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THE ELECTROMAGNETIC FIELD

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INTRODUCTION

In a detailed theory of a group of physical phenomena an analogy is exhibited between observed facts and the logical consequences of a self-consistent mathematical structure. The analogy constitutes the theory. A theory is valued for the diversity of the phenomena which it describes and correlates, and for the stimulus it furnishes for the discovery of new experimental facts. The mathematical structure should be of such nature that the analytical manipulations required in the derivation of the theorems can be readily performed.

Judged in this light, the Maxwell-Lorentz theory has attained a degree of success little short of marvelous. Its triumphs have forced it into the position of an ultimate theory. With Maxwell, light became an electro-magnetic phenomenon, and the subject of electromagnetic radiation was opened for investigation. With the electron theory of Lorentz, matter became an electrical complex. Through the immediate success of this theory, and influenced by a vast amount of experimental evidence that was soon forthcoming, the conviction grew that all physical phenomena were electromagnetic. The viewpoint of physicists toward electromagnetic theory today is determined by their belief in the basic rôle played by the theory. Its fundamental character makes it necessary that the definitions and concepts of the theory be stated with special care and precision. Its concepts are the most fundamental concepts of physics. To "explain" electromagnetic action is meaningless. Simplicity of statement and recognition of the fundamental character of the concepts are the demands. The basic mathematical structure of the theory consists of a set of vector differential equations, the "field equations" of the electron theory. This is the form of statement a century after the science of electrodynamics was born. The viewpoint toward these equations is, like the equations themselves, a result of evolution during that century.

The first half of the century saw the acceptance of the conception of action at a distance for electromagnetic phenomena, and action in a medium for optical phenomena. With the names of Ampère, Weber, Grassmann, Gauss, Reimann, Neumann, Kirchoff, Helmholz, Clausius, and Betti is associated the development of mathematical expressions for laws of electrodynamic action. Forces between current elements or moving charges were either expressed directly or through the aid of auxiliary

vectors, or as Lagrangian derivatives of a function playing the rôle of a kinetic potential. These laws were all point laws, giving the action of charges on charges with no attempt to describe the way in which these forces, acting at different points of space, could arise. In the earlier theories there was no suggestion of an actual propagation of effects from one point to another, but a suggestion of this nature was contained in a letter which Gauss wrote to Weber in 1845. He mentioned that he had himself (in 1835) attempted to deduce the fundamental law for electrodynamic action, but had never published his results because he had failed to accomplish that which seemed to him the real task—the derivation of the law from a consideration of the propagation of effects with a finite velocity. Kirchoff in 1857 noticed the coincidence between the value of the velocity of light and that of the ratio of the electrical units. In 1858 Riemann presented a paper to the Göttingen Academy in which he assumed a finite velocity of propagation, and deduced that this must be equal to the ratio of the units, and hence to the velocity of light. In 1867, Lorenz, of Copenhagen, extended the theory of Neumann, obtained expressions for the retarded vector and scalar potentials which are equivalent to the forms commonly used today, and was led independently of Maxwell to the conception of light as an electromagnetic phenomenon. The difference in viewpoints is, however, striking; for Lorentz considered that if light were shown to be electromagnetic in nature there was no longer the necessity for maintaining the hypothesis of an aether. The action at a distance theory was thus moving certainly toward the discovery of time lag in effects and toward the electromagnetic theory of light. Maxwell reached this goal, however, by an attack from quite a different angle, and in the glare caused by his brilliant investigations much of the work just mentioned was lost sight of.

Impressed by Faraday's conception of lines of magnetic and electric force, and by Kelvin's analogies of the electric and magnetic field of force with heat flow, elastic deformation, and fluid motion, Maxwell turned his attention aside from elements of current or charge, and conceived of all phenomena as due to conditions existing in a mechanical medium. From his equations there resulted the determination of the velocity of propagation of effects. Maxwell at once identified the mechanical medium of his theory with the aether which optical phenomena had long since led physicists to consider, and founded the electromagnetic theory of light. Light became an electromagnetic phenomenon, but electromagnetism an aether phenomenon. The vectors of Maxwell's theory expressed the state of the aether. Confidence was not lacking that the specification of the

aether as an elastic medium could be obtained, so that the field equations would follow from the laws of mechanics. Heaviside and Hertz, avoiding discussion of the detailed mechanical models which Maxwell considered in the derivation of his equations, simplified the analytical statement of the theory. The resultant field equations were universally accepted as the basis of electrodynamic theory. The psychological effect of Maxwell's work was also far reaching in character. Many of his outstanding results were certainly correct, and these successes, together with the recognized genius of the man himself, naturally impressed upon the future development of the subject not only the analytical expressions for which he was responsible but also his methods of thought and his point of view. He gave to physicists a more systematic treatment of the subject than they had had, a treatment capable of bolder extensions, a theory amazingly successful in explaining old results and predicting new ones; and behind it all was the idea, so comforting to the English physicists, of a mechanical analogy. If there were difficulty or dissatisfaction because of vagueness of definition and complexity of the underlying concepts, it was overwhelmed by the prestige obtained by the great achievements of the theory. Through the following years the concepts of the Maxwell theory became firmly fixed in the mind of each student of physics. There was so much talk about lines of force, tubes of force, stresses in the medium, and localized energy that an easy familiarity with the terms began to carry with it a sense of understanding and reality, and curiosity became dulled as the years passed by. The idea of a medium whose state was expressed through the equations of the field was fundamental to the theory, and the idea of action at a distance seemed to retain a historical interest only.

The next great advance was the formulation of the electron theory of Lorentz. He conceived that all electromagnetic and optical interactions of matter were due to the presence of corpuscular charges, "electrons," within the matter. This is a partial return to the earlier viewpoint, in which the action of charge on charge played the entire rôle. But with Lorentz, the Maxwell theory is preserved. Instead of direct description of the action of charge on charge, the theory is phrased in terms of the action of medium on charge, and charge on medium. Electrons produce a "field" which is propagated in the medium, and which acts on all other electrons. The rôle of the medium, in Lorentz' theory, becomes far more clearly that of an intermediary only. With the negative result of all aether drag experiments, the proof of the covariance of the field equations under the Lorentz transformation, and the statement of the

theory of relativity, the aether, as a mechanical concept, vanished. To it was not even left the rôle of determining a system of reference. The idea of the field remained, however, as its trace, and electromagnetic theory remained a field theory, whether the field was thought of in terms of its components with their energy densities or as a world-tensor.

The end result of Lorentz' theory is the direct description, through the retarded potentials, of the action of charge on charge. Thus, though this is not at all the viewpoint of Lorentz' own presentation, we may conceive that we are back in spirit to action at a distance, but action after a lapse of time. Whether we use the language of action at a distance or action in the medium is obviously a matter of words only, if the analytical formulations are really equivalent; but it is not a matter of indifference if the question becomes one of extension or modification of the theory. In such attempts, intuition is led by the picture accepted as fundamental. If the electrodynamic field be considered as fundamental, such concepts as the localization of energy in space and flux of energy density seem a compelling, not an arbitrary, assumption. Certainly, in consideration of such a searching question as the reconciliation of quantum ideas on energy interchanges with general theory, the type of attempted modification will depend upon the choice of viewpoint in this particular.

The great scientific task of the next fifty years is the development of a new "electromagnetic" theory. It is impossible to forecast the form such a theory will take, so greatly are we prejudiced by our present views. It will, however, doubtless be based on a quantitative description of the individual behavior of charges, and will yield as statistical concepts such ideas as inertia, force, and even length and time. Thus it will explain the mechanical behavior of ponderable matter, rather than be itself "explained" by mechanics. That this new theory should yield ideas such as those just mentioned, rather than depend upon them, is but consistent with the basic rôle this theory must play.

Pending the advent of this new theory, it is essential to have a knowledge of that system of equations which constitutes the present electromagnetic theory. Remembering that the future of the theory lies in the hands of the present students, it seems of the greatest importance to arrive at these equations in a way which will excite, rather than dull, curiosity, and which tends to produce that attitude toward fundamentals which must prevail before a real electron theory of electricity replaces the present electrical theory of electrons.

The present volume is an introduction to the mathematical field theory of electrodynamics, written in an attempt to keep clear the relation

between the mathematical mechanism and the physical reality. It is hoped that a student who gains his familiarity with the Maxwell field equations from this book will have an open mind, ready to evaluate without prejudice those fundamental new developments in electrical theory which are bound to come. The authors realize that this volume closes just at that interesting point where many valuable developments begin. However, just as, according to the behaviorists, one's psychology is largely "set" before he reaches the age of three, so one's viewpoint toward electrodynamics is a fixed and settled matter by the time one has acquired the field equations and has discussed certain of their most fundamental applications. It is with this viewpoint that the authors are primarily concerned.

The authors wish to acknowledge their indebtedness to Mr. Ivan Sokolnikoff, Instructor of Mathematics at the University of Wisconsin, for his kind and efficient assistance in the reading of proof.

CHAPTER I

COULOMB'S LAW AND SOME ANALYTIC CONSEQUENCES

Part I. Discrete Charges

INTRODUCTION

Part I of this chapter introduces the fundamental inverse square law for the electrostatic action between two concentrated charges. This relationship is then extended, by means of the superposition principle, to cover the case of any number of discrete charges. The mutual electrostatic energy Ψ of a configuration of charges is defined as the amount of work which must be done against the electrostatic forces to produce the given configuration, starting from a configuration in which each charge is very far distant from any other charge. This quantity Ψ depends, for its value, on the location of every charge, and is thus a function of the co-ordinates of each charge. If the co-ordinates $x_\epsilon, y_\epsilon, z_\epsilon$ of one charge ϵ be allowed to change, all the other charges remaining fixed, the function Ψ varies, the change corresponding physically to the fact that work must be done to move the charge ϵ. The rate at which Ψ varies when ϵ is moved in any direction—mathematically expressed, the directional derivative, taken at the position of ϵ, of the function $\Psi(x_\epsilon, y_\epsilon, z_\epsilon)$—gives the component of the electrostatic force on ϵ in that direction. This leads to the introduction of a vector $\nabla\Psi$, called "nabla psi," whose component in any direction is the rate of change Ψ in that direction, the force on any charge then being given in terms of this vector $\nabla\Psi$. In computing such rates of change of Ψ it is clearly possible to disregard portions of Ψ which do not depend upon $x_\epsilon, y_\epsilon, z_\epsilon$. The extraneous part of Ψ is therefore discarded, and the remaining portion is ϵ times a quantity which is defined to be the "electrostatic potential" at $x_\epsilon, y_\epsilon, z_\epsilon$. The negative nabla of this electrostatic potential is also given a special name—the "electrostatic intensity."

---

§ 1. *Coulomb's Law and Superposition of Effects.*—The basis for the analytic study of electrostatics is Coulomb's law of force. This law was suggested and even roughly verified before the experiments of Coulomb (1785), but belief in its validity was not general previous to that time. Using a torsion balance, he found that the force between two small charged bodies was along the line joining them, and varied inversely as the square of the distance, provided this distance was large compared to

the linear dimensions of the two bodies. For such distances the extension of the bodies may be disregarded, a single point may be taken as giving the location of each body, and the law of force may be written in the vector form*

$$\boldsymbol{F}_2 = K \frac{e_1 e_2}{r_{12}^2} \boldsymbol{r}'_{12} .$$

In this equation $\boldsymbol{r}'_{12}$ is a vector of unit length pointing from one body having the charge e_1, toward the other body, of charge e_2. $\boldsymbol{F}_2$ is the vector force acting on body 2, and r_{12} is the distance between the bodies. The measurement of the charges e_1 and e_2 depends upon the value chosen for the proportionality factor K. In this book, following Heaviside and Lorentz, the value of K is taken as $1/4\pi$, and the law of force becomes

$$\boldsymbol{F}_2 = \frac{1}{4\pi} \frac{e_1 e_2}{r_{12}^2} \boldsymbol{r}'_{12} . \tag{1}$$

Like unit charges at a distance of 1 centimeter thus repel each other with a force of $1/4\pi$ dynes. This "rational" unit of charge is smaller than the older electrostatic unit, corresponding to the choice $K = 1$, in the ratio $1/\sqrt{4\pi}$. Thus if e' be the measure of a charge in the older electrostatic units, and e the measure of the same charge in the units here adopted, then

$$\sqrt{4\pi}\, e' = e .$$

The modern theory interprets a charged body as one with an excess or deficit in the number of its electrons, the total charge being the deviation from the normal number times the electronic charge

$$\sqrt{4\pi}\ (4.77 \times 10^{-10}).$$

The charge of an electron itself is considered negative, on account of earlier unfortunate conventions, so that a body with a deficit of electrons is called "positively charged."

The force between charged bodies may then be considered as due to forces between the elementary charges, an electron being an elementary negative charge, and the nucleus of an atom being an elementary positive charge. It is obviously in accord with the experiments which establish (1) to assume: first, that at distances large compared to atomic dimen-

* It is assumed, in the early sections of this chapter, that the charges under consideration are located in "free space." The modification necessary when the charges are located in a polarizable medium will be discussed later. All vectors are represented by bold face.

sions there is an electrostatic force between each pair of elementary charges which obeys Coulomb's law; and, second, that these forces superpose without alteration, i.e., that the force on a charge due to several charges is the vector sum of the forces which would be exerted by these charges separately.

It should not be concluded that these assumptions are the only ones in accord with the experimental facts of electrostatics. Such experiments deal with the average value, taken over appreciable times, of the gross effect of large groups of charges. The actual force between pairs of elementary charges might follow any law which gives the inverse square law as a statistical result, i.e., as the time mean of the sum of the forces between an enormous number of pairs of charges. It is, however, these gross effects which are under discussion in electrostatics, and for such purposes it suffices to assume Coulomb's law between elementary charges at large distances and vector superposition of forces.

It is a matter of some practical importance to note that for the purposes of electrostatics the nature of the law for very small distances is a matter of indifference, except for the fact that one form of the law may be more convenient than another for calculation. It would be quite in keeping with the experimental facts to assume, for example, the law

$$F_2 = \frac{1}{4\pi} \frac{e_1 e_2}{r_{12}^2} \beta(r_{12}) r'_{12} ,$$

provided that $\beta(r_{12})$ is a factor which reduces effectively to unity for all except very small distances r_{12}. If, in particular, it be desired to have a form which reduces to zero, instead of becoming infinite when $r_{12} = 0$, then β could be chosen as

$$\beta(r_{12}) = e^{-\alpha/r_{12}^2} ,$$

where α is sufficiently small, an example of which is plotted in Figure 1.

§ 2. *Mutual Electrostatic Energy Ψ of a Configuration of Elementary Charges.*—The mutual electrostatic energy of a set of elementary charges is defined as the work which must be done against the electrostatic forces to bring the charges to their given positions, starting from a configuration of "infinite sepa-

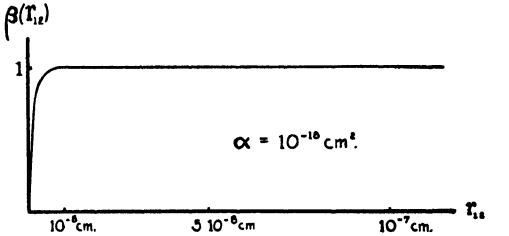

FIG. 1.—Graph of $\beta(r_{12})$

ration," i.e., a configuration in which all the charges are very far apart. The amount of work is, of course, the same as that done by the electrostatic forces when the charges are allowed to separate from their configuration to one of infinite mutual separation. It will be seen that this work is independent of the paths along which the charges move, and is thus completely determined by the configuration. Consider first the case of two charges e_1 and e_2, located at points P_1 and P_2. Let e_1 be kept fixed, while e_2 is allowed to recede along some definite path. Let s_2 measure length from P_2 along this path, and let r_{12} be the distance from the charge e_1 to the charge e_2, so that r_{12} is a function of s_2. Then the component of electrostatic force acting on e_2 tangent to the path at any point is

$$(\boldsymbol{F}_2)_{s_2} = \frac{e_1 e_2}{4\pi r_{12}^2} \cos(r_{12}, s_2) ,$$

but since

$$\cos(r_{12}, s_2) = \frac{dr_{12}}{ds_2} ,$$

this may be written

$$(\boldsymbol{F}_2)_{s_2} = \frac{e_1 e_2}{4\pi r_{12}^2} \frac{dr_{12}}{ds_2} = -\frac{e_1 e_2}{4\pi} \frac{d}{ds_2}\left(\frac{1}{r_{12}}\right) , \tag{2}$$

the value of the derivative being taken at the point where the force component is desired.

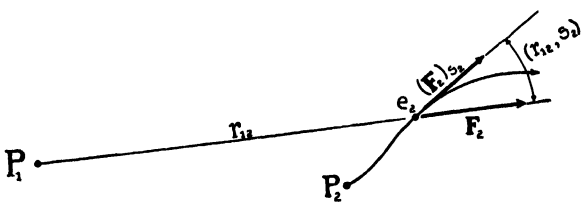

FIG. 2

The work done by the electrostatic force when e_2 recedes is therefore

$$\left\{ \begin{aligned} \Psi_{12} &= -\frac{e_1 e_2}{4\pi} \int_0^\infty \frac{d}{ds_2}\left(\frac{1}{r_{12}}\right) ds_2 , \\ &= \frac{e_1 e_2}{4\pi} \frac{1}{r_{12}} , \end{aligned} \right. \tag{3}$$

where, in the last expression, r_{12} is the distance between the charges when they are in the configuration whose energy is being reckoned. The form

of the integral giving the mutual electrostatic energy shows at once that the energy does not depend on the path over which e_2 moves; for the integrand is a derivative with respect to s_2, whose integral with respect to s_2 depends only upon the limits, i.e., only upon the initial and final positions of the charge e_2.

The mutual electrostatic energy Ψ_{12} of the two charges is a number characteristic of the mutual configuration. If the position of either charge be changed, this number changes, and Ψ_{12} becomes a function of the parameters which give the location of the charge in question. If, as was supposed above, e_2 moves on a path along which s_2 measures distance, the mutual energy Ψ_{12} for any position s_2 of e_2 is a function of s_2 (since r_{12} depends upon s_2), and the component of force in the direction s_2 at any point on the path is

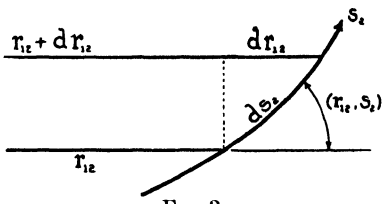

FIG. 3

$$-\frac{e_1 e_2}{4\pi}\frac{d}{ds_2}\left(\frac{1}{r_{12}}\right) = -\frac{d\Psi_{12}}{ds_2},$$

the value of the derivative being taken at the point where the force is desired. In particular, the component of force in the direction of s_2 at the point P_2 is

$$(F_2)_{s_2} = -\left(\frac{d\Psi_{12}}{ds_2}\right)_{s_2=0}.$$

In computing a component of electrostatic force on a charge, the idea of the motion of the charge on a given path is obviously extraneous, and there is, indeed, nothing in the analytical form of the last equation which need involve the idea of a path or motion. The expression*

$$\left(\frac{d\Psi_{12}}{ds_2}\right)_{s_2=0}$$

is a directional derivative of the function Ψ_{12}, giving its rate of change at the point P_2 in the direction s_2. Thus (2) may be re-written simply

$$(F_2)_{s_2} = -\frac{d\Psi_{12}}{ds_2},$$

* See Appendix, § 3, A, for a discussion of directional derivatives.

where, on the right, it is understood that the rate of change of Ψ_{12} in a direction s_2 is calculated for the point P_2. In calculating this derivative one end of r_{12} is fixed at P_1, while the other end is variable.

It follows at once from the principle of superposition that the work done when n charges e_i, located at points $P_1, P_2, \ldots . P_n$, are allowed to recede to a state of infinite mutual separation, is given by

$$\Psi = \frac{1}{4\pi} \sum \frac{e_i e_j}{r_{ij}} , \tag{4}$$

each pair of charges appearing once in the summation. The component of force in a direction s_k on any one charge e_k due to the others is given by

$$(F_k)_{s_k} = -\frac{d\Psi}{ds_k} .$$

In forming this directional derivative, all the r_{ij}, $j \neq k$, are constant, while each r_{ik} is a function of s_k, one end, P_i, being fixed, while the other end is variable.

§ 3. *The Vector Force* $\boldsymbol{F}$ *Expressed as* $-\nabla\Psi$.—The equation expressing the component in a direction $\boldsymbol{s}$ of a vector

$$\boldsymbol{C} = \boldsymbol{i}C_x + \boldsymbol{j}C_y + \boldsymbol{k}C_z ,$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors in the directions of the co-ordinate axes, is

$$C_s = C_x \cos(s,x) + C_y \cos(s,y) + C_z \cos(s,z) .$$

By comparison of this equation with the equation

$$\frac{d\Phi}{ds} = \frac{\partial\Phi}{\partial x}\frac{dx}{ds} + \frac{\partial\Phi}{\partial y}\frac{dy}{ds} + \frac{\partial\Phi}{\partial z}\frac{dz}{ds} ,$$

$$= \frac{\partial\Phi}{\partial x}\cos(s,x) + \frac{\partial\Phi}{\partial y}\cos(s,y)\,\frac{\partial\Phi}{\partial z}\cos(s,z) ,$$

for the directional derivative of any function $\Phi(x,y,z)$, it is seen that $d\Phi/ds$ is the component in the direction of $\boldsymbol{s}$ of a vector whose components in the co-ordinate directions are $\partial\Phi/dx$, $\partial\Phi/\partial y$, $\partial\Phi/\partial z$. This vector is written $\nabla\Phi$ (read "nabla phi"), so that

$$\nabla\Phi \equiv \boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z} .$$

Thus*

$$\frac{d\Phi}{ds} = (\nabla\Phi)_s .$$

Now the maximum component of a vector is the component in the direction of the vector itself. Therefore ∇, operating on a scalar-point function, produces a vector-point function which at every point has the direction of the greatest rate of increase of the scalar function, and whose magnitude measures that rate of increase. Thus if $z=f(x,y)$ be thought of as a surface, z being measured vertically, the two dimensional vector ∇f is given by

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} .$$

At any point x,y this vector has the direction of the projection on the xy-plane, of that tangent line to the surface at the point $x,y,f(x,y)$ which points "up hill" on the surface, i.e., which makes a greater angle with the horizontal than any other tangent line at this point. The magnitude of ∇f is, moreover, the slope of this direction of steepest ascent.

As a second example of the use of this operator, let r be the distance between $P(x,y,z)$ and $P'(x', y', z')$. Then

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 ,$$

and

$$\nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} .$$
$$= i\frac{(x-x')}{r} + j\frac{(y-y')}{r} + k\frac{(z-z')}{r} .$$

The length of the vector ∇r is

$$|\nabla r| = \frac{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{r} = 1 ,$$

and it points, as is seen from its components (or directly from the meaning of the operator nabla), from P' toward P. Thus

$$\nabla r = r' ,$$

* The partial-derivative notation is sometimes used for directional derivatives See remarks at the end of Appendix § 3, A.

where $\mathbf{r}'$ is a unit vector from P' toward P. Moreover,

$$-\nabla \frac{1}{r} = \frac{\nabla r}{r^2} = \frac{\mathbf{r}'}{r^2} .$$

When the function upon which the nabla operates contains more than one set of variables, the set with respect to which the differentiation involved in ∇ is to be taken can be indicated by means of a subscript or superscript on the symbol ∇. Thus

$$\nabla' r = \mathbf{i} \frac{\partial r}{\partial x'} + \mathbf{j} \frac{\partial r}{\partial y'} + \mathbf{k} \frac{\partial r}{\partial z'} ,$$

$$\nabla' r = -\nabla r = -\mathbf{r}' ,$$

and obviously

$$\nabla f(r) = -\nabla' f(r) \tag{5}$$

for any function f.

Now the vector force on any one of a group of elementary charges may be expressed in terms of the nabla of the function giving the mutual electrostatic energy of the configuration. For since

$$(\mathbf{F}_k)_{s_k} = -\frac{d\Psi}{ds_k} ,$$

it is evident from the foregoing discussion that

$$\mathbf{F}_k = -\nabla_k \Psi , \tag{6}$$

where the differentiation indicated by the differential operator ∇_k is to be taken with respect to the variables x_k, y_k, z_k fixing the position of the charge e_k. The energy is indeed a function of x_k, y_k, z_k, all the r_{ij}, $j \neq k$ being constant, while each r_{ik} is a function of x_k, y_k, z_k.

§ 4. *Electrostatic Potential and Electrostatic Intensity.*—Consider a configuration made up of a charge ϵ and n other charges e_i. The total mutual electrostatic energy of this configuration is conveniently written, when one is interested in the force on the charge ϵ due to the other charges e_i, in the form

$$\Psi \equiv \Psi_\epsilon + \Psi_i ,$$

$$= \frac{\epsilon}{4\pi} \Sigma \frac{e_i}{r_{\epsilon i}} + \frac{1}{4\pi} \Sigma \frac{e_i e_j}{r_{ij}} ,$$

where $r_{\epsilon i}$ is the distance from ϵ to e_i, and where, in the second sum, every pair e_i and e_j of the n charges is taken once. The portion Ψ_i of the total energy may be called the internal electrostatic energy of the configuration formed by the charges e_i. This internal electrostatic energy, however, is, in general, dependent upon the presence and location of the charge ϵ, for the positions of the charges e_i are, in general, dependent upon the location of the charge ϵ. The portion Ψ_ϵ of the total energy may be called the electrostatic energy of the charge ϵ when in the presence of the charges e_i. The advantage of separating the total mutual electrostatic energy into these terms is one of mathematical convenience in reckoning the force $\boldsymbol{F}_\epsilon$ on the charge ϵ. For the portion Ψ_ϵ contains all the terms of Ψ which depend upon x_ϵ, y_ϵ, z_ϵ, the co-ordiordinates of the charge ϵ. Therefore

$$\boldsymbol{F}_\epsilon = -\nabla_\epsilon \Psi = -\nabla_\epsilon \Psi_\epsilon ,$$

where, as above,

$$\Psi_\epsilon = \frac{\epsilon}{4\pi} \Sigma \frac{e_i}{r_{\epsilon i}} .$$

The coefficient of ϵ in this expression is called the electrostatic potential Φ at the position ϵ due to the charges e_i. Thus

$$\Phi \equiv \frac{1}{4\pi} \Sigma \frac{e_i}{r_i} , \tag{7}$$

where, for simplicity of notation, the distance to e_i from the point at which the potential is being calculated is now written r_i. Accordingly,

$$\boldsymbol{F}_\epsilon = -\epsilon \nabla \Phi .$$

The coefficient of ϵ in this expression—i.e., the force per unit charge acting on ϵ—is also given a special name. It is called the electrostatic intensity $\boldsymbol{E}$ due to the charges e_i. That is,

$$\boldsymbol{E} \equiv -\nabla \Phi , \tag{8}$$

and

$$\boldsymbol{F}_\epsilon = \epsilon \boldsymbol{E} .$$

The electrostatic potential Φ and the electrostatic intensity $\boldsymbol{E}$, considered from a purely analytical point of view as the scalar and vector point functions defined by the foregoing equations, can obviously be

calculated at any point in space. It is clear, however, from their definitions as coefficients of ϵ that these quantities have direct physical significance only at points at which charges are located.

PROBLEMS FOR PART I, CHAPTER I

1. Given
$$u = xyz^2 ,$$
find du/ds, if s measures distance in a direction which makes equal angles with the positive co-ordinate axes.

2. Referring to Problem 1, what is the numerical value of du/ds at a point P of co-ordinates (1,2,1)? What is the approximate difference in the values of u at P and at a second point P' which is 0.01 units of distance from P in the direction s?

3. Given:
$$r^2 = x^2 + y^2 + z^2 ,$$
what is $\partial r/\partial x$ and $\partial(1/r)/\partial x$?

4. A vector $\boldsymbol{A}$ is 5 units long and has direction cosines $1/2$, $-1/2$, $1/2\sqrt{2}$. What are its components? Draw the vector.

5. Referring to Problems 1 and 4, what is the rate of change of u in the direction of $\boldsymbol{A}$?

6. A vector $\boldsymbol{B}$ has a length of 5 units and makes equal angles with the negative x-axis, the positive y-axis, and the negative z-axis. What are its direction cosines and its components?

7. Referring to Problems 5 and 6, what is the component of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$?

8. If $\boldsymbol{A}$ represents a force and $\boldsymbol{B}$ a displacement, what is the work done?

9. Given:
$$p = xy^2z;$$
what is ∇p?

10. Given:
$$r^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 ,$$
compute and discuss $\nabla \frac{1}{r}$ and $\nabla_1 \frac{1}{r}$.

11. Given a surface

$$z = x^2 + 3xy ,$$

z being pointed upward; in what direction will a particle slide if placed on this surface at the point (1,2,7) ?

12. Assuming the surface of Problem 11 to be smooth, what are the x- and y-components of the horizontal force which would just prevent the particle from moving?

13. Show that the operator

$$\nabla \equiv \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z}$$

is distributive.

14. Interpret the equation

$$q = -k \nabla \theta ,$$

where k is thermal conductivity and θ is temperature.

15. If u is a function of $x,y.z$, show that ∇u is normal to the surfaces $u = \text{constant}$.

16. Show that the electrostatic intensity at a point $P(x,y,z)$ due to charges e_i located at points (x', y_i', z_i') is

$$\boldsymbol{E} = \frac{1}{4\pi} \Sigma e_i \nabla' \frac{1}{r_i} .$$

Part II. Complexes of Charge

Introduction

As was seen above, the total electrostatic force between two small charged bodies is, in some cases, to be described by Coulomb's law, in which the spatial distribution of the bodies is neglected. The total charge of each body and the distance between the bodies determine the force. A body may, however, be in electrostatic interaction even if it have no total charge. For example, a small uncharged piece of glass is acted upon by a small charged body with a force which does not obey Coulomb's law as applied to the bodies as a whole. The total effect, in this case is due to the spatial distribution of the positive and negative elementary charges.

In any case the total electrostatic force between groups of charges is given exactly by summing the action of each elementary charge on every other elementary charge. As a step in carrying out this summation process, it is found convenient to calculate the potential due to each group of charges.

In the following pages the potential due to the charges forming a small group or "complex of charges" is represented in terms of a convergent series in inverse powers of the distance to the complex. The first term of this series depends only on the total charge of the complex. This corresponds to the Coulomb law applied to the complex as a whole. But even in very small complexes it is not sufficient to consider this term alone, for the effect of spatial extension may yield a term just as important as the Coulomb term. In fact, there are important cases in which the Coulomb term is zero, so that the second term of the series, which depends upon spatial extension of the complex of charges, is then the leading term. This second term depends upon the quantity $\Sigma e_i l_i$, where l_i is the vector locating e_i with respect to some point within the complex. This vector quantity is called the "polarization" of the complex, and can clearly have a non-vanishing value when the total charge of the complex (and hence the first term of the series) is zero.

It is clear that in determining the action between two complexes of charges, the forces on the charges of one complex will affect their positions, and thus their actions on the charges of the second complex. This, in turn, affects the positions of this second set of charges, and so on. To resolve this problem of interaction, it is necessary to have a relationship between the polarization of a complex and the intensity due to the

distant charges. It is assumed that when a neutral and unpolarized complex is subjected to the electrostatic influence of distant charges, the polarization produced in the complex by the differential shift of positive and negative charge is proportional to the intensity due to the distant charges. Such would be the case, for example, if the displacements of all charges be small, and if they be opposed by forces which, like elastic restraining forces, are themselves proportional to displacement.

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§ 5. *Force between a Single Charge and a Complex.*—A group of charges will be referred to as a complex when these charges are contained within a volume whose dimensions are small compared to the distance to the charge or charges whose interaction with the complex is under consideration. The force on a charge ϵ due to any set of charges e_i is given, in terms of the potential due to the charges, by the equation

$$F_\epsilon = -\epsilon \nabla \Phi ,$$

where

$$\Phi = \frac{1}{4\pi} \Sigma \frac{e_i}{r_i} .$$

If the set of charges e_i form a complex, an approximate expression for this potential may be obtained by expanding the quantities $1/r_i$ in converging power series in l_i/r, where l_i are the distances of e_i from a point O within the complex, and where r is the distance from O to the location of ϵ. Indeed (see Fig. 4),

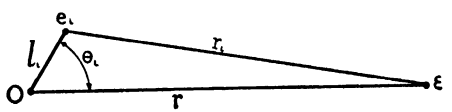

FIG. 4

$$r_i^{-1} = \{r^2 + l_i^2 - 2rl_i \cos\theta_i\}^{-\frac{1}{2}} = \frac{1}{r}\left\{1 - \frac{2l_i\cos\theta_i}{r} + \left(\frac{l_i}{r}\right)^2\right\}^{-\frac{1}{2}},$$

$$= \frac{1}{r} + \frac{l_i\cos\theta_i}{r^2} + \frac{l_i^2}{2r^3}(3\cos^2\theta_i - 1) + \ldots ,$$

$$= \frac{1}{r}\sum_0^\infty \left(\frac{l_i}{r}\right)^n P_n(\cos\theta_i) ,$$

where P_n is the so-called Legendre polynomial of degree n, i.e.,*

$$P_n(x) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{\lfloor n}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\ldots(2n-1)(2n-3)}x^{n-4} - \ldots\right.$$

* See, e g, Byerly's *Fourier Series and Spherical Harmonics*, p. 10.

Thus the potential at ϵ due to the complex of charges e_i may be written

$$(9) \quad 4\pi\Phi = \frac{1}{r}\Sigma e_i + \frac{1}{r^2}\Sigma e_i l_i \cos\theta_i + \frac{1}{r^3}\Sigma e_i l_i^2 \tfrac{3}{2}(\cos^2\theta_i - \tfrac{1}{3}) + \ldots\ldots$$

The first term of this expansion of Φ is the value this function would have, were a total charge Σe_i concentrated at O. The variation of this term, when the co-ordinates x_ϵ, y_ϵ, z_ϵ of ϵ are allowed to vary, corresponds to a force depending upon the charge ϵ, the concentrated charge Σe_i, and the inverse square of the distance. It is obvious that if all powers of l_i/r, including the first, be dropped, so that the spatial extension of the complex be entirely disregarded, the expression for the potential would reduce to this Coulomb-law term which involves simply the charges ϵ and Σe_i, and the mutual distance r.

In the second term of the expansion, the coefficient of $1/r^2$ appears as the sum of the projections on r of vectors l_i, each multiplied by the charge e_i. The coefficient is, then, the same as the projection on r of the vector sum

$$(10) \qquad \boldsymbol{p} \equiv \Sigma e_i \boldsymbol{l}_i \,,$$

which will be called the "polarization of the complex" relative to the point O. The coefficient of $1/r^2$ in the second term may thus be written*

$$p \cos\theta \,,$$

where θ is the angle between the vector $\boldsymbol{p}$ and the direction from O to the location of the charge ϵ. This second "polarization" term in the expansion for the potential may appear even if the total charge Σe_i of the complex be zero. The term is entirely characterized by the value of the vector sum $\boldsymbol{p}$, i.e., by the polarization of the complex; and it is evident that different configurations of the charges e_i, simple or complicated, having the same polarization, are entirely equivalent as far as this term is concerned. In a discussion which involves this term only, it is often convenient to adopt a simplified picture of the complex. Consider a configuration consisting of two equal and opposite charges $+e$ and $-e$, the position of $+e$ with respect to $-e$ being given by a vector $\boldsymbol{l}$. Then if

$$e\boldsymbol{l} = \boldsymbol{p} \,,$$

it is evident that the term of lowest order due to this pair of charges is identical with the polarization term due to the complex e_i.

* See also Part II, Problem 5, of this chapter.

If one is willing to abandon a physically possible model, this pair of charges can be idealized into a so-called "doublet," by permitting l to approach zero, while the magnitude of the charges increases indefinitely in such a way that the product el is always equal to $\boldsymbol{p}$. It is evident that the total effect of such a doublet is equal to the polarization term due to the complex; for if l approaches zero while el remains constant, all the other terms in the expansion of Φ for the doublet approach zero, since they contain higher powers of l. The magnitude p of $\boldsymbol{p}$ is called the "moment" of the polarization, or of the equivalent doublet, and the direction of $\boldsymbol{p}$ is called the "axis" of the polarization, or of the equivalent doublet.

As regards the first two terms in the foregoing expansion for the potential, it is clear that any complex is equivalent to a single charge of magnitude Σe_i and a doublet of polarization $\boldsymbol{p}$. Both the Coulomb term and the polarization term depend upon the choice of location of O within the complex. The nature of this dependence is indicated in number 11 of the problems listed next below.

The foregoing remarks have been restricted to the first two terms of the series for Φ. It is not possible, on purely mathematical grounds, to decide how many terms of this expansion are required to obtain a sufficiently accurate value for the potential. A physical argument will be given, in § 9, to show that in ordinary problems only the first two terms are to be retained. Were all charges of the same sign, the second term would always be much smaller than the first, since the ratio of the contribution a given charge makes to the second term and to the first term is the small quantity $l_i \cos \theta_i / r$. Thus, in the case of charges of one sign, the rapidity of convergence of the series, and, accordingly, the approximation obtained by breaking off at any given point, is determined in a simple way by the smallness of the ratio l_i/r. This is why, in the theory of gravitational potential, it is customary to retain only the first term. The theory of electrostatic potential is, then, fundamentally more complicated, since Σe_i may vanish so as to make the first term zero, while, at the same time, the second term is not zero. In the same way it is conceivable that the first two terms vanish, and the third be the leading term. The number of terms to be retained would thus have to be determined by direct examination in each special case were it not for the physical arguments, referred to above and to be given later, which eliminates the necessity of considering terms beyond the second.

The nature of the polarization term in the expression for the potential may be better understood by considering the case of a complex whose

total charge is zero, but which is polarized; and for which all the terms beyond the second in the expansion of Φ are zero. The force $\boldsymbol{F}_\epsilon$ on a charge ϵ due to this complex may then be written

$$\boldsymbol{F}_\epsilon = -\epsilon\, \nabla_\epsilon \Phi = -\frac{\epsilon p}{4\pi} \nabla_\epsilon \frac{\cos\theta}{r^2}$$

and the components of this force in the directions of increasing r and θ respectively are*

$$(11) \qquad (F_\epsilon)_r = -\frac{\epsilon p}{4\pi}\left(\nabla_\epsilon \frac{\cos\theta}{r^2}\right)_r = -\frac{\epsilon p}{4\pi}\frac{\partial}{\partial r}\frac{\cos\theta}{r^2} = \frac{2\epsilon p}{4\pi}\frac{\cos\theta}{r^3},$$

$$(12) \qquad (F_\epsilon)_\theta = -\frac{\epsilon p}{4\pi}\left(\nabla_\epsilon \frac{\cos\theta}{r^2}\right)_\theta = -\frac{\epsilon p}{4\pi}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\cos\theta}{r^2} = \frac{\epsilon p}{4\pi}\frac{\sin\theta}{r^3}.$$

The force on the complex as a whole is, of course, equal and opposite to $\boldsymbol{F}_\epsilon$. The angle ψ between the direction of r and the direction of the force $\boldsymbol{F}_\epsilon$ is given by

$$\tan\psi = \frac{(F_\epsilon)_\theta}{(F_\epsilon)_r} = \frac{\sin\theta}{2\cos\theta} = \tfrac{1}{2}\tan\theta.$$

In Figure 6 curves are drawn having at every point the direction that $\boldsymbol{F}_\epsilon$ would have if, keeping the configuration of the complex itself constant, the charge ϵ were located at that point.

§ 6. *Force and Torque between Two Complexes.*—The exact expressions for the effect of one group of charges e_j on a second group of

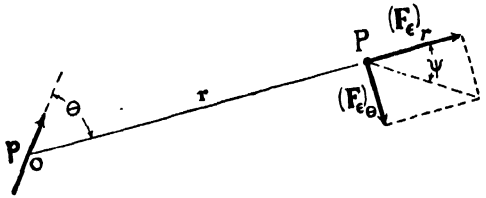

FIG. 5.—The components of force on a charge ϵ due to a complex of polarization p

charges e_i may also be given simpler approximate forms when each of the groups is a complex, i.e., when the distances between the charges of one group are small compared to the least distance to any charge of the other group. The force due to the first complex on a charge e_i of the second complex is given by

$$-e_i \nabla\Phi = e_i \boldsymbol{E}_i,$$

* See Part II, Problem 4 of this chapter.

where Φ is the potential due to the charges e_j, and where $\boldsymbol{E}_i$ is the value at the location of e_i of the electrostatic intensity $\boldsymbol{E}$ due to the charges e_j. The force $\boldsymbol{F}$ on the second complex due to the first complex is thus given by

$$\boldsymbol{F} = \Sigma e_i \boldsymbol{E}_i .$$

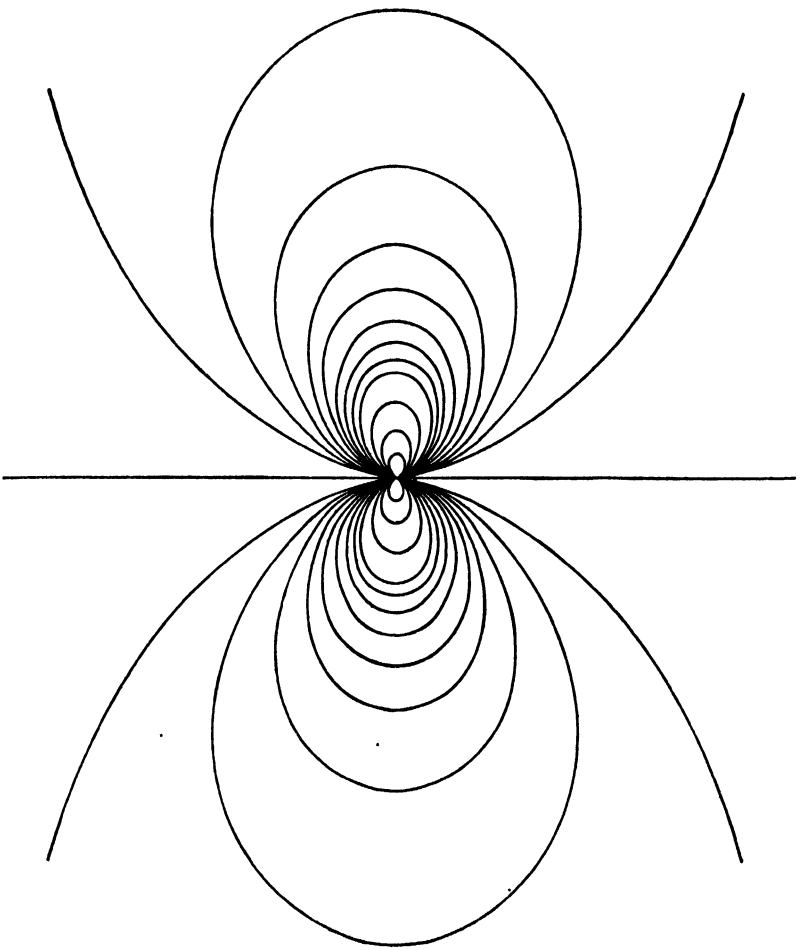

FIG. 6. (Taken by permission from Jeans, *Electricity and Magnetism*)

Let O be some point within the second complex, and let $\boldsymbol{l}_i$ be vectors locating e_i with respect to O. Then this sum may be simplified by expanding $\boldsymbol{E}$ in a Taylor's series about the point O, thus obtaining the

special values $\boldsymbol{E}_i$ of the function $\boldsymbol{E}$, in terms of the value of $\boldsymbol{E}$ and its successive derivatives at O, and in terms of the distances l_i from O to e_i. In fact,

$$\Sigma e_i \boldsymbol{E}_i = (\boldsymbol{E})_0 \Sigma e_i + \Sigma e_i l_i \left(\frac{d\boldsymbol{E}}{dl_i}\right)_0 + \tfrac{1}{2}\Sigma e_i l_i^2 \left(\frac{d^2\boldsymbol{E}}{dl_i^2}\right)_0 + \ldots,$$

where the subscript 0 indicates that the value of the quantity is to be taken at O.

It follows from the definition of $\boldsymbol{E}$ that its magnitude cannot exceed, at any point P, the value $\Sigma|e_j|/4\pi r_0^2$, where r_0 is the least distance from P to e_j. It is, in fact, evident that the value of $\boldsymbol{E}$ at P would be increased (or unaltered) by making all the charges positive and moving them all to the position of the one nearest P. Similar upper bounds may be obtained for the successive directional derivatives of $\boldsymbol{E}$. In fact, for a single charge,

$$\left|\frac{d\boldsymbol{E}}{dl}\right| \leq \left|\frac{d\boldsymbol{E}}{dr}\right| = \frac{2e}{4\pi r_0^3}.$$

Thus the rate of change, in any direction, of the intensity due to a group of charges is increased (or unaltered) by making all the charges positive and moving them to the position of the nearest. That is,

$$\left|\frac{d\boldsymbol{E}}{dl}\right| \leq \frac{2\Sigma|e_j|}{4\pi r_0^3}.$$

In the same way $|d^2\boldsymbol{E}/dl^2|$ cannot exceed $6\Sigma|e_j|/4\pi r_0^4$, etc. Thus the terms of

$$e_i \boldsymbol{E}_i = e_i(\boldsymbol{E})_0 + e_i l_i \left(\frac{d\boldsymbol{E}}{dl_i}\right)_0 + \frac{e_i l_i^2}{\lfloor 2} \left(\frac{d^2\boldsymbol{E}}{dl_i^2}\right)_0 + \ldots.$$

are equal to or less than the corresponding terms of the series

$$\frac{e_i \Sigma|e_j|}{4\pi r_0^2} + \frac{\lfloor 2\, e_i l_i \Sigma|e_j|}{4\pi r_0^3} + \frac{\lfloor 3\, e_i l_i^2 \Sigma|e_j|}{\lfloor 2\, 4\pi r_0^4} + \ldots,$$

which is convergent, since the limiting ratio of successive terms is l_i/r_0. Hence the series for $e_i\boldsymbol{E}_i$ is convergent, and the series $\Sigma e_i \boldsymbol{E}_i$, which contains a finite number of terms $e_i\boldsymbol{E}_i$, is also convergent. The question, however, as to how many terms one must use to obtain a good approximation must be answered, as in the case of the series expansion for Φ, by a closer examination. It does not follow, for example, from the upper

bounds just found that the second term is necessarily smaller than the first, it is merely stated that the first term is smaller than a certain quantity A_1 and that the second term is smaller than a certain quantity $A_2 < A_1$. The argument, however, to be given in § 9, which justifies discarding terms beyond the second in the expression for Φ, also applies to this equation.

If, then, only the first two terms of this series be retained, the force on the second complex is given by

$$\boldsymbol{F} = \Sigma e_\iota \boldsymbol{E}_i = (\boldsymbol{E})_0 \Sigma e_\iota + \Sigma e_\iota l_\iota \left(\frac{d\boldsymbol{E}}{dl_\iota}\right)_0 .$$

In the second term of this expression, each charge e_ι is multiplied by its distance l_ι from O, and by the rate of change at the point O and in the direction $\boldsymbol{l}_i$ of the vector $\boldsymbol{E}$. The rate of change of a vector in any direction is itself a vector whose components are the rates of change in this direction of the components of the vector. The rate of change in the direction $\boldsymbol{l}_i$ of a component, say E_x, of $\boldsymbol{E}$ is given by*

$$\frac{1}{l_i} (\boldsymbol{l}_i, \nabla E_x) .$$

The vector rate of change of $\boldsymbol{E}$ in the direction $\boldsymbol{l}_i$ is thus given by

$$\frac{1}{l_i} \{(\boldsymbol{l}_i, \nabla E_x)\boldsymbol{i} + (\boldsymbol{l}_i, \nabla E_y)\boldsymbol{j} + (\boldsymbol{l}_i, \nabla E_z)\boldsymbol{k}\} \equiv \frac{1}{l_i} (\boldsymbol{l}_i, \nabla)\boldsymbol{E} ,$$

where the right-hand side of this equation is an abbreviation for the left-hand side.† The force $\boldsymbol{F}$ may thus be written, making use of the notation just introduced, and dropping for simplicity the zero subscript on $\boldsymbol{E}$ and its derivatives,

$$(13) \qquad \begin{cases} \boldsymbol{F} = \boldsymbol{E}\, \Sigma e_\iota + \Sigma e_\iota (\boldsymbol{l}_\iota, \nabla)\boldsymbol{E} , \\ \quad = \boldsymbol{E}\, \Sigma e_\iota + \Sigma (e_\iota \boldsymbol{l}_i, \nabla)\boldsymbol{E} , \\ \quad = \boldsymbol{E}\, \Sigma e_\iota + (\boldsymbol{p}, \nabla)\boldsymbol{E} , \end{cases}$$

where

$$\boldsymbol{p} = \Sigma e_\iota \boldsymbol{l}_i$$

is the polarization of the second complex with respect to the point O.

* See the Mathematical Appendix for a discussion of scalar and vector products

† See also Part II, Problem 18, of this chapter.

The vector torque $\boldsymbol{T}$ on the second complex is given by

$$\boldsymbol{T}=\Sigma[\boldsymbol{l}_i, e_i\boldsymbol{E}_i]=\Sigma[e_i\boldsymbol{l}_i, \boldsymbol{E}_i] .$$

If $\boldsymbol{E}$ be again expanded in a Taylor's series, this becomes

$$\boldsymbol{T}=\Sigma[e_i\boldsymbol{l}_i(\boldsymbol{E})_0]+ \ldots . =[\Sigma e_i\boldsymbol{l}_i, (\boldsymbol{E})_0]=[\boldsymbol{p}, \boldsymbol{E}]+ \ldots . \quad (14)$$

In equation (14) as in (13), the zero subscript has been omitted. In both these equations, then, it is understood that the value of $\boldsymbol{E}$ and of its derivatives are to be taken at that point within the complex relative to which the polarization is defined.

§ 7. *The Polarization of a Complex.*—It is customary, in electrostatics, to speak of the "location" of the elementary charges under consideration. This idea of the location of a charge is a complicated one and involves the use of time and space averages. No one believes that the individual charges forming a body which is in electrostatic equilibrium are, in reality, stationary. The experimental evidence for Coulomb's law teaches that the (time-average) force on a small charged body A_1 due to a small charged body A_2 may, for certain separation distances, be written as a constant times the inverse square of a distance r, one end of which is within A_1 and one end of which is within A_2. As has been pointed out above, it is possible to assume, for a law of action between elementary charges, such an expression as will yield, on resynthesis, this experimentally checked law for the time-average effects between large numbers of charges. A given charge e_1 may, in fact, move in a very complicated way always remaining, however, within a small volume ΔA_1 of body A_1, while a second charge e_2 is moving within a small volume ΔA_2 of A_2. It is consistent with Coulomb's law to assign, for the action of e_2 on e_1, a value proportional to the inverse square of a distance r, one end of which is within ΔA_1 and the other end within ΔA_2. The two ends of this line would then be called, for electrostatic purposes, the "locations" or better the "effective positions" of charges e_1 and e_2, respectively. It is clear that to a certain approximation involving the ratio of the dimensions of ΔA_1 or ΔA_2 and r, it is immaterial where, within ΔA_1 and ΔA_2, the charge is said to be located. It is also clear that a more exact determination of this "location" would necessitate an averaging process for which one would require a knowledge of the actual force between the actually moving charges, and a knowledge of their paths. The statistical nature of the electrostatic problem for ponderable bodies fortunately permits one to disregard such difficult details.

When the charges forming a complex are subjected to the electrostatic effect of other charges, forces arise which change the effective positions of all charges unless restraints of some nature prevent these changes. If the charges forming the complex are, at the same time, those which constitute an atom, the charges are constrained to remain within a very small volume, which may be thought of as the "volume" of the atom. If no exterior charges are present, the electrons and nucleus making up the atom would form a certain configuration which from the point of view of electrostatics would be described, as is indicated above, by giving a fixed effective position of each charge. The presence of exterior charges causes an excess force to act on each of the charges of the atom. The actual effect of this excess force is undoubtedly highly complicated. If the electrons are in orbital motion about the nucleus, these orbits will be warped and shifted. The gross effect, however, of all such changes is a simple shift of the effective time-mean position of each charge, and of this alone does electrostatics take account. The total disturbance to the atom is small when, as is actually the case, the excess forces are small in comparison to the normal forces acting on the charges making up the atom. Thus the electrostatic effect of exterior charges is to produce a small differential shift of charge within the atom, the electrons and nucleus being oppositely affected. If the exterior charges are sufficiently distant (or if they are distributed with the necessary symmetry), the force per unit charge on each of the elementary charges e_i of the atom due to these exterior charges will be the same. That is, the force on each e_i due to the exterior charges e_j will be given by

$$F_i = e_i E , \tag{15}$$

where E, the electrostatic intensity due to the exterior charges e_j, is the same for all the charges e_i. It is assumed that the electrons and nucleus are bound by intra-molecular forces in such a way that an excess force produces a displacement proportional to its magnitude and in its direction. If all the electronic displacements are small with respect to the distances between the electrons, the reactions of these electrons on each other will not be sensibly changed, and the force producing the displacements will be solely the force due to the exterior charges. Thus the effective displacements, and, accordingly, the resulting or induced polarization, will be proportional to the force per unit charge due to the exterior charges, i.e.,

$$p = kE . \tag{16}$$

Any atom or complex which polarizes according to this equation will be called an "isotropic" atom or complex, and the assumption is then that there are actual atoms or complexes which are isotropic. It is evident that a ponderable body may be statistically isotropic without being formed of isotropic atoms.

Equation (16) gives the induced polarization due to an impressed intensity. A molecule may have, when not under the influence of other external charges, an inherent polarization $\boldsymbol{p}_0$. For such a molecule or complex, the total polarization, in the presence of a field, would be the sum of the inherent polarization and the polarization induced by the field.

As an example of the polarization caused by a given exterior charge, consider an isotropic uncharged atom, and a charge ϵ whose distance from the atom, r, is large compared to the dimensions of the atom. Under these circumstances the charges forming the atom constitute a complex. The force per unit charge on the charges of the complex is sensibly the same for the different charges, and

$$p = kE = k\frac{\epsilon}{4\pi r^2}, \tag{17}$$

the direction of $\boldsymbol{p}$ being the direction from the charge ϵ to the atom.

PROBLEMS FOR PART II, CHAPTER I

1. Referring to § 5, expand r_i^{-1}, viewed as a function of l_i, in a Taylor's series in the neighborhood of $l_i = 0$, and thus establish the equation

$$r_i^{-1} = \frac{1}{r} + \frac{l_i \cos\theta_i}{r^2} + \frac{l_i^2(3\cos^2\theta_i - 1)}{2r^3} + \cdots .$$

The Taylor's series derivation of this equation furnishes more than a mere check of the result found in § 5 by trigonometry and algebra. See the first footnote to § 9.

2. Show that $(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{C}) = (\boldsymbol{A},\boldsymbol{B}) + (\boldsymbol{A},\boldsymbol{C})$.

3. Interpret geometrically the equation

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2(\boldsymbol{A},\boldsymbol{B}) + \boldsymbol{B}^2$$

4. Given a function $f(r,\theta)$ of the polar variables r and θ; explain why the component of ∇f in the direction of increasing θ is given by

$$\frac{1}{r}\frac{\partial f}{\partial \theta}.$$

5. Show that the potential at the point $P(x,y,z)$ due to a doublet of polarization $\boldsymbol{p}$ located at the point $O(x',y',z')$ may be written

$$\left(\boldsymbol{p},\nabla'\frac{1}{r}\right).$$

6. If $\boldsymbol{F}$ is a force applied at a point P and $\boldsymbol{r}$ is the vector to P from a point O, interpret physically the expression $[\boldsymbol{r},\boldsymbol{F}]$.

7. The vector $\boldsymbol{S}$ makes equal angles with the positive co-ordinate axes What is the rate of change at the point (1,2,1) in the direction of $\boldsymbol{S}$ of the vector

$$\boldsymbol{i}xy+\boldsymbol{j}zy^2+\boldsymbol{k}zx^2\,?$$

8. Given:

$$\boldsymbol{A}=\boldsymbol{i}2-\boldsymbol{j}3+\boldsymbol{k}\,,$$
$$\boldsymbol{B}=\boldsymbol{i}-\boldsymbol{j}2-3\boldsymbol{k}\,,$$
$$\boldsymbol{C}=\boldsymbol{i}3+\boldsymbol{j}-2\boldsymbol{k}\,;$$

what is $(\boldsymbol{AB})$, $(\boldsymbol{AC})$, $(\boldsymbol{BC})$, $(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{C})$, $(\boldsymbol{C},\boldsymbol{A}-\boldsymbol{B})$, $[\boldsymbol{AB}]$, $[\boldsymbol{AC}]$, $[\boldsymbol{CB}]$, $[\boldsymbol{BA}]$, $(\boldsymbol{A},[\boldsymbol{BC}])$?

9. Given:

$$\boldsymbol{A}=\boldsymbol{i}xz^2y+\boldsymbol{j}y^2x^2z+\boldsymbol{k}z^2xy\,,$$
$$\boldsymbol{B}=\boldsymbol{i}x-\boldsymbol{j}z+\boldsymbol{k}y\,;$$

what is the rate of change, at the point $(1,1,-1)$, of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$?

10. Given a function $F(r,\theta,\varphi)$ of the polar variables r, θ, and φ; what are the components, in the directions of increasing r, θ, and φ, of ∇F?

11. *a*) If $\boldsymbol{p}_0$ be the polarization of a complex relative to a reference point O, and $\boldsymbol{d}$ be the vector from O to an alternative reference point Q, show that

$$\boldsymbol{p}_Q=\boldsymbol{p}_0-\boldsymbol{d}\epsilon\,,$$

where ϵ is the total charge of the complex, and where $\boldsymbol{p}_Q$ is the polarization relative to Q. When is the polarization of a complex independent of the reference point?

b) Show that the first or Coulomb terms in the two expressions for the potential at any point P which correspond to two choices,

O and Q, of a reference point within the complex, differ by an amount whose leading term is $\left(\epsilon \boldsymbol{d}, \nabla' \frac{1}{r}\right)$. How might this term be interpreted?

c) Let, as before, r be the scalar distance from P to O, and let R be the scalar distance from P to Q. Then if O and Q be chosen as a reference point, the first two terms in the expansion for $4\pi\Phi$ are

$$\frac{\epsilon}{r}+\left(\boldsymbol{p}_0, \nabla' \frac{1}{r}\right),$$

and

$$\frac{\epsilon}{R}+\left(\boldsymbol{p}_Q, \nabla' \frac{1}{R}\right),$$

respectively. By (*a*) and (*b*) the latter expression is equal to

$$\frac{\epsilon}{r}+\left(\epsilon \boldsymbol{d}, \nabla' \frac{1}{r}\right)+\left(\boldsymbol{p}_0-\epsilon \boldsymbol{d}, \nabla' \frac{1}{R}\right).$$

Show that $\left(\boldsymbol{p}_0-\epsilon \boldsymbol{d}, \nabla' \frac{1}{R}\right)$ and $\left(\boldsymbol{p}_0-\epsilon \boldsymbol{d}, \nabla' \frac{1}{r}\right)$ differ by terms which are of higher order than "polarization" terms, and hence show that the sum of the first two terms in the expression for the potential is independent of the location of the reference point, in so far as Coulomb and polarization terms are concerned.

12. If an atom is polarized by a charge ϵ at a distance r from it, the magnitude of the polarization is

$$p=kE=\frac{k\epsilon}{4\pi r^2},$$

while the direction is along r. The potential at the position of ϵ due to the polarized atom is

$$\Phi=-\frac{p}{4\pi r^2}$$

Explain why the force on ϵ cannot be found by substituting the foregoing value for p in the expression for Φ, and then computing $-\epsilon \nabla\Phi$.

13. Two neutral but polarizable atoms are located a distance a apart. A distance r (large compared to a) from these two atoms is a charge ϵ. What is the magnitude and direction of the polarization of the two atoms if: (a) the charge ϵ is located on the line joining the two atoms; (b) the charge ϵ is located on the perpendicular to the line which joins the atoms?

14. A positive charge $+e$ is located and fixed at the point $x=a$, $y=0$. Two equal negative charges $-e$ are placed on the y-axis, on which they are free to slide. What is the equilibrium configuration?

15. A charge $+e$ is located a distance $3a$ from the center of a circle of radius a. Two equal negative charges $-e$ are free to move on the circumference of the circle. Show that equilibrium exists when the angle between the radius which points toward $+e$ and the radius to one of the negative charges is approximately 63°50′.

16. Three isotropic neutral atoms are located at the vertices of an equilateral triangle. They are polarized by their own interaction and by an electrostatic intensity of magnitude E whose direction is normal to one of the sides of the triangle. What is the polarization of each atom?

17. A charge ϵ is brought from infinity to a position P, while a neutral molecule is fixed at a second point Q. Suppose, first, that the molecule has a fixed inherent polarization and no induced polarization; and, second, that it has no inherent polarization, but the induced polarization, as given by equation (17), due to the presence of the charge ϵ. Show that if the charge ϵ be brought to such a position that the actual final polarization of the molecule is the same in the two cases, the work done in the first is twice the work done in the second.

18. Show that by interpreting ∇ as a vector operator the symbol $(\boldsymbol{A}\nabla)\boldsymbol{B}$ has precisely the meaning assigned to it in § 3, D, of the Appendix.

19. The preceding problem suggests the possibility of viewing $(\boldsymbol{A}\nabla)\boldsymbol{B}$ as a scalar $(\boldsymbol{A}\,\nabla)$ times the vector $\boldsymbol{B}$. This cannot, in general, be done since a scalar times a vector has the direction of the vector, while $(\boldsymbol{A}\,\nabla)\boldsymbol{B}$ does not have the direction of $\boldsymbol{B}$. Nevertheless, show that

$$[(\boldsymbol{A},\nabla)\boldsymbol{B}]_s=(\boldsymbol{A},\nabla)\boldsymbol{B}_s\,,$$

and state this result in terms of rates of change.

20. Show that

$$(\boldsymbol{A}_1, \nabla)\boldsymbol{B}+(\boldsymbol{A}_2, \nabla)\boldsymbol{B}=(\boldsymbol{A}_1+\boldsymbol{A}_2, \nabla)\boldsymbol{B}\,,$$

and explain how this fact is used in obtaining equation (13).

21. Given the vector $\boldsymbol{i} \sin x+\boldsymbol{j} \cos x$; show that at the origin this vector is of unit length and points in the y-direction, while its rate of change in the direction of a positive x-axis is a unit vector which points in the positive x-direction.

Part III. Ponderable Bodies

Introduction

In Part III of chapter i are developed the analytical methods suitable for the electrostatic treatment of ponderable bodies. Coulomb's law and the principle of superposition still form the basis of the development, the characteristic difference between the purely academic theory for isolated charges and the practical theory for ponderable bodies being that the latter is essentially a statistical theory. Account is no longer taken of the separate charges, but certain averaged-over characteristic measures of the charged condition of the body in question are introduced; viz., the volume densities of charge and polarization, and, as limiting cases of these, surface densities of charge and polarization. These densities are continuous functions of position within the body. The possibility of formulating the electrostatic problem for ponderable bodies in terms of such continuous densities and the consequent enormous gain in analytical simplicity is directly due to the fact that it is possible to subdivide a body into volume cells whose dimensions are very small compared to the body itself and to its distances to other bodies, but which, nevertheless, contain a very great number of individual charges. Two important points must be noted in connection with these densities: (1) that they are introduced for a definite purpose, viz., as convenient means of expressing the potential due to a charged body at points not near that body; (2) that they are inescapably connected with a definite subdivision of the body. The final sections of Part III are closely connected with the two points just emphasized, and they can be more profitably discussed, after the sections have been studied, in the concluding remarks at the end of the chapter.

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§ 8. *Forces Due to Large Groups of Atoms. The Statistical Nature of the Problem.*—In the preceding sections, methods have been given by means of which one can obtain the forces on the elementary charges making up a configuration. Particular study has been given to the case where certain of the charges, say those which make up an atom, form a complex whose greatest dimension is small compared to the distance of this complex to any other charge. Such a theory is sufficient if one is content to

consider only problems involving small numbers of charges in known positions, but such a theory is not at all able to solve a physical problem dealing with ponderable bodies. If the bodies involved in some physical problem were small compared to the distances between them, then they would each act as a simple charge, if charged, or as a doublet, if polarized. But bodies must be treated which are not small compared to their separation distances. This first distinction between the electrostatic problem for elementary charges or single atoms, and the electrostatic problem for ponderable bodies, thus leads at once to the question of distribution of charge. For, if the bodies involved are large enough so that a single point does not effectively locate each body, then evidently one cannot obtain the forces and torques acting unless the distribution of charge on each body is known.

This question of the distribution of charge on ponderable bodies is, in reality, an extremely complicated one. Vast numbers of charges, composing one body, react on vast numbers of charges composing other bodies, while at the same time any one charge of a certain body is also acted upon by all the other charges of this same body. The equilibrium configuration of the charges within the bodies is thus a compromise between the interactions of the charges of a body and the forces due to charges on other bodies. The complexity of the problem is partially indicated by a consideration of the number of terms which enter into the expression for the mutual electrostatic energy of several ponderable bodies. The number of terms in the sum

$$\Psi = \frac{1}{4\pi} \Sigma \frac{e_i e_j}{r_{ij}}$$

is roughly indicated by the fact that there are 6.06×10^{23} molecules in a mol of a substance and several elementary charges in each molecule.

The analytical mechanism previously developed is obviously unsuitable for such a problem, and the method of attack is suggested by the remark, just above, concerning the number of elementary charges involved. Questions which deal with an exceedingly large number of individual cases are treated by statistical methods, and one loses sight of the individual case in a study of the average behavior and the average result. Advantage is taken, in this way, of the very complexity of the problem, and the great numbers involved become a distinct help rather than a hindrance. A similar situation is met, for example, in the kinetic theory of gases. There are in each cubic centimeter of gas a vast number of molecules. These molecules collide with one another, and the individual

history of each molecule is immensely complicated. If the gas is contained in some vessel, the various molecules collide with the walls of the vessel. One can conceive of the problem of studying the frequency, speed, and direction with which a particular molecule would strike a given area of the wall, but such a study is obviously out of the question; indeed, it would be useless to carry it out even were it possible to do so. For there are so many molecules present that one may confidently expect that the predictions of probability will be fulfilled, and that the average pressure due to all the individual collisions will be the "expected" result which a statistical study furnishes. That is to say, the data are sufficiently numerous so that the statistics will be regular. If there were only, say, two hundred molecules in a vessel, the problem would actually be exceedingly more difficult, for then the individual description of the motion of each molecule would have to be made. Mechanics describes definitely the phenomenon of the impact of two perfectly elastic spheres (just as electrostatics has a definite description for the force between two elementary charges), but this mechanics of the individual case must be incorporated in a statistical theory dealing with expected averages before progress can be made concerning the behavior of a vast aggregate of molecules.

§ 9. *The Concentration Method for a Single Complex.*—It has been seen above that the force on a charge is given in terms of the potential due to other charges; and it is evident from the remarks just made that progress in the electrostatic problem for ponderable bodies depends upon the possibility of expressing, by a method that will take into proper account the statistical nature of the problem, the potential

$$\Phi = \frac{1}{4\pi} \Sigma \frac{e_i}{r_i}$$

in some form which is suitable for this case where the number of charges involved makes the direct summation process a practical impossibility. As regards the analytical expression of the potential, two alternative procedures are available which correspond, physically, to a hypothetical concentration or spreading of the charges present. The first, or concentration method, is useful when all the charges e_i are located within a volume $\Delta\tau$ whose greatest linear dimension l is small compared to the least distance of any of the charges from P, the point at which the potential is being calculated. The process consists of choosing a fixed point of reference within $\Delta\tau$, and placing there fictitious elements which serve to represent the potential to any desired order of approximation.

This method was used above in discussing the force on a single charge due to a complex, and it was there found that

$$4\pi\Phi = \frac{\Sigma e_i}{r} + \frac{\Sigma e_i l_i \cos\theta_i}{r^2} + \cdots ,$$

$$= \frac{\epsilon}{r} + \frac{p\cos\theta}{r^2} + \cdots ,$$

$$= \frac{\epsilon}{r} + \left(\boldsymbol{p}, \nabla' \frac{1}{r}\right) + \cdots ,$$

where r is the distance between $P(x,y,z)$ and $O(x',y',z')$, the chosen reference point within the complex; where

$$\epsilon = \Sigma e_i \; ; \tag{18}$$

and where θ is the angle between O–P and the polarization vector $\boldsymbol{p}$ given by

$$\boldsymbol{p} \equiv \Sigma e_i \boldsymbol{l}_i \, .$$

The potential of this set of charges e_i is thus approximately equal to the potential due to a single concentrated charge of magnitude ϵ and a doublet of polarization $\boldsymbol{p}$, each located at the point O.

Although the $(n+1)$st term in the series expansion for the potential due to a complex contains the factor $(l_i/r)^n$, which is less than the factor $(l/r)^n$, nevertheless the rapidity of convergence cannot be estimated from the smallness of the ratio l/r; neither is it correct to think of l/r as the ratio of a term to the preceding term. The expression for the potential may be written as

$$4\pi\Phi = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \cdots ,$$

where

$$\begin{aligned} A_1 &= \Sigma e_i \, , \\ A_2 &= \Sigma e_i l_i \cos\theta_i \, , \\ A_3 &= \Sigma e_i l_i^2 \frac{(3\cos^2\theta_i - 1)}{2} \, , \\ &\vdots \end{aligned}$$

and a comparison of the relative magnitudes of the various terms in the expansion must be based upon a consideration of the relative size of the coefficients A_i. There is no reason, mathematically, why an arrange-

ment of charges could not occur which would make A_4, say, so large that the fourth term would be the predominating term of the expansion. There are sensible physical reasons, however, for restricting attention to the first two terms. Suppose, for example, that the set of charges e_i are those composing a molecule. The existence of polarized but neutral molecules evidently makes necessary a consideration of at least the second term. It may easily be seen, however, that just as a polarization depends upon a varying distribution of charge, so the third term of this series is a sort of "differential polarization" which depends upon a varying polarization.* Now, in any actual electrostatic problem the charges e_i are located in positions determined by an interplay of external forces and the interactions of the charges themselves. A uniform external force, since it acts oppositely on negative and positive charges, can evidently give them a relative shift and produce polarization. To produce a "differential polarization" of appreciable magnitude it would be necessary that the positive and negative charges in one portion of $\Delta\tau$ be shifted relatively to each other in a way sensibly different from that which obtains in the other portions of $\Delta\tau$. This, in turn, demands either that the external forces vary over the small extension of $\Delta\tau$ more rapidly than is physically reasonable, or that the binding forces which affect the interactions of the charges e_i have a more rapid non-uniformity than is believed to actually obtain. It must be remembered, moreover, that even if the various quantities $A_1, A_2, A_3, \ldots$, were of the same order of magnitude, the terms of the series would rapidly decrease on account of the increasing power of r in the denominator. In case $\epsilon = 0$ and $\rho = 0$, the third term is, of course, the leading term, and there is no reason, other than that just given, why it should not be considered. Furthermore, since the degree of approximation furnished by a given number of terms is dependent on the distance from the complex to the point at which the electrostatic effect is to be determined, it is evident that in cases where this distance is not large compared to l it is not to be expected that two terms will give an adequate approximation, and the term involving zonal harmonics of higher order must be included. A similar extension should presumably be made in the study of the reaction of an atom to an electric field which varies appreciably over the atom. Variations of this order of magnitude are in fact present in the case of the sharp electric pulses which constitute X-rays. It has been customary, however, to concentrate attention so exclusively on the charge and polari-

* See Part II, Problem 1, of this chapter; also Maxwell, *Electricity and Magnetism*, I (1873), 157–63.

zation terms that the fact is often lost sight of that they are but the two leading terms of a convergent series representation.* It is, in general, impossible to compare the magnitude of these two leading terms

$$\frac{\epsilon}{r} \quad \text{and} \quad \frac{p\cos\theta}{r^2}.$$

If, however, $\boldsymbol{p}=0$ when referred to some point within $\nabla\tau$, then the first term is the predominating term, for then

$$p<l\Sigma e_i,$$

regardless of the choice of O, and the ratio of the second term to the first term is less than l/r. In fact, if the polarization relative to some point O is zero, and $\boldsymbol{l}_0$ is a vector from O' to O, then $l_0 \leq l$, while

$$\boldsymbol{p}_0=\Sigma e_i(\boldsymbol{l}_0+\boldsymbol{l}_i)=\Sigma e_i\boldsymbol{l}_0=\boldsymbol{l}_0\Sigma e_i,$$

where $\boldsymbol{l}_i$ are the vectors locating e_i relative to O.

§ 10. *The Spreading Method for a Single Complex.*—The concentration method is a natural one when dealing with the reaction between an electron and an atom, or between two atoms; for the spatial extension of the groups of charges making up each atom is so small that the series will converge rapidly for reasonable distances r.

Through the "spreading method," an approximate representation of quite different character may be obtained for the potential due to a complex of charges. In the concentration method, a single point of reference is arbitrarily chosen, and the potential approximately represented by a series of fictitious elements placed at that point. In the spreading method, on the other hand, a definite volume $\Delta\tau$ is arbitrarily chosen, and an approximation to the potential of the complex is sought in the form of an integral†

$$\frac{1}{4\pi}\int\frac{\rho(x',y',z')dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

* In recent work, by Debye and others, on the magnetic and dielectric properties of liquids, it has been found necessary to use third-, fourth-, and even fifth-order terms in the expansion of the potential.

† The integration variables in this and the following integrals are x',y',z', the potential being calculated for the point $P(x,y,z)$. In any case where confusion could arise the volume and surface elements of integration will be written with a prime, as $d\tau'$, $d\sigma'$, to remind that the primed variables are the integration variables

carried over the volume $\Delta\tau$, where r is the distance from the point $P(x,y,z)$ to the infinitesimal volume element $d\tau' = dx'dy'dz'$. The conditions that ρ must satisfy in order that this integral furnish, with desired accuracy, an approximation to the potential, are obtained by expanding $1/r$ in the foregoing integrand about a point O within the volume, and comparing the expression so obtained with the expansion, given by equation (9) and taken with respect to the same point O, of the potential due to the complex. Thus

$$\int_{\Delta\tau} \frac{\rho d\tau}{r} = \frac{1}{r_0}\int_{\Delta\tau} \rho d\tau + \frac{1}{r_0^2}\int_{\Delta\tau} \rho l \cos\varphi d\tau + \dots ,$$
$$\equiv \frac{\epsilon}{r_0} + \frac{p\cos\theta}{r_0^2} + \dots ,$$

where ϵ is the total charge Σe_i, where r_0 is the distance from O to P, and where φ is the angle between O–P and the vector l from O to $d\tau$, and where θ is the angle between O-P and $\mathfrak{p}$. The desired approximation will thus be obtained if the function ρ is chosen to satisfy the two conditions

$$\int_{\Delta\tau} \rho d\tau = \epsilon , \tag{19}$$

$$\int_{\Delta\tau} \rho l d\tau = \mathfrak{p} , \tag{20}$$

the second of these two conditions resulting from the fact that the equation

$$\int_{\Delta\tau} \rho l \cos\varphi d\tau = p\cos\theta$$

must be satisfied for every direction of r_0.

These two conditions may be satisfied by a function ρ which varies linearly throughout the given volume $\Delta\tau$. Assume, in fact, the form

$$\rho = \rho_0 + \alpha x' + \beta y' + \gamma z' ,$$

where the origin for x', y', z', is at O, so that ρ_0 is the value of ρ at that point. Then

$$\rho_0 \Delta\tau + \alpha\int x'd\tau + \beta\int y'd\tau + \gamma\int z'd\tau = \epsilon ,$$

and

$$\begin{cases} \rho_0 \int x' d\tau + \int x'(\alpha x' + \beta y' + \gamma z') d\tau = p_{x'} , \\ \rho_0 \int y' d\tau + \int y'(\alpha x' + \beta y' + \gamma z') d\tau = p_{y'} , \\ \rho_0 \int z' d\tau + \int z'(\alpha x' + \beta y' + \gamma z') d\tau = p_{z'} . \end{cases}$$

These equations reduce to simpler form when the point O is chosen as the center of volume of $\Delta\tau$. The conditions (19) and (20) then become independent conditions on the value of ρ at O and on the part of ρ which varies linearly and vanishes at O, viz.,

$$\rho_0 \, \Delta\tau = \epsilon ,$$

$$\int \boldsymbol{l}(\alpha x' + \beta y' + \gamma z') d\tau = \boldsymbol{p} .$$

The vector equation gives a unique determination of α, β, γ in all cases, the values being most simply expressed when the x,y,z axes are chosen in the direction of the principal axes of inertia of the volume $\Delta\tau$. Then

$$\int y'z' d\tau = 0 , \quad \text{etc.},$$

and

$$\alpha = \frac{p_{x'}}{I_{y'z'}} , \qquad \beta = \frac{p_{y'}}{I_{z'x'}} , \qquad \gamma = \frac{p_{z'}}{I_{x'y'}} ,$$

where

$$I_{y'z'} = \int x'^2 d\tau , \quad \text{etc.}$$

Since, in general, $I_{y'z'} \neq I_{x'y'}$, it is seen, from the foregoing equation, that the direction of the linear variation of ρ is not, in general, that of the polarization vector $\boldsymbol{p}$.

The potential due to a complex of charges which lie in a volume $\Delta\tau$ is thus approximately represented by the integral

$$\frac{1}{4\pi} \int_{\Delta\tau} \frac{\rho d\tau}{r} ,$$

taken over $\Delta\tau$, where ρ varies linearly. When the reference point O is taken at the center of volume of $\Delta\tau$, as will always be done, the integral

(21) $$\frac{1}{4\pi} \int \frac{\rho_0 d\tau}{r}$$

represents, to terms of the order $1/r_0^3$, the part $\epsilon/4\pi r_0$ of the potential; by the integral which arises from the variable part of ρ, namely,

$$\frac{1}{4\pi}\int\frac{\alpha x'+\beta y'+\gamma z'}{r}\,d\tau\,, \tag{22}$$

the part $p\cos\theta/r_0^2$ is also represented to terms of the order $1/r_0^3$. This linear function is the most simple function which will give the required degree of approximation in terms of a single integral. If approximation of this same type but of higher order were required, other than linear variation of ρ would be necessary, there being no limit to the degree of approximation obtainable by allowing ρ to be sufficiently complicated.

On the other hand, a representation giving the same degree of approximation may be obtained in terms of two functions, each even more simple than the linear function ρ. The expression (21), where ρ has a constant value ρ_0, represents, as has just been noted, the first term of (9) to terms of order $1/r_0^3$. The second term of (9), namely, $p\cos\theta/r_0^2$, has heretofore been represented by the term (22) which arises from a uniform variation of ρ. The second term of (9), however, can be represented to terms of order $1/r_0^3$ by the integral

$$\int\frac{P_0\cos\psi}{r^2}\,d\tau\,, \tag{23}$$

where ψ is the angle between the direction from $d\tau$ to P and the constant vector $\boldsymbol{P}_0$. In fact, expanding the integrand of (23),

$$\int\frac{P_0\cos\psi}{r^2}\,d\tau=\frac{P_0\cos\psi_0}{r_0^2}\,\Delta\tau+\frac{1}{r_0^3}(\quad)+\ldots\,,$$

$$\equiv\frac{p\cos\theta}{r_0^2}+\ldots\,,$$

where ψ_0 is the angle between $\boldsymbol{P}_0$ and the direction from O to P. The last equation, since it holds for all locations of the point at which the potential is calculated, demands that the projection of $\Delta\tau\boldsymbol{P}$ on any direction equal the projection of $\boldsymbol{p}$ on this same direction, i.e., it demands that

$$\Delta\tau\boldsymbol{P}=\boldsymbol{p}\,.$$

The potential of the complex of charges which lie in a volume $\Delta\tau$ may therefore be approximately represented by the two integrals

$$\frac{1}{4\pi}\int\frac{\rho_0 d\tau}{r}+\frac{1}{4\pi}\int\frac{P_0\cos\psi}{r^2}\,d\tau\,,$$

where ρ_0 and P_0 have the constant values given by

$$\Delta\tau\rho_0=\epsilon\,,$$
$$\Delta\tau P_0=p\,.$$

It is evident, therefore, that the spreading method may itself proceed in either of two ways. A representation, holding up to nth-order terms, may be obtained by means of a single integral, in which case conditions on the value at O of ρ and its $(n-1)$ successive derivatives arise from comparison of the expansion of this integral with (9). Or a representation holding up to nth-order terms may be obtained by means of n separate integrals. In this latter case functions appear in each integral which may have a constant value over $\Delta\tau$. These two procedures correspond, physically, to a hypothetical spreading of charge, on the one hand, and a hypothetical spreading of charge, polarization, etc., on the other. If charge alone be spread, it must be spread in a more and more complicated way as higher approximation is demanded; if charge, polarization, etc., be spread, each may be spread uniformly.

§ 11. *The Potential Due to a Body.*—In seeking a representation for the potential due to the charges which form a ponderable body, it would be naturally suggested by the foregoing treatment to consider all the charges as forming a single complex, and to represent the potential due to this complex either by the concentration or the spreading method. However, since electrostatic problems often demand the value of the potential at points whose least distance from some body is not large compared to the dimensions of that body, it is evident that such series representations would, when applied to the body as a whole, converge very slowly if at all, and even in the more favorable event, many terms would be necessary to furnish a good approximation.* This difficulty can be escaped by subdividing the body into volume cells $\Delta\tau_i$, of linear dimensions l small compared to the least distance from the body to the point P at which the electrostatic effect is to be measured, and treating

* The concentration method, with several points of concentration, is, in fact, used in the "method of images," to be studied later.

the charges in each of these volume cells $\Delta\tau_i$ as a separate complex. The potential due to the charges in these cells could then be represented either by the concentration or the spreading method. The concentration method, however, does not effect as great a gain in simplicity as is desired, since the potential of the whole body, if this method be used on the various subcomplexes, consists of a sum of as many terms as there are volume cells $\Delta\tau_i$.

These facts suggest that the spreading method be used for the representation of the potential due to the subcomplexes. If the contribution to the total potential due to the charges in each $\Delta\tau_i$ is to be represented by a single integral

$$\int_{\Delta\tau_i} \frac{\rho_i d\tau}{r},$$

each function ρ_i must, as was seen above, be the sum of a constant part and a linear variation which vanishes at O_i, the center of volume of $\Delta\tau_i$. The potential of the whole body would then be given by

$$\Sigma_i \int_{\Delta\tau_i} \frac{\rho_i d\tau}{r}.$$

This expression again involves the sum of as many terms as there are volume cells $\Delta\tau_i$, and no gain in simplicity results from the use of the spreading method unless this sum of integrals can be written as a single integral. The union of all the functions ρ_i, each defined over its own volume cell $\Delta\tau_i$, gives a function ρ, defined over the whole body. Then

$$\Sigma \int_{\Delta\tau_i} \frac{\rho_i d\tau}{r} = \int \frac{\rho d\tau}{r}, \tag{24}$$

the last integral being extended over the whole body. But the simplicity gained by using ρ instead of ρ_i is apparent rather than actual, for the function ρ is a "patchy" discontinuous function, and the right side of (24) can be evaluated only by evaluating the left side. Indeed, even in the idealized case where the subcomplexes in $\Delta\tau_i$ are identical (each with respect to its own reference point), they do not join to form a continuous function ρ, the situation being as shown in the following schematic drawing.

It is thus apparent that this representation, by integrals involving a single function, of the potential due to the subcomplexes in the volume elements $\Delta\tau_i$ is not a convenient one. The spreading method, however,

can proceed in either of two ways, and it is evident that the situation just considered, namely, that of identical subcomplexes in $\Delta\tau_i$, will be treated very simply if the potential is represented by means of two integrals, rather than one, these two corresponding physically to the uniform

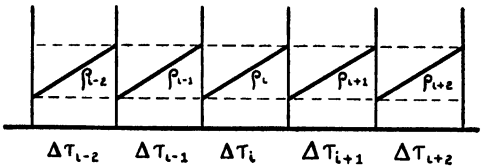

FIG. 7.—Schematic diagram showing the densities in adjoining cells containing identical complexes of charge.

spreading of both charge and polarization. The potential due to the charges in $\Delta\tau_i$ will thus be written

$$\frac{1}{4\pi}\int_{\Delta\tau_i}\frac{\rho_0^i\,d\tau}{r}+\frac{1}{4\pi}\int_{\Delta\tau_i}\frac{P_0^i\cos\psi}{r^2}d\tau\,,$$

where ρ_0^i and $\boldsymbol{P}_0^i$ are scalar and vector functions, constant over their region of definition $\Delta\tau_i$, and given by

$$\Delta\tau_i\,\rho_0^i=\epsilon_i\,,$$
$$\Delta\tau_i\,\boldsymbol{P}_0^i=\boldsymbol{p}_i\,,$$

where ψ is the angle between $\boldsymbol{P}_0^i$ and the direction from $d\tau$ to P, where ϵ_i is the sum of the charges in $\Delta\tau_i$, and where $\boldsymbol{p}_i$ is the polarization with respect to O_i, due to the charges in $\Delta\tau_i$. If all the $\Delta\tau_i$ contain identical subcomplexes, however, it is evident that ρ_0^i and $\boldsymbol{P}_0^i$ are the same in each $\Delta\tau_i$, so that the potential of the whole body may be written

$$\frac{1}{4\pi}\int\frac{\rho_0\,d\tau}{r}+\frac{1}{4\pi}\int\frac{P_0\cos\psi d\tau}{r^2}\,,$$

where ρ_0 is a function, constant over the whole body, formed by the union of ρ_0^i; and $\boldsymbol{P}_0$ is a function, constant over the whole body, formed by the union of $\boldsymbol{P}_0^i$.

The case in which the subcomplexes in $\Delta\tau_i$ are identical is, of course, a trivial one, but the possibility of a simple representation of the potential due to the charges forming a ponderable body depends (as will be seen presently) upon the fact that it is possible to choose the volume cells

$\Delta\tau_i$ in such a way that the subcomplexes in neighboring cells, while not identical, are nearly the same. It is in fact possible to choose volume cells of dimensions which are small compared to the distance from any cell to P, but which are large enough so that each $\Delta\tau_i$ will contain very many elementary charges. If each volume cell contains many charges, the statistics are regular, i.e., any characteristic of charge when averaged over the volume element will be uninfluenced by the particular circumstances of individual charges, and will be a true measure of the slowly varying

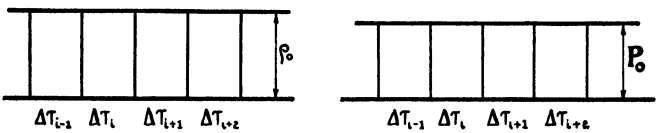

FIG. 8.—Schematic diagrams showing the densities ρ and P of adjoining cells containing identical complexes

average trend of this characteristic. Such a subdivision then leads to a simple description of the potential in those cases in which the average characteristics of charge experience very small percentage changes over distances of the order of the dimensions of $\Delta\tau_i$. The possibility of such a subdivision of a body definitely depends upon the physical fact that it is possible to pick out three distinct orders of distances—molecular distances, distances used for dimensions of volume cells, and experimental distances to those exterior points at which the electrostatic effect is to be measured.

Suppose, for example, that a body is so subdivided, and consider the values of $\epsilon_i/\Delta\tau_i$ associated with all the volume elements $\Delta\tau_i$ cut by some straight line m–n through the body. Figure 9 shows a possible set of

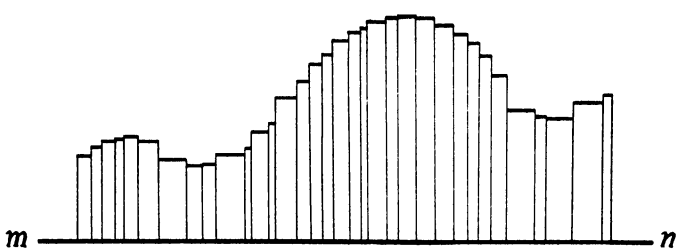

FIG. 9.—Schematic diagram of the values $\epsilon_i/\Delta\tau_i$

such values of $\epsilon_i/\Delta\tau_i$. If now the body were redivided into volume elements one-half as large as before, a new set of values of $\epsilon_i/\Delta\tau_i$ would re-

sult. But since the volume elements are in either case large enough to contain so many charges that individual eccentricities are averaged out, the set of values in this second case would have the same general aspect as before (see Fig. 10). If this redivision process were repeated too many times, a cell size would be reached such that too few charges would lie in each cell, and the neighboring values of $\epsilon_i/\Delta\tau_i$ would differ widely, each being appreciably influenced by the particular circumstances of individual charges rather than by the general trend of the average number of charges. On the other hand, it is obvious that the volume elements $\Delta\tau_i$ should be chosen as small as is consistent with the requirement that each contain many charges, both so that the dimensions of $\Delta\tau_i$ shall be as small as possible as compared to r, and so that general tendencies will not be masked.

These remarks can be illustrated by means of an example. Suppose ten thousand coins are tossed simultaneously and the number of heads and tails counted, and suppose that this experiment is repeated a very large number of times. There would thus be obtained data from which could be made a table giving the number of times f which a given number of heads n actually appeared. A superficial study based directly upon these individual number pairs (n,f) might be very misleading. It would be possible, for example, that a portion of the table be as follows:

n	f
4,900	17
4,901	6
4,902	0
4,903	9
4,904	26
4,905	20
4,906	16
.	.
.	.
.	.

No clear indication of a general trend in the frequency f is furnished by such a set of values. Individual eccentricities could be averaged out, however, by preparing an auxiliary table giving the frequency f of the instances in which the number n of heads lay in a range 0 to k, k to $2k$, $2k$ to $3k$, ... In choosing the magnitude of k, questions arise which are entirely similar to those considered above in connection with the choice of the size of the volume elements $\Delta\tau_i$. If k be taken as small as unity, the individual eccentricities play too great a rôle. If k is as

large as ten thousand (to take a ridiculous extreme), the general trend of the frequency table is completely masked by too gross an averaging process. The choice of k must then be a compromise one, it being taken as large with respect to unity as is consistent with the requirement that it

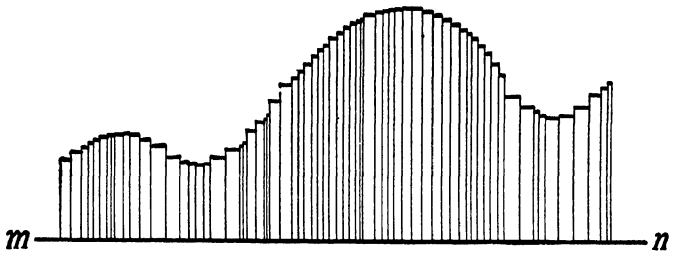

FIG. 10.—Schematic diagram of the values $\epsilon_i/\Delta\tau_i$ using a finer subdivision than that shown in Figure 9.

be small with respect to ten thousand. If, for example, k were chosen to be one hundred, a new table of one hundred entries would be formed giving the number of times f which the number of heads lay in the range 0–100, 100–200, 200–300, ... If these values of f are used to construct a frequency polygon, a figure similar to the following will result.

In this figure the characteristic behavior of the frequency is shown, freed of the confusing eccentricities of the individual case. But in attempting to formulate a theory of this experiment, and in extending this

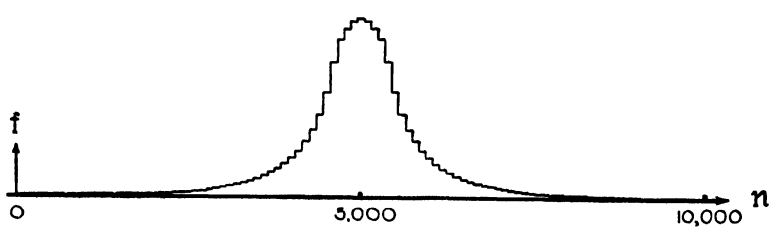

FIG. 11.—Frequency polygon for heads in an hypothetical coin-tossing experiment

theory to include other more general cases, a further degree of simplification is exceedingly convenient; namely, it simplifies matters greatly to consider not the discrete set of values f, but rather a smooth curve which is sensibly equivalent to the actual frequency polygon. In choosing such a curve it is convenient to establish some definite procedure; for example, a smooth curve can be drawn, by interpolation, through the midpoints of each horizontal segment of the frequency polygon, and the further

discussion of the experiment would be on the basis of a continuous function $F(x)$ which for the special integral arguments

$$x = 50,\ 150,\ 250,\ \ldots\ ,$$

takes on the values f. It would also be possible to introduce a continuous function whose mean values over the ranges 0–100, 100–200, 200–300, ... , equal the values of f. If there were small percentage variations of f from one to another of these ranges of n, all such methods of choosing the smooth curve would furnish sensibly the same result. This function $F(x)$ has a definite value for every value of the argument, whether integral or not, although there is obviously no interpretation for the value of F for a non-integral argument. Indeed, it is evident that the value of F for an integral argument has no immediate relation to the actual frequency with which this number of heads appeared; the usefulness of the function F depends rather upon the fact that the frequency of the cases in which the number of heads was between n_1 and n_2 is given, to a high degree of accuracy, by the expression

$$\frac{1}{(n_2 - n_1)} \int_{n_1}^{n_2} F(x)dx\ .$$

The introduction of this continuous function makes available, at one stroke, all the mechanism of calculus, and it is the introduction of such a continuous function, indeed, which makes possible a statistical theory conveniently applicable to vast numbers of individual cases.

The possibility of the introduction of a continuous function in the electrostatic case, as will be seen presently, is directly dependent upon the fact that the $\Delta\tau_i$ can be chosen so that the percentage variation in any quantity characteristic of the circumstances of charge is very small in neighboring volume cells. The condition of statistical regularity and slow variation of the polarization may be made precise by writing

$$\mathfrak{p}_i \gg \Delta\mathfrak{p}_i \gg \Delta^2\mathfrak{p}_i \gg \ldots\ , \tag{25}$$

where $\mathfrak{p}_i$ is the polarization, with respect to O_i, due to the charges in the ith cell, and where $\Delta\mathfrak{p}_i$, $\Delta^2\mathfrak{p}_i$, ... , are the first, second, ... differences between the successive values of $\mathfrak{p}_i$ in neighboring cells.*

* The symbol $\gg$ means "very large compared with," and usually indicates that any one quantity is negligible compared with its predecessor in such an inequality

Slightly less severe conditions can be imposed in the case of the total charges ϵ_i. In applying the spreading method to a single complex, it has been found that a linear variation in the continuous density ρ was sufficient to represent the polarization term of an expansion, of type (9), for the potential due to the complex. In general, a rate of change of density is interpretable in terms of polarization, and vice versa. It will be seen later that it is possible to choose volume cells $\Delta\tau_i$ in such a way that the separate cells appear polarized, or in such a way that they do not appear polarized. In the second instance, however, a rate of change of charge is found to exist which, just as in the case already considered in § 10, gives rise to those terms in the potential which were previously accounted for by polarization terms. Thus, since both charge and polarization terms are to be retained, it is not possible to neglect the first differences $\Delta\epsilon_i$ in the values of the total charges ϵ_i of the volume cells. No higher-order effects, however, are to be considered, so that

$$\Delta\epsilon_i \gg \Delta^2\epsilon_i \gg \Delta^3\epsilon_i \gg \ldots\ldots \tag{26}$$

It is then clear that if charge and polarization are both spread over each volume cell $\Delta\tau_i$, there will result functions ρ_0^i and $\boldsymbol{P}_0^i$, each constant over its region of definition $\Delta\tau_i$, and satisfying the inequalities

$$\Delta\rho_0^i \gg \Delta^2\rho_0^i \gg \ldots\ldots,$$

$$\boldsymbol{P}_0^i \gg \Delta\boldsymbol{P}_0^i \gg \Delta^2\boldsymbol{P}_0^i \gg \ldots\ldots$$

Now the unions of the functions ρ_0^i and $\boldsymbol{P}_0^i$ do not produce functions ρ and $\boldsymbol{P}$ continuous over the whole body, since the functions ρ and $\boldsymbol{P}$ experience discontinuous jumps in passing from one volume cell to another. Since matters have been arranged, however, so that these jumps in both ρ and $\boldsymbol{P}$ are very small, the sensible procedure is obviously to smooth out these small discontinuities and employ continuous functions ρ and $\boldsymbol{P}$. It is thus natural to attempt to represent the potential due to all the charges forming the body by means of the expression

$$\Phi = \frac{1}{4\pi}\int\frac{\rho d\tau}{r} + \frac{1}{4\pi}\int\frac{P\cos\psi}{r^2}d\tau\,, \tag{27}$$

where the integrals extend over the whole body, and where ρ and $\boldsymbol{P}$ are no longer constant over $\Delta\tau_i$, but are slowly varying continuous functions. Since these functions ρ and $\boldsymbol{P}$ are the functions ρ_0^i and $\boldsymbol{P}_0^i$ with

the sudden jumps smoothed out, the continuous functions will satisfy the inequalities

$$l\frac{d\rho}{dl} \gg \frac{l^2}{2}\frac{d^2\rho}{dl^2} \gg \cdots\cdots, \tag{28}$$

$$\boldsymbol{P} \gg l\frac{d\boldsymbol{P}}{dl} \gg \frac{l^2}{2}\frac{d^2\boldsymbol{P}}{dl^2} \gg \cdots\cdots, \tag{29}$$

l being a distance of the order of the dimensions of $\Delta\tau_i$. For reasons entirely similar to those stated in connection with (26), it is not permissible to state in (28) that $l\frac{d\rho}{dl}$ is necessarily very small compared to ρ itself.

The essential distinction between the problem of representing the potential due to a single small complex and the problem of representing the potential due to a ponderable body thus arises from the necessity, in the latter case, of using functions ρ_i and $\boldsymbol{P}_i$, each defined over $\Delta\tau_i$, which join on to one another to form functions ρ and $\boldsymbol{P}$, continuous over the whole body. What conditions must be satisfied by the functions ρ and $\boldsymbol{P}$, continuous over the whole body, in order that the two leading terms in the potential due to the body be given by the expression (27) at least to terms of order $1/r^3$, where r is the distance from P to any point of the body?

§ 12. *The Approximate Representation of the Potential of a Body in Terms of Continuous Functions* ρ *and* $\boldsymbol{P}$.—The potential due to the body is to be represented in the form (27) where the integrals are extended over the whole body, and where ρ and $\boldsymbol{P}$ are continuous scalar and vector functions which will be called the "volume density of charge" and the "volume density of polarization," respectively. The expression (27) can obviously be written

$$\sum_i \left[\int_{\Delta\tau_i}\frac{\rho d\tau}{r} + \int_{\Delta\tau_i}\frac{P\cos\psi}{r^2}\,d\tau\right],$$

where $\Delta\tau_i$ are the volume cells, discussed above, into which the body is divided. Then (27) will approximate, to terms of order $1/r^3$, the first two terms in the potential due to the whole body provided each term in the foregoing sum approximates, to the desired order, the first two terms in an expansion of type (9) of the potential due to the charges ϵ_i in $\Delta\tau_i$, i.e., provided

$$\frac{\Sigma e_j}{r_i} + \frac{\Sigma e_j l_j\cos\theta_j}{r_i^2} \equiv \int_{\Delta\tau_i}\frac{\rho d\tau}{r} + \int_{\Delta\tau_i}\frac{P\cos\psi}{r^2}\,d\tau\,, \tag{30}$$

where r_i is now the distance from P to O_i, the center of volume of $\Delta\tau_i$, where l_j are the vectors locating the charges e_j, within $\Delta\tau_i$, with respect to O_i, and where θ_j is the angle between $O_i - P$ and l_j. The conditions which ρ and P must satisfy are now to be obtained by expanding the integrands on the right side of (30). Thus,

$$\int_{\Delta\tau_i} \frac{\rho d\tau}{r} = \int_{\Delta\tau_i} \left[\left(\frac{\rho}{r}\right)_i + s \left(\frac{d}{ds}\frac{\rho}{r}\right)_i + \frac{s^2}{2}\left(\frac{d^2}{ds^2}\frac{\rho}{r}\right)_i + \cdots \right] d\tau , \tag{31}$$

$$= \int_{\Delta\tau_i} \left[\left(\frac{\rho}{r}\right)_i + \rho_i s \left(\frac{d}{ds}\frac{1}{r}\right)_i + s \left(\frac{1}{r}\frac{d\rho}{ds}\right)_i + s^2 \left(\frac{d\rho}{ds}\frac{d}{ds}\frac{1}{r}\right)_i + \cdots \right] d\tau ,$$

$$= \int_{\Delta\tau_i} \left(\frac{\rho}{r}\right)_i d\tau + \int_{\Delta\tau_i} \left(\frac{\rho}{r^2}\right)_i s\cos\varphi \, d\tau + \int_{\Delta\tau_i} \left(\frac{1}{r}\right)_i s \left(\frac{d\rho}{ds}\right)_i d\tau , \tag{32}$$

$$+ \int_{\Delta\tau_i} \left(\frac{1}{r^2}\right)_i s^2 \left(\frac{d\rho}{ds}\right)_i \cos\varphi \, d\tau + \cdots ,$$

where the subscript i on a parenthesis indicates that the value of the quantity is to be taken at the point O_i, where s measures the distance

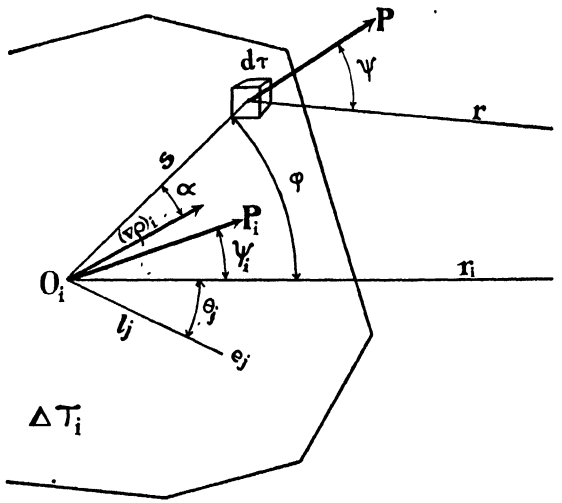

FIG. 12

from O_i in any direction to (x', y', z'), and where φ is the angle between the direction of s and the direction from O_i to P. Two terms

are omitted in (32) which are explicitly included in equation (31). These two terms contain, as factors of the integrand, the expressions

$$\rho_i \left(\frac{d^2}{ds^2}\frac{1}{r}\right)_i \text{ and } \left(\frac{1}{r}\right)_i \left(\frac{d^2\rho}{ds^2}\right)_i .$$

The first is to be neglected since a second derivative of $1/r$ is an inverse cube term. The second is to be neglected on account of (28). For similar reasons all further terms are to be neglected in (32). The second and third terms of (32) may be written

$$\left(\frac{\rho}{r^2}\right)_i \int_{\Delta\tau_i} s\cos\varphi\, d\tau + \left(\frac{|\nabla\rho|}{r}\right)_i \int_{\Delta\tau_i} s\cos\alpha\, d\tau ,$$

where α is the angle between $\boldsymbol{s}$ and the direction of $\nabla\rho$ at O_i. Each of these integrals is zero, since s is measured from O_i, the center of volume of $\Delta\tau_i$.

Equation (31) thus reduces to

$$\int_{\Delta\tau_i} \frac{\rho d\tau}{r} = \frac{\rho_i \Delta\tau_i}{r_i} + \frac{1}{r_i^2}\int_{\Delta\tau_i} s^2 \left(\frac{d\rho}{ds}\right)_i \cos\varphi\, dt + \ldots ,$$

where ρ_i is the value of ρ at O_i.

Similarly,

$$\int_{\Delta\tau_i} \frac{P\cos\psi}{r^2}\, d\tau = \frac{P_i \cos\psi_i}{r_i^2}\,\Delta\tau_i + \ldots ,$$

where P_i is the value of P at O_i, and where no terms beyond the first need be retained since $\boldsymbol{P}$ satisfies equation (29), and since only second-order terms in l/r_i are being retained.

If these values are substituted in (30), the following equation is obtained:

$$\frac{\Sigma e_j}{r_i} + \frac{\Sigma e_j l_j \cos\theta_j}{r_i^2} \equiv \frac{\rho_i \Delta\tau_i}{r_i} + \frac{1}{r_i^2}\int_{\Delta\tau_i} s^2 \left(\frac{d\rho}{ds}\right)_i \cos\varphi\, d\tau + \frac{P_i\cos\psi_i}{r_i^2}\,\Delta\tau_i + \ldots .$$

This equation will be satisfied, provided ρ_i satisfies the condition

$$\rho_i = \frac{\Sigma e_j}{\Delta\tau_i} , \tag{33}$$

where a single cell thick. The first integral of (36) is, to a high degree of approximation, equal to the integral

$$\int \frac{\eta d\sigma}{r},$$

where

$$\eta \equiv h\rho$$

is the so-called "surface density" of charge, and where $d\sigma$ is an element of the surface area of the body. Indeed, the two integrands

$$\frac{\rho d\tau}{r} \text{ and } \frac{\eta d\sigma}{r} = \frac{\rho h d\sigma}{r}$$

are the same, except that the value of r in one differs from its value in the other by an amount less than h.

The continuous density of polarization $\boldsymbol{P}$ to be determined throughout the exterior volume cells $\Delta\tau_j$ is, as before, obtained by interpolation from the values of the polarization vectors for the individual cells, diminished by the polarizations already represented by the variation in the volume density of charge ρ. The asymmetrical circumstances which necessitate the separate consideration of the surface cells suggest breaking up the cell polarizations into two components: $(\boldsymbol{p}_j)_n$, the component in the direction of the exterior normal to the body, and $(\boldsymbol{p}_j)_t$, the component in the tangent plane.* Now as regards directions lying in a plane tangent to the surface, a surface charge is surrounded by other charges in as symmetrical a way as is an interior charge. Hence the component $(\boldsymbol{p}_j)_t$ due to a given number of charges lying "on" the surface is of the same order as any component of the polarization due to the same number of interior charges. On the other hand, the component $(\boldsymbol{p}_j)_n$ due to a certain number of charges lying "on" the surface may be of the same order as a component of the polarization due to a much larger number of interior charges.

The number of charges available, in a cell $\Delta\tau_i$, to contribute to the value of the polarization of that cell cannot be estimated by dividing the total net charge ϵ_i of the cell by the elementary charge. Neutral polarizable atoms, for example, may produce large values of $\boldsymbol{p}_i$ and, at the same time, a zero value for ϵ_i. In fact, it will be clear, when illustra-

* The polarizations $\boldsymbol{p}_j$ are referred to O_j, the centers of volume of $\Delta\tau_j$. Let lines through O_j normal to the surface cut the surface at O'_j. The tangent planes referred to are tangent to the surface at the points O'_j.

tive examples have been studied,* that the value (when a physically reasonable mode of subdivision is used) of $\boldsymbol{P}_i$ for a cell depends directly on the number of polarizable units in this cell, while the polarization effects due to a changing value, from cell to cell, of the total charge ϵ_i do not normally influence the value of $\boldsymbol{P}_i$. Changing values of ϵ_i will be found to contribute equally to the two terms on the right side of equation (35). Thus, in actual electrostatic problems, non-vanishing values of $\boldsymbol{P}_i$ arise when a body is formed of polarizable units of some sort. The number density of these units is a physical characteristic of the material forming the body, and is the same at points near the surface and points in the interior. Hence, the number of charges, in a surface cell $\Delta\tau_j$, available to contribute to the final value of $\boldsymbol{P}$ is smaller than the corresponding value for an interior cell $\Delta\tau_i$ in the ratio of the volumes; that is to say, in the ratio h/l. Since this ratio is small, the polarization effects for surface elements would be negligible if each polarizable unit of a surface element contributed to $\boldsymbol{p}_j$ sensibly the same amount that each interior polarizable unit contributes to $\boldsymbol{p}_i$. As regards all but the normal component of polarization, the contribution, per unit, is the same near the surface as within the body. The normal component $(\boldsymbol{p}_j)_n$ may, however, be abnormally large. Hence only this component need be considered. Moreover, since the continuous volume density of charge in $\Delta\tau_j$ has no variation in this normal direction, the whole burden for the representation of the polarization rests upon the function $\boldsymbol{P}$, and it is therefore to be determined by interpolation from the values

$$P_j = (P_j)_n = \frac{1}{\Delta\tau_j}\,(\boldsymbol{p}_j)_n\,.$$

The vector $\boldsymbol{P}$ thus determined is, of course, everywhere normal to the surface of the body, as the foregoing equation implies.

The second integral of (36) can be written as a surface rather than as a volume integral, the procedure being entirely similar to that used in the case of the first integral of (36). In fact,

$$\int \frac{P\cos\psi}{r^2}\,d\tau = \int \frac{\mu\cos(nr)}{r^2}\,d\sigma = \int \left(\mu,\ \nabla'\frac{1}{r}\right) d\sigma\,,$$

where

$$\mu \equiv h\boldsymbol{P}$$

* These examp'es occur in the next section.

is the surface density of polarization of the superficial layer. Thus (36) can be written

$$\frac{1}{4\pi}\int\frac{\eta d\sigma}{r}+\frac{1}{4\pi}\int\left(\mu,\nabla'\frac{1}{r}\right)d\sigma\,,$$

where η is a continuous scalar function, defined at every point on the surface of the body; and where μ is a continuous vector function, also defined at every point on the surface of the body, and directed along the exterior normal.

The representation of the potential due to the charges forming a ponderable body is, then, obtained as follows. A very thin shell at the surface of the body is to be divided into volume elements $\Delta\tau_j$ of lateral dimensions l and thickness h, where h is very small compared to l, and where l, while small compared to the distance to the exterior point where the potential is desired, is large enough so that a cell of linear dimensions l contains very many charges. The remaining portion of the body is to be divided into volume cells $\Delta\tau_i$ of linear dimensions of the order of l. The total charge ϵ and the polarization $\boldsymbol{p}$ with respect to the center of volume O is reckoned for each volume cell. The volume density of charge is obtained by interpolation from

$$\rho_i=\frac{1}{\Delta\tau_i}\epsilon_i\,, \tag{37}$$

the values of ρ at O_i, the centers of volume of $\Delta\tau_i$. The volume density of polarization $\boldsymbol{P}$ is then obtained by interpolation from

$$\boldsymbol{P}_i=\frac{1}{\Delta\tau_i}\boldsymbol{p}_i-\frac{1}{\Delta\tau_i}\int s\,\boldsymbol{s}\left(\frac{d\rho}{ds}\right)_i d\tau\,, \tag{38}$$

the values of $\boldsymbol{P}$ at O_i. The surface density of charge η is obtained by interpolation from

$$\eta_j=\frac{\epsilon_j h}{\Delta\tau_j}\,, \tag{39}$$

the values of η at O_i', the points on the surface of the body which lie on the same normals as do O_j, the centers of volume of $\Delta\tau_j$. The surface density of polarization μ is obtained by interpolation from

$$\mu_j=(\mu_j)_n=\frac{h}{\Delta\tau_j}(\boldsymbol{p}_j)_n\,, \tag{40}$$

the values of μ_j at O_j'. The two leading terms in the expansion of type (9) for the potential due to all charges forming the body are then given by the two volume and the two surface integrals

$$(41)\quad \Phi=\frac{1}{4\pi}\int\frac{\rho' d\tau'}{r}+\frac{1}{4\pi}\int\left(\boldsymbol{P}',\nabla'\frac{1}{r}\right)d\tau'+\frac{1}{4\pi}\int\frac{\eta' d\sigma'}{r}+\frac{1}{4\pi}\int\left(\boldsymbol{\mu}',\nabla'\frac{1}{r}\right)d\sigma$$

In the fundamental equation attention is explicitly called to the fact that the densities are functions of the integration variables x',y',z'. The distance r is, of course, a function of both the primed and unprimed variables. The potential Φ is a function of x,y,z only.

§ 14. *The Physical Meaning of the Densities* $\rho, \eta, \boldsymbol{P}, \boldsymbol{\mu}$.—By considering the idealized case of a rectangular parallelopiped V, formed by the planes $x=0, x=2a, y=0, y=2b, z=0, z=2c$, and within which are various simple configurations of charge, there may be seen the relations between the four densities defined above, and what one is accustomed to think of as the physical concepts of charge, polarization, etc. The configurations of charge to be considered may be most briefly described by giving the locations of the charges with respect to a standard set of points within V. Thus suppose V be subdivided by a series of planes, parallel to the co-ordinate planes and spaced a distance k (of the order of molecular dimensions and hence very small compared to a, b, or c). The points M located by the intersections of these planes form a cubical space lattice. The points M_0 lie on the back face $x=0$ of the parallelopiped, the points M_k on the next parallel plane, the points M_{2a} on the front face $x=2a$, etc.

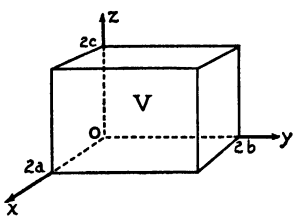

FIG. 13

Suppose first that at all the points M are located equal charges $+e$. If equal cubes of side l be chosen for the volume elements $\Delta\tau_i$, they will all contain the same total amount of charge $\epsilon_i=\left(\frac{l}{k}\right)^3 e$. Then the function ρ, interpolated from the equal values $\rho_i=\frac{e}{k^3}$, is a constant whose volume integral, extended over any portion of V, gives to a high degree of accuracy the total charge lying in that portion. The values of $\boldsymbol{P}_i$, since ρ is a constant, are given by

$$\boldsymbol{P}_i=\frac{1}{\Delta\tau_i}\boldsymbol{p}_i\,.$$

If the cells $\Delta\tau_i$ are chosen with their faces parallel to, but midway between, the planes of the cubical space lattice formed by the charges, then $p_i=0$ and the potential of the body is given by

$$4\pi\Phi=\int\frac{\rho d\tau}{r}\,.$$

It is not necessary to use any surface elements $\Delta\tau_j$ for, under the conditions assumed, equations (25) and (26) are satisfied for all volume cells, including the outermost ones. If the volume cells $\Delta\tau_i$ be shifted to the left, say, a distance smaller than k, each $\Delta\tau_i$ would be polarized as well as charged. The maximum polarization so introduced, however, is equal to $k\epsilon_i/2$. Regardless of the choice of $\Delta\tau_i$, therefore, the polarization term is negligible compared to the charge term, and the potential of the body is given by the foregoing integral in terms of the volume density of charge ρ. The definition of the densities is, then, in this special case independent of the choice of the subdivision of the body into volume cells $\Delta\tau_i$.

As a second example, suppose that the charges $-e$ are located at the points M_{2nk}, $n=0,1,2,\ldots$, while charges $+e$ are located at all other M-points. Then along lines L parallel to the x-axis will be found charges $-e$, $+e$, $-e$, $+e$, etc., spaced a distance k, the last charge being $+e$, provided k is chosen so that $2a$ is an odd multiple of k. Then move the charges which lie on alternate horizontal planes a distance, to the front,

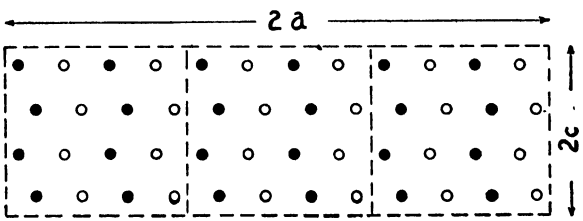

FIG. 14.—Arrangement of charges used in illustrations. Subdivision No. 1

which is a small fraction of k. The charges lying in any plane parallel to xz-plane would then be located as shown in the following sketch, where the relative size of the distance k is enormously enlarged, and where solid dots are used to represent negative charges; circles, positive charges. It is then possible to choose a set of volume cells $\Delta\tau_i$ (see dotted lines in

Fig. 14; this will be referred to as subdivision I) for which $\rho_i=0$ while p_i has the value $l^3e/2k^2$. Then

$$P_i=P_x=\frac{1}{\Delta\tau_i}\,p_i=e/2k^2\,,$$

and the potential due to the whole configuration of charges is given by

$$4\pi\Phi=\int\frac{e}{2k^2}\,\frac{\cos\psi}{r^2}\,d\tau\,. \tag{42}$$

Just as in the first example, and for the same reason, surface densities play no rôle in this representation. Having chosen the volume cells $\Delta\tau_i$ in this way, it is natural to say that the body is polarized, but uncharged. It is possible, however, to obtain, by choosing the $\Delta\tau_i$ otherwise, an alternative representation which leads to an alternative physical interpretation. Suppose, in fact, that the cells are chosen as shown in Figure 15, where very thin surface elements $\Delta\tau_j$ are chosen first, and the re-

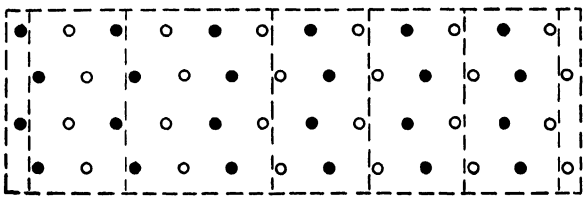

FIG. 15.—Subdivision No. II

mainder of the body then subdivided. (This will be referred to as subdivision II.) It is clear from the figure that the surface elements $\Delta\tau_j$ are charged, while the interior volume elements $\Delta\tau_i$ are both uncharged and unpolarized. (In the figure the second volume element $\Delta\tau_j$ from the left has $\epsilon_i=0$, but the polarization p_i is not zero. This is a result, however, of the gross misrepresentation of relative dimensions. In the case where k is very small compared to the dimensions of the volume elements, the number of uncompensated pairs of charges is vanishingly small as compared to the total number, and the polarization is negligible.) The charge ϵ_j in each exterior volume element $\Delta\tau_j$ is equal to $(l/2k)(l/k)(\pm e)$. Thus

$$\eta_j=\frac{\epsilon_j h}{\Delta\tau_j}=\pm\frac{l^2e}{2k^2l^2}=\pm\frac{e}{2k^2}\,,$$

where the plus and minus signs hold on the faces x equals $2a$ and 0, respectively. Then the continuous function η interpolated from this set of equal values is given by

$$\eta = \pm \frac{e}{2k^2},$$

and the potential of the whole configuration of charges is given by

$$4\pi\Phi = \int \frac{\eta d\sigma}{r} = \int_{x=0} \left(-\frac{e}{2k^2}\right) \frac{d\sigma}{r} + \int_{x=2a} \left(+\frac{e}{2k^2}\right) \frac{d\sigma}{r}. \tag{43}$$

Having subdivided the body in this way, it is natural to say that there is no volume density of charge or polarization, and that there is a surface density of charge on the two faces $x=0$, $x=2a$. That this representation is mathematically equivalent to the one given by (42) is seen at once by treating (42) by integration by parts. Indeed, the general formula*

$$\int \left(\boldsymbol{P}, \nabla \frac{1}{r}\right) d\tau = \int \frac{P_n d\sigma}{r} + \int -\frac{\operatorname{div} \boldsymbol{P}}{r} d\tau \tag{44}$$

reduces, for the special value $P = P_x = e/2k^2$, to the equation

$$\int \frac{e}{2k^2} \frac{\cos\psi}{r^2} d\tau = \int_{x=0} \left(-\frac{e}{2k^2}\right) \frac{d\sigma}{r} + \int_{x=2a} \left(\frac{e}{2k^2}\right) \frac{d\sigma}{r}.$$

It is impossible to choose, on mathematical grounds, between the two descriptions; and it is quite meaningless to argue as to whether there "really is, or is not", a volume density of polarization. A volume density of polarization can be defined only with reference to a definite volume, and whether or not such a volume is polarized depends on how it is chosen. There may be, in certain cases, a physical reason for preferring the representation given by (42), on account of the physical interpretation to which this equation most naturally leads. If, for example, a body is thought of as composed of molecules each one of which is neutral but polarized, it might be convenient to think of the body as possessing a volume density of polarization, and equation (42) would be, at least in the first instance, preferable to (43).

As a third example, suppose that at the points M_0, M_k, M_{2k}, ... , are located charges e, $e+e_1$, $e+2e_1$, ... , where e_1 is small compared

* See Part III, Problem 15, of this chapter.

to e. If equal cubes of side l are then chosen for $\Delta\tau_i$, a cell which is ith from the left end of V contains total charge

$$\epsilon_i = \frac{l^3}{k^3}\left[e + \frac{l}{k}\left(i-\frac{1}{2}\right)e_1\right],$$

so that

$$\rho_i = \frac{1}{k^3}\left[e + \frac{l}{k}\left(i-\frac{1}{2}\right)e_1\right],$$

and the volume density of charge is given by the continuous function

$$\rho = \frac{1}{k^3}\left(e + \frac{e_1 x}{k}\right).$$

The polarization $\boldsymbol{p}_i$ of this ith cell is given by

$$p_i = p_x = \frac{l^2}{k^2}\sum_{n=0}^{l/k}\left[\frac{(i-1)l}{k} + n\right]\left[nk - \frac{l}{2}\right]e_1,$$

$$= \frac{e_1 l^5}{12k^4}\left[1 + \frac{3k}{l} + \frac{2k^2}{l^2}\right],$$

$$= \frac{e_1 l^5}{12k^4} + \cdots\cdots.$$

The volume density of polarization $\boldsymbol{P}$ would be interpolated from the values

$$P_i = P_x = \frac{e_1 l^2}{12k^4} - \frac{1}{l^3}\int s_x^2\left(\frac{d\rho}{dx}\right)_i d\tau.$$

However,

$$\frac{1}{l^3}\int s_x^2\left(\frac{d\rho}{dx}\right)_i d\tau = \frac{2e_1}{lk^4}\int_0^{l/2}\xi^2 d\xi$$

$$= \frac{e_1 l^2}{12k^4},$$

where ξ is a distance, parallel to the direction of x, measured from the center of volume of the cell $\Delta\tau_i$. Thus, to a high order of approximation, the values P_i vanish, and the potential due to this configuration of charges is given by

$$4\pi\Phi = \int\frac{\rho d\tau}{r} = \frac{1}{k^3}\int\frac{\left(e + \frac{e_1 x}{2k}\right)}{r}d\tau$$

in terms of the volume density of charge. This example illustrates the fact that the complexes within the volume elements $\Delta\tau_i$ can be polarized without there being a volume density of polarization. The variation in ρ which is introduced by the interpolation is just sufficient to represent the second term in the expansion of type (9), and the polarization term vanishes.

As a fourth example, consider a configuration which is the same as that of the second example, except that the charges found along a line L parallel to the x-axis are not equal in magnitude (and alternating in sign), but are equal to $-e$, $+e$, $-(e+e_1)$, $+(e+e_1)$, $-(e+2e_1)$, $+(e+2e_1)$, etc. Then if subdivision I of the second example be used, the total charge ϵ_i in $\Delta\tau_i$ is again zero, while the polarization p_i is now given by

$$p_i = p_x = \frac{l^3}{2k^2}\left[e + \frac{\left(i-\frac{1}{2}\right)l}{2k}\, e_1\right].$$

Therefore,

$$P_i = \frac{1}{2k^2}\left[e + \frac{\left(i-\frac{1}{2}\right)l}{2k}\, e_1\right],$$

and the volume density of polarization is given by the continuous function

$$P = P_x = \frac{1}{2k^2}\left[e + \frac{e_1 x}{2k}\right].$$

The potential due to the configuration of charges is thus

$$4\pi\Phi = \int \frac{1}{2k^2}\frac{\left(e + \frac{e_1 x}{2k}\right)}{r^2} \cos(r,x)\, d\tau \tag{45}$$

in terms of the volume density of polarization. Having chosen the volume cells $\Delta\tau_i$ in this way, the volume and surface densities of charge are zero, and it is natural to say that the body is uncharged but polarized, the volume density of polarization being variable.

If the subdivision II of the second example be used, an alternative representation of the potential is obtained. Each interior volume element $\Delta\tau_i$ is found to have a negligible polarization p_i. The total charge ϵ_i in $\Delta\tau_i$, however, differs from its value zero in the preceding mode of subdivision, due to the fact that $l^2/2k^2$ positive charges are cut off by the

new position of the right face of $\Delta\tau_i$, and an equal number of positive charges added (or negative charges subtracted) by the new position of the left face $\Delta\tau_i$. That is,

$$\epsilon_i = -\frac{l^2}{2k^2}\left[e+\frac{il}{2k}e_1\right]+\frac{l^2}{2k^2}\left[e+\frac{(i-1)l}{2k}e_1\right],$$

$$= -\frac{l^3 e_1}{4k^3}.$$

Therefore,

$$\rho_i = -\frac{e_1}{4k^3},$$

and the volume density of charge is given by the continuous function

$$\rho = -\frac{e_1}{4k^3}.$$

The surface elements on the face $x=2a$ have a total charge

$$\epsilon_j = \frac{l^2}{2k^2}\left[e+\frac{ae_1}{2k}\right],$$

so that the surface density of charge on this face is

$$\eta = \frac{e}{2k^2}+\frac{ae_1}{4k^3}.$$

Similarly,

$$\left.\begin{aligned}\epsilon_j &= -\frac{l^2 e}{2k^2}\\ \eta &= -\frac{e}{2k^2}\end{aligned}\right\} x=0.$$

The potential of the body is therefore given by

$$(46)\quad 4\pi\Phi = \int\frac{\left(-\frac{e_1}{4k^3}\right)}{r}d\tau+\int_{x=2a}\frac{\left(\frac{e}{2k^2}+\frac{ae_1}{4k^3}\right)}{r}d\sigma+\int_{x=0}\frac{\left(-\frac{e}{2k^2}\right)}{r}d\sigma.$$

Just as in the second example, the representations (45) and (46) are analytically equivalent. Indeed, if (45) be treated, as before, by integration by parts, the equivalence of the two formulas is immediately estab-

lished. It is thus legitimate to say that the body is uncharged and non-uniformly polarized, and use (45) for the potential; or to say that the body is unpolarized and has constant volume and surface densities of charge, and use (46) for the potential. But it is not legitimate to say that the body is non-uniformly polarized, treat (45) by integration by parts, and then, still calling it a "polarized body," say that there is a volume and surface density of charge. The integration by parts carries with it, if one insists on a physical interpretation of the result, a changed point of view. That is, it is not correct to say that a non-uniformly polarized body has a volume density of charge given, at any point, by $-\operatorname{div} \boldsymbol{P}$, and a surface density of charge given by P_n; but it is true that any polarized body can be viewed as a non-polarized body having a volume density of charge $-\operatorname{div} \boldsymbol{P}$ and a surface density of charge P_n.

These examples show the relationships between the mathematically defined continuous densities for idealized cases which would be recognized, physically, as being instances of uniform charge, uniform polarization, non-uniform charge, and non-uniform polarization. Although the examples are very simple and highly idealized, they exhibit the essential character of the relations which exist in the more complicated and general cases, and certain general conclusions can be drawn which are of importance in the classification to be made presently, of bodies as conductors and dielectrics. When a body is formed of neutral polarizable units whose dimensions are very small compared to the dimensions of $\Delta\tau_i$, it is possible to choose the cells $\Delta\tau_i$ in such a way that $\rho_i = \rho = 0$. The potential due to such a body is then described in terms of the volume density of polarization $\boldsymbol{P}$, as interpolated from the values $\boldsymbol{P}_i = \boldsymbol{p}_i/\Delta\tau_i$. It is natural, and usual, to refer to such a body as a polarized body. In case the polarization of each unit within $\Delta\tau_i$ is the same, the polarization $\boldsymbol{p}_i$ of $\Delta\tau_i$ is equal to the number of units times the polarization per unit. This neutral polarizable unit may be a molecule, or, as in a crystal, simply an arbitrary group of charges which may be viewed as a unit of construction of the crystal structure.

In the case of a body in which there are no such neutral polarized units whose dimensions are small compared to the dimensions of $\Delta\tau_i$, the polarization of $\Delta\tau_i$ has the value which arises from the varying number density of charges across $\Delta\tau_i$. From the statistical regularity of the problem, it is evident that this variation is small, and that this polarization $\boldsymbol{p}_i$ will be canceled (as in example 2) by that polarization which is introduced by the interpolated variation of ρ, so that the values of $\boldsymbol{P}_i$ will vanish. That is to say, when rapid fluctuations in sign are ruled out

(such as would occur, for example, were small neutral polarized units present), the slowly varying continuous density ρ is capable of completely describing the charged condition of the volume cells. For such a body, the problem of representing the potential is quite the same as in the gravitational case. The potential of such a body is thus described in terms of a charge density, and it is natural and usual, to refer to such a body as charged, but unpolarized.

§ 15. *Mutual Electrostatic Energy in Terms of the Continuous Densities.*—It has been seen above that the mutual electrostatic energy of a charge e in the presence of a ponderable body (i.e., the total mutual energy of the whole configuration minus the internal electrostatic energy of the charges forming the body itself) is given by

$$\Psi_e = e\Phi \,, \tag{47}$$

where Φ, the potential due to the charges forming the body, is given, by (41), as a function of x, y, z. Then the force $\boldsymbol{F}_e$ on the charge e is

$$\boldsymbol{F}_e = -\nabla_e \Psi_e = -e\,\nabla\Phi \,, \tag{48}$$

the indicated differentiation being carried out with respect to the coordinates of e. The total force $\boldsymbol{F}$ on a set of charges e_i is then given by

$$\boldsymbol{F} = -\Sigma e_i\,\nabla\Phi_i = -\nabla\Sigma e_i\Phi_i \,, \tag{49}$$

where Φ_i is the value of Φ at the position of e_i. If this set of charges e_i forms a second body (as distinguished from the first body, to which Φ is due), the total force on this second body is thus expressed as the negative nabla of

$$\Psi = \Sigma e_i\Phi_i = \frac{1}{4\pi}\,\Sigma\,\frac{e_i\,e_{i'}}{r_{ii'}} \,, \tag{50}$$

the mutual electrostatic energy of one body in the presence of the other. In this sum the charges $e_{i'}$ are those forming the body 1, and the summation is to be carried out first with respect to i' for a fixed i, and then with respect to i; so that it does not contain the internal electrostatic energy of either body.

It is obviously out of the question actually to compute $\Sigma e_i\Phi_i$ for a ponderable body, and the procedure, as before, is to express this sum in terms of integrals, of continuous functions, over the volume and surface of the body. The charges forming body 2 are divided, by means of volume cells $\Delta\tau_i$ of linear dimensions of the order of l, into subcom-

plexes which satisfy the conditions of statistical regularity and slow variation (25) and (26). Then the sum

$$\Delta\Psi = \sum_{\Delta\tau_i} e_i \Phi_i ,$$

where e_i are now the charges in $\Delta\tau_i$, is to be approximately represented by means of integrals of continuous functions. This problem is, analytically, entirely similar to the one, just considered, which has resulted in definitions of ρ and $\boldsymbol{P}$, the only difference being that in the sum now being considered Φ appears where $1/r$ appeared before. It is thus evident, by direct analogy, that the foregoing sum can be represented by two integrals of the form

$$\int_{\Delta\tau_i} \rho\phi d\tau + \int_{\Delta\tau_i} (\boldsymbol{P}, \nabla\Phi) d\tau ,$$

where ρ and $\boldsymbol{P}$ are continuous functions which, as before, are to be characterized by comparing the terms in the expansions of these two integrals with the terms obtained by expanding directly the sum $\Sigma e_i\Phi_i$. The value of Φ, however, cannot exceed $\Sigma|e|/r_0$, where the summation extends over all the charges of body 1, and where r_0 is the minimum distance from $\Delta\tau_i$ to body 1. Similarly, $\partial\Phi/\partial x$ cannot exceed $\Sigma|e|/r_0^2$, $\partial^2\phi/\partial x^2$ cannot exceed $2\Sigma|e|/r_0^3$, etc. That is, the quantities

$$\Phi , \qquad s\left(\frac{d\Phi}{ds}\right) , \qquad s^2\left(\frac{d^2\Phi}{ds^2}\right) , \ldots ,$$

are of decreasing magnitude, and the ratio of successive terms is of the same order of magnitude as the ratio of successive terms of

$$\frac{1}{r} , \qquad s\left(\frac{d\ 1/r}{ds}\right) , \qquad s^2\left(\frac{d^2\ 1/r}{ds^2}\right) , \ldots .$$

Thus, in characterizing ρ and $\boldsymbol{P}$ by comparing terms in the expansions just mentioned, the same terms can be neglected now as were neglected previously, and the whole computation can be obtained from the previous equations by simply replacing $1/r$ by Φ. The definitions of the functions ρ and $\boldsymbol{P}$ are then the same, in terms of the total charge and polarization of the volume elements of the body 2, as the definitions of ρ and $\boldsymbol{P}$ for body 1. That is, the densities, as defined above, are available for use both in finding the potential due to a body and in finding the

total force on a body. Indeed, the mutual electrostatic energy of the two bodies, from the variation of which the force can be found, is given by

$$\Psi = \int \rho\Phi d\tau + \int (\boldsymbol{P}, \nabla\Phi) d\tau + \int \eta\Phi d\sigma + \int (\boldsymbol{\mu}, \nabla\Phi) d\sigma \,, \tag{51}$$

the integrals being extended over the volume and surface of body 2, and Φ being the potential due to body 1, as given by (41). Note that the variables of integration in these integrals are the unprimed variables x,y,z; and that ρ, η, $\boldsymbol{P}$, $\boldsymbol{\mu}$, and Φ are all functions of x,y,z.

§ 16. *The Force and Torque on a Body.*—The total force on a body can be found by calculating the variation in the mutual electrostatic energy Ψ when the body is shifted but all the charges forming it are maintained in the same relative configuration. That is, it can be obtained by computing the negative nabla of Ψ, holding ρ, η, $\boldsymbol{P}$, and $\boldsymbol{\mu}$ constant and, accordingly, allowing only Φ to vary. To make clear the fact that only Φ is to vary, the equation for the force may be written

$$\boldsymbol{F} = -\nabla_\Phi \Psi \,.$$

Thus,

$$\boldsymbol{F} = \int \rho(-\nabla_\Phi\Phi) d\tau + \int \eta(-\nabla_\Phi\Phi) d\sigma - \int \nabla_\Phi(\boldsymbol{P}, \nabla\Phi) d\tau - \int \nabla_\Phi(\boldsymbol{\mu}, \nabla\Phi) d\sigma \,.$$

The last two integrals can be simplified by means of the identity

$$\nabla_B(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}, \nabla_B)\boldsymbol{B} + [\boldsymbol{A} \text{ curl } \boldsymbol{B}] \,,$$

which, for the special case $\boldsymbol{A} = \boldsymbol{P}$, $\boldsymbol{B} = \nabla\Phi$, reduces to

$$\nabla_\Phi(\boldsymbol{P}, \nabla\phi) = (\boldsymbol{P}, \nabla)\nabla\Phi \,,$$

since

$$\text{curl } \nabla\phi \equiv 0 \,.$$

In the expression $(\boldsymbol{P}, \nabla)\ \nabla\Phi$, it is not necessary to retain a subscript on either operator ∇. Making use of this result, and substituting the electrostatic intensity $\boldsymbol{E}$ for $-\nabla\phi$, the foregoing equation reduces to

$$\boldsymbol{F} = \int \rho\boldsymbol{E} d\tau + \int \eta\boldsymbol{E} d\sigma + \int (\boldsymbol{P}, \nabla)\boldsymbol{E} d\tau + \int (\boldsymbol{\mu}, \nabla)\boldsymbol{E} d\sigma \,. \tag{52}$$

The $\boldsymbol{E}$ vector in this equation is that due to all the bodies except the one on which the force is being determined. The electrostatic intensity $\boldsymbol{E}$ and the densities ρ, η, $\boldsymbol{P}$, $\boldsymbol{\mu}$ are functions of x,y,z, the variables of the integration.

In a similar way the torque on a body may be obtained by calculating the variation in Ψ corresponding to a rotation of the body, the relative configuration of the charges remaining unchanged during the rotation. The x-component of the vector torque $\boldsymbol{T}$ is thus given by

$$T_x = -\frac{\partial \Psi}{\partial \theta},$$

where θ measures an angular displacement of the body about the

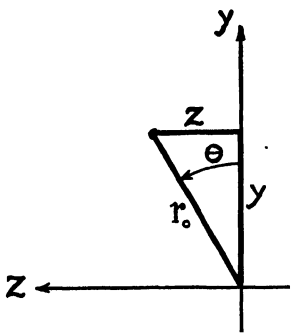

FIG. 16

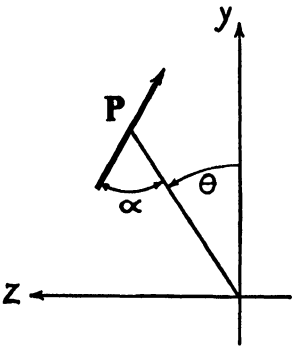

FIG. 17

x-axis. As the body rotates, the y- and z-co-ordinates of a fixed point of the body change according to the equations

$$y = r_0 \cos\theta,$$
$$z = r_0 \sin\theta,$$

so that

$$\frac{\partial}{\partial\theta} = \frac{dy}{d\theta}\frac{\partial}{\partial y} + \frac{dz}{d\theta}\frac{\partial}{\partial z} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Thus

$$-\frac{\partial}{\partial\theta}\int \rho\phi\, d\tau = -\int \rho \frac{\partial\phi}{\partial\theta}\, d\tau = -\int \rho \left| y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y}\right| d\tau,$$

which is the x-component of the vector

$$\int [\boldsymbol{r}, \rho\boldsymbol{E}]\, d\tau,$$

where $\boldsymbol{r}$ is a vector from the origin of the co-ordinates x,y,z to the infinitesimal element $d\tau$. Likewise,

$$-\frac{\partial}{\partial\theta}\int\eta\Phi d\sigma=-\int\eta\left[y\frac{\partial\phi}{\partial z}-z\frac{\partial\phi}{\partial y}\right]d\sigma$$

is the x-component of

$$\int[\boldsymbol{r},\,\eta\boldsymbol{E}]d\sigma\,.$$

Since the relative configuration of the charges is unaltered, the vector $\boldsymbol{P}$, at a fixed point in the body, has a constant magnitude, but a direction which rotates with the body. Thus

$$P_y=P'\cos(\alpha-\theta)\,,$$
$$P_z=-P'\sin(\alpha-\theta)\,,$$

where P' is the component of $\boldsymbol{P}$ in the y–z-plane. Thus

$$\frac{\partial P_y}{\partial\theta}=-P_z\,,$$
$$\frac{\partial P_z}{\partial\theta}=P_y\,.$$

Therefore,

$$-\frac{\partial}{\partial\theta}\int(\boldsymbol{P},\nabla\Phi)d\tau=\int\frac{\partial}{\partial\theta}(\boldsymbol{P},\,\boldsymbol{E})d\tau=\int\left(P_x\frac{\partial E_x}{\partial\theta}+P_y\frac{\partial E_y}{\partial\theta}+E_y\frac{\partial P_y}{\partial\theta}\right.$$
$$\left.+P_z\frac{\partial E_z}{\partial\theta}+E_z\frac{\partial P_z}{\partial\theta}\right)d\tau\,,$$
$$=\int(E_zP_y-E_yP_z)d\tau+\int\left[P_x\left(y\frac{\partial E_x}{\partial z}-z\frac{\partial E_x}{\partial y}\right)+P_y\left(y\frac{\partial E_y}{\partial z}-z\frac{\partial E_y}{\partial y}\right)\right.$$
$$\left.+P_z\left(y\frac{\partial E_z}{\partial z}-z\frac{\partial E_z}{\partial y}\right)\right]d\tau\,.$$

Since $\boldsymbol{E}=-\nabla\Phi$, however, the curl of $\boldsymbol{E}$ vanishes identically, so that

$$\frac{\partial E_z}{\partial y}=\frac{\partial E_y}{\partial z}\,;\quad\frac{\partial E_x}{\partial z}=\frac{\partial E_z}{\partial x}\,;\quad\frac{\partial E_x}{\partial y}=\frac{\partial E_y}{\partial x}\,.$$

Using these equalities, the foregoing integrals reduce to

$$\int (E_z P_y - E_y P_z) d\tau + \int \left[y\left(P_x \frac{\partial E_z}{\partial x} + P_y \frac{\partial E_z}{\partial y} + P_z \frac{\partial E_z}{\partial z} \right) - z\left(P_x \frac{\partial E_y}{\partial x} + P_y \frac{\partial E_y}{\partial y} + P_z \frac{\partial E_y}{\partial z} \right) \right] d\tau ,$$

which is the x-component of the vector

$$\int [\boldsymbol{P}, \boldsymbol{E}] d\tau + \int [\boldsymbol{r}, (\boldsymbol{P}, \nabla)\boldsymbol{E}] d\tau .$$

In the same way it may be seen that

$$-\frac{\partial}{\partial \theta} \int (\boldsymbol{\mu}, \nabla \Phi) d\sigma$$

is the x-component of

$$\int [\boldsymbol{\mu}, \boldsymbol{E}] d\sigma + \int [\boldsymbol{r}, (\boldsymbol{\mu}, \nabla)\boldsymbol{E}] d\sigma .$$

Thus, collecting terms,

$$(53) \quad \boldsymbol{T} = \int [\boldsymbol{r}, \rho \boldsymbol{E}] d\tau + \int [\boldsymbol{r}, \eta \boldsymbol{E}] d\sigma + \int [\boldsymbol{P}, \boldsymbol{E}] d\tau + \int [\boldsymbol{\mu}, \boldsymbol{E}] d\sigma + \int [\boldsymbol{r}, (\boldsymbol{P}, \nabla)\boldsymbol{E}] d\tau + \int [\boldsymbol{r}, (\boldsymbol{\mu}, \nabla)\boldsymbol{E}] d\sigma$$

Here, as in (52), the vector $\boldsymbol{E}$ is that due to all the bodies except the one on which the torque is being determined. The electrostatic intensity $\boldsymbol{E}$, and the densities $\rho, \eta, \boldsymbol{P}, \boldsymbol{\mu}$ are functions of x,y,z, the variables of integration.

§ 17. *The Essential Electrostatic Characteristics of a Ponderable Body.* —This section is concerned with the relations between ρ, η, and $\boldsymbol{P}$, and not at all with the density of superficial polarization $\boldsymbol{\mu}$. If the body under consideration has a polarized surface layer, the density $\boldsymbol{\mu}$ is, of course, an essential electrostatic characteristic of the body which must be considered along with the two other essential characteristics treated in this section.

In § 14 it was noted, in connection with the illustrative examples, that if a body be subdivided and the resulting densities ρ, η, and $\boldsymbol{P}$ be determined, the expression for the potential due to the body, namely,

$$\Phi = \frac{1}{4\pi} \int \frac{\rho}{r} d\tau + \frac{1}{4\pi} \int \frac{\eta}{r} d\sigma + \frac{1}{4\pi} \int \left(\boldsymbol{P}, \nabla \frac{1}{r} \right) d\tau ,$$

can be transformed to read:

$$\Phi=\frac{1}{4\pi}\int\frac{(\rho-\operatorname{div}\boldsymbol{P})}{r}\,d\tau+\frac{1}{4\pi}\int\frac{(\eta+P_n)}{r}\,d\sigma\,.$$

It was furthermore seen, in the examples there considered, that it was possible to subdivide the body in different ways, giving rise to different densities; so that, for example, a body could legitimately be considered to have no volume or surface density of charge and a volume density of polarization, or to be unpolarized, but charged on the surface and throughout the interior. The questions "Is the body polarized?" and "Is the volume density of charge zero?" have, without further specification, no meaning. It was found, however, in the examples considered that the quantities

$$(\rho-\operatorname{div}\boldsymbol{P}) \qquad \text{and} \qquad (\eta+P_n)$$

were independent of the mode of subdivision. The examples thus suggest that these two combined quantities, rather than the three densities ρ, η, $\boldsymbol{P}$ themselves, are the essential measures of the electrostatic characteristics of a body; and that, in any given case, it is possible to subdivide in a way that throws the burden of the description on ρ and η only, on $\boldsymbol{P}$ only, or, as an intermediate step on ρ, η, and $\boldsymbol{P}$, the quantities $(\rho-\operatorname{div}\boldsymbol{P})$ and $(\eta+P_n)$ remaining unchanged throughout.

It would clearly be difficult to argue directly, in a general case, that it is always possible to subdivide a body into volume cells $\Delta\tau_i$ and surface cells $\Delta\tau_j$ for which $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ are, for example, zero, while ρ_i and ρ_j are, in general, not zero; and also possible to subdivide this same body into a set of volume cells $\Delta\tau_i$ for which $\rho_i\equiv 0$, while $\boldsymbol{p}_i\neq 0$. But this direct argument can fortunately be avoided. Suppose that a first division of a body A gives rise to densities ρ_1, η_1, and $\boldsymbol{P}_1$, and thus to the following expression for the potential due to the body:

$$\Phi=\frac{1}{4\pi}\int\frac{(\rho_1-\operatorname{div}\boldsymbol{P}_1)}{r}\,d\tau+\frac{1}{4\pi}\int\frac{(\eta_1+P_{1_n})}{r}\,d\sigma\,.$$

The force on this body, when in an external field $\boldsymbol{E}$, is

$$\boldsymbol{F}_A=\int\rho_1\boldsymbol{E}d\tau+\int\eta_1\boldsymbol{E}d\sigma+\int(\boldsymbol{P}_1,\nabla)\boldsymbol{E}d\tau\,,$$

and the torque is

$$\boldsymbol{F}_A=\int[\boldsymbol{P}_1,\boldsymbol{E}]d\tau+\int[\boldsymbol{r},(\boldsymbol{P}_1,\nabla)\boldsymbol{E}]d\tau+\int[\boldsymbol{r},\rho_1\boldsymbol{E}]d\tau+\int[\boldsymbol{r},\eta_1\boldsymbol{E}]d\sigma\,.$$

These expressions for force and torque have been obtained by calculating appropriate rates of change of the mutual electrostatic energy

$$\Psi = \int \Phi \rho_1 d\tau + \int \Phi \eta_1 d\sigma + \int (\boldsymbol{P}_1, \nabla\Phi) d\tau \ .$$

The last written integral can be transformed by (44) to read

$$\int \Phi P_{1_n} d\sigma - \int \Phi \ \mathrm{div}\ \boldsymbol{P}_1 d\tau \ ,$$

and hence

$$\Psi = \int \Phi(\rho_1 - \mathrm{div}\ \boldsymbol{P}_1) d\tau + \int \Phi(\eta_1 + P_{1_n}) d\sigma \ .$$

If, now, the force and torque be calculated from this last expression for Ψ, the following expressions will obviously be obtained:

$$\boldsymbol{F}_A = \int \boldsymbol{E}(\rho_1 - \mathrm{div}\ \boldsymbol{P}_1) d\tau + \int \boldsymbol{E}(\eta_1 + P_{1_n}) d\sigma \ ,$$
$$\boldsymbol{T}_A = \int [\boldsymbol{r}, \boldsymbol{E}(\rho_1 - (\mathrm{div}\ \boldsymbol{P}_1)] d\tau + \int [\boldsymbol{r}, \boldsymbol{E}(\eta_1 + P_{1_n}) d\sigma \ .$$

Thus the potential due to the body and the force and torque on the body depend only upon the two particular combinations

$$(\rho - \mathrm{div}\ \boldsymbol{P}) \qquad \text{and} \qquad (\eta + P_n) \ .$$

Thus two bodies, of the same size and shape, for which the values of $(\rho - \mathrm{div}\ \boldsymbol{P})$ and the values of $(\eta + P_n)$ are identical, are electrostatically indistinguishable. That is, if a second body B have volume and surface densities ρ_2 and η_2, and a polarization $\boldsymbol{P}_2$, satisfying the relations

$$\rho_2 - \mathrm{div}\ \boldsymbol{P}_2 \equiv \rho_1 - \mathrm{div}\ \boldsymbol{P}_1 \ ,$$
$$\eta_2 + P_{2_n} \equiv \eta_1 + P_{1_n} \ ,$$

then

$$\Phi_A \equiv \Phi_B \ , \qquad \boldsymbol{F}_A \equiv \boldsymbol{F}_B \ , \qquad \boldsymbol{T}_A \equiv \boldsymbol{T}_B \ ,$$

and the two bodies are electrostatically equivalent. Thus, aside from the question as to whether A can be so subdivided as to give rise to densities ρ_2, η_2, $\boldsymbol{P}_2$, it is clear that another distribution of charges, entirely equivalent (to the order of the approximation involved in all this treatment) to the body A, and which may therefore be substituted for A, can be divided so as to result in densities ρ_2, η_2, $\boldsymbol{P}_2$. In particular, either ρ_2 and η_2 or $\boldsymbol{P}_2$, might be zero; so that body A can always be considered to be charged but not polarized, or polarized but not charged. The choice of the description is a matter of viewpoint, and the choice of viewpoint must be made on physical rather than mathematical grounds.

PROBLEMS FOR PART III, CHAPTER I

1. State in detail the argument by which equation (20) follows from the equation just below it.
2. By expanding $1/r$ in the integrand of expression (21) in § 10, prove the statement that this quantity represents, to terms of order $1/r_0^3$, the Coulomb term in the potential due to the complex.
3. See Figure 12 and justify the sign of the fourth term of equation (32).
4. Given a scalar u and a vector A; show that

$$\text{div } (u,A) = u \text{ div } A + (A, \nabla u) .$$

5. Show that div $\nabla u = \nabla^2 u$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
6. Calculate div $u \nabla v$ where u and v are scalar functions.
7. Calculate the values of

$$\iint x^2 \, dy \, dx \qquad \text{and} \qquad \int \frac{x^3 \cos (nx)}{3} \, ds ,$$

the first integral being extended over the area of a circle, and the second around the boundary of the same circle. Explain the result.

8. Integrate x^2 throughout the volume of a sphere, and $[x^3 \cos (nx)]/3$ over the surface of the same sphere.
9. Calculate the divergence of the vectors

$$A = ix + jy + kz ,$$

$$B = i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r} ,$$

$$C = i\frac{x}{r^3} + j\frac{y}{r^3} + k\frac{z}{r^3} .$$

10. Show that div $\frac{r_1}{r^3} = 0$.
11. Calculate the curl of the vectors

$$A = ix + jy + kz ,$$

$$B = i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r} ,$$

$$C = ixz - jyx^2 + kxyz.$$

12. Show that curl $\nabla\Phi \equiv 0$.

13. Show that div curl $\boldsymbol{A} \equiv 0$.

14. In the general formula

$$\int \text{div } \boldsymbol{A} d\tau = \int A_n d\sigma \ ,$$

substitute for $\boldsymbol{A}$ the vectors $u\,\nabla u$, $u\,\nabla v$, $-v\,\nabla u$, where u and v are scalar functions. Add the two last formulas.

15. In the same general formula, substitute for $\boldsymbol{A}$ the vector $\boldsymbol{P}/r$. Cf. with equation (44); cf. § 14.

16. Calculate $\nabla(\boldsymbol{A},\boldsymbol{B})$ and, as a special case, $\nabla_B(\boldsymbol{A},\boldsymbol{B})$.

17. Show that $(\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]) = (\boldsymbol{B}[\boldsymbol{C},\boldsymbol{A}])$.

18. Show that $[\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]] = \boldsymbol{B}(\boldsymbol{A},\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A},\boldsymbol{B})$.

19. What condition is imposed on the vector $iM+jN$ by the fact that $Mdx+Ndy$ is an exact differential?

20. If the total charge of a complex is not zero, there is always some point relative to which the polarization of the complex is zero. Show how to locate this point.

21. Why is it impossible to use the result of the last problem and the argument at the end of § 9 to attempt to show that the Coulomb term is always the leading term in the potential due to a complex whose total charge does not vanish?

Conclusion to Chapter I

Coulomb's law, as experimentally deduced, is essentially a statistical law, applying to the action between complexes of charge. It is consistent with the experimental facts to idealize this law to cover the case of a single pair of charges, so long as this idealized form of the law is used, along with the principle of superposition, only in cases where large numbers of charge are present. There is some evidence that Coulomb's law does hold between a single pair of charges, but it is interesting to note that in the only cases where the direct action between two charges produces an observable effect, namely, in those cases where the charges come close together, question has often arisen concerning the exactness of the inverse square law.

It is hoped that the definitions here given for the electrostatic potential and the electrostatic intensity are such as will emphasize the fact that these functions have physical significance only at points where charge is located, although, considered formally as mathematical functions, their values can of course be calculated at any point in space. The intensity at an "empty" point in space is often defined as the force which would act on a unit charge if it were placed there without disturbing the positions of any of the existing charges. The objection that a unit charge cannot be introduced without, in general, altering the whole configuration is sometimes met by defining the intensity as the limiting value of the force per unit charge which would act on a charge were it located there, the limit being taken as the magnitude of the charge approaches zero. Such a limiting process is clearly incompatible with the modern concept of a minimum indivisible charge—the electron. The present authors object to basing physical theories on definitions which involve what would happen if an impossible thing were true. The most significant trend of modern theoretical physics is the general insistence that theories must be based on the facts of experience. The theory of relativity, for example, grew out of an insistent demand that we substitute, for the old hypothetical simultaneity, a concept of simultaneity based on a physically realizable process. A similar viewpoint is found in the more recent quantum dynamics, where we also find the proposal to discard from physical theories quantities which are not subject to experimental measurement.

The definition of intensity discussed in the preceding paragraph is subject, moreover, to an objection other than that which relates to an impossible limiting process. The point involved is a rather trivial case of a general question which has been of fundamental significance in the development of electrical theories, namely, the general question of the significance of the electric and magnetic vectors at points in empty space. Physics clearly has no legitimate concern with a theory that cannot be tested by experience, and it is equally clear that, at present at any rate, such appeal to the senses as forms the basis of experience can only be effected by material agencies. The authors feel, therefore, that a physical theory has no concern with "conditions" at an empty point in space, and for the precise reason that the point is empty. The English school of physicists have been chiefly responsible for the inclusion of all points of space within the legitimate field of inquiry of a physical theory, this extension being based upon the hypothesis that there is something at an empty point in space—the aether. This hypothetical aether has, however, consistently refused to make any direct appeal to our senses, and it becomes more and more clear that even were there such a medium, its essential nature lies in the fact that it cannot make such an appeal. It would, therefore, seem to furnish physics with no excuse for concerning itself with conditions at points where there is nothing to be conditioned.

This does not at all mean that a mathematical theory of electricity should refuse to consider values of such quantities as electrostatic intensity or electrostatic potential at points where there is no charge. Indeed, the great service which a mathematical theory renders to the physical theory consists of the imbedding, so to speak, of the discrete physical quantities in a mathematical continuum, the physical quantities then appearing as special instances (taken at points where there is charge) of a mathematical continuous function. This mathematical function is found to obey a certain differential equation, and certain boundary conditions, and thus the physical problem is brought under convenient and powerful methods of analysis. The complete lack of physical significance of the mathematical function at points in free space is evident from the fact that it is perfectly conceivable to imbed the physical values in two different mathematical continua, satisfying different differential equations and boundary conditions, but physically equivalent since they lead to the same values at all points where there is charge.

It will perhaps be felt that the continuous densities are introduced with more care and at greater length than is warranted. The authors feel, however, that confusion has existed here which cannot be met by a brief

treatment. A procedure often followed is, in fact, this: In the case of conducting material the volume density ρ (and as a limiting case, the surface density η) is either not defined at all or is defined as a sort of smoothed-over continuous density of such a sort that its volume integral throughout any volume is equal to the actual total charge in this volume. The actual definition of this density ρ is often open to serious criticism. Such a definition, for example, as

$$\rho = \lim_{\Delta\tau \doteq 0} \frac{\Sigma e}{\Delta\tau}$$

does not produce, in the case of a concentrated charge, a continuous (or finite) function ρ; while the concept of an extended charge presupposes the concept of density. Other writers borrow, without examination, the notion of a continuous density from potential theory; but even granting that potential theory has such a concept to lend, the difficulty remains that the electrical problem differs in an essential way from the gravitational problem, since in the former negative as well as positive elements appear.

When dielectrics are treated, they are often idealized into electrostatically equivalent bodies formed of doublets; the volume polarization $\boldsymbol{P}$ is defined (although not always in such a way as to make it a continuous function); and the potential of a dielectric is then expressed, in terms of $\boldsymbol{P}$, as

$$\frac{1}{4\pi}\int \left(\boldsymbol{P}, \nabla \frac{1}{r}\right) d\tau \,.$$

It is then shown that this integral can be transformed to read

$$\frac{1}{4\pi}\int \frac{-\operatorname{div} \boldsymbol{P}}{r}\, d\tau + \frac{1}{4\pi}\int \frac{P_n}{r}\, d\sigma \,,$$

and it appears that the potential is the same "as though" the dielectric had a volume charge of density $-\operatorname{div} \boldsymbol{P}$ and a surface charge P_n. The idealized doublet counterpart of the body is then used to show that the body actually does have such volume and surface densities of charge. All of this, while slightly confusing, is not serious until one asks for the representation of the potential due to a body which it is desirable (for physical reasons as, for example, in the case of a leaky dielectric) to de-

scribe as both polarized and charged. If the densities ρ, η, and $\boldsymbol{P}$ are determined as they were in the separate cases above, do the terms

$$\int \frac{\rho}{r}\, d\tau + \int \frac{\eta}{r}\, d\sigma$$

contain some portion of the term

$$\int \left(\boldsymbol{P}, \nabla \frac{1}{r}\right) d\tau$$

or do they not? That is to say, has there been a double count, certain charges appearing in two rôles?

It is hoped that the relationship between ρ, η, and $\boldsymbol{P}$ is made clear by the treatment given here. The notion that the functions ρ, η, and $\boldsymbol{P}$ result from a particular subdivision of the body leads, without surprise, to the fact that different subdivisions might lead to different functions ρ, η, and $\boldsymbol{P}$. This physical possibility for a shift in the description finds its mathematical counterpart in equation (44), which shows how the volume polarization term can be mathematically changed into volume and surface density terms. The final answer to the question of the possibility of shifting the burden of the description from $\boldsymbol{P}$ on to ρ and η or vice versa comes in § 17. There it is shown that, since these densities are introduced for the specific purpose of obtaining the potential due to a body or the force and torque on a body, the body may be subdivided in various ways that will result in it being judged charged but not polarized, polarized but not charged, etc.; but that the essential electrostatic characteristics ($\rho - \operatorname{div} \boldsymbol{P}$) and ($\eta + P_n$) are independent of the mode of subdivision.

The four examples used to illustrate the physical meaning of the continuous densities are unfortunately long. They are, however, as simple as the actual situation being studied permits. Cases of uniform charge, uniform polarization, varying charge, and varying polarization must be analyzed, and linear variation is the simplest choice. Although these examples are as simple as possible, an added importance results from the fact that they are, at the same time, as complicated as possible. The volume V may be thought of as composing a small part of a body. The assumed conditions of statistical regularity then limit consideration to cases of linear variation; and a superposition of the four examples represents as complex a situation as electrostatics, in its present form, is competent to deal with.

CHAPTER II

THE ELECTROSTATIC PROBLEM FOR CONDUCTORS AND DIELECTRICS

Part I. The Distribution Problem: Conductors

Introduction

The formulation of the preceding chapter enables one to determine the potential due to a body or the force on a body when the distribution of charge is known. The present chapter is primarily concerned with the question of distribution of charge. The obvious mode of attack is to consider the average of all forces acting on charges, and to equate (since an electrostatic condition is by definition one of statistical equilibrium) this sum to zero. The condition thus obtained is the essential characterization of the distribution. In considering all the forces on charge, the theory of the previous chapter is not sufficient, since it applies only to the action of distant groups of charge; whereas the charges within a body are subject to the action of both distant and neighboring charges. A sphere of radius δ is therefore drawn about the charge, the forces on which are under consideration, so as to separate the distant from the neighboring charges. The force due to the charges outside this sphere can be treated by the methods of the last chapter. Specific assumption must, on the other hand, be made concerning the force due to charges within this sphere, the behavior of this force furnishing a discriminating characteristic of various types of materials.

Before assumptions are made, however, concerning the force due to the neighboring charges within the δ-sphere, a function Φ^* is introduced, this being defined, at any interior point, as the potential due to all charges outside a sphere of radius δ drawn about this point. The rate of change of this function Φ^* does not give (at least directly) the force due to the charges without the δ-sphere, since in the variation of Φ^* the deleting sphere moves, and certain charges are removed from consideration while other new ones are added; while in obtaining force from the variation of potential, all charges must be held fixed. It is found that if the densities ρ and $\boldsymbol{P}$ have negligible variation throughout the interior of the δ-sphere, the force due to outside charges is given by $\boldsymbol{E}+\boldsymbol{P}/3$, where $\boldsymbol{E}$ is the negative nabla of a function which is computed, at interior points, from the same formula which, at outside points, gives the electrostatic potential, and where $\boldsymbol{P}$ is the polarization at the point in question.

The total force on an interior charge is thus $\boldsymbol{E}+\boldsymbol{P}/3+\boldsymbol{f}$, where $\boldsymbol{f}$ is

the force due to the charges within δ. The term f is further resolved into that portion f_1 due to the charges forming the few atoms immediately neighboring the point in question, and f_2 due to the remaining charges within δ. A conductor is then defined as a body for which the force f_1, which is characteristic of the detailed structure of the body, is zero. It is argued, on physical grounds, that P and f_2 are both zero. And since the total force vanishes when there is equilibrium, it follows that, at such interior points, $E=0$, or what is equivalent, Φ is constant. The constancy of Φ at interior points is, together with a knowledge of the total charge and of the behavior of Φ at infinitely distant points, sufficient to characterize uniquely the distribution. The first two of these conditions are, however, in integral form and are not convenient to apply; and an equivalent set of equations is obtained ([I], p. 99) in the form of a partial differential equation and boundary conditions. From these conditions Φ may be found, and hence the distribution of charge and the forces and torques may be determined. It is shown that the solution of conditions (I) is unique, so that however a solution be obtained, one is assured that it is the only one and hence that it furnishes a correct solution for the physical problem.

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§ 18. *The Force on an Interior Charge Due to Non-neighboring Charges.* —Equations (52) and (53) above express the total force and torque on a body in terms of the densities ρ, η, P, μ of that body and in terms of the potential due to all charges not located on the body in question. But the integrals involved cannot be evaluated unless the continuous density functions ρ, η, P, μ are known for all bodies present. These densities, depending as they do upon the distribution of charges, are determined by the exceedingly complicated interaction of all the charges present, and a new difficulty arises in connection with these interactions; for each charge of a certain body is affected not only by distant charges on other bodies and distant charges on the body in question, but also by charges in the immediate neighborhood of the one being considered. Now it is evident from the assumptions made when the densities were defined that the expression for Φ in terms of these densities is valid only when the point at which Φ is being calculated is far from all the charges which contribute to the densities ρ, η, P, μ which appear in the formula for Φ. To emphasize this restricted availability of the function Φ, it will be referred to as a "group potential." The group potential due to a group of charges has thus no significance at a point near the group of charges.

The solution of the distribution problem evidently involves a knowledge of the total force on each charge. The force due to all those charges which lie outside a sphere of radius δ about the charge in question (δ large compared to molecular and atomic dimensions) can be expressed in terms of the group-potential function due to these distant charges. The first step in the solution of the distribution problem thus involves a separation of the force on a charge into two parts—one due to the distant charges, and the other due to the neighboring charges within the δ-sphere.

In different types of bodies different assumptions are made concerning the force due to neighboring charges. But before any such separation of the general problem is made, it is possible to discuss, in terms still entirely general, the force due to non-neighboring charges.

Thus, consider a charge which is within a body of volume τ and surface Σ, and let Φ^* be the potential, at $O(x,y,z)$, the position of this charge, due to all the charges which lie outside a sphere of radius δ drawn around O. That is, let

$$(54)\quad 4\pi\Phi^* = \int_{\tau-\delta} \frac{\rho' d\tau'}{r} + \int_{\tau-\delta} \left(\boldsymbol{P}', \nabla' \frac{1}{r}\right) d\tau' + \int_{\Sigma} \frac{\eta' d\sigma'}{r} + \int_{\Sigma} \left(\boldsymbol{\mu}', \nabla' \frac{1}{r}\right) d\sigma' \,.$$

where ρ', η', $\boldsymbol{P}'$, $\boldsymbol{\mu}'$ are functions of the integration variables $x'y'z'$. Although it has direct physical meaning only at a point where a charge is located, this function Φ^* can be calculated at any interior point, but the negative nabla of this scalar function Φ^* is not the intensity $\boldsymbol{E}^*$ due to all charges outside the δ-sphere, because in calculating the intensity from the variation of the potential, all charges contributing to the potential are to be held fixed. In considering the variation of Φ^* from point to point, however, the δ-sphere is carried with the variable point, so that in the process of variation certain charges are removed from consideration, while other new ones are added. Thus

$$\boldsymbol{E}^* = -\nabla^* \Phi^* \,,$$

where ∇^* takes account of the variation from point to point within a fixed δ-sphere of the potential due to the charges outside the sphere. That is,

$$(55)\quad -4\pi \boldsymbol{E}^* = 4\pi \nabla^* \Phi^* = \int_{\tau-\delta} \rho' \nabla \frac{1}{r} d\tau' + \int_{\tau-\delta} \nabla \left(\boldsymbol{P}', \nabla' \frac{1}{r}\right) d\tau' + \int_{\Sigma} \eta' \nabla \frac{1}{r} d\sigma' + \int_{\Sigma} \nabla \left(\boldsymbol{\mu}', \nabla' \frac{1}{r}\right) d\sigma' \,.$$

The analytic distinction between $\nabla\Phi^*$ and $\nabla^*\Phi^*$ will be made clear by considering the variation of an integral of the type

$$u^* = \int_{-\tau} f(x,y,z,x',y',z')d\tau' ,$$

where x,y,z are the co-ordinates of the center of the sphere δ. Then the change in this function, corresponding to an increment Δx in x, arises from two sources: first, from the fact that the parameter x in the integrand has the new value $x+\Delta x$; and, second, from the fact that the integration region has been slightly altered by the shift of the sphere which is to be deleted from τ. The portion of the rate of change of u^* with respect to x which results from the first-named source is obviously

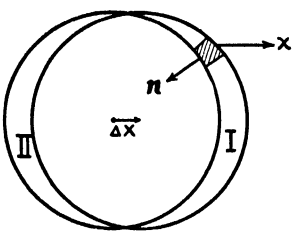

FIG. 18

$$\int_{\tau-\delta} \frac{\partial f}{\partial x} d\tau .$$

On account of the shift in the δ-sphere, on the other hand, elements of volume $\Delta\tau' = d\sigma' \cos(n,x)\Delta x$ are added to the integration region, where $d\sigma'$ is an element of the surface s of the δ-sphere and $\boldsymbol{n}$ is an interior normal to the δ-sphere at $d\sigma'$. The sign of the factor $\cos(nx)$ takes account of the fact that in the shift of the δ-sphere the region I (see Fig. 18) is to be removed from, and the region II added to, the region of integration $\tau-\delta$. Thus the portion of the rate of change of u^* with respect to x which results from the shift of the δ-sphere is

$$\int_s f \cos(nx) d\sigma'$$

so that

$$\frac{\partial u^*}{\partial x} = \int_{\tau-\delta} \frac{\partial f}{\partial x} d\tau' + \int_s f \cos(nx) d\sigma' . \tag{56}$$

Apply this equation to (54). The result, namely,

$$\nabla\Phi^* = \nabla^*\Phi^* + \frac{1}{4\pi}\int_s \frac{\rho'}{r} \boldsymbol{n} d\sigma' + \frac{1}{4\pi}\int_s \left(\boldsymbol{P}', \nabla' \frac{1}{r}\right)\boldsymbol{n} d\sigma' \tag{57}$$

shows that the difference between the two modes of variation indicated by ∇ and ∇^* appears as two integrals taken over the surface s of the δ-sphere.

These two integrals can be simplified in case the physical conditions of the problem permit the assumption that the functions ρ' and $\boldsymbol{P}'$ are so slowly varying across the range of the δ-sphere as to be practically constant. This is a more stringent condition of slow variation than has previously been assumed for these functions, since δ must be large enough to contain many volume cells $\Delta\tau_i$ if the group potential is to be legitimately used for all charges outside δ. Under such an assumption the values ρ' and $\boldsymbol{P}'$ in the foregoing integrals can be replaced by ρ_0 and $\boldsymbol{P}_0$, the values of ρ' and $\boldsymbol{P}'$ at 0†. These integrals then become

$$\frac{\rho_0}{4\pi\delta}\int_s \boldsymbol{n}\,d\sigma' + \frac{1}{4\pi}\int_s \left(\boldsymbol{P}_0, \nabla' \frac{1}{r}\right)\boldsymbol{n}\,d\sigma' ,$$

the first of which is clearly zero. The second can be written

$$-\frac{P_0}{4\pi\delta^2}\int \cos\varphi\ \boldsymbol{n}\ d\sigma' ,$$

where φ is the angle between $\boldsymbol{P}_0$ and a line from O to $d\sigma'$. It is then clear from symmetry that this integral has no component perpendicular to the direction of $\boldsymbol{P}_0$, while the component parallel to $\boldsymbol{P}_0$ is given by

$$+\frac{P_0}{4\pi\delta^2}\int_0^\pi \cos^2\varphi\, 2\pi\delta^2 \sin\varphi\, d\varphi = +\frac{P_0}{3} .$$

The value of the integral is thus $+\frac{1}{3}\boldsymbol{P}_0$, and equation (57) can be rewritten

$$\nabla\Phi^* = \nabla^*\Phi^* + \frac{\boldsymbol{P}_0}{3} ,$$

or

$$\boldsymbol{E}^* = -\nabla\Phi^* + \frac{\boldsymbol{P}}{3} , \tag{58}$$

where $\boldsymbol{E}^*$ and $\boldsymbol{P}$ are both measured at O, and where $\boldsymbol{E}^*$, Φ^*, and $\boldsymbol{P}$ are all functions of x,y,z, the co-ordinates of O.

Now the mathematical function $\boldsymbol{E}^*$ defined by (55) is, when ρ' and $\boldsymbol{P}'$ are sensibly constant over the sphere, independent of the size δ. For two different values of $\boldsymbol{E}^*$, corresponding to two different choices of

† Note that this makes ρ_0 and $\boldsymbol{P}_0$ functions of x,y,z, the co-ordinates of O.

δ, differ by the intensity due to the uniformly charged and uniformly polarized spherical shell contained between the two δ-spheres in question; and the intensity due to this completely symmetrical distribution is zero.† It is customary to take advantage of this mathematical fact by letting δ approach zero. Thus

$$\boldsymbol{E}^* = \lim_{\delta \doteq 0} \boldsymbol{E}^* = -\lim_{\delta \doteq 0} \nabla\Phi^* + \lim_{\delta = 0} \frac{\boldsymbol{P}}{3},$$

or

$$\boldsymbol{E}^* = -\lim_{\delta \doteq 0} \nabla\Phi^* + \frac{\boldsymbol{P}}{3}. \tag{59}$$

It should be carefully realized that allowing δ to approach zero is a purely mathematical device, and in no way affects the fact that $\boldsymbol{E}^*$ is still the force due to the non-neighboring charges. The advantage, referred to above, of this last equation over (58) depends upon the fact that it is possible to prove that the first term of the right member of (59) is equal to the negative nabla of a function Φ given by

$$\Phi = \lim_{\delta \doteq 0} \Phi^*, \tag{60}$$

i.e., a function which is the analytical extension, for points within a body, of the same function which gives, at exterior points, the potential due to the body. The proof just mentioned will now be given.

The function Φ^* may be written

$$\Phi^* = \Phi_1^* + \Phi_2^* + \Phi_3^* + \Phi_4^*,$$

the four terms Φ_1^*, Φ_2^*, Φ_3^*, and Φ_4^* being abbreviations for the four integrals on the right side of (54). Since Φ_3^* and Φ_4^* are regular integrals which do not depend in any way on the value of δ, it is clear that only Φ_1^* and Φ_2^* require consideration.

It follows from (56) that

$$\frac{\partial \Phi_1^*}{\partial x} = \frac{1}{4\pi}\int_{\tau-} \rho' \frac{\partial \frac{1}{r}}{\partial x}\, d\tau' + \frac{1}{4\pi}\int_{s} \rho' \frac{\cos(nx)}{r}\, d\sigma',$$

$$= \frac{1}{4\pi}\int_{\tau-\delta} \rho' \frac{\partial \frac{1}{r}}{\partial x}\, d\tau',$$

† This portion of this statement which refers to polarization may not seem obvious. It is easy to check, by direct calculation, that the intensity at the center of a uniformly polarized spherical shell is zero. See Part I, Problem 9, of this chapter

the last step depending upon the fact that the surface integral over s has been shown to be zero when ρ' is constant. Then

$$\lim_{\delta=0} \frac{\partial \Phi_1^*}{\partial x} = \frac{1}{4\pi} \lim_{\delta=0} \int_{\tau-\delta} \rho' \frac{\partial \frac{1}{r}}{\partial x} d\tau' = \frac{1}{4\pi} \int_{\tau} \rho' \frac{\partial \frac{1}{r}}{\partial x} d\tau' ,$$

provided the integral on the right exists; indeed, this equation is the definition of the improper integral on the right. That the improper integral does exist is made evident by writing $d\tau' = r^2 dr\, d\omega$, where $d\omega$ is an element of solid angle; for, when the element of volume is thus written, the singularity is removed. Then the fact that

$$\int_{\tau} \rho' \frac{\partial \frac{1}{r}}{\partial x} d\tau' = \frac{\partial}{\partial x} \int_{\tau} \frac{\rho'}{r} d\tau'$$

depends upon the theorem that an improper integral may be differentiated with respect to a parameter under the sign of integration, provided the resulting integral is uniformly convergent with respect to the parameter. The uniform convergence with respect to x follows at once from the remarks, made above, concerning the existence of the integral. It is therefore concluded that

$$\lim_{\delta=0} \frac{\partial \Phi_1^*}{\partial x} = \frac{\partial \Phi_1}{\partial x} .$$

The function Φ_2^* requires slightly different treatment, owing to the fact that the integrand of the improper integral,

$$\Phi_2^* = \frac{1}{4\pi} \int_{\tau} \left(\boldsymbol{P}', \nabla' \frac{1}{r} \right) d\tau' ,$$

becomes infinite at $r=0$ as $1/r^2$, rather than as $1/r$. This difficulty can be avoided by the procedure (familiar from the theory of the Newtonian potential) of reducing the order of the singularity by an integration by parts carried out before δ is allowed to approach zero, i.e., before the integral becomes improper. Thus

$$\Phi_2^* = \frac{1}{4\pi} \int_{\tau-\delta} \left(\boldsymbol{P}', \nabla' \frac{1}{r} \right) d\tau' = \frac{1}{4\pi} \int_{\tau-\delta} \frac{-\operatorname{div}' \boldsymbol{P}'}{r} d\tau' + \frac{1}{4\pi} \int_{\Sigma} \frac{P_n'}{r} d\sigma' + \frac{1}{4\pi} \int_{s} \frac{P_n}{r} d\sigma'$$

the integration by parts being based upon the identity.

$$\text{(61)} \qquad \operatorname{div}'\left(\frac{\boldsymbol{P}'}{r}\right)=\frac{\operatorname{div}'\boldsymbol{P}'}{r}+\left(\boldsymbol{P}', \nabla'\frac{1}{r}\right).$$

The function Φ_2^* can therefore be written as the sum of three integrals, the first of which is entirely analogous to Φ_1^*, and the second of which is entirely analogous to Φ_3^*. It is therefore evident that

$$\text{(62)} \qquad \lim_{\delta \doteq 0}\frac{\partial \Phi_2^*}{\partial x}=\frac{\partial \Phi_2}{\partial x},$$

provided

$$\lim_{\delta \doteq 0}\frac{\partial}{\partial x}\int_s \frac{P_n'}{r}\,d\sigma'=\frac{\partial}{\partial x}\lim_{\delta \doteq 0}\int_s \frac{P_n'}{r}\,d\sigma'.$$

But since $\boldsymbol{P}'$ is constant over s, the integral

$$\int_s \frac{P_n}{r}\,d\sigma'$$

is zero from symmetry, so that the last equation is satisfied.

It has been shown, therefore, that

$$\lim_{\delta \doteq 0}\nabla\Phi^*=\nabla\,\Phi,$$

where

$$\Phi=\lim_{\delta=0}\Phi^*.$$

Equation (59) may thus be written

$$\text{(63)} \qquad \boldsymbol{E}^*=\boldsymbol{E}+\frac{\boldsymbol{P}}{3},$$

where

$$\text{(64)} \qquad \boldsymbol{E}=-\nabla\Phi.$$

In these last two equations all quantities written are functions of the variables x,y,z.

§ 19. *Definition of a Conductor.*—The expression for the force on a charge due to the non-neighboring charges has just been obtained. The distribution of charges is determined, however, by the total force on

each charge, so that it is now necessary to discuss the force due to the charges within the δ-sphere. This force is affected by the microscopic details of construction of the body, and is different for different types of bodies.

Bodies are classified, as regards their electrostatic behavior, into two groups: conductors and dielectrics. A conductor is a body whose atoms are so constructed and combined that electrons are free to wander about in the body, moving from atom to atom. In other words, those forces of constitution, operative over ranges of atomic dimensions, which are characteristic of the detailed structure of the body are such that the electrons behave, for purposes of electrostatics, as if they were in neutral equilibrium under these forces. When the condition of a body is that of electrostatic equilibrium, the total force on each charge may be said to be zero; and when a conductor is placed under the electrostatic influence of another body, the charges of the conductor will be acted on by forces in excess of the normal forces of constitution, and the charges will shift about into a new equilibrium configuration, i.e., into such new positions that the total force on each charge again vanishes. An idealization of the actual physical problem is evidently involved in the statement that the total force on each charge vanishes when electrostatic equilibrium prevails. It is of course recognized that a conductor is never, microscopically, in equilibrium. The charges are moving about, but with a random motion that does not cause any drift of charge from one volume element to another. A microscopic motion of charge which does not affect the constant value of the total charge ϵ_i and the polarization $\boldsymbol{p}_i$ of each volume cell $\Delta\tau_i$ will not affect the densities ρ, η, $\boldsymbol{P}$, μ, and is therefore of no consequence as regards the electrostatic behavior of the body. Each charge can thus be thought of as stationary at an effective position, and as acted upon by zero force.

Now it was observed, when the polarization $\boldsymbol{p}_i$ of a volume cell $\Delta\tau_i$ was discussed, that this polarization is contributed to meagerly by the slow variation across $\Delta\tau_i$ of the number density of charges; but that the major portion of $\boldsymbol{p}_i$, in a case where $\boldsymbol{p}_i$ is large enough to be important, arises from the existence within $\Delta\tau_i$, of neutral but polarized units. Such units might be, for example, polarized atoms or molecules. But such units can exist only if there is, within each unit, some restraint which allows a small differential shift of the positive and negative charges, but which prevents these charges from wandering away from the unit in question. A conductor is, by definition, a body in which such restraints do not exist, so that the polarization $\boldsymbol{p}_i$ of a volume cell in a

conductor arises solely from the slow variation across the cell of the number density of charges. It has been assumed above, however, that the density ρ is sensibly constant not only across a cell, but also across the many cells contained in a sphere of radius δ. It is thus evident that the volume polarization term should play no rôle in the discussion of the electrostatic behavior of an idealized conductor, so that the force on an interior charge, due to the non-neighboring charges, is, from equation (64), given by

$$\boldsymbol{E}^* = \boldsymbol{E} = -\nabla\Phi \,. \tag{65}$$

The force due to the charges within the δ-sphere will be denoted by f, and will be divided into that part f_1 due to the charges within a few molecular diameters of the center of the sphere, and the part f_2 due to all the other charges in the δ-sphere. The present state of knowledge concerning the forces acting between charges very near one another, and concerning the detailed construction of a body, makes impossible any analysis of the forces f_1 and f_2, and, in general, assumptions must be made concerning each of these portions of the total force. However, the charges whose distances from the one in question are of the order of atomic dimensions give rise to the above-mentioned force of constitution, and in the case of a conductor, this force is zero. Thus the force f_1 vanishes by virtue of the definition of a conductor. There remains for consideration the force f_2 due to those charges which are within the δ-sphere, but whose distances from the center of the sphere is greater than the range of the intra-molecular forces of constitution. It is assumed that this force f_2 is also zero. This assumption is a natural one in view of the fact that the density ρ is practically constant over the interior of the δ-sphere. Under any law of force which depends upon the distance between charges, and acts along the line joining them, the force on a charge due to a symmetrical spherical distribution about it would be zero. The charges within the δ-sphere do not necessarily form an absolutely symmetrical configuration, but they do give rise to a density ρ which is completely symmetrical. The foregoing assumption is thus based upon the reasonable notion that if the charges give rise to a completely symmetrical density, they do not themselves sensibly deviate from a completely symmetrical distribution. Those slight deviations from complete symmetry which exist in the immediate neighborhood of the charge in question would be the most likely to give rise to a non-vanishing force, but these immediately neighboring charges have already been disposed of when considering the force f_1.

If, then, the total force on a charge, and the force due to all the charges within the δ-sphere are each, in the case of a conductor, equal to zero, it follows that their difference, i.e., the force due to the non-neighboring charges, must equal zero. A conductor is thus characterized by the equation

$$\boldsymbol{E}^* = \boldsymbol{E} = -\nabla\Phi = 0 \,, \tag{66}$$

which holds at all interior points. It follows directly from this equation that the function Φ must be constant throughout the interior of a conductor.

§ 20. *The Surface Polarization of a Conductor.*—The surface atoms of a body are, unlike those in the interior, under conditions which are essentially one sided. It is natural, therefore, that a charge which is very near the surface of a body be subjected to restraints quite unlike those which govern interior charges. And it is natural that the "one-sided" electrical force, which acts on charges near the surface, produce a relative shift of the unlike charges, and hence a polarization $\boldsymbol{p}_j$ of the thin volume cells $\Delta\tau_j$ which lie along the surface of the body. It should be noted that the surface polarization now being discussed is one which results not from any special charged condition of the body, nor from its being under the influence of any other charged body, but simply from the inherent one-sidedness of the conditions that obtain at the surface. That is, it is an intrinsic surface polarization, characteristic of the electrical structure of the body in question. At a point on the surface the conditions are asymmetrical with respect to the tangent plane, but symmetrical with respect to the normal. The intrinsic polarization is thus taken as directed along the normal, and, since it is determined only by the characteristic structure of the body, it is assumed to be constant in magnitude. It has been noted that an intrinsic surface polarization is to be expected on the basis of the Rutherford atom, and quantitative deductions have been made on the basis of this model*

It is customary to assume that, in the case of a conductor, there is no surface polarization other than the intrinsic surface polarization. This assumption is a reasonable one, since the forces due to distant charges are surely small compared to the intra-atomic forces of constitution. When these intra-atomic forces are in balance, as they are in the interior of a conductor, the forces due to distant charges may produce observable results; but when the intra-atomic forces are out of balance, as

* J. Frenkel, *Philosophical Magazine*, XXXIII (April, 1917), 297–322.

they are near the surface of a body, a possible polarization produced by the small external forces would be negligible compared to the intrinsic polarization produced by the much larger forces of constitution.

The portion of the potential Φ which corresponds to this constant intrinsic surface polarization is given by

$$\frac{1}{4\pi}\int\left(\mu',\nabla'\frac{1}{r}\right)d\sigma' = \frac{\mu}{4\pi}\int\frac{\cos\theta\, d\sigma'}{r^2}$$

where θ is the angle between μ' and the direction from $d\sigma'$ to P, of co-ordinates x,y,z, the point at which the potential is being measured. But $\cos\theta\, d\sigma'$ is numerically equal to the projection of the area $d\sigma'$ on a plane normal to the direction of r, so that

$$\frac{\cos\theta\, d\sigma'}{r^2} = \pm d\omega\ ,$$

where $d\omega$ is the solid angle subtended by $d\sigma'$ at the point where the potential is being calculated, the upper or lower sign being used according as the angle between μ' and the direction from $d\sigma'$ to P is acute or obtuse. The foregoing integral may thus be written in the form

$$\frac{\mu}{4\pi}\int(\pm d\omega)\ .$$

If this integral be extended over the surface of a body, and if the point at which the potential is being measured be outside the body, a line having the direction of r cuts the surface at an even number of points, at half of which $d\omega$ is positive, and at half of which $d\omega$ is negative. Thus the potential due to this normal surface polarization of constant magnitude is, at all points outside the surface, equal to zero. If, on the other hand, the point at which the potential is being measured is inside the surface, a line having the direction of r cuts the surface at an odd number of points; say $2n+1$ points. At $2n$ of these points the values of $d\omega$ cancel, while at the remaining point $d\omega$ is positive or negative, according as μ points along the interior or exterior normal to the surface. Thus the potential due to this polarized layer is, at interior points, given by

$$\pm\frac{\mu}{4\pi}\int d\omega = \pm\mu\ .$$

To avoid carrying the double sign, it can be assumed that the polarization is directed along a certain one of the normals, say the interior normal.

Then the magnitude itself is to be considered negative if the polarization is actually oppositely directed. The potential at interior points would then be given always by $+\mu$.

It should be noted that the potential due to this polarized layer is discontinuous across the surface, the constant "inside" value differing from the constant "outside" value by the amount μ. In the more general case of a variable polarized surface layer, the potential has this same discontinuity characteristic, i.e., the value which Φ approaches as the surface is approached from without differs from the corresponding inside limit by an amount μ, where μ is the magnitude of the surface density of polarization at the point approached.*

§ 21. *The Distribution Problem for Conductors.*—It follows from the defining characteristic of a conductor, discussed above, that the charge on a conductor is distributed in such a way as to satisfy the two equations:

$$\int \rho \frac{d\tau}{r} + \int \eta \frac{d\sigma}{r} = \text{Constant at interior points,} \tag{67}$$

$$\int \rho \, d\tau + \int \eta \, d\sigma = \text{Total charge on conductor.}$$

The functions ρ and η are to be determined from these simultaneous integral equations. In the special case of a spherical conductor of radius a and of total charge ϵ, the symmetry of the problem aids one in guessing possible functions ρ and η which will satisfy these equations. For example, if the volume density of charge ρ be zero, while the surface density be uniform and given by $\epsilon/4\pi a^2$, the second of these equations is obviously satisfied. Moreover, the value of the function

$$\int \eta \frac{d\sigma}{r} = \frac{\epsilon}{4\pi a^2} \int \frac{d\sigma}{r}$$

is constant within the sphere. In fact, let P be a point within the sphere whose distance from the center is b, and let θ be the angle between the radius through P and any other radius. Then the surface area

$$2\pi a^2 \sin\theta \, d\theta$$

lying between cones of generating angles θ and $\theta + d\theta$ may be chosen for the surface element $d\sigma$, and the integral may be written

$$\frac{\epsilon}{2} \int^{\pi} \frac{\sin\theta \, d\theta}{r}.$$

* See, e.g., Poincaré, *Théorie du Potentiel Newtonien*, pp. 218 ff.

If the variable of integration be now changed, according to the scheme

$$r^2 = a^2 + b^2 - 2ab\cos\theta\,,$$

$$r\,dr = ab\sin\theta\,d\theta\,,$$

the value of the integral is at once seen to be independent of the position within the sphere of P. In fact,

$$\frac{\epsilon}{2ab}\int_{a-b}^{a+b} dr = \frac{\epsilon}{a}\,.$$

It is obvious, however, that in a less simple case the determination of the functions ρ and η from equations (67) would offer great practical difficulties, owing to the fact that such pairs of integral equations have not received much study. The method of procedure, by means of which the problem is reduced to more familiar analytic form, is a common one, often used in evaluating difficult integrals. Consider, for example, the definite integral

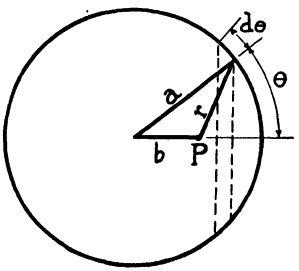

FIG. 19

$$u = \int_0^\infty e^{-x^2 - \frac{a^2}{x^2}}\,dx\,.$$

This integral can be evaluated by differentiating under the sign, with respect to the parameter a, and by forming a differential equation which u satisfies. Thus

$$\frac{du}{da} = -2\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,\frac{a\,dx}{x^2} = -2\int_0^\infty e^{-z^2-\frac{a^2}{z^2}}\,dz\,,$$

the last step resulting from the substitution $x = \frac{a}{z}$. Then

$$\frac{du}{da} = -2u\,,$$

and hence

$$u = Ce^{-2a}\,.$$

When $a = 0$, u reduces to the probability integral

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}\,,$$

so that the constant C equals $\sqrt{\pi}/2$, and the value of the definite integral is

$$u=\frac{\sqrt{\pi}}{2}e^{-2a}.$$

A similar procedure will be followed here. The integrals

$$\Phi=\frac{1}{4\pi}\int\rho'\frac{d\tau'}{r}+\frac{1}{4\pi}\int\eta'\frac{d\sigma'}{r}+\frac{1}{4\pi}\int\mu'\frac{\partial\frac{1}{r}}{\partial n}d\sigma'$$

will be differentiated with respect to the co-ordinates x,y,z of the point at which Φ is being measured, and a differential equation for Φ will be derived. This differential equation, together with boundary conditions, will then be shown to have a unique solution, from which the densities ρ' and η' can be deduced.

The differential equation for Φ, just referred to, is obtained, as a matter of fact, by computing the second rather than the first derivatives of Φ. The integrals

$$\frac{1}{4\pi}\int\eta'\frac{d\sigma'}{r}+\frac{1}{4\pi}\int\mu'\frac{\partial\frac{1}{r}}{\partial n}d\sigma'$$

are regular, and can be differentiated with respect to x, y, or z under the sign. Thus:

$$\frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\int\eta'\frac{d\sigma'}{r}+\frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\int\mu'\frac{\partial\frac{1}{r}}{\partial n}d\sigma'=\frac{1}{4\pi}\int\eta'\frac{\partial^2\frac{1}{r}}{\partial x^2}d\sigma'$$
$$+\frac{1}{4\pi}\int\mu'\frac{\partial}{\partial n}\frac{\partial^2\frac{1}{r}}{\partial x^2}d\sigma$$

If these terms be added to the corresponding second partials with respect to y and z, the result is*

$$\nabla^2\frac{1}{4\pi}\int\eta'\frac{d\sigma'}{r}+\nabla^2\frac{1}{4\pi}\int\mu'\frac{\partial\frac{1}{r}}{\partial n}d\sigma'=\frac{1}{4\pi}\int\eta'\nabla^2\frac{1}{r}d\sigma'$$
$$+\frac{1}{4\pi}\int\mu'\frac{\partial}{\partial n}\nabla^2\frac{1}{r}d\sigma'=0$$

*See Appendix, § 4, (41), for the operator $\nabla^2\equiv$"nabla square."

since

$$\nabla^2 \frac{1}{r} = 0 .$$

If the point x,y,z is a point in empty space, i.e., a point which is not within some body, the integral

$$\frac{1}{4\pi} \int \rho' \frac{d\tau'}{r}$$

is also regular, and

$$\nabla^2 \frac{1}{4\pi} \int \rho' \frac{d\tau'}{r} = \frac{1}{4\pi} \int \rho' \nabla^2 \frac{1}{r} d\tau' = 0 .$$

Therefore at all points in empty space the function Φ satisfies the linear partial-differential equation

$$\nabla^2 \Phi = 0 .$$

At points within a conductor the integral

$$\Phi_1 = \frac{1}{4\pi} \int \rho' \frac{d\tau'}{r}$$

is improper, and care must be used in carrying out the differentiation. It has already been seen that an improper integral of the type

$$\int \frac{f}{r} d\tau'$$

can be differentiated, with respect to x, y, or z, under the sign. Thus:*

$$\frac{\partial \Phi_1}{\partial x} = \frac{1}{4\pi} \lim_{\epsilon \doteq 0} \left\{ \int_{\tau - \epsilon} \rho' \frac{\partial \frac{1}{r}}{\partial x} d\tau' \right\} = -\frac{1}{4\pi} \lim_{\epsilon \doteq 0} \left\{ \int_{\tau - \epsilon} \rho' \frac{\partial \frac{1}{r}}{\partial x'} d\tau' \right\} ,$$

the integral being carried out over all of the body τ which is exterior to

* One should distinguish carefully between this equation and equation (56). The sphere, of radius ϵ, here under consideration has a fixed center.

a sphere, of radius ϵ, drawn about the point x,y,z. If the expression within the bracket be treated by integration by parts, the result is

$$\frac{\partial\Phi_1}{\partial x}=\frac{1}{4\pi}\lim_{\epsilon\doteq 0}\left\{\int_{\tau-\epsilon}\frac{\partial\rho'}{\partial x'}\frac{1}{r}\,d\tau'-\int_S\frac{\rho'\cos(nx')}{r}\,d\sigma'-\int_\Sigma\frac{\rho'\cos(nx')}{r}\,d\sigma'\right\}$$

$$=\frac{1}{4\pi}\lim_{\epsilon\doteq 0}\left\{\int_{\tau-\epsilon}\frac{\partial\rho'}{\partial x'}\frac{1}{r}\,d\tau'-\int_\Sigma\frac{\rho'\cos(nx')}{r}\,d\sigma'\right\},$$

where Σ is the surface of the body τ, and where S is the surface of the ϵ-sphere, the limit of the integral over S being clearly zero.

Both these integrals again satisfy the conditions that permit differentiation under the sign; thus

$$\frac{\partial^2\Phi_1}{\partial x^2}=-\frac{1}{4\pi}\lim_{\epsilon\doteq 0}\left\{\int_{\tau-\epsilon}\frac{\partial\rho'}{\partial x'}\frac{\partial\frac{1}{r}}{\partial x'}\,d\tau'-\int_\Sigma\frac{\partial\frac{1}{r}}{\partial x'}\rho'\cos(nx')d\sigma'\right\},$$

or, integrating by parts again,

$$=\frac{1}{4\pi}\lim_{\epsilon\doteq 0}\left\{\int_{\tau-\epsilon}\rho'\frac{\partial^2\frac{1}{r}}{\partial x'^2}\,d\tau'-\int_S\rho'\frac{\partial\frac{1}{r}}{\partial x'}\cos(nx')d\sigma'\right\}$$

If this expression be added to the corresponding second partials with respect to y and z, the result is

$$\nabla^2\Phi=\frac{1}{4\pi}\lim_{\epsilon\doteq 0}\left\{\int_{\tau-\epsilon}\rho'\nabla'^2\frac{1}{r}\,d\tau'-\int_S\rho'\left[\frac{\partial\frac{1}{r}}{\partial x'}\cos(nx')+\frac{\partial\frac{1}{r}}{\partial y'}\cos(ny')+\frac{\partial\frac{1}{r}}{\partial z'}\cos(nz')\right]d\sigma'\right\}$$

But

$$\nabla'^2\frac{1}{r}=0,$$

and

$$\frac{\partial\frac{1}{r}}{\partial x'}\cos(nx')+\frac{\partial\frac{1}{r}}{\partial y'}\cos(ny')+\frac{\partial\frac{1}{r}}{\partial z'}\cos(nz')=\frac{\partial\frac{1}{r}}{\partial n}=-\frac{\partial\frac{1}{r}}{\partial r}=\frac{1}{r^2}.$$

Therefore,

$$\nabla^2\Phi = -\frac{1}{4\pi}\lim_{\epsilon=0}\int_S \frac{\rho'}{r^2}\,d\sigma' .$$

By the theorem of the mean for integrals, this reduces to

$$\nabla^2\Phi = -\frac{1}{4\pi}\lim_{\epsilon=0}\frac{\bar{\rho}}{\epsilon^2}\int d\sigma' = -\lim_{\epsilon=0}\bar{\rho} ,$$

where $\bar{\rho}$ is the value of ρ at some point on the sphere of radius ϵ. Thus

$$\nabla^2\Phi = -\operatorname{div} \boldsymbol{E} = -\rho , \tag{68}$$

where the value of ρ is taken at the point x,y,z. Since, however, the potential is known to be constant at points within a conductor, and since $\nabla^2\Phi$ is therefore zero at all interior points, it follows that the volume density of charge must be zero, so that, in the electrostatic problem for conductors, the potential function Φ satisfies, at every point in space, the differential equation

$$\nabla^2\Phi = 0 . \tag{69}$$

The variation, from point to point, of the potential Φ is governed by this equation. The total function Φ is a sort of "patchwork" function, built out of parts which are constant within conductors, and the part which varies from point to point in empty space. A vast number of functions vary in the way prescribed by the foregoing equation, and to pick out the one which, in a given problem, will fit properly on to the constant portions within the conductors, it is necessary to investigate the continuity, across the boundaries of the conductors, of the potential Φ and its derivatives.

It follows from the remark just above concerning the vanishing of ρ that the potential is given by the two terms

$$\Phi = \frac{1}{4\pi}\int \eta' \frac{d\sigma'}{r} + \frac{1}{4\pi}\int \mu' \frac{\partial \frac{1}{r}}{\partial n}\,d\sigma' .$$

The second term, namely, that due to the intrinsic surface polarization, has been shown to be zero at all exterior points. In determining Φ at such points this term may therefore be disregarded, but it should be remembered, in connection with the boundary conditions, that the portion

of Φ due to this term is discontinuous across the surface of a conductor, the discontinuity having the constant value μ.

The first term is continuous across the surface of a conductor. It is, in fact, clear that any discontinuity in this function can arise only from the integral over a very small region of the surface which contains the point at which the boundary is to be crossed. A sufficiently small portion of the surface may be regarded as a plane surface over which η is sensibly constant. It is thus sufficient to consider the behavior in the potential due to a circular disk with a constant surface density η.* The potential due to such a disk, at a point on the axis whose distance from the disk is x, is given by

$$\frac{\eta}{4\pi}\int_0^a \frac{2\pi\, y\, dy}{\sqrt{x^2+y^2}} = \frac{\eta}{2}\left[\sqrt{x^2+a^2} - \sqrt{x^2}\right],$$

where a is the radius of the disk. As x approaches zero, either in the negative or the positive direction, this expression approaches the value $a\eta/2$, so that the potential is continuous across the disk, and hence across the surface. Since this term is continuous across the surface of a conductor, and since the intrinsic polarization term has the constant discontinuity μ, it follows that the potential Φ, at points outside a conductor, must approach, as the surface of the conductor is approached, a constant value equal to Φ_i, so that $\Phi_i+\mu_i$ is the constant interior value of the potential of the ith conductor, where μ_i is the magnitude of the intrinsic polarization.

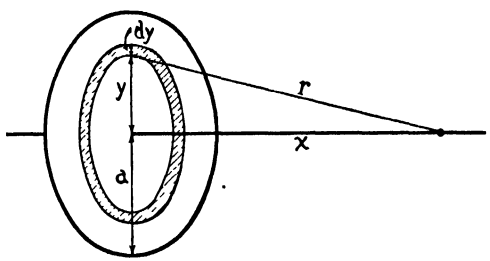

FIG. 20.—The potential at points on the axis of a circular disk.

Although the potential due to this circular disk is continuous, the normal derivative of the potential is discontinuous. Indeed, from the foregoing expression

$$\frac{\partial}{\partial x}\frac{\eta}{2}\left[\sqrt{x^2+a^2} - \sqrt{x^2}\right] = \frac{\eta}{2}\left[\frac{x}{\sqrt{x^2+a^2}} - \frac{x}{\sqrt{x^2}}\right],$$

* For a detailed and rigorous treatment, see Poincaré, *op. cit.*, pp 92 ff.

so that

$$\lim_{x \doteq 0} \frac{\partial \Phi}{\partial x} = -\frac{\eta}{2}, \qquad x>0$$

$$\lim_{x \doteq 0} \frac{\partial \Phi}{\partial x} = +\frac{\eta}{2}, \qquad x<0 .$$

That the values of $\partial\Phi/\partial x$ just below and just above the disk must be different is obvious physically. If, for example, the surface density η be positive, $-\partial\Phi/\partial x$ just above the disk is the upward intensity, which would be clearly positive; while $-\partial\Phi/\partial x$ just below the disk is again the upward intensity, which would clearly be negative at such a point. Then if $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are oppositely directed normals to the surface at the point in question,

$$\left(\frac{\partial \Phi}{\partial n_1}\right)_1 + \left(\frac{\partial \Phi}{\partial n_2}\right)_2 = -\eta ,$$

where, for example, $(\partial\Phi/\partial n_1)_1$ means the limit of $\partial\Phi/\partial n_1$ as the point at which the derivative is taken approaches the surface from the side into which $\boldsymbol{n}_1$ points. It is customary to abbreviate this notation and write simply

$$\frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = -\eta , \tag{70}$$

the meaning, of course, being the same as before. Although this equation has been obtained by considering only the potential due to the surface charge η, it is true if the potential due to the intrinsic polarized layer be included in Φ, for the normal derivatives of this latter contribution are both zero. This equation can be further simplified on account of the fact that the potential is constant within a conductor. Thus if $\boldsymbol{n}_1$ is an interior normal, and $\boldsymbol{n}_2 = \boldsymbol{n}$ an exterior normal,

$$\frac{\partial \Phi}{\partial n_1} = 0 ,$$

so that*

$$\frac{\partial \Phi}{\partial n_2} = \frac{\partial \Phi}{\partial n} = -\eta . \tag{71}$$

* The density η has been considered a function of x',y',z', while Φ is a function of x,y,z. This equation means, as its derivation clearly indicates, that the value of η at x',y',z' is equal to the values of $\partial\Phi/\partial n$ at $x=x'$, $y=y'$, $z=z'$. Thus when $\partial\Phi/\partial n$ is substituted for η, x,y,z in $\partial\Phi/\partial n$ must be replaced by x',y',z'; and vice versa

A still further specializing characteristic of the potential Φ may be obtained by considering the nature of the limiting value of this function as the distance from any body to the point at which Φ is measured increases indefinitely. It is supposed that all the bodies under considera-

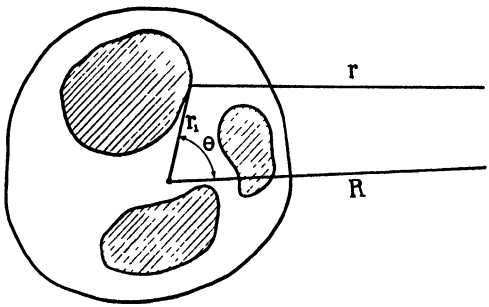

FIG. 21

tion can be contained within some finite volume. Then at any point without this volume

$$\Phi = \frac{1}{4\pi}\int \frac{\eta' d\sigma'}{r} ,$$

the integral being extended over the surface of all conductors. This integral may be re-written as

$$\Phi = \frac{1}{4\pi}\int \frac{\eta'}{(R^2+r_1^2-2Rr_1\cos\theta)^{1/2}}\, d\sigma' ,$$

$$= \frac{1}{R4\pi}\int \eta' \left(1+\frac{r_1^2}{R^2}-2\frac{r_1}{R}\cos\theta\right)^{-1/2} d\sigma' ,$$

so that

$$\lim_{R=\infty} R\Phi = \frac{1}{4\pi}\int \eta' d\sigma' = \frac{\text{Total charge}}{4\pi} .$$

In the same way it may be shown that

$$\lim_{R=\infty} R^2 \frac{\partial \Phi}{\partial s}$$

is also finite. When it is desired to refer to the characteristic behavior at infinity which is expressed by these relations, the potential will be said to be "regular at infinity."

The various points just discussed give rise to the following schedule of conditions on the function Φ:

(I)
- *a*) $\nabla^2\Phi = 0$, at all points.
- *b*) Either the constant potential Φ_i or the total charge e_i of each conductor must be known. Thus Φ reduces, on each conductor, to a known constant Φ_i or Φ reduces to an unknown constant, while on the surface Σ_i of the ith conductor*
$$\int_{\Sigma_i} \eta' d\sigma' = -\int_{\Sigma_i} \frac{\partial \Phi}{\partial n} d\sigma' = e_i .$$
- *c*) Φ is regular at infinity.

This schedule of conditions will be referred to collectively by the Roman numeral (I). The mathematical formulation of the electrostatic problem will now be completed by proving that there exists but one function Φ satisfying these conditions. This theorem will then guarantee that a function, obtained in any way whatsoever, which satisfies these conditions, is the actual solution of the physical problem.

§ 22. *The Uniqueness of the Solution.*—Before giving the proof of the uniqueness of the solution of (I), it is necessary to establish an identity which is a form of Green's theorem. Integration furnishes the equation†

$$\int_\tau \frac{\partial A_x}{\partial x} d\tau = \int_\Sigma A_x \cos(nx) d\sigma , \tag{72}$$

where Σ is the surface of the volume τ, and where n is an exterior normal. If this equation be added to the two corresponding equations in y and z, the result is

$$\int_\tau \operatorname{div} \boldsymbol{A}\, d\tau = \int_\Sigma A_n d\sigma , \tag{73}$$

where $\boldsymbol{A}$ is the vector whose components are A_x, A_y, and A_z. If $\boldsymbol{A}$ be given the special form $U\nabla U$, this equation reduces to

$$\int_\tau U\nabla^2 U\, d\tau - \int_\Sigma U \frac{\partial U}{\partial n} d\sigma + \int_\tau \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right] d\tau = 0 , \tag{74}$$

which is the identity sought.

* See last footnote.

† See Appendix, § 5, A, for more complete discussion

The uniqueness will first be proved for a closed region which is a cavity within a conductor whose potential Φ_0 is supposed known. Suppose that there are two functions Φ_1 and Φ_2 satisfying (I), and set

$$\Phi_1 - \Phi_2 = U .$$

Then from (I)

$$\nabla^2 U = 0 ,$$

at all points within the cavity; while on the walls of the cavity, since Φ_1 and Φ_2 reduce, at all such points, to the same constant Φ_0,

$$U = 0 .$$

If these values be substituted in (74), the result is

$$\int_\tau \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 \right] d\tau = 0 .$$

But since the integrand is essentially positive, it follows that it must vanish identically, i.e., it follows that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0 .$$

Thus the rate of change of U in any direction is zero, and since U is zero on the boundary of the cavity, it is zero throughout the interior. The difference between the two solutions Φ_1 and Φ_2 therefore vanishes identically, and the solution is unique.

Consider next the potential at any point in free space due to a set of conductors whose potentials are known. Apply formula (74) to the volume whose interior boundary is the exterior surface of all the conductors, and whose exterior boundary is a large sphere, of radius R, which contains all the conductors present. Call U, as before, the difference of two possible solutions. Then from (I), U satisfies the conditions

$$\begin{cases} \nabla^2 U = 0, \\ U = 0 \text{ on the interior boundary,} \\ U \text{ is regular at infinity.} \end{cases}$$

The first volume integral of (74) vanishes as before. The surface integral must now be extended over both the interior and exterior boundaries.

The surface integral over the interior boundary vanishes, as before, since U is zero on this boundary. Let the radius R of the exterior boundary approach infinity. Then, from the regularity at infinity, U vanishes as $1/R$, and $\partial U/\partial n = \partial U/\partial R$ vanishes as $1/R^2$. Thus if $d\sigma = R^2 d\omega$ is an element of surface on the sphere,

$$\int U \frac{\partial U}{\partial n}\, d\sigma < F \int \frac{d\sigma}{R^3} = F \int \frac{R^2 d\omega}{R^3} = \frac{F}{R} \int d\omega ,$$

where F is some finite number. Therefore, as R approaches infinity, the surface integral over the exterior surface vanishes. Then, as before,

$$\int \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 \right] d\tau = 0 ,$$

the region of integration now having become all space exterior to the conductors, and it follows as before that the potential is unique.

If, in either of these proofs, the potential of each conductor is not known, but its total charge e_i is given, the proof must be modified. In proving that the surface integral over the conductors is zero, it is not now known that U vanishes on the surface of each conductor. However, since U is the difference of two solutions Φ_1 and Φ_2, each of which must reduce to a constant on each conductor, it follows that U is constant over the surface of each conductor. Thus

$$\int U \frac{\partial U}{\partial n}\, d\sigma = U \int \frac{\partial U}{\partial n}\, d\sigma = U \left[\int \frac{\partial \Phi_1}{\partial n}\, d\sigma - \int \frac{\partial \Phi_2}{\partial n}\, d\sigma \right] = U(-e_i + e_i) = 0 ,$$

and the remainder of the proof, in either instance, goes exactly as before.

§ 23. *Gauss's Theorem.*—In the previous section it was shown that the surface density η on a conductor is given by $-\partial\Phi/\partial n$, so that

$$\int \frac{\partial \Phi}{\partial n}\, d\sigma = -e , \tag{75}$$

where e is the total charge on the surface of the conductor, the integral being extended over that surface. It is possible, however, to assign a more general meaning to this equation. In fact, if the surface integral be extended over any closed surface Σ whatever, the foregoing equation is still correct provided e is interpreted as the total charge contained

within the surface. To prove this statement, apply (73) to the region in question, and let $\boldsymbol{A}$ have the special value $\nabla\Phi$. The formula can be applied only to a region throughout which the components $\partial\Phi/\partial x$, $\partial\Phi/\partial y$, $\partial\Phi/\partial z$ of $\nabla\Phi$ are continuous. It has been shown that these derivatives are discontinuous across any surface S_i where $\eta \neq 0$. Such surfaces of

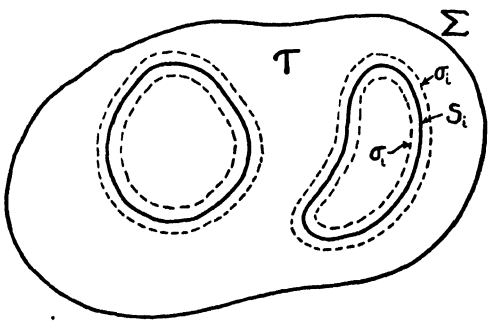

FIG. 22

discontinuity S_i must therefore be excluded from the region of integration by inclosing them in surfaces σ_i, such as are shown dotted in the figure. Let the volume within Σ be called τ, and the volume inclosed by the surfaces σ_i be called δ. Then (74) reads:

$$\int_\Sigma \frac{\partial\Phi}{\partial n}\,d\sigma + \sum_i \int_{\sigma_i} \frac{\partial\Phi}{\partial n}\,d\sigma = \int_{\tau-\delta} \nabla\Phi\, d\tau = -\int_{\tau-\delta} \rho\, d\tau\ .$$

As an inclosing surface σ_i shrinks down on the surface of discontinuity S_i, the integral

$$\int_{\sigma_i} \frac{\partial\Phi}{\partial n}\,d\sigma$$

approaches the value

$$-\int_{S_i}\left[\left(\frac{\partial\Phi}{\partial n_1}\right)_1 + \left(\frac{\partial\Phi}{\partial n_2}\right)_2\right] d\sigma\ ,$$

$$= \int_{S_i} \eta\, d\sigma\ .$$

Thus, as all the inclosing surfaces σ_i shrink down on the surfaces S_i, the foregoing equation becomes

$$\int_\Sigma \frac{\partial \Phi}{\partial n}\, d\sigma = -\int_\tau \rho d\tau - \sum_i \int_{S_i} \eta d\sigma ,$$

$$= -e ,$$

where e is the total charge within Σ.* Equation (75), with the extended significance just given it, is sometimes called "Gauss's theorem." The derivation just given includes the case of both surface and volume densities of charge. In an electrostatic problem for conductors, ρ is zero, and only the integral involving η remains.

§ 24. *Applications of the Uniqueness Theorem.*—The uniqueness theorem may be used to establish the fundamental additive character of potentials and charge densities. Suppose, for example, that a set of conductors be raised to potentials Φ_i by charges e_i, and that Φ is the potential at any point in free space. Then if the charges are given new values me_i, each charge being the same fractional part m of its original value, the potential at any point will be $m\Phi$. For the conditions

$$\nabla^2 m\Phi = m\nabla^2\Phi = 0 ,$$

$$\int \frac{\partial m\Phi}{dn}\, d\sigma = m\int \frac{\partial \Phi}{dn}\, d\sigma = -me_i ,$$

$m\Phi$ is regular at infinity,

are satisfied; and therefore by the uniqueness theorem $m\Phi$ is the only and hence the correct solution. Since the potential at any point in space is m times its original value, it follows directly that the potentials of the conductors themselves are given by $m\Phi_i$. At any point on the surface of a conductor the surface density of charge is given by the negative rate of change of the potential in the direction of the exterior normal. Hence if η is the surface density when the charges are e_i and the potential Φ, the surface density when the charges are me_i and the potential $m\Phi$ is given by

$$-\frac{\partial m\Phi}{\partial n} = -m\frac{\partial \Phi}{\partial n} = m\eta .$$

* It is easily seen that if there be charge on the surface Σ itself, the integration under consideration gives the charge on Σ as well as that within Σ.

Increasing or decreasing all the charges in a given ratio thus increases or decreases (in the same given ratio) both the potential at any point, and the surface density on all the conductors.

Suppose further that a set of conductors be raised to potentials Φ_i' by charges e_i', and to potentials Φ_i'' by charges e_i''; then they are raised to potentials $(\Phi_i'+\Phi_i'')$ by charges $(e_i'+e_i'')$. For since charges e_i' raise the conductors to potentials Φ_i',

$$\nabla^2\Phi' = 0 ,$$

$$\int \frac{\partial \Phi_i'}{\partial n}\, d\sigma = -e_i' ,$$

Φ' is regular at infinity,

and since the charges e_i'' raise the conductors to potentials Φ_i'',

$$\nabla^2\Phi'' = 0 ,$$

$$\int \frac{\partial \Phi_i''}{\partial n}\, d\sigma = -e_i'' ,$$

Φ'' is regular at infinity,

it follows by addition that

$$\nabla^2(\Phi'+\Phi'') = 0 ,$$

$$\int \frac{\partial(\Phi_i'+\Phi_i'')}{\partial n}\, d\sigma = \int \frac{\partial \Phi_i'}{\partial n}\, d\sigma + \int \frac{\partial \Phi_i''}{\partial n}\, d\sigma = -(e_i'+e_i'') ,$$

$\Phi'+\Phi''$ is regular at infinity,

which, by the uniqueness theorem, prove that $\Phi'+\Phi''$ is the potential when the charges are $e_i'+e_i''$. When the total charges on the conductors are $e_i'+e_i''$, the surface densities of charge η_i can be shown, in the same manner as above, to be

$$\eta_i = \eta_i' + \eta_i'' ,$$

where η_i' and η_i'' are the surface densities when the total charges are e_i' and e_i'', respectively.

As a further illustration of the usefulness of the uniqueness theorem, consider the problem of determining the potential at any point due to a conductor which is itself uncharged, but which contains a cavity within which is located a charge $+e$.

The potential at any point exterior to the conductor, due to a charge e located in the cavity, satisfies the schedule of conditions

$$\begin{cases} \nabla^2\Phi = 0 \text{ in free space,} \\ \int \frac{\partial\Phi}{\partial n}\, d\sigma = -e \text{ over exterior boundary of the conductor,} \\ \Phi \text{ is regular at infinity,} \end{cases}$$

the second equation resulting from an application of Gauss's theorem to the external surface of the conductor.* These conditions are identical, however, with those which hold when a solid conductor has a charge e. It follows from the uniqueness theorem, therefore, that the potential at any point in the space outside the conductor is the same in the two cases. In the case of a solid conductor there is a surface charge given by

$$\eta = -\frac{\partial\Phi}{\partial n},$$

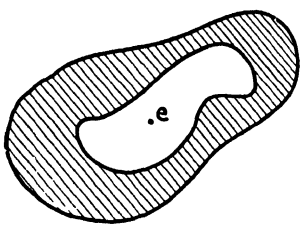

FIG. 23.—A charge e located in a cavity of a conductor.

and since the potential at every point, and hence the normal derivative of the potential, is the same in the two cases, there is the same surface distribution on the exterior surface of the hollow conductor. Since, however, the total charge of the hollow conductor is zero, and since the volume density of charge is also zero, it follows that there is a surface distribution on the walls of the cavity of total amount $-e$. In the case of the hollow conductor, the surface distributions on the exterior surface, and on the walls of the cavity, are known as "induced charges."

PROBLEMS FOR PART I, CHAPTER II

1. Under what circumstances can an integral be differentiated with respect to a parameter which occurs in the integrand?
2. The function x^2+y^2 is integrated throughout the interior of the parallelopiped bounded by the planes $x=\pm a$, $y=\pm b$, $z=\pm c$. What

* See preceding footnote.

functions, integrated over the surface of this same body, would produce the same final answer?

3. At points outside the sphere $x^2+y^2+z^2=4$, the intensity

$$\boldsymbol{E}=i3x+j3y+k3z\ .$$

At points inside the sphere, $\boldsymbol{E}=0$. What value does η have on the sphere?

4. For $x>0$, the intensity vector is given by the equation

$$\boldsymbol{E}=i(2+x)+j(3+y)+k(3+z)\ ,$$

while for $x<0$,

$$\boldsymbol{E}=i(3y+4z-x)+j(3+y+x)+k(3+z-x).$$

What is η on the plane $x=0$?

5. On one side of the plane $x+y+z=0$,

$$\boldsymbol{E}=i\left(2+\frac{x}{\sqrt{3}}-\frac{y}{\sqrt{2}}-\frac{z}{\sqrt{6}}\right)+j\left(3+\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{2}}-\frac{z}{\sqrt{6}}\right)+k\left(3+\frac{x}{\sqrt{3}}+\frac{2z}{\sqrt{6}}\right)\ .$$

while on the other side

$$\boldsymbol{E}=i\left(\frac{6x}{\sqrt{3}}+\frac{4y}{\sqrt{2}}+\frac{6z}{\sqrt{6}}\right)+j\left(3+\frac{2x}{\sqrt{3}}-\frac{2z}{\sqrt{6}}\right)+k\left(3+\frac{y}{\sqrt{2}}+\frac{3z}{\sqrt{6}}\right)\ .$$

What is the surface density of charge on the plane?

6. Prove from the uniqueness theorem that the charge on an isolated spherical conductor is uniformly distributed.

7. Prove that the potential is constant at points within a cavity located in a conductor.

8. A conductor, on which is a total charge e, contains a cavity within which is located an additional amount of charge e'. Use Gauss's theorem to find the total charge on the wall of the cavity, and on the exterior surface of the body.

9. Suppose a spherical shell whose interior and exterior radii are r_1 and r_2 to have a uniform volume density of polarization whose magnitude

is P and whose direction is that of the x-axis. Show that the x-component of the intensity at the center of the sphere is

$$P\int_0^{\pi}\int_{r_1}^{r_2}\frac{(2\cos^2\theta-\sin^2\theta)}{r^3}2\pi r^2\sin\theta\,dr\,d\theta=0\,.$$

10. In § 20 it was shown that the normal derivative of the potential due to a circular disk is discontinuous as the disk is pierced. Show that the tangential derivative of the potential is continuous.

11. As is indicated by the previous problem, the tangential component of the electrostatic intensity is continuous as one pierces a charged surface. Do the fields given in Problems 3 and 4 conform to this demand?

12. Show that Problem 5 is obtained from Problem 4 by rotating the x-axis of Problem 4 into the position of a normal to the plane $x+y+z=0$, and the y-axis of Problem 4 into a position lying in the x-y-plane of Problem 5.

Part II. Special and General Methods of Solution

Introduction

The schedule of conditions (I) above has just been shown to determine uniquely the electrostatic potential. For various types of problems special methods have been developed which lead conveniently and simply to the function which satisfies the schedule of conditions (I). Part II of this chapter exhibits and explains some of these methods. The image method, for example, furnishes a solution to many problems. For other problems it is convenient to transform the conditions (I) into the form they assume when curvilinear co-ordinates are used. The essential advantage in the use of such co-ordinates is that it is possible (as, for example, in the case of the sphere or ellipsoid) to choose such co-ordinates that the equation of the surface of the conductor in question is $\lambda = a$, where a is a constant and λ is one of the curvilinear co-ordinates. Then the condition " Φ is constant on a conductor" is expressed as " Φ = constant, when $\lambda = a$"—a formulation of this condition which is more simple and more easy to apply than it would be in case other less suitable co-ordinates were used.

This section also introduces the concept of the capacity of a conductor, and contains a brief treatment of logarithmic potential.

§ 25. *The Method of Images.*—An important special method of determining the potential and the distribution of charge in certain electrostatic problems is the so-called "method of images." This method will be illustrated by several problems, the most simple of which is the determination of the potential due to a charge e located a distance a from an infinite conducting plane whose potential is zero. It is convenient to separate the potential Φ into the sum of two terms, one of which, $e/4\pi r$, is the potential due to a charge e; while the other, Φ_1, is the potential due to the charge which is induced on the surface of the conductor. The potential Φ_1 must satisfy the conditions

$$\nabla^2 \Phi_1 = 0 \text{ in free space,}$$

$$\Phi_1 = -\frac{e}{4\pi r} \text{ on the surface of the conductor,}$$

the latter condition arising from the fact that on the conductor

$$\Phi=\frac{e}{4\pi r}+\Phi_1=0\,.$$

If, by inspection or otherwise, a function Φ_1, regular at infinity, can be obtained which satisfies those conditions, then the potential Φ is the physical potential sought. It is easily seen, however, that

$$\Phi_1=-\frac{e}{4\pi r'}$$

is such a function, r' being the distance to O', the image point of O, where e is located. Thus the complete solution is

$$\Phi=\frac{e}{4\pi}\left(\frac{1}{r}-\frac{1}{r'}\right),$$

and the effect of the charge induced on the surface of the conductor is exactly equivalent, at exterior points, to an imaginary charge $-e$ located at O'.

Since the intensity within the conductor is zero, the surface density of charge is given by $-\partial\Phi/\partial n$, and it is easily calculated from the last equation that

$$\eta=-\frac{ae}{2\pi r^3}\,.$$

The surface charge is thus heaped up under the inducing charge e, as is shown in Figure 24.

The method of images may be stated as follows: Having a given distribution of charges and conductors, one seeks an arrangement of imaginary charges not located within the region in which the potential is desired, which imaginary distribution together with the actual specified distribution will make the potential reduce to suitable constants on all conductors. Then this imaginary distribution may be discarded; for by the uniqueness theorem the solution obtained is the only one, and the actual induced surface charge on all conductors may be calculated as usual from the normal derivative of the potential. The word "image" is used in describing the method since, as is illustrated by the problem just discussed, the notion of a geometrical image is often involved in the location of the imaginary charges.

Consider next a pair of conducting planes intersecting normally, with a charge e located within the right angle. The potential at any point in the angle between the planes may be obtained at once by the image method, the previous illustration suggesting the proper imaginary distribution of image charges. Reflect $+e$ at O in plane B and plane A; and reflect the last charge so obtained in plane B, changing the sign of the charge at each reflection. There are thus obtained the three imaginary image charges located at O_1, O_2, and O_3 (see Fig. 25). The one actual

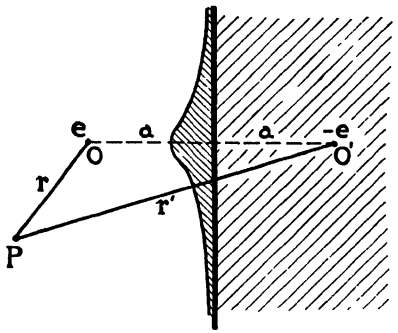

FIG. 24.—A charge e located a distance a from the plane face of an infinite conductor.

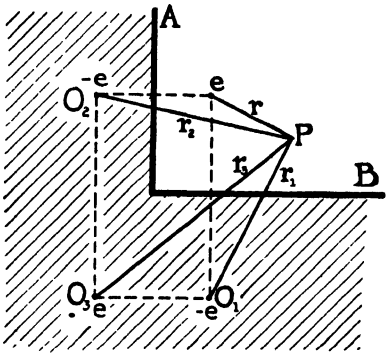

FIG. 25.—A charge e located in the right-angled corner of an infinite conductor.

and the three imaginary charges have a combined potential which is obviously zero on planes A and B. Therefore, by the uniqueness theorem the potential sought is

$$\Phi = \frac{e}{4\pi}\left(\frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_3} - \frac{1}{r_2}\right).$$

The surface charges induced on the planes and effectively equivalent to the three imaginary charges may be determined by an examination of the normal derivatives of the potential Φ. The total surface charge on plane B turns out to be

$$-\frac{2e}{\pi}\tan^{-1}\frac{a}{b},$$

where a and b are the distances from the charge e to the planes A and B, respectively.

The case of the three planes intersecting at right angles can be treated in the same way. The case of two planes intersecting at an angle π/n, where n is an integer, can also be treated by the method of images. Figure 26, for example, shows the necessary distribution of the image charges for the case $n=3$.

The image method may also be used to obtain the potential due to a charge e which is located a distance b from the center of a conducting sphere. Let the radius of the sphere be a, and suppose first that the sphere be grounded so that its potential is zero. If the image method is to

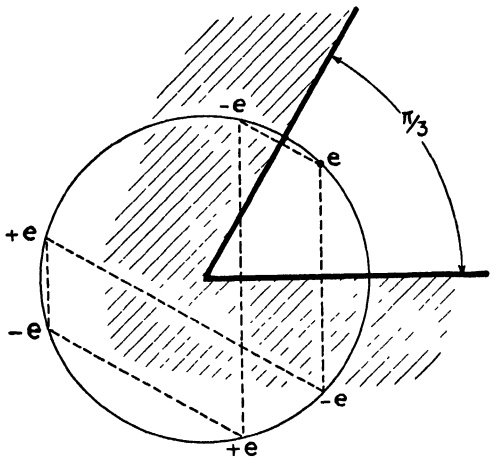

FIG. 26.—A charge e located between two perfectly conducting planes intersecting at an angle $\pi/3$.

furnish a solution it must be possible to locate within the sphere a charge e' of such a magnitude and at such a distance x from the center that the potential due to e' will cancel the potential due to the charge e at every point on the surface of the sphere, i.e., so that

$$\frac{e}{r}+\frac{e'}{r'}\equiv 0\ ,$$

at every point on the surface, r being the distance to e, and r' the distance to e'. Transposing and squaring, this relation becomes

$$e^2r'^2\equiv e'^2r^2\ ,$$

$$e^2(a^2+x^2-2ax\cos\theta)\equiv e'^2(a^2+b^2-2ab\cos\theta)\ .$$

* For a detailed discussion of the number of reflections necessary, see T. Oryng, *Phys. Zt*, January, 1928, p. 41, and H. Mauer, *ibid.*, March, 1928, p. 147.

This expression must be an identity in θ, so that, equating coefficients,

$$xe^2 = be'^2 ,$$

or

$$x = bp ,$$

where

$$p = e'^2/e^2 .$$

Also

$$e^2a^2 + e^2b^2p^2 = e'^2a^2 + e'^2b^2 ,$$

$$b^2p^2 - (a^2+b^2)p + a^2 = 0 ,$$

$$p = 1 \text{ or } a^2/b^2 .$$

If $p=1$, then $e' = \pm e$, and $x = \pm b$: a possibility that is discarded since the charge e' must be within the sphere. Thus

$$p = a^2/b^2 ,$$

$$x = a^2/b ,$$

$$e' = -ea/b .$$

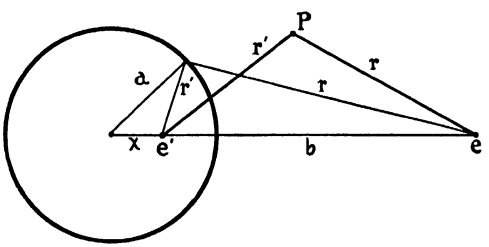

FIG. 27.—A charge e located a distance b from the center of a perfectly conducting sphere of radius a

The sign of e' is so chosen as to make the potential vanish on the surface of the sphere, the double sign having been introduced by squaring. Then

$$\Phi = \frac{1}{4\pi}\left(\frac{e}{r} + \frac{e'}{r'}\right) = \frac{e}{4\pi}\left(\frac{1}{r} - \frac{a}{br'}\right) ,$$

where r' is measured to the point $x = a^2/b$ at which e' is located.

If the potential of the sphere is to be a constant Φ_0, instead of zero, this modified problem can be solved at once by assuming a second fictitious charge, which obviously must be placed at the center of the sphere in order that the potential due to it be symmetrical over the surface. In order that this second charge raise the sphere to potential Φ_0, the magnitude of the charge must be $4\pi\Phi_0 a$. Thus the potential

$$\Phi = \frac{1}{4\pi}\left\{\frac{e}{r} - \frac{ea}{br'} + \frac{4\pi\Phi_0 a}{r''}\right\}$$

satisfies all the conditions of the problem, and is the unique solution, r'' being the distance from the point at which Φ is measured to the center of the sphere.

If the sphere has a total charge $\bar{e}$, a further modification is necessary. An amount e' of this charge must be placed at the image point in order to nullify, at points on the surface, the potential due to e. There remains an amount $\bar{e}-e'$ which, if thought of as located at the center of the sphere, is available to raise the potential of the sphere to the value $(\bar{e}-e)/4\pi a$. Substituting this for Φ_0 in the previous solution, the potential is given by

$$\Phi = \frac{1}{4\pi}\left\{\frac{e}{r} - \frac{ea}{br'} + \frac{\bar{e}-e'}{r''}\right\},$$

$$= \frac{1}{4\pi}\left\{\frac{e}{r} - \frac{ea}{br'} + \frac{b\bar{e}+ea}{br''}\right\}.$$

The potential as given by this expression is constant over the surface of the sphere; the surface integral of the negative normal derivative of the potential is $\bar{e}$; the equation $\nabla^2\Phi = 0$ is satisfied at every exterior point (except at the location of e); and Φ is regular at infinity: this expression is, therefore, the only solution. The distribution of charge assumed in obtaining the solution is, of course, only hypothetical; and if the actual distribution be desired, it may be calculated from the equation

$$\eta = -\frac{\partial \Phi}{\partial n}.$$

Using the hypothetical but equivalent distribution of charge, the force $\boldsymbol{F}$ between the charge e and the sphere can be readily calculated. It is given by

$$|\boldsymbol{F}| = F_x = \frac{e}{4\pi}\left\{-\frac{ea}{b}\frac{1}{(b-x)^2} + \frac{\bar{e}+ae/b}{b^2}\right\},$$

$$= \frac{e^2}{4\pi}\left\{\frac{\bar{e}b+ea}{eb^3} - \frac{ab}{(b^2-a^2)^2}\right\}.$$

The method of images may be applied to many problems. A finite number of image charges suffices in any case where the region in which the potential is to be determined is bounded by:

1. A single spherical surface or a plane.
2. Two planes, a sphere and a plane, or two spheres, the intersections being at an angle π/n, n integral.

3. The two surfaces of 2 and a third, which may be either plane or spherical, which cuts both orthogonally.
4. The three surfaces of 3 and a fourth, cutting the first two orthogonally and the third at an angle π/n, n integral. Of these four surfaces at least one must be spherical.*

It is possible, however, to apply the method of images to cases where a finite number of images will not suffice. The case of a charge located between two planes which intersect at an angle not commensurable with π may, for example, be solved by the image method. An infinite number of image charges is required, and the potential is expressed by means of an infinite series.†

An infinite number of image charges is also required in the case of two conducting spheres, and on account of its importance, this problem will be discussed briefly. Let the two spheres A and B be of radii a and b, and let the distance between their centers be c. Suppose that A be at potential Φ_0, and that B be grounded. If a charge $4\pi\Phi_0 a$ be located at A (the center of sphere A), this charge produces the required potential Φ_0 over the surface of sphere A; but the potential over the surface of B is not zero. From the last problem discussed, however, it follows that an image charge $-4\pi\Phi_0 ab/c$ at B', where $BB'=b^2/c$, will reduce the potential of sphere B to zero. This charge, in turn, disturbs the value of the potential over sphere A, which must be restored to the constant value Φ_0 by an image charge

$$\frac{4\pi\Phi_0 ab}{c}\,\frac{a}{c-\frac{b^2}{c}},$$

at A', where

$$AA'=\frac{a^2}{c-\frac{b^2}{c}}.$$

* Maxwell, *Electricity and Magnetism*, 1, 206.

† The method of images may also be applied to problems in which the image points fall within the region in which the potential is desired. In this case one makes use of superimposed Riemann spaces, comparable to the many-leaved Riemann surfaces of analytic-function theory See Riemann-Weber, *Differentialgleichungen der Physik*, Vol. II, chap. xiii (1927).

To obtain again zero potential over sphere B requires a charge

$$-\frac{4\pi\Phi_0 ab}{c}\cdot\frac{a}{c-\frac{b^2}{c}}\cdot\frac{b}{c-\frac{a^2}{c-\frac{b^2}{c}}}$$

located at B'', where

$$BB'=\frac{b^2}{c-\frac{a^2}{c-\frac{b^2}{c}}}\cdot$$

This process must be carried on indefinitely, the value and position of each successive charge being given by the results of the previous problem. The successive charges, moreover, crowd closer and closer together, so that their effect at exterior points becomes less and less. In the case of spheres of equal radii, separated a distance greater than a diameter, it is not necessary to use more than two or three successive image charges to obtain a very close approximation. The potential is, in the general case, given by an infinite series. The analysis involved in the summation of this series is complicated, and will not be given here.

§ 26. *Curvilinear Co-ordinates.*—In locating a point in space by means of rectangular co-ordinates, the equations $x=$constant, $y=$constant, $z=$constant each determine a plane. The intersection of two of these planes is a straight line which intersects the third plane at the point in question. A more general system of co-ordinates can be obtained by using not plane but curved surfaces. For example, if special values u_0, v_0, w_0 be assigned to three functions U, V, and W of x, y, z—that is, if one sets,

$$u_0=U(x,y,z)\ ,$$
$$v_0=V(x,y,z)\ ,$$
$$w_0=W(x,y,z)\ ,$$

then each of these equations determines a surface. Two of these surfaces intersect in a space curve which intersects the third surface at a point. The three numbers u_0, v_0, w_0 which thus determine this point are called "curvilinear co-ordinates" of the point. The surfaces $u=$constant, $v=$constant, $w=$constant are known as the "co-ordinate surfaces"; their intersections, pair by pair, are known as the "co-ordinate lines"; and the tangents to the co-ordinate lines at a point P are called the "co-ordinate

axes" at P. If such functions are chosen as make the co-ordinate axes at every point in space mutually perpendicular, the co-ordinates are known as "orthogonal curvilinear co-ordinates." Most physical problems are best handled by means of orthogonal co-ordinates, and only such will be considered here.

The polar co-ordinates ρ and θ which locate a point in a plane are a special example of orthogonal curvilinear co-ordinates in two dimensions. Since one less dimension is involved than in the general case considered just above, the co-ordinate surfaces reduce to lines—the radial lines $\theta = \text{constant}$, and the concentric circles $\rho = \text{constant}$. These intersect

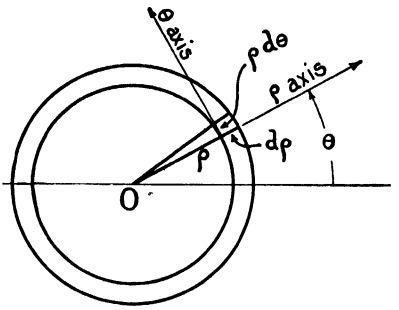

FIG. 28.—Polar co-ordinates in the plane.

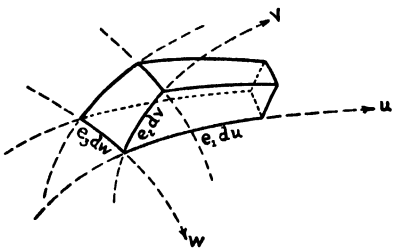

FIG. 29.—Element of volume using curvilinear co-ordinates.

everywhere at right angles, the co-ordinate axes at a point P being the radius and tangent to that circle, with center at the origin, which passes through P. The distance ds_ρ, measured along the ρ-axis, from the circle whose radius is ρ to the circle whose radius is $\rho+d\rho$, is clearly $d\rho$. The distance ds_θ, measured along the θ-axis, from the radius whose angle with the polar axis is θ to the radius whose angle with the polar axis is $\theta+d\theta$, is, however, $\rho d\theta$. In general, the distance ds_u measured along the u-axis between two surfaces for which u differs by an amount du is a function of du which vanishes when $du=0$ so that the first term in the Taylor expansion for ds_u as a power series in du is

$$ds_u = e_1 du ,$$

where e_1 is some function of u, v, w. In the same way

$$ds_v = e_2 dv ,$$

$$ds_w = e_3 dw .$$

To calculate these quantities e_1, e_2, and e_3, it is necessary, in general, to obtain by differentiation the values in terms of u,v,w of the increments dx_u, dy_u, dz_u which correspond to an increment du in u, v and w being held constant. Then

$$e_1^2 du^2 = dx_u^2 + dy_u^2 + dz_u^2 ,$$

etc., from which e_1, e_2, and e_3 may be obtained. This calculation is seldom necessary, however, and in specific cases the three functions

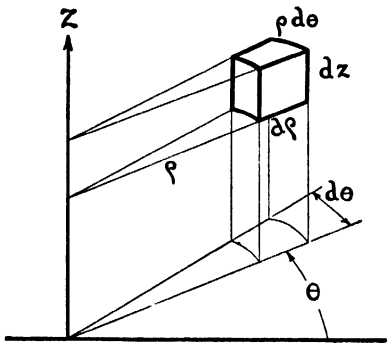

FIG. 30.—Cylindrical co-ordinates

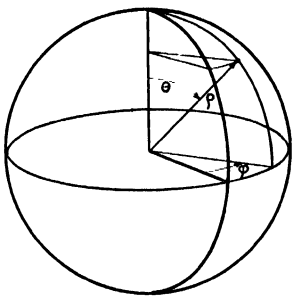

FIG. 31.—Polar co-ordinates in space.

e_1, e_2, e_3 may usually be determined by inspection, they being those quantities by which increments in the variables must be multiplied to give distances measured along the co-ordinate axes.

In cylindrical co-ordinates, for example, a point is located by means of the three variables ρ, θ, and z. If one chooses

$$u = \rho , \qquad v = \theta , \qquad w = z ,$$

then from the figure

$$ds_\rho = d\rho , \qquad ds_\theta = \rho d\theta , \qquad ds_z = dz ,$$
$$e_1 = 1 \qquad e_2 = \rho , \qquad e_3 = 1 .$$

In polar co-ordinates a point is located by giving the radius of the sphere on which it lies, and its co-latitude and longitude on that sphere, i.e., it is located by the three variables ρ, θ, φ, as shown on the figure. Then if one choses

$$u = \rho , \qquad v = \theta , \qquad w = \varphi ,$$

it follows by inspection from the figure that

$$ds_\rho = d\rho\ , \qquad ds_\theta = \rho d\theta\ , \qquad ds = \rho \sin\theta\, d\varphi\ ,$$
$$e_1 = 1\ , \qquad e_2 = \rho\ , \qquad e_3 = \rho \sin\theta\ .$$

The expression for the divergence of a vector A in terms of general orthogonal curvilinear co-ordinates is most easily obtained from the equation

$$\operatorname{div} A = \lim_{d\tau \to 0} \frac{\int A_n d\sigma}{d\tau}\ ,$$

which is obtained at once by applying the mean-value theorem to the formula

$$\int \operatorname{div} A\ d\tau = \int A_n d\sigma\ ,$$

the volume integral being extended over the infinitesimal volume $d\tau$. This equation is to be applied to a volume element whose edges have lengths

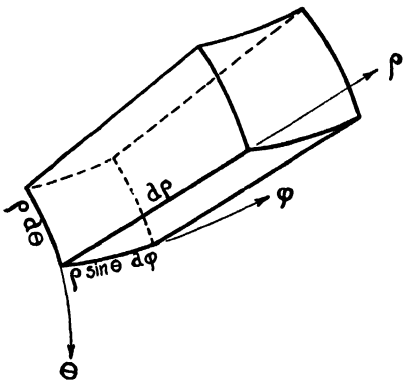

FIG. 32.—Element of volume using polar co-ordinates in space.

$$ds_u = e_1 du\ , \qquad ds_v = e_2 dv\ , \qquad ds_w = e_3 dw\ ,$$

and whose faces have areas

$$d\sigma_{uv} = e_1 e_2 du\ dv\ , \qquad d\sigma_{vw} = e_2 e_3 dv\ dw\ , \qquad d\sigma_{uw} = e_1 e_3 du\ dw\ .$$

The surface integral of the normal component of a vector is often referred to as the "total normal flux" of that vector through the surface in question. The total normal flux of the vector A out through the surface of the volume element here considered may be written as the sum of three terms, each of which represents the contribution from a pair of opposite faces. The total normal flux out through a pair of opposite faces is equal to the flux out through one face, minus the flux in through the other, i.e., it is equal to, for example,

$$\left[A_u d\sigma_{vw} + \frac{\partial(A_u d\sigma_{vw})}{\partial u}\, du\right] - A_u d\sigma_{vw} =$$

$$\left[A_u e_2 e_3 dv\ dw + \frac{\partial(A_u e_2 e_3 dv\ dw)}{\partial u}\, du\right] - A_u e_2 e_3 dv\ dw = \frac{\partial(A_u e_2 e_3 dv\ dw)}{\partial u}\, du\ ,$$

where A_u is the component of $\boldsymbol{A}$ in the direction of increasing u. The expression for div $\boldsymbol{A}$ therefore takes the form

$$\operatorname{div} \boldsymbol{A} = \lim \frac{1}{e_1 e_2 e_3 du\, dv\, dw}\left[\frac{\partial(A_u e_2 e_3 dv\, dw)}{\partial u} du + \frac{\partial(A_v e_1 e_3 du\, dw)}{\partial v} dv + \frac{\partial(A_w e_1 e_2 du\, dv)}{\partial w} dw\right],$$

or, canceling,

$$\operatorname{div} \boldsymbol{A} = \frac{1}{e_1 e_2 e_3}\left[\frac{\partial(A_u e_2 e_3)}{\partial u} + \frac{\partial(A_v e_1 e_3)}{\partial v} + \frac{\partial(A_w e_2 e_1)}{\partial w}\right].$$

If the values of e_1, e_2, e_3 for various systems of co-ordinates be substituted in this formula, the following values for div $\boldsymbol{A}$ result:

a) Polar co-ordinates in the plane:

$$\operatorname{div} \boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial(A_\theta)}{\partial \theta}\right];$$

b) Cylindrical co-ordinates:

$$\operatorname{div} \boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial(A_\theta)}{\partial \theta} + \frac{\partial(\rho A_z)}{\partial z}\right],$$

$$= \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z};$$

c) Polar co-ordinates in space:

$$\operatorname{div} \boldsymbol{A} = \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial(\rho^2 \sin\theta A_\rho)}{\partial \rho} + \frac{\partial(\rho \sin\theta A_\theta)}{\partial \theta} + \frac{\partial(\rho A_\varphi}{\partial \varphi}\right],$$

$$= \frac{1}{\rho_2}\frac{\partial(\rho^2 A_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial(A_\varphi)}{\partial \varphi}.$$

At any point the space rate of change of a scalar function Φ in the direction of the positive u-axis is given by

$$\lim \frac{\Phi(u+du, v, w) - \Phi(u, v, w)}{\text{Distance from } u, v, w \text{ to } u+du, v, w} = \frac{1}{e_1}\frac{\partial \Phi}{\partial u}.$$

Let $\boldsymbol{u'}, \boldsymbol{v'}, \boldsymbol{w'}$ be unit vectors which, at every point in space, are directed along the u-, v-, w-axes at that point. Then if nabla of the scalar func-

tion Φ be expressed in terms of the curvilinear co-ordinates, the scalar coefficient of $\boldsymbol{u}'$ is

$$\frac{1}{e_1}\frac{\partial \Phi}{\partial u},$$

since this scalar coefficient is the rate of change of Φ in the u-direction. Hence this expression for $\nabla\Phi$ in curvilinear co-ordinates is

$$\nabla\Phi = \frac{1}{e_1}\frac{\partial \Phi}{\partial u}\boldsymbol{u}' + \frac{1}{e_2}\frac{\partial \Phi}{\partial v}\boldsymbol{v}' + \frac{1}{e_3}\frac{\partial \Phi}{\partial w}\boldsymbol{w}' .$$

By combining the foregoing expressions by means of the identity

$$\operatorname{div} \nabla\Phi = \nabla^2\Phi ,$$

the following expression for $\nabla^2\Phi$ in general orthogonal curvilinear co-ordinates is obtained:

$$\nabla^2\Phi = \frac{1}{e_1 e_2 e_3}\left[\frac{\partial}{\partial u}\frac{e_2 e_3}{e_1}\frac{\partial \Phi}{\partial u} + \frac{\partial}{\partial v}\frac{e_1 e_3}{e_2}\frac{\partial \Phi}{\partial v} + \frac{\partial}{\partial w}\frac{e_1 e_2}{e_3}\frac{\partial \Phi}{\partial w}\right]$$

From the values of e_1, e_2, e_3 given above, the following formulas result:
a) Polar co-ordinates in the plane:

$$\nabla^2\Phi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}\right)\right] ;$$

b) Cylindrical co-ordinates:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial\left(\rho\,\dfrac{\partial \Phi}{\partial \rho}\right)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial \theta^2} + \frac{\partial^2\Phi}{\partial z^2} ;$$

c) Polar co-ordinates in space:

$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{\partial\left(\rho^2\,\dfrac{\partial \Phi}{\partial \rho}\right)}{\partial \rho} + \frac{1}{\rho^2 \sin\theta}\frac{\partial\left(\sin\theta\,\dfrac{\partial \Phi}{\partial \theta}\right)}{\partial \vartheta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2\Phi}{\partial \varphi^2} .$$

§ 27. *Potential Due to a Charged Sphere: the Capacity of a Conductor.*— As a first illustration of the use of curvilinear co-ordinates, consider the problem of determining the potential at any point in space due to a conducting sphere which carries a charge e. In choosing the type of co-ordinates to be used, one seeks such co-ordinates that the boundary will

be expressed by as simple an equation as possible. In this example, if polar co-ordinates be chosen, the boundary of the sphere is given by the equation $\rho = a$, where a is the radius of the sphere. The potential must therefore reduce to a constant value for $\rho = a$, and since this condition is independent of the variables θ and φ, it is sensible to assume that the potential itself is, at any point, independent of θ and φ, and that it thus depends only on ρ. The nature of the boundary conditions having suggested a simplifying assumption, one proceeds on the basis of this assumption, and if a solution is obtained which satisfies all the conditions, it is known, from the uniqueness theorem, that this solution is the only and hence the correct one. That there is a potential, due to a sphere containing charge, which depends only on ρ and is independent of θ and φ, is also clearly indicated by the physical symmetry of the problem, but the assumption is again justified only by the fact that the solution to which it leads is known to be unique. It will be seen in other problems that the boundary conditions suggest simplifying assumptions of a similar nature.

The last written equation for $\nabla^2\Phi$ is thus reduced, by the assumption that Φ is independent of θ and φ, to the form.

$$\frac{1}{\rho^2}\frac{\partial\left(\rho^2\dfrac{\partial\Phi}{\partial\rho}\right)}{\partial\rho} = 0 ,$$

so that

$$\rho^2\frac{\partial\Phi}{\partial\rho} = \alpha , \qquad \frac{\partial\Phi}{\partial\rho} = \frac{\alpha}{\rho^2} , \qquad \Phi = -\frac{\alpha}{\rho} + \beta ,$$

where α and β are constants of integration.

It remains to satisfy the two conditions

$$\int \frac{\partial\Phi}{\partial n}\, d\sigma = \int \frac{\partial\Phi}{\partial\rho}\, d\sigma = -e ,$$

Φ is regular at infinity .

The second of these two conditions clearly demands that $\beta = 0$; while from the first

$$\int \frac{\partial\Phi}{\partial\rho}\, d\sigma = \int \frac{\alpha}{\rho^2}\, d\sigma = \alpha \int d\omega = 4\pi\alpha = -e ,$$

$$\alpha = -\frac{e}{4\pi} .$$

The unique solution of the problem is thus given by

$$\Phi = \frac{e}{4\pi\rho},$$

and the potential is seen to be the same as if the charge e were concentrated at the center of the sphere. At points within the conductor the potential is constant. If there be no polarized layer on the surface, so that the constant value of the inside potential is equal to the value approached on the surface by the outside potential, the potential at all points within the sphere is given by

$$(\Phi)_a = \frac{e}{4\pi a}.$$

The ratio of the charge on an isolated conductor to the potential to which this charge raises the conductor is called the "capacity" of the conductor. The capacity C of an isolated sphere, namely,

$$C = \frac{e}{\frac{e}{4\pi a}} = 4\pi a,$$

thus varies directly as the radius of the sphere.

§ 28. *Concentric Spherical Shells: Potential and Capacity.*—Consider next a sphere of radius a surrounded by a spherical shell of inner radius b_1 and outer radius b_2. The co-ordinates just used are again obviously suitable, and, the symmetry being the same as before, the potential in the space between the two conductors is given by

$$\Phi_1 = -\frac{\alpha_1}{\rho} + \beta_1,$$

where α_1 and β_1 are constants which now depend upon the potentials and charges of the two conductors. A similar expression

$$\Phi_2 = -\frac{\alpha_2}{\rho} + \beta_2$$

holds for points outside the shell, the regularity at infinity demanding that $\beta_2 = 0$.

Suppose first that the inner sphere has charge e, and that the shell

is grounded so that its potential is zero. Then Φ_1 and Φ_2 must vanish for $\rho = b_1$ and $\rho = b_2$, respectively, while

$$\int \frac{\partial \Phi_1}{\partial n}\, d\sigma = -e\,, \qquad a_1 = -\frac{e}{4\pi}\,, \qquad \Phi_1 = \frac{e}{4\pi\rho} + \beta_1\,.$$

It is readily checked that the potential is zero at all points outside the shell, while the potential at points between the two spheres is given by

$$\Phi_1 = \frac{e}{4\pi}\left(\frac{1}{\rho} - \frac{1}{b_1}\right).$$

The potential of the inner sphere is obtained by setting $\rho = a$. Its value is

$$(\Phi_1)_a = \frac{e}{4\pi}\left(\frac{1}{a} - \frac{1}{b_1}\right) = \frac{e}{4\pi}\,\frac{b_1 - a}{ab_1}\,,$$

which is smaller than the potential to which the sphere would be raised by the charge e if the surrounding shell were not present. In fact, the capacity of the sphere, when the shell is present, namely,

$$C = \frac{e}{\dfrac{e}{4\pi}\,\dfrac{b_1 - a}{ab_1}} = \frac{4\pi a b_1}{b_1 - a}\,,$$

may be made as large as desired by making $b_1 - a$ small enough, i.e., by having the outer shell close enough to the sphere. It thus appears that it is possible greatly to increase the capacity of a conductor by having a second conductor in close proximity to it. Such a combination is called a "condenser," and the capacity of the condenser is defined as the ratio of the charge to the difference of potential of the two conductors. Since the outer shell is grounded, and thus at zero potential, the capacity of the spherical condenser just described is given by

$$C = \frac{e}{\dfrac{e}{4\pi}\,\dfrac{b_1 - a}{ab_1} - 0} = \frac{4\pi a b_1}{b_1 - a}\,,$$

and is numerically equal to the capacity of the inner sphere when the shell is present.

If the inner sphere have charge e but the outer shell be not grounded, it is possible to obtain the expression for the capacity of the spherical

condenser thus formed without knowing either the potential or charge of the outer shell. For the difference of potential of the two conductors is equal to the difference in the values for $\rho=a$ and for $\rho=b_1$ of the expression

$$\Phi_1=\frac{e}{4\pi\rho}+\beta_1 .$$

Thus

$$(\Phi_1)_a-(\Phi_1)_{b_1}=\frac{e}{4\pi}\left(\frac{1}{a}-\frac{1}{b_1}\right)=\frac{e}{4\pi}\frac{b_1-a}{ab_1},$$

so that the capacity of the spherical condenser is

$$C=\frac{e}{\frac{e}{4\pi}\frac{b_1-a}{ab_1}}=\frac{4\pi ab_1}{b_1-a},$$

the same value as that obtained above when the outer shell was grounded.

As regards the distribution of charge, it is evident from the expression for Φ that in either of the two cases considered the charge e on the sphere is uniformly distributed as a surface charge of density $e/4\pi a^2$. If the outer shell be grounded, the surface charge on its inner face is given by

$$\eta=-\left(\frac{\partial\Phi}{\partial n}\right)_{\rho=b_1}=\left(\frac{\partial\Phi}{\partial\rho}\right)_{\rho=b_1}=-\frac{e}{4\pi b_1^2}.$$

There is no surface charge on its outer face, since the potential at all exterior points is zero. There is, thus, a total induced charge on the shell equal to

$$\int_{\rho=b_1}\eta d\sigma=-\frac{e}{4\pi b_1^2}\int_{\rho=b_1}d\sigma=-e .$$

Suppose, however, that the outer shell be not grounded, and that its total charge be zero. Whatever surface charges may be induced on the inner and outer walls of the shell will be uniformly distributed so that their effect, at outside points, will be the same as if the charge were concentrated at the center of the shell. The total charge on the shell is, however, zero. Therefore the potential at outside points is given by

$$\Phi_2=\frac{e}{4\pi\rho},$$

and the potential of the shell is

$$(\Phi_2)_{b_2}=\frac{e}{4\pi b_2}.$$

Hence the expression

$$\Phi_1=\frac{e}{4\pi\rho}+\beta_1$$

must reduce to $e/4\pi b_2$ when $\rho=b_1$; so that

$$\beta_1=\frac{e}{4\pi}\left(\frac{1}{b_2}-\frac{1}{b_1}\right),$$

$$\Phi_1=\frac{e}{4\pi}\left(\frac{1}{\rho}+\frac{1}{b_2}-\frac{1}{b_1}\right).$$

It follows that there is an induced charge on the inner face of the shell whose density is given by

$$\eta=-\left(\frac{\partial\Phi}{\partial n}\right)_{\rho=b_1}=\left(\frac{\partial\Phi}{\partial\rho}\right)_{\rho=b_1}=-\frac{e}{4\pi b_1^2},$$

while on the outer face of the shell

$$\eta=-\left(\frac{\partial\Phi}{\partial n}\right)_{\rho=b_2}=-\left(\frac{\partial\Phi}{\partial\rho}\right)_{\rho=b_2}=\frac{e}{4\pi b_2^2}.$$

From these values it may easily be checked that the total charge on the shell is zero.

§ 29. *Ellipsoidal Co-ordinates: the Potential Due to a Charged Ellipsoid.*—An important physical problem which further illustrates the use of curvilinear co-ordinates is the determination of the potential due to an ellipsoid which carries a charge e. Let the equation of the boundary of the ellipsoid be, in rectangular co-ordinates,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{76}$$

It is desirable to choose co-ordinates in which the boundary is given by a simple expression involving, if possible, only one of the co-ordinates. Now the equation

$$\frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}=1 \tag{77}$$

represents a family of quadric surfaces confocal with the foregoing ellipsoid (76), the squares of the distances from the origin to the foci being given, for any value of t, by

$$d_1^2=(a^2+t)-(b^2+t)=a^2-b^2\,,$$
$$d_2^2=(b^2+t)-(c^2+t)=b^2-c^2\,,$$
$$d_3^2=(a^2+t)-(c^2+t)=a^2-c^2\,,$$

if $a>b>c$. Direct examination of (77) then shows that the surface is an ellipsoid if $t>-c^2$. For $-c^2>t>-b^2$, the surface is a hyperboloid of one sheet; and when $-b^2>t>-a^2$, the surface is a hyperboloid of two sheets. The locus is imaginary for $-a^2>t$. The equation (77) is, for a given point, x_0, y_0, z_0, a cubic in t, and may be seen to have three real roots u, v, w satisfying the inequalities

$$-a^2<w<-b^2<v<-c^2<u\,.$$

Indeed, this cubic in t may be written

$$f(t)=(b^2+t)(c^2+t)x_0^2+(a^2+t)(c^2+t)y_0^2+(a^2+t)(b^2+t)z_0^2 \\ -(a^2+t)(b^2+t)(c^2+t)=0\,,$$

so that

when $t=+\infty$,	$f(t)=-\infty$,	sign is $-$
when $t=-c^2$,	$f(t)=z_0^2(a^2-c^2)(b^2-c^2)$,	sign is $+$
when $t=-b^2$,	$f(t)=-y_0^2(b^2-c^2)(a^2-b^2)$,	sign is $-$
when $t=-a^2$,	$f(t)=x_0^2(a^2-b^2)(a^2-c^2)$,	sign is $+$

From this table it is evident that the graph of $f(t)$ crosses the axis at three points which satisfy the inequality given above. Thus the quadric surface passing through the point x_0, y_0, z_0 is an ellipsoid if the parameter t be chosen equal to u, a hyperboloid of one sheet if $t=v$, and a hyperboloid of two sheets if $t=w$. These three surfaces, moreover, may be shown to have mutually perpendicular normals at the point in question. The three numbers u, v, w thus determine three curved surfaces which intersect normally at the point x_0, y_0, z_0, so that u, v, w may be taken as orthogonal curvilinear co-ordinates. Such co-ordinates are known as "ellipsoidal co-ordinates." It is evident that the boundary of the ellipsoidal conductor under consideration is given by the relation $u=0$. It is the simplicity of the expression which indicates the probable advisability of these co-ordinates.

In order to make use of these ellipsoidal co-ordinates, it is necessary to determine the quantities e_1, e_2, e_3, which cannot, in this case, be obtained by inspection. The calculation may be considerably shortened by a consideration of the identity

$$(a^2+t)(b^2+t)(c^2+t)\left[\frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}-1\right]=-(t-u)(t-v)(t-w)\ .$$

That this is indeed an identity is evident from the fact that each side of the equation vanishes for t equal to u, v, or w, and that the coefficient of t^3 on each side is equal to -1. If, in this expression, t be set equal to $-a^2$, the result is

$$x^2=\frac{(a^2+u)(a^2+v)(a^2+w)}{(b^2-a^2)(c^2-a^2)}\ .$$

From this equation there is obtained, by logarithmic differentiation, the value of dx_u, the increment in x which corresponds to an increment du in u, when v and w are held constant; namely,

$$2\frac{dx_u}{x}=\frac{du}{a^2+u}\ ,$$

similar values holding for dy_u and dz_u. Hence

$$e_1^2du^2=\frac{1}{4}\,du^2\left[\left(\frac{x}{a^2+u}\right)^2+\left(\frac{y}{b^2+u}\right)^2+\left(\frac{z}{c^2+u}\right)^2\right]\ .$$

Now from the identity above:

$$\left[\frac{\dfrac{x^2}{a^2+t}+\dfrac{y^2}{b^2+t}+\dfrac{z^2}{c^2+t}-1}{t-u}\right]_{t=u}=-\frac{(u-v)(u-w)}{(a^2+u)(b^2+u)(c^2+u)}\ .$$

If the indeterminate form on the left be evaluated by the ordinary process of differentiating numerator and denominator separately, the equation takes the form

$$(78)\qquad -\left(\frac{x}{a^2+u}\right)^2-\left(\frac{y}{b^2+u}\right)^2-\left(\frac{z}{c^2+u}\right)^2=-\frac{(u-v)(u-w)}{(a^2+u)(b^2+u)(c^2+u)}\ .$$

Therefore,

$$e_1=\frac{1}{2}\sqrt{\frac{(u-v)(u-w)}{(a^2+u)(b^2+u)(c^2+u)}}\ ,$$

with similar expressions for e_2 and e_3; so that, for example,

$$\frac{e_2e_3}{e_1}=\frac{1}{2}\frac{(v-w)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}{\sqrt{(a^2+v)(b^2+v)(c^2+v)(a^2+w)(b^2+w)(c^2+w)}}.$$

These expressions for e_1, e_2, e_3 are now to be substituted in the general formula for $\nabla^2\Phi$ in orthogonal curvilinear co-ordinates; but since the boundary condition in the ellipsoidal co-ordinates being used involves only the co-ordinate u, it will be assumed that the potential itself depends, at any point, only upon u, and is independent of the values of v and w. If this assumption lead to a solution which satisfies all the conditions of the problem, this solution, since it is known to be unique, is the potential sought. It will be assumed, therefore, that the potential satisfies the equation

$$e_1e_2e_3\,\nabla^2\Phi=\frac{\partial}{\partial u}\frac{e_2e_3}{e_1}\frac{\partial\Phi}{\partial u}=0,$$

which reduces, after the removal of the factor independent of u, to the equation

$$\frac{\partial}{\partial u}\sqrt{(a^2+u)(b^2+u)(c^2+u)}\,\frac{\partial\Phi}{\partial u}=0.$$

The conditions on the potential Φ are, then, when expressed in ellipsoidal co-ordinates,

$$\frac{\partial}{\partial u}\sqrt{(a^2+u)(b^2+u)(c^2+u)}\,\frac{\partial\Phi}{\partial u}=0 \text{ in free space},$$

$$\left.\begin{array}{l}\Phi \text{ is constant, and}\\ \displaystyle\int\frac{\partial\Phi}{\partial n}\,d\sigma=-e\end{array}\right\}\text{on surface of conductor } u=0,$$

$$\Phi \text{ is regular at infinity.}$$

The equation which the potential satisfies in free space may be integrated at once to give

$$\Phi=a\int_u^\infty\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

where $-a$ is the integration constant of the first integration. The infinite upper limit of the integral assures the vanishing of Φ at infinity

That Φ vanishes at infinity in the required manner will be shown later. Since Φ is constant on the ellipsoids $u=$ constant, its maximum rate of change is in the direction of the normal to $u=$ constant, i.e., in the direction of increasing u. The value of this normal derivative is

$$\frac{1}{e_1}\frac{d\Phi}{du}=-2\sqrt{\frac{(a^2+u)(b^2+u)(c^2+u)}{(u-v)(u-w)}}\cdot\frac{a}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$=-\frac{2a}{\sqrt{(u-v)(u-w)}}.$$

Now from the equation

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1,$$

it is clear that as x, y, or z increase without limit so does u, and since

$$\frac{x^2}{1+\frac{a^2}{u}}+\frac{y^2}{1+\frac{b^2}{u}}+\frac{z^2}{1+\frac{c^2}{u}}=u,$$

it follows that the co-ordinate u of a point very far away from the origin is approximately equal to the square of the distance R to the point. That is,

$$u=R^2(1+\epsilon_1),$$

where ϵ_1 denotes, as does any ϵ_i used below, a quantity which approaches zero as R increases beyond limit. Since v and w remain finite, being always included in the range $-a^2$ to $-c^2$, it follows that for points very far away from the origin

$$\frac{1}{e_1}\frac{d\Phi}{du}=-\frac{2a}{R^2}(1+\epsilon_2).$$

For such points the potential is given by

$$\Phi=a\int_u^\infty\frac{du}{u^{3/2}(1+\epsilon_3)}=\frac{2a}{u^{1/2}}(1+\epsilon_4)=\frac{2a}{R}(1+\epsilon_5),$$

and it is seen that Φ is regular at infinity.

The constant a must now be chosen in accordance with the condi-

tion that the charge on the conductor is e. Since any body containing a charge e produces at a very large distance R a potential of the form

$$\Phi = \frac{e}{4\pi R}(1+\epsilon_6),$$

it follows from the preceding equation that

$$a = \frac{e}{8\pi},$$

and therefore

$$\Phi = \frac{e}{8\pi}\int_u^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

The surface density of charge on the ellipsoid is given by

$$\eta = -\left(\frac{\partial \Phi}{\partial n}\right)_{u=0} = -\left(\frac{1}{e_1}\frac{d\Phi}{du}\right)_{u=0} = \frac{2a}{\sqrt{vw}} = \frac{e}{4\pi\sqrt{vw}}.$$

Now from equation (78) above, for $u=0$,

$$\left(\frac{vw}{a^2b^2c^2}\right)_{u=0} = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4},$$

and it follows that

$$\eta = \frac{e}{4\pi abc}\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}.$$

The equation of the plane tangent to the ellipsoid at the point x_0, y_0, z_0 is

$$X\frac{x_0}{a^2}+Y\frac{y_0}{b^2}+Z\frac{z_0}{c^2}=1,$$

where X, Y, Z are running co-ordinates in the plane. After dividing through by the square root of the sum of the squares of the coefficients of X, Y, and Z, the right member is the distance D from the origin to the tangent plane. That is,

$$D = \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}},$$

so that

$$\eta = \frac{e}{4\pi abc} D .$$

In other words, the surface density at any point on a charged ellipsoidal conductor is proportional to the perpendicular distance from the center of the ellipsoid to the plane tangent to the ellipsoid at the point. The charge is thus greater on the more sharply rounded ends farther away from the origin.

If $a=b$ and a be greater than c, the ellipsoid is an oblate spheroid, and

$$\Phi = \frac{e}{8\pi} \int_u^\infty \frac{du}{(a^2+u)\sqrt{c^2+u}},$$

$$= \frac{e}{4\pi \sqrt{a^2-c^2}} \tan^{-1} \sqrt{\frac{a^2-c^2}{c^2+u}} .$$

Thus the potential to which a charge e raises an oblate spheroid of semi-axes a and c is

$$(\Phi)_{u=0} = \frac{e}{4\pi\sqrt{a^2-c^2}} \tan^{-1} \sqrt{\frac{a^2}{c^2}-1} = \frac{e}{4\pi\sqrt{a^2-c^2}} \cos^{-1} \frac{c}{a},$$

and the capacity of the oblate spheroid is given by

$$C = \frac{4\pi\sqrt{a^2-c^2}}{\cos^{-1} \frac{c}{a}} .$$

If, on the other hand, $a=b$, and a be less than c,

$$\Phi = \frac{e}{8\pi \sqrt{c^2-a^2}} \log \frac{\sqrt{c^2+u}+\sqrt{c^2-a^2}}{\sqrt{c^2+u}-\sqrt{c^2-a^2}},$$

$$(\Phi)_{u=0} = \frac{e}{8\pi\sqrt{c^2-a^2}} \log \frac{c+\sqrt{c^2-a^2}}{c-\sqrt{c^2-a^2}},$$

$$C = \frac{8\pi\sqrt{c^2-a^2}}{\log \frac{c+\sqrt{c^2-a^2}}{c-\sqrt{c^2-a^2}}} .$$

A very thin circular disk corresponds to the values $a=b$, $c=0$. Then

$$\Phi=\frac{e}{4\pi a}\tan^{-1}\frac{a}{\sqrt{u}}.$$

$$(\Phi)_{u=0}=\frac{e}{8a},$$

$$C=8a.$$

The charge distribution on a circular disk can be obtained most readily from the general expression above. If z be eliminated by means of the equation of the ellipsoid, the general expression is

$$\eta=\frac{e}{4\pi ab\sqrt{1-\frac{x^2}{a^2}\left(1-\frac{c^2}{a^2}\right)-\frac{y^2}{b^2}\left(1-\frac{c^2}{b^2}\right)}},$$

which, for $c=0$, gives

$$\eta=\frac{e}{4\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}.$$

Or, if $a=b$ and $x^2+y^2=r^2$,

$$\eta=\frac{e}{4\pi a\sqrt{a^2-r^2}}.$$

Since this density occurs on both sides of the ellipsoid, it is necessary, in the limit $c=0$, to double this value, so that, finally,

$$\eta=\frac{e}{2\pi a\sqrt{a^2-r^2}}.$$

A round rod corresponds to the values $a=b$, $c\gg a$. Then

$$\Phi=\frac{e}{8\pi c\sqrt{1-\frac{a^2}{c^2}}}\log\frac{\sqrt{1+\frac{u}{c^2}}+\sqrt{1-\frac{a^2}{c^2}}}{\sqrt{1+\frac{u}{c^2}}-\sqrt{1-\frac{a^2}{c^2}}},$$

$$=\frac{e\left(1+\frac{a^2}{2c^2}\right)}{8\pi c}\log\frac{4c^2+u-a^2}{u+a^2},$$

terms of higher order than $(a/c)^2$ or u/c^2 being neglected in the last expression.

§ 30. *Two-dimensional (Logarithmic) Potential.*—In many cases of practical importance the charged conductors, whose potentials are to be investigated, are long parallel cylinders. The distribution of charge on these conductors may be expected to be nearly uniform (as regards displacement along the cylinder) except near the ends. The value of the potential at a point P outside the conductors will be practically independent of a translation of P parallel to the axes of the cylinders, as long as P lies in a region R near the middle portion of the conductors. In this region R the potential is thus effectively a function of but two variables. If a knowledge of the behavior of the potential in the region R is alone required, the problem may be simplified by restricting the consideration to this region, thus reducing the mathematical problem, from the beginning, to one of two dimensions.

This approximate two-dimensional behavior of the problem is illustrated by the case of a long prolate ellipsoid of revolution. The potential has just been found to be

$$\Phi=\frac{e}{8\pi c}\left(1+\frac{a^2}{2c^2}\right)\log\frac{4c^2+u-a^2}{u+a^2},$$

where a is the radius of a cross-section of the ellipsoid, c is the half-length, and u is the ellipsoidal co-ordinate, determined for any point x, y, z by the equation

$$\frac{r^2}{a^2+u}+\frac{z^2}{c^2+u}=1,$$

where

$$r^2=x^2+y^2.$$

If the ellipsoid be very long, and if the consideration be restricted to a region R near the center, then a/c, z/c, and u/c are all small. Since z/c is small, it follows that

$$u=r^2-a^2.$$

On substituting this value for u, and neglecting squares of the small quantities, it is seen that the potential is approximately represented in the region R by the function

$$\Phi=\frac{e}{4\pi c}\left[\log\frac{1}{r}+\log 2c\right],$$

or

$$\Phi = \frac{e}{4\pi c} \log \frac{1}{r} + \text{Constant} .$$

The perpendicular distance D from the center of the ellipsoid to a plane tangent to the ellipsoid is approximately equal to $a = b$ if the point of tangency lies in the region R. Thus the surface charge in the region under consideration is given by

$$\eta = \frac{e}{4\pi abc} D = \frac{e}{4\pi ac} ,$$

so that

$$e' = 2\pi a \eta = \frac{e}{2c} ,$$

where e' is the total charge per unit length. The approximate potential at any point in R can thus be written

$$\Phi = \frac{e'}{2\pi} \log \frac{1}{r} + \frac{e'}{2\pi} \log 2c ,$$

and the potential of the ellipsoid itself is approximately

$$\Phi_0 = \frac{e'}{2\pi} \log \frac{1}{a} + \frac{e'}{2\pi} \log 2c .$$

This logarithmic law of variation of the potential in the region R may be directly obtained without knowledge of the potential of a long prolate ellipsoid if the assumption, discussed above, be made that the potential in R is sufficiently well represented by a function Φ independent of z.

Laplace's equation in cylindrical co-ordinates reduces, for the case of a function independent of both z and the angle θ, to

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\Phi}{dr} \right) = 0 .$$

Then

$$r \frac{d\Phi}{dr} = a ,$$

$$\Phi = a \log r + \beta ,$$

where α and β are constants of integration. If the charge per unit length of the cylinder be e', then, for $r=a$,

$$e' = -\int \frac{d\Phi}{dr}\, d\sigma = -\int_0^{2\pi} \frac{\alpha}{a}\, a\, d\theta ,$$
$$= -2\pi\alpha ,$$

so that

$$\Phi = \frac{e'}{2\pi} \log \frac{1}{r} + \beta .$$

§ 31. *Cylindrical Condenser.*—A cylindrical condenser is formed of two co-axial conducting cylinders. Let the external radius of the smaller cylinder be r_0, and the internal radius of the large cylinder be r_1, and suppose that the condenser be provided at the ends with guard cylinders to eliminate end effects, i.e., to extend the region R in which the potential is independent of z to include the total length of the cylinders under consideration. Then the potential at points between the two cylinders may be assumed to have radial symmetry so that, from the foregoing result,

$$\Phi = \frac{e'}{2\pi} \log \frac{1}{r} + \beta .$$

Then the potentials Φ_0 and Φ_1 of the inner and outer cylinders are given by

$$\Phi_0 = \frac{e'}{2\pi} \log \frac{1}{r_0} + \beta ,$$

$$\Phi_1 = \frac{e'}{2\pi} \log \frac{1}{r_1} + \beta ,$$

and the capacity per unit length is

$$C = \frac{e'}{\Phi_0 - \Phi_1} = \frac{2\pi}{\log \frac{r_1}{r_0}} ,$$

where e' is the charge per unit length.

§ 32. *Capacity of a Wire to Earth.*—Let r_0 be the radius of a long wire which is at distance b from a plane conductor, and suppose r_0 small as compared to b. Let e' be the charge per unit length of the wire, and

assume, as above, that the potential is independent of distance parallel to the wire. The potential due to the wire alone, at a distance r, is

$$\frac{e'}{2\pi}\log\frac{1}{r}+\text{Constant.}$$

If r be large compared to r_0, this expression is valid regardless of the distribution of the charge on the wire. Let the potential of the plane conductor (the earth) be zero. The charge induced on the plane conductor then nullifies, for points on the plane, the effect of the logarithmic variation of the potential due to the wire. It follows, as in the corresponding three-dimensional case, that the image method gives an immediate solution, the potential being

$$\frac{e'}{2\pi}\log\frac{r'}{r},$$

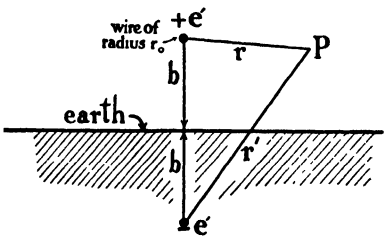

FIG. 33.—The electrostatic problem for a long wire of radius r_0 located b units above the surface of the earth, and having a charge e' per unit length of wire.

where r' is the distance to the image wire. The potential of the wire is then

$$\frac{e'}{2\pi}\log\frac{2b}{r_0},$$

and the capacity per unit length of the wire against the earth is

$$C=\frac{2\pi}{\log\frac{2b}{r_0}}.$$

PROBLEMS FOR PART II, CHAPTER II

1. Find the potential due to an isolated spherical conductor whose total charge e is known, by solving the equation $\nabla^2\Phi=0$ in spherical polar co-ordinates.
2. The Newtonian potential due to a solid homogeneous sphere of volume density ρ satisfies Poisson's equation $\nabla^2\Phi=-\rho$. Use the form this equation takes in spherical polar co-ordinates to find the Newtonian potential, at external points, due to such a sphere. Evaluate the con-

stants of integration from a consideration of regularity at infinity, and from the fact that the expression for the potential must approach, for distant points, the form $M/4\pi r$, where M is the mass of the sphere. Obtain an expression for the potential at inside points, evaluating the constants by making use of the fact that the potential and its normal derivative are both continuous at the surface of the sphere.

3. Show that the induced surface density of charge on a plane, due to the presence of a charge e located a units from the plane, is

$$-\frac{ae}{2\pi r^3},$$

where r is the distance to the point on the plane under consideration from the charge e.

4. Referring to the preceding problem, find the force on the charge e due to the induced surface charge.
5. Consider two very long cylindrical conductors, of radius a, placed parallel to one another, a distance c apart, and each a height b above the earth. Show that, if a be small compared to b and c, the capacity, per unit length, between the two conductors, when they are equally but oppositely charged, is

$$\frac{2\pi}{\log \frac{4b^2c^2}{a^2(c^2+4b^2)}}.$$

6. Referring to the preceding problem, show that, when the two conductors are charged with the same sign, their capacity, per unit length, with the earth is

$$\frac{4\pi}{\log \frac{2b\sqrt{c^2+4b^2}}{ac}}.$$

7. By analytical methods, find the quantities e_1, e_2, and e_3 for polar co-ordinates in the plane, and for polar co-ordinates in space.
8. Derive an expression for the capacity of three condensers in series.
9. An infinite conductor occupies all of the region where x and y are not both positive. A charge e is located at the point x, y, 0 where x and y are both positive. What is the magnitude and direction of the force on the charge e?

Part III. Dielectrics

Introduction

In Part I of the present chapter the force on an interior charge was found to be $\boldsymbol{E}+\boldsymbol{P}/3+\boldsymbol{f}_1+\boldsymbol{f}_2$ where the first two terms accounted for the charges outside a sphere of radius δ, where $\boldsymbol{f}_1$ was the constitutive force due to the charges of the few molecules immediately neighboring the point in question, and where $\boldsymbol{f}_2$ was due to all other charges within the δ sphere. A conductor, it will be remembered, is defined as a body for which the characteristic constitutive force $\boldsymbol{f}_1$ is always zero, so that if the other components of the total force do not vanish or cancel, movement of charge from atom to atom will result. A dielectric, on the other hand, is a body so constructed that attempted movements of charge are opposed. As soon as a charge shifts its position when acted on by an excess force, this shift produces an alteration in the constitutive force $\boldsymbol{f}_1$, and a new balance is reached in a way analogous to the mechanical case of an elastic displacement. To lead to a definite quantitative relation upon which the analytical theory of dielectrics can be based, it is necessary to consider in detail the various forces which an interior charge experiences, both under normal circumstances (i.e., when the body is uncharged and unpolarized) and when the body is under the influence of other charges or charged bodies. It is first argued that, just as in the theory of a conductor, the force $\boldsymbol{f}_2$ is always zero. It follows from this that the normal constitutive force $\boldsymbol{f}_1^0$ is zero, the upper index indicating the value of the constitutive force when the body is uncharged and unpolarized. When a body is charged and polarized, an interior charge experiences a total force equal, since $\boldsymbol{f}_2=0$, to

$$\boldsymbol{E}+\frac{\boldsymbol{P}}{3}+\boldsymbol{f}_1=0\,.$$

Thus when a body is polarized, the shift of charge brings into play a nonvanishing constitutive force $\boldsymbol{f}_1$ which, reversed, balances the force $\boldsymbol{E}+\boldsymbol{P}/3$ due to the charges outside the δ-sphere. The analytical theory of dielectrics is based upon the assumption that this constitutive force is proportional to the shift which gives rise to it, and hence proportional to $\boldsymbol{P}$. It follows at once that $\boldsymbol{E}$ is proportional to $\boldsymbol{P}$. With this proportionality as a basis, the electrostatic problem for dielectrics is quickly

and easily formulated in much the same way as was previously done for conductors.

An example is given of the use of the image method when dielectrics are present, and two special cases are treated and illustrated by examples; namely, the case of the polarization of a body in an external field which is not appreciably affected by the polarization which it causes, and the case of a body which polarizes uniformly in a given field.

§ 33. *The Definition of a Dielectric: the Mathematical Formulation of the Electrostatic Problem for Dielectrics.*—A conductor has been defined as a body whose atoms are so constructed and combined that electrons are free to pass from atom to atom. A dielectric, on the other hand, is a body whose atoms are so constructed and combined that the effective position of each electron remains near the normal position which it occupies when the body is not under the electrostatic influence of any other body. When acted on by a force in excess of the normal forces of constitution, an electron is influenced in such a way that its effective position shifts slightly from its normal location, and assumes a new position where it is in stable equilibrium under all the forces now acting on it.

In order to make a definite quantitative assumption upon which an analytical theory of dielectrics can be based, it is necessary to consider, as in a theory of conductors, the total force which an interior charge experiences. This total force may be written

$$E+\frac{P}{3}+f_1+f_2\ ,$$

where the first two terms represent the force due to the charges outside a sphere of radius δ and center at the charge in question; where f_1 is the constitutive force due to the charges of the few molecules immediately neighboring the point in question; and where f_2 is due to the remaining charges within the δ-sphere.

Consider first the force f_2. It is due to charges lying in a spherical shell of exterior radius δ and an interior radius which is a small multiple of the average distance between molecules. The charges lying in this spherical shell are located in a way which is sufficiently symmetrical, about the center, to give rise to continuous densities ρ and P which are (since they are constant within δ) perfectly symmetrical about the center of the δ-sphere. Now a perfectly symmetrical distribution of charges, located in this spherical shell, would produce a zero force at the center. It is reasonable to assume that a distribution of charges which gives rise to symmetrical functions ρ and P is itself so nearly symmetri-

cal as to permit the conclusion that the force f_2 is zero. It should be noted that this argument applies to the value of f_2 when the body is not charged or polarized, and also to the value of f_2 when the body is charged and polarized, just so long as the continuous densities ρ and P are sensibly constant over the δ-sphere.* This argument for the vanishing of f_2 is independent of the type of body (conductor or dielectric) under consideration. In the theory of the conductor, given in Part I of this chapter, the vanishing of f_2 was argued on the same grounds; the reasoning is repeated here merely for the sake of completeness.

Consider, now, an uncharged and unpolarized body in electrostatic equilibrium. The total force on each interior charge is, since E and P are zero,

$$f_1^0+f_2^0=0\,,$$

where the upper indices are used to indicate the "normal" values of these forces, i.e., the values when the body is not under the electrostatic influence of any other charges or charged bodies. Since, however, the force f_2^0 is always zero, it follows that the normal constitutive force f_1^0 is also zero.

When this body is under the electrostatic influence of other charges or charged bodies, the force on an interior charge is

$$E+\frac{P}{3}+f_1=0\,,$$

where the force $f_2=0$ has been omitted. Thus, in equilibrium, the force $E+P/3$, due to the charges outside the δ-sphere, is balanced by the reversed constitutive force, $-f_1$.

The normal constitutive force has just been seen to be.zero. When a dielectric is brought under the influence of other charges or charged bodies, the charges forming the dielectric in question move, and shift their effective position in such a way that a non-vanishing constitutive force f_1 is brought into play to balance the force $E+P/3$. The analytical theory of dielectrics is based upon the assumption that the constitutive force f_1 which exists when the charges are in this shifted position is itself proportional to the shift, and hence proportional to the resulting polarization. If, then, f_1 is proportional to P, it follows that

$$E=-\frac{P}{3}-f_1$$

* See footnote to § 18 and Part I, Problem 9, of this chapter.

is also proportional to $\boldsymbol{P}$. Using $(\epsilon-1)$ as the proportionality factor,*

$$\boldsymbol{P}=(\epsilon-1)\boldsymbol{E}\,, \tag{79}$$

where ϵ is the so-called "dielectric constant" of the material.†

Since the displacement of all charges is small, and since the body is supposed to be uncharged when in its normal unpolarized condition, it is evident that the volume cells $\Delta\tau_i$ may be chosen so that $\rho_i=0$. It is therefore natural to say that such a body is uncharged but polarized, and to describe the potential due to it in terms of the volume density of polarization $\boldsymbol{P}$ as obtained by interpolation from the values

$$\boldsymbol{P}_i=\boldsymbol{p}_i/\Delta\tau_i\,,$$

and in terms of a surface density of polarization μ. That is,

$$\Phi=\frac{1}{4\pi}\int\left(\boldsymbol{P}',\nabla'\frac{1}{r}\right)d\tau'+\frac{1}{4\pi}\int\left(\mu',\nabla'\frac{1}{r}\right)d\sigma'\,. \tag{80}$$

Now (80) may be written (see equation [44]),

$$\Phi=\frac{1}{4\pi}\int\frac{-\operatorname{div}'\boldsymbol{P}'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{P_n}{r}\,d\sigma'+\frac{1}{4\pi}\int\left(\mu',\nabla'\frac{1}{r}\right)d\sigma'\,. \tag{81}$$

This equation is exactly similar to the one studied in § 21, except that $-\operatorname{div}'\boldsymbol{P}'$ and P_n' now play the rôles previously played by ρ' and η'. It therefore follows from (68) that‡

$$\operatorname{div}\boldsymbol{E}=-\operatorname{div}\boldsymbol{P}\,,$$

or

$$\operatorname{div}(\boldsymbol{E}+\boldsymbol{P})=0\,. \tag{82}$$

* This relationship holds for an isotropic body. For an anisotropic body each component of $\boldsymbol{P}$ is assumed to be a linear function of the components of $\boldsymbol{E}$. See, e.g. Livens, *The Theory of Electricity* (1926), art. 79.

† The dielectric constant ϵ is also sometimes called the "specific inductive capacity." There is, in fact, some objection to calling ϵ a "constant" since, in general, its value for a given body depends upon frequency. This consideration does not, of course, enter into electrostatics.

‡ Equation (68) states, in fact, that div $\boldsymbol{E}$ at x,y,z is equal to the value of $-\operatorname{div}'\boldsymbol{P}'$ at $x'=x,\ y'=y,\ z'=z$, i.e., the value of $-\operatorname{div}\boldsymbol{P}$, where $\boldsymbol{P}$ is the same function of x,y,z as is $\boldsymbol{P}'$ of x',y',z'

On account of the linear relation between $\boldsymbol{P}$ and $\boldsymbol{E}$, this last equation reduces to the form

$$\operatorname{div} \epsilon \boldsymbol{E} = 0 .$$

For a homogeneous substance, moreover, the dielectric constant ϵ has the same value at all points of the body, so that

$$\operatorname{div} \epsilon \boldsymbol{E} = \epsilon \operatorname{div} \boldsymbol{E} ,$$

and it follows that the potential due to a dielectric satisfies, at all points, the partial differential equation

$$\operatorname{div} \boldsymbol{P} = -\operatorname{div} \boldsymbol{E} = \nabla^2 \Phi = 0 .$$

It should be especially noted that, in the case of an ideal dielectric, the divergence of the polarization vanishes.

Further conditions on Φ and its derivatives may be obtained by modifying those previously found for the potential due to a conductor, by replacing η by the function which now plays the rôle previously played by η. Thus suppose that two dielectrics, 1 and 2, are in contact along the surface S. The potential due to these two dielectrics may then be written, as in (81), in terms of the volume integral of $-\operatorname{div} \boldsymbol{P}$ and in terms of the surface integrals of the normal component of $\boldsymbol{P}$. On the surface S, the function

$$-(P_{n_1} + P_{n_2})$$

plays the rôle previously played by η, the negative sign resulting from the fact that $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ point into the regions 1 and 2, while P_n, for either dielectric, is the component of $\boldsymbol{P}$ along the normal exterior to that dielectric. Thus the previous equation

$$\frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = -\eta$$

becomes, in the present case,

$$\frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = P_{n_1} + P_{n_2} ,$$

or

$$\text{(83)} \qquad \begin{cases} -E_{n_1} - E_{n_2} = P_{n_1} + P_{n_2} , \\ (E+P)_{n_1} + (E+P)_{n_2} = 0 , \end{cases}$$

where, according to the usual convention, $(E+P)_{n_1}$ means the limit, as the surface is approached from the side 1, of the component of $\boldsymbol{E}+\boldsymbol{P}$ along the normal directed into 1. From the linear relation between $\boldsymbol{E}$ and $\boldsymbol{P}$ this last equation becomes

$$\epsilon_1 E_{n_1}+\epsilon_2 E_{n_2}=0\,,$$

or

$$\epsilon_1 \frac{\partial \Phi}{\partial n_1}+\epsilon_2 \frac{\partial \Phi}{\partial n_2}=0\,, \tag{84}$$

where ϵ_1 and ϵ_2 are the dielectric constants of substances 1 and 2, respectively. From equation (79) it is clear that empty space may be re-

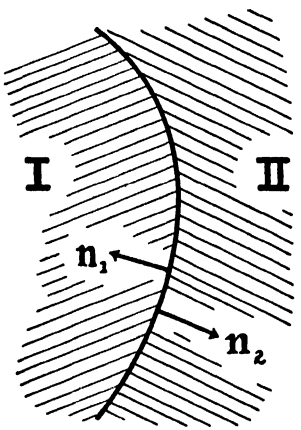

FIG. 34.—The boundary between two regions

garded as a dielectric whose dielectric constant has the value unity. Hence this last equation may be written in the form

$$\epsilon \frac{\partial \Phi}{\partial n_1}+\frac{\partial \Phi}{\partial n_2}=0\,, \tag{85}$$

for application to the surface between a dielectric and space.

The behavior of the normal derivative of Φ at the boundary between a conductor and a dielectric can be obtained in a manner similar to that used above. Indeed, if both conductors and dielectrics are pres-

ent the potential is given by the general expression (41), which, by (44), can be written

$$\Phi=\frac{1}{4\pi}\int\frac{(\rho'-div'\,\boldsymbol{P}')}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{(\eta'+P'_n)}{r}\,d\sigma'+\frac{1}{4\pi}\int\left(\boldsymbol{\mu}',\nabla'\,\frac{1}{r}\right)d\sigma'\,,$$

where $\boldsymbol{n}$ is a normal exterior to the dielectric. It therefore follows from the analytical study of § 22 that

$$\frac{\partial\Phi}{\partial n_1}+\frac{\partial\Phi}{\partial n_2}=-\eta-P_n\,,$$

or if 1 is a conductor and 2 is a dielectric,

$$-E_{n_1}-E_{n_2}+P_n=-\eta\,,$$

$$(E+P)_{n_2}=\eta\,,$$

since $\boldsymbol{E}$ is zero in the conductor, and since P_{n_1} and P_{n_2} are components along oppositely directed normals. Thus,

$$(\epsilon E)_{n_2}=\eta\,,$$

or

$$\epsilon\frac{\partial\Phi}{\partial n}=-\eta\,,$$

where $\boldsymbol{n}$ is, of course, a normal exterior to the conductor. Equation (75) is therefore replaced, when the conductor is surrounded by a dielectric, by the equation

$$\int\epsilon\frac{\partial\Phi}{\partial n}\,d\sigma=-e\,, \tag{86}$$

where n, now, is measured along the normal exterior to the conductor, and where the subscript can be dropped from the n without confusion, since at the surface of the conductor only one of the normal derivatives of Φ is different from zero. This more general condition to be applied to the surface of a conductor includes the previous condition, as is seen at once by giving ϵ the value characteristic of free space.

The foregoing discussion gives rise to the following schedule of conditions on the potential Φ, in which, for completeness, the conditions which apply to conductors are also given. These conditions constitute

the mathematical formulation of the electrostatic problem when both conductors and dielectrics are present.

(II)

a) $\nabla^2\Phi = 0$ at all points in free space and at all points within conductors or dielectrics;

b) Φ is continuous everywhere except across surfaces where $\mu \neq 0$ (across such surfaces $\Phi_1 - \Phi_2 = \mu$);

c) Across a surface between two dielectrics

$$\epsilon_1 \frac{\partial \Phi}{\partial n_1} + \epsilon_2 \frac{\partial \Phi}{\partial n_2} = 0 \ ;$$

d) Across a surface between a conductor and a dielectric

$$\epsilon \frac{\partial \Phi}{\partial n} = -\eta \ ;$$

e) On the surface of a conductor:

i) Φ is a known constant Φ_i, or

ii) Φ is an unknown constant, and

$$\int \epsilon \frac{\partial \Phi}{\partial n} \, d\sigma = -e_i \ ;$$

f) Φ is regular at infinity.

The proof of the uniqueness of the solution when dielectrics are present is left to the reader (see Part III, Problem 30, of this chapter).

On the basis of the uniqueness theorem, the following theorem can easily be established: Given a configuration of conductors C_i of total charges e_i. If the potential at any point, when these conductors are located in empty space, is Φ, then the potential when these same conductors are located in a medium of dielectric constant ϵ is Φ/ϵ. For, in the first instance, the conditions on Φ are

$\nabla^2\Phi = 0$ all points,

Φ = Constant on each conductor,

Φ is regular at ∞,

$$\int_{c_i} \frac{\partial \Phi}{\partial n} \, d\sigma = -e_i \ .$$

In the second case only the last condition is changed. It is, in fact, replaced by

$$\int_{c_1} \epsilon \frac{\partial \Phi}{\partial n} d\sigma = -e_1 ,$$

and the theorem stated is obviously true.

Thus the potential due to a conducting sphere of total charge e located in a medium of dielectric constant ϵ is

$$\Phi = \frac{e}{4\pi r \epsilon} .$$

It is instructive to interpret this expression in terms of the actual charges present. From symmetry, the charge ϵ on the sphere would be uniformly spread on its surface. The dielectric medium will be polarized according to the equation

$$P = P_r = (\epsilon - 1)\left[\frac{e}{4\pi r^2} + \frac{4\pi P_n a^2}{4\pi r^2}\right] ,$$

since the total electrostatic intensity at any point is that due to the charge e and that due to the uniform surface density P_n on the internal bounding surface of the dielectric. When $r = a$, $P_r = -P_n$, so that

$$-P_n = (\epsilon - 1)\left[\frac{e}{4\pi a^2} + P_n\right] ,$$

or

$$P_n = -\frac{\epsilon - 1}{\epsilon} \frac{e}{4\pi a^2} ,$$

and the electrostatic intensity at any point is

$$E = E_r = \frac{e}{4\pi r^2} - \frac{4\pi a^2}{4\pi r^2} \frac{(\epsilon - 1)e}{\epsilon \, 4\pi \, a^2} ,$$

$$= \frac{e}{4\pi r^2} - \frac{(\epsilon - 1)}{\epsilon} \frac{e}{4\pi r^2} = \frac{e}{4\pi r^2 \epsilon} .$$

Thus the intensity $e/4\pi r^2$ due to the charge e is reduced by the negative surface charge on the internal bounding surface of the dielectric, the part subtracted being just enough to reduce the total intensity to the value $e/4\pi r^2 \epsilon$. A similar compensation operates in the general case, as the foregoing theorem shows.

§ 34. *The Image Method for Dielectrics.*—The image method, which has been seen to furnish solutions of electrostatic problems involving conductors, may also be used when dielectrics are present. A simple case will illustrate the way in which the method is applied. Consider, for example, a charge e located a distance a from the plane surface of a dielectric whose dimensions are so great that, for purposes of analysis, it may be considered as occupying all of space to the right of the plane which coincides with the plane face in question. Since div $\boldsymbol{P}$ is zero, the potential due to the dielectric has the form

$$4\pi\Phi = \int \frac{P'_n}{r}\, d\sigma' ,$$

the integral being extended over the plane face. It is clear from the symmetry of this expression that if the potential due to the dielectric can, at outside points, be duplicated by an image charge e' within the dielectric and at a distance x from the plane face, then the potential due to the dielectric can, at inside points, be duplicated by a similar and similarly located image charge outside the dielectric. Thus for points P_1 in free space

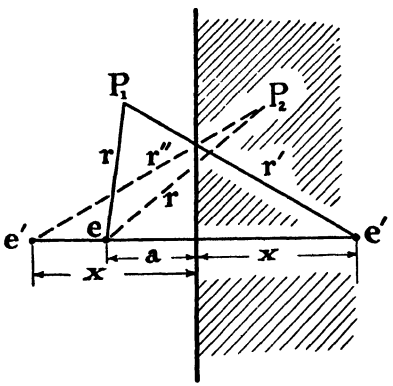

FIG. 35.—A charge e located a units from the plane surface of an infinite dielectric. The figure, for clarity, shows the distances a and x as different, although they are actually equal.

$$4\pi\Phi = \frac{e}{r} + \frac{e'}{r'} , \tag{87}$$

where r and r' are the distances from P_1 to e and to the image charge e' which is located a distance x within the dielectric; while for points P_2 in the dielectric

$$4\pi\Phi = \frac{e}{r} + \frac{e'}{r''} , \tag{88}$$

where r'' is the distance from P_2 to the charge e' outside the dielectric. It is evident that each of these expressions satifies the equation $\nabla^2\Phi = 0$, and is regular at infinity; and that the potential, as given by these two

expressions, is continuous across the boundary of the dielectrics. It remains to satisfy the condition

$$\epsilon_1 \frac{\partial \Phi}{\partial n_1} + \epsilon_2 \frac{\partial \Phi}{\partial n_2} = 0 ,$$

which, in this case, may be written

$$\frac{\partial \Phi}{\partial n_1} + \epsilon \frac{\partial \Phi}{\partial n_2} = 0 ,$$

or

$$E_{n_1} + \epsilon E_{n_2} = 0 .$$

Now, from (87) it follows that E_{n_1} at a point whose distance, along the surface of the dielectric, from the foot of the perpendicular through e, is δ, is given by

$$E_{n_1} = \frac{1}{4\pi}\left[\frac{e}{r^2}\cos(r,n) + \frac{e'}{r'^2}\cos(r',n)\right] = \frac{1}{4\pi}\left[-\frac{ae}{(a^2+\delta^2)^{3/2}} + \frac{e'x}{(x^2+\delta^2)^{3/2}}\right].$$

Likewise, from (88)

$$E_{n_2} = \frac{1}{4\pi}\left[\frac{ae}{(a^2+\delta^2)^{3/2}} + \frac{e'x}{(x^2+\delta^2)^{3/2}}\right],$$

so that the foregoing condition becomes

$$e(\epsilon-1)\frac{a}{(a^2+\delta^2)^{3/2}} + e'(\epsilon+1)\frac{x}{(x^2+\delta^2)^{3/2}} \equiv 0 ,$$

which must hold identically in δ. Hence, comparing coefficients of like powers of δ,

$$x = a ,$$

$$e' = -\frac{\epsilon-1}{\epsilon+1}\, e .$$

The image charge is thus located at the same distance from the dielectric as is the charge e, and the magnitude of the image charge depends upon the dielectric constant of the material. It may be noted that as ϵ increases without limit the solution of the corresponding problem involving an infinite conductor is approached.

§ 35. *The Polarization of a Dielectric in a Given External Field.*— The problem of determining the polarization of a dielectric and the potential due to a dielectric is, in general, a complicated one; but it is considerably simplified providing the dielectric in question is far enough from all other bodies so that changes in its polarization do not sensibly affect the configuration of charges on these other bodies, i.e., provided $\boldsymbol{E}^0$, the intensity due to everything except the dielectric in question, may be regarded as unaffected by the presence of the dielectric. Then

$$\boldsymbol{P}=(\epsilon-1)\boldsymbol{E}$$

may be written

$$\boldsymbol{P}=(\epsilon-1)\,[\boldsymbol{E}^0-\nabla U]\,,$$

where U is the potential due to the dielectric alone. If U can be determined, $\boldsymbol{P}$ may be obtained from this last equation. The potential U must satisfy the conditions

a) $\nabla^2 U=0$,

b) U is continuous,*

c) $\epsilon\left(\dfrac{\partial U}{\partial n}\right)_i-\left(\dfrac{\partial U}{\partial n}\right)_0=(\epsilon-1)E^0_n$,

d) U is regular at infinity.

Condition (*c*) is obtained from the relation

$$\epsilon\, E_{n_1}+E_{n_2}=0\,,$$

where region 1 is the dielectric and region 2 is the surrounding space. Then

$$E_{n_1}=E^0_{n_1}-\frac{\partial U}{\partial n_1}=-E^0_n+\left(\frac{\partial U}{\partial n}\right)_i,$$

$$E_{n_2}=E^0_{n_2}-\frac{\partial U}{\partial n_2}=E^0_n-\left(\frac{\partial U}{\partial n}\right)_0,$$

where n is measured in the direction of an exterior normal to the dielectric, and where the subscript i indicates the limiting value of the derivative as the surface is approached from inside. Similarly, the subscript 0 indicates a limiting outside value. If these expressions be substituted in (II), (*c*) of § 33, condition (*c*) is obtained.

* It is assumed here that there is no superficial polarized layer.

That the solution of these conditions is unique follows at once from the more general proof for the schedule (II) of § 33.

To illustrate the use of these equations the solution will be obtained for the case of a dielectric sphere of radius a, placed in a uniform external field $\boldsymbol{E}^0$. Choosing spherical co-ordinates with the line $\theta=0$ in the direction of $\boldsymbol{E}^0$, it is clear from symmetry that the potential U is independent of φ. The general equation $\nabla^2 U=0$ therefore reduces to the form

$$r\frac{\partial^2(rU)}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\,\dfrac{\partial U}{\partial\theta}\right)}{\partial\theta}=0\,. \tag{89}$$

It is necessary to obtain two special solutions U_1 and U_0 of this equation (to be used for the potential inside and outside the sphere, respectively) which satisfy the conditions

$$\left.\begin{aligned} U_1&=U_0\\ \epsilon\left(\frac{\partial U_1}{\partial n}\right)_1-\left(\frac{\partial U_0}{\partial n}\right)_0&=(\epsilon-1)E^0\cos\theta \end{aligned}\right\}\ \text{at } r=a\,,$$

U is regular at infinity.

If it be assumed that U is the product of a function of r alone and a function of θ alone, the following solution of (89) is readily obtained:*

$$U=\sum_{n=0}^{n=n}\left(b_n r^n+\frac{c_n}{r^{n+1}}\right)P_n(\cos\theta)\,,$$

where n is a positive integer, where b_n and c_n are arbitrary constants, and where $P_n(\cos\theta)$ are the surface zonal harmonics or Legendre polynomials previously referred to in § 5.

It is clearly not possible to use a negative exponent for r at points within the sphere, since this would give rise to an infinite value of U; and it is also clear that a positive exponent for r cannot be used at outside points, since this would interfere with the regularity at infinity. The fact that only the first power of $\cos\theta$ enters the boundary conditions leads one to attempt to satisfy the conditions by the use of

$$P_1(\cos\theta)\equiv\cos\theta$$

* See, e.g., Byerly, *Fourier Series and Spherical Harmonics*, arts. 74–77.

alone. These remarks suggest the particular solutions

$$U_i = br\cos\theta\,,$$

$$U_0 = \frac{c\cos\theta}{r^2}\,,$$

which will now be tested to determine whether the constants b and c can be given such values as to make these solutions of (89) conform to all the conditions of the problem. The continuity of U across the surface of the sphere demands that

$$ba\cos\theta = \frac{c}{a^2}\cos\theta\,,$$

or

$$b = \frac{c}{a^3}\,,$$

so that

$$U_0 = \frac{ba^3\cos\theta}{r^2}\,.$$

Thus

$$\left(\frac{\partial U_i}{\partial r}\right)_{r=a} = b\cos\theta\,,$$

$$\left(\frac{\partial U_0}{\partial r}\right)_{r=a} = -2b\cos\theta\,,$$

so that, to satisfy the remaining condition, one must have

$$\epsilon b\cos\theta + 2b\cos\theta = (\epsilon - 1)E^0\cos\theta\,,$$

or

$$b = \frac{\epsilon - 1}{\epsilon + 2}E^0\,.$$

All of the requirements of the problem are thus met by the functions

$$U_i = \frac{\epsilon - 1}{\epsilon + 2}E^0 r\cos\theta\,,$$

or

$$U_i = \frac{\epsilon - 1}{\epsilon + 2}E^0 x\,,$$

where $x=r\cos\theta$, and

$$U_0=\frac{\epsilon-1}{\epsilon+2}\frac{a^3}{r^2}E^0\cos\theta\,.$$

The potential at outside points is of the same type as that due to a doublet.

The state of polarization of the sphere can now be determined from the equation

$$P=(\epsilon-1)E=(\epsilon-1)\left(E^0-\frac{dU_i}{dx}\right).$$

Thus

$$\boldsymbol{P}=(\epsilon-1)\left[\boldsymbol{E}^0-\left(\frac{\epsilon-1}{\epsilon+2}\right)\boldsymbol{E}^0\right]=\frac{3(\epsilon-1)}{\epsilon+2}\,\boldsymbol{E}^0\,,$$

so that the polarization of the sphere is seen to be uniform.

If the intensity at points within the sphere be written in the form

$$\boldsymbol{E}=\boldsymbol{E}^0-L\boldsymbol{P}\,,$$

it is easily checked from the foregoing expressions that L, which is known as the "depolarizing factor," has the value 1/3. The significance of the name is evident from the fact that the field due to the polarization is such as to weaken the total field, and hence such as to weaken the polarization itself.

§ 36. *The Potential Due to a Uniformly Polarized Dielectric.*—It has just been seen that a dielectric sphere is uniformly polarized by a constant external field. It is sometimes possible to determine the polarization of a dielectric in a constant external field and the potential U due to the dielectric by assuming that it is uniformly polarized, and then showing that the potential U which results from this assumption satisfies all the conditions of the problem. Since $\operatorname{div}\boldsymbol{P}=0$ for an ideal dielectric, the potential U can be written

$$U=\frac{1}{4\pi}\int\frac{P'_n d\sigma'}{r}\,. \tag{90}$$

A function U determined from this equation satisfies $\nabla^2U=0$, is continuous, and is regular at infinity; its derivatives, moreover, satisfy the relation

$$\left(\frac{\partial U_i}{\partial n}\right)_1-\left(\frac{\partial U_0}{\partial n}\right)_0=P_n\,, \tag{91}$$

where $\boldsymbol{n}$ is the exterior normal to the dielectric. If $\boldsymbol{P}$ also satisfies the relation

$$\boldsymbol{P}=(\epsilon-1)(\boldsymbol{E}^0-\nabla U)\,, \tag{92}$$

so that

$$P_n=(\epsilon-1)\left[E_n^0-\left(\frac{\partial U_i}{\partial n}\right)_i\right],$$

equation (91) becomes

$$\epsilon\left(\frac{\partial U_i}{\partial n}\right)_i-\left(\frac{\partial U_0}{\partial n}\right)_0=(\epsilon-1)E_n^0\,. \tag{93}$$

That is, if a function U is obtained from (90) under the assumption that $\boldsymbol{P}$ is a constant, this function U satisfies all the necessary conditions provided the constant value of $\boldsymbol{P}$ be determined from (92). Whether a constant value of $\boldsymbol{P}$ can be determined from (92) clearly depends upon whether ∇U is a constant. Thus if the assumption of a constant $\boldsymbol{P}$ leads, by (90), to a U whose rate of change in any direction is constant at points within the dielectric, the constant value of $\boldsymbol{P}$ can be determined from (92), and all the conditions of the problem are satisfied.

If $\boldsymbol{P}$ is a constant, equation (90) can be written in a more useful form. Indeed, this surface integral is equal, since div $\boldsymbol{P}=0$, to the volume integral

$$U=\frac{1}{4\pi}\int\left(\boldsymbol{P}',\nabla'\frac{1}{r}\right)d\tau'\,,$$

or

$$U=\frac{P}{4\pi}\int\frac{d'\dfrac{1}{r}}{dh}\,d\tau'\,,$$

where h is measured in the direction of $\boldsymbol{P}$, and where the prime on the symbol of differentiation indicates that the variable end of r has coordinates x', y', z'. This equation can then be re-written in the form

$$U=-\frac{P}{4\pi}\int\frac{\partial\dfrac{1}{r}}{\partial h}\,d\tau'=-\frac{P}{4\pi}\frac{\partial}{\partial h}\int\frac{d\tau'}{r}=-\frac{1}{4\pi}\left(\boldsymbol{P},\nabla\int\frac{d\tau'}{r}\right), \tag{94}$$

where the end of r whose co-ordinates are x, y, z is now variable. The potential due to a uniformly polarized dielectric can thus be expressed in terms of the constant polarization vector and the Newtonian potential function due to the body. This form is useful since the Newtonian potential function is known for many geometrical forms, and it may be easily tested whether or not the assumption of a constant $\boldsymbol{P}$ leads to a U whose rate of change is constant.

This method would clearly be successful in the case of a sphere. It is also successful in the case of an ellipsoid. For an ellipsoid of semi-axes a, b, and c, for example, the Newtonian potential function for interior points is given by*

$$\frac{1}{4\pi}\int\frac{d\tau'}{r} = \text{Constant} - \frac{1}{2}\left[Lx^2+My^2+Nz^2\right],$$

where M and N are obtained from

$$L = \frac{abc}{2}\int_0^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

by cyclic permutation of the letters a, b, c. Then

$$U = -\left(\boldsymbol{P}, \nabla\,\frac{1}{4\pi}\int\frac{d\tau'}{r}\right) = (\boldsymbol{P}, \boldsymbol{i}Lx+\boldsymbol{j}My+\boldsymbol{k}Nz) = xLP_x+yMP_y+zNP_z,$$

so that

$$-\nabla U = -(\boldsymbol{i}LP_x+\boldsymbol{j}MP_y+\boldsymbol{k}NP_z)$$

is indeed constant if $\boldsymbol{P}$ is constant. The components of the polarization, are then given, from (92), as

$$\begin{aligned} P_x &= (\epsilon-1)(E_x^0-LP_x), \\ P_y &= (\epsilon-1)(E_y^0-MP_y), \\ P_z &= (\epsilon-1)(E_z^0-NP_z). \end{aligned}$$

Since the factors which multiply $(\epsilon-1)$ in these equations are the components of the total field, it is clear that the three constants L, M, and

* See, e.g., A. G. Webster, *The Dynamics of Particles, etc*, p. 418.

N are the depolarization factors for the three components of the polarization. If these equations are solved for the components of the polarization, the results are

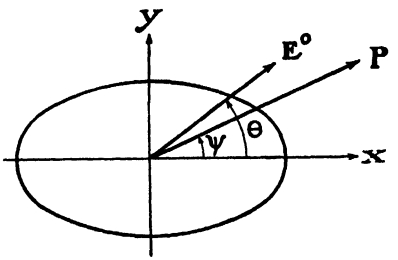

FIG. 36.—The polarization of an ellipsoidal dielectric in a uniform external field E^0.

$$P_x = \frac{E_x^0}{L + \frac{1}{\epsilon - 1}},$$

$$P_y = \frac{E_y^0}{M + \frac{1}{\epsilon - 1}},$$

$$P_z = \frac{E_z^0}{N + \frac{1}{\epsilon - 1}}.$$

As an example of the use of these equations, consider an ellipsoid of revolution for which $b = c$, the E^0 vector lying in the x–y plane. Then

$$E_x^0 = E^0 \cos\theta,$$
$$E_y^0 = E^0 \sin\theta,$$
$$E_z^0 = 0.$$

These values give, upon substitution,

$$P_x = \frac{E^0 \cos\theta}{L + \frac{1}{\epsilon - 1}},$$

$$P_y = \frac{E^0 \sin\theta}{M + \frac{1}{\epsilon - 1}},$$

$$P_z = 0,$$

so that (see Fig. 36)

$$\tan\psi = \frac{P_y}{P_x} = \frac{L + \frac{1}{\epsilon - 1}}{M + \frac{1}{\epsilon - 1}} \tan\theta.$$

If $b < a$, then $\tan\psi$ is less than $\tan\theta$, i.e., the ellipsoid tends to polarize along its major axis.

§ 37. *The Mutual Electrostatic Energy of a System of Conductors and Dielectrics.*—An expression has been formed, in an earlier chapter, for the mutual electrostatic energy of a set of charges. An analogous expression will now be obtained for the energy necessary to form a configuration of conductors and dielectrics, starting from a condition of infinite separation of the various bodies involved, and with each conductor already charged with the same total charge which it has in the final configuration.

This energy is clearly equal to the work done by the electrostatic forces when the bodies recede, from their given configuration, to positions of infinite mutual separation. Let all the bodies except one, say body A_1, remain fixed, and let A_1 move to an infinitely distant position. The work done by the electrostatic forces when A_1 moves off is independent of the path on which it moves, so long as its final state is, in any instance, non-rotating and "isolated." The theorem of conservation of energy would, in fact, be violated if it were possible to move this body off along two paths with different total work to positions which are indistinguishable. It is thus simplest to let A_1 move with pure translation on a straight line. Since all other bodies under consideration remain fixed while A_1 moves off, one can think of them for the moment as forming a single other body A^1. That is, A^1 consists of all the bodies present except A_1. Choose a point P^1 fixed in A^1, and a point P_1 fixed in A_1. Join them by a straight line. Mark a point O on this line between P^1 and P_1, and let s_1 measure distance from O to P_1, while s^1 measures distance from O to P^1. Now let A_1 move a distance ds_1. The charge $\eta_1 d\sigma_1$ located on a surface element $d\sigma_1$ of A_1 experiences a force, due to A^1, equal to

$$-\frac{d\Phi^1}{ds_1}\,\eta_1 d\sigma_1\,,$$

where Φ^1 is the potential at $d\sigma_1$ due to A^1. That is,

$$\frac{d\Phi^1}{ds_1}=\frac{1}{4\pi}\int_{A^1}\eta^1\,\frac{d\,\frac{1}{r}}{ds_1}\,d\sigma^1\,.$$

Thus the work dW_1 done by A^1 on A_1 during this displacement is given by

$$4\pi dW_1=-ds_1\int_{A_1}\eta_1\int_{A^1}\eta^1\,\frac{d\,\frac{1}{r}}{ds_1}\,d\sigma^1\,d\sigma_1\,.$$

This work is clearly also equal to the work A_1 would do on A^1 if A^1 were to be displaced a distance $ds^1 = ds_1$; and hence dW_1 is equal to the half-sum of these two equivalent expressions. That is,

$$4\pi dW_1 = -\frac{ds_1}{2}\int_{A_1}\eta_1\int_{A^1}\eta^1 \frac{d\frac{1}{r}}{ds_1} d\sigma^1 d\sigma_1 - \frac{ds_1}{2}\int_{A^1}\eta^1\int_{A_1}\eta_1 \frac{d\frac{1}{r}}{ds_1} d\sigma_1 d\sigma^1 .$$

Now the quantity

$$-\frac{ds_1}{2}\int_{A_1}\eta_1\int_{A_1}\eta_1 \frac{d\frac{1}{r}}{ds_1} d\sigma_1 d\sigma_1$$

has the value zero, since, in the double summation involved, the term $\eta\eta \cos\theta \, d\sigma \, d\sigma/r^2$ for two surface elements is canceled by the term

$$\eta\eta \cos(\pi-\theta) d\sigma \, d\sigma/r^2$$

for these same two elements with interchanged rôles. The angles θ and $\pi-\theta$ are, in each instance, between the direction of s and the direction from the fixed surface element to the variable surface element. If this last written quantity and an analogous integral over A^1 be added to the expression for dW_1, then the work which A^1 does on A_1 during the displacement ds_1 can be written in the symmetrical form

$$4\pi dW_1 = -\frac{ds_1}{2}\int_{A_1+A^1}\eta\int_{A_1+A^1}\eta \frac{d\frac{1}{r}}{ds_1} d\sigma \, d\sigma .$$

In the foregoing expressions only surface integrals are used, even though dielectrics be present, since the latter may be considered bodies of volume density $-\operatorname{div} \boldsymbol{P} = 0$ and surface density P_n.

Now the rate of change of W_1 with respect to s_1 is not expressed, through the foregoing formula, as the total derivative of some quantity with respect to s_1. In fact, as A_1 moves off, the surface densities η change continuously, having at any stage of the separation the electrostatic values characteristic of that instantaneous configuration. Hence the total work cannot be calculated from the foregoing expression without a knowledge of the η as functions of s_1. It is possible, however, to transform the foregoing expression into one which (when both sides are di-

vided by ds_1) is a total derivative with respect to s_1, and which may thus be integrated directly. In fact, if

$$\Phi=\frac{1}{4\pi}\int_{A_1+A^1}\frac{\eta}{r}\,d\sigma$$

is the total potential at any point, then

$$\frac{1}{2}\frac{d}{ds_1}\int\eta\Phi d\sigma=\frac{1}{8\pi}\int\eta\int\eta\,\frac{d\frac{1}{r}}{ds_1}\,d\sigma\,d\sigma+\frac{1}{8\pi}\int\eta\int\frac{1}{r}\frac{d\eta}{ds_1}\,d\sigma\,d\sigma$$
$$+\frac{1}{8\pi}\int\frac{d\eta}{ds_1}\int\frac{\eta}{r}\,d\sigma\,d\sigma\,,$$

all integrals being extended over all bodies present. The last two integrals on the right have the same value, as is obvious if one views each as the limit of a double sum. Hence, replacing these two integrals by twice the latter,

$$\frac{1}{2}\frac{d}{ds_1}\int\eta\Phi\,d\sigma=\frac{1}{8\pi}\int\eta\int\eta\,\frac{d\frac{1}{r}}{ds_1}\,d\sigma\,d\sigma+\int\frac{d\eta}{ds_1}\,\Phi d\sigma\,.$$

Thus,

$$\frac{dW_1}{ds_1}=-\frac{1}{2}\frac{d}{ds_1}\int\eta\Phi d\sigma+\int\frac{d\eta}{ds_1}\,\Phi d\sigma\,.$$

Now Φ is a constant over the surface of each conductor, and the total charge of each conductor remains constant, so that, over the surface of any conductor,

$$\int\frac{d\eta}{ds_1}\,\Phi d\sigma=\Phi\int\frac{d\eta}{ds_1}\,d\sigma=\Phi\frac{d}{ds_1}\int\eta d\sigma=0\,.$$

Hence that portion of the second term of dW_1/ds_1 which is due to conductors vanishes. Furthermore, for a dielectric,

$$\frac{1}{2}\frac{d}{ds_1}\int\eta\Phi d\sigma=\int\frac{d\eta}{ds_1}\,\Phi d\sigma\,.$$

In fact, since $\eta=P_n$ and div $\boldsymbol{P}=0$,

$$\int\eta\Phi d\sigma=\int P_n\Phi d\sigma=\int(\boldsymbol{P},\,\nabla\Phi)d\tau\,.$$

But in a dielectric $\boldsymbol{P} = -(\epsilon - 1)\nabla\Phi$, so that

$$\frac{d}{ds_1}\int \eta\Phi d\sigma = -\frac{1}{\epsilon-1}\int \frac{d}{ds_1}(\boldsymbol{P},\boldsymbol{P})d\tau = -\frac{2}{\epsilon-1}\left(\int \frac{d\boldsymbol{P}}{ds_1}, \boldsymbol{P}\right)d\tau ,$$

$$= 2\int \left(\frac{d\boldsymbol{P}}{ds_1}, \nabla\Phi\right)d\tau = 2\int \frac{dP_n}{ds_1}\Phi d\sigma = 2\int \frac{d\eta}{ds_1}\Phi d\sigma$$

The second term in dW_1/ds_1 therefore disappears, the terms for conductors vanishing of themselves, and the terms for dielectrics canceling against the corresponding contributions to the first term. There remains, therefore, only the expression

$$\frac{dW_1}{ds_1} = \frac{1}{2}\frac{d}{ds_1}\int \eta\Phi d\sigma ,$$

where the integral is extended over the surfaces of conductors only; or, since Φ is constant over each conductor, and since the surface integral of η gives the total charge on each conductor,

$$\frac{dW_1}{ds_1} = -\frac{1}{2}\frac{d}{ds_1}\Sigma\Phi_i e_i ,$$

the sum extending over all the conductors. The work done when A_1 moves from an initial position $s = s$ to a final position $s = \infty$ is, then, to be found by integrating the foregoing value for dW_1/ds_1 between the given limits. Thus,

$$W_1 = \frac{1}{2}\Sigma e_i\Phi_i - \frac{1}{2}\Sigma e_i\Phi_i' ,$$

where the index i runs over all integral values from 1 to n, the number of conductors initially present; and where Φ_i' is the potential of the ith conductor after body A_1 has been moved off. One of the bodies, namely A_1, having been removed from the original configuration, one may now move another, say A_2, out along a straight line, and calculate the work W_2 done on A_2. In this way the bodies may be removed one at a time, and the total work

$$W = W_1 + W_2 + \ldots .$$

calculated. It somewhat simplifies the calculation of W to suppose that the n conductors are removed first. Then

$$W = W_1 + W_2 + \ldots . + W_n .$$

In fact, the foregoing expression for the work done in removing one body from the configuration shows that if there are no conductors in the configuration this work is zero. If the expression for dW_1/ds_1 be integrated between two arbitrary values of s_1, the result, if there are no conductors present in the configuration, is also zero; which proves that the force on a dielectric, due to a configuration of dielectrics only, is zero. Thus if there are no conductors present the dielectrics will be unpolarized, and will not react on each other.

If, as is assumed, body A_1 is a conductor, then one may write

$$\Sigma e_i \Phi_i' = \sum_{i=2}^{n} e_i \Phi_i' + e_1 \Phi_1^0 ,$$

where Φ_1^0 is the potential to which the charge e_1 raises the isolated conductor A_1. The remaining dielectrics and $n-1$ conductors now form a new configuration, for which Φ_i' gives the initial potentials of the conductors. Thus, as above,

$$W_2 = \frac{1}{2} \sum_{i=2}^{n} e_i \Phi_i' - \frac{1}{2} \sum_{i=3}^{n} e_i \Phi_i' - e_2 \Phi_2^0 ,$$

where Φ_i'' are the potentials of the remaining $n-2$ conductors after A_1 and A_2 have been moved off, and Φ_2^0 is the potential to which the charge e_2 raises the isolated conductor A_2. Thus, by addition,

$$W = \frac{1}{2} \Sigma e_i \Phi_i - \frac{1}{2} \Sigma e_i \Phi_i^0 , \tag{95}$$

where Φ_i^0 is the potential of conductor i, when isolated, due to its own charge e_i, and Φ_i is the potential of conductor i in the final configuration.

In the expression just obtained, the value of W appears as a discrete sum, there being one term for each conductor. It is possible to transform the expression so that W be given by a volume integral extended over all the space not occupied by conductors. In fact, since

$$e_i = \int_i \epsilon E_n d\sigma ,$$

where ϵ is the dielectric constant (unity for empty space) of the medium surrounding the ith conductor, it follows that

$$W=\frac{1}{2}\Sigma\int\epsilon E_n\Phi d\sigma=-\frac{1}{2}\int_T \text{div }\epsilon\,\Phi\boldsymbol{E}dT\ ,$$

the last integral being extended throughout all the space T not occupied by conductors. The negative sign arises from the fact that the exterior normal of the region T is an interior normal to a conductor, while E_n is the component of $\boldsymbol{E}$ along the normal exterior to the conductor. Moreover,

$$\begin{aligned}\text{div }(\epsilon\Phi\boldsymbol{E})&=\Phi\text{ div }\epsilon\boldsymbol{E}+(\nabla\Phi,\ \epsilon\boldsymbol{E})\\&=-\epsilon E^2\ ,\end{aligned}$$

since div $\epsilon\boldsymbol{E}=\epsilon$ div $\boldsymbol{E}=0$, and $\nabla\Phi=-\boldsymbol{E}$.
Thus

$$W=\frac{1}{2}\epsilon\int E^2d\tau\ . \tag{96}$$

In the classical electromagnetic theory of Maxwell, this equation is regarded as especially significant since it has suggested the hypothesis that the energy W is distributed throughout T with volume density $\epsilon E^2/2$. The energy of the configuration here appears as a volume integral, and it is therefore clearly compatible with this equation to assume that the electrostatic energy of the configuration is distributed throughout the space T. Such a statement, however, involves the assumption that electrostatic energy is, in its nature, something which can be spatially distributed. And it is hardly necessary to point out that while the foregoing equation can suggest such a hypothesis, it in no way demands it. The hypothesis of a spatially distributed electrostatic energy of volume density $\epsilon E^2/2$ has, however, played a large rôle in the development of electromagnetic theory. Applications of this hypothesis will appear in later chapters.

Problems for Part III, Chapter II

1. Let e be the measure of a given charge in the units used in this book; let $e_{e.s.u.}$ be the measure of the same charge in electrostatic units, and e_c be the measure of this charge in Coulombs. Show that

$$e=\sqrt{4\pi}\,e_{e.s.u.}=\tfrac{1}{3}\sqrt{4\pi}\cdot 3\cdot 10^9\,e_c\ .$$

2. Let Φ, $\Phi_{e.s.u.}$, and Φ_v be the measures in the units here used, in electrostatic units, and in volts, respectively, of the potential due to a given configuration of charge. Show that

$$\Phi = \frac{1}{\sqrt{4\pi}}\Phi_{e.s.u.} = \frac{1}{300\sqrt{4\pi}}\Phi_v .$$

3. In electrostatic units, the relation between intensity and polarization is

$$\boldsymbol{P}_{e.s.u} = \frac{(\epsilon_{e.s.u.}-1)}{4\pi}\boldsymbol{E}_{e.s.u.} .$$

Find the relation between $\boldsymbol{E}$ and $\boldsymbol{E}_{e.s.u.}$, and between $\boldsymbol{P}$ and $\boldsymbol{P}_{e.s.u.}$, and hence show that the numerical measure of a dielectric constant is the same in the two systems.

4. What differential equation does Φ satisfy in a dielectric where ϵ is not a constant, but is a specified function of the co-ordinates?

5. A spherical conductor, with a charge e, is surrounded by a shell of dielectric, of external radius a, and dielectric constant ϵ. Show that, at points in free space,

$$\Phi = \frac{e}{4\pi r},$$

while at points within the dielectric,

$$\Phi = \frac{e}{4\pi\epsilon}\left[\frac{1}{r} + \frac{\epsilon-1}{a}\right] .$$

6. The two plates of a large parallel plate condenser are a distance d apart. Between them are two slabs of dielectric, one of thickness $d/2$ and dielectric constant ϵ_1, the other of thickness $d/2$ and dielectric constant ϵ_2. One plate of the condenser has a charge density $+\eta$, and the other $-\eta$. Obtain expressions for the potential at points within both dielectrics. Show that the capacity of the condenser, per unit area, is

$$\frac{2\epsilon_1\epsilon_2}{d(\epsilon_1+\epsilon_2)},$$

and that the charge density on the face between the two dielectrics is

$$\eta\,\frac{\epsilon_2-\epsilon_1}{\epsilon_1\epsilon_2} .$$

7. It can be shown (see Poincaré, *Théorie du Potentiel Newtonien* [1899], p. 111) that the tangential derivatives of the potential are continuous when an attracting surface is pierced. Use this fact, and knowledge concerning the behavior of the normal derivatives, to show how the direction of the $\boldsymbol{E}$ vector is refracted upon passing from one dielectric to another.
8. A charge e is distant b units from the plane face of an infinite dielectric. Show that the induced surface density of charge on the face of the dielectric is equal to

$$+\frac{eb}{2\pi\rho^3}\frac{1-\epsilon}{1+\epsilon},$$

where ρ is the distance to the point in question from e.
9. Referring to the preceding problem, what is the force on the charge ϵ?
10. Use the method of § 36 to find the polarization of a spherical dielectric when placed in a uniform field.
11. A sphere, of dielectric constant ϵ_1, is located in an infinite medium of dielectric constant ϵ_2. Find the polarization of the sphere due to a constant and uniform field $\boldsymbol{E}_0$.
12. A dielectric sphere is polarized by the action of a point charge e located b units from the center of the sphere. Find the state of polarization of the sphere.
13. Find the state of polarization of an infinite dielectric cylinder, placed in a constant and uniform field $\boldsymbol{E}^0$ which is normal to the axis of the cylinder. Use the method of § 35, and the method of § 36. What is the depolarizing factor for a cylinder?
14. An infinite row of doublets, equally spaced a distance a apart, are polarized by their mutual action and by a uniform field normal to the row. What is the polarization of each doublet?
15. Show that the torque on an ellipsoidal dielectric in a uniform field is

$$\boldsymbol{T}=\frac{4}{3}\pi\, abc[\boldsymbol{P}, \boldsymbol{E}^0].$$

16. Show that for an ellipsoid of revolution ($b=c$), polarized by a constant and uniform field $\boldsymbol{E}^0$ which lies in the x–y plane (see Fig. 36), that T_z is a maximum when $\theta=45^0$.
17. A doublet has a fixed polarization $\boldsymbol{P}$ which makes an angle θ with the line joining this doublet and a glass sphere of radius a. What is the magnitude and direction of the force on the sphere?

18. The region x negative is filled with a dielectric ϵ_1, while the region x positive is filled with a dielectric ϵ_2. At the point $x=-a$, $y=z=0$, there is located a small conducting sphere of charge e. What is the potential at every point, and what is the charge density on the interface between the two dielectrics? For what limiting ratio of ϵ_1/ϵ_2 does the region x positive act like a conductor?
19. Prove by Gauss's theorem that Φ cannot have a maximum or minimum at any point unless this point is occupied by charge.
20. If $F(z)=f_1(x,y)+if_2(x,y)$ is any analytic function of the complex variable $z=x+iy$, show that $f_1(x,y)$ and $f_2(x,y)$ are solutions of $\nabla^2 f=0$. This fact is the basis of the method of conjugate functions, which is useful when only two dimensions are involved. The student should consult a standard text, such as Webster or Jeans, for a detailed discussion of this method.
21. Show that

$$C=4\pi C_{e.s.u.}=4\pi\cdot 9\cdot 10^5 C_{m.f.},$$

where C, $C_{e.s.u.}$, and $C_{m.f.}$ are the measures of a capacity in the units here used, in electrostatic units, and in microfarads.
22. What is the radius of a conducting sphere whose capacity is 1 microfarad?
23. By the use of the symmetrical form of Green's theorem, prove the following theorem, which is also due to Green: If charges $e_1, e_2, \ldots$, on conductors 1, 2, ..., raise them to potentials $\Phi_1, \Phi_2, \ldots$, and charges $e_1', e_2', \ldots$, raise them to potentials $\Phi_1', \Phi_2', \ldots$, then

$$\Sigma e_i\Phi_i'=\Sigma e_i'\Phi_i.$$

24. Show, as a special case of the preceding theorem, that the potential to which conductor 1 is raised when unit charge is placed on conductor 2, the other conductors present being uncharged, is equal to the potential to which conductor 2 is raised when unit charge is placed on 1, the rest remaining uncharged.
25. A conductor in the form of a prolate ellipsoid or revolution for which $a=b=5$ cm., $c=10$ cm., is charged with 1 Coulomb of electricity. To what potential (measured in volts) is it raised?
26. Why is the direction of $\boldsymbol{E}$, just outside a conductor, normal to the surface of the conductor?
27. If u is a scalar function which satisfies $\nabla^2 u=0$ at all points except on certain surfaces of discontinuity S, and which vanishes at infin-

ity as $1/r$, and which is continuous as the surfaces S are crossed; and if $\boldsymbol{A}$ is a vector which vanished at infinity as $1/r^2$, and whose normal component is continuous as the surfaces S are crossed, show that

$$\int(\nabla u, \boldsymbol{A})d\tau ,$$

extended over all space, is zero. Under what circumstances would Φ and $\epsilon\boldsymbol{E}$ satisfy the conditions on u and $\boldsymbol{A}$? Interpret the vanishing of the integral in this case.

28. A conducting sphere of radius a is placed in a constant and uniform field $\boldsymbol{E}$. If the total charge on the sphere be zero, what is the induced density of charge at points on the surface of the sphere? Show that the polarization $\boldsymbol{p}$ of the sphere, considered as one complex, is $4\pi a^3\boldsymbol{E}$. Hence show that

$$\boldsymbol{p}_{e.s.u.} = a^3\boldsymbol{E}_{e.s.u.} .$$

29. A rigid molecule of fixed and inherent polarization $\boldsymbol{p}$ is located in a field whose potential is Φ. The potential energy of the molecule is then $(\boldsymbol{p}, \nabla\Phi) = pE\cos\theta$, where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{E}$. In a gas composed of such molecules, the number of molecules whose polarization makes an angle with the field lying within a solid angle $d\omega$ about the direction specified by θ is then, by the Maxwell-Boltzman law, proportional to

$$e^{\frac{pE\cos\theta}{kT}}\, d\omega ,$$

where T is the absolute temperature and $k = 1.37\cdot10^{-16}$ erg. Show that the mean value of the component of the polarization parallel to the field is

$$p\left[\coth\beta + \frac{1}{\beta}\right] ,$$

where

$$\beta = \frac{pE}{kT} .$$

30. Prove that only one function Φ satisfies the conditions (II) of § 33.

31. Consider a body which has densities $\rho';\eta';\boldsymbol{P}'$. The potential due to it may be written

$$\Phi=\frac{1}{4\pi}\int\frac{\rho'-\operatorname{div}'\boldsymbol{P}'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{\eta'+P'_n}{r}\,d\sigma' .$$

Show that the calculation which produced equation (68) of § 21 results, when applied to the present equation, in the relations

$$\nabla^2\Phi=-\operatorname{div}\boldsymbol{E}=-\rho+\operatorname{div}\boldsymbol{P} ,$$

$$\operatorname{div}(\boldsymbol{E}+\boldsymbol{P})=\rho .$$

32. The vector $\epsilon\boldsymbol{E}=\boldsymbol{E}+\boldsymbol{P}$ was denoted, by Maxwell, by the letter $\boldsymbol{D}$ and called the "electric induction." Show that the boundary relations on the surface between the two media can be phrased as follows:
   a) The normal component of $\boldsymbol{D}$ is continuous.
   b) The tangential component of $\boldsymbol{E}$ is continuous.

Conclusion to Chapter II

The definitions of conductors and dielectrics, and the treatment of the force on an interior charge which precedes and makes possible these definitions, may seem unnecessarily detailed and cumbersome. The authors would cheerfully scrap such computations and discussions if they could find a shorter, reasonably logical way to proceed. The continuous densities ρ, $\boldsymbol{P}$, etc., were defined, in this chapter, so as to furnish a convenient method of approximating the potential due to distant complexes of charge. The value of the function

$$\Phi = \frac{1}{4\pi} \int \rho \frac{d\tau}{r}$$

can, since the resulting improper integral is convergent, be calculated at points of the region where ρ differs from zero; but it is perfectly clear from the original meaning of ρ that at such points these values have no immediate physical significance. However, in considering the total force on an interior charge (as a necessary step in solving the problem of distribution of charge), assumptions concerning the nature of the force due to very near charges have led to relationships between the total force and the nabla of the foregoing function Φ. The region of physical usefulness of this function has thus been extended beyond what was immediately anticipated at the time ρ was defined. One can conceive of an "interior potential" function of a quite different sort, which would take into account the actual effective position of each charge, and whose difference at two interior points would accordingly measure directly the work necessary to move a charge from one to the other of the points. Such a potential would fluctuate widely from point to point, would necessitate a knowledge of the action of charges at small distances, and would clearly necessitate the use of far more precise approximations than those used in defining ρ. The gap between the rather roughly macroscopic potential Φ defined in terms of the continuous smoothed over ρ, and the definitely microscopic quantity, the force on an interior charge, is bridged by means of the assumptions concerning the constitutive force characteristic of various types of bodies.

The authors have not felt it practicable or necessary to include much of the standard classical theory of systems of conductors, nor any except the most simple and fundamental of the special methods of solution. The standard texts contain a great fund of such analytical material which there is no need to duplicate.

CHAPTER III

MAGNETOSTATICS

Part I. The Fundamental Law

INTRODUCTION

Electrostatics deals with the actions between "stationary" charges. Magnetostatics, on the other hand, treats of the actions of moving charges, or currents, and, in particular, of what are known as "steady-state" configurations of moving charge. A steady state of motion prevails when at any fixed point any quantity which describes the state of motion at that point is itself independent of time. This definition requires obvious modification when it is to be applied to a statistical problem. This modification is of the same sort as that previously required in the definition of a static configuration. The electric current is thought of as the steady drift of moving charges which, besides this steady drift, have complicated motions which may be called random in the sense that they average out and contribute nothing to the steady drift. To set up a suitable definition of a steady state of motion in terms of these individual motions of charges, one deals not with the instantaneous state of motion at any instant, but with time averages taken over times which are long reckoned from an atomic point of view (i.e., long compared to the period of an orbital electron, the time between two impacts of a molecule, etc.), but very short from an experimental point of view.

There is an obvious analogy between this situation and the motion of a gas whose molecules have, besides the general velocity of translation, heat motions. The analysis of the present situation, however, has to be fine grained enough to take account of small circulatory motions of charge, even though these do not contribute to the steady drift of charge from one place to another. Indeed, the averaging process suggested above should remove from consideration truly random motions of charge, but should not wipe out a steady circulation of charge, however small scale it may be.

The development of the mathematical theory of electrostatics in the two preceding chapters began: first, with a statement of the fundamental law of electrostatic action; second, it was shown that electrostatic forces are derivable from a potential; third, the sums in the expression for this scalar potential were replaced by integrals of continuous functions. The concept of continuous densities (volume or surface) of charge and polari-

zation arose in connection with the problem of properly determining the integrands of the potential integrals. Fourth, it was shown, from these integral expressions, that the potential satisfies the differential equation $\nabla^2\Phi = 0$, and such boundary conditions were obtained as make the problem unique.

The same four steps will be followed in this chapter in the development of magnetostatics. The final formulation of the general problem bears a very close analogy to the previous formulation of the general problem of electrostatics. The details of the development, however, differ from that of the previous chapters in two principal ways. It will be noted, for one thing, that in carrying out the third and fourth steps mentioned above, it is necessary, in magnetostatics, to impose restrictions on the theory somewhat more severe than those imposed in electrostatics. And it will also be noted that the situation with respect to the fundamental law is not so simple as it was before. Experimentation on the forces due to moving charges is necessarily more difficult than electrostatic experimentation. The empirical basis, here given, for the fundamental law is found in the celebrated researches of Ampère. Compared to the experiment of Coulomb, it seems complicated and indirect. Four experimental laws are first stated. To these are joined the principle of superposition of effects and the assumption that the total force and torque are zero on a system consisting of two current elements. On this basis it is possible to deduce a definite law for the action of one current element on another. The experimental basis does not pretend, however, to justify this expression except in so far as it is used to find, by integration, the action of a closed circuit on an element. Indeed, strictly speaking, one has no right in magnetostatics to consider anything other than the interaction of closed circuits. For magnetostatics deals only with steady-state distributions of current. In a steady state, the current, at any fixed point in space, must be independent of the time. Thus if attention is restricted to finite current distributions, only closed currents can enter a legitimate problem of magnetostatics. It will be seen that the actual steps in the derivation of the fundamental law are: (1) the statement of the experimental basis; (2) a derivation of an expression for the action of an element of a closed circuit on a current element; (3) the addition, to this expression, of terms whose total contribution is zero, when the action of the whole closed circuit is calculated. This third step results in a second expression for the force with which an element of a closed circuit acts on a current element. It is clear that the experimental basis, which deals only with the action of closed circuits, affords no basis

for a choice between these two expressions. Indeed, it is clear that these two expressions are equivalent from the point of view of magnetostatics. The second form, since it is simpler and more naturally connected with the more general laws of later developments of the theory, is retained. The fundamental law of magnetostatics is, however, not to be expressed in terms of current elements, but in terms of moving charges. So this law is still further decomposed, and an expression written for the action of one moving charge on another. It must again be emphasized that this last expression has validity only when it is properly used. In particular, it must be emphasized that this expression does not give the actual action between two isolated moving charges. There is no evidence, in the method of derivation here given, that the expression is anything more than such an amount, to be ascribed to the action of one moving charge on another, as will lead to the correct expression for the action of a closed circuit. One should not be disturbed at the observation that the fundamental law of magnetostatics does not apply to the case of two isolated moving charges. The reason is, of course, the simple one that two isolated moving charges cannot form a steady-state configuration. In just the same way, the fundamental law of electrostatic action does not apply, in reality, to two isolated charges, since two isolated charges cannot form a static configuration.

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§ 38. *The Fundamental Law of Magnetostatics: the Researches of Ampère.*—The previous chapters have been concerned with a study of the forces acting between charges and charged bodies, all of the charges being statistically at rest. When charges are in motion new phenomena are met with, and the description of the forces acting is more complicated and difficult. Steady states of motion will first be studied, as a step in the passage from the static to the general case.

When, in electrostatics, a group of electrons is said to be at relative rest the description, as has been pointed out, is not intended to apply to each electron individually, but to the group as a whole. It is understood, for example, that the individual electrons may and probably do have very lively motions. These motions need not necessarily be thought of as consisting solely of small fluctuations of each electron about a mean position of rest, but it is possible that some given electron located on a conductor which is in electrical equilibrium may wander a considerable distance in the conductor, in a way analogous to that in which some given molecule in a gas, devoid of mass motion, may wander off

through the gas. One speaks of a group of electrons as being at rest in the same sense that one speaks of the population of a country as being fixed when (the numerical population remaining constant) the center of population remains at a fixed point. The individual members of the population move about within their towns or cities, or even wander off on long trips, but there are compensating trips of other individuals, so that the center of population does not move.

It is evident, moreover, that the process of readjustment and compensation will not likely be complete at any one instant of time, so that there will be small fluctuations, with time, of the mean position of the group. On the other hand, only time averages are experimentally observable. Thus when one studies, in what is described as an "electrostatic problem," the force due to a group of charges said to be in static equilibrium or "at rest," he is in reality studying the time average, over a long time, of the force due to a group of electrons moving in such a way that their average position fluctuates about a fixed point, and which exert a force whose time average is constant.

It is necessary to recall this viewpoint in order to be clear as to what is meant by "charges in motion." A group of charges will be said to be in motion when their mean position is not stationary, small fluctuations with time being disregarded. There is no attempt to describe the motions of the separate charges, but only the drift of the group as a whole. The center of population of the United States has moved, in the last century, over 500 miles westward, almost exactly along the thirty-ninth parallel of latitude. Such a motion, if sensibly uniform, would be described as a steady westward motion of the population in just the same sense as one speaks of the steady motion of a group of electrons. In particular, a steady state of motion will be said to exist when, at every point, the time average of those functions which describe the motion of charge are independent of the time. At various points in the development, the statistical nature of the theory finds explicit form in the analysis, but it is essential to a clear understanding of the limitations of the results that this underlying microscopic viewpoint be borne constantly in mind.

Magnetostatics is primarily concerned with the actions between currents of electricity. Just as there is, in general, a volume density of electricity when charges are present, so there is a current when moving charges are present. The precise definition of electric-current density, however, will not be given at this point, just as the precise definition of ρ was not given, in electrostatics, until one sought to express, by means of integrals, the potential due to ponderable charged matter. The experi-

ments upon which the fundamental law of magnetostatics is based are, however, experiments with currents flowing in wires; and so it will be convenient to adopt a preliminary definition for such a current. If the cross-section of the wire be negligibly small compared to the other dimensions involved, and if the wire be formed of a non-magnetic conducting material, such as copper, a very simple definition of current can be given which, after the complete definition of current is later given, will be found, indeed, to be quite satisfactory for this special case. The magnitude of the current flowing in a wire is, in fact, defined as the total amount of charge which crosses a given section of the wire per unit time. When the charge is measured in rational or Heaviside electrostatic units the current, thus defined, is measured in rational electrostatic units. It is convenient, however, to introduce a new unit of current, which is larger than the rational electrostatic unit in the ratio of $c=3\times10^{10}:1$. This unit of current is called the "rational electromagnetic unit." If the measure of a given current in this rational electromagnetic unit be I, then it follows directly from the definition just given that

$$\boldsymbol{Ids}=\frac{\mathrm{N}\epsilon}{c}\boldsymbol{V}\,, \tag{97}$$

where the current I is flowing in a circuit of which $\boldsymbol{ds}$ is a vector element of arc, and where N is the number of charges ϵ, moving with velocity $\boldsymbol{V}$, present in the element $\boldsymbol{ds}$ at any time. The fact that $\boldsymbol{ds}$ and $\boldsymbol{V}$ necessarily have the same direction is due to the negligibly small cross-sectional dimension of the wire. This equality permits one to interpret experiments on currents in terms of the actions between moving charges. It is clear that the quantities in equation (97) involve, in a somewhat vague way, certain approximations and averaging processes. Similarly, vague approximations and averages were employed in passing from the Coulomb experiments on small charged bodies to the electrostatic law of force between two charges. In both cases the procedure is to arrive, boldly if need be, at a reasonable law, and then use care and precision in constructing from this law a theory. The test of the law is then to be found in the success of the theory.

The early part of the nineteenth century saw the origin of the science of electromagnetism in what is known as "Oersted's experiment." In 1819, while trying to show that there was no effect of an electric current on a magnetic needle, he accidentally discovered that there was an effect at right angles to the current. Oersted stopped with the discovery itself, but in the years between 1820 and 1825, Ampère, by a series of wonder-

fully painstaking experiments, arrived at a set of relations giving the action of one current on another. His extraordinary researches on this subject were reported in various papers, the chief of these being the famous *Mémoire sur la Théorie mathématique des Phénomènes electrodynamiques*, which was characterized by Maxwell as "perfect in form and unassailable in accuracy." The conclusions of Ampère were based upon the following experimental results:

1. *The action of a current on another current or current element is unchanged in magnitude but reversed in direction when the direction of the current is reversed.*

This followed from the experimental fact that the force exerted by two straight parallel currents of equal magnitude but opposite direction is zero when the currents are very near one another. The total force exerted on these two parallel currents is also zero.

2. *The effect of a conductor bent or twisted in any manner is equivalent to that of a straight one, provided that the two are traversed by equal currents, and that the former nearly coincides with the latter.*

This was experimentally proved by observing that the action of a closed circuit is zero provided that it consist of a straight portion A–B and a return portion from B to A which zigzags about the straight portion A–B in any manner, always, however, keeping within a small distance of the straight line A–B.

3. *The action of a closed circuit on a current element is always normal to the latter.*

A short length of wire was bent into an arc of a circle and fastened, with its plane horizontal, on the end of a needle which, in turn, was suspended by a vertical thread. The needle was counterbalanced so that the needle and wire arc were in a horizontal plane, the point of suspension of the needle being the center of curvature of the arc. The two ends of the wire were touching mercury, so that a current could be caused to flow through it, and it be free to move. It was found that when a closed circuit was brought near this apparatus there was no tendency to rotate about the vertical suspension, showing that the action on the arc was normal to it, and hence passed through the axis of suspension. Capillary action prevented this experiment from being carried out with the accuracy with which the two previous laws were tested.

4. *In similar and similarly situated circuits traversed by equal currents the forces are equal.*

Ampère tested this law by the use of three circular closed circuits, whose radii were in the ratio 1:2:4, the distance from center to center

of the first two and the distance from center to center of the last two being in the ratio 1:2. The three were traversed by the same current, and the actions of the two circuits whose radii were respectively 1 and 4 on the circuit whose radius was 2 were found to be equal (but opposite on account of the direction of the current). This test is obviously unsatisfactory due to its lack of generality.

These four laws are concerned with the action of closed circuits, or, at most, the action of a closed circuit on an element of current. But it is clearly consistent (and indeed quite equivalent, considering how the result is to be used) to assume that they hold for the action of one current element on a second current element. The forces between the current elements are, further, to satisfy the ordinary mechanical requirements for equilibrium that (*a*) there is no total force on the system formed by the two elements of current; and that (*b*) there is no total torque on the whole system. It follows from (*a*) that action and reaction are equal in magnitude and opposite in direction, and thus from (*b*) that the force between the two elements must be directed along the line joining them. For if the force on one element of current has a component perpendicular to the line joining them, the force on the other element must, by (*a*), have an equal and opposite component. But these two components form a couple the possibility of which is excluded by (*b*).

To arrive at an analytical expression for the action between two elements of current, suppose that currents I and I' are flowing along the elementary arcs ds and ds'. For briefness the elements of arc ds and ds' will themselves be referred to as the "elements of current." The action of ds' on ds may, by the second law, be obtained by considering the action of the components of ds' on the components of ds. If ds' be resolved into a component ds'_r in the direction of $\boldsymbol{r}$, the vector from ds' to ds, and a component ds'_n normal to $\boldsymbol{r}$, and if ds be resolved into a component ds_r along $\boldsymbol{r}$ and a component ds_n normal to $\boldsymbol{r}$, this latter being further resolved into a component $(ds_n)_p$ parallel to ds'_n, and a component $(ds_n)_n$ normal to ds'_n, then the interactions to be considered are the forces exerted by

(1) ds'_r on ds_r ,
(2) ds'_r on $(ds_n)_p$,
(3) ds'_r on $(ds_n)_n$,
(4) ds'_n on ds_r ,
(5) ds'_n on $(ds_n)_p$,
(6) ds'_n on $(ds_n)_n$.

Let the forces due to the interactions of these various components be denoted by $\boldsymbol{dF}_1 \ldots . \boldsymbol{dF}_6$, the direction of each of these being along $\boldsymbol{r}$.

Each of these forces will depend, in some way, upon r, the distance between the elements, and will be, assuming the superposition of effects, proportional to the magnitudes of the component arcs concerned and to the current strengths. Thus the first pair of components listed will give rise to a force

$$dF_1 = rII'a(r)ds_r ds'_r .$$

This expression is clearly consistent with Ampère's first law. It is also consistent with the principle of action and reaction, since the force with which ds acts on ds' would be given by an expression which differs

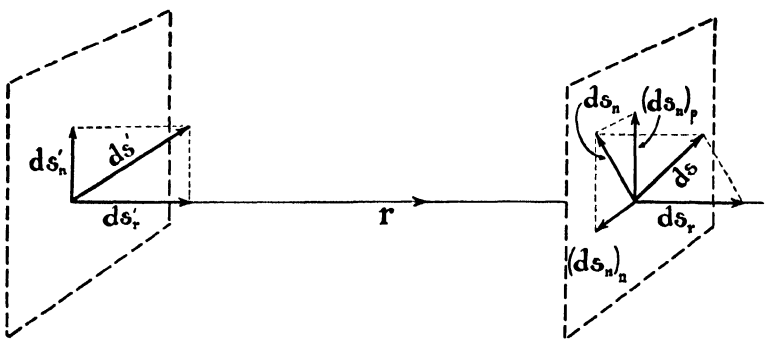

FIG. 37.—The resolution into components of the acting element ds' and the element ds being acted on.

from the foregoing only in that the vector r, which points toward the element acted on, would be reversed in direction.

The forces dF_2, dF_3, and dF_4 can be treated together, since they are each due to the action between a component along the joining line and a component at right angles to that line. It may be easily argued that these forces are zero. For if there were, for example, a repulsion dF between two such elements it would, by the first law, change to an attraction $-dF$ if the current were reversed in the perpendicular component. There is, however, nothing to distinguish the relative configuration of r and the component arcs in the two cases, a rotation of 180° about r bringing the two into coincidence. Therefore $dF = -dF = 0$.

The fifth pair of components, however, give rise to a force which may be written

$$dF_5 = rII'\beta(r)ds'_n(ds_n)_p ,$$

and the sixth pair of components to a force,

$$d\boldsymbol{F}_6 = rII'\gamma(r)ds_n(ds_n)_n ,$$

both of these forces obeying the first law and the principle of action and reaction. The last expression may be given a different form if the components of arc are written as vectors in the directions along which the original elements ds and ds' have been resolved. Indeed, the scalar product

$$([\boldsymbol{ds},\boldsymbol{r}],\boldsymbol{ds}')$$

is equal to

$$|[\boldsymbol{ds},\boldsymbol{r}]|ds' \cos\theta = |[\boldsymbol{ds},\boldsymbol{r}]|ds'_n \cos\varphi ,$$

where θ is the angle between $\boldsymbol{ds}'$ and $[\boldsymbol{ds},\boldsymbol{r}]$, and φ is the angle (lying in a plane normal to $\boldsymbol{r}$) between $\boldsymbol{ds}'_n$ and $[\boldsymbol{ds},\boldsymbol{r}]$. But the magnitude of the vector product $[\boldsymbol{ds},\boldsymbol{r}]$ is rds_n. Thus

$$([\boldsymbol{ds},\boldsymbol{r}]\boldsymbol{ds}') = rds_n \cos\varphi\, ds'_n = rds'_n(ds_n)_n ,$$

or, by cyclic permutation of the vectors entering into the scalar product,

$$(\boldsymbol{r}[\boldsymbol{ds}',\boldsymbol{ds}]) = rds'_n(ds_n)_n .$$

The force due to the sixth pair of components may thus be re-written

$$d\boldsymbol{F}_6 = rII'\frac{\gamma(r)}{r}(\boldsymbol{r}[\boldsymbol{ds}',\boldsymbol{ds}]) = rII'C(r)(\boldsymbol{r}[\boldsymbol{ds}',\boldsymbol{ds}]) .$$

The total force exerted by ds' on ds then takes the form

$$(98) \qquad d\boldsymbol{F} = rII'[\alpha(r)ds'_r ds_r + \beta(r)ds'_n(ds_n)_p] + rII'C(r)(\boldsymbol{r}[\boldsymbol{ds}',\boldsymbol{ds}]) .$$

The bracketed expression is a linear function of the two products $ds'_r\, ds_r$ and $ds'_n(ds_n)_p$. The two expressions

$$(\boldsymbol{ds}',\boldsymbol{ds}) = (\boldsymbol{ds}'_r+\boldsymbol{ds}'_n,\boldsymbol{ds}_r+(\boldsymbol{ds}_n)_p+(\boldsymbol{ds}_n)_n) = ds'_r ds_r + ds'_n(ds_n)_p ,$$
$$(\boldsymbol{ds}',\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r}) = r^2 ds'_r ds_r ,$$

are also linear functions of these same quantities. Hence the term within the brackets in equation (98) may be written as a linear function of $(\boldsymbol{ds}',\boldsymbol{ds})$ and $(\boldsymbol{ds}',\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r})$, i.e.,

$$[\alpha(r)ds'_r ds_r + \beta(r)ds'_n(ds_n)_p] = A(r)(\boldsymbol{ds}',\boldsymbol{ds}) + B(r)(\boldsymbol{ds}',\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r}) ,$$

* See Appendix, § 4 (35).

and the total force exerted by ds' on ds may be re-written in the form

$$\boldsymbol{dF} = \boldsymbol{r}II'\{A(r)(\boldsymbol{ds'},\boldsymbol{ds}) + B(r)(\boldsymbol{ds'},\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r}) + C(r)(\boldsymbol{r}[\boldsymbol{ds'},\boldsymbol{ds}])\} .$$

Ampère's fourth law says that the force will be unchanged when all the dimensions of the ciruits are changed in the same ratio, the currents being held constant. Thus when r becomes kr, ds becomes $k\,ds$ and ds' becomes $k\,ds'$, the force $\boldsymbol{dF}$ being unaltered. This condition may be applied to the separate terms in the force expression, since special cases may be constructed in which the total force consists of only one of these terms. If this condition is, then, applied to the first term in the expression for $\boldsymbol{dF}$, the result is the following identity in k:

$$\boldsymbol{r}A(r)(\boldsymbol{ds'},\boldsymbol{ds}) = k^3\boldsymbol{r}A(kr)(\boldsymbol{ds'},\boldsymbol{ds}) ,$$

or

$$A(kr) = \frac{1}{k^3} A(r) .$$

Differentiating with respect to k,

$$A'(kr)r = -\frac{3}{k^4} A(r) .$$

Thus if $k=1$,

$$\frac{A'(r)}{A(r)} = -\frac{3}{r} ,$$

and, integrating,

$$\log A(r) = -3 \log r + \log a ,$$

so that

$$A(r) = \frac{a}{r^3} .$$

In the same way the functions $B(r)$ and $C(r)$ may be determined, so that the expression for $\boldsymbol{dF}$ is reduced to the form

$$\boldsymbol{dF} = \boldsymbol{r}II'\left\{\frac{a}{r^3}(\boldsymbol{ds'},\boldsymbol{ds}) + \frac{b}{r^5}(\boldsymbol{ds'},\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r}) + \frac{c}{r^4}(\boldsymbol{r}[\boldsymbol{ds'},\boldsymbol{ds}])\right\} . \tag{99}$$

There are as yet no restrictions on the values of a, b, and c in this expression.

Use has been made up to this point, however, of only the first, second, and fourth laws of Ampère. If dF_s denotes the component of force parallel to ds exerted by ds' on ds, then, according to the third law,

$$\int dF_s = 0 ,$$

the integration being carried out with respect to s' around an arbitrary closed path. This identity can be used to evaluate the constants a, b, and c. If Cartesian co-ordinates x', y', z' are introduced with origin at ds and with the x-axis parallel to ds, then

$$-\boldsymbol{r} = i x' + j y' + k z' ,$$
$$\boldsymbol{ds'} = i dx' + j dy' + k dz' ,$$
$$\boldsymbol{ds} = i ds .$$

If these expressions be substituted in (99), the component of $\boldsymbol{dF}$ parallel to ds, i.e., the x-component of $\boldsymbol{dF}$, is seen to be

$$II' ds \left\{ -\frac{ax'dx'}{r^3} - \frac{bx'^2}{r^5}(x'dx' + y'dy' + z'dz') + \frac{cx'}{r^4}(y'dz' - z'dy') \right\} .$$

Thus, rearranging terms, the line integral

$$(100) \qquad II' ds \int \left\{ -\left(\frac{ax'}{r^3} + \frac{bx'^3}{r^5}\right)\frac{dx'}{ds'} - \left(\frac{bx'^2 y'}{r^5} + \frac{cz'x'}{r^4}\right)\frac{dy'}{ds'} - \left(\frac{bz'x'^2}{r^5} - \frac{cx'y'}{r^4}\right)\frac{dz'}{ds'} \right\} ds'$$

must, by the third law, vanish when the integration is carried out around any closed curve.

The conditions imposed by this requirement upon the terms of the integrand may be conveniently obtained by the use of an important identity, known as "Stokes's law," between the line integral, around a closed path, of the tangential component of a vector, and the surface integral, extended over any surface terminating in this closed path, of the normal component of the curl of the same vector. This relation, namely,

$$(101) \qquad \int A_s ds \equiv \int (\text{curl } \boldsymbol{A})_n d\sigma ,$$

where $\boldsymbol{A}$ is any vector, and where curl $\boldsymbol{A}$ is defined by the equation

$$\text{curl } \boldsymbol{A} \equiv i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) ,$$

is essentially a reduction of a double integral to a single integral,* the reduction being possible since the integrand of the double integral consists of sums and differences of partial derivatives. It is clear from this identity that if the left member be zero when integrated around any closed curve, then the surface integral of the normal component of curl vanishes over every surface, so that the curl of the vector A must itself vanish identically, i.e.,

$$\frac{\partial A_z}{\partial y} \equiv \frac{\partial A_y}{\partial z}, \qquad \frac{\partial A_x}{\partial z} \equiv \frac{\partial A_z}{\partial x}, \qquad \frac{\partial A_y}{\partial x} \equiv \frac{\partial A_x}{\partial y}.$$

Now any line integral of the form

$$\int \left(P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds} \right) ds$$

may be thought of as the line integral of the tangential component of the vector whose rectangular components are P, Q, and R, respectively, so that if such a line integral vanishes for every closed curve it follows, from the discussion above, that

$$\frac{\partial R}{\partial y} \equiv \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} \equiv \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y}.$$

If the first of these conditions be applied to (100), the result is

$$+\frac{5bx'^2y'z'}{r^7} - \frac{4cx'y'^2}{r^6} + \frac{cx'}{r^4} \equiv +\frac{5bx'^2y'z'}{r^7} + \frac{4cx'z'^2}{r^6} - \frac{cx'}{r^4},$$

or

$$\frac{2cx'}{r^4} = \frac{4cx'y'^2}{r^6} + \frac{4cx'z'^2}{r^6}.$$

Since this relation must hold identically, it follows that $c=0$.

If the second condition be applied, the result is

$$-\frac{3ax'z'}{r^5} - \frac{5bx'^3z'}{r^7} = -\frac{5bx'^3z'}{r^7} + \frac{2bx'z'}{r^5} - \frac{4cx'^2y'}{r^6} + \frac{cy'}{r^4},$$

or, canceling and setting $c=0$,

$$-3ax'z' = 2bx'z',$$

$$b = -\frac{3}{2}a.$$

* See Appendix, § 5, C.

If c and b be given the values just found, the third condition is identically satisfied. These results give, upon substitution, the following final form for the force exerted on ds by ds'.

$$dF = aII'r\left\{\frac{2(ds',ds)}{r^3} - \frac{3(ds',r)(ds,r)}{r^5}\right\} .$$

The choice of the constant in this relation fixes the unit of current. If the force be measured in dynes, and the currents in rational electromagnetic units, the constant is experimentally found to have the value $-1/4\pi$. The equation then reads

$$dF = -\frac{II'}{4\pi r^3} r\left\{2(ds',ds) - \frac{3}{r^2}(ds',r)(ds,r)\right\} . \tag{102}$$

The experimental basis for this expression only justifies its use to find the force due to a closed circuit S' on ds; and it is thus permissible to add to the expression for dF any terms which vanish when integrated with respect to s' around the closed curve S'. That is, terms may be added which are exact differentials. Thus it is permissible to add the expression

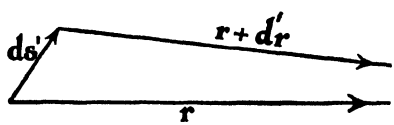

FIG. 38

$$d'\{r(ds,r)\mathrm{X}(r)\} \tag{103}$$

in which ds is a constant, the end of r which describes the closed circuit S' being variable, and in which $\mathrm{X}(r)$ is some function of r. Performing the indicated differentiation, this becomes

$$(ds,r)\mathrm{X}(r)d'r + r(ds,r)\mathrm{X}'(r)d'r + r\mathrm{X}(r)d'(ds,r) .$$

However,

$$d'r = -ds' ,$$

and

$$d'(ds,r) = (ds,d'r) = -(ds,d's) .$$

Also

$$d'r = \frac{r}{r}\,ds'\,\frac{d'r}{ds'} = -\frac{1}{r}\left\{r\,ds'\cos(r,ds')\right\} = -\frac{1}{r}(r,ds') .$$

Upon substituting these values, the original differential (103) becomes

$$(104) \qquad -\boldsymbol{ds}'(\boldsymbol{ds},\boldsymbol{r})\mathrm{X}(r)-\boldsymbol{r}\mathrm{X}(r)(\boldsymbol{ds}',\boldsymbol{ds})-\frac{\mathrm{X}'(r)}{r}(\boldsymbol{ds}',\boldsymbol{r})(\boldsymbol{ds},\boldsymbol{r})\boldsymbol{r}\,.$$

Now choose

$$\mathrm{X}(r)=-\frac{II'}{4\pi}\frac{1}{r^3},$$

and add (104) to (102) above. The result is

$$(105) \qquad \frac{II'}{4\pi r^3}\left\{(\boldsymbol{ds},\boldsymbol{r})\boldsymbol{ds}'-(\boldsymbol{ds},\boldsymbol{ds}')\boldsymbol{r}\right\}.$$

The advantage of the particular choice which has been made in the terms which have been added to $\boldsymbol{dF}$ is now apparent; for the two terms of (105) may by comparison with the vector identity

$$[\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]]\equiv\boldsymbol{B}(\boldsymbol{A},\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A},\boldsymbol{B})$$

be thrown into the form

$$[\boldsymbol{ds}[\boldsymbol{ds}',\boldsymbol{r}]]\,.$$

Thus the force on ds due to the element ds' of the closed circuit S' can be written

$$\boldsymbol{dF}=\frac{II'}{4\pi r^3}[\boldsymbol{ds}[\boldsymbol{ds}',\boldsymbol{r}]]\,.$$

It should be clearly recognized that this expression for $\boldsymbol{dF}$ is not equal to that given by (102), but is equivalent to it, in the sense that the two lead to the same result for the force on a closed circuit. From this expression can be written, at once, the fundamental law of magnetostatic action between moving charges. The force $\boldsymbol{dF}$ is, in fact, acting on the moving charges ϵ present in the current element Ids at any time. The force on a single charge ϵ due to a single charge e_i moving with a velocity $\boldsymbol{v}_i$ in ds' is thus, by (97) and the expression just written,

$$\boldsymbol{F}_i=\frac{\epsilon}{4\pi c r^3}\left[\boldsymbol{v}\left[\frac{e_i\boldsymbol{v}_i}{c},\boldsymbol{r}_i\right]\right]$$

$$=\frac{\epsilon}{c}[\boldsymbol{v},\boldsymbol{B}_i]\,,$$

where $\boldsymbol{v}$ is the velocity of the charge ϵ being acted on, where $\boldsymbol{r}_i$ points from e_i to ϵ, and where the vector $\boldsymbol{B}_i$ is defined by the equation

$$\boldsymbol{B}_i=\frac{1}{4\pi c}\left[e_i\boldsymbol{v}_i, \frac{\boldsymbol{r}_i}{r^3}\right].$$

Thus the magnetostatic force on a charge ϵ moving with a velocity $\boldsymbol{v}$ due to a group of charges e_i moving with velocity $\boldsymbol{v}_i$ is

$$\boldsymbol{F}=\frac{\epsilon}{c}[\boldsymbol{v},\boldsymbol{B}], \tag{106}$$

where

$$\begin{cases} \boldsymbol{B}=\dfrac{1}{4\pi c}\Sigma\left[e_i\boldsymbol{v}_i, \dfrac{\boldsymbol{r}_i}{r^3}\right] \\ \quad=\dfrac{1}{4\pi c}\Sigma\left[e_i\boldsymbol{v}_i, \nabla'\dfrac{1}{r_i}\right], \end{cases} \tag{107}$$

the co-ordinates of e_i being x_i', y_i', z_i', while the co-ordinates of ϵ are x, y, z. The law stated in (107) is often referred to as the law of Biot and Savart. They established this law, however, only for straight currents.*

§ 39. *The Vector Potential.*—In electrostatics the form of the law of force was such that it was possible to obtain the force from a scalar potential. An analogous vector potential will now be obtained for the law of force in magnetostatics.

If the scalar u and the vector $\boldsymbol{C}$ are functions of the two sets of variables x, y, z and x', y', z', then the identity

$$\text{curl } u\,\boldsymbol{C}=u \text{ curl } \boldsymbol{C}+[\nabla u, \boldsymbol{C}]$$

reduces, if u be set equal to $1/r$ and if $\boldsymbol{C}$ be set equal to $e_i\boldsymbol{v}_i$, to the equation

$$\text{curl } \frac{e_i\boldsymbol{v}_i}{r_i}=\left[\nabla\frac{1}{r_i}, e_i\boldsymbol{v}_i\right]=\left[e_i\boldsymbol{v}_i, \nabla'\frac{1}{r_i}\right],$$

the other term dropping out since the velocities $\boldsymbol{v}_i$ are functions of x', y', z' only. Thus from equation (107)

$$\boldsymbol{B}=\frac{1}{4\pi c}\Sigma \text{ curl } \frac{e_i\boldsymbol{v}_i}{r_i}=\text{curl}\left\{\frac{1}{4\pi c}\Sigma\frac{e_i\boldsymbol{v}_i}{r_i}\right\}$$

* *Ann. chim. phys.*, XV (1820), 222.

This last equation can be written

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \,, \tag{108}$$

where

$$\boldsymbol{A} = \frac{1}{4\pi c} \sum \frac{e_i \boldsymbol{v}_i}{r_i} \,. \tag{109}$$

This auxiliary vector $\boldsymbol{A}$ is called the "vector potential." In the electrostatic case, the force was expressed directly in terms of the gradient of the scalar potential; in the magnetostatic case, the force on a moving charge is expressed in terms of the vector product of the velocity of the charge and the curl of the vector potential.

PROBLEMS FOR PART I, CHAPTER III

1. Given the vector

$$\boldsymbol{C} = \boldsymbol{i}xy + \boldsymbol{j}z + \boldsymbol{k}y \,.$$

Form and compare the line integral of C_s around a circle of radius a in the x–y-plane, and the surface integral, over this same circle, of $(\operatorname{curl} \boldsymbol{C})_z$.

2. Given

$$\boldsymbol{C} = \boldsymbol{j}mx \,;$$

find curl $\boldsymbol{C}$ by means of the definition, and by means of the relation

$$(\operatorname{curl} \boldsymbol{C})_n = \lim_{d\sigma \doteq 0} \frac{1}{d\sigma} \int C_s ds \,.$$

3. Compare (107), § 38, with the ordinary expression of the Biot-Savart law.

Part II. Complexes of Charge

INTRODUCTION

Part II of this chapter contains the calculation of the vector potential due to a complex of moving charges—that is to say, due to a group of moving charges whose interdistances are small compared to the distance from any one of them to the point at which the potential is being calculated. This one section is set off as a part to emphasize the analogy between the first three parts of this chapter, and the first three parts of chapter i.

§ 40. *The Vector Potential Due to a Complex of Charges.*—The value at a point $P(x,y,z)$ of the vector potential $\boldsymbol{A}$ due to a complex of charges e_i moving with velocities $\boldsymbol{v}_i$ is given by (109). The problem of approximating this vector expression in terms of quantities characteristic of the complex as a whole is similar to the previous problem of approximating, by the concentration method, the corresponding scalar potential.

$$\Phi=\frac{1}{4\pi}\Sigma\frac{e_i}{r_i}.$$

Choose, as before, a point 0 of co-ordinates x',y',z' within the complex, and denote by r the distance from P to O. Then, expanding $1/r_i$:

$$(110)\qquad \frac{1}{4\pi c}\Sigma\frac{e_i\boldsymbol{v}_i}{r_i}=\frac{1}{4\pi c}\left(\frac{\Sigma e_i\boldsymbol{v}_i}{r}\right)_O+\frac{1}{4\pi c}\Sigma l_i\left(\frac{\partial\dfrac{e_i\boldsymbol{v}_i}{r}}{\partial l_i}\right)_O+\ldots\ldots,$$

$$=\frac{1}{4\pi c}\frac{\Sigma e_i\boldsymbol{v}_i}{r}+\frac{1}{4\pi cr^2}\Sigma e_i\boldsymbol{v}_i(\boldsymbol{l}_i,\boldsymbol{r}')+\ldots\ldots,$$

where $\boldsymbol{l}_i$ are the vectors locating e_i with respect to the fixed point O, and where $\boldsymbol{r}'$ is a unit vector pointing from O to P. It is customary, as

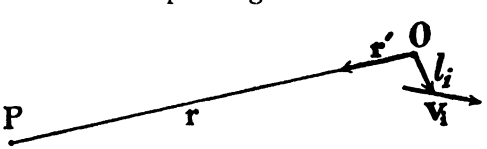

Fig. 39

in the previous case, to consider only two terms of this expansion. The very small ratio l_i/r tends, of course, to make the successive terms de-

crease rapidly in magnitude; and there is in this case, as well as in electrostatics, a physical basis, to be mentioned presently, for stopping with the second term.

To write the second term in a form more easily interpreted, it may be noted that

$$\frac{d}{dt}\, e_i l_i(r', l_i) = e_i v_i(r', l_i) + l_i(r', e_i v_i) \,.$$

Also

$$[r'[l_i, e_i v_i]\,] = l_i(r', e_i v_i) - e_i v_i(r', l_i) \,,$$

so that

$$e_i v_i(l_i, r') = -\frac{1}{2}\,[r'[l_i, e_i v_i]\,] + \frac{1}{2}\,\frac{d}{dt}\, e_i l_i(r', l_i) \,.$$

Thus

$$\Sigma e_i v_i(r', l_i) = -c[r', m] + \frac{1}{2}\,\frac{d}{dt}\,\Sigma e_i l_i(r', l_i) \,,$$

where

$$m \equiv \frac{1}{2c}\,\Sigma[l_i, e_i v_i] \,. \tag{111}$$

The vector m, defined by this equation, is called the "magnetization" of the complex. The magnitude of $[l_i, e_i v_i]$ is the product of $e_i v_i$ and the perpendicular distance from O to v_i, so that m measures a sort of angular momentum of the charge about O; "momentum" in this case being the product of charge by velocity rather than, as in the case of mechanical momentum, the product of mass by velocity.

Now the quantity

$$\frac{d}{dt}\,\Sigma e_i l_i(r', l_i)$$

is statistically equal to zero for the cases with which magnetostatics deals. It can, in fact, be easily argued, for three important cases, that zero is the time average of this quantity over intervals long from the point of view of atomic phenomena but short from the point of view of ordinary experimentation. These three cases are: (1) a sensibly uniform drift of the charges of the complex; (2) any periodic motion of the charges; (3) a random motion of the charges. The first covers the case of that component of velocity which results in a conduction current; the second, the case of any sort of orbital motion in a closed path; the third, the case of possible

random motion, such as heat motions, which may be superposed on the other motions. The arguments for these three cases are indicated in Problems 1, 2, and 3 at the end of this part.

Making use of these simplifications, and the further definition,

$$\Sigma \frac{e_i v_i}{c} \equiv j \,, \tag{112}$$

the vector potential due the complex takes the form

$$A = \frac{1}{4\pi} \frac{j}{r} - \frac{1}{4\pi r^2} [r', m] \,,$$

or

$$A = \frac{1}{4\pi} \frac{j}{r} + \frac{1}{4\pi} \left[m, \nabla' \frac{1}{r} \right] . \tag{113}$$

The vector j is called the "current" of the complex.

The leading term in the expansion is due to the movement of the complex as a whole; the second, or magnetization term, is due to differential movement, or circulation, within the complex. The magnetization of the complex would have a non-vanishing value if the velocities of the charges were such that the complex turns as it drifts as a whole. The first term would vanish and the second term would give the principal effect if the complex consisted, for example, of a ring of charge rotating about a fixed point. If such a complex were subdivided into many complexes, however, it is clear that the description would then be thrown back again on to the first term, except in those cases which would require, for this purpose, a finer subdivision into cells than is permitted by statistical considerations. The description in two terms is thus, just as before in electrostatics, a result of the order of magnitude of the dimensions of the complexes considered. The third term of this expansion (which would correspond to a differential circulation) and the following terms are considered negligible for physical reasons analogous to those which permitted the two-term expansion of the scalar potential.

As an illustration, consider a complex composed of a single orbital electron ϵ moving with angular velocity ω in a circle of radius a. Then

$$|m| = \left| \frac{1}{2c} \Sigma [l_i, e_i v_i] \right| = + \frac{1}{2c} \epsilon a^2 \omega = + \frac{\epsilon}{c} \left(\frac{1}{2} a^2 \omega \right) ,$$

so that the magnetization of the complex is numerically equal to ϵ/c times the sectorial velocity of the rotating charge.

PROBLEMS FOR PART II, CHAPTER III

1. Referring to the quantity

$$\frac{d}{dt}\Sigma e_i \boldsymbol{l}_i(\boldsymbol{r}',\boldsymbol{l}_i) = \Sigma e_i \boldsymbol{v}_i(\boldsymbol{r}',\boldsymbol{l}_i) + \Sigma e_i \boldsymbol{l}_i(\boldsymbol{r}_i',\boldsymbol{v}_i) ,$$
$$= \Sigma e_i \boldsymbol{v}_i (l_i)_r + \Sigma e_i \boldsymbol{l}_i (v_i)_r ,$$

of § 40, where $(l_i)_r$ and $(v_i)_r$ are the components of $\boldsymbol{l}_i$ and $\boldsymbol{v}_i$ along the direction OP; show that when all the charges have the same velocity this quantity has a time average of zero, if the average be computed over an interval which is long compared to the time required for a charge to move a distance equal to the mean distance between charges. In fact, to compute the time average of this quantity over such an interval, one may simply compute the instantaneous value this quantity would have if, at the given instant, a charge moving with the common constant velocity were located at every point of the volume element in question. Then if the reference point within the element is its center of gravity, simple considerations of symmetry show that the two terms written above are each zero.

2. Show in the same way that this quantity is also zero, on the average, if the velocities $\boldsymbol{v}_i$ are random velocities.

3. Since the time average $\overline{F(t)}$ of any quantity $F(t)$ over an interval $t_1 \leq t \leq t_2$ is

$$\overline{F(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} F(t)dt ,$$

show that the time average of the quantity referred to in the last two problems is zero for any periodic motion, the interval $t_2 - t_1$ being long compared to the period.

Part III—Ponderable Bodies

INTRODUCTION

The opening sections of Part III of this chapter follow the same general lines as those of Part III of chapter i. It is not necessary or desirable to repeat here the considerations which affect the introduction of those continuous densities in terms of which may be written the (vector) potential due to ponderable bodies. The present calculation draws heavily on the former one, and is hence considerably condensed. The striking analogy between electrostatics and magnetostatics is exhibited in § 45. This part concludes, as does its counterpart in chapter i, with a calculation of the force and torque on a ponderable body.

§ 41. *The Vector Potential Due to a Ponderable Body.*—It would be possible, just as in the previous development, to represent the potential due to a single complex, using the spreading method, in terms of two integrals, extended over an arbitrary volume containing the complex, these integrals involving vector functions j and m which are constant over the volume in question. Since the development is closely analogous to that given before, however, the intermediate details will be passed over, and the vector potential due to a ponderable body will be considered at once. Guided by the expression, found above, for the vector potential due to a single complex, and by the previous development, the vector potential due to the body τ will be represented by means of the two integrals

$$\frac{1}{4\pi}\int_\tau \frac{\boldsymbol{i}}{r}\,d\tau' + \frac{1}{4\pi}\int_\tau \left[\boldsymbol{M}, \nabla' \frac{1}{r}\right] d\tau' , \tag{114}$$

where $\boldsymbol{i}$ and $\boldsymbol{M}$ are continuous vector functions of the variables of integration x', y', z'—the co-ordinates* of the point P, at which $\boldsymbol{A}$ is being calculated, being x, y, z. The definitions of the functions $\boldsymbol{i}$ and $\boldsymbol{M}$ are to be obtained, as before, by a comparison of the expansion of these two integrals with an expansion of type (110). The vectors $\boldsymbol{i}$ and $\boldsymbol{M}$ will be called the "volume densities" of current and magnetization, respectively.

Let the body be divided into volume cells $\Delta\tau_i$, the order of whose di-

* Primes will not be written on i and $\boldsymbol{M}$ in this section.

mensions is fixed by the statistical considerations discussed at length in chapter ii. Then

$$\int_\tau \frac{\boldsymbol{i}}{r}\,d\tau' + \int_\tau \left[\boldsymbol{M}, \nabla' \frac{1}{r}\right] d\tau' = \Sigma \left\{ \int_{\Delta\tau_i} \frac{\boldsymbol{i}}{r}\,d\tau' + \int_{\Delta\tau_i} \left[\boldsymbol{M}, \nabla' \frac{1}{r}\right] d\tau' \right\}.$$

However,

$$\int_{\Delta\tau_i} \frac{\boldsymbol{i}}{r}\,d\tau' = \int_{\Delta\tau_i} \left\{ \left(\frac{\boldsymbol{i}}{r}\right)_i + s \left(\frac{\partial \frac{\boldsymbol{i}}{r}}{\partial s}\right)_i + \frac{s^2}{2} \left(\frac{\partial^2 \frac{\boldsymbol{i}}{r}}{\partial s^2}\right)_i + \dots . \right\} d\tau',$$

$$= \left(\frac{\boldsymbol{i}}{r}\right)_i \Delta\tau_i + \left(\frac{\boldsymbol{i}}{r^2}\right)_i \int_{\Delta\tau_i} s \cos\varphi\, d\tau' + \frac{1}{r_i} \int_{\Delta\tau_i} s \left(\frac{\partial \boldsymbol{i}}{\partial s}\right)_i d\tau' + \int_{\Delta\tau_i} s^2 \left(\frac{\partial \boldsymbol{i}}{\partial s}\right)_i \left(\frac{1}{r_i}\right)^2 \cos\varphi\, d\tau',$$

where the subscript i indicates that the value of a quantity is to be taken at O_i, a point within $\Delta\tau_i$, and where φ is the angle between $\boldsymbol{s}$ and the direction from O_i to P. If the point O_i be chosen at the center of the volume of $\Delta\tau_i$, the first integral (the second term just above) vanishes.

The second integral (the third term) may be written in the form

$$\frac{1}{r_i} \int_{\Delta\tau_i} s \frac{1}{s} (\boldsymbol{s}, \nabla) \boldsymbol{i}\, d\tau',$$

the x-component of which, for example, is

$$\frac{1}{r_i} \int_{\Delta\tau_i} s\, |(\nabla i_{x'})_i| \cos(h, s) d\tau' = \frac{|(\nabla i_{x'})_i|}{r_i} \int_{\Delta\tau_i} s \cos(h, s) d\tau',$$

where $\boldsymbol{h}$ has the direction of $(\nabla i_{x'})_i$. This integral, and the other two components in the same way, vanishes since O_i is the center of volume of $\Delta\tau_i$. Thus

$$\int_{\Delta\tau_i} \frac{\boldsymbol{i}}{r}\,d\tau' = \left(\frac{\boldsymbol{i}}{r}\right)_i \Delta\tau_i + \frac{1}{r_i^2} \int_{\Delta\tau_i} s^2 \left(\frac{\partial \boldsymbol{i}}{\partial s}\right)_i \cos\varphi\, d\tau' + \dots .$$

Similarly,

$$\int_{\Delta\tau_i} \left[\boldsymbol{M}, \nabla' \frac{1}{r}\right] d\tau' = \left[\boldsymbol{M}_i, \nabla' \frac{1}{r_i}\right] \Delta\tau_i + \dots ,$$

all other terms being dropped on account of the small variation, across $\Delta\tau_i$, of $\boldsymbol{M}$, and on account of the powers of r_i in the denominator. The vector potential due to the charges e_j within $\Delta\tau_i$ may, however, be written, according to (110), as

$$\frac{1}{4\pi c}\frac{\Sigma e_j \boldsymbol{v}_j}{r_i}+\frac{1}{8\pi c}\left[\Sigma[\boldsymbol{l}_j, e_j\boldsymbol{v}_j],\ \nabla'\frac{1}{r_i}\right].$$

Therefore, identifying the coefficients of like powers of r_i, the value at O_i of the vector current density $\boldsymbol{i}$ is given by the equation

$$\boldsymbol{i}_i \equiv \frac{\Sigma e_j \boldsymbol{v}_j}{c\Delta\tau_i}, \tag{115}$$

while the value, at O_i, of the volume density of magnetization $\boldsymbol{M}$ is to be determined from the equation

$$\Delta\tau_i\left[\boldsymbol{M}_i, \nabla'\frac{1}{r_i}\right]=\frac{1}{2c}\left[\Sigma[\boldsymbol{l}_j, e_j\boldsymbol{v}_j],\ \nabla'\frac{1}{r_i}\right]-\int_{\Delta\tau_i} s^2\left(\frac{\partial \boldsymbol{i}}{\partial s}\right)_i \cos\varphi\, d\tau'.$$

The left member of this equation and the first term of the right member are both vectors which are normal to $\boldsymbol{r}_i$. The direction of the last term, however, depends upon the variation of $\boldsymbol{i}$. It therefore follows that the type of representation attempted will furnish the desired approximation to the vector potential only in case the statistical regularity is such that this last term is sensibly zero, i.e., only in case that the values (115) calculated for the various volume cells $\Delta\tau_i$ differ so little, from cell to neighboring cell, that the continuous vector function $\boldsymbol{i}$, interpolated from these values, has a negligible variation across any one cell. This condition is analogous to that previously stated for the variation in the continuous density function ρ, there being this difference: in the electrostatic case if the variation of ρ across a volume cell is not negligible, the definition of the polarization is affected, while in the present case, if the variation of $\boldsymbol{i}$ across the volume cell is not negligible, the assumed representation is itself inadequate. In case this last term does indeed sensibly vanish, the value at O_i of the magnetization density is to be determined from the equation

$$\boldsymbol{M}_i \equiv \frac{1}{\Delta\tau_i 2c}\Sigma[\boldsymbol{l}_j, e_j\boldsymbol{v}_j]. \tag{116}$$

The continuous vector functions $\boldsymbol{M}$ and $\boldsymbol{i}$ are, as in the previous case, to be determined from the values $\boldsymbol{M}_i$ and $\boldsymbol{i}_i$ at O_i by interpolation, and the vector potential due to the body is given by

$$\boldsymbol{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{i}}{r}\,d\tau'+\frac{1}{4\pi}\int\left[\boldsymbol{M},\nabla'\frac{1}{r}\right]d\tau' . \tag{117}$$

By the use of a vector identity,* the last term in this equation may be rewritten

$$\frac{1}{4\pi}\int\frac{\operatorname{curl}'\boldsymbol{M}}{r}\,d\tau'-\frac{1}{4\pi}\int\operatorname{curl}'\frac{\boldsymbol{M}}{r}\,d\tau' . \tag{118}$$

The second of these integrals may be still further transformed by means of the identity†

$$\int\operatorname{curl}'\boldsymbol{C}d\tau=\int[\boldsymbol{n},\boldsymbol{C}]d\sigma , \tag{119}$$

where $\boldsymbol{n}$ is a unit vector directed along the exterior normal to the body. Thus substituting,

$$\boldsymbol{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{i}'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{\operatorname{curl}'\boldsymbol{M}'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{[\boldsymbol{M}',\boldsymbol{n}]}{r}\,d\sigma' . \tag{120}$$

In this fundamental equation all primes are inserted to indicate that $\boldsymbol{i}$ and $\boldsymbol{M}$ are functions of x', y', z', the variables of integration.

If one defines, for the volume element $\Delta\tau_i$, the vector $\boldsymbol{u}_i$ by the following equation:

$$\boldsymbol{i}_i=\frac{\Sigma e_j\boldsymbol{v}_j}{c\Delta\tau_i}\equiv\frac{\boldsymbol{u}_i\Sigma e_{j'}}{c\Delta\tau_i} ,$$

where $e_{j'}$ are the charges within $\Delta\tau_i$ whose velocities $\boldsymbol{v}_{j'}$ do not vanish, then

$$\boldsymbol{i}_i=\boldsymbol{u}_i\frac{\rho_i}{c} ,$$

where ρ_i is the density of "moving charge."

If, as in electrostatics, a continuous scalar function ρ be interpolated from the values ρ_i, and a "velocity of moving charge" vector $\boldsymbol{u}$ be interpolated from the values $\boldsymbol{u}_i$, then

$$\boldsymbol{i}=\rho\frac{\boldsymbol{u}}{c} .$$

* See Appendix, § 4, (38). † *Ibid.*, § 5, D.

This relation will often be used when it is desirable to bring into explicit evidence the fact that current is moving charge. The foregoing equations clearly define a velocity of moving charge only when the density of moving charge does not vanish.

Following the previous development, the next step in the representation of the vector potential due to a ponderable body would be to construct a thin shell of volume elements on the surface of the body, and to represent the vector potential due to the charges in these elements by means of surface integrals. It must be remembered, however, that the only reason for so doing is an expectation, based upon physical grounds, that the demand of statistical regularity in the quantities involved forces a separate consideration of these exterior volume elements. There is no point in using these thin exterior volume elements if it be possible to divide the whole of the body into ordinary volume elements whose characteristics vary slowly, from element to element, throughout the entire group. It was found, for example, in electrostatics, that it was possible to subdivide a body in such a way that there was a volume density ρ' and a surface density η', giving rise to the two terms

$$\frac{1}{4\pi}\int \frac{\rho'}{r}\,d\tau' + \frac{1}{4\pi}\int \frac{\eta'}{r}\,d\sigma' ,$$

and it was found possible to subdivide, if not this same body, then another body electrostatically equivalent to it, in such a way that the consideration of a surface density of charge did not enter at all, there being a new volume density ρ_1' and a volume density of polarization $\boldsymbol{P}_1'$, giving rise to the terms

$$\frac{1}{4\pi}\int \frac{\rho_1'}{r}\,d\tau' + \frac{1}{4\pi}\int \left(\boldsymbol{P}_1', \nabla' \frac{1}{r}\right) d\tau' . \tag{121}$$

This last expression could, however, be transformed into the two new terms

$$\frac{1}{4\pi}\int \frac{(\rho_1' - \operatorname{div} \boldsymbol{P}')}{r}\,d\tau' + \frac{1}{4\pi}\int \frac{(P_1)_n}{r}\,d\sigma' , \tag{122}$$

these two integrals being the equivalents of the previous two integrals (121) which result from the first method of subdivision. Thus in the second method of subdivision, the surface density of charge term, which does not appear, is absorbed in the volume-polarization term. The two representations, as has been pointed out, are analytically equivalent, but

correspond to different modes of subdivision and different physical points of view. The surface density of charge plays such a fundamental rôle from the physical point of view in the electrostatic problem for conductors that it is customary to use the mode of subdivision which keeps the surface density of charge in specific evidence.

A similar analytical situation obtains in the case at hand. The contributions to the vector potential due to the charges in a very thin exterior volume element would be represented by a term

$$\frac{1}{4\pi}\int \frac{\boldsymbol{l}}{r}\, d\sigma' ,$$

where $\boldsymbol{l}$ is a surface density of current interpolated from

$$\boldsymbol{l}_j = h\, \boldsymbol{i}_j ,$$

the values of $\boldsymbol{l}$ at O_j, the centers of volume of the cells $\Delta\tau_j$ whose thickness is h. It is clear that in the steady state the direction, at any point on the surface, of $\boldsymbol{l}$ must be tangent to the surface at that point. It often happens, however, that from a physical point of view the current flowing in these surface elements does not arise from charge which is actually moving in this thin surface shell, but is a superficial aspect of a volume distribution of elementary closed circuits or "current whirls." Thus (see Fig. 40) if an exterior shell is separated from the rest of the body, this shell will cut through many of the current whirls and, if it is thin enough, contain few that are not cut; so that, viewed apart from the rest of the body, this shell contains a moving sheet of charge. If the surface shell were not used, however, this moving sheet of charge would not exist as such, the moving charges which formed it now being recognized as belonging to a volume distribution of current whirls. Such current whirls bear the same relation to the volume density of magnetization $\boldsymbol{M}$ as do polarized molecules to the volume density of polarization $\boldsymbol{P}$. It is customary, in magnetostatics, to throw the emphasis on the $\boldsymbol{M}$ vector, rather than to bring surface currents into explicit prominence. Physically this is based upon a belief in the actual existence of such current whirls; analytically it implies a mode of subdivision that makes unnecessary the use of a thin surface shell, and which gives rise to a non-vanishing value for the volume density of magnetization. A simple picture of a polarized molecule indicates, in electrostatics, how it is that the two integrals

$$\int \frac{\eta}{r}\, d\sigma \text{ and } \int \frac{P_n}{r}\, d\sigma$$

may correspond to the same physical situation. That the two integrals

$$\int \frac{\boldsymbol{l}}{r}\, d\sigma' \text{ and } \int \frac{[\boldsymbol{M},\boldsymbol{n}]}{r}\, d\sigma'$$

can also correspond to the same physical situation may be indicated by an analogous argument, using a simple picture of a current whirl (see Figs. 40 and 41). The details of such an argument will not be given; indeed,

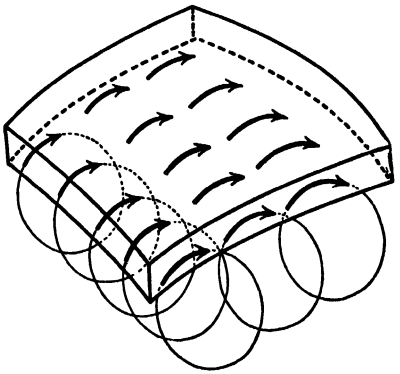

FIG. 40.—Figure showing how interior whirls of current give rise, when only a thin shell of surface is viewed, to a surface current sheet.

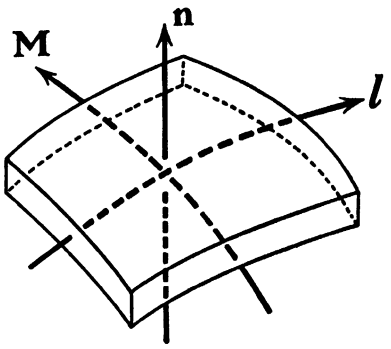

FIG. 41.—This figure, together with Figure 40, indicates the geometrical relationship between the magnetization $\boldsymbol{M}$, the surface density of current $\boldsymbol{l}$ and an external normal $\boldsymbol{n}$.

there is little interest in them, the identification which arises from the transformation

$$\int \left[\boldsymbol{M}',\nabla' \frac{1}{r}\right] d\tau' = \int \frac{\operatorname{curl}' \boldsymbol{M}'}{r}\, d\tau' + \int \frac{[\boldsymbol{M}',\boldsymbol{n}]}{r}\, d\sigma'$$

being exact and convincing.

In electrostatics it has been seen that it is possible to use all three densities ρ, η, $\boldsymbol{P}$ or to use less than three densities, so long as one can assign suitable values to

$$\rho - \operatorname{div} \boldsymbol{P} \text{ and } \eta + P_n .$$

This could clearly be done by choosing ρ and η, ρ and $\boldsymbol{P}$, or $\boldsymbol{P}$ alone. In discussing perfect conductors one chooses ρ and η. In discussing ideal dielectrics one chooses $\boldsymbol{P}$. The transformation written just above

indicates, in just the same way that one may use, in magnetostatics, i and l, i and M, or M alone. The choice actually made is precisely the one not used in either branch of electrostatics, namely, i and M. This non-analogous choice of densities partially obscures the actual analogy between electrostatics and magnetostatics and makes it necessary, when that analogy is exhibited (§ 45) to generalize somewhat the customary equations so as to use, in both instances, all of the available densities.

The foregoing discussion has been limited to but one of the two terms which would enter were a thin exterior shell used in subdividing the body. Besides the surface current term, there might also be a surface integral arising from a superficial distribution of current whirls, corresponding to the normally polarized layer on a conductor. There are, however, no convincing reasons why abnormalities should exist, near the surface, in the values m_i, and since there is no experimental evidence of such a layer, this term is customarily disregarded. The potential due to a ponderable body is, then, described in terms of the volume current density and the volume density of magnetization by the equation

$$A=\frac{1}{4\pi}\int\frac{i'}{r}\,d\tau'+\frac{1}{4\pi}\int\left[M',\nabla'\,\frac{1}{r}\right]d\tau', \tag{123}$$

or by the equivalent equation

$$A=\frac{1}{4\pi}\int\frac{i'+\operatorname{curl}' M'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{[M',n]}{r}\,d\sigma'. \tag{124}$$

In these last two equations primes have been written on the densities to remind that they are functions of x',y',z', the variables of integration. The vector potential A is, of course, a function of x,y,z. The vector n is a unit vector directed along the exterior normal.

At the beginning of this chapter, the current flowing along a wire was defined as the amount of charge which, per unit time, passes a given cross-section of the wire. Similarly, the current passing through any area or across any line is the amount of charge which, per unit time, passes through this area or crosses this line. Current thus defined has the dimensions of a charge divided by a time. The volume current density i, however, has the dimensions of charge divided by the product of time and area. It is thus dimensionally evident that, with the foregoing definitions, the volume density of current has to be multiplied by area to obtain current. It is, in fact, evident from the equation which defines i that $i_n d\sigma$ is the charge per unit time (i.e., the current) which passes through

$d\sigma$. Similarly, if one considers a surface density $\boldsymbol{l}$ of current, the current passing across a given line element ds on this surface is given by $l_n ds$, where l_n is the component of $\boldsymbol{l}$ normal to ds. It is clear that, inasmuch as current is charge per unit time, the terminology of $\boldsymbol{i}$ and $\boldsymbol{l}$ as volume and surface densities of current is not strictly logical.

The equations given above express the vector potential $\boldsymbol{A}$ in terms of the volume densities of current and magnetization. When the $\boldsymbol{B}$ field due to a given configuration of currents is desired, it is sometimes more convenient to use formulas which express $\boldsymbol{B}$ directly. For example, from the equation

$$\boldsymbol{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{i}'}{r}\,d\tau'$$

one calculates

$$B_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=\frac{1}{4\pi}\int\left(i_z'\frac{\partial\frac{1}{r}}{\partial y}-i_y'\frac{\partial\frac{1}{r}}{\partial z}\right)d\tau',$$

$$=-\frac{1}{4\pi}\int\left[\boldsymbol{i}',\nabla\frac{1}{r}\right]_x d\tau'=\frac{1}{4\pi}\int\left[\boldsymbol{i}',\nabla'\frac{1}{r}\right]_x d\tau'.$$

Hence, in general,

$$\boldsymbol{B}=\frac{1}{4\pi}\int\left[\boldsymbol{i}'+\operatorname{curl}'\boldsymbol{M}',\nabla'\frac{1}{r}\right]d\tau'+\int\left[[\boldsymbol{M}',\boldsymbol{n}],\nabla'\frac{1}{r}\right]d\sigma'. \tag{125}$$

If only volume densities of currents are present,

$$\boldsymbol{B}=\frac{1}{4\pi}\int\left[\boldsymbol{i}',\nabla'\frac{1}{r}\right]d\tau'; \tag{126}$$

while if only linear circuits are present,

$$\boldsymbol{I}d\boldsymbol{s}'=\boldsymbol{i}q\,d\boldsymbol{s}'=\boldsymbol{i}d\tau',$$

so that

$$\boldsymbol{B}=\frac{I}{4\pi}\int\left[d\boldsymbol{s}',\nabla'\frac{1}{r}\right]. \tag{127}$$

The expressions for $\boldsymbol{B}$ just obtained from the general expression for $\boldsymbol{A}$ could also be obtained directly from (107).

§ 42. *The Divergence and Curl of the Vector Potential: Distant Points.* —The foregoing analysis applies only to points P which are sufficiently removed from all parts of the body under consideration to insure the convergence of the approximating expansion. At all such points, $\boldsymbol{A}$ is given by the equations just above, while

$$\text{curl } \boldsymbol{A}=\boldsymbol{B} ,$$

so that

$$\text{div } \boldsymbol{B}=0 .$$

It may be shown that the vector potential $\boldsymbol{A}$ also has zero divergence at points in free space. For in the region under consideration the integrals are regular, and may be differentiated under the sign of integration. Moreover, the identity*

$$\text{div } [\boldsymbol{C}, \boldsymbol{D}]\equiv(\boldsymbol{D}, \text{ curl } \boldsymbol{C})-(\boldsymbol{C}, \text{ curl } \boldsymbol{D})$$

gives, in this case,

$$\text{div}\left[\boldsymbol{M}', \nabla' \frac{1}{r}\right]=-\text{div}\left[\boldsymbol{M}', \nabla \frac{1}{r}\right]=0 ,$$

since $\boldsymbol{M}'$ is not a function of $x, y, z,$ and since the curl of a nabla is zero. Also the identity†

$$\text{div } u\boldsymbol{C}\equiv u \text{ div } \boldsymbol{C}+(\boldsymbol{C}, \nabla u)$$

gives, since $\boldsymbol{i}'$ is not a function of $x, y, z,$ the relation

$$\text{div } \frac{\boldsymbol{i}'}{r}=\left(\boldsymbol{i}', \nabla \frac{1}{r}\right)=-\left(\boldsymbol{i}', \nabla' \frac{1}{r}\right)=-\text{div}' \frac{\boldsymbol{i}'}{r}+\frac{1}{r} \text{ div}' \boldsymbol{i}'$$

$$=-\text{div}' \frac{\boldsymbol{i}'}{r} ,$$

the last step following from the fact that, in the steady state, there is no heaping up of charges at any place, so that $\text{div}' \boldsymbol{i}'=0$. Thus, susbtituting these results,

$$\text{div } \boldsymbol{A}=-\frac{1}{4\pi}\int \text{div}' \frac{\boldsymbol{i}'}{r} d\tau'=-\frac{1}{4\pi}\int \frac{i_n'}{r} d\sigma' .$$

* See Appendix, § 4, (43).

† *Ibid.*, (37).

Since the integration is supposed extended over the whole of the body containing the current, there is no normal flow across the boundary, and $i'_n=0$. Thus the last integral vanishes and div $A=0$. Thus

$$\text{div } A=0 .$$

Also at distant points,

$$\nabla^2 A_x=\frac{1}{4\pi}\int i'_x\nabla^2\frac{1}{r}\,d\tau'+\frac{1}{4\pi}\int\left[M',\nabla\nabla^2\frac{1}{r}\right]_x d\tau'=0 ,$$

so that

$$\nabla^2 A=0 . \tag{128}$$

However,†

$$\text{curl curl } A=-\nabla^2 A+\nabla \text{ div } A ,$$

so that, since the vector A has zero divergence,

$$\text{curl curl } A=\text{curl } B=-\nabla^2 A=0 .$$

At points removed from any body the vectors A and B thus satisfy the following relations, set here for reference:

$$\left\{\begin{aligned} \text{div } A&=0 ,\\ \text{curl } A&=B ,\\ \text{div } B&=0 ,\\ \text{curl } B&=\text{curl curl } A=0 . \end{aligned}\right. \tag{129}$$

§ 43. *The Vector Potential at Points within a Body.*—The force on moving charge is expressed in terms of an auxiliary vector B which itself, in turn, is expressed as the curl of a second auxiliary vector A. This vector potential A is directly available for computing the force on moving charge only at points removed from any body; but, just as was the case with the scalar potential Φ in electrostatics, the formula from which A is computed at distant points may also be used to compute the values at points within the body. There is thus obtained a vector function which has a value at every point, exterior or interior, and which is the analytical extension, beyond its region of original physical significance, of the exterior vector potential. This interior vector potential will now be introduced and investigated.

† See Appendix, § 4, (42), and §7

The total force on a moving charge e located at a point within a body τ will be written as

$$F=\frac{e}{c}[v,B^*+N],$$

where B^* is due to all the charges outside a sphere of radius δ drawn around the charge in question, and where N is due to the neighboring charges within this sphere. This equation assumes the extension, to cover the case of very near charges, of the law that the force on a moving charge is always normal and proportional to its velocity. It makes no further assumption, however, concerning the force due to nearby charges, since nothing is said concerning N itself. If δ is sufficiently large, the vector B^* will be calculated, as above, from the vector potential A^* due to all the charges of the body not within the sphere. Just as in the analogous electrostatic case, however, special care must be used. The vector A^* is, at every point within the body, the vector potential due to all the charges lying outside a sphere of radius δ drawn about that point. In the variation, from point to point, of this function A^*, the deleted sphere is carried with the variable point, so that certain charges are removed from consideration, and certain other new charges appear. In obtaining the B vector by taking the curl of the A vector, however, the position and velocity configuration of charges must not be altered. Thus

$$\text{div}^*\ A^*=0,$$
$$\text{curl}^*\ A^*=B^*,$$

where curl* and div* are formed by taking the variation, within a fixed sphere, of the vector potential due to the charges without this sphere, the special notation being necessary, in the case in hand, to assure the differentiation being carried out in the proper way. These values will now be compared with the values curl A^* and div A^*, in the calculation of which the δ-sphere moves.

It has been previously shown† that if

$$\Phi^*=\frac{1}{4\pi}\int_{\tau-\delta}\frac{\rho'}{r}\,d\tau',$$

† See § 18, equation, (56).

then

$$\frac{\partial \Phi^*}{\partial x} = \frac{1}{4\pi}\int_{\tau-\delta} \rho' \frac{\partial \frac{1}{r}}{\partial x} d\tau' + \frac{1}{4\pi}\int_s \frac{\rho'}{r} \cos(n,x') d\sigma' , \tag{130}$$

where s is the surface of the δ-sphere, and where n is a normal interior to the sphere. Thus,

$$\frac{\partial A_x^*}{\partial x} = \frac{1}{4\pi}\int_{\tau-\delta} i'_x \frac{\partial \frac{1}{r}}{\partial x} d\tau' + \frac{1}{4\pi}\int_{\tau-\delta} \left[M', \nabla' \frac{\partial \frac{1}{r}}{\partial x}\right]_x d\tau'$$

$$+ \frac{1}{4\pi}\int_s \frac{i'_x}{r} \cos(n,x') d\sigma' + \frac{1}{4\pi}\int_s \left[M', \nabla' \frac{1}{r}\right]_x \cos(n,x') d\sigma' ,$$

so that

$$\operatorname{div} A^* = \operatorname{div}^* A^* + \frac{1}{4\pi}\int_s \frac{i'_n}{r} d\sigma' + \frac{1}{4\pi}\int_s \left[M', \nabla' \frac{1}{r}\right]_n d\sigma' .$$

However,

$$\int_s \frac{i'_n}{r} d\sigma' = \frac{1}{\delta}\int_s i'_n d\sigma' = 0 ,$$

since, in the steady state, there can be no net flow into or out of the sphere. The other integral over the surface s also vanishes, since the vector $[M', \nabla' 1/r]$ has no component along the normal to the sphere. Thus,

$$\operatorname{div} A^* = \operatorname{div}^* A^* = 0 .$$

The same method yields the value of curl A^*. Indeed, making use of the differentiation-scheme (130),

$$\operatorname{curl}_x A^* = \operatorname{curl}_x^* A^* + \frac{1}{4\pi}\int_s \frac{1}{r} [i'_z \cos(n,y') - i'_y \cos(n,z')] d\sigma' ,$$

$$+ \frac{1}{4\pi}\int_s \left\{ \left[M', \nabla' \frac{1}{r}\right]_z \cos(n,y') - \left[M', \nabla' \frac{1}{r}\right]_y \cos(n,z') \right\} d\sigma' ,$$

so that

$$\operatorname{curl} A^* = \operatorname{curl}^* A^* + \frac{1}{4\pi}\int_s \left[n, \frac{i'}{r}\right] d\sigma' + \frac{1}{4\pi}\int_s \left[n \left[M', \nabla' \frac{1}{r}\right]\right] d\sigma' .$$

If the vector i' be sensibly constant over the interior of the δ-sphere, the first surface integral vanishes, as may be easily seen from considerations of symmetry. In the second surface integral, the nabla operator is taken with respect to the variables of integration x', y', z', so that

$$\nabla' \frac{1}{r} = -\frac{\nabla' r}{r^2} = \frac{n}{r^2}.$$

Thus,

$$\left[n\left[M', \nabla' \frac{1}{r}\right]\right] = \frac{1}{r^2}\left[n\left[M', n\right]\right] = \frac{1}{r^2}\left[M'(n,n) - n(n,M')\right] = \frac{M' + nM'_r}{r^2}, \tag{131}$$

where M'_r is the component of M' in the direction of $\nabla' r$, i.e., the direction opposite to n. This last vector is to be summed over the surface of the δ-sphere. If θ be the angle between M' and r, then at A and B (see Fig. 42) the values of M'_r are numerically equal but opposite in sign, while the components of n normal to M' are equal. Since all the surface elements can be paired in this way, it is seen that the component normal to M' arising from the integration of the term nM'_r/r^2 is zero. The integration of the vector (131) over the surface of the sphere thus produces a vector in the direction of M'. The component of (131) along M' is

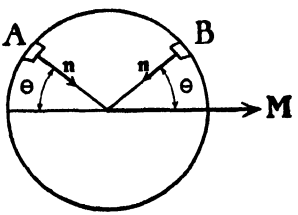

FIG 42

$$\frac{M'}{r^2}(1 - \cos^2\theta).$$

Thus

$$\frac{1}{4\pi}\int_s \left[n\left[M', \nabla' \frac{1}{r}\right]\right] d\sigma' = \frac{M}{4\pi\delta^2}\int_s (1 - \cos^2\theta) d\sigma',$$

where M has been removed from under the sign of integration since it, as well as i, is assumed sensibly constant throughout the δ-sphere.† Since

$$d\sigma' = 2\pi\delta^2 \sin\theta \, d\theta,$$

† At this point, the prime on M is dropped, since the quantity removed from under the integral sign is the value of M at the center of the sphere, i.e., at the point x,y,z.

the value of the foregoing integral is

$$\frac{M}{2}\int_0^{\pi}(1-\cos^2\theta)\,\sin\theta\,d\theta=\frac{2M}{3}\,.$$

Thus

$$\text{curl}\,\boldsymbol{A}^*=\text{curl}^*\,\boldsymbol{A}+\frac{2\boldsymbol{M}}{3}\,,$$

or

$$\boldsymbol{B}^*=\text{curl}\,\boldsymbol{A}^*-\frac{2\boldsymbol{M}}{3}\,.$$

Consider now the vector, referred to at the beginning of this section, which is the extension, within the body, of the vector function $\boldsymbol{A}$. That is, consider the vector defined, at interior points, by the equation

$$\boldsymbol{A}\equiv\lim_{\delta\doteq 0}\boldsymbol{A}^*\,,$$

and let, for these interior points as well as for exterior points, the equation

$$\boldsymbol{B}\equiv\text{curl}\,\boldsymbol{A}$$

be the definition of $\boldsymbol{B}$ within the body. The relation

$$\text{div}\,\boldsymbol{A}=\text{div}\lim_{\delta\doteq 0}\boldsymbol{A}^*=\lim_{\delta\doteq 0}\text{div}\,\boldsymbol{A}^*=0$$

then follows from the calculations which establish the relation

$$\lim_{\delta\doteq 0}\nabla\Phi^*=\nabla\lim_{\delta\doteq 0}\Phi^*=\nabla\Phi\,,$$

since it was there shown that for integrals of the type which occur both here and in the previous instance, the operators $\partial/\partial x$ and $\lim_{\delta\doteq 0}$ can be permuted. For the same reason the relation

$$\lim_{\delta\doteq 0}\boldsymbol{B}^*=\lim_{\delta\doteq 0}\text{curl}\,\boldsymbol{A}^*-\lim_{\delta\doteq 0}\frac{2}{3}\boldsymbol{M}$$

reduces to the form

$$\lim_{\delta\doteq 0}\boldsymbol{B}^*=\text{curl}\,\boldsymbol{A}-\lim_{\delta\doteq 0}\frac{2}{3}\boldsymbol{M}\,,$$

or, since M is assumed sensibly constant throughout the δ-sphere,

$$\lim_{\delta \doteq 0} B^* = \text{curl } A - \frac{2}{3} M .$$

The vectors B^* corresponding to two different choices of δ differ by the effect of all the charges contained in the spherical shell bounded by the two spheres. The contribution to B^* from the moving charges in this spherical shell can be calculated directly from the fundamental equation

$$B = \frac{1}{4\pi c} \Sigma \left[e_i v_i, \frac{r_i}{r^3} \right] ,$$

where r_i points from e_i to the place where B is being calculated. When the summation is extended over the symmetrical arrangement of moving charges which (since i and M are assumed sensibly constant over this region) exists within the spherical shell in question, the result is zero. The value of B^* is thus independent of δ, and

$$B^* = \text{curl } A - \frac{2}{3} M .$$

If the vector A be written in the form

$$A = \frac{1}{4\pi} \int_\tau \frac{i' + \text{curl } M'}{r} \, d\tau' + \frac{1}{4\pi} \int_\Sigma [M', n] d\sigma' ,$$

the first integral is, at interior points, improper. However, the equation

$$-\nabla^2 A = i + \text{curl } M \tag{132}$$

may be obtained, a component at a time, by applying the differentiation scheme developed in § 21 for the analogous integral

$$\Phi = \frac{1}{4\pi} \int \frac{\rho'}{r} \, d\tau' .$$

Moreover, since,

$$\text{curl curl } A = -\nabla^2 A + \nabla \text{ div } A ,$$

and since the vector A has zero divergence everywhere, equation (132) may be written in the form

$$\text{curl } B = \text{curl curl } A = i + \text{curl } M ,$$

a relationship which corresponds to the second-order partial-differential equation

$$\nabla^2\Phi = -\rho$$

in electrostatics.

The relationships just developed are collected here for reference. At all points, both interior and exterior, the vector potential $\boldsymbol{A}$ is given by (123) or (124), and

(133) $$\operatorname{div} \boldsymbol{A} = 0 ,$$

(134) $$\operatorname{curl} \boldsymbol{A} = \boldsymbol{B} ,$$

(135) $$\operatorname{div} \boldsymbol{B} = 0 ,$$

(136) $$\operatorname{curl} \boldsymbol{B} = \operatorname{curl} \operatorname{curl} \boldsymbol{A} = \boldsymbol{i} + \operatorname{curl} \boldsymbol{M} ,$$

(137) $$\boldsymbol{B}^* = \operatorname{curl} \boldsymbol{A} - \frac{2}{3} \boldsymbol{M} .$$

The components of $\boldsymbol{A}$ are given by integrals entirely similar, analytically, to those occurring in the expression for the electrostatic potential Φ. It thus follows from previous arguments that $\boldsymbol{A}$ is continuous everywhere, while across the surface of a body

$$\left(\frac{\partial \boldsymbol{A}}{\partial n_1}\right)_1 + \left(\frac{\partial \boldsymbol{A}}{\partial n_2}\right)_2 = -[\boldsymbol{M},\boldsymbol{n}] ,$$

where, for example, $(\partial \boldsymbol{A}/\partial n_1)_1$ means the limit of $\partial \boldsymbol{A}/\partial n_1$, as the point at which the derivative is taken approaches the surface from the side into which n_1 points; and where $\boldsymbol{n}$ is the exterior normal to the body. This equation is usually written in the abbreviated form

(138) $$\frac{\partial \boldsymbol{A}}{\partial n_1} + \frac{\partial \boldsymbol{A}}{\partial n_2} = -[\boldsymbol{M},\boldsymbol{n}].$$

From an argument similar to that used for Φ, it also follows that the vector $\boldsymbol{A}$ is regular at infinity, vanishing as $1/r$. The previous proof that there is but one function satisfying the schedule of conditions (I) of

§ 21 also proves, considering one component at a time, that there is but one vector satisfying the conditions

$$\text{(III)} \quad \begin{cases} a)\ \nabla^2 \boldsymbol{A} = -(\boldsymbol{i} + \text{curl}\ \boldsymbol{M}) , \\ b)\ \boldsymbol{A} \text{ is continuous,} \\ c)\ \dfrac{\partial \boldsymbol{A}}{\partial n_1} + \dfrac{\partial \boldsymbol{A}}{\partial n_2} = -[\boldsymbol{M}, \boldsymbol{n}] , \\ d)\ \boldsymbol{A} \text{ is regular at infinity.} \end{cases}$$

It therefore also follows that if a vector be found from the conditions

$$\text{(III')} \quad \begin{cases} a)\ \text{curl curl}\ \boldsymbol{A} = \boldsymbol{i} + \text{curl}\ \boldsymbol{M} , \\ b)\ \boldsymbol{A} \text{ is continuous} , \\ c)\ \dfrac{\partial \boldsymbol{A}}{\partial n_1} + \dfrac{\partial \boldsymbol{A}}{\partial n_2} = -[\boldsymbol{M}, \boldsymbol{n}] , \\ d)\ \boldsymbol{A} \text{ is regular at infinity,} \end{cases}$$

this vector, if it also satisfy the condition that its divergence vanish, is the unique solution and hence the vector potential sought; for when div $\boldsymbol{A} = 0$, the two quantities curl curl $\boldsymbol{A}$ and $-\nabla^2 \boldsymbol{A}$ are the same.

§ 44. *The Vector Potential Due to an Infinite Cylindrical Shell of Current.*—As an illustration of the use of the foregoing equations, the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ will be calculated for the case of an infinitely long cylinder of radius R on which there is a surface current of density $\boldsymbol{l}$ per unit area, directed parallel to the axis of the cylinder. This surface current may be thought of as an actual moving sheet of charge, or as the superficial aspect of a volume magnetization, directed, at every point on the surface, normal to the radius at that point.

For the sake of variety, the schedule (III') will be solved. This necessitates writing the components, in co-ordinates other than rectangular, of the curl of a vector. Now from Stokes's law may be written at once the relation

$$\text{(139)} \qquad \text{curl}_n\ \boldsymbol{A} = \lim_{\sigma \doteq 0} \left\{ \frac{1}{\sigma} \int A_s ds \right\} ,$$

where $\text{curl}_n\ \boldsymbol{A}$ is the component of curl $\boldsymbol{A}$ normal to the elementary area σ around the perimeter of which the line integral is extended. The direction along this perimeter used in calculating the tangential compo-

nent A_s is related to the direction of n by the right-hand screw rule. The foregoing equation is analogous to the equation

$$\operatorname{div} A = \lim_{\tau \doteq 0} \left\{ \frac{1}{\tau} \int A_n d\sigma \right\} ,$$

and, like the latter, is independent of any system of co-ordinates. It may be used, therefore, just as was the latter, to determine the components in general orthogonal curvilinear co-ordinates of the vector which it defines. The equations so obtained, namely,

$$(140) \qquad \begin{cases} \operatorname{curl}_u A = \dfrac{1}{e_2 e_3} \left(\dfrac{\partial e_3 A_w}{\partial v} - \dfrac{\partial e_2 A_v}{\delta w} \right) , \\ \operatorname{curl}_v A = \dfrac{1}{e_3 e_1} \left(\dfrac{\partial e_1 A_u}{\partial w} - \dfrac{\partial e_3 A_w}{\partial u} \right) , \\ \operatorname{curl}_w A = \dfrac{1}{e_1 e_2} \left(\dfrac{\partial e_2 A_v}{\partial u} - \dfrac{\partial e_1 A_u}{\partial v} \right) , \end{cases}$$

reduce, for cylindrical co-ordinates, r, θ, z, to the equations

$$\operatorname{curl}_r A = \frac{1}{r} \left(\frac{\partial A_z}{\partial \theta} - \frac{\partial r A_\theta}{\partial z} \right) ,$$

$$\operatorname{curl}_\theta A = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} ,$$

$$\operatorname{curl}_z A = \frac{1}{r} \left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) .$$

In the case here considered it is clear, from symmetry, that

$$\frac{\partial A}{\partial \theta} = 0 , \qquad \frac{\partial A}{\partial z} = 0 ,$$

so that

$$\operatorname{curl}_r A = 0 ,$$

$$\operatorname{curl}_\theta A = - \frac{\partial A_z}{\partial r} ,$$

$$\operatorname{curl}_z A = \frac{1}{r} \frac{\partial r A_\theta}{\partial r} .$$

Hence, applying these equations again,

$$\text{curl}_r \text{ curl } \boldsymbol{A} = \frac{1}{r}\left(\frac{\partial}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r} rA_\theta + \frac{\partial}{\partial z} r \frac{\partial}{\partial r} A_z\right),$$

$$\text{curl}_\theta \text{ curl } \boldsymbol{A} = -\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} rA_\theta,$$

$$\text{curl}_z \text{ curl } \boldsymbol{A} = -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} A_z.$$

Now since $\boldsymbol{A}$ may be expressed, through the integral form, as the sum of vectors all of which are parallel to z, it follows that A_z is the only component of $\boldsymbol{A}$. Thus,

$$\text{curl}_r \text{ curl } \boldsymbol{A} = 0,$$

$$\text{curl}_\theta \text{ curl } \boldsymbol{A} = 0,$$

$$\text{curl}_z \text{ curl } \boldsymbol{A} = -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} A_z.$$

The schedule of conditions (III′) thus reduces, since one suposes that $i+\text{curl } \boldsymbol{M}$ is zero, to

$$a) \quad -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} A_z = 0,$$

$$b) \quad A_z \text{ is continuous,}$$

$$c) \quad \left(\frac{\partial A_z}{\partial r}\right)_o - \left(\frac{\partial A_z}{\partial r}\right)_i = -l,$$

$$d) \quad \text{div } \boldsymbol{A} = 0.$$

For a cylinder whose length is actually infinite, the regularity of $\boldsymbol{A}$ at infinity cannot be assumed, since the proof of regularity depends upon the possibility of inclosing all the charges within some finite volume. In the equations just written the subscripts o and i refer to the values of A_z outside and inside the cylinder. From (a),

$$A_z = a \log r + b,$$

which, together with $A_r = A_\theta = 0$, satisfies (d). From (b), it follows that inside the cylinder a must be zero. Thus

$$(A_z)_i = b_i,$$

$$(A_z)_o = a \log r + b_o,$$

so that, from (*c*),

$$\frac{a}{R} = -l .$$

Thus

$$(A_z)_o = Rl \log \frac{1}{r} + b_o = \frac{2I}{4\pi} \log \frac{1}{r} + b_o ,$$

where $I = 2\pi Rl$ is the total current passing along the cylinder. Using (*b*) again,

$$(A_z)_i = b_i = (A_z)_o = \frac{2I}{4\pi} \log \frac{1}{r} + b_o ,$$

when $r = R$, so that

$$(A_z)_i = b_i = \frac{2I}{4\pi} \log \frac{1}{R} + b_o .$$

Now $\boldsymbol{B} = \text{curl } \boldsymbol{A}$, so that $\boldsymbol{B}_i$ is zero, while at outside points the only component of $\boldsymbol{B}$ is

$$(B_\theta)_o = -\left(\frac{\partial A_z}{\partial r}\right)_o = \frac{2I}{4\pi r} ,$$

so that the magnitude of $\boldsymbol{B}$ is constant on circles concentric with the cylinder, the direction of $\boldsymbol{B}$ being tangential to these circles.

§ 45. *The Schedule of Conditions on* $\boldsymbol{B}$*: Analogy between Electrostatics and Magnetostatics.*—The vector potential $\boldsymbol{A}$ has been seen to be analogous to the scalar potential Φ. The conditions on the vector $\boldsymbol{A}$, which have been found above, will now be translated, through the relationship $\boldsymbol{B} = \text{curl } \boldsymbol{A}$, over into the vector $\boldsymbol{B}$, in order further to exhibit the close analytical analogy between the fundamental equations of electrostatics and magnetostatics.

Since the vector $\boldsymbol{A}$ is continuous across any surface, it follows that the derivatives of $\boldsymbol{A}$ in directions tangential to the surface must also be continuous. For suppose that in some direction $\boldsymbol{h}$ tangential to the surface the derivative is not continuous across the surface. At two points P_1 and P_2, lying very near each other, but on opposite sides of the surface, the values $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ of $\boldsymbol{A}$ may, since $\boldsymbol{A}$ is continuous, be made as nearly equal as is desired by choosing P_1 and P_2 near enough together. Starting at these points, and going a distance d in the direction $\boldsymbol{h}$, the values $\boldsymbol{A}_1'$ and $\boldsymbol{A}_2'$ which will be reached cannot, since the

derivatives in this direction on the two sides of the surface are supposed unequal, be made to differ by as little as desired. The known continuity of $\boldsymbol{A}$ is thus contradicted, and the assumption that the tangential derivative is discontinuous is proved untenable.

The component normal to the surface of curl $\boldsymbol{A}$ is made up, however, of derivatives taken along directions tangential to the surface. It therefore follows that the normal component of $\boldsymbol{B}$ is continuous across the surface. Thus

$$B_{n_1}+B_{n_2}=0\,,$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are oppositely directed normals to the surface; or, in vector form

$$(\boldsymbol{B}, \boldsymbol{n}_1)+(\boldsymbol{B}, \boldsymbol{n}_2)=0\,.$$

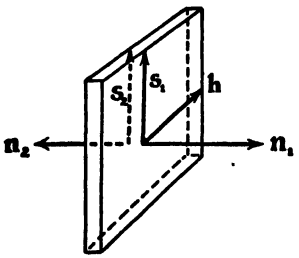

FIG. 43.—The surface separating regions 1 and 2 is drawn with a finite thickness, so as to separate the actually coincident vectors S_1 and S_2.

Let $\boldsymbol{h}$ be tangential to the surface and in the direction of $[\boldsymbol{M},\boldsymbol{n}]$, and let s_1 and s_2 be tangential to the surface, on its two sides, in a direction $\boldsymbol{s}$ which is normal to $\boldsymbol{h}$, and so chosen that $\boldsymbol{n}$, $\boldsymbol{h}$, and $\boldsymbol{s}$ form a right system (see Fig. 43). Then

$$\operatorname{curl}_s \boldsymbol{A}=\frac{\partial A_h}{\partial n_1}-\frac{\partial A_{n_1}}{\partial h}\,,$$

so that

$$B_{s_1}=\frac{\partial A_h}{\partial n_1}-\frac{\partial A_{n_1}}{\partial h}\,,$$

$$B_{s_2}=-\frac{\partial A_h}{\partial n_2}-\frac{\partial A_{n_1}}{\partial h}\,,$$

where B_{s_1} and B_{s_2} are the limiting values, on the two sides of the surface, of the component of $\boldsymbol{B}$ in the direction $\boldsymbol{s}$. Hence, subtracting, and noting that two terms cancel on account of the continuity, across the surface, of the tangential derivative of any component of $\boldsymbol{A}$,

$$B_{s_2}-B_{s_1}=-\left(\frac{\partial A_h}{\partial n_1}+\frac{\partial A_h}{\partial n_2}\right)=|[\boldsymbol{M}, \boldsymbol{n}]|\,, \tag{141}$$

the last step following from the general relation

$$\frac{\partial A}{\partial n_1}+\frac{\partial A}{\partial n_2}=-[M, n] .$$

If one now writes, for the values of B on the two sides of the surface,

$$B_1=B_n+B_{s_1}+B_h ,$$
$$B_2=B_n+B_{s_2}+B_h ,$$

where the notation itself expresses the fact that the components of B normal to the surface, and in the direction of $[M,n]$, are continuous across the surface,* then

$$\begin{aligned}[n_1,B]&=[n_1,B_n]+[n_1,B_{s_1}]+[n_1,B_h] ,\\ &=-[n_2,B_n]-[n_2,B_{s_1}]-[n_2,B_h] ,\\ [n_2,B]&=[n_2,B_n]+[n_2,B_{s_2}]+[n_2,B_h] ,\end{aligned}$$

so that, by adding,

$$[n_1,B]+[n_2,B]=[n_2,B_{s_2}]-[n_2,B_{s_1}] .$$

The right-hand member of this equation is a vector whose magnitude is $B_{s_2}-B_{s_1}$ and whose direction is the direction of $[M,n]$. Hence (141) may be written

$$[n_1,B]+[n_2,B]=[M,n] .$$

Finally, since A vanishes at infinity as $1/R$, the vector B, which is formed from the derivatives of A, is regular at infinity, and vanishes as $1/R^2$. The conditions on B are thus

(IV)
$$\begin{cases} \text{curl } B=i+\text{curl } M , \\ \text{div } B=0 , \\ (n_1,B)+(n_2,B)=0 , \\ [n_1, B]+[n_2, B]=[M,n] , \\ \quad B \text{ is regular at infinity,} \end{cases}$$

the first two of these equations following directly from the relations $B=\text{curl } A$ and $\text{curl curl } A=i+\text{curl } M$.

* See Part III, Problem 7, of this chapter.

In order that one may exhibit, in as complete a form as possible, the analogy between the electrostatics and magnetostatics it is necessary, at the beginning, to subdivide the bodies in an "analogous" way,* i.e., so as to throw the burden of the description, in each case, on the same set of densities. For this reason the electrostatic equations will be written down as though the bodies had been so subdivided as to exhibit volume densities of charge and polarization and a surface density of charge, the polarization term then being analytically merged with the volume and surface density of charge terms. The formula for Φ then would read†

$$\Phi=\frac{1}{4\pi}\int\frac{\rho-\operatorname{div}\boldsymbol{P}}{r}\,d\tau+\frac{1}{4\pi}\int\frac{\eta+P_n}{r}\,d\sigma\,. \tag{142}$$

Analogously the body, in the magnetostatic case, will be so subdivided as to exhibit the three analogous densities, namely, volume densities of current and magnetization, and a surface density of current. As above, the magnetization term is then to be analytically merged with the other two terms, and the formula for $\boldsymbol{A}$ reads

$$\boldsymbol{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{i}+\operatorname{curl}\boldsymbol{M}}{r}\,d\tau+\frac{1}{4\pi}\int\frac{\boldsymbol{l}+[\boldsymbol{M},\boldsymbol{n}]}{r}\,d\sigma\,. \tag{143}$$

Formulas (142) and (143) may now be re-written so as to furnish the basis for the formal analogy by writing

$$\operatorname{div}\boldsymbol{P}\equiv(\nabla,\boldsymbol{P})\,,$$
$$P_n\equiv(\boldsymbol{n},\boldsymbol{P})\,,$$
$$\operatorname{curl}\boldsymbol{M}=[\nabla,\boldsymbol{M}]\,.$$

Then

$$4\pi\Phi=\int\frac{\rho-(\nabla,\boldsymbol{P})}{r}\,d\tau+\int\frac{\eta+(\boldsymbol{n},\boldsymbol{P})}{r}\,d\sigma\,, \tag{144}$$

$$4\pi\boldsymbol{A}=\int\frac{\boldsymbol{i}+[\nabla,\boldsymbol{M}]}{r}\,d\tau+\int\frac{\boldsymbol{l}-[\boldsymbol{n},\boldsymbol{M}]}{r}\,d\sigma\,. \tag{145}$$

* See remarks just below Fig. 41 of § 40.

† All primes will be omitted in these formulas, to exhibit as simply as possible the analogy of form.

One can pass from one of these equations to the other by interchanging

$\boldsymbol{A}$ and Φ ,	$\boldsymbol{B}$ and $\boldsymbol{E}$,
i and ρ ,	[ ] and ( ) ,
$-\boldsymbol{M}$ and $\boldsymbol{P}$,	( ) and [ ] .
$\boldsymbol{l}$ and η ,	

Since the mathematical formulation of electrostatics and magnetostatics is largely obtained from equations (144) and (145), respectively, the analogy just set forth persists throughout. In fact,

$\nabla^2\Phi = -\{\rho - (\nabla,\boldsymbol{P})\}$,	$\nabla^2\boldsymbol{A} = -\{\boldsymbol{i} + [\nabla,\boldsymbol{M}]\}$,
Φ is continuous	$\boldsymbol{A}$ is continuous,
$\frac{\partial\Phi}{\partial n_1} + \frac{\partial\Phi}{\partial n_2} = -\{\eta + (\boldsymbol{n},\boldsymbol{P})\}$,	$\frac{\partial\boldsymbol{A}}{\partial n_1} + \frac{\partial\boldsymbol{A}}{\partial n_2} = -\{\boldsymbol{l} - [\boldsymbol{n},\boldsymbol{M}]\}$,
Φ vanishes at ∞ as $1/R$;	$\boldsymbol{A}$ vanishes at ∞ as $1/R$,
$(\nabla,\boldsymbol{E}) = \rho - (\nabla,\boldsymbol{P})$,	$[\nabla,\boldsymbol{B}] = \boldsymbol{i} + [\nabla,\boldsymbol{M}]$,
$[\nabla,\boldsymbol{E}] = 0$,	$(\nabla,\boldsymbol{B}) = 0$,
$(\boldsymbol{n}_1\boldsymbol{E}) + (\boldsymbol{n}_2,\boldsymbol{E}) = \eta + (\boldsymbol{n},\boldsymbol{P})$,	$[\boldsymbol{n}_1,\boldsymbol{B}] + [\boldsymbol{n}_2,\boldsymbol{B}] = \boldsymbol{l} - [\boldsymbol{n},\boldsymbol{M}]$,
$[\boldsymbol{n}_1\boldsymbol{E}] + [\boldsymbol{n}_2\boldsymbol{E}] = 0$	$(\boldsymbol{n}_1,\boldsymbol{B}) + (\boldsymbol{n}_2,\boldsymbol{B}) = 0$,
$\boldsymbol{E}$ vanishes at ∞ as $1/R^2$;	$\boldsymbol{B}$ vanishes at ∞ as $1/R^2$.

The analogy can be artificially made more complete by interchanging the two vectors which appear in every vector and scalar product. This, since it changes the signs of the vector products and does not affect the scalar products, brings all signs into agreement without requiring that $-\boldsymbol{M}$ be analogous to $\boldsymbol{P}$. Such an expression, however, as $(\boldsymbol{P},\nabla)$ is objectional from an operational point of view; and the fact is, as appears from a comparison of the equations,

$$\boldsymbol{E}^* = \boldsymbol{E} + \frac{\boldsymbol{P}}{3} ,$$

$$\boldsymbol{B}^* = \boldsymbol{B} - \frac{2\boldsymbol{M}}{3} ,$$

that $-\boldsymbol{M}$, and not $\boldsymbol{M}$, is actually analogous to $\boldsymbol{P}$.

§ 46. *The Force and Torque on a Complex and on a Ponderable Body.*— The force on a single moving charge due to a complex of charges is ex-

pressed, through the foregoing equations for $\boldsymbol{F}$, $\boldsymbol{B}$, and $\boldsymbol{A}$, in terms of vectors $\boldsymbol{j}$ and $\boldsymbol{m}$, which are characteristic of the complex. The total force and torque on a complex 2 due to a complex 1 can be expressed in terms of the $\boldsymbol{j}$ and $\boldsymbol{m}$ vectors characteristic of complex 2, and the $\boldsymbol{B}$ vector due to complex 1. Let charges e_i be moving with velocities $\boldsymbol{v_i}$ in complex 2. Then the total force on 2 is

$$\boldsymbol{F}=\frac{1}{c}\Sigma[e_i\boldsymbol{v}_i,\ \boldsymbol{B}_i]\ ,$$

where $\boldsymbol{B}_i$ are the values at the positions of the charges e_i of the $\boldsymbol{B}$ vector due to 1. These values $\boldsymbol{B}_i$ may be expressed in terms of the value $\boldsymbol{B}_0$ of $\boldsymbol{B}$ at a point O within 2, and in terms of the successive derivatives of $\boldsymbol{B}$ taken at this same point and in various directions. Indeed,

$$\boldsymbol{B}_i=\boldsymbol{B}_0+l_i\left(\frac{\partial \boldsymbol{B}}{\partial l_i}\right)_0+\ \ldots\ ,$$

where $\boldsymbol{l_i}$ are the vectors which locate the charges e_i with respect to O. If the complexes are separated a distance which is large compared to their own dimensions, the foregoing expression will be rapidly convergent, and the higher-order terms may be disregarded. This value for $\boldsymbol{B}_i$ gives, upon substitution,

$$\boldsymbol{F}=\Sigma\left[\frac{e_i\boldsymbol{v}_i}{c}\ ,\boldsymbol{B}_0\right]+\frac{1}{c}\Sigma\left[e_i\boldsymbol{v}_i,\ l_i\left(\frac{\partial \boldsymbol{B}}{\partial l_i}\right)_0\right]\ .$$

The term $l_i(\partial \boldsymbol{B}/\partial l_i)_0$ may be written in the equivalent form $(\boldsymbol{l_i}\nabla)\boldsymbol{B}$, so that the second term in this expression for the force has an x-component equal to

$$\frac{1}{c}\Sigma e_i\dot{y}_i(\boldsymbol{l}_i,\nabla B_z)-\frac{1}{c}\Sigma e_i\dot{z}_i(\boldsymbol{l}_i,\nabla B_y)\ , \tag{146}$$

where $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ are the components of $\boldsymbol{v}_i$. It has been previously shown in § 40, however, that

$$\Sigma e_i\boldsymbol{v}_i(\boldsymbol{l}_i,\boldsymbol{r}')=-c[\boldsymbol{r}',\boldsymbol{m}]\ . \tag{147}$$

so that, for example,

$$\Sigma e_i\dot{y}_i(\boldsymbol{l}_i,\boldsymbol{r}')=-c\{r_z m_x-r'_x m_z\}\ .$$

Each term of (146) is of the same form as the first member of this last equation. Thus, (146) reduces to

$$-m_x\left(\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)+m_y\frac{\partial B_y}{\partial x}+m_z\frac{\partial B_z}{\partial x}\,.$$

Since, however, both div $\boldsymbol{B}$ and curl $\boldsymbol{B}$ are zero, this may be re-written as

$$m_x\frac{\partial B_x}{\partial x}+m_y\frac{\partial B_x}{\partial y}+m_z\frac{\partial B_x}{\partial z}\,,$$

which is the x-component of

$$(\boldsymbol{m},\nabla)\boldsymbol{B}\,.$$

The force on complex 2 due to complex 1 is thus

$$\boldsymbol{F}=[\boldsymbol{j},\boldsymbol{B}]+(\boldsymbol{m},\nabla)\boldsymbol{B}\,, \tag{148}$$

where the value of $\boldsymbol{B}$ is taken at the point with respect to which $\boldsymbol{m}$ is defined and is due to all charges except those of the complex the force on which is being calculated.

The torque on complex 2 is

$$T=\Sigma[l_i,F_i]\,,$$

where

$$\boldsymbol{F}_i=\frac{1}{c}\,[e_i\boldsymbol{v}_i,\,\boldsymbol{B}_i]\,.$$

If, as a first approximation, $\boldsymbol{B}_i$ is set equal to $\boldsymbol{B}_0$, it will be found that the resulting value for T involves $\boldsymbol{m}$, the magnetization of complex 2. It is therefore not necessary to take into account the variation of $\boldsymbol{B}$, the term which would result from such a calculation being a negligibly small one involving the "supermagnetization" of complex 2. Thus,

$$\boldsymbol{T}=\Sigma\left[l_i,\left[\frac{e_i\boldsymbol{v}_i}{c}\,,\boldsymbol{B}\right]\right]\,,$$

or

$$\boldsymbol{T}=\Sigma\frac{e_i\boldsymbol{v}_i}{c}\,(l_i,\boldsymbol{B})-\boldsymbol{B}\Sigma\left(l_i,\frac{e_i\boldsymbol{v}_i}{c}\right)\,.$$

The coefficient of $\boldsymbol{B}$ in the last term is one-half the time derivative of $\Sigma(\boldsymbol{l}_i,e,\boldsymbol{l}_i/c)$, and can, by arguments similar to those indicated for a similar expression in § 40, be shown to vanish. The first term is similar to the left member of (147), with $\boldsymbol{B}$ written for $\boldsymbol{r}'$. Thus

$$\boldsymbol{T} = -[\boldsymbol{B},\boldsymbol{m}] = [\boldsymbol{m},\boldsymbol{B}] \,. \tag{149}$$

It has just been seen that the torque and force on a complex is given in terms of the $\boldsymbol{j}$ and $\boldsymbol{m}$ characteristic of the complex. It may be shown (the argument being closely analogous to that previously used in the electrostatic case) that the continuous densities $\boldsymbol{i}$ and $\boldsymbol{M}$, characteristic of a ponderable body and obtained by an averaging process from the vectors $\boldsymbol{j}_i$ and $\boldsymbol{m}_i$ characteristic of the subcomplexes of the body, are useful not only for describing the $\boldsymbol{A}$ or $\boldsymbol{B}$ due to the body, but also the total force and torque on the body. Indeed, the extension of the foregoing expressions to the integral forms expressing the total force and torque on a ponderable body clearly leads to the following equations:

$$\boldsymbol{F} = \int [\boldsymbol{i},\boldsymbol{B}] d\tau + \int (\boldsymbol{M},\nabla)\boldsymbol{B} d\tau \,, \tag{150}$$

$$\boldsymbol{T} = \int [\boldsymbol{M},\boldsymbol{B}] d\tau + \int [\boldsymbol{r},[\boldsymbol{i},\boldsymbol{B}]] d\tau + \int [\boldsymbol{r},\ (\boldsymbol{M},\nabla)\boldsymbol{B}] d\tau \,, \tag{151}$$

where $\boldsymbol{r}$ is the vector ending at $d\tau$ and beginning at the point to which the torque is referred, and where $\boldsymbol{B}$ is due to all bodies except the one the force and torque on which is being calculated. The densities $\boldsymbol{M}$ and $\boldsymbol{i}$ as well as the field vector $\boldsymbol{B}$ are all functions of x,y,z, the variables of integration. The first term in the expression for $\boldsymbol{T}$ is the sum of the vector torque on the individual subcomplexes into which the body is divided. The second and third terms represent the contribution to the total torque due to the forces on the individual subcomplexes.

PROBLEMS FOR PART III, CHAPTER III

1. Find the $\boldsymbol{B}$ vector due to an infinitely long straight wire, of small cross-section, carrying a current I.
2. What is the $\boldsymbol{B}$ vector at any point on the axis of a plane circular coil of wire carrying a current I?
3. What is the $\boldsymbol{B}$ vector at any point on the line perpendicular at its center of gravity to a rectangular coil of wire?
4. What is the $\boldsymbol{B}$ vector at points on the axis of a long solenoidal coil of wire carrying a current I, there being n turns of wire per unit length of the coil?

5. Equation (127) may (see Appendix, § 5, D) be written:

$$\boldsymbol{B} = -\nabla \frac{1}{4\pi} \int \frac{\partial \frac{1}{r}}{\partial n} d\sigma' ,$$

so that the $\boldsymbol{B}$ vector appears as the negative nabla of the potential due to a certain double distribution over a surface terminated by the current-carrying circuit. This distribution is called a "magnetic double shell," and is treated at length in many texts.

6. Show (see Appendix, § 5, D) that the vector potential due to a linear circuit may be written:

$$\boldsymbol{A} = \frac{I}{4\pi} \int \left[\boldsymbol{n}, \nabla' \frac{1}{r} \right] d\sigma' ,$$

where the surface integral is carried over a surface terminated by the circuit in question.

7. Referring to § 45, show that:

$$B_{h_1} - B_{h_2} = \frac{\partial A_s}{\partial n_1} + \frac{\partial A_s}{\partial n_2} ,$$

and hence argue that the component of $\boldsymbol{B}$ in the direction of $[\boldsymbol{M},\boldsymbol{n}]$ is continuous as one passes across the surface between two bodies.

8. Many writers use, besides $\boldsymbol{B}$, a second magnetic vector $\boldsymbol{H} \equiv \boldsymbol{B}/\mu$. Show that, across the boundary between two magnetizable bodies:
   a) The normal component of $\boldsymbol{B}$ is continuous.
   b) The tangential component of $\boldsymbol{H}$ is continuous.

9. In § 37 the energy of a configuration of conductors and dielectrics was calculated. The form of the final result has served as a basis for adopting $\boldsymbol{E}^2/2$ as the "volume density" of electrostatic energy. By the aid of the above analogy, show how $\boldsymbol{B}^2/2$ is similarly related to magnetostatic energy.

PART IV—MAGNETISM

INTRODUCTION

In the same sense that the first three parts of this chapter correspond to the three parts of chapter i, the present part corresponds to the treatment, in chapter ii, to the polarization of a dielectric and the relation between polarization and electrostatic intensity.

The last two sections of Part IV call attention to the fact that in a general magnetostatic problem the Coulomb forces have to be taken into account, as well as the forces specifically associated with the motion of the charges. This inclusion of both types of force is illustrated by two problems.

§ 47. *The Relation between* $\boldsymbol{B}$ *and* $\boldsymbol{M}$*: Diamagnetism, Paramagnetism, and Ferromagnetism.*—If a dielectric which is not inherently and permanently polarized be placed under the influence of charged bodies, this dielectric will become polarized, the vector polarization being proportional to the total electrostatic intensity, namely,

$$\boldsymbol{P} = (\epsilon - 1)\boldsymbol{E} \,. \tag{152}$$

Analogously, if a body be not inherently and permanently "magnetized" (i.e., if the magnetization $\boldsymbol{M}$ vanish at every point when the body is not in the neighborhood of moving charge), it may happen that this body, when placed near moving charge, will become magnetized. The way in which the resulting value, at any interior point, of the magnetization $\boldsymbol{M}$ depends upon the value, at that point, of the $\boldsymbol{B}$ vector can be determined experimentally. The torque on a magnetized ellipsoid, for example, can be observed experimentally, and, $\boldsymbol{B}$ being known, the value of $\boldsymbol{M}$ computed from equation (151).* It results that if the relationship be written in the form

$$\boldsymbol{M} = \left(1 - \frac{1}{\mu}\right)\boldsymbol{B} \,, \tag{153}$$

* This calculation is greatly simplified by the fact that a homogeneous ellipsoid magnetizes uniformly in a uniform $\boldsymbol{B}$ field, just as a homogeneous dielectric ellipsoid polarizes uniformly in a uniform $\boldsymbol{E}$ field.

or, more closely analogous to (158),*

$$-\boldsymbol{M}=\left(\frac{1}{\mu}-1\right)\boldsymbol{B}\,,$$

there is a class of bodies, called "paramagnetic," for which μ is a constant greater than 1; a class, called "diamagnetic," for which μ is constant and less than 1; and a class, called "ferromagnetic," for which μ is variable and very much greater than 1. The quantity μ is called the magnetic permeability of the substance in question.†

In the case of a magnetized body carrying no true volume current,

$$\boldsymbol{A}=\frac{1}{4\pi}\int\frac{\text{curl}'\,\boldsymbol{M}'}{r}\,d\tau'+\frac{1}{4\pi}\int\frac{[\boldsymbol{M}',\boldsymbol{n}]}{r}\,d\sigma'\,,$$

so that

$$\nabla^2\boldsymbol{A}=-\text{curl}\,\boldsymbol{M}\,.$$

But, since div $\boldsymbol{A}=0$, curl curl $\boldsymbol{A}$ and $-\nabla^2\boldsymbol{A}$ are equivalent, so that

$$\text{curl curl}\,\boldsymbol{A}=\text{curl}\,\boldsymbol{M}\,,$$

or

$$\text{curl}\,(\boldsymbol{B}-\boldsymbol{M})=0\,.$$

Since, however, for either paramagnetic or diamagnetic bodies $\boldsymbol{B}$ is proportional to $\boldsymbol{M}$, it follows that

$$\text{curl}\,\boldsymbol{B}=0\,,$$
$$\text{curl}\,\boldsymbol{M}=0\,.$$

Thus paramagnetic or diamagnetic bodies are magnetized in such a way that the curl of the magnetization vanishes, just as, in electrostatics, the divergence of the polarization vanishes.

If a paramagnetic or diamagnetic body be placed in a uniform external field equal, say, to $\boldsymbol{B}^0$, then

$$\boldsymbol{M}=\left(1-\frac{1}{\mu}\right)(\boldsymbol{B}^0+\text{curl}\,\boldsymbol{A})\,,$$

* This is a further illustration of the fact, pointed out in § 45, that the vector $-\boldsymbol{M}$ is analogous to the electrostatic vector $\boldsymbol{P}$.

† When one views $-\boldsymbol{M}$ as analogous to $\boldsymbol{P}$, then *diamagnetic* bodies are analogous to dielectrics.

where $\boldsymbol{A}$ is the vector potential due to the magnetized body itself. Then

$$\text{curl } \boldsymbol{M} = 0 = \left(1 - \frac{1}{\mu}\right) \text{curl } (\boldsymbol{B}^0 + \text{curl } \boldsymbol{A}) = \left(1 - \frac{1}{\mu}\right) \text{curl curl } \boldsymbol{A},$$

so that

$$\nabla^2 \boldsymbol{A} = 0 ,$$

and the analysis of § 35 and § 36 may be used, with a change in notation, for the case of the magnetization of a body in a given uniform external field, and for the cases in which a body magnetizes uniformly. In particular, it follows, as was just mentioned in a footnote, that an ellipsoid magnetizes uniformly in a uniform field.

§ 48. *A Rotating Conducting Sphere in a Uniform Magnetic Field.*— This chapter has been primarily concerned, thus far, with the forces between charges which are due to their motions. It is, in general, necessary to consider not only these "magnetic" forces, but also the Coulomb forces. In fact, the Coulomb forces, even in so-called "electrostatic problems," are acting on moving charges. An electrostatic problem is not, strictly speaking, a static problem, but merely one in which the effects which are caused by the motions may be disregarded.* If one examined, for instance, a very small element of volume he would observe within this element moving charges. He could not decide, from an examination of this one element, whether the motional effects could be disregarded or not. But, in any event, he would take the Coulomb forces into account.

The Coulomb forces have not been in specific evidence so far in this chapter, since it has dealt only with the motional forces. If, for example, the Ampère experiments could be and had been carried out on current elements consisting of flights of electrons, it would have been necessary to consider Coulomb forces from the beginning. The Ampère experiments were, however, carried out with currents flowing in metal wires, where the amount of moving negative charges in every volume element is sensibly equal to the amount of comparatively stationary positive charge. The Coulomb forces are, under these circumstances, zero.

This and the following section deal with problems which require the consideration of both the Coulomb and the motional forces. The first of these problems is that of a conducting sphere rotating uniformly in a uniform magnetic field.

* Except, of course, in so far as the motional effects may be partly responsible for the fundamental assumptions which are used.

From the definition of current as moving charge it follows at once that when a body is uniformly rotated about a fixed axis, the charges of the body form closed circuits of current, and if the body be symmetrical about the axis of rotation, and if the body be located in a uniform external $\boldsymbol{B}$ field which is parallel to the axis of rotation, it is clear that there will be a symmetrical distribution of the charge of the body, and accordingly a steady state of current distribution. Any charge e of the body will be acted on by a force $e[\boldsymbol{v},\boldsymbol{B}]/c$ and also by the electrostatic force $e\boldsymbol{E}$ which arises from the abnormal distribution of the charge. The potential and the distribution of charge will now be obtained for the case of a conducting sphere of radius R, rotating with an angular velocity $\boldsymbol{\omega}$ which is in the direction of the uniform external field $\boldsymbol{B}$. The external field will be assumed to be large compared to the change in it caused by the rotating charges of the body, so that this change will be neglected.

Certain points involved in this section deserve special attention. In the first place, in a problem involving both Coulomb and magnetostatic forces, many, but not all, of the equations of chapter ii are available. The Coulomb force is the negative nabla of the scalar potential Φ. This potential is given by the integrals, written in chapter ii, in terms of ρ, η, $\boldsymbol{P}$, and $\boldsymbol{\mu}$. Thus, all the analytic consequences of the integral representation are still available for use in this more general situation. The only relations which are no longer available are those which result from the electrostatic assumption that the total force on a charge is zero, and that this force is given, per unit charge, by the electrostatic intensity $\boldsymbol{E}$ (corrected by the addition of $\boldsymbol{P}/3$ in the case of dielectrics). Thus, the equation $\nabla^2\Phi = -\rho$ is still available. In the case of conductors, however, the interior condition $\boldsymbol{E}=0$ and its consequences $\Phi =$ constant and $\rho=0$ are no longer valid.

In the second place, the equilibrium relation

$$\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v},\boldsymbol{B}]=0$$

requires careful consideration. In chapter iv it will be seen that the division of forces into "electrical" and "mechanical" is a choice resulting from method of treatment, rather than from basic difference in kind. Individual electrical forces combine to produce macroscopic forces whose nature permits simplified treatment, and which are called "mechanical" forces. In the problem here considered, two forces on charge are recognized as electrical in nature—the force due to movement in the external

$\boldsymbol{B}$ field and that due to abnormal distribution of charge on the sphere. If this sphere were rotating, with the same angular velocity ω, and the $\boldsymbol{B}$ field were not present, then the problem would be a purely mechanical one. Any "particle" of the sphere would be acted on by such a "mechanical" force as would cause it to move with constant speed in a circle. In the actual problem, then, the two foregoing electrical forces act in addition to the mechanical force. If the two electrical forces have a zero resultant, the sphere acts exactly as it would if no field were present; i.e., the mechanical forces cause each particle of the sphere to move with constant speed in a circle. And once rotating with the angular velocity ω, the sphere continues indefinitely to do so. To sum up, then: there are present both electrical and mechanical forces. The latter are just such as to produce the desired motion, and the former thus have a zero resultant.

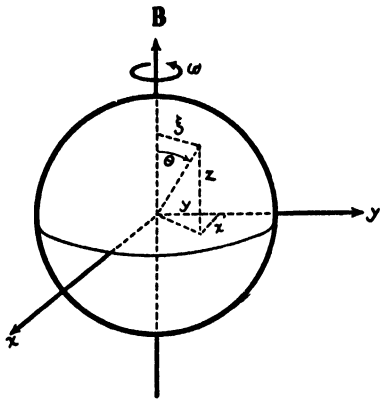

FIG. 44.—A conducting sphere rotating with angular velocity w about the direction of an applied external field B.

Let ξ be the distance of a point of the sphere from the axis of rotation, so that, if z be measured along this axis, $\xi^2=x^2+y^2$. Then the force per unit charge $[v,B]/c$, which, on account of its origin, may be called a "motional intensity," is in the direction of ξ, so that, under equilibrium conditions,

$$E+\frac{1}{c}[v,B]=0\ ,$$

one has

$$E=E_\xi=-\frac{\xi\omega B}{c}\ .$$

Hence

$$\frac{d\Phi_i}{d\xi}=\frac{\xi\omega B}{c}\ ,$$

so that

$$\Phi_i=\Phi_0+\frac{\xi^2\omega B}{2c}\ ,$$

or

$$\Phi_i = \Phi_0 + \frac{\omega B r^2 \sin^2 \theta}{2c},$$

where Φ_0 is the potential at the axis. Then

$$\rho = -\nabla^2 \Phi_i = -\frac{2\omega B}{c},$$

as is obtained at once by differentiating. There is, thus, a uniform negative volume charge throughout the interior of the sphere. To obtain the surface distribution of charge from the equation

$$\frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = -\eta, \tag{154}$$

it is necessary to know the potential Φ_e at exterior points. This exterior potential satisfies the relations,

$$\begin{cases} \nabla^2 \Phi_e = 0, \\ \Phi_e = \Phi_i, \qquad r = R, \\ \Phi_e \text{ is regular at infinity.} \end{cases}$$

The symmetry of the problem clearly indicates that the equation $\nabla^2 \Phi_e = 0$ is to be expressed in spherical co-ordinates, and that Φ_e depends only upon r and θ, so that the solid zonal harmonics

$$r^n P_n(\cos \theta),$$

$$\frac{1}{r^{n+1}} P_n(\cos \theta),$$

are available solutions (see § 35). It will, therefore, be convenient, in order to obtain suggestions as to the proper choice of terms for the exterior potential, to express the interior potential in terms of the surface zonal harmonics $P_n(\cos \theta)$. Now

$$P_2(\cos \theta) = \frac{3}{2}\left(\cos^2 \theta - \frac{1}{3}\right),$$

so that

$$\sin^2 \theta = 1 - \left\{\frac{2}{3} P_2(\cos \theta) + \frac{1}{3}\right\} = \frac{2}{3} - \frac{2}{3} P_2(\cos \theta);$$

thus,

$$\Phi_i = \Phi_0 + \frac{\omega B r^2}{3c} - \frac{\omega B r^2}{3c} P_2 (\cos \theta) .$$

Now every term of the sum

$$\Sigma A_n P_n (\cos \theta) r^{-n-1}$$

satisfies $\nabla^2 \Phi = 0$ and is regular at infinity. The form of the interior potential clearly suggests that the remaining condition

$$\Phi_i = \Phi_e , \qquad r = R ,$$

can be met by choosing

$$\Phi_e = \frac{A_0}{r} + \frac{A_2 P_2 (\cos \theta)}{r^3} .$$

The values of A_0 and A_2 are then easily calculated, the resulting value being

$$\Phi_e = \frac{\Phi_0 R + \frac{1}{3} \omega B R^3}{rc} - \frac{\omega B R^5}{3 r^3 c} P_2 (\cos \theta) .$$

If the total charge e of the sphere be known, rather than the potential at the axis, the condition*

$$\int \frac{\partial \Phi}{\partial n} d\sigma = -e$$

gives, since†

$$\int P_n (\cos \theta) d\sigma = 0 , \qquad n \neq 0 , \tag{155}$$

the result

$$\frac{e}{4\pi} = \frac{1}{c} \left(\Phi_0 R + \frac{1}{3} \omega B R^3 \right) .$$

* Note that this is an application of Gauss's Theorem.

† This equation is obtained from the fundamental relation $\int_{-1}^{+1} P_n(x) P_m(x) dx = 0$, $m \neq n$ (see Byerly, *Fourier Series and Spherical Harmonics*, p. 171, § 91) by setting $m = 0$

Thus, if $e=0$,

$$\Phi_0 = -\frac{\omega}{3c} BR^2 .$$

If, on the other hand, Φ_0 be zero, the total amount of charge brought to the sphere by grounding the axis is

$$e = \frac{4\pi}{3c} \omega BR^3 .$$

Since both the interior and exterior potentials are now known, the surface density of charge can be found at once from (154). Thus, for the case of zero total charge,

$$\eta = \frac{\omega BR}{2c} \{3 - 5 \cos^2 \theta\} .$$

The surface density changes sign at the zone

$$\cos^2 \theta = \frac{3}{5}, \qquad \theta = 39°14' .$$

Since η may be written in the form

$$\eta = \frac{2}{3} \frac{\omega BR}{c} - \frac{5}{3} \frac{\omega BR}{c} P_2 (\cos \theta) ,$$

the total surface charge is, remembering (155),

$$4\pi R^2 \left(\frac{2}{3} \frac{\omega BR}{c} \right) = \frac{8\pi \omega BR^3}{3c} ,$$

while the total volume charge is

$$\frac{4}{3} \pi R^3 \left(-\frac{2\omega B}{c} \right) = -\frac{8\pi \omega BR^3}{3c} .$$

§ 49. *A Dielectric Sphere Rotating in a Uniform Magnetic Field.*— As a further illustration, consider a problem which differs from the one just solved only in that the sphere is now formed of a dielectric material. The polarization at any point is proportional to the total force per unit charge, i.e.,*

$$\boldsymbol{P} = (\epsilon - 1) \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{v}, \boldsymbol{B}] \right\} . \tag{156}$$

* The calculation in the preceding section indicates that the solution to such problems is to be obtained by setting equal to zero the resultant of the electrical

However, for an ideal dielectric,

$$\text{div } (\boldsymbol{E}+\boldsymbol{P})=0 ,$$

so that, substituting

$$\text{div}\left\{ \epsilon\boldsymbol{E}+\frac{(\epsilon-1)}{c}\,[\boldsymbol{v},\boldsymbol{B}]\right\}=0 ,$$

or

$$\text{div } \boldsymbol{E}=-\nabla^2\Phi=-\frac{\epsilon-1}{c\epsilon}\,\text{div}\,[\boldsymbol{v},\boldsymbol{B}] .$$

As noted before, moreover, the direction of $[\boldsymbol{v},\boldsymbol{B}]$ is normal to the axis of rotation, and its magnitude is $\omega B\xi$ or $\omega B\sqrt{x^2+y^2}$, so that

$$\text{div}\,[\boldsymbol{v},\boldsymbol{B}]=2\omega B .$$

Thus the interior potential satisfies the equation

$$\nabla^2\Phi_i=\frac{\epsilon-1}{c\epsilon}\,2\omega B , \tag{157}$$

and the exterior potential, the equation

$$\nabla^2\Phi_e=0 . \tag{158}$$

It remains to find the conditions that Φ_i and Φ_e satisfy on the boundary of the sphere. Equation (83), namely,

$$(\boldsymbol{P}+\boldsymbol{E})_{n_1}+(\boldsymbol{P}+\boldsymbol{E})_{n_2}=0 ,$$

forces, the required motion then being caused by the mechanical forces. A casual reading of the present section might lead one to think that, in the case of a dielectric sphere, this vanishing of the electrical resultant is not necessary. To be sure, there is not written in this section an equation which states explicitly that the total electrical force vanishes. However, the proportionality between polarization and electrical intensity is based upon the idea of the balance of two electrical forces—the "external" electric force and the "internal" electric force which opposes the shift of charge. Thus in this problem three forces are recognized as electrical: the two of the previous section and the internal electrical force just mentioned. The proportionality between polarization and the total external electrical force is, in point of fact, equivalent to the vanishing of the total electrical force.

reduces, since $\boldsymbol{P}$ is zero at exterior points, to

$$(E_r)_e-(P_r+E_r)_i=0\ .$$

However,

$$[\boldsymbol{v},\boldsymbol{B}]_r=\omega B\xi\ \sin\theta=\omega Br\ \sin^2\theta\ ,$$

so that, from (156),

$$P_r=(\epsilon-1)\left\{E_r+\frac{\omega rB\ \sin^2\theta}{c}\right\}\ ,$$

and, substituting,

$$(E_r)_e-\epsilon(E_r)_i=\frac{\epsilon-1}{c}\ \omega RB\ \sin^2\theta\ ,$$

or

$$\epsilon\left(\frac{\partial\Phi_i}{\partial r}\right)_{r=R}-\left(\frac{\partial\Phi_e}{\partial r}\right)_{r=R}=\frac{\epsilon-1}{c}\ \omega RB\ \sin^2\theta\ . \tag{159}$$

The potential is continuous, so that

$$\Phi_i=\Phi_e\ ,\qquad r=R\ ,$$

and is regular at infinity. These last two conditions, together with (157), (158), and (159) above, serve to determine uniquely the potential.

The boundary condition (159) indicates the use of the surface zonal harmonic $P_2(\cos\theta)$. The expression

$$\Phi_i=A_0+A_2r^2P_2\ (\cos\theta)+Cr^2$$

will therefore be assumed for interior points, the term Cr^2 being added since $\nabla^2\Phi_i$ is not zero. From the foregoing expression it follows that

$$\nabla^2\Phi_i=6C\ ,$$

so that

$$C=\frac{\epsilon-1}{\epsilon}\ \frac{\omega B}{3c}\ .$$

For the exterior potential will be assumed the expression

$$\Phi_e=\frac{B_1}{r}+\frac{B_2}{r^3}\ P_2\ (\cos\theta)\ ,$$

the form of which is indicated by the interior potential. Suppose that the total charge of the sphere be zero. Then

$$\int \frac{\partial \Phi_i}{\partial r}\, d\sigma = 0 ,$$

or, from (155),

$$B_1 = 0 .$$

This condition that Φ is continuous across the surface of the sphere gives the results

$$A_0 = -CR^2 ,$$
$$A_2 = \frac{B_2}{R^5} .$$

Also,

$$\epsilon \left(\frac{\partial \Phi_i}{\partial r}\right)_{r=R} = 2\epsilon R A_2 P_2 (\cos\theta) + 2\epsilon RC ,$$
$$= 2\epsilon R A_2 \left(1 - \frac{3}{2}\sin^2\theta\right) + 2\epsilon RC ,$$
$$\left(\frac{\partial \Phi_e}{\partial r}\right)_{r=R} = -\frac{3B_2}{R^4} P_2 (\cos\theta) ,$$
$$= -\frac{3B_2}{R^4}\left(1 - \frac{3}{2}\sin^2\theta\right) .$$

Substituting these values in (159) and equating the coefficients of like powers of $\sin\theta$,

$$B_2 = \frac{2}{3}\,\frac{1-\epsilon}{c}\,\frac{R^5 \omega B}{2\epsilon+3} ,$$
$$A_2 = \frac{2}{3}\,\frac{1-\epsilon}{c}\,\frac{\omega B}{2\epsilon+3} ,$$

so that

$$\Phi_i = \frac{\omega B(\epsilon-1)}{c}\left[\frac{r^2-R^2}{3\epsilon} + \frac{r^2}{2\epsilon+3}\left(\sin^2\theta - \frac{2}{3}\right)\right] ,$$
$$\Phi_e = \frac{\omega B(\epsilon-1)}{c}\,\frac{R^5}{r^3}\,\frac{\left(\sin^2\theta - \frac{2}{3}\right)}{(2\epsilon+3)} .$$

PROBLEMS FOR PART IV, CHAPTER III

1. Calculate and discuss the volume and surface densities of charge $-\operatorname{div} \boldsymbol{P}$, and P_n for the rotating dielectric sphere whose potentials are given by the last two equations of § 49.
2. Given that, in ordinary or c.g.s. electromagnetic units,

$$M_{rmu} = \frac{1}{4\pi}\left(1 - \frac{1}{\mu_{rmu}}\right) B_{rmu} ,$$

show that the measure of permeability is the same in rational and in c.g.s. electromagnetic units.

PART V—STEADY CURRENTS: OHM'S LAW

INTRODUCTION

This part contains the equations which govern the flow of currents in matter that is not sensibly polarizable or magnetizable. The importance of Ohm's law as applied to linear circuits is too well known to require emphasis. Problems of current flow in volume conductors have recently acquired new and larger interest and importance on account of their applications to geophysics. The analogy between such problems and the polarization of dielectrics is pointed out. The part concludes by listing the solutions of several classic problems in three-dimensional flow.

§ 50. *Steady-State Currents in Conductors: Ohm's Law.*—The following two sections contain those further developments of the theorem of magnetostatics which apply to bodies which carry currents, but for which the value of μ is very nearly or exactly equal to 1, so that magnetization effects play no rôle. It has been found, in the electrostatic case, that all points of a conductor are at the same potential. If, however, two points of a conductor are maintained at different potentials (the cause of the potential difference not being important for the present), the situation is no longer a static one, there being a movement of electrons through the conductor from the point of lower to the point of higher potential. Such a movement of elementary charges gives rise to a current, as has been seen in the previous sections, the direction of the current being opposite to the direction of flow of electrons. When a current flows in a conductor there are always two effects produced: (1) the so-called "magnetic forces," which have been considered above, are experienced; and (2) heat is developed in the conductor.

Suppose that the conductor under consideration be a wire of small cross-section q, and that one end A of the wire be maintained at a higher potential than the other end B. Under these circumstances electrons will move along the wire from B to A, and if the potential difference has existed sufficiently long for a steady state to be set up, the same number of charges pass, in unit time, any cross-section of the wire. If the average drift velocity of the charges be v, and if i be the volume density of current measured in electromagnetic units (namely, $1/c$ times

the amount of charge in electrostatic units passing along the wire per unit cross-section per unit time) then

$$cqi = N\epsilon qv ,$$

where N is the number of moving electrons ϵ per unit volume.

Now when a charge e is moved through a potential difference $d\Phi$, it follows, from the definition of the potential, that the amount of work W done on the charge is given by

$$W = ed\Phi .$$

Therefore, since the moving electrons which constitute the current move, per second, through a potential difference given by

$$v\frac{d\Phi}{ds} ,$$

the work done by the charges per second per centimeter of wire is

$$-N\epsilon q\frac{d\Phi}{ds}v = qiE_s c .$$

In the last two expressions for the work done by the charges, the forces considered are those due to distant charges. The expressions thus tacitly assume that the forces due to the nearby charges do, on the average, no work. Those moving charges which constitute the current would also give rise to magnetic forces; but these forces are always normal to the velocity of the charges acted upon, and hence do no work. Accordingly, they do not need to be considered in the foregoing expressions. Thus Q, the work done per second per unit volume, is given by*

$$Q = icE .$$

Experimentally, however, it is found that this amount of work, which appears as heat, is proportional to E^2, so that, writing σ as the factor of proportionality,†

$$Q = icE = \sigma E^2 ,$$

* Since a linear circuit is under consideration, it is unnecessary to distinguish between E_s and E.

† The Greek letter σ is used very widely for conductivity. It is also used, in this book, for surface area. The two uses are so obviously distinct that no confusion should result.

or

$$i=\frac{\sigma E}{c}. \tag{160}$$

This last equation may be written in the form

$$\frac{ds}{q\sigma}\, cqi = Eds\,,$$

where ds is an element of length along the wire. Then

$$\frac{cqi}{q\sigma}\int_0^l ds=\frac{lc}{q\sigma}\, qi=-\int_0^l \frac{d\Phi}{ds}\, ds=\Phi_A-\Phi_B\,,$$

so that writing R for $l/q\sigma$, and I for qi,

$$\Phi_A-\Phi_B=cRI\,. \tag{161}$$

The relationship (160), between the current density i and the potential gradient E, or its equivalent (161), between the total current and the total potential difference $\Phi_A-\Phi_B$, is known as Ohm's law; R is called the resistance of the wire, so that the resistance of the wire is seen to depend directly upon its length, inversely upon its cross-section, and inversely upon the value of σ, which quantity is known as the specific conductivity of the material. According to (160), σ is the proportionality factor between the electrostatic intensity E and the current; in rational electrostatic units, $ci=\rho u$. That is to say, σ is measured in rational electrostatic units (see Part V, Problem 1 of this chapter).

Ohm's law has been stated above in scalar form as applied to a linear conductor. It is found experimentally, however, that in a uniform isotropic medium an immediate extension of the same law, namely,

$$\boldsymbol{i}=\frac{\sigma \boldsymbol{E}}{c}\,, \tag{162}$$

holds in vector form, where σ is a scalar constant characteristic of the conducting material, and where $\boldsymbol{i}$ is the volume density of current. In the steady state of volume distribution of current, the net rate at which charge is leaving any closed region T of surface Σ is zero, since charge cannot heap up in any part of the conductor, i.e.,

$$\int_\Sigma i_n d\sigma=\int \operatorname{div} \boldsymbol{i} d\tau=0\,.$$

Since this equation can be applied to an arbitrary volume T, it follows that the steady state is characterized by the relation

$$\operatorname{div} \boldsymbol{i} = 0 ,$$

or

$$\operatorname{div} \frac{\sigma \boldsymbol{E}}{c} = -\frac{\sigma}{c} \operatorname{div} \nabla\Phi = 0 ,$$

$$\nabla^2\Phi = 0 . \tag{163}$$

This last equation furnishes a basis for the determination of Φ, $\boldsymbol{E}$, and $\boldsymbol{i}$ for steady currents. To characterize the potential uniquely, however, it is necessary to add to the relation (163) the relations which Φ must satisfy on the boundaries of the region under consideration. Across the boundary between a conductor and a dielectric no flow of charge can take place, so that $i_n = 0$, or

$$\frac{\partial\Phi}{\partial n} = 0 , \tag{164}$$

the quantity on the left meaning, as usual, the limiting value of the normal derivative as the boundary is approached (in this instance, from within). The normal flow of charge across the boundary between two conductors must be, in the steady state, continuous, so that across such a boundary

$$i_{n_1} + i_{n_2} = 0$$

or

$$\sigma_1 \frac{\partial\Phi}{\partial n_1} + \sigma_2 \frac{\partial\Phi}{\partial n_2} = 0 , \tag{165}$$

where $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ are the normals to the surface pointing into the conductors whose specific conductivities are σ_1 and σ_2. There may also be portions of the boundary on which the potential is a known constant, as would be the case, for example, if electrodes be maintained at known potentials.

It is important to note that the problem of determining the steady-state distribution of current in an infinite conducting medium when electrodes are maintained at given potentials is analytically equivalent to the electrostatic problem of determining the potential due to conductors having the form and potentials of the electrodes. If the conducting medium in which the steady currents exist is not infinite in extent in all direc-

tions, then additional conditions, which have no counterpart in the analogous electrostatic problem, must be satisfied on the boundary of the conducting medium. This additional boundary condition, which expresses the fact that the flow of current on the boundary must take place in tangential directions, can often be met by introducing image electrodes, or image conductors, in the analogous electrostatic problem. This method will be illustrated below.

Similarly, the problem of determining the steady-current flow when a body of one conductivity is immersed in a body of a second conductivity is analogous to the problem of the polarization of a body of one dielectric constant immersed in a body of a second dielectric constant. For the steady-current boundary condition on the surface between the two conductors is reduced to that which holds in electrostatics at the boundary between two dielectrics by the interchange of σ and ϵ. For example, suppose that a charge e be located in a medium of dielectric constant ϵ_1, a distance b from a sphere of radius a and dielectric constant ϵ_2. If one determines the potential Φ at all points for this electrostatic problem, he has, at the same time, solved the following steady-state current problem: a sphere of one conductivity is immersed in a medium of a second conductivity; at a distance b from the center of the sphere is located a steady source of current.

§ 51. *The Distribution of Currents in Volume Conductors.**—The experimentally important problems concerning current distribution in volume conductors can be divided into three types:

I. The case in which a portion of the boundary of the region under consideration consists of electrodes A, B, ... , at known potentials Φ_A, Φ_B, ... The potential Φ is then to be determined from the following conditions

$$\nabla^2\Phi = 0 \text{ in the interior,}$$
$$\Phi = \Phi_A \text{ on the electrode } A, \text{ etc.,}$$
$$\frac{\partial\Phi}{\partial n} = 0 \text{ on the remainder of the boundary .}$$

Let the total current passing through electrode A be denoted by I_A, etc., so that

$$I_A = \int_A i_n d\sigma = -\int_A \frac{\sigma}{c}\frac{\partial\Phi}{\partial n}\, d\sigma .$$

* A considerable portion of the material of this section is taken from the article of Debye, *Encyl. der math. Wiss.*, Vol. V, art. 17. See also Riemann-Weber, *Die Part. Diff.-Gleich.*, I, 429.

Then if the integral

$$\int \frac{\sigma}{c} \Phi \nabla^2 \Phi d\tau = 0$$

be treated by integration by parts, the result is

$$\int \frac{\sigma}{c} \Phi \frac{\partial \Phi}{\partial n} d\sigma - \int \frac{\sigma}{c} (\nabla \Phi, \nabla \Phi) d\tau = 0 ,$$

where the surface integral is extended over the complete boundary. The values of Φ and $\partial\Phi/\partial n$ on the boundary, however, reduce this equation to the form

$$\Phi_A I_A + \Phi_B I_B + \ldots . = \int \frac{\sigma}{c} E^2 d\tau = \frac{Q}{c} ,$$

where Q is the total heat developed in the conductor by the passage of the current. Similarly

$$0 = \int \frac{\sigma}{c} \nabla^2 \Phi d\tau = \int div \frac{\sigma \nabla \Phi}{c} d\tau = \int \left(\frac{\sigma}{c} \nabla \Phi \right)_n d\sigma = \int \frac{\sigma}{c} \frac{d\Phi}{\partial n} d\sigma ,$$

so that

$$I_A + I_B + \ldots . = 0 ,$$

an equation which expresses the fact that a steady state prevails.

If there be but two electrodes A and B,

$$I_A = -I_B = I ,$$

and

$$I(\Phi_A - \Phi_B) = \frac{Q}{c} .$$

The constant R defined by the equation

$$\Phi_A - \Phi_B = cRI ,$$

or its equivalent

$$Q = c^2 R I^2 ,$$

is, by analogy with the case of a linear conductor, known as the gross or effective resistance of the volume conductor. The notion of gross resistance clearly does not apply* to the case of more than two electrodes.

II. The case just considered may be solved more simply provided the electrodes (which may now be either interior or on the surface) are of dimensions small enough compared to their distances apart so that they may be considered isolated point sources of current. In the immediate neighborhood of one of these point sources, where the influence of the other electrodes may be disregarded, the flow of current is symmetrical about the point, and the total flow out through a small inclosing sphere is

$$-\int \frac{\sigma}{c}\frac{\partial \Phi}{\partial r}\, d\sigma = -\frac{\sigma}{c}\frac{\partial \Phi}{\partial r} 4\pi r^2 = I\,,$$

so that the behavior of Φ in the neighborhood of the electrode is found by integration, to be

$$\Phi = \frac{Ic}{4\pi\sigma r}\,.$$

In the case of a surface electrode, the integration is extended over half a sphere, so that the factor 2π occurs in the place of 4π. That the electrodes would constitute singular points of this type is also clear by analogy with the electrostatic case.

The conditions on Φ are thus

$$\nabla^2\Phi = 0$$

at all points except at certain interior points A, B, ... , and at certain points A', B', ... , on the boundary, where Φ becomes infinite as

$$\frac{cI_A}{4\pi\sigma r}\,,\quad \frac{cI_B}{4\pi\sigma r}\,,\ \ldots\ldots\,,\quad \frac{cI_{A'}}{2\pi\sigma r}\,,\quad \frac{cI_{B'}}{2\pi\sigma r}\,,\ \ldots\ldots$$

The steady-state relation still holds, namely,

$$I_A + I_B + \ldots\ldots + I_{A'} + I_{B'} + \ldots\ldots = 0\,.$$

It is necessary, however, to take into account the dimensions of the electrodes in order to calculate the heat developed or, in the case of two electrodes, the gross resistance.

* That is, without modification. *Partial* gross resistances may be defined.

III. In case there are circular surface electrodes whose dimensions are small compared to their separation distances, the behavior of the potential Φ in the neighborhood of one of these disk electrodes is subject to the same mathematical conditions as is the electrostatic potential in the neighborhood of an isolated charged circular disk. The distribution of current over the electrode is thus the same as the distribution of charge on a circular disk in the electrostatic case. In fact, the electrostatic problem when a very thin circular disk conductor carries a total charge e is given by the equations

$$a) \qquad \nabla^2\Phi = 0\,,$$

$$b) \qquad \Phi = \text{constant on the disk},$$

$$c) \qquad \int \frac{\partial \Phi}{\partial n} d\sigma' = -e\,,$$

the integration being over the surface of the disk. It is evident from symmetry that the solution of this problem satisfies the additional condition,

$$\frac{\partial \Phi}{\partial n} = 0\,,$$

at all points of the infinite plane which contains the plane of the disk (points on the disk being, of course, excluded), the derivative being taken along either normal to this plane. This electrostatic problem was solved in § 29, and it was found that, across the face of the disk,

$$\frac{\partial \Phi}{\partial n} = -\eta = \frac{-e}{2\pi a \sqrt{a^2 - r^2}}\,,$$

where a is the radius of the disk, r the distance from any point on the face of the disk to the center of the face, and where the disk has been considered infinitely thin (and one-sided) so that the total charge is to be found by integrating over only one face of the disk. In fact,

$$\frac{e}{2\pi a} \int_0^a \frac{2\pi r\, dr}{\sqrt{a^2 - r^2}} = e\,.$$

Now let a circular disk electrode of radius a be in contact with a volume conductor of conductivity σ. Let a be small compared to the distance to any other electrode, and compared to the radius of curvature

of the surface of the conductor at the place in question. One may then assume that the current flow in the neighborhood of the electrode is the same as it would be if the electrode were in contact with an infinite half-space of conductivity σ. The analytical conditions for this problem are

$$a') \qquad \nabla^2\Phi = 0\,,$$

$$b') \qquad \Phi = \text{constant on the disk},$$

$$c') \qquad \frac{\sigma}{c}\int \frac{\partial\Phi}{\partial n}\,d\sigma' = -I\,,$$

where I is the current entering the conductor through the electrode in question, the normal n pointing into the electrode; and,

$$\frac{\partial\Phi}{\partial n} = 0\,,$$

at all points on the plane of the half-space except at points on the disk. It is thus seen at once that the two problems are entirely similar. In fact, the potential for the electrostatic problem is determined, except for a constant factor, by the equations a) and b), the constant then being determined from c). Thus the solution of the current-flow problem is to be found at once by redetermining this constant, using c') instead of c). One thus assumes

$$\frac{\partial\Phi}{\partial n} = \mathrm{K}\frac{1}{\sqrt{a^2 - r^2}}\,.$$

and it is found at once that $\mathrm{K} = cI/2\pi\sigma a$. Thus to pass from the electrostatic to the current problem, one first replaces e by $2e$ (which makes the disk effectively one-sided), and then replaces e by cI/σ. This can be accomplished in one step by replacing e by $2cI/\sigma$. Thus for the case of surface circular electrodes;

$$\nabla^2\Phi = 0 \text{ at all interior points,}$$

$$\partial\Phi/\partial n = 0 \text{ on the boundary except at the electrodes,}$$

$$\frac{\partial\Phi}{\partial n} = \frac{cI_k}{2\pi\sigma a_k\sqrt{a_k^2 - r_k^2}} \text{ on the } k\text{th electrode.}$$

In the last equation I_k is the total current passing through the kth electrode, a_k is the radius of this electrode, and r_k is the distance from the center of this electrode to the point under consideration.

Many special problems of current distribution have been solved. The approximate solution for the case of two small spherical electrodes A and B, of radius a and b located a distance l apart in an infinite conductor, can be obtained at once. From the remarks made under II above it follows that

$$\Phi=\frac{cI}{4\pi\sigma}\left(\frac{1}{r_a}-\frac{1}{r_b}\right),$$

where r_a is the distance to the sphere A at which the current I enters the conductor, and r_b is the distance to the sphere B at which the current leaves. By direct substitution it follows that

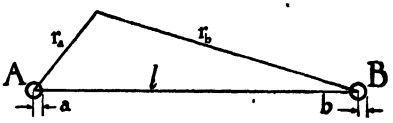

FIG. 45.—Two spherical electrodes A and B of radius a and b respectively, located a distance l apart in an infinite conductor.

$$\Phi_a-\Phi_b=\frac{cI}{4\pi\sigma}\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{l}\right),$$

so that the effective resistance of the infinite conductor is given by

$$R=\frac{1}{4\pi\sigma}\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{l}\right).$$

The method of images may be used for the case of a conducting half-space, one electrode (a small sphere of radius a) being located a distance h from the face of the conductor, the other electrode (a plane parallel to the face of the conductor) being infinitely large and infinitely far removed. The spherical conductor may be replaced by a singular point, and the potential written as

$$\Phi=\frac{cI}{4\pi\sigma}\left(\frac{1}{r_1}+\frac{1}{r_2}\right),$$

where r_1 is the distance to the spherical electrode, and r_2 is the distance to the image electrode (see Fig. 46). It will be noted that the singularity and its image appear with the same sign rather than with opposite sign, as in the electrostatic case, since here the normal derivative has to vanish on the face of the conductor rather than the potential itself. The potential of the spherical electrode, and hence the potential difference between it and the infinitely distant electrode, is given by

$$\frac{cI}{4\pi\sigma}\left(\frac{1}{a}+\frac{1}{2h}\right)=\frac{cI}{4\pi a\sigma}\quad\text{for } h\gg a,$$

so that the effective resistance of the half-space is

$$R=\frac{1}{4\pi\sigma}\left(\frac{1}{a}+\frac{1}{2h}\right)=\frac{1}{4\pi\sigma a}\,.$$

If the electrode is on the plane face of the conducting half-space, and may be regarded as a half-sphere, the potential is

$$\Phi=\frac{cI}{2\pi\sigma r}\,,$$

so that the potential of the electrode is

$$\frac{cI}{2\pi\sigma a}\,,$$

and

$$R=\frac{1}{2\pi\sigma a}\,.$$

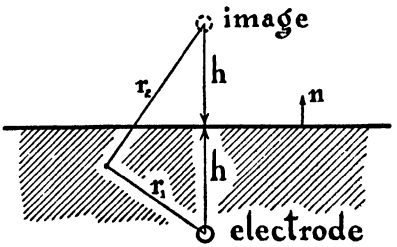

FIG. 46.—A single spherical electrode located a distance n below the plane surface of a conducting half-space.

If the surface electrode be disk shaped rather than spherical, the solution may be obtained at once from the expression for the potential to which a circular disk is raised by an electrostatic charge. To pass from the electrostatic to the current-flow problem it is only necessary, as indicated above, to replace e by $2cI/\sigma$. The potential of the disk is thus

$$\frac{2Ic}{8\sigma a}=\frac{Ic}{4\sigma a}\,,$$

so that

$$R=\frac{1}{4\sigma a}\,.$$

Suppose now that two conducting half-spaces 1 and 2 of specific conductivities σ_1 and σ_2 be in contact along their plane surfaces, a spherical electrode of radius a being located in 1 a distance h from the separating plane. Then

$$\nabla^2\Phi_1=0\,,$$

$$\nabla^2\Phi_2=0\,,$$

$$\sigma_1\frac{\partial\Phi_1}{\partial n_1}+\sigma_2\frac{\partial\Phi_2}{\partial n_2}=0$$

while Φ_1 becomes infinite, at the electrode, as $Ic/4\pi\sigma_1 r$. The boundary condition may be written

$$\frac{\partial\Phi_1}{\partial n_1}+\sigma\frac{\partial\Phi_2}{\partial n_2}=0\,,$$

where

$$\sigma=\frac{\sigma_2}{\sigma_1}\,,$$

and is then analogous to the equation

$$\frac{\partial\Phi}{\partial n_1}+\epsilon\frac{\partial\Phi}{\partial n_2}=0\,,$$

which holds on the boundary between free space and a dielectric of constant ϵ. The electrostatic problem of an inducing charge $+e$ a distance h from the plane face of a dielectric half-space has been previously solved,* and it was found that

$$\Phi_1=\frac{e}{4\pi}\left[\frac{1}{r}+\frac{1-\epsilon}{1+\epsilon}\cdot\frac{1}{r'^2}\right]\,,$$

where r is the distance to e, and r' the distance to the image of e in the plane face (see Fig. 47); while in the dielectric

$$\Phi_2=\frac{2e}{4\pi(1+\epsilon)r}\,.$$

These two potentials, with $\epsilon=\sigma=\sigma_2/\sigma_1$, satisfy all the demands of the present problem except the demand that Φ_1 become infinite at the electrode as

$$\frac{Ic}{4\pi\sigma_1 r}\,.$$

This final condition can be met by multiplying both Φ_1 and Φ_2 by the constant $cI/\sigma_1 e$. Hence,

$$\Phi_1=\frac{Ic}{4\pi\sigma_1}\left[\frac{1}{r}+\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\frac{1}{r'}\right]\,,$$

$$\Phi_2=\frac{Ic}{2\pi(\sigma_1+\sigma_2)r}\,.$$

* See § 34.

Hence,

$$R=\frac{1}{4\pi\sigma_1}\left(\frac{1}{a}+\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\cdot\frac{1}{2h}\right).$$

It may be readily calculated that, of the total current I which is at the electrode, an amount $I\sigma_2/(\sigma_1+\sigma_2)$ flows to an infinitely distant plane, parallel to the separation plane, and located in region 2, and an amount $I\sigma_1/(\sigma_1+\sigma_2)$ to a similarly located plane in region 1.

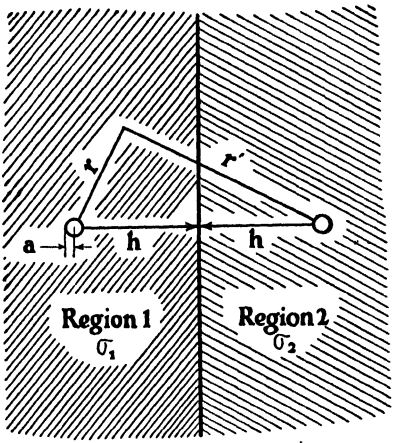

FIG. 47.—A spherical electrode of radius a located a distance n from the plane face separating one half-space of conductivity σ_1 and a second half-space of conductivity σ_2.

Similarly, if two electrodes of radius a and b are located distances h_1 and h_2 from the plane surface of a conducting

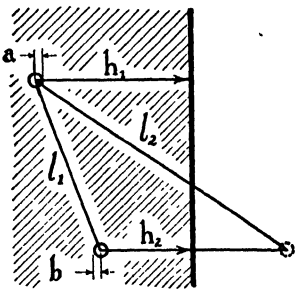

FIG. 48.—Two spherical electrodes of radius a and b located distances h_1 and h_2 from the face of a conducting half-space, the distance between electrodes being l_1.

half-space, the distance of electrode 1 from electrode 2 and to the image of electrode 2 being l_1 and l_2, respectively, it may be shown that

$$R=\frac{1}{4\pi\sigma}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{2h_1}+\frac{1}{2h_2}-\frac{2}{l_1}-\frac{2}{l_2}\right).$$

PROBLEMS FOR PART V, CHAPTER III

1. The conductivity is the factor by which intensity is to be multiplied to give current. If σ, $\sigma_{e.s.u.}$, $\sigma_{e.m.u.}$ denote the conductivity measured in rational electrostatic units, in electrostatic units, and in electromagnetic units, respectively, show that

$$\sigma=4\pi\sigma_{e.s.u.}=c^2\cdot 4\pi\sigma_{e.m.u.}.$$

2. The reciprocal of σ is called the resistivity r. Show that

$$\sigma = \frac{4\pi \cdot 9 \cdot 10^{11}}{r},$$

where r is the resistivity in ohm-centimeters.

3. A source and a sink (point) electrode are located on a conducting plane. Show that the current flows, from one to the other, along arcs of circles.
4. A source electrode and a sink electrode are located at the points x_1,y_1 and x_2,y_2 on a plane conductor which occupies the region $y>0$. What are the lines of current flow?
5. Consider a volume conductor R with electrodes A and B. Bring a second conductor R' into contact with R along a surface S'. Prove that the resistance between electrodes is lowered, and that the heat developed in R alone is greater than before.*
6. What equation does Φ satisfy in a conductor for which σ is not a constant, but a function of x,y,z, the co-ordinates of a variable point of the body?

* Pierce, *Annals of Mathematics* (1904), p 153.

Conclusion to Chapter III

In this chapter the fundamental equations for magnetostatics have been developed. The theory proceeds along lines closely similar to those followed in electrostatics. A fundamental law of magnetostatic action is first obtained, which plays the same rôle that Coulomb's law plays in electrostatics. In the derivation of this law from experimental evidence, a certain amount of temporary vagueness of concept is necessary. It is not possible at the outset, for example, to say accurately what one means by a "current." This situation is not at all different from or worse than that met at the beginning of electrostatics, where it is not possible to specify accurately a density of charge. A charged body is one possessing an excess or deficit of electrons; a current is moving charge. These general statements are enough, in each case, to permit one to describe the experimental basis, idealize to the law which expresses the action between units, and then, in the process of analytical resynthesis, accurately define the densities of charge and polarization in electrostatics, and current and magnetization in magnetostatics.

The analytical discussion necessary to formulate, for the experimental evidence, the fundamental law of magnetostatics has been seen to be somewhat complicated. The experimental basis is not sufficient for a unique deduction of a differential law for the force between current elements. This is explicitly recognized when one arbitrarily adds, to one deduced form for a differential law, further terms. It is, in fact, meaningless to inquire if a certain one is the differential law for magnetostatics, because it is logically improper to distinguish between laws which give the same results in any conceivable magnetostatic problem. It may be noted, in passing, that although equality of action and reaction was demanded, at the outset, for the action between current elements, this demand was later dropped, for current elements, although not, of course, for closed circuits. It seems advisable, analytically, to first demand equality of action and reaction for the elements and then add such terms as destroy this equality for the elements but not for closed circuits, and which, at the same time, simplify the expressions.

Having once obtained a satisfactory differential law—that is to say, one which is handy in form and which is consistent with the experimental facts for closed circuits—the remainder of the development is largely

a matter of translation, into new notation, of the analytical results developed in electrostatics. The arguments of the first two chapters show that if

$$F(x,y,z)=\frac{1}{4\pi}\int\frac{f_1(x',y',z')d\tau'}{r}+\frac{1}{4\pi}\int\frac{f_2(x',y',z')}{r}\,d\sigma' ,$$

then F is continuous at all points of space, while

$$\frac{\partial F}{\partial n_1}+\frac{\partial F}{\partial n_2}=f_2 ,$$

when one crosses one of the surfaces over which the second integral is extended. Furthermore, at all points,

$$\nabla^2 F=-f_1 .$$

These general facts, once one has obtained an expression for the vector potential which is analogous to the expression for Φ in electrostatics, furnish the analytical skeleton for the theory.

One statistical consideration is in more specific evidence, now, than in electrostatics; namely, the use of time averages. Time averages do play a rôle in electrostatics as when, for example, one speaks of a charge as "at rest." But in magnetostatics terms automatically appear, when one seeks to define "current" and "magnetization," which are removed by explicit recognition of the fact that time averages, only, are "stationary" in a magnetostatic problem. The three distinct orders of time intervals which make possible the present form of macroscopic magnetostatic theory are thus as much emphasized as are the corresponding three distinct orders of distances.

Finally, it is necessary to note that the motional intensity introduced in magnetostatics will, in general, be experienced in addition to the Coulomb forces. The Coulomb forces are thus to be thought of not simply as the forces which act when charges are at rest, but rather as that part of the force, which acts between any charges, which is independent of the state of motion. When the motions are such as to produce, of themselves, no actions, the problem reduces to an electrostatic one.

CHAPTER IV

THE MAXWELL FIELD EQUATIONS

Introduction

The previous three chapters contain the analysis of those forces, on charges or charged bodies, which are independent of motion and those which are due to a steady state of motion. It remains to consider the general case of the force on charges which are moving in any way whatsoever. The development of this more general theory does not follow the steps used in both electrostatics and magnetostatics. The passage from the special to the general equation is one characterized by great boldness in generalization, and there is less direct dependence on fundamental experiments. It is clear that this must be the case, because it is scarcely possible to frame an experiment whose circumstances are as general as the desired conclusions.

Electrostatics makes use of the equations

$$\begin{aligned} \operatorname{div} \boldsymbol{E} &= \rho \,, \\ \operatorname{curl} \boldsymbol{E} &= 0 \,, \\ \boldsymbol{F}_\epsilon &= \epsilon \boldsymbol{E} \,; \end{aligned}$$

while magnetostatics, in its general form which includes both Coulomb and motional intensities, is based on the relations

$$\begin{aligned} \operatorname{div} \boldsymbol{E} &= \rho \,, \\ \operatorname{curl} \boldsymbol{E} &= 0 \,, \\ \operatorname{div} \boldsymbol{B} &= 0 \,, \\ \operatorname{curl} \boldsymbol{B} &= \boldsymbol{i}, \\ \boldsymbol{F}_\epsilon &= \epsilon \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{v}, \boldsymbol{B}] \right\} \,. \end{aligned}$$

The equations of magnetostatics are, thus, a generalization of those of electrostatics. The first step in the next generalization is suggested by an experiment with moving circuits. This experiment is, from one point of view, a magnetostatic experiment; while, from a second and equally valid point of view, the currents involved are not steady. Thus the force equation of magnetostatics has to be generalized so as to be consistent with this experiment. This generalization of the force leads, in turn, to a non-vanishing value for the curl of the new generalized electric

intensity vector $\boldsymbol{E}$. The resulting equation expresses the Faraday law of induction. The experiment which leads to the generalization is not a satisfactory one on which to base a general theory, since it is far too special in its nature. Nevertheless, it is assumed that the generalization so obtained is adequate to cover the most general situation.

Not only must a new value be assigned to the curl of the electric vector $\boldsymbol{E}$, but also a new term must be added to the expression for the curl of the magnetic vector $\boldsymbol{B}$. An easy calculation suggests the most simple value to adopt for this new term. The curl of the $\boldsymbol{B}$ vector then is given by two terms: the first, viz., $\boldsymbol{i}$, being that given by the Ampère laws; while the second, namely, $\dot{\boldsymbol{B}}/c$, is the new term just mentioned. This new term is often called the "Maxwell term," since its suggestion constitutes one of the greatest contributions to electromagnetic theory made by this great physicist. Having obtained a set of equations for the behavior, in a general case, of the $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors at points in empty space, the analogous generalizations are then made for points within matter.

The remainder of the chapter contains certain of the most fundamental applications of these, the Maxwell field equations, and the method of solving these equations in terms of scalar and vector potentials. By studying the rate of doing work of the forces exerted on all the charges within a certain volume, and by modifying the form of the analytical expression for this rate, one is led to the concepts of the spatial density of electric and magnetic energy, and of the Poynting vector which measures the flux of energy at any point. The authors do not pretend to understand these concepts, but discuss them as adequately as they are able. The same remarks apply to the closely related concepts of Maxwell stresses and of electromagnetic momentum, which are obtained in an analogous way from the expression for the force acting on all the charges of a region.

The four field equations give (when suitable boundary conditions are also present) the values of the field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ when the positions and motions of all charges are known as functions of time. Instead of working with these four simultaneous partial-differential vector equations, it is possible to introduce two potentials, one scalar and one vector, in terms of which one can express $\boldsymbol{E}$ and $\boldsymbol{B}$. Each of these potentials, moreover, is then to be determined from a single equation, there being the additional advantage that these two equations for the two potentials have exactly the same analytical form, that of a so-called "wave-equation." The solution of the four field equations is thus reduced to the solu-

tion of a wave-equation. A suitable solution for the wave-equation is then obtained. The potentials, as given by this solution, are referred to as "retarded" potentials, since their values at time t and place O appear as the sum of effects which arise at the various points P of space at previous times $t-r/c$ (where r is the distance between O and P), which travel with velocity c, and which then arrive at the point O at the instant t in question. The actual computation of these retarded potentials is somewhat complicated, and a considerable amount of analysis (§ 57 and § 58) is necessary to modify the integrals to a form more suitable for calculation.

These expressions are then used to obtain the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields due to a uniformly moving point charge and a uniformly moving extended charge. The case of the extended charge is carried through for two different shapes—a rigid sphere and a deformable oblate spheroid. The rigid sphere is often referred to as the "Abraham electron," while the deformable spheroid is referred to as the "Lorentz electron." The total electromagnetic momentum associated with these two different shapes is calculated. These expressions refer to a uniformly moving charge, but it is assumed that for a slowly changing velocity the momentum has at any instant the value given by this formula. From the time rate of change of this momentum, one calculates the forces acting on the charge due to the reaction of its own field. This reaction may also be viewed as being the result of an "electromagnetic mass" possessed by the charge. This electromagnetic mass turns out to be different for accelerations parallel to and perpendicular to the velocity of the charge. The two values are hence called the "longitudinal" and the "transverse" electromagnetic mass. Not only is the electromagnetic mass dependent upon the orientation of the acceleration relative to the velocity, but it also depends upon the velocity itself, becoming indefinitely large as the velocity of the charge approaches that of light. As the velocity of the charge approaches zero, the mass approaches the so-called "rest-mass" of the charge. The dependence of mass on velocity is somewhat different for the two shapes assumed above; and experiments have indicated (although not entirely conclusively) that actual electrons correspond (at least in this respect) more nearly to the Lorentz electron than to the Abraham electron. Other considerations (now quite invalid) have led to the further conclusion that the mass of an electron is entirely electromagnetic. This conclusion is, from the present point of view, not supported in any way by the experiments in question, but, as a pure assumption, the statement continues to be a part of modern electromagnetic theory.

As a further illustration of the use of the general solutions referred to above, the case of an oscillating dipole or Herzian oscillator is studied in some detail. The field is analyzed both "near" the dipole and in the more distant region, or "wave-zone." The results are of basic importance in connection with the theory of wireless telegraphy or telephony. The chapter concludes with a brief consideration of the field due to an arbitrarily moving-point charge.

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§ 52. *The Maxwell Field Equations.*—The previous chapters have been concerned with a study of the forces between charges at relative rest, and the forces on charges which are moving in a magnetic field whose value $\boldsymbol{B}$ at every point is independent of the time t. Thus, the theory, as developed to this point, clearly has no immediate answer concerning the force on a charge when the steady state does not prevail, i.e., when the $\boldsymbol{B}$ vector is a function of t as well as of x, y, z. A definite suggestion as to how the theory is to be extended to cover the general case can, however, be obtained from the study of an experiment which, from one point of view, comes under the previous theory but which, from an alternative point of view, is an example of the general case.* Consider, in fact, two closed circuits 1 and 2, and suppose first that 2 is stationary and is traversed by a current which is maintained constant by some outside influence, while circuit 1 moves with a velocity $\boldsymbol{v}$. The magnetic field due to the current in circuit 2 is then constant at any point, and the intensity due to this motion on the charges of circuit 1 is

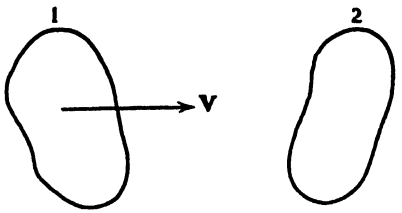

FIG. 49.—Two circuits in uniform relative motion.

$$\boldsymbol{F}=\frac{1}{c}[\boldsymbol{v},\boldsymbol{B}]$$

In order that the experiment be rigorously magnetostatic, one may suppose that a second external influence prevents the motion of charges in circuit 1. If the total "electromotive force" or *E.M.F.* seeking to

* Although not an entirely satisfactory example, as will appear later.

cause current to flow in a circuit be defined as the line integral around the circuit of the tangential component of the intensity, then

$$E.M.F. \equiv \int_1 F_s ds = \int \text{curl}_n \, \boldsymbol{F} \, d\sigma \, ,$$

the last integral being extended over any surface terminated in the closed circuit 1. However

$$\text{curl } \boldsymbol{F} = \frac{1}{c} \text{ curl } [\boldsymbol{v},\boldsymbol{B}] = -\frac{1}{c} (\boldsymbol{v},\nabla)\boldsymbol{B} \, ,$$

since, in the general identity,*

$$\text{curl } [\boldsymbol{v},\boldsymbol{B}] = (\boldsymbol{B},\nabla)\boldsymbol{v} - (\boldsymbol{v},\nabla)\boldsymbol{B} + \boldsymbol{v} \text{ div } \boldsymbol{B} - \boldsymbol{B} \text{ div } \boldsymbol{v} \, ,$$

the divergence of $\boldsymbol{B}$ is zero, and $\boldsymbol{v}$ is a constant.

Suppose now, on the other hand, that circuit 1 is fixed, and circuit 2 (in which the current is maintained at its previous constant value) moves with a velocity $-\boldsymbol{v}$. The actual physical situation, according to a simple relativity principle, is the same in the two cases; in either instance one circuit moves with respect to the other with a velocity of magnitude v, and it is a mere peculiarity of the method of description which one is said to be still and which moving. In the latter case, however, the electrons of circuit 1, the forces on which are being investigated, are at rest. Thus the motional intensity is zero, as is also the ordinary electrostatic force, since both wires are supposed uncharged. But since the two cases are in reality identical, it must be concluded that there is a force, causing the electrons of circuit 1 to move, the total $E.M.F.$ around this circuit being the same as before.

The expression for the total intensity must thus be amended to include a term, additional to the electrostatic and motional terms, which will give the force on a charge at rest due to an unsteady state of neighboring moving charges. Thus let the total intensity be

$$\frac{1}{c} [\boldsymbol{v},\boldsymbol{B}] + \boldsymbol{E} + \boldsymbol{E}' \, ,$$

where the last term, which may be called the "induced intensity," is the

* See Appendix, § 4, (44).

new additional term just referred to. It follows from the foregoing discussion that when circuit 2 moves and circuit 1 is stationary,

$$\int_1 E'_s ds = \int_1 F_s ds = \int \text{curl}_n \, \boldsymbol{F} \, d\sigma \, ,$$

where, as was shown above,

$$\text{curl} \, \boldsymbol{F} = -\frac{1}{c} (\boldsymbol{v}, \nabla) \boldsymbol{B} \, .$$

But an observer located on the stationary circuit 1 would calculate, at any point, a changing value of the $\boldsymbol{B}$ vector due to the moving circuit 2, the rate of change of $\boldsymbol{B}$ being given by the expression

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{v}, \nabla) \boldsymbol{B} \, ,$$

since the value of $\boldsymbol{B}$ is constant at any point moving, relative to 1, with the velocity $-\boldsymbol{v}$. Thus

$$\text{curl}_n \, \boldsymbol{F} = -\frac{1}{c} \frac{\partial B_n}{\partial t} \, ,$$

and

$$\int_1 E'_s ds = -\frac{1}{c} \int \frac{\partial B_n}{\partial t} \, d\sigma \, ,$$

or

$$\int \left(\text{curl} \, \boldsymbol{E}' + \frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} \right)_n d\sigma = 0 \, .$$

Since this equation holds for an arbitrary circuit and hence for an arbitrary surface, it follows that

$$\text{curl} \, \boldsymbol{E}' = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} \, . \tag{166}$$

To a large school of physicists the state of the "field" at a given point in space has a definite and describable reality (a reality aided by certain mechanical conceptions of the aether) apart from the nature of that which is producing the "field." That is to say, a given value of $\boldsymbol{B}$ at a certain point is taken to be descriptive of some condition obtaining at that point, and the emphasis is so thrown over on to the importance of

this condition and its description by means of $\boldsymbol{B}$ that one is not to be concerned particularly with what has produced the value of $\boldsymbol{B}$. One who adopts this point of view will say that the last equation above states that the curl of the $\boldsymbol{E}'$ vector is given by $1/c$ times the negative rate of change of the $\boldsymbol{B}$ vector, and will not feel it necessary to qualify this statement at all; in particular, he will not feel it necessary to distinguish between two cases, in which the negative rates of change of $\boldsymbol{B}$ are equal, the cause of the change being quite different in the two cases. Such an interpretation of the foregoing equation, however, does not follow from the derivation of it here given, for the $\boldsymbol{B}$ vector can be made to change in a way not contemplated in the experiment here considered, namely, by holding both circuits fixed in space but changing the current in circuit 2. Suppose, indeed, that the current in circuit 2 be changed in such a way that, at a certain point, the rate of change of the $\boldsymbol{B}$ vector due to 2 has the same value it had in the previous case, when the current in 2 was held steady but the circuit was moved as a whole. From equation (166) it would follow as a formal result that the value of curl $\boldsymbol{E}'$ would be the same in the two cases; and this result would be entirely reasonable to an exponent of the theory that all one needs to know, at a point, is the "state of the aether" at that point. From the more direct point of view that considers the things acting, as well as the things acted on, it is clear that the foregoing result, if true, is remarkable and one not to be expected; for the two cases differ in a fundamental way. In the case of a moving circuit traversed by a constant current, it is possible to choose an unaccelerated set of axes (namely, a set moving with the circuit) with respect to which the state of motion of the acting electrons is steady, in the sense that at any point the average velocity of charge is independent of the time. In the case of a fixed circuit traversed by a changing current, no unaccelerated set of axes can be chosen with respect to which the state of motion of the acting electrons is steady. This distinction might well prove to be an essential one, and it is reasonable to suspect that the values of curl $\boldsymbol{E}'$ are not actually the same in the two cases. It cannot be urged that it has been shown experimentally that moving circuits and changing currents are rigorously equivalent as regards induced electromotive forces. Roughly this must be the case, for the predictions of a theory based upon this assumption have had general experimental success; but there appears never to have been an experiment of sufficient accuracy to prove that the two values do not differ by second- or higher-order terms in such a ratio as v/c. It is very easy to let the notation carry the burden of the argument, to neglect this

discussion, and to hold that the value of curl $\boldsymbol{E}'$ is related to the rate of change of $\boldsymbol{B}$ in every case in the way stated by the last equation. It is important to point out, however, that by so doing one may be overlooking something of fundamental physical significance, and it is desirable to insist upon the potential importance of keeping in mind the auxiliary nature of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, and the necessity of always going back to the physical case. It is not the purpose of this book to revamp electromagnetic theory, but to try to present the existing theory in as logical a form as it permits, emphasizing but leaving open those questions which seem actually to be, up to the present, unanswered. Although further investigation of the point in question would seem highly desirable, the present discussion, in accordance with the plan just stated, will be dropped at this point, and it will be definitely assumed that equation (166) holds, without restriction, for any case.

The foregoing considerations have led to a value for curl $\boldsymbol{E}'$. A similar calculation can be made to obtain the value of the divergence of $\boldsymbol{E}'$. Indeed,

$$\text{div } \boldsymbol{E}' = \text{div } \boldsymbol{F} = \text{div } \frac{1}{c} [\boldsymbol{v},\boldsymbol{B}] .$$

However,*

$$\text{div } [\boldsymbol{v},\boldsymbol{B}] = (\boldsymbol{B}, \text{ curl } \boldsymbol{v}) - (\boldsymbol{v}, \text{ curl } \boldsymbol{B}) ,$$

so that, since $\boldsymbol{v}$ is constant,

$$\text{div } \boldsymbol{E}' = -\frac{1}{c} (\boldsymbol{v}, \text{ curl } \boldsymbol{B}) ,$$

or, setting curl $\boldsymbol{B} = \boldsymbol{i} = \rho\boldsymbol{u}/c$, where $\boldsymbol{u}$ is the mean velocity of the charges which give rise to the density ρ and the current $\boldsymbol{i}$,

$$\text{div } \boldsymbol{E}' = -\frac{\rho}{c^2} (\boldsymbol{v},\boldsymbol{u}) .$$

From these values for the curl and divergence of $\boldsymbol{E}'$ and the previously obtained values for curl and divergence of $\boldsymbol{E}$, it follows that

$$\text{curl } (\boldsymbol{E}+\boldsymbol{E}') = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} ,$$

$$\text{div } (\boldsymbol{E}+\boldsymbol{E}') = \rho\left(1 - \frac{(\boldsymbol{v},\boldsymbol{u})}{c^2}\right) .$$

* See Appendix, § 4, (43).

Since, however, the velocities ordinarily met with are exceedingly small compared with $c=3\cdot10^{10}$, the last equation can be written simply

$$\text{div } (\boldsymbol{E}+\boldsymbol{E}')=\rho \ .$$

It is thus clear that it is simpler to use a single vector $\boldsymbol{E}$ which will give, in all cases, the force per unit charge on a stationary charge, whether this force be due to stationary or moving charges. If only stationary charges be present,

$$\text{div } \boldsymbol{E}=\rho \ ,$$

$$\text{curl } \boldsymbol{E}=0 \ ,$$

and $\boldsymbol{E}$ is identical with the electrostatic intensity previously discussed. The same equations and statement hold for magnetostatics. When, however, a non-steady state prevails,

$$\text{div } \boldsymbol{E}=\rho$$

$$\text{curl } \boldsymbol{E}=-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \ ,$$

and $\boldsymbol{E}$ includes both the electrostatic term and the new term which has just been introduced. *From this point forward, the vector* $\boldsymbol{E}$ *will be used with this extended meaning.*

The assumption just made has not entirely prepared the ground for a study of the general case, for the second of the equations,

$$\text{curl } \boldsymbol{E}=-\frac{1}{c}\dot{\boldsymbol{B}} \ ,$$

$$\text{curl } \boldsymbol{B}=\boldsymbol{i} \ ,$$

$$\text{div } \boldsymbol{E}=\rho \ ,$$

$$\text{div } \boldsymbol{B}=0 \ ,$$

carries with it a restriction upon the vector current $\boldsymbol{i}$. In fact, this equation can only be true under the special condition that $\text{div } \boldsymbol{i}=0$, since the divergence of a curl is identically zero. If this equation be amended by adding a vector $\mathbf{X}$ so that

$$\text{curl } \boldsymbol{B}=\boldsymbol{i}+\mathbf{X} \ ,$$

then $\mathbf{X}$ must satisfy the relation

$$\text{div } \mathbf{X}=-\text{div } \boldsymbol{i} \ .$$

However, for any region,

$$\int \operatorname{div} \boldsymbol{i}\, d\tau = \int i_n\, d\sigma = -\frac{\partial}{\partial t}\int \frac{\rho}{c}\, d\tau = -\frac{\partial}{\partial t}\int \frac{\operatorname{div} \boldsymbol{E}}{c}\, d\tau = -\int \frac{\operatorname{div} \dot{\boldsymbol{E}}}{c}\, d\tau ,$$

so that

$$\operatorname{div} \boldsymbol{i} = -\frac{\operatorname{div} \dot{\boldsymbol{E}}}{c} .$$

Thus, substituting,

$$\operatorname{div} \mathbf{X} = \frac{\operatorname{div} \dot{\boldsymbol{E}}}{c} ,$$

so that the most simple and most directly suggested choice for $\mathbf{X}$ is

$$\mathbf{X} = \frac{\dot{\boldsymbol{E}}}{c} .$$

The revised equation then reads

$$\operatorname{curl} \boldsymbol{B} = \boldsymbol{i} + \frac{\dot{\boldsymbol{E}}}{c} .$$

The first term on the right side of this equation is, as has been seen, due to Ampère. The second term is due to Maxwell who, making a bold generalization based upon the hint that a changing intensity causes, in the case of a dielectric, a shifting of charge and thus a type of current, adopted as his definition of current a quantity which is the sum of $\dot{\boldsymbol{E}}$ and the ordinary convection current $\boldsymbol{i} = \rho \boldsymbol{u}/c$. The analysis to follow will be based upon the equations obtained above, i.e., upon the four so-called "field equations"

$$\operatorname{curl} \boldsymbol{B} = \boldsymbol{i} + \frac{\dot{\boldsymbol{E}}}{c} = \frac{\rho \boldsymbol{u} + \dot{\boldsymbol{E}}}{c} , \tag{167}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{B}} , \tag{168}$$

$$\operatorname{div} \boldsymbol{E} = \rho , \tag{169}$$

$$\operatorname{div} \boldsymbol{B} = 0 , \tag{170}$$

and upon the force equation

$$\boldsymbol{F} = \boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}, \boldsymbol{B}] . \tag{171}$$

The first two of the foregoing field equations are sometimes referred to as the "circuital relations," the names of Ampère and Faraday being associated with the first and second, respectively. The second equation is, indeed, the mathematical statement of Faraday's law of induction. The last equation gives the force $\boldsymbol{F}$ per unit charge in terms of the velocity $\boldsymbol{v}$ of the charge acted on, and in terms of the auxiliary vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, which are to be determined, through the field equations,* from the supposed known values of ρ and $\rho\boldsymbol{u}/c=\boldsymbol{i}$.

A vector field, it will be remembered, is simply a region at every point of which a vector is defined. Thus one speaks of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, meaning the totality of values which these vectors assume at the different points of the region being considered; and the field equations, as has just been stated, are the fundamental differential equations from which these vectors are to be determined. The expressions "$\boldsymbol{E}$ and $\boldsymbol{B}$ fields" and "field equations" will be used here in this sense, there being no implication, arising through a non-technical connotation of the word "field," of the existence of any physical reality which is being measured by the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$.†

There is one further equation which, although it is a consequence of the field equations, is of sufficient physical importance to warrant its being written here as one of the fundamental equations of electrodynamics. It is the equation of conservation of charge, or "equation of continuity," which states that, since no charge is created or destroyed, the rate at which charge leaves a certain volume element must be equal to the negative time rate of change of the total charge within, i.e.,

$$\text{div}\,(\rho\boldsymbol{u})+\dot{\rho}=0\,. \tag{172}$$

This same principle of the conservation of charge was used to obtain the Maxwell term $\dot{\boldsymbol{E}}$, in equation (167); and, indeed, the equation of continuity can be obtained from (167) by taking the divergence of both sides, and substituting $\dot{\rho}$ for div $\dot{\boldsymbol{E}}$.

In chapter ii, Part III, Problem 32, and chapter iii, Part III, Problem 8, it was pointed out that in electrostatics and magnetostatics the boundary conditions on the surface between two media can be written:

a) The tangential component of $\boldsymbol{E}$ is continuous.

b) The normal component of $\epsilon\boldsymbol{E}$ is continuous.

* Which, in a definite case, must be supplemented by boundary conditions.

† See, in this connection, A. Einstein, *Aether und Relativitätstheorie*. Berlin: J. Springer, 1920.

c) The tangential component of $\boldsymbol{B}/\mu$ is continuous.

d) The normal component of $\boldsymbol{B}$ is continuous.

These same conditions are assumed for the general $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors. These conditions may be deduced from the field equations themselves.*

It may readily be shown† that if $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{B}_2$ are two solutions of the Maxwell field equations which are regular at infinity and for which, at a given instant, $\boldsymbol{E}_1=\boldsymbol{E}_2$ and $\boldsymbol{B}_1=\boldsymbol{B}_2$ at all points of space, then $\boldsymbol{E}_1\equiv\boldsymbol{E}_2$ and $\boldsymbol{B}_1\equiv\boldsymbol{B}_2$ at all later instants. This uniqueness proof does not take account of the fact, which will be discovered in § 56, that the field equations imply a finite velocity of effects. Thus the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ within a certain closed region τ and at a time t should follow uniquely from the initial values of $\boldsymbol{E}$ and $\boldsymbol{B}$ at a time t_0 not throughout all space, but only in that region T, containing τ, which contains all those points which can communicate, so to speak, during the interval from t_0 to t, with the points of τ. A uniqueness proof which thus takes account of the finite velocity of electrodynamic action has been given.‡

§ 53. *The Field Equations within Matter.*—The field equations, in the form (167)–(170) written in the previous section, apply only to empty space. The generalizations necessary for points within matter are easily obtained. In the argument by means of which equations (167)–(170) were obtained, use was made of the equations

$$\text{div } \boldsymbol{E}=\rho\ ,$$

$$\text{curl } \boldsymbol{B}=\boldsymbol{i}\ .$$

At points within polarizable and magnetizable matter, these equations must be replaced by the more general relations (see chap. ii, Part III, Problem 31; and equation (136), § 43).

$$\text{div } \boldsymbol{E}=\rho-\text{div } \boldsymbol{P}\ , \tag{173}$$

$$\text{curl } \boldsymbol{B}=\boldsymbol{i}+\text{curl } \boldsymbol{M}\ . \tag{174}$$

Now since

$$\boldsymbol{P}=(\epsilon-1)\boldsymbol{E}\ ,$$

the first of these equations can be re-written

$$\text{div } \epsilon\boldsymbol{E}=\rho\ ;$$

* See H. Bateman, *Electrical and Optical Wave Motion* (1914), p. 17.

† See Riemann-Weber, *Differentialgleichungen der Physik*, II (1927), 393.

‡ See A. Rubinowicz, *Phys. Zeits.*, XXVII (1926), 707.

and since

$$\boldsymbol{M}=\left(1-\frac{1}{\mu}\right)\boldsymbol{B}\,,$$

the second becomes

$$\operatorname{curl}\frac{\boldsymbol{B}}{\mu}=\boldsymbol{i}\,.$$

Hence the argument which, in § 52, generalized the equation

$$\operatorname{curl}\boldsymbol{B}=\boldsymbol{i}$$

to read

$$\operatorname{curl}\boldsymbol{B}=\boldsymbol{i}+\frac{\dot{\boldsymbol{E}}}{c}\,,$$

generalizes the equation

$$\operatorname{curl}\frac{\boldsymbol{B}}{\mu}=\boldsymbol{i}$$

to read

$$\operatorname{curl}\frac{\boldsymbol{B}}{\mu}=\boldsymbol{i}+\frac{\epsilon\dot{\boldsymbol{E}}}{c}\,,$$

or, using (173) and (174),

$$\operatorname{curl}\boldsymbol{B}=\boldsymbol{i}+\frac{\dot{\boldsymbol{E}}}{c}+\frac{\dot{\boldsymbol{P}}}{c}+\operatorname{curl}\boldsymbol{M}\,.$$

Thus, if μ be a constant,

$$\operatorname{curl}\boldsymbol{B}=\mu\boldsymbol{i}+\frac{\mu\epsilon\dot{\boldsymbol{E}}}{c}\,.$$

Hence, within homogeneous matter, equations (167)–(170) are to be replaced by

$$\operatorname{curl}\boldsymbol{B}=\mu\boldsymbol{i}+\frac{\mu\epsilon\dot{\boldsymbol{E}}}{c}\,, \tag{175}$$

$$\operatorname{curl}\boldsymbol{E}=-\frac{1}{c}\dot{\boldsymbol{B}}\,, \tag{176}$$

$$\operatorname{div}\epsilon\boldsymbol{E}=\rho\,, \tag{177}$$

$$\operatorname{div}\boldsymbol{B}=0\,. \tag{178}$$

When the current i is a "conduction" current in a conducting body, then

$$i=\frac{\sigma}{c}E\,.$$

When the current i arises from the motion of charged bodies or the motion of ions in space, the current is called a "convection" current, and is written

$$i=\frac{\rho u}{c}\,.$$

In the case of a conducting body, (175) can be written

$$\text{curl } B=\frac{\mu\sigma}{c}E+\frac{\mu\epsilon}{c}\dot{E}\,. \tag{179}$$

In many important applications, the dependence of the vectors E and B upon time is known to be sinusoidal. Thus E and B would be equal to the real parts of the complex expressions $E_1e^{i\omega t}$ and $B_1e^{i\omega t}$, respectively, where E_1 and B_1 are now functions of x, y, z only; and where ω, the angular frequency, is related to the period T by means of the equation

$$T=\frac{2\pi}{\omega}\,.$$

Then

$$\dot{E}=i\omega E_1e^{i\omega t}\,,$$

etc., so that equations (179) and (176) become

$$\text{curl } B_1=\left(\frac{\mu\sigma+i\omega\mu\epsilon}{c}\right)E_1\,, \tag{180}$$

$$\text{curl } E_1=-\frac{i\omega}{c}B_1\,. \tag{181}$$

It is to be understood that, when these equations (together with [177] and [178]) are solved, E and B are then given by the real part of the product of these solutions with $e^{i\omega t}$.

§ 54. *The Activity Equation.*—Using the field equations as an analytical basis, certain integral transformations can be made which have

played an important rôle in the development of electrodynamics, especially in connection with the so-called "dynamics" of the electron. Suppose that within a region τ there exist charges, giving rise to a density ρ, moving with velocity $\boldsymbol{u}$. If

$$\boldsymbol{F}=\boldsymbol{E}+\frac{1}{c}[\boldsymbol{u},\boldsymbol{B}]$$

be the total force per unit charge acting on these charges, then the activity of these forces (i.e., the rate at which these forces do work) is

$$\int_\tau (\boldsymbol{u},\rho\boldsymbol{F})d\tau=\int(\rho\boldsymbol{u},\boldsymbol{F})d\tau\,,$$

or, substituting the foregoing expression for $\boldsymbol{F}$, and (167) for $\rho\boldsymbol{u}$, and noting that $\boldsymbol{u}$ and $[\boldsymbol{u},\boldsymbol{B}]$ are perpendicular,

$$\int_\tau (\boldsymbol{u},\rho\boldsymbol{F})d\tau=c\int\left(\boldsymbol{E},\ \text{curl}\ \boldsymbol{B}-\frac{1}{c}\dot{\boldsymbol{E}}\right)d\tau\,.$$

The general vector relation*

$$\int\{(\boldsymbol{B},\ \text{curl}\ \boldsymbol{E})-(\boldsymbol{E},\ \text{curl}\ \boldsymbol{B})\}d\tau=\int[\boldsymbol{E},\boldsymbol{B}]_n d\sigma$$

furnishes, since $\text{curl}\ \boldsymbol{E}=-1/c\ \dot{\boldsymbol{B}}$, the equation

$$\int(\boldsymbol{E},\ \text{curl}\ \boldsymbol{B})d\tau=-\int[\boldsymbol{E},\boldsymbol{B}]_n d\sigma-\frac{1}{c}\int(\boldsymbol{B},\dot{\boldsymbol{B}})d\tau\,,$$

so that, finally,

$$\begin{aligned}(182)\qquad \int(\boldsymbol{u},\rho\boldsymbol{F})d\tau&=-\int\{(\boldsymbol{B},\dot{\boldsymbol{B}})+(\boldsymbol{E},\dot{\boldsymbol{E}})\}d\tau-c\int[\boldsymbol{E},\boldsymbol{B}]_n d\sigma\,,\\&=-\frac{1}{2}\frac{\partial}{\partial t}\int(\boldsymbol{B}^2+\boldsymbol{E}^2)d\tau+c\int[\boldsymbol{E},\boldsymbol{B}]_{n'}d\sigma\,,\end{aligned}$$

where, in the last equation, $\boldsymbol{n}'$ is an interior normal.

Having obtained this equation, one seeks to interpret its various terms. It will perhaps be least confusing if the ordinary interpretation be stated at once, without comment. This statement will be followed by remarks concerning this or other interpretations.

* See Problem 1 of this chapter.

The left member of equation (182) is the rate at which the "forces of the field" do work on the charges or charged bodies within τ. The volume integral on the right is considered to give the rate of loss of the spatially distributed energy* within τ. It follows, if the principle of conservation of energy be assumed, that the rate of doing work on the charged bodies within τ is equal to the total rate of loss of energy of the field. If the rate of loss of energy of the field within τ is not sufficient to account for the calculated activity of the forces, then this deficiency must be met by bringing energy across the boundary of τ from without. Thus the surface integral is interpreted as measuring the rate at which localized energy flows across the boundary of τ from without, there being, on this basis, a "flux of energy" per unit time and per unit normal area, given by

$$\boldsymbol{S} = c[\boldsymbol{E},\boldsymbol{B}] \,. \tag{183}$$

The conception of such a flow of energy was first formulated by Poynting,† and the vector $\boldsymbol{S}$ is known as the "Poynting vector." The Poynting vector has played an important part in the development of modern electrodynamics; indeed, H. A. Lorentz has said: "Other examples might likewise show us how Poynting's theorem throws a clear light on many questions. Indeed its importance can hardly be overestimated, and it is now difficult to recall the state of electromagnetic theory of some thirty years ago, when we had to do without this beautiful theorem.‡

The interpretation just given suggests several comments. In the first place, the present authors are not able to ascribe any significance whatever to the phrase "localized energy." They do not believe that "Where?" is a fair or sensible question to ask concerning energy.§ Energy

* See problem 9, p. 219.

† J. H. Poynting, *London Transactions*, CLXXV (1884), 343.

‡ *The Theory of Electrons* (1916), p. 25.

§ There is one situation which, in a loose sense, is an exception to the general statement made here. Suppose one consider a region of space τ which is so small that its dimensions may be entirely neglected relative to the other dimensions under consideration Then τ may be said to be located "at" P, where P is any point within τ. Now suppose a configuration of charges to be located within τ and suppose the field of this configuration, however intense it may be within τ, to fall off toward zero very rapidly as one leaves τ. An outside charge does not then feel the effect of this configuration until the outside charge comes very close to τ—that is, until the charge is "at" P. The energy of this configuration "belongs" to the configuration, but since the configuration belongs to the point P, the energy may be said to belong to P, or to be "at" P.

The theory of relativity furnishes a basis for considering mass and energy to be

is a function of configuration, just as the beauty of a certain black-and-white design is a function of configuration. The authors see no more reason or excuse for speaking of a spatial energy density than they would for saying, in the case of a design, that its beauty was distributed over it with a certain density. Such a view would lead one to assign to a perfectly blank square inch in one portion of the design a certain amount of beauty, and to an equally blank square inch in another portion a certain different amount of beauty.*

In the preceding paragraph the phrase "reason or excuse" was used Some readers will argue that there may be no impelling or very logical reason for adopting a concept of localized energy, but that the excuse lies in the usefulness of the concept. There seems little doubt that the concept has in the past played a useful rôle in suggesting calculations and developments of theory. Such of these as are, in actual fact, independent of this concept (and this covers a large number of instances) should now be freed of this suspicious connection. Calculations which depend essentially on the concept of spatially distributed density may be retained tentatively.

As a matter of history, the notion of the location of energy in space was the natural result of a supposed understanding of where and how mechanical energy is stored. But if one is really to believe that matter is electrically constituted, it will not do to pretend to understand "where" electrical energy is, on the basis of an illusory conception of "where" ordinary mechanical energy is. To say that the potential energy of a spring is "stored in the spring" may perhaps give one some comfort

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essentially the same thing. From this viewpoint, when one remarks that the concept of location does not, in general, apply to energy, he is at the same time making a similar statement about mass. Until recently this would have seemed heresy indeed; but such a conclusion is in agreement with the recent viewpoint of wave mechanics. When a wave actually pervades all space one speaks of "where" it is only when its amplitude has markedly high values within a small region, and falls off rapidly outside this region.

* In a book published just as this volume was going to press A. S. Eddington (*The Nature of the Physical World*, p. 103), speaking of entropy, says: "The conception of entropy . . . . marked a reaction from the view that everything to which science need pay attention is discovered by a microscopic dissection of objects. It provided an alternative standpoint in which the center of interest is shifted from the entities reached by the customary analysis (atoms, electric potentials, etc.) to qualities possessed by the system as a whole, *which cannot be split up and located—a bit here and a bit there.*" (The italics do not occur in the original.) Entropy, like energy, is a function of configuration.

so long as he considers the situation from the gross viewpoint of continuously distributed matter. The modern theory of the electrical constitution of matter, however, recognizes this spring to be but an assembly of electrons and protons, with comparatively vast regions of "free space" between them. It is thus not possible to use the vague macroscopic concept of localized mechanical energy to help one understand where is the energy of those charges of which the spring is solely composed.

In the second place, the process of replacing volume by surface integrals, and vice versa, is so familiar that one hardly need remark that equation (182) suggests the interpretation given, rather than demands it. This is, of course, recognized by every careful writer on electrodynamics. In the rather extensive discussion of this theorem given in his text, Livens presents the customary interpretation, and also an alternative theory due to McDonald, which ascribes a quite other value to the Poynting vector.

In the third place, it is clear that a physical interpretation of any one term in (182) involves either knowledge or assumption concerning the nature of the other two terms. The usual analysis, given above, is essentially an interpretation of the third term, based upon a supposed knowledge of the first term, and an assumption concerning the second.

The phrase "a supposed knowledge of the first term" may seem harsh criticism when this first term is taken as the start of the calculation, and is deliberately set up to represent a certain physical rate of doing work. The significance of this quantity is not, however, clear until one states what is doing the work in question, and in what form is the energy reappearing. Suppose, for example, that the force

$$\boldsymbol{F}=\boldsymbol{E}+\frac{1}{c}[\boldsymbol{u},\boldsymbol{B}]$$

is calculated from the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ due to all charges, both those within and without τ. Then the middle term of (182) is the rate of change of total electromagnetic field energy, and the usual interpretation of the equation implies that the work being done on the charges within τ is all converted into some other form, such as chemical, thermal, or ordinary mechanical energy. In fact, if the total work per second, say W, done by the total field on the charges within τ is converted, through the action of the field on the charges, into an equal amount

$$W_1+W_2=W\ ,$$

where W_1 is, say, thermal in nature, while W_2 is electromagnetic, then the right side of equation (182), which supposedly measures the net rate of gain of the field, should equal $W - W_2 = W_1$, rather than W.

It cannot be said, moreover, that the nature of the middle term is really understood apart from this equation itself. In both electrostatics and magnetostatics, energy densities in space have, to be sure, been calculated. The situation with which (182) deals is now, however, a general one, and it is hardly to be expected that in terms of the generalized vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, the energy densities will take their previous simple forms. If one asks for a perfectly general calculation, analogous, for example, to that which indicated, in electrostatics, the value $\boldsymbol{E}^2/2$ the answer is that the derivation given above of equation (182) is precisely the requested calculation. This equation, in distinction to the special forms found earlier, contains two terms on the right side, rather than one; so that there is, inherently, less compulsion in the separate interpretation of them. Some writers have attempted to remove this difficulty by the assumption that the third term would vanish when extended over an infinitely distant surface. McDonald has pointed out* the impropriety of this assumption.

Some of the difficulties just raised are met, if one adopts the hypothesis, which will be formally introduced and discussed at a later point, that every charge moves in such a way that the total force on it is zero, the force due to its own field just canceling the force due to all other charges. To understand the effect this hypothesis has on an interpretation of the foregoing energy relation, it will be necessary to consider, in a more general way, the relation between electrodynamics and mechanics.

What is referred to as "classical electrodynamics" began its real development in about 1825. The concepts and analytical framework of ordinary dynamics were, by that time, essentially complete. It was inevitable that the new science would build on the old, borrowing its concepts and its methods. Science now, however, has come to believe in an electrical theory of the constitution of matter. Electrodynamics is now the basic science; and in terms of its fundamental concepts must all more special theories be stated. This ideal has, however, by no means been realized as yet, and the basic laws of electrodynamics are phrased in terms of concepts which properly belong to one of its own by-products. It may eventually turn out that concepts to which the names "force," "mass," "momentum," etc., will be attached will continue to be used in electrodynamics. It may, however, be that an attempt to retain these

* *Electric Waves* (1902), p. 33.

terms will be a serious handicap. To discuss the behavior of a single charge in the language of mechanics may turn out to be analogous to an attempt to discuss in detail the impact of two molecules in terms of thermal concepts.*

Science has been very progressive in adopting the new viewpoint that electrodynamics is fundamental, but it has been exceedingly conservative in its attempt to retain the dynamical concepts. The hypothesis, referred to above, that a charge moves so that the total force on it is zero results from acceptance of the further hypothesis that the ordinary mechanical mass of each charge is zero. Thus a charge is admitted to be a purely electrical entity, and what mechanics views to be its "mass" is now explained† as the reaction with which its own field opposes its acceleration. This "electromagnetic explanation" of mechanical mass is viewed to be one of the great triumphs of electromagnetic theory. One surely has here a glimpse of the power of the theory, but it is at best a shadowy glimpse. The idea of mass is used to explain away mass, and the mechanical concept of force is left, parentless, upon our hands.

The quantities with which mechanics deals are supposedly the statistical aspects of an underlying fine-grained electrodynamics. It is perhaps too much to say that the foregoing hypotheses of the non-existence of mechanical mass and the vanishing of total force constitute an admission that the mechanical concepts entirely collapse in the case of microscopic electrodynamics, but these hypotheses seem, at least, to be the last stand of the mechanical nomenclature. It is probable that we ought merely to say that preliminary calculations, in which mechanical concepts have been used, have indicated that each charge moves in such a way that a certain quantity is always zero. This quantity is calculated in a specified way from the positions and motions of all charges, and does not, in any ordinary sense, deserve the name "force."

At various places in electrodynamics one takes up positions which, in varying degrees, admit the fundamental nature of electrodynamical processes. When one treats of totally uncharged bodies, a roughly macro-

* In his latest book (*The Nature of the Physical World*) referred to just above, Eddington speaks (p. 75) of "primary laws" which control the behavior of individual units, and "secondary laws" which control the group behavior of vast numbers of individual units. He says: "It has been the conviction of nearly all physicists that at the root of everything there is a complete scheme of primary law governing the career of every particle or constituent of the world with an iron determinism." He adds in a footnote the remark: "There are, however, others besides myself who have recently begun to question it."

† The details of this argument will be found in § 60.

scopic viewpoint is satisfactory, and one deals with mechanical concepts alone. For such bodies, in fact, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are sensibly zero at all points outside of a certain surface, which is therefore recognized as the surface of the body. It is then convenient to annihilate mentally the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields at all points, and to replace the statistical electrodynamic properties of this assemblage of charges by mechanical properties. Thus the mechanical mass of the body is the gross evidence of the individual reactions, on the charges, which result from a change in their state of motion and hence in their fields. The mechanical elastic properties of the body result statistically from the electrodynamic interactions when the relative configuration is altered, etc.

If bodies are charged, it is customary to use both mechanical and electrodynamical concepts. When, however, a single charge is under consideration, the nature of the problem indicates an exclusively electrical viewpoint and only electrodynamical considerations should enter.

Thus, whenever one discusses the force on charge due to all charges, this "force" is zero if all actions are being recognized as electrodynamical. If one considers a non-vanishing force on charge due to all charges, he thereby discloses that he is viewing as mechanical some of the actions involved. He is, by virtue of his viewpoint, splitting up the total zero force into two equal and opposite portions

$$\boldsymbol{F}_e+\boldsymbol{F}_m=0\,,$$

one of which, $\boldsymbol{F}_e$, he recognizes as electrical, and the other of which, $\boldsymbol{F}_m$, he calls mechanical.

It has been seen above that it is not possible to discuss such an equation as (182) intelligently unless one states where energy is coming from and in what form it reappears. In the light of the remarks just made, this means that one must, before interpreting such an equation, tell more about the charges or charged bodies under consideration so that it will be clear by inference, if not otherwise, what particular mixture of electrodynamics and dynamics one is using. Otherwise the situation is similar to an attempt to discuss the energy relations when two bodies are rubbed together without saying whether the point of view is microscopic (so that heat motions of the molecule are recognized as contributing to mechanical energy) or macroscopic. For example, if, in (182), the left side be admitted to be zero, then all actions are electrical and all energies electrodynamical. The zero value of the right-hand side of the equation then is interpreted through the theorem of conservation of energy as

saying that the gain in energy of the field within τ must be accounted for through the flux of energy into the region across its bounding surface. If, on the other hand, the left side of the equation is claimed to be non-vanishing, this claim carries with it a completely different point of view. The non-vanishing electrodynamic force $\boldsymbol{F_e}$ is equal and opposite to a force $\boldsymbol{F_m}$ now being called mechanical. Thus the work done by the electrodynamic forces is equal to the work done against the mechanical forces. The left-hand side of (182) thus must measure the amount of energy which is being converted, per second, over into a form not recognized as electrical.

These considerations remove the formal difficulties in the interpretation of (182) so long as only the total field vectors enter. It remains to see whether similar interpretation be possible when partial field vectors are involved in the energy relation. It follows from the linearity of the field equations that any partial field $\boldsymbol{E_1}$, $\boldsymbol{B_1}$, due to a partial group of charges which give rise to the density ρ_1, satisfy the field equation with ρ_1 written in place of ρ. Then just as above

$$\int (\boldsymbol{u}_1,\rho_1\boldsymbol{F}_1)d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\int (\boldsymbol{E}_1^2+\boldsymbol{B}_1^2)d\tau - c\int [\boldsymbol{E}_1,\boldsymbol{B}_1]_n d\sigma ,$$

where $\boldsymbol{F_1}$ is the force due to the group 1 of charges. If the group 1 consists of a single charge e_1 moving with velocity $\boldsymbol{u_1}$, this equation takes the form

$$(\boldsymbol{u}_1,e_1\boldsymbol{F}_1) = -\frac{1}{2}\frac{\partial}{\partial t}\int (\boldsymbol{E}_1^2+\boldsymbol{B}_1^2)d\tau - c\int [\boldsymbol{E}_1,\boldsymbol{B}_1]_n d\sigma ,$$

or, from the hypothesis that the total force on e_1 is zero,

$$(\boldsymbol{u}_1,e_1\boldsymbol{F}_2) = +\frac{1}{2}\frac{\partial}{\partial t}\int (\boldsymbol{E}_1^2+\boldsymbol{B}_1^2)d\tau + c\int [\boldsymbol{E}_1,\boldsymbol{B}_1]_n d\sigma ,$$

where $\boldsymbol{F_2}$ is the force due to all other charges. This equation is then normally interpreted to mean that the rate at which the external forces (the forces due to all other charges) do work on e_1 is equal to the rate of gain of the energy of e_1's field within τ, plus the flow out through the bounding surface of τ of energy associated with the field of e_1. Further discussion of the activity equation will be found in the conclusion to this chapter.

§ 55. *The Maxwell Stresses and Electromagnetic Momentum.*—The preceding section has dealt with the flow of localized energy into a region

τ. The force on the charges in τ is due to the action of charge both within and without τ, and the "medium" theory introduced by Maxwell, since it does not recognize action at a distance, interprets these forces in terms of stresses in the medium. This interpretation results from a transformation of the expression

$$\boldsymbol{F}=\int_{\tau}\rho\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v},\boldsymbol{B}]\right\}d\tau\,, \tag{184}$$

for the total force on all the charges within τ. Using the relations $\rho=\operatorname{div}\boldsymbol{E}$ and $\rho\boldsymbol{v}=\operatorname{curl}\boldsymbol{B}-\dot{\boldsymbol{E}}/c$, equation (184) may be written

$$\boldsymbol{F}=\int\left\{\boldsymbol{E}\operatorname{div}\boldsymbol{E}+[\operatorname{curl}\boldsymbol{B},\boldsymbol{B}]-\frac{1}{c}[\dot{\boldsymbol{E}},\boldsymbol{B}]\right\}d\tau\,,$$

or, adding and subtracting the term,

$$\frac{1}{c^2}\dot{\boldsymbol{S}}=\frac{1}{c}[\dot{\boldsymbol{E}},\boldsymbol{B}]+\frac{1}{c}[\boldsymbol{E},\dot{\boldsymbol{B}}]\,,$$

$$=\frac{1}{c}[\dot{\boldsymbol{E}},\boldsymbol{B}]-[\boldsymbol{E},\operatorname{curl}\boldsymbol{E}]\,,$$

and adding, for the sake of symmetry, the term $\boldsymbol{B}\operatorname{div}\boldsymbol{B}\equiv 0$, the expression for the total force takes the form

$$\boldsymbol{F}=\int\{\boldsymbol{E}\operatorname{div}\boldsymbol{E}+\boldsymbol{B}\operatorname{div}\boldsymbol{B}+[\operatorname{curl}\boldsymbol{E},\boldsymbol{E}]+[\operatorname{curl}\boldsymbol{B},\boldsymbol{B}]\}d\tau-\frac{1}{c^2}\int\dot{\boldsymbol{S}}d\tau\,. \tag{185}$$

The first of these two integrals can be transformed into an integral extended over the surface of τ. The form of the terms containing $\boldsymbol{E}$ and $\boldsymbol{B}$ being the same, the calculation will be carried through only for those containing $\boldsymbol{E}$. In fact, if the x-component of

$$\int\{\boldsymbol{E}\operatorname{div}\boldsymbol{E}+[\operatorname{curl}\boldsymbol{E},\boldsymbol{E}]\}d\tau$$

be written out at length, and the terms rearranged, the result is:

$$\int\left\{\left(E_x\frac{\partial E_x}{\partial x}-E_y\frac{\partial E_y}{\partial x}-E_z\frac{\partial E_z}{\partial x}\right)+\left(E_x\frac{\partial E_y}{\partial y}+E_y\frac{\partial E_x}{\partial y}\right)+\left(E_x\frac{\partial E_z}{\partial z}+E_z\frac{\partial E_x}{\partial z}\right)\right\}d\tau$$

$$=\int\left\{\frac{1}{2}\frac{\partial}{\partial x}(E_x^2-E_y^2-E_z^2)+\frac{\partial}{\partial y}(E_xE_y)+\frac{\partial}{\partial z}(E_xE_z)\right\}d\tau\,,$$

or, by partial integration,

$$= \int \left\{ \frac{1}{2} (E_x^2 - E_y^2 - E_z^2) \cos(nx) + E_x E_y \cos(ny) + E_x E_z \cos(nz) \right\} d\sigma ,$$

$$= \int \left\{ E_x [E_x \cos(nx) + E_y \cos(ny) + E_z \cos(nz)] - \frac{1}{2} (E_x^2 + E_y^2 + E_z^2) \cos(nx) \right\} d\sigma ,$$

$$= \frac{1}{2} \int \{2E_x E_n - E^2 \cos(nx)\} d\sigma .$$

Thus, writing the similar terms in $\boldsymbol{B}$, and returning to vector form,

$$\boldsymbol{F} = \frac{1}{2} \int \{2E_n \boldsymbol{E} + 2B_n \boldsymbol{B} - (E^2 + B^2)\boldsymbol{n}\} d\sigma - \frac{1}{c^2} \int \dot{\boldsymbol{S}} d\tau , \tag{186}$$

where $\boldsymbol{n}$ is a unit vector in the direction of the outward drawn normal to the bounding surface of τ. In the case of a steady state of current distribution, the volume integral vanishes. The average value of the volume integral would also clearly be zero when the field vectors are periodic functions of the time. In either of these two instances the force is given by the surface integral only, and is expressible in terms of stresses,* known as the "Maxwell stresses," exerted across the elements of the surface of τ by that portion of the medium which is without. The stress across an element of the surface which is normal to the axis of x is, for example,

$$\frac{1}{2} [2E_x \boldsymbol{E} + 2B_x \boldsymbol{B} - (E^2 + B^2)\boldsymbol{x}'] ,$$

where $\boldsymbol{x}'$ is a unit vector in the x-direction, so that

$$X_x = \frac{1}{2} (E_x^2 - E_y^2 - E_z^2) + \frac{1}{2} (B_x^2 - B_y^2 - B_z^2) , \tag{187}$$

$$Y_x = X_y = E_x E_y + B_x B_y , \tag{188}$$

$$Z_x = X_z = E_x E_z + B_x B_z , \tag{189}$$

the other three components of the stress being obtainable from these by cyclic permutation. A more detailed investigation, based upon more

* For a short discussion of the customary method of specifying the stress at a point within an elastic solid, see Jeans, *Electricity and Magnetism* (1911), p. 142.

general assumptions as to the character of the medium involved, furnishes the result* that in the case of a homogeneous but anisotropic medium and a steady state of currents, the force on the changes within τ is again expressible in terms of stresses exerted across the boundary of τ, this stress having the nine components

$$X_x=\frac{1}{2}\left[\epsilon_{11}E_x^2-\epsilon_{22}E_y^2-\epsilon_{33}E_z^2+E_xE_y(\epsilon_{12}-\epsilon_{21})+E_xE_z(\epsilon_{13}-\epsilon_{31})-E_yE_z(\epsilon_{23}+\epsilon_{32})\right]$$

$$+\frac{1}{2}\left[\mu_{11}B_x^2-\mu_{22}B_y^2-\mu_{33}B_z^2+B_xB_y(\mu_{12}-\mu_{21})+B_xB_z(\mu_{13}-\mu_{31})-B_yB_z(\mu_{23}+\mu_{32})\right],$$

$$X_y=E_x(\epsilon_{11}E_x+\epsilon_{12}E_y+\epsilon_{13}E_z)+B_y(\mu_{11}B_x+\mu_{12}B_y+\mu_{13}B_z).$$

Thus, in the case of an anisotropic medium the stress system is not of the ordinary mechanical type, which is always self-conjugate—that is, in which $X_y=Y_x$, etc. In the case of a non-homogeneous medium further terms enter, such as, for example,

$$x'\int E^2\frac{\partial\epsilon}{\partial x}\,d\tau.$$

Therefore, only provided the second integral of (186) vanish and only in the case of a homogeneous isotropic medium can the force on the charges within τ be attributed to an ordinary stress system such as exists in an elastic medium in equilibrium.

To obtain an interpretation of the volume integral

$$-\frac{1}{c^2}\int\dot{\mathbf{S}}d\tau,$$

suppose that the region τ is infinite, but that all the charges are contained within some finite region within τ, and suppose that a steady state exists. Then since the steady-state vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ each vanish at infinity as $1/r^2$, the surface integral in (186) vanishes. If this steady state could then be disturbed in any way, the surface integral would retain the value zero for an infinite time, since the altered values of $\boldsymbol{E}$ and $\boldsymbol{B}$ are propagated outward at a finite velocity c. Under these circumstances the total force $\boldsymbol{F}$ on all charges would be given by

$$\boldsymbol{F}=-\frac{1}{c^2}\int\dot{\mathbf{S}}d\tau,$$

* G. H. Livens, *The Theory of Electricity* (1918), pp. 591, 592, and 200.

or by

$$\boldsymbol{F} = -\frac{d\boldsymbol{G}}{dt}, \tag{190}$$

if $\boldsymbol{G}$ be defined by the equation

$$\boldsymbol{G} \equiv \frac{1}{c^2}\int \boldsymbol{S} d\tau, \tag{191}$$

the integral being extended over all space. If the total force $\boldsymbol{F}$ act to increase the mechanical momentum $\boldsymbol{M}$ of the charged bodies present, then

$$\boldsymbol{F} = \frac{d\boldsymbol{M}}{dt},$$

so that

$$\frac{d}{dt}(\boldsymbol{M}+\boldsymbol{G}) = 0. \tag{192}$$

The vector $\boldsymbol{G}$ is called, following Abraham, the "electromagnetic momentum vector," and equation (192) thus states that the sum of the mechanical and electromagnetic momentum is conserved. The viewpoint furnishes a more complete interpretation of equation (186). In fact, the force exerted across the boundary of τ is considered to be used partly in increasing the electromagnetic momentum within τ; the remainder, namely,

$$\frac{1}{2}\int \{2E_n\boldsymbol{E} + 2B_n\boldsymbol{B} - (E^2+B^2)\boldsymbol{n}\} d\sigma - \frac{1}{c^2}\int \dot{\boldsymbol{S}} d\tau,$$

is available to act upon the charges or charged bodies present. The concept of a spatially distributed electromagnetic momentum demands, as has been pointed out by Lorentz,* an exceedingly great "density" for the hypothetical medium or "aether." The discussion of the previous section applies directly to the obvious questions involved in this viewpoint.† For example, the relation

$$\boldsymbol{F} = \frac{d\boldsymbol{M}}{dt}$$

* *Op. cit.*, p. 31.

† Livens, *The Theory of Electricity* (1918), p. 592; Abraham and Föppl, *Theorie der Elektrizität*, II (1918), 28; Richardson, *The Electron Theory of Matter* (1918), p. 216.

is true only provided the force F is engaged solely in altering the velocities of mechanical masses, and not at all in doing work of a purely electrical character. Even if, however, the concepts of an energy flow, of stress in the medium, and of an electromagnetic momentum should turn out to be, from a strict point of view, mere figures of speech which themselves become vague when closely examined, still they are important historically and because of the possibility of their indicating the first rough approach to new bits of theory. They should be retained as long as any advantage remains in their picturesque suggestiveness, and should then be discarded without surprise or reluctance.

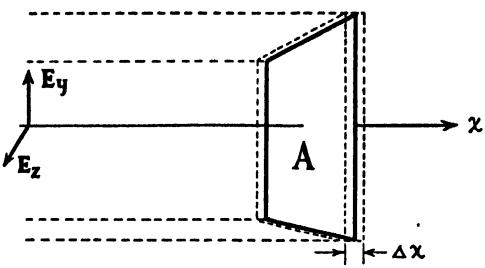

FIG. 50.—The radiation pressure on a perfectly black and absorbing plane disk of area A.

As an example illustrative of the use of the concepts of aether stresses and electromagnetic momentum, consider the problem of determining the pressure exerted on a perfectly opaque, perfectly black, plane disk by a beam of light. Suppose that the disk is of area A, and is normal to the x-axis, and that the beam of light also has cross-section A, is plane polarized, and is traveling in the x-direction (to the right). The electromagnetic theory of light will not be discussed here. It is only necessary to know that such a plane-polarized beam as is here considered is represented by the equations

$$\text{(193)} \qquad \begin{cases} E_y = a \cos \omega(t - x/c) , \\ B_z = a \cos \omega(t - x/c) . \end{cases}$$

Let the disk be contained in a flat volume τ whose ends are of area A and parallel to the disk, and whose sides are of vanishing length Δx. In computing, by means of the Maxwell stresses, the force on all the charge within this volume τ, it should be remembered that, since the disk is perfectly black, there is no reflected light and the field to the left of the disk consists only of the incident light (193); while since the disk is perfectly opaque, the total field to the right of the disk is zero. Moreover, since the expressions (193) are periodic, the average value is zero for that portion of the total force which is due to the volume term

in (186). Therefore the force on the disk can be found by considering the Maxwell stress across the left face of τ only. From (187)–(189)

$$Y_x = Z_x = 0 ,$$

$$X_x = -\frac{a^2}{2} [\cos^2 \omega(t-x/c) + \cos^2 \omega(t-x/c)] .$$

This last expression is the force exerted by the region on the positive size of a surface normal to the x-axis. Thus the force exerted by the region to the left of the left face of τ is, indicating average values by means of bars,

$$-\bar{X}_x = a^2 \overline{[\cos^2 \omega(t-x/c)]}$$
$$= \frac{a^2}{2} ,$$

since the average value of the cosine squared is one-half. There is, thus, a normal pressure on the disk of magnitude $a^2/2$. Since the total electromagnetic energy, per unit volume of the incident field, is, on the average,

$$\frac{1}{2} \overline{[E^2+B^2]} = \frac{a^2}{2} \overline{[\cos^2 \omega(t-x/c) + \cos^2 \omega(t-x/c)]}$$
$$= \frac{a^2}{2} ,$$

it appears that the pressure on the disk is numerically equal to the average energy density in the incident beam.

This same result can be obtained by means of the concept of electromagnetic momentum by supposing that, at a certain instant, the source of the beam of light is removed or destroyed. Then the rear of the beam of light (that is to say, the plane to the right of which the field is given by [193] and to the left of which the field vanishes) advances to the right with the velocity c. The total electromagnetic momentum is decreased, each second, by the amount contained in a parallelopiped of length c and cross-section A; or, since the momentum per unit volume within the beam is given by

$$\frac{1}{c^2} \mathbf{S} = \frac{1}{c} [\boldsymbol{E},\boldsymbol{B}] = \boldsymbol{x}' \frac{a^2 \cos^2 \omega(t-x/c)}{c} ,$$

by an amount

$$\frac{cAa^2 \cos^2\omega(t-x/c)}{c} .$$

Since the total momentum is conserved, the average value of this expression measures the average increase per second of mechanical momentum. The total force on the disk is therefore $x'Aa^2/2$, and the pressure caused by the light, $a^2/2$ as before. This so-called "radiation pressure" has been experimentally measured by Lebedew,* and later, checking the result just given to within 1 per cent, by Nichols and Hull.†

Although this result can be obtained, as above, by the use of stresses in the medium or by the notion of an electromagnetic momentum, nevertheless it is quite clear that it can be obtained directly from the equation (184) for the total force, without any necessity for interpretation of the various terms on the right side of that equation. Indeed, the fact that it can be so obtained would seem to be, in the last analysis, the only justification‡ for the two calculations given above.§

§ 56. *The Solution of the Field Equations for Free Space: the Wave-Equation.* It is required to determine the electric and magnetic vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ through the field equations (167)–(170) of § 52. It will be assumed, as an initial condition, that the charges which give rise to the density ρ and whose field is to be determined were at rest in a finite region of space until after a time t, the field having been electrostatic up to that time. The equation of conservation of charge is, of course, to be satisfied at all times.

In this general case, just as in the restricted case of magnetostatics, the equation $\operatorname{div} \boldsymbol{B}=0$ indicates the assumption of a vector potential $\boldsymbol{A}$ satisfying the relation

$$\boldsymbol{B} \equiv \operatorname{curl} \boldsymbol{A} .$$

It follows from substitution in (168) that the vector $\boldsymbol{E}+\dot{\boldsymbol{A}}/c$ has zero

* P. Lebedew, *Ann. Phys.*, Series G (1901), 436.

† E. F. Nichols and G. F. Hull, *Astrophysical Journal*, XVII (1903), 315; *Ann. Phys.*, XII (1903), 225.

‡ I.e., the only theoretical justification; the final result may, of course, be tested experimentally.

§ For a detailed study of the radiation pressure on a perfectly absorbing disk see W. F. G. Swann, *Philosophical Magazine* (7th ser., 1926), I, 584. He considers partial fields, the momentum associated with the field due to the disk itself, as well as forces of non-electromagnetic nature.

curl, so that* this vector can be determined from a scalar potential Φ, i.e.,†

$$\boldsymbol{E}+\frac{1}{c}\dot{\boldsymbol{A}}\equiv-\nabla\Phi\,,$$

or

$$\boldsymbol{E}=-\nabla\Phi-\frac{1}{c}\dot{\boldsymbol{A}}\,. \tag{194}$$

The assumption of an electrostatic condition up to time t_0 gives rise to the following initial conditions on Φ and $\boldsymbol{A}$:

$$(\Phi)_{t\leq t_0}=\Phi_0\,,\qquad \left(\frac{\partial\Phi}{\partial t}\right)_{t\leq t_0}=0\,,$$

$$(\boldsymbol{A})_{t\leq t_0}=0\,,\qquad \left(\frac{\partial A}{\partial t}\right)_{t\leq t}=0\,.$$

If the expressions for $\boldsymbol{A}$ and Φ be introduced into equations (167) and (169), each of the resulting equations involves both Φ and $\boldsymbol{A}$, viz.,

$$-\nabla^2\boldsymbol{A}+\nabla\,\mathrm{div}\,\boldsymbol{A}=-\frac{1}{c}\nabla\Phi-\frac{1}{c^2}\boldsymbol{A}+\frac{1}{c}\rho\boldsymbol{u}\,, \tag{195}$$

$$-\nabla^2\Phi-\frac{1}{c}\,\mathrm{div}\,\dot{\boldsymbol{A}}=\rho\,. \tag{196}$$

However, only one essential characteristic of the vector potential $\boldsymbol{A}$ is controlled by the relation $\boldsymbol{B}=\mathrm{curl}\,\boldsymbol{A}$, and, since it is possible‡ to specify the divergence of this vector without in any way affecting its curl, it is natural to assume such a value for the divergence as will simplify the problem by separating the potentials. The assumption

$$\mathrm{div}\,\boldsymbol{A}=-\frac{1}{c}\dot{\Phi} \tag{197}$$

* See Appendix, § 6, D.

† The notation used here for the scalar and vector potentials is justified by the fact that these potentials are direct generalizations of the scalar and vector potentials previously used in electrostatics and magnetostatics.

‡ See Appendix, § 6, C, and the paragraph at the end of § 6

accomplishes such a separation. Equations (195) and (196) then become, in fact,

$$\nabla^2 \boldsymbol{A} - \frac{\ddot{\boldsymbol{A}}}{c^2} = -\frac{1}{c}\rho \boldsymbol{u}\,, \tag{198}$$

$$\nabla^2 \Phi - \frac{1}{c^2}\ddot{\Phi} = -\rho\,. \tag{199}$$

Conversely, if solutions of these equations are obtained, the resulting vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ will satisfy the field equations (167)–(170) provided that Φ and $\boldsymbol{A}$ satisfy (197). However, adding the divergence of (198) and $1/c$ times the time derivative of (199),

$$\nabla^2 W - \frac{1}{c^2}\ddot{W} = 0\,, \tag{200}$$

where

$$W = \operatorname{div} \boldsymbol{A} + \frac{1}{c}\dot{\Phi}\,.$$

Also, from the initial conditions,

$$(W)_{t \le t_0} = \left(\frac{\partial W}{\partial t}\right)_{t \le t_0} = 0\,. \tag{201}$$

It will be shown later that these conditions force W to be identically zero, so that the solutions Φ and $\boldsymbol{A}$ of equations (198) and (199) and the initial conditions also satisfy the condition of connection (197). The vectors determined from the solutions of (198) and (199) therefore satisfy the field equations.

The problem of obtaining $\boldsymbol{E}$ and $\boldsymbol{B}$ is thus reduced to the equivalent problem of obtaining Φ and $\boldsymbol{A}$ from (198) and (199). These last two equations are of the same analytical form, the right member being a known function in either case. Any equation of the form

$$\nabla^2 \Psi - \frac{1}{k^2}\ddot{\Psi} = \chi$$

is called a "wave-equation," homogeneous or non-homogeneous according as χ is or is not zero. Such an equation is met with in a study of the propagation of waves along a string, of longitudinal waves on a rod, of sound waves of small amplitude, and, in fact, in any problem in which the disturbances are propagated with a constant finite velocity. Since the

two wave-equations above are analytically similar, attention will be directed to Φ. The problem* may be reduced to one of two independent variables, in the manner of Poisson, by the introduction of a function U defined as

$$U(r,t)=\frac{1}{r}\int \Phi d\sigma\,, \tag{202}$$

where the integration is taken over a sphere of radius r about the point P at which the potential Φ is to be determined, $d\sigma=r^2d\omega$ being the element of area. It is clear from this definition that U vanishes for $r=0$; namely,

$$U(0,t)=0\,, \tag{203}$$

and that, from the initial conditions,

$$U(r,\,t_0)=\frac{1}{r}\int \Phi_0 d\sigma\,, \tag{204}$$

$$\dot{U}_2(r,t_0)=0\,, \tag{205}$$

where the subscript 2 denotes the differentiation of U with respect to its second argument. If U be found, the value of Φ at the point in question is given by

$$4\pi\Phi=\lim_{r\doteq 0}\frac{U(r,t)}{r}=\lim_{r\doteq 0}\frac{U(r,t)-U(0,t)}{r}=U_1(0,t)\,, \tag{206}$$

where the subscript denotes differentiation of $U(r,\,t)$ with respect to its first argument.

Multiply (199) by $1/r$ and integrate each term over the surface of a sphere of radius r about P. Then

$$\frac{1}{r}\int \nabla^2\Phi d\sigma-\frac{1}{rc^2}\int \ddot{\Phi}d\sigma=-\int\frac{\rho}{r}\,d\sigma\,,$$

or

$$\frac{1}{r}\int \nabla^2\Phi d\sigma-\frac{1}{c^2}U_{22}=-\int\frac{\rho}{r}\,d\sigma\,.$$

* M. Abraham, *Theorie der Elektrizität*, II (1918), 39.

However,

$$\int r^2 dr \int \nabla^2 \Phi d\omega = \int \nabla^2 \Phi d\tau = \int \operatorname{div} \nabla \Phi d\tau = \int \frac{\partial \Phi}{\partial r} d\sigma = r^2 \frac{\partial}{\partial r} \int \Phi d\omega ,$$

where the integration is throughout the volume and over the surface of a sphere of radius r, $d\omega$ being an element of solid angle. Hence, differentiating with respect to r,

$$r^2 \int \nabla^2 \Phi d\omega = \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \int \Phi d\omega ,$$

$$= 2r \frac{\partial}{\partial r} \int \Phi d\omega + r^2 \frac{\partial^2}{\partial r^2} \int \Phi d\omega ,$$

or

$$r \int \nabla^2 \Phi d\omega = \frac{1}{r} \int \nabla^2 \Phi d\sigma = r \frac{\partial^2}{\partial r^2} \int \Phi d\omega + 2 \frac{\partial}{\partial r} \int \Phi d\omega ,$$

$$= \frac{\partial^2}{\partial r^2} r \int \Phi d\omega ,$$

$$= \frac{\partial^2}{\partial r^2} \frac{1}{r} \int \Phi d\sigma = U_{11} .$$

Hence

$$U_{11}(r,t) - \frac{1}{c^2} U_{22}(r,t) = - \int_r \frac{\rho}{r} d\sigma . \tag{207}$$

Abraham has applied a method of Riemann for the determination of U and thus of Φ. His analysis is, however, unnecessarily complicated, for it is not necessary to find U itself, since, according to equation (206), the potential Φ is determined by $U_1(0,t)$, i.e., by the value at $r=0$ of the derivative of U with respect to r.

The differential equation (207), holding for all values of its arguments, remains an identity in r and t when $t-r/c$ is substituted for t, i.e.,

$$U_{11}\left(r,t-\frac{r}{c}\right) - \frac{1}{c^2} U_{22}\left(r,t-\frac{r}{c}\right) = - \int_r \frac{\{\rho\}}{r} d\sigma ,$$

where $\{\rho\}$, the so-called "retarded value" of ρ, is given by

$$\{\rho\} = \rho(x,y,z,t-r/c) .$$

Now

$$\frac{\partial}{\partial r}\left[U_1\left(r,t-\frac{r}{c}\right)+\frac{1}{c}U_2\left(r,t-\frac{r}{c}\right)\right]=U_{11}-\frac{1}{c}U_{12}+\frac{1}{c}U_{21}-\frac{1}{c^2}U_{22}$$

$$=U_{11}-\frac{1}{c^2}U_{22}\,.$$

Consequently integration of the foregoing equation with respect to r from $r=0$ to $r=c(t-t_0)$ gives

$$\left[U_1\left(r,t-\frac{r}{c}\right)+\frac{1}{c}U_2\left(r,t-\frac{r}{c}\right)\right]\Bigg|_{r=0}^{r=c(t-t_0)}=-\int_0^{c(t-t_0)}\int\frac{\{\rho\}}{r}\,d\sigma\,dr\,,$$

or

$$U_1(c(t-t_0),t_0)-U_1(0,t)+\frac{1}{c}U_2(c(t-t_0),t_0)-\frac{1}{c}U_2(0,t)=$$

$$-\int_{c(t-t_0)}\frac{\{\rho\}}{r}\,d\tau\,, \tag{208}$$

the last integral on the right being extended throughout the interior of a sphere of radius $c(t-t_0)$.

From equation (204), which holds identically in r, it follows on differentiating with respect to r and substituting $c(t-t_0)$ for r, that, writing $d\sigma=r^2d\omega$,

$$U_1(c(t-t_0),t_0)=\left[\frac{\partial}{\partial r}\int\Phi_0 r\,d\omega\right]_{r=c(t-t_0)}=\int_{c(t-t_0)}\left(r\frac{\partial\Phi_0}{\partial r}+\Phi_0\right)d\omega\,.$$

The third and fourth terms of the left member of equation (208) vanish on account of the initial conditions (205) and (203), respectively, so that, using (206),

$$4\pi\Phi=\int_{c(t-t_0)}\frac{\{\rho\}}{r}\,d\tau+\int_{c(t-t_0)}\left(r\frac{\partial\Phi_0}{\partial r}+\Phi_0\right)d\omega\,.$$

It has been assumed that the initial potential Φ_0 due to the charges in question was, until after the time t_0, the electrostatic potential due to these charges, which were confined to a finite volume of space. This potential is consequently regular at infinity, so that, however large r becomes,

$$\left|r\frac{\partial\Phi_0}{\partial r}+\Phi_0\right|<\frac{J}{r}\,,$$

where f is a finite number. If then the initial time t_0 be allowed to approach $-\infty$, the equation results,

$$\Phi = \frac{1}{4\pi}\int \frac{\{\rho\}}{r}\, d\tau\,, \tag{209}$$

the integral being taken over all space.

In the same manner the equation

$$\boldsymbol{A} = \frac{1}{4\pi c}\int \frac{\{\rho \boldsymbol{u}\}}{r}\, d\tau \tag{210}$$

is obtained.

It follows as a result of the unique integral representation that the function

$$W = \operatorname{div} \boldsymbol{A} + \frac{1}{c}\dot{\Phi}\,,$$

mentioned above, is identically zero, since it satisfies the homogeneous equation

$$\nabla^2 W - \frac{1}{c^2}\ddot{W} = 0$$

and the initial equations

$$(W)_{t\leq t_0} = \left(\frac{\partial W}{\partial t}\right)_{t\leq t_0} = 0\,.$$

The required equation of connection between the potentials Φ and $\boldsymbol{A}$ is thus satisfied.

§ 57. *The Retarded Potentials.*—In the discussion which is to follow it will be convenient to denote by t_1 that particular instant of time at which the value of Φ or $\boldsymbol{A}$ is desired. It is clear from the solution given above that the value of the scalar (or vector) potential Φ (or $\boldsymbol{A}$) at a point O and at a time t_1 is to be obtained by summing, for all volume elements $d\tau$ of space, $1/r$ times the value which the function ρ (or $\rho\boldsymbol{u}$) had at these volume elements at times $t-r/c$; these times being earlier than t_1 by just such intervals as would be sufficient, in the case of each volume element, to permit a disturbance arising at $d\tau$ at time $t-r/c$ and traveling with velocity c, to arrive at O at time t_1. It is thus clear that if the charges producing the potential are all located within a moving region V with respect to which they are stationary, then the region of

space which furnishes non-vanishing contributions to the value of Φ at time t_1 does not coincide with the position of V at that or any other instant. A simple example will make clear that there will be points within V at which the retarded ρ vanishes, and points without V at which the retarded ρ does not vanish. Thus, suppose, in a problem involving only two dimensions, that V be a rectangle of length b moving with speed v directly toward the point O at which Φ is desired; and suppose that within V, ρ has the constant value ρ_0. Then the retarded value of ρ at a point P (see Fig. 51) is ρ_0 or zero according as

$$v\frac{1}{c}\sqrt{(a-x)^2+y^2}\lesssim -x\,,$$

where a is the distance, at time t, from O to M, the midpoint of the front face of V; and where x, y are the co-ordinates of P referred to M. It is then easily calculated that the front and rear faces of the effective integration region are portions of hyperbolae whose asymptotes have, to a second-order approximation in v/c, the slope $\pm c/v$; and whose vertices are displaced behind the corresponding faces of V a distance equal, in each instance, to v/c times the distance of that face from O. Thus the "length" of the effective integration region, namely,

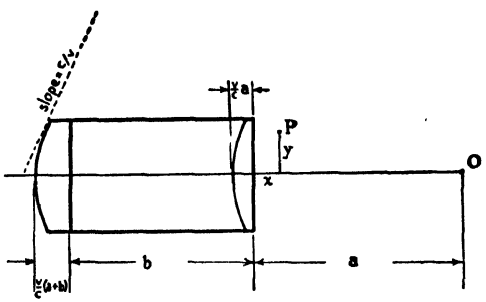

FIG. 51.—The effective shape of a rectangular charged area moving with velocity v toward o.

$$\frac{v}{c}(a+b)-\frac{v}{c}(a)+b=b\left(1+\frac{v}{c}\right),$$

is greater than the length b of V. The effectiveness of the charge in producing potential at O is thus increased by virtue of the motion of the charge toward O, and would be, conversely, decreased if the charge moved away from O.

This simple example indicates that the calculation of the retarded values of ρ would be, in a more general case, somewhat difficult. But in any case where the charges under consideration are stationary with respect to a moving region V, an alternative interpretation of (209) leads to a method of calculation which is often more convenient. For the sake of clearness in the presentation of this new viewpoint, it is useful to

refer to points within V and fixed relative to V as "material" points, to distinguish these moving material points, at which the density of charge is independent of t, from the fixed and purely geometrical points of space. The point O at which the potential Φ is being calculated will be located for convenience, at the origin; and Φ will be calculated at the instant t_1. In the discussion above, the reciprocal of r, the distance to a point P, has been associated with the retarded density at P. The alternative viewpoint is based upon the observation that it is equivalent, in the case under discussion, to associate the reciprocal of r, the distance to P, with the value which the density has, at time $t=t_1$, at a material

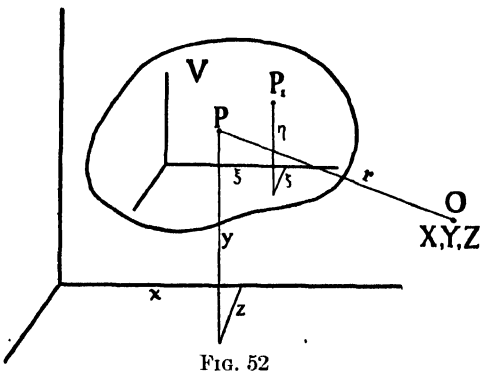

FIG. 52

point P_1 which was at P at time $t=t_1-r/c$; for the density at P_1 is the same as the density at P at $t=t_1-r/c$. The points P are the retarded positions in space of the material points P_1, since a disturbance arising at P when P_1 was there arrives at O at the instant t_1 under consideration. To take advantage of this viewpoint, it is only necessary to transform the expression for Φ so that the integration be performed with respect to the co-ordinates of the material points P_1 rather than with respect to x, y, z, the co-ordinates of P; for then the integration will be extended over V itself.

Let ξ, η, ζ be the co-ordinates, at the instant $t=t_1$, of a certain material point P_1 of V. Each material point of V can be identified at this or any other time by the fixed values ξ, η, ζ characteristic of that point. The co-ordinates of a material point of V at any other instant t clearly depend upon what material point of V is under consideration (i.e., upon ξ, η, ζ) and upon t. Thus, let the co-ordinates at time t of the material point $P_1(\xi, \eta, \zeta)$ be

$$f(t,\xi,\eta,\zeta)\,, \qquad g(t,\xi,\eta,\zeta)\,, \qquad h(t,\xi,\eta,\zeta)\,,$$

so that the co-ordinates of P, the retarded position of $P_1(\xi,\eta,\zeta)$, are

$$\text{(211)} \qquad \begin{cases} x=f(\tau,\xi,\eta,\zeta)\,, \\ y=g(\tau,\xi,\eta,\zeta)\,, \\ z=h(\tau,\xi,\eta,\zeta)\,, \end{cases}$$

where

$$\tau=t_1-r/c\,, \qquad r^2=x^2+y^2+z^2\,.$$

These equations give the analytical relation between the co-ordinates of P and P_1 which enable one to introduce ξ, η, ζ as new integration variables in (209), namely,

$$4\pi\Phi=\int\frac{\{\rho\}}{r}\,d\tau=\int\frac{\rho(\xi,\eta,\zeta)}{r}\,d\tau=\int_V\frac{\rho(\xi,\eta,\zeta)d\xi d\eta d\zeta}{r\,\dfrac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}}\,.$$

The Jacobian occurring in the denominator may be readily calculated. In fact, differentiating the first equation (211) with respect to x,

$$1=\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}+\frac{\partial f}{\partial\zeta}\frac{\partial\zeta}{\partial x}+\dot{f}\,\frac{\partial\tau}{\partial x}\,.$$

From this, and the eight similar equations obtained by differentiating (211) with respect to x, y, and z, the partial derivatives

$$\frac{\partial\xi}{\partial x}\,,\ \frac{\partial\xi}{\partial y}\,,\ \frac{\partial\xi}{\partial z}\,,\ \frac{\partial\eta}{\partial x}\,,\ \ldots\,,\ \frac{\partial\zeta}{\partial z}$$

may be calculated in terms of the quantities

$$\frac{\partial f}{\partial\xi}\,,\ \frac{\partial f}{\partial\eta}\,,\ \frac{\partial f}{\partial\zeta}\,,\ \frac{\partial g}{\partial\xi}\,,\ \ldots\,,\ \frac{\partial h}{\partial\zeta}\,.$$

Any one of these latter quantities, as, for example, $\partial f/\partial\xi$, may be expanded as a Taylor's series in the neighborhood of the value $t=t_1$. Thus

$$\frac{\partial f}{\partial\xi}=\left(\frac{\partial f}{\partial\xi}\right)_{t=t_1}+(t-t_1)\left(\frac{\partial^2 f}{\partial t\partial\xi}\right)_{t=t_1}+\frac{(t-t_1)^2}{\lfloor 2}\left(\frac{\partial^3 f}{\partial^2 t\partial\xi}\right)_{t=t_1}+\cdots\cdots$$

Since, however, the equation

$$\xi=f(t_1,\xi,\eta,\zeta)$$

holds identically in ξ, η, ζ,

$$\left(\frac{\partial f}{\partial \xi}\right)_{t=t_1} = 1 .$$

Also

$$(\tau - t_1) = -r/c ,$$

so that

$$\left(\frac{\partial f}{\partial \xi}\right)_{t=\tau} = 1 - \frac{r}{c}\left(\frac{\partial \dot{f}}{\partial \xi}\right)_{t=t_1} + \frac{r^2}{2c^2}\left(\frac{\partial \ddot{f}}{\partial \xi}\right)_{t=t_1} - \cdots\cdots$$

Similarly,

$$\left(\frac{\partial f}{\partial \eta}\right)_{t=\tau} = -\frac{r}{c}\left(\frac{\partial \dot{f}}{\partial \eta}\right)_{t=t_1} + \frac{r^2}{2c^2}\left(\frac{\partial \ddot{f}}{\partial \eta}\right)_{t=t_1} - \cdots\cdots$$

Thus, from the nine linear equations indicated above, one calculates the partial derivatives which enter the required Jacobian, each of these derivatives being given as a power series in r/c, the coefficients in these series involving the spatial variation, relative to the body, of the components of velocity, acceleration, etc.

If the region V be moving with constant velocity w in some direction, say that of the x-axis, then

$$f(t,\xi,\eta,\zeta) = \xi + (t - t_1)w ,$$
$$g(t,\xi,\eta,\zeta) = \eta ,$$
$$h(t,\xi,\eta,\zeta) = \zeta ,$$

and

$$\left(\frac{\partial f}{\partial \xi}\right)_{t=\tau} = 1 , \qquad \left(\frac{\partial f}{\partial \eta}\right)_{t=\tau} = 0 , \qquad \left(\frac{\partial f}{\partial \zeta}\right)_{t=\tau} = 0 ,$$

etc. Then

$$1 = \frac{\partial \xi}{\partial x} + \dot{f}\,\frac{\partial \tau}{\partial x} ,$$

so that

$$\frac{\partial \xi}{\partial x} = 1 + \frac{x}{cr}\,\dot{f} ,$$

while

$$\frac{\partial \xi}{dy}=\frac{y}{cr}\dot{f},$$

etc. The functional determinant has then, for this case, the value

$$\frac{1}{abc}\begin{vmatrix} \frac{1+a}{a} & 1 & 1 \\ 1 & \frac{1+b}{b} & 1 \\ 1 & 1 & \frac{1+c}{c} \end{vmatrix}$$

where

$$a=\frac{1}{c}\frac{x}{r}\dot{f}(\tau,\xi,\eta,\zeta), \qquad b=\frac{1}{c}\frac{y}{r}\dot{g}(\tau,\xi,\eta,\zeta),$$

$$c=\frac{1}{c}\frac{z}{r}\dot{h}(\tau,\xi,\eta,\zeta).$$

Thus

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}=1+\frac{1}{c}\left[\frac{x}{r}\dot{f}(\tau,\xi,\eta,\zeta)+\frac{y}{r}\dot{g}(\tau,\xi,\eta,\zeta)+\frac{z}{r}\dot{h}(\tau,\xi,\eta,\zeta)\right]$$

The quantity within the brackets is the component of the velocity of the material point $P_1(\xi, \eta, \zeta)$ in the direction from O to P, taken at the retarded time $\tau=t_1-r/c$, i.e., it is the negative of the retarded radial component u_r of the velocity of $P_1(\xi,\eta,\zeta)$, if this radial component be reckoned positive from P_1 to O. Thus

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}=\left\{1-\frac{u_r}{c}\right\}.$$

This value for the functional determinant has been derived under the assumption that the body has a constant velocity in the x-direction. However, if the body moves so that

$$f(t,\xi,\eta,\zeta)=\xi+(t-t_1)w+\frac{(t-t_1)^2}{2}a+\ldots,$$

where $w,a,\ldots,$ are the velocity, acceleration, $\ldots,$ in the x-direction, the foregoing calculation is unchanged. If the body rotates as it is

translated, on the other hand, the functional determinant has the more general value, to be calculated as indicated above.

In changing the variables of integration in the expression for Φ, it should be further noted that

$$r(x,y,z) = \{r(\xi,\eta,\zeta)\} ,$$

where $r(x,y,z)$ is the distance from O to P, and $r(\xi,\eta,\zeta)$ is the distance from O to P_1. Finally, then,

$$\Phi = \frac{1}{4\pi}\int_V \frac{\rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}} . \tag{212}$$

It should be remembered that this equation gives the value at O and at time t_1, of the potential Φ due to a configuration of charges which, considered as a whole, has rigid body motion of translation with no rotation. The quantities ξ, η, ζ, r, and u_r specify (at the time t_1) the position with respect to O, the distance to O, and the radial component of the velocity, respectively, of a material point within this moving configuration. It may be, at first sight, doubted whether this expression is more simple than (209), especially since the integral has not been freed of retarded quantities. But in the new expression (212) the dependence upon time of the quantities which are retarded is explicitly expressed. To obtain the retarded quantities in a specific case it is only necessary to solve equations (211) for x, y, z in terms of ξ, η, ζ and substitute these values in

$$\{r\} = r(x,y,z) = (x^2+y^2+z^2)^{\frac{1}{2}} ,$$

$$\{u_r\} = -\frac{x}{\{r\}}\dot{f}(\tau,\xi,\eta,\zeta) - \frac{y}{\{r\}}\dot{g}(\tau,\xi,\eta,\zeta) - \frac{z}{\{r\}}\dot{h}(\tau,\xi,\eta,\zeta) .$$

In making the analogous transformation of the integral which gives the vector potential $\boldsymbol{A}$, one new consideration enters. The vector function $\boldsymbol{u}$ cannot be treated as was the scalar function ρ, since it is not true that the velocity at the point P at time $\tau = t_1 - r/c$ is equal to the velocity at P_1 at time t_1. Since, however, the material point $P_1(\xi,\eta,\zeta)$ was at P at the time τ, the velocity at P at time τ is simply the velocity, at time τ, of the material point $P_1(\xi, \eta, \zeta)$, i.e., it is equal to

$$\boldsymbol{i}\dot{f}(\tau,\xi,\eta,\zeta) + \boldsymbol{j}\dot{g}(\tau,\xi,\eta,\zeta) + \boldsymbol{k}\dot{h}(\tau,\xi,\eta,\zeta) .$$

Denoting this quantity by $\{u\}$, it follows, the other details of the transformation being identical with the former case, that

$$A=\frac{1}{4\pi c}\int\frac{\rho(\xi,\eta,\zeta)\{u\}d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}}. \tag{213}$$

It should be carefully noted that the identification of the density at P_1 at time t_1 with the density at P at time τ is true only for rigid body motion, so that the resulting formulas apply only to such motion.

§ 58. *The* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *Fields Due to a Moving Configuration of Relatively Stationary Charges.*—By means of the relations

$$\boldsymbol{B}=\operatorname{curl}\boldsymbol{A}\,,$$

$$\boldsymbol{E}=-\nabla\Phi-\frac{1}{c}\dot{\boldsymbol{A}}\,,$$

and the expressions (209), (210) or (212), (213) for the retarded potentials Φ and $\boldsymbol{A}$, the field due to any system of charges moving with rigid body motion can be calculated.* The integrals (209), (210) and (212), (213) give the values of Φ and $\boldsymbol{A}$ at the origin. Before differentiating Φ and $\boldsymbol{A}$ to obtain $\boldsymbol{B}$ and $\boldsymbol{E}$, it is necessary to modify these integrals so as to bring into explicit evidence the co-ordinates upon which Φ and $\boldsymbol{A}$ depend. Let the values of Φ and $\boldsymbol{A}$ be sought at the point X, Y, Z and at the time t rather than at the origin at the time t_1. Then clearly

$$\Phi(X,Y,Z,t)=\frac{1}{4\pi}\int\frac{\rho(\xi,\eta,\zeta)d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}}, \tag{214}$$

$$A(X,Y,Z,t)=\frac{1}{4\pi c}\int\frac{\rho(\xi,\eta,\zeta)\{u\}d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}}, \tag{215}$$

where now

$$\{r\}=r(x,y,z)=[(X-x)^2+(Y-y)^2+(Z-z)^2]^{\frac{1}{2}},$$

$$\{u_r\}=\frac{X-x}{\{r\}}\dot{f}(\tau,\xi,\eta,\zeta)+\frac{Y-y}{\{r\}}\dot{g}(\tau,\xi,\eta,\zeta)+\frac{Z-z}{\{r\}}\dot{h}(\tau,\xi,\eta,\zeta).$$

* There are, of course, restrictions on the use of (212) and (213). These restrictions were met in connection with computing the Jacobian in § 57.

It will also be recalled that

$$x=f(\tau,\xi,\eta,\zeta)\,, \qquad y=g(\tau,\xi,\eta,\zeta)\,, \qquad z=h(\tau,\xi,\eta,\zeta)\,,$$

$$\tau=t-\frac{\{r\}}{c}\,.$$

The expression for $\{r\}$ has been written so as to be consistent with the definition

$$\{\boldsymbol{r}\}=\boldsymbol{i}(X-x)+\boldsymbol{j}(Y-y)+\boldsymbol{k}(Z-z)\,.$$

This vector points to X, Y, Z from $P(x,y,z)$, the retarded position of the material point ξ,η,ζ. The definition is chosen thus to conform with the previous definition of $\{u_r\}$, the component of $\{\boldsymbol{u}\}$ in the direction of $\{\boldsymbol{r}\}$. In the discussion which follows the braces will be omitted from the expressions $\{\boldsymbol{u}\}$, $\{u_r\}$, and $\{r\}$, and it will be understood, wherever $\boldsymbol{u}$, u_r, and r occur, that the retarded values of these quantities are to be used. No confusion can arise from this convention since only retarded values of $\boldsymbol{u}$ and u_r occur, and since a different notation will be introduced for the distance from ξ,η,ζ to X, Y, Z when this unretarded distance occurs.

The equations just given form the analytical basis for the calculation of $\boldsymbol{E}$ and $\boldsymbol{B}$. In differentiating with respect to X, for example, it must be noted that x, y, z are functions of X, Y, Z, since x, y, z are functions of τ, which, in turn, depends upon X, Y, Z. That this must be so is clear physically since x, y, z are the co-ordinates of the retarded position of the material point ξ,η,ζ; and the retarded position obviously depends on the point X, Y, Z with respect to which the retardation is carried out. Since the equations involved are rather long, it will be convenient, at certain points, temporarily to replace x, y, z by x_i $(i=1,2,3)$; X, Y, Z by X_i; and f, g, h by f_i. As just stated, x_i are functions of X_i; in fact,

$$\frac{\partial x_i}{\partial X_j}=f_i\,\frac{\partial \tau}{\partial X_j}\,,$$

whereas

$$\frac{\partial \tau}{\partial X_i}=\sum_{j=1}^{3}\frac{(X_j-x_j)}{cr}\,\frac{\partial x_j}{\partial X_i}-\frac{X_i-x_i}{cr}\,,$$

or, substituting from the previous equation and solving for $\partial\tau/\partial X_i$,

$$\frac{\partial\tau}{\partial X_i}=\frac{-(X_i-x_i)}{cr\left(1-\frac{u_r}{c}\right)},$$

thus, introducing the abbreviation $(1-u_r/c)=H$,

$$\frac{\partial\tau}{\partial X_i}=\frac{-(X_i-x_i)}{crH}.$$

With these preliminaries disposed of, the calculation of $\boldsymbol{E}$ and $\boldsymbol{B}$ is easily accomplished. In fact,

$$\boldsymbol{B}=\operatorname{curl}\boldsymbol{A}=\frac{1}{4\pi c}\int\operatorname{curl}\frac{\rho\boldsymbol{u}}{rH}\,d\xi d\eta d\zeta,$$

the differentiation under the sign being unquestionably legitimate so long as X, Y, Z is not within the region of integration. However, v being a scalar and $\boldsymbol{V}$ a vector,

$$\operatorname{curl} v\boldsymbol{V}=v\operatorname{curl}\boldsymbol{V}+[\nabla v,\ \boldsymbol{V}].$$

Hence

$$\boldsymbol{B}=\frac{1}{4\pi c}\int\frac{\rho}{rH}\operatorname{curl}\boldsymbol{u}\,d\xi d\eta d\zeta+\frac{1}{4\pi c}\int\rho\left[\nabla\frac{1}{rH},\ \boldsymbol{u}\right]d\xi d\eta d\zeta.$$

Now,

$$\begin{aligned}\operatorname{curl}_X\boldsymbol{u}&=\frac{\partial}{\partial Y}\dot{h}(\tau,\xi,\eta,\zeta)-\frac{\partial}{\partial Z}\dot{g}(\tau,\xi,\eta,\zeta),\\&=\ddot{h}\frac{\partial\tau}{\partial Y}-\ddot{g}\frac{\partial\tau}{\partial Z}=\frac{1}{cH}\left[\ddot{g}\frac{Z-z}{r}-\ddot{h}\frac{Y-y}{r}\right],\\&=\frac{1}{cH}[\dot{\boldsymbol{u}},\boldsymbol{r}_1]_X,\end{aligned}$$

where $\boldsymbol{r}_1$ is a unit vector in the direction of the retarded vector $\boldsymbol{r}$. Thus

$$\operatorname{curl}\boldsymbol{u}=\frac{1}{cH}[\dot{\boldsymbol{u}},\boldsymbol{r}_1].$$

Turning now to the second integral in the expression for $\boldsymbol{B}$,

$$\nabla\frac{1}{rH}=\frac{1}{H}\nabla\frac{1}{r}+\frac{1}{r}\nabla\frac{1}{H}=\frac{1}{H}\nabla\frac{1}{r}-\frac{\nabla H}{rH^2}.$$

But, from the definition of H,

$$\nabla H = -\frac{1}{c}\cdot\nabla u_r\,,$$

$$= -\frac{1}{c}\nabla\frac{1}{r}\sum_i \dot{f}_i(X_i-x_i) - \frac{1}{cr}\nabla\sum_i \dot{f}_i(X_i-x_i)\,.$$

Therefore, substituting,

$$\nabla\frac{1}{rH} = \left[\frac{1}{H} + \frac{\sum_i \dot{f}_i(X_i-x_i)}{crH^2}\right]\nabla\frac{1}{r} + \frac{1}{cr^2H^2}\nabla\sum_i \dot{f}_i(X_i-x_i)\,,$$

$$= \frac{1}{H^2}\nabla\frac{1}{r} + \frac{1}{cr^2H^2}\nabla\,\Sigma\dot{f}_i(X_i-x_i)\,.$$

Now

$$\left(\nabla\frac{1}{r}\right)_X = -\frac{1}{r^2}(\nabla r)_X = -\frac{X-x}{r^3} - \frac{X-x}{cr^3H}\,\Sigma\frac{X_i-x_i}{r}\dot{f}_i\,.$$

$$= -\frac{X-x}{r}\left[\frac{1}{r^2} + \frac{u_r}{cr^2H}\right] = -\frac{X-x}{r^3H}\,,$$

so that

$$\nabla\frac{1}{r} = -\frac{r_1}{r^2H}\,.$$

Also

$$(\nabla\Sigma\dot{f}_i(X_i-x_i))_X = \Sigma(X_i-x_i)(\nabla\dot{f}_i)_X + \Sigma\dot{f}_i(\nabla(X_i-x_i))_X\,,$$

$$= \frac{\partial\tau}{\partial X}\,\Sigma\ddot{f}_i(X_i-x_i) - \Sigma f_i\frac{\partial x_i}{\partial X} + f\,,$$

$$= -\frac{X-x}{cH}\,\Sigma\frac{X_i-x_i}{r}\ddot{f}_i + \frac{X-x}{crH}\,\Sigma f^2 + \dot{f}\,,$$

$$= -\frac{X-x}{cH}(r_1,\dot{u}) + \frac{X-x}{crH}(u,u) + \dot{f}\,,$$

so that

$$\frac{\nabla\Sigma\dot{f}_i(X_i-x_i)}{cr^2H^2} = -\frac{r_1}{c^2rH^3}(r_1,u) + \frac{r_1}{c^2r^2H^3}(u,u) + \frac{u}{cr^2H^2}$$

Hence, finally,

$$\nabla \frac{1}{rH} = -\frac{\boldsymbol{r}_1}{r^2H^3} - \frac{\boldsymbol{r}_1(\boldsymbol{r}_1,\dot{\boldsymbol{u}})}{c^2rH^3} + \frac{\boldsymbol{r}_1(\boldsymbol{u},\boldsymbol{u})}{c^2r^2H^3} + \frac{\boldsymbol{u}}{cr^2H^2}$$

and

$$\left[\nabla \frac{1}{rH}, \boldsymbol{u}\right] = \frac{[\boldsymbol{u},\boldsymbol{r}_1]\,[(\boldsymbol{r},\dot{\boldsymbol{u}}) + c^2 - (\boldsymbol{u},\boldsymbol{u})]}{c^2r^2H^3}.$$

Substituting this value, the value of $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \frac{1}{4\pi}\int \left\{ \frac{[\dot{\boldsymbol{u}},\boldsymbol{r}_1]}{c^2rH^2} + \frac{[\boldsymbol{u},\boldsymbol{r}_1]\,[(\boldsymbol{r},\dot{\boldsymbol{u}}) + c^2 - (\boldsymbol{u},\boldsymbol{u})]}{c^3r^2H^3} \right\} \rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta \,. \tag{216}$$

To avoid confusion when reference is made to this important equation, the braces are written in the integrand to indicate that the retarded values of $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, r, $\boldsymbol{r}_1$, and H are to be used. To carry out the integration in a given case, it would be necessary to eliminate the variables τ, x, y, z from the integrand, and express it as a function of X, Y, Z, t and the parameters of integration ξ, η, ζ. This would be accomplished by solving equations (211) for x, y, z, τ in terms of ξ, η, ζ, X, Y, Z, t and substituting.

In the analogous calculation of $\boldsymbol{E}$ only one new point arises. In differentiating $\boldsymbol{A}$ with respect to t, it is necessary to note that x, y, z are functions of t. In fact,

$$\frac{\partial x_i}{\partial t} = \dot{f}_i \frac{\partial \tau}{\partial t},$$

whereas

$$\frac{\partial \tau}{\partial t} = 1 + \frac{1}{cr} \sum \frac{\partial x_i}{\partial t}(X_i - x_i),$$

or, substituting from the previous relation and solving for $\partial\tau/\partial t$,

$$\frac{\partial \tau}{\partial t} = \frac{1}{1 - \frac{1}{c}\sum \dot{f}_i \frac{(X_i - x_i)}{r}} = \frac{1}{H}.$$

Thus,

$$\frac{\partial \dot{\boldsymbol{u}}}{\partial t} = \dot{\boldsymbol{u}} \frac{\partial \tau}{\partial t} = \frac{\dot{\boldsymbol{u}}}{H}.$$

The other details of the calculation are entirely similar to those just given, and it is found that

$$(217)\qquad E=\frac{1}{4\pi}\int\left\{-\frac{\dot{u}}{c^2rH^2}+\frac{\left(r_1-\frac{u}{c}\right)[(\dot{u},r)+c^2-(u,u)]}{c^3r^2H^3}\right\}\rho(\xi,\eta,\zeta)d\xi d\eta d\zeta\ .$$

§ 59. *The Field of a Uniformly Moving Point Charge.*—The general formulas just found give the field due to any system of charges moving with rigid body motion. Certain special cases which come under the foregoing formulas will now be discussed. Consider first a single point charge e moving with uniform speed w in the z-direction. For a point charge (or for an extended charge provided the distance to the point X, Y, Z is large compared to the dimensions of the charge) the quantities in the brackets in (216) and (217) may be removed from under the sign of integration, the integration of ρ then giving simply the total charge e.

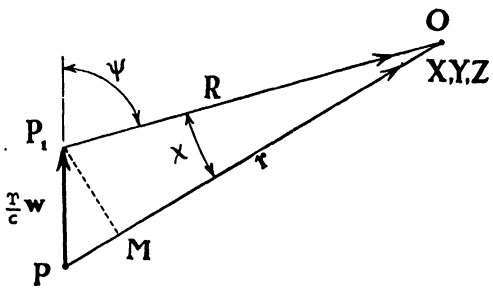

FIG. 53.—A point charge moving with velocity w. P_1 is the instantaneous position of the charge, and P the retarded position, relative to the observation point O.

In the case of an extended charge the approximation involved in thus removing from under the sign the terms involving r (the retarded distance to X, Y, Z from the various portions of the charge) can be definitely estimated, in any given case, by expanding the integrand in a power series in r_0, the distance to X, Y, Z from the retarded position of some one definite point of the charge. In the case here considered, since $\dot{u}=0$, $u=w$,

$$(218)\qquad E=\frac{e(1-\beta^2)}{4\pi}\left\{\frac{r_1-\frac{w}{c}}{r^2H^3}\right\}=\frac{e(1-\beta^2)}{4\pi}\left\{\frac{r-w\frac{r}{c}}{r^3H^3}\right\},$$

where β is the ratio of the velocity of the charge to c, i.e., where

$$\beta=\frac{w}{c}\ .$$

Now (see Fig. 53),

$$\boldsymbol{r}-\boldsymbol{w}\frac{r}{c}=\boldsymbol{R}\,,$$

where $\boldsymbol{R}$ is the unretarded vector to X,Y,Z from P_1, the position of the charge at time t. From the figure,

$$PM=w_r\frac{r}{c}\,,$$

so that

$$MO=r\left(1-\frac{w_r}{c}\right)=rH\,.$$

Also

$$MO=R\cos\chi=R\sqrt{1-\sin^2\chi}\,.$$

But, from the triangle $O\,P\,P_1$,

$$\frac{\sin\chi}{\sin\psi}=\frac{w}{c}\beta\,,$$

so that

$$MO=rH=R\sqrt{1-\beta^2\sin^2\psi}\,.$$

Thus,

$$\boldsymbol{E}=\frac{e(1-\beta^2)}{4\pi}\,\frac{\boldsymbol{R}}{R^3(1-\beta^2\sin^2\psi)^{3/2}}\,, \tag{219}$$

where the value of the vector $\boldsymbol{E}$ at time t is now described entirely in terms of the position of the charge at the same time t. It is important to note that the magnitude of $\boldsymbol{E}$ varies inversely as the square of the distance from the charge.

In just the same way, from (216)

$$\boldsymbol{B}=\frac{e(1-\beta^2)}{4\pi c}\,\frac{[\boldsymbol{w},\boldsymbol{r}]}{r^3H^3}\,, \tag{220}$$

$$=\frac{e(1-\beta^2)}{4\pi c}\,\frac{[\boldsymbol{w},\boldsymbol{r}]}{R^3(1-\beta^2\sin^2\psi)^{3/2}}\,.$$

But, since $\boldsymbol{w}r/c+\boldsymbol{R}=\boldsymbol{r}$,

$$[\boldsymbol{w},\boldsymbol{r}]=[\boldsymbol{w},\boldsymbol{R}]\,,$$

so that

$$\boldsymbol{B}=\frac{e(1-\beta^2)}{4\pi c}\frac{[\boldsymbol{w},\boldsymbol{R}]}{R^3(1-\beta^2\sin^2\psi)^{3/2}} \tag{221}$$
$$=\frac{1}{c}[\boldsymbol{w},\boldsymbol{E}]\,.$$

The electric vector $\boldsymbol{E}$ is thus, at any instant, directed from the charge toward the point at which $\boldsymbol{E}$ is being measured, while the magnetic vector $\boldsymbol{B}$ is normal to $\boldsymbol{E}$ and to the velocity of the charge.

The moving charge in question would exert a force of

$$\boldsymbol{F}=\boldsymbol{E}+\frac{1}{c}[\boldsymbol{w},\boldsymbol{B}]$$

per unit charge on any other charge which also has a velocity $\boldsymbol{w}$, so that the two are fixed relative to each other. From the expressions just found for $\boldsymbol{E}$ and $\boldsymbol{B}$ it follows that

$$F_x=(1-\beta^2)E_x\,,$$
$$F_y=(1-\beta^2)E_y\,,$$
$$F_z=E_z\,.$$

However, introducing the co-ordinates X,Y,Z of the point at which the force is being determined, and taking, for convenience, the origin of co-ordinates for X,Y,Z at P_1, the position of the charge at time t,

$$\sin^2\psi=\frac{X^2+Y^2}{X^2+Y^2+Z^2}\,,$$

so that

$$rH=R\sqrt{1-\beta^2\sin^2\psi}=\sqrt{Z^2+(1-\beta^2)(X^2+Y^2)}\,. \tag{222}$$

It is then readily seen that the three components of the force $\boldsymbol{F}$ are the negative partial derivatives with respect to X,Y,Z of the scalar function

$$\Theta=\frac{e(1-\beta^2)}{4\pi rH}=\frac{e(1-\beta^2)}{4\pi\sqrt{Z^2+(1-\beta^2)(X^2+Y^2)}}\,. \tag{223}$$

That is,

$$\boldsymbol{F}=-\nabla\Theta\,. \tag{224}$$

This function Θ, which is called "the convection-potential," plays a rôle in the case of a charge moving with constant velocity which is analogous to the rôle played by the electrostatic potential in the case of a stationary charge.

§ 60. *The Field Due to a Uniformly Moving Extended Charge.*—Suppose now that the charge whose field is to be determined moves with uniform velocity $\boldsymbol{w}$, as in the previous case, but is not small enough to permit the identification of the distances from X,Y,Z to its various points. It will be remembered that ξ,η,ζ are the co-ordinates of the material points within the charged region at some definite time, and it is convenient, for present purposes, to choose for this definite time the instant t at which the values of the field vectors are desired. Then the vector from the volume element $d\xi\, d\eta\, d\zeta$ (located at ξ,η,ζ) to the point X,Y,Z has scalar components $X-\xi$, etc.; and, just as was shown above in equation (220),

$$rH = R\sqrt{1-\beta^2\sin^2\psi} = \sqrt{(Z-\zeta)^2+(1-\beta^2)[(X-\xi)^2+(Y-\eta)^2]}\,,$$

the modification in the last expression arising from the fact that P_1 is no longer the origin of co-ordinates for X,Y,Z. The expression for Φ is, thus, from (214),

$$\Phi = \frac{1}{4\pi}\int \frac{\rho(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta}{\sqrt{1-\beta^2}\left[\dfrac{(Z-\zeta)^2}{1-\beta^2}+(X-\xi)^2+(Y-\eta)^2\right]^{1/2}}\,.$$

Now if new variables Z' and ζ' be introduced, according to the relations

$$Z = \sqrt{1-\beta^2}\,Z'\,, \qquad \zeta = \sqrt{1-\beta^2}\,\zeta'\,,$$

the integral becomes

$$\frac{1}{4\pi}\int \frac{\rho(\xi,\eta,\sqrt{1-\beta^2}\zeta')\,d\xi\,d\eta\,d\zeta'}{[(X-\xi)^2+(Y-\eta)^2+(Z'-\zeta')^2]^{1/2}}\,.$$

This, however, is the expression for the electrostatic potential at a point X,Y,Z' due to a charged volume which is obtained from the instantaneous volume, occupied at time t by the actual charge in question, by stretching this actual volume in the Z-direction in the ratio $1:\sqrt{1-\beta^2}$.

The charge density at corresponding points in the actual and fictitious stretched volume is, as the integral now stands, the same; so that the total charge in the fictitious stretched volume is larger than the actual charge in the ratio $1:\sqrt{1-\beta^2}$. Thus if a new volume density ρ' be defined by the equation

$$\rho'(\xi,\eta,\zeta') = \sqrt{1-\beta^2}\,\rho(\xi,\eta,\sqrt{1-\beta^2}\,\zeta') ,$$

the total charge in the fictitious volume is the same as the actual total charge, and the fictitious distribution of charge would be producible by an actual stretch of the actual charge. Then

$$\Phi = \frac{\Phi'}{\sqrt{1-\beta^2}} , \tag{225}$$

where

$$\Phi' = \frac{1}{4\pi}\int \frac{\rho'\,d\xi\,d\eta\,d\zeta'}{[(X-\xi)^2+(Y-\eta)^2+(Z'-\zeta')^2]^{1/2}} . \tag{226}$$

It is in no wise necessary, and from one point of view is unfortunate, to introduce the density ρ'. The relation (223) would be simpler if the original density $\rho(\xi,\eta,\sqrt{1-\beta^2}\,\zeta')$ were retained, since the factor $\sqrt{1-\beta^2}$ would then not be present. The only possible advantage of introducing ρ' is that then the fictitious distribution is producible by an actual stretch, holding the total charge constant, of the actual distribution This in itself may easily be, however, a disadvantage, since it tends to conceal, rather than emphasize, the fact that the stretched configuration of charge is purely fictitious, and is introduced for purposes of mathematical convenience.

It follows at once from (215) that for the case here considered of uniform velocity in the Z-direction,

$$\begin{cases} A_X = A_Y = 0 , \\ A_Z = \beta\Phi \end{cases} \tag{227}$$

It has thus been shown that the scalar and vector potentials at a point X,Y,Z due to the moving charge can be obtained by calculating the electrostatic potential of a fictitious distribution of charge which is

obtained from the actual instantaneous distribution at time t by stretching in the Z-direction in the ratio $1:\sqrt{1-\beta^2}$, holding the total charge constant. It is evident from an inspection of (226) and (227) that the moving charge carries its field with it, i.e., that Φ and $\boldsymbol{A}$ have values independent of the time at points which move so as to be fixed relative to the moving charge. It follows directly from this that, for example,

$$\frac{\partial A_Z}{\partial t}\,\Delta t+\frac{\partial A_Z}{\partial Z}\,w\Delta t=0\,,$$

or

$$\frac{\partial A_Z}{\partial t}=-w\,\frac{\partial A_Z}{\partial Z}\,, \tag{228}$$

since the first of these two equations expresses the total increment A_Z experiences in moving from a given point to the position which this point would occupy, after an interval Δt, if it moved with the charge.

The components of $\boldsymbol{E}$ may be written at once as

$$\begin{cases} E_X=-\dfrac{\partial\Phi}{\partial X}=-(1-\beta^2)^{-1/2}\,\dfrac{\partial\Phi'}{\partial X}\,, \\ E_Y=-\dfrac{\partial\Phi}{\partial Y}=-(1-\beta^2)^{-1/2}\,\dfrac{\partial\Phi'}{\partial Y}\,, \\ E_Z=-\dfrac{\partial\Phi}{\partial Z}-\dfrac{1}{c}\,\dfrac{\partial A_Z}{\partial t}=-(1-\beta^2)\dfrac{\partial\Phi}{\partial Z}=-\dfrac{\partial\Phi'}{\partial Z'}\,, \end{cases} \tag{229}$$

the relationships (227) and (228) being used in obtaining the value of E_Z. Also, from

$$\begin{cases} B_X=\beta\dfrac{\partial\Phi}{\partial Y}=\beta(1-\beta^2)^{-1/2}\,\dfrac{\partial\Phi'}{\partial Y}\,, \\ B_Y=-\beta\,\dfrac{\partial\Phi}{\partial X}=-\beta(1-\beta^2)^{-1/2}\,\dfrac{\partial\Phi'}{\partial X}\,, \\ B_Z=0\,. \end{cases} \tag{230}$$

The Poynting vector $\boldsymbol{S}$ and the electromagnetic momentum vector $\boldsymbol{G}$ can now be written at once in terms of the function Φ'. Since by definition

$$\boldsymbol{S}=c[\boldsymbol{E},\,\boldsymbol{B}]\,,$$

it follows that

$$(231)\qquad \begin{cases} S_X = -c\beta(1-\beta^2)^{-1/2}\dfrac{\partial\Phi'}{\partial X}\dfrac{\partial\Phi'}{\partial Z'}, \\ S_Y = -c\beta(1-\beta^2)^{-1/2}\dfrac{\partial\Phi'}{\partial Y}\dfrac{\partial\Phi'}{\partial Z'}, \\ S_Z = \dfrac{c\beta}{1-\beta^2}\left[\left(\dfrac{\partial\Phi'}{\partial X}\right)^2+\left(\dfrac{\partial\Phi'}{\partial Y}\right)^2\right], \end{cases}$$

and writing $d\tau' = dXdYdZ' = (1-\beta^2)^{-1/2}dXdYdZ = (1-\beta^2)^{-1/2}\,d\tau$,

$$(232)\qquad \begin{cases} G_X = \dfrac{1}{c^2}\int S_X d\tau = -\dfrac{\beta}{c}\int \dfrac{\partial\Phi'}{\partial Z'}\dfrac{\partial\Phi'}{\partial X}\,d\tau', \\ G_Y = \dfrac{1}{c^2}\int S_Y d\tau = -\dfrac{\beta}{c}\int \dfrac{\partial\Phi'}{\partial Z'}\dfrac{\partial\Phi'}{\partial Y}\,d\tau', \\ G_Z = \dfrac{1}{c^2}\int S_Z d\tau = \dfrac{\beta}{c\sqrt{1-\beta^2}}\int\left[\left(\dfrac{\partial\Phi'}{\partial X}\right)^2+\left(\dfrac{\partial\Phi'}{\partial Y}\right)^2\right]d\tau', \end{cases}$$

where the integration is carried out over all space.

A calculation equivalent to the foregoing has been used by various writers to investigate what properties of the elementary charge, an electron, follow from various assumptions concerning its shape. Thus Abraham* considered the case of a very thin spherical shell of charge, or what amounts to the same thing, a sphere with uniform surface density of charge. The fictitious stationary configuration is then a prolate ellipsoidal homeoid, or, in the case of a surface density of charge, a prolate ellipsoid whose surface charge turns out to be distributed in its equilibrium configuration. The field Φ' of such a static distribution being known, the fields, $\boldsymbol{E},\boldsymbol{B}$ and the vector $\boldsymbol{G}$ due to the moving spherical shell can be calculated. Abraham found, for example,

$$G_X = G_Y = 0,$$

$$(233)\qquad G_Z = \frac{e^2}{16\pi ac}\left[\frac{1+\beta^2}{\beta^2}\log\frac{1+\beta}{1-\beta}-\frac{2}{\beta}\right],$$

where a is the radius of the sphere and e is its total charge.

* M. Abraham, "Prinzipien der Dynamik des Elektrons," *Ann. der Physik, 10* (1903), 105.

Before investigating the consequences of this expression for the electromagnetic momentum due to a moving sphere, the corresponding expression will be obtained for an electron of slightly different shape. Since the fictitious fixed configuration is found from the instantaneous actual configuration by stretching, in the direction of the motion, in the ratio $1:\sqrt{1-\beta^2}$, it is clear that if the actual configuration were an oblate spheroid of semi-axes a, a, $a\sqrt{1-\beta^2}$ in the X-, Y-, Z-directions, respectively, then the auxiliary fixed body would be simply a sphere of radius a, the potential Φ' due to which can be written down at once. Now in an attempt to explain the celebrated Michelson-Morley experiment, Fitzgerald and Lorentz independently suggested, that when a body is in motion with respect to a system of co-ordinates this body is contracted, all dimensions (as observed in this system of co-ordinates) parallel to the motion being made shorter in the ratio $1:\sqrt{1-\beta^2}:1$, while the other dimensions are unchanged. Assuming such a contraction, a spherical electron of radius a would become, if it were in motion, an oblate spheroid of semi-axes a, a, $a\sqrt{1-\beta^2}$; so that the associated electrostatic figure would be, as was just mentioned, a sphere of radius a. An electron which is, when at rest, a sphere, but whose dimensions change, when it is in motion, in the way just discussed, is called a "Lorentz deformable electron," to contrast it with the Abraham "rigid" electron. For a Lorentz electron, at outside points,

$$\Phi' = \frac{e}{4\pi R'},$$

where R' is the distance from the position of the center of the sphere to the point X,Y,Z'; while at points inside the sphere, Φ has the constant value $e/4\pi a$. If these values are substituted in equations (232) for the components of $\boldsymbol{G}$, it is clear that the integration is to be extended over all space exterior to the sphere and that, from symmetry, the X- and Y-components of $\boldsymbol{G}$ vanish. Moreover, since

$$\int\left(\frac{\partial\Phi'}{\partial X}\right)^2 d\tau = \int\left(\frac{\partial\Phi'}{\partial Y}\right)^2 d\tau = \int\left(\frac{\partial\Phi'}{\partial Z'}\right)^2 d\tau,$$

when these integrals are extended over the region exterior to the sphere, the Z-component of $\boldsymbol{G}$ is given by

$$G_Z = \frac{2}{3}\frac{\beta}{c\sqrt{1-\beta^2}}\int\left[\left(\frac{\partial\Phi'}{\partial X}\right)^2+\left(\frac{\partial\Phi'}{\partial Y}\right)^2+\left(\frac{\partial\Phi'}{\partial Z'}\right)^2\right]d\tau,$$

$$= \frac{2\beta e^2}{48\pi^2 c\sqrt{1-\beta^2}} \int \frac{X^2+Y^2+Z'^2}{R'^6}\, d\tau ,$$

$$= \frac{\beta e^2}{24\pi^2 c\sqrt{1-\beta^2}} \int \frac{4\pi R'^2 dR'}{R'^4} = \frac{e^2\beta}{6\pi c a\sqrt{1-\beta^2}} . \tag{234}$$

Now since

$$\boldsymbol{F} = -\frac{d\boldsymbol{G}}{dt} ,$$

the force on either the Abraham rigid electron or the Lorentz contractile electron can be found by calculating the time derivative of the expressions which give, in the two instances, the electromagnetic momentum associated with the electron's field. It is clear that when $\boldsymbol{w}$ is constant, this force is zero in either case. But it has been assumed that for very small percentage changes in velocity, the expressions (233) and (234) give the proper instantaneous values of the varying $\boldsymbol{G}$ vector. Motions in which the changes of velocity are small enough to permit this approximation have been called "quasi-stationary" motions. Since the electromagnetic momentum vector has been found to be parallel to the velocity vector $\boldsymbol{w}$, it is possible to write

$$\boldsymbol{G} = \frac{G}{w}\boldsymbol{w} ,$$

so that

$$\boldsymbol{F} = -\frac{d\boldsymbol{G}}{dt} = -\frac{d}{dt}\left(\frac{G}{w}\boldsymbol{w}\right) = -\frac{d\frac{G}{w}}{dt}\boldsymbol{w} - \frac{G}{w}\frac{d\boldsymbol{w}}{dt} .$$

The time derivative of vector velocity is vector acceleration, so that, resolving the acceleration into a tangential component $\boldsymbol{j}'$ and a normal component $\boldsymbol{j}''$,

$$\frac{d\boldsymbol{w}}{dt} = \boldsymbol{j}' + \boldsymbol{j}'' .$$

Moreover, since the magnitude of the tangential component of the acceleration is the time derivative of the scalar speed,

$$\frac{dw}{dt}\boldsymbol{w} = w\boldsymbol{j}' .$$

Thus

$$\boldsymbol{F}=-\frac{d\frac{G}{w}}{dt}\,w-\frac{G}{w}\,(j'+j'')=-\frac{d\frac{G}{w}}{dw}\,wj'-\frac{G}{w}\,(j'+j'')\,,$$

$$=j'\left(-\frac{G}{w}\right)-w\,\frac{d\frac{G}{w}}{dw}+j''\left(-\frac{G}{w}\right),$$

or, replacing w by its value $c\beta$,

$$\boldsymbol{F}=-j'\,\frac{1}{c}\,\frac{dG}{d\beta}-j''\,\frac{G}{c\beta}\,. \tag{235}$$

If there be acting on the charge in question an external force $\boldsymbol{F}_e$ in addition to the force due to its own field, then the total force $\boldsymbol{F}+\boldsymbol{F}_e$ is related to the ordinary mechanical mass of the charge by the equation

$$\boldsymbol{F}+\boldsymbol{F}_e=m\boldsymbol{j}=m(j'+j'')\,,$$

so that

$$\boldsymbol{F}_e=j'\left(m+\frac{1}{c}\,\frac{dG}{d\beta}\right)+j''\left(m+\frac{G}{c\beta}\right). \tag{236}$$

Thus, when an external force is applied to the charge, it reacts to this force as would a neutral body possessing, with respect to tangential accelerations, a "longitudinal" mass

$$m+m'=m+\frac{1}{c}\,\frac{dG}{d\beta}\,, \tag{237}$$

which is different from the "transverse" mass

$$m+m''=m+\frac{G}{c\beta}\,, \tag{238}$$

which resists normal acceleration. For this reason the quantities

$$m'=\frac{1}{c}\,\frac{dG}{d\beta}\,, \tag{239}$$

$$m''=\frac{G}{c\beta}\,, \tag{240}$$

which have to be added to the ordinary mechanical mass to obtain the effective longitudinal and transverse masses, have been called the "longitudinal" and "transverse" electromagnetic mass of the charge. Such an increase in effective mass due to the presence of charge was worked out by J. J. Thomson for the case of a charged sphere before the concept of an electron had been advanced. The situation is analogous to the case of a sphere moving in an incompressible perfect fluid which extends to infinity in all directions. If a force operates on such a sphere to change its state of motion, the sphere reacts to this force, due to the fact that the state of motion of the surrounding fluid must also be changed, as if it had a mass greater than its actual mass. In this hydrodynamical analogy, however, the addition to the mechanical mass turns out to be a constant independent of the velocity, and no distinction exists between the transverse and longitudinal masses. From equations (233) and (234) and from the definitions just given, the longitudinal and transverse electromagnetic masses of the Abraham rigid electron and the Lorentz contractile electron can be easily calculated. In fact, for the Lorentz electron,

$$(241)\qquad \begin{cases} m' = \dfrac{e^2}{6\pi a c^2}(1-\beta^2)^{-3/2} = \dfrac{e^2}{6\pi a c^2}\left[1+\dfrac{3}{2}\beta^2+\ldots.\right], \\ m'' = \dfrac{e^2}{6\pi a c^2}(1-\beta^2)^{-1/2} = \dfrac{e^2}{6\pi a c^2}\left[1+\dfrac{1}{2}\beta^2+\ldots\right], \end{cases}$$

while for the Abraham electron

$$(242)\qquad \begin{cases} m' = \dfrac{e^2}{4\pi a c^2}\left[\dfrac{2}{3}+\dfrac{4}{5}\beta^2+\dfrac{6}{7}\beta^4+\ldots.\right], \\ m'' = \dfrac{e^2}{8\pi a c^2}\left[\left(1+\dfrac{1}{3}\right)+\left(\dfrac{1}{3}+\dfrac{1}{5}\right)\beta^2+\left(\dfrac{1}{5}+\dfrac{1}{7}\right)\beta^4+\ldots\right], \end{cases}$$

where these quantities have been expanded in a power series in β for ease in comparison with the preceding values. It is seen that for very small velocities the longitudinal and transverse masses are equal, and are, moreover, the same for the rigid and the deformable charge. This zero-velocity electromagnetic mass is, in fact, given by

$$m_0 = \frac{e^2}{6\pi a c^2},$$

while in general the longitudinal mass is larger than the transverse mass. For small velocities the two values

$$m'' = m_0\left(1+\frac{1}{2}\beta^2+\ldots.\right),$$

$$m'' = m_0\left(1+\frac{2}{5}\beta^2+\ldots.\right),$$

which hold for the Lorentz and Abraham electrons, respectively, are so nearly the same that only very careful experimentation can decide between them. In fact, vacuum-tube experiments which show the variation of mass with velocity have not decided with absolute certainty between the two hypotheses, but the evidence at present is strongly in favor of the Lorentz contractile electron.

The results just found for the force on an electron due to its own field have been obtained by using the vectors $\boldsymbol{S}$ and $\boldsymbol{G}$. In connection with previous discussion concerning the concepts of the Poynting vector and the electromagnetic momentum vector it is worth while to note that the present results could be obtained directly from the fundamental equation

$$\boldsymbol{F} = \int_{\cdot}\rho\left[\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v},\boldsymbol{B}]\right]d\tau$$

without utilizing either $\boldsymbol{S}$ or $\boldsymbol{G}$. Indeed, the identification of this expression with the quantity

$$-\frac{d\boldsymbol{G}}{dt} = -\frac{d}{dt}\int \boldsymbol{S}\,d\tau\,,$$

where $\boldsymbol{S}=[\boldsymbol{E},\boldsymbol{B}]/c$ depends, it will be remembered, only upon the vanishing of the surface integral

$$\int[2E_n\boldsymbol{E}+2B_n\boldsymbol{B}-(E^2+B^2)\boldsymbol{n}]d\sigma$$

as the surface over which the integral is extended recedes to infinity so as to inclose all of space. In this particular instance it may be easily seen that this surface integral vanishes for all times, since each component of $\boldsymbol{E}$ or $\boldsymbol{B}$ vanishes as $1/r^2$. It may be, in a definite case, that the calculation of the integral giving the vector $\boldsymbol{G}$ is simpler than the corresponding direct calculation of $\boldsymbol{F}$; but it is clear that in this important example, at least, there is no necessity for attempting to interpret physically the

intermediate stages of the calculation. The original expression gives the force; any quantity into which this can be legitimately transformed also gives the force; and the intermediate steps are of analytical, rather than necessarily of physical, interest.

In equations (237) and (238) the expressions for longitudinal and transverse masses have each been written as the sum of two terms, a "mechanical mass" term and an "electromagnetic mass" term. The electromagnetic mass depends, in each instance, on velocity. At the time these formulas were originally developed, it was assumed that the mechanical mass was independent of velocity. Thus when the experiments revealed an actual change of mass with velocity as predicted by the electromagnetic-mass terms, and no residual mass independent of velocity, it was natural to conclude that the entire mass of the electron was electromagnetic in character. This argument no longer has force, for it has subsequently developed, as an essential part of the restricted theory of relativity, that ordinary mechanical mass, if there be any such thing, varies with velocity in just the same way as does electromagnetic mass. The justification for continuing to assume that the mass of the electron is entirely electromagnetic is that it is the simplest assumption, and, if tenable, therefore the most desirable.

The assumption that the mass of the elementary charge is entirely electromagnetic is more than a mere choice of name. For this assumption carries with it an assumption concerning the motion of elementary charges. In fact, if the ordinary mechanical mass be zero, then the equation just before (236) shows that the total force on the charge is also zero. That is to say, *when an external force acts on an electron, it moves in such a way that the reaction of its own field on itself exactly cancels the applied force.* This conclusion concerning the motion of a charge is, in one sense at least, a very curious one. It is the result, on the one hand, of an assumption that an electron is quite unmechanical in that it has no mechanical mass, and, on the other hand, of an unwillingness to give up the fundamental equation of mechanics. It can hardly turn out to be a satisfactory principle unless it be indeed true that the equations of mechanics are more fundamental than the concepts of mechanics.

§ 61. *The Field Due to an Oscillating Dipole.*—An important case which can be treated by means of equations (216) and (217) is that of a positive charge $+e$ stationary at a point P accompanied by an oscillating negative charge $-e$ which vibrates about P. The two charges then form a dipole whose vector moment $\boldsymbol{p}$ is a function of time. The case ordinarily considered is that of simple harmonic oscillations, and the

vibrating doublet is sometimes called a "Hertzian oscillator." The field due to such an oscillator will be found under the restricting assumptions that the velocity $\boldsymbol{u}$ of the negative charge is small compared to the velocity of light, so that only leading terms in the ratio u/c will be retained;* and that the dimensions of the charges and the displacement between the two charges is small compared to the distance to the point O at which the field is being investigated. All quantities except ρ can then be removed from under the integration sign in (216) and (217), the integration then giving the magnitude of the charge concerned. The retarded values of $\boldsymbol{r}$, $\boldsymbol{r}_1$, or r may, on account of the foregoing assumption, be identified with $\boldsymbol{R}$, $\boldsymbol{R}_1$, or R, where $\boldsymbol{R}$ is the vector from P to O. Further, since the moment $\boldsymbol{p}$ of the doublet is the product of $-e$ and the vector displacement which locates $+e$ with respect to $-e$,

$$\text{(243)} \qquad \begin{cases} \dot{\boldsymbol{p}} = -e\boldsymbol{u} , \\ \ddot{\boldsymbol{p}} = -e\dot{\boldsymbol{u}} . \end{cases}$$

The vector $\boldsymbol{B}$ due to the doublet is clearly all due to the oscillating negative charge, since the velocity of the positive charge is zero. Since only leading terms in the ratio u/c are to be retained, it is clear, from (216), that the portion of $\boldsymbol{B}$ depending on the inverse first power from the dipole is

$$-\frac{e}{4\pi cR}\left\{\frac{\left[\frac{\dot{\boldsymbol{u}}}{c}, \boldsymbol{R}_1\right]}{H^2}\right\} = -\frac{e}{4\pi cR}\left\{\left[\frac{\dot{\boldsymbol{u}}}{c}, \boldsymbol{R}_1\right]\left[1+2\frac{u_r}{c}+\ldots.\right]\right\},$$

$$\text{(244)} \qquad = -\frac{e\{[\dot{\boldsymbol{u}}, \boldsymbol{R}_1]\}}{4\pi c^2 R},$$

while the leading term depending upon the inverse second power of the distance is

$$-\frac{e}{4\pi R^2}\left\{\frac{\left[\frac{\boldsymbol{u}}{c}, \boldsymbol{R}_1\right]}{H^3}\right\} = -\frac{e}{4\pi R^2}\left\{\left[\frac{\boldsymbol{u}}{c}, \boldsymbol{R}_1\right]\left[1+3\frac{u_r}{c}+\ldots.\right]\right\},$$

$$\text{(245)} \qquad = -\frac{e\{[\boldsymbol{u}, \boldsymbol{R}_1]\}}{4\pi cR^2}.$$

* This statement also furnishes a criterion for dropping higher powers of $\dot{u}/c$ since in the case of simple harmonic motion the magnitude of $\dot{u}$ is of the order of νu, where ν is the frequency.

Thus

$$\boldsymbol{B}=-\frac{e}{R}\frac{\{[\dot{\boldsymbol{u}},\boldsymbol{R}_1]\}}{4\pi c^2}-\frac{e}{R^2}\frac{\{[\boldsymbol{u},\boldsymbol{R}_1]\}}{4\pi c}, \tag{246}$$

which can also be written, from (243),

$$\boldsymbol{B}=\frac{1}{R}\frac{\{[\ddot{\boldsymbol{p}},\boldsymbol{R}_1]\}}{4\pi c^2}+\frac{1}{R^2}\frac{\{[\dot{\boldsymbol{p}},\boldsymbol{R}_1]\}}{4\pi c}. \tag{247}$$

The fixed positive charge $+e$ contributes to the $\boldsymbol{E}$ vector the one term

$$\frac{e\boldsymbol{R}_1}{4\pi R^2}$$

obtained from (217) by setting $\boldsymbol{u}=0$, or directly from Coulomb's law. The leading term in $1/R$ due to the negative charge is

$$\frac{e}{4\pi cR}\left\{\frac{\dot{\boldsymbol{u}}}{c}-\boldsymbol{R}_1\left(\frac{\dot{\boldsymbol{u}}}{c},\boldsymbol{R}_1\right)\right\}=-\frac{e}{4\pi c^2R}[\boldsymbol{R}_1[\boldsymbol{R}_1,\dot{\boldsymbol{u}}]],$$

$$=\frac{[\boldsymbol{R}[\boldsymbol{R},\ddot{\boldsymbol{p}}]]}{4\pi c^2R^3},$$

the factor H^2 disappearing, just as above, since its inclusion would involve higher powers of u/c. The leading term in $\boldsymbol{E}$ due to $-e$ and varying as $1/R^2$ is

$$-\frac{e\{\boldsymbol{r}_1\}}{4\pi R^2},$$

where, for a reason which will appear presently, the retarded vector $\{\boldsymbol{r}_1\}$ has not been set equal to $\boldsymbol{R}_1$, as was done in the other term. Then

$$\boldsymbol{E}=\frac{\{[\boldsymbol{R}_1[\boldsymbol{R}_1,\ddot{\boldsymbol{p}}]]\}}{4\pi c^2R}+\frac{e}{4\pi R^2}(\boldsymbol{R}_1-\{\boldsymbol{r}_1\}). \tag{248}$$

In the neighborhood of the vibrating doublet it is clear that the term depending upon $1/R^2$ will predominate over the term in $1/R$, while for more distant points the term in $1/R^2$ will be negligible compared to the term in $1/R$. The latter more distant region, in which the $1/R^2$

term may be neglected, is called the "wave-zone." In the neighborhood of the charge the field is given by

$$\boldsymbol{B}=\frac{1}{4\pi cR^2}\{[\dot{\boldsymbol{p}},\boldsymbol{R}_1]\},$$

(249)

$$\boldsymbol{E}=\frac{e}{4\pi R^2}(\boldsymbol{R}_1-\{\boldsymbol{r}_1\}).$$

The $\boldsymbol{B}$ vector has the value which, according to the Biot-Savart law (107), would arise from a steady-current element

$$Ids=\frac{\{\dot{\boldsymbol{p}}\}}{c}=-\frac{e}{c}\{\boldsymbol{u}\}.$$

This current $-\frac{e}{c}\{\boldsymbol{u}\}$ is the current due to the motion which the charge did have at time $t-R/c$: i.e., in the neighborhood of the vibrator, the Biot-Savart steady-state values of $\boldsymbol{B}$ are propagated outward with constant velocity c. The $\boldsymbol{E}$ vector in this region is clearly equal to the electrostatic $\boldsymbol{E}$ vector due to the instantaneous moment which the dipole possessed at the retarded time $t-R/c$ (it is clear that the identification of $\boldsymbol{R}_1$ and $\{\boldsymbol{r}_1\}$ would conceal this fact; this explains why the two vectors were not identified above). That is, the electrostatic values $\boldsymbol{E}$ characteristic of the instantaneous state of the dipole are propagated outward with constant velocity c.

In the wave-zone the field is given by

(250)

$$\begin{cases}\boldsymbol{E}=\dfrac{\{[\boldsymbol{R}_1[\boldsymbol{R}_1,\ddot{\boldsymbol{p}}]]\}}{4\pi c^2R},\\[2ex] \boldsymbol{B}=\dfrac{\{[\ddot{\boldsymbol{p}},\boldsymbol{R}_1]\}}{4\pi c^2R}.\end{cases}$$

From these values it follows that

$$\boldsymbol{E}=[\boldsymbol{B},\boldsymbol{R}_1],$$

so that, since $\boldsymbol{B}$ is normal to $\boldsymbol{R}_1$, the magnitudes of $\boldsymbol{B}$ and $\boldsymbol{E}$ are equal. This magnitude, moreover, depends only on the component $\ddot{\boldsymbol{p}}_1$ of $\ddot{\boldsymbol{p}}$ normal to $\boldsymbol{R}$, since $\ddot{\boldsymbol{p}}$ may be written as the sum of this normal component and a component $\ddot{\boldsymbol{p}}_{11}$ parallel to $\boldsymbol{R}$; and

$$[\ddot{\boldsymbol{p}}_{11}+\ddot{\boldsymbol{p}}_1,\boldsymbol{R}_1]=[\ddot{\boldsymbol{p}}_1,\boldsymbol{R}_1].$$

If θ be the angle between $\ddot{\boldsymbol{p}}$ and $\boldsymbol{R}$, then

$$E=B=\frac{\{\ddot{p}\}\sin\theta}{4\pi c^2 R}. \tag{251}$$

The three vectors $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{R}$ are mutually perpendicular and form a right system, the orientation of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the plane normal to $\boldsymbol{R}$ being fixed by the fact that $\boldsymbol{B}$ is normal to the projection of $\ddot{\boldsymbol{p}}$ on this plane. It should be noted that the portion of the total $\boldsymbol{E}$ and $\boldsymbol{B}$ fields which varies as $1/R$, that is to say, the field in the wave-zone, is due not to the velocity of the charge, but to its acceleration, and, in particular, to the component of this acceleration normal to the line drawn to the oscillator. The electromagnetic theory of light teaches that a plane-polarized beam of light moving in the x-direction is nothing other than the electromagnetic field

$$\begin{cases} B_z=E_y=a\cos n\left(t-\dfrac{x}{c}\right), \\ B_x=B_y=E_x=E_z=0. \end{cases} \tag{252}$$

In such a field $\boldsymbol{E}$, $\boldsymbol{B}$, and the direction of propagation form a mutually perpendicular right system, just as do $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{R}$ above. Indeed, it is this fact that has led to the use of the term "wave-zone" for the region in which the $1/R$ terms predominate. In the immediate neighborhood of a point in the wave-zone, the field is entirely similar in character to the plane-polarized wave (252). However, the amplitude a is not uniform for points on a sphere about the oscillator, but varies, according to (251), as the sine of the angle between $\boldsymbol{R}$ and the axis of the oscillator. The amplitude is thus a maximum on the great circle whose plane is normal to $\boldsymbol{p}$, and is zero at the points in which $\boldsymbol{p}$ (extended) cuts the sphere in question. Further, the amplitude falls off as the inverse first power of R.

The Poynting vector

$$\boldsymbol{S}=c[\boldsymbol{E},\boldsymbol{B}]$$

has the direction of $\boldsymbol{R}$, and a magnitude (since $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular)

$$S=\frac{\{\ddot{p}\}^2\sin^2\theta}{16\pi^2 c^3 R^2}.$$

This expression indicates a maximum energy flow in the direction normal to the axis of the oscillator. The total flux U of energy out through a sphere of radius R is, using the zonal area between θ and $\theta+d\theta$ as the element of area,

$$U=2\pi R^2\int_0^\pi S\sin\theta\,d\theta=\frac{\{\ddot{p}\}^2}{8\pi c^3}\int_0^\pi \sin^3\theta\,d\theta=\frac{\{\ddot{p}\}^2}{6\pi c^3},$$

this expression giving the instantaneous value of this energy flux at time t. If the negative charge of the oscillator vibrates with simple harmonic motion of frequency $\omega/2\pi$, so that

$$p=p_0\cos\omega t,$$

p_0 being the amplitude of the vibration, then the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are both simple harmonic functions of the time, and

$$\lambda=\frac{2\pi c}{\omega}$$

is the wave-length of the resulting disturbance. The wave-zone region may be characterized somewhat more definitely than previously in terms of this wave-length λ. In fact, going back to equation (247), the ratio of the $1/R$ term in $\boldsymbol{B}$ to the $1/R^2$ term is

$$\frac{\ddot{p}}{Rc^2}\cdot\frac{cR^2}{\dot{p}}=\frac{R}{c}\frac{\ddot{p}}{\dot{p}}=\frac{R\omega}{c}=2\pi\frac{R}{\lambda},$$

so that the wave-zone is the region for which R is large compared to λ. The same characterization results from a comparison of the terms of $\boldsymbol{E}$.

The time average of the energy flux out through a sphere of radius R may now be written at once. In fact,

$$\ddot{p}=-\omega^2 p,$$

whereas, since the variation with time is sinusoidal,

$$\overline{\ddot{p}^2}=\frac{\omega^4}{2}p_0^2,$$

so that

$$\overline{U}=\frac{\omega^4 p_0^2}{12\pi c^3}=\frac{4\pi^3 c p_0^2}{3\lambda^4}.$$

It should be noted that U and $\overline{U}$ are independent of the radius R of the sphere used in reckoning the energy flux, so that there is a flow of energy continuously away from the oscillator. If the average energy of the oscillator be W, then since it is the source of the energy flow just calculated,

$$-\frac{dW}{dt}=\overline{U}=\frac{4\pi^3 c p_0^2}{3\lambda^4}. \tag{253}$$

If the negative charge of the oscillator be considered to have a material mass m and if it oscillate about its equilibrium position under the influence of some elastic restraining force, then since, in such a motion, the average energy is half-kinetic and half-potential,

$$W=2\cdot\overline{\frac{1}{2}mv^2}=\frac{m}{e^2}\overline{p}^2=\frac{m}{e^2}\omega^2\overline{p}^2=\frac{m}{e^2}\frac{\omega^2}{2}p_0^2,$$

$$=\frac{2\pi^2 mc^2}{e^2\lambda^2}p_0^2. \tag{254}$$

Solving (253) and (254) for p_0^2 and equating,

$$\frac{dW}{dt}=-\frac{2e^2\pi}{3\lambda^2 cm}W,$$

or

$$W=W_0 e^{-kt},$$

where

$$k=-\frac{2e^2\pi}{3\lambda^2 cm}.$$

This equation must clearly be used only over intervals of time which include enough oscillations of the doublet to permit the use of the average value $\overline{U}$. This diminution in the energy of an oscillator due to its radiation of energy is known as "radiation damping," and the constant k, as given above, is the damping factor. If this damping factor be very small, the oscillations of the negative charge are, over short periods of time, sensibly simple harmonic. Neglecting the radiation damping, the equation of motion of the negative charge would be

$$\ddot{p}+h^2 p=0,$$

or

$$m\frac{dv}{dt}=\frac{m}{e}h^2p\,,$$

h^2 being the stiffness coefficient corresponding to the elastic restoring force. Including the back action of the charge's own field on itself, the equation would be of the form

$$m\frac{dv}{dt}=\frac{m}{e^2}h^2p+K\,,$$

where K is a dissipative force. By a comparison of the work done by this force and the energy radiated, it may be argued that

$$K=\frac{8\pi e^2}{3c^3}\frac{d^2v}{dt^2}=-\frac{8\pi e}{3c^2}\ddot{p}\,,$$

so that the complete equation of motion is a third-order differential equation. For a further discussion of this matter, a consideration of the magnitude of radiation damping from the viewpoint of interference phenomena, and a treatment of a moving positive charge about which a negative charge oscillates (a "moving light-source") see Abraham und Föppl, *Theorie der Elektrizität*, II (1918), 97. The theory of the Hertzian oscillator has been extensively used in connection with wireless telegraphy.

§ 62. *An Arbitrarily Moving Point Charge.*—The special examples of the effects of moving charge will be concluded with a brief consideration of the general case* (216), (217). The vector

$$\boldsymbol{R}=\left\{\boldsymbol{r}-\boldsymbol{v}\frac{r}{c}\right\}$$

is not, as in the case of uniform motion, the vector drawn to the observer from the position of the charge at time t, but is the vector drawn to the observer from the position the charge would occupy, at time t, if it moved in the interval from τ to t, with a constant velocity equal to the instantaneous velocity at time τ. The vector $\boldsymbol{E}$ has a component which is parallel to the retarded acceleration of the charge; and a component along the direction of $\boldsymbol{R}$. The vector $\boldsymbol{B}$ may be written, as in (253),

$$\boldsymbol{B}=\{[\boldsymbol{r}_1,\boldsymbol{E}]\}\,,$$

* Since (216) and (217) were obtained from (212) and (213), the equations are all subject to the restrictions mentioned, in § 57, in connection with the evaluation of the Jacobian.

and is, accordingly, normal to $\{r_1\}$ and E. In the wave-zone, where the terms depending on the inverse second power of the distance may be neglected,

$$E = -\left\{ \frac{e}{4\pi} \frac{\dot{u}}{c^2 H^2 r} + \frac{e(\dot{u},r)\left(r_1 - \frac{u}{c}\right)}{4\pi c^2 H^3 r^2} \right\} .$$

Now

$$(\{r\} \; R) = \{r\}^2 - \left\{ (u,r) \frac{r}{c} \right\} = \left\{ r^2 \left(1 - \frac{u_r}{c}\right) \right\} ,$$

so that

$$E = \left\{ \frac{e}{4\pi r^3 c^2 H^3} [-\dot{u}(r,R) + R(\dot{u},r)] \right\} ,$$

$$= \left\{ \frac{e}{4\pi c^2 r^3 H^3} [r[R,\dot{u}]] \right\} . \tag{255}$$

Thus the E vector is in the plane of R and $\{\dot{u}\}$, as stated above in general, it further appearing from this last equation that in the wave-zone the E vector is normal to $\{r\}$, the vector drawn to the observer from the position of the charge at the retarded time τ. In the wave-zone, then, since B is always normal to $\{r\}$ and E, the three vectors $\{r\}$, E, B form a mutually perpendicular right system, so that the flux of energy

$$S = c[E,B] ,$$
$$= cE^2\{r_1\} ,$$

is directed away from the retarded position of the charge.

Problems for Chapter IV

1. Prove the relation

$$\int \{(B, \text{ curl } E) - (E, \text{ curl } B)\} d\tau = \int [E,B]_n d\sigma ,$$

by considering the integral

$$\int \text{div } [E,B] \, d\tau .$$

2. Check the series expansions given (§ 60, equations [241] and [242]) for the longitudinal and transverse electromagnetic masses of the Lorentz and Abraham electrons.

3. What is the electromagnetic rest-mass, in grams, of an electron? Assume the "radius" of the electron to be 1.4×10^{-13} cm.
4. What is the ratio of the longitudinal to the transverse electromagnetic mass of a Lorentz electron moving with one-quarter the velocity of light? With one-half the velocity of light?
5. Referring to chapter ii, Part III, Problems 1, 2, and 3, show that

$$\rho = \sqrt{4\pi}\,\rho_{e.s.u.},$$

$$\boldsymbol{E} = \frac{1}{\sqrt{4\pi}}\,\boldsymbol{E}_{e.s.u.} = \frac{1}{300\sqrt{4\pi}}\,\boldsymbol{E}_{\text{volts per cm.}},$$

$$\boldsymbol{P} = \sqrt{4\pi}\,\boldsymbol{P}_{e.s.u.}.$$

6. The $\boldsymbol{B}$ vector due to a volume distribution of currents is

$$\boldsymbol{B} = \frac{1}{4\pi}\int\left[\boldsymbol{i},\nabla\frac{1}{r}\right]d\tau$$

(see equation [126]), where $\boldsymbol{i}$ is measured in rational electromagnetic units. Also, in c.g.s. electromagnetic units,

$$\boldsymbol{B}_{e.m.u.} = \int\left[\boldsymbol{i}_{e\ m.u.},\nabla\frac{1}{r}\right]d\tau.$$

Show that

$$\boldsymbol{i} = \sqrt{4\pi}\,\boldsymbol{i}_{e.m.u.},$$

$$\boldsymbol{B} = \frac{1}{\sqrt{4\pi}}\,\boldsymbol{B}_{e.m.u.},$$

$$\boldsymbol{A} = \frac{1}{\sqrt{4\pi}}\,\boldsymbol{A}_{e.m.u.}.$$

7. In rational electromagnetic units, the portion of the vector potential $\boldsymbol{A}$ which is due to a volume magnetization is given by (see equation [117])

$$\boldsymbol{A} = \frac{1}{4\pi}\int\left[\boldsymbol{M},\nabla\frac{1}{r}\right]d\tau,$$

while in c.g.s. electromagnetic units

$$A_{e.m.u.} = \int \left[M_{e.m.u.}, \nabla \frac{1}{r} \right] d\tau \, .$$

Show that

$$M = \sqrt{4\pi} M_{e.m.u.} \, ,$$

and hence that the numerical measure of the magnetic permeability μ is the same in rational and c.g.s. electromagnetic units.

8. The vector $E+P=\epsilon E$ was called, by Maxwell, the "electric induction" and was denoted by the letter D. Similarly, the vector $B-M=B/\mu$ was given the symbol H, the name of the H vector being different for different writers. Show that the field equations at points in matter are, in terms of these vectors,

$$\text{curl } H = i + \frac{\dot{D}}{c} \, ,$$

$$\text{curl } E = -\frac{1}{c} \dot{B} \, ,$$

$$\text{div } D = \rho \, ,$$

$$\text{div } B = 0 \, .$$

9. The field equations in the form given by equations (167)–(170) make use of more than one system of units. The quantities ρ and E are measured in rational electrostatic units; the quantities B and i are measured in rational electromagnetic units. Use the relations developed in the problems above to show that

$$\text{curl } B_{e.m.u.} = 4\pi i_{e.m.u.} + \frac{\dot{E}_{e.s.u.}}{c} \, ,$$

$$\text{curl } E_{e.s.u.} = -\frac{1}{c} \dot{B}_{e.m.u.} \, ,$$

$$\text{div } E_{e.s.u.} = 4\pi \rho_{e.s.u.} \, ,$$

$$\text{div } B_{e.m.u.} = 0 \, .$$

Note that the change from the field equations in rational units to the field equations in c.g.s. electrostatic and electromagnetic units

involves, besides the mere change of units, changes in the definitions of potential and a resulting change in the relation between polarization and intensity. That is,

$$\Phi = \frac{1}{4\pi}\int \frac{\rho\, d\tau}{r} \qquad \Phi_{e.s.u} = \int \frac{\rho_{e.s.u}\, d\tau}{r},$$

$$\boldsymbol{P} = (\epsilon - 1)\boldsymbol{E} \qquad \boldsymbol{P}_{e.s.u.} = \frac{(\epsilon-1)}{4\pi}\boldsymbol{E}_{e.s.u.}.$$

10. Show that, at points in matter,

$$\text{curl } \boldsymbol{B}_{e.m.u.} = 4\pi\mu \boldsymbol{i}_{e.m.u.} + \frac{\mu\epsilon \dot{\boldsymbol{E}}_{e.s.u.}}{c},$$

$$\text{curl } \boldsymbol{E}_{e.s.u.} = -\frac{1}{c}\dot{\boldsymbol{B}}_{e.m.u.},$$

$$\text{div } \epsilon \boldsymbol{E}_{e.s.u.} = 4\pi\rho,$$

$$\text{div } \boldsymbol{B}_{e.m.u.} = 0.$$

11. Some authors (see, e.g., J. R. Carson, *Philosophical Magazine*, XLI [1921], 607) have found it convenient to use an exclusively electromagnetic system of units. One may readily check that using c.g.s. electrostatic and electromagnetic units

$$e_{e.s.u} = c\, e_{e.m.u},$$

$$\boldsymbol{E}_{e.s.u.} = \frac{1}{c}\boldsymbol{E}_{e.m.u.},$$

$$\boldsymbol{P}_{e.s.u.} = c\boldsymbol{P}_{e.m.u}.$$

The second of these relations, indeed, follows from the first by virtue of the fact that the product of charge by potential must be independent of the units chosen for charge. Polarization, moreover, must transform in the same way as charge, since it is defined as the product of charge by distance. One must then write, for the relation between polarization and intensity,

$$\boldsymbol{P}_{e.m.u.} = \frac{1}{4\pi}\left(\epsilon_{e.m.u.} - \frac{1}{c^2}\right)\boldsymbol{E}_{e.m.u.}.$$

In fact, this relation, when one defines

$$\epsilon_{e.m.u.} \equiv \frac{\epsilon_{c.s.u.}}{c^2},$$

reduces to the proper relation

$$\boldsymbol{P}_{e.s.u.} = \frac{(\epsilon_{e.s.u.} - 1)}{4\pi} \boldsymbol{E}_{e.s.u.}.$$

Thus the numerical measure of the dielectric constant (which is the same in rational electrostatic and c.g.s electrostatic units) is, in c.g.s. electromagnetic units, equal to $1/c^2$ times its numerical measure in c.g.s. electrostatic units.

Using these relations one readily finds

$$\begin{aligned} \text{curl } \boldsymbol{B} &= 4\pi \boldsymbol{i} + \epsilon \dot{\boldsymbol{E}}, \\ \text{curl } \boldsymbol{E} &= -\dot{\boldsymbol{B}}, \\ \text{div } \epsilon \boldsymbol{E} &= 4\pi\rho, \\ \text{div } \boldsymbol{B} &= 0, \end{aligned}$$

where all quantities are measured in c.g.s. electromagnetic units, or

$$\begin{aligned} \text{curl } \boldsymbol{B} &= \boldsymbol{i} + \epsilon \dot{\boldsymbol{E}}, \\ \text{curl } \boldsymbol{E} &= -\dot{\boldsymbol{B}}, \\ \text{div } \epsilon \boldsymbol{E} &= \rho, \\ \text{div } \boldsymbol{B} &= 0, \end{aligned}$$

where all quantities are in rational electromagnetic units. The last four equations are entirely free from scale factors, and are therefore (especially for theoretical work) in particularly convenient form.

12. Show that

$$\Psi = \frac{e(t - r/c)}{r}$$

is a solution of

$$\nabla^2 \Psi - \frac{1}{c^2} \ddot{\Psi} = 0,$$

and hence obtain the retarded potential solution of the equation

$$\nabla^2\Phi - \frac{1}{c^2}\ddot{\Phi} = -\rho\ .$$

See Jeans, *Electricity and Magnetism* (1911), § 645.

13. Show by differentiation of the expressions

$$\Phi = \frac{1}{4\pi}\int \frac{\{\rho\}}{r}\, d\tau\ ,$$

$$\boldsymbol{A} = \frac{1}{4\pi c}\int \frac{\{\rho\boldsymbol{u}\}}{r}\, d\tau\ ,$$

that the equation

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c}\dot{\Phi} = 0$$

is satisfied.

Conclusion to Chapter IV

In the previous chapters the authors have aimed to present in the actual text as straightforward as possible a presentation of classical electrodynamical theory, relegating to the chapter conclusions critical remarks on controversial matters. In this last chapter it is scarcely possible to follow this plan. Many of the topics treated are controversial from the very start, and it has been necessary to include in the text a considerable amount of critical matter that would otherwise be found in this conclusion. The discussions already given, however, by no means cover the desirable ground, and this chapter will conclude with a rather disjointed collection of further observations.

It is worth while, first of all, to reconsider briefly the way in which the Maxwell field equations have been obtained,* for such is the fundamental rôle they play that any insight into their significance cannot fail to be useful and important. The five equations have been obtained as generalizations of the laws of electrostatics and magnetostatics, which laws rest primarily on the experiments of Coulomb and Ampère. The generalization has three significant features. The first of these is concerned with the replacement of the old equation

$$\text{curl } \boldsymbol{E}=0$$

by the more general relation

$$\text{curl } \boldsymbol{E}=-\frac{\dot{\boldsymbol{B}}}{c}.$$

The second replaces the old equation

$$\text{curl } \boldsymbol{B}=\boldsymbol{i}$$

by the equation

$$\text{curl } \boldsymbol{B}=\boldsymbol{i}+\frac{\dot{\boldsymbol{E}}}{c}.$$

* The reader who is interested in various other ways of obtaining the field equations should consult *An Introduction to Electrodynamics*, by L. Page (1922), and "New Deductions of the Electromagnetic Equations," by W. F. G. Swann, in *Physical Review*, XXVIII (1926), 531.

The third generalization is one of significance rather than form. It consists of the assumption that the expression

$$F = E + \frac{1}{c}[v,B],$$

which gives the force per unit charge in magnetostatics, continues to be valid for the generalized field vectors.

It has been seen that the first of these generalizations, which furnishes the Faraday induction law, is in a restricted sense but a rephrasing of the old Ampère law of action between currents. Having indicated the nature of the generalization by means of a special experiment involving moving circuits, it is assumed that the new expression holds in all cases. This mode of obtaining the Faraday equation is one that sharply focuses attention on that mysterious aether whose properties the E and B vectors are supposed to measure. It has, in fact, been pointed out that one who believes in the aether finds it comparatively easy to accept, as a universal truth, the equation

$$\text{curl } E' = -\frac{\dot{B}}{c}$$

once he has shown this equation true in a particular case.

A decreasing number of physicists, however, finds the hypothesis of an aether tenable or desirable. It is beyond the purpose of this volume to consider the way in which the "aether drift" experiments and the theory of relativity have removed the last excuses for continuing to assume such a substance. It has not seemed possible or desirable to include in this volume any treatment of relativity, although the authors realize that many will regard this an inexcusable omission. There is current a somewhat vague impression that relativity rests, to a considerable extent, upon electrodynamics; while electrodynamics rests upon relativity. (Neither dependence has, to be sure, been completely analyzed.) While no one would doubt the close interlocking of these two subjects, there is, nevertheless, some danger of assuming that each "proves" the other. The authors feel that a student must know a certain basic amount of electrodynamics before he can intelligently study relativity, and they have attempted to set down that basic amount of electrodynamics without invoking any but the most elementary relativity principle. This, to be sure, leaves the development of the theory somewhat rougher (super-

ficially rougher) in spots than it would otherwise be. The term $(\boldsymbol{u},\boldsymbol{v})/c^2$ neglected, for example, in passing from the electrostatic relation

$$\text{div } \boldsymbol{E} = \rho$$

to the general relation

$$\text{div } (\boldsymbol{E}+\boldsymbol{E}') = \rho$$

is of precisely the order with which relativity deals.

The aether physicists seem at last to be willing,* in their final philosophical paragraphs, to give up the aether; but they continue to use it throughout the body of their texts, compelling its unsubstantial texture to bear the brunt of many an argument. This is by no means the only instance in electrical theory of the essential use of a concept finally discredited and discarded. The theory frequently has been developed around certain properties or concepts, only to have the culmination of the theory be the proof that the property is non-existent or that the concept is untenable. Examples of this are frequent and familiar: "the state of the medium" (when there is no medium), "the force on a magnetic pole" (when there is no such thing as a magnetic pole), "the fictitious density $-\text{div } \boldsymbol{P}$" (which is zero), "the force on a charge" (which always vanishes). The present authors have by no means avoided all such difficulties. They wish to point them out and thus draw attention to the rather obvious warning that it is deceivingly easy to make correct statements about free space or about non-existent properties. Even the great physicist Lorentz has said: "The formulae for the aether constitute the part of electromagnetic theory that is most firmly established. Though perhaps the way in which they are deduced will be changed in future years, it is hardly conceivable that the equations themselves will be altered. It is only when we come to consider phenomena in ponderable bodies that we are led into uncertainties and doubts." The inference is plain:

Mother, may I go out to swim?
Yes, my darling daughter.
Hang your clothes on a hickory limb,
But don't go near the water.

The activity equation and the so-called "spatial density" of energy have been discussed at considerable length in the text itself. The aether

* See, e.g., J. H. Jeans, *op. cit.* (1925), p. 618.

theory was doubtless responsible for the idea of spatially distributed energy. The aether has unobtrusively slipped away; and those who, if there were an aether, would choose to think of the energy of a configuration of charges as being a strain energy in this medium must, it would seem, now abandon this view. The authors, however, have an attitude toward spatially distributed density which is quite independent of the existence or non-existence of an aether.

Some optimistic pragmatist may urge, to counter all these objections to spatially distributed energy, that the idea has always "worked" and so deserves a place in our philosophy. The answer is that the idea has not always worked. In one of the most conspicuous fields of modern physical research, one comes, if he believes that energy can be located spatially, face to face with a baffling paradox. How large is a quantum? It is as "big," as Eddington points out, as the mirror of the 100-inch telescope at Mount Wilson, and it is as "small" as the atom.* It reminds one (and quite properly so) of the frothy, infinitely rigid, perfectly tenuous, exceedingly dense, and quite impalpable aether. We should, by this time, have learned that all statements are true if they are made about nothing.

The authors have already indicated their belief that the conclusion, "the total force on an elementary charge is zero," is not so much a statement about the vanishing of a quantity as it is the admission of the collapse of a concept. There seem to be, fundamentally, but two sorts of "forces" in the universe—electrical "forces" and the "force" of gravitation. The general theory of relativity has completely changed our description of gravitational effects. A body A no longer tends to go in a straight line, and is corrupted away from that tendency by the gravitational force due to a second body B. Body A simply and always travels on a geodesic in a four-dimensional space-time manifold. If no other body B is present, the space aspect of that geodesic is a straight line. If another body B is present, its presence affects the metric properties of the space, and the space aspect of the geodesic is a curved line. Thus

* Eddington's explanation (*op. cit.* [1928], chap. x) is that we must not think of space and time at all in connection with an individual quantum. The present authors have vainly speculated, for roughly ten years, concerning a possible law for the unit electrical "action-response" that takes place between two electrical elements, this action-response correlation to be arithmetic in nature, and quite free of any notion of either time or space. Macroscopic time, geometry, and mechanics should then emerge statistically.

The quantum theory has evaded most of the difficulties involved in a spatial density of energy by substituting, in its place, a spatial density for the probability of energy.

the old gravitational force is geometrized away. A body is not acted on by a force. It goes where it will. Its mass imposes on the space-time manifold such a metric as leads us to locate its successive positions in accordance with what we would hitherto have called a law for its motion under a force. A similar geometrization of electrical forces has been widely attempted, but without complete success. The concept of a macroscopic force has been of untold value to science; and macroscopic electrical forces will doubtless, with entire success, be geometrized under the relativity scheme. The authors believe that microscopic "forces" do not exist. If all macroscopic forces are geometrized away, we shall not feel so bad over the lack of microscopic forces.

The collapse of microscopic dynamics which, the authors believe, was so clearly foreshadowed by the paradoxical character of the "dynamics" of the electron—this collapse is now definitely recognized. The "wave-mechanics" of the newer quantum theory constitutes the first attempt to construct a microscopic dynamical theory for individual actions. The early success of this wave-mechanics should be encouragement enough to convince us that the general tendency of this theory is a correct and worthy one. The limitations of the present wave-mechanics arise, doubtless, from the fact that it is still too mechanical in nature. For this reason, the earlier matrix theory, and the more abstract formulation of wave-mechanics at the hands of Dirac, seem to be the more promising.

There remains, finally, one relatively trivial matter which should be mentioned. Except for the linkage with more ordinary notations which occurs in the problems, use has been made of only two field vectors, $\boldsymbol{E}$ and $\boldsymbol{B}$. When in free space, the use of one electric and one magnetic vector, rather than of the four vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$, is an obviously desirable simplification; and within matter, it is useful to have in explicit evidence the electrical properties ϵ and μ of the matter. The choice of $\boldsymbol{B}$ as the fundamental magnetic vector, rather than $\boldsymbol{H}$, rests on the occurrence of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the equation for force on a charge. The subject of magnetostatics has been developed in as close analogy as possible with electrostatics; and the fundamental magnetic vector—the counterpart of $\boldsymbol{E}$—must clearly be the vector which, in the basic law for magnetostatic action, plays the same rôle as does $\boldsymbol{E}$ in electrostatics. The choice of $\boldsymbol{B}$ rather than $\boldsymbol{H}$ is also clearly indicated by the fact that the divergence of $\boldsymbol{E}$ gives the total charge, while the curl of $\boldsymbol{B}$ (not of $\boldsymbol{H}$) gives the total current. The confusion which results from the choice of $\boldsymbol{H}$ as the fundamental magnetic vector is, perhaps, most clearly illustrated by the equations which arise when one considers the relation be-

tween the so-called microscopic and macroscopic field equations. Lorentz, for example, takes as microcopic equations, valid everywhere,

$$\operatorname{curl} \boldsymbol{e} = -\frac{1}{c}\frac{\partial \boldsymbol{h}}{\partial t}, \qquad \operatorname{curl} \boldsymbol{h} = \frac{1}{c}(\rho \boldsymbol{v} + \dot{\boldsymbol{e}})$$

$$\operatorname{div} \boldsymbol{e} = \rho, \qquad \operatorname{div} \boldsymbol{h} = 0,$$

and finds that the average values, over physically small-volume elements, of $\boldsymbol{e}$ and $\boldsymbol{h}$ are given by

$$\bar{\boldsymbol{e}} = \boldsymbol{E}, \qquad \bar{\boldsymbol{h}} = \boldsymbol{B},$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the ordinary macroscopic field vectors used in this volume. The last equation indicates that $\boldsymbol{B}$ is the fundamental macroscopic vector, and that the fundamental microscopic vector should be designated by $\boldsymbol{b}$ rather than $\boldsymbol{h}$.

MATHEMATICAL APPENDIX

OUTLINE

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330 § 1. *Definition* of vector and scalar. Components of a vector. Magnitude and direction cosines. Angle between two vectors. Component of a vector in any direction. Addition. Representation by means of fundamental unit vectors.

332 § 2. *Multiplication* of vectors.

A. Scalar product: $(\boldsymbol{AB}) \equiv A_x B_x + A_y B_y + A_z B_z$.

B. Vector product: $[\boldsymbol{AB}]_x \equiv A_y B_z - A_z B_y$;

$$[\boldsymbol{AB}] \equiv \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} .$$

334 § 3. *Differential operators.*

334 A. The operator ∇. Directional derivative of a scalar. $(\boldsymbol{n}, \nabla\Phi) \equiv \partial\Phi/\partial n$.

336 B. $(\nabla, \boldsymbol{A}) \equiv \text{div } \boldsymbol{A}$. Interpretation as strength of a source.

338 C. $[\nabla, \boldsymbol{A}] \equiv \text{curl } \boldsymbol{A}$. Kinematic interpretation.

341 D. Directional derivative of a vector $(\boldsymbol{n}, \nabla)\boldsymbol{B}$; $(\boldsymbol{A}, \nabla)\boldsymbol{B}$.

342 E. Line integrals. If $\int A_s ds = 0$, then $\boldsymbol{A} \equiv \nabla\Phi$.

345 § 4. *Relations* between scalar and vector products, divergence, and curl.

$(\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]) = (\boldsymbol{B}[\boldsymbol{A},\boldsymbol{C}]) = (\boldsymbol{C}[\boldsymbol{A},\boldsymbol{B}])$,

$[\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]] = \boldsymbol{B}(\boldsymbol{A},\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A},\boldsymbol{B})$,

$\text{div } \Phi\boldsymbol{A} = \Phi \text{ div } \boldsymbol{A} + (\boldsymbol{A}, \nabla\Phi)$,

$\text{curl } \Phi\boldsymbol{A} = \Phi \text{ curl } \boldsymbol{A} + [\nabla\Phi, \boldsymbol{A}]$,

$\text{div curl } \boldsymbol{A} = 0$,

$\text{curl } \nabla\Phi = 0$,

$\text{div } \nabla\Phi = \nabla^2\Phi$,

$\text{curl curl } \boldsymbol{A} = \nabla \text{ div } \boldsymbol{A} - \nabla^2\boldsymbol{A}$,

$\text{div } [\boldsymbol{A},\boldsymbol{B}] = (\boldsymbol{B}, \text{ curl } \boldsymbol{A}) - (\boldsymbol{A}, \text{ curl } \boldsymbol{B})$,

$\text{curl } [\boldsymbol{A},\boldsymbol{B}] = \boldsymbol{A} \text{ div } \boldsymbol{B} - \boldsymbol{B} \text{ div } \boldsymbol{A} + (\boldsymbol{B}, \nabla)\boldsymbol{A} - (\boldsymbol{A}, \nabla)\boldsymbol{B}$,

$\nabla(\boldsymbol{A},\boldsymbol{B}) = (\boldsymbol{A}, \nabla)\boldsymbol{B} + (\boldsymbol{B}, \nabla)\boldsymbol{A} + [\boldsymbol{A}, \text{ curl } \boldsymbol{B}] + [\boldsymbol{B}, \text{ curl } \boldsymbol{A}]$.

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347 § 5. *Integral transformations.*
347 A. Integration by parts.
348 B. Green's theorem.
349 C. Stokes's theorem.
351 D. $\int \text{curl } \boldsymbol{B} d\tau = \int [\boldsymbol{n}, \boldsymbol{B}] d\sigma$,

$$\int u \, \boldsymbol{ds} = \int [\boldsymbol{n}, \nabla u] d\sigma ,$$

$$\int \left[[\boldsymbol{ds}, \nabla \frac{1}{r} \right] = -\nabla \int \frac{\partial \frac{1}{r}}{\partial n} d\sigma .$$

352 § 6. *Vector fields.*
352 A. Solution of the equation $\nabla^2 u = -\rho$.
355 B. If curl $\boldsymbol{A} = 0$, then $\boldsymbol{A} = \nabla \Phi$. If div $\boldsymbol{A} = 0$, then $\boldsymbol{A} = \text{curl } \boldsymbol{B}$.
357 C. A vector field is uniquely determined by the specification of divergence and curl.
359 D. Solution when div $\boldsymbol{A} = \rho$, curl $\boldsymbol{A} = 0$. Discontinuities.
363 E. Solution when curl $\boldsymbol{A} = \boldsymbol{i}$, div $\boldsymbol{A} = 0$.
364 F. General solution; an arbitrary vector is the sum of a potential and a solenoidal vector.
365 § 7. *Curvilinear co-ordinates.*

§ 1. *Definition of Vector: Components: Addition.*—Of the measurable quantities with which one deals in physics many are completely characterized by a magnitude alone; that is to say, they are completely characterized by a single number (to which must be assigned appropriate dimensions). Temperature, volume density, mass, potential, etc., are such quantities. *Such magnitudes are known as scalar magnitudes, or simply as scalars.* There is an important class of quantities, however, which possess direction as well as magnitude, and which require, for their complete characterization, the specification of three quantities. These three quantities may be the three projections of this directed magnitude on the directions of the three axes of an orthogonal system of co-ordinates; or they may consist of two direction cosines and a third number measuring magnitude alone; or the direction and magnitude may be specified in any other way by means of three numbers. *Such quantities, which require for their characterization both a magnitude and a direction, are known as vector magnitudes, or as vectors.* Velocity, force, acceleration, for example, are vector quantities.

To distinguish between vector and scalar quantities, vectors will be

denoted by letters in bold-face type. The absolute magnitude of a vector, that is to say, its scalar magnitude regardless of its direction, will be denoted by $|\boldsymbol{A}|$ or by A, the corresponding letter in ordinary type. Thus the absolute magnitude of the vector $\boldsymbol{A}$ is $|\boldsymbol{A}| \equiv A$.

Since a vector quantity possesses both length and direction, it may be graphically represented by means of a directed line segment whose length is a measure of the absolute magnitude of the vector, and whose direction is parallel to the direction of the vector quantity.

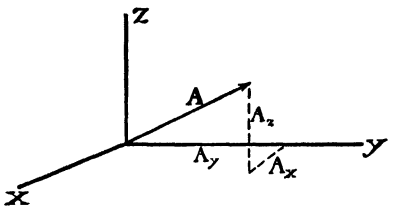

FIG. 1.—A vector $\boldsymbol{A}$ and its rectangular components.

Let the scalar projections of the vector $\boldsymbol{A}$ upon the directions of the three co-ordinate axes be A_x, A_y, A_z. Let the angles which $\boldsymbol{A}$ makes with the x-, y-, z-axes be denoted by (Ax), (Ay), (Az), respectively. The length of the vector is then obviously given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,, \tag{1}$$

while the direction cosines of $\boldsymbol{A}$ are

$$\cos(Ax) = \frac{A_x}{A}\,, \qquad \cos(Ay) = \frac{A_y}{A}\,, \qquad \cos(Az) = \frac{A_z}{A}\,; \tag{2}$$

and satisfy the relation

$$\cos^2(Ax) + \cos^2(Ay) + \cos^2(Az) = 1\,, \tag{3}$$

as is seen at once from (1). Also, from (1) and (2),

$$A = A_x \cos(Ax) + A_y \cos(Ay) + A_z \cos(Az)\,. \tag{4}$$

For the component A_s of a vector $\boldsymbol{A}$ along any direction $\boldsymbol{s}$, one has

$$A_s = A_x \cos(sx) + A_y \cos(sy) + A_z \cos(sz)\,, \tag{5}$$

since the sum of the projections, on any direction, of the sides of an unclosed polygon is obviously equal to the projection, on this same direction, of the closing side of the polygon. But since

$$A_s = A \cos(As)\,, \tag{6}$$

it follows from (2) that the cosine of the angle between two directions is

$$\cos(As) = \cos(Ax)\cos(sx) + \cos(Ay)\cos(sy) + \cos(Az)\cos(sz)\,. \tag{7}$$

By the sum of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is meant the geometrical sum, i.e., the initial point of $\boldsymbol{B}$ is placed in coincidence with the end point of $\boldsymbol{A}$; then the sum $\boldsymbol{C}$ is defined as the vector whose initial point is the initial point of $\boldsymbol{A}$, and whose end point is the end point of $\boldsymbol{B}$.

Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be vectors of unit length (usually called "unit vectors") in the direction of the three axes x, y, and z, respectively. Then by $A_x\boldsymbol{i}$ is meant a vector whose direction is the direction of $\boldsymbol{i}$, and whose magnitude is the product of the magnitude of $\boldsymbol{i}$ and the scalar A_x. Thus a scalar h multiplying a vector stretches the length of the vector in the ratio h to unity, and does not affect the vector's direction. $A_x\boldsymbol{i}$ is thus a vector pointing in the positive x-direction, and of magnitude A_x. It then follows from the definition of addition given above that

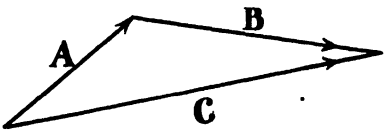

FIG. 2.—Vector addition

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}\,, \tag{8}$$

so that a vector may be represented by means of its scalar components along three orthogonal directions, and unit vectors along these same directions.

§ 2. *Multiplication of Vectors.*—

A. Two sorts of vector multiplication are in common use, scalar and vector. By the scalar product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is meant a scalar quantity equal to the product of the magnitude of one vector and the scalar projection of the second upon the first. The scalar product of $\boldsymbol{A}$ and $\boldsymbol{B}$ will be denoted by $(\boldsymbol{A},\boldsymbol{B})$. Thus

$$(\boldsymbol{A},\boldsymbol{B}) = |\boldsymbol{A}| \cdot |\boldsymbol{B}| \cdot \cos(A,B)\,, \tag{9}$$

or, from (2) and (7),

$$(\boldsymbol{A},\boldsymbol{B}) = A_xB_x + A_yB_y + A_zB_z\,. \tag{10}$$

From the definition it follows at once that

$$(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{C}) = (\boldsymbol{A},\boldsymbol{B}) + (\boldsymbol{A},\boldsymbol{C})\,, \tag{11}$$

and

$$(A,B) = (B,A) \,. \tag{12}$$

By definition, also,

$$A^2 \equiv (A,A) = A^2 \,.$$

B. By the vector product of two vectors A and B is meant a third vector C which is normal to the plane of A and B, so directed that A,B,C form a right-handed system*, and of absolute magnitude equal to the product of the magnitudes of A and B by the sine of the angle between them, i.e., equal to the area of the parallelogram formed on A

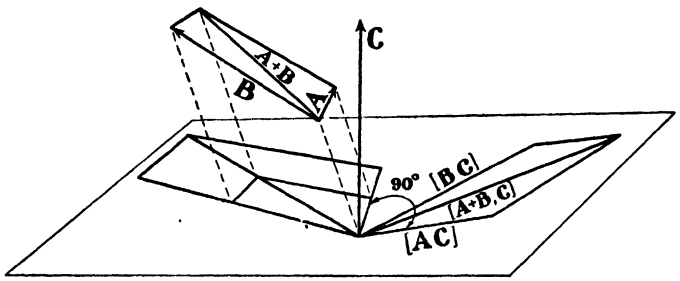

FIG. 3.—The distributive law for the vector product of the sum of two vectors and a third vector.

and B. The vector product of A and B will be denoted by $[A,B]$. Thus if

$$[A,B] = C \,,$$

then

$$|C| = |A| \cdot |B| \cdot \sin (A,B) \,. \tag{13}$$

A distributive law for vector multiplication can be proved, similar to (11) for scalar multiplication. Thus let it be required to prove that

$$[A+B,C] = [A,C] + [B,C] \,. \tag{14}$$

Project A, B, and $A+B$ on a plane perpendicular to C. The three projected vectors are of absolute magnitude $A \sin (A,C)$, $B \sin (B,C)$, and

* A right-hand screw directed along C advances in the positive direction of C by a rotation from A toward B through the smaller angle.

$(A+B)$ sin $(A+B,C)$. Now enlarge in the ratio of C to unity, and rot the three vectors thus obtained 90° in their own plane. The three p jected, enlarged, and rotated vectors then represent $[A,C]$, $[B,C]$, a $[A+B,C]$, and the foregoing identity becomes obvious. Then, since

$$[i,j]=k, \qquad [j,k]=i, \qquad [k,i]=j, \tag{15}$$
$$[i,i]=[j,j]=[k,k]=0 ,$$

it follows that

$$[A,B]=[iA_x+jA_y+kA_z,\ iB_x+jB_y+kB_z] , \tag{16}$$
$$=i(A_yB_z-A_zB_y)+j(A_zB_x-A_xB_z)+k(A_xB_y-A_yB_x) .$$

This result may be expressed in determinant form, viz.,

$$[A,B]=\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} . \tag{17}$$

It should be noted that $(A,B)=(B,A)$ while $[A,B]=-[B,A]$.

§ 3. *Differential Operators.*—

A. Let u be a scalar-point function, i.e., let u be a function of the three variables x, y, z which assigns a scalar to every point x, y, z of space. Consider a line through the point x, y, z and let s be distance measured along this line. Then the values of u for the points of the line form a function of s. The derivative of this function, with respect to s, measures the rate of change of u along the line. The value of this derivative at the point x, y, z is thus the rate of change of the function u in the direction s at the point x, y, z. This directional derivative is given by

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}+\frac{\partial u}{\partial z}\frac{dz}{ds} ,$$

$$=\frac{\partial u}{\partial x}\cos(sx)+\frac{\partial u}{\partial y}\cos(sy)+\frac{\partial u}{\partial z}\cos(sz) .$$

From a comparison of this equation with equation (5), it appears that a vector with x-, y-, and z-components

$$\frac{\partial u}{\partial x} , \frac{\partial u}{\partial y} , \frac{\partial u}{\partial z}$$

would have a component in any direction s equal to the rate of change of u in that direction. Then since its maximum component is in its own direction, the vector itself would measure the rate of change of u in the direction of greatest increase, and it would point in this direction. Such a vector is called ∇u (read "nabla u"). Thus.

$$\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}. \tag{18}$$

Since the component of ∇u in any direction s is equal to the rate of change of u in that direction,

$$(\nabla u)_s = (s, \nabla u) = \frac{du}{ds}, \tag{19}$$

where s is a unit vector in the direction s.

The differential operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

operating on a scalar-point function thus produces a vector-point function which gives the rate of change and direction of greatest rate of change of the scalar. It is a fundamentally important operator.

When the function upon which ∇ operates is a function of more than one set of variables, the set with respect to which the differentiation involved in ∇ is to be taken can be indicated by means of a subscript. Thus, if r be the distance from a point O of co-ordinates x_0, y_0, z_0 to a point P of co-ordinates x_p, y_p, z_p,

$$r^2 = (x_p - x_0)^2 + (y_p - y_0)^2 + (z_p - z_0)^2 .$$

Then

$$\nabla_0 r = i\frac{\partial r}{\partial x_0} + j\frac{\partial r}{\partial y_0} + k\frac{\partial r}{\partial z_0} \tag{20}$$

while

$$\nabla_p r = i\frac{\partial r}{\partial x_p} + j\frac{\partial r}{\partial y_p} + k\frac{\partial r}{\partial z_p}, \tag{21}$$

and obviously

$$\nabla_0 f(r) = -\nabla_p f(r) . \tag{22}$$

In the discussion just given, the directional derivative has been written as a total derivative of u with respect to s. Many books, however, write

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}+\frac{\partial u}{\partial z}\frac{dz}{ds}$$

for the directional derivative. There are arguments for and against both notations. For example, when the direction of s coincides with that of x, the total derivative notation produces the rather unfortunate formula

$$\frac{du}{dx}=\frac{\partial u}{\partial x}.$$

On the other hand, the partial derivative notation describes the total change in the function u resulting from a displacement ds by the equally unfortunate equation

$$du=\frac{\partial u}{\partial s}\,ds\,.$$

In this book both notations will be used. In certain calculations, such as that given in § 2 of chapter i the total derivative notation is clearly preferable. On the other hand, when one writes the rate of change of a function u along a direction n normal to a given surface, it seems preferable to write this as $\partial u/\partial n$. In fact, when one speaks of the normal to a surface, it is very natural to think also of the tangent plane and an orthogonal trihedral of axes located at the point in question, one directed along the normal and the other two lying in the tangent plane. Having all these axes in mind, one writes $\partial u/\partial n$ rather than du/dn for the same reasons one writes $\partial u/\partial x$ rather than du/dx.

B. The differential operator ∇ can be considered formally as a vector of components $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, so that its scalar and vector products with another vector may be taken. For example,

$$(\nabla,\boldsymbol{A})=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\,. \tag{23}$$

This latter quantity is denoted by div $\boldsymbol{A}$ (read "divergence $\boldsymbol{A}$"). Thus

$$\text{div }\boldsymbol{A}=(\nabla,\boldsymbol{A})=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\,. \tag{24}$$

A physical illustration will indicate the significance of this quantity. Suppose that V is the vector velocity at any point of an incompressible fluid. Consider a parallelopiped with center at the origin, and of small sides $2\delta x$ $2\delta y$, $2\delta z$ parallel to the co-ordinate axes. Denote by $(V_x)_0$ the value of the x-component of velocity at the origin. Then at a neighboring point x, y, z,

$$(25)\qquad \begin{cases} V_x=(V_x)_0+\left(\dfrac{\partial V_x}{\partial x}\right)_0\cdot x+\left(\dfrac{\partial V_x}{\partial y}\right)_0\cdot y+\left(\dfrac{\partial V_x}{\partial z}\right)_0\cdot z\,, \\ V_y=(V_y)_0+\left(\dfrac{\partial V_y}{\partial x}\right)_0\cdot x+\left(\dfrac{\partial V_y}{\partial y}\right)_0\cdot y+\left(\dfrac{\partial V_y}{\partial z}\right)_0\cdot z\,, \\ V_z=(V_z)_0+\left(\dfrac{\partial V_z}{\partial x}\right)_0\cdot x+\left(\dfrac{\partial V_z}{\partial y}\right)_0\cdot y+\left(\dfrac{\partial V_z}{\partial z}\right)_0\cdot z\,, \end{cases}$$

the Maclaurin series expansions being broken off with the linear terms since they are to be used for values of x, y, and z less than or equal to

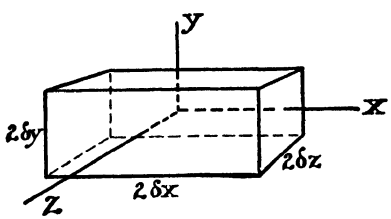

FIG. 4

the small values δx, δy, δz. Then the flow of liquid out of the parallelopiped through the right-hand face is

$$\int_{-\delta z}^{+\delta z}\int_{-\delta y}^{+\delta y} V_x\,dy\,dz=\int_{-\delta z}^{+\delta z}\int_{-\delta y}^{+\delta y}\left\{(V_x)_0+\left(\frac{\partial V_x}{\partial x}\right)_0\cdot x+\left(\frac{\partial V_x}{\partial y}\right)_0\cdot y+\left(\frac{\partial V_x}{\partial z}\right)_0\cdot z\right\}dy\,dz\,,$$

$$=4\delta y\,\delta z\left\{(V_x)_0+\left(\frac{\partial V_x}{\partial x}\right)\cdot\delta x\right\}.$$

Similarly, the flow of liquid out through the left-hand face is

$$-4\delta y\,\delta z\left\{(V_x)_0-\left(\frac{\partial V_x}{\partial x}\right)_0\cdot\delta x\right\},$$

so that the total loss of liquid per second through these two faces is

$$8\delta x\ \delta y\ \delta z\left(\frac{\partial V_x}{\partial x}\right)_0 . \tag{26}$$

Thus the total flow of liquid per second out of the parallelopiped is

$$8\delta x\ \delta y\ \delta z\left\{\left(\frac{\partial V_x}{\partial x}\right)_0+\left(\frac{\partial V_y}{\partial y}\right)_0+\left(\frac{\partial V_z}{\partial z}\right)_0\right\}=d\tau(\text{div } \boldsymbol{V})_0 , \tag{27}$$

$d\tau$ being the volume of the parallelopiped.

The flow of liquid out of the parallelopiped per second per unit volume is thus

$$\text{div } \boldsymbol{V},$$

the value being taken at the point under consideration—in this example, the origin. If there are no sources or sinks in the region under consideration, i.e., if there are no points at which liquid is being introduced or removed, then obviously the total outward flow through any closed surface must be zero, and accordingly the divergence would be zero. If the divergence is not zero, it gives the rate at which liquid is being introduced (or removed) and is a measure of what is called the "strength" of the point source (or sink) at the point. It is seen from this example that one could set

$$\text{div } \boldsymbol{V}=\lim_{d\tau \doteq 0}\frac{\int V_n d\sigma}{d\tau} , \tag{28}$$

$d\sigma$ being an element of surface of the volume element $d\tau$, and V_n the component of $\boldsymbol{V}$ directed along the exterior-pointing normal to the surface of $d\tau$.

C. One may also take the vector product of ∇ and a vector $\boldsymbol{A}$. Thus

$$[\nabla,\boldsymbol{A}]=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} . \tag{29}$$

This quantity is known as curl $\boldsymbol{A}$. That is,

$$[\nabla,\boldsymbol{A}]\equiv\text{curl } \boldsymbol{A} . \tag{30}$$

The components of curl A are

$$(31)\qquad \begin{cases} (\text{curl } A)_x \equiv \text{curl}_x A = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}, \\ (\text{curl } A)_y \equiv \text{curl}_y A = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}, \\ (\text{curl } A)_z \equiv \text{curl}_z A = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}. \end{cases}$$

An illustration will again help to make clear the meaning of this quantity. Suppose that one has an incompressible liquid of uniform density ρ. Consider a small sphere of the liquid and, at a given instant, choose the origin of co-ordinates at the center of this sphere. What will be the location and the shape, after a small interval of time, of the portion of liquid originally contained within this sphere? In the first place, the sphere may have been translated as a whole; in the second place, it may have undergone a strain which leaves it no longer a sphere; and, in the third place, it may have been rotated into a new orientation. The curl of the vector velocity furnishes a measure of this last type of motion, as will now be shown.

The components of velocity at the origin being $(V_x)_0$, $(V_y)_0$, $(V_z)_0$, the components of velocity at points near the origin are given by (25) above. Then the velocity of a point x, y, z near the origin, relative to the velocity of the center of the sphere, has components

$$\left(\frac{\partial V_x}{\partial x}\right)_0 \cdot x + \left(\frac{\partial V_x}{\partial y}\right)_0 \cdot y + \left(\frac{\partial V_x}{\partial z}\right)_0 \cdot z,$$
$$\left(\frac{\partial V_y}{\partial x}\right)_0 \cdot x + \left(\frac{\partial V_y}{\partial y}\right)_0 \cdot y + \left(\frac{\partial V_y}{\partial z}\right)_0 \cdot z,$$
$$\left(\frac{\partial V_z}{\partial x}\right)_0 \cdot x + \left(\frac{\partial V_z}{\partial y}\right)_0 \cdot y + \left(\frac{\partial V_z}{\partial z}\right)_0 \cdot z,$$

so that the moment of momentum of the sphere about the x-axis is

$$\iiint \left\{ \left[\left(\frac{\partial V_z}{\partial x}\right)_0 \cdot x + \left(\frac{\partial V_z}{\partial y}\right)_0 \cdot y + \left(\frac{\partial V_z}{\partial z}\right)_0 \cdot z \right] \cdot y - \left[\left(\frac{\partial V_y}{\partial x}\right)_0 \cdot x + \left(\frac{\partial V_y}{\partial y}\right)_0 \cdot y + \left(\frac{\partial V_y}{\partial z}\right)_0 \cdot z \right] \cdot z \right\} \rho \, dx \, dy \, dz.$$

Noticing that, from symmetry,

$$\iiint xy\, dx\, dy\, dz = 0$$

and

$$\iiint x^2\, dx\, dy\, dz = \iiint y^2\, dx\, dy\, dz\,,$$

the moment of momentum becomes

$$\rho\left[\left(\frac{\partial V_z}{\partial y}\right)_0 - \left(\frac{\partial V_y}{\partial z}\right)_0\right]\iiint y^2\, dx\, dy\, dz\,.$$

The moment of inertia of the sphere about the same axis is

$$\rho\iiint (y^2+z^2)dx\, dy\, dz = 2\rho\iiint y^2\, dx\, dy\, dz\,.$$

If the moment of momentum be divided by the moment of inertia, a sort of effective average value is obtained for the angular velocity about the x-axis which, as the radius of the sphere is allowed to approach zero, becomes the angular velocity about the x-axis at the point in question (in this case, the origin). That is to say, the angular velocity at this point has the x-component

$$\frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right).$$

Hence, if W denotes the vector angular velocity at any point in the liquid,

$$W = \tfrac{1}{2}\ \text{curl}\ V\,. \tag{32}$$

A simpler, but less general, interpretation of the curl of a vector may be obtained from considering the vector velocity of any point P of a rigid body which is in motion. If V_0 be the velocity of some point O of the body, w the angular velocity of the body relative to O, and r the vector from O to P, then the velocity V_P of P is

$$V_P = V_0 + [w,r]\,.$$

Then from (44) below (and the fact that V_0 and w are constant vectors) it follows that

$$\begin{aligned}\text{curl}\ V_P &= w\ \text{div}\ r - (w,\nabla)r\,,\\ &= 3w - w\,,\\ &= 2w\,.\end{aligned}$$

D. Equation (20) gives the rate of change of a scalar u in a direction s as

$$(\nabla u)_s = (s, \nabla u) = \frac{du}{ds},$$

provided s be a unit vector in the direction s. In general, one might say that

$$(A, \nabla u),$$

often written

$$(A, \nabla)u,$$

gives the rate of change of the scalar u in the direction of the vector A multiplied by the absolute magnitude of A. This leads one to consider assigning a suitable meaning to the symbol

$$(A, \nabla)B.$$

A vector field is a region of space at every point of which a vector is given, say by expressing its components as functions of x, y, and z. Suppose that at a point x, y, z in such a vector field the vector has the value

$$B = iB_x + jB_y + kB_z,$$

and at a neighboring point $x+dx$, $y+dy$, $z+dz$ the value

$$B' = iB'_x + jB'_y + kB'_z.$$

Then the change which the vector B experiences in going from the first to the second point is

$$B' - B = i(B'_x - B_x) + j(B'_y - B_y) + k(B'_z - B_z),$$

and is obviously a vector whose components are the changes which the components of B experience. But the components are scalar quantities, and the rate of change which they experience in going from one point to another may be expressed as

$$(a, \nabla)B_x, \qquad (a, \nabla)B_y, \qquad \text{and} \qquad (a, \nabla)B_z,$$

where a is a unit vector pointing in the direction from x, y, z to $x+dx$, $y+dy$, $z+dz$. Thus the rate of change which the vector B experiences in departing from the first point in the direction of the second point is

$$i(a, \nabla)B_x + j(a, \nabla)B_y + k(a, \nabla)B_z.$$

This last expression is adopted as the meaning of the symbol

$$(\boldsymbol{a},\nabla)\boldsymbol{B}\ .$$

Thus the vector

$$(\boldsymbol{a},\nabla)\boldsymbol{B}=\boldsymbol{C}$$

of components

$$\text{(33)}\qquad \begin{cases} C_x=a_x\dfrac{\partial B_x}{\partial x}+a_y\dfrac{\partial B_x}{\partial y}+a_z\dfrac{\partial B_x}{\partial z}\ , \\ C_y=a_x\dfrac{\partial B_y}{\partial x}+a_y\dfrac{\partial B_y}{\partial y}+a_z\dfrac{\partial B_y}{\partial z}\ , \\ C_z=a_x\dfrac{\partial B_z}{\partial x}+a_y\dfrac{\partial B_z}{\partial y}+a_z\dfrac{\partial B_z}{\partial z} \end{cases}$$

measures the rate of change of the vector $\boldsymbol{B}$ in the direction of the unit vector $\boldsymbol{a}$. Then obviously

$$(\boldsymbol{A}\nabla)\boldsymbol{B}$$

measures the rate of change of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$, multiplied by the absolute magnitude of $\boldsymbol{A}$. This notation furnishes a short way of expressing the directional derivative of a vector.

E. This section will conclude with a theorem which involves the concept of the line integral of the tangential component of a vector. Before this theorem is stated and proved line integrals will be briefly discussed.

In mathematical physics one deals constantly with integrals of the type $\int f\,d\tau$, $\int f\,d\sigma$, and $\int f\,ds$, where f is a scalar-point function and where $d\tau$ is an element of volume, $d\sigma$ an element of surface, and ds an element of line. In the first of these integrals, which is extended over a volume contained within a closed surface, it is not necessary to postulate any ordering of the elements of the integral, and $d\tau$ is intrinsically positive, being simply the numerical measure of the size of the volume of the element. Similarly, we may consider that the second of the foregoing integrals means the limit of the sum of products of essentially positive elements of area $d\sigma$ by the value of f at some point of $d\sigma$. These three fundamental types of integrals are viewed similarly, therefore, only when we think of ds in the third as intrinsically positive—as a mere measure of the length of a line element.

Consider, for example, the line integral of the tangential component of a force. The expression "tangential component" is obviously ambiguous until it be stipulated which direction along the curve is to be taken

as the positive direction. Suppose the integration be carried out along a curve whose end points are A and B. If it be specified that the direction along the curve from A to B is to be considered the direction of the positive tangent, then the tangential component F_t is thereby defined at every point of the curve. If it be desirable to indicate, by means of the notation, that the direction from A to B is considered positive in reckoning the sign of F_t, this may be done by using the letters A and B as upper and lower limits of the definite integral. Then

$$\int_A^B F_t\, ds = -\int_B^A F_t\, ds\ ,$$

but the reason for the change in sign is not that ds has changed sign, nor is it because the "limits of integration" have been interchanged. The reason is, of course, that F_t in the second integral is equal but opposite to F_t in the first. In neither integral are the letters A and B to be interpreted as limits in the ordinary sense. Each integral requires the summing of the product of every positive ds by the associated value of F_t. The letters A and B lay down the convention according to which F_t is to be calculated. The letters A and B also, of course, indicate that all elements ds lying between A and B are to be included in the integration process.

To illustrate this viewpoint further, and show that the distinction being insisted on is not an artificial one, consider Gauss's theorem for a closed curve in a plane. Namely,

$$\oint E_n ds = e\ ,$$

where e is the sum of all the charge on the plane within the closed curve and where E_n is the component, along the external normal to the curve, of the electrostatic intensity. Having specified that $\boldsymbol{n}$ is the exterior normal to the curve, E_n is defined at every point. Then this integral has, and should have, the same value e regardless of the direction of transcription of the closed curve. If $\boldsymbol{n}$ be specified as the exterior normal, but ds given sign, this integral has the value e only when the integration is carried out in a certain manner, and has the value $-e$ if this order be reversed.

The expression

$$\int F_x\, dx + F_y\, dy$$

is to be viewed as an abbreviation for the line integral,

$$\int \left(F_x \frac{dx}{ds} + F_y \frac{dy}{ds} \right) ds .$$

The signs of dx/ds and dy/ds are indeterminate until a sense of transcription has been specified. The element ds, however, is to be considered as essentially positive, here as always.

This section will now conclude with a proof of the following theorem: If the line integral of the tangential component A_s of a vector around any closed circuit be zero, then the vector $\boldsymbol{A}$ is expressible as the nabla of a scalar-point function, and vice versa.

First, suppose that

$$\oint A_s \, ds = 0 . \tag{34}$$

Then, if

$$\Phi = \Phi_0 + \int_0^P A_s \, ds ,$$

Φ_0 being the value assigned to Φ at the point O, the function Φ is, by virtue of (34), a uniquely defined function of position. The amount, $d\Phi$, by which this function changes in going from a given point a distance ds to a neighboring point is

$$d\Phi = A_s \, ds ,$$

and, accordingly,

$$A_s = \frac{d\Phi}{ds} \; ; \; \boldsymbol{A} = \nabla \Phi .$$

Conversely, suppose

$$\boldsymbol{A} = \nabla \Phi .$$

Then

$$A_s = \frac{d\Phi}{ds} ,$$

and obviously the integral

$$\int A_s \, ds = \int \frac{d\Phi}{ds} \, ds$$

will be zero around any closed path.

§ 4. *Relations between Scalar and Vector Products, Divergence, and Curl.*—In this section are listed certain important vector relationships, together with a brief indication of their proofs.

(35) $$(\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]) \equiv (\boldsymbol{B}[\boldsymbol{C},\boldsymbol{A}]) \equiv (\boldsymbol{C}[\boldsymbol{A},\boldsymbol{B}]) \,.$$

$[\boldsymbol{B},\boldsymbol{C}]$ is numerically equal to the area of the parallelogram formed on the vectors $\boldsymbol{B}$ and $\boldsymbol{C}$, and is normal to the plane of $\boldsymbol{B}$ and $\boldsymbol{C}$. Hence $(\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}])$ is numerically equal to the volume of the parallelopiped formed on the three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. The other two scalar products may be likewise interpreted as being equal numerically to this same volume.

(36) $$[\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]] = \boldsymbol{B}(\boldsymbol{A},\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A},\boldsymbol{B}) \,.$$

The x-component of $[\boldsymbol{A}[\boldsymbol{B},\boldsymbol{C}]]$ is by definition

$$A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z) \,,$$

which may be written

$$B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z) \,,$$

which is the x-component of

$$\boldsymbol{B}(\boldsymbol{A},\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A},\boldsymbol{B}) \,.$$

(37) $$\operatorname{div} \Phi\boldsymbol{A} = \Phi \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, \nabla\Phi) \,,$$

$$\operatorname{div} \Phi\boldsymbol{A} = \Phi \operatorname{div} \boldsymbol{A} + A_x \frac{\partial\Phi}{\partial x} + A_y \frac{\partial\Phi}{\partial y} + A_z \frac{\partial\Phi}{\partial z} \,,$$

$$= \Phi \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, \nabla\Phi) \,.$$

(38) $$\operatorname{curl} \Phi\boldsymbol{A} = \Phi \operatorname{curl} \boldsymbol{A} + [\nabla\Phi, \boldsymbol{A}] \,,$$

$$\operatorname{curl}_x \Phi\boldsymbol{A} = \frac{\partial}{\partial y}(\Phi A_z) - \frac{\partial}{\partial z}(\Phi A_y) = \Phi\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + A_z \frac{\partial\Phi}{\partial y} - A_y \frac{\partial\Phi}{\partial z} \,,$$

$$= \Phi \operatorname{curl}_x \boldsymbol{A} + [\nabla\Phi, \boldsymbol{A}]_x \,.$$

(39) $$\operatorname{div} \operatorname{curl} \boldsymbol{A} \equiv 0 \,,$$

$$\operatorname{div} \operatorname{curl} \boldsymbol{A} = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \,,$$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_z}{\partial y\,\partial x} - \frac{\partial^2 A_y}{\partial x\,\partial z} + \frac{\partial^2 A_y}{\partial z\,\partial x} + \frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_x}{\partial z\,\partial y} \,,$$

$$= 0 \,.$$

(40) $$\text{curl}\ \nabla\Phi \equiv 0\ ,$$

$$\text{curl}_x\ \nabla\Phi = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial y}\right) \equiv 0\ ;$$

(41) $$\text{div}\ \nabla\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \nabla^2\Phi\ ,$$

$$\text{div}\ \nabla\Phi = \frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial z}\right)\ ,$$

$$= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \nabla^2\Phi\ .$$

(42) $$\text{curl curl}\ \boldsymbol{A} = \nabla\ \text{div}\ \boldsymbol{A} - \nabla^2\boldsymbol{A}\ ,$$

$$\text{curl}_x\ \text{curl}\ \boldsymbol{A} = \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\ ,$$

$$= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\,\partial y} + \frac{\partial^2 A_z}{\partial x\,\partial z} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}\ ,$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \nabla^2 A_x\ ,$$

$$= \nabla_x\ \text{div}\ \boldsymbol{A} - \nabla^2 A_x\ .$$

It is to be noted that this is a vector formula, $\nabla^2\boldsymbol{A}$ denoting in Cartesian co-ordinates the vector of components $\nabla^2 A_x$, $\nabla^2 A_y$, $\nabla^2 A_z$, respectively. For $\nabla^2\boldsymbol{A}$ in other co-ordinate systems see § 7 of this Appendix

(43) $$\text{div}\ [\boldsymbol{A},\boldsymbol{B}] = (\boldsymbol{B},\ \text{curl}\ \boldsymbol{A}) - (\boldsymbol{A},\ \text{curl}\ \boldsymbol{B})\ ,$$

$$\text{div}\ [\boldsymbol{A},\boldsymbol{B}] = \frac{\partial}{\partial x}(A_y B_z - A_z B_y) + \frac{\partial}{\partial y}(A_z B_x - A_x B_z) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x)\ .$$

$$= -A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$+ B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\ .$$

$$= (\boldsymbol{B},\ \text{curl}\ \boldsymbol{A}) - (\boldsymbol{A},\ \text{curl}\ \boldsymbol{B})\ .$$

(44) $$\text{curl}\ [\boldsymbol{A},\boldsymbol{B}] = \boldsymbol{A}\ \text{div}\ \boldsymbol{B} - \boldsymbol{B}\ \text{div}\ \boldsymbol{A} + (\boldsymbol{B},\nabla)\boldsymbol{A} - (\boldsymbol{A},\nabla)\boldsymbol{B}\ ,$$

$$\text{curl}_x\ [\boldsymbol{A},\boldsymbol{B}] = \frac{\partial}{\partial y}(A_x B_y - A_y B_x) - \frac{\partial}{\partial z}(A_z B_x - A_x B_z)\ ,$$

$$= A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)$$
$$+B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}$$
$$-A_x\frac{\partial B_x}{\partial x}-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}.$$
$$= A_x \operatorname{div} \boldsymbol{B}-B_x \operatorname{div} \boldsymbol{A}+(\boldsymbol{B},\nabla)A_x-(\boldsymbol{A},\nabla)B_x .$$

(45) $$\nabla(\boldsymbol{A},\boldsymbol{B}) = (\boldsymbol{A},\nabla)\boldsymbol{B}+(\boldsymbol{B},\nabla)\boldsymbol{A}+[\boldsymbol{A}, \operatorname{curl} \boldsymbol{B}]+[\boldsymbol{B}, \operatorname{curl} \boldsymbol{A}]$$

$$\frac{\partial}{\partial x}(A_xB_x+A_yB_y+A_zB_z)=A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_y}{\partial x}+A_z\frac{\partial B_z}{\partial x}$$
$$+B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_y}{\partial x}+B_z\frac{\partial A_z}{\partial x},$$
$$= A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}+B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}$$
$$+B_z\frac{\partial A_x}{\partial z}+A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\cdot$$
$$+B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right).$$

§ 5. *Integral Transformations.*—

A. Consider the expression

$$\int \frac{\partial W}{\partial x}\, d\sigma ,$$

where this surface integral is to be extended over the plane area inclosed by the curve shown in Figure 5. This integral may be written as

$$\iint \frac{\partial W}{\partial x}\, dx\, dy = \int_{y_1}^{y_2}(W_2-W_1)dy ,$$

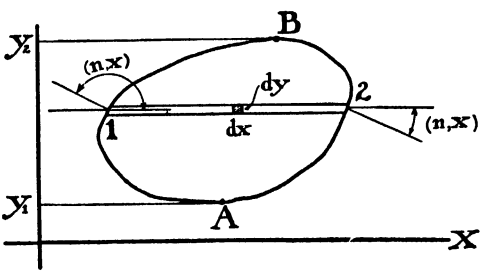

FIG. 5

where W_1 and W_2 are the values of W at the points marked 1 and 2 on the figure. Then according to the convention here

adopted that ds is intrinsically positive,

$$dy = ds \cos(nx) \qquad \text{at the point marked 2,}$$
$$dy = -ds \cos(nx) \qquad \text{at the point marked 1,}$$

so that this integral may be written:

$$\int_A^B W_2 \cos(nx) ds + \int_A^B W_1 \cos(nx) ds .$$

Thus

$$\int \frac{\partial W}{\partial x} d\sigma = \int W \cos(nx) ds , \tag{46}$$

taken over the whole curve.

In just the same way it may be seen that

$$\int \frac{\partial W}{\partial x} d\tau = \int W \cos(nx) d\sigma , \tag{47}$$

where the integral on the left is extended throughout the volume contained within a closed surface. The integral on the right is extended over this closed surface. It follows at once from this last equation that

$$\int \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau = \int [A_x \cos(nx) + A_y \cos(ny) + A_z \cos(nz)] d\sigma ,$$

or

$$\int \operatorname{div} A \, d\tau = \int A_n \, d\sigma . \tag{48}$$

B. *Green's Theorem.*—If, in (48), one substitutes for A the vector $u, \nabla v$ it follows, from (37) and (20), that

$$\int u \nabla^2 v \, d\tau = \int u \frac{\partial v}{\partial n} d\sigma - \int \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] d\tau , \tag{49}$$

or, if

$$u = v ,$$

$$\int u \nabla^2 u \, d\tau = \int u \frac{\partial u}{\partial n} d\sigma - \int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] d\tau . \tag{50}$$

Both formula (49) and formula (50) are referred to as "Green's theorem." A further formula, also called "Green's theorem," but usually called the "symmetrical form of Green's theorem" to distinguish between it and the two forms just given, is obtained by interchanging u and v in (49) and subtracting the resulting equations. Thus:

$$\int [u\nabla^2 v - v\nabla^2 u]d\tau = \int \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]d\sigma \,. \tag{51}$$

C. *Stokes's Theorem.*—In A of this section it has been seen that a three-dimensional integral whose integrand is formed of partial derivatives can be reduced to a two-dimensional integral. A partial integration, of this same sort, will reduce a two-dimensional (i.e., a surface) integral whose integrand involves derivatives, to a one-dimensional (i.e., a line) integral. Consider an open surface, whose equation is $z=f(x,y)$, bounded by the closed contour S (see Fig. 6). Let S' be the projection of S on the x–y-plane, and let $P(x,y,z)$ be a given function whose values at points on the surface in question are

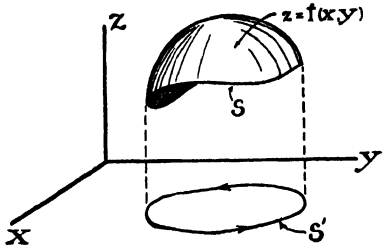

FIG. 6

$$P[x,y,f(x,y)] = P_1(x,y) \,.$$

Consider the line integral

$$\int_S P\frac{dx}{ds}\,ds \,,$$

where, in determining the sign of dx/ds, the curve S is to be traversed in the direction judged counterclockwise by a man who stands on the surface with his head in the direction of that normal to the surface which we will call n. This integral is equal to

$$\int_{S'} P_1\frac{dx}{ds}\,ds \,,$$

since, on the surface $P \equiv P_1$, and since the projection on the x-axis of an element ds of S is equal to the projection on the x-axis of the projection of ds on the x–y-plane. Now by (46)

$$\int_{S'} \frac{\partial P_1}{\partial y}\, dy\, dx = \int_{S'} P_1 \cos(ny) ds\,,$$

and, ds being essentially positive, $dx = \cos(ny) ds$, so that

$$\int_S P \frac{dx}{ds}\, ds = \int_{S'} P_1 \frac{dx}{ds}\, ds = \int_{S'} \frac{\partial P_1}{\partial y}\, dy\, dx\,. \tag{52}$$

Now

$$\frac{\partial P_1}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\,,$$

while, if the point x, y, z and the point $x+dx, y+dy, z+dz$ are both on the surface $z = f(x,y)$,

$$\cos(nx)dx + \cos(ny)dy + \cos(nz)dz = 0\,,$$

so that

$$\frac{\partial z}{\partial y} = -\frac{\cos(ny)}{\cos(nz)}\,.$$

Further, if $d\sigma$ be an element of the surface $z = f(x,y)$,

$$dx\, dy = \cos(nz) d\sigma, \quad dx\, dz = \cos(ny) d\sigma\,, \quad dy\, dz = \cos(nx) d\sigma\,.$$

Substituting these relations into (52),

$$\int_S P \frac{dx}{ds}\, ds = \int \left[\frac{\partial P}{\partial z}\cos(ny) - \frac{\partial P}{\partial y}\cos(nz)\right] d\sigma\,.$$

Analogous relations can be written down at once for the line integrals

$$\int_S Q \frac{dy}{ds}\, ds\,, \qquad \int_S R \frac{dz}{ds}\, ds\,,$$

so that, adding,

$$\int \left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds}\right) ds = \int \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos(nx) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos(ny) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos(nz)\right] d\sigma\,,$$

or, in vector notation,

$$\int A_s\, ds = \int \mathrm{curl}_n \boldsymbol{A}\, d\sigma\,, \tag{53}$$

where $\boldsymbol{A}$ is any vector. This relationship is known as "Stokes's theorem."

D. This section contains the statement and proof of three special integral transformations

$$\int \mathrm{curl}\ \boldsymbol{A}\, d\tau = \int [\boldsymbol{n},\boldsymbol{A}] d\sigma\,. \tag{54}$$

Consider the x-component of the left member:

$$\int \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) d\tau = \int [A_z \cos(ny) - A_y \cos(nz)] d\sigma\,,$$

$$= \int [\boldsymbol{n},\boldsymbol{A}]_x\, d\sigma\,.$$

$$\int u\, \boldsymbol{ds} = \int [\boldsymbol{n}, \nabla u] d\sigma\,, \tag{55}$$

where u is any scalar function, where the integral on the right is extended over the open surface which spans the closed curve over which the integral on the left is extended, and where the directions of $\boldsymbol{n}$ and $\boldsymbol{ds}$ are as in Stokes's theorem.

Consider the x-component of the left member:

$$\int u \frac{dx}{ds}\, ds = \int C_s\, ds\,,$$

where $\boldsymbol{C} = i u$.

Thus, by Stokes's theorem,

$$\int C_s\, ds = \int \mathrm{curl}_n\, \boldsymbol{C}\, d\sigma = \int \left[\cos(ny)\, \frac{\partial u}{\partial z} - \cos(nz)\, \frac{\partial u}{\partial y}\right] d\sigma\,,$$

which is the x-component of the right member.

$$\int \left[\boldsymbol{ds}', \nabla' \frac{1}{r}\right] = -\nabla \int \frac{\partial \frac{1}{r}}{\partial n}\, d\sigma'\,, \tag{56}$$

where the surface and line integrals and the direction of $\boldsymbol{n}$ and $\boldsymbol{ds}'$ are related as in Stokes's theorem, where the integration variables are

primed, and where the differentiation involved in ∇ is with respect to the unprimed variables.

Consider the x-component of the left member

$$\int \left(\frac{dy'}{ds'} \frac{\partial \frac{1}{r}}{\partial z'} - \frac{dz'}{ds'} \frac{\partial \frac{1}{r}}{\partial y'} \right) ds' = - \int C_s ds' ,$$

where

$$\boldsymbol{C} = -j \frac{\partial \frac{1}{r}}{\partial z'} + k \frac{\partial \frac{1}{r}}{\partial y'} .$$

Hence by Stokes's theorem, the component in question is equal to

$$- \int \operatorname{curl}_n \boldsymbol{C} \, d\sigma' = - \int \left[\cos(nx) \left(\frac{\partial^2 \frac{1}{r}}{\partial y'^2} + \frac{\partial^2 \frac{1}{r}}{\partial z'^2} \right) + \cos(ny) \left(- \frac{\partial^2 \frac{1}{r}}{\partial x' \partial y'} \right) + \cos(nz) \left(- \frac{\partial^2 \frac{1}{r}}{\partial x' \partial z'} \right) \right] d\sigma'$$

or, making use of the relation $\nabla'^2 (1/r) = 0$,

$$= \int \left[\cos(nx) \frac{\partial}{\partial x'} \frac{\partial \frac{1}{r}}{\partial x'} + \cos(ny) \frac{\partial}{\partial y'} \frac{\partial \frac{1}{r}}{\partial x'} + \cos(nz) \frac{\partial}{\partial z'} \frac{\partial \frac{1}{r}}{\partial x'} \right] d\sigma' ,$$

$$= \int \frac{\partial}{\partial n} \frac{\partial \frac{1}{r}}{\partial x'} \, d\sigma' = \int \frac{\partial}{\partial x'} \frac{\partial \frac{1}{r}}{\partial n} \, d\sigma' = - \frac{\partial}{\partial x} \int \frac{\partial \frac{1}{r}}{\partial n} \, d\sigma' .$$

§ 6. *Vector Fields.*—

A. A vector field is a region of space at every point of which a vector is defined. The vector might be, for example, the velocity at any point in a field, or the vector of electric force at any point in the neighborhood of charge. Mathematical physics is much concerned with the determination of vector fields of various sorts under given boundary conditions, and it is the purpose of this section to give some of the fundamental theorems concerning vector fields.

If, at every point of its field, a vector has zero divergence, it is known as a "solenoidal vector"; while any vector which can be represented as the nabla of some scalar-point function is known as a "potential vector." Thus a solenoidal vector has everywhere zero divergence, while a potential vector has everywhere zero curl. It will be shown that any vector which is solenoidal, and hence of zero divergence, can be represented as the curl of some other vector; and that any vector which has zero curl is necessarily a potential vector. It will further be shown that an arbitrary vector can be represented as the sum of a solenoidal vector and a potential vector, and the solutions for these two types will be obtained.

THEOREM 1. *A solution, continuous together with its first derivatives and vanishing at infinity as $1/r$, of the equation*

$$\nabla^2 u = -\rho$$

is given by

$$u = \frac{1}{4\pi}\int \frac{\rho'}{r}\, d\tau' .$$

Throughout the remainder of this Appendix, in accordance with the practice used in the text, the variables of integration are, in general, the co-ordinates x', y', z' of a point P. The distance r is measured from a point O, of co-ordinates x, y, z to P, so that r is a function of both the primed and the unprimed variables. Other quantities which occur under integral signs, such as ρ' in the last equation, are supposed expressed as functions of x', y', z'. The values of the integrals are, of course, functions of x, y, z only.

Proof: Consider a region contained within a sphere Σ of radius R. Let an arbitrary point O of the volume V contained within Σ be chosen as origin, and in Green's formula

$$\int \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\sigma' = \int (v\nabla'^2 u - u\nabla'^2 v) d\tau' ,$$

write $v = 1/r$, r being the distance from the point O to any point P of co-ordinates x', y', z'. Then since the foregoing formula is valid only for use in a region throughout which u and v, together with their first derivatives, are finite and continuous, the origin O must be excluded from the region V by deleting from it a small sphere of radius δ and

center at O. Call the surface of this small sphere σ, and the region contained between σ and Σ, V_*. Then

$$(57)\quad \int_\sigma \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial \frac{1}{r}}{\partial n}\right) d\sigma' + \int_\Sigma \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial \frac{1}{r}}{\partial n}\right) d\sigma' = \int_{V_*} \left(\frac{1}{r}\nabla'^2 u - u\nabla'^2\frac{1}{r}\right) d\tau'$$

Consider the first of these integrals,

$$\int_\sigma \frac{1}{r}\frac{\partial u}{\partial n}\, d\sigma' = r\int \frac{\partial u}{\partial n}\, d\omega\,,$$

where $d\omega$ is an element of solid angle. Then, since $\partial u/\partial n$ is finite,

$$\lim_{r=0} r\int \frac{\partial u}{\partial n}\, d\omega = 0\,.$$

Also, since on σ

$$\frac{\partial \frac{1}{r}}{\partial n} = -\frac{\partial \frac{1}{r}}{\partial r} = \frac{1}{r^2}\,,$$

$$-\int u\frac{\partial \frac{1}{r}}{\partial n}\, d\sigma' = -\int u\, d\omega\,,$$

$$\lim_{r=0} -\int u d\omega = -4\pi u_0\,,$$

u_0 being the value of u at O.

The second integral of (57) involves derivatives, along the direction of the external normal to V, of u and of $1/r$. As R approaches infinity, these derivatives become more and more nearly equal to the derivatives of u and $1/r$ with respect to r. Thus the quantity within the parenthesis vanishes at infinity as $1/r^3$, since u vanishes by hypothesis as $1/r$, so that $\partial u/\partial r$ vanishes as $1/r^2$. The element of surface, however, increases as r^2. Hence the integral approaches zero as R approaches infinity.

Since $\nabla'^2(1/r)=0$ throughout V_*, the third integral of (58) reduces to

$$\int \frac{\nabla' u}{r}\, d\tau' .$$

Hence

$$-4\pi u_0 = \int \frac{\nabla'^2 u}{r}\, d\tau' = -\int \frac{\rho'}{r} d\tau' ,$$

$$u_0 = \frac{1}{4\pi}\int \frac{\rho'}{r} d\tau' .$$

The origin having been arbitrarily chosen, this is the value of u at any point.

B. In § 4 were established the following equations:

$$\text{div curl } \boldsymbol{A} \equiv 0 ,$$
$$\text{curl } \nabla\Phi = 0 .$$

The following more general theorems will now be proved:

THEOREM 2. *The necessary and sufficient condition that the curl of a vector vanish identically is that the vector be the nabla of some function.*

THEOREM 3. *The necessary and sufficient condition that the divergence of a vector vanish identically is that the vector be the curl of some other vector.*

These theorems will now be proved in order. First assume

$$\boldsymbol{A} = \nabla\Phi ,$$

then

$$\text{curl } \boldsymbol{A} = \text{curl } \nabla\Phi = 0 .$$

Next assume

$$\text{curl } \boldsymbol{A} = 0 .$$

By Stokes's theorem and the last equation,

$$\int A_s\, ds = \int \text{curl}_n \boldsymbol{A}\, d\sigma = 0 .$$

Then, by theorem of § 3, E,

$$\boldsymbol{A} = \nabla\Phi .$$

The first of the foregoing theorems is thus established. To prove the second theorem, assume first that

$$\boldsymbol{A} = \text{curl } \boldsymbol{B} .$$

Then, by (39),

$$\text{div } \boldsymbol{A} = \text{div curl } \boldsymbol{B} = 0 .$$

Next assume

$$\text{div } \boldsymbol{A} = 0 . \tag{58}$$

It is desired to determine, if possible, a vector $\boldsymbol{B}$ such that

$$\boldsymbol{A} = \text{curl } \boldsymbol{B} .$$

It will, in fact, be shown that it is possible to obtain a third vector $\boldsymbol{C}$ whose curl is the desired vector $\boldsymbol{B}$. That is, a vector $\boldsymbol{C}$ will be determined from the relations

$$\begin{aligned} \boldsymbol{B} &= \text{curl } \boldsymbol{C} , \\ \boldsymbol{A} &= \text{curl curl } \boldsymbol{C} , \\ &= \nabla \text{ div } \boldsymbol{C} - \nabla^2 \boldsymbol{C} . \end{aligned}$$

Thus

$$\nabla^2 C_x = -A_x , \tag{59}$$

$$\nabla^2 C_y = -A_y , \tag{60}$$

$$\nabla^2 C_z = -A_z , \tag{61}$$

provided

$$\text{div } \boldsymbol{C} = 0 . \tag{62}$$

These four differential equations for the determination of C_x, C_y, C_z would be independent and hence, in general, would not possess a solution were it not for the condition expressed in (58). Indeed, by virtue of this equation the four differential equations above are dependent. The first three of them give, using (58),

$$\nabla^2 \text{ div } \boldsymbol{C} = 0 ,$$

which is satisfied by (62).

By the first theorem of this section, C_x and C_y may be determined from (59) and (60). Then (62) gives C_z except for an arbitrary function of x and y.

$$C_z = C_z' + \mathrm{X}(x,y) , \tag{63}$$

where C_z' is determined from C_x, C_y, and (62). But from (61)

$$\frac{\partial \nabla^2 C_z}{\partial z} = \frac{\partial \nabla^2 C_z'}{\partial z} = -\frac{\partial A_z}{\partial z},$$

so that

$$\nabla^2 C_z' = -A_z + \Phi(x,y). \tag{64}$$

Thus (61) will be satisfied by (63) provided X is a solution of

$$\nabla^2 \mathrm{X} = -\Phi(x,y).$$

The equations (59), (60), (61), and (62) may thus be replaced by

$$\nabla^2 C_x = -A_x,$$
$$\nabla^2 C_y = -A_y,$$
$$\nabla^2 \mathrm{X} = -\Phi.$$

These equations, which by the first theorem of this section are known to possess solutions, together with (63) and (64), serve to determine $\boldsymbol{C}$ and hence $\boldsymbol{B}$. The proof of the second of the two theorems is thus complete.

C. The importance of the concepts of divergence and curl is made clearer by:

THEOREM 4. *A vector is uniquely determined if the divergence and curl be specified, and if the normal component of the vector be known over a closed surface, or if the vector vanish as $1/r^2$ at infinity. In the former case the vector is determined within the closed surface.*

Proof: First suppose the normal component to be known over some closed surface. Suppose there are two vectors $\boldsymbol{A}$ and $\boldsymbol{A}'$, each having the same divergence and curl throughout the interior of the closed surface, and with $A_n = A_n'$ on the surface. Then if $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{A}'$,

$$\operatorname{div} \boldsymbol{B} = 0,$$
$$\operatorname{curl} \boldsymbol{B} = 0, \tag{65}$$
$$B_n = 0. \tag{66}$$

Now from (65) and Theorem 2,

$$\boldsymbol{B} = \nabla u, \tag{67}$$

so that

$$\nabla^2 u = 0.$$

Also, from (66) and (67),

$$\frac{\partial u}{\partial n}=0\,.$$

If these values be now inserted in

$$(68)\qquad \int\left[\left(\frac{\partial u}{\partial x'}\right)^2+\left(\frac{\partial u}{\partial y'}\right)^2+\left(\frac{\partial u}{\partial z'}\right)^2\right]d\tau'+\int u\nabla'^2u\,d\tau'=\int u\,\frac{\partial u}{\partial n}\,d\sigma'\,,$$

one has

$$\int\left(\frac{\partial u}{\partial x'}\right)^2+\left(\frac{\partial u}{\partial y'}\right)^2+\left(\frac{\partial u}{\partial z'}\right)^2\Big]\,d\tau'=0\,,$$

or, since the integrand is essentially positive,

$$\frac{\partial u}{\partial x'}=\frac{\partial u}{\partial y'}=\frac{\partial u}{\partial z'}=0\;;\qquad \nabla' u=0\;;\qquad \boldsymbol{B}=0\,.$$

Second, suppose that $\boldsymbol{A}$ and $\boldsymbol{A}'$ vanish at infinity as $1/r^2$. If the vector is to be determined throughout all space, the surface integral of (68) is to be extended over the surface of a sphere of indefinitely large radius. This integral may be written

$$\int u\,\frac{\partial u}{\partial r}\,r^2\,d\omega\,,$$

where $d\omega$ is an element of solid angle. Then since $\boldsymbol{B}$ vanishes at infinity as $1/r^2$, u will vanish as $1/r$, $\partial u/\partial r$ will vanish as $1/r^2$, and $u\,\dfrac{\partial u}{\partial r}$ as $1/r^3$, so that the value of the integral approaches zero as the radius of the sphere becomes indefinitely great. This integral again being zero, the proof goes exactly as above.

It has just been proved that a vector field is uniquely determined by the specification of divergence and curl. Two special cases of this will now be carried out to completion, i.e., the vector field will be actually obtained in terms of the specified divergence and curl. In the first example the divergence will be an arbitrary function of x, y, and z, while the curl will be identically zero. The first example will thus furnish an explicit representation of a general potential vector. In the second example the curl will be an arbitrary function of x, y, and z, while the divergence will be identically zero. Thus the second example will furnish explicit representation of a general solenoidal vector. The actual generality of these apparently special cases follows from the fact, proved at

the end of this section, that any vector is the sum of a potential vector and a solenoidal vector.

D. First example: It is desired to determine a vector $\boldsymbol{A}$ from the data,

$$\operatorname{div} \boldsymbol{A} = \rho \,, \qquad \operatorname{curl} \boldsymbol{A} = 0 \,,$$

ρ being a given function of x, y, and z. It is further assumed that the vector to be determined vanishes at infinity as $1/r^2$. Then since the curl of $\boldsymbol{A}$ vanishes

$$\boldsymbol{A} = -\nabla u \,,$$

so that

$$\operatorname{div} \boldsymbol{A} = +\rho = -\nabla^2 u \,.$$

Since $\boldsymbol{A}$ vanishes at infinity as $1/r^2$, u vanishes as $1/r$, and by Theorem 1

$$u = \frac{1}{4\pi} \int \frac{\rho'}{r} d\tau' \,,$$

so that

$$\boldsymbol{A} = -\nabla \frac{1}{4\pi} \int \frac{\rho'}{r} d\tau' \,. \tag{69}$$

This solution tacitly assumes that u and its first derivatives are everywhere continuous, since use is made of Theorem 1 where this is necessary. That is, the solution just given assumes the continuity of $\boldsymbol{A}$. In important cases, however, there are discontinuities. Discontinuities of a certain type will be assumed in u, and it will be made clear later to what important types of distribution of sources each corresponds. Let it then be assumed that there is a surface f_{12}, closed or unclosed, and illustrative of whatever other surfaces of the same sort there may be, on which

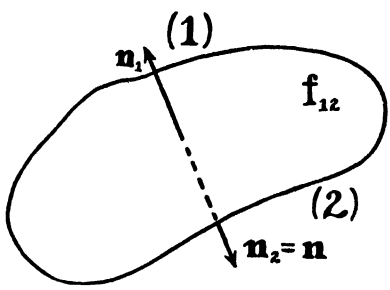

FIG. 7

$$\begin{cases} \dfrac{\partial u}{\partial n_1} + \dfrac{\partial u}{\partial n_2} = -\eta \,, \\ A_{n_1} + A_{n_2} = \eta \,. \end{cases} \tag{70}$$

The subscripts 1 and 2 denote the two sides of the surface, arbitrarily chosen; n_1 is the normal pointing from the surface toward the side 1;

$\boldsymbol{n}_2$ a normal pointing from the surface toward the side 2; and $\boldsymbol{n}$ a unit vector pointing along $\boldsymbol{n}_2$. Thus the component of $\boldsymbol{A}$ normal to the surface experiences a sudden jump in value upon crossing the surface.

Suppose also that there is a surface F_{12}, illustrative of a second sort, across which

$$u_1 - u_2 = p\,, \tag{71}$$

u_1 and u_2 indicating the values of u on the two sides of the surface. Thus u itself experiences a sudden jump p upon crossing such a surface as F_{12}. Then obviously the surfaces f_{12} and F_{12} must be excluded from the region throughout which the volume integrals of Green's formula are extended. This can be done by surrounding f_{12} and F_{12} by closed surfaces f and F which lie everywhere very close to f_{12} and F_{12}, respectively. Then the interior of these closed surfaces f and F will be excluded, and they themselves must be included in the total boundary over which the surface integrals are extended. It is to be noted that $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are opposite to the exterior normals of f and F. Then, as in Theorem 1, writing $v = 1/r$ in Green's formula,

$$\int_\sigma \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial \frac{1}{r}}{\partial n}\right) d\sigma' + \int_f (\quad) d\sigma' + \int_F (\quad) d\sigma' + \int_\Sigma (\quad) d\sigma' = \int \left(\frac{1}{r}\nabla'^2 u - u\nabla'^2 \frac{1}{r}\right) d\tau' \tag{72}$$

the integrand in the four surface integrals on the left being the same. The volume integral is to be extended over the interior of Σ with the exception of the interiors of σ, f, and F. Then, as in Theorem 1,

$$\lim_{r=0} \int_\sigma (\quad) d\sigma' = -4\pi u_0\,,$$

$$\lim_{R=\infty} \int_\Sigma (\quad) d\sigma' = 0\,,$$

$$\lim \int \left(\frac{1}{r}\nabla'^2 u - u\nabla'^2 \frac{1}{r}\right) d\tau' = \int \frac{\nabla'^2 u}{r}\, d\tau'\,.$$

Also

$$\lim \int_f (\quad) d\sigma' = -\int_{f_{12}} \frac{1}{r}\left(\frac{\partial u}{\partial n_1}+\frac{\partial u}{\partial n_2}\right) d\sigma' ,$$

$$= \int_{f_{12}} \frac{\eta'}{r} d\sigma' .$$

And

$$\lim \int_F (\quad) d\sigma' = -\int_{F_{12}} \left(u_1 \frac{\partial \frac{1}{r}}{\partial n_1} + u_2 \frac{\partial \frac{1}{r}}{\partial n_2} \right) d\sigma' ;$$

but

$$\frac{\partial \frac{1}{r}}{\partial n_1} = -\frac{\partial \frac{1}{r}}{\partial n_2} = -\left(\boldsymbol{n}, \nabla \frac{1}{r}\right) = +\left(\boldsymbol{n}, \nabla' \frac{1}{r}\right) .$$

Thus

$$\lim \int_F (\quad) d\sigma' = -\int_{F_{12}} (u_1 - u_2)\left(\boldsymbol{n}, \nabla' \frac{1}{r}\right) d\sigma' ,$$

$$= -\int_{F_{12}} \left(p\boldsymbol{n}, \nabla' \frac{1}{r}\right) d\sigma' ;$$

so that

$$u = \frac{1}{4\pi}\int \frac{\rho'}{r} d\tau' + \frac{1}{4\pi}\int_{f_{12}} \frac{\eta'}{r} d\sigma' - \frac{1}{4\pi}\int_{F_{12}} \left(p\boldsymbol{n}, \nabla' \frac{1}{r}\right) d\sigma' , \tag{73}$$

$$\boldsymbol{A} = -\nabla \frac{1}{4\pi}\int \frac{\rho'}{r} d\tau' - \nabla \frac{1}{4\pi}\int_{f_{12}} \frac{\eta'}{r} d\sigma' + \nabla \frac{1}{4\pi}\int_{F_{12}} \left(p\boldsymbol{n}, \nabla' \frac{1}{r}\right) d\sigma' . \tag{74}$$

These three terms will now be examined and interpreted. The first integral checks with the original solution, and is obviously the part of $\boldsymbol{A}$ due to a volume distribution of sources, i.e., a volume distribution of div $\boldsymbol{A} = \rho$.

To see the meaning of the second integral, consider a small flat element of volume extending half on either side of the surface f_{12}, and cutting from this surface an element df whose dimensions are large with respect to the perpendicular dimension of the volume element. Then for

the total normal flux of the A vector cut through the surface of this element one has

$$\int A_n \, d\sigma = (A_{n_1} + A_{n_2}) df_{12} ,$$

since the contribution from the narrow band that forms the sides is negligible compared to the contributions from the relatively large top and bottom. But if there were spread over f_{12} a surface distribution of sources of strength $(A_{n_1} + A_{n_2})$ per unit area, the flux per second out from an element df_{12} would be $(A_{n_1} + A_{n_2}) df_{12}$. That is, $(A_{n_1} + A_{n_2})$ is a measure of the strength per unit area of sources distributed on f_{12}, and is therefore called the "surface divergence." Thus the discontinuity in the normal component tells the per unit area strength of surface distribution of sources, just as the divergence tells the per unit volume strength of volume distribution of sources. The second integral of (74) therefore gives the part of A due to surface distributions of sources. The word "source" is nowhere here to be thought of too narrowly. One can always form a definite physical picture by thinking of the sources as actual sources, in the ordinary sense, of liquid, and A as a velocity vector. But since div $E = \rho$, E being the electrostatic intensity and ρ volume density of charge, "charge" plays the rôle of "source" when the vector under consideration is electrostatic intensity, and surface distributions of charge would give rise to integrals such as the one here being considered.

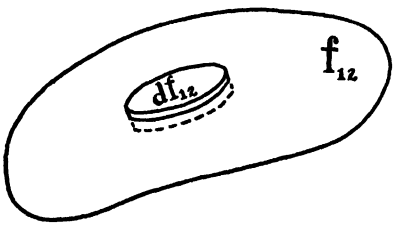

FIG. 8

The third integral gives the part of A due to a surface distribution of doublets, i.e., a surface one side of which has a distribution of sources, the other a distribution of sinks, the strength per unit area on the two sides being the same. This can be seen at once from results obtained in chapter ii, where the potential due to a doublet was considered. For there it was seen that a surface distribution of doublets whose polarization is μ per unit area of surface gives rise to an intensity

$$E = -\nabla \frac{1}{4\pi} \int \left(\mu, \nabla' \frac{1}{r} \right) d\sigma' .$$

Thus this third integral is due to a double sheet of moment pn per unit area.

Volume distributions of sources and sinks, surface distributions of sources and sinks, and double-sheet distributions of opposed sources and sinks are thus responsible for the three terms of (74).

E. Second example: It is now desired to determine a vector $\boldsymbol{A}$ from the data,

$$\text{curl } \boldsymbol{A}=\boldsymbol{i}\,, \qquad \text{div } \boldsymbol{A}=0\,, \tag{75}$$

$\boldsymbol{i}$ being a given vector-point function. It is further assumed, as before, that the vector vanishes at infinity as $1/r^2$. Then, since the divergence of $\boldsymbol{A}$ vanishes, it is possible to write

$$\boldsymbol{A}=\text{curl } \boldsymbol{B}\,. \tag{76}$$

Now the auxiliary vector $\boldsymbol{B}$ has its curl specified by the equation just given, but to specify it completely one may independently assign any convenient value to its divergence; in particular, one may assume

$$\text{div } \boldsymbol{B}=0\,. \tag{77}$$

Indeed, to any given vector $\boldsymbol{B}$ may be added a potential vector ∇u which will not disturb the value of curl $\boldsymbol{B}$, and such that

$$\text{div } (\boldsymbol{B}+\nabla u)=\text{div } \boldsymbol{B}+\nabla^2 u=0\,,$$

provided that

$$\nabla^2 u=-\text{div } \boldsymbol{B}\,,$$

an equation which has been shown to possess a solution. Then, from (75), (76), and (77),

$$\text{curl curl } \boldsymbol{B}=\boldsymbol{i}=\nabla \text{ div } \boldsymbol{B}-\nabla^2\boldsymbol{B}=-\nabla^2\boldsymbol{B}\,,$$

$$\nabla^2\boldsymbol{B}=-\boldsymbol{i}\,.$$

That is,

$$\nabla^2 B_x=-i_x\,,$$

$$\nabla^2 B_y=-i_y\,,$$

$$\nabla^2 B_z=-i_z\,.$$

Each of these three equations is known to possess a solution, it being possible to express the solution of all three by means of the vector equation

$$\boldsymbol{B}=\frac{1}{4\pi}\int\frac{\boldsymbol{i}'}{r}\,d\tau' , \tag{78}$$

where $\boldsymbol{i}$ is now expressed as a function of x', y', z'. Then,

$$\boldsymbol{A}=\operatorname{curl}\frac{1}{4\pi}\int\frac{\boldsymbol{i}'}{r}\,d\tau' . \tag{79}$$

F. General solution: Now from (69)

$$\boldsymbol{B}=-\nabla\frac{1}{4\pi}\int\frac{\operatorname{div}'\boldsymbol{B}'}{r}\,d\tau' ,$$

if curl $\boldsymbol{B}=0$, and where, under the sign, div′ $\boldsymbol{B}'$ is expressed as a function of x', y', z'; while from (78)

$$\boldsymbol{C}=\operatorname{curl}\frac{1}{4\pi}\int\frac{\operatorname{curl}'\boldsymbol{C}'}{r}\,d\tau' ,$$

if div $\boldsymbol{C}=0$, and where, under the sign, curl′ $\boldsymbol{C}'$ is expressed as a function of x', y', z'. Then if one sets $\boldsymbol{A}=\boldsymbol{B}+\boldsymbol{C}$,

$$\operatorname{div}\boldsymbol{A}=\operatorname{div}\boldsymbol{B} ,$$
$$\operatorname{curl}\boldsymbol{A}=\operatorname{curl}\boldsymbol{C} ,$$

and

$$\boldsymbol{A}=-\nabla\frac{1}{4\pi}\int\frac{\operatorname{div}'\boldsymbol{A}'}{r}\,d\tau'+\operatorname{curl}\frac{1}{4\pi}\int\frac{\operatorname{curl}'\boldsymbol{A}'}{r}\,d\tau' . \tag{80}$$

Equation (80) is an identity satisfied by any vector-point function $\boldsymbol{A}$, continuous together with its first derivatives. It expresses the fact, proved in Theorem 4, that a vector is determined by means of its divergence and curl, and gives explicitly a vector in terms of specified divergence and curl. It shows, moreover, that an arbitrary vector is expressible in terms of a potential vector and a solenoidal vector. This last can be seen directly. If, $\boldsymbol{A}$ being an arbitrary vector, one determines a function u from the equation

$$\nabla^2 u=\operatorname{div}\boldsymbol{A} ;$$

sets

$$\boldsymbol{B} = \nabla u \,,$$

and then writes

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C} \,,$$

it follows that

$$\text{div } \boldsymbol{B} = \text{div } \nabla u = \nabla^2 u = \text{div } \boldsymbol{A} \,,$$

from which it results at once that the divergence of $\boldsymbol{C}$ is zero, i.e., it is a solenoidal vector.

The purpose of this section has been chiefly to show the importance of the concepts of divergence and curl; to show that they can be specified independently, and that their specification determines uniquely a vector; and to indicate the method of determining the vector when the divergence and curl are specified.

§ 7. *Curvilinear Co-ordinates.*—The general theory of curvilinear co-ordinates is developed in the text (§ 26), and the expressions for div $\boldsymbol{A}$, curl $\boldsymbol{A}$, $\nabla\Phi$, and $\nabla^2\Phi$ in general orthogonal curvilinear co-ordinates are calculated (§§ 26 and 44). These equations will be re-written, near the end of this section, for convenience in reference. Before this is done, however, the method of obtaining these relations will be briefly discussed.

Let two vectors be associated with a point P in space (or, more generally, with every point in space). Let the two vectors be represented by directed line segments each placed with its initial point at P. Then the two vectors (whose magnitudes are assumed to be of the same kind, as regards physical dimensions) are said to be the "same" vector provided the end points of the two directed line segments also coincide. Now a vector may be analytically specified by giving, at each point of space, the three components of the vector along the co-ordinate axes at the point in question. In this sense, a vector is an ordered set of three functions. But two vector fields which, at every point of space, are the "same" may be analytically specified by two entirely different ordered triples of functions, provided the two vector fields are referred to two different co-ordinate systems. Thus, at every point P of space, let a vector point directly away from a fixed point O, and have a length equal to the distance $O-P$. This vector, if one uses polar co-ordinates ρ, θ, φ, and a set of unit vectors $\boldsymbol{\rho}', \boldsymbol{\theta}', \boldsymbol{\varphi}'$, corresponds to the ordered set of functions $\rho, 0, 0$, while if one uses Cartesian co-ordinates x, y, z with origin at O, and unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, the same vector corresponds to the ordered set x, y, z.

Now one originally defines the scalar quantity div $\boldsymbol{A}$ in Cartesian co-ordinates. The operator "div" associates with the given vector field $\boldsymbol{A}$ a scalar field. Then one wishes, whatever system of co-ordinates he may be using, to be able to calculate at any point of space, the same numerical value div $\boldsymbol{A}$, given the same vector field $\boldsymbol{A}$. As a matter of direct computations, this is a long and unpleasant task. The work is greatly simplified by noting that (as the divergence theorem shows) the numerical value of div $\boldsymbol{A}$ at any point P is the limit of the total normal flux per unit volume of the vector $\boldsymbol{A}$ out through the surface of any small volume containing P, the limit being taken as this volume becomes smaller and smaller. This numerical value can be calculated directly in general orthogonal curvilinear co-ordinates, and the desired expression for div $\boldsymbol{A}$ is thus found very simply.

Second, given a vector field $\boldsymbol{A}$, one knows how, in Cartesian co-ordinates, to calculate an associated vector field curl $\boldsymbol{A}$. The problem now is, given this same vector field $\boldsymbol{A}$ in orthogonal curvilinear co-ordinates, to obtain the rule for calculating the same associated vector field curl $\boldsymbol{A}$. Again the task of direct transformation is avoided, this time by Stokes's theorem, rather than by the divergence theorem (see § 44).

Given a scalar field Φ, one can associate with it a vector field $\nabla\Phi$ by means of the operator ∇. One knows how to compute $\nabla\Phi$ from Φ in Cartesian co-ordinates. To learn how to compute the same vector $\nabla\Phi$ from Φ, using general orthogonal curvilinear co-ordinates, one might proceed to a direct examination of the general relations between differentiation with respect to x, y, z, and with respect to u, v, w; and the relations between the two ordered triples of functions which, in Cartesian and in curvilinear co-ordinates, furnish the analytical specification of the vector. Here this unpleasant task is avoided by merely noting that $\nabla\Phi$ is a vector whose component in any direction measures the rate of change of the scalar Φ in that direction. Such a vector can be set up directly in the curvilinear co-ordinate system.

The calculations just described lead to the formulas:

$$\operatorname{div} \boldsymbol{A} = \frac{1}{e_1 e_2 e_3}\left[\frac{\partial(A_u e_2 e_3)}{\partial u} + \frac{\partial(A_v e_1 e_3)}{\partial v} + \frac{\partial(A_w e_1 e_2)}{\partial w}\right];$$

$$\operatorname{curl} \boldsymbol{A} = \boldsymbol{u}'\frac{1}{e_2 e_3}\left[\frac{\partial(e_3 A_w)}{\partial v} - \frac{\partial(e_2 A_v)}{\partial w}\right] + \boldsymbol{v}'\frac{1}{e_3 e_1}\left[\frac{\partial(e_1 A_u)}{\partial w} - \frac{\partial(e_3 A_w)}{\partial u}\right]$$
$$+ \boldsymbol{w}'\frac{1}{e_1 e_2}\left[\frac{\partial(e_2 A_v)}{\partial u} - \frac{\partial(e_1 A_u)}{\partial v}\right];$$

$$\nabla\Phi = u' \frac{1}{e_1}\frac{\partial\Phi}{\partial u} + v' \frac{1}{e_2}\frac{\partial\Phi}{\partial v} + w' \frac{1}{e_3}\frac{\partial\Phi}{\partial w} ;$$

$$\nabla^2\Phi = \frac{1}{e_1 e_2 e_3}\left[\frac{\partial}{\partial u}\left(\frac{e_2 e_3}{e_1}\frac{\partial\Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{e_1 e_3}{e_2}\frac{\partial\Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{e_1 e_2}{e_3}\frac{\partial\Phi}{\partial w}\right)\right] .$$

The foregoing remarks have been made chiefly to prepare the ground for a discussion of a vector relation which has, apparently, not always been clearly understood. By checking components in Cartesian co-ordinates it was proved in § 4 of this Appendix that

$$\text{curl curl } \boldsymbol{A} = -\nabla^2\boldsymbol{A} + \nabla \text{ div } \boldsymbol{A} ,$$

where

$$\nabla^2\boldsymbol{A} = \boldsymbol{i}\nabla^2 A_x + \boldsymbol{j}\nabla^2 A_y + \boldsymbol{k}\nabla^2 A_z .$$

If any orthogonal curvilinear system of co-ordinates be used, one knows, from the equation written just above, how to calculate at every point the "same" vector curl curl $\boldsymbol{A}$ and ∇ div $\boldsymbol{A}$. But how is one to compute the "same" vector $\nabla^2\boldsymbol{A}$? One may very easily check that the vector

$$\boldsymbol{u}'\nabla^2 A_u + \boldsymbol{v}'\nabla^2 A_v + \boldsymbol{w}'\nabla^2 A_w$$

is not the "same" as the vector

$$\boldsymbol{i}\nabla^2 A_x + \boldsymbol{j}\nabla^2 A_y + \boldsymbol{k}\nabla^2 A_z .$$

If, for example,

$$\boldsymbol{A} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z ,$$

then

$$\boldsymbol{i}\nabla^2 A_x + \boldsymbol{j}\nabla^2 A_y + \boldsymbol{k}\nabla^2 A_z = 0 ,$$

whereas

$$\boldsymbol{\rho}'\nabla^2 A_\rho + \boldsymbol{\theta}'\nabla^2 A_\theta + \boldsymbol{\varphi}'\nabla^2 A_\phi = \boldsymbol{\rho}'\nabla^2\rho ,$$

$$= \boldsymbol{\rho}' \frac{2}{\rho} .$$

That is to say, the simple way in which one computes $\nabla^2\boldsymbol{A}$ from $\boldsymbol{A}$ in Cartesian co-ordinates does not, if carried formally over to general orthogonal curvilinear co-ordinates, produce the (same) desired vector. There is, of course, no reason to expect that so simple a rule would be found. The obvious way to learn how to compute $\nabla^2\boldsymbol{A}$ in any co-ordinate system is to make use of the fact that

$$\nabla^2\boldsymbol{A} = \nabla \text{ div } \boldsymbol{A} - \text{curl curl } \boldsymbol{A}$$

in Cartesian co-ordinates, and the further fact that one knows how, in any co-ordinate system, to compute the quantities on the right. Thus, from direct substitution of the results given above it follows that $\nabla^2 \boldsymbol{A}$ is a vector whose component in the u-direction is given by

$$\frac{1}{e_1}\frac{\partial}{\partial u}\left\{\frac{1}{e_1e_2e_3}\left[\frac{\partial}{\partial u}(e_2e_3A_u)+\frac{\partial}{\partial v}(e_3e_1A_v)+\frac{\partial}{\partial w}(e_1e_2A_w)\right]\right\}$$

$$-\frac{1}{e_2e_3}\left\{\frac{\partial}{\partial v}\left[\frac{e_3}{e_1e_2}\left(\frac{\partial(e_2A_v)}{\partial u}-\frac{\partial(e_1A_u)}{\partial v}\right)\right]-\frac{\partial}{\partial w}\left[\frac{e_2}{e_3e_1}\left(\frac{\partial(e_1A_u)}{\partial w}-\frac{\partial(e_3A_w)}{\partial u}\right)\right]\right\}.$$

The other two components of $\nabla^2 \boldsymbol{A}$ can be written at once by cyclic permutation of the indices 1,2,3, and of the letters u, v, w. It is clear that the expression for $\nabla^2 \boldsymbol{A}$ is much simpler in Cartesian co-ordinates than in other systems. In space polar co-ordinates, for example, the component of $\nabla^2 \boldsymbol{A}$ in the ρ-direction consists of eighteen terms, three of which involve A_ρ, A_θ, and A_φ; eight of which involve first derivatives of these quantities; and seven of which involve second derivatives of these quantities.

At various points in the text it is calculated, for a certain vector $\boldsymbol{A}$, that

$$i\nabla^2 A_x+j\nabla^2 A_y+k\nabla^2 A_z=iC_x+jC_y+kC_z,$$

where $\boldsymbol{C}$ is some given vector. One is then justified in concluding that, in general,

$$\nabla^2 \boldsymbol{A}=\boldsymbol{C},$$

or that

$$\nabla \text{ div } \boldsymbol{A}-\text{curl curl } \boldsymbol{A}=\boldsymbol{C}.$$

On account, however, of the complexity in the relations written just above, one is not often able to make convenient use of these equations to calculate $\boldsymbol{A}$ from a knowledge of $\boldsymbol{C}$ in co-ordinates other than Cartesian

TABLE FOR CHANGE OF UNITS

Explanation

A student is usually told the relations between various units. The only relation one ever actually uses, however, is not the relation between units, but rather the relation between measures of quantities when certain units are used. Thus one actually requires such equations as

$$e = \sqrt{4\pi}\, e_{\text{e s.u.}} ,$$

which states that the measure e of a certain charge in rational electrostatic units is $\sqrt{4\pi}$ times the measure of this same charge in c.g.s. electrostatic units.* Since only relations between measures are actually used, and since a statement of the (inverse) relation between the units themselves only adds confusion, the table will deal exclusively with measures.

The following table gives the numerical factors required for the various relations between measures. The first two columns contain the ordinary electrical entities listed under the heading "Rational Electrostatic" or "Rational Electromagnetic" according as one or the other of these rational units is used for the quantity in question in this book. Any number listed in the body of the table is the factor by which one must multiply the measure, in the vertically corresponding units, of the horizontally corresponding quantity in order to obtain the measure of this same quantity in the appropriate rational units. For example, suppose one requires the relationship between P, the polarization measured in rational electrostatic units, and polarization measured in c.g.s. electrostatic units. By observing the numerical factor written horizontally opposite P and vertically under the c.g.s. electrostatic heading, one writes

$$P = \sqrt{4\pi}\, P_{\text{e.s.u.}} .$$

Similarly,

$$\rho = [1.063 \times 10^{10}]\, \rho_{\text{coulombs per cu. cm.}} ,$$

$$B = \frac{1}{\sqrt{4\pi}}\, B_{\text{e m u}} .$$

* The absence of subscripts will indicate that the quantity in question is measured in rational units. When it is desirable to distinguish between rational electrostatic and rational electromagnetic, one may use the subscripts *r.e.s* and *r.e.m.*

Throughout the table the letter c stands for the ratio between electrostatic and electromagnetic measures of charge. This ratio is known to be numerically equal to the velocity of light. Thus $c=3\times10^{10}$. The factors for reduction to practical units are given in reduced numerical form, since when one uses these he is likely to want a definite numerical,

TABLE FOR CHANGE OF UNITS

	Rational Electrostatic	Rational Electromagnetic	C.G.S. Electrostatic	C.G.S. Electromagnetic	Practical	Rational Electrostatic	Rational Electromagnetic
Charge	e	.....	$\sqrt{4\pi}$	$c\sqrt{4\pi}$	$1\ 063\times10^{10}$	1	c
Continuous densities	ρ, η	.....	$\sqrt{4\pi}$	$c\sqrt{4\pi}$	$1\ 063\times10^{10}$	1	c
	....	i, l	$\frac{\sqrt{4\pi}}{c}$	$\sqrt{4\pi}$	0 3545	$\frac{1}{c}$	1
	P, μ	.....	$\sqrt{4\pi}$.	$c\sqrt{4\pi}$	.....	1	c
	...	M	$\frac{\sqrt{4\pi}}{c}$	$\sqrt{4\pi}$	.....	$\frac{1}{c}$	1
Electrical properties of matter	ϵ	...	1	c^2	.....	1	c^2
	..	μ	.....	1	...	.....	1
	σ	.	4π	$4\pi c^2$	$1\ 131\times10^{11}$	1	c^2
The electromagnetic field	Φ	.....	$\frac{1}{\sqrt{4\pi}}$	$\frac{1}{c\sqrt{4\pi}}$	$9\ 403\times10^{-1}$	1	c
	...	A	$\frac{1}{c\sqrt{4\pi}}$	$\frac{1}{\sqrt{4\pi}}$	.....	$\frac{1}{c}$	1
	E	.....	$\frac{1}{\sqrt{4\pi}}$	$\frac{1}{c\sqrt{4\pi}}$	$9\ 403\times10^{-1}$	1	c
	....	B	$\frac{1}{c\sqrt{4\pi}}$	$\frac{1}{\sqrt{4\pi}}$	0 2821	$\frac{1}{c}$	1
Capacity	C	.....	4π	$4\pi c^2$	$1\ 131\times10^{7}$	1	c^2

rather than a literal, answer. The units used, as "practical," are: coulombs for charge and charge densities; ampères for current densities; volts for potentials; volts per centimeter for electrical intensities; gausses for magnetic intensities; microfarads for capacity; and reciprocal ohm-centimeters for conductivity. The reciprocal of the conductivity in reciprocal ohm-centimeters is the resistivity in ohm-centimeters, i.e., the quantity which, when multiplied by length in centimeters and

divided by cross-section in square centimeters, gives resistance in ohms.

This table gives, in the first instance, the relations between measures in rational and in other units. It can be used, however, to obtain relations between measures in any units. Thus suppose one requires the relationship between measures of electrical intensity in c.g.s. electromagnetic and in volts per centimeter. One writes from the table,

$$E=\frac{E_{\text{e m.u.}}}{c\sqrt{4\pi}}=[9.403\times10^{-4}]E_{\text{volts per cm.}},$$

from which one obtains at once as the desired result,

$$E_{\text{e.m.u.}}=c\sqrt{4\pi}[9.403\times10^{-4}]E_{\text{volts per cm.}}.$$

FORMULA INDEX

All of the more important formulas of the text are included in this Index. The numbers on the left are the equation numbers, as they occur in the text. The numbers on the right are the numbers of the pages on which these equations first occur. Any question concerning the meaning of notations should be settled by reference to the text page indicated.

CHAPTER I

The force on a second charge due to a first:

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(12) $(\boldsymbol{F}_\epsilon)_\theta=\frac{\epsilon p}{4\pi}\frac{\sin\theta}{r^3}$ 18

The force on a complex:

(13) $\boldsymbol{F}=\boldsymbol{E}\Sigma e_i+(\boldsymbol{p},\nabla)\boldsymbol{E}$ 21

The torque on a complex

(14) $\boldsymbol{T}=[\boldsymbol{p},\boldsymbol{E}]$ 22

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(16) $\boldsymbol{p}=k\boldsymbol{E}$ 23

The potential due to a body: . . . . . .

(41) $4\pi\Phi=\int\frac{\rho' d\tau'}{r}+\int\left(\boldsymbol{P}',\nabla'\frac{1}{r}\right)d\tau'+\int\frac{\eta' d\sigma'}{r}+\int\left(\mu',\nabla'\frac{1}{r}\right)d\sigma'$ . . 54

(44) $=\int\frac{(\rho'-\operatorname{div}'\boldsymbol{P}')}{r}d\tau'+\int\frac{(\eta'+P_n')}{r}d\sigma'+\int\left(\mu',\nabla'\frac{1}{r}\right)d\sigma'$ . . 57

The mutual electrostatic energy of a set of bodies:

(51) $\left.\begin{aligned}\Psi&=\int\rho\Phi d\tau+\int(\boldsymbol{P},\nabla\Phi)d\tau+\int\eta\Phi d\sigma+\int(\mu,\nabla\Phi)d\sigma\\&=\int\Phi(\rho-\operatorname{div}\boldsymbol{P})d\tau+\int\Phi(\eta+P_n)d\sigma+\int(\mu,\nabla\Phi)d\sigma\end{aligned}\right\}$ 64

The force on a body:

(52) $\left.\begin{aligned}\boldsymbol{E}&=\int\rho\boldsymbol{E}d\tau+\int\eta\boldsymbol{E}d\sigma+\int(\boldsymbol{P},\nabla)\boldsymbol{E}d\tau+\int(\mu,\nabla)\boldsymbol{E}d\sigma\\&=\int\boldsymbol{E}(\rho-\operatorname{div}\boldsymbol{P})d\tau+\int\boldsymbol{E}(\eta+P_n)d\sigma+\int(\mu,\nabla)\boldsymbol{E}d\sigma\end{aligned}\right\}$ 64

The torque on a body:

(53) $\left.\begin{aligned}\boldsymbol{T}&=\int[\boldsymbol{r},\rho\boldsymbol{E}]d\tau+\int[\boldsymbol{r},\eta\boldsymbol{E}]d\sigma+\int[\boldsymbol{P},\boldsymbol{E}]d\tau+\int[\mu,\boldsymbol{E}]d\sigma+\int[\boldsymbol{r},(\boldsymbol{P},\nabla)\boldsymbol{E}]d\tau\\&\qquad+\int[\boldsymbol{r},(\mu,\nabla\boldsymbol{E})]d\sigma\\&=\int[\boldsymbol{r},\boldsymbol{E}(\rho-\operatorname{div}\boldsymbol{P})]d\tau+\int[\boldsymbol{r},\boldsymbol{E}(\eta+P_n)]d\sigma+\int[\mu,\boldsymbol{E}]d\tau\\&\qquad+\int[\boldsymbol{r},(\mu,\nabla)\boldsymbol{E}]d\sigma\end{aligned}\right\}$ . . 67

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The force on an interior charge due to non-neighboring charges:

(63) $\boldsymbol{E}^*=\boldsymbol{E}+\frac{1}{3}\boldsymbol{P}=-\nabla\Phi+\frac{1}{3}\boldsymbol{P}$ 86

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The potential due to volume and surface distributions of charge satisfies, at interior points, the equation:

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On any charged surface, the normal derivatives of the potential satisfy the boundary condition:

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The electrostatic problem for conductors:

(I) 100

a) $\nabla^2\Phi=0$ at all points

b) Either the constant potential Φ_i or the total charge e_i of each conductor must be known; thus Φ reduces, on the surface Σ_i of the ith conductor, to a known constant Φ_i or Φ reduces to an unknown constant, while

$$\int_{\Sigma_i}\eta d\sigma=-\int_{\Sigma i}\frac{\partial\Phi}{\partial n}d\sigma=e_i$$

c) Φ is regular at infinity

Gauss's theorem: The total charge e on and within any surface Σ is given by

(75) $-\int_{\Sigma}\frac{\partial\Phi}{\partial n}d\sigma=e$ 102

The divergence of a vector in curvilinear co-ordinates:

$$\text{div } A=\frac{1}{e_1e_2e_3}\left[\frac{\partial(A_ue_2e_3)}{\partial u}+\frac{\partial(A_ve_1e_3)}{\partial v}+\frac{\partial(A_we_1e_2)}{\partial w}\right]$$ 120

The nabla of a scalar in curvilinear co-ordinates:

$$\nabla\Phi=\frac{1}{e_1}\frac{\partial\Phi}{\partial u}\boldsymbol{u}'+\frac{1}{e_2}\frac{\partial\Phi}{\partial v}\boldsymbol{v}'+\frac{1}{e_3}\frac{\partial\Phi}{\partial w}\boldsymbol{w}'$$ 121

The Laplace operator in curvilinear co-ordinates:

$$\nabla^2\Phi=\frac{1}{e_1e_2e_3}\left[\frac{\partial}{\partial u}\frac{e_2e_3}{e_1}\frac{\partial\Phi}{\partial u}+\frac{\partial}{\partial v}\frac{e_1e_3}{e_2}\frac{\partial\Phi}{\partial v}+\frac{\partial}{\partial w}\frac{e_1e_2}{e_3}\frac{\partial\Phi}{\partial w}\right]$$ 121

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On a surface between two dielectrics:

$$\epsilon_1 \frac{\partial \Phi}{\partial n_1} + \epsilon_2 \frac{\partial \Phi}{\partial n_2} = 0 \qquad (84) \quad 144$$

On a surface between a dielectric and free space:

$$\epsilon \frac{\partial \Phi}{\partial n_1} + \frac{\partial \Phi}{\partial n_2} = 0 \qquad (85) \quad 144$$

On a surface between a conductor and a dielectric:

$$\epsilon \frac{\partial \Phi}{\partial n} = -\eta \qquad 145$$

The electrostatic problem for conductors and dielectrics:

(11)
- *a)* $\nabla^2 \Phi = 0$ at all points
- *b)* Φ is continuous everywhere except across surfaces where $\mu = 0$; across such surfaces $\Phi_1 - \Phi_2 = \mu$
- *c)* Across a surface between two dielectrics
$$\epsilon_1 \frac{\partial \Phi}{\partial n_1} + \epsilon_2 \frac{\partial \Phi}{\partial n_2} = 0$$
- *d)* Across a surface between a conductor and a dielectric
$$\epsilon \frac{\partial \Phi}{\partial n} = -\eta$$
- *e)* On the surface of a conductor
  - i) Φ is a known constant Φ_i, or
  - ii) Φ is an unknown constant, while
$$\int \epsilon \frac{\partial \Phi}{\partial n} \, d\sigma = -e_i$$
- *f)* Φ is regular at infinity

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The polarization of a dielectric in a given external field:

$$\boldsymbol{P} = (\epsilon - 1)[\boldsymbol{E}^0 - \nabla U] ,$$

where

- *a)* $\nabla^2 U = 0$
- *b)* U is continuous
- *c)* $\epsilon \frac{\partial U}{\partial n_i} - \frac{\partial U}{\partial n_0} = (\epsilon - 1) E_n^0$
- *d)* U is regular at infinity

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The potential due to a uniformly charged dielectric. Compute:

$$U = -\frac{1}{4\pi} \left(\boldsymbol{P}, \nabla \int \frac{d\tau}{r} \right) \qquad (94) \quad 154$$

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The magnetostatic vector potential:

(109) $A=\frac{1}{4\pi c}\sum\frac{e_i v_i}{r_i}$ 186

(108) $B=\text{curl } A$ 186

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(113) $A=\frac{1}{4\pi}\frac{j}{r}+\frac{1}{4\pi}\left[m,\nabla'\frac{1}{r}\right]$ 189

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(117) $A=\frac{1}{4\pi}\int\frac{i'}{r}d\tau'+\frac{1}{4\pi}\int\left[M',\nabla'\frac{1}{r}\right]d\tau'$ 194

(124) $=\frac{1}{4\pi}\int\frac{i'+\text{curl}' M'}{r}d\tau'+\frac{1}{4\pi}\int\frac{[M',n]}{r}d\sigma'$ 198

The magnetostatic vector B due to a body:

(125) $B=\frac{1}{4\pi}\int\left[i'+\text{curl}' M',\nabla'\frac{1}{r}\right]d\tau'+\frac{1}{4\pi}\int\left[[M',n],\nabla'\frac{1}{r}\right]d\sigma'$ . . . 199

The magnetostatic vector B due to volume currents only:

(126) $B=\frac{1}{4\pi}\int\left[i',\nabla'\frac{1}{r}\right]d\tau'$ 199

The magnetostatic vector B due to linear currents only:

(127) $B=\frac{I}{4\pi}\int\left[ds',\nabla'\frac{1}{r}\right]$ 199

At points in free space:

$$(129)\quad \left\{\begin{array}{r} \operatorname{div} A=0 \\ \operatorname{curl} A=0 \\ \operatorname{div} B=0 \\ \operatorname{curl} B=\operatorname{curl}\ \operatorname{curl} A=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 201$$

The magnetostatic vector B at a point in a body and due to non-neighboring charges:

$$(137)\quad B^{*}=\operatorname{curl} A-\frac{2}{3}M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ 207$$

At points within a body, the magnetostatic vector potential satisfies the differential equation:

$$(136)\quad \operatorname{curl}\operatorname{curl} A=i+\operatorname{curl} M \quad \ldots\ldots\ldots\ldots\ 207$$

On the surface of a magnetized body, the normal derivatives of the magnetostatic vector potential satisfy the boundary relation:

$$(138)\quad \frac{\partial A}{\partial n_1}+\frac{\partial A}{\partial n_2}=-[M,n] \quad \ldots\ldots\ldots\ldots\ 207$$

The magnetostatic problem:

$$(\mathrm{III})\quad \left\{\begin{array}{ll} a)^{*} & \nabla^2 A=-(i+\operatorname{curl} M) \\ b) & A \text{ is continuous} \\ c) & \dfrac{\partial A}{\partial n_1}+\dfrac{\partial A}{\partial n_2}=-[M,n] \\ d) & A \text{ is regular at infinity} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ 208$$

The magnetostatic force on a complex:

$$(148)\quad F=[j,B]+(m,\nabla)B \quad \ldots\ldots\ldots\ldots\ 217$$

The magnetostatic torque on a complex:

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The magnetostatic torque on a body:

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$$\text{curl curl } \boldsymbol{A} = \text{curl } \boldsymbol{B} = \text{curl } \boldsymbol{M} = 0 \quad \ldots \quad 221$$

The electrostatic and magnetostatic motional intensity:

$$\text{Total intensity} = \boldsymbol{E} + \frac{1}{c}[\boldsymbol{v},\boldsymbol{B}] \quad \ldots \quad 223$$

Ohm's law:

(160)
$$\begin{cases} i = \dfrac{\sigma E}{c} \\ \Phi_A - \Phi_B = cRI \end{cases} \text{ for linear circuits} \quad \ldots \quad 234$$

$$i = \frac{\sigma E}{c} \quad \text{for volume conductors} \quad \ldots \quad 234$$

The steady state of currents in volume conductors is characterized by

(163) *a*) $\nabla^2\Phi = 0$

(164) *b*) $\partial\Phi/\partial n = 0$ on the boundary between a conductor and a dielectric or free space

(165) *c*) $\sigma_1 \dfrac{\partial\Phi}{\partial n_1} + \sigma_2 \dfrac{\partial\Phi}{\partial n_2} = 0$ on the boundary between two conductors

235

For a body equipped with surface electrodes:

a) $\nabla^2\Phi = 0$ in the interior 236

b) $\Phi = \Phi_A$ on electrode A, etc. 236

c) $\dfrac{\partial\Phi}{\partial n} = 0$ on the remainder of the boundary 236

d) $\Phi_A I_A + \Phi_B I_B + \ldots = Q/c$ 237

e) $I_A + I_B + \ldots = 0$ 237

For a body equipped with very small electrodes:

a) $\nabla^2\Phi = 0$ everywhere except at interior points $A \ldots$ where interior electrodes are located, and at points $A' \ldots$ on the boundary, where surface electrodes are located

b) At $A \ldots$ and $A' \ldots$ Φ becomes infinite as

$$\frac{cI_A}{4\pi\sigma r} \cdots, \qquad \frac{cI_{A'}}{2\pi\sigma r} \cdots$$

c) $I_A + \ldots + I_{A'} + \ldots = 0$

238

For a body equipped with small circular surface electrodes:

$$\left.\begin{array}{ll} a) & \nabla^2\Phi=0 \text{ at all interior points} \\ b) & \dfrac{\partial\Phi}{\partial n}=\dfrac{cI_k}{2\pi\sigma a_k\sqrt{a_k^2-r_k'}} \text{ on the } k\text{th electrode} \\ c) & \dfrac{\partial\Phi}{\partial n}=0 \text{ on the remainder of the boundary} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad 240$$

CHAPTER IV

The field equations for free space:

(167) $\operatorname{curl} B=i+\dfrac{\dot{E}}{c}=\dfrac{\rho u+\dot{E}}{c}$ 260

(168) $\operatorname{curl} E=-\dfrac{\dot{B}}{c}$ 260

(169) $\operatorname{div} E=\rho$ 260

(170) $\operatorname{div} B=0$ 260

(171) $F=E+\dfrac{1}{c}[v,B]$ 260

The equation of continuity:

(172) $\operatorname{div}(\rho u)+\dot{\rho}=0$ 261

The field equations within matter:

(175) $\operatorname{curl} B=\dfrac{\mu\sigma E+\mu\epsilon\dot{E}}{c}$ 263

(176) $\operatorname{curl} E=-\dfrac{\dot{B}}{c}$ 263

(177) $\operatorname{div}\epsilon E=\rho$ 263

(178) $\operatorname{div} B=0$ 263

The circuital relations when the dependence on time is sinusoidal:

(180) $\operatorname{curl} B_1=\dfrac{(\mu\sigma+i\omega\mu\epsilon)}{c}E_1$ 264

(181) $\operatorname{curl} E_1=-\dfrac{i\omega}{c}B_1$ 264

FORMULA INDEX

The Poynting vector:

(183) $\boldsymbol{S}=c[\boldsymbol{E},\boldsymbol{B}]$ 266

The electromagnetic momentum:

(191) $\boldsymbol{G}=\frac{1}{c^2}\int \boldsymbol{S}\,d\tau$ 276

The solution of the field equations for free space—the retarded potentials:

(209) $\Phi=\frac{1}{4\pi}\int\frac{\{\rho\}}{r}\,d\tau$ 285

(210) $\boldsymbol{A}=\frac{1}{4\pi c}\int\frac{\{\rho\boldsymbol{u}\}}{r}\,d\tau$ 285

$\boldsymbol{B}=\text{curl }\boldsymbol{A}$ 186

(194) $\boldsymbol{E}=-\nabla\Phi-\frac{\dot{\boldsymbol{A}}}{c}$ 280

Alternative expression for the retarded potentials:

(212) $\Phi=\frac{1}{4\pi}\int\frac{\rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}}$ 291

(213) $\boldsymbol{A}=\frac{1}{4\pi c}\int\frac{\rho(\xi,\eta,\zeta)\,\{\boldsymbol{u}\}\,d\xi d\eta d\zeta}{\{r\}\left\{1-\frac{u_r}{c}\right\}}$ 292

The fields due to a moving set of relatively fixed charges:

(216) $\boldsymbol{B}=\frac{1}{4\pi}\int\left\{\frac{[\dot{\boldsymbol{u}},\boldsymbol{r}_1]}{c^2rH}+\frac{[\boldsymbol{u},\boldsymbol{r}_1][(\boldsymbol{r},\dot{\boldsymbol{u}})+c^2-(\boldsymbol{u},\boldsymbol{u})]}{c^3r^2H^3}\right\}\rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$ 296

(217) $\boldsymbol{E}=\frac{1}{4\pi}\int\left\{-\frac{\dot{\boldsymbol{u}}}{c^2rH}+\frac{\left(\boldsymbol{r}_1-\frac{\boldsymbol{u}}{c}\right)[(\dot{\boldsymbol{u}},\boldsymbol{r})+c^2-(\boldsymbol{u},\boldsymbol{u})]}{c^3r^2H^3}\right\}\rho(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$. 297

The field of a uniformly moving point charge:

(219) $\boldsymbol{E}=\frac{e(1-\beta^2)}{4\pi}\frac{\boldsymbol{R}}{R^3(1-\beta^2\sin^2\psi)^{3/2}}$ 298

(220) $\boldsymbol{B}=\frac{e(1-\beta^2)}{4\pi}\frac{[\boldsymbol{w},\boldsymbol{r}]}{R^3(1-\beta^2\sin^2\psi)^{3/2}}$ 298

The field of a uniformly moving extended charge:

$$(226)\qquad \Phi' = \frac{1}{4\pi}\int \frac{\rho' d\xi d\eta d\zeta'}{[(X-\xi)^2+(Y-\eta)^2+(Z'-\zeta')^2]^{1/2}} \quad \ldots\ldots\ldots \quad 301$$

$$(229)\qquad \left\{\begin{aligned} E_x &= -(1-\beta^2)^{-1/2}\frac{\partial \Phi'}{\partial X} \\ E_y &= -(1-\beta^2)^{-1/2}\frac{\partial \Phi'}{\partial Y} \\ E_z &= -\frac{\partial \Phi'}{\partial Z'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad 302$$

$$(230)\qquad \left\{\begin{aligned} B_x &= \beta(1-\beta^2)^{-1/2}\frac{\partial \Phi'}{\partial Y} \\ B_y &= -\beta(1-\beta^2)^{-1/2}\frac{\partial \Phi'}{\partial X} \\ B_z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad 302$$

The longitudinal and transverse electromagnetic mass:

$$(239)\qquad m' = \frac{1}{c}\frac{\partial G}{\partial \beta} \quad \ldots\ldots\ldots\ldots \quad 306$$

$$(240)\qquad m'' = \frac{G}{c\beta} \quad \ldots\ldots\ldots\ldots \quad 306$$

The longitudinal and transverse electromagnetic mass of the Lorentz electron:

$$(241)\qquad \left\{\begin{aligned} m' &= \frac{e^2}{6\pi a c^2}(1-\beta^2)^{-1/2} \quad \ldots\ldots\ldots\ldots \quad 307 \\ m'' &= \frac{e^2}{6\pi a c^2}(1-\beta^2)^{-1/2} \quad \ldots\ldots\ldots\ldots \quad 307 \end{aligned}\right.$$

The longitudinal and transverse electromagnetic mass of the Abraham electron:

$$(242)\qquad \left\{\begin{aligned} m' &= \frac{e^2}{4\pi a c^2}\left(\frac{2}{3}+\frac{4}{5}\beta^2+\frac{6}{7}\beta^4+\ldots.\right) \quad \ldots\ldots\ldots \quad 307 \\ m'' &= \frac{e^2}{8\pi a c^2}\left[\left(1+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{5}\right)\beta^2+\left(\frac{1}{5}+\frac{1}{7}\right)\beta^4+\ldots.\right] \quad \ldots \quad 307 \end{aligned}\right.$$

The rest mass of the Abraham or Lorentz electrons:

$$m_0 = \frac{e^2}{6\pi a c^2} \quad \ldots\ldots\ldots\ldots \quad 307$$

The field due to an oscillating dipole:

(247) $B=\frac{1}{R}\frac{\{[\ddot{p},R_1]\}}{4\pi c^2}+\frac{1}{R^2}\frac{\{[\dot{p},R_1]\}}{4\pi c}$ 311

(248) $E=\frac{1}{R}\frac{\{[R_1[R_1,\ddot{p}]]\}}{4\pi c^2}+\frac{1}{R^2}\frac{e}{4\pi}(R_1-\{r_1\})$ 311

The wave-zone field due to an arbitrarily moving point charge:

$B=\{[r_1,E]\}$ 316

(254) $E=\left\{\frac{e}{4\pi c^2 r^3 H^3}[r[R,\dot{u}]]\right\}$ 317

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