty$. If we wish to know more exactly how rapidly it converges to zero, then we can use the above theorem. Setting

$$\begin{aligned} \frac{t}{t^2 + \omega^2} &= \frac{t}{\omega^2} \frac{t}{1 + \left(\frac{t}{\omega}\right)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{\omega^{2n+2}} \\ &= \frac{t}{\omega^2} - \frac{t^3}{\omega^4} + \frac{t^5}{\omega^6} - + \dots \quad \text{for } |t| < \omega, \\ \sin x \sqrt{t} &= \frac{x}{1!} t^{1/2} - \frac{x^3}{3!} t^{3/2} + \frac{x^5}{5!} t^{5/2} - + \dots \quad \text{for all } t, \end{aligned}$$

we have

$$(40.2) \quad \frac{t}{t^2 + \omega^2} \sin x \sqrt{t} = \frac{x}{\omega^2} t^{3/2} - \frac{x^3}{3! \omega^2} t^{5/2} + \left(\frac{x^5}{5! \omega^2} - \frac{x}{\omega^4} \right) t^{7/2} + \dots$$

for $|t| < \omega$.

Consequently, for $s \rightarrow \infty$, $f(s)$ has the asymptotic expansion

$$(40.3) \quad f(s) \approx -\frac{1}{\pi} \left\{ \frac{x}{\omega^2} \frac{\Gamma(5/2)}{s^{5/2}} - \frac{x^3}{3! \omega^2} \frac{\Gamma(7/2)}{s^{7/2}} + \left(\frac{x^5}{5! \omega^2} - \frac{x}{\omega^4} \right) \frac{\Gamma(9/2)}{s^{9/2}} + \dots \right\}.$$

From this we see that $f(s)$ tends to zero as $\frac{\text{const}}{s^{5/2}}$: this is not obvious from the definition of $f(s)$. Using this comparison function the remainder tends to zero as $\frac{\text{const}}{s^{7/2}}$, etc.

§ 41. Asymptotic Expansion of the Original Function

The problem of determining the original function from the image function occurs much more frequently than the converse problem because in the treatment of functional equations with the $\mathfrak{L}$ -transformations the image function of the solution is always found first and then the original function is obtained. We have considered this problem in detail in Chapter 7. When the methods given there fail, then we must be satisfied if we can find at least an asymptotic expansion of the original function. In very many cases the engineer is not interested in the complete solution, but is satisfied when he knows only how the solution behaves for large t ; for example, whether or not it remains bounded (stability). For such complicated image functions as occur in boundary value problems for partial differential equations the next theorem, in which the assumptions are relatively simple, gives what is often the only method by which we can obtain information about the solution, and it should be employed in practice much more often than it has been.

If we wish to obtain information about $Y(t)$ from $y(s)$, then first we must represent $Y(t)$ by means of $y(s)$ using the complex inversion formula

$$(41.1) \quad Y(t) = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi j} \int_{x-j\omega}^{x+j\omega} e^{ts} y(s) ds.$$

As an $\mathfrak{L}$ -transform, $y(s)$ is analytic in a right half-plane, in which the path of integration also lies. The singular points of $y(s)$ all lie in the corresponding left half-plane; the one situated furthest to the right will be denoted by α_0 (we assume first that there is only one, i.e., that there do not exist several singular points with the same greatest real part).

It may now be possible to replace the vertical straight line ($x = \alpha$) which has been used as the path of integration by a path $\mathfrak{C}$ of "angular form", which consists of a circular arc around the singular point on the right-hand side and two rays which are inclined to the positive real axis at angles

$\pm \vartheta \left(\frac{\pi}{2} < \vartheta \leq \pi \right)$ (Fig. 41.1). This is certainly possible if $y(s)$ tends uniformly to 0 in the region between the old and the new paths of integration when $s \rightarrow \infty$. Because, if we think of the two paths as connected by two large circular arcs above and below, then the function $e^{ts}y(s)$ is analytic in the region enclosed and the integral along the boundary is therefore 0. As the radii of the inserted circular arcs tend to ∞ , the integrals along these arcs tend to 0 by Theorem 34.2. Therefore it follows that the integral along the former path, traversed from below to above, plus the integral along the new path, traversed from above to below, is equal to 0.

This means that the integrals are equal when both paths are traversed from below to above.

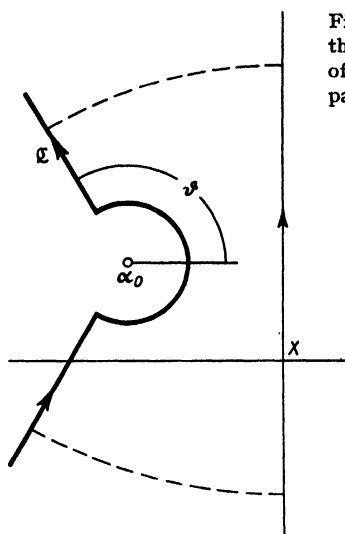


Figure 41.1 The replacement of the original straight line path of integration by an angular path.



Figure 41.2 An angular path for more than one singularity.

We assume therefore in what follows that $Y(t)$ can be represented by

$$(41.2) \quad Y(t) = \frac{1}{2\pi j} \int_{\mathcal{C}} e^{ts} y(s) ds$$

instead of by (41.1). Then the following theorem is valid (EINF. p. 237).

Theorem 41.1. *If $y(s)$ can be expanded in a neighbourhood of α_0 in an absolutely convergent power series with arbitrary (perhaps non-integral) exponents:*

$$(41.3) \quad y(s) = \sum_{\nu=0}^{\infty} c_{\nu}(s - \alpha_0)^{\lambda_{\nu}}, \quad -N < \lambda_0 < \lambda_1 < \dots \rightarrow \infty,$$

then the following asymptotic expansion for $Y(t)$ is valid for $t \rightarrow \infty$:

$$(41.4) \quad Y(t) \approx e^{\alpha_0 t} \sum_{\nu=0}^{\infty} \frac{c_{\nu}}{\Gamma(-\lambda_{\nu})} t^{-\lambda_{\nu}-1}.$$

In this $\frac{1}{\Gamma(-\lambda_{\nu})} = 0$, if λ_{ν} takes one of the values $0, 1, 2, \dots$ (As is well known, $\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \dots = \infty$).

Positive exponents in the expansion of $y(s)$ do not contribute anything to the asymptotic expansion of $Y(t)$; this is explained by the fact that in their totality the corresponding powers represent an analytic function which does not contribute to the singular behaviour of $y(s)$ at α_0 (which is the source of the asymptotic behaviour of $Y(t)$).

The first λ , can be negative; for them we have the correspondence

$$(s - \alpha_0)^{\lambda} \leftrightarrow \frac{t^{-\lambda} - 1}{\Gamma(-\lambda)} e^{\alpha_0 t}.$$

For positive λ , (which are of principal interest here) this correspondence does not exist (then $t^{-\lambda} - 1$ is not integrable at $t = 0$). If we assume that the correspondence is formally valid for these values too, then we can say that (41.4) is the term-by-term transformation of (41.3).

In the case which occurs most frequently in applications where the λ , are half-integral, the following formula can be used to calculate more easily the coefficients in (41.4):

$$(41.5) \quad \frac{1}{\Gamma\left(\frac{1}{2} - \nu\right)} = \frac{(-1)^\nu}{\pi} \Gamma\left(\nu + \frac{1}{2}\right) = \frac{(-1)^\nu (2\nu)!}{4^\nu \nu! \sqrt{\pi}}.$$

We shall consider now the case in which there are *several singular points of $y(s)$ with the same greatest real part*, for example, $\alpha_0, \alpha_1, \alpha_2$. Then first we must see whether or not the straight line path of integration in (41.1) can be replaced by an angular path which passes round these singular points along small circular arcs (Fig. 41.2). As previously, the sufficient condition for this is that between the two paths, $y(s)$ tends uniformly to zero for $s \rightarrow \infty$. When $y(s)$ can be expanded in a series of powers of $s - \alpha_0$ at α_0 , of $s - \alpha_1$ at α_1 , and of $s - \alpha_2$ at α_2 , then these series are to be transformed separately by the above theorem and superimposed to obtain the asymptotic expansion of $Y(t)$ for $t \rightarrow \infty$. It should be noted that the exponential factors $e^{\alpha_0 t}$, $e^{\alpha_1 t}$, $e^{\alpha_2 t}$, which multiply the series, are different, but have the same absolute value because $\Re \alpha_0 = \Re \alpha_1 = \Re \alpha_2$. These exponential factors increase or decrease as $t \rightarrow +\infty$, according as $\Re \alpha_0 > 0$ or < 0 , and in fact much faster than the powers $t^{-\lambda} - 1$ increase ($\lambda, < -1$) or decrease ($\lambda, > -1$). Therefore they decide the asymptotic behaviour of $Y(t)$.

If $y(s)$ has other singularities which lie further to the left these do not contribute to the asymptotic expansion of $Y(t)$ for $t \rightarrow +\infty$.

To demonstrate this method we use again the example of the one-sided infinitely long heat conductor on whose boundary the temperature is put equal to $\cos \omega t$. From (34.2) its temperature is

$$U(x, t) = \frac{1}{2\pi i} \int_{\alpha-j\infty}^{\alpha+j\infty} e^{ts} \frac{s}{s^2 + \omega^2} e^{-s\sqrt{x}} ds \quad (\alpha > 0).$$

If we cut out a narrow angle around the negative real axis, then the function $e^{-s\sqrt{x}}$ with $x > 0$ tends uniformly to 0 in the rest of the plane as $s \rightarrow \infty$, because there $\Re \sqrt{s} > 0$. The other factor $\frac{s}{s^2 + \omega^2}$ tends uniformly to 0

in the entire plane. Therefore we can replace the straight line path of integration by an angular path which goes along circular arcs around the three singular points of $y(s)$, namely the two poles at $s = \pm j\omega$ and the branch point $s = 0$, all of which have real part 0. In the neighbourhood of these points $y(s)$ can be expanded in a power series. If we first consider the point $s = j\omega$ and set

$$y(s) = \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}} = \frac{1}{s - j\omega} \frac{s}{s + j\omega} e^{-x\sqrt{s}} = \frac{1}{s - j\omega} z(s),$$

then $z(s)$ is analytic in the neighbourhood of $s = j\omega$ and therefore can be expanded in a Taylor series

$$z(s) = z(j\omega) + \frac{z'(j\omega)}{1!} (s - j\omega) + \frac{z''(j\omega)}{2!} (s - j\omega)^2 + \dots$$

If we divide this by $s - j\omega$ to obtain $y(s)$, then all the terms with the exception of the first have non-negative integral exponent. By transforming according to (41.4) they can all be omitted so that we have to consider only the first term

$$\frac{z(j\omega)}{s - j\omega} = \frac{j\omega}{j\omega + j\omega} e^{-x\sqrt{j\omega}} \frac{1}{s - j\omega} = \frac{e^{-x\sqrt{j\omega}}}{2(s - j\omega)}.$$

This gives ($\lambda_0 = -1$)

$$(41.6) \quad \frac{1}{2} e^{-x\sqrt{j\omega}} e^{j\omega t}.$$

Analogously, the expansion at $s = -j\omega$ gives

$$(41.7) \quad \frac{1}{2} e^{-x\sqrt{-j\omega}} e^{-j\omega t}.$$

At $s = 0$ we have (cf. p. 211)

$$\frac{s}{s^2 + \omega^2} = \frac{s}{\omega^2} - \frac{s^3}{\omega^4} + \frac{s^5}{\omega^6} - \dots,$$

$$e^{-x\sqrt{s}} = 1 - \frac{x}{1!} s^{1/2} + \frac{x^2}{2!} s - \frac{x^3}{3!} s^{3/2} + \frac{x^4}{4!} s^2 - \dots,$$

and therefore

$$\frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}} = \frac{1}{\omega^2} s - \frac{x}{1! \omega^2} s^{3/2} + \frac{x^2}{2! \omega^2} s^2 - \frac{x^3}{3! \omega^2} s^{5/2} + \left(\frac{x^4}{4! \omega^2} - \frac{1}{\omega^4} \right) s^3 + \dots$$

All terms with integral exponents can be omitted and we obtain

$$- \frac{x}{1! \omega^2} \frac{1}{\Gamma(-3/2)} t^{-5/2} - \frac{x^3}{3! \omega^2} \frac{1}{\Gamma(-5/2)} t^{-7/2} + \dots,$$

and thus by formula (41.5) we can write

$$(41.8) \quad -\frac{1}{\pi} \left\{ \frac{x}{\omega^2} \frac{\Gamma(5/2)}{t^{5/2}} - \frac{x^3}{3! \omega^2} \frac{\Gamma(7/2)}{t^{7/2}} + \dots \right\}.$$

The superposition of (41.6, 7, 8) gives the complete asymptotic expansion of $U(x, t)$ for $t \rightarrow \infty$. The functions (41.6) and (41.7) can be combined, as on p. 189, into the real function

$$e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2})$$

and we obtain the same representation of the solution $U(x, t)$ as found previously in (34.4) and (40.3). However, the present derivation is much more methodical and simpler.

§ 42. Investigation of Stability

If the behaviour of a physical system is controlled by certain functional equations (differential, difference or a combination of these) then in many cases (for example, in automatic control) it is of great importance whether or not the solution is stable, i.e., whether it approaches a finite value or at least remains bounded with increasing t . This question is answered by the asymptotic behaviour of the solution for $t \rightarrow \infty$. If we apply the $\mathfrak{L}$ -transformation to the problem, we try to determine the asymptotic behaviour of the original function directly from the image function without obtaining the original function explicitly. For this purpose Theorem 41.1 is very useful, and it would be sufficient simply to refer to it. However, since the consideration of stability on the basis of the image function is rather carelessly treated in the literature in most cases, a few words are added here.

When the physical system is controlled by an ordinary differential equation, then the matter is extremely simple. If we assume, for the sake of definiteness, that the system is initially at rest, i.e., that all initial values $Y(0+)$, $Y'(0+)$, ..., $Y^{(n-1)}(0+)$ vanish, and that the excitation or input is the unit step function $U(t)$ (the solution is the step response) then by (13.2) we have

$$y(s) = \frac{1}{s} \frac{1}{p(s)},$$

where $p(s)$ is the characteristic polynomial of the differential equation. $y(s)$ is a rational function which, when expanded in partial fractions, has the form

$$(42.1) \quad y(s) = \sum_{\nu} \frac{b_{\nu}}{s - \alpha_{\nu}},$$

if the zeros of the denominator are simple, and the form

$$(42.2) \quad y(s) = \sum_{\nu} \left(\frac{b_{\nu}^{(1)}}{s - \alpha_{\nu}} + \cdots + \frac{b_{\nu}^{(l_{\nu})}}{s - \alpha_{\nu}} \right),$$

if the zeros have multiplicity l_{ν} . In this form $y(s)$ can be transformed immediately to

$$(42.3) \quad Y(t) = \sum_{\nu} b_{\nu} e^{\alpha_{\nu} t}$$

or

$$(42.4) \quad Y(t) = \sum_{\nu} \left(b_{\nu}^{(1)} + \frac{b_{\nu}^{(2)}}{1!} t + \cdots + \frac{b_{\nu}^{(l_{\nu})}}{(l_{\nu}-1)!} t^{l_{\nu}-1} \right) e^{\alpha_{\nu} t}.$$

The solution $Y(t)$ is called *stable* if for $t \rightarrow \infty$ it tends to the initial value 0 (proper stability) or to a different fixed value (improper stability) or oscillates between finite bounds (quasi-stability). In every other case it is called *unstable*. The behaviour of $Y(t)$ is determined by the zero with the greatest real part (we assume first that there exists only one of this kind), because

$$|e^{\alpha_{\nu} t}| = e^{\Re \alpha_{\nu} t}$$

and

$$|e^{\alpha_{\nu} t}| < |e^{\alpha_0 t}| \text{ for } \nu \neq 0.$$

$Y(t)$ is

for $\Re \alpha_0 > 0$ *unstable*

for $\Re \alpha_0 < 0$ *stable*

for $\Re \alpha_0 = 0$ in the case $\alpha_0 = 0$ *improper stable*, when α_0 is simple
unstable, when α_0 is multiple

in the case $\Im \alpha_0 \neq 0$ *quasi-stable*, when α_0 is simple
unstable, when α_0 is multiple.

The discussion is similar when several zeros occur with the same greatest real part. *The solution $Y(t)$ is only stable (in any sense), when no zero of the denominator of $y(s)$, i.e., pole of $y(s)$, lies in the right half-plane and when those on the imaginary axis are simple.*

To establish this, we do not require to write down $Y(t)$ explicitly, since all this can be seen from the image function alone.

Exactly the same is true for the case in which the physical occurrence is controlled by a system of ordinary differential equations because, from § 16 only rational functions occur here also (if the inputs are unit step functions).

However, more complicated cases can occur in which $y(s)$ is not a rational, but a *meromorphic* function or even a *multi-valued* function; for example, when the basic differential equation is partial as in the examples in Chapter 5. Also, for example, transcendental functions occur with servomechanisms which are controlled by ordinary differential equations, if the servomechanism does not react to a change immediately, but only after a "dead time". It is now customary in engineering to use simply the same criterion as above with the rational functions: the singularities with the greatest real part are determined and the stability of the original function is judged by their position relative to the imaginary axis. At the basis of this behaviour is the same naive idea which was mentioned already on p. 194: it is assumed as self-evident that $y(s)$ can be expanded in a series of partial fractions which makes the poles obvious, and it is then transformed term-by-term into a series of exponential functions. It has been stressed already that an entire function can occur in addition to the partial fraction series. An entire function is analytic in the entire plane, but has a singularity at infinity (generally a difficult one), the character of which can influence decisively the asymptotic behaviour of $Y(t)$.

This can be illustrated by a striking example. It can be shown that

$$(42.5) \quad Y(t) = \sin t^\alpha \quad (\alpha > 1)$$

has an image function $y(s)$, which is an entire function and therefore possesses no singularities in the finite plane. To the original function

$$(42.6) \quad t^n Y(t) = t^n \sin t^\alpha \quad (n = 1, 2, \dots)$$

there corresponds by (3.1) the image function $(-1)^n y^{(n)}(s)$. Because $y(s)$ is an entire function, the same is true for $y^{(n)}(s)$. The function $t^n \sin t^\alpha$, assumes arbitrarily large values for $t \rightarrow \infty$ and therefore describes an instable occurrence. There corresponds to it an image function which possesses no singularities at all in the finite plane. For such a function the criterion cited above is completely useless. If such a function forms part of the image function found for a problem, then the other terms simulate stability if they possess only poles with negative real parts, while the instability produced by that function remains entirely unobserved.

Moreover, function (42.5) is interesting from the physical standpoint, because α can be chosen so near to 1 that $\sin t^\alpha$ differs from $\sin t$ in an arbitrarily large interval by an arbitrarily small amount: but $\mathfrak{L}\{\sin t\} = \frac{1}{s^2 + 1}$ has the singular points $s = \pm j$.

The representation by the partial fraction expansion is also wrong if $y(s)$ is a multi-valued function, as in the example considered in § 41; this example does not arise from mathematical subtlety, but in a reasonable physical problem. Here no partial fraction expansion of $y(s)$ can exist and therefore no exponential expansion of $Y(t)$.

In all cases in which $y(s)$ is not a rational function only theorems such as theorem 41.1⁶⁸⁾ can give a sure test of the asymptotic behaviour and thus of the stability of the original function. Using these theorems it becomes obvious also that not only is the behaviour of $y(s)$ at its singular points in the finite plane of importance but also its *behaviour at infinity*, since on the latter depends whether or not the straight line path of integration in the complex inversion formula can be replaced by an angular path.

⁶⁸⁾ There exist further theorems of this kind which are based upon more complicated singularities. See also the *Handbuch* quoted in footnote 62, 2. Band, Chapters 6. and 7.

Subject Index

- Admittance 90
- Alternating current theory, complex 68, 120
- Amplifying factor of pulse former 138
- Amplitude of a real oscillation 12, of a complex oscillation 15
- Amplitude density 17
- Amplitude characteristic 69
- Asymptotic representation 208, expansion 209
- Basic region for a partial differential equation 161
- Bessel function J_0 185, 192, J_ν 192
- Block, representation by a 56, 90
- Boundary conditions in partial differential equations 162
- Boundary value problem for difference equations 119, 126
- Calculus of residues 188, 195
- Cardinal series 145
- Characteristic equation 54
- Characteristic oscillations of a differential equation 74
- Characteristic solutions of a difference equation 113
- Compatibility relations 95, 100
- Complex A.C. theory 68, 120
- Complex current and voltage 121
- Complex frequency 26, 122
- Control area, quadratic 43
- Convergence half-plane 27
- Convolution 41, differentiation of 183
- Convolution sum 130
- Convolution theorem for the $\mathfrak{D}$ -transformation 130, the $\mathfrak{L}$ -transformation 40, the $\mathfrak{B}$ -transformation 135, complex convolution theorem 41
- Correspondence 32, symbol 32
- Current loop 89
- $\mathfrak{D}$ $\mathfrak{D}$ 127
- Damping ratio 49
- Damping theorem for the $\mathfrak{D}$ -transformation 130, the $\mathfrak{L}$ -transformation 36, the $\mathfrak{B}$ -transformation 135
- Deformation of the path of integration in the inversion formula 187, 212
- Differentiation theorems 37, 39
- Dirac δ -function 21
- Discrete $\mathfrak{L}$ -transformation 127
- Distribution function 17
- Distortion in transmission 179
- Distortionless conductor 177
- Double source function 168
- Duhamel's formula 62
- Elastance 96
- Excitation 56
- Exponential type, an entire function of 191
- Factorial series 199
- Fourier integral 16
- Fourier series 11, complex form 12
- Fourier-Stieltjes integral 23
- Four-terminal network 96, 122
- Frequency, positive and negative 15, complex 26, 122
- Frequency response 68, 71
- Grammatical rules for the transformation as a translation 34
- Green's function 56, 170
- Hakenintegral 29
- Heat explosion 170
- Heaviside's expansion theorem 62
- Hold circuit 141
- Image equation 44
- Image function 31, 127
- Image space 31

- Impedance 90
- Impulse 139
- Impulse function 21
- Impulse lengthener 141
- Impulse response 66, 68, 93, 112
- Initial conditions in partial differential equations 162
- Initial values 44, 77, 94, left- and right-sided 75, 95, 97, 98
- Input 56
- Instability 217
- Integration theorems 39
- Inverse transformation 32
- Inversion of the $\mathfrak{D}$ -transformation 130, 136, 141, the $\mathfrak{L}$ -transformation 32, the $\mathfrak{g}$ -transformation 136
- Jump-capable system 98, 104
- Kernel 182, reciprocal 182
- Kirchhoff's current law 89
- $\mathfrak{L}$ $\mathfrak{L}$ 31, $\mathfrak{L}^{-1}$ 32
- Ladder network 122
- Laplace integral 26, inversion 26, as an analytic function 27, 33, as a transformation 31
- Laurent-transform 134
- Lumped parameters 89
- Mapping, $\mathfrak{L}$ -transformation as a 31, 33, 34
- Maximum of an original function 203
- Meromorphic function, inverse transformation 195
- Modulating function for a pulse 140
- Network 90
- Neumann series 182
- Original equation 44
- Original function 31
- Original sequence 127
- Original space 31
- Oscillations harmonic 12, complex 15
- Output 56
- Parseval equation 42
- Partial fraction expansion 47, 54, 115, 132, 137
- Periodic original function 160
- Phase of a real oscillation 12, complex oscillation 15, 17
- Phase characteristic 69
- Poisson integral for J_0 186
- Power series in $1/s$, inverse transformation 191, 192
- Pseudo-function 21
- Pulse 138
- Pulse former 138
- Pulse length 138
- Pulse period 138
- Recurrence relation 109
- Response 56
- Sampled-data 140, 142
- Sampling device 138
- Sampling, periodic 138
- Sampling theorem 144
- Series of functions, inverse transformation 198, 200
- Similarity theorem 35
- Sinusoidal oscillation 46
- Source function 168
- Spectral density 17, 71, of a damped function 26
- Spectral distribution 17, of $e^{j\omega t}$ 19
- Spectral line 19
- Spectral representation 17, of a damped function 26
- Spectral thinking 32, 121
- Spectrum of a periodic function 15
- Stability 217
- Steady state 70, 120, 171
- Step response 60, 66, 71
- Stieltjes integral 22, 64, 128
- System function 56, 93
- Taylor transform 137
- Telegraph equation 174
- Thomson cable 166, 174
- Total energy 43

- Transient 190, 211
- Translation, $\mathfrak{L}$ -transformation as 34
- Translation theorems for the $\mathfrak{D}$ -transformation 130, the $\mathfrak{L}$ -transformation 35, 36, step functions 111, the $\mathfrak{F}$ -transformation 135
- T-structure 122
- U $U(t)$ 28
- Uniqueness properties of the inverse $\mathfrak{L}$ -transformation 32
- Unit step function 28
- Weight function 56, 73, 94, 112
- $\mathbf{Z}$ $\mathfrak{F}$ 134

APPENDIX

Table of Laplace Transforms

No.	$f(s)$	$F(t)$
1	$f(as) \quad (a > 0)$	$\frac{1}{a} F\left(\frac{t}{a}\right)$
2	$f(s-a)$	$\left. \begin{matrix} e^{at} F(t) \\ e^{-at} F(t) \end{matrix} \right\} \text{1st shifting theorem.}$
3	$f(s+a)$	
4	$f(as-\beta) \quad (a > 0, \beta \text{ complex})$	$\frac{1}{a} e^{\frac{\beta}{a}t} F\left(\frac{t}{a}\right)$
5	$\frac{1}{2j} [f(s-ja) - f(s+ja)]$	$F(t) \sin at$
6	$\frac{1}{2} [f(s-ja) + f(s+ja)]$	$F(t) \cos at$
7	$\frac{1}{2} [f(s-a) - f(s+a)]$	$F(t) \sinh at$
8	$\frac{1}{2} [f(s-a) + f(s+a)]$	$F(t) \cosh at$
9	$e^{-as} f(s)$	$\left\{ \begin{matrix} F(t-a) & \text{for } t > a \geq 0 \\ 0 & \text{for } t < a \end{matrix} \right\}$ ~~not~~ shifftby u
10	$\frac{1}{a} e^{-\frac{b}{a}s} f\left(\frac{s}{a}\right) \quad (a, b > 0)$	$\left\{ \begin{matrix} F(at-b) & \text{for } t > \frac{b}{a} \\ 0 & \text{for } t < \frac{b}{a} \end{matrix} \right.$
11	$e^{as} \left[f(s) - \int_0^a e^{-s\tau} F(\tau) d\tau \right]$	$F(t+a) \quad (a \geq 0)$
12	$e^{ks} \left[y(s) - \frac{1-e^{-s}k-1}{s} \sum_{\nu=0}^{k-1} Y_{\nu} e^{-\nu s} \right]$	$Y(t+k)$ with $Y(t) = Y_n$ for $n \leq t < n+1$ +
13	$\frac{df(s)}{ds}$	$-tF(t)$

No.	$f(s)$	$F(t)$
14	$\frac{d^n f(s)}{ds^n}$	$(-t)^n F(t)$
15	$sf(s) - F(+0)$	$\frac{dF(t)}{dt}$
16	$s^n f(s) - \sum_{k=0}^{n-1} F^{(k)}(+0) s^{n-k-1}$	$\frac{d^n F(t)}{dt^n}$
17	$\int_s^\infty f(\sigma) d\sigma$	$\frac{F(t)}{t}$
18	$\left(\int_s^\infty d\sigma\right)^n f(\sigma)$	$\frac{F(t)}{t^n}$
19	$\frac{1}{s} \int_s^\infty f(\sigma) d\sigma$	$\int_0^t \frac{F(\tau)}{\tau} d\tau$
20	$\frac{1}{s} \int_0^s f(\sigma) d\sigma$	$\int_t^\infty \frac{F(\tau)}{\tau} d\tau$
21	$\frac{1}{s} f(s)$	$\int_0^t F(\tau) d\tau = F(t) * 1$
22	$\frac{1}{s^n} f(s)$	$\left(\int_0^t d\tau\right)^n F(\tau)$
23	$f_1(s) \cdot f_2(s)$	$F_1(t) * F_2(t) = \int_0^t F_1(\tau) F_2(t-\tau) d\tau$
24	$\frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} f_1(\sigma) f_2(s-\sigma) d\sigma$	$F_1(t) \cdot F_2(t)$

No.	$f(s)$	$F(t)$
25	$\frac{1}{\sqrt{s}} f\left(\frac{1}{s}\right)$	$\int_0^{\infty} \frac{\cos(2\sqrt{t}\tau)}{\sqrt{\pi t}} F(\tau) d\tau$
26	$\frac{1}{s\sqrt{s}} f\left(\frac{1}{s}\right)$	$\int_0^{\infty} \frac{\sin(2\sqrt{t}\tau)}{\sqrt{\pi \tau}} F(\tau) d\tau$
27	$\frac{1}{s} f\left(\frac{1}{s}\right)$	$\int_0^{\infty} J_0(2\sqrt{t}\tau) F(\tau) d\tau$
28	$\frac{1}{s} f\left(s + \frac{1}{s}\right)$	$\int_0^t J_0\left(2\sqrt{(t-\tau)\tau}\right) F(\tau) d\tau$
29	$f(\sqrt{s})$	$\int_0^{\infty} \frac{\tau}{2\sqrt{\pi t^{3/2}}} e^{-\frac{\tau^2}{4t}} F(\tau) d\tau$
30	$f(\log s)$	$\int_0^{\infty} \frac{t^{r-1}}{\Gamma(r)} F(r) dr$
31	$\frac{1}{s} (\log s)$	$\int_0^{\infty} \frac{t^r}{\Gamma(r+1)} F(r) dr$
32	$\frac{1}{1-e^{-as}} \int_0^a e^{-sr} F(r) dr$	$F(t) = F(t+a)$

No.	$f(s)$	$F(t)$
33	1	$\delta(t)$
34	$\frac{1}{s}$	1
35	$\frac{1}{s-a}$	e^{at}
36	$\frac{1}{1+as}$	$\frac{1}{a}e^{-\frac{t}{a}}$
37	$\frac{1}{s^2}$	t
38	$\frac{a}{s^2+a^2}$	$\sin at$
39	$\frac{a}{s^2-a^2}$	$\sinh at$
40	$\frac{1}{s(s-a)}$	$\frac{1}{a}(e^{at}-1)$
41	$\frac{1}{s(1+as)}$	$1-e^{-\frac{t}{a}}$
42	$\frac{1}{(s-a)^2}$	te^{at}
43	$\frac{1}{(1+as)^2}$	$\frac{1}{a^2}te^{-\frac{t}{a}}$
44	$\frac{1}{(s-a)(s-b)}$	$\frac{e^{at}-e^{bt}}{a-b}$
45	$\frac{1}{(1+as)(1+bs)}$	$\frac{e^{-\frac{t}{a}}-e^{-\frac{t}{b}}}{a-b}$

No.	$f(s)$	$F(t)$
46	$\frac{1}{s^2 + c_1 s + c_0}$ $\left(D = c_0 - \frac{c_1^2}{4}\right)$	$\begin{cases} \frac{1}{\sqrt{-D}} e^{-\frac{c_1}{2} t} \sinh \sqrt{-D} t & (D < 0) \\ \frac{1}{\omega} e^{-\frac{c_1}{2} t} \sin \omega t & (D > 0, \sqrt{-D} = j\omega) \end{cases}$
47	$\frac{1}{s^2 + 2\delta s + (\delta^2 + \omega^2)}$	$\frac{1}{\omega} e^{-\delta t} \sin \omega t$
48	$\frac{s}{s^2 + a^2}$	$\cos at$
49	$\frac{s}{s^2 - a^2}$	$\cosh at$
50	$\frac{s}{(s-a)^2}$	$(1+at) e^{at}$
51	$\frac{s}{(1+as)^2}$	$\frac{1}{a^3} (a-t) e^{-\frac{t}{a}}$
52	$\frac{s}{(s-a)(s-b)}$	$\frac{ae^{at} - be^{bt}}{a-b}$
53	$\frac{s}{(1+as)(1+bs)}$	$\frac{ae^{-\frac{t}{b}} - be^{-\frac{t}{a}}}{ab(a-b)}$
54	$\frac{1}{s^3}$	$\frac{1}{2} t^2$
55	$\frac{1}{s^2(s-a)}$	$\frac{1}{a^2} (e^{at} - 1 - at)$
56	$\frac{1}{s^2(1+as)}$	$ae^{-\frac{t}{a}} + t - a$
57	$\frac{1}{s(s-a)^2}$	$\frac{1}{a^2} [1 + (at-1)e^{at}]$

No.	$f(s)$	$F(t)$
58	$\frac{1}{s(1+as)^2}$	$1 - \frac{a+t}{a} e^{-\frac{t}{a}}$
59	$\frac{1}{s(s-a)(s-b)}$	$\frac{1}{ab} + \frac{be^{at} - ae^{bt}}{ab(a-b)}$
60	$\frac{1}{s(1+as)(1+bs)}$	$1 + \frac{ae^{-\frac{t}{a}} - be^{-\frac{t}{b}}}{b-a}$
61	$\frac{1}{(s-a)(s-b)(s-c)}$	$\frac{(c-b)e^{at} + (a-c)e^{bt} + (b-a)e^{ct}}{(a-b)(a-c)(c-b)}$
62	$\frac{1}{(1+as)(1+bs)(1+cs)}$	$\frac{a(b-c)e^{-\frac{t}{a}} + b(c-a)e^{-\frac{t}{b}} + c(a-b)e^{-\frac{t}{c}}}{(a-b)(a-c)(b-c)}$
63	$\frac{1}{(s-a)(s-b)^2}$	$\frac{e^{at} - [1 + (a-b)t]e^{bt}}{(a-b)^2}$
64	$\frac{1}{(1+as)(1+bs)^2}$	$\frac{abe^{-\frac{t}{a}} - [ab + (a-b)t]e^{-\frac{t}{b}}}{b(a-b)^2}$
65	$\frac{1}{(s-a)^3}$	$\frac{1}{2} t^2 e^{at}$
66	$\frac{1}{(1+as)^3}$	$\frac{1}{2a^3} t^2 e^{-\frac{t}{a}}$
67	$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2} (1 - \cos at)$
68	$\frac{1}{s(s^2-a^2)}$	$\frac{1}{a^2} (\cosh at - 1)$
69	$\frac{s}{(s-a)(s-b)(s-c)}$	$\frac{a(b-c)e^{at} + b(c-a)e^{bt} + c(a-b)e^{ct}}{(a-b)(b-c)(a-c)}$
70	$\frac{c^2}{(s-a)(s-b)(s-c)}$	$\frac{a^2(b-c)e^{at} + b^2(c-a)e^{bt} + c^2(a-b)e^{ct}}{(a-b)(b-c)(a-c)}$

No.	$f(s)$	$F(t)$
71	$\frac{s}{(1+as)(1+bs)(1+cs)}$	$\frac{(c-b)e^{-\frac{t}{a}} + (a-c)e^{-\frac{t}{b}} + (b-a)e^{-\frac{t}{c}}}{(a-b)(b-c)(a-c)}$
72	$\frac{s^2}{(1+as)(1+bs)(1+cs)}$	$\frac{bc(b-c)e^{-\frac{t}{a}} + ac(c-a)e^{-\frac{t}{b}} + ab(a-b)e^{-\frac{t}{c}}}{abc(a-b)(b-c)(a-c)}$
73	$\frac{s}{(s-a)(s-b)^2}$	$\frac{ae^{at} - [a + b(a-b)t]e^{bt}}{(a-b)^2}$
74	$\frac{s^2}{(s-a)(s-b)^2}$	$\frac{a^2e^{at} - [2ab - b^2 + b^2(a-b)t]e^{bt}}{(a-b)^2}$
75	$\frac{s}{(1+as)(1+bs)^2}$	$\frac{-b^2e^{-\frac{t}{a}} + [b^2 + (a-b)t]e^{-\frac{t}{b}}}{b^2(a-b)^2}$
76	$\frac{s^2}{(1+as)(1+bs)^2}$	$\frac{b^3e^{-\frac{t}{a}} + [ab(a-2b) - (a-b)at]e^{-\frac{t}{b}}}{ab^3(a-b)^2}$
77	$\frac{s}{(s-a)^3}$	$\left(t + \frac{1}{2}at^2\right)e^{at}$
78	$\frac{s^2}{(s-a)^3}$	$\left(1 + 2at + \frac{1}{2}a^2t^2\right)e^{at}$
79	$\frac{s}{(1+as)^3}$	$\left(\frac{t}{a^3} - \frac{t^2}{2a^4}\right)e^{-\frac{t}{a}}$
80	$\frac{s^2}{(1+as)^3}$	$\left(\frac{1}{a^3} - \frac{2t}{a^4} + \frac{t^2}{2a^5}\right)e^{-\frac{t}{a}}$
81	$\frac{s^2 + 2a^2}{s(s^2 + 4a^2)}$	$\cos^2 at$
82	$\frac{s^2 - 2a^2}{s(s^2 - 4a^2)}$	$\cosh^2 at$

No.	$f(s)$	$F(t)$
83	$\frac{2a^2}{s(s^2 + 4a^2)}$	$\sin^2 at$
84	$\frac{2a^2}{s(s^2 - 4a^2)}$	$\sinh^2 at$
85	$\frac{a^3}{s^4 + a^4}$	$\frac{1}{\sqrt{2}} \left(\cosh \frac{a}{\sqrt{2}} t \sin \frac{a}{\sqrt{2}} t - \sinh \frac{a}{\sqrt{2}} t \cos \frac{a}{\sqrt{2}} t \right)$
86	$\frac{a^2 s}{s^4 + a^4}$	$\sin \frac{a}{\sqrt{2}} t \sinh \frac{a}{\sqrt{2}} t$
87	$\frac{as^2}{s^4 + a^4}$	$\frac{1}{\sqrt{2}} \left(\cos \frac{a}{\sqrt{2}} t \sinh \frac{a}{\sqrt{2}} t + \sin \frac{a}{\sqrt{2}} t \cosh \frac{a}{\sqrt{2}} t \right)$
88	$\frac{s^3}{s^4 + a^4}$	$\cos \frac{a}{\sqrt{2}} t \cosh \frac{a}{\sqrt{2}} t$
89	$\frac{a^3}{s^4 - a^4}$	$\frac{1}{2} (\sinh at - \sin at)$
90	$\frac{a^2 s}{s^4 - a^4}$	$\frac{1}{2} (\cosh at - \cos at)$
91	$\frac{as^2}{s^4 - a^4}$	$\frac{1}{2} (\sinh at + \sin at)$
92	$\frac{s^3}{s^4 - a^4}$	$\frac{1}{2} (\cosh at + \cos at)$
93	$\frac{2a^2 s}{s^4 + 4a^4}$	$\sin at \sinh at$
94	$\frac{a(s^2 - 2a^2)}{s^4 + 4a^4}$	$\cos at \sinh at$
95	$\frac{a(s^2 + 2a^2)}{s^4 + 4a^4}$	$\sin at \cosh at$
96	$\frac{s^3}{s^4 + 4a^4}$	$\cos at \cosh at$

No.	$f(s)$	$F(t)$
97	$\frac{a^3}{(s^2 + a^2)^2}$	$\frac{1}{2} (\sin at - at \cos at)$
98	$\frac{as}{(s^2 + a^2)^2}$	$\frac{t}{2} \sin at$
99	$\frac{as^2}{(s^2 + a^2)^2}$	$\frac{1}{2} (\sin at + at \cos at)$
100	$\frac{s^3}{(s^2 + a^2)^2}$	$\cos at - \frac{at}{2} \sin at$
101	$\frac{a^3}{(s^2 - a^2)^2}$	$\frac{1}{2} (at \cosh at - \sinh at)$
102	$\frac{as}{(s^2 - a^2)^2}$	$\frac{t}{2} \sinh at$
103	$\frac{as^2}{(s^2 - a^2)^2}$	$\frac{1}{2} (\sinh at + at \cosh at)$
104	$\frac{s^3}{(s^2 - a^2)^2}$	$\cosh at + \frac{at}{2} \sinh at$
105	$\frac{ab}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{a \sin bt - b \sin at}{a^2 - b^2}$
106	$\frac{s}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{\cos bt - \cos at}{a^2 - b^2}$
107	$\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{a \sin at - b \sin bt}{a^2 - b^2}$
108	$\frac{s^3}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{a^2 \cos at - b^2 \cos bt}{a^2 - b^2}$
109	$\frac{ab}{(s^2 - a^2)(s^2 - b^2)}$	$\frac{b \sinh at - a \sinh bt}{a^2 - b^2}$
110	$\frac{s}{(s^2 - a^2)(s^2 - b^2)}$	$\frac{\cosh at - \cosh bt}{a^2 - b^2}$

No.	$f(s)$	$F(t)$
111	$\frac{s^2}{(s^2 - a^2)(s^2 - b^2)}$	$\frac{a \sinh at - b \sinh bt}{a^2 - b^2}$
112	$\frac{s^3}{(s^2 - a^2)(s^2 - b^2)}$	$\frac{a^2 \cosh at - b^2 \cosh bt}{a^2 - b^2}$
113	$\frac{a^2}{s^2(s^2 + a^2)}$	$t - \frac{1}{a} \sin at$
114	$\frac{a^2}{s^2(s^2 - a^2)}$	$\frac{1}{a} \sinh at - t$
115	$\frac{a^4}{s(s^2 + a^2)^2}$	$1 - \cos at - \frac{at}{2} \sin at$
116	$\frac{a^4}{s(s^2 - a^2)^2}$	$1 - \cosh at + \frac{at}{2} \sinh at$
117	$\frac{a^2 b^2}{s(s^2 + a^2)(s^2 + b^2)}$	$1 + \frac{b^2 \cos at - a^2 \cos bt}{a^2 - b^2}$
118	$\frac{a^2 b^2}{s(s^2 - a^2)(s^2 - b^2)}$	$1 + \frac{b^2 \cosh at - a^2 \cosh bt}{a^2 - b^2}$
119	$\frac{a^5}{(s^2 + a^2)^3}$	$\frac{1}{8} [(3 - a^2 t^2) \sin at - 3 at \cos at]$
120	$\frac{a^3 s}{(s^2 + a^2)^3}$	$\frac{t}{8} (\sin at - at \cos at)$
121	$\frac{a^3 s^2}{(s^2 + a^2)^3}$	$\frac{1}{8} [(1 + a^2 t^2) \sin at - at \cos at]$
122	$\frac{a^5}{(s^2 - a^2)^3}$	$\frac{1}{8} [(3 + a^2 t^2) \sinh at - 3 at \cosh at]$
123	$\frac{a^3 s}{(s^2 - a^2)^3}$	$\frac{t}{8} (at \cosh at - \sinh at)$
124	$\frac{a^3 s^2}{(s^2 - a^2)^3}$	$\frac{1}{8} [at \cosh at - (1 - a^2 t^2) \sinh at]$

No.	$f(s)$	$F(t)$
125	$\frac{1}{s^n} \quad (n \text{ a positive integer})$	$\frac{1}{(n-1)!} t^{n-1}$
126	$\frac{(s-1)^n}{s^{n+1}}$	$L_n(t)$
127	$\frac{\left(s - \frac{1}{2}\right)^n}{\left(s + \frac{1}{2}\right)^{n+1}}$	$e^{-\frac{t}{2}} L_n(t)$
128	$\frac{1}{s(as+1)\dots(as+n)}$	$\frac{1}{n!} \left(1 - e^{-\frac{t}{a}}\right)^n$
129	$\frac{s \sin b + a \cos b}{s^2 + a^2}$	$\sin(at + b)$
130	$\frac{s \cos b - a \sin b}{s^2 + a^2}$	$\cos(at + b)$
131	$\frac{1}{p(s)}$	$\sum_{k=1}^n \frac{1}{p'(\alpha_k)} e^{\alpha_k t}$
132	$\frac{1}{s p(s)}$	$\frac{1}{p(0)} + \sum_{k=1}^n \frac{1}{\alpha_k p'(\alpha_k)} e^{\alpha_k t}$
133	$\frac{g(s)}{p(s)}$ $p(s) = (s - \alpha_1) \dots (s - \alpha_n),$ with α_k all different	$\sum_{k=1}^n \frac{g(\alpha_k)}{p'(\alpha_k)} e^{\alpha_k t}$

No.	$f(s)$	$F(t)$
134	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
135	$\frac{1}{s \sqrt{s}}$	$2 \sqrt{\frac{t}{\pi}}$
136	$\frac{s+a}{s \sqrt{s}}$	$\frac{1+2at}{\sqrt{\pi t}}$
137	$\frac{1}{\sqrt{s+a}}$	$\frac{e^{-at}}{\sqrt{\pi t}}$
138	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2t \sqrt{\pi t}} (e^{bt} - e^{at})$
139	$\sqrt{\sqrt{s^2+a^2}-s}$	$\frac{\sin at}{t \sqrt{2\pi t}}$
140	$\sqrt{\frac{\sqrt{s^2+a^2}-s}{s^2+a^2}}$	$\sqrt{\frac{2}{\pi t}} \sin at$
141	$\sqrt{\frac{\sqrt{s^2+a^2}+s}{s^2+a^2}}$	$\sqrt{\frac{2}{\pi t}} \cos at$
142	$\sqrt{\frac{\sqrt{s^2-a^2}-s}{s^2-a^2}}$	$\sqrt{\frac{2}{\pi t}} \sinh at$
143	$\sqrt{\frac{\sqrt{s^2-a^2}+s}{s^2-a^2}}$	$\sqrt{\frac{2}{\pi t}} \cosh at$
144	$\frac{1}{\sqrt{s^2+1}}$	$J_0(t)$
145	$\frac{(\sqrt{s^2+1}-s)^\nu}{\sqrt{s^2+1}}$	$J_\nu(t) \quad (\Re \nu > -1)$
146	$\frac{(2a)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}(s^2+a^2)^{\nu+1/2}}$	$t^\nu J_\nu(at) \quad \left(\Re \nu > -\frac{1}{2}\right)$

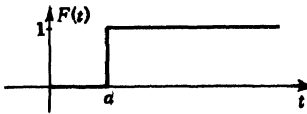
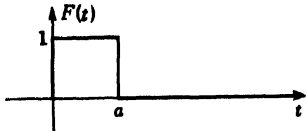
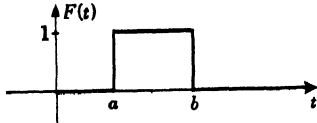
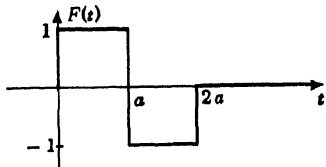
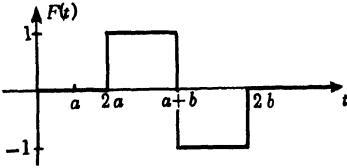
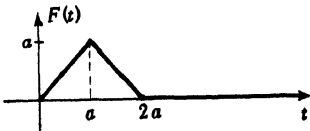
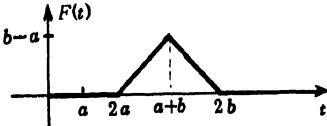
No.	$f(s)$	$F(t)$
147	$\frac{1}{s^\nu}$	$\frac{t^{\nu-1}}{\Gamma(\nu)} \quad (\Re \nu > 0)$
148	$\frac{1}{s^n \sqrt{s}}$	$\frac{4^n n!}{(2n)! \sqrt{\pi}} t^{n-\frac{1}{2}}$
149	$\frac{1}{(s+a)^\nu}$	$\frac{t^{\nu-1}}{\Gamma(\nu)} e^{-at}$

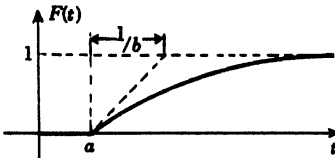
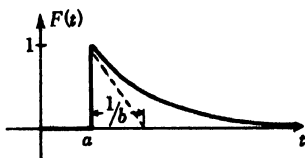
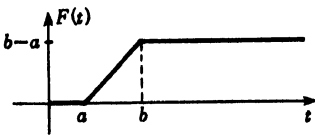
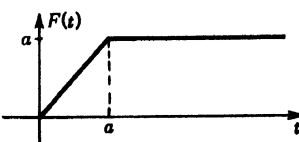
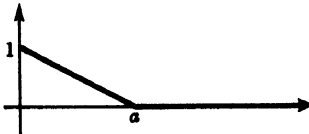
No.	$f(s)$	$F(t)$
150	$\frac{\log s}{s}$	$-\log t - C$
151	$-\sqrt{\frac{\pi}{s}} (\log 4s + C)$	$\frac{\log t}{\sqrt{t}}$
152	$\frac{(\log s)^2}{s}$	$(\log t + C)^2 - \frac{\pi^2}{6}$
153	$\frac{1}{\log s}$	$\int_0^{\infty} \frac{t^{r-1}}{\Gamma(r)} dr$
154	$\log \frac{s-a}{s}$	$\frac{1 - e^{at}}{t}$
155	$\log \frac{s-a}{s-b}$	$\frac{e^{bt} - e^{at}}{t}$
156	$\log \frac{s+a}{s-a}$	$\frac{2}{t} \sinh at$
157	$\log \frac{s^2 + a^2}{s^2}$	$\frac{2}{t} (1 - \cos at)$
158	$\log \frac{s^2 + a^2}{s^2 + b^2}$	$\frac{2}{t} (\cos bt - \cos at)$
159	e^{-as}	$\delta(t-a)$
160	$\frac{e^{-as}}{s}$	$U(t-a)$
161	$\frac{e^{-\frac{a^2}{4s}}}{s}$	$J_0(a\sqrt{t})$
162	$\frac{a^\nu}{2^\nu s^{\nu+1}} e^{-\frac{a^2}{4s}}$	$t^{\frac{\nu}{2}} J_\nu(a\sqrt{t}) \quad (\Re \nu > -1)$

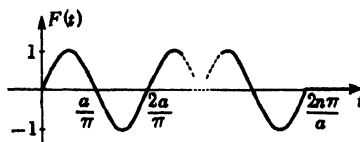
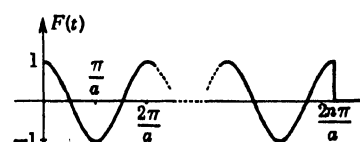
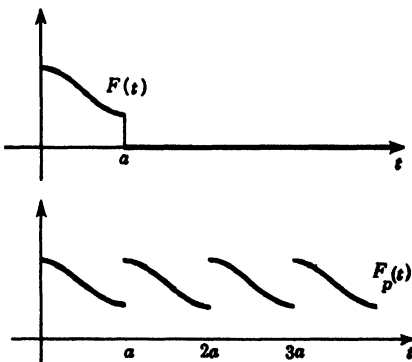
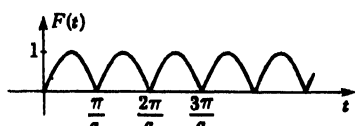
No.	$f(s)$	$F(t)$
163	$\chi(a, s)$	$\frac{\cos a \sqrt{t}}{\pi \sqrt{t}}$
164	$\psi(a, s)$	$\frac{\sin a \sqrt{t}}{\pi}$
165	$\frac{1}{e^s \sqrt{s}}$	$\frac{\cosh 2 \sqrt{t}}{\sqrt{\pi t}}$
166	$\frac{1}{s \sqrt{s} e^s}$	$\frac{\sinh 2 \sqrt{t}}{\sqrt{\pi}}$
167	$e^{-a\sqrt{s}}$	$\psi(a, t)$
168	$\frac{e^{-a\sqrt{s}}}{s}$	$\operatorname{erfc} \frac{a}{2 \sqrt{t}}$
169	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\chi(a, t)$
170	$\frac{1}{s(b + \sqrt{s})} e^{-a\sqrt{s}}$	$\frac{1}{b} \operatorname{erfc} \left(\frac{a}{2 \sqrt{t}} \right) - \frac{1}{b} e^{b^2 t + ab} \operatorname{erfc} \left(\frac{a}{2 \sqrt{t}} + b \sqrt{t} \right)$
171	$\frac{1}{\sqrt{s}(b + \sqrt{s})} e^{-a\sqrt{s}}$	$e^{b^2 t + ab} \operatorname{erfc} \left(\frac{a}{2 \sqrt{t}} + b \sqrt{t} \right)$
172	$e^{-x\sqrt{as^2+bs+c}} - e^{-(b/2\sqrt{a})x} e^{-\sqrt{a}xz}$ $(d = \left(\frac{b}{2}\right)^2 - ac)$	$\begin{cases} 0 & \text{for } 0 \leq t \leq \sqrt{a}x \\ \sqrt{\frac{d}{a}} x e^{-(b/2\sqrt{a})t} \frac{I_1 \left(\frac{\sqrt{d}}{a} \sqrt{t^2 - ax^2} \right)}{\sqrt{t^2 - ax^2}} & \text{for } t \geq \sqrt{a}x \end{cases}$
173	$\frac{1}{\sqrt{s}} \sin \frac{a}{s}$	$\frac{\sinh \sqrt{2at} \sin \sqrt{2at}}{\sqrt{\pi t}}$
174	$\frac{1}{s \sqrt{s}} \sin \frac{a}{s}$	$\frac{\cosh \sqrt{2at} \sin \sqrt{2at}}{\sqrt{a\pi}}$

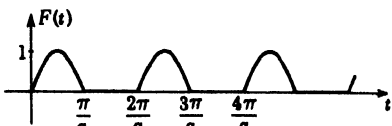
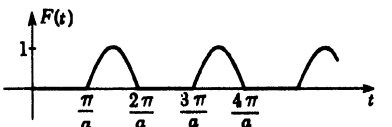
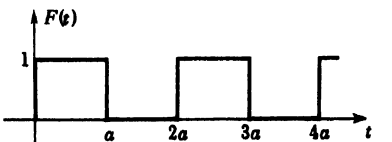
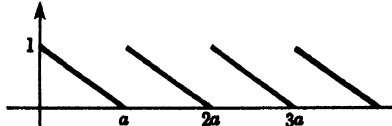
No.	$f(s)$	$F(t)$
175	$\frac{1}{\sqrt{s}} \cos \frac{a}{s}$	$\frac{\cosh \sqrt{2at} \cos \sqrt{2at}}{\sqrt{\pi t}}$
176	$\frac{1}{s\sqrt{s}} \cos \frac{a}{s}$	$\frac{\sinh \sqrt{2at} \cos \sqrt{2at}}{\sqrt{a\pi}}$
177	$\frac{1}{\sqrt{s}} \sinh \frac{a}{s}$	$\frac{\cosh 2\sqrt{at} - \cos 2\sqrt{at}}{2\sqrt{\pi t}}$
178	$\frac{1}{s\sqrt{s}} \sinh \frac{a}{s}$	$\frac{\sinh 2\sqrt{at} - \sin 2\sqrt{at}}{2\sqrt{a\pi}}$
179	$\frac{1}{\sqrt{s}} \cosh \frac{a}{s}$	$\frac{\cosh 2\sqrt{at} + \cos 2\sqrt{at}}{2\sqrt{\pi t}}$
180	$\frac{1}{s\sqrt{s}} \cosh \frac{a}{s}$	$\frac{\sinh 2\sqrt{at} + \sin 2\sqrt{at}}{2\sqrt{a\pi}}$
181	$\operatorname{arctg} \frac{a}{s}$	$\frac{\sin at}{t}$
182	$\operatorname{arctg} \frac{2as}{s^2 - a^2 + b^2}$	$\frac{2}{t} \sin at \cos bt$
183	$\operatorname{arctg} \frac{s^2 - as + b}{ab}$	$\frac{e^{at} - 1}{t} \sin bt$
184	$\frac{\cosh(2v-1)\sqrt{s}}{\sqrt{s} \sinh \sqrt{s}} \quad (0 \leq v \leq 1)$	$\theta_3(v, t)$
185	$\frac{\sinh x\sqrt{s}}{\sinh l\sqrt{s}} \quad (-l < x < +l)$	$\frac{1}{l} \frac{\partial}{\partial x} \theta_3\left(\frac{l-x}{2l}, \frac{t}{l^2}\right) = \sum_{n=-\infty}^{+\infty} \psi(2nl+l-x, t)$ $= -\frac{2\pi}{l^2} \sum_{n=1}^{\infty} (-1)^n n \sin n \frac{\pi}{l} x e^{-n^2 \frac{\pi^2}{l^2} t}$

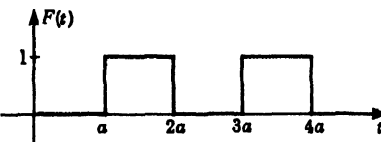
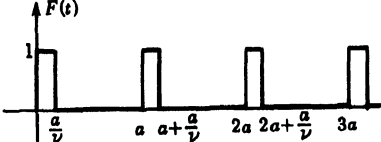
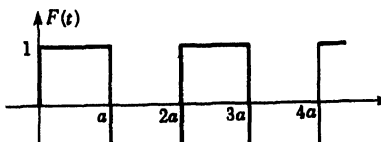
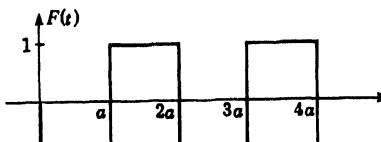
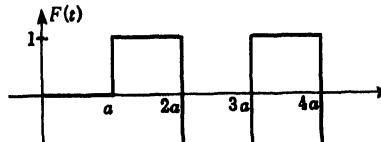
No.	$f(s)$	$F(t)$
186	$\frac{\sinh(l-x)\sqrt{s}}{\sinh l\sqrt{s}} \quad (0 < x < 2l)$	$-\frac{1}{l} \frac{\partial}{\partial x} \theta_3\left(\frac{x}{2l}, \frac{t}{l^2}\right) = \sum_{n=-\infty}^{+\infty} \psi(2nl+x, t)$ $= \frac{2\pi}{l^2} \sum_{n=1}^{\infty} n \sin n \frac{\pi}{l} x e^{-n^2 \frac{\pi^2}{l^2} t}$
187	$\begin{cases} \frac{1}{\sqrt{s}} e^{-x\sqrt{s}} \sinh \xi \sqrt{s} & (0 \leq \xi \leq x) \\ \frac{1}{\sqrt{s}} e^{-\xi\sqrt{s}} \sinh x \sqrt{s} & (x \leq \xi < \infty) \end{cases}$	$\frac{1}{2} [\chi(x-\xi, t) - \chi(x+\xi, t)]$
188	$\begin{cases} \frac{\sinh \xi \sqrt{s} \sinh(l-x)\sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & (0 \leq \xi \leq x) \\ \frac{\sinh x \sqrt{s} \sinh(l-\xi)\sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & (x \leq \xi \leq l) \end{cases}$	$\frac{1}{2} \sum_{n=-\infty}^{+\infty} [\chi(2nl+x-\xi, t) - \chi(2nl+x+\xi, t)]$ $= \frac{2}{l} \sum_{n=1}^{\infty} e^{-n^2 \frac{\pi^2}{l^2} t} \sin n \frac{\pi}{l} x \sin n \frac{\pi}{l} \xi$
189	$\frac{\sqrt{\pi}}{2} e^{\left(\frac{s}{2}\right)^2} \operatorname{erfc} \frac{s}{2}$	e^{-t^2}

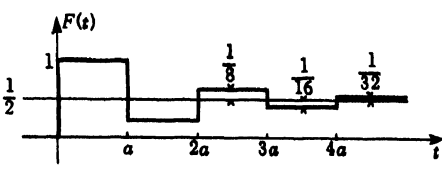
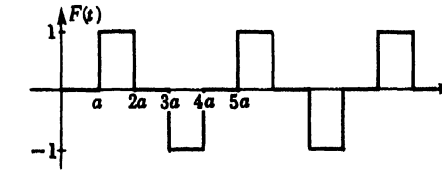
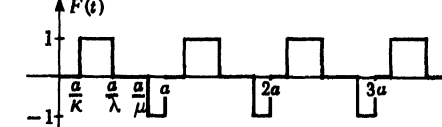
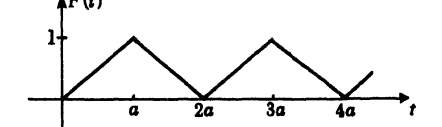
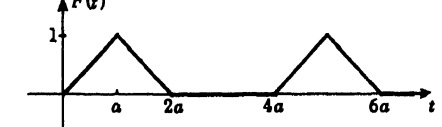
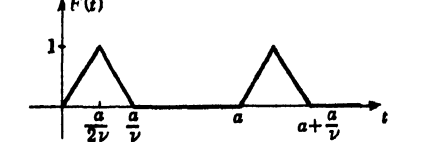
No.	$f(s)$	$F(t)$
190	$\frac{e^{-as}}{s}$	
191	$\frac{1 - e^{-as}}{s}$	
192	$\frac{e^{-as} - e^{-bs}}{s}$	
193	$\frac{(1 - e^{-as})^2}{s}$	
194	$\frac{(e^{-as} - e^{-bs})^2}{s}$	
195	$\frac{(1 - e^{-as})^2}{s^2}$	
196	$\frac{(e^{-as} - e^{-bs})^2}{s^2}$	

No.	$f(s)$	$F(t)$
197	$\frac{be^{-as}}{s(s+b)}$	 $\begin{cases} 0 & \text{for } 0 < t < a \\ 1 - e^{-b(t-a)} & \text{for } a < t \end{cases}$
198	$\frac{e^{-as}}{s+b}$	 $\begin{cases} 0 & \text{for } 0 < t < a \\ e^{-b(t-a)} & \text{for } a < t \end{cases}$
199	$\frac{e^{-as} - e^{-bs}}{s^2}$	
200	$\frac{1 - e^{-as}}{s^2}$	
201	$\frac{e^{-as} + as - 1}{as^2}$	

No.	$f(s)$	$F(t)$
202	$a \frac{\left(1 - e^{-\frac{2n\pi s}{a}}\right)}{s^2 + a^2}$	 $\begin{cases} \sin at & \text{for } 0 < t < \frac{2n\pi}{a} \\ 0 & \text{for } \frac{2n\pi}{a} < t \end{cases}$
203	$s \frac{\left(1 - e^{-\frac{2n\pi s}{a}}\right)}{s^2 + a^2}$	 $\begin{cases} \cos at & \text{for } 0 < t < \frac{2n\pi}{a} \\ 0 & \text{for } \frac{2n\pi}{a} < t \end{cases}$
204	$f_p(s) = \frac{f(s)}{1 - e^{-as}}$	
205	$\frac{a}{s^2 + a^2} \cdot \frac{1 + e^{-\frac{\pi s}{a}}}{1 - e^{-\frac{\pi s}{a}}}$	 $ \sin at $

No.	$f(s)$	$F(t)$
206	$\frac{a}{s^2 + a^2} \quad \frac{1}{1 - e^{-\frac{\pi}{a}s}}$	 $\left\{ \begin{array}{l} \sin at \quad \text{for } \frac{2n\pi}{a} < t < \frac{(2n+1)\pi}{a} \\ 0 \quad \text{for } \frac{(2n+1)\pi}{a} < t < \frac{(2n+2)\pi}{a} \end{array} \right.$ $n = 0, 1, \dots$
207	$\frac{a}{s^2 + a^2} \quad \frac{1}{e^{\frac{\pi}{a}s} - 1}$	 $\left\{ \begin{array}{l} 0 \quad \text{for } \frac{2n\pi}{a} < t < \frac{(2n+1)\pi}{a} \\ -\sin at \quad \text{for } \frac{(2n+1)\pi}{a} < t < \frac{(2n+2)\pi}{a} \end{array} \right.$ $n = 0, 1, \dots$
208	$\frac{1}{s(1 + e^{-as})}$	
209	$\frac{e^{-as} + as - 1}{as^2(1 - e^{-as})}$	

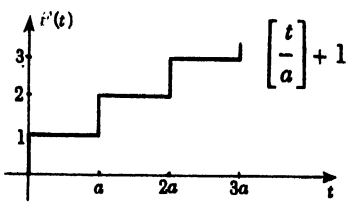
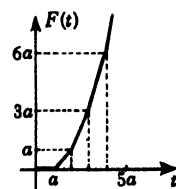
No.	$f(s)$	$F(t)$
210	$\frac{1}{s(1+e^{as})}$	
211	$\frac{1 - e^{-\frac{a}{\nu}s}}{s(1 - e^{-as})}$	
212	$\frac{1 - e^{-as}}{s(1 + e^{-as})}$	
213	$\frac{e^{-as} - 1}{s(1 + e^{-as})}$	
214	$\frac{1 - e^{-as}}{s(1 + e^{as})}$	

No.	$f(s)$	$F(t)$
215	$\frac{4 - e^{-as}}{s(4 + 2e^{-as})}$	 $\frac{1}{2} + (-1)^n \frac{1}{2^{n+1}} \quad \text{for } na < t < (n+1)a,$
216	$\frac{1 - e^{-as}}{s(e^{as} + e^{-as})}$	
217	$\frac{e^{-\frac{as}{\kappa}} - e^{-\frac{as}{\lambda}} + e^{-as} - e^{-\frac{as}{\mu}}}{s(1 - e^{-as})}$	
218	$\frac{1 - e^{-as}}{as^2(1 + e^{-as})}$	
219	$\frac{(1 - e^{-as})^2}{as^2(1 - e^{-4as})}$	
220	$\frac{2\nu(1 - e^{-\frac{as}{2\nu}})^2}{as^2(1 - e^{-as})}$	

No.	$f(s)$	$F(t)$
221	$\frac{\left(1 - e^{-\frac{as}{\mu}}\right)\left(1 - e^{-(2 - \frac{1}{\mu})as}\right)}{as^2(1 - e^{-2as})}$	
222	$\frac{\left(1 - e^{-\frac{as}{\mu}}\right)\left(1 - e^{-(2 - \frac{1}{\mu})as}\right)}{as^2(1 - e^{-4as})}$	
223	$\frac{2\nu\left(1 - e^{-\frac{as}{2\nu\mu}} - e^{-\frac{(2\mu-1)as}{2\nu\mu}} + e^{-\frac{as}{\nu}}\right)}{as^2(1 - e^{-as})}$	
224	$\frac{\nu(\nu-1) + \nu e^{-as} - \nu^2 e^{-\frac{as}{\nu}}}{(\nu-1)as^2(1 - e^{-as})}$	
225	$\frac{\nu(\nu-1) + \nu e^{-as} - \nu^2 e^{-\frac{as}{\nu}}}{(\nu-1)as^2(1 - e^{-2as})}$	
226	$\frac{\lambda\nu(\nu-1) + \lambda\nu e^{-\frac{as}{\lambda}} - \lambda\nu^2 e^{-\frac{as}{\lambda\nu}}}{as^2(\nu-1)(1 - e^{-as})}$	
227	$\frac{as + 1 - e^{as}}{as^2(1 - e^{as})}$	

No.	$f(s)$	$F(t)$
228	$\frac{1 - (1 + as)e^{-as}}{as^2(1 - e^{-2as})}$	
229	$\frac{\nu - (\nu + as)e^{-\frac{as}{\nu}}}{as^2(1 - e^{-as})}$	
230	$\frac{\mu - \mu e^{-\frac{as}{\mu}} - ase^{-as}}{a\mu s^2(1 - e^{-as})}$	
231	$\frac{\mu - \mu e^{-\frac{as}{\mu}} - ase^{-as}}{a\mu s^2(1 - e^{-2as})}$	
232	$\frac{\mu\nu - \mu\nu e^{-\frac{as}{\nu\mu}} - ase^{-\frac{as}{\nu}}}{a\mu s^2(1 - e^{-as})}$	
233	$\frac{2 - as - (2 + as)e^{-as}}{as^2(1 - e^{-as})}$	
234	$\frac{2\mu e^{-\frac{as}{2}} \left(e^{\frac{as}{2\mu}} - e^{-\frac{as}{2\mu}} \right) - as(1 + e^{-as})}{a\mu s^2(1 - e^{-as})}$	

No.	$f(s)$	$F(t)$
235	$\frac{2(1 - e^{-as})}{as^2(1 + e^{-as})} - \frac{1}{s}$	
236	$\frac{2\left(e^{-\frac{\mu-1}{2\mu}as} - e^{-\frac{\mu+1}{2\mu}as}\right)}{as^2(1 + e^{-as})} - \frac{1}{\mu s}$	
237	$\frac{1}{s(e^s - 1)}$	
238	$\frac{1}{s(e^{as} - 1)}$	
239	$\frac{1}{s(1 - e^{-s})}$	

No.	$f(s)$	$F(t)$
240	$\frac{1}{s(1 - e^{-as})}$	
241	$\frac{1}{s^2(e^{as} - 1)}$	 $n \left(t - \frac{(n+1)a}{2} \right) \text{ for } na < t < (n+1)a$
242	$\frac{1}{s(e^s - \alpha)} (\alpha + 1)$	$\frac{\alpha^{[t]} - 1}{\alpha - 1}$
243	$\frac{e^s - 1}{s} \frac{1}{e^s - \alpha}$	$\alpha^{[t]}$
244	$\frac{e^s - 1}{s} \frac{1}{(e^s - \alpha)^2}$	$[t] \alpha^{[t]-1}$
245	$\frac{e^s - 1}{s} \frac{1}{(e^s - \alpha)^3}$	$\frac{1}{2} [t] ([t] - 1) \alpha^{[t]-2}$
246	$\frac{e^s - 1}{s} \frac{1}{(e^s - \alpha)(e^s - \beta)}$	$\frac{\alpha^{[t]} - \beta^{[t]}}{\alpha - \beta} \quad (\alpha \neq \beta)$

No.	$f(s)$	$F(t)$
247	$\frac{e^s - 1}{s} \frac{e^s - (a + \beta)}{(e^s - a)(e^s - \beta)}$	$-\alpha\beta \frac{a^{[t]-1} - \beta^{[t]-1}}{a - \beta} (a + \beta)$
248	$\frac{e^s - 1}{s} \frac{e^{\mu s}}{p(e^s)} \quad (\mu = 0, 1, \dots, r-1)$ $(p(z) = z^r + c_{r-1}z^{r-1} + \dots + c_0)$	$Q^{([t] + \mu)}(0)$ $\left(Q(t) \sim \frac{1}{p(s)} \right)$
249	$\frac{1 - e^{-s}}{s} \sum_{k=0}^{\infty} F_k e^{-ks}$	$F([t])$
250	$\frac{e^s - 1}{s} \frac{e^s + 1}{(e^s - 1)^3}$	$[t]^2$
251	$\frac{e^s - 1}{s} \frac{e^s + a}{(e^s - a)^3}$	$[t]^2 a^{[t]-1}$
252	$\frac{e^s - 1}{s} \frac{(c - d)e^s - (c\beta - da)}{(e^s - a)(e^s - \beta)}$	$ca^{[t]} - d\beta^{[t]}$
253	$\frac{e^s - 1}{s} \frac{\sin \beta}{e^{2s} - 2e^s \cos \beta + 1}$	$\sin \beta [t]$
254	$\frac{e^s - 1}{s} \frac{e^s - \cos \beta}{e^{2s} - 2e^s \cos \beta + 1}$	$\cos \beta [t]$
255	$\frac{e^s - 1}{s} \frac{a \sin \beta}{e^{2s} - 2ae^s \cos \beta + a^2}$	$a^{[t]} \sin \beta [t]$
256	$\frac{e^s - 1}{s} \frac{e^s - a \cos \beta}{e^{2s} - 2ae^s \cos \beta + a^2}$	$a^{[t]} \cos \beta [t]$

$$U(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}$$

$$\delta(t) = \text{Dirac-function, Impulse-function}$$

$$L_n(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!} \quad (\text{Laguerre Polynomial})$$

$$\chi(a, t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}$$

$$\psi(a, t) = \frac{a}{2\sqrt{\pi t^{3/2}}} e^{-\frac{a^2}{4t}}$$

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$$

$$J_0(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{t}{2}\right)^{2k}}{(k!)^2}$$

$$J_{\nu}(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{t}{2}\right)^{2k+\nu}}{k! \Gamma(\nu+k+1)}$$

$$I_1(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2k+1}}{k! (k+1)!}$$

$$\theta_3(v, t) = 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t} \cos 2k\pi v = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{+\infty} e^{-\frac{(v+n)^2}{t}}$$

$$[t] = \text{largest integer} \leq t$$

$$C = 0,577 \dots \quad (\text{Euler's constant})$$

