DAMAGE BOOK

UNIVERSAL
LIBRARY
OU_160295
UNIVERSAL
LIBRARY

HIGHER ALGEBRA

BY

S. BARNARD, M.A.

FORMERLY ASSISTANT MASTER AT RUGBY SCHOOL, LATE FELLOW AND LECTURER AT EMMANUEL COLLEGE, CAMBRIDGE

AND

J. M. CHILD, B.A., B.Sc.

FORMERLY LECTURER IN MATHEMATICS IN THE UNIVERSITY OF MANCHESTER
LATE HEAD OF MATHEMATICAL DEPARTMENT, TECHNICAL COLLEGE, DERBY
FORMERLY SCHOLAR AT JESUS COLLEGE, CAMBRIDGE

LONDON
MACMILLAN & CO LTD
NEW YORK · ST MARTIN'S PRESS
1959

This book is copyright in all countries which are signatories to the Berne Convention

First Edition 1936
Reprinted 1947, 1949, 1952, 1955, 1959

MACMILLAN AND COMPANY LIMITED
London Bombay Calcutta Madras Melbourne

THE MACMILLAN COMPANY OF CANADA LIMITED
Toronto

ST MARTIN'S PRESS INC
New York

PRINTED IN GREAT BRITAIN BY
LOWE AND BRYDONE (PRINTERS) LIMITED, LONDON, N.W.10

CHAPTER

EXERCISE XV (128).

Minors, Expansion in Terms of Second Minors (132, 133). Product of Two Determinants (134). Rectangular Arrays (135). Reciprocal Determinants, Two Methods of Expansion (136, 137). Use of Double Suffix, Symmetric and Skew-symmetric Determinants, Pfaffian (138-143).

EXERCISE XVI (143)

X. SYSTEMS OF EQUATIONS.

Definitions, Equivalent Systems (149, 150). Linear Equations in Two Unknowns, Line at Infinity (150-152). Linear Equations in Three Unknowns, Equation to a Plane, Plane at Infinity (153-157).

EXERCISE XVII (158).

Systems of Equations of any Degree, Methods of Solution for Special Types (160-164).

EXERCISE XVIII (164).

XI. RECIPROCAL AND BINOMIAL EQUATIONS.

Reduction of Reciprocal Equations (168-170). The Equation $x^n - 1 = 0$, Special Roots (170, 171). The Equation $x^n - A = 0$ (172). The Equation $x^{17} - 1 = 0$, Regular 17-sided Polygon (173-176).

EXERCISE XIX (177).

XII. CUBIC AND BIQUADRATIC EQUATIONS.

The Cubic Equation (roots α, β, γ), Equation whose Roots are $(\beta - \gamma)^2$, etc., Value of Δ, Character of Roots (179, 180). Cardan's Solution, Trigonometrical Solution, the Functions $\alpha + \omega\beta + \omega^2\gamma$, $\alpha + \omega^2\beta + \omega\gamma$ (180, 181). Cubic as Sum of Two Cubes, the Hessian (182, 183). Tschirnhausen's Transformation (186).

EXERCISE XX (184).

The Biquadratic Equation (roots α, β, γ, δ) (186). The Functions $\lambda = \beta\gamma + \alpha\delta$, etc., the Functions I, J, Δ, Reducing Cubic, Character of Roots (187-189). Ferrari's Solution and Deductions (189-191). Descartes' Solution (191). Conditions for Four Real Roots (192-194). Transformation into Reciprocal Form (194). Tschirnhausen's Transformation (195).

EXERCISE XXI (197).

XIII. THEORY OF IRRATIONALS.

Sections of the System of Rationals, Dedekind's Definition (200, 201). Equality and Inequality (202). Use of Sequences in defining a Real Number, Endless Decimals (203, 204). The Fundamental Operations of Arithmetic, Powers, Roots and Surds (204-209). Irrational Indices, Logarithms (209, 210). Definitions, Interval, Steadily Increasing Functions (210). Sections of the System of Real Numbers, the Continuum (211, 212). Ratio and Proportion, Euclid's Definition (212, 213).

EXERCISE XXII (214).

CHAPTER

XIV. INEQUALITIES.

Weierstrass' Inequalities (216). Elementary Methods (216, 217)

$(a_1b_1+a_2b_2+\ldots)^2 \leqslant (a_1^2+a_2^2+\ldots)(b_1^2+b_2^2+\ldots)$, (218).

For n Numbers $a_1, a_2 \ldots$

$$\frac{\Sigma a_r^{x+y+\ldots}}{n} \gtrless \frac{\Sigma a_r^x}{n} \cdot \frac{\Sigma a_r^y}{n} \cdot \ldots, \quad (218).$$

$$(a^x-1)/x \gtrless (a^y-1)/y, \quad (219).$$

$$xa^{x-1}(a-b) \lesseqgtr a^x-b^x \lesseqgtr xb^{x-1}(a-b), \quad (219).$$

$$(1+x)^n \gtrless 1+nx, \quad (220).$$

Arithmetic and Geometric Means (221, 222).

$$\left(\frac{ax+by+\ldots}{x+y+\ldots}\right)^{x+y+\ldots} \gtrless a^x b^y \ldots, \quad (222).$$

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n} \gtrless \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m,$$

and Extension (223). Maxima and Minima (223, 224).

EXERCISE XXIII (224).

XV. SEQUENCES AND LIMITS.

Definitions, Theorems, Monotone Sequences (228-232). Exponential Inequalities and Limits,

$$\left(1+\frac{1}{m}\right)^m > \left(1+\frac{1}{n}\right)^n \text{ and } \left(1-\frac{1}{m}\right)^{-m} < \left(1-\frac{1}{n}\right)^{-n} \text{ if } m>n,$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim\left(1-\frac{1}{n}\right)^{-n} = e, \quad (232, 233).$$

EXERCISE XXIV (233).

General Principle of Convergence (235-237). Bounds of a Sequence. Limits of Indetermination (237-240).

Theorems:

(1) Increasing Sequence (u_n), where $u_n - u_{n-1} < k(u_{n-1} - u_{n-2})$

(2) If $u_n>0$ and $u_{n+1}/u_n \to l$, then $u_n^{\frac{1}{n}} \to l$.

(3) If $\lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} (u_1+u_2+\ldots+u_n)/n = l$.

(4) If $\lim_{n\to\infty} a_n = a$, and $\lim_{n\to\infty} b_n = b$, then

$$\lim_{n\to\infty} (a_nb_1+a_{n-1}b_2+\ldots+a_1b_n)/n = ab, \quad (240\text{-}243).$$

Complex Sequences, General Principle of Convergence (243, 244).

EXERCISE XXV (244).

CHAPTER

XVI. CONVERGENCE OF SERIES (1).

Definitions, Elementary Theorems, Geometric Series (247, 248). *Series of Positive Terms.* Introduction and Removal of Brackets, Changing Order of Terms, Comparison Tests, $\Sigma 1/n^p$, D'Alembert's and Cauchy's Tests (248-254).

EXERCISE XXVI (254).

Series with Terms alternately Positive and Negative (256).

Series with Terms Positive or Negative. Absolute Convergence, General Condition for Convergence, Pringsheim's Theorem, Introduction and Removal of Brackets, Rearrangement of Terms, Approximate Sum, Rapidity of Convergence or Divergence (256-261).

Series of Complex Terms. Condition of Convergence, Absolute Convergence, Geometric Series, $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$.

If $u_n/u_{n+1} = 1 + a_n/n$, where $a_n \to a > 0$, then $u_n \to 0$.

Convergence of Binomial Series (261-263).

EXERCISE XXVII (264).

XVII. CONTINUOUS VARIABLE.

Meaning of Continuous Variation, Limit, Tending to $\pm\infty$, Theorems on Limits and Polynomials (266-268). Continuous and Discontinuous Functions (269, 270). Continuity of Sums, Products, etc., Function of a Function, $\lim_{x \to a} \phi\{f(x)\}$, Rational Functions, x^n (271). Fundamental Theorems (272). Derivatives, Tangent to a Curve, Notation of the Calculus, Rules of Differentiation (273-277). Continuity of $\phi\{f(x)\}$, Derivatives of $\phi\{f(x)\}$ and x^n (278, 279). Meaning of Sign of $f'(x)$ (279). Complex Functions, Higher Derivatives (279, 280). Maxima and Minima, Points of Inflexion (280-282).

EXERCISE XXVIII (282).

Inverse Functions, Bounds of a Function, Rolle's Theorem, Mean-Value Theorem (284-288). Integration (289). Taylor's Theorem, Lagrange's Form of Remainder (290, 291). Function of a Complex Variable, Continuity (291, 292).

EXERCISE XXIX (293).

XVIII. THEORY OF EQUATIONS (2), POLYNOMIALS (2), RATIONAL FRACTIONS (1).

Multiple Roots, Rolle's Theorem, Position of Real Roots of $f(x) = 0$ (295, 296). Newton's Theorem on Sums of Powers of the Roots of $f(x) = 0$ (297). Order and Weight of Symmetric Functions (298, 299). Partial Derivatives, Taylor's Theorem for Polynomials in x and in $x, y, \ldots$. Euler's Theorem for Polynomials, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \ldots = nu$ (299-302). A Theorem on Partial Fractions (302). The Equation

$$a_0\frac{\partial v}{\partial a_1} + 2a_1\frac{\partial v}{\partial a_2} + \ldots = 0, \quad (303).$$

EXERCISE XXX (304).

CHAPTER

XIX. EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND SERIES.

Continuity, Inequalities and Limits (306, 307). The Exponential Theorem, Series for a^x (307, 308). Meaning of an Irrational Index, Derivatives of a^x, $\log x$ and x^n (309). Inequalities and Limits, the way in which e^x and $\log x$ tend to ∞, Euler's Constant γ, Series for $\log 2$ (310-312). The Exponential Function $E(z)$, Complex Index (312, 313). Series for $\sin x$, $\cos x$ and Exponential Values (313). Use of Exponential Theorem in Summing Series (314).

EXERCISE XXXI (315).

Logarithmic Series and their Use in Summation of Series, Calculation of Logarithms (315-319). The Hyperbolic Functions (319-321).

EXERCISE XXXII (321).

XX. CONVERGENCE (2).

Series of Positive Terms. Cauchy's Condensation Test, Test Series $\Sigma \frac{1}{n(\log n)^p}$, (325). Kummer's, Raabe's and Gauss's Tests (326-328). Binomial and Hyper-geometric Series (328, 329). De Morgan and Bertrand's Tests (330).

Series with Terms Positive or Negative. Theorem, Abel's Inequality, Dirichlet's and Abel's Tests (330, 331). Power Series, Interval and Radius of Convergence, Criterion for Identity of Power Series (332-334). Binomial Series $1+nz+\ldots$ when z is complex (334). Multiplication of Series, Merten's and Abel's Theorems (335-338).

EXERCISE XXXIII (338).

XXI. BINOMIAL AND MULTINOMIAL THEOREMS.

Statement, Vandermonde's Theorem (340). *Binomial Theorem.* Euler's Proof, Second Proof, Particular Instances (341-345). Numerically Greatest Term, Approximate Values of $\sqrt[n]{1+x}$ (345-348).

EXERCISE XXXIV (349).

Use of Binomial Theorem in Summing Series, (Multinomial) Theorem (351-355).

EXERCISE XXXV (355).

XXII. RATIONAL FRACTIONS (2), (RECURRING) SERIES AND DIFFERENCE EQUATIONS.

Expansion of a Rational Fraction (357-359).

EXERCISE XXXVI (359).

Expansions of $\cos n\theta$ and $\sin n\theta/\sin\theta$ in Powers of $\cos\theta$ (360). Recurring Series, Scale of Relation, Convergence, Generating Function, Sum (360-363). Linear Difference Equations with Constant Coefficients (363-365).

EXERCISE XXXVII (365).

Difference Equations, General and Particular Solutions (367-370).

EXERCISE XXXVIII (370).

CHAPTER
XXIII. THE OPERATORS Δ, E, D. INTERPOLATION.

The Operators Δ, E, Series for $\Delta^r u_x$, u_{x+r}; $u_1+u_2+u_3+\ldots$ in Terms of u_1, Δu_1, $\Delta^2 u_1$, ... (373-379). Interpolation, Lagrange's and Bessel's Formulae (379-382). The Operator D, Value of $\left(\frac{d}{dx}\right)^n (uv)$ (382, 383).

EXERCISE XXXIX (384).

XXIV. CONTINUED FRACTIONS (1).

Definitions, Formation of Convergents, Infinite Continued Fractions (388-391). Simple and Recurring Continued Fractions (391-394).

EXERCISE XL (394).

Simple Continued Fractions, Properties of the Convergents, an Irrational as a Simple Continued Fraction (396-401). Approximations, Miscellaneous Theorems (402-406). Symmetric Continued Fractions, Application to Theory of Numbers (406-409). Simple Recurring Continued Fractions (409-411).

EXERCISE XLI (411).

XXV. INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

Solutions of $ax \pm by = \pm c$ (414-417). Two Equations in x, y, z; the Equation $ax \pm by \pm cz \pm \ldots = k$ (417-419).

EXERCISE XLII (419).

XXVI. THEORY OF NUMBERS (2).

Congruence, Numbers less than and prime to n, Value of $\varphi(n)$; $\Sigma\varphi(d_r) = n$, where d_r is any Divisor of n (421-424). Fermat's Theorem, Euler's Extension, Wilson's and Lagrange's Theorems (424-427).

EXERCISE XLIII (427).

Roots of a Congruence, the Linear Congruence, Simultaneous Congruences (428-431). Theorem on Fractions (431). The General Congruence, Division (mod n), Congruences to a Prime Modulus (432-434).

EXERCISE XLIV (434).

XXVII. RESIDUES OF POWERS OF A NUMBER, RECURRING DECIMALS.

Residues of a, ag, ag^2, ... (mod n), the Congruence $g^t \equiv 1 \pmod{n}$, (436, 437). An Odd Prime Modulus, Primitive Roots (438, 439). Decimal Equivalent of m/n, Number of Figures in the Period, Short Methods of Reckoning, Prime Factors of $10^t - 1$,

($t = 1, 2, \ldots 10$), (439-444).

EXERCISE XLV (442).

The Congruence $x^n \equiv a \pmod{p}$, Methods of Solution (444-445).

EXERCISE XLVI (445).

CHAPTER

Exercise XLVII (447).

Separation of the Roots, Sturm's Theorem, Fourier's Theorem (447-456).

Exercise XLVIII (456).

Newton's Method of Approximating to a Root, Fourier's Rule, Nearly Equal Roots (456-460). Horner's Method (461-465).

Exercise XLIX (466).

XXIX. Implicit Functions, Curve Tracing.

Implicit Functions, Rule for Approximations (467-469). Series for Roots of a Cubic Equation (when all Real) (470).

Exercise L (471).

Tangents, Asymptotes, Intersection of a Straight Line and Curve (473). Curve Tracing, Newton's Parallelogram (474-480).

Exercise LI (480).

XXX. Infinite Products.

Convergence, Absolute Convergence, Derangement of Factors, Expansion as a Series (485-488). Convergence discussed by Use of Logarithms (490).

Exercise LII (491).

XXXI. Permutations, Combinations and Distributions.

Combinations with Repetitions, Things not all Different (493). Distributions, Arrangement in Groups, Distribution in Parcels (494-497). Derangements, General Theorem (498, 499). Partition of Numbers, Table of p Partitions of n, Euler's Use of Series (500-504).

Exercise LIII (504).

Exercise LIV (506).

XXXII. Probability.

First Principles, Exclusive Events, Independent and Interdependent Events (508-513). Probability estimated by Frequency, Expectation, Successive Events (513-518). Probability of Causes, Value of Testimony, Application of Geometry (518-522).

Exercise LV (523).

XXXIII. Continued Fractions (2).

Expression of a Quadratic Surd as a Simple Continued Fraction, the Form $(\pm\sqrt{N}\pm b_1)/r_1$ (527-530). The form $\sqrt{A/B}$, Cycle of Quotients, the b and r Cycles, Solution in Integers of $Bx^2 - Ay^2 = M$ (530-534). The Form $\sqrt{N}$, Integral Solutions of $x^2 - Ny^2 = M$ (534-538). The Cycle belonging to $(\sqrt{N}+b)/r$ found by the G.C.M. Process (538).

Exercise LVI (540).

Miscellaneous Exercises A (543), B (555).

HIGHER ALGEBRA

CHAPTER I

THEORY OF NUMBERS (1)

In this section we discuss properties peculiar to whole numbers. The word 'number' is taken to mean 'whole number,' and, unless otherwise stated, 'positive whole number.'

1. Division. (1) Let b be any positive whole number, and consider the sequence

$$\ldots -3b,\ -2b,\ -b,\ 0,\ b,\ 2b,\ 3b, \ldots$$

continued indefinitely both ways. Any whole number a (positive, negative or zero) is either a term of this sequence, or it lies between two consecutive terms. Thus two numbers q and r can be determined uniquely so that

$$a = bq + r \quad \text{and} \quad 0 \leqslant r < b. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

To *divide* a by b is to find the numbers q and r which satisfy these conditions; q is called the *quotient* and r the *remainder*.

If $r = 0$, we say that a is *divisible* by b or is a *multiple* of b, and that b is a *divisor* or a *factor* of a. Among the divisors of a number we count the number itself and 1.

Whatever b may be, $0 = b \,.\, 0 + 0$; hence zero must be regarded as divisible by every whole number.

(2) If $r = b - r'$, we have

$$a = b(q+1) - r' \quad \text{and} \quad 0 \leqslant r' < b;$$

also if $r \geqslant \frac{1}{2}b$ then $r' \leqslant \frac{1}{2}b$. Hence it is always possible to find numbers Q and R such that

$$a = bQ + R \quad \text{and} \quad |\, R \,| \leqslant \tfrac{1}{2}b.$$

Ex. 1. *Every number is of one of the forms* $5n$, $5n \pm 1$, $5n \pm 2$.

For if any number is divided by 5, the remainder is one of the numbers 0, 1, 2, 5 − 2, 5 − 1.

Ex. 2. *Every square number is of one of the forms* $5n$, $5n \pm 1$.

The square of every number is of one of the forms $(5m)^2$, $(5m \pm 1)^2$, $(5m \pm 2)^2$. If these are divided by 5, the remainders are 0, 1, 4; and, since $4 = 5 - 1$, the forms are $5n$, $5n+1$, and $5n-1$.

Ex. 3. *The square of every odd number is of the form* $8n+1$.

For $(2k+1)^2 = 4k(k+1)+1$, and either k or $k+1$ must be even, so that $k(k+1)$ is divisible by 2.

2. Theorems on Division.

(1) *If both* a *and* b *are divisible by* c, *so also is* $ma \pm nb$.

(2) *If* r *is the remainder when* a *is divided by* b, *then* cr *is the remainder when* ca *is divided by* cb.

For if $a = bq + r$ and $0 \leqslant r < b$,

then $ca = (cb)q + cr$ and $0 \leqslant cr < cb$.

(3) *If both* a *and* b *are divisible by* c, *and* r *is the remainder when* a *is divided by* b, *then* (i) r *is divisible by* c *and* (ii) r/c *is the remainder when* a/c *is divided by* b/c.

For if $a = bq + r$ and $0 \leqslant r < b$, then since a and b are divisible by c, so also is r, which is equal to $a - bq$. Thus we have

$$a/c = (b/c)q + r/c \quad \text{where} \quad 0 \leqslant r/c < b/c.$$

3. Theorems on the Greatest Common Divisor.

(1) *If* r *is the remainder when* a *is divided by* b, *the common divisors of* a *and* b *are the same as those of* b *and* r.

For, since $a = bq + r$, every common divisor of b and r is a divisor of a; and, since $r = a - bq$, every common divisor of a and b is a divisor of r. This proves the statement in question.

(2) *If* a *and* b *are any two numbers, there exists a number* g, *and one only, such that the common divisors of* a *and* b *are the same as the divisors of* g.

For numbers (q_1, r_1), (q_2, r_2), etc., can always be found in succession, and uniquely, so that

$$a = bq_1 + r_1, \quad b = r_1 q_2 + r_2, \quad r_1 = r_2 q_3 + r_3, \text{ etc., } \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

where $$b > r_1 > r_2 > r_3 \ldots \geqslant 0.$$

Since the number of positive integers less than b is limited, a zero remainder must occur; and, supposing that $r_{n+1} = 0$, the process terminates with

$$r_{n-2} = r_{n-1} q_n + r_n, \quad r_{n-1} = r_n q_{n+1}.$$

Hence, by (**A**), the following pairs of numbers have the same common divisors :

$$(a, b) \quad (b, r_1), \quad (r_1, r_2), \quad \ldots, \quad (r_{n-1}, r_n);$$

and, since $r_{n-1} = r_n q_{n+1}$, the common divisors of r_{n-1} and r_n are the divisors of r_n. Hence the number g exists, and its value is r_n.

Since g is divisible by any common divisor of a and b, it is the *greatest common divisor* of these numbers and is known in elementary arithmetic as the greatest common measure (G.C.M.) of a and b.*

(3) *If a and b are each multiplied by any number m, or are divided by a common divisor m, then g, the greatest common divisor of a and b, is multiplied or divided by m.*

For each of the r's in the proof of (2) is multiplied or divided by m.

(4) If $a, b, c, \ldots$, are several numbers, and g_1 is the greatest common divisor of a and b, g_2 that of g_1 and c, g_3 that of g_2 and d, and so on, then the following sets of numbers have the same common divisors :

$$(a, b, c, d, \ldots), \quad (g_1, b, c, d, \ldots), \quad (g_2, c, d, \ldots), \text{ etc.}$$

Hence we arrive eventually at a number g, whose divisors are the common divisors of $a, b, c, \ldots$; also g is uniquely determined. This number g is the *greatest common divisor*, or the *greatest common measure* of $a, b, c, \ldots$.

4. Numbers Prime to each other. Two numbers, a and b, are said to be prime to each other when their greatest common divisor is 1, so that they have no common divisor except 1. This is often expressed by saying that a is prime to b, or that b is prime to a. The following theorems are fundamental.

(1) *If the product ab is divisible by a number m, and m is prime to one factor a, then m is a divisor of the other factor b.*

For, if a is prime to m, the greatest common divisor of a and m is 1 ; hence the greatest common divisor of ab and mb is b. But, by hypothesis, m is a divisor of ab, and is therefore a common divisor of ab and mb ; hence m is equal to, or is a divisor of, b.

(2) *If a is prime to b, and each of these numbers is a divisor of N, then ab is a divisor of N.*

Suppose that $N = aq$, then b is a divisor of aq, and since b is prime to one factor a, it must be a divisor of the other factor q. Let $q = mb$, then $N = mab$, and ab is a divisor of N.

* The process, here stated algebraically, is that used in Elementary Arithmetic.

(3) *If a is prime to b, positive integers x, y can be found such that*

$$ax - by = \pm 1.$$

For it follows from equations (A) of Art. 3, (2), that

$$r_1 = a - bq_1, \quad r_2 = -aq_2 + b(1 + q_1q_2),$$
$$r_3 = a(1 + q_2q_3) - b(q_1 + q_3 + q_1q_2q_3).$$

Continuing thus, *every remainder can be expressed in the form* $\pm(ax - by)$, *where x, y are positive integers.*

If a is prime to b, the last remainder is 1, and the theorem follows.

5. Theorems on Prime Numbers. A number which has no divisors except itself and 1 is called a *prime number*, or simply a *prime*. Numbers which are not prime are said to be *composite*.

(1) *A prime number, p, is prime to every number which is not a multiple of p.*

For, if a is any such number, q and r can be found such that $a = pq + r$ where $0 < r < p$. Now p and 1 are the only divisors of p, and as $r < p$, the only common divisor of p and r is 1, that is to say, p is prime to r and therefore to a.

(2) *If a prime p is a divisor of a product $abcd \ldots hk$, it is a divisor of at least one of the factors a, b, $\ldots$ k.*

For if the prime p is not a divisor of a, by Theorem (1) it is prime to a, hence it is a divisor of $bcd \ldots k$. If, in addition, p is not a divisor of b, it must be a divisor of $cd \ldots k$. Continuing thus, it can be shown that if p is not a divisor of any of the numbers a, b, c, $\ldots$ h, it must be a divisor of k.

6. Theorems on Numbers, Prime or Composite.

(1) *Every composite number N has at least one prime divisor.*

For, since N is not a prime, it has a divisor, n, different from N and from 1, which is *not greater than any other divisor*. Further, this divisor must be a prime ; for, otherwise, it would have a divisor *less* than itself and greater than 1, and this latter would be a divisor of N. This contradicts the hypothesis that n is not greater than any other divisor.

It follows that every composite number, N, can be expressed as the product of prime factors.

For, since N has at least one prime factor, p, we have $N = pa$, where $1 < p < N$. If a is not a prime, it has at least one prime factor, q ; and $a = qb$, where $1 < q < a$.

Thus, $N = pa = pqb$; and so on.

But the numbers less than N are limited; and $N>a>b>\dots$, therefore the set $N, a, b, \dots$ must finally end in a prime. Hence, N can be expressed in the form, $N=pqr \dots u$, where $p, q, r, \dots, u$ are all primes, not necessarily all different.

That is to say, any composite number, N, can be expressed as

$$N=p^a \cdot q^b \cdot r^c \cdot \dots \cdot u^s,$$

where $p, q, r, \dots, u$ are all different primes.

(2) *A composite number can be expressed as the product of prime factors in one way only.*

For suppose that $N=p^aq^br^c\dots=P^AQ^BR^C\dots$ where $p, q, r, \dots, P, Q, R, \dots$, are primes; then, since the prime P is a divisor of the product $p^aq^br^c\dots$, it is a divisor of one of the factors $p, q, r, \dots$, and is therefore equal to one of them. In the same way each of the set $P, Q, R, \dots$ is equal to one of the set $p, q, r, \dots$, and no prime factor can occur in one of the expressions for N which does not occur in the other. Suppose then that

$$N=p^aq^br^c\dots=p^Aq^Br^C\dots .$$

If $a \neq A$, one of them must be the greater. Let $A>a$, and suppose that $A=a+e$, then $q^b \cdot r^c \cdot \dots t^m=p^e \cdot q^B \cdot r^C \cdot \dots t^M$; but this is impossible, as the left-hand side of the equality is a number prime to p, and the right-hand side is divisible by p.

Hence, a must be equal to A; and similarly, $b=B, \dots, m=M$; and thus the two expressions for N are identical.

The above theorem is one of the most important in the Theory of Numbers, and the following propositions are immediately deducible.

(3) *If m is prime to each of the numbers, $a, b, \dots, k$, it is prime to the product $ab \dots k$.*

(4) *If a is prime to b, then a^n is prime to b^n, where n is any integer; and conversely.*

(5) *A number N is a prime, if it is not divisible by any prime number greater than* 1 *and less than, or equal to, $\sqrt{N}$.*

For, if $N=ab$, where $b \geqslant a$, then $N \geqslant a^2$; that is, $a \leqslant \sqrt{N}$.

(6) *The sequence of primes is endless.*

For, if p is any prime, the number $\lfloor p+1$ is greater than p and is not divisible by p or by any smaller prime. If then $\lfloor p+1$ is not a prime, it must have a prime divisor greater than p, and in either case a prime greater than p exists.

All the primes less than a given number N can be obtained in order by a process called the *Sieve of Eratosthenes*. The process consists in writing down in order all the numbers from 1 to $N-1$, and erasing all multiples of the primes 2, 3, 5, 7, ... which are less than $\sqrt{N}$. Nevertheless, the problem of discovering whether a large number is prime or composite is one of great difficulty.

Ex. 1. *If n is any number, prove that $n(n+1)(n+2)$ is divisible by* 6.

Of the two consecutive numbers, n and $n+1$, one is divisible by 2; and one of the three consecutive numbers, n, $n+1$, $n+2$, is divisible by 3. Hence the product $n(n+1)(n+2)$ is divisible by 2 and by 3; and, since 2 is prime to 3, $n(n+1)(n+2)$ is divisible by 6.

Ex. 2. *Prove that $3^{2n+1}+2^{n+2}$ is divisible by* 7.

We have $3^{2n+1}=3\,.\,9^n=3(7+2)^n=7k+3\,.\,2^n$, by the Binomial theorem,

and $2^{n+2}=4\,.\,2^n$;

$$\therefore\ 3^{2n+1}+2^{n+2}=7k+2^n(3+4)=7(k+2^n).$$

7. The Divisors of a Given Number N. *Let $N=p^a\,.\,q^b\,.\,r^c\,.\,\ldots$, where p, q, r, ..., are primes; and let n be the number, and s the sum, of the divisors of N, including* 1 *and* N.

Then, (1) $$n=(a+1)(b+1)(c+1)\ \ldots;$$

$$s=\frac{p^{a+1}-1}{p-1}\cdot\frac{q^{b+1}-1}{q-1}\cdot\frac{r^{c+1}-1}{r-1}\cdot\ \ldots\ .$$

For the divisors of N are the terms in the expansion of

$$(1+p+p^2+\ldots+p^a)(1+q+q^2+\ldots+q^b)(1+r+r^2+\ldots+r^c)\ \ldots;$$

and the expressions for n and s follow immediately.

(2) *The number of ways in which N can be expressed as the product of two factors, including N and* 1, *is $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, according as N is, or is not, a perfect square.*

For, if N is a perfect square, each of the numbers, a, b, c, ..., is even, and therefore n is odd; but if N is not a perfect square, at least one of these numbers is odd, and therefore n is even.

Further, if $d_1(=1)$, d_2, d_3, ..., d_{n-2}, d_{n-1}, $d_n(=N)$, are the divisors of N in ascending order, the different ways of expressing N as the product of two factors are:

$$d_1d_n,\quad d_2d_{n-1},\quad d_3d_{n-2},\quad \ldots,\quad d_xd_x,\qquad \text{when}\quad n=2x-1,$$

and $$d_1d_n,\quad d_2d_{n-1},\quad d_3d_{n-2},\quad \ldots,\quad d_yd_{y+1},\qquad \text{when}\quad n=2y.$$

Hence, the number of ways is either x or y, i.e. either $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, according as N is, or is not, a perfect square.

(3) *The number of ways in which N can be expressed as the product of two factors, which are prime to one another, is 2^{m-1}, where m is the number of different prime factors of N.*

For, such factors are the terms in the expansion of

$$(1+p^a)(1+q^b)(1+r^c)\dots;$$

and their number is 2^m. Hence the number of pairs is 2^{m-1}.

For example, if $N=2^2\cdot 3^3\cdot 5=540$, $n=(2+1)(3+1)(1+1)=24$,

$$s=\frac{2^3-1}{2-1}\cdot\frac{3^4-1}{3-1}\cdot\frac{5^2-1}{5-1}=1680, \quad \text{and} \quad m=3,\ 2^{m-1}=4.$$

8. The Symbol I[x/y]. If a is a fraction or an irrational number, the symbol $I(a)$ will be used to denote the integral part of a. Thus if $x=qy+r$ where $0\leqslant r<y$, then $I(x/y)=q$.

(1) *If n_1, n_2, n_3, ..., are any integers, and s is their sum and a is any number, then*

$$I[s/a]\geqslant I[n_1/a]+I[n_2/a]+I[n_3/a]+\dots.$$

Let $n_1=aq_1+r_1$, $n_2=aq_2+r_2$, $n_3=aq_3+r_3$, etc.; then

$$s=a(q_1+q_2+q_3+\dots)+(r_1+r_2+r_3+\dots).$$

Hence, $$I[s/a]=(q_1+q_2+q_3+\dots)+I[(r_1+r_2+r_3+\dots)/a];$$

and, since $q_1=I[n_1/a]$, $q_2=I[n_2/a]$, ..., the result follows.

(2) *The highest power of a prime p which is contained in $\lfloor\underline{n}$ is*

$$I[n/p]+I[n/p^2]+I[n/p^3]+\dots.$$

For, of the numbers from 1 to n inclusive, there are $I[n/p]$ which are divisible by p; of these $I[n/p^2]$ are divisible by p^2; and so on; hence the result follows.

9. Theorems. (1) *The product of any n consecutive integers is divisible by $\lfloor\underline{n}$.*

For $(m+1)(m+2)\dots(m+n)/\lfloor\underline{n}=\lfloor\underline{m+n}/\lfloor\underline{m}\,\lfloor\underline{n}$, and to show that the last expression is an integer it is sufficient to show that any prime p which occurs in $\lfloor\underline{m}\,\lfloor\underline{n}$ occurs to at least as high a power in $\lfloor\underline{m+n}$. Thus we have to show that

$$\begin{aligned}I[(m+n)/p]+I[(m+n)/p^2]+I[(m+n)/p^3]+\dots\\ \geqslant I[m/p]+I[m/p^2]+I[m/p^3]+\dots\\ +I[n/p]+I[n/p^2]+I[n/p^3]+\dots.\end{aligned}$$

Now $I[(m+n)/p]\geqslant I[m/p]+I[n/p]$, and the same is true if we replace p by p^2, p^3, ..., in succession: hence the result in question.

(2) *If n is a prime, C_r^n is divisible by n.*

For by the preceding $n(n-1)(n-2)\ldots(n-r+1)$ is divisible by $\lfloor r$, and since n is a prime and r is supposed to be less than n, $\lfloor r$ is prime to n.

Hence, $\lfloor r$ is a divisor of $(n-1)(n-2)\ldots(n-r+1)$

and $n(n-1)\ldots(n-r+1)/\lfloor r$ is divisible by n.

Thus *if n is a prime, all the coefficients in the expansion of $(1+x)^n$, except the first and last, are divisible by n.*

NOTE. The reader is supposed to be acquainted with what is said in elementary text-books about 'permutations and combinations' and the 'binomial theorem for a positive integral index.' In what follows, P_r^n denotes the number of permutations, and C_r^n the number of combinations, of n things taken r at a time.

Ex. 1. *Find the highest power of* 5 *contained in* $\lfloor 158$.

We have $I[158/5]=31$, $I[158/5^2]=I[31/5]=6$, $I\lfloor 158/5^3\rfloor=I\lfloor 6/5\rfloor=1$: therefore the required power has an index $=31+6+1=38$.

Ex. 2. *If n is an odd prime, the integral part of $(\sqrt{5}+2)^n-2^{n+1}$ is divisible by $20n$.*

Let $(\sqrt{5}+2)^n=N+f$ where $0<f<1$; and let $(\sqrt{5}-2)^n=f'$. Then, since $0<\sqrt{5}-2<1$, we have also $0<f'<1$.

Again, since n is odd, $N+f-f'=(\sqrt{5}+2)^n-(\sqrt{5}-2)^n=$an integer; hence, since f and f' are positive and less than 1, $f=f'$, and thus

$$N=2(C_1^n\,.\,2\,.\,5^{\frac{1}{2}(n-1)}+C_3^n\,.\,2^3\,.\,5^{\frac{1}{2}(n-3)}+\ldots+C_2^n\,.\,2^{n-2}\,.\,5+2^n).$$

Moreover, since n is a prime, C_r^n is divisible by n, and therefore $N-2^{n+1}$ is divisible by $20n$; which is the required result.

10. Numbers in Arithmetical Progression.

(1) *Let a be prime to n, then if the first n terms of the arithmetical progression x, $x+a$, $x+2a$, ..., are divided by n, the remainders are the numbers* 0, 1, 2, ..., $n-1$, *taken in a certain order.*

If we suppose that two of the terms as $x+ma$, $x+m'a$ leave equal remainders, then their difference $(m-m')a$ would be divisible by n. This is impossible, for a is prime to n and $|\,m-m'\,|<n$.

Therefore the remainders are all different, and as each is less than n, they must be the numbers 0, 1, 2, ..., $n-1$, taken in some order or other.

If the progression is continued beyond the n-th term, the remainders recur in the same order.

For the terms $x+ma$, $x+m'a$ leave the same remainder if $m=m'+qn$.

(2) *If a and n are not prime to one another and g is their greatest common divisor, the remainders recur in a cycle of n/g numbers.*

For let $a=ga'$, $n=gn'$, so that a' is prime to n'. The terms $x+ma$,

$x+m'a$ leave the same remainder if, and only if, $a(m-m')$ is divisible by n, that is, if $a'(m-m')$ is divisible by n'. Since a' is prime to n', this can only happen when $m-m'$ is divisible by n'. Thus the first n' terms leave different remainders and, after that, they recur in order.

11. Method of Induction. Many theorems relating to whole numbers can be proved by a process known as *mathematical induction*. In some cases this is the only method available. The method may be described as follows.

Let $f(n)$ be a function of an integral variable n. Suppose that a certain statement S, relating to $f(n)$, is true when $n=a$.

Further, suppose we can prove that, if S is true when $n=m$, it is also true when $n=m+1$.

Then since S is true when $n=a$, it is true when $n=a+1$, $a+2$, $a+3$, ... in succession, that is, when $n \geqslant a$.

Ex. 1. *Show that* $2^{2n}-3n-1$ *is divisible by* 9.

Let $f(n)=2^{2n}-3n-1$, then $f(1)=0$ and 0 is divisible by 9, so the theorem is true for $n=1$.

Again, $$f(n+1)-f(n)=2^{2(n+1)}-2^{2n}-3=3(2^{2n}-1).$$

Also $2^{2n}-1=(3+1)^n-1=3k$, where k is an integer,

$$\therefore\ f(n+1)-f(n)=9k.$$

Hence if $f(n)$ is divisible by 9, so is $f(n+1)$; and, since $f(1)$ is so divisible, it follows in succession that $f(2)$, $f(3)$, $f(4)$, etc., are so divisible; that is, the theorem holds for all values of n.

Or more easily, using the method of (9) Ex. 2, $f(n)=(3+1)^n-3n-1$, etc.

Ex. 2. *If* n *is a positive integer, prove that*

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}.$$

If $$u_n=\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1},$$

then $$u_{n+1}=\frac{1}{n+1}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}+\frac{1}{2n+1};$$

$$\therefore\ u_{n+1}-u_n=\frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{n}=-\frac{1}{2n}+\frac{1}{2n+1}.$$

Again, if $$v_n=1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2n-1},$$

then $$v_{n+1}=1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2n+1};$$

$$\therefore\ v_{n+1}-v_n=-\frac{1}{2n}+\frac{1}{2n+1}=u_{n+1}-u_n.$$

Hence if $u_n=v_n$, then $u_{n+1}=v_{n+1}$. Now the statement holds for $n=1$; hence, in succession, it is true for $n=2, 3, 4, \ldots$, that is, for any value of n.

EXERCISE I

1. If q is the quotient and r the remainder when a is divided by b, show that q is the quotient when a is divided by $b+1$, provided that $r \geqslant q$.

2. If a and b are prime to each other, show that

(i) $a+b$ and $a-b$ have no common factor other than 2;

(ii) a^2-ab+b^2 and $a+b$ have no common factor other than 3.

3. If a and b are prime to each other, and n is a prime, prove that

$$(a^n+b^n)/(a+b) \text{ and } a+b$$

have no common factor, unless $a+b$ is a multiple of n.

4. If a is prime to b and y, and b is prime to x, then $ax+by$ is prime to ab.

5. If $X=ax+by$ and $Y=a'x+b'y$, where $ab'-a'b=1$, the greatest common divisor of X and Y is the same as that of x and y.

6. If p is a prime, and $p=a^2-b^2$, then $a=\frac{1}{2}(p+1)$, $b=\frac{1}{2}(p-1)$.

7. Express 55 in the form a^2-b^2 in two ways.

8. If x takes the values 1, 2, 3, ..., in the expressions

$$\text{(i) } x^2+x+17, \quad \text{(ii) } 2x^2+29, \quad \text{(iii) } x^2+x+41,$$

the resulting values of the expressions are primes, provided that in (i) $x<16$, in (ii) $x<29$, and in (iii) $x<40$.

Verify for $x=15$, 28, 39 respectively.

9. If $f(x)$ is a polynomial, it cannot represent primes only.

[Let $u=f(x)$ and $v=f(x+ku)$, where k is any integer. Prove that u is a factor of v.]

10. For the values 2, 3, 4, ... 10 of x, the number $2 \cdot 3 \cdot 5 \cdot 7+x$ is composite. Hence write down nine consecutive numbers none of which is a prime.

11. If n is any odd number, then $n(n^2-1)$ is divisible by 24; and if n is an odd prime greater than 3, then n^2-1 is divisible by 24.

12. Show that 2^n+1 or 2^n-1 is divisible by 3, according as n is odd or even.

13. If n is prime to 5, then n^2+1 or n^2-1 is divisible by 5, and therefore n^4 is of the form $5m+1$.

14. If n is prime to 5, then n^5-n is divisible by 30; hence the fifth power of any number has the same right-hand digit as the number itself.

15. Show that $2^{2n}+1$ or $2^{2n}-1$ is divisible by 5, according as n is odd or even.

16. Show that $5^{2n}+1$ or $5^{2n}-1$ is divisible by 13, according as n is odd or even.

17. If 3^n-1 is divided by 13, show that the remainder is either 0, 2, or 8, according as n is of the form $3m$, $3m+1$, or $3m-1$; hence prove that 3^n-1 or $3^{2n}+3^n+1$ is divisible by 13, according as n is, or is not, a multiple of 3.

18. If 2^n+1 is a prime, then n must be a power of 2.

19. Show that $7^{2n}-48n-1$ is divisible by 2304.

20. Show that $7^{2n}+16n-1$ is divisible by 64.

21. Prove that $2^{2n+1}-9n^2+3n-2$ is divisible by 54.

22. Find the number of divisors of 2000, and their sum.

23. Let s be the sum of the divisors of N, excluding N itself; if $s=N$, then N is called a *perfect number*.
Show that, if 2^n-1 is a prime, then $2^{n-1}(2^n-1)$ is a perfect number; and find the three least numbers given by this formula.

24. If n, r, s are the numbers, and P, R, S the products, of the divisors of N, L, M respectively, where $N=LM$, and L and M are prime to one another, prove that (i) $n=r\,.\,s$, (ii) $P=R^s\,.\,S^r$.

25. If n is the number, and P the product, of the divisors of N, prove that $P^2=N^n$.

26. If the product of the divisors of N, excluding N itself, is equal to N, then N is the product of two primes or the cube of a prime.

27. If N has 16 divisors, it cannot have more than 4 prime factors, a, b, c, d, and it must be of one of the forms $abcd$, a^3b^3, a^3bc, a^7b, a^{15}. Hence find the smallest number having 16 divisors.

28. Find the smallest number with 24 divisors.

29. Prove by induction that $n(n+1)(n+2)\ldots(n+r-1)$ is divisible by $\lfloor\underline{r}$.

30. Find the highest power of the prime p in $\lfloor\underline{N}$, when

$$\text{(i) } p^r-1<N<p^r+p\,; \quad \text{(ii) } p^r-p<N<p^r.$$

31. If N is expressed as a polynomial in a prime p, with each of the coefficients less than p, and s is the sum of these coefficients, prove that the power of p contained in $\lfloor\underline{N}$ is $(N-s)/(p-1)$

32. Show that the index of the highest power of 2 contained in $\lfloor\underline{N}$ is $N-1$ when N is a power of 2, and $N-r$ when N is equal to 2^r-1.

33. Show that $\lfloor\underline{2n-1}/(\lfloor\underline{n}\,\lfloor\underline{n-1})$ is odd or even according as n is, or is not, a power of 2.

34. Prove that $\lfloor\underline{2n}$ is divisible by $\lfloor\underline{n}\,.\,\lfloor\underline{n+1}$.

35. Prove that, if g is the greatest common divisor of m and $n+1$, then $g\,.\,\lfloor\underline{m+n}$ is divisible by $\lfloor\underline{m}\,.\,\lfloor\underline{n+1}$.

36. Prove that the power of 2 in $\lfloor\underline{3n}$ is greater than or equal to the power of 2 in $\lfloor\underline{n}\,.\,\lfloor\underline{n+1}\,.\,\lfloor\underline{n+2}$.

37. Prove that, if n is greater than 2, then $\lfloor\underline{3n}$ is divisible by $\lfloor\underline{n}\,.\,\lfloor\underline{n+1}\,.\,\lfloor\underline{n+2}$.

38. If a, b, c, ..., are numbers whose sum is a prime, p, then

$$\lfloor\underline{p}/(\lfloor\underline{a}\,.\,\lfloor\underline{b}\,.\,\lfloor\underline{c}\ldots)$$

is an integer divisible by p.

39. If $u_n=(3+\sqrt{5})^n+(3-\sqrt{5})^n$, show that u_n is an integer, and that

$$u_{n+1}=6u_n-4u_{n-1}.$$

Hence prove that the integer next greater than $(3+\sqrt{5})^n$ is divisible by 2^n.

40. The integer next greater than $(\sqrt{7}+\sqrt{3})^{2n}$ is divisible by 2^{2n}.

41. If $2^n+1=xy$, prove that $x-1$ and $y-1$ are divisible by the same power of 2.

CHAPTER II

RATIONALS AND IRRATIONALS

1. Rational Numbers. Starting with the natural numbers, the number system is enlarged by

(i) the introduction of fractions, in order that division may be always possible ;

(ii) the introduction of negative numbers and zero, so that subtraction may be always possible.

We thus obtain the *system of rational numbers* or *rationals*, consisting of positive and negative integers and fractions, with the number zero.

This system can be arranged in a definite order so as to form the *rational scale.*

With reference to any two rationals x and y, the terms 'greater than,' 'equal to' and 'less than' are defined as referring to their relative positions on the rational scale.

To say that $x>y$ or $y<x$ is to say that x follows y on the scale. To say that $x=y$ is to say that x and y stand for the same number.

2. The Fundamental Laws of Order are as follows : (1) If $a=b$ then $b=a$. (2) If $a=b$ and $b=c$, then $a=c$. (3) If $a\geqslant b$ and $b>c$, or if $a>b$ and $b\geqslant c$, then $a>c$. (4) If $a>b$ then $-a<-b$.

Hence we deduce the following rules for equalities and inequalities, where it is assumed that zero is not used as a divisor :

(5) If $a=b$ then

$$a+x=b+x, \quad a-x=b-x, \quad ax=bx, \quad a/x=b/x.$$

(6) If $a=b$ and $x=y$, then

$$a+x=b+y, \quad a-x=b-y, \quad ax=by, \quad a/x=b/y.$$

(7) If $a>b$ then

$$a+x>b+x \quad \text{and} \quad a-x>b-x.$$

Also $ax\gtrless bx$ and $a/x\gtrless b/x$ according as x is positive or negative.

(8) If $a>b$ and $x>y$, then $a+x>b+y$, and if a and y or b and x ar both positive, then $ax>by$.

3. Fundamental Laws of Arithmetic. Any two rationals can be combined by the operations of addition, subtraction, multiplication and division, the result in each case being a definite rational number, excepting that zero cannot be used as a divisor. This is what is meant when it is said that the system of rationals is *closed* for these operations.

The fundamental laws of addition and multiplication are

(1) $a+b=b+a$, (2) $(a+b)+c=a+(b+c)$,

(3) $ab=ba$, (4) $(a+b)c=ac+bc$,

and (5) $(ab)c=a(bc)$.

The first and third constitute the *Commutative Law*, the second and fifth the *Associative Law*, the fourth the *Distributive Law*.

4. Theorem of Eudoxus.* *If a and b are any two positive rationals, an integer n exists such that $nb>a$.* This simply amounts to saying that an integer exists which is greater than a/b.

5. Representation of Numbers by Points on a Line. Take a straight line $X'OX$ as axis; in OX take a point 1 so that the segment $O1$ contains the unit of length. To find the point a which is to represent any positive rational a, let $a=m/n$. Divide the segment $O1$ into n equal parts and set off a length Oa, along OX, equal to m of these parts. The point which represents $-a$ is in OX', at the same distance from O as the point a.

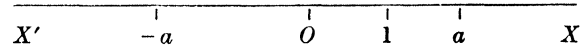


FIG. 1.

Points constructed thus represent the rational numbers in the following respects: (i) For every number there is one point and one only.

(ii) The points occur in the order in which the corresponding numbers stand on the rational scale.

The point X is generally taken to the right of O so that, if $a>b$, the point a is to the right of the point b.

6. Absolute Values. The *absolute* or *numerical* value of a is $+a$ or $-a$, according as a is positive or negative, and is denoted by $|a|$. Thus $|a-b|=|b-a|$. It is obvious that

and $|ab|=|a|.|b|$, $|a+b|\leqslant|a|+|b|$, $|a+b|\geqslant|a|-|b|$.

a is positive, to say that $|x|<a$ is the same as saying that $-a<x<a$.

If $|x|>a$, then $x>a$ or $x<-a$.

* Sometimes ascribed to Archimedes.

7. Large and Small Numbers. Whether a number is regarded as large or small depends on the purpose to which numbers are applied. An error of 6 inches would be large in measuring a table, but small in setting out a mile course.

If for purposes of comparison we choose some positive number ϵ which we regard as small, then any number x is said to be *small* if $|x| < \epsilon$.

Any number x is said to be *large* if $|x| > N$ where N is some *previously chosen* positive number, which is regarded as large.

We say that x is large or small compared with y when x/y is large or small. If x/y is neither large nor small, x and y are said to be *large or small numbers of the same order.*

We use the abbreviation 'y *is* $O(x)$' to indicate that y is of the same order as x.

If ϵ is small, numbers which are of the same order as ϵ, ϵ^2, ϵ^3, ... are called *small numbers of the first, second, third, ... orders* respectively.

If N is large, numbers of the same order as N, N^2, N^3, ... are called *large numbers of the first, second, third, ... orders* respectively.

If $x - y$ is small, we say that x and y are *nearly equal.*

8. Meaning of 'Tends.' To say that x *tends to zero* is to say that x varies in such a way that its numerical value becomes and remains less than any positive number that we may choose, no matter how small. This is expressed by writing $x \to 0$.

If $x - a$ tends to zero, where a is constant, we say that x *tends to* a, and we write $x \to a$.

If x tends to a and is always greater than a, we say that x tends to a *from above*, or *from the right.* This is expressed by writing $x \to a + 0$.

If x tends to a and is always less than a, we say that x tends to a *from below*, or *from the left*, and we write $x \to a - 0$.

To say that x *tends to infinity* ($x \to \infty$) is to say that x becomes and remains greater than any positive number that we may choose, however great that number may be.

In this case we also say that $-x$ *tends to* $-\infty$. Thus, as x tends to zero from above, $1/x$ tends to ∞, and $-1/x$ tends to $-\infty$.

9. Aggregate. Any collection of numbers is called an *aggregate* or *set*; the numbers themselves are the *elements* of the set.

If the number of elements exceeds any positive integer we choose, however great, we say that their number is *infinite*, and have what is called an *infinite set.*

10. System Everywhere Dense. An important property of the system of rationals is that *between any two rationals a and b there are infinitely many rationals.*

For if $a<b$ and k is any positive rational, it is easily seen that

$$a<(a+kb)/(1+k)<b.$$

This fact is sometimes expressed by saying that *the system of rationals is everywhere dense.*

11. Sequence. A succession of numbers $u_1, u_2, u_3, \ldots u_n, \ldots$, formed according to some definite rule, is called a *sequence*, which is generally denoted by (u_n). Such a rule defines u_n as a *function of the positive integral variable n.*

The rule may be quite arbitrary, and it is unnecessary that we should be able to express u_n in terms of n by an algebraical formula. For instance, u_n may denote the nth prime number, or the integral part of $\sqrt{n}$.

A sequence in which each term is followed by another term is called an *infinite sequence.*

12. Approximate Values. Suppose that the object of an experiment is to determine a certain number represented by A. No matter what care may be taken to ensure accuracy, it is unlikely that the *exact* value of A can be found. All that can be done in most cases is to find two numbers a and a' between which A must lie.

If we find that $a<A<a'$, we call a a *Lower Limit* and a' an *Upper Limit* to the value of A ; a and a' are also called *approximate values* of A.

Definition. If a is an approximate value of A, $A-a$ is called the (*Absolute*) *Error*, and $(A-a)/A$ the *Relative Error*, in taking a to represent A.

If $a<A<a'$, we say that *a and a' are approximate values of A, the first* **in defect** *and the second* **in excess** *with errors numerically less than $a'-a$.*

Observe that the error $A-a$ cannot exceed $a'-a$; therefore $a'-a$ is an upper limit to the error in taking a to represent A.

Again, if a is positive, we have $1/A<1/a$ and $A-a<a'-a$; hence

$$(A-a)/A<(a'-a)/a;$$

and $(a'-a)/a$ is an upper limit to the relative error.

We estimate the *comparative accuracy* of different experiments by comparing the *relative* errors : *the smaller the relative error, the greater is the degree of accuracy.*

In writing down decimal approximations of a number, we adopt the following scheme:

(i) If we write $\pi = 3{\cdot}1416$ (*approx.*), we shall mean that π is some number lying between 3·14155 and 3·14165, and we say that 3·1416 is the value of π *correct to* 5 *significant figures, or to* 4 *places of decimals.*

Since $3{\cdot}14155 < \pi < 3{\cdot}14165$, the error in representing π by 3·1416 is numerically less than 0·00005, or $\frac{1}{2} \cdot \frac{1}{10^4}$.

(ii) If we write $\pi = 3{\cdot}14159 \ldots$, we shall mean that the figures 3, 1, 4, ... 9 are those which actually occur in the decimal representation of π, and that π is some number lying between 3·14159 and 3·14160. The error in representing π by 3·14159 is positive and is less than $1/10^5$.

13. Fundamental Theorems. In theoretical and practical work, the following theorems are of fundamental importance.

If a, a', b, b' are given numbers and A, B are any numbers such that

$$a < A < a' \quad \text{and} \quad b < B < b',$$

then (i) $a + b < A + B < a' + b'$; (ii) $a - b' < A - B < a' - b$;

and if all the letters denote positive numbers,

(iii) $ab < AB < a'b'$; (iv) $a/b' < A/B < a'/b$.

The truth of (i) and (iii) follows from Art. 2, (8).

Proof of (ii). Since $a < A$ and $B < b'$, $\therefore\ a + B < A + b'$.

Subtracting $B + b'$ from each member of the last inequality,

$$(a + B) - (B + b') < (A + b') - (B + b'); \quad \therefore\ a - b' < A - B.$$

Similarly it can be shown that $A - B < a' - b$.

Proof of (iv). Since $a < A$ and $B < b'$, and a, A, B, b' are positive,

therefore $$aB < Ab'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

Dividing each side of (**A**) by Bb', which is positive,

$$\frac{aB}{Bb'} < \frac{Ab'}{Bb'}; \quad \therefore\ \frac{a}{b'} < \frac{A}{B}.$$

Similarly it can be shown that $A/B < a'/b$.

Ex. 1. *Show that the error in representing* $\frac{1234567}{3456789}$ *by* $\frac{12}{35}$ *is less than* 0·05.

We have $\frac{12}{35} < \frac{1234567}{3456789} < \frac{13}{34}$. Also $\frac{12}{35} = 0{\cdot}34 \ldots$ and $\frac{13}{34} = 0{\cdot}38 \ldots$. Hence,

$$\frac{13}{34} - \frac{12}{35} < 0{\cdot}39 - 0{\cdot}34 = 0{\cdot}05;$$

therefore the error is less than 0·05.

Ex. 2. *If* $0<x<1$, *then* $1-2x$ *is an approximate value of* $1/(1+x)^2$ *with an error in defect less than* $5x^2$.

By division we find that $\dfrac{1}{(1+x)^2} = 1-2x+\dfrac{3x^2+2x^3}{(1+x)^2} < 1-2x+5x^2$;

$$\therefore\ 1-2x < 1/(1+x)^2 < 1-2x+5x^2.$$

14. Non-perfect Powers. *If* A *is an integer which is not a perfect* n*-th power, then no rational* x *exists such that* $x^n = A$.

For, by hypothesis, x cannot be an integer. If possible, let $x = p/q$, where p/q is a fraction in lowest terms. Then $(p/q)^n$ is a fraction in lowest terms, and so it cannot be equal to the integer A.

15. Need for New Numbers. There are many purposes for which rational numbers are inadequate; for example: (i) there is no rational whose square equals 7. (ii) if the diameter of a circle is 1 inch and we denote the length of the circumference by x inches, it can be shown that x is not a rational number.

We extend our number-system by inventing a new class of numbers called *Irrationals*.

An irrational is defined by some rule which gives it ordinal rank with the rational numbers. Such a rule is the following.

Meaning of $\sqrt[n]{A}$. We consider a positive number A which is not a perfect nth power, so that no rational exists whose nth power is A.

The symbol $\sqrt[n]{A}$ (the nth root of A) is used to denote the irrational whose place among the numbers on the rational scale is determined by the following rule: $\sqrt[n]{A}$ *is to follow every positive rational whose* n*-th power is less than* A, *and to precede every rational whose* n*-th power is greater than* A. Thus if a, a' are positive rationals such that

$$a^n < A < a'^n,$$

then a, a' are *approximate values of* $\sqrt[n]{A}$ with errors, respectively in defect and excess, less than $a'-a$.

16. Representation of a Number by an Endless Decimal. Take the following instance. No rational exists whose square is 7. Hence 7 lies between two consecutive terms of the sequence $1^2, 2^2, 3^2, 4^2, \ldots$. Now $2^2<7<3^2$, therefore 7 lies between two consecutive terms of $(2{\cdot}0)^2$, $(2{\cdot}1)^2$, $(2{\cdot}2)^2$, ... $(3{\cdot}0)^2$. We find that $(2{\cdot}6)^2<7<(2{\cdot}7)^2$, therefore 7 lies between consecutive terms of $(2{\cdot}60)^2$, $(2{\cdot}61)^2$, ... $(2{\cdot}70)^2$, and we find that $(2{\cdot}64)^2<7<(2{\cdot}65)^2$.

However far this process is carried on, it can be continued further; for otherwise it would be possible to find a rational whose square is 7.

The process therefore gives rise to an *endless decimal* 2·6457 ... which is such that 7 lies between the two classes of numbers

$$2^2,\ (2{\cdot}6)^2,\ (2{\cdot}64)^2,\ (2{\cdot}645)^2 \ldots \quad \text{and} \quad \ldots (2{\cdot}646)^2,\ (2{\cdot}65)^2,\ (2{\cdot}7)^2,\ 3^2,$$

and $\sqrt{7}$ lies between the two classes of numbers

$$2,\ 2{\cdot}6,\ 2{\cdot}64,\ 2{\cdot}645 \ldots \quad \text{and} \quad \ldots 2{\cdot}646,\ 2{\cdot}65,\ 2{\cdot}7,\ 3.$$

This is what is meant when we write $\sqrt{7} = 2{\cdot}6457 \ldots$.

Observe that $\sqrt{7}$ is related to the decimal 2·6457 ... in the same way that 2/3 is related to 0·6666 ... .

For if we say that 2/3 *is equal to* the endless decimal 0·6666 ... , this is only an abbreviation for saying that 2/3 lies between the two classes of numbers

$$0{\cdot}6,\ 0{\cdot}66,\ 0{\cdot}666, \ldots \quad \text{and} \quad \ldots 0{\cdot}667,\ 0{\cdot}67,\ 0{\cdot}7.$$

It should further be noticed that the decimal equivalent of $\sqrt{7}$ cannot recur, for if it did, $\sqrt{7}$ would be a rational number.

By a process similar to that just described we can find the decimal representation of any root of a given number; but, of course, in practice other methods would be used.

17. Real Numbers. The rational and irrational numbers together form the system of *Real Numbers*. In Ch. XIII the rules for equalities and inequalities and the definitions of addition, subtraction, multiplication and division, already given for rationals, are extended so as to apply to all real numbers.

Theoretically it is essential to make this extension, but it is unnecessary for purposes of practical reckoning, where we replace any irrational which may occur by a sufficiently close rational approximation.

Ex. 1. *Given that* $\sqrt[3]{2} = 1{\cdot}259921 \ldots$, *prove that*

$$2 - d_n^3 < 6/10^n,$$

where d_n *is the decimal continued to n places.*

Let $d'_n = d_n + 1/10^n$, then $d_n^3 < 2 < d'^3_n$;

therefore $$2 - d_n^3 < d'^3_n - d_n^3 = (d'_n - d_n)(d_n^2 + d_n d'_n + d'^2_n).$$

Now $$d_n < d'_n < 1{\cdot}3;$$

$$\therefore\ d_n^2 + d_n d'_n + d'^2_n < (1{\cdot}3)^2 \,.\, 3 < 6;$$

$$\therefore\ 2 - d_n^3 < 6/10^n.$$

It follows that $$2 - d_n^3 \to 0 \text{ as } n \to \infty .$$

Hence we are justified in writing $(\sqrt[3]{2})^3 = 2$.

Ex. 2. Given that $\sqrt{5}=2{\cdot}236\ldots$, $\sqrt[3]{6}=1{\cdot}817\ldots$, *find a decimal approximation to* $\sqrt{5}/\sqrt[3]{6}$. *Find also an upper limit to the error.*

We have $$2{\cdot}236/1{\cdot}818<\sqrt{5}/\sqrt[3]{6}<2{\cdot}237/1{\cdot}817.$$

Now $2{\cdot}236/1{\cdot}818=1{\cdot}229\ldots$ and $2{\cdot}237/1{\cdot}817=1{\cdot}231\ldots$;

$$\therefore\ 1{\cdot}229<\sqrt{5}/\sqrt[3]{6}<1{\cdot}232.$$

$\therefore$ 1·229 is an approximate value of $\sqrt{5}/\sqrt[3]{6}$ with an error in defect less than 0·003.

Ex. 3. If a *is positive and less than* 1, *show that* $1-a/2$ *is an approximate value of* $\sqrt{(1-a)}$ *with an error in excess less than* $a^2/2$.

Applying the square root process to $1-a$, we have

$$\begin{array}{rl} N= & 1-a\ (\ 1-\frac{1}{2}a-\frac{1}{2}a^2 \\ & \overline{1} \\ 2-\frac{1}{2}a & \big|\ -a \\ & \big|\ -a+\frac{1}{4}a^2 \\ \hline 2-a-\frac{1}{2}a^2 & \big|\ -\frac{1}{4}a^2 \qquad\qquad =N-(1-\frac{1}{2}a)^2 \\ & \big|\ -\ a^2+\frac{1}{2}a^3+\frac{1}{4}a^4 \\ \hline & \qquad \frac{3}{4}a^2-\frac{1}{2}a^3-\frac{1}{4}a^4=N-(1-\frac{1}{2}a-\frac{1}{2}a^2)^2. \end{array}$$

Now $\frac{3}{4}a^2-\frac{1}{2}a^3-\frac{1}{4}a^4=\frac{1}{4}a^2(3-2a-a^2)=\frac{1}{4}a^2(3+a)(1-a)$; and since $0<a<1$,

$$\therefore\ a^2(3+a)(1-a)>0$$

and

$$(1-\tfrac{1}{2}a-\tfrac{1}{2}a^2)^2<1-a<(1-\tfrac{1}{2}a)^2.$$

18. The Function a^x. (1) *If* $a>1$ *and* n *is a positive integer, then*

$$a^n>1+n(a-1).$$

For, if $a=1+b$, by the Binomial theorem, $a^n=(1+b)^n>1+nb$.

(2) *If* $a>0$, *then* $a^n\gtrless 1$ *and* $a^{\frac{1}{n}}\gtrless 1$ *according as* $a\gtrless 1$.

For if $a>1$, then $a^n>1$. If $a<1$, let $a=1/b$; then $b>1$ and $a^n=1/b^n<1$.

Again, if $a>1$, then $a^{\frac{1}{n}}>1$; for otherwise we should have $a=(a^{\frac{1}{n}})^n<1$. Similarly, if $a<1$, then $a^{\frac{1}{n}}<1$.

(3) *If* $a>0$ *and* x *is a positive rational, then* $a^x\gtrless 1$ *according as* $a\gtrless 1$.

For if $x=p/q$, $a^x=\sqrt[q]{a^p}\gtrless 1$, according as $a\gtrless 1$.

(4) *If* $a>0$ *and* $x>y$, *then* $a^x\gtrless a^y$ *according as* $a\gtrless 1$.

For $a^x/a^y=a^{x-y}\gtrless 1$ according as $a\gtrless 1$.

(5) *If* $a>1$ *and* N *is any positive number, however great, a positive integer* m *can be found such that*

$$a^n>N \quad \textit{for} \quad n\geqslant m.$$

For we can choose m so that $m(a-1)>N-1$, and then by (1),

$$a^n\geqslant a^m>1+m(a-1)>N.$$

(6) *If a is positive and not equal to 1, then $a^x \to \infty$ or 0 as $x \to \infty$ according as $a \gtrless 1$.*

First let $a>1$. Then in (5), if $x>m$, $a^x>a^m>N$, and so $a^x \to \infty$.

If $a<1$, let $a=1/b$. Then $b>1$, and as $x \to \infty$, $b^x \to \infty$ and $a^x = 1/b^x \to 0$.

(7) *If $a>0$, a positive integer m can be found such that*

$$| a^{\frac{1}{n}} - 1 | < \epsilon \quad \text{for} \quad n \geqslant m,$$

where ϵ is any positive number, however small.

If $a>1$, as in (5), m can be chosen so that $(1+\epsilon)^m>a$. Hence if $n \geqslant m$,

$$(1+\epsilon)^n>a \quad \text{and} \quad a^{\frac{1}{n}} - 1 < \epsilon.$$

If $a<1$, assuming that $0<\epsilon<1$, m can be chosen so that $\left(\frac{1}{1-\epsilon}\right)^m > \frac{1}{a}$, that is, $(1-\epsilon)^m<a$. Hence, if $n \geqslant m$, $(1-\epsilon)^n \leqslant (1-\epsilon)^m < a$, and therefore $1 - a^{\frac{1}{n}} < \epsilon$.

If $a=1$, then $a^{\frac{1}{n}} - 1 = 0$ for all values of n.

(8) *If a is positive and $x \to 0$, then $a^x \to 1$.*

First, let $x \to 0$ through positive values. If $a>1$, choose m as in the first part of (7); then if $x<1/m$, $a^x - 1 < a^{\frac{1}{m}} - 1 < \epsilon$.

If $a<1$, choose m as in the second part of (7); then if $x<1/m$, $a^x > a^{\frac{1}{m}}$ and $1 - a^x < 1 - a^{\frac{1}{m}} < \epsilon$.

Thus in both cases, $a^x - 1 \to 0$ and $a^x \to 1$. If $x \to 0$ through negative values, let $a=1/b$ and $x=-y$. Then $a^x = 1/b^y \to 1$.

EXERCISE II

APPROXIMATE VALUES

1. If $a<A<a'$, where a, a' are known numbers, and any number c lying between a and a' is chosen as an approximate value of A, the numerical value of the error in representing A by c cannot exceed the greater of the numbers $(c-a)$, $(a'-c)$.

Hence show that, if $\frac{1}{2}(a+a')$ is chosen as an approximate value of A, the greatest numerical value of the error is less than it would be for any other approximation.

2. If $f(x) = 2+3x-4x^2-5x^3+8x^4$ and x is small, find an upper limit to the numerical value of the error in representing $f(x)$ by $2+3x$.

[If e is the error, $|e| = |-4x^2-5x^3+8x^4| < |4x^2| + |5x^3| + |8x^4| < 17x^2$, when $|x|<1$.]

3. The length of a man's step is approximately 30 in., this value being correct to the nearest inch. If the number of steps which he takes per mile is calculated on the assumption that he steps exactly 30 in., show that the error in the result may be as much as 35, but cannot exceed 36.

4. If $0{\cdot}2<x<0{\cdot}3$ and $0{\cdot}6<y<0{\cdot}7$, show that

$$\text{(i) } 0{\cdot}3<y-x<0{\cdot}5\text{; (ii) } 2<\frac{1}{y-x}<4\text{; (iii) } 0{\cdot}4<\frac{x^2+y^2}{(x+y)^2}<0{\cdot}91.$$

5. If x, y, z are all positive and $x<y<z$, show that

$$\frac{x^2}{z}<\frac{x^2+y^2+z^2}{x+y+z}<\frac{z^2}{x}.$$

6. If the numerical value of the error in taking a to represent A cannot exceed α, and the numerical value of the error in taking a' to represent a cannot exceed α', then $\alpha+\alpha'$ is an upper limit of the error in taking a' for A. [i.e. given that $a-\alpha<A<a+\alpha$ and $a'-\alpha'<a\leqslant a'+\alpha'$, it is required to show that

$$a'-(\alpha+\alpha')<A<a'+(\alpha+\alpha').]$$

7. If a, b are observed values of two numbers A and B, subject to errors (positive or negative), whose numerical values cannot exceed α and β respectively, show that an upper limit to the numerical value of the error in taking

(i) $a+b$ to represent $A+B$ is $\alpha+\beta$;

(ii) $a-b$ to represent $A-B$ is $\alpha+\beta$;

(iii) ab to represent AB is $\alpha b+\beta(a+\alpha)$;

(iv) a/b to represent A/B is $\frac{1}{b'}\left(\alpha+\frac{a}{b}\beta\right)$, where b' is any positive number less than $b-\beta$.

8. If α is small, show that $1/(1+\alpha)^3$ is nearly equal to $1-3\alpha$. Also show that, if $0<\alpha<0{\cdot}01$, then the error $<0{\cdot}0007$.

9. If $0<x<1$, show that the error in taking $\frac{2}{3}+\frac{1}{9}x$ as the value of $\frac{2+3x}{3+4x}$ is numerically less than $\frac{1}{6}x^2$.

10. If $x>100$, the error in taking $\frac{3}{4}$ as the value of $\frac{2+3x}{3+4x}$ is less than $0{\cdot}0007$.

[Divide $3x+2$ by $4x+3$ in *descending* powers of x.]

11. If x is small, show that (i) $\frac{1+ax}{1+bx}$ is nearly equal to $1+(a-b)x-b(a-b)x^2$;

(ii) if $a=5$, $b=2$ and $x<0{\cdot}1$, the error in this approximation is less than $0{\cdot}01$.

12. By the division transformation, show that

$$\frac{1+ax}{(1+bx)(1+cx)}=1+(a-b-c)x+\frac{R}{(1+bx)(1+cx)}\cdot x^2,$$

where $R=(b+c-a)(b+c+bcx)-bc$.

13. By means of Ex. 12, show that if x is small, $\frac{1+ax}{(1+bx)(1+cx)}$ is nearly equal to $1+(a-b-c)x$.

Show also that if $a=2$, $b=3$, $c=4$ and $x<0{\cdot}01$, the error in this approximation is less than $0{\cdot}003$.

14. The focal length (f) of a lens is given by the formula $1/f=1/v-1/u$. The values of u and v may be in error by as much as 2 per cent. of the corresponding true values. If the true values of u and v are 20 and 13, show that the value of f as calculated from the observed values may be in error as much as 9·9 per cent. of its true value.

15. The weight (w grammes) of water displaced by a solid is given by the formula $w = w_1 - w_2$, where w_1, w_2 are the weights (in grammes) of the solid (i) *in vacuo*, (ii) in water. In determining the values of w_1, w_2, the error in each case may be as much as 1 per cent. of the true value. If the true values of w_1, w_2 are 13 and 10, show that the value of w calculated from the observed values of w_1, w_2 may be in error by as much as 7 per cent. of its true value.

[The greatest and least possible values of w_1 are 13·13 and 12·87, those of w_2 are 10·1 and 9·9, hence those of $w_1 - w_2$ are 3·23 and 2·77.]

16. Given that the endless decimal corresponding to $\sqrt[5]{123}$ is $2{\cdot}618060\ldots$, prove that $123 - (2{\cdot}618060)^5 < 0{\cdot}0003$.

[Proceed as in Ex. 1 of Art. 17, using the identity

$$a'^5 - a^5 = (a' - a)(a'^4 + a'^3a + a'^2a^2 + a'a^3 + a^4).]$$

17. If x and y are positive and $x > y$, then $x - y^2/(2x)$ is an approximate value of $\sqrt{(x^2 - y^2)}$ with an error in excess less than $y^4/(2x^3)$.

[In Ex. 3 of Art. 17, put $a = y^2/x^2$.]

18. If a and b are nearly equal, show that $\frac{1}{2}(a+b) - \frac{(a-b)^2}{4(a+b)}$ is an approximate value of $\sqrt{ab}$ with an error in excess less than $\frac{(a-b)^4}{4(a+b)^3}$. [Use Ex. 17.]

19. Given that $\sqrt{14} = 3{\cdot}741\ldots$, $\sqrt[3]{3} = 1{\cdot}442\ldots$, for each of the following obtain a decimal approximation with an error in defect, finding also an upper limit to the error:

(i) $\sqrt{14} \times \sqrt[3]{3}$; (ii) $\sqrt{14}/\sqrt[3]{3}$; (iii) $\sqrt{\sqrt{14} + \sqrt[3]{3}}$.

20. If $0 < a^3 - A < \epsilon$, then $a - \sqrt[3]{A} < \frac{\epsilon}{3(\sqrt[3]{A})^2}$.

[Put $y = \sqrt[3]{A}$ in the identity $a^3 - y^3 = (a-y)(a^2 + ay + y^2)$.]

21. If x is small, prove that (i) $\sqrt[3]{1+x}$ is nearly equal to $1 + x/3$; (ii) if x is positive, the error in taking $1 + x/3$ to represent $\sqrt[3]{1+x}$ is negative, and is numerically less than $x^2/8$.

[Prove that if $x < 1$, then $\left(1 + \frac{x}{3}\right)^3 - (1+x) < \frac{10x^2}{27}$, and use the last example.]

22. Approximate values of A, B are $a = 3083$, $b = 8377$, each correct to the nearest digit. Using Ex. 7, show that an upper limit to the error in taking a/b for A/B is $1/10^4$.

[The error $< \frac{1}{8000}(0{\cdot}5 + 0{\cdot}4 \times 0{\cdot}5) < \frac{1}{10^4}$.]

23. In the last example, suppose a/b expressed as a decimal to n places, to the nearest digit. Show that the numerical value of the error in taking this as the value of A/B is less than

$$\frac{1}{10^4} + \frac{1}{2} \cdot \frac{1}{10^n}.$$

Hence show that it is useless to carry on the division to more than four places of decimals.

24. If $A = 3{\cdot}141592\ldots$, $B = 1{\cdot}414213\ldots$, by using Ex. 7 find how many figures must be kept in A, B to find an approximate value of AB with an error numerically less than $1/10^3$.

[Choose a, b so that $\alpha b + \beta(a+\alpha) < 1/10^3$. Now $b < 2$ and $a + \alpha < 4$, $\therefore$ we may take α, β such that $2\alpha + 4\beta < 1/10^3$. This is satisfied if $\alpha < \frac{1}{6} \cdot \frac{1}{10^3}$, $\beta < \frac{1}{6} \cdot \frac{1}{10^3}$. We may therefore take $a = 3{\cdot}1416$, $b = 1{\cdot}4142$.]

CHAPTER III

POLYNOMIALS (1)

1. Definitions. An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n,$$

where a_0, a_1, ... a_n are independent of x, is called a *polynomial in x of the n-th degree.*

A polynomial in x and y is the sum of a number of terms of the form ax^my^n, where a is independent of x, y, and m, n are positive integers.

A polynomial in any number of variables is similarly defined, and is sometimes called a *rational integral function* of the variables: its *degree* is that of its highest term.

If the sum of the indices of the variables in each term of a polynomial is a constant number, the polynomial is said to be *homogeneous.* For instance,

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n,$$

is a homogeneous polynomial in x, y of degree n.

A *constant* may be regarded as a polynomial of degree *zero.*

Polynomials of the first, second, third, fourth, ... degrees are known as linear, quadratic, cubic, quartic, ... functions respectively.

In some parts of Algebra, a homogeneous polynomial is called a *quantic.*

A homogeneous polynomial in x, y of degree n is often written in the form

$$a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{\lfloor 2}a_2x^{n-2}y^2 + \ldots + a_ny^n,$$

where binomial coefficients are introduced for convenience. This is shortly denoted by

$$(a_0, a_1, a_2, \ldots a_n\text{⟩}x, y)^n.$$

Using this notation, cubic and quartic functions of x may be denoted by

$$(a, b, c, d\text{⟩}x, 1)^3 = ax^3 + 3bx^2 + 3cx + d,$$

$$(a, b, c, d, e\text{⟩}x, 1)^4 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e.$$

If A, B, C are polynomials and $A = BC$, then B and C are called *factors* or *divisors* of A, and A is said to be (*exactly*) *divisible* by B or by C.

2. Division. Let A and B be polynomials in x, A being of higher degree than B, or of the same degree.

To 'divide A by B' is to find an identity of the form

$$A = BQ + R,$$

where Q and R are polynomials (or in special cases constants) and R is of lower degree than B. Such division is always possible: the result is *unique* and is called the *division transformation*, Q being the *quotient* and R the *remainder*.

3. Synthetic Division. In dividing ax^3+bx^2+cx+d by $x-h$, the reckoning may be arranged as on the left:

$$\begin{array}{cccc} a & +b & +c & +d \\ & ah & +ph & +qh \\ \hline a & +p & +q & +r \end{array},$$

where $p=ah+b$, $q=ph+c$, $r=qh+d$;

and then, the quotient $=ax^2+px+q$,

and the remainder $=r$.

$$\begin{array}{rl} x-h\,)\,ax^3+ & bx^2+\ cx+\ d\,(\,ax^2+px+q \\ & \underline{ax^3-ahx^2} \\ & \quad px^2+\ cx \\ & \quad \underline{px^2-phx} \\ & \qquad qx+\ d \\ & \qquad \underline{qx-qh} \\ & \qquad\quad r \end{array}$$

This process is known as *synthetic division*. It is justified by a comparison with the reckoning on the right.

It is evident that the method can be used to divide any polynomial in x by $x-h$. If any powers of x are missing, these must be supplied with zero coefficients.

Ex. 1. *Divide* $3x^4-x^3+2x^2-2x-1$ *by* $x+2$.

We divide by $x-(-2)$ as follows:

$$\begin{array}{c|ccccc} 1-(-2) & 3 & -1 & +2 & -2 & -1 \\ & & -6 & +14 & -32 & +68 \\ \hline & 3 & -7 & +16 & -34 & +67 \end{array}$$

$\therefore$ Quotient $=3x^3-7x^2+16x-34$; remainder $=67$.

To divide a polynomial A by $ax-b$, we may use the following rule:

Find the quotient Q and the remainder R in the division of A by $x-b/a$, then Q/a and R are respectively the quotient and remainder in the division of A by $ax-b$.

For $$A=\left(x-\frac{b}{a}\right)Q+R=(ax-b)\frac{Q}{a}+R.$$

Ex. 2. *Divide* x^3+2x^2-3x-4 *by* $2x-1$.

Dividing by $x-\frac{1}{2}$, as on the right, it is seen that

$1-\frac{1}{2}$	1	$+2$	-3	-4
		$+\frac{1}{2}$	$+\frac{5}{4}$	$-\frac{7}{8}$
	1	$+\frac{5}{2}$	$-\frac{7}{4}$	$-\frac{39}{8}$

the quotient $=x^2+\frac{5}{2}x-\frac{7}{4}$

and the remainder $=-\frac{39}{8}$.

$\therefore$ the required quotient $=\frac{1}{2}(x^2+\frac{5}{2}x-\frac{7}{4})$; remainder $=-\frac{39}{8}$.

If the divisor is of the second or higher degree, the process is as follows: to divide $4x^4+3x^3+x-1$ by x^2-2x+3, write the divisor in the form $x^2-(2x-3)$ and proceed thus,

$1-(2-3)$	4	$+3$	$+0$	$+1$	-1	
			-12	-33	-30	(a)
		8	$+22$	$+20$		(b)
	4	$+11$	$+10$	-12	-31	(c)

The quotient is $4x^2+11x+10$ and the remainder $-12x-31$. The first term in line (c) is 4; $4(2-3)=8-12$; put 8 in line (b) and -12 in (a); $8+3=11$; $11(2-3)=22-33$; put 22 in (b) and -33 in (a); and so on.

The reckoning will be understood if it is compared with ordinary division.

Ex. 3. *Express* $f(x)\equiv 4x^4+3x^3+x-1$ *in the form*

$$a(x^2-2x+3)^2+(bx+c)(x^2-2x+3)+dx+e,$$

where $a, b, \ldots e$ *are independent of* x.

Divide $f(x)$ by x^2-2x+3 as above. The remainder is $-12x-31$ and the quotient $4x^2+11x+10$.

Divide $4x^2+11x+10$ by x^2-2x+3. The remainder is $19x-2$ and the quotient 4. Hence

$f(x)=4(x^2-2x+3)^2+(19x-2)(x^2-2x+3)-12x-31.$

$1-(2-3)$	4	$+11$	$+10$
			-12
		8	
	4	$+19$	-2

4. Applications of Synthetic Division. The following transformations are often required.

Let $f(x)$ be a polynomial in x.

(1) *To express* $f(x)$ *as a polynomial in* $x-h$. Suppose that

$$f(x)=a_0(x-h)^n+a_1(x-h)^{n-1}+\ldots+a_{n-1}(x-h)+a_n.$$

Then a_n is the remainder when $f(x)$ is divided by $x-h$. If Q is the quotient, a_{n-1} is the remainder when Q is divided by $x-h$. If Q' is the quotient, a_{n-2} is the remainder when Q' is divided by $x-h$. Continuing thus, we can find all the coefficients.

The reckoning is arranged as in Ex. 1, overleaf.

Ex. 1. *Express* $2x^3+x^2-5x-3$ *as a polynomial in* $x-2$.

$1-2$	2	$+1$	-5	-3
		$+4$	$+10$	$+10$
$1-2$	2	$+5$	$+5$	$+7$
		$+4$	$+18$	
$1-2$	2	$+9$	$+23$	
		$+4$		
	2	$+13$		

$$\therefore\ 2x^3+x^2-5x-3=2(x-2)^3+13(x-2)^2+23(x-2)+7.$$

(2) *To expand* $f(x+h)$ *in powers of* x, we express $f(x)$ as a polynomial in $x-h$, and then change x into $x+h$.

Ex. 2. *If* $f(x)=2x^3+x^2-5x-3$, *find* $f(x+2)$.

Proceed as in Ex. 1 and substitute $x+2$ for x in the result, thus

$$f(x+2)=2x^3+13x^2+23x+7.$$

(3) *If* $f(n)$ *is a polynomial in* n *of degree* r, *to express* $f(n)$ *in the form*

$$a_0+a_1n+a_2n(n+1)+a_3n(n+1)(n+2)+\ldots \\ +a_rn(n+1)(n+2)\ldots(n+r-1).$$

This can be done by dividing by n, $n+1$, $n+2$, ... in succession.

Ex. 3. *Express* n^4+3n^2+2 *in the form*

$$a+bn+cn(n+1)+dn(n+1)(n+2)+en(n+1)(n+2)(n+3).$$

Divide n^4+3n^2+2 by n, the quotient by $n+1$, the quotient thus obtained by $n+2$, and finally the last quotient by $n+3$. The reckoning is shown below, and the required coefficients are the numbers in thick type.

$1+1$	1	$+0$	$+3$	$+0$	$+2$
		-1	$+1$	-4	
$1+2$	1	-1	$+4$	-4	
		-2	$+6$		
$1+3$	1	-3	$+10$		
		-3			
	1	-6			

$$\therefore\ n^4+3n^2+2=2-4n+10n(n+1)-6n(n+1)(n+2) \\ +n(n+1)(n+2)(n+3).$$

5. The Remainder Theorem. *If* $f(x)$ *is a polynomial, then* $f(h)$ *is the remainder when* $f(x)$ *is divided by* $x-h$.

This follows on substituting h for x in the identity

$$f(x)=(x-h)Q+R,$$

where Q and R are respectively the quotient and remainder in the division of $f(x)$ by $x-h$; for R is independent of x.

Corollary. *If $f(h)=0$, then $x-h$ is a factor of $f(x)$.*
For, in the preceding, $R=f(h)=0$.

Ex. 1. *Show that, if n is odd, $x+1$ is a factor of x^n+1.*
Since n is odd, $(-1)^n=-1$, hence if $x=-1$,

$$x^n+1=(-1)^n+1=-1+1=0.$$

$\therefore$ $x+1$ is a factor of x^n+1.

Ex. 2. *If $f(x)=3x^4+2x^3-6x-4$, find the value of $f(0{\cdot}2)$.*
$f(0{\cdot}2)$ is the remainder in the division of $f(x)$ by $x-0{\cdot}2$. Performing the division:

$1-0{\cdot}2$	3	$+2$	$+0$	-6	-4
		$+0{\cdot}6$	$+0{\cdot}52$	$+0{\cdot}104$	$-1{\cdot}1792$
	3	$+2{\cdot}6$	$+0{\cdot}52$	$-5{\cdot}896$	$-5{\cdot}1792$

Remainder $=-5{\cdot}1792=f(0{\cdot}2)$.

6. Theorem. *If $f(x)$ is a polynomial which vanishes when x has the values $\alpha_1, \alpha_3, \ldots \alpha_n$, no two of which are equal, then the product*

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$$

is a factor of $f(x)$.

For since $f(\alpha_1)=0$, by the Remainder theorem $f(x)=(x-\alpha_1)\,.f_1(x)$, where $f_1(x)$ is a polynomial. Therefore $f(\alpha_2)=(\alpha_2-\alpha_1)f_1(\alpha_2)$.

Now $f(\alpha_2)=0$ and $\alpha_2-\alpha_1\neq 0$, therefore $f_1(\alpha_2)=0$, and consequently $x-\alpha_2$ is a factor of $f_1(x)$. Hence

$$f_1(x)=(x-\alpha_2)f_2(x)\cdot \quad \text{and} \quad f(x)=(x-\alpha_1)(x-\alpha_2)f_2(x),$$

where $f_2(x)$ is a polynomial.

Proceeding in this way, we can show that $(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ is a factor of $f(x)$.

7. Theorem. *A polynomial $f(x)$ of the n-th degree cannot vanish for more than n values of x unless all its coefficients are zero.*

For otherwise (by Art. 6) the product of more than n expressions of the form $x-\alpha$ would be a factor of the polynomial. That is to say, a polynomial of the nth degree would have a factor of degree higher than n, which is impossible.

If all the coefficients of $f(x)$ are zero, $f(x)$ is said to *vanish identically*. Thus $(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b)$ vanishes identically; for it is of the first degree in x, and it vanishes for the values a, b, c of x.

Corollary. *An equation of the n-th degree cannot have more than n roots.*

NOTE. Later it will be shown that every equation of the nth degree has exactly n roots, not necessarily all different, which may be real or imaginary.

8. Theorem. *If for more than n values of x,*

$$ax^n + bx^{n-1} + \ldots + hx + k = a'x^n + b'x^{n-1} + \ldots + h'x + k',$$

then $a = a'$, $b = b'$, ..., $h = h'$, $k = k'$; that is to say the polynomials are identically equal.

For by the given conditions the polynomial

$$(a - a')x^n + (b - b')x^{n-1} + \ldots + (h - h')x + (k - k')$$

vanishes for more than n values of x; hence, by the preceding article, it follows that

$$a = a', \quad b = b', \quad \ldots, \quad h = h', \quad k = k'.$$

9. Polynomials in Two or More Variables.

(1) *If $f(x, y)$ is a polynomial in x and y which vanishes for all values of x and y, then all the coefficients of $f(x, y)$ are zero.*

The following method applies in all cases. Let the polynomial be

$$u = ax^2 + bxy + cy^2 + dx + ey + f,$$

which is supposed to vanish for all values of x and y; then

$$u = ax^2 + x(by + d) + (cy^2 + ey + f),$$

and for any given value of y, the last expression is a polynomial in x, which vanishes for all values of x;

hence $$a = 0, \quad by + d = 0, \quad cy^2 + ey + f = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Also the relations (A) hold for all values of y;

$$\therefore \; a = 0, \quad b = 0, \quad d = 0, \quad c = 0, \quad e = 0, \quad f = 0.$$

(2) *If for all values of x and y the polynomials $f(x, y)$ and $F(x, y)$ have equal values, then the coefficients of like terms in the polynomials are equal.*

This follows immediately from (1).

NOTE. The fact that two polynomials, u, v, are identically equal is often expressed by writing $u \equiv v$.

10. The Method of Undetermined Coefficients. It is often necessary to enquire if a given function can be expressed in a certain specified form. In cases of this kind we may use the method of 'Undetermined Coefficients,' which may be described as follows.

Assume that the function is expressed in the given form, where some of the coefficients are unknown. In order that the identity involved in this assumption may be true, the unknown coefficients must satisfy certain equations. If these equations have a solution, the function can be expressed in the specified form.

Ex. 1. *Show that values of* a, b, c *exist such that*

$$2x^2-5x-1\equiv a(x+1)(x-2)+b(x-2)(x-1)+c(x-1)(x+1),$$

and find these values.

The right-hand side $=a(x^2-x-2)+b(x^2-3x+2)+c(x^2-1)$

$$=(a+b+c)x^2-(a+3b)x-(2a-2b+c).$$

This is equal to the left-hand side for all values of x, if values for a, b, c can be found such that $a+b+c=2$, $a+3b=5$, $2a-2b+c=1$.

These equations have one solution, namely, $a=2$, $b=1$, $c=-1$.

In a question of this kind, *if it has been shown that an identity of the specified form exists, the values of the constants may be found by giving special values to the variables.*

Thus, assuming that an identity of the specified form exists, we can find a, b, c, by putting $x=1$, -1, 2 in succession.

Ex. 2. *Search for factors of* $S=2x^2-5xy-3y^2-x+10y-3$.

Since $2x^2-5xy-3y^2=(2x+y)(x-3y)$ we assume that

$$S=(2x+y+p)(x-3y+q)=2x^2-5xy-3y^2+x(p+2q)+y(q-3p)+pq.$$

Equating coefficients, we have the *three* conditions:

$$p+2q=-1, \quad -3p+q=10, \quad pq=-3.$$

From the first two, $p=-3$, $q=1$, and these values *happen* to satisfy the third condition.

11. Quadratic Functions of x and y. Every homogeneous function of x, y, z of the second degree is of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy. \quad \text{.....(A)}$$

When $z=1$ this reduces to

$$ax^2+2hxy+by^2+2gx+2fy+c, \quad \text{.....(B)}$$

which is the general form of a quadratic function of x and y.

From Ex. 2 of Art. 10 it appears that the cases in which an expression of the form (B) can be resolved into factors are *exceptional.* The condition that this may be possible is investigated in the next article.

12. Theorem. *The necessary and sufficient condition that the quadratic function*

$$S=ax^2+2hxy+by^2+2gx+2fy+c$$

can be expressed as the product of two linear factors is

$$\Delta=abc+2fgh-af^2-bg^2-ch^2=0.$$

Assume that $S=(px+qy+r)(p'x+q'y+r')$ and consider the equation $S=0$; this can be written

$$ax^2+2x(hy+g)+(by^2+2fy+c)=0,$$

giving

$$x=\frac{1}{a}\left\{-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}\right\}. \quad \text{.....(C)}$$

These values of x must be the same as $-\frac{1}{p}(qy+r)$ and $-\frac{1}{p'}(q'y+r')$; they are therefore rational functions of y. Hence the expression under the root sign in (C) is a perfect square. Now

$$(hy+g)^2-a(by^2+2fy+c)=y^2(h^2-ab)+2y(hg-af)+(g^2-ac),$$

and the condition that this may be a perfect square is

$$(hg-af)^2-(h^2-ab)(g^2-ac)=0$$

or

$$a(abc+2fgh-af^2-bg^2-ch^2)=0.$$

The following cases must now be considered:

(i) *If a is not zero*, it follows that

$$abc+2fgh-af^2-bg^2-ch^2=0. \quad \text{.....(D)}$$

(ii) *If $a=0$* the reasoning fails; but, except when $b=0$, we shall be led to the condition (D) by solving for y and proceeding as before.

(iii) *If both a and b are zero, and $h\neq 0$*, then

$$\begin{aligned} S &= 2hxy+2gx+2fy+c \\ &= 2x(hy+g)+\frac{2f}{h}(hy+g)+c-\frac{2fg}{h} \\ &= 2(hy+g)\left(x+\frac{f}{h}\right)-\frac{\Delta}{h^2}, \end{aligned}$$

for in this case $\Delta=2fgh-ch^2$. Hence S can be resolved into linear factors if, and only if, $\Delta=0$.

(iv) *If $a=b=h=0$*, S is not a quadratic function. Hence, in all cases, if S can be resolved into linear factors, then $\Delta=0$.

Conversely, if $\Delta=0$, then S can be resolved into linear factors.

This is evident in all cases, from the preceding.

NOTE. *The expression $abc+2fgh-af^2-bg^2-ch^2$ is generally denoted by Δ, and is called the* **discriminant** *of S. Observe that p, q, r, p', q', r', are not necessarily all real.*

Ex. 1. *If, in the above, p, q, r, p', q', r' are all real, prove that*

$$bc-f^2, \quad ca-g^2, \quad ab-h^2,$$

are all negative; and vice versa.

The square root of $(hy+g)^2-a(by^2+2fy+c)$, or

$$y^2(h^2-ab)+2y(hg-af)+g^2-ac,$$

is real if, and only if

$$h^2-ab \quad \text{and} \quad g^2-ac \quad \text{are positive.}$$

Similarly, by solving $S=0$ as a quadratic in y, we find that f^2-bc must be positive.

EXERCISE III

DIVISION AND FACTORS

In Exx. 1-6 find by synthetic division the quotient and remainder when the first expression is divided by the second.

1. $5x^4+6x+2$; $x+4$.
2. $x^5-5x^4+7x^2-1$; $x-2$.
3. $x^4-4x^3+6x^2-4x+1$; $3x-2$.
4. x^5-1 ; $2x+1$.
5. $3x^4-5x^3-11x^2+x-1$; x^2-2x-2.
6. $3x^5+6x^4-2x^3-x^2-2x+4$; x^2+2x-1.
7. If $4x^4-(a-1)x^3+ax^2-6x+1$ is divisible by $2x-1$, find the value of a.
8. Express x^3+2x^2+x+80 as a polynomial in (i) $x+5$; (ii) $x+1$; (iii) $x+\frac{1}{3}$.
9. If x is given by the equation $x^3-x^2-8x+3=0$, find the equation giving y when (i) $y=x-2$; (ii) $y=3x-1$; (iii) $y=3x+4$.
10. Express n^4 in the form

$$a+bn+cn(n+1)+dn(n+1)(n+2)+en(n+1)(n+2)(n+3).$$

11. If $f(x)=6x^3+4x^2+3x+2$, find the values of $f(-\frac{3}{2}), f(-\frac{2}{3}), f(\frac{1}{3})$.

Factorise the expressions in Exx. 12, 13.

12. $x^4-2x^3-7x^2+8x+12$.
13. $2x^4-3x^3y-6x^2y^2-8xy^3-3y^4$.
14. Find a polynomial in x of the third degree which shall vanish when $x=1$ and when $x=-2$, and shall have the values 4 and 28 when $x=-1$ and $x=2$ respectively. [The polynomial is of the form $(x-1)(x+2)(ax+b)$.]
15. Find a quadratic function of x which shall vanish when $x=-\frac{3}{7}$ and have the values 24 and 62 when $x=3$ and $x=4$ respectively.
[The function is of the form $(7x+3)(ax+b)$.]
16. Find a homogeneous function of x and y of the second degree which shall vanish when $x=y$ and also when $x=4$ and $y=3$, and have the value 2 when $x=2$ and $y=1$.
[The function is of the form $(x-y)(ax+by)$.]
17. If n is odd, show that x^m+1 is a factor of $x^{mn}+1$.
18. Express $x^{15}+1$ as the product of four factors.
[It follows from Ex. 17 that x^3+1 and x^5+1 are factors.]
19. Prove that the condition that ax^2+bx+c and $a'x^2+b'x+c'$ may have a common linear factor is

$$(ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

[$x-\alpha$ will be a common factor if $a\alpha^2+b\alpha+c=0$ and $a'\alpha^2+b'\alpha+c'=0$. Eliminating α between these equations, we obtain the required condition.]
20. Find the condition that ax^3+bx+c and $a'x^3+b'x+c'$ may have a common linear factor.
21. If n is any number, show that n^2 can be expressed in the form

$$a(n-1)^2+b(n-2)^2+c(n-3)^2,$$

and find the values of a, b, c.

22. If p, q, r, s are four consecutive numbers, show that p^2, q^2, r^2, s^2 are connected by a linear equation, and find this equation.
[This follows from Ex. 21.]

23. If x, y, z, w are four consecutive terms of an arithmetical progression, prove that x^2, y^2, z^2, w^2 are connected by a linear equation with constant coefficients. Find this equation.
[Write $x=p-3q$, $y=p-q$, $z=p+q$, $w=p+3q$, and find b, c, d such that $(p-3q)^2+b(p-q)^2+c(p+q)^2+d(p+3q)^2\equiv 0$.]

24. Show that constants a, b, c can be found such that

$$3x^2+16x+23\equiv a(x+2)(x+3)+b(x+3)(x+1)+c(x+1)(x+2)$$

and find the values of a, b, c.

In Exx. 25, 26 show that values of a, b can be found for which the first expression is divisible by the second, and find these values.

25. $x^5+2x^3+ax^2+b$; x^3+1.
[Find the remainder in the division of the first expression by the second, and make this identically zero.]

26. $ax^4+bx^3-12x^2+21x-5$; $2x^2+3x-1$.
[Arrange in ascending powers of x before dividing.]

Resolve the expressions in Exx. 27, 28 into factors.

27. $3x^2+xy-4y^2+8x+13y-3$.

28. $3x^2+2xy-8y^2-7x+16y-6$.

29. Show that $x^6-6x^5+24x^4-56x^3+96x^2-96x+64$ is a perfect cube, and find the cube root.

30. If x^3+3x^2-9x+c is the product of three factors, two of which are identical, show that c is either 5 or -27 ; and resolve the given expression into factors in each case.

31. (i) Find the remainder in the division of

$$x^3+3px+q \quad \text{by} \quad x^2-2ax+a^2.$$

(ii) If $x^3+3px+q$ has a factor of the form $(x-a)^2$, show that $q^2+4p^3=0$.

(iii) If the equation $x^3+3px+q=0$ has two equal roots, what is the relation connecting p and q ?

32. If x^4+px^2+qx+r can be expressed in the form $(x-a)^3(x-b)$, show that

$$8p^3=-27q^2 \quad \text{and} \quad p^2=-12r.$$

In Exx. 33-34 find the value, or the values, of λ for which the given expression can be resolved into factors, and for each value of λ resolve the expression into factors.

33. $x^2-xy+3y-2+\lambda(x^2-y^2)$.

34. $x^2+y^2-6y+4+\lambda(x^2-3y+2)$.

35. If $ax^2+2hxy+by^2+2gx+2fy+c$ is a perfect square, show that $f=\pm\sqrt{bc}$, $g=\pm\sqrt{ca}$, $h=\pm\sqrt{ab}$, provided that the negative sign does not precede an odd number of the radicals.

36. If $5x^2+4xy+y^2-24x-10y+24=0$, where x and y are real numbers, show that x must lie between $2-\sqrt{5}$ and $2+\sqrt{5}$ inclusive, and y between -4 and $+6$ inclusive.
[Solve for x and y in succession.]

37. If $2x^2+4xy+y^2-12x-8y+15=0$, where x and y are real, show that x cannot lie between $1\pm\sqrt{\frac{1}{2}}$ and y cannot lie between 1 and 3.

38. If $x^2+xy-y^2+2x-y+1=0$, where x and y are real, show that y cannot lie between 0 and $-\frac{8}{5}$, but x can have any real value whatever.

39. If $f(x)$ is a rational integral function of x and a, b are unequal, show that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is

$$[(x-a)f(b)-(x-b)f(a)]/(b-a).$$

[The remainder is a linear function of x, and may therefore be assumed to be $A(x-a)+B(x-b)$.]

40. (i) If ax^3+bx+c has a factor of the form x^2+px+1, show that $a^2-c^2=ab$.

(ii) In this case, prove that ax^3+bx+c and cx^3+bx^2+a have a common quadratic factor.

[For (ii), in the identity $ax^3+bx+c=(x^2+px+1)(ax+c)$ put $1/x$ for x and multiply each side by x^3.]

41. (i) If ax^5+bx^2+c has a factor of the form x^2+px+1, prove that

$$(a^2-c^2)(a^2-c^2+bc)=a^2b^2.$$

(ii) In this case, prove that ax^5+bx^2+c and cx^5+bx^3+a have a common quadratic factor.

13. Expansion of Products.

To expand the product

$$(a+b)(p+q+r)(x+y+z+w), \quad \text{.....(A)}$$

we multiply each term of $(a+b)$ by each term of $(p+q+r)$; we then multiply each of the products so formed by each term of $(x+y+z+w)$, and add the results.

Hence the expression (A) is equal to the sum of all the products which can be formed by choosing any term out of each of the factors and multiplying them together.

In general, *the product of any number of polynomials is equal to the sum of all the products which can be formed by choosing any term out of each polynomial and multiplying them together.*

14. Theorem.

If p_1, p_2, p_3, ... denote the sums of the products of a_1, a_2, a_3, ... , a_n, taken one, two, three, ... at a time, respectively, then

$$(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n.$$

For $(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)$ is equal to the sum of all the products which can be formed by choosing a term out of each of the factors and multiplying these terms together.

Choosing x out of each factor, we obtain the term x^n of the expansion.

Choosing x out of any $(n-1)$ of the factors and the a out of the remaining factor, we obtain

$$x^{n-1}(a_1+a_2+a_3+\ldots+a_n) \quad \text{or} \quad p_1x^{n-1}.$$

Choosing x out of any $(n-2)$ of the factors and an a from each of the two remaining factors, we obtain

$$x^{n-2}(a_1a_2+a_1a_3+\ldots+a_2a_3+\ldots) \quad \text{or} \quad p_2x^{n-2}.$$

Choosing x out of any $(n-r)$ of the factors and an a from each of the r remaining factors, we obtain p_rx^{n-r}.

Finally, choosing an a out of each of the factors, we obtain $a_1a_2a_3\ldots a_n$ or p_n;

$$\therefore\ (x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n.$$

In particular, if $a_1=a_2=a_3=\ldots=a_n=a$, we have

$$(x+a)^n=x^n+C_1^nx^{n-1}a+C_2^nx^{n-2}a^2+\ldots C_r^nx^{n-r}a^r+\ldots+a^n,$$

which is the *Binomial Theorem for a positive integral index.*

Many identities can be obtained by expanding a function of x in two different ways by the Binomial theorem and equating coefficients.

Ex. 1. *Use the identity* $(1+x)^{2n}=(1+2x+x^2)^n$ *to prove that*

$$2^n+\frac{n(n-1)}{\lfloor\underline{1}\rfloor\underline{1}}\cdot 2^{n-2}+\frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{2}\rfloor\underline{2}}\cdot 2^{n-4}+\ldots=\frac{\lfloor\underline{2n}}{\lfloor\underline{n}\rfloor\underline{n}}.$$

We have

$$(1+x)^{2n}=(x^2+2x)^n+C_1^n(x^2+2x)^{n-1}+C_2^n(x^2+2x)^{n-2}+\ldots+1.$$

The coefficient of x^n in $(1+x)^{2n}$ is C_n^{2n},

the coefficient of x^n in $(x^2+2x)^n$ is 2^n,

the coefficient of x^n in $C_1^n(x^2+2x)^{n-1}$ is $C_1^n\cdot C_1^{n-1}\cdot 2^{n-2}$,

the coefficient of x^n in $C_2^n(x^2+2x)^{n-2}$ is $C_2^n\cdot C_2^{n-2}\cdot 2^{n-4}$,

and so on. The result follows by equating coefficients in the identity.

The use of the Multinomial theorem, explained in Art. 16 on the opposite page, can sometimes be avoided, as in Ex. 2.

Ex. 2. *Find the coefficient of* x^r *in* $(1+x+x^2+x^3)^n$.

Here $1+\Sigma a_rx^r\equiv(1+x+x^2+x^3)^n=(1+x)^n(1+x^2)^n$

$$=(1+\Sigma C_s^nx^s)(1+\Sigma C_t^nx^{2t}).$$

Hence, if $r=2m$, then $a_r=C_m^n+C_{m-1}^nC_2^n+C_{m-2}^nC_4^n+\ldots$;

and, if $r=2m+1$, $a_r=C_m^nC_1^n+C_{m-1}^nC_3^n+C_{m-2}^nC_5^n+\ldots$.

15. Expansion of f(x+h) in Powers of x. Here $f(x)$ is supposed to be a polynomial in x.

When the coefficients in $f(x)$ are known numbers, we use synthetic division, as in Art. 3.

The following general theorem is required later : the notation is explained in Art. 1 of this chapter.

If $$f(x)=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{\lfloor 2}a_2x^{n-2}+\ldots+a_n,$$

then $$f(x+h)=a_0x^n+nA_1x^{n-1}+\frac{n(n-1)}{\lfloor 2}A_2x^{n-2}+\ldots+A_n,$$

where $A_1=a_0h+a_1$, $A_2=a_0h^2+2a_1h+a_2$, $A_3=a_0h^3+3a_1h^2+3a_2h+a_3$,

$$A_r=(a_0, a_1, a_2, \ldots a_r\}h, 1)^r.$$

For by the Binomial theorem,

$$\begin{aligned}f(x+h)&=a_0(x+h)^n+C_1^na_1(x+h)^{n-1}+C_2^na_2(x+h)^{n-2}+\ldots+a_n\\&=a_0x^n+k_1x^{n-1}+k_2x^{n-2}+\ldots+k_rx^{n-r}+\ldots+k_n,\end{aligned}$$

where $$k_r=a_0C_r^nh^r+a_1C_1^nC_{r-1}^{n-1}h^{r-1}+a_2C_2^nC_{r-2}^{n-2}h^{r-2}+\ldots$$

to $r+1$ terms.

Now $$C_p^nC_{r-p}^{n-p}=\frac{\lfloor n}{\lfloor p\,\lfloor n-p}\cdot\frac{\lfloor n-p}{\lfloor r-p\,\lfloor n-r}$$

$$=\frac{\lfloor n}{\lfloor r\,\lfloor n-r}\cdot\frac{\lfloor r}{\lfloor p\,\lfloor r-p}$$

$$=C_r^n\,.\,C_p^r.$$

Thus $$k_r=C_r^n(a_0h^r+C_1^ra_1h^{r-1}+C_2^ra_2h^{r-2}+\ldots+a_r)=C_r^nA_r,$$

which proves the theorem.

16. Multinomial Theorem for a Positive Integral Index.

(1) *Expansion of* $(a+b+c+\ldots+k)^n$. The product

$$(a+b+c+\ldots+k)(a+b+c+\ldots+k)\ldots \text{ to } n \text{ factors}$$

is the sum of all the products which can be formed by choosing a term out of each of the n factors and multiplying these terms together.

The expansion is therefore the sum of a number of terms of the form $a^\alpha b^\beta c^\gamma\ldots k^\kappa$, where each index may have any of the values 0, 1, 2, ... , n, subject to the condition

$$\alpha+\beta+\gamma+\ldots+\kappa=n. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Choose any positive integral or zero values for $\alpha, \beta, \gamma, \ldots, \kappa$ which satisfy this condition. To obtain the coefficient of the term $a^\alpha b^\beta c^\gamma \ldots k^\kappa$, divide the n factors into groups containing $\alpha, \beta, \gamma, \ldots, \kappa$ factors, respectively. Take a out of each factor of the first group, b out of each factor of the second group, and so on.

The number of ways in which this can be done is

$$\frac{\lfloor n}{\lfloor\alpha \lfloor\beta \lfloor\gamma \ldots \lfloor\kappa}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

and this is the coefficient of $a^\alpha b^\beta c^\gamma \ldots k^\kappa$ in the expansion (i.e. the number of times which this particular term occurs). Hence

$$(a+b+c+\ldots+k)^n = \Sigma \frac{\lfloor n}{\lfloor\alpha \lfloor\beta \lfloor\gamma \ldots \lfloor\kappa} \cdot a^\alpha b^\beta c^\gamma \ldots k^\kappa, \quad \ldots\ldots\ldots\text{(C)}$$

where $\alpha, \beta, \gamma, \ldots, \kappa$ are to have all possible sets of values which can be chosen from the numbers 0, 1, 2, ... , n, *subject to the condition*

$$\alpha+\beta+\gamma+\ldots+\kappa = n.$$

This result is called the *Multinomial Theorem* for a positive integral index.

Thus in the expansion of $(a-b-c+d)^6$ the coefficient of a^3b^2c is

$$(-1)^3 \frac{\lfloor 6}{\lfloor 3 \lfloor 2 \lfloor 1 \lfloor 0} = -60.$$

(2) *If there are m numbers $a, b, c, \ldots k$, the greatest coefficient in the expansion of $(a+b+c+\ldots+k)^n$ is*

$$\lfloor n / (\lfloor q)^{m-r} (\lfloor q+1)^r,$$

where q is the quotient and r the remainder when n is divided by m.

The coefficient $\lfloor n / \lfloor\alpha \lfloor\beta \lfloor\gamma \ldots \lfloor\kappa$ has its greatest value when $\lfloor\alpha \lfloor\beta \ldots \lfloor\kappa$ has its least value. Denote this value by v, then v is the least value of $\lfloor\alpha \lfloor\beta \ldots \lfloor\kappa$ if α and β alone vary and $\gamma, \ldots \kappa$ are constant.

Thus $\lfloor\alpha \lfloor\beta$ must have its least value subject to the condition $\alpha+\beta=C$, where C is a constant.

Let $u_\alpha = \lfloor\alpha \lfloor\beta = \lfloor\alpha \lfloor C-\alpha$, then $\frac{u_\alpha}{u_{\alpha-1}} = \frac{\alpha}{C-\alpha+1}$ and $u_\alpha \lessgtr u_{\alpha-1}$ according as $\alpha \lessgtr C-\alpha+1$, i.e. as $\alpha \lessgtr \frac{1}{2}(C+1)$. Hence u_α has its least value when $\alpha=\beta=\frac{1}{2}C$ if C is even, and when $\alpha=\frac{1}{2}(C+1)$, $\beta=\frac{1}{2}(C-1)$ if C is odd.

Thus α and β must be equal or differ by 1, and the same is true for any pair of the numbers $a, b, \ldots k$.

We conclude that some (possibly all) of the set $a, b, \ldots k$ are equal to a certain number q, each of the rest being equal to $q+1$.

Let r of the numbers be equal to $q+1$ and $m-r$ of them equal to q, then $0 \leqslant r < m$ and $r(q+1)+(m-r)q = n$, that is $qm+r=n$.

Hence q is the quotient and r the remainder when n is divided by m, which proves the theorem.

(3) *If there are m numbers $a, b, c, \ldots k$, the coefficient of x^r in the expansion of $(a+bx+cx^2+\ldots+kx^m)^n$ is $\Sigma \dfrac{\lfloor n}{\lfloor\alpha \lfloor\beta \lfloor\gamma \ldots \lfloor\kappa} . a^\alpha b^\beta c^\gamma \ldots k^\kappa$, where $\alpha, \beta, \gamma, \ldots \kappa$ have any of the values 0, 1, 2, ... n subject to the conditions*

$$\alpha+\beta+\gamma+\ldots+\kappa = n \quad \text{and} \quad \beta+2\gamma+\ldots+m\kappa = r.$$

This follows from (1) by putting $bx, cx^2, \ldots kx^m$ for $b, c, \ldots k$ in equation (C).

Ex. 1. *Find the coefficient of x^4 in the expansion of*

$$(1+2x+3x^2)^5.$$

$$\text{The coefficient} = \Sigma \frac{\lfloor 5}{\lfloor\alpha \lfloor\beta \lfloor\gamma} . 1^\alpha . 2^\beta . 3^\gamma,$$

where α, β, γ have any of the values 0, 1, 2, ... 5 such that

$$\alpha+\beta+\gamma = 5 \quad \text{and} \quad \beta+2\gamma = 4.$$

α	β	γ
1	4	0
2	2	1
3	0	2

The solutions of the second equation are (4, 0), (2, 1), (0, 2), and the possible values of α, β, γ are shown in the margin;

$$\therefore \text{ the coefficient} = \frac{\lfloor 5}{\lfloor 1 \lfloor 4 \lfloor 0} . 2^4 . 3^0 + \frac{\lfloor 5}{\lfloor 2 \lfloor 2 \lfloor 1} . 2^2 . 3 + \frac{\lfloor 5}{\lfloor 3 \lfloor 0 \lfloor 2} . 2^0 . 3^2$$

$$= 80 + 360 + 90 = 530.$$

Ex. 2. *Find the greatest coefficient in the expansion of*

$$(a+b+c)^7.$$

Here $m=3$ and $7 = 2 . 3 + 1$, so that $q=2, r=1$;

$$\therefore \text{ greatest coefficient} = \lfloor 7 / (\lfloor 2)^2 . \lfloor 3.$$

EXERCISE IV

BINOMIAL COEFFICIENTS

If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, prove the statements in Exx. 1-12.

1. $c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1} = n(1+x)^{n-1}$.

2. $c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.

3. $c_0+2c_1x+3c_2x^2+\ldots+(n+1)c_nx^n=\{1+(n+1)x\}(1+x)^{n-1}$.

4. $c_0-2c_1+3c_2-\ldots+(-1)^n(n+1)c_n=0$.

5. $c_2+2c_3+3c_4+\ldots+(n-1)c_n=1+(n-2)2^{n-1}$.

6. $c_0c_1+c_1c_2+c_2c_3+\ldots+c_{n-1}c_n=\dfrac{\lfloor 2n}{\lfloor n+1 \rfloor n-1}$.

7. $c_0c_r+c_1c_{r+1}+c_2c_{r+2}+\ldots+c_{n-r}c_n=\dfrac{\lfloor 2n}{\lfloor n+r \rfloor n-r}$.

8. $\dfrac{c_1}{c_0}+\dfrac{2c_2}{c_1}+\dfrac{3c_3}{c_2}+\ldots+\dfrac{nc_n}{c_{n-1}}=\dfrac{1}{2}n(n+1)$.

9. Show that $c_0^2-c_1^2+c_2^2-\ldots+(-1)^nc_n^2=\dfrac{\lfloor n}{\lfloor \frac{1}{2}n \rfloor \frac{1}{2}n}$ or zero, according as n is even or odd.

10. Show that the sum of the products of $c_0, c_1, c_2, \ldots, c_n$ taken two together is equal to $2^{2n-1}-\dfrac{\lfloor 2n}{2\lfloor n \lfloor n}$.

[The required sum $=\frac{1}{2}\{(c_0+c_1+c_2+\ldots+c_n)^2-(c_0^2+c_1^2+c_2^2+\ldots+c_n^2)\}$.]

11. Using the result of Ex. 1, show that

$$c_1^2+2c_2^2+3c_3^2+\ldots+nc_n^2$$

is equal to the coefficient of x^{n-1} in the expansion of $n(1+x)^{2n-1}$. Hence show that the sum of this series is $\lfloor 2n-1/(\lfloor n-1)^2$.

12. Using the result of Ex. 3, show that

$$c_0^2+2c_1^2+3c_2^2+\ldots+(n+1)c_n^2$$

is equal to the coefficient of x^n in the expansion of

$$[1+(n+1)x](1+x)^{2n-1}.$$

Hence show that the sum of this series is $\dfrac{(n+2)\lfloor 2n-1}{\lfloor n \lfloor n-1}$.

13. Prove that if $n>1$,

$$\frac{1}{\lfloor 1 \lfloor n-1}+\frac{1}{\lfloor 2 \lfloor n-2}+\frac{1}{\lfloor 3 \lfloor n-3}+\ldots+\frac{1}{\lfloor n-1 \lfloor 1}=\frac{2^n-2}{\lfloor n}.$$

14. If the terms of the series

$$(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+\ldots+(x+2)^{n-1}$$

are expanded and the whole is arranged in powers of x, show that the coefficient of x^r is $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}(3^{n-r}-2^{n-r})$.

15. Show that $\Sigma_{r=1}^{r=n-1}r(n-r)(C_r^n)^2=n^2C_n^{2n-2}$.

[Using Ex. 11, the sum of the series is equal to the coefficient of x^n in the expansion of $n^2(1+x)^{2n-2}$.]

In Ex. 16–18 find the coefficients of the products stated:

16. $abcde$ in the expansion of $(a+b+c+d+e)^5$.

17. $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$.
[Write $a^3b^4c^5=(bc)^x(ca)^y(ab)^z$ and find x, y, z.]

18. a^2b^2cd in the expansion of $(ab+ac+ad+bc+bd+cd)^3$.

19. If $(1+x+x^2)^n=c_0+c_1x+c_2x^2+\ldots+c_rx^r+\ldots+c_{2n}x^{2n}$, prove that

(i) $c_0=c_{2n}$, $c_1=c_{2n-1}$, $\ldots$ $c_r=c_{2n-r}$ $\ldots$.

(ii) $c_0^2-c_1^2+c_2^2-\ldots+c_{2n}^2=c_n$.

(iii) $c_0+c_2+c_4+\ldots+c_{2n}=1+c_1+c_3+c_5+\ldots+c_{2n-1}$.

(iv) $c_0-nc_1+\dfrac{n(n-1)}{\lfloor 2}c_2-\dfrac{n(n-1)(n-2)}{\lfloor 3}c_3+\ldots+(-1)^nc_n$ is equal to $\dfrac{\lfloor n}{\lfloor \frac{1}{3}n \lfloor \frac{2}{3}n}$ or zero, according as n is or is not a multiple of 3.

(v) $c_n=1+\dfrac{n(n-1)}{1^2}+\dfrac{n(n-1)(n-2)(n-3)}{1^2\,.\,2^2}+\ldots$, and write down the last term of this series when n is even and when n is odd.

(vi) $c_0c_2-c_1c_3+c_2c_4-\ldots+c_{2n-2}c_{2n}=c_{n+1}$.

[In (i) put $1/x$ for x and multiply by x^{2n}. In (ii) put $-x$ for x, and observe that
$(1+x+x^2)(1-x+x^2)=1+x^2+x^4$.]

20. If $(1+x+x^2+x^3)^n=c_0+c_1x+c_2x^2+\ldots+c_{3n}x^{3n}$:

(i) Find the value of $c_0+c_1+c_2+\ldots+c_{3n}$.

(ii) Prove that $c_0+c_2+c_4+c_6+\ldots=c_1+c_3+c_5+c_7+\ldots$.

(iii) Prove that $c_0=c_{3n}$, $c_1=c_{3n-1}$, $c_2=c_{3n-2}$, etc.

21. In the expansion of
$(1+x)(1+x+x^2)(1+x+x^2+x^3)\ldots(1+x+x^2+\ldots+x^n)$:

(i) What is the term containing the highest power of x?

(ii) Find the sum of the coefficients.

(iii) Show that the sum of the coefficients of the even terms is equal to that of the odd terms.

(iv) Show that the coefficients of the terms equidistant from the beginning and end are equal.

22. In the expansion of $(1+x)^n(1+y)^n(1+z)^n$, show that the sum of the coefficients of the terms of degree r is $\dfrac{\lfloor 3n}{\lfloor r \lfloor 3n-r}$.

23. By expanding the two sides of the identity
$$(2x+x^2)^n=\{(1+x)^2-1\}^n$$
and equating the coefficients of x^n, show that
$$C_n^{2n}-C_1^nC_n^{2n-2}+C_2^nC_n^{2n-4}-\ldots=2^n.$$
Also write down the last term of the series on the left (i) when n is even, (ii) when n is odd.

24. By expanding $\{(1+x)^2-2x\}^m$ in two different ways, prove that, if $m>n$,

$$C_{2n}^{2m}-2\,.\,C_1^m C_{2n-1}^{2m-2}+2^2\,.\,C_2^m C_{2n-2}^{2m-4}-\ldots=C_n^m.$$

25. If $(1+2x+2x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$,

prove that (i) $a_r=2^{r-n}\{C_r^{2n}+C_1^n C_r^{2n-2}+C_2^n C_r^{2n-4}+\ldots\}$,

the last term in the bracket being $C_{\frac{1}{2}r}^n$ or $(r+1)C_{\frac{1}{2}(r+1)}^n$, according as r is even or odd.

In particular, show that

(ii) $a_2=4n^2-2^{2-n}n(1+3C_1^{n-1}+5C_2^{n-1}+\ldots$ to n terms).

(iii) $a_4=\frac{1}{3}2^{3-n}n(n-1)\{1\,.\,3+3\,.\,5C_1^{n-2}+5\,.\,7C_2^{n-2}+\ldots$ to $(n-1)$ terms$\}$.

[We have $(1+2x+2x^2)=2(x+\frac{1}{2})^2+\frac{1}{2}$;

$$\therefore\ (1+2x+2x^2)^n=2^n(x+\tfrac{1}{2})^{2n}+2^{n-2}C_1^n(x+\tfrac{1}{2})^{2n-2}+\ldots.]$$

17. Highest Common Factor (H.C.F.). We consider polynomials in a single variable x. These will be denoted by capital letters.

(1) *If $A\equiv BQ+R$, where A, B, Q, R are polynomials, the common factors of A and B are the same as those of B and R.*

For since $A=BQ+R$, every common factor of B and R is a factor of A and, since $R=A-BQ$, every common factor of A and B is a factor of R.

(2) *If B is not of higher degree than A, polynomials or constants (Q_1, R_1), (Q_2, R_2), (Q_3, R_3) ... can be found such that*

$$A=BQ_1+R_1,\quad B=R_1Q_2+R_2,\quad R_1=R_2Q_3+R_3,\ \textit{etc.},$$

where the degree of any one of the set B, R_1, R_2, ... is less than that of the preceding.

It follows from the last theorem that

(*a*) if one of the pairs (A, B), (B, R_1), (R_1, R_2), (R_2, R_3) have a common factor, this is a common factor of every pair;

(*b*) if one pair have no common factor, the same is true for every pair.

Moreover, the process must terminate and that in one of two ways: the last remainder, say R_n, must vanish identically, or be independent of x.

If $R_n=0$, then $R_{n-2}=R_{n-1}Q_n$, so that R_{n-1} is a common factor of (R_{n-2}, R_{n-1}), and therefore also of (A, B). Moreover, every common factor of (A, B) is a factor of R_{n-1}, which is therefore the H.C.F. of A, B.

If R_n is independent of x, then (R_{n-2}, R_{n-1}) have no common factor, and the same is true of (A, B).

(3) **The H.C.F. Process.** Equations (1) of the last section are found by successive divisions, as represented below.

In practice, the reckoning is generally arranged as on the right.

$$\begin{array}{l} B)\ A\ (Q_1 \\ \underline{BQ_1} \\ R_1)\ B\ (Q_2 \\ \underline{R_1Q_2} \\ R_2)\ R_1\ (Q_3 \\ \qquad \text{etc.} \end{array} \qquad \begin{array}{c|c|c|c} Q_2 & B & A & Q_1 \\ & \underline{R_1Q_2} & \underline{BQ_1} & \\ & R_2 & R_1 & Q_3 \end{array} \quad \text{etc.}$$

The process may often be shortened by noticing that :

(i) We may multiply or divide any of the set A, B, R_1, R_2, ... by any constant or by any polynomial which is not a factor of the preceding member of the set.

(ii) If we arrive at a remainder R_n which can be completely factorised, the process need not be continued. For if we find which, if any, of these factors is a factor of R_{n-1}, we can write down all the common factors of (R_{n-1}, R_n).

(iii) We may use the following theorem :

If $X=lA+mB$ *and* $Y=l'A+m'B$, *where* l, m, l', m' *are constants different from zero and such that* $lm'-l'm\neq 0$, *then the* H.C.F. *of* X *and* Y *is the same as that of* A *and* B.

For every common factor of A and B is a common factor of X and Y. Moreover,

$$A(lm'-l'm)=m'X-mY \quad \text{and} \quad B(lm'-l'm)=lY-l'X,$$

therefore every common factor of X and Y is a common factor of A and B.

Ex. 1. *Find the* H.C.F. *of* $A=3x^3+x+4$ *and* $B=2x^3-x^2+3$.

Paying attention to the remark (i) and using detached coefficients, the reckoning is as follows :

$$\begin{array}{r|l||l|l} & & 3+0+1+4 & \\ & & \underline{\qquad\qquad 2} & \\ & 2-1+0+3 & 6+0+2+8 & 3 \\ & \underline{\qquad\quad 3} & 6-3+0+9 & \\ 2 & 6-3+0+9 & \underline{\qquad 3+2-1} & (a) \\ & 6+4-2 & \qquad\qquad 7 & \\ -7 & \underline{-7+2+9} & 21+14-7 & -3 \\ & -7-7 & 21-6-27 & \\ 9 & \underline{\qquad 9+9} & 20\,)\,20+20 & \\ & \underline{\qquad 9+9} & \qquad\qquad 1+1 & \end{array}$$

Hence the H.C.F. $=x+1$. *But it is unnecessary to go beyond the step marked* (a), which shows that $R_1=3x^2+2x-1=(3x-1)(x+1)$.

Now $x+1$ is, and $3x-1$ is not, a factor of B. Therefore $x+1$ is the H.C.F. of B, R_1, and consequently that of A and B.

Ex. 2. *Find the* H.C.F. *of* A *and* B *where*

$$A = 12x^5 + 5x^3 + 8 \quad \textit{and} \quad B = 8x^5 + 5x^2 + 12.$$

We have
$$2A - 3B = 10x^3 - 15x^2 - 20 = 5(2x^3 - 3x^2 - 4),$$
$$3A - 2B = 20x^5 + 15x^3 - 10x^2 = 5x^2(4x^3 + 3x - 2).$$

Putting $C = 2x^3 - 3x^2 - 4$ and $D = 4x^3 + 3x - 2$, it follows that the H.C.F. of A and B is the same as that of C and D.

Further,
$$C - 2D = 6x^2 + 3x + 6 = 3(2x^2 + x + 2),$$
$$2C - D = 6x^3 + 3x^2 + 6x = 3x(2x^2 + x + 2).$$

Hence the H.C.F. of $C - 2D$ and $2C - D$ is $2x^2 + x + 2$: this is therefore the H.C.F. of C and D, and also that of A and B.

18. Prime and Composite Functions. If a polynomial has no factors except itself (and constants) it is said to be *prime* : otherwise it is said to be *composite.*

Thus $x^2 + 3x + 2$ is a composite function whose *prime factors* are $x + 1$ and $x + 2$.

Polynomials which have no common factor (except constants) are said to be *prime to each other.* Such expressions *have no* H.C.F.

Thus $2(x + 1)$ and $4(x^2 + 1)$ are prime to each other.

We can prove theorems for polynomials analogous to those of Ch. I, Arts. 3, 4 and 5, relating to whole numbers.

Remembering the distinction between arithmetical and algebraical primeness, the reader should have no difficulty in making the necessary verbal alterations. Thus, the theorem corresponding to that of 4 (i) is as follows :

If A, B, M *are polynomials and* M *is a factor of* AB *and prime to* A, *then* M *is a factor of* B.

For if M is prime to A, these two have no common factor. Hence the H.C.F. of MB and AB is B. But M is a factor of AB, and is therefore a common factor of MB and AB. Hence M is equal to, or is a factor of, B.

Theorem. *If* A *and* B *are polynomials in* x, *then polynomials* X, Y, *prime to each other, can be found such that*

$$AX + BY = 1 \quad \textit{or} \quad AX + BY = G,$$

according as A *is or is not prime to* B, G *being the* H.C.F. *of* A *and* B *in the latter case.*

For if Q_1, Q_2, ... are the quotients and R_1, R_2, ... the remainders in the process of searching for the H.C.F. of A and B,

$$A = BQ_1 + R_1, \quad B = R_1Q_2 + R_2, \quad R_1 = R_2Q_3 + R_3, \text{ etc.}$$

Therefore we obtain in succession the following equations :

$$R_1 = A - BQ_1,$$
$$R_2 = -AQ_2 + B(1 + Q_1Q_2),$$
$$R_3 = A(1 + Q_2Q_3) - B(Q_1 + Q_3 + Q_1Q_2Q_3).$$

Continuing thus, we can express any remainder in the form $AX + BY$, where X and Y are polynomials.

Now the last remainder is either G, the H.C.F.; or it is a constant c. Hence we can find X, Y such that

$$AX + BY = G \quad \text{or} \quad AX + BY = c.$$

In either case X is prime to Y, for the first equation may be written $\frac{A}{G} . X + \frac{B}{G} . Y = 1$ and G is a factor of A and of B.; thus in either case any common factor of X and Y would be a factor of a constant.

In the second case, the polynomials in question are X/c and Y/c.

NOTE. This theorem is fundamental in the Theory of Partial Fractions, which are dealt with in Ch. VII.

EXERCISE V

H.C.F. AND ITS USE

Find the H.C.F. of the functions in Exx. 1-6.

1. $3x^4 - 4x^3 + 1$, $4x^4 - 5x^3 - x^2 + x + 1$.
2. $4x^4 + 5x^2 + 7x + 2$, $16x^3 + 10x + 7$.
3. $2x^4 - 13x^2 + x + 15$, $3x^4 - 2x^3 - 17x^2 + 12x + 9$.
4. $x^5 + 5x^2 - 2$, $2x^5 - 5x^3 + 1$.
5. $2x^5 - 5x^2 + 3$, $3x^5 - 5x^3 + 2$.
6. $12x^3 + 2x^2 - 21x - 4$, $6x^3 + 7x^2 - 14x - 8$, $21x^4 - 28x^3 + 9x^2 - 16$.

In Exx. 7-9, find polynomials X and Y for which the statements are identically true.

7. $(x-1)^2X - (x+1)^2Y = 1$.
8. $(x^3 - 1)X - (x^5 - 1)Y = x - 1$.
9. $(x^3 - 3x^2 + 2x - 1)X - (x^3 - 2x^2 - 2)Y = 1$.
10. Use the H.C.F. process to obtain

$$3(x^2 + 3x + 4) - (3x^2 + 2x + 1) = 7x + 11,$$
$$x(3x^2 + 2x + 1) - (3x - 7)(x^2 + 3x + 4) = 10x + 28 ;$$

and hence find A, B, C, D, such that

$$\frac{2x - 3}{(x^2 + 3x + 4)(3x^2 + 2x + 1)} = \frac{Ax + B}{x^2 + 3x + 4} + \frac{Cx + D}{3x^2 + 2x + 1} .$$

[Multiply the first identity by λ, the second by μ, and add : find λ and μ such that $(7\lambda + 10\mu)/(11\lambda + 28\mu) = 2/(-3)$; and thus obtain

$$(Ax + B)(3x^2 + 2x + 1) + (Cx + D)(x^2 + 3x + 4) = 2x - 3.]$$

CHAPTER IV

SYMMETRIC AND ALTERNATING FUNCTIONS, SUBSTITUTIONS

1. Symmetric Functions. A function which is unaltered by the interchange of any two of the variables which it contains is said to be *symmetric with regard to these variables.*

Thus $yz+zx+xy$ and $(x^2y+y^2z+z^2x)(x^2z+y^2x+z^2y)$ are symmetric with regard to x, y, z. (In the second expression, the interchange of any two letters transforms one factor into the other.)

The interchange of any two letters, x, y, is called the *transposition* (xy).

Terms of an expression which are such that one can be changed into the other by one or more transpositions are said to be *of the same type.*

Thus all the terms of $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y$ are of the same type, and the expression is symmetric with regard to x, y, z.

A symmetric function which is the sum of a number of terms of the same type is often written in an abbreviated form thus: *Choose any one of the terms and place the letter* Σ (*sigma*) *before it.* For instance,

$$x+y+z \text{ is represented by } \Sigma x \text{ and } yz+zx+xy \text{ by } \Sigma xy.$$

Again, $(x+y+z)^2=x^2+y^2+z^2+2yz+2zx+2xy=\Sigma x^2+2\Sigma yz.$

It is obvious that

(i) *If a term of some particular type occurs in a symmetric function, then all the terms of this type occur.*

(ii) *The sum, difference, product and quotient of two symmetric functions are also symmetric functions.*

Considerations of symmetry greatly facilitate many algebraical processes, as in the following examples.

Ex. 1. *Expand* $(y+z-x)(z+x-y)(x+y-z)$.

This expression is symmetric, homogeneous, and of the third degree in x, y, z. We may therefore assume that $(y+z-x)(z+x-y)(x+y-z)=a\,.\,\Sigma x^3+b\,.\,\Sigma x^2y+cxyz$, where a, b, c, are independent of x, y, z.

In this assumed identity,

(i) put $x=1$, $y=0$, $z=0$; then $-1=a$;
(ii) put $x=1$, $y=1$, $z=0$; then $0=2a+2b$, $\therefore\ b=1$;
(iii) put $x=1$, $y=1$, $z=1$; then $1=3a+6b+c$, $\therefore\ c=-2$.

Hence the required product is $-x^3-y^3-z^3+y^2z+yz^2+z^2x+zx^2+x^2y+xy^2-2xyz.$

Ex. 2. *Expand* $(a+b+c+d)(ab+ac+ad+bc+bd+cd)$. *Test the result by putting* $a=b=c=d=1$.

The product is the sum of all the products which can be obtained by multiplying any term of the first expression by any term of the second. Hence the terms in the product are of one of the types a^2b, abc.

The coefficient of a^2b in the product is 1 ; for this term is obtained as the product of a and ab, and in no other way.

The coefficient of abc is 3 ; for this term is obtained in each of the three ways $a(bc)$, $b(ac)$, $c(ab)$.

Hence, the required product is $\Sigma a^2b+3\Sigma abc$.

Test. The number of terms of the type a^2b is 12, and the number of terms of the type abc is 4 ; hence, if $a=b=c=d=1$,

$$\Sigma a \,.\, \Sigma ab = 4 \,.\, 6 = 24 \quad \text{and} \quad \Sigma a^2b + 3\Sigma abc = 12+3 \,.\, 4 = 24,$$

so that the test is satisfied.

Ex. 3. *Factorise* $(x+y+z)^5-x^5-y^5-z^5$.

Denote the given expression by E. Since $E=0$ when $x=-y$, it follows that $x+y$ is a factor of E ; similarly, $y+z$ and $z+x$ are factors. The remaining factor is a homogeneous symmetric function of x, y, z, of the second degree. We therefore assume that

$$E=(y+z)(z+x)(x+y)\{a(x^2+y^2+z^2)+b(yz+zx+xy)\},$$

where a and b are independent of x, y, z, and proceed thus :

$$\left.\begin{array}{ll} \text{put } x=1,\ y=1,\ z=0\ ; & \text{then } 2a+b=15. \\ \text{put } x=1,\ y=1,\ z=1\ ; & \text{then } a+b=10. \end{array}\right\} \quad \therefore \left.\begin{array}{l} a=5 \\ b=5 \end{array}\right\};$$

$$\therefore\ E=5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy).$$

2. Alternating Functions. If a function E of x, y, z, ... is transformed into $-E$ by the interchange of any two of the set x, y, z, ... , then E is called an *alternating function of* x, y, z, ...

Such a function is $x^n(y-z)+y^n(z-x)+z^n(x-y)$; for the interchange of any two letters, say x and y, transforms it into

$$y^n(x-z)+x^n(z-y)+z^n(y-x) \;=\; -E.$$

Observe that *the product and the quotient of two alternating functions are symmetric functions.*

Thus $\{x^3(y-z)+y^3(z-x)+z^3(x-y)\}/(y-z)(z-x)(x-y)$ is symmetric with regard to x, y, z.

Ex. 1. *Factorise* $x^3(y-z)+y^3(z-x)+z^3(x-y)$.

Denote the expression by E. Since $E=0$ when $x=y$, it follows that $x-y$ is a factor. Similarly $y-z$ and $z-x$ are factors, thus

$$E=(y-z)(z-x)(x-y) \,.\, F,$$

where F is symmetric, homogeneous and of the first degree in x, y, z. Hence

$$E=k(y-z)(z-x)(x-y)(x+y+z),$$

where k is independent of x, y, z. Equating the coefficients of x^3y on each side, we find that $k=-1$, thus

$$E=-(y-z)(z-x)(x-y)(x+y+z).$$

3. Cyclic Expressions. An expression is said to be *cyclic with regard to the letters* $a, b, c, d, \ldots h, k$, *arranged in this order*, when it is unaltered by changing a into b, b into c, c into d, ... , h into k, and k into a.

This interchange of the letters is called *the cyclic substitution* $(abc \ldots k)$.

Thus $a^2b+b^2c+c^2d+d^2a$ is cyclic with regard to a, b, c, d (in this order), for the cyclic substitution $(abcd)$ changes the first term into the second, the second into the third, ... , and the last into the first.

It is obvious that

(i) *If a term of some particular type occurs in a cyclic expression, then the term which can be derived from this by the cyclic interchange, must also occur; and the coefficients of these terms must be equal.*

(ii) *The sum, difference, product, and quotient, of two cyclic expressions are also cyclic.*

In writing a cyclic expression, it is unnecessary to write the whole. Thus $x^2(y-z)+y^2(z-x)+z^2(x-y)$ is often denoted by $\Sigma x^2(y-z)$, where the meaning of Σ must not be confused with that in Art. 1.

Again, we sometimes denote $x^2(y-z)+y^2(z-x)+z^2(x-y)$ by

$$x^2(y-z)+\ldots+\ldots,$$

it being understood that the second term is to be obtained from the first and the third from the second by cyclic interchange.

The student should be familiar with the following important identities.

1. $(b-c)+(c-a)+(a-b)=0.$
2. $a(b-c)+b(c-a)+c(a-b)=0.$
3. $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b).$
4. $bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b).$
5. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b).$
6. $a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$
7. $(a+b+c)(bc+ca+ab)=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc.$
8. $(b+c)(c+a)(a+b)=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc.$
9. $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$
10. $(b-c)^2+(c-a)^2+(a-b)^2=2(a^2+b^2+c^2-bc-ca-ab).$
11. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)=-a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2.$

It will be proved later (pp. 95, 96) that

(i) Any symmetric function of α, β, γ can be expressed in terms of $\Sigma\alpha, \Sigma\alpha\beta, \alpha\beta\gamma$.

(ii) Any symmetric function of $\alpha, \beta, \gamma, \delta$ can be expressed in terms of $\Sigma\alpha, \Sigma\alpha\beta, \Sigma\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$.

This mode of expression is extremely useful in factorising symmetric functions, and in proving identities.

Ex. 1. *Factorise* $a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)-4abc.$

Denoting the given expression by E, we have

$$\begin{aligned} E &= a\{1-(b^2+c^2)+b^2c^2\}+\ldots\ldots+\ldots\ldots-4abc \\ &= \Sigma a-\Sigma ab^2+abc\Sigma ab-4abc. \end{aligned}$$

Now $$\Sigma ab^2=\Sigma a\,.\,\Sigma ab-3abc\,;$$

$$\begin{aligned} \therefore\ E &= \Sigma a-\Sigma a\,.\,\Sigma ab-abc+abc\Sigma ab \\ &= \Sigma a(1-\Sigma ab)-abc(1-\Sigma ab) \\ &= (1-bc-ca-ab)(a+b+c-abc). \end{aligned}$$

4. Substitutions. (1) We consider processes by which one arrangement of a set of elements may be transformed into another.

Taking the permutations *cdba*, *bdac* of *a*, *b*, *c*, *d*, the first is changed into the second by replacing *a* by *c*, *b* by *a*, *c* by *b* and leaving *d* unaltered. This process is represented by the *operator*

$$\begin{pmatrix} abcd \\ cabd \end{pmatrix} \text{ or } \begin{pmatrix} abc \\ cab \end{pmatrix}, \qquad \text{and we write} \qquad \begin{pmatrix} abc \\ cab \end{pmatrix} cdba = bdac.$$

Such a process and also the operator which effects it is called a *substitution.*

As previously stated, the interchange of two elements *a*, *b* is called *the transposition* (*ab*).

Also a substitution such as $\begin{pmatrix} abcd \\ bcda \end{pmatrix}$, in which each letter is replaced by the one immediately following it and the last by the first, is called a *cyclic substitution* or *cycle,* and is denoted by (*abcd*).

If two operators are connected by the sign $=$, the meaning is that one is equivalent to the other, thus $(abcd)=(bcda)$.

(2) Two or more substitutions may be applied successively. This is indicated as follows, *the order of operations being from right to left.*

Let $S=(ab)$, $T=(bc)$, then

$$STabcd = Sacbd = bcad, \text{ and } TSabcd = Tbacd = cabd.$$

Thus $$ST=\begin{pmatrix} abcd \\ bcad \end{pmatrix}, \quad TS=\begin{pmatrix} abcd \\ cabd \end{pmatrix}.$$

This process is called *multiplication of substitutions,* and the resulting substitution is called the *product.*

Multiplication of this kind is not necessarily commutative, but *if the substitutions have no common letter, it is commutative.*

The operation indicated by $(ab)(ab)$, in which (ab) is performed twice, produces no change in the order of the letters, and is called an *identical substitution.*

(3) *Any substitution is cyclic, or is the product of two or more cyclic substitutions which have no common element.*

As an instance, consider the substitution

$$S=\begin{pmatrix} abcdefghk \\ chfbgaedk \end{pmatrix}.$$

Here a is changed to c, c to f, f to a, thus completing the cycle (acf). Also b is changed to h, h to d, d to b, making the cycle (bhd). Next, e is changed to g and g to e, giving the cycle (eg). The element k is unchanged, and we write

$$S=(acf)(bhd)(eg)(k) \quad \text{or} \quad S=(acf)(bhd)(eg).$$

This expression for S is unique, and the order of the factors is indifferent. Moreover, the method applies universally, for in effecting any substitution we must arrive at a stage where some letter is replaced by the first, thus completing a cycle. The same argument applies to the set of letters not contained in this cycle.

(4) *A cyclic substitution of n elements is the product of $n-1$ transpositions.*

For $(abc)=(ab)(bc)$, $(abcd)=(abc)(cd)=(ab)(bc)(cd)$,
$(abcde)=(abcd)(de)=(ab)(bc)(cd)(de)$, and so on.

We also have equalities such as

$$(ae)(ad)(ac)(ab)=(abcde), \qquad (ab)(ac)(ad)(ae)=(edcba).$$

(5) *A substitution which deranges n letters and which is the product of r cycles is equivalent to $n-r$ transpositions.*

This follows at once from (3) and (4). Thus if $S=\begin{pmatrix} abcdefgh \\ chfbgaed \end{pmatrix}$,

then $$S=(acf)(bhd)(eg)=(ac)(cf)(bh)(hd)(eg).$$

If we introduce the product $(ab)(ab)$, S is unaltered and the number of transpositions is increased by 2.

Thus, if a given substitution is equivalent to j transpositions, the number j *is not unique.* We shall prove that

$$j=n-r+2s \text{ where } s \text{ is a positive integer or zero.}$$

This is a very important theorem, and to prove it we introduce the notion of '*inversions.*'

(6) Taking the elements a, b, c, d, e, choose some arrangement, as $abcde$, and call it the *normal arrangement.*

Consider the arrangement $bdeac$. Here b precedes a, but follows it in the normal arrangement. On this account we say that the pair ba con-

stitutes an *inversion*. Thus *bdeac* contains five inversions, namely,

$$ba, \quad da, \quad dc, \quad ea, \quad ec.$$

Theorem 1. *If i is the number of inversions which are introduced or removed by a substitution which is equivalent to j transpositions, then i and j are both even or both odd.*

For consider the effect of a single transposition (fg).

If f, g are consecutive elements, the transposition (fg) does not alter the position of f or of g relative to the other elements. It therefore introduces or removes a single inversion due to the interchange of f, g.

If f, g are separated by n elements p, q, r, ... x, then f can be moved to the place occupied by g by $n+1$ interchanges of consecutive elements, and then g can be moved to the place originally occupied by f by n such interchanges. (The steps are shown in the margin.)

$$\begin{aligned} &\dots fpq \dots xg \dots \\ &\dots pq \dots xgf \dots \\ &\dots gpq \dots xf \dots \end{aligned}$$

Thus the transposition (fg) can be effected by $2n+1$ interchanges of consecutive elements. Therefore *any* transposition introduces or removes an odd number of inversions, and the theorem follows.

Again, for a given substitution, i is a fixed number, and therefore whatever value j may have, it must be even or odd, according as i is even or odd. Hence the following.

Theorem 2. *If one arrangement A of a given set of elements is changed into another B by j transpositions, then j is always even or always odd.*

In other words :

The number of transpositions which are equivalent to a given substitution is not unique, but is always even or always odd.

Referring to (5), the minimum value of j is $n-r$, and so we have the theorem stated at the end of that section.

Thus substitutions may be divided into two distinct classes. We say that *a substitution is even or odd according as it is equivalent to an even or an odd number of transpositions.*

Rule. To determine the class of a substitution S we may express it as the product of cycles, and count the number of cycles with an even number of elements : then S is even or odd according as this number is even or odd.

Or we can settle the question by counting the number of inversions, but this generally takes longer.

Thus we see at once that the substitution in (5) is odd. Also it has 17 inversions.

EXERCISE VI

SYMMETRIC AND CYCLIC FUNCTIONS

1. Show that $(bc-ad)(ca-bd)(ab-cd)$ is symmetric with regard to a, b, c, d.

2. Show that the following expressions are cyclic with regard to a, b, c, d, taken in this order:

$$\text{(i) } (a-b+c-d)^2; \quad \text{(ii) } (a-b)(c-d)+(b-c)(d-a).$$

Expand the expressions in Exx. 3, 4.

3. $(y+z-2x)(z+x-2y)(x+y-2z)$.

4. $(x+y+z)^3+(y+z-x)^3+(z+x-y)^3+(x+y-z)^3$.

Prove the identities in Exx. 5-14.

5. $(\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2)(\alpha+\beta+\gamma)=\Sigma\alpha^3\beta^2+\alpha\beta\gamma\Sigma\alpha\beta$.

6. $(\alpha-\beta)(\alpha-\gamma)+(\beta-\gamma)(\beta-\alpha)+(\gamma-\alpha)(\gamma-\beta)=\Sigma\alpha^2-\Sigma\alpha\beta$.

7. $(\beta-\gamma)(\beta+\gamma-\alpha)+(\gamma-\alpha)(\gamma+\alpha-\beta)+(\alpha-\beta)(\alpha+\beta-\gamma)=0$.

8. $\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2=\Sigma\alpha^2\beta-6\alpha\beta\gamma$.

9. $(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2)(\alpha+\beta+\gamma)=\Sigma\alpha^3\beta+2\Sigma\alpha^2\beta^2+2\alpha\beta\gamma\Sigma\alpha$.

10. $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)=\Sigma a^2\,.\,\Sigma ab$.

11. $(a+b-c)(a^2+b^2-c^2)+(b+c-a)(b^2+c^2-a^2)+(c+a-b)(c^2+a^2-b^2)$
$=3\Sigma a^3-\Sigma a^2b$.

12. $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2+(bz-cy)^2+(cx-az)^2+(ay-bx)^2$.

13. $(b^2-ca)(c^2-ab)+(c^2-ab)(a^2-bc)+(a^2-bc)(b^2-ca)$
$=-(bc+ca+ab)(a^2+b^2+c^2-bc-ca-ab)$.

14. $(a^2-bc)(b^2-ca)(c^2-ab)=abc(a^3+b^3+c^3)-(b^3c^3+c^3a^3+a^3b^3)$.

15. If one of the numbers a, b, c is the geometric mean between the other two, use Ex. 14 to show that

$$b^3c^3+c^3a^3+a^3b^3=abc(a^3+b^3+c^3).$$

16. If the numbers x, y, z, taken in some order or other, form an arithmetical progression, use Ex. 3 to show that

$$2(x+y+z)^3+27xyz=9(x+y+z)(yz+zx+xy).$$

17. (i) Express $2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$ as the sum of three squares.

(ii) Hence show that $(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$ is negative for all real values of a, b, c, except when $a=b=c$.

[Put $b-c=x$, $c-a=y$, $a-b=z$, and notice that

$$x^2+y^2+z^2+2yz+2zx+2xy=(x+y+z)^2=0.]$$

18. If $x+y+z=0$, show that

(i) $2yz=x^2-y^2-z^2$;

(ii) $(y^2+z^2-x^2)(z^2+x^2-y^2)(x^2+y^2-z^2)+8x^2y^2z^2=0$;

(iii) $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ can be expressed in the form $px^2+qy^2+rz^2$; and find p, q, r in terms of a, b, c, f, g, h.

Prove the identities in Exx. 19-23, where $\Sigma\alpha$, $\Sigma\alpha\beta$, etc., denote symmetric functions of α, β, γ, δ: also verify each by putting $\alpha=\beta=\gamma=\delta=1$.

19. $(\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2)=\Sigma\alpha^3+\Sigma\alpha^2\beta$.

20. $(\alpha+\beta+\gamma+\delta)(\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma)=\Sigma\alpha^2\beta\gamma+4\alpha\beta\gamma\delta$.

21. $(\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma)^2=\Sigma\alpha^2\beta^2\gamma^2+2\alpha\beta\gamma\delta\Sigma\alpha\beta$.

22. $(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)^2=\Sigma\alpha^2\beta^2+2\Sigma\alpha^2\beta\gamma+6\alpha\beta\gamma\delta$.

23. $\Sigma\alpha\beta\,.\,\Sigma\alpha\beta\gamma=\Sigma\alpha^2\beta^2\gamma+3\alpha\beta\gamma\delta\Sigma\alpha$.

Simplify the expressions in Exx. 24-28.

24. $(b^{-1}+c^{-1})(b+c-a)+(c^{-1}+a^{-1})(c+a-b)+(a^{-1}+b^{-1})(a+b-c)$.

25. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.

26. $\dfrac{b^2+c^2-a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-b^2}{(b-c)(b-a)}+\dfrac{a^2+b^2-c^2}{(c-a)(c-b)}$.

27. $\dfrac{b-c}{1+bc}+\dfrac{c-a}{1+ca}+\dfrac{a-b}{1+ab}$.

28. $\dfrac{a(b-c)}{1+bc}+\dfrac{b(c-a)}{1+ca}+\dfrac{c(a-b)}{1+ab}$.

Factorise the expressions in Exx. 29-35.

29. $(b-c)^2(b+c-2a)+(c-a)^2(c+a-2b)+(a-b)^2(a+b-2c)$.
[Put $b-c=x$, $c-a=y$, $a-b=z$, noting that $b+c-2a=y-z$.]

30. $(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2+(a-b)(a+b-2c)^2$.

31. $(b-c)(b+c-a)^3+(c-a)(c+a-b)^3+(a-b)(a+b-c)^3$.
[Put $b+c-a=x$, $c+a-b=y$, $a+b-c=z$.]

32. $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$.
[Put $b+c=x$, $c+a=y$, $a+b=z$.]

33. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

34. $(a+b)^2(a+c)^2(b-c)+(b+c)^2(b+a)^2(c-a)+(c+a)^2(c+b)^2(a-b)$.
[Put $b+c=x$, $c+a=y$, $a+b=z$.]

35. $(1-a^2)(1-b^2)(1-c^2)+(a-bc)(b-ca)(c-ab)$.

36. Express the following substitutions as the product of transpositions:

$$\text{(i)}\begin{pmatrix}123456\\654321\end{pmatrix};\quad \text{(ii)}\begin{pmatrix}123456\\246135\end{pmatrix};\quad \text{(iii)}\begin{pmatrix}123456\\641235\end{pmatrix}.$$

CHAPTER V

COMPLEX NUMBERS

1. Preliminary. In order that the operation of taking the square root of a number may be always possible, we extend the number system so as to include numbers of a new class known as *imaginary* numbers.

Consider the equation

$$x^2 - 2x + 5 = 0.$$

Proceeding in the usual way, we obtain the *formal* solution,

$$x = 1 \pm \sqrt{-4}, \quad \text{or} \quad x = 1 \pm 2\sqrt{-1},$$

which at present has no meaning.

Suppose the symbol ι to possess the following properties :

(i) *It combines with itself and with real numbers according to the laws of algebra.*

(ii) *We may substitute (-1) for ι^2, wherever this occurs.*

The reader can verify for himself that (at any rate so far as addition, subtraction, multiplication and division are concerned) reckoning under these rules will not lead to results which are mutually inconsistent.

Ex. 1. *Show that if $x = 1 + 2\iota$, then $x^2 - 2x + 5 = 0$.*

We have
$$x^2 = 1 + 4\iota + 4\iota^2 = 1 + 4\iota - 4 = -3 + 4\iota;$$
$$\therefore\ x^2 - 2x + 5 = -3 + 4\iota - 2(1 + 2\iota) + 5 = 0.$$

Immediate consequences of the supposition are

(i) $$(x + \iota y) + (x' + \iota y') = (x + x') + \iota(y + y').$$

(ii) $$(x + \iota y) - (x' + \iota y') = (x - x') + \iota(y - y').$$

(iii) $$(x + \iota y)(x' + \iota y') = xx' - yy' + \iota(xy' + x'y).$$

(iv) $$\frac{x + \iota y}{x' + \iota y'} = \frac{(x + \iota y)(x' - \iota y')}{(x' + \iota y')(x' - \iota y')} = \frac{xx' + yy'}{x'^2 + y'^2} + \iota \frac{yx' - xy'}{x'^2 + y'^2}.$$

This reckoning is at present meaningless, and can only be justified by an extension of the number system.

2. A Complex Number is represented by an expression of the form $x+\iota y$, where x, y are real numbers. *The sign + does not indicate addition as hitherto understood, nor does the symbol ι (at present) denote a number.* These things are parts of the scheme used to express numbers of a new class, and they signify that *the pair of real numbers (x, y) are united to form a single complex number.*

The complex number $x+\iota 0$ is to be regarded as identical with the real number x, and we write

$$x+\iota 0=x, \quad \text{.....(A)}$$

and in particular

$$0+\iota 0=0. \quad \text{.....(B)}$$

Thus the system of complex numbers includes all the real numbers, together with new numbers which are said to be *imaginary.*

For shortness we write

$$0+\iota y=\iota y, \quad \text{.....(C)}$$

and in particular

$$0+\iota 1=\iota, \quad \text{.....(D)}$$

so that *as an abbreviation for $0+\iota 1$, the symbol ι denotes a complex number.*

Thus *two* real numbers are required to express a single complex number. This may be compared with the fact that *two* whole numbers are required to express a fraction.

In order that expressions of the form $x+\iota y$ may be regarded as denoting *numbers*, we have to say what is meant by 'equality' and to define the fundamental operations.

3. Definition of Equality. We say that

$$x+\iota y=x'+\iota y' \text{ if and only if } x=x' \text{ and } y=y'.$$

In applying this definition, we are said *to equate real and imaginary parts.* It should be noticed that the terms '*greater than*' and '*less than*' have no significance in connection with imaginary numbers.

4. The Fundamental Operations. In defining these we are guided by the formal reckoning of Art. 1, and the equations (i)-(iv) are taken as *defining* addition, subtraction, multiplication and division.

5. Addition. The sum of the complex numbers $x+\iota y$ and $x'+\iota y'$ is defined as

$$(x+x')+\iota(y+y'),$$

and we write

$$(x+\iota y)+(x'+\iota y')=(x+x')+\iota(y+y'). \quad \text{.....(E)}$$

It follows that

$$x+\iota y=(x+\iota 0)+(0+\iota y), \quad \text{.....(F)}$$

so that $x+\iota y$ is the *sum* of the numbers denoted by x and ιy.

6. Subtraction is defined by the equation

$$(x+\iota y)-(x'+\iota y')=(x-x')+\iota(y-y'). \quad \text{.....(G)}$$

Hence *subtraction is the inverse of addition*, for $x-x'+\iota(y-y')$ is the number to which, if $x'+\iota y'$ is added, the result is $x+\iota y$.

The meaning of $-(x+\iota y)$ is defined by the equation

$$-(x+\iota y)=(0+\iota 0)-(x+\iota y).$$

Hence

$$-(x+\iota y)=(0-x)+\iota(0-y)=(-x)+\iota(-y). \quad \text{.....(H)}$$

In particular, putting $x=0$, we have

$$-\iota y=\iota(-y). \quad \text{.....(I)}$$

7. Multiplication is defined by the equation

$$(x+\iota y)(x'+\iota y')=xx'-yy'+\iota(xy'+x'y). \quad \text{.....(J)}$$

It follows that multiplication is commutative.

Putting $x'=k$, $y'=0$ in the last equation, we see that if k is a real number,

$$(x+\iota y)k=xk+\iota yk=k(x+\iota y). \quad \text{.....(K)}$$

Again, if we put $x=x'=0$ and $y=y'=1$, we have

$$(0+\iota 1)^2=-1 \quad \text{or} \quad \iota^2=-1. \quad \text{.....(L)}$$

8. Division is the inverse of multiplication, that is to say, the equations

$$\frac{x+\iota y}{x'+\iota y'}=X+\iota Y \quad \text{and} \quad (X+\iota Y)(x'+\iota y')=x+\iota y$$

mean the same thing. From the last equation we have

$$Xx'-Yy'+\iota(Xy'+Yx')=x+\iota y.$$

Equating real and imaginary parts, we have

$$Xx'-Yy'=x \quad \text{and} \quad Xy'+Yx'=y.$$

Therefore

$$X=\frac{xx'+yy'}{x'^2+y'^2}, \quad Y=\frac{yx'-xy'}{x'^2+y'^2},$$

unless x' and y' are both zero. Thus *the equation defining division is*

$$\frac{x+\iota y}{x'+\iota y'}=\frac{xx'+yy'}{x'^2+y'^2}+\iota\frac{yx'-xy'}{x'^2+y'^2}, \quad \text{.....(M)}$$

excepting the case in which $x'+\iota y'=0$.

Putting $x'=k$, $y'=0$, we see that if k is a real number different from zero,

$$\frac{1}{k}(x+\iota y)=\frac{x}{k}+\iota\frac{y}{k}. \quad \text{.....(N)}$$

It is easy to see that addition, subtraction, multiplication and division (as defined above) are subject to the same laws as the corresponding operations for real numbers.

We have therefore justified, and attached a definite meaning to, the process of reckoning described in Art. 1.

9. Zero Products. For complex, as for real numbers, *if a product is zero, then one of the factors of the product is zero.* For let

$$(x+\iota y)(x'+\iota y')=0,$$

where x, y, x', y' are real numbers. Then

$$xx'-yy'+\iota(xy'+x'y)=0.$$

Hence, by the rule of equality,

$$xx'-yy'=0 \quad \text{and} \quad xy'+x'y=0;$$
$$\therefore\ x(x'^2+y'^2)=0 \quad \text{and} \quad y(x'^2+y'^2)=0.$$

If $x'^2+y'^2$ is not zero, it follows that $x=0$, $y=0$, and therefore $x+\iota y=0$.

If $x'^2+y'^2=0$, then, since x', y' are real they must both vanish, and therefore $x'+\iota y'=0$.

10. Examples.

Ex. 1. *Express* $(1+2\iota)^2/(2+\iota)^2$ *in the form* $X+\iota Y$.

$$\frac{(1+2\iota)^2}{(2+\iota)^2}=\frac{1+4\iota-4}{4+4\iota-1}=\frac{-3+4\iota}{3+4\iota}=\frac{(-3+4\iota)(3-4\iota)}{(3+4\iota)(3-4\iota)};$$
$$\therefore \text{ the expression}=\frac{-9+16+24\iota}{9+16}=\frac{7}{25}+\iota\frac{24}{25}.$$

Ex. 2. *Solve the equation* $x^3=1$.

The equation can be written $(x-1)(x^2+x+1)=0$, giving $x=1$ or $x^2+x+1=0$. If $x^2+x+1=0$, we have in the usual way

$$x=\tfrac{1}{2}(-1\pm\sqrt{-3})=\tfrac{1}{2}(-1\pm\iota\sqrt{3}).$$

Hence the numbers 1, $\frac{1}{2}(-1\pm\iota\sqrt{3})$ *are called the three cube roots of unity.*

Ex. 3. *Verify that* $\frac{1}{2}(-1+\iota\sqrt{3})$ *is a root of* $x^3=1$.

Denoting the given number by ω, we have

$$\omega^3=\tfrac{1}{8}\{-1+3\iota\sqrt{3}-3(\iota\sqrt{3})^2+(\iota\sqrt{3})^3\}=\tfrac{1}{8}(-1+3\iota\sqrt{3}+9-3\iota\sqrt{3})=1.$$

11. Geometrical Representation of Complex Numbers.

(1) *In naming line-segments and angles* the order of the letters will denote the direction of measurement. Thus AB denotes the distance travelled by a point in moving *from* A *to* B, and we write

$$AB+BA=0, \quad \text{and} \quad BA=-AB.$$

Again, we use '$\angle BAC$' to denote the *least* angle through which AB must turn in order that it may lie along AC.

Thus $\angle CAB = -\angle BAC$, and we shall suppose that

$$-\pi < \angle BAC \leqslant \pi.$$

(2) If $z = x + \iota y$, we say that the number z is *represented* by the point whose coordinates referred to rectangular axes are x, y.

When there is no risk of confusion, this will be called *the point* z.

A diagram showing points which represent complex numbers is called an *Argand diagram*.

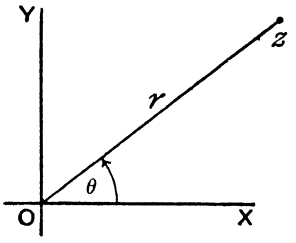


Fig. 2.

Let (r, θ) be the polar coordinates of the point z, then

$$z = x + \iota y = r(\cos\theta + \iota \sin\theta),$$

where $x = r\cos\theta, \quad y = r\sin\theta,$

and hence $r = \sqrt{(x^2 + y^2)}, \quad \tan\theta = y/x.$

Here r, which is essentially positive, is called the *modulus* of z and is denoted by $|z|$ or by $\operatorname{mod} z$. Thus

$$\operatorname{mod} z = |z| = r = \sqrt{(x^2 + y^2)}.$$

It follows that *if* $|z| = 0$ *then* $x = 0$, $y = 0$, *and consequently* $z = 0$.

The angle θ is called the *amplitude* of z, and is denoted by $\operatorname{am} z$. This angle has infinitely many values differing by multiples of 2π. The value of θ such that

$$-\pi < \theta \leqslant \pi$$

is called the *principal value of the amplitude.*

With the convention of § (1), it will be seen that *the principal value of* $\operatorname{am} z$ *is* $\angle XOz$.

Unless otherwise stated, $\operatorname{am} z$ will mean the principal value of the amplitude of z.

12. Addition and Subtraction.

(1) Let $z = x + \iota y$, $z' = x' + \iota y'$. Draw the parallelogram $Ozsz'$, then s *represents* $z + z'$.

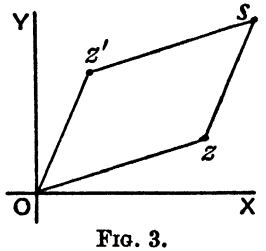


Fig. 3.

For if (X, Y) are the coordinates of s,

then X = sum of projections of Oz, zs on OX

= sum of projections of Oz, Oz' on OX

$= x + x'$.

Similarly $Y=y+y'$, therefore s represents $z+z'$. It is to be observed that

$$|z+z'| = \text{the } \textit{length } Os, \quad \text{and} \quad \text{am}\,(z+z') = \angle XOs.$$

Now $Os \leqslant Oz+Oz'$, the sign of equality occurring if $\angle XOz = \angle XOz'$ i.e. if $\text{am}\,z = \text{am}\,z'$. Therefore

$$|z+z'| \leqslant |z|+|z'|.$$

To prove this algebraically, we have to show that

$$\sqrt{\{(x+x')^2+(y+y')^2\}} \leqslant \sqrt{(x^2+y^2)}+\sqrt{(x'^2+y'^2)},$$

where each root is to have its positive value. This will be the case if

$$(x+x')^2+(y+y')^2 \leqslant x^2+y^2+x'^2+y'^2+2\sqrt{(x^2+y^2)(x'^2+y'^2)},$$

i.e. if $$xx'+yy' \leqslant \sqrt{(x^2+y^2)(x'^2+y'^2)},$$

or if $$(xx'+yy')^2 \leqslant (x^2+y^2)(x'^2+y'^2),$$

or if $$(xy'-x'y)^2 \geqslant 0,$$ which is the case.

(2) Draw the parallelogram $Oz'zd$ (Fig. 4), then

d represents $z-z'$.

For, by the last construction, the number represented by $d+z'=z$.

Observe that

$$|z-z'| = \text{the } \textit{length } z'z,$$

and $$\text{am}\,(z-z') = \angle XOd.$$

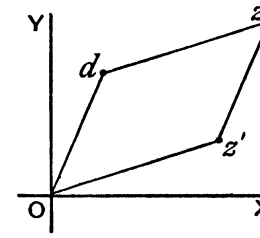


FIG. 4.

(3) *If* $s_n = z_1+z_2+\ldots+z_n$, *where* $z_1, z_2, \ldots z_n$ *are given complex numbers, it is required to find the point* s_n.

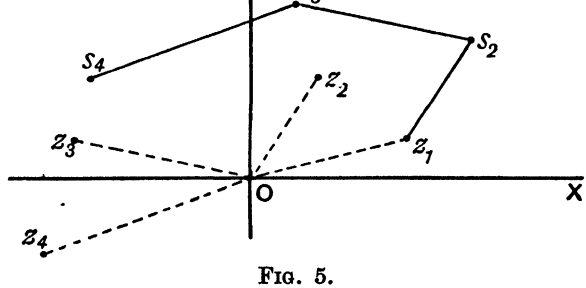


FIG. 5.

Draw in succession z_1s_2, s_2s_3, s_3s_4, ... equal to, parallel to, and in the same sense as Oz_2, Oz_3, Oz_4, ... , then s_2, s_3, ... are the points representing z_1+z_2, $z_1+z_2+z_3$, etc.

This is a continued application of the first construction of this article.

(4) *The modulus of the sum of any number of complex numbers is less than or equal to the sum of the moduli of the numbers.*

For in the last figure, the moduli of z_1, z_2, z_3, ... are the lengths Oz_1, z_1s_2, s_2s_3, ... , and that of s_n is Os_n. Also

$$Os_n \leqslant Oz_1 + z_1s_2 + s_2s_3 + \ldots + s_{n-1}s_n.$$

Therefore
$$|s_n| \leqslant |z_1| + |z_2| + \ldots + |z_n|.$$

NOTE. The sign of equality is to be taken if, and only if, the points O, z_1, s_2 ... s_n, are in the same straight line *and occur in this order*, that is, *if* z_1, z_2, ... z_n *have the same amplitude.*

To prove this algebraically, we have by (1) of this article,

$$|s_n| \leqslant |s_{n-1}| + |z_n| \leqslant |s_{n-2}| + |z_{n-1}| + |z_n|,$$

and so on.

13. Products and Quotients.

(1) Let
$$z = x + \iota y = r(\cos\theta + \iota \sin\theta),$$
$$z' = x' + \iota y' = r'(\cos\theta' + \iota \sin\theta').$$

By the definition of multiplication,

$$zz' = xx' - yy' + \iota(xy' + yx')$$
$$= rr'\{\cos\theta\cos\theta' - \sin\theta\sin\theta' + \iota(\cos\theta\sin\theta' + \cos\theta'\sin\theta)\};$$
$$\therefore\ zz' = rr'\{\cos(\theta+\theta') + \iota\sin(\theta+\theta')\}. \quad \ldots\ldots\ldots\

Ex. 1. *Show that* $\tan^{-1}\frac{1}{5}$ *is nearly equal to* $\frac{\pi}{16}$.

We have $(5+\iota)=\sqrt{26}(\cos\theta+\iota\sin\theta)$, where $\tan\theta=1/5$,
and therefore $(5+\iota)^4=676(\cos 4\theta+\iota\sin 4\theta)$, by (E).
But $(5+\iota)^4=(24+10\iota)^2=476+480\iota$; hence we have

$$\cos 4\theta=476/676,\quad \sin 4\theta=480/676,\quad \text{and}\quad \tan 4\theta=1,\ \text{nearly.}$$

$$\therefore\ 4\theta=\pi/4 \text{ approximately.}$$

(2) **Modulus and Amplitude.** From (D) we see that if Z is the product of the complex numbers $z_1, z_2, \ldots, z_n$,

$$|Z|=r_1r_2\ldots r_n=|z_1|\,.\,|z_2|\ldots|z_n|, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(G)}$$

and
$$\operatorname{am} Z=\theta_1+\theta_2+\ldots+\theta_n=\operatorname{am} z_1+\operatorname{am} z_2+\ldots+\operatorname{am} z_n; \qquad \ldots\ldots\ldots\text{(H)}$$

but note that the last equation does not give the *principal* value of am Z unless

$$-\pi<\theta_1+\theta_2+\ldots+\theta_n\leqslant\pi.$$

Again, from (B) we have

$$\left|\frac{z}{z'}\right|=\frac{r}{r'}=\frac{|z|}{|z'|}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

and
$$\operatorname{am}\frac{z}{z'}=\theta-\theta'=\operatorname{am} z-\operatorname{am} z', \qquad \ldots\ldots\ldots\ldots\ldots\text{(J)}$$

although the last equation does not give the *principal* value of am z/z' unless

$$-\pi<\theta-\theta'\leqslant\pi.$$

It is important to notice that the amplitude of z/z' is the angle through which Oz' must be turned in order that it may lie along Oz; and that, with the convention of Art. 11, (1),

the principal value of $\operatorname{am}\frac{z}{z'}=\angle z'Oz$.

Again, if z, a, a' are any numbers,

$$\left|\frac{z-a}{z-a'}\right|=\frac{\text{the length } az}{\text{the length } a'z},$$

and
$$\operatorname{am}\frac{z-a}{z-a'}=\angle a'za.$$

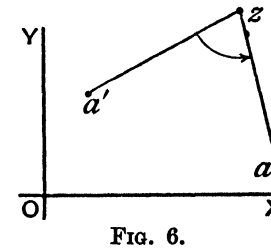


FIG. 6.

(3) **De Moivre's Theorem.** From (E) and (F) it follows that *if n is a positive or negative integer*

$$(\cos\theta+\iota\sin\theta)^n=\cos n\theta+\iota\sin n\theta.$$

This, with an extension to be given later (Art. 16), is known as De Moivre's Theorem.

14. Conjugate Numbers. (1) *If* $z=x+\iota y$ *and* $z'=x-\iota y$, *then* z, z' *are called conjugate numbers.*

Their product is x^2+y^2, and if r, θ are the polar coordinates of the point z, those of the point z' are r, $-\theta$. Hence

$$|x+\iota y|=r=\sqrt{(x^2+y^2)}=|x-\iota y|,$$
$$\text{am}\,(x+\iota y)=\theta=-\text{am}\,(x-\iota y).$$

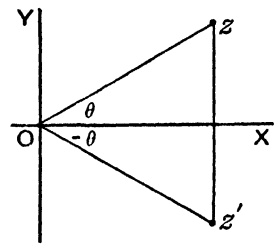


FIG. 7.

(2) *If* $f(z)$ *is a polynomial in* z *with real coefficients, and* $f(z)=f(x+\iota y)=X+\iota Y$, *where* X, Y *are real, then* $f(x-\iota y)=X-\iota Y$.

We have $f(x+\iota y)=a_0+a_1(x+\iota y)+a_2(x+\iota y)^2+\ldots$ where a_0, a_1, ... are real.

We can therefore obtain expansions of the form

$$f(x+\iota y)=\Sigma a x^m(\iota y)^n,\quad f(x-\iota y)=\Sigma a x^m(-\iota y)^n,$$

where every a is real and m, n are positive integers or zero.

Every term of $f(x+\iota y)$ in which n is even or zero is real, and occurs with the same sign in $f(x-\iota y)$.

Every term of $f(x+\iota y)$ in which n is odd is imaginary, and occurs with the opposite sign in $f(x-\iota y)$.

Hence the result follows.

(3) *If an equation with real coefficients has complex roots, then these occur in conjugate pairs.*

Let $\alpha+\iota\beta$ be a root of $f(x)=0$, where α, β, as well as the coefficients of the polynomial $f(x)$, are real.

Let $f(\alpha+\iota\beta)=A+\iota B$, where A, B are real. Since $f(\alpha+\iota\beta)=0$, we have $A=0$, $B=0$. Hence

$$f(\alpha-\iota\beta)=A-\iota B=0.$$

Therefore $\alpha-\iota\beta$ is also a root of $f(x)=0$.

EXERCISE VII

COMPLEX NUMBERS

1. Find the modulus and amplitude of $\sqrt{3}+\iota$, $\sqrt{3}-\iota$, $-\sqrt{3}+\iota$, $-\sqrt{3}-\iota$.
2. The condition that $(a+\iota b)/(a'+\iota b')$ may be real is $ab'-a'b=0$.
3. Find z so that $z(3+\iota 4)=2+3\iota$.
4. Express in the form $X+\iota Y$,

$$\text{(i)}\ \frac{1+\iota}{2+3\iota};\quad \text{(ii)}\ \frac{1}{(1-\iota)^3};\quad \text{(iii)}\ \frac{1}{1-(\cos\theta+\iota\sin\theta)}.$$

5. If $z=x+\iota y$, express in the form $X+\iota Y$

$$\text{(i) } z^n; \quad \text{(ii) } \frac{1}{z}; \quad \text{(iii) } \frac{z+1}{z-1}.$$

6. Find the moduli of (i) $(2+3\iota)^2(3-4\iota)^3$; (ii) $(2+3\iota)^2/(3+4\iota)^3$.

7. What is the principal value of the amplitude of $(\cos 50^0+\iota \sin 50^0)^6$?

8. (i) If $|z|=|z'|=1$ and $\operatorname{am} z=-\operatorname{am} z'$, then $z'=\frac{1}{z}$.

(ii) If $|z|=|z'|$ and $\operatorname{am} z$, $\operatorname{am} z'$ differ by π, then $z'=-z$.

9. Let $z=x+\iota y$ where x, y are variables. Prove that

(i) if $2|z-1|=|z-2|$, then $3(x^2+y^2)=4x$;

(ii) if $|2z-1|=|z-2|$, then $x^2+y^2=1$.

Thus, in either case, the point z describes a circle.

10. Prove that $|z+z'|^2+|z-z'|^2=2|z|^2+2|z'|^2$.

11. Prove that $|2z\cos\alpha+z^2|<1$ if $|z|<0{\cdot}41$.

[Let $|z|=r$, then $|2z\cos\alpha+z^2|<2|z\cos\alpha|+|z|^2<2r+r^2<1$ if $r^2+2r-1<0$.]

12. If $z=x+\iota y$ and $Z=X+\iota Y$, show that

$$\text{(i) if } z=\frac{Z-1}{Z+1}, \text{ then } x^2+y^2=\frac{(X-1)^2+Y^2}{(X+1)^2+Y^2};$$

$$\text{(ii) if } z=\frac{\iota(Z+1)}{Z-1}, \text{ then } x^2+y^2-x=\frac{X^2+Y^2+2X-2Y+1}{(X-1)^2+Y^2}.$$

13. Given that $1+2\iota$ is one root of the equation, $x^4-3x^3+8x^2-7x+5=0$, find the other three roots.

14. From the formula $\cos n\theta+\iota\sin n\theta=(\cos\theta+\iota\sin\theta)^n$, where n is a positive integer, deduce that

$$\cos n\theta=\cos^n\theta-C_2^n\cos^{n-2}\theta\sin^2\theta+C_4^n\cos^{n-4}\theta\sin^4\theta-\dots,$$

$$\sin n\theta=C_1^n\cos^{n-1}\theta\sin\theta-C_3^n\cos^{n-3}\theta\sin^3\theta+\dots,$$

$$\tan n\theta=\frac{C_1^n\tan\theta-C_3^n\tan^3\theta+\dots}{1-C_2^n\tan^2\theta+C_4^n\tan^4\theta-\dots}.$$

15. By putting $z=r(\cos\theta+\iota\sin\theta)$ in the identity

$$1+z+z^2+\dots+z^{n-1}=(1-z^n)/(1-z),$$

prove that

$$1+r\cos\theta+r^2\cos 2\theta+\dots+r^{n-1}\cos(n-1)\theta$$
$$=\frac{1-r\cos\theta-r^n\cos n\theta+r^{n+1}\cos(n-1)\theta}{1-2r\cos\theta+r^2},$$

$$r\sin\theta+r^2\sin 2\theta+\dots+r^{n-1}\sin(n-1)\theta$$
$$=\frac{r\sin\theta-r^n\sin n\theta+r^{n+1}\sin(n-1)\theta}{1-2r\cos\theta+r^2}.$$

16. If $z=\cos\theta+\iota\sin\theta$ and n is a positive integer, prove that

$$z^n+z^{-n}=2\cos n\theta,\quad z^n-z^{-n}=2\iota\sin n\theta.$$

Hence show that

$$2^{n-1}\cos^n\theta=\cos n\theta+C_1^n\cos(n-2)\theta+C_2^n\cos(n-4)\theta+\dots,$$

and if n is even

$$(-1)^{\frac{n}{2}}2^{n-1}\sin^n\theta=\cos n\theta-C_1^n\cos(n-2)\theta+C_2^n\cos(n-4)\theta-\dots,$$

and if n is odd

$$(-1)^{\frac{n-1}{2}}2^{n-1}\sin^n\theta=\sin n\theta-C_1^n\sin(n-2)\theta+C_2^n\sin(n-4)\theta-\dots.$$

[$(z+z^{-1})^n=z^n+z^{-n}+C_1^n(z^{n-2}+z^{-(n-2)})+\dots$; similarly for $(z-z^{-1})^n$.]

17. By the method of Ex. 16, prove that

$$16\cos^3\theta\sin^2\theta=-\cos 5\theta-\cos 3\theta+2\cos\theta.$$

18. If $\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$, prove that

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$$

and

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$$

[Put $a=\cos\alpha+\iota\sin\alpha$, $b=\cos\beta+\iota\sin\beta$, $c=\cos\gamma+\iota\sin\gamma$, then $a+b+c=0$ and $a^3+b^3+c^3=3abc$, etc.]

19. From the identity

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2$$

deduce the identities

$$\cos 2(\theta+\alpha)\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\dots+\dots=\cos 4\theta,$$

$$\sin 2(\theta+\alpha)\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\dots+\dots=\sin 4\theta.$$

[Put $x=\cos 2\theta+\iota\sin 2\theta$, $a=\cos 2\alpha+\iota\sin 2\alpha$, etc.]

15. Roots of Complex Numbers. (1) If n is a positive integer, any number whose nth power is equal to z is called an *n-th root of z.* Let

$$z=r(\cos\theta+\iota\sin\theta),$$

where $r>0$ and $-\pi<\theta\leqslant\pi$. Let $\sqrt[n]{r}$ denote the positive arithmetical nth root of r. Consider the values of

$$u=\sqrt[n]{r}\left\{\cos\frac{\theta+2k\pi}{n}+\iota\sin\frac{\theta+2k\pi}{n}\right\},$$

where k is any integer or zero. By Art. 13 we have

$$u^n=r\{\cos(\theta+2k\pi)+\iota\sin(\theta+2k\pi)\}=z;$$

from which it follows that u is an nth root of z.

Again, two values of u will be equal if, and only if, the corresponding angles have the same sine and cosine. That this may be the case, the angles must differ by a multiple of 2π, and the corresponding values of k must differ by a multiple of n. Therefore u has n distinct values, namely those given by

$$k = 0, 1, 2, \ldots, n-1.$$

Thus *the n-th roots of* $z = r(\cos\theta + \iota \sin\theta)$ *are the values of*

$$\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + \iota\sin\frac{\theta+2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ldots, n-1.$$

If $-\pi < \theta \leqslant \pi$, then $\sqrt[n]{r}\,.\left\{\cos\frac{\theta}{n} + \iota\sin\frac{\theta}{n}\right\}$ is called the *principal value* of $\sqrt[n]{z}$.

(2) *To find the points representing the n values of* $\sqrt[n]{z}$, we must be able to find the length represented by $\sqrt[n]{r}$ and to divide the angle θ into n equal parts.

Ex. 1. *Find the points* Q_1, Q_2, Q_3 *representing the values of* $\sqrt[3]{z}$, *where* $z = \sqrt{5} + \iota\sqrt{3}$.

Here $Oz = r = \sqrt{5+3} = \sqrt{8}$

and $\tan XOz = \tan\theta = \sqrt{3}/\sqrt{5}$.

Also $\sqrt[3]{r} = \sqrt{2}$, and so Q_1, Q_2, Q_3 are points on the circle with centre O and radius $\sqrt{2}$, such that

$\angle XOQ_1 = \theta/3$, $\angle Q_1OQ_2 = \angle Q_2OQ_3 = 2\pi/3$.

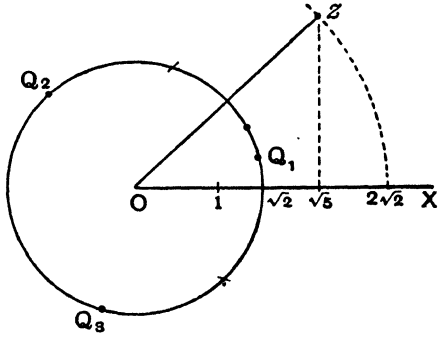


FIG. 8.

16. General Form of De Moivre's Theorem. *If* x *is rational, then* $\cos x\theta + \iota \sin x\theta$ *is one of the values of* $(\cos\theta + \iota \sin\theta)^x$.

The case in which x is a positive or negative integer has been considered in Art. 13.

Let $x = p/q$ where p, q are integers and q is positive, then since

$$(\cos\theta/q + \iota\sin\theta/q)^q = \cos\theta + \iota\sin\theta,$$

therefore

$$\cos\theta/q + \iota\sin\theta/q \text{ is a value of } (\cos\theta + \iota\sin\theta)^{\frac{1}{q}};$$

also, since p is an integer,

$$(\cos\theta/q + \iota\sin\theta/q)^p = \cos p\theta/q + \iota\sin p\theta/q;$$

therefore

$$\cos p\theta/q + \iota\sin p\theta/q \text{ is a value of } (\cos\theta + \iota\sin\theta)^{\frac{p}{q}}.$$

17. The n-th Roots of Unity. In Art. 15, let $r=1$ and $\theta=0$, then since $\cos 0+\iota \sin 0=1$, *the n-th roots of unity are the values of*

$$\cos\frac{2k\pi}{n}+\iota\sin\frac{2k\pi}{n} \text{ where } k=0, 1, 2, \ldots n-1.$$

The *principal n-th root,* given by $k=0$, is 1.

If n is even, the root -1 is given by $k=\frac{1}{2}n$.

Since $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\ldots+1)$, it follows that if α is any root, other than 1, of $x^n-1=0$, then

$$\alpha^{n-1}+\alpha^{n-2}+\ldots+\alpha+1=0.$$

18. The n-th Roots of (-1). These are the values of

$$\cos\frac{(2k+1)\pi}{n}+\iota\sin\frac{(2k+1)\pi}{n} \text{ where } k=0, 1, 2, \ldots n-1.$$

This follows from Art. 15 by putting $r=1$, $\theta=\pi$.

The *principal n-th root of* (-1) is $\cos\frac{\pi}{n}+\iota\sin\frac{\pi}{n}$, corresponding to $k=0$.

If n is odd, the root (-1) is given by $k=\frac{1}{2}(n-1)$.

19. Factors of x^n-1 and x^n+1.

(1) *The n factors of* x^n-1 *are*

$$x-\{\cos 2r\pi/n+\iota\sin 2r\pi/n\} \text{ where } r=0, 1, 2, \ldots n-1.$$

This follows from Art. 15.

(i) *If n is even,* the factors $x-1$ and $x+1$ are given by $r=0$, $r=n/2$. The remaining $(n-2)$ factors can be grouped in pairs as follows :

Since $2r\pi/n+2(n-r)\pi/n=2\pi$, the factors corresponding to r and $n-r$ are

$$x-\left\{\cos\frac{2r\pi}{n}+\iota\sin\frac{2r\pi}{n}\right\} \quad \text{and} \quad x-\left\{\cos\frac{2r\pi}{n}-\iota\sin\frac{2r\pi}{n}\right\}.$$

And the product of these is

$$\left(x-\cos\frac{2r\pi}{n}\right)^2+\sin^2\frac{2r\pi}{n}=x^2-2x\cos\frac{2r\pi}{n}+1;$$

therefore

$$x^n-1=(x-1)(x+1)\Pi_{r=1}^{r=\frac{1}{2}(n-2)}\left(x^2-2x\cos\frac{2r\pi}{n}+1\right), \quad \ldots\ldots \text{(A)}$$

where Π denotes the product of the factors as indicated.

(ii) *If n is odd,* the factor $x-1$ is given by $r=0$. The remaining $n-1$ factors can be grouped in pairs, and

$$x^n-1=(x-1)\Pi_{r=1}^{r=\frac{1}{2}(n-1)}\left(x^2-2x\cos\frac{2r\pi}{n}+1\right). \quad \ldots\ldots\ldots\ldots \text{(B)}$$

(2) Similarly, the n factors of x^n+1 are

$$x-\{\cos(2r+1)\pi/n+\iota\sin(2r+1)\pi/n\},\quad \text{where } r=0,1,2,\ldots(n-1).$$

(i) *If n is even*

$$x^n+1=\Pi_{r=0}^{r=\frac{1}{2}(n-2)}\left(x^2-2x\cos\frac{(2r+1)\pi}{n}+1\right). \quad \ldots\ldots\ldots\ldots(\mathbf{C})$$

(ii) *If n is odd,*

$$x^n+1=(x+1)\Pi_{r=0}^{r=\frac{1}{2}(n-3)}\left(x^2-2x\cos\frac{(2r+1)\pi}{n}+1\right). \quad \ldots\ldots\ldots\ldots(\mathbf{D})$$

20. The Imaginary Cube Roots of Unity. These are the roots of $x^2+x+1=0$. Their actual values are

$$\cos\frac{2\pi}{3}\pm\iota\sin\frac{2\pi}{3}\quad\text{or}\quad\tfrac{1}{2}(-1\pm\iota\sqrt{3}).$$

Denoting either of these by ω, the other is ω^2, for $(\omega^2)^3=(\omega^3)^2=1$. Moreover,

$$1+\omega+\omega^2=0. \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

Again, except when r is a multiple of 3, ω^r and ω^{2r} are the imaginary cube roots of 1, and so

$$1+\omega^r+\omega^{2r}=0. \quad \ldots\ldots\ldots\ldots(\mathbf{B})$$

The following identities are important.

$$x^3-y^3=(x-y)(x-\omega y)(x-\omega^2y). \quad \ldots\ldots\ldots\ldots(\mathbf{C})$$

$$x^3+y^3=(x+y)(x+\omega y)(x+\omega^2y). \quad \ldots\ldots\ldots\ldots(\mathbf{D})$$

$$x^2+y^2+z^2-yz-zx-xy=(x+\omega y+\omega^2z)(x+\omega^2y+\omega z). \quad \ldots\ldots(\mathbf{E})$$

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z). \quad \ldots\ldots(\mathbf{F})$$

Ex. 1. *Prove that* $(x^3+y^3+z^3-3xyz)^2=X^3+Y^3+Z^3-3XYZ$, *where*

$$X=x^2+2yz,\quad Y=y^2+2zx,\quad Z=z^2+2xy.$$

This follows from (**F**) on observing that

$$(x+\omega y+\omega^2z)^2=X+\omega^2Y+\omega Z,\quad (x+\omega^2y+\omega z)^2=X+\omega Y+\omega^2Z$$

and

$$(x+y+z)^2=X+Y+Z.$$

Ex. 2. *If* $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$ *and*

$$S_1=c_0+c_3+c_6+\ldots,\quad S_2=c_1+c_4+c_7+\ldots,\quad S_3=c_2+c_5+c_8+\ldots,$$

each of the series being continued as far as possible, show that the values of S_1, S_2, S_3 *are* $\frac{1}{3}\left(2^n+2\cos\frac{r\pi}{3}\right)$, *where* $r=n$, $n-2$, $n+2$, *respectively.*

Putting 1, ω, ω^2 for x, in succession,

$$(1+1)^n=c_0+c_1+\ldots+c_r+\ldots+c_n, \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

$$(1+\omega)^n=c_0+c_1\omega+\ldots+c_r\omega^r+\ldots+c_n\omega^n, \quad \ldots\ldots\ldots\ldots(\mathbf{B})$$

$$(1+\omega^2)^n=c_0+c_1\omega^2+\ldots+c_r\omega^{2r}+\ldots+c_n\omega^{2n}. \quad \ldots\ldots\ldots\ldots(\mathbf{C})$$

If r is not a multiple of 3, $1+\omega^r+\omega^{2r}=0$; hence, adding (A), (B), (C) together, we have $3(c_0+c_3+c_6+\ldots)=2^n+(1+\omega)^n+(1+\omega^2)^n$.

Now $1+\omega=1+\cos\frac{2\pi}{3}+\iota\sin\frac{2\pi}{3}=2\cos\frac{\pi}{3}\left(\cos\frac{\pi}{3}+\iota\sin\frac{\pi}{3}\right)$,

and, since $\cos\frac{\pi}{3}=\frac{1}{2}$, we have $1+\omega=\cos\frac{\pi}{3}+\iota\sin\frac{\pi}{3}$;

$$\therefore\ (1+\omega)^n=\cos\frac{n\pi}{3}+\iota\sin\frac{n\pi}{3}.$$

Also $1+\omega^2=1+\cos\frac{2\pi}{3}-\iota\sin\frac{2\pi}{3}$, and $(1+\omega^2)^n=\cos\frac{n\pi}{3}-\iota\sin\frac{n\pi}{3}$.

$$\therefore\ 3(c_0+c_3+c_6+\ldots)=2^n+2\cos\frac{n\pi}{3}.$$

Again, multiply (B) by ω^2, (C) by ω, and add to (A); then

$$3(c_1+c_4+c_7+\ldots)=2^n+\omega^2(1+\omega)^n+\omega(1+\omega^2)^n.$$

Also $\omega^2(1+\omega)^n=\left(\cos\frac{2\pi}{3}-\iota\sin\frac{2\pi}{3}\right)\left(\cos\frac{n\pi}{3}+\iota\sin\frac{n\pi}{3}\right)$

$$=\cos\frac{(n-2)\pi}{3}+\iota\sin\frac{(n-2)\pi}{3},$$

and $\omega(1+\omega^2)^n=\cos\frac{(n-2)\pi}{3}-\iota\sin\frac{(n-2)\pi}{3}$;

$$\therefore\ 3(c_1+c_4+c_7+\ldots)=2^n+2\cos\frac{(n-2)\pi}{3}.$$

Finally, multiply (B) by ω, (C) by ω^2, and add to (A); it will be found that

$$3(c_2+c_5+c_8+\ldots)=2^n+2\cos\frac{(n+2)\pi}{3}.$$

EXERCISE VIII

COMPLEX ROOTS

1. Prove that the values of $\sqrt{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm\iota)$.

2. Find the values of $\sqrt{(1+\iota)}$ and $\sqrt{(1-\iota)}$.

3. Use Art. 19 to show that

$$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1),$$

$$x^5-1=(x-1)\left(x^2+2x\cos\frac{2\pi}{5}+1\right)\left(x^2-2x\cos\frac{2\pi}{5}+1\right).$$

4. Prove that

$$x^5-1=(x-1)\{x^2+\tfrac{1}{2}(\sqrt{5}+1)x+1\}\{x^2-\tfrac{1}{2}(\sqrt{5}-1)x+1\}.$$

$[x^4+x^3+x^2+x+1=x^2\{(x+x^{-1})^2+(x+x^{-1})-1\}$

$$=x^2(x+x^{-1}-\alpha)(x+x^{-1}-\beta),$$

where α, β are the roots of $z^2+z-1=0$.]

5. Give a geometrical construction to find the points z_1, z_2 corresponding to the values of $\sqrt{z}$.

[If $O1$ is the unit of length, Oz_1 bisects the angle XOz, and is a mean proportional to $O1$ and Oz; and z_2 is on z_1O produced, so that $Oz_2=z_1O$.]

6. If a, b are complex numbers, show that

$$|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}|=|a+b|+|a-b|.$$

[Let $z_1=a+\sqrt{a^2-b^2}$, $z_2=a-\sqrt{a^2-b^2}$, then by Exercise VII, Ex. 10,

$$|z_1|^2+|z_2|^2=\tfrac{1}{2}|z_1+z_2|^2+\tfrac{1}{2}|z_1-z_2|^2=2|a|^2+2|a^2-b^2|;$$

$$\therefore \{|z_1|+|z_2|\}^2=2\{|a|^2+|a^2-b^2|+|b|^2\}$$

$$=\{|a+b|^2+2|a+b|.|a-b|+|a-b|^2\}$$

$$=\{|a+b|+|a-b|\}^2.]$$

7. Solve the equation

$$z^2+2(1+2\iota)z-(11+2\iota)=0.$$

Verify that the sum of the roots is $-2(1+2\iota)$ and the product $-(11+2\iota)$.

[Put $z=x+\iota y$, equate real and imaginary parts to zero and solve for x, y.]

8. Prove that, with regard to the quadratic

$$z^2+(p+\iota p')z+q+\iota q'=0,$$

(i) if the equation has one real root, then

$$q'^2-pp'q'+qp'^2=0;$$

(ii) if the equation has two equal roots, then

$$p^2-p'^2=4q \quad \text{and} \quad pp'=2q'.$$

[(i) If α is a real root, by the rule of equality $\alpha^2+p\alpha+q=0$, $p'\alpha+q'=0$. Eliminate α.

(ii) In this case $(p+\iota p')^2=4(q+\iota q')$, etc.]

9. If $z=x+\iota y=r(\cos\theta+\iota\sin\theta)$, prove that

$$\sqrt{z}=\pm\frac{1}{\sqrt{2}}\{\sqrt{r+x}+\iota\sqrt{r-x}\} \quad \text{or} \quad \pm\frac{1}{\sqrt{2}}\{\sqrt{r+x}-\iota\sqrt{r-x}\},$$

according as y is positive or negative.

$$[\sqrt{z}=\pm\sqrt{r}\left(\cos\frac{\theta}{2}+\iota\sin\frac{\theta}{2}\right) \quad \text{and} \quad \cos\theta=\frac{x}{r};$$

$$\therefore \cos\frac{\theta}{2}=\pm\sqrt{\frac{r+x}{2r}}, \quad \sin\frac{\theta}{2}=\pm\sqrt{\frac{r-x}{2r}}.$$

Also if $y>0$ then $0<\theta<\pi$, and if $y<0$ we have $-\pi<\theta<0$, etc.]

10. Find the roots of $z^n=(z+1)^n$, and show that the points which represent them are collinear.

[The roots are $-\frac{1}{2}\left(1+\iota\cot\frac{r\pi}{n}\right)$, where $r=0, 1, 2, \ldots n-1$. The corresponding points lie on the line $x+\frac{1}{2}=0$.]

11. Show that the roots of $(1+z)^n=(1-z)^n$ are the values of $\iota \tan\frac{r\pi}{n}$, where $r=0, 1, 2, \ldots n-1$, but omitting $n/2$ if n is even.

12. Prove that

(i) $x^{2n}-2x^n\cos\theta+1=\Pi_{r=0}^{r=n-1}\left(x^2-2x\cos\frac{\theta+2r\pi}{n}+1\right)$;

(ii) $x^n+x^{-n}-2\cos\theta=\Pi_{r=0}^{r=n-1}\left(x+x^{-1}-2\cos\frac{\theta+2r\pi}{n}\right)$;

(iii) $\cos n\phi-\cos n\theta=2^{n-1}\Pi_{r=0}^{r=n-1}\left\{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}$.

[(i) $x^{2n}-2x^n\cos\theta+1=\{x^n-(\cos\theta+\iota\sin\theta)\}\{x^n-(\cos\theta-\iota\sin\theta)\}$. Now use Art. 15. (iii) Put $x=\cos\phi+\iota\sin\phi$, and for θ put $n\theta$.]

13. If n is odd and not a multiple of 3, prove that $x(x+1)(x^2+x+1)$ is a factor of $(x+1)^n-x^n-1$.

[Put $x=0, -1, \omega$, where ω is an imaginary cube root of unity.]

14. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, prove that

$$a_0+a_3+a_6+\ldots=a_1+a_4+a_7+\ldots=a_2+a_5+a_8+\ldots.$$

15. If

$$u=x+y+z+a(y+z-2x),$$
$$v=x+y+z+a(z+x-2y),$$
$$w=x+y+z+a(x+y-2z),$$

prove that $27a^2(x^3+y^3+z^3-3xyz)=u^3+v^3+w^3-3uvw$.

16. If

$$(a+\omega)^{-1}+(b+\omega)^{-1}+(c+\omega)^{-1}+(d+\omega)^{-1}=2\omega^{-1},$$
$$(a+\omega')^{-1}+(b+\omega')^{-1}+(c+\omega')^{-1}+(d+\omega')^{-1}=2\omega'^{-1},$$

where ω and ω' are the imaginary cube roots of unity, prove that

$$(a+1)^{-1}+(b+1)^{-1}+(c+1)^{-1}+(d+1)^{-1}=2.$$

[Consider the equation $(a+x)^{-1}+(b+x)^{-1}+(c+x)^{-1}+(d+x)^{-1}=2x^{-1}$.]

17. If $z_1^2+z_2^2+z_3^2-z_2z_3-z_3z_1-z_1z_2=0$, prove that

$$|z_2-z_3|=|z_3-z_1|=|z_1-z_2|.$$

[For $z_1+\omega z_2+\omega^2 z_3=0$, $\therefore\ z_3-z_1=\omega(z_2-z_3)$;
$\therefore\ |z_3-z_1|=|\omega|\,.\,|z_2-z_3|=|z_2-z_3|$.]

18. If $\omega=\frac{1}{2}(-1+\iota\sqrt{3})$, being an imaginary cube root of unity, and a, b, c are real, then

$$\sqrt{(a+\omega b+\omega^2 c)}=\pm\tfrac{1}{2}\{\sqrt{2D+L}+\iota\sqrt{2D-L}\},$$
$$\sqrt{(a+\omega^2 b+\omega c)}=\pm\tfrac{1}{2}\{\sqrt{2D+L}-\iota\sqrt{2D-L}\},$$

where

$$D=\sqrt{(a^2+b^2+c^2-bc-ca-ab)},$$
$$L=2a-b-c,$$

provided that $b>c$. If $b<c$, the sign of ι must be changed.

[Use Ex. 9. For a positive number x, $\sqrt{x}$ denotes the positive square root.]

21. Points representing the Product and Quotient of Two Given Numbers. Let $z=r(\cos\theta+\iota\sin\theta)$, $z'=r'(\cos\theta'+\iota\sin\theta')$.

(1) *Construction for the point representing the product zz'.*

Let 1 be the point on OX which represents unity, so that $O1=1$. Draw the triangle $Oz'P$ directly similar to the triangle $O1z$.

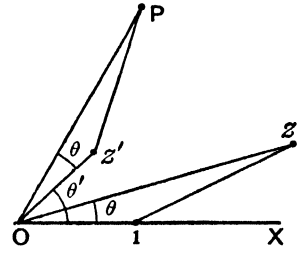


FIG. 9.

Then P represents the product zz'.

For by similar triangles

$$\frac{OP}{Oz'}=\frac{Oz}{O1}, \text{ that is, } \frac{OP}{r'}=\frac{r}{1}, \quad \therefore\ OP=rr';$$

also $\angle z'OP=\angle 1Oz=\theta$, $\therefore\ \angle 1OP=\theta+\theta'$.

But $zz'=rr'\{\cos(\theta+\theta')+\iota\sin(\theta+\theta')\}$.

Therefore P represents zz'.

(2) *Construction for the point representing the quotient z/z'.*

Draw the triangle OzQ directly similar to the triangle $Oz'1$.

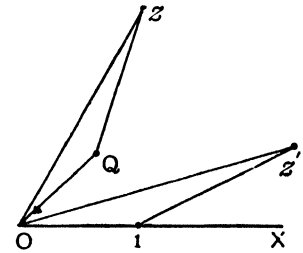


FIG. 10.

Then Q represents the quotient z/z'.

For, by the last construction,

$$(\text{number represented by } Q)\ .\ z'=z.$$

(3) *If k is constant, and z varies so that*

$$\left|\frac{z-a}{z-a'}\right|=k,$$

then the point z describes a circle of which a, a' are inverse points; unless $k=1$, in which case z describes the perpendicular bisector of aa'.

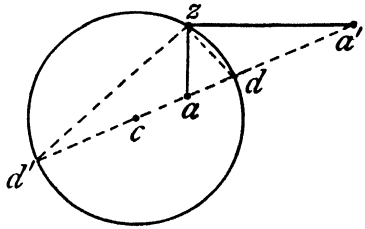


FIG. 11.

For let the bisectors of $\angle aza'$ meet aa' at d, d'. Then, by Art. 13, (2), $az:a'z=k:1$.

Therefore d, d' divide aa' internally and externally in the ratio $k:1$, and are *fixed* points.

Therefore z describes the circle on dd' as diameter.

Also if c is the mid-point of dd', we have $ca \,.\, ca' = cd^2$; hence a, a' are inverse points with regard to the circle.

If $k=1$, then $az=a'z$; and z lies on the perpendicular bisector of aa'.

Conversely, if the point z describes a circle of which a, a' are inverse points, then $\left|\dfrac{z-a}{z-a'}\right| = k$.

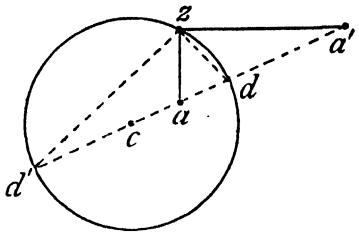


FIG. 11.

For, since $ca \,.\, ca' = cd^2$, then dd' is divided internally and externally in the same ratio, k to 1, say.

Hence, $az : a'z = k : 1$.*

(4) *If z varies so that*

$$\operatorname{am}\frac{z-a}{z-a'} = \phi,$$

where ϕ is a constant angle, then the point z describes an arc of a segment of a circle on aa', containing an angle ϕ.

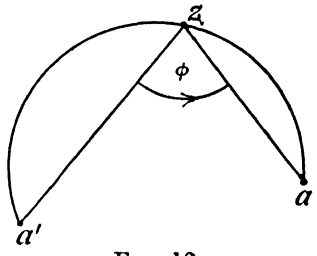


FIG. 12.

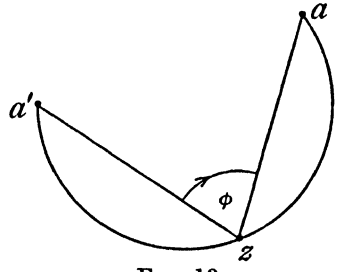


FIG. 13.

For by Art. 13, $\angle a'za = \phi$. The sign of ϕ determines the side of aa' on which the segment lies. Thus ϕ is positive in Fig. 12 and negative in Fig. 13.

22. Displacements and Vectors.

(1) In connection with the geometrical representation of complex numbers, we introduce the notions of *displacement* and *directed length* or *vector*.

(2) **Displacements in a given Plane.** Let P, Q be two points in the plane OXY. The change of position which a point undergoes in moving from P to Q is called the *displacement* $\overline{PQ}$.

* See *Elements of Geometry*, Barnard and Child, p. 316: 'Circle of Apollonius.'

If any straight line $P'Q'$ is drawn equal to, parallel to, and in the same sense as PQ, the displacements $\overline{PQ}$, $\overline{P'Q'}$ are said to be *equal*.

To specify completely a displacement $\overline{PQ}$ we must know :

(i) its magnitude, i.e. the length PQ ;

(ii) its direction ;

(iii) its sense, denoted by the order of the letters, and if necessary by an arrow.

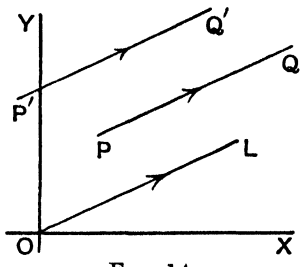


FIG. 14.

If we draw OL equal and parallel to PQ and in the same sense, we say that $\angle XOL$ is the angle which PQ makes with OX.

This angle determines the direction and sense of $\overline{PQ}$.*

(3) *Vectors.* An expression (such as $\overline{PQ}$) used to denote a line-segment with reference to its length, direction and sense, the actual *position* of the line being indifferent, is called a *vector*.

Quantities which can be represented by lines used in this way are called *vector quantities*. Velocities and accelerations are vector quantities.

A force can be represented by a vector '*localised*' to lie in the line of action of the force.

Quantities (such as mass) which do not involve the idea of direction are called *scalar*.

(4) **Connection with Complex Number.** If $z=x+\iota y$ and P is the point (x, y), a one-to-one correspondence exists between the number z and any of the following : (i) the point P ; (ii) the displacement $\overline{OP}$; (iii) the vector (or directed length) $\overline{OP}$.

Any one of these three things may therefore be said to represent z, *or to be represented by* z.

23. Addition of Displacements and of Vectors.

(1) Let P, Q, R be any three points. If a point moves from P to Q and then from Q to R, the resulting *change of position* is the same as if it had moved directly from P to R. We therefore define the *addition of displacements* as follows :

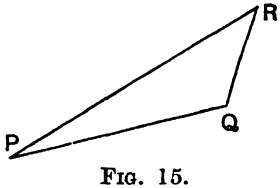


FIG. 15.

* Some writers use an underline instead of an overline.

The result of adding $\overline{QR}$ to $\overline{PQ}$ is defined to be $\overline{PR}$; and this is expressed by writing $\overline{PQ}+\overline{QR}=\overline{PR}$. .....(A)

This equation is also taken as defining the *addition of vectors.*

If PQ, QR, RS are any three displacements or vectors (Fig. 17),

$$\overline{PQ}+\overline{QR}+\overline{RS}=\overline{PR}+\overline{RS}=\overline{PS}.$$

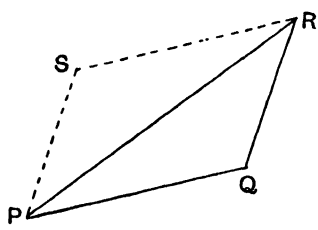


FIG. 16.

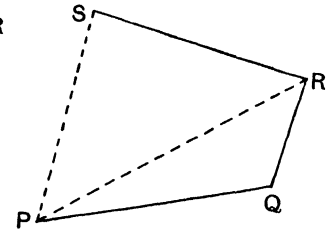


FIG. 17.

(2) *The Commutative and Associative Laws hold for the addition of displacements and vectors.*

(i) In Fig. 16, complete the parallelogram $PQRS$; then by Arts. 23, (1), and 22,

$$\overline{PQ}+\overline{QR}=\overline{PR}=\overline{PS}+\overline{SR}=\overline{QR}+\overline{PQ}.$$

Hence the commutative law holds, and $\overline{PR}$ is called the *sum* of $\overline{PQ}$ and $\overline{QR}$.

(ii) In Fig. 17,

$$(\overline{PQ}+\overline{QR})+\overline{RS}=\overline{PR}+\overline{RS}=\overline{PS},$$
$$\overline{PQ}+(\overline{QR}+\overline{RS})=\overline{PQ}+\overline{QS}=\overline{PS},$$

therefore $$(\overline{PQ}+\overline{QR})+\overline{RS}=\overline{PQ}+(\overline{QR}+\overline{RS}),$$

and the associative law holds.

NOTE. In Fig. 16, $\overline{PQ}+\overline{PS}=\overline{PR}$, a fact which is expressed by saying that *displacements and vectors are added by the parallelogram law.*

24. Zero and Negative Displacements and Vectors. If after two or more displacements the moving point returns to its initial position, we say that the resulting displacement is *zero.* Thus we write

$$\overline{PQ}+\overline{QP}=0.$$

This equation is also written in the form $-\overline{PQ}=\overline{QP}$, which *defines* the meaning of $-\overline{PQ}$.

These equations are also taken as defining the meaning of *zero and negative vectors.*

25. Subtraction. For displacements and vectors the meaning of $\overline{PQ} - \overline{QR}$ is defined by

$$\overline{PQ} - \overline{QR} = \overline{PQ} + (-\overline{QR}) = \overline{PQ} + \overline{RQ}.$$

Thus $$\overline{OP} - \overline{OQ} = \overline{OP} + \overline{QO} = \overline{QO} + \overline{OP} = \overline{QP}.$$

26. Definition of Multiplication by a Real Number. To multiply a displacement or a vector $\overline{PQ}$ by a positive number k is to multiply its length by k, leaving its direction unaltered. The resulting displacement or vector is denoted by $k\overline{PQ}$ or by $\overline{PQ} \,.\, k$.

Further, we define $(-k)PQ$ by the equation

$$(-k)\overline{PQ} = k(-\overline{PQ}) = k\overline{QP}.$$

In particular,

$$(-1)\overline{PQ} = \overline{QP}.$$

So that *to multiply a vector by* (-1) *is to turn it through two right angles.*

27. The Distributive Law. *We shall prove that if k is a real number, then*

$$k(\overline{PQ} + \overline{QR}) = k\overline{PQ} + k\overline{QR}.$$

(i) *Let k be positive.* Along PQ set off $PQ' = kPQ$. Draw $Q'R'$ parallel to QR to meet PR in R'. By similar triangles,

$$PR' = kPR \quad \text{and} \quad Q'R' = kQR;$$

$$\therefore\ k(\overline{PQ} + \overline{QR}) = k\overline{PR} = \overline{PR'},$$

and $$k\overline{PQ} + k\overline{QR} = \overline{PQ'} + \overline{Q'R'} = \overline{PR'}.$$

Hence the theorem holds for positive numbers.

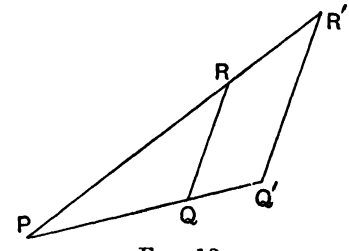


FIG. 18.

(ii) *For a negative number* $(-k)$, we have

$$(-k)(\overline{PQ} + \overline{QR}) = (-k)\overline{PR} = \overline{R'P},$$

and $$(-k)\overline{PQ} + (-k)\overline{QR} = k\overline{QP} + k\overline{RQ} = \overline{Q'P} + \overline{R'Q'} = \overline{R'P};$$

hence the theorem holds for negative numbers.

Thus *the distributive law holds for the multiplication of displacements and vectors by real numbers.*

NOTE. The diagram of Fig. 18 is drawn for a value of k greater than unity: the student should see that the same result follows from a diagram in which k is less than unity.

28. Complex Numbers represented by Vectors. It will now be seen that, so far as addition, subtraction and multiplication by real numbers are concerned, complex numbers are subject to the same laws as the vectors which represent them. This fact is fundamental in theory and very useful in practice.

It should be noticed that if a number z is represented by a vector $\overline{AB}$, then $|z|$ is the length AB and am z is the angle which the directed line AB makes with the directed line Ox.

Theorem 1. *If C divides AB in the ratio $n:m$ and O is any point, then*

$$(m+n)\overline{OC}=m\overline{OA}+n\overline{OB}.$$

For $$m\overline{OC}=m\overline{OA}+m\overline{AC},$$

$$n\overline{OC}=n\overline{OB}+n\overline{BC}.$$

Also $$m\overline{AC}=n\overline{CB}=-n\overline{BC};$$

whence the result follows by addition.

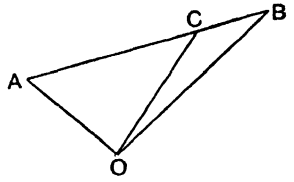


FIG. 19.

Theorem 2. *If z is the point which divides the straight line joining z_1, z_2 in the ratio $n:m$, then the corresponding numbers are connected by the relation*

$$(m+n)z=mz_1+nz_2.$$

This follows from Theorem 1, in accordance with the principle stated in this article.

In particular, *if z is the mid-point of z_1z_2, then* $z=\frac{1}{2}(z_1+z_2)$.

Ex. 1. *If a, b are complex numbers, prove geometrically that*

$$|a+b|^2+|a-b|^2=2|a|^2+2|b|^2.$$

Let A, B be the points which represent a, b.

Bisect AB at C, then

$$\overline{OA}+\overline{OB}=2\overline{OC}$$

and $$\overline{OA}-\overline{OB}=\overline{BA}=2\overline{CA}.$$

Therefore $a+b$ and $a-b$ are represented by $2\overline{OC}$ and $2\overline{CA}$; hence

$$|a+b|=2OC, \quad |a-b|=2CA.$$

Now, since C is mid-point of base AB,

$$OA^2+OB^2=2OC^2+2CA^2;$$

$$\therefore\ 2|a|^2+2|b|^2=|a+b|^2+|a-b|^2.$$

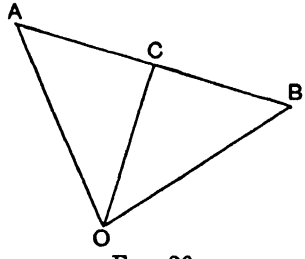


FIG. 20.

Ex. 2. *If* $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ *are connected by the relation*

$$a\overline{OA}+b\overline{OB}+c\overline{OC}=0, \quad \textit{where} \quad a+b+c=0,$$

then A, B, C are collinear.

[This is the converse of Theorem 1 of this article. We have

$$(a+c)\overline{OB}=a\,.\,\overline{OA}+c\,.\,\overline{OC}\,; \text{ hence } \quad a\,.\,\overline{AB}=c\,.\,\overline{BC}.]$$

29. The Symbol ι as an Operator.* Along two straight lines at right angles set off, consecutively, equal lengths OP, OQ, OP', OQ' in the positive direction of rotation.

Let the symbol ι *applied to a vector denote the operation of turning it through a right angle in the positive direction of rotation.*

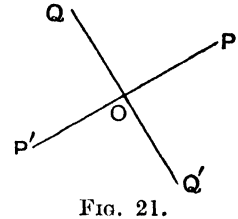


FIG. 21.

To bring our language into conformity with that of algebra, we say that *to multiply a vector by* ι *is to turn it through a right angle in the positive sense.*

Thus in Fig. 21, $\overline{OQ}=\iota\overline{OP}$ and $\overline{OP'}=\iota\overline{OQ}$.

Therefore $$\overline{OP'}=\iota(\iota\overline{OP})=\iota^2\overline{OP},$$

where $\iota^2\overline{OP}$ is an abbreviation for $\iota(\iota\overline{OP})$. Hence $\iota^2\overline{OP}=(-1)\overline{OP}$.

Thus ι^2 and -1 denote the same operation, and in this sense we write $\iota^2=-1$.

Again, $\overline{OQ'}=\iota\overline{OP'}=\iota(-1)\overline{OP}$ and $\overline{OQ'}=-\overline{OQ}=(-1)\iota\overline{OP}$.

Either of these results is written in the form $(-\iota)\,\overline{OP}=\overline{OQ'}$, so that *to multiply a vector by* $(-\iota)$ *is to turn it through a right angle in the negative sense.*

Again, if $\iota^3\,.\,\iota^2\overline{OP}$ is taken to mean $\iota^3(\iota^2\overline{OP})$, it is obvious that

$$\iota^3\,.\,\iota^2\overline{OP}=\iota^5\overline{OP}=\iota^2\,.\,\iota^3\overline{OP}=\overline{OQ}.$$

Ex. 1. *If* a, b *are complex numbers, find numbers* z, z' *so that the points* z, z' *and* a, b *may be opposite corners of a square.*

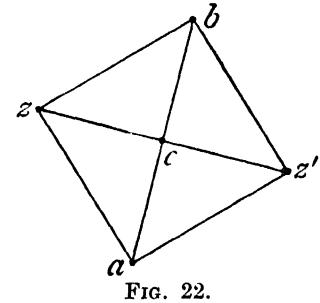


FIG. 22.

Let c be the mid-point of ab, then

$$\overline{Oz}=\overline{Oc}+\overline{cz}=\overline{Oc}+\iota\overline{cb}\,;$$

$$\therefore\ Oz=\tfrac{1}{2}(Oa+Ob)+\tfrac{1}{2}\iota(Ob-Oa)\,;$$

$$\therefore\ z=\tfrac{1}{2}(a+b)-\tfrac{1}{2}\iota(a-b).$$

Similarly, $z'=\tfrac{1}{2}(a+b)+\tfrac{1}{2}\iota(a-b)$.

* For the moment, the reader should forget his conception of ι as denoting a *number*.

30. The Operator $\cos\theta + \iota \sin\theta$. Draw two equal straight lines OP, OP' inclined at an angle θ. Draw $P'N$ perpendicular to OP. Along NP set off NQ equal to NP'. Then

$$\overline{OP'} = \overline{ON} + \overline{NP'} = \overline{ON} + \iota\overline{NQ}.$$

Also $$\overline{ON} = \frac{ON}{OP}\overline{OP} = \cos\theta \,.\, \overline{OP},$$

$$NQ = \frac{NQ}{OP}OP = \sin\theta \,.\, OP;$$

$$\therefore\ \overline{OP'} = \cos\theta \,.\, \overline{OP} + \iota \sin\theta \,.\, \overline{OP},$$

which we write in the form

$$\overline{OP'} = (\cos\theta + \iota \sin\theta)\overline{OP},$$

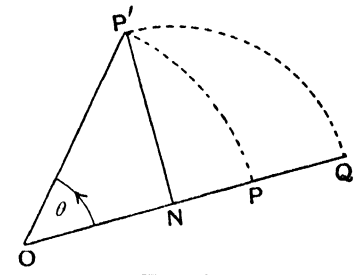


FIG. 23.

and we say that *to multiply a vector by $\cos\theta + \iota \sin\theta$ is to turn it through the angle θ.*

31. Multiplication and Division of a Vector by a Complex Number. In accordance with Arts. 26, 29 and 30 of this chapter, we say that *to multiply a vector OP by the complex number $r(\cos\theta + \iota \sin\theta)$ is to multiply its length by r and turn the resulting vector through the angle θ.*

Here r is the *stretching* factor and $\cos\theta + \iota \sin\theta$ the *turning* factor. These are independent of each other, and the *order* in which they are applied is indifferent.

If $\overline{OQ}$ is the vector obtained by multiplying $\overline{OP}$ by the complex number z, we write

$$\overline{OQ} = z\overline{OP} \quad \text{and} \quad \overline{OQ}/\overline{OP} = z\,;$$

we also say that the *ratio* of $\overline{OQ}$ to $\overline{OP}$ is the number z.

Division is the inverse of multiplication, so that if $\overline{OQ} = z\overline{OP}$, then $\overline{OP}$ is the result of dividing $\overline{OQ}$ by z.

Therefore *to divide a vector OQ by z is to divide its length by r and turn the resulting vector through the angle $(-\theta)$.*

The result is the same as that obtained by multiplying $\overline{OQ}$ by $1/z$.

32. Product of Complex Numbers.

Let $$\overline{OQ} = z'\overline{OP} \quad \text{and} \quad \overline{OR} = z\overline{OQ},$$

then we write

$$\overline{OR} = z(z'\overline{OP}) = zz'\overline{OP},$$

where zz' applied to $\overline{OP}$ denotes that the operators z', z are to be applied *in succession, in this order.*

Since the stretching and turning factors may be applied in any order, $\overline{OP}$ may be transformed into $\overline{OR}$ by multiplying its length by rr' and turning the resulting vector through the angle $(\theta+\theta')$.

Hence the operations denoted by

$$zz', \quad z'z \quad \text{and} \quad rr'\{\cos(\theta+\theta')+\iota\sin(\theta+\theta')\}$$

are equivalent.

Again, if we take $(\cos\theta+\iota\sin\theta)^n$ to mean that the operation

$$(\cos\theta+\iota\sin\theta)$$

is to be applied n times, the result is the same as that given by the operator

$$\cos n\theta+\iota\sin n\theta.$$

In this sense then

$$(\cos\theta+\iota\sin\theta)^n=\cos n\theta+\iota\sin n\theta.$$

It will be seen that *complex numbers used as operators on vectors conform to the laws of algebra.*

EXERCISE IX

ARGAND DIAGRAMS: VECTORS

1. If $z=3+2\iota$, $z'=1+\iota$, mark the points z, z' in an Argand diagram, and find by geometrical construction, the points representing

$$z+z', \quad z-z', \quad zz', \quad z/z'.$$

2. Let z, a, b be complex numbers of which a, b are constant and z varies. If Z is given in terms of z by one of the following equations, it is required to find the point Z corresponding to a given point z. Explain the constructions indicated in the diagrams, $O1$ being the unit of length.

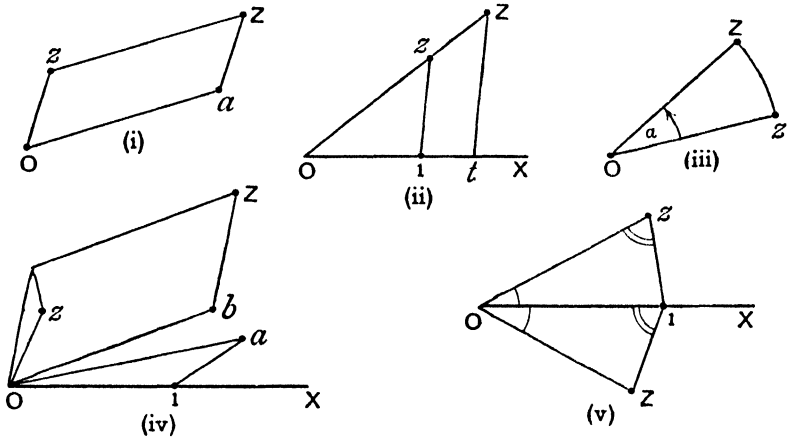


FIG. 24.

(i) $Z=z+a$. (ii) $Z=tz$ where t is real.

(iii) $Z=(\cos\alpha+\iota\sin\alpha)z$. (iv) $Z=az+b$. (v) $Z=1/z$.

3. If $Z=(1+z)/(1-z)$, find the point Z corresponding to a given point z, and show that if $|z|=1$, the point Z is on the y-axis.

4. If a, b are given complex numbers and

$$z=a+(b-a)t,$$

find the point z corresponding to any given real value of t, and prove that as t varies from $-\infty$ to $+\infty$, z describes the entire line which passes through a, b, the segment ab corresponding to the values from 0 to 1 of t.

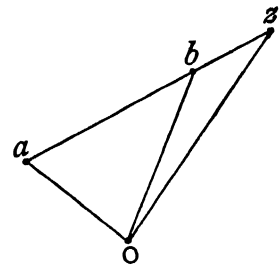


FIG. 25.

[Along the line ab set off the length $az=t\,.\,ab$, then $\overline{Oz}=\overline{Oa}+\overline{az}=\overline{Oa}+t\,.\,\overline{ab}$; $\therefore\ z=a+(b-a)t$, etc.]

5. If $z=a(1+\iota t)$ where t is a real number, prove that as t varies z describes the line through the point a perpendicular to Oa.

6. If c, a are given numbers, a being real, and,

(i) if $$z=c+a(\cos\phi+\iota\sin\phi),$$

find the point z corresponding to a given value of ϕ;

(ii) if $$z=c+a(1+\iota t)/(1-\iota t),$$

where t is real, show that as t varies from $-\infty$ to $+\infty$, the point z describes once the circle with centre c and radius a. [Put $t=\tan\frac{1}{2}\phi$.]

7. If A, B, C, D are any four points in a plane, then

$$AD\,.\,BC\leqslant BD\,.\,AC+CD\,.\,AB.$$

Show that this theorem is an immediate consequence of the identity

$$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)=0.$$

8. If G is the centroid of particles of mass m_1, m_2, m_3, ... at A_1, A_2, A_3, ... and O is any point, then $(m_1+m_2+m_3+\ldots)\overline{OG}=m_1\overline{OA}_1+m_2\overline{OA}_2+m_3\overline{OA}_3+\ldots$.

To include cases where m_1, m_2, ... are not all of the same sign, we may suppose G to be the centre of parallel forces, proportional to m_1, m_2, acting at A_1, A_2 ... .

9. If z is the centroid of particles of mass m_1, m_2, m_3, ... at z_1, z_2, z_3, ..., then

$$(m_1+m_2+m_3+\ldots)z=m_1z_1+m_2z_2+m_3z_3+\ldots.$$

10. *Any three coplanar and non-parallel vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are connected by a relation of the form $p\overline{OA}+q\overline{OB}+r\overline{OC}=0$, where p, q, r are real numbers.*

Moreover, $$p:q:r=\triangle OBC:\triangle OCA:\triangle OAB,$$

where the signs of the areas are determined by the usual rule.

Also the points A, B, C are collinear if $p+q+r=0$, and conversely.

[For let $$p:q:r=\triangle OBC:\triangle OCA:\triangle OAB,$$

and let G be the centre of parallel forces, acting at A, B, C, and proportional to p, q, r, then G coincides with O, and $p\overline{OA}+q\overline{OB}+r\overline{OC}=(p+q+r)OG=0$.]

11. (i) *If α, β are non-parallel vectors and p, q, p', q' are real numbers, an equation of the form*

$$p\alpha+q\beta=p'\alpha+q'\beta$$

involves the two equations $p=p'$, $q=q'$.

(ii) If α, β, γ are non-parallel coplanar vectors connected by the equations

$$p\alpha+q\beta+r\gamma=0 \quad \text{and} \quad p'\alpha+q'\beta+r'\gamma=0,$$

where p, q, r, p', q', r' are real, then these are one and the same equation, that is to say

$$p/p'=q/q'=r/r'.$$

[These theorems follow at once from Ex. 10.]

12. Any three complex numbers z_1, z_2, z_3 are connected by a relation of the form

$$pz_1+qz_2+rz_3=0,$$

where p, q, r are real numbers.

Moreover, $$p:q:r=\triangle Oz_2z_3:\triangle Oz_3z_1:\triangle Oz_1z_2,$$

the signs of the areas being determined as usual.

Also the points z_1, z_2, z_3 are collinear if $p+q+r=0$, and conversely.

Prove this algebraically, and deduce Ex. 11.

[If $z_1=x_1+\iota y_1$, $z_2=x_2+\iota y_2$, $z_3=x_3+\iota y_3$, we have

$$z_1(x_2y_3-x_3y_2)+\ldots+\ldots=0,$$

whence the results follow immediately.]

13. If the points A, B, C are collinear, and O is any point, then

$$\overline{OA}\,.\,BC+\overline{OB}\,.\,CA+\overline{OC}\,.\,AB=0.$$

[In Art. 28, Theorem 1, $m:n:m+n=CB:AC:AB$.]

14. Let z_1, z_2, z_3 be complex numbers, no two of which are equal, then

(i) If the points z_1, z_2, z_3 are collinear,

$$z_1\,|\,z_2-z_3\,|\pm z_2\,|\,z_3-z_1\,|\pm z_3\,|\,z_1-z_2\,|=0.$$

Also, if the point z_1 lies between z_2 and z_3, the ambiguous signs are both minus.

(ii) If the above equation holds, then either z_1, z_2, z_3 are collinear, or else O is the centre of a circle which touches the sides of the triangle $z_1z_2z_3$.

15. If $A_1A_2A_3$ is an equilateral triangle, the vertices occurring in the positive direction of rotation, prove that

$$\overline{A_3A_1}=\left(\cos\frac{2\pi}{3}+\iota\sin\frac{2\pi}{3}\right)\overline{A_2A_3}=\omega\overline{A_2A_3},$$

where ω is an imaginary cube root of unity.

Also, if z_1, z_2, z_3 are the numbers corresponding to A_1, A_2, A_3, prove that $z_1+\omega z_2+\omega^2z_3=0$, and consequently

$$z_1^2+z_2^2+z_3^2-z_2z_3-z_3z_1-z_1z_2=0.$$

16. If $AXYA'X'Y'$ is a regular hexagon and A, A' represent given complex numbers a, a', then the numbers represented by X, X', Y, Y' are given by

$$\tfrac{1}{2}(a+a')+\tfrac{1}{2}(a-a')(\cos\theta+\iota\sin\theta),$$

where θ has the values $\pm\frac{\pi}{3}$, $\pm\frac{2\pi}{3}$.

17. The triads of points A, B, C and X, Y, Z are the vertices of directly similar triangles if the corresponding complex numbers a, b, c and x, y, z are connected by the relation

$$x(b-c)+y(c-a)+z(a-b)=0.$$

[The triangles are directly similar if $\frac{\overline{AB}}{\overline{AC}}=\frac{\overline{XY}}{\overline{XZ}}$, i.e. if $\frac{b-a}{c-a}=\frac{y-x}{z-x}$.]

18. If ABC is a triangle and triangles BCX, CAY, ABZ are drawn on BC, CA, AB, directly similar to one another, the centroids of XYZ and ABC coincide.

[By Ex. 17, $\frac{x-c}{b-c}=\frac{y-a}{c-a}=\frac{z-b}{a-b}$. Hence show that $x+y+z=a+b+c$.]

19. Equilateral triangles are drawn on the sides of a given triangle ABC, all outwards or all inwards. Prove that their centroids form an equilateral triangle.

[Let P, Q, R be the centroids of the triangles drawn outwards on BC, CA, AB. Prove that

$$\overline{QA}=\overline{CA}\cdot\frac{1}{\sqrt{3}}(\cos 30+\iota\sin 30),\quad \overline{AR}=\overline{AB}\cdot\frac{1}{\sqrt{3}}(\cos 30-\iota\sin 30).$$

Hence show that $\overline{QR}=\frac{1}{2}\overline{CB}+\frac{\iota}{2\sqrt{3}}(\overline{CA}-\overline{AB})$, and that $\overline{RP}$ has a similar value. Hence show that $\overline{RQ}=\overline{RP}(\cos 60+\iota\sin 60)$, and use Ex. 15.]

'TRANSFORMATIONS'

20. If Z, z are connected by any of the relations in Ex. 2 and if z describes a given curve s, then Z will describe a curve S.

Explain the following, where a, b are given complex numbers.

(i) If $Z=z+a$, S can be obtained from s by a *translation*.

(ii) If $Z=tz$ where t is real, the curves s, S are similar and similarly situated, O being the centre of similitude. In this case we say that S is a *magnification* of s.

(iii) If $Z=(\cos\alpha+\iota\sin\alpha)z$, S can be obtained from s by a *rotation* about O through $\angle\alpha$.

(iv) If $Z=az+b$, S can be obtained from s by *a rotation, a magnification and a translation.*

(v) If $Z=1/z$, S is the reflection in OX of the inverse of s, O being the centre of inversion.

21. Show that each of the substitutions in Ex. 20 converts a circle c into a circle C, except that in (v), if c passes through O, then C is a straight line.

22. *Show that the substitution*

$$Z=\frac{az+b}{a'z+b'}$$

converts a circle into a circle or, in an exceptional case, into a straight line.

[$Z=\frac{a}{a'}+\frac{ba'-b'a}{a'}\cdot\frac{1}{a'z+b'}$, therefore the transformation is equivalent to one or more of those in Ex. 20.]

CHAPTER VI

THEORY OF EQUATIONS (1)

1. Roots of Equations. Under this heading we consider equations of the type $f(x)=0$ where $f(x)$ is a polynomial, and ' equation ' will mean an equation of this kind.

The general equation of the nth degree will be written in one of the forms :

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0;$$

or,
$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{\lfloor 2}a_2x^{n-2}+\ldots+a_n=0,$$

where binomial coefficients are introduced. The last equation is written briefly in the form

$$(a_0, a_1, a_2, \ldots a_n\}x, 1)^n=0.$$

For the present, we assume that *every equation has one root.* This is the fundamental theorem of the Theory of Equations, and will be proved in another volume.

(1) It follows that *every equation of the n-th degree has exactly n roots.*

For let $f(x)\equiv x^n+p_1x^{n-1}+\ldots+p_n$ and let α be a root of $f(x)=0$. By the Remainder theorem, $f(x)$ is divisible by $x-\alpha$; we may therefore assume that

$$f(x)\equiv(x-\alpha)(x^{n-1}+q_1x^{n-2}+\ldots+q_{n-1})\equiv(x-\alpha)\,.\,\phi(x).$$

Let β be a root of $\phi(x)=0$; as before, $\phi(x)$ is divisible by $x-\beta$, and we may assume that

$$\phi(x)\equiv(x-\beta)(x^{n-2}+r_1x^{n-3}+\ldots+r_{n-2});$$

$$\therefore\ f(x)\equiv(x-\alpha)(x-\beta)(x^{n-2}+r_1x^{n-3}+\ldots+r_{n-2}).$$

Proceeding in this way, we can show that

$$f(x)\equiv(x-\alpha)(x-\beta)\ldots(x-\lambda),$$

where there are n linear factors on the right. Hence $f(x)=0$ has n roots $\alpha, \beta, \gamma, \ldots, \lambda$, and no others.

(2) **Imaginary Roots.** Let the coefficients of $f(x)$ be real; then, if $\alpha+\iota\beta$ is a root, so is $\alpha-\iota\beta$ (Ch. V, 14). Therefore $f(x)$ is divisible by

$$(x-\alpha-\iota\beta)(x-\alpha+\iota\beta),$$

that is, by $(x-\alpha)^2+\beta^2$.

Thus a polynomial in x with real coefficients can be resolved into factors which are linear or quadratic functions of x with real coefficients.

(3) **Multiple Roots.** If $f(x)=(x-\alpha)^r \cdot \phi(x)$ where $\phi(x)$ is not divisible by $x-\alpha$, α is called an *r-multiple root* of $f(x)=0$.

When $r=2$, we say that α is a *double root*.

2. Relations connecting the Roots and Coefficients of an Equation.

Theorem. *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

then the sums of the products of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ taken one, two, three, ..., n at a time, are respectively equal to

$$-p_1, \quad p_2, \quad -p_3, \quad \ldots, \quad (-1)^n p_n.$$

For $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

$$=x^n-\Sigma\alpha_1 \cdot x^{n-1}+\Sigma\alpha_1\alpha_2 \cdot x^{n-2}+\ldots$$
$$+(-1)^n\alpha_1\alpha_2\ldots\alpha_n, \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

and, equating coefficients, we have

$$\Sigma\alpha_1=-p_1, \quad \Sigma\alpha_1\alpha_2=p_2, \quad \ldots\ \alpha_1\alpha_2\ldots\alpha_n=(-1)^n p_n. \quad \ldots\ldots\ldots\ldots(\mathbf{B})$$

Conversely, if $\alpha_1, \alpha_2, \ldots \alpha_n$ satisfy the equations (**B**), *they are the roots of*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0.$$

For under these circumstances the identity (**A**) holds.

It follows that the result of eliminating $\alpha_2, \alpha_3, \ldots \alpha_n$ from equations (**B**) is $\alpha_1^n+p_1\alpha_1^{n-1}+\ldots+p_n=0$.

Ex. 1. *If α, β, γ are the roots of $2x^3+x^2-2x-1=0$, write down the values of*

$$\alpha+\beta+\gamma, \quad \beta\gamma+\gamma\alpha+\alpha\beta, \quad \alpha\beta\gamma.$$

Write the equation in the form $x^3+\frac{1}{2}x^2+(-1)x+(-\frac{1}{2})=0$;

$$\therefore\ \alpha+\beta+\gamma=-\tfrac{1}{2}, \quad \beta\gamma+\gamma\alpha+\alpha\beta=-1, \quad \alpha\beta\gamma=\tfrac{1}{2}.$$

3. Transformation of Equations.

Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x)=0$, and suppose that we require the equation whose roots are $\phi(\alpha), \phi(\beta), \phi(\gamma), \ldots$ where $\phi(x)$ is a given function of x.

Let $y=\phi(x)$ and suppose that from this equation we can find x as a single-valued function of y, which we denote by $\phi^{-1}(y)$.

Transforming the equation $f(x)=0$ by the substitution $x=\phi^{-1}(y)$, we obtain $f\{\phi^{-1}(y)\}=0$, which is the equation required.

A case in which x is not a single-valued function of y is given in Ex. 2.

Ex. 1. *If* α, β, γ *are the roots of* $x^3-x-1=0$, *find the equation whose roots are* $\frac{1+\alpha}{1-\alpha}$, $\frac{1+\beta}{1-\beta}$, $\frac{1+\gamma}{1-\gamma}$. *Hence write down the value of* $\Sigma(1+\alpha)/(1-\alpha)$.

Let $y=\dfrac{1+x}{1-x}$, then $x=\dfrac{y-1}{y+1}$; and, if $x^3-x-1=0$, then y is given by

$$\left(\frac{y-1}{y+1}\right)^3-\frac{y-1}{y+1}-1=0,$$

which is equivalent to

$$y^3+7y^2-y+1=0.$$

This is the required equation. Hence also

$$\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}=-7.$$

Ex. 2. *If* α, β, γ *are the roots of* $f(x)=x^3+p_1x^2+p_2x+p_3=0$, *find the equation whose roots are* α^3, β^3, γ^3.

Let $y=x^3$, then $x=y^{\frac{1}{3}}$ where $y^{\frac{1}{3}}$ denotes *any* cube root of y. The required equation is therefore obtained by rationalising

$$f(y^{\frac{1}{3}})=y+p_1y^{\frac{2}{3}}+p_2y^{\frac{1}{3}}+p_3=0;$$

for the result will be the same whichever cube root $y^{\frac{1}{3}}$ stands for. To rationalise the equation, let $y+p_3=l$, $p_1y^{\frac{2}{3}}=m$, $p_2y^{\frac{1}{3}}=n$, then $l+m+n=0$, and therefore

$$l^3+m^3+n^3-3lmn=0.$$

Hence the required equation is

$$(y+p_3)^3+p_1^3y^2+p_2^3y-3(y+p_3)\cdot p_1y^{\frac{2}{3}}\cdot p_2y^{\frac{1}{3}}=0,$$

which is the same as

$$y^3+(p_1^3-3p_1p_2+3p_3)y^2+(p_2^3-3p_1p_2p_3+3p_3^2)y+p_3^3=0.$$

4. Special Cases. The following transformations are often required. Let α, β, γ, ... be the roots of $f(x)=0$, then

(1) the equation whose roots are $-\alpha$, $-\beta$, $-\gamma$, ... is $f(-x)=0$;

(2) the equation whose roots are $1/\alpha$, $1/\beta$, $1/\gamma$, ... is $f(1/x)=0$;

(3) the equation whose roots are $k\alpha$, $k\beta$, $k\gamma$, ... is $f(x/k)=0$.

This transformation is called '*multiplying the roots of* $f(x)=0$ *by* k.'

(4) The equation whose roots are $\alpha-h$, $\beta-h$, $\gamma-h$ is $f(x+h)=0$.

We find $f(x+h)$ as in Ch. III, 4, and the transformation is called '*diminishing the roots of* $f(x)=0$ *by* h.'

(5) *The second term of* $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ *can be removed by diminishing the roots by* $-a_1/na_0$.

For $$f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + \ldots,$$

so that the coefficient of x^{n-1} in $f(x+h)$ is $na_0h + a_1$, and this is zero when $h = -a_1/na_0$.

(6) *To transform* $f(x) = 0$ *into an equation in which the coefficient of the first term is* 1 *and the other coefficients are the least possible integers,* proceed as follows:

Ex. 1. *Consider the equation of*

$$72x^3 - 54x^2 + 45x - 7 = 0.$$

Write this $$x^3 - \tfrac{3}{4}x^2 + \tfrac{5}{8}x - \tfrac{7}{72} = 0.$$

Putting $x = y/k$ and multiplying by k^3, the equation becomes

$$y^3 - \frac{3}{4}ky^2 + \frac{5}{8}k^2y - \frac{7}{72}k^3 = 0.$$

or $$y^3 - \frac{3}{2^2}ky^2 + \frac{5}{2^3}k^2y - \frac{7}{2^3 \cdot 3^2}k^3 = 0.$$

The least value of k which will make every coefficient an integer is 12, so that the required equation is

$$y^3 - 9y^2 + 90y - 168 = 0 \quad \text{where} \quad y = 12x.$$

Ex. 2. *Transform* $x^3 - 6x^2 + 5x + 12 = 0$ *into an equation lacking the second term.*

We diminish the roots by $6/3 = 2$.

The reckoning on the right shows that the resulting equation is

$$x^3 - 7x + 6 = 0,$$

which is the one required.

1 − 2)	1	−6	+5	+12
		+2	−8	−6
	1	−4	−3	+6
		+2	−4	
	1	−2	−7	
		+2		
	1	0		

Ex. 3. *If* α, β, γ *are the roots of* $2x^3 + 3x^2 - x - 1 = 0$, *find the equation whose roots are* $2\alpha + 3, 2\beta + 3, 2\gamma + 3$.

The equation whose roots are $2\alpha, 2\beta, 2\gamma$ is

$$2 \cdot \frac{x^3}{8} + 3 \cdot \frac{x^2}{4} - \frac{x}{2} - 1 = 0, \quad \text{or} \quad x^3 + 3x^2 - 2x - 4 = 0.$$

Increasing the roots of this by 3 (as in the margin, where we divide successively by $x+3$), the required equation is

$$x^3 - 6x^2 + 7x + 2 = 0.$$

1 − (−3))	1	+3	−2	−4
		−3	+0	+6
	1	+0	−2	+2
		−3	+9	
	1	−3	+7	
		−3		
	1	−6		

Ex. 4. *If two roots* α *and* β *of* $f(x) = 0$ *are connected by the relation* $\beta = \phi(\alpha)$ we can generally find them as follows: The equations $f(x) = 0$ and $f\{\phi(x)\} = 0$ have a common root, namely α. Therefore $x - \alpha$ is a common factor of $f(x)$ and $f\{\phi(x)\}$, and may be found by the H.C.F. process.

If however $f(x)$ and $f\{\phi(x)\}$ are identical, then the method fails.

Ex. 5. *Solve* $2x^3+x^2-7x-6=0$, *given that the difference between two of the roots is* 3.

Let the roots be α, $\alpha+3$, β. Find the equation whose roots are $\alpha-3$, α, $\beta-3$. The reckoning on the right shows that this equation is $2x^3+19x^2+53x+36=0$.

Hence, $x-\alpha$ is a common factor of $2x^3+19x^2+53x+36$ and $2x^3+x^2-7x-6$.

2	+ 1	− 7	− 6
	+ 6	+21	+42
	+ 7	+14	+36
	+ 6	+39	
	+13	+53	
	+ 6		
	+19		

Find the H.C.F. of these expressions; this proves to be $x+1$, therefore $\alpha=-1$; and hence -1 and 2 are roots.

Again, the product of the three roots is 3; and the third remaining root is $-3/2$.

5. The Cubic Equation.

We take as the standard form

$$u\equiv ax^3+3bx^2+3cx+d=0. \quad \text{.....(A)}$$

If $x=y+h$, this becomes

$$ay^3+3By^2+3Cy+D=0,$$

where $B=ah+b$, $C=ah^2+2bh+c$, $D=ah^3+3bh^2+3ch+d$.

If $h=-b/a$, then $B=0$, and the equation is

$$y^3+\frac{3H}{a^2}y+\frac{G}{a^3}=0, \quad \text{.....(B)}$$

where $H=ac-b^2$, $G=a^2d-3abc+2b^3$.

Or, if we write $z=ay=ax+b$, the equation is

$$z^3+3Hz+G=0. \quad \text{.....(C)}$$

If α, β, γ are the roots of (A), those of (B) are $\alpha+b/a$, $\beta+b/a$, $\gamma+b/a$, and the roots of (C) are $a\alpha+b$, $a\beta+b$, $a\gamma+b$.

6. The Biquadratic Equation.

We take as the standard form

$$u\equiv ax^4+4bx^3+6cx^2+4dx+e=0. \quad \text{.....(A)}$$

The function u is called a *quartic*. If $x=y-b/a$ the equation becomes

$$y^4+\frac{6H}{a^2}y^2+\frac{4G}{a^3}y+\frac{K}{a^4}=0, \quad \text{.....(B)}$$

where $H=ac-b^2$, $G=a^2d-3abc+2b^3$ (as for the cubic),

and $K=a^3e-4a^2bd+6ab^2c-3b^4$.

If $z=ay=ax+b$, the equation is

$$z^4+6Hz^2+4Gz+K=0. \quad \text{.....(C)}$$

If α, β, γ, δ are the roots of (A), those of (B) are $\alpha+b/a$, etc., and those of (C) are $a\alpha+b$, etc.

EXERCISE X

TRANSFORMATION OF EQUATIONS

1. If α, β, γ are the roots of $2x^3+3x^2-x-1=0$, find the equations whose roots are (i) $1/2\alpha$, $1/2\beta$, $1/2\gamma$; (ii) $\alpha-1$, $\beta-1$, $\gamma-1$; (iii) $1/(1-\alpha)$, $1/(1-\beta)$, $1/(1-\gamma)$; (iv) $\alpha+2$, $\beta+2$, $\gamma+2$; (v) α^2, β^2, γ^2.

[(iii) should be deduced from (ii).]

2. If α, β, γ are the roots of $8x^3-4x^2+6x-1=0$, find the equations whose roots are (i) $\alpha+\frac{1}{2}$, $\beta+\frac{1}{2}$, $\gamma+\frac{1}{2}$; (ii) $2\alpha+1$, $2\beta+1$, $2\gamma+1$.

[For (ii) use the result of (i).]

3. Solve $x^4-2x^3-3x^2+4x-1=0$, given that the product of two roots is unity.

[If α, α^{-1} are these roots, $(x-\alpha)(x-\alpha^{-1})$ is a common factor of

$$x^4-2x^3-3x^2+4x-1 \quad \text{and} \quad x^4-4x^3+3x^2+2x-1.]$$

4. Solve $4x^4-4x^3-13x^2+9x+9=0$, given that the sum of two roots is zero.

[If α, $-\alpha$ are these roots, $(x-\alpha)(x+\alpha)$ is a common factor of

$$4x^4-4x^3-13x^2+9x+9 \quad \text{and} \quad 4x^4+4x^3-13x^2-9x+9.]$$

5. Solve $4x^3-24x^2+23x+18=0$, given that the roots are in arithmetical progression.

[Let the roots be $\alpha-\delta$, α, $\alpha+\delta$, $\therefore\ 3\alpha=\frac{24}{4}$.]

6. Solve $6x^3-11x^2-3x+2=0$, given that the roots are in harmonical progression.

[Solve $2y^3-3y^2-11y+6=0$, whose roots are in A.P.]

7. Solve $x^4-8x^3+14x^2+8x-15=0$, whose roots are in A.P.

[Let the roots be $\alpha-3\delta$, $\alpha-\delta$, $\alpha+\delta$, $\alpha+3\delta$.]

8. Solve $2x^4-15x^3+35x^2-30x+8=0$, whose roots are in G.P.

[Let the roots be $\alpha\rho^{-3}$, $\alpha\rho^{-1}$, $\alpha\rho$, $\alpha\rho^3$.]

9. Solve $x^3-7x^2+36=0$, given that the difference between two of the roots is 5.

10. Solve $4x^4-4x^3-25x^2+x+6=0$, given that the difference between two of the roots is unity.

11. Transform into equations lacking the second term:

(i) $x^3-6x^2+4x-7=0$;

(ii) $x^4+8x^3+x^2-x-10=0$.

12. Transform into equations with integral coefficients:

(i) $x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0$;

(ii) $x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{7}{150}x-\frac{13}{900}=0$.

13. If n is odd, prove that

$$\tan n\theta=(-1)^{\frac{1}{2}(n-1)}\tan\theta\tan\left(\theta+\frac{\pi}{n}\right)\tan\left(\theta+\frac{2\pi}{n}\right)\ldots\tan\left(\theta+\frac{(n-1)\pi}{n}\right),$$

$$n\tan n\theta=\tan\theta+\tan\left(\theta+\frac{\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)+\ldots+\tan\left(\theta+\frac{(n-1)\pi}{n}\right).$$

[Regard the equation of Exercise VII, 14, as giving $\tan\theta$ in terms of $\tan n\theta$.]

7. Character and Position of the Roots of an Equation. We say that the *character* of the roots is known when we know how many are real and how many are imaginary.

The *position* of a real root is its position on the scale of real numbers, and is determined roughly for a non-integral root by finding two consecutive integers between which the root lies. For a complete discussion, we require Sturm's Theorem (Ch. XXVIII), but a good deal of information can be derived from the elementary theorems which follow.

8. Some general Theorems. Here we suppose that

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n,$$

where p_1, p_2, ... are real numbers.

If $l + \iota m$ is a root of $f(x) = 0$, then $l - \iota m$ is also a root (Ch. V, 14) and $f(x)$ has the quadratic factor, $\{x - (l + \iota m)\}\{x - (l - \iota m)\} = (x - l)^2 + m^2$.

The last expression is positive for all real values of x. Thus if $\alpha, \beta, \ldots \kappa$ are the real roots of $f(x) = 0$, these being not necessarily all different,

$$f(x) = (x - \alpha)(x - \beta)\ldots(x - \kappa) \,.\, \phi(x),$$

where $\phi(x)$ is positive for all real values of x. Also this expression in factors is *unique*.

Hence, if x varies, the sign of $f(x)$ can change only when x passes through a real root of $f(x) = 0$, and we draw the following conclusions.

(1) *If x is greater than any of the roots, $f(x)$ is positive. This is also the case when all the roots are imaginary.*

Whence it follows that *for sufficiently large values of x, any polynomial has the same sign as its highest term.*

(2) Let a and b be any real numbers: then

(i) *If $f(a)$ and $f(b)$ have like signs, an even number of roots of $f(x) = 0$ lie between a and b, or else there is no root between a and b.*

(ii) *If $f(a)$ and $f(b)$ have unlike signs, an odd number of roots of $f(x) = 0$ lie between a and b.*

(3) (i) *If $f(x) = 0$ is an equation of odd degree, it has at least one real root.*

(ii) *If $f(x) = 0$ is of even degree and p_n is negative, the equation has at least one positive root and at least one negative root.*

For, by (1), a positive number a can be found so large that $f(a)$ is positive and $f(-a)$ has the sign of $(-1)^n$.

Hence if n is odd, $f(a)$ and $f(-a)$ have unlike signs and at least one root lies between a and $-a$. If n is even and p_n negative, $f(a)$ and $f(-a)$ are positive and $f(0)$ is negative, so that at least one root lies between 0 and a and at least one between 0 and $-a$.

9. Descartes' Rule of Signs. In what follows, every equation is supposed to have a term independent of x, so that zero roots do not occur.

In considering the signs of the terms of a polynomial taken in order from left to right, we say that a *continuation*, or a *change, of sign* occurs at any particular term according as that term has the same sign as the preceding, or the opposite sign.

Thus $x^7-2x^5-3x^4-4x^3+5x^2-6x+7$ has 2 continuations, and 4 changes of sign, the continuations occurring at the terms $-3x^4$, $-4x^3$, and the changes at $-2x^5$, $+5x^2$, $-6x$, $+7$.

The equation $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$ is said to be *complete* when no coefficient is zero. If $a_r=0$ we say that the corresponding term is *missing*.

In a complete equation: (i) *If x is changed into $-x$, a change of sign becomes a continuation and* vice versa.

(ii) *If μ is the number of changes, and μ' the number of continuations, of sign, then $\mu+\mu'=n$, where n is the degree of the equation.*

Descartes' Rule is as follows: *The equation $f(x)=0$ cannot have more positive roots than $f(x)$ has changes of sign, or more negative roots than $f(-x)$ has changes of sign.*

To prove the first part, we shall show that if u is any polynomial and $v=u(x-a)$, where a is positive, then v, when expanded, has at least one more change of sign than u.

First suppose that no term is missing in u and consider the following instance:

Signs of terms of u,	+	−	−	−	+	+	−	+	
		−	+	+	+	−	−	+	−
Signs of terms of v,	+	−	±	±	+	±	−	+	−

where $\pm$ indicates that the sign may be $+$ or $-$, or that the corresponding term is zero.

In the diagram of corresponding signs, observe that

(i) If the rth sign of u is a continuation, the rth sign of v is ambiguous.

(ii) Unlike signs precede and follow a single ambiguity or a group of ambiguities.

(iii) A change of sign is introduced at the end of v.

On account of (i) and (ii), v has at least as many changes of sign as u, even in the most unfavourable case in which all the ambiguities are continuations: and on account of (iii) v has certainly one more change of sign than u.

That no changes of sign are lost on account of any terms which may be missing from u appears on considering such instances as

+	+		0	0	−					−	+		0	0	−				
		−	−		0	0		+				+	−		0	0		+
+	±	−		0	−		+			−	+	−		0	−		+		

Thus v has at least one more change of sign than u.

Next let $f(x)=\phi(x)\,.\,(x-\alpha)(x-\beta)\ldots$ where $\alpha, \beta, \ldots$ are the positive roots of $f(x)=0$. If $\phi(x)$ is multiplied in succession by $x-\alpha$, $x-\beta$, ..., each multiplication introduces at least one change of sign.

Hence $f(x)$ has at least as many changes of sign as $f(x)=0$ has positive roots.

Again, the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$, with their signs changed. Hence the second part of the theorem follows from the first.

10. Corollaries. Let n be the degree of $f(x)$, and

μ the number of changes of sign in $f(x)$,
μ' the number of changes of sign in $f(-x)$,
m the number of positive roots of $f(x)=0$,
m' the number of negative roots of $f(x)=0$;

then (i) *if* $\mu+\mu'<n$, *the equation* $f(x)=0$ *has at least* $n-(\mu+\mu')$ *imaginary roots;*

(ii) *if all the roots of* $f(x)=0$ *are real, then* $m=\mu$ *and* $m'=\mu'$.

For by Descartes' rule, $m\leqslant\mu$ and $m'\leqslant\mu'$. Hence

$$m+m'\leqslant\mu+\mu'\leqslant n, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

and the number of imaginary roots $=n-(m+m')\geqslant n-(\mu+\mu')$.

Again, if all the roots are real $m+m'=n$, therefore by (**A**) $m+m'=\mu+\mu'$. Now $m\leqslant\mu$; and, if $m<\mu$, it follows that $m'>\mu'$, which is impossible. Therefore $m=\mu$ and $m'=\mu'$.

Ex. 1. *If* q, r, s, *are positive, show that the equation*

$$f(x)=x^4+qx^2+rx-s=0$$

has one positive, one negative and two imaginary roots.

By Art. 8, (3), $f(x)=0$ has a positive and a negative root. Also $\mu=1$ and $\mu'=1$, therefore these are the only real roots.

Ex. 2. *Show that* $x^5-2x^2+7=0$ *has at least two imaginary roots.*

Here $\mu=2$, $\mu'=1$; therefore $\mu+\mu'=3$. Hence there cannot be more than three real roots, namely, two positive and one negative.

Therefore the equation has at least two imaginary roots.

11. De Gua's Rule. *If a group of r consecutive terms is missing from $f(x)=0$, then*

(i) *if r is even, the equation has at least r imaginary roots:*

(ii) *if r is odd, there are at least $r+1$ or at least $r-1$ imaginary roots, according as the terms which immediately precede and follow the group have like or unlike signs.*

Suppose that the r terms between hx^m and kx^{m-r-1} are missing from $f(x)$.

Let $f(x)=hx^m+kx^{m-r-1}+\phi(x)$, and $f_1(x)=\psi(x)+\phi(x)$,

where $\psi(x)=hx^m+c_1x^{m-1}+c_2x^{m-2}+\dots c_rx^{m-r}+kx^{m-r-1}$,

none of the set, $c_1, c_2, \dots, c_r$, being zero.

The number of changes of sign in $\psi(x)$ + the number of changes of sign in $\psi(-x)$ is $r+1$.

Let μ = number of changes of sign in hx^m+kx^{m-r-1},

μ' = number of changes of sign in $h(-x)^m+k(-x)^{m-r-1}$.

Thus the total number of changes of sign in $f(x)$ and $f(-x)$ is less than the number in $f_1(x)$ and $f_1(-x)$ by $r+1-(\mu+\mu')$.

Hence $f(x)=0$ has at least $r+1-(\mu+\mu')$ imaginary roots.

First suppose that r is even, then $(-x)^m$ and $(-x)^{m-r-1}$ have opposite signs, and

(i) if h, k have the same signs, $\mu=0$ and $\mu'=1$;

(ii) if h, k have unlike signs, $\mu=1$ and $\mu'=0$, and in both cases

$$r+1-(\mu+\mu')=r.$$

If r is odd, $(-x)^m$ and $(-x)^{m-r-1}$ have the same sign, and

(i) if h, k have the same sign, $\mu=0$, $\mu'=0$ and $r+1-(\mu+\mu')=r+1$.

(ii) if h, k have unlike signs, $\mu=1$, $\mu'=1$ and $r+1-(\mu+\mu')=r-1$.

This completes the proof.

Ex. 1. *If $H=ac-b^2>0$, the equation*

$$(a, b, c, \dots k)(x, 1)^n=0$$

has at least two imaginary roots.

For by the substitution $x=y-b/a$, the equation becomes $x^n+\frac{nH}{a^2}x^{n-2}+\dots=0$, and by De Gua's rule this equation has at least two imaginary roots if $H>0$.

12. Limits to the Roots of an Equation. (1) In this article, whatever is said about the roots of an equation refers to the *real* roots only. Also, the equation is supposed to be written with its first term positive.

In searching for the roots of an equation, it is advisable to begin by finding two numbers between which the real roots lie. If h, l are two such numbers and $h>l$, then h is called an *upper limit* and l a *lower limit* to the roots.

(2) **Upper Limits.** Any number h is an upper limit to the roots of $f(x)=0$, provided that $f(x)>0$ when $x\geqslant h$.

Method of grouping terms. The process consists in arranging the terms of the equation $f(x)=0$ in groups, so that we can find by inspection a number h such that the sum of the terms in each group is $\geqslant 0$ for $x\geqslant h$. To distribute the terms conveniently, it is often advisable to multiply $f(x)$ by some positive integer. With a little ingenuity, this method can be made to yield quite close limits, as in the examples at the end of this article. Other methods are given in the next exercise, and in Ch. XXVIII.

(3) **Lower Limits.**

(i) *If h' is an upper limit to the roots of $f(-x)=0$, then $-h'$ is a lower limit to the roots of $f(x)=0$.*

For if α is the least root of $f(x)=0$, then $-\alpha$ is the greatest root of $(-x)=0$. Hence, $h'>-\alpha$, and therefore $-h'<\alpha$.

(ii) *If h'' is an upper limit to the positive roots of $f(1/x)=0$, then $1/h''$ is a lower limit to the positive roots of $f(x)=0$.*

For if α' is the least positive root of $f(x)=0$, then $1/\alpha'$ is the greatest positive root of $f(1/x)=0$. Hence $h''>1/\alpha'$ and so $1/h''<\alpha'$.

Ex. 1. *Find an upper and a lower limit to the positive roots of*

$$f(x)=x^3-10x^2-11x-100=0.$$

Here $6f(x)=6x^3-60x^2-66x-600 = 4x^2(x-15)+x(x^2-66)+(x^3-600)$.

Thus $f(x)>0$ for $x\geqslant 15$ and 15 is an upper limit to the roots.

Again, putting $x=1/y$, the equation becomes $\phi(y)=100y^3+11y^2+10y-1=0$.

Now $$\phi(y)=5y^2(20y-1)+(16y^3+10y-1).$$

The greater root of $16y^2+10y-1=0$ is $\dfrac{1\cdot 4\ \ldots}{16}$; and if y has this value, $20y-1>0$.

Hence $\phi(y)>0$ for $y\geqslant \dfrac{1\cdot 5}{16}$, i.e. for $x=\dfrac{1}{y}\leqslant \dfrac{16}{1\cdot 5}$, i.e. for $x=10\cdot 6$.

Thus 10·6 is a lower limit to the positive roots.

Ex. 2. *Find an upper and a lower limit to the roots of*

$$f(x)=3x^4-61x^3+127x^2+220x-520=0.$$

We have $f(x)=x^2(3x^2-61x+127)+(220x-520)$.

The greater root of $3x^2-61x+127=0$ is $17\cdot 9\ldots$. Thus 18 is an upper limit to the roots.

Again, $f(-x)=(3x^4-520)+x(61x^2+127x-220)$, and each group of terms is positive if $x\geqslant 4$. Hence -4 is a lower limit to the roots of $f(x)=0$.

13. To find the Rational Roots of an Equation. Any such roots which may exist can be found by using the following theorem: *If $px-q$ is a factor of $ax^n+bx^{n-1}+\ldots+hx+k$, where $a, b, \ldots h, k$ are integers or zero and p, q are integers prime to one another, then p is a factor of a, and q is a factor of k.*

For, denoting the polynomial by $f(x)$, if $px-q$ is a factor of $f(x)$, then $f(q/p)=0$, and therefore

$$aq^n+bq^{n-1}p+\ldots+hqp^{n-1}+kp^n=0.$$

Hence $aq^n=p\times$ an integer, so that aq^n is divisible by p. Now p is prime to q, therefore p is prime to q^n. Hence a is divisible by p. Similarly k is divisible by q.

Hence *the rational roots of $ax^n+bx^{n-1}+\ldots+k=0$ must be included among the values of $\pm q/p$, where p is prime to q, p is a factor of a and q a factor of k.*

We can test the values of $\pm q/p$ by synthetic division, or we may use the method of divisors given below.

Ex. 1. *Search for rational roots of* $f(x)=2x^3-5x^2+5x-3=0$.

We have $f(x)=2x^2(x-\frac{5}{2})+5(x-\frac{3}{5})$, therefore $\frac{5}{2}$ is an upper limit of the roots. Also $f(-x)=0$ has no positive roots, therefore $f(x)=0$ has no negative roots.

If q/p is a root, $p=1$ or 2, $q=1$ or 3, and the only values of q/p which lie between 0 and $\frac{5}{2}$ are 1, $\frac{1}{2}$, $\frac{3}{2}$. Testing these by synthetic division, we find that $\frac{3}{2}$ is a root; and it is the only rational root.

14. Newton's Method of Divisors. Let the given equation be transformed into

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

where $p_1, p_2, \ldots$ are the least possible integers. (See Art. 4, (6).)

By the last theorem, if α is a rational root of (**A**), then α is an integer and p_n is divisible by α.

Let h be any factor of p_n. If $x-h$ is a factor of $f(x)$, and we divide $f(x)$ by $x-h$, the usual reckoning is as follows:

$1-h)$	$1+p_1$	$+p_2$	$+\ldots+p_{n-2}$	$+p_{n-1}$	$+p_n$
	h	$+q_1h$	$+q_{n-3}h$	$+q_{n-2}h$	$+q_{n-1}h$
	q_1	q_2	q_{n-2}	q_{n-1}	0

where $q_1=p_1+h$, $q_2=p_2+q_1h$, $\ldots$ $p_{n-1}+q_{n-2}h=q_{n-1}$, $p_n+q_{n-1}h=0$.

Newton's method consists in performing these operations in the reverse order, thus

p_n+p_{n-1}	$+p_{n-2}$	$+\ldots+p_2$	$+p_1$	$+1$
$-q_{n-1}$	$-q_{n-2}$	$-q_2$	$-q_1$	-1
$-q_{n-2}h$	$-q_{n-3}h$	$-q_1h$	$-h$	0

Here p_n is divided by h, the quotient is $-q_{n-1}$; this is added to p_{n-1}, giving $-q_{n-2}h$. Dividing this by h, we get $-q_{n-2}$, which is added to p_{n-2}, and so on.

If any q in the process is not an integer, $f(x)$ is not divisible by $x-h$ and the reckoning need not be continued.

Herein lies the advantage of Newton's method.

Further, the last number in the bottom line must be 0.

To lessen the number of trials, choose any number a, then if $x-h$ is a factor of $f(x)$, $f(a)$ is divisible by $a-h$. We generally take $a=\pm1$.

Ex. 1. *Find the rational roots of* $f(x)=x^4-39x^2+46x-168=0$.

Since $$f(x)=x^2\,.\,(x^2-39)+2(23x-84),$$

and $$4f(-x)=2x^2(x^2-78)+(x^4-672)+x(x^3-184),$$

it follows that all the real roots lie between -9 and 7.

If h is a rational root, it is an integer and $\pm h$ is a factor of $168=2^3\,.\,3\,.\,7$.

The possible values of h are therefore

$$\pm1,\quad \pm2,\quad \pm3,\quad \pm4,\quad \pm6,\quad -7,\quad -8. \qquad \text{.....(A)}$$

A number of these can be excluded at once, for

$$f(1)=-160=-2^5\,.\,5\,;\quad f(-1)=-252=-2^2\,.\,3^2\,.\,7,$$

so that ±1 are not roots.

Also if h is a root, $1-h$ is a factor of 160, thus we can exclude -2, 4, -6, -8.

Again, $-1-h$ must be a factor of 252, and so we can exclude 4. The remaining numbers are

$$2,\quad +3,\quad -3,\quad +6,\quad -7.$$

Testing these as below by Newton's method, we find that 6 and -7 are the only rational roots.

1 − 2) −168	+46	−39	+ 0	+1,	1 − 3) −168	+46			
	−84	−19	−29			−56			
	−38	−58	−29			−10 *			
1 + 3) −168	+46	−39,			1 − 6) −168	+46	−39	+0	+1
	+56	−34				−28	+ 3	−6	−1
	102	−73 *				+18	−36	−6	0
1 + 7) −168	+46	−39	+0	+1					
	+24	−10	+7	−1					
	70	−49	+7	0					

EXERCISE XI

REAL ROOTS

1. Use Descartes' rule of signs to show that:

(i) If q is positive, $x^3+qx+r=0$ has only one real root.

(ii) $x^7-3x^4+2x^3-1=0$ has at least four imaginary roots.

* The reckoning stops at these stages because 10 and 73 are not divisible by 3.

2. Show that the roots of $(a-x)(b-x)-h^2=0$ are real and are separated by a and by b.

[Denoting the left-hand side by $f(x)$, we have

$$f(\infty)>0, \quad f(a)=f(b)<0, \quad f(-\infty)>0.]$$

By grouping terms, find an upper and a lower limit to the roots of the equations in Exx. 3-9.

3. $x^4-3x^3-2x^2+7x+3=0$. **4.** $x^5-10x^4-5x^3+6x^2-11x-350=0$.

[For an upper limit in Ex. 4, group thus: $x^3(x^2-10x-5)+(6x^2-11x-350)$.
For a lower limit, write $x=-y$ and group thus:

$$y(y^4-11)+y^3(9y-5)+(y^4-6y^2+350).]$$

5. $x^4+4x^3-11x^2-9x-50=0$. **6.** $x^4-2x^3+3x^2-5x+1=0$.

7. $2x^3-11x^2-10x-1=0$.

[Group thus: $x(2x^2-11x-11)+(x-1)$.]

8. $x^5+x^4-6x^3-8x^2-15x-10=0$.

[Group thus: $x(x^4-6x^2-15)+(x^4-8x^2-10)$.]

9. $x^5-3x^4-24x^3+95x^2-46x-101=0$.

10. For the equation $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, if the numerically greatest negative coefficient is equal to $-p$, then $p+1$ is an upper limit to the roots.

[Let $x>1$, then $f(x)>0$ if $x^n>p(x^{n-1}+x^{n-2}+\ldots+1)$, i.e. if $x^n>p\cdot\dfrac{x^n-1}{x-1}$. This holds if $x^n>p\cdot\dfrac{x^n}{x-1}$, i.e. if $x-1>p$.]

11. For the equation $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, if the numerically greatest negative coefficient is equal to $-p$ and the first negative coefficient is p_r, then $\sqrt[r]{p}+1$ is an upper limit to the roots.

[Let $x>1$, then $f(x)>0$ if $x^n>p(x^{n-r}+x^{n-r-1}+\ldots+1)$, i.e. if

$$x^n>p\cdot\frac{x^{n-r+1}-1}{x-1}.$$

This holds if $x^n>\dfrac{px^{n-r+1}}{x-1}$, i.e. if $(x-1)x^{r-1}>p$, which holds if $(x-1)^r>p$, i.e. if $x>\sqrt[r]{p}+1$.]

12. If the rule in Ex. 11 is applied to the equations in Exx. 3-7, show that the limits are (1) 4, −4; (2) 351, −5; (3) 9, −51; (4) 6, −1; (5) 7, −6.

13. By using Ex. 11, find an upper limit to the roots of $x^7-100x^2-237=0$.

Find the rational roots of the equations in Exx. 14-19.

14. $x^3-9x^2+22x-24=0$. **15.** $x^3-5x^2-18x+72=0$.

16. $3x^3-2x^2-6x+4=0$. **17.** $4x^4-36x^3+45x^2+54x-81=0$.

18. $6x^4-25x^3+26x^2+4x-8=0$. **19.** $6x^4+53x^3-95x^2-25x+42=0$.

15. Symmetric Functions of the Roots of an Equation.

Let $\alpha, \beta, \gamma, \delta, \epsilon, \ldots$ be the roots of

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0, \quad \ldots\ldots\ldots\

16. Symmetric Functions involving only the Differences of the Roots of $f(x)=0$.

These are unaltered if the roots are diminished by any number h. Hence they may be calculated by using the equation $f(y+h)=0$, where h is chosen so as to remove the second term.

A useful check is provided by the theorem in Ex. 1 below. The functions in Exx. 2, 3 are important.

Ex. 1. *Let v be any symmetric function of the differences of the roots of*

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

then if v is expressed in terms of $a_0, a_1, a_2, \ldots$, the sum of the numerical coefficients is zero.

The sum of the numerical coefficients is obtained by putting $a_0=a_1=a_2=\ldots=1$ in v. But in this case the given equation becomes $(x+1)^n$, every root of which is -1, and since v involves only the differences of the roots, in this case, $v=0$.

Ex. 2. *If α, β, γ are the roots of $ax^3+3bx^2+3cx+d=0$, find the values of*

(i) $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)$,

(ii) $(\beta-\gamma)(\gamma-\alpha)+(\gamma-\alpha)(\alpha-\beta)+(\alpha-\beta)(\beta-\gamma)$.

(i) We have $a(x-\alpha)(x-\beta)(x-\gamma)=ax^3+3bx^2+3cx+d$.

Substituting $+\iota$ and $-\iota$ for x in this identity, we find that

$$a(\alpha-\iota)(\beta-\iota)(\gamma-\iota)=\iota(a-3c)+(3b-d),$$

$$a(\alpha+\iota)(\beta+\iota)(\gamma+\iota)=-\iota(a-3c)+(3b-d),$$

therefore $$a^2(\alpha^2+1)(\beta^2+1)(\gamma^2+1)=(a-3c)^2+(3b-d)^2.$$

(ii) $\Sigma(\beta-\gamma)(\gamma-\alpha)=\Sigma\beta\gamma-\Sigma\alpha^2=3\Sigma\beta\gamma-(\Sigma\alpha)^2=\frac{9}{a^2}(ac-b^2)$.

Ex. 3. *If $\alpha, \beta, \gamma, \delta$ are the roots of*

$$ax^4+4bx^3+6cx^2+4dx+e=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

find the values of

(i) $\Sigma(\alpha-\beta)^2$ (ii) $\Sigma(\alpha-\beta)^2\gamma^2\delta^2$,

(iii) $(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)$.

(i) $\Sigma(\alpha-\beta)^2=3\Sigma\alpha^2-2\Sigma\alpha\beta=3(\Sigma\alpha)^2-8\Sigma\alpha\beta=\frac{48}{a^2}(b^2-ac)$.

(ii) The equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ is obtained by interchanging a and e, b and d, thus

$$\Sigma(\alpha-\beta)^2\gamma^2\delta^2=\alpha^2\beta^2\gamma^2\delta^2\,.\,\Sigma\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$$

$$=\frac{e^2}{a^2}\cdot\frac{48}{e^2}(d^2-ec) \qquad =\frac{48}{a^2}(d^2-ce).$$

(iii) Denote the function by v. Since v is a function of the differences of $\alpha, \beta, \gamma, \delta$, its value may be found by using the equation

$$y^4+\frac{6H}{a^2}y^2+\frac{4G}{a^3}y+\frac{K}{a^4}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

obtained by the substitution $x=y-b/a$.

Let α', β', γ', δ' be the roots of **(B)**. Then since $\Sigma\alpha'=0$,

$$\beta+\gamma-\alpha-\delta=\beta'+\gamma'-\alpha'-\delta'=-2(\delta'+\alpha'),$$

and
$$\begin{aligned}-v/8&=(\delta'+\alpha')(\delta'+\beta')(\delta'+\gamma')\\&=\delta'^3+\delta'^2(\alpha'+\beta'+\gamma')+\delta'(\beta'\gamma'+\gamma'\alpha'+\alpha'\beta')+\alpha'\beta'\gamma'\\&=\delta'^2\,.\,\Sigma\alpha'+\Sigma\alpha'\beta'\gamma';\\&\therefore\ v=-8\Sigma\alpha'\beta'\gamma'=32G/a^3.\end{aligned}$$

Ex. 4. *If* $\alpha+\beta+\gamma+\delta=0$, *prove that*

$$\alpha^5+\beta^5+\gamma^5+\delta^5=-5\Sigma\alpha\beta\,.\,\Sigma\alpha\beta\gamma.$$

Let α, β, γ, δ be the roots of $x^4+p_2x^2+p_3x+p_4=0$, and let $s_r=\Sigma\alpha^r$, then by Art. 15, $s_1=0$, $s_2=-2p_2$, $s_3=-3p_3$. Also $\alpha^5+p_2\alpha^3+p_3\alpha^2+p_4\alpha=0$, with three similar equations. Hence by addition, $s_5+p_2s_3+p_3s_2+p_4s_1=0$;

$$\therefore\ s_5=-p_2(-3p_3)-p_3(-2p_2)=5p_2p_3=-5\Sigma\alpha\beta\,.\,\Sigma\alpha\beta\gamma.$$

17. Equations whose Roots are Symmetric Functions of α, β, γ. The following are typical examples.

Ex. 1. *If* α, β, γ *are the roots of* $x^3+qx+r=0$, *find the equation whose roots are* $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$.

Let $z=(\beta-\gamma)^2$, then since $\Sigma\alpha=0$ and $\alpha\beta\gamma=-r$,

$$z=(\beta+\gamma)^2-4\beta\gamma=\alpha^2+4r/\alpha.$$

Hence the required equation can be found by eliminating α from

$$\alpha^3+q\alpha+r=0,\quad \alpha^3-z\alpha+4r=0.$$

By subtraction,
$$(z+q)\alpha=3r.$$

Substituting for α in the first equation and simplifying, we have

$$(z+q)^3+3q(z+q)^2+27r^2=0,$$

which on expansion becomes

$$z^3+6qz^2+9q^2z+4q^3+27r^2=0.$$

The artifice employed in the next example is often useful.

Ex. 2. (i) *Find the condition that the sum of two roots* α, β *of*

$$x^4+p_1x^3+p_2x^2+p_3x+p_4=0 \quad\text{.....(A)}$$

may be zero.

(ii) *Use the result to find the equation whose roots are the six values of* $\frac{1}{2}(\alpha+\beta)$, *where* α, β *are any two roots of*

$$ax^4+4bx^3+6cx^2+4dx+e=0. \quad\text{.....(B)}$$

(i) Since α and $-\alpha$ are roots of **(A)**,

$$\begin{aligned}&\alpha^4+p_1\alpha^3+p_2\alpha^2+p_3\alpha+p_4=0,\\&\alpha^4-p_1\alpha^3+p_2\alpha^2-p_3\alpha+p_4=0;\\&\therefore\ \alpha^4+p_2\alpha^2+p_4=0 \quad\text{and}\quad \alpha(p_1\alpha^2+p_3)=0.\end{aligned}$$

Now $\alpha\neq0$ unless $p_4=0$, therefore $p_1^2p_4-p_1p_2p_3+p_3^2=0$, ...**(C)** which is the required condition.

(ii) Let $z=\frac{1}{2}(\alpha+\beta)$, then we have $(\alpha-z)+(\beta-z)=0$.

If then we diminish the roots of (**B**) by z, writing $x=y+z$, the sum of two roots of the resulting equation in y will be zero. This equation is

$$ay^4+4By^3+6Cy^2+4Dy+E=0, \quad \text{.....(D)}$$

where $B=az+b, \quad C=az^2+2bz+c,$

$$D=(a, b, c, d)(z, 1)^3, \quad E=(a, b, c, d, e)(z, 1)^4.$$

For equation (**D**), the condition (**C**) gives

$$B^2E-6BCD+aD^2=0,$$

an equation of the sixth degree in z, whose roots are the values of $\frac{1}{2}(\alpha+\beta)$.

EXERCISE XII

SYMMETRIC FUNCTIONS OF ROOTS

If α, β, γ are the roots of $ax^3+3bx^2+3cx+d=0$ and

$$H=ac-b^2, \quad G=a^2d-3abc+2b^3,$$

prove that:

1. $a^3\Sigma\alpha^3=-3(a^2d-9abc+9b^3)$.
2.* $a^3\Sigma\beta^3\gamma^3=3(d^2a-9dcb+9c^3)$.
3. $a^4(\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2)(\alpha\gamma^2+\beta\alpha^2+\gamma\beta^2)=a^4(3\alpha^2\beta^2\gamma^2+\alpha\beta\gamma\Sigma\alpha^3+\Sigma\beta^3\gamma^3)$
$=9(a^2d^2-6abcd+3ac^3+3b^3d)$.
4. $a^2\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}=18(b^2-ac)$.
5.* $a^2\{\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2\}=18(c^2-db)$.

*Explain how to derive (2) from (1) and (5) from (4).

6. $a^2\{\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\}=9(ad-bc)$.
7. $a^4\{(\beta-\gamma)^4+(\gamma-\alpha)^4+(\alpha-\beta)^4\}=162H^2$.
8. $a^3(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma)=8a^2d-36abc+27b^3$.
9. $a^3(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=-27G$.
10. (i) The condition that α, β, γ are in A.P. is $G=0$.
(ii) The condition that they are in G.P. is $ac^3=b^3d$.
(iii) The condition that they are in H.P. is $d^2a-3dcb+2c^3=0$.

[(i) If $2\alpha=\beta+\gamma$, then $\alpha=-b/a$. (ii) Eliminate x between $u=0$ and $ax^3+d=0$. (iii) Put $x=1/y$ and use condition (i).]

11. The condition that α, β, γ may be connected by the equation

$$(\alpha-h)(\beta-h)=(\gamma-h)^2$$

is $aC^3=B^3D$, where $B=ah+b$, $C=ah^2+2bh+c$, $D=ah^3+3bh^2+3ch+d$.

12. The equation whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}, \quad \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}, \quad \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

is $$a(ax^2+2bx+c)^3=(ax^3+3bx^2+3cx+d)(ax+b)^2.$$

Show that this reduces to a cubic equation. [Use Ex. 11.]

13. If α, β, γ are the roots of $x^3+qx+r=0$, prove that the equation whose roots are $\frac{\beta}{\gamma}+\frac{\gamma}{\beta}$, $\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}$, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ is $r^2(z+1)^3+q^3(z+1)+q^3=0$.

[Let $z=\frac{\beta}{\gamma}+\frac{\gamma}{\beta}$. Prove that $\alpha^3+2q\alpha-rz=0$. Eliminate α between this equation and $\alpha^3+q\alpha+r=0$.]

14. If α, β are any two roots of $x^3+qx+r=0$, the equation whose roots are the six values of α/β is

$$r^2(z^2+z+1)^3+q^3z^2(z+1)^2=0.$$

[Let $z=\frac{\alpha}{\beta}$; $\therefore\ \alpha=\beta z$, hence the required equation is got by eliminating β from

$$\beta^3+q\beta+r=0 \quad \text{and} \quad z^3\beta^3+qz\beta+r=0.]$$

15. If $a+b+c=0$ show that

(i) $a^5+b^5+c^5=5abc(bc+ca+ab)$; (ii) $a^7+b^7+c^7=7abc(bc+ca+ab)^2$.

[Let a, b, c be the roots of $x^3+qx+r=0$.]

16. Use the last example to show that

$$(x+a)^5-x^5-a^5=5ax(x+a)(x^2+ax+a^2),$$
$$(x+a)^7-x^7-a^7=7ax(x+a)(x^2+ax+a^2)^2.$$

17. Solve $(x+a+b)^5-x^5-a^5-b^5=0$.

[Observe that $x+a+b+(-x)+(-a)+(-b)=0$, and use Ex. 4, p. 97.]

18. If α, β, γ, δ are the roots of $x^4+p_2x^2+p_3x+p_4=0$, prove that

(i) $\Sigma\alpha^4=2(p_2^2-2p_4)$, (ii) $\Sigma\alpha^7=-7p_3(p_2^2-p_4)$.

If α, β, γ, δ are the roots of $ax^4+4bx^3+6cx^2+4dx+e=0$ and $H=ac-b^2$, $G=a^2d-3abc+2b^3$, prove that:

19. $a^2\Sigma\alpha^2=4(4b^2-3ac)$.

20.* $a^2\Sigma\alpha^2\beta^2\gamma^2=4(4d^2-3ec)$.

21. $a^2\Sigma\alpha^2\beta=12(ad-2bc)$.

22.* $a^2\Sigma\alpha\beta^2\gamma^2=12(eb-2dc)$.

* Note that (20) follows from (19) and (22) from (21).

23. $a^2\Sigma(\alpha-\beta)^2(\gamma^2+\gamma\delta+\delta^2)=144(c^2-bd)$.

24. $a^3\Sigma(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)=-32G$.

25. If α, β, γ, ... are the roots of $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, prove that

$$\Sigma(\alpha-\beta)^2=(n-1)p_1^2-2np_2,$$
$$\Sigma\alpha^3\beta\gamma=-p_1^2p_3+2p_2p_3+p_1p_4-5p_5,$$
$$\Sigma\alpha^3\beta^2=-p_1p_2^2+2p_1^2p_3+p_2p_3-5p_1p_4+5p_5.$$

26. Show that the squares of the roots of

$$a_0x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots+(-1)^na_n=0$$

are the roots of

$$b_0x^n-b_1x^{n-1}+b_2x^{n-2}-\ldots+(-1)^nb_n=0,$$

where

$$b_0=a_0^2,\quad b_1=a_1^2-2a_0a_2,\quad b_2=a_2^2-2a_1a_3+2a_0a_4,\ \ldots$$
$$b_r=a_r^2-2a_{r-1}a_{r+1}+2a_{r-2}a_{r+2}-2a_{r-3}a_{r+3}+\ldots.$$

27. If the equation whose roots are the squares of the roots of the cubic

$$x^3-ax^2+bx-1=0$$

is identical with this cubic, prove that either $a=b=0$ or $a=b=3$, or a, b are the roots of $z^2+z+2=0$.

CHAPTER VII

PARTIAL FRACTIONS

1. Rational Fractions. An expression of the form P/Q, where P and Q are polynomials in x, is called a *rational fraction.* If P is of lower degree than Q, P/Q is called a *proper fraction.* If P is not of lower degree than Q, P/Q is called an *improper fraction.* By means of the division transformation, *an improper fraction can be expressed as the sum of an integral function and a proper fraction.* Thus

$$\frac{x^2+1}{x+1}=(x-1)+\frac{2}{x+1}.$$

Theorem. *If* $A+\frac{X}{Y}\equiv A'+\frac{X'}{Y'}$, *where the letters denote polynomials in* x *and* X/Y, X'/Y' *are proper fractions, then* $A\equiv A'$ *and* $X/Y\equiv X'/Y'$.

For
$$A-A'=\frac{X'}{Y'}-\frac{X}{Y}=\frac{YX'-XY'}{YY'}.$$

Also X and X' are respectively of lower degrees than Y and Y', and therefore $YX'-XY'$ is of lower degree than YY'.

Hence $A-A'\equiv 0$; for otherwise, the polynomial $A-A'$ would be identically equal to a proper fraction, which is impossible. Thus $A\equiv A'$, and consequently $X/Y\equiv X'/Y'$.

2. Partial Fractions. To resolve a given fraction into *partial fractions* is to express it as the sum of two or more simpler fractions.

Fundamental Theorem. *If* C/AB *is a proper fraction and the factors* A, B *are prime to each other, proper fractions* X/A *and* Y/B *can be found such that*

$$\frac{C}{AB}=\frac{X}{A}+\frac{Y}{B}.$$

For since A is prime to B, polynomials X', Y' can be found so that

$$BX'+AY'=1, \qquad \text{(Ch. III, 18.)}$$

and therefore
$$\frac{C}{AB}=\frac{CX'}{A}+\frac{CY'}{B}.$$

If CX'/A and CY'/B are not proper fractions, by division, we can find polynomials Q, Q', X, Y, such that

$$\frac{CX'}{A}=Q+\frac{X}{A} \quad \text{and} \quad \frac{CY'}{B}=Q'+\frac{Y}{B},$$

where X/A and Y/B are proper fractions, and then

$$\frac{C}{AB}=Q+Q'+\frac{X}{A}+\frac{Y}{B}=Q+Q'+\frac{BX+AY}{AB}.$$

Now X and Y are respectively of lower degrees than A and B, therefore $BX+AY$ is of lower degree than AB. Thus $(BX+AY)/AB$ is a proper fraction: so also is C/AB, therefore

$$Q+Q'\equiv 0 \quad \text{and} \quad \frac{C}{AB}=\frac{X}{A}+\frac{Y}{B}.$$

3. To resolve a Proper Fraction P/Q into its simplest set of Partial Fractions. We shall prove that

(i) *To a non-repeated factor $x-a$ of Q there corresponds a fraction of the form* $\dfrac{A}{x-a}$.

(ii) *To a factor $(x-b)^n$ of Q there corresponds a group of the form*

$$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\ldots+\frac{B_n}{(x-b)^n}.$$

(iii) *To a non-repeated quadratic factor x^2+px+q of Q there corresponds a fraction of the form* $\dfrac{Cx+D}{x^2+px+q}$.

(iv) *To a factor $(x^2+px+q)^n$ of Q there corresponds a group of the form*

$$\frac{C_1x+D_1}{x^2+px+q}+\frac{C_2x+D_2}{(x^2+px+q)^2}+\ldots+\frac{C_nx+D_n}{(x^2+px+q)^n}.$$

Here A, B_1, B_2, ... , C_1, C_2, ... are independent of x.

Proof (i). Let $Q=(x-a)\,.\,B$, then since $x-a$ is a non-repeated factor, B is prime to $x-a$, and so

$$P/Q=X/(x-a)+Y/B,$$

where the fractions on the right are proper fractions. Hence X is a constant.

(ii) Let $Q=(x-b)^n\,.\,B$. It is assumed that $x-b$ is not a factor of B, so that B is prime to $(x-b)^n$, and consequently

$$P/Q=X/(x-b)^n+Y/B,$$

where the fractions on the right are proper fractions.

Hence X is of degree $n-1$ at most, and it can be put in the form

$$B_1(x-b)^{n-1}+B_2(x-b)^{n-2}+\ldots+B_n, \qquad \text{(Ch. III, 4.)}$$

where B_1, B_2, ... are constants, which proves the statement in question.

(iii) The proof is similar to that of (i), but X is of the form $Cx+D$.

(iv) The proof is similar to that of (ii), but each of the set B_1, B_2, ... is of the form $Cx+D$.

Various ways of finding the constants are explained in the following examples.

Ex. 1. *Express* $\dfrac{x^4}{(x-a)(x-b)(x-c)}$ *as the sum of an integral function of* x *and three proper fractions.*

We have $(x-a)(x-b)(x-c)=x^3-x^2\,.\,\Sigma a+x\Sigma ab-abc$.

Hence it is easy to see that the quotient in the division of x^4 by $(x-a)(x-b)(x-c)$ is $x+a+b+c$. We may therefore assume that

$$\frac{x^4}{(x-a)(x-b)(x-c)}=x+a+b+c+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}. \qquad \ldots\ldots\ldots\ldots\text{(A)}$$

To find A, multiply each side of (A) by $x-a$, and put $x=a$.

$$\therefore\ A=\frac{a^4}{(a-b)(a-c)}.$$

The values of B and C can be found in the same way;

$$\therefore\ \frac{x^4}{(x-a)(x-b)(x-c)}=x+a+b+c+\frac{a^4}{(a-b)(a-c)}\cdot\frac{1}{x-a}+\ldots+\ldots.$$

Ex. 2. *Resolve into the simplest possible partial fractions*

(i) $\dfrac{x^3-5}{(x-1)^3(x-2)}$; (ii) $\dfrac{3x^3-2x^2-1}{(x^3+x+1)(x^2-x+1)}$; (iii) $\dfrac{1}{(x-1)(x^2-1)(x^3-1)}$.

(i) Let $$\frac{x^3-5}{(x-1)^3(x-2)}=\frac{a}{x-1}+\frac{b}{(x-1)^2}+\frac{c}{(x-1)^3}+\frac{d}{x-2}. \qquad \ldots\ldots\ldots\ldots\text{(A)}$$

Multiply each side by $x-2$, and then put $x=2$;

$$\therefore\ d=\frac{2^3-5}{(2-1)^3}=3.$$

Now $$\frac{x^3-5}{(x-1)^3(x-2)}-\frac{3}{x-2}=\frac{-2x^2+5x+1}{(x-1)^3}=\frac{-2(x-1)^2+(x-1)+4}{(x-1)^3}.$$

Therefore $$\frac{x^3-5}{(x-1)^3(x-2)}=-\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}+\frac{3}{x-2}.$$

Second method. Find d as before. Next, multiply each side of (A) by $(x-1)^3$, and then put $x=1$;

$$\therefore\ c=\frac{1^3-5}{1-2}=4.$$

Next, multiply each side of (A) by x, and let $x\to\infty$;

$$\therefore\ 1=a+d;\quad \therefore\ a=-2.$$

Finally, put $x=0$ in (A), $\therefore\ -\dfrac{5}{2}=-a+b-c-\dfrac{d}{2}$, $\therefore\ b=1$.

Particular attention is called to the last two steps. In this way two equations connecting the coefficients can *always* be written down. The steps are equivalent to the following: Multiply each side of (**A**) by $(x-1)^3(x-2)$ and equate the coefficients of x^3 (the highest power of x) and the absolute terms on each side.

(ii) Let $$\frac{3x^3-2x^2-1}{(x^2+x+1)(x^2-x+1)}=\frac{ax+b}{x^2+x+1}+\frac{cx+d}{x^2-x+1}. \quad \text{.....(A)}$$

Multiplying by $(x^2+x+1)(x^2-x+1)$, we find that

$$3x^3-2x^2-1=(a+c)x^3+(-a+b+c+d)x^2+(a-b+c+d)x+b+d,$$

and, equating coefficients,

$$a+c=3, \quad -a+b+c+d=-2, \quad a-b+c+d=0, \quad b+d=-1. \quad \text{.....(B)}$$

Whence $$a=2, \quad b=1, \quad c=1, \quad d=-2.$$

Note that the first and last equations in (**B**) can be found by (i) multiplying (**A**) by x and letting $x\to\infty$, and (ii) putting $x=0$. Two other equations may be quickly obtained by putting $x=1$ and $x=-1$.

The following method is sometimes useful. From (**A**),

$$\frac{3x^3-2x^2-1}{x^2-x+1}=ax+b+\frac{cx^3+(c+d)x^2+(c+d)x+d}{x^2-x+1},$$

whence by division,

$$3x+1+\frac{-2x-2}{x^2-x+1}=ax+b+cx+(2c+d)+\frac{(2c+2d)x-2c}{x^2-x+1}.$$

The integral functions and also the fractions are identically equal, and so

$$a+c=3, \quad b+2c+d=1, \quad 2c+2d=-2, \quad -2c=-2,$$

giving $a=2$, $b=1$, $c=1$, $d=-2$.

(iii) Let $$\frac{1}{(x+1)(x-1)^3(x^2+x+1)}=\frac{a}{x+1}+\frac{b}{x-1}+\frac{c}{(x-1)^2}+\frac{d}{(x-1)^3}+\frac{ex+f}{x^2+x+1}. \quad \text{...(A)}$$

We first find e and f. Multiply both sides of (**A**) by x^2+x+1, and then let

$$x^2+x+1=0.$$

This gives

$$\frac{1}{(x+1)(x-1)^3}=ex+f, \quad \text{when} \quad x^2+x+1=0.$$

We have $x^2=-x-1$ and $x^3=1$, whence we find that

$$(x-1)^3=x^3-3x^2+3x-1=3(x+1)+3x=3(2x+1)$$

and $$(x+1)(x-1)^3=3(2x^2+3x+1)=3(x-1).$$

It follows that

$$1=3(x-1)(ex+f)=3\{ex^2-(e-f)x-f\}.$$

$$\therefore \quad 3(f-2e)x-3(e+f)=1.$$

This linear relation holds for *two* values of x (namely, the roots of $x^2+x+1=0$). It is therefore an identity, and so

$$f-2e=0 \quad \text{and} \quad 1=-3(e+f), \quad \text{giving} \quad e=-1/9 \quad \text{and} \quad f=-2/9.$$

Next, multiply (A) by $x+1$, and then let $x=-1$;

$$\therefore\ a=\frac{1}{(-2)^3\,.\,1}=-\frac{1}{8}.$$

Multiply (A) by $(x-1)^3$, and then let $x=1$;

$$\therefore\ d=\frac{1}{2\,.\,3}=\frac{1}{6}.$$

Multiply (A) by x, and let $x\to\infty$; $\therefore\ 0=a+b+e$, giving $b=\frac{1}{8}+\frac{1}{9}=\frac{17}{72}$.

In (A), let $x=0$, $\therefore\ -1=a-b+c-d+f$, giving $c=-\frac{1}{4}$;

$$\therefore\ \frac{1}{(x-1)(x^2-1)(x^3-1)}$$
$$=-\frac{1}{8}\cdot\frac{1}{x+1}+\frac{17}{72}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{(x-1)^2}+\frac{1}{6}\cdot\frac{1}{(x-1)^3}-\frac{1}{9}\cdot\frac{x+2}{x^2+x+1}.$$

EXERCISE XIII

PARTIAL FRACTIONS

Resolve into partial fractions

1. $\dfrac{1}{(x-1)^2(x-2)}$.
2. $\dfrac{3(x-6)}{x^3(x+3)}$.
3. $\dfrac{x^2+1}{(x-2)^3(x-3)}$.
4. $\dfrac{2x^2+1}{x^3-1}$.
5. $\dfrac{(x+2)^2}{(x-2)(x^2+x+2)}$.
6. $\dfrac{x^3+x^2+1}{(x^2+2)(x^2+3)}$.
7. $\dfrac{x^5+1}{(x^2-x+1)^3}$.
8. $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}$.
9. $\dfrac{x^3-19x-15}{(x+2)^3(x^2+1)}$.
10. $\dfrac{x^4}{(x^2+1)^3(x^2+2)}$.
11. $\dfrac{x^3+x+1}{(x^2+1)^2(x^2+2)}$.
12. $\dfrac{(x^2+1)^2}{(x^3+1)^2}$.
13. $\dfrac{(x^2+1)^2}{(x^2-1)^3}$.
14. $\dfrac{1}{x^4+1}$.

15. Express $\dfrac{1-abx^2}{(1-ax)(1-bx)}$ as the sum of a constant and two proper fractions.

Hence show that if $1-abx^2$ is divided by $1-(a+b)x+abx^2$, so as to obtain a quotient of $(n+1)$ terms, this quotient is

$$1+(a+b)x+(a^2+b^2)x^2+\ldots+(a^n+b^n)x^n,$$

and find the corresponding remainder.

16. Express $\dfrac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)}$ as the sum of a constant and three proper fractions.

17. Express $\dfrac{x^3}{(x-a)(x-b)(x-c)}$ as the sum of a constant and three proper fractions.

18. Hence show that

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}=1.$$

19. Express $\dfrac{x^4}{(x-a)(x-b)(x-c)}$ as the sum of an integral function of x and three proper fractions.

Hence show that

$$\frac{a^4}{(a-b)(a-c)(a-d)}+\frac{b^4}{(b-c)(b-d)(b-a)}+\frac{c^4}{(c-d)(c-a)(c-b)}+\frac{d^4}{(d-a)(d-b)(d-c)}$$

is equal to $a+b+c+d$.

20. If $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$, show that

$$\frac{\underline{|n}}{x(x+1)(x+2)(x+3)\ldots(x+n)}=\frac{c_0}{x}-\frac{c_1}{x+1}+\frac{c_2}{x+2}-\ldots+(-1)^n\frac{c_n}{x+n}.$$

[Assume that $\dfrac{\underline{|n}}{x(x+1)(x+2)\ldots(x+n)}=\dfrac{a_0}{x}+\dfrac{a_1}{x+1}+\ldots+\dfrac{a_r}{x+r}+\ldots+\dfrac{a_n}{x+n}$.]

21. Hence show that

(i) $$\frac{c_0}{1}-\frac{c_1}{2}+\frac{c_2}{3}-\ldots+(-1)^n\frac{c_n}{n+1}=\frac{1}{n+1}.$$

(ii) $$\frac{c_0}{2}-\frac{c_1}{3}+\frac{c_2}{4}-\ldots+(-1)^n\frac{c_n}{n+2}=\frac{1}{(n+1)(n+2)}.$$

22. Use Ex. 21 and similar identities to prove that

(i) $$\frac{c_0}{1.2}-\frac{c_1}{2.3}+\frac{c_2}{3.4}-\ldots+(-1)^n\frac{c_n}{(n+1)(n+2)}=\frac{1}{n+2}.$$

(ii) $$\frac{c_0}{2.3}-\frac{c_1}{3.4}+\frac{c_2}{4.5}-\ldots+(-1)^n\frac{c_n}{(n+2)(n+3)}=\frac{1}{(n+2)(n+3)}.$$

(iii) $$\frac{c_0}{1.2.3}-\frac{c_1}{2.3.4}+\ldots+(-1)^n\frac{c_n}{(n+1)(n+2)(n+3)}=\frac{1}{2(n+3)}.$$

(iv) $$\frac{c_0}{2.3.4}-\frac{c_1}{3.4.5}+\ldots+(-1)\frac{c_n}{(n+2)(n+3)(n+4)}=\frac{1}{2(n+3)(n+4)}.$$

23. Prove that

$$\frac{x^{n+1}}{(x-a_1)(x-a_2)\ldots(x-a_n)}=x+a_1+a_2+\ldots+a_n+\Sigma\frac{a_1^{n+1}}{(a_1-a_2)\ldots(a_1-a_n)}\cdot\frac{1}{x-a_1}.$$

24. Use Ex. 23 to prove that

$$c_n(n+1)^{n+1}-c_{n-1}n^{n+1}+c_{n-2}(n-1)^{n+1}-\ldots+(-1)^nc_0\,1^{n+1}=\tfrac{1}{2}\underline{|n+2}.$$

[In Ex. 23, write $n+1$ for n, and put

$$a_1=1,\ a_2=2,\ \ldots,\ a_n=n,\quad \text{and}\quad x=0.]$$

CHAPTER VIII

SUMMATION OF SERIES

1. Meaning of Summation. Let u_n be a function of the positive integral variable, and let

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

The function s_n possesses this peculiarity—it is the sum of n terms, and these terms cannot be added up, unless the value of n is specified.

Sometimes it is possible to express s_n as the sum of p terms, *where p is a number which does not depend on n.* For example,

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n,$$

and the number of terms on the right (namely two) is independent of n.

To sum a series to n terms or *to find the sum of the first n terms of a series* is to express s_n in the form just described ; but this is not always possible.

Among the series capable of summation are arithmetic and geometric series ; a harmonic series cannot be summed.

The sum of the first n terms of the series

$$u_1 + u_2 + u_3 + \ldots + u_r + \ldots$$

is often denoted by $\Sigma_{r=1}^{r=n} u_r$ and sometimes by Σu_n, or simply by s_n.

2. Method of Differences. *If we are able to express u_n in the form $v_n - v_{n-1}$, where v_n is some function of n, then we can sum the series to n terms.*

For, by hypothesis,

$$\begin{aligned} u_n &= v_n - v_{n-1}, \\ u_{n-1} &= v_{n-1} - v_{n-2}, \\ &\ldots\ldots\ldots\ldots\ldots, \\ u_2 &= v_2 - v_1, \\ u_1 &= v_1 - v_0 ; \end{aligned}$$

whence by addition

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n = v_n - v_0.$$

Ex. 1. *Sum the series* $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ *to* n *terms.*

Here $u_n = n(n+1)(n+2)$

$= \frac{1}{4}n(n+1)(n+2)\{(n+3) - (n-1)\}$*

$= \frac{1}{4}n(n+1)(n+2)(n+3) - \frac{1}{4}(n-1)n(n+1)(n+2)$;

$\therefore\ u_n = v_n - v_{n-1}$, where $v_n = \frac{1}{4}n(n+1)(n+2)(n+3)$;

and since $v_0 = 0$, we have by the preceding,

$$s_n = v_n - v_0 = \tfrac{1}{4}n(n+1)(n+2)(n+3).$$

3. Series in which u_n is the Product of r successive terms of an A.P., beginning with the n-th.

The last example is an instance of this type, and the same method can be applied to the general case, which is as follows:

It is required to sum to n *terms the series in which*

$$u_n = (a+nb)(a+\overline{n+1}\ .\ b)\ldots(a+\overline{n+r-1}\ .\ b),$$

where a, b, r *are constants.*

We have

$$u_n = \frac{u_n\{(a+\overline{n+r}\ .\ b) - (a+\overline{n-1}\ .\ b)\}}{(r+1)b}$$

$$= \frac{u_n(a+\overline{n+r}\ .\ b)}{(r+1)b} - \frac{(a+\overline{n-1}\ .\ b)u_n}{(r+1)b}.$$

Therefore $u_n = v_n - v_{n-1}$, where $v_n = \dfrac{u_n(a+\overline{n+r}\ .\ b)}{(r+1)b}$;

and consequently $s_n = v_n - v_0$.

Here v_0 is independent of n, hence the sum to n terms may be found by the following rule: *On the right of* u_n *introduce as a factor the next term of the* A.P.*; divide by the number of factors so increased and by the common difference of the* A.P. *and add a constant.*

The constant is found by putting $n = 1$, as in the next example; or by substituting 0 for n in v_n.

Ex. 1. *Sum to* n *terms* $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \ldots$.

Here $u_n = (2n-1)(2n+1)(2n+3)$, and applying the rule,

$$s_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{4 \cdot 2} + C. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

To find the constant C, put $n = 1$ in (**A**), noting that $s_1 = 1 \cdot 3 \cdot 5$;

$$\therefore\ 1 \cdot 3 \cdot 5 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 2} + C, \quad \therefore\ C = 15(1 - \tfrac{7}{8}) = \tfrac{15}{8};$$

$$\therefore\ s_n = \tfrac{1}{8}\{(2n-1)(2n+1)(2n+3)(2n+5) + 15\}.$$

* The expression in the bracket { } is formed by subtracting the term which precedes n from that which follows $(n+2)$ in the A.P. 1, 2, 3, ...

4. Series in which u_n is a Rational Integral Function of n. In Ch. III, 4, (3), it has been shown that a rational integral function of n of degree r can be expressed in the form

$$a+bn+cn(n+1)+dn(n+1)(n+2)+\dots \text{ to } (r+1) \text{ terms},$$

where a, b, c, ... are constants whose values may be found by synthetic division. We can therefore use the rule of the last article to sum series of this type, as in the next example.

Ex. 1. *Sum the series*

$$2\,.\,3+3\,.\,6+4\,.\,11+\dots+(n+1)(n^2+2).$$

Here $u_n=(n+1)(n^2+2)$

$=n^3+n^2+2n+2.$

$1+1)$	1	$+1$	$+2$ $+2$
		-1	$+0$
$1+2)$	1	$+0$	$+2$
		-2	
	1	-2	

Dividing by n, $n+1$, $n+2$ in succession, as on the right, the reckoning shows that

$$u_n=n(n+1)(n+2)-2n(n+1)+2n+2;$$

therefore

$$\begin{aligned}\Sigma u_n&=\Sigma n(n+1)(n+2)-2\Sigma n(n+1)+2\Sigma n+2n\\&=\tfrac{1}{4}n(n+1)(n+2)(n+3)-\tfrac{2}{3}n(n+1)(n+2)+n(n+1)+2n\\&=\tfrac{1}{12}n(3n^3+10n^2+21n+38),\end{aligned}$$

the 'constant' clearly being zero.

5. Series in which u_n is the Reciprocal of the Product of r successive terms of an A.P., beginning with the n-th. A simple instance is the following:

Ex. 1. *Sum to n terms* $\dfrac{1}{1\,.\,3\,.\,5}+\dfrac{1}{3\,.\,5\,.\,7}+\dfrac{1}{5\,.\,7\,.\,9}+\dots$.

Here $u_n=\dfrac{1}{(2n-1)(2n+1)(2n+3)}=\dfrac{1}{4}\cdot\dfrac{(2n+3)-(2n-1)}{(2n-1)(2n+1)(2n+3)}$ *;

$$\therefore\ u_n=\frac{1}{4}\cdot\frac{1}{(2n-1)(2n+1)}-\frac{1}{4}\cdot\frac{1}{(2n+1)(2n+3)}=v_{n-1}-v_n,$$

where $v_n=\dfrac{1}{4}\cdot\dfrac{1}{(2n+1)(2n+3)}$;

$$\therefore\ s_n=v_0-v_n=\frac{1}{12}-\frac{1}{4}\cdot\frac{1}{(2n+1)(2n+3)}.$$

The general case is as follows: *It is required to sum to n terms the series in which u_n is the reciprocal of*

$$(a+nb)(a+\overline{n+1}\,.\,b)\dots(a+\overline{n+r-1}\,.\,b),$$

where a, b, r are constants.

* The numerator is the difference between the first and last factors of the denominator.

Proceeding as in the last example, we have

$$u_n=\frac{1}{(r-1)b}\cdot\frac{(a+\overline{n+r-1}\,.\,b)-(a+nb)}{(a+nb)(a+\overline{n+1}\,.\,b)\ldots(a+\overline{n+r-1}\,.\,b)}.$$

Therefore $u_n=v_{n-1}-v_n$, and $s_n=v_0-v_n$, where

$$v_n=\frac{1}{(r-1)b}\cdot\frac{1}{(a+\overline{n+1}\,.\,b)(a+\overline{n+2}\,.\,b)\ldots(a+\overline{n+r-1}\,.\,b)}.$$

Now v_0 is independent of n, hence the sum to n terms may be found by the following rule : *Remove the left-hand factor from the denominator of u_n ; divide by the number of factors so diminished and by the common difference of the* A.P. *and subtract the result from a constant.*

The constant is found either by putting $n=1$, or by substituting 0 for n in v_n. Thus, in the last example, by the rule

$$s_n=C-\frac{1}{2\,.\,2}\cdot\frac{1}{(2n+1)(2n+3)},$$

and putting $n=1$, $\frac{1}{1\,.\,3\,.\,5}=C-\frac{1}{2\,.\,2}\cdot\frac{1}{3\,.\,5}$, $\therefore\ C=\frac{1}{12}$,

or,
$$v_0=\frac{1}{2\,.\,2}\cdot\frac{1}{1\,.\,3}=\frac{1}{12}.$$

Ex. 2. *Sum the series* $\frac{1}{1\,.\,4}+\frac{1}{2\,.\,5}+\frac{1}{3\,.\,6}+\ldots+\frac{1}{n(n+3)}$.

$$u_n=\frac{1}{n(n+3)}=\frac{n^2+3n+2}{n(n+1)(n+2)(n+3)}=\frac{n(n+3)+2}{n(n+1)(n+2)(n+3)}.$$

$$\therefore\ u_n=\frac{1}{(n+1)(n+2)}+\frac{2}{n(n+1)(n+2)(n+3)}.$$

$$\therefore\ s_n=C-\frac{1}{n+2}-\frac{2}{3(n+1)(n+2)(n+3)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

To find the constant C, put $n=1$ in (A), observing that $s_1=\frac{1}{4}$.

$$\therefore\ \frac{1}{1\,.\,4}=C-\frac{1}{3}-\frac{2}{3\,.\,2\,.\,3\,.\,4},\quad\therefore\ C=\frac{11}{18}.$$

Alternative method, by Partial Fractions. We have

$$u_n=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right);$$

$$\therefore\ 3s_n=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\ldots-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3};$$

$$\therefore\ s_n=\frac{1}{3}\left\{1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right\}$$

$$=\frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right);$$

which is the same result as before, in a different form.

The last example is an instance of *a series whose n-th term is*

$$P/(a+nb)(a+n+1b)\ldots(a+n+r-1b),$$

where P is a polynomial in n of degree m.

If $m<r-1$, this can be summed by expressing P in the form

$$a_0+a_1(a+nb)+a_2(a+nb)(a+n+1b)+\ldots.$$

If $m=r-1$, the series cannot be summed, for to do this we should have to sum a harmonical progression.

If $m>r-1$, the nth term consists of an integral and a fractional part which can be dealt with separately.

6. Another important Type. *If u_n is the n-th term of*

$$\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots+\frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)},$$

then
$$s_n=\frac{1}{a-b+1}\{u_n(a+n)-a\}.$$

For
$$u_n=u_n\cdot\frac{(a+n)-(b+n-1)}{a-b+1}=v_n-v_{n-1},$$

where
$$v_n=u_n(a+n)/(a-b+1)\quad(n\geqslant 1).$$

Also
$$u_1=u_1\cdot\frac{(a+1)-b}{a-b+1}=v_1-\frac{a}{a-b+1}.$$

Therefore
$$s_n=v_n-\frac{a}{a-b+1}=\frac{1}{a-b+1}\{u_n(a+n)-a\}.$$

7. The Series $u_0+u_1x+u_2x^2+\ldots+u_nx^n$ where u_n is a Polynomial in n of Degree m.

If $x=1$, this series can be summed as in Art. 4. If $x\neq 1$, let

$$s=u_0+u_1x+u_2x^2+\ldots+u_nx^n,$$

then
$$s(1-x)=u_0+v_1x+v_2x^2+\ldots+v_nx^n-u_nx^{n+1},$$

where
$$v_1=u_1-u_0,\quad v_2=u_2-u_1,\ \ldots\ v_n=u_n-u_{n-1}.$$

Now v_n is a polynomial in n of degree $m-1$; thus by repeatedly multiplying by $1-x$ (m multiplications in all), we can reduce the problem of finding s to that of summing a geometric series.

It follows that $s=P/(1-x)^{m+1}$ where P is a polynomial in x.

Ex. 1. *Sum the series* $1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}$.

Denoting the sum by s, we have

$$s(1-x)=1+3x+5x^2+\ldots+(2n-1)x^{n-1}-n^2x^n.$$

Let
$$s'=1+3x+5x^2+\ldots+(2n-1)x^{n-1},$$

then $$s'(1-x)=1+2x+2x^2+\ldots+2x^{n-1}-(2n-1)x^n$$
$$=\frac{2(1-x^n)}{1-x}-1-(2n-1)x^n;$$
$$\therefore\ s=\frac{2(1-x^n)}{(1-x)^3}-\frac{1+(2n-1)x^n}{(1-x)^2}-\frac{n^2x^n}{1-x}.$$

8. The Series $1^r+2^r+3^r+\ldots+n^r$. We shall write

$$S_r=1^r+2^r+3^r+\ldots+n^r=f(n),$$

and $$\sigma=n(n+1),\quad \sigma'=2n+1.$$

(1) **Theorem.** *If r is a positive integer, S_r can be expressed as a polynomial in n of which the highest term is $n^{r+1}/(r+1)$.*

For we can find $a_1, a_2, \ldots a_r$, independent of n, such that

$$n^r=a_1n+a_2n(n-1)+\ldots+a_rn(n-1)(n-2)\ldots(n-r+1), \quad \ldots\ldots\text{(A)}$$

where obviously $a_r=1$. Writing $n-1, n-2, \ldots 1$ in succession for n and adding, by Art. 3, we have

$$S_r=\tfrac{1}{2}a_1(n+1)n+\tfrac{1}{3}a_2(n+1)n(n-1)+\ldots$$
$$+\frac{1}{r+1}a_r(n+1)n(n-1)\ldots(n-r+1). \quad \ldots\ldots\text{(B)}$$

The right-hand side, when expanded, is a polynomial in n of which the highest term is $n^{r+1}.a_r/(r+1)$, that is, $n^{r+1}/(r+1)$.

Ex. 1. *Show that* $S_2=\frac{1}{6}n(n+1)(2n+1)$, $S_3=\frac{1}{4}n^2(n+1)^2$.

We have $$n^2=n+n(n-1);$$
$$\therefore\ S_2=\tfrac{1}{2}(n+1)n+\tfrac{1}{3}(n+1)n(n-1)=\tfrac{1}{6}n(n+1)(2n+1).$$

Also $$n^3=n+3n(n-1)+n(n-1)(n-2);$$
$$\therefore\ S_3=\tfrac{1}{2}(n+1)n+(n+1)n(n-1)+\tfrac{1}{4}(n+1)n(n-1)(n-2)=\tfrac{1}{4}n^2(n+1)^2.$$

(2) If $S_r=b_1n+b_2n^2+b_3n^3+\ldots+b_{r+1}n^{r+1}=f(n)$, the values of b_{r+1}, b_r, b_{r-1}, ... may be found as follows.

$$f(n)=1^r+2^r+\ldots+n^r, \quad \text{and} \quad f(n-1)=1^r+2^r+\ldots+(n-1)^r,$$

therefore $$n^r=f(n)-f(n-1),$$

so that $$n^r=b_{r+1}\{n^{r+1}-(n-1)^{r+1}\}+b_r\{n^r-(n-1)^r\}+\ldots+b_1.$$

Expanding and equating coefficients, it will be found that

$$b_{r+1}=\frac{1}{r+1},\quad b_r=\frac{1}{2},\quad b_{r-1}=\frac{r}{12},\quad b_{r-2}=0,\quad b_{r-3}=-\frac{r(r-1)(r-2)}{720},$$

and so on. If r is large, $b_1, b_2, \ldots$ cannot be found conveniently in this way, and we use other methods.

(3) The values of $S_1, S_3, S_5, \ldots$, may be found in succession by considering the function $v_n = n^r(n+1)^r - (n-1)^r n^r$.

We have $v_1 + v_2 + \ldots + v_n = n^r(n+1)^r$,

and, expanding by the Binomial theorem,

$$\tfrac{1}{2}v_n = C_1^r n^{2r-1} + C_3^r n^{2r-3} + C_5^r n^{2r-5} + \ldots .$$

Putting $n-1, n-2, \ldots 1$ in succession for n and adding,

$$\tfrac{1}{2}\sigma^r = C_1^r S_{2r-1} + C_3^r S_{2r-3} + C_5^r S_{2r-5} + \ldots , \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

where $\sigma = n(n+1)$. The last term is rS_{r+1} or S_r, according as r is even or odd.

When $r = 1, 2, 3, \ldots$ this formula gives

$\tfrac{1}{2}\sigma = S_1$	
$\tfrac{1}{2}\sigma^2 = 2S_3$	$S_3 = \tfrac{1}{4}\sigma^2$
$\tfrac{1}{2}\sigma^3 = 3S_5 + S_3$	$S_5 = \tfrac{1}{12}\sigma^2(2\sigma - 1)$
$\tfrac{1}{2}\sigma^4 = 4S_7 + 4S_5$	$S_7 = \tfrac{1}{24}\sigma^2(3\sigma^2 - 4\sigma + 2)$
$\tfrac{1}{2}\sigma^5 = 5S_9 + 10S_7 + S_5$	$S_9 = \tfrac{1}{20}\sigma^2(2\sigma^3 - 5\sigma^2 + 6\sigma - 3)$
$\tfrac{1}{2}\sigma^6 = 6S_{11} + 20S_9 + 6S_7$	$S_{11} = \tfrac{1}{24}\sigma^2(2\sigma^4 - 8\sigma^3 + 17\sigma^2 - 20\sigma + 10)$.

Thus if r is odd and >1 and $S_r = f(n)$, then $n^2(n+1)^2$ is a factor of $f(n)$, and the remaining factor is a rational integral function of $n(n+1)$. Of course, it does not follow that S_r is *arithmetically* divisible by $n^2(n+1)^2$.

(4) The values of $S_2, S_4, S_6, \ldots$, may be found as follows.

Let $w_n = n^r(n+1)^r(2n+1) - (n-1)^r n^r(2n-1)$;

then $w_1 + w_2 + \ldots + w_n = n^r(n+1)^r(2n+1) = \sigma^r\sigma'$,

where $\sigma = n(n+1)$ and $\sigma' = 2n+1$.

On expansion, we have

$$\begin{aligned}\tfrac{1}{2}w_n &= (2C_1^r + 1)n^{2r} + (2C_3^r + C_2^r)n^{2r-2} + (2C_5^r + C_4^r)n^{2r-4} + \ldots \\ &= (2r+1)n^{2r} + \tfrac{1}{3}C_2^r(2r-1)n^{2r-2} + \tfrac{1}{5}C_4^r(2r-3)n^{2r-4} + \ldots .\end{aligned}$$

Putting $n-1, n-2, \ldots 1$ in succession for n and adding,

$$\tfrac{1}{2}\sigma^r\sigma' = (2r+1)S_{2r} + \tfrac{1}{3}C_2^r(2r-1)S_{2r-2} + \tfrac{1}{5}C_4^r(2r-3)S_{2r-4} + \ldots , \quad \ldots\ldots (\mathbf{B})$$

the last term being S_r or $(r+2)S_{r+1}$, according as r is even or odd.

When $r = 1, 2, 3, \ldots$ this formula gives

$r=1$	$\tfrac{1}{2}\sigma\sigma' = 3S_2$	$S_2 = \tfrac{1}{6}\sigma\sigma'$
$r=2$	$\tfrac{1}{2}\sigma^2\sigma' = 5S_4 + S_2$	$S_4 = \tfrac{1}{30}\sigma\sigma'(3\sigma - 1)$
$r=3$	$\tfrac{1}{2}\sigma^3\sigma' = 7S_6 + 5S_4$	$S_6 = \tfrac{1}{42}\sigma\sigma'(3\sigma^2 - 3\sigma + 1)$
$r=4$	$\tfrac{1}{2}\sigma^4\sigma' = 9S_8 + 14S_6 + S_4$	$S_8 = \tfrac{1}{90}\sigma\sigma'(5\sigma^3 - 10\sigma^2 + 9\sigma - 3)$
$r=5$	$\tfrac{1}{2}\sigma^5\sigma' = 11S_{10} + 30S_8 + 7S_6$	$S_{10} = \tfrac{1}{66}\sigma\sigma'(3\sigma^4 - 10\sigma^3 + 17\sigma^2 - 15\sigma + 5)$.

Thus if r is even and $S_r = f(n)$, then $n(n+1)(2n+1)$ is a factor of $f(n)$, and the remaining factor is a rational integral function of $n(n+1)$.

It will presently appear that the formula **(B)** is immediately deducible from **(A)** of the last section.

(5) **Bernoulli's Numbers.** We *define* B_r as the coefficient of n in S_r, when this is expressed as a polynomial in n.

Thus $B_1 = \frac{1}{2}$, and B_3, B_5, B_7, ... are all zero; for $S_1 = \frac{1}{2}n(n+1)$ and S_3, S_5, S_7, ... all contain the factor $n^2(n+1)^2$. The numbers B_2, $-B_4$, B_6, $-B_8$, ... are the famous *numbers of Bernoulli*, in terms of which a large number of functions can be expanded. For convenience we shall refer to the whole set B_1, B_2, B_3, ... as Bernoulli's numbers.

A great many formulae can be given for calculating these numbers; one of the best is the following.

If $r > 1$, *then*

$$(2r+1)B_{2r} + \tfrac{1}{3}C_2^r(2r-1)B_{2r-2} + \tfrac{1}{5}C_4^r(2r-3)B_{2r-4} + \ldots = 0, \quad \ldots\ldots\textbf{(C)}$$

the last term on the left being B_r *or* $(r+2)B_{r+1}$, *according as* r *is even or odd. If* $r = 1$, *the zero on the right of* **(C)** *is to be replaced by* $\frac{1}{2}$.

For B_{2m} is the coefficient of n in S_{2m}. Hence by the formula **(B)** of Art. 8, (4), the left-hand side of **(C)** is the coefficient of n in the expansion of $\frac{1}{2}n^r(n+1)^r(2n+1)$.

Putting $r = 1, 2, 3, \ldots$, in **(C)**,

$r=1$	$3B_2 = \frac{1}{2}$	$B_2 = \frac{1}{6}$
$r=2$	$5B_4 + B_2 = 0$	$B_4 = -\frac{1}{30}$
$r=3$	$7B_6 + 5B_4 = 0$	$B_6 = \frac{1}{42}$
$r=4$	$9B_8 + 14B_6 + B_4 = 0$	$B_8 = -\frac{1}{30}$
$r=5$	$11B_{10} + 30B_8 + 7B_6 = 0$	$B_{10} = \frac{5}{66}$

Continuing thus, we find that

$$B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798}, \quad B_{20} = -\frac{174611}{330}.$$

A few more of the extraordinary properties possessed by the sequence $B_1, B_2, B_3, \ldots$ are given below.

By the Binomial theorem,

$$(n+1)^{r+1} - n^{r+1} = C_1^{r+1}n^r + C_2^{r+1}n^{r-1} + \ldots + C_r^{r+1}n + 1.$$

Putting $n-1, n-2, \ldots 1$ in succession for n and adding,

$$(n+1)^{r+1} - 1 = C_1^{r+1}S_r + C_2^{r+1}S_{r-1} + \ldots + C_r^{r+1}S_1 + n.$$

Equating the coefficients of n on each side, we have

$$C_1^{r+1}B_r + C_2^{r+1}B_{r-1} + \ldots + C_r^{r+1}B_1 + 1 = r + 1. \quad \ldots\ldots\ldots\ldots\textbf{(D)}$$

Similarly, from the equation

$$n^{r+1}-(n-1)^{r+1}=C_1^{r+1}n^r-C_2^{r+1}n^{r-1}+\ldots+(-1)^r,$$

it follows that

$$n^{r+1}=C_1^{r+1}S_r-C_2^{r+1}S_{r-1}+\ldots+(-1)^{r-1}C_r^{r+1}S_1+(-1)^r n,$$

and consequently

$$C_1^{r+1}B_r-C_2^{r+1}B_{r-1}+\ldots+(-1)^{r-1}C_r^{r+1}B_1+(-1)^r=0. \quad \ldots\ldots\ldots(\mathbf{E})$$

From (**D**) and (**E**) by addition and subtraction,

$$C_1^{r+1}B_r \quad +C_3^{r+1}B_{r-2}+C_5^{r+1}B_{r-4}+\ldots=\tfrac{1}{2}(r+1), \quad \ldots\ldots\ldots\ldots(\mathbf{F})$$

$$C_2^{r+1}B_{r-1}+C_4^{r+1}B_{r-3}+C_6^{r+1}B_{r-5}+\ldots=\tfrac{1}{2}(r+1), \quad \ldots\ldots\ldots\ldots(\mathbf{G})$$

where each series continues till B_1, or $B_0=1$, appears.

Formulae (**F**), (**G**) are equivalent to equations 10′, 11′ in Chrystal's *Algebra*, Vol. II, p. 231. They may be used for calculating B_2, B_4, etc., but their use requires twice the amount of labour required by (**C**).

Symbolic notation. Equations (**D**), (**E**), (**F**), (**G**) can be written symbolically thus :

$$(B+1)^{r+1}-B^{r+1}=r+1, \quad \ldots\ldots\ldots\ldots(\mathbf{D})$$

$$B^{r+1}-(B-1)^{r+1}=0, \quad \ldots\ldots\ldots\ldots(\mathbf{E})$$

$$(B+1)^{r+1}-(B-1)^{r+1}=(r+1), \quad \ldots\ldots\ldots\ldots(\mathbf{F})$$

$$(B+1)^{r+1}+(B-1)^{r+1}-2B^{r+1}=(r+1), \quad \ldots\ldots\ldots\ldots(\mathbf{G})$$

where, after expansion, each index is to be changed into a suffix, i.e. B_m *is to be written for* B^m.

(6) **Theorem.** *If* $S_r=1^r+2^r+\ldots+n^r$ *where* r *is a positive integer, then*

$$\text{(i) } \frac{d}{dn}S_r=rS_{r-1}+B_r \quad \textit{and} \quad \text{(ii) } S_r=r\int S_{r-1}dn+nB_r$$

where B_r *is independent of* n *and is, in fact, the Bernoullian number as defined in Art.* 8, (5).

For S_r can be expressed as a polynomial in n of degree $r+1$. Denoting this polynomial by $f(n)$, we have

$$n^r=f(n)-f(n-1).$$

This relation involves only positive integral powers of n up to n^r, and holds for all positive integral values of n. Hence it is an identity, and we may differentiate both sides with regard to n. Therefore

$$rn^{r-1}=f'(n)-f'(n-1).$$

Writing $n-1$, $n-2$, ... 1 in succession for n, and adding,

$$f'(n)=r\{n^{r-1}+(n-1)^{r-1}+\ldots+2^{r-1}+1^{r-1}\}+B_r,$$

where B_r is independent of n.

Moreover, B_r is the coefficient of n in $f(n)$, and is therefore the Bernoullian number defined in Art. 8, (5). The second statement follows from the first. No constant of integration is added, for S_r is divisible by n.

Ex. 1. *Deduce formula* (**B**) *of Art.* 8, (4), *from* (**A**) *of Art.* 8, (3).
Putting $r+1$ for r in (**A**), we have

$$\tfrac{1}{2}\sigma^{r+1} = C_1^{r+1}S_{2r+1} + C_3^{r+1}S_{2r-1} + C_5^{r+1}S_{2r-3} + \dots .$$

Differentiating with regard to n,

$$\tfrac{1}{2}(r+1)\sigma^r\sigma' = C_1^{r+1}(2r+1)S_{2r} + C_3^{r+1}(2r-1)S_{2r-2} + C_5^{r+1}(2r-3)S_{2r-4} + \dots .$$

No constant is added, for every term is divisible by n.
Dividing by $r+1$, we obtain the formula (**B**).

(7) Using the relation $S_r = r\int S_{r-1}dn + nB_r$, we can find the values of S_1, S_2, S_3 ... by successive integrations. (At any stage, the constant B_r can be found by putting $n=1$.) Thus $S_0 = n$, $S_1 = \dfrac{n^2}{2} + B_1 n$,

$$S_2 = \frac{2n^3}{2\,.\,3} + B_1\frac{2n^2}{2} + B_2 n, \qquad S_3 = \frac{2\,.\,3n^4}{2\,.\,3\,.\,4} + B_1\frac{2\,.\,3n^3}{2\,.\,3} + B_2\frac{3n^2}{2} + B_3 n.$$

Continuing the process, we arrive at the equation

$$\frac{S_r}{\lfloor r} = \frac{n^{r+1}}{\lfloor r+1} + B_1\frac{n^r}{\lfloor 1 \lfloor r} + B_2\frac{n^{r-1}}{\lfloor 2 \lfloor r-1} + B_3\frac{n^{r-2}}{\lfloor 3 \lfloor r-2} + \dots + B_r\frac{n}{\lfloor r},$$

which may be written

$$(r+1)S_r = n^{r+1} + B_1C_1^{r+1}n^r + B_2C_2^{r+1}n^{r-1} + \dots + B_rC_r^{r+1}n, \quad \dots\dots(\mathbf{H})$$

and expressed symbolically, as above, in the form

$$(r+1)S_r = (n+B)^{r+1} - B^{r+1}. \quad \dots\dots(\mathbf{H})$$

This result is known as *Bernoulli's Theorem.*

Putting $n=1$, we have equation (**D**) of Art. 8, (5).

Ex. 1. *Verify the following, either by substitution in* (**H**), *or, starting with*

$$S_3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4},$$

by successive integration.

$$S_4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$S_5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$$

$$S_6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$$

$$S_7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12}$$

$$S_8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}$$

$$S_9 = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{n^2}{20}$$

$$S_{10} = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}.$$

EXERCISE XIV

SUMMATION OF SERIES

A (Arts. 3, 4.)

Sum to n terms the series in Exx. 1-9.

1. $1.2.3.4+2.3.4.5+3.4.5.6+\dots$.

2. $2.5.8+5.8.11+8.11.14+\dots$.

3. $1^2+3^2+5^2+\dots$.

4. $1.5+2.6+3.7+\dots$.

5. $1^2.2+2^2.3+3^2.4+\dots$.

6. $1.2^2+2.3^2+3.4^2+\dots$.

7. $1.2.4+2.3.5+3.4.6+\dots$.

8. $1.2.3+4.5.6+7.8.9+\dots$.

9.* $2.1+5.3+8.5+\dots$.

Find the value of

10. $11^2+12^2+13^2+\dots+21^2$.

11. $11^2-12^2+13^2-\dots-20^2+21^2$.

12. $11^3+12^3+13^3+\dots+21^3$.

13. $20^3-19^3+18^3-\dots+2^3-1^3$.

Sum to n terms the series in Exx. 14-16.

14. $1^2.2^2+2^2.3^2+3^2.4^2+\dots$.

15.† $1.n+2(n-1)+3(n-2)+\dots$.

16. $1.n^2+2(n-1)^2+3(n-2)^2+\dots$.

17. Show that whether n is odd or even,

(i) $1^2-2^2+3^2-4^2+\dots$ to $(n-1)$ terms $=\frac{1}{2}(-1)^n n(n-1)$.

(ii) $1^4-2^4+3^4-4^4+\dots$ to $(n-1)$ terms $=\frac{1}{2}(-1)^n n(n-1)(n^2-n-1)$.

18. Show that the sum of the products, taken two together, of the first n natural numbers is $\frac{1}{24}n(n^2-1)(3n+2)$.

19. Find (i) the sum of the squares, (ii) the sum of the products taken two together of the numbers $1, 4, 7, \dots (3n-2)$.

20. Find the number of shot which can be arranged in a pyramidal pile on a triangular base, each side of the base containing 10 shot.

21. Find the number of shot in a pile of 8 courses on a triangular base, each side of the base containing 12 shot.

22. Show that the number of shot in a pile of n courses on a square base, each side of the base containing $2n$ shot, is

$$\tfrac{1}{6}n(2n+1)(7n+1).$$

23. Find the number of shot in a pile of a rectangular base, the sides of the base containing 16 and 10 shot respectively, and the top course consisting of a single row of shot.

24. A pile of shot of n courses stands on a rectangular base, and the top course consists of a single row of x shot. Show that the number of shot in the pile is

$$\tfrac{1}{6}n(n+1)(2n+3x-2).$$

* Here the nth term is the product of the nth terms of the arithmetic series, $2, 5, 8, \dots$, and $1, 3, 5, \dots$.

† Here $u_r=r(n-r-1)=nr-(r-1)r$.

25. Show that if n is of the form $5m+1$ or $5m+3$, where m is a natural number or zero, then

$$1^4+2^4+3^4+\ldots+n^4 \text{ is divisible by } 1^2+2^2+3^2+\ldots+n^2.$$

26. Show that if n is of the form $3m+1$, then

$$1^5+2^5+3^5+\ldots+n^5 \text{ is divisible by } 1^3+2^3+3^3+\ldots+n^3.$$

B (Arts. 5, 6, 7.)

Sum to n terms the series in Exx. 1-24.

1. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots.$

2. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots.$

3. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots.$

4. $\frac{1}{2.5.8}+\frac{1}{5.8.11}+\frac{1}{8.11.14}+\ldots.$

5. $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots.$

6. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots.$

7. $\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\ldots.$

8. $\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\ldots.$

9. $\frac{1}{2.4}+\frac{1.3}{2.4.6}+\frac{1.3.5}{2.4.6.8}+\ldots.$

10. $\frac{4}{5}+\frac{4.7}{5.8}+\frac{4.7.10}{5.8.11}+\ldots.$

11. $1+2x+3x^2+4x^3+\ldots.$

12. $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\ldots.$

13. $a+(a+d)r+(a+2d)r^2+\ldots.$

14. $1^2+\frac{3^2}{2}+\frac{5^2}{2^2}+\frac{7^2}{2^3}+\ldots.$

15. $\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\ldots.$

16. $1.\lfloor\underline{1}+2.\lfloor\underline{2}+3.\lfloor\underline{3}+\ldots.$ [Here $u_n=n\lfloor\underline{n}=\lfloor\underline{n+1}-\lfloor\underline{n}.$]

17. $\frac{1}{1.3}+\frac{2}{1.3.5}+\frac{3}{1.3.5.7}+\ldots.$ [Here $u_n=\frac{1}{2}\cdot\frac{(2n+1)-1}{1.3.5\ldots(2n+1)}$.]

18. $\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots.$ [Here $u_n=\frac{2n+1}{n^2(n+1)^2}=\frac{(n+1)^2-n^2}{n^2(n+1)^2}$.]

19. $\frac{1}{3.9.11}+\frac{1}{5.11.13}+\frac{1}{7.13.15}+\ldots.$

20. $\frac{1}{x+1}+\frac{2x}{(x+1)(x+2)}+\frac{3x^2}{(x+1)(x+2)(x+3)}+\ldots.$
[Observe that $nx^{n-1}=x^{n-1}(x+n)-x^n$.]

21. $1+(1+2)x+(1+2+3)x^2+(1+2+3+4)x^3+\ldots.$
[The value of $s_n(1-x)$ can be found by Ex. 11.]

22. $\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+\dots$. 23. $\frac{1}{2.3}+2.\frac{2}{3.4}+2^2.\frac{3}{4.5}+\dots$.

24. $\frac{1}{\lfloor 3}+\frac{5}{\lfloor 4}+\frac{11}{\lfloor 5}+\dots+\frac{n^2+n-1}{\lfloor n+2}$.

25. Prove that

$$1+n+\frac{n(n+1)}{\lfloor 2}+\frac{n(n+1)(n+2)}{\lfloor 3}+\dots \text{ to } r \text{ terms}=\frac{\lfloor n+r-1}{\lfloor n \lfloor r-1}.$$

Sum to n terms

26. $\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\dots$. 27. $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\dots$.

28. $\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\dots$.

C (Art. 8.)

1. If $1^r+2^r+3^r+\dots+n^r=f(n)$, show that

$$f(-n)=(-1)^{r+1}f(n-1).$$

[Put $-n+1$, $-n+2$, $\dots$ -1 for n in the identity $f(n)-f(n-1)=n^r$.]

2. If $S_r=b_1n+b_2n^2+b_3n^3+\dots+b_{r+1}n^{r+1}$, show that

$$b_1-b_2+b_3-b_4+-\dots=0,$$
$$2b_2-3b_3+4b_4-5b_5+\dots=0,$$
$$3.2b_3-4.3b_4+5.4b_5-\dots=0.$$

[Use Art. 8, (2).]

3. Given that $S_9=\frac{1}{20}\sigma^2(2\sigma^3-5\sigma^2+6\sigma-3)$ where $\sigma=n(n+1)$, deduce the value of S_8 by differentiation.

4. Continue the reckoning in Art. 8, (5), to find the values of B_{12} and B_{14}.

5. Show that formula (C) of Art. 8, (5), may be written symbolically in the form

$$B^r(B+1)^r(2B+1)=B^r(B-1)^r(2B-1),$$

where after expansion B_m is to be written for B^m.

6. Find the value of B_4 by using formulae (F) and (G) of Art. 8, (5).

7. Show that

$$C_1^{r+1}2^rB_r+C_3^{r+1}2^{r-2}B_{r-2}+C_5^{r+1}2^{r-4}B_{r-4}+\dots=r+1.$$

[Expand $(2n+1)^{r+1}-(2n-1)^{r+1}$.]

CHAPTER IX

DETERMINANTS

1. Notation. Many complicated expressions can be easily handled if they are expressed as ' determinants.' The theory which we are about to explain arises in connection with linear equations.

If the equations $a_1x+b_1=0$, $a_2x+b_2=0$ are satisfied by the same value of x, then $a_1b_2-a_2b_1=0$. The expression $a_1b_2-a_2b_1$ is called a *determinant of the second order*, and is denoted by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \text{or by} \quad (a_1b_2).$$

Next consider the system

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0, \quad a_3x+b_3y+c_3=0.$$

If these equations are satisfied by the same values of x and y, it is easy to show that

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$$

The expression on the left is called a *determinant of the third order*, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \text{or by} \quad (a_1b_2c_3).$$

This determinant in its expanded form is

$$a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1$$

In considering the form of this expression, observe that:

(i) *Disregarding the signs*, the terms can be obtained by writing the letters a, b, c in their natural order and arranging the suffixes in all possible ways. The number of arrangements is $\lfloor 3$, giving six terms.

(ii) *The sign* of any particular term depends on *the order of the suffixes.*

As in Ch. IV, 4, (1), the interchange of two suffixes, 2, 3, for example, is called the transposition (2 3).

The term $a_1b_2c_3$, in which the suffixes occur in *their natural order*, has the sign +.

The sign of any other term is + or −, according as the arrangement of the suffixes in that term is derived from the arrangement 1 2 3 by an even or by an odd number of transpositions. Thus, considering the second and third terms, 1 2 3 is changed to 1 3 2 by the *single* transposition (2 3). Also 1 2 3 is changed to 2 3 1 by the *two* transpositions (1 2), (1 3) performed in this order. These ideas will now be generalised.

2. Definition of a Determinant. By means of n letters $a, b, c, \ldots l$, and n suffixes $1, 2, 3, \ldots n$, we can represent n^2 numbers thus,

$$\begin{array}{llll} a_1 & b_1 & c_1 \ldots & l_1, \\ a_2 & b_2 & c_2 \ldots & l_2, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_n & b_n & c_n \ldots & l_n. \end{array}$$

The determinant of the nth order, denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots & l_1 \\ a_2 & b_2 & c_2 \ldots & l_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_n & b_n & c_n \ldots & l_n \end{vmatrix}, \quad \text{or by} \quad (a_1 b_2 c_3 \ldots l_n),$$

is defined as the sum of all the products, each with the sign determined by the rule below, which can be formed by writing the letters $a, b, c, \ldots l$ in their natural order and arranging the suffixes in all possible orders.

Rule of Signs. The sign of the term $a_1 b_2 c_3 \ldots l_n$, where the suffixes occur in their natural order, is +. The sign of any other term is + or −, according as the arrangement of suffixes in that term is derived from the arrangement 1 2 3 ... n by an even or by an odd number of transpositions (that is, by an even or an odd substitution).

This rule is justified by Theorem 2 of Ch. IV, 4.

Each of the n^2 numbers $a_1, b_1 \ldots a_2, b_2, \ldots$ is called an *element* of the determinant.

The diagonal through the left-hand top corner, which contains the elements $a_1, b_2, c_3, \ldots l_n$, is called the *leading* or *principal diagonal*, and $a_1 b_2 c_3 \ldots l_n$ is called the *leading term*.

The expanded form of the determinant has $\lfloor n$ terms: half of these have the sign + and half the sign −.

For suppose that there are p terms of the first sort and q of the second. Since the interchange of the two suffixes transforms a positive into a negative term, $p \leqslant q$. Similarly $q \leqslant p$, and therefore $p = q$.

Every term contains one element and one only from each row, and one element and one only from each column. We can find the sign of any particular term by the rule at the end of Art. 4, Ch. IV.

Ex. 1. *In the expanded form of* $(a_1b_2c_3d_4e_5)$, *find the signs of* (i) $a_2b_4c_5d_1e_3$, (ii) $a_5b_4c_3d_2e_1$.

(i) $\begin{pmatrix}1&2&3&4&5\\2&4&5&1&3\end{pmatrix}=(1\,2\,4)(3\,5)=(1\,2)(2\,4)(3\,5)$, and the sign is $-$. (The last step is unnecessary if we use the rule referred to.)

Or, we may make single transpositions, as follows,

$$2\,4\,5\,1\,3,\quad 1\,4\,5\,2\,3,\quad 1\,2\,5\,4\,3,\quad 1\,2\,3\,4\,5,$$

showing that *three* of these are required, and so the sign is $-$.

(ii) $\begin{pmatrix}1&2&3&4&5\\5&4&3&2&1\end{pmatrix}=(1\,5)(2\,4)$, and the sign is $+$.

Or if, as some text-books advise, we count the number of 'inversions' (Ch. IV, 4) in the arrangements of the suffixes in (i) and (ii), these are 5 and 10 respectively—the first odd and the second even, giving the same results. But this takes much longer.

3. Theorem. *A determinant is unaltered by changing its rows into columns and its columns into rows.*

Let $$\Delta=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix},\quad \Delta'=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}.$$

The leading term of each determinant is $a_1b_2c_3$.

The remaining terms of Δ are derived from $a_1b_2c_3$ by keeping the letters in their natural order and arranging the suffixes in all possible ways, an interchange of two suffixes producing a change of sign.

The remaining terms of Δ' are derived from $a_1b_2c_3$ by keeping the suffixes in their natural order and arranging the letters in all possible ways, an interchange of two letters producing a change of sign.

The results in the two cases are identical. The same argument applies to a determinant of any order.

4. Theorem. *The interchange of two rows, or of two columns, changes the sign of a determinant without altering its numerical value.*

For the interchange of two rows is equivalent to the interchange of the suffixes, and the interchange of two columns is equivalent to the interchange of two letters. Hence in either case the sign of every term of the determinant is changed.

5. Theorem. *A determinant in which two rows or two columns are identical is equal to zero.*

If two rows or two columns of a determinant Δ are identical, the value of Δ is unaltered by the interchange of these rows or columns.

But this interchange transforms Δ into $-\Delta$, therefore

$$\Delta=-\Delta,\quad \text{and}\quad \Delta=0.$$

6. Expansion of a Determinant in Terms of the Elements of any Row or Column.

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Every term in the expansion of Δ contains one element and one only from each row, and one element and one only from each column.

Hence, Δ can be expressed in any of the six forms :

$$a_1A_1 + b_1B_1 + c_1C_1, \quad a_1A_1 + a_2A_2 + a_3A_3,$$
$$a_2A_2 + b_2B_2 + c_2C_2, \quad b_1B_1 + b_2B_2 + b_3B_3,$$
$$a_3A_3 + b_3B_3 + c_3C_3, \quad c_1C_1 + c_2C_2 + c_3C_3,$$

where A_1 contains no element from the row or column containing a_1, a similar remark applying to every A, B and C.

To determine A_1, every term of Δ which contains a_1 can be obtained from $a_1b_2c_3$ by keeping the letters in the natural order, retaining 1 as the suffix of a and arranging the other suffixes in all possible orders, the interchange of two suffixes producing a change of sign.

Hence A_1 is equal to the determinant (b_2c_3), obtained by erasing the row and column in Δ which contain a_1.

To determine any other coefficient, say C_3. By the interchange of *consecutive* rows, bring the row containing c_3 to the top. By the interchange of *consecutive* columns, bring the column containing c_3 to the extreme left.

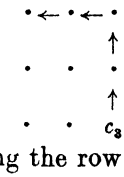
Thus we have in succession

$$\begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} c_3 & a_3 & b_3 \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix},$$

and now c_3 occupies the top left-hand corner, and the relative positions of the elements not belonging to the row and column containing c_3 is unaltered.

Every interchange of rows or of columns introduces a change of sign.

The number of interchanges is equal to the number of vertical and horizontal 'moves' (indicated by arrows) required to bring c_3 to the left-hand top corner, which in this case is 4.



Hence, C_3 is equal to the determinant obtained by erasing the row and column containing c_3, multiplied by $(-1)^4$.

Similar reasoning applies to a determinant of any order.

Ex. 1. *Expand the determinant* $\Delta=(a_1b_2c_3)$ *with reference to the elements* (i) *in the first row,* (ii) *in the second column.*

The results are

$$\text{(i)}\ \Delta= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

$$\text{(ii)}\ \Delta= -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Ex. 2. *Prove that*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2.$$

Expanding with reference to the first row,

$$\Delta=a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$=a(bc-f^2)-h(hc-fg)+g(hf-bg)$$

$$=abc+2fgh-af^2-bg^2-ch^2.$$

7. Minors and Cofactors.

For the determinant

$$\Delta=(a_1b_2\ldots l_n),$$

the *minor* of any element is the determinant obtained by omitting from Δ the row and column containing the element.

The *cofactor* (H_r) of an element (h_r) is the coefficient of that element in the expanded form of Δ, and is therefore equal to the corresponding minor with the proper sign prefixed.

The sign is determined as follows: Count the number of interchanges of consecutive rows and columns (i.e. of vertical and horizontal 'moves') required to bring the element to the left-hand top corner. The sign is + or −, according as this number is even or odd. The rule may be stated thus. *For the element* h_r *which is in the r-th row and the s-th column,*

$$\textit{the cofactor } H_r=(-1)^{r+s}\,.\,(\textit{the minor of } h_r).$$

For the number of 'moves' required to bring h_r to the left-hand top corner is $(r-1)+(s-1)$.

Ex. 1. *Find the cofactors of* d_2 *and* c_4 *in the determinant* $\Delta=(a_1b_2c_3d_4)$.

Omitting the row and column which contain d_2, it will be seen that

$$\text{the minor of } d_2=(a_1b_3c_4).$$

Denote the cofactors in question by D_2, C_4; by the rule given above

$$D_2=(-1)^{2+4}(a_1b_3c_4)=(a_1b_3c_4).$$

Similarly $C_4=(-1)^{4+3}(a_1b_2d_3)=-(a_1b_2d_3)$.

The student who is acquainted with 'partial differentiation' should observe that

$$H_r=\frac{\partial\Delta}{\partial h_r}.$$

8. Important Identities. For the determinant

$$\Delta = (a_1 b_2 c_3)$$

we have a number of identities of the following types:

$$\text{(i) } a_2A_1 + b_2B_1 + c_2C_1 = 0, \quad \text{(ii) } b_1A_1 + b_2A_2 + b_3A_3 = 0.$$

where A_1, B_1, ... are the cofactors of a_1, b_1, ...

For

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1,$$

where A_1, B_1, C_1 are independent of a_1, b_1, c_1. Hence the expression

$$a_2A_1 + b_2B_1 + c_2C_1$$

is the result of substituting a_2, b_2, c_2 for a_1, b_1, c_1 in Δ, and is therefore equal to a determinant with two rows identical, therefore

$$a_2A_1 + b_2B_1 + c_2C_1 = 0.$$

The second identity is obtained by putting b_1, b_2, b_3 for a_1, a_2, a_3 in the equality

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3.$$

Similar identities hold for determinants of any order.

9. Theorem. *If every element in any row, or in any column, of a determinant is multiplied by a number k, then the determinant is multiplied by k.*

For every term in the expansion of a determinant Δ contains one element and one only out of each row, and one element and one only out of each column.

Thus

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}.$$

Conversely, if all the elements in any row, or in any column, have a common factor k, then k is a factor of the determinant, and can be taken outside.

10. Theorem. *A determinant can be expressed as the sum of two determinants by expressing every element in any row or column as the sum of two numbers.*

For example, we shall prove that

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

Denote the determinants on the right by Δ, Δ', and the one on the left by Δ''.

We have $$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3$$

where A_1, A_2, A_3 are independent of a_1, a_2, a_3.

Now Δ' is obtained from Δ by substituting α_1, α_2, α_3 for a_1, a_2, a_3.

And Δ'' is obtained from Δ by substituting $a_1+\alpha_1$, $a_2+\alpha_2$, $a_3+\alpha_3$ for a_1, a_2, a_3.

Hence $$\Delta' = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3,$$

and $$\Delta'' = (a_1+\alpha_1) A_1 + (a_2+\alpha_2) A_2 + (a_3+\alpha_3) A_3 ;$$

$$\therefore\ \Delta'' = \Delta + \Delta'.$$

Similar reasoning applies in all cases.

11. Theorem. *A determinant is unaltered by adding to the elements of any row (or column) k times the corresponding elements of any other row (or column), where k is any given number.*

For example, we shall prove that

$$\begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Denoting the determinant on the right by Δ and that on the left by Δ', we have

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix}.$$

Now $$\begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0.$$

$$\therefore\ \Delta' = \Delta.$$

Note that by a second application of the above,

$$\begin{vmatrix} ha_1+kb_1+lc_1 & b_1 & c_1 \\ ha_2+kb_2+lc_2 & b_2 & c_2 \\ ha_3+kb_3+lc_3 & b_3 & c_3 \end{vmatrix} = h \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

NOTE. In simplifying determinants the following points should be carefully observed.

(i) When, as in the last part of Art. 11, a column (or row) is replaced by one which contains the elements of that column (or row) all multiplied by a number h, then the determinant is multiplied by h. For instance, if the second column is subtracted from the first, and the remainders are put in the *second* column, the sign of the determinant is changed.

(ii) When more than one column (or row) is changed by a succession of changes of this kind, *at least one column (or row) must be left unaltered,* as in Art. 13, Ex. 1.

12. Theorem. *If the elements of a determinant Δ are rational integral functions of x and two rows (or columns) become identical when $x=a$, then $x-a$ is a factor of Δ.*

For Δ can be expressed as a polynomial in x, and $\Delta=0$ when $x=a$; so that the result follows by the Remainder theorem.

(Again, *if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.*)

For $r-1$ of these rows can be replaced by rows in which the elements are the differences of corresponding elements in the original rows; hence $x-a$ is a factor of all the elements in each of these $(r-1)$ substituted rows; and the result follows.

13. Examples. The determinant in question is denoted by Δ.

Ex. 1. *Find the value of the determinant on the right.* $\begin{vmatrix} 6 & 8 & 7 & 11 \\ 13 & 9 & 12 & 4 \\ 7 & 10 & 3 & 5 \\ 14 & 1 & 6 & 15 \end{vmatrix}$.

We have

$$\Delta = \begin{vmatrix} 6 & 1 & 1 & -1 \\ 13 & -3 & -1 & -22 \\ 7 & 7 & -4 & -9 \\ 14 & -5 & -8 & -13 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 19 & -2 & -23 & -22 \\ 31 & 11 & -13 & -9 \\ 62 & 3 & -21 & -13 \end{vmatrix};$$

$$\therefore \Delta = \begin{vmatrix} 19 & -2 & -23 \\ 31 & 11 & -13 \\ 62 & 3 & -21 \end{vmatrix} = \begin{vmatrix} 19 & -2 & -23 \\ 31 & 11 & -13 \\ 0 & -19 & 5 \end{vmatrix}.$$

Expanding with reference to the bottom row,

$$\Delta = 19(-19 \cdot 13 + 23 \cdot 31) + 5(19 \cdot 11 + 2 \cdot 31) = 10209.$$

[In the first step, the 3rd column is taken from the 2nd to form a new 2nd column, the 1st column is taken from the 3rd to form a new 3rd column, and twice the 1st column is taken from the 4th to form a new 4th column. In the second step, six times the 3rd column is taken from the 1st, the 3rd is subtracted from the 2nd, and the 4th added to the 3rd.

Ex. 2. *Prove that*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

If $\alpha=\beta$, two columns in Δ are identical, and consequently $\alpha-\beta$ is a factor of Δ. Similarly $\beta-\gamma$, etc., are factors. Hence

$$\Delta = k(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

Since Δ is of the sixth degree in $\alpha, \beta, \ldots, k$ is independent of these quantities. Now the term $\beta\gamma^2\delta^3$ occurs in Δ, and on the right-hand side the coefficient of this term is $-k$, therefore $k=-1$.

Ex. 3. *If ω is a cube root of unity, prove that $a+b\omega+c\omega^2$ is a factor of*

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Hence show that the determinant is equal to

$$-(a^3+b^3+c^3-3abc).$$

To the first column add ω times the second and ω^2 times the third, therefore

$$\Delta = \begin{vmatrix} a+b\omega+c\omega^2 & b & c \\ b+c\omega+a\omega^2 & c & a \\ c+a\omega+b\omega^2 & a & b \end{vmatrix} = (a+b\omega+c\omega^2) \begin{vmatrix} 1 & a & c \\ \omega^2 & b & a \\ \omega & a & b \end{vmatrix};$$

$\therefore$ $a+b\omega+c\omega^2$ is a factor of Δ, similarly $a+b\omega^2+c\omega$ and $a+b+c$ are factors. The rest follows as in Ex. 2.

This method applies to a determinant of any order in which the second row is derived from the first, the third from the second, and so on, by a cyclic substitution.

Ex. 4. *Prove that*

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3,$$

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix} = (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}.$$

Subtracting the sum of the second and third rows from the first and taking 2 outside,

$$\Delta = 2(a+b+c)^2 \begin{vmatrix} bc & -c & -b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix} = \frac{2(a+b+c)^2}{bc} \begin{vmatrix} bc & 0 & 0 \\ b^2 & bc+ba & b^2 \\ c^2 & c^2 & ac+bc \end{vmatrix}.$$

In the last step the first column is added to b times the second and to c times the third, *which is equivalent to multiplying by* bc;

$$\therefore \Delta = 2(a+b+c)^2\{bc(c+a)(a+b)-b^2c^2\}$$
$$\equiv 2abc(a+b+c)^3.$$

Alternatively. If we substitute 0 for a,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0;$$

hence a is a factor; and similarly b and c are factors.

Again, if $-(b+c)$ is substituted for a,

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix};$$

and in this there are three identical columns; hence, by Art. 12, $(a+b+c)^2$ is a factor of Δ.

Hence, since Δ is of the sixth degree,

$$\Delta = Nabc(a+b+c)^3 ;$$

the remaining factor necessarily being of the first degree and symmetrical, since the cyclic substitution (a, b, c) gives a determinant equal to Δ.

Putting $a=1$, $b=1$, $c=1$, we have

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27N,$$

whence $N=2$.

EXERCISE XV

A. SIMPLE EVALUATIONS

1. The positive terms in the expansion of $(a_1b_2c_3)$ are

$$a_1b_2c_3, \quad b_1c_2a_3, \quad c_1a_2b_3.$$

Hence the rule of Sarrus, which is as follows:

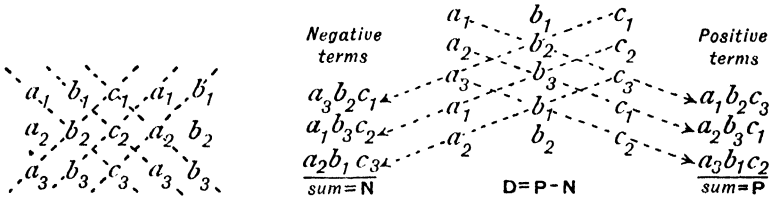


FIG. 26.

The determinant $(a_1b_2c_3)$ is the sum of the products of the three elements lying on each of the six lines in the above diagram on the left, the sign of the product being + or − according as the line which determines it is drawn downwards or upwards, proceeding from left to right.

A more convenient way of using the rule in numerical cases is shown on the right.

2. Apply the rule of Ex. 1 to show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

3. Show that, in the expansion of $(a_1b_2c_3d_4)$, the following terms occur:

$$+b_1c_2a_3d_4, \quad +c_1a_2b_3d_4, \quad +a_1b_2c_3d_4,$$
$$+a_1d_2b_3c_4, \quad +b_1d_2c_3a_4, \quad +c_1d_2a_3b_4,$$

and that the other positive terms are obtained from these by the substitution **(12)(34)**.

4. If T is any term in the expansion of $(a_1b_2c_3\ldots)$ and another term T' is derived from T by a cyclic substitution involving r letters, the order of the suffixes being unchanged, then T and T' have the same sign, or opposite signs, according as r is odd or even. [Exx. 1 and 3 are instances.]

5. In the determinant $(a_1b_2c_3\ldots l_n)$, the sign of the term $a_nb_{n-1}c_{n-2}\ldots l_1$, forming the 'second diagonal' has the sign + if and only if n is of the form $4k$ or $4k+1$.

6. In the expansion of $(a_1b_2c_3d_4e_5)$ show that the terms $+a_4b_3c_1d_2e_3$, $-a_3b_4c_5d_2e_1$ occur.

7. In the expansion of $(a_1b_2c_3d_4e_5f_6)$ show that the terms $+a_2b_4c_6d_1e_3f_5$, $+a_6b_4c_1d_2e_3f_5$ occur.

8. Expand the following:

(i) $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}$. (ii) $\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix}$. (iii) $\begin{vmatrix} 7 & 11 & 13 \\ 17 & 19 & 23 \\ 29 & 31 & 37 \end{vmatrix}$.

9. Prove that

(i) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = -8$. (ii) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = 0$.

(iii) $\begin{vmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix} = 1+a^4+a^8$. (iv) $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ a & a & b & a \\ b & b & b & a \end{vmatrix} = -(a-b)^4$.

(v) $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

10. Prove that

$$\begin{vmatrix} l_1-t & m_1 & n_1 \\ l_2 & m_2-t & n_2 \\ l_3 & m_3 & n_3-t \end{vmatrix}$$

is equal to $-t^3+t^2(l_1+m_2+n_3)-t(L_1+M_2+N_3)+(l_1m_2n_3)$,

where L_1, M_2, N_3 are the cofactors of l_1, m_2, n_3 in $(l_1m_2n_3)$.

11. Prove that

$$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma),$$

where l, m, n have any values whatever.

[From the first, second and third columns respectively take α, β, γ times the fourth column.]

12. Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

[Multiply the rows by a, $-b$, $-c$, $-d$ to form a new top row.]

13. Prove that

$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right),$$

and that a similar result holds for a determinant of the nth order and of this form.

[Subtract the top row from each of the others and expand with reference to the top row.]

14. If $u=ax^4+4bx^3+6cx^2+4dx+e$ and $u_{11}=ax^2+2bx+c$, $u_{12}=bx^2+2cx+d$, $u_{22}=cx^2+2dx+e$, prove that

$$\begin{vmatrix} a & b & c & u_{11} \\ b & c & d & u_{12} \\ c & d & e & u_{22} \\ u_{11} & u_{12} & u_{22} & 0 \end{vmatrix} = -u\,.\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

[From the fourth column subtract x^2 times the first column, $2x$ times the second, and the third.]

15. If $u=ax^2+2bxy+cy^2$, $u'=a'x^2+2b'xy+c'y^2$, prove that

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix} = -\frac{1}{y}\cdot\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix}.$$

[Add x times the second column to y times the third: then add x times the first column to y times the second.]

16. Prove that

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = -(ac-b^2)(ax^2+2bxy+cy^2).$$

[Multiply the first column by x, the second by y, and subtract from the third.]

17. Prove that the determinant

$$\begin{vmatrix} a-x & b & c & d \\ b & c-x & d & a \\ c & d & a-x & b \\ d & a & b & c-x \end{vmatrix}$$

is equal to $(x-a-b-c-d)(x-a+b-c+d)\{x^2-(a-c)^2-(b-d)^2\}$.

B. METHOD OF FACTORS

18. Prove that $\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3(x+a+b+c+d)$.

19. If $2s = a+b+c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c).$$

[Show that s^2, $s-a$, $s-b$, $s-c$ are factors: deduce that Ns is the remaining factor, and find N by putting $a=2$, $b=0$, $c=0$.]

20. Express the following in factors:

(i) $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$. (ii) $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$. (iii) $\begin{vmatrix} \mu\nu & \mu+\nu & 1 \\ \nu\lambda & \nu+\lambda & 1 \\ \lambda\mu & \lambda+\mu & 1 \end{vmatrix}$.

21. Prove that

$$\begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix} = (bc')(ca')(ab').$$

[Divide the first row by $b'c'$, the second by $c'a'$, the third by $a'b'$. Put $a/a' = \lambda$, $b/b' = \mu$, $c/c' = \nu$, and use the last example.]

22. If $\lambda = \beta\gamma + \alpha\delta$, $\mu = \gamma\alpha + \beta\delta$, $\nu = \alpha\beta + \gamma\delta$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

Hence show that the following determinant has the same value:

$$-\begin{vmatrix} \beta^2\gamma^2+\alpha^2\delta^2 & \beta\gamma+\alpha\delta & 1 \\ \gamma^2\alpha^2+\beta^2\delta^2 & \gamma\alpha+\beta\delta & 1 \\ \alpha^2\beta^2+\gamma^2\delta^2 & \alpha\beta+\gamma\delta & 1 \end{vmatrix}.$$

23. Prove that

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

By equating the coefficients of x on each side, show that

$$\begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\beta\gamma+\gamma\alpha+\alpha\beta).$$

24. If ω is a root of $x^4=1$, show that $a+b\omega+c\omega^2+d\omega^3$ is a factor of

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}.$$

Hence show that the determinant is equal to

$$-(a+b+c+d)(a-b+c-d)\{(a-c)^2+(b-d)^2\}.$$

14. Minors of a Determinant. The determinant obtained by omitting any number of rows and the same number of columns from a given determinant Δ is called a *minor* of Δ.

If we omit two rows and two columns, the resulting determinant is called a *second minor*, and so on.

If from Δ we omit the rows and columns which contain every element of a given minor, we get another minor which is said to be *complementary* to the first. Thus (a_1b_2) and $(c_3d_4e_5)$ are complementary minors of

$$(a_1b_2c_3d_4e_5).$$

By the interchange of rows and columns, any minor of Δ can be put into the left-hand top corner, and then the complementary minor can be seen at a glance.

15. Expansion of a Determinant by means of its Second Minors. Consider the determinant

$$\Delta=(a_1b_2c_3d_4e_5).$$

In the expansion of Δ, the sum of the terms which contain a_1b_2 is

$$a_1b_2(c_3d_4e_5),$$

and the sum of those which contain a_2b_1 is $-a_2b_1(c_3d_4e_5)$. Hence every term in the product $(a_1b_2)(c_3d_4e_5)$ occurs in the expansion.

Again, every term in Δ contains one element out of the first row and one out of the second, and these elements have different suffixes.

Hence Δ is equal to the sum of all the products, each with its proper sign, of the form

$$(a_1b_2)(c_3d_4e_5),$$

where the first factor is any minor formed from elements of the first two rows (or of the first two columns), and the second is the complementary minor.

The sign of a product is determined as follows. Take, for example,

$$(b_1d_2)(a_3c_4e_5).$$

In the expansion of this product, the term $b_1d_2a_3c_4e_5$ occurs. Keeping the suffixes in the same order, observe that

$$\begin{pmatrix} a & b & c & d & e \\ b & d & a & c & e \end{pmatrix} = (abdc) = (ab)(bd)(dc).$$

Therefore, in the expansion of Δ, this term occurs with a negative sign, and the sign of the product is $-$.

More generally, if any two rows of a determinant Δ are

$$a_r \quad b_r \quad c_r \ldots h_r \ \ldots k_r \ \ldots,$$
$$a_{r'} \quad b_{r'} \quad c_{r'} \ldots h_{r'} \ldots k_{r'} \ldots,$$

where h_r, $h_{r'}$ belong to the sth column and k_r, $k_{r'}$ to the s'th column, then

$$\Delta = \Sigma(-1)^{r+r'+s+s'}(h_rk_{r'}) \,.\, (\text{the complementary minor of } (h_r\, k_{r'})).$$

The summation is to include all the minors which can be formed from the two rows. *The sign of the product* may be determined as above, or as in Art. 7 by counting the number of 'moves' required to bring h_r to the place occupied by a_1 and $k_{r'}$ to the place occupied by b_2.

Thus, in the instance given above, the sign of $(b_1d_2)(a_3c_4e_5)$ is given by $(-1)^{1+2+2+4} = -1$.

In a similar way we can expand a determinant in terms of the rth minors which can be formed from any r rows (or columns).

NOTE. In the 'double-suffix' notation of Art. 20, the rule takes the following convenient form: In $(a_{pq}\, a_{rs} \ldots)$(complementary minor), the sign is $(-1)^{p+q+r+s+\cdots}$.

Ex. 1. *Expand $\Delta = (a_1b_2c_3d_4)$ in terms of the minors of the rows containing a_1 and a_2.* It is easy to verify that

$$\Delta = (b_1c_2)(a_3d_4) + (c_1a_2)(b_3d_4) + (a_1b_2)(c_3d_4) + (a_1d_2)(b_3c_4) + (b_1d_2)(c_3a_4) + (c_1d_2)(a_3b_4).$$

Ex. 2. *Prove that if*

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ p_1 & q_1 & r_1 & l_1 & m_1 & n_1 \\ p_2 & q_2 & r_2 & l_2 & m_2 & n_2 \\ p_3 & q_3 & r_3 & l_3 & m_3 & n_3 \end{vmatrix}, \quad \textit{and} \quad \Delta' = \begin{vmatrix} 0 & 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 \\ l_1 & m_1 & n_1 & p_1 & q_1 & r_1 \\ l_2 & m_2 & n_2 & p_2 & q_2 & r_2 \\ l_3 & m_3 & n_3 & p_3 & q_3 & r_3 \end{vmatrix}$$

then $\Delta = (a_1b_2c_3) \,.\, (l_1m_2n_3)$, *and* $\Delta' = (-1)^3(a_1b_2c_3) \,.\, (l_1m_2n_3)$.

Considering the minors of Δ, which can be formed from the elements of the first three rows, the only one not equal to zero is $(a_1b_2c_3)$, and the complementary minor is $(l_1m_2n_3)$. Moreover, the term $a_1b_2c_3l_1m_2n_3$ occurs in the expansion of Δ, therefore

$$\Delta = (a_1b_2c_3) \,.\, (l_1m_2n_3).$$

Again, Δ is transformed into Δ' by the interchange of the first and fourth, the second and fifth and the third and sixth columns. Whence the second result follows.

16. Product of Two Determinants of the same Order. As a typical case we shall prove that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1m_1 & a_1l_2+b_1m_2 \\ a_2l_1+b_2m_1 & a_2l_2+b_2m_2 \end{vmatrix}.$$

Denote the determinants on the left by Δ and Δ'; then we have

$$\Delta\Delta' = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ -1 & 0 & l_1 & l_2 \\ 0 & -1 & m_1 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_1l_1+b_1m_1 & a_1l_2+b_1m_2 \\ a_2 & b_2 & a_2l_1+b_2m_1 & a_2l_2+b_2m_2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

where the second determinant is obtained from the first as follows: To the third column add l_1 times the first and m_1 times the second: to the fourth column add l_2 times the first and m_2 times the second.

Hence, as in the last example,

$$\Delta\Delta' = (-1)^2 \cdot \begin{vmatrix} a_1l_1+b_1m_1 & a_1l_2+b_1m_2 \\ a_2l_1+b_2m_1 & a_2l_2+b_2m_2 \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix},$$

whence the result.

The product of two determinants of the n-th order ($n>2$) can be expressed in a similar form, and by the same method.

Second method. It is required to show that $(a_1b_2c_3)\,.(l_1m_2n_3) = \Delta$, where

$$\Delta = \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}.$$

The determinant Δ is the sum of 27 determinants, two of which are

$$\begin{vmatrix} b_1m_1 & a_1l_2 & c_1n_3 \\ b_2m_1 & a_2l_2 & c_2n_3 \\ b_3m_1 & a_3l_2 & c_3n_3 \end{vmatrix} \text{ and } \begin{vmatrix} b_1m_1 & b_1m_2 & a_1l_3 \\ b_2m_1 & b_2m_2 & a_2l_3 \\ b_3m_1 & b_3m_2 & a_3l_3 \end{vmatrix},$$

which are the same as

$$m_1l_2n_3 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \text{ and } m_1m_2l_3 \begin{vmatrix} b_1 & b_1 & a_1 \\ b_2 & b_2 & a_2 \\ b_3 & b_3 & a_3 \end{vmatrix}.$$

The second of these is zero, and the first is equal to

$$m_1l_2n_3(b_1a_2c_3) = -m_1l_2n_3(a_1b_2c_3).$$

With regard to the sign, observe that $l_1m_2n_3$ is transformed into $m_1l_2n_3$ by the same number of interchanges of letters as those required to change $a_1b_2c_3$ into $b_1a_2c_3$. This number is odd, and therefore the sign is $-$.

Of the 27 determinants, all are equal to zero except the six of the first type, in which a's are in one column, b's in another and c's in the third.

Moreover, for reasons similar to the above, $(a_1b_2c_3)$ is a factor of each of the six, and the other factor is a term (with the proper sign) of $(l_1m_2n_3)$, therefore

$$\Delta = (a_1b_2c_3) \cdot (l_1m_2n_3).$$

A similar result holds for determinants of the nth order ($n = 4, 5 \ldots$).

17. Rectangular Arrays. A number of symbols arranged in rows and columns is called a *rectangular array.*

If the number of rows is not equal to the number of columns, the array is generally enclosed by double lines, and is sometimes called a *matrix.*

(1) *Let the number of columns exceed the number of rows.*

Consider the arrays $\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$, $\begin{Vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{Vmatrix}$.

Proceeding as in the last article, we form the determinant

$$\Delta = \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \end{vmatrix}.$$

By reasoning similar to that in Art. 16 it will be seen that

$$\Delta = (a_1b_2)(l_1m_2) + (b_1c_2)(m_1n_2) + (c_1a_2)(n_1l_2).$$

This process is called *multiplying the arrays.*

In general, *the determinant formed thus from two such arrays is equal to the sum of the products obtained by multiplying every determinant which can be formed from one array by the corresponding determinant formed from the other array.*

(2) *Let the number of rows exceed the number of columns.*

If from the arrays $\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix}$, $\begin{Vmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{Vmatrix}$,

we form the determinant

$$\Delta = \begin{vmatrix} a_1l_1 + b_1m_1 & a_1l_2 + b_1m_2 & a_1l_3 + b_1m_3 \\ a_2l_1 + b_2m_1 & a_2l_2 + b_2m_2 & a_2l_3 + b_2m_3 \\ a_3l_1 + b_3m_1 & a_3l_2 + b_3m_2 & a_3l_3 + b_3m_3 \end{vmatrix},$$

it will be seen that $\Delta = 0$. This follows, as in Art. 16, or from the fact that

$$\Delta = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \cdot \begin{vmatrix} l_1 & m_1 & 0 \\ l_2 & m_2 & 0 \\ l_3 & m_3 & 0 \end{vmatrix}.$$

In general, *the determinant formed by multiplying two such arrays is zero.*

Ex. 1. *If* $s_r=\alpha^r+\beta^r+\gamma^r+\delta^r$, *prove that*

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}=\Sigma(\alpha-\beta)^2, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}=\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2,$$

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}=(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\delta)^2(\beta-\delta^2)(\gamma-\delta)^2.$$

The first two results are obtained by squaring the arrays

$$\begin{Vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \end{Vmatrix}.$$

The last is got by squaring the determinant in Ex. 2 of Art. 13.

Ex. 2. *Prove that*

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$$

is equal to

$$2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1).$$

This result is obtained by forming the product

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \end{vmatrix}.$$

18. Reciprocal Determinants. Let $\Delta=(a_1b_2\ldots k_n)$ and

$$\Delta'=(A_1B_2\ldots K_n),$$

where A_1, B_1, ... are the cofactors of a_1, b_1, ... in Δ. For reasons connected with linear transformations, A_1, B_1, ... are sometimes called *inverse* elements, and Δ' is called the *reciprocal* determinant.

(1) *If* Δ *is a determinant of the n-th order and* Δ' *is the reciprocal determinant, then* $\Delta'=\Delta^{n-1}$.

The following method applies in all cases. Let $\Delta=(a_1b_2c_3)$, then using identities of the type

$$a_1A_1+b_1B_1+c_1C_1=\Delta,$$
$$a_1A_2+b_1B_2+c_1C_2=0,$$
$$a_1A_3+b_1B_3+c_1C_3=0,$$

it will be seen that

$$\Delta\Delta'=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=\begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix};$$

whence it follows that $\Delta'=\Delta^2$.

(2) *Any minor of Δ' of order r is equal to the complement of the corresponding minor of Δ multiplied by Δ^{r-1}, provided that $\Delta \neq 0$.*

As a typical instance, let $\Delta = (a_1 b_2 c_3 d_4)$. We shall prove that, if $\Delta \neq 0$, then

$$(A_1 B_2) = (c_3 d_4) \,.\, \Delta \quad \text{and} \quad (A_1 B_2 C_3) = d_4 \Delta^2.$$

We have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & c_1 & d_1 \\ 0 & \Delta & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & c_4 & d_4 \end{vmatrix};$$

therefore

$$\Delta (A_1 B_2) = \Delta^2 (c_3 d_4) \quad \text{and} \quad (A_1 B_2) = \Delta \,.\, (c_3 d_4).$$

The second identity is obtained by multiplying Δ by

$$\cdot \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

19. Two Methods of Expansion.

(1) *With reference to the elements of a row and column.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and $K = \begin{vmatrix} a_1 & b_1 & c_1 & l \\ a_2 & b_2 & c_2 & m \\ a_3 & b_3 & c_3 & n \\ l' & m' & n' & r \end{vmatrix}$.

Let A_1, B_1, ... be the cofactors of a_1, b_1, ... in Δ.

In the expansion of K, the sum of the terms containing r is Δr: every other term contains one of the three l, m, n and one of the three l', m', n'.

Again, $\begin{vmatrix} a_1 & l \\ l' & r \end{vmatrix}$ and $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ are complementary minors of Δ;

hence, coefficient of ll' in K $= -$ coefficient of $a_1 r$ in K

$= -$ coefficient of a_1 in $\Delta = -A_1$;

and similarly,

coefficient of mn' in K $= -$ coefficient of $c_2 r$ in K

$= -$ coefficient of c_2 in $\Delta = -C_2$.

Thus it will be seen that

$$K = \Delta r - \{A_1 ll' + B_2 mm' + C_3 nn' + C_2 mn' + B_3 m'n + A_3 nl' + C_1 n'l + B_1 lm' + A_2 l'm\}.$$

(2) *With reference to a diagonal.*

Let $\Delta = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}$, and $K = \begin{vmatrix} x_1 & b_1 & c_1 & d_1 \\ a_2 & x_2 & c_2 & d_2 \\ a_3 & b_3 & x_3 & d_3 \\ a_4 & b_4 & c_4 & x_4 \end{vmatrix}$.

Let A_1, B_2, C_3, D_4 be the cofactors of the elements in the leading diagonal of Δ, then the value of K is

$$x_1x_2x_3x_4 - (x_2x_3d_1a_4 + x_3x_1d_2b_4 + x_1x_2d_3c_4 + x_1x_4b_3c_2 + x_2x_4c_1a_3 + x_3x_4a_2b_1) + x_1A_1 + x_2B_2 + x_3C_3 + x_4D_4 + \Delta.$$

Note that the expression in brackets is the sum of $x_2x_3d_1a_4$, $x_1x_4b_3c_2$ and terms derived from these by the cyclic transpositions (abc), (123).

The verification is left to the reader.

20. Definitions. A determinant of the nth order is often written in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ a_{31} & a_{32} & a_{33} \dots a_{3n} \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{n1} & a_{n2} & a_{n3} \dots a_{nn} \end{vmatrix} \equiv (a_{11}a_{22} \dots a_{nn}).$$

Denoting any element by a_{rs}, the determinant is said to be *symmetric* if $a_{sr} = a_{rs}$. If $a_{sr} = -a_{rs}$, the determinant is *skew-symmetric*: it is implied that all the elements in the leading diagonal are zero.

For example, if

$$K = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}, \quad L = \begin{vmatrix} 0 & z & y \\ -z & 0 & x \\ -y & -x & 0 \end{vmatrix},$$

the determinant K is symmetric and L is skew-symmetric. We also say that the first determinant is *bordered* by l, m, n.

21. Symmetric Determinants.

(1) *If A_{rs}, A_{sr} are the cofactors of the elements a_{rs}, a_{sr} of a symmetric determinant Δ, then $A_{rs} = A_{sr}$.*

For A_{rs} is transformed into A_{sr} by changing rows into columns.

Thus if $\Delta = (a_{11} \quad a_{22} \quad a_{33})$,

$$A_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = A_{32}.$$

(2) For the determinants $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, $K = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}$,

we have the following important relations :

(i) $BC - F^2 = \Delta a$, $GH - AF = \Delta f$, etc. ;

(ii) $K = -(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm)$,

where A, B, ... are the cofactors of a, b, ... in Δ.

These follow from Arts. 18, 19.

(3) Consider the following symmetric determinants :

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad K = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & r \end{vmatrix}.$$

Let A, B, ... be the cofactors of a, b, ... in Δ. We shall prove that, *if* $\Delta = 0$, *then*

$$AK = -(Al + Hm + Gn)^2.$$

Because $\Delta = 0$, the coefficient of r in the expansion of K is zero, therefore

$$-AK = A(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm).$$

Moreover, $AB - H^2 = \Delta c$, $CA - G^2 = \Delta b$, $GH - AF = \Delta f$, etc.

Hence $AB = H^2$, $AC = G^2$, $AF = GH$, etc., and therefore

$$-AK = A^2l^2 + H^2m^2 + G^2n^2 + 2GHmn + 2AGnl + 2AHlm = (Al + Hm + Gn)^2.$$

In the same way we can prove that

$$-BK = (Hl + Bm + Fn)^2 \quad \text{and} \quad -CK = (Gl + Fm + Cn)^2.$$

More generally, using the notation of Art. 20, *let*

$$\Delta_n = (a_{11}a_{22} \dots a_{nn}), \quad \Delta_{n-1} = (a_{11}a_{22} \dots a_{n-1,\,n-1}),$$

the determinants being symmetric, and let A_{11}, A_{12}, ... *be the cofactors of* a_{11}, a_{12}, ... *in* Δ_{n-1}. *Then, if* $\Delta_{n-1} = 0$,

$$A_{11}\Delta_n = -(A_{11}a_{1n} + A_{12}a_{2n} + \dots + A_{1,\,n-1}a_{n-1,\,n})^2,$$

with similar values for $A_{22}\Delta_n$, $A_{33}\Delta_n$, ...

For, expanding as in Art. 19, (1), since $a_{sr} = a_{rs}$, $A_{sr} = A_{rs}$ and $\Delta_{n-1} = 0$, we have $\Delta_n = -(A_{11}a_{1n}^2 + A_{22}a_{2n}^2 + \dots + 2A_{12}a_{1n}a_{2n} + \dots)$.

Now, since $\Delta_{n-1} = 0$, every minor of the reciprocal determinant

$$(A_{11}A_{22} \dots A_{n-1,\,n-1})$$

is zero. Hence, by Art. 18, (2), $A_{11}A_{pp} = A_{1p}^2$ and $A_{11}A_{pq} = A_{1p}A_{1q}$.

Using such identities, it follows that $A_{11}\Delta_n$ has the value stated.

(4) *The following result is of great importance.*

Consider the symmetric determinants

$$\begin{vmatrix} a-x & h \\ h & b-x \end{vmatrix}, \quad \begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix}, \quad \begin{vmatrix} a-x & h & g & l \\ h & b-x & f & m \\ g & f & c-x & n \\ l & m & n & d-x \end{vmatrix}$$

Denote these by Δ_2, Δ_3, Δ_4. *Let* $\Delta_1 = a-x$, *and suppose the sequence* $\Delta_1, \Delta_2, \Delta_3 \ldots$ *to be continued indefinitely by constructing determinants of the above type. Then the roots of the equations*

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0, \ldots,$$

are all real, and the roots of any one of them are separated by those of the preceding equation.

If λ_1, λ_2 $(\lambda_1 < \lambda_2)$ are the roots of $\Delta_2 = 0$, then $\lambda_1 < a < \lambda_2$ (see Exercise XI, 2).

Also, as in the last section, if $\Delta_2 = 0$, then $\Delta_1 \Delta_3$ is negative. Hence if x has the values $-\infty$, λ_1, λ_2, ∞, the signs of Δ_3 are $+ \; - \; + \; -$.

Therefore the roots of $\Delta_3 = 0$ are real, and denoting them in ascending order by μ_1, μ_2, μ_3, we have $\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \mu_3$.

Again, as in the last section, if $\Delta_3 = 0$ then $\Delta_2 \Delta_4$ is negative; hence,

if x has the values $\quad -\infty, \; \mu_1, \; \mu_2, \; \mu_3, \; \infty,$

the signs of Δ_4 are $\quad + \; , \; - , \; + , \; - , \; + \; ;$

therefore the roots of $\Delta_4 = 0$ are real and are separated by those of $\Delta_3 = 0$; and so on without end.

22. Skew-symmetric Determinants.

(1) *For the skew-symmetric determinant* $(a_{11} a_{22}, \ldots a_{nn})$, *if* A_{rs}, A_{sr} *are the cofactors of* a_{rs}, a_{sr}, *then* $A_{sr} = (-1)^{n-1} A_{rs}$.

For A_{rs} is transformed into A_{sr} by changing the sign of every element and then changing rows into columns; and since A_{rs} is a determinant of $(n-1)$ rows, the result follows.

Thus in the determinant

$$\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$$

the cofactors of c and $-c$ are

$$-\begin{vmatrix} -c & a \\ -b & 0 \end{vmatrix} \quad \text{and} \quad -\begin{vmatrix} c & b \\ -a & 0 \end{vmatrix},$$

and one of these is $(-1)^2$ times the other.

(2) *Every skew-symmetric determinant of odd order is equal to zero.*

For if Δ is a skew-symmetric determinant of odd order n, and Δ' is the determinant obtained by changing the sign of every element in Δ,

$$\Delta' = (-1)^n \Delta = -\Delta.$$

But Δ' can also be obtained from Δ by changing rows into columns, so that $\Delta' = \Delta$: whence the result follows. For example,

$$\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} = - \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}.$$

(3) *Every skew-symmetric determinant of even order is a perfect square.*

(i) Let $\Delta_n = (a_{11} a_{22} \dots a_{nn})$, $\Delta_{n-1} = (a_{11} a_{22} \dots a_{n-1,\, n-1})$, where n is even and the determinants are skew-symmetric. For $n = 2$,

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = a_{12}^2.$$

Let $A_{11}, A_{12}, \dots$ be the cofactors of $a_{11}, a_{12}, \dots$ in Δ_{n-1}. Since n is even, by the preceding $\Delta_{n-1} = 0$ and $A_{sr} = (-1)^{n-2} A_{rs} = A_{rs}$. Remembering that $a_{sr} = -a_{rs}$, exactly as in Art. 21, (3), we can show that

$$A_{11} \Delta_n = (A_{11} a_{1n} + A_{12} a_{2n} + \dots + A_{1,\, n-1}\, a_{n-1,\, n})^2. \quad \dots\dots\dots\dots\dots\dots (\mathbf{A})$$

Hence if A_{11} is a perfect square, so also is Δ_n ; moreover, A_{11} is a skew-symmetric determinant of order $n-2$; and it has been shown that such a determinant of order two is a perfect square, therefore the same is true for those of orders four, six, etc.

(ii) If n is even and $\Delta_n = \delta_n^2$, the value of δ_n* is given by the following rule.

Write down all the arrangements in pairs of the numbers 1, 2, 3, ... n. *Denote any such arrangement by* (hk), (lm), ... (uv), *then*

$$\delta_n = \Sigma (\pm a_{hk} a_{lm} \dots a_{uv}),$$

where the positive or negative sign is to be taken according as the arrangement $hklm \dots uv$ *is derived from* $1234 \dots n$ *by an even or by an odd number of transpositions. The term* $a_{12} a_{34} \dots a_{n-1,\, n}$ *is to have the sign* $+$.

(α) The rule is suggested by the case when $n = 4$. By equation (**A**),

$$A_{11} \Delta_4 = (A_{11} a_{14} + A_{12} a_{24} + A_{13} a_{34})^2,$$

where A_{11}, A_{12}, A_{13} are the cofactors of a_{11}, a_{12}, a_{13} in Δ_3, so that $A_{11} = a_{23}^2$, $A_{12} = -a_{13} a_{23}$, $A_{13} = a_{12} a_{23}$. Whence we find that

$$\Delta_4 = (a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23})^2,$$

which agrees with the rule.

* Named the Pfaffian, by Cayley, after J. F. Pfaff.

(β) *Assuming that the rule holds for determinants of order* $n-2$, we shall prove that it holds for those of order n. Then, since it holds when $n=4$, it will hold for $n=6$, 8, etc. By equation (A),

$$\delta_n = a_{1n}\sqrt{A_{11}} + a_{2n}\frac{A_{12}}{\sqrt{A_{11}}} + \ldots + a_{n-1,n}\frac{A_{1,n-1}}{\sqrt{A_{11}}}.$$

Now $\Delta_{n-1}=0$, therefore by Art. 18, (2), $A_{12}^2 = A_{11}A_{22}$, $A_{13}^2 = A_{11}A_{33}$, etc. Hence

$$\delta_n = a_{1n}\sqrt{A_{11}} \pm a_{2n}\sqrt{A_{22}} \pm \ldots \pm a_{n-1,n}\sqrt{A_{n-1,n-1}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

where the values of $\sqrt{A_{11}}$, $\sqrt{A_{22}}$, etc., are given by the rule. We shall show that *the signs on the right are alternately positive and negative.*

Consider the product $P = a_{1n}\sqrt{A_{11}} \cdot a_{pn}\sqrt{A_{pp}}$. The values of $\sqrt{A_{11}}$ and $\sqrt{A_{pp}}$ are obtained by the rule from the sets

$$2, 3, 4, \ldots n-1 \quad \text{and} \quad 1, 2, 3, \ldots p-1, p+1, \ldots n-1.$$

Moreover, every term in the expansion of P, with the proper sign, must occur in Δ_n.

For clearness, the element a_{rs} will be denoted by (rs), so that $(sr) = -(rs)$.

If p *is even*, the product P contains a term M, which is the product of

$$\underline{(1n)}(23)(45)\ldots(p-2, p-1)\underline{(p, p+1)}(p+2, p+3)\ldots(n-2, n-1)$$

and

$$(pn)(12)(34)\ldots(p-3, p-2)(p-1, p+1)\underline{(p+2, p+3)\ldots(n-2, n-1)}.$$

By changing the order of the numbers in each of the $\frac{1}{2}(n-p)+1$ pairs which are underlined, it will be seen that $(-1)^{\frac{1}{2}(n-p)+1}M$ is the product of elements (rs) in which corresponding values of r, s are as below.

r	1,	2,	3, … $p-2$,	$p-1$,	p,	$p+1$,	$p+2$,	$p+3$, … $n-2$,	$n-1$,	n,
s	2,	3,	4, … $p-1$,	$p+1$,	n,	p,	$p+3$,	$p+2$, … $n-1$,	$n-2$,	1.

The numbers in the second line can be written in the same order as those in the first by $\frac{1}{2}(n+p)$ transpositions, and since

$$\tfrac{1}{2}(n-p)+1+\tfrac{1}{2}(n+p) = n+1, \text{ an odd number,}$$

it follows that $-M$ is a term in Δ_n, so that $-a_{pn}\sqrt{A_{pp}}$ occurs in the value of δ_n.

If p *is odd*, the corresponding process requires changes of order in $\frac{1}{2}(n-p+1)$ pairs, and $\frac{1}{2}(n+p-1)$ transpositions. Since the sum of these numbers is n, which is even, $+a_{pn}\sqrt{A_{pp}}$ occurs in δ_n; thus

$$\delta_n = \Sigma_{p=1}^{p=n-1}(-1)^{p-1}a_{pn}\sqrt{A_{pp}}.$$

It remains to be shown that the value of δ_n can be written down by the rule. It is obviously necessary only to consider the first term in the value of $a_{pn}\sqrt{A_{pp}}$.

This is $(pn)(12)(34)\ldots(p-1, p+1)(p+2, p+3)\ldots(n-2, n-1)$,

or $(pn)(12)(34)\ldots(p-2, p-1)(p+1, p+2)\ldots(n-2, n-1)$,

according as p is even or odd. Now the arrangement

$$p, n, 1, 2, 3, \ldots p-1, p+1, \ldots n-1$$

can be derived from 1, 2, 3, ... n by $p+n-1$ transpositions. The rule asserts that the sign of the term is + or − according as $p+n-1$ is even or odd, that is according as p is odd or even, which agrees with what has been proved.

EXERCISE XVI

A. MISCELLANEOUS METHODS

1. If $a+b+c=0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$$

2. Prove that

$$(b_1c_2)(a_1d_2)+(c_1a_2)(b_1d_2)+(a_1b_2)(c_1d_2)=0.$$

[Use the identities

$$a_1(b_1c_2)+b_1(c_1a_2)+c_1(a_1b_2)=0, \quad a_2(b_1c_2)+b_2(c_1a_2)+c_2(a_1b_2)=0.]$$

3. If $(a_1b_2c_3)=0$ and $(d_1b_2c_3)=0$, prove that

$$\frac{(b_2c_3)}{(a_2d_3)}=\frac{(b_3c_1)}{(a_3d_1)}=\frac{(b_1c_2)}{(a_1d_2)}.$$

4. Prove that

$$\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = 2(a_1b_2c_3).$$

5. Prove that

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ (a+c)(a+b) & (a+b)(b+c) & (b+c)(c+a) \end{vmatrix}$$
$$=3(b-c)(c-a)(a-b)(a+b+c)(bc+ca+ab).$$

6. If

$$\begin{vmatrix} 1/(a+x) & 1/(b+x) & 1/(c+x) \\ 1/(a+y) & 1/(b+y) & 1/(c+y) \\ 1/(a+z) & 1/(b+z) & 1/(c+z) \end{vmatrix} = \frac{P}{Q},$$

where Q is the product of the denominators, prove that

$$P=(b-c)(c-a)(a-b)(y-z)(z-x)(x-y).$$

7. Show that $\begin{vmatrix} 0 & c & b & a \\ c & 0 & a & b \\ b & a & 0 & c \\ a & b & c & 0 \end{vmatrix} = -(a+b+c)(b+c-a)(c+a-b)(a+b-c).$

8. Prove that the 'skew' determinant $\begin{vmatrix} x & c & b & l \\ -c & x & a & m \\ -b & -a & x & n \\ -l & -m & -n & x \end{vmatrix}$

is equal to $x^4+x^2(a^2+b^2+c^2+l^2+m^2+n^2)+(al-bm+cn)^2.$

9. Prove that $\begin{vmatrix} 0 & c & b & b-c \\ c & 0 & a & c-a \\ b & a & 0 & a-b \\ a & b & c & a+b+c \end{vmatrix} = 2(a+b+c)\{abc-(b-c)(c-a)(a-b)\}.$

10. Prove that $\begin{vmatrix} -1 & b & c & d \\ a & -1 & c & d \\ a & b & -1 & d \\ a & b & c & -1 \end{vmatrix}$

is equal to $(a+1)(b+1)(c+1)(d+1)\left\{1-\frac{a}{a+1}-\frac{b}{b+1}-\frac{c}{c+1}-\frac{d}{d+1}\right\}.$

[This follows from Ex. XV, 13, by putting $1+a_1=-1/a$, $1+a_2=-1/b$, etc.; or, directly, by the same method.]

11. Prove the following:

(i) $\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix} = 1-ax-by-cz.$

(ii) $\begin{vmatrix} 0 & -c & b & x \\ c & 0 & -a & y \\ -b & a & 0 & z \\ x' & y' & z' & w \end{vmatrix} = -(ax+by+cz)(ax'+by'+cz').$

(iii) $\begin{vmatrix} 0 & c^2 & b^2 & 1 \\ c^2 & 0 & a^2 & 1 \\ b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -(a+b+c)(b+c-a)(c+a-b)(a+b-c).$

(iv) $\begin{vmatrix} 0 & c^2 & b^2 & d^2 \\ c^2 & 0 & a^2 & e^2 \\ b^2 & a^2 & 0 & f^2 \\ d^2 & e^2 & f^2 & 0 \end{vmatrix} = -16s(s-ad)(s-be)(s-cf)$, where $2s=ad+be+cf.$

12. Denoting the squares of the differences $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ by l, m, n and those of $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$ by l', m', n' respectively, prove that

$$\begin{vmatrix} 0 & n & m & l' \\ -n & 0 & l & m' \\ -m & -l & 0 & n' \\ -l' & -m' & -n' & 0 \end{vmatrix} = 4lnl'n'.$$

[If $\beta\gamma+\alpha\delta=\lambda$, $\gamma\alpha+\beta\delta=\mu$, $\alpha\beta+\gamma\delta=\nu$, then $ll'=(\mu-\nu)^2$, etc.]

13. If $\Delta = \begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$, $\Delta' = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 & x \\ 2^2 & 3^2 & 4^2 & 5^2 & y \\ 3^2 & 4^2 & 5^2 & 6^2 & z \\ 4^2 & 5^2 & 6^2 & 7^2 & w \\ x & y & z & w & 0 \end{vmatrix}$,

prove that

$$\Delta=(x-2y+z)^2,$$

$$\Delta'=8(x-3y+3z-w)^2.$$

14. If in a determinant Δ, when $x=a$:

(i) p rows become the same, then $(x-a)^{p-1}$ is a factor of Δ;

(ii) p rows become the same and q columns become the same, then $(x-a)^{p+q-2}$ is a factor, provided that the common elements are zero or are divisible by $(x-a)^2$.

B. PRODUCTS

15. Write down as a determinant the product

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Hence show that the product of two expressions of the form $a^3+b^3+c^3-3abc$ is of the same form.

16. Prove that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

[Consider the product $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$.]

17. By forming the product on the left, prove the identity on the right,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \\ c_3 & -2b_3 & a_3 \end{vmatrix}; \quad 2(a_1b_2c_3)^2 = \begin{vmatrix} 2\Delta_1 & I_{12} & I_{31} \\ I_{12} & 2\Delta_2 & I_{23} \\ I_{31} & I_{23} & 2\Delta_3 \end{vmatrix},$$

where $\Delta_1=a_1c_1-b_1^2$, $I_{12}=a_1c_2+a_2c_1-2b_1b_2$, etc.

18. By forming the product

$$\begin{vmatrix} a+\iota b & c+\iota d \\ -c+\iota d & a-\iota b \end{vmatrix} . \begin{vmatrix} b'+\iota a' & d'+\iota c' \\ -d'+\iota c' & b'-\iota a' \end{vmatrix},$$

prove that

$$(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2) \equiv (aa'+bb'+cc'+dd')^2+(l+l')^2+(m+m')^2+(n+n')^2,$$

where $l=(bc')$, $m=(ca')$, $n=(ab')$, $l'=(ad')$, $m'=(bd')$, $n'=(cd')$.

19. Express $(1^2+2^2+3^2+5^2)^2$ as the sum of four squares by taking 1, 2, 3, 5 for a, b, c, d and 2, 5, 1, 3 for a', b', c', d'.

20. By multiplying the arrays on the left, prove that the determinant Δ is zero.

$$\left\| \begin{matrix} l & l' \\ m & m' \\ n & n' \end{matrix} \right\|, \quad \left\| \begin{matrix} l' & l \\ m' & m \\ n' & n \end{matrix} \right\|; \quad \Delta = \begin{vmatrix} 2ll' & lm'+l'm & nl'+n'l \\ lm'+l'm & 2mm' & mn'+m'n \\ nl'+n'l & mn'+m'n & 2nn' \end{vmatrix}.$$

Hence prove that if $ax^2+2hxy+by^2+2gx+2fy+c$ is the product of two linear factors, then

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

21. Prove that

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix} = 0.$$

[Multiply the two arrays of four rows

$$\left\| \begin{matrix} a_1^2 & a_1 & 1 \\ & \text{etc.} & \end{matrix} \right\|, \quad \left\| \begin{matrix} 1 & -2b_1 & b_1^2 \\ & \text{etc.} & \end{matrix} \right\|.]$$

C. MISCELLANEOUS THEOREMS AND EXAMPLES

22. Prove that

$$\text{(i)} \quad \begin{vmatrix} x & 1 & 1 & \dots \\ 1 & x & 1 & \dots \\ 1 & 1 & x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = (x-1)^{n-1}(x+n-1),$$

the determinant being of the nth order.

$$\text{(ii)} \quad \begin{vmatrix} a & a+b & a+2b & \dots & a+nb \\ a+nb & a & a+b & \dots & a+(n-1)b \\ a+(n-1)b & a+nb & a & \dots & a+(n-2)b \\ \dots & \dots & \dots & \dots & \dots \\ a+b & a+2b & a+3b & \dots & a \end{vmatrix} = (-b)^n(n+1)^n(a+\tfrac{1}{2}nb).$$

23. If $u_n = \begin{vmatrix} a_1 & 1 & 0 & 0 & \dots & \dots & 0 \\ -1 & a_2 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & a_3 & 1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & -1 & a_{n-1} & 1 \\ 0 & 0 & \dots & \dots & 0 & -1 & a_n \end{vmatrix}$,

prove that $u_n = a_n u_{n-1} + u_{n-2}$.

[Expand with reference to the bottom row. Such determinants are called *continuants*.]

24. If $\begin{vmatrix} a & b & c & d & 0 & 0 \\ a' & b' & c' & d' & 0 & 0 \\ a'' & b'' & c'' & d'' & 0 & 0 \\ 0 & a & b & 0 & c & d \\ 0 & a' & b' & 0 & c' & d' \\ 0 & a'' & b'' & 0 & c'' & d'' \end{vmatrix} = 0$,

prove that $\begin{vmatrix} a & c & d \\ a' & c' & d' \\ a'' & c'' & d'' \end{vmatrix}^2 = \begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix} \times \begin{vmatrix} b & c & d \\ b' & c' & d' \\ b'' & c'' & d'' \end{vmatrix}$.

25. (i) By using the fact that the determinant on the right is zero, prove that

$$t(y-z)(z-x)(x-y) + x(y-z)(y-t)(z-t) + y(z-x)(z-t)(x-t) + z(x-y)(x-t)(y-t) \equiv 0.$$

$$\begin{vmatrix} t & x & y & z \\ 1 & 1 & 1 & 1 \\ t & x & y & z \\ t^2 & x^2 & y^2 & z^2 \end{vmatrix}$$

(ii) Hence show that, if α, β, γ, δ are any angles,

$$\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta) + \sin(\beta-\gamma)\sin(\beta-\delta)\sin(\gamma-\delta) + \sin(\gamma-\alpha)\sin(\gamma-\delta)\sin(\alpha-\delta) + \sin(\alpha-\beta)\sin(\alpha-\delta)\sin(\beta-\delta) \equiv 0.$$

(iii) Use the last result to show that if a, b, c, d and a', b', c', d' are the lengths of the sides of two quadrilaterals such that the sides a, b, c, d are parallel to a', b', c', d', respectively, then $LMN + Lmn + Mnl + Nlm = 0$,

where $l = (bc')$, $m = (ca')$, $n = (ab')$, $L = (ad')$, $M = (bd')$, $N = (cd')$.

[(ii) In (i) put $x = \cos 2\alpha + \iota \sin 2\alpha$, $y = \cos 2\beta + \iota \sin 2\beta$,
$z = \cos 2\gamma + \iota \sin 2\gamma$, $t = \cos 2\delta + \iota \sin 2\delta$.

(iii) Let a, b, c, d make angles α, β, γ, δ with a fixed line. By projecting the circuits a, b, c, d and a', b', c', d' on lines perpendicular to d, a, b, c, prove that

$$\frac{\sin(\alpha-\delta)}{l} = \dots = \dots = \frac{\sin(\beta-\gamma)}{L} = \dots = \dots .]$$

26. Prove that the equations (i), (ii) are respectively the equation to the line joining (x_1y_1), (x_2y_2) and the equation to the plane through $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$.

$$\text{(i)} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad \text{(ii)} \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Show also that the last equation may be written in the form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

27. If A, B, C, D are four points in a plane and the lengths BC, CA, AB, AD, BD, CD are denoted by a, b, c, x, y, z, prove that these are connected by the equation

$$\begin{vmatrix} 0 & c^2 & b^2 & x^2 & 1 \\ c^2 & 0 & a^2 & y^2 & 1 \\ b^2 & a^2 & 0 & z^2 & 1 \\ x^2 & y^2 & z^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

[Let (x_1, y_1), (x_2, y_2), ... be the coordinates of the points. Multiply the arrays

$$\left\| \begin{matrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ \cdots\cdots\cdots & & & \\ 1 & 0 & 0 & 0 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1 & -2x_1 & -2y_1 & x_1^2+y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2+y_2^2 \\ & \cdots\cdots\cdots & & \\ 0 & 0 & 0 & 1 \end{matrix} \right\|.]$$

28. In a tetrahedron $ABCD$, the lengths BC, CA, AB, AD, BD, CD are denoted by a, b, c, d, e, f respectively; the volume is V, the radius of the circumscribing sphere is R.

Assuming equation (i), where (x_1, y_1, z_1), (x_2, y_2, z_2), ... are the coordinates of A, B ..., prove equation (ii).

$$\text{(i)}\ 6V = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \quad \text{(ii)}\ -36R^2V^2 = \tfrac{1}{16} \begin{vmatrix} 0 & c^2 & b^2 & d^2 \\ c^2 & 0 & a^2 & e^2 \\ b^2 & a^2 & 0 & f^2 \\ d^2 & e^2 & f^2 & 0 \end{vmatrix}.*$$

[Take the origin at the centre of the sphere. Multiply the determinants

$$6RV = \begin{vmatrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ \cdots\cdots\cdots & & & \end{vmatrix}, \quad -6RV = \begin{vmatrix} x_1 & y_1 & z_1 & -R \\ x_2 & y_2 & z_2 & -R \\ \cdots\cdots\cdots & & & \end{vmatrix},$$

and use the equations

$$x_1^2+y_1^2+z_1^2 = R^2, \quad (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = c^2, \text{ etc.}]$$

* Note the value of this determinant given in Ex. 11, (iv).

CHAPTER X

SYSTEMS OF EQUATIONS

1. Systems of Equations. If $u=0$ is an equation connecting the variables $x, y, \ldots$, which is satisfied when $x=x_1, y=y_1, \ldots$, then $(x_1, y_1, \ldots)$ is called a *solution* of $u=0$.

Let $u=0, v=0, \ldots$ be a system of equations connecting $x, y, \ldots$. If the equations have a common solution $(x_1, y_1, \ldots)$, they are said to be *consistent*, and $(x_1, y_1, \ldots)$ is called a *solution of the system* $u=0, v=0, \ldots$. Equations which have no common solution are said to be *inconsistent.*

Equations which are *homogeneous* in $x, y, \ldots$ have the common solution $(0, 0, \ldots)$, in which each of the variables is zero : *but they are not said to be consistent unless some other common solution exists.*

If one equation of a system is satisfied by *all* the values of $x, y, \ldots$ which satisfy the others, the equations are said to be *not independent.*

To *eliminate* a variable x from two equations is to obtain from these two an equation free from x. Any result so obtained must hold if the equations are consistent.

To *solve* a system of equations is to obtain all the solutions (if any exist). This is done by a process of elimination.

Illustrations. The equations $x+y=1$, $x+y=2$, which represent parallel lines (or planes), are *inconsistent.* So also are the equations $u=0, v=0, 2u+3v=4$.

The equations $u=0, v=0, lu+mv=0$ are *not independent.*

2. Equivalent Systems. (1) Two systems of equations are said to be *equivalent* when every solution of either system is a solution of the other. In such cases we may say that one system *is the same* as the other.

(2) *If* $a_1, a_2, \ldots$ *are constants and* $a_1 \neq 0$, *the system of equations*

$$u_1=0, \quad u_2=0, \quad \ldots \quad u_n=0,$$

is equivalent to the system

$$a_1u_1+a_2u_2+\ldots+a_nu_n=0, \quad u_2=0, \quad \ldots \quad u_n=0.$$

For any solution of the first system is a solution of the second and, since a_1 is a constant (not zero), any solution of the second system is a solution of the first.

(3) *If l, m, l', m' are constants such that $lm'-l'm\neq 0$, then the pair of equations $u=0$, $v=0$ is equivalent to the pair*

$$lu+mv=0, \quad l'u+m'v=0.$$

For any values of the variables x, y, ... which satisfy $u=0$, $v=0$ also satisfy the second pair.

Moreover, any values of x, y, ... which satisfy the second pair also satisfy

$$m'(lu+mv)-m(l'u+m'v)=0,$$
$$-l'(lu+mv)+l(l'u+m'v)=0,$$

which are the same as

$$(lm'-l'm)u=0, \quad (lm'-l'm)v=0,$$

and since $lm'-l'm\neq 0$, these are the same as $u=0$, $v=0$.

It is left to the reader to supply proofs of the following :

(4) *The pair of equations $uv=0$, $w=0$ is equivalent to the two pairs*

$$u=0, \ w=0 \quad \textit{and} \quad v=0, \ w=0.$$

(5) *Any solution of the system*

$$u_1=v_1, \quad u_2=v_2$$

is a solution of

$$u_1u_2=v_1v_2, \quad u_2=v_2.$$

But the second pair of equations is equivalent to the two pairs

$$u_1=v_1, \ u_2=v_2 \quad \textit{and} \quad u_2=0, \ v_2=0.$$

3. Systems of Linear Equations. In considering this subject, the chief difficulty is to include all the special cases which may arise.

(1) *The equation $ax+by+c=0$ has infinitely many solutions, unless $a=0$, $b=0$, $c\neq 0$, in which case there is no solution.*

A formal proof is hardly required.

(2) Consider the equations

$$u_1\equiv a_1x+b_1y+c_1=0$$
$$u_2\equiv a_2x+b_2y+c_2=0,$$

excluding the case in which either is of the form $0x+0y+c=0$.

(i) *If $(a_1b_2)\neq 0$, there is an unique solution.*

For by Art. 2, (3), the equations are equivalent to

$$u_1b_2-u_2b_1=0, \quad -u_1a_2+u_2a_1=0,$$

which are the same as

$$x(a_1b_2)=(b_1c_2), \quad y(a_1b_2)=(c_1a_2).$$

(ii) *If* $(a_1b_2)=0$, *the equations are inconsistent unless also* $(b_1c_2)=0$ *and* $(c_1a_2)=0$.

For if $(a_1b_2)=0$, we have $u_1b_2-u_2b_1\equiv-(b_1c_2)$, $u_1a_2-u_2a_1\equiv(c_1a_2)$.

Hence the equations cannot have a common solution unless (b_1c_2) and (c_1a_2) are both zero.

(iii) *If* (a_1b_2), (b_1c_2), (c_1a_2) *are all zero, the two equations are one and the same, and there are infinitely many solutions.*

For in this case

$$u_1b_2\equiv u_2b_1 \quad \text{and} \quad u_1a_2\equiv u_2a_1.$$

Hence any solution of one equation is a solution of the other, or else an a is zero *and* a b is zero. The latter alternative cannot occur.

For example, suppose that $a_1=0$; then by hypothesis $b_1\neq0$ and $a_2b_1=a_1b_2=0$, therefore $a_2=0$ and $b_2\neq0$.

Hence the equations are one and the same.

(iv) *The case in which the equations are inconsistent* may be regarded from another point of view.

Suppose the coefficients of x, y to vary in such a way that $(a_1b_2)\to0$ while (b_1c_2) and (c_1a_2) remain finite; then $x\to\infty$, $y\to\infty$ and the fractions x/y, $-b_1/a_1$, $-b_2/a_2$ tend to equality.

We shall say that *in the limiting case when* $(a_1b_2)=0$, *the equations are satisfied by infinite values of* x, y *which are in the ratio* $-b_1:a_1$ *or* $-b_2:a_2$.

(3) *If* $a_1x+b_1y=0$ *and* $a_2x+b_2y=0$, *then either* $x=0$, $y=0$ *or else* $(a_1b_2)=0$.

Conversely, if $(a_1b_2)=0$ *these equations have solutions other than* $(0,0)$, i.e. *they are consistent.*

This is a particular case of the preceding.

The following results are required later.

Ex. 1. *If* $(a_1b_2)=0$ *and* $(a_1c_2)=0$, *then* $(b_1c_2)=0$ *or else* $a_1=0$, $a_2=0$.

For a_1, a_2 satisfy the equations $a_1b_2-a_2b_1=0$, $a_1c_2-a_2c_1=0$.

Ex. 2. *If* $(a_1b_2)=0$, $(b_1c_2)=0$ *and* $(c_1a_2)=0$, *where* a_1, b_1, c_1 *are not all zero and* a_2, b_2, c_2 *are not all zero, then of the pairs* (a_1,a_2), (b_1,b_2), (c_1,c_2) *there is at least one in which neither quantity is zero.*

For suppose that $a_1\neq0$, then if $a_2=0$ we should have $a_1b_2=0$, $a_1c_2=0$; and therefore $b_2=0$, $c_2=0$, which contradicts the hypothesis. Hence, if $a_1\neq0$, then $a_2\neq0$.

Ex. 3. *Let* $\Delta=(a_1b_2c_3)$ *and let* A_1, B_1, ... *be the cofactors of* a_1, b_1, ... *Prove that if* $\Delta=0$ *and* $C_3=0$, *then either* $C_1=0$, $C_2=0$ *or else* $A_3=0$, $B_3=0$.

For we have

$$a_1C_1+a_2C_2=0, \quad b_1C_1+b_2C_2=0, \quad c_1C_1+c_2C_2=0,$$

therefore either $C_1=0$, $C_2=0$, or else $(b_1c_2)=0$, $(c_1a_2)=0$, i.e. $A_3=0$, $B_3=0$.

Or thus: Using identities of the type $(B_2C_3)=\Delta a_1$, it will be seen that (C_1A_3), (C_1B_3), (C_2A_3), (C_2B_3) are all zero. Whence the result follows.

4. Three Linear Equations. Consider the equations

$$u_1 \equiv a_1x + b_1y + c_1 = 0,$$
$$u_2 \equiv a_2x + b_2y + c_2 = 0,$$
$$u_3 \equiv a_3x + b_3y + c_3 = 0.$$

Let $\Delta = (a_1b_2c_3)$, and let A_1, B_1, ... be the cofactors of a_1, b_1, ...

(1) *If the equations are consistent, then* $\Delta = 0$.

For on account of the identities

$$a_1C_1 + a_2C_2 + a_3C_3 = 0,$$
$$b_1C_1 + b_2C_2 + b_3C_3 = 0,$$
$$c_1C_1 + c_2C_2 + c_3C_3 = \Delta,$$

we have $$C_1u_1 + C_2u_2 + C_3u_3 \equiv \Delta.$$

Hence, if the equations are satisfied by the same values of x, y, we must have $\Delta = 0$.

(2) *If* $\Delta = 0$ *the equations are consistent and have an unique solution except when every C is zero; in which case each of the equations (assumed to be distinct) is inconsistent with either of the others.*

For by the preceding, $u_1C_1 + u_2C_2 + u_3C_3 \equiv 0$.

(i) *If no C is zero,* the equations are not independent.

(ii) *If* $C_1 = 0$, $C_2 \neq 0$, $C_3 \neq 0$, then $u_2C_2 + u_3C_3 \equiv 0$ and $u_2 = 0$, $u_3 = 0$ are the same equation.

In both of these cases, since $C_3 \neq 0$, the equations $u_1 = 0$, $u_2 = 0$ have an unique solution, which is the solution of the system.

(iii) *If* $C_1 = 0$, $C_2 = 0$, $C_3 \neq 0$, then $u_3 \equiv 0$. This case is excluded.

(iv) *If every C is zero* and the equations are distinct, each is inconsistent with either of the others. In this case we may say that *the equations are satisfied by infinite values of x, y which are in the ratio* $-b_1 : a_1$.

5. The Line at Infinity. For the sake of complete generality we say that the ' equation,'

$$0x + 0y + c = 0 \quad \text{where} \quad c \neq 0,$$

represents the *line at infinity.* The *ideal* point in which a straight line $u = 0$ meets the line at infinity is called the *point at infinity* on that line.

Parallel straight lines are to be regarded as meeting on the line at infinity. For the lines $u = 0$, $u + k = 0$ intersect on the line $0x + 0y + k = 0$. These conventions enable us to include various special cases under one heading.

Illustration. *If* $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ *(as above) are the equations to three straight lines, then the necessary and sufficient condition that they may meet at a point (which may be at infinity) is* $\Delta = 0$.

6. Two Equations in Three Unknowns. Consider the equations

$$u_1 \equiv a_1x + b_1y + c_1z + d_1 = 0, \quad \text{.....(A)}$$

$$u_2 \equiv a_2x + b_2y + c_2z + d_2 = 0, \quad \text{.....(B)}$$

excluding the case in which a_1, b_1, c_1 or a_2, b_2, c_2 are all zero.

(1) *Of the three* (a_1b_2), (b_1c_2), (c_1a_2), *suppose that at least one, say* (a_1b_2), *is not zero*, then the equations are equivalent to

$$-u_1a_2 + u_2a_1 = 0, \quad u_1b_2 - u_2b_1 = 0,$$

which are the same as

$$-(a_1b_2)y + (c_1a_2)z = (a_1d_2), \quad \text{.....(C)}$$

$$-(b_1c_2)z + (a_1b_2)x = (b_1d_2). \quad \text{.....(D)}$$

Corresponding to any value of z, there are definite values of x and y, provided that (b_1c_2) and (c_1a_2) are not both zero.

If $(b_1c_2) = 0$ and $(c_1a_2) = 0$, then x and y have definite values and z may have any value. (In fact, we have $c_1 = 0$, $c_2 = 0$.)

Thus in every case, by Art. 3, Ex. 1, there are infinitely many solutions.

(2) *Suppose that* (a_1b_2), (b_1c_2), (c_1a_2) *are all zero.* Then of the pairs (a_1, a_2), (b_1, b_2), (c_1, c_2) there is at least one, say (a_1, a_2), in which neither quantity is zero. (See Art. 3, Ex. 2.) Also we have

$$u_1a_2 - u_2a_1 \equiv -(a_1d_2). \quad \text{.....(E)}$$

Thus the equations are of the form $u_1 = 0$, $u_1 + k = 0$, where $k = (a_1d_2)/a_2$, and they are inconsistent unless $k = 0$.

(3) *In particular* it follows that the equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0,$$

are always consistent (i.e. they have solutions other than (0, 0, 0)). They are equivalent to $x/(b_1c_2) = y/(c_1a_2) = z/(a_1b_2)$,

provided that no denominator is zero.

If $(b_1c_2) = 0$, they are equivalent to $x = 0$, $y/(c_1a_2) = z/(a_1b_2)$.

If $(b_1c_2) = 0$ and $(c_1a_2) = 0$, then $x = 0$, $y = 0$, and z may have any value.

(4) If (α, β, γ) is any solution of (A), (B), these may be written

$$a_1(x-\alpha) + b_1(y-\beta) + c_1(z-\gamma) = 0, \quad a_2(x-\alpha) + b_2(y-\beta) + c_2(z-\gamma) = 0,$$

which are equivalent to

$$(x-\alpha)/(b_1c_2) = (y-\beta)/(c_1a_2) = (z-\gamma)/(a_1b_2), \quad \text{.....(F)}$$

provided that no denominator is zero.

If $(b_1c_2) = 0$, they are equivalent to

$$x = \alpha, \quad (y-\beta)/(c_1a_2) = (z-\gamma)/(a_1b_2). \quad \text{.....(G)}$$

If $(b_1c_2) = 0$ and $(c_1a_2) = 0$, then $x = \alpha$, $y = \beta$, and z may have any value.

NOTE. Much labour can be saved by the following considerations.

Equations (A), (B) are unchanged by the simultaneous cyclic substitutions $(a_1b_1c_1)$, $(a_2b_2c_2)$, (xyz). If then we make these changes in any equation derived from (A) and (B), the resulting equation will also hold. For example, equation (D) can be derived from (C) in this way.

7. The Equation to a Plane. (1) The equation $ax+by+cz+d=0$ represents a plane unless a, b, c are all zero.

For the sake of complete generality we say that the ' equation '

$$0x+0y+0z+d=0 \quad \text{where} \quad d\neq 0$$

represents an *ideal* plane called the *plane at infinity.*

(2) **Two Planes.** Referring to Art. 6, let $u_1=0$, $u_2=0$ be the equations to two planes. In general these intersect in a line, and we say that *the equations to the line* are $u_1=0$, $u_2=0$. The equations may be written in one of the forms described in Art. 6, (4), where (α, β, γ) is any point on the line.

The equation to any plane through the line $u_1=0$, $u_2=0$ is of the form $u_1+ku_2=0$, where k is a constant.

Parallel Planes. If (a_1b_2), (b_1c_2), (c_1a_2) are all zero, the planes $u_1=0$, $u_2=0$ do not intersect, that is to say they are parallel.

Hence the equation to any plane parallel to the plane $u_1=0$ is of the form $u_1+k=0$, where k is a constant. All this follows from Art. 6, (2).

We therefore say that *parallel planes intersect in a line on the plane at infinity.*

Ex. 1. *The condition that the line whose equations are*

$$(x-\alpha)/l=(y-\beta)/m=(z-y)/n \quad \text{.....(A)}$$

may be parallel to the plane

$$ax+by+cz+d=0$$

is $la+mb+nc=0$. *If also* $a\alpha+b\beta+c\gamma+d=0$, *then the line is in the plane.*

If the line meets the plane at (x, y, z) and each fraction in (A) is denoted by r, then

$$x=lr+\alpha, \quad y=mr+\beta, \quad z=nr+\gamma,$$

where $$r(la+mb+nc)+a\alpha+b\beta+c\gamma+d=0.$$

If $la+mb+nc=0$, no value of r exists which satisfies the last equation unless $a\alpha+b\beta+c\gamma+d=0$, in which case r may have any value. Hence the results follow.

Ex. 2. *The condition that the lines whose equations are*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'},$$

may meet in a point is $(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(nl'-n'l)+(\gamma-\gamma')(lm'-l'm)=0$, *and the lines are parallel if* $l/l'=m/m'=n/n'$.

If the lines meet at (x, y, z) and $(x-\alpha)/l=r$, $(x-\alpha')/l'=r'$, we have

$$lr-l'r'+(\alpha-\alpha')=0, \qquad mr-m'r'+(\beta-\beta')=0, \qquad nr-n'r'+(\gamma-\gamma')=0.$$

These equations are satisfied by the same (finite) values of r, r' if

$$\begin{vmatrix} \alpha-\alpha' & l & l' \\ \beta-\beta' & m & m' \\ \gamma-\gamma' & n & n' \end{vmatrix}=0,$$

provided that (mn'), (nl'), (lm') are not all zero: but if these three are all zero, the values of r, r' are infinite. In the latter case equations (A) are satisfied by infinite values of x, y, z and the lines are parallel.

8. Three Equations in Three Unknowns. Consider the equations

$$\left.\begin{aligned} u_1 &\equiv a_1x+b_1y+c_1z+d_1=0 \\ u_2 &\equiv a_2x+b_2y+c_2z+d_2=0 \\ u_3 &\equiv a_3x+b_3y+c_3z+d_3=0 \end{aligned}\right\}. \quad \text{.....(A)}$$

Let $\Delta=(a_1b_2c_3)$ and let A_1, B_1, ... be the cofactors of a_1, b_1, ... in Δ. Also let $\quad \Delta_1=(d_1b_2c_3), \quad \Delta_2=(a_1d_2c_3), \quad \Delta_3=(a_1b_2d_3).$

On account of the identities

$$\left.\begin{aligned} a_1A_1+a_2A_2+a_3A_3&=\Delta \\ b_1A_1+b_2A_2+b_3A_3&=0 \\ c_1A_1+c_2A_2+c_3A_3&=0 \\ d_1A_1+d_2A_2+d_3A_3&=\Delta_1 \end{aligned}\right\}, \quad \text{.....(B)}$$

we have $\quad u_1A_1+u_2A_2+u_3A_3\equiv\Delta x+\Delta_1;$

and similarly, $\quad u_1B_1+u_2B_2+u_3B_3\equiv\Delta y+\Delta_2,$

and $\quad u_1C_1+u_2C_2+u_3C_3\equiv\Delta z+\Delta_3.$ $\quad$ .....(C)

(1) *Suppose that* $\Delta\neq0$. The last identities show that *if a solution exists*, it is given by $\quad x=-\Delta_1/\Delta, \quad y=-\Delta_2/\Delta, \quad z=-\Delta_3/\Delta.$ .....(D)

We can show by actual substitution that these values of x, y, z satisfy the equations. Or we may use the following method, in which *every step is reversible, so that verification is unnecessary.*

Because $(A_1B_2C_3)=\Delta^2\neq0$, therefore at least one term of $(A_1B_2C_3)$ is not zero. Suppose that $A_1B_2C_3\neq0$, so that no one of A_1, B_2, C_3 is zero.*

Because $A_1\neq0$, the equations are equivalent to

$$u_2=0, \quad u_3=0, \quad A_1u_1+A_2u_2+A_3u_3=0,$$

that is to

$$u_2=0, \quad u_3=0, \quad \Delta x=-\Delta_1.$$

In the same way, because $B_2\neq0$ and $C_3\neq0$, the equations are equivalent to each of the sets, $u_3=0$, $u_1=0$, $\Delta y=-\Delta_2$; $\quad u_1=0$, $u_2=0$, $\Delta z=-\Delta_3$.

Hence the equations have the unique solution, $(-\Delta_1/\Delta, -\Delta_2/\Delta, -\Delta_3/\Delta)$.

* It is essential to show that suffixes p, q, r exist, all different, such that none of A_p, B_q, C_r is zero. This step has been omitted in the text-books. See Chrystal, *Algebra*, Vol. I, p. 360.

(2) *If* $\Delta = 0$ *and* $\Delta_1, \Delta_2, \Delta_3$ *are not all zero, the equations are inconsistent.*

For example, suppose that $\Delta_1 \neq 0$, then since

$$u_1A_1 + u_2A_2 + u_3A_3 \equiv \Delta_1, \quad \text{.....(E)}$$

it follows that the equations have no common solution.

If we suppose the coefficients to vary so that $\Delta \to 0$, and at least one of $\Delta_1, \Delta_2, \Delta_3$ remains finite, then at least one of x, y, z tends to infinity. Neglecting d_1, d_2, d_3 in comparison with the large terms, we see that in general the following ratios tend to equality :

$$x : y : z, \quad A_1 : B_1 : C_1, \quad A_2 : B_2 : C_2, \quad A_3 : B_3 : C_3.$$

In the limiting case *when* $\Delta = 0$, *we say that the equations have a solution* (x, y, z) *such that at least one of* x, y, z *is infinite and*

$$x : y : z = A_1 : B_1 : C_1 = A_2 : B_2 : C_2 = A_3 : B_3 : C_3.$$

(3) *If* $\Delta, \Delta_1, \Delta_2, \Delta_3$ *are all zero, the equations are not all independent and there are infinitely many solutions ; or else each of the equations (assumed to be distinct) is inconsistent with either of the others.*

For we have $u_1A_1 + u_2A_2 + u_3A_3 \equiv 0$. .....(F)

(i) *If no* A *is zero,* the equations are not independent.

(ii) *If* $A_1 = 0$, $A_2 \neq 0$, $A_3 \neq 0$, then $u_2A_2 + u_3A_3 \equiv 0$; and $u_2 = 0, u_3 = 0$ are the same equation.

In both cases, since $A_3 \neq 0$, the equations $u_1 = 0$, $u_2 = 0$ have infinitely many solutions which are solutions of the system.

(iii) *If* $A_1 = 0$, $A_2 = 0$, $A_3 \neq 0$, we have $u_3 \equiv 0$, which case is excluded.

(iv) If, in the above, every A is replaced by the corresponding B or C, similar results follow. Therefore the only remaining case is that in which *every minor of* Δ *is zero,* and then every two of the equations are inconsistent ; or they are one and the same.

Ex. 1. *If* a, b, c *are unequal, solve the equations* $x + y + z = 1$, $ax + by + cz = d$, $a^2x + b^2y + c^2z = d^2$.

We have $$x \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix}, \quad \therefore x = \frac{(d-b)(d-c)}{(a-b)(a-c)},$$

and the values of y, z are obtained by the cyclic substitutions (abc), (xyz).

9. Three Planes. Referring to 8, let $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ be the equations to three planes, then :

(i) *If* $\Delta \neq 0$ *they meet in the point given by the equations in Art.* 8, (1).

(ii) *If* $\Delta = 0$ *and* $\Delta_1, \Delta_2, \Delta_3$ *are not all zero, two of the planes meet in a line which is parallel to the third plane.*

(iii) *If* Δ, Δ_1, Δ_2, Δ_3 *are all zero, the planes have a common line of intersection or they are all parallel.*

Proof of (ii). Suppose that $\Delta_1 \neq 0$. We have

$$d_1A_1 + d_2A_2 + d_3A_3 = \Delta_1, \quad \text{.....(A)}$$

$$u_1A_1 + u_2A_2 + u_3A_3 \equiv \Delta_1. \quad \text{.....(B)}$$

On account of (**A**), at least one A is not zero. Let $A_1 \neq 0$, then the planes $u_2 = 0$, $u_3 = 0$ are not parallel and, on account of (**B**), their line of intersection does not meet the plane $u_1 = 0$.

Observe that all the planes are parallel to the line

$$x/A_1 = y/B_1 = z/C_1,$$

which is parallel to the line, $u_2 = 0$, $u_3 = 0$.

The truth of (iii) follows from Art. 8, (3).

Hence we conclude that, *if* $\Delta = 0$, *the planes either meet in a point at infinity, or else they have a common line of intersection, which may be at infinity.*

10. Four Equations in Three Unknowns. Consider the equations

$$u_1 \equiv a_1x + b_1y + c_1z + d_1 = 0,$$
$$u_2 \equiv a_2x + b_2y + c_2z + d_2 = 0,$$
$$u_3 \equiv a_3x + b_3y + c_3z + d_3 = 0,$$
$$u_4 \equiv a_4x + b_4y + c_4z + d_4 = 0.$$

Let $\Delta = (a_1b_2c_3d_4)$ and let A_1, B_1, ... be the cofactors of a_1, b_1 ... .

(1) *If the equations are consistent, then* $\Delta = 0$.

For, as in Art. 4, (1), we have

$$u_1D_1 + u_2D_2 + u_3D_3 + u_4D_4 \equiv \Delta.$$

(2) *If* $\Delta = 0$, *the equations are consistent and have an unique solution except when every* D *is zero.*

This follows from the identity

$$u_1D_1 + u_2D_2 + u_3D_3 + u_4D_4 \equiv 0.$$

(3) *If every* D *is zero, the equations are inconsistent, but they are to be regarded as having a common infinite solution, unless every minor of* Δ *is zero, in which case there are infinitely many solutions.*

This is explained with reference to four planes in Exercise XVII, 17.

The methods already described apply to *systems of linear equations connecting more than three variables*, and lead to similar results.

EXERCISE XVII

SIMULTANEOUS LINEAR EQUATIONS

1. Solve by means of determinants :

(i) $x-y+z=0$, $2x+3y-5z=7$, $3x-4y-2z=-1$.

(ii) $x-y+z-w=-1$, $3x+2y+4z+2w=1$, $2x+4y-5z=-1$, $3y-2z+3w=5$.

2. With regard to the equations

$$4x+7y-14z=10, \quad 2x+3y-4z=-4, \quad x+y+z=6,$$

show that the first can be expressed in the form

$$l(2x+3y-4z)+m(x+y+z)=n,$$

and find $l : m : n$. Hence show that the equations are inconsistent.

3. With regard to the equations

$$4x+7y-14z=-24, \quad 2x+3y-4z=-4, \quad x+y+z=6,$$

show that the first can be put in the form

$$l(2x+3y-4z+4)+m(x+y+z-6)=0,$$

and find $l : m$. Hence show that the equations are not independent.

4. If x, y, z are not all zero and

$$ax+by+cz=bx+cy+az=cx+ay+bz=0,$$

prove that

$$a^3+b^3+c^3+3abc=0$$

5. Having given the equations

$$x=cy+bz, \quad y=az+cx, \quad z=bx+ay$$

where x, y, z are not all zero, prove that

$$a^2+b^2+c^2+2abc=1$$

and

$$x^2/(1-a^2)=y^2/(1-b^2)=z^2/(1-c^2).$$

6. Having given the equations

$$a=b\cos C+c\cos B, \quad b=c\cos A+a\cos C, \quad c=a\cos B+b\cos A,$$

deduce the following :

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$

$$2bc\cos A=b^2+c^2-a^2, \quad a/\sin A=b/\sin B=c/\sin C.$$

7. Having given that

$$x+y+z=1, \quad ax+by+cz=d, \quad a^2x+b^2y+c^2z=d^2,$$

prove that

$$a^3x+b^3y+c^3z=d^3-(d-a)(d-b)(d-c)$$

and

$$a^4x+b^4y+c^4z=d^4-(d-a)(d-b)(d-c)(a+b+c+d).$$

8. If the equations

$$\frac{cy+bz}{b-c}=\frac{az+cx}{c-a}=\frac{bx+ay}{a-b}=\frac{ax+by+cz}{a+b+c}$$

are consistent, prove that either

$$a+b+c=0 \quad \text{or} \quad (b-c)(c-a)(a-b)=abc.$$

[Put each fraction equal to k and eliminate x, y, z, k.]

9. If the three equations of Art. 8 are distinct and if $\Delta=0$ and $\Delta_1=0$, show that $\Delta_2=0$ and $\Delta_3=0$ unless $b_1:c_1=b_2:c_2=b_3:c_3$.

[Since $\Delta=0$, $A_1:A_2:A_3=B_1:B_2:B_3$, and since $d_1A_1+d_2A_2+d_3A_3=\Delta_1=0$, therefore $\Delta_2=d_1B_1+d_2B_2+d_3B_3=0$ unless A_1, A_2, A_3 are all zero.]

10. If the equations $x=by+cz+du$, $y=cz+ax+du$,

$z=ax+by+du$, $u=ax+by+cz$

are consistent and no one of a, b, c, d is equal to -1, then

$$a/(a+1)+b/(b+1)+c/(c+1)+d/(d+1)=1.$$

Also if $a=-1$, then one of b, c, d is equal to -1. [Ex. XVI, 10.]

11. Using the notation of Art. 8, show that if $u_1=0$, $u_2=0$, $u_3=0$ are the equations to three straight lines, the coordinates of the vertices of the triangle formed by them are $\left(\frac{A_1}{C_1}, \frac{B_1}{C_1}\right)$, $\left(\frac{A_2}{C_2}, \frac{B_2}{C_2}\right)$, $\left(\frac{A_3}{C_3}, \frac{B_3}{C_3}\right)$, and that the area of the triangle is $\frac{1}{2}\Delta^2/C_1C_2C_3$.

12. Prove that if $c_1\neq0$, the equations $u_1=0$, $u_2=0$, $u_3=0$ of Art. 8 are equivalent to $u_1=0$, $B_3x-A_3y=(d_1c_2)$, $-B_2x+A_2y=(d_1c_3)$;

and if $\Delta\neq0$ deduce the solution in the form already given.

13. Show that the condition that the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) may be in the same plane is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$$

14. Using the notation of Art. 10, consider the lines whose equations are $u_1=0$, $u_2=0$ and $u_3=0$, $u_4=0$. It is required to find the equation to a plane which passes through either of these and is parallel to the other.

[We have $u_1D_1+u_2D_2+u_3D_3+u_4D_4\equiv\Delta$, therefore the required planes are $u_1D_1+u_2D_2=0$ and $u_3D_3+u_4D_4=0$.]

15. Consider the lines whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}.$$

Show that the equation to the plane which contains the first line and is parallel to the second is

$$(x-\alpha)(mn'-m'n)+(y-\beta)(nl'-n'l)+(z-\gamma)(lm'-l'm)=0.$$

Deduce the condition that the lines may meet in a point or be parallel.

[The required equation is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$ where $al+bm+cn=0$ and $al'+bm'+cn'=0$.]

16. Using the notation of Art. 10, show that if $u_1=0$, $u_2=0$, $u_3=0$, $u_4=0$ are the equations to four planes, the vertices of the tetrahedron bounded by them are the points $\left(\frac{A_1}{D_1}, \frac{B_1}{D_1}, \frac{C_1}{D_1}\right)$, $\left(\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right)$, etc., and that the volume of the tetrahedron is $\frac{1}{6}\Delta^3/D_1D_2D_3D_4$.

[*It is shown in works on Coordinate Geometry that the volume of the tetrahedron whose vertices are* (x_1, y_1, z_1), *etc., is one-sixth of the determinant in Ex.* 13.]

17. Using the notation of Art. 10, let $u_1=0$, $u_2=0$, $u_3=0$, $u_4=0$ be the equations to four planes, and suppose that $\Delta=0$, then

(i) If at least one D, say D_1, is not zero, the planes meet at the point

$$(-A_1/D_1,\quad -B_1/D_1,\quad -c_1/D_1).$$

(ii) Suppose that every D is zero, then

(α) If all the minors of Δ are not zero, the four planes are parallel to a certain line, that is to say, they have a common point at infinity.

(β) If every minor of Δ is zero, the planes have a common line of intersection or they are all parallel.

For example, if $A_4\neq 0$ and we consider the planes u_1, u_2, u_3. Since $D_4=0$, two of these, say u_1, u_2, meet in a line which is parallel to the third.

Also, considering the planes, u_1, u_2, u_4, since $D_3=0$ the plane u_4 contains or is parallel to the line $u_1=0$, $u_2=0$. Hence all the planes are parallel to the line

$$x/(b_1c_2)=y/(c_1a_2)=z/(a_1b_2).$$

18. Given the equations

$$a_2c_3+a_3c_2-2b_2b_3=0,\quad a_3c_1+a_1c_3-2b_1b_3=0,\quad a_1c_2+a_2c_1-2b_1b_2=0,$$

prove that

$$\text{(i)}\quad \frac{a_2b_3-a_3b_2}{a_1(a_1c_1-b_1^2)}=\frac{a_3b_1-a_1b_3}{a_2(a_2c_2-b_1^2)}=\frac{a_1b_2-a_2b_1}{a_3(a_3c_3-b_3^2)}.$$

$$\text{(ii)}\quad \frac{a_1^2}{a_1c_1-b_1^2}+\frac{a_2^2}{a_2c_2-b_2^2}+\frac{a_3^2}{a_3c_3-b_3^2}=0.$$

[Prove that $\dfrac{a_1}{(a_2b_3)}=\dfrac{-2b_1}{(c_2a_3)}=\dfrac{c_1}{(b_2c_3)}=\dfrac{2(a_1c_1-b_1^2)}{(a_1b_2c_3)}$.]

19. If $x=\dfrac{l_1X+m_1Y+n_1}{l_3X+m_3Y+n_3}$, $\quad y=\dfrac{l_2X+m_2Y+n_2}{l_3X+m_3Y+n_3}$,

prove that $X=\dfrac{L_1x+L_2y+L_3}{N_1x+N_2y+N_3}$, $\quad Y=\dfrac{M_1x+M_2y+M_3}{N_1x+N_2y+N_3}$,

where L_1, M_1, etc., are the cofactors of l_1, m_1, etc., in the determinant $(l_1m_2n_3)$.

20. If x, y are connected with X, Y by the equations of Ex. 19, and X, Y are given by

$$N_1X-N_2Y=L_1-M_2,\quad N_2X+N_1Y=L_2+M_1,$$

prove that the corresponding values of x, y are given by

$$l_3x-m_3y=l_1-m_2,\quad m_3x+l_3y=m_1+l_2.$$

11. Systems of Equations of any Degree. The question of elimination is considered in detail in another volume. Here we illustrate various methods of dealing with systems of special types.

(1) *Systems which are symmetrical with regard to x, y, ...*

Ex. 1. *Find the rational solutions of*

$$x^2+y^2+z^2=50,\quad (y+z)(z+x)(x+y)=14,$$
$$x^3+y^3+z^3-3xyz=146.$$

If $\Sigma x=p$, $\Sigma yz=q$, $xyz=r$, the given equations are equivalent to

$$p^2-2q=50,\quad pq-r=14,\quad p^3-3pq=146;$$

whence we find that $p^3-150p+292=0$, giving $p=2$ or $-1\pm\sqrt{147}$.

If $p=2$, then $q=-23$, $r=-60$ and x, y, z are the roots of

$$\theta^3-2\theta^2-23\theta+60=0.$$

Thus x, y, z may have the values 3, 4, -5 taken in any order, and there is no other rational solution.

(2) *A solution may sometimes be discovered by considering the properties of a particular determinant.*

Ex. 2. *Solve*

$$x^2-yz=a, \quad y^2-zx=b, \quad z^2-xy=c.$$

Consider the determinant

$$\Delta=\begin{vmatrix} x & z & y \\ z & y & x \\ y & x & z \end{vmatrix}=3xyz-x^3-y^3-z^3.$$

The cofactors of x, y, z, in the top line, are $yz-x^2$, $zx-y^2$, $xy-z^2$;

$$\therefore\ (zx-y^2)(xy-z^2)-(yz-x^2)^2=\Delta x.$$

Hence *if* (x, y, z) *is a solution,*

$$bc-a^2=\Delta x, \text{ and similarly, } ca-b^2=\Delta y, \quad ab-c^2=\Delta z.$$

Multiplying these by a, b, c, and adding,

$$3abc-a^3-b^3-c^3=\Delta(ax+by+cz)=-\Delta^2.$$

Therefore the only possible solutions are given by

$$\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}=\pm\frac{1}{\sqrt{(a^3+b^3+c^3-3abc)}}.$$

It is easily verified by substitution that these are solutions unless $a^3+b^3+c^3=3abc$, in which there is no solution unless a, b, c are all zero.

(3) *A change of the variables.*

Ex. 3. *Solve*

$$x(x-a)=yz, \quad y(y-b)=zx, \quad z(z-c)=xy.$$

Let $x=1/X$, $y=1/Y$, $z=1/Z$, then the equations are equivalent to

$$\frac{X^2-YZ}{a}=\frac{Y^2-ZX}{b}=\frac{Z^2-XY}{c}=-XYZ.$$

Therefore, as in Ex. 2,

$$X/A=Y/B=Z/C \quad \text{or} \quad Ax=By=Cz=k,$$

where $A=a^2-bc$, $B=b^2-ca$, $C=c^2-ab$, and k is to be determined. Substituting in the first equation

$$\frac{k}{A}\left(\frac{k}{A}-a\right)=\frac{k^2}{BC},$$

therefore, either $k=0$, giving the solution $(0, 0, 0)$,

or $k(BC-A^2)=aABC$, giving $k(a^3+b^3+c^3-3abc)=-ABC$;

$$\therefore\ x=-(b^3-ca)(c^2-ab)/(a^3+b^3+c^3-3abc),$$

with similar values for y, z.

(4) *In the case of two variables x, y the substitution $y=tx$ is sometimes useful. In equations involving x, y, z, we may write $y=tx$, $z=t'x$.*

Ex. 4. *Solve*

$$ax^2+bxy+cy^2=lx+my, \quad a'x^2+b'xy+c'y^2=l'x+m'y.$$

One solution is (0, 0), and if $y=tx$ the equations become

$$x^2(a+bt+ct^2)=x(l+mt), \quad x^2(a'+b't+c't^2)=x(l'+m't).$$

Hence if $x\neq0$, the solutions are given by

$$x=(l+mt)/(a+bt+ct^2), \quad y=tx,$$

where

$$(a+bt+ct^2)(l'+m't)=(a'+b't+c't^2)(l+mt).$$

If $x=0$ *and* $y\neq0$, the equations become $cy^2=my$, $c'y^2=m'y$, $\therefore\ cm'-c'm=0$ and $y=-m/c$. In this one value of t is infinite.

(5) *If a solution $x=x_1$, $y=y_1$, ... can be guessed, then a suitable substitution is*

$$x=x_1+X, \quad y=y_1+Y, \ \ldots\ .$$

Ex. 5. *Solve*

$$(y-c)(z-b)=a^2, \quad (z-a)(x-c)=b^2, \quad (x-b)(y-a)=c^2.$$

One solution is $(b+c, c+a, a+b)$, and if we write

$$x=b+c+X, \quad y=c+a+Y, \quad z=a+b+Z,$$

the equations become

$$YZ+a(Y+Z)=0, \quad ZX+b(Z+X)=0, \quad XY+c(X+Y)=0.$$

Hence *if none of the three, X, Y, Z, is zero*, we have

$$\frac{1}{Y}+\frac{1}{Z}=-\frac{1}{a}, \quad \frac{1}{Z}+\frac{1}{X}=-\frac{1}{b}, \quad \frac{1}{X}+\frac{1}{Y}=-\frac{1}{c};$$

$$\therefore\ \frac{2}{X}=\frac{1}{a}-\frac{1}{b}-\frac{1}{c},$$

with similar equations. This gives the only solution, beside the obvious one.

Special cases. If $X=0$, $Y=0$ and $Z\neq0$, then $a=0$, $b=0$ and Z may have any value.

If $X=0$, $Y\neq0$, $Z\neq0$, then $b=0$, $c=0$ and Y, Z may have any values such that $YZ+a(Y+Z)=0$.

Ex. 6. *The eight solutions of*

$$x^2+2yz=a, \quad y^2+2zx=b, \quad z^2+2xy=c, \ \ldots\ldots\ldots\

Adding the second and third of equations (**A**) and subtracting the first, we easily find that

$$(y-z)^2+x(k+X)=b+c-a\,;$$

$$\therefore\ \left(\frac{b-c}{X}\right)^2+\frac{k^2-X^2}{3}=b+c-a\,;$$

$$\therefore\ X^4-2X^2(2a-b-c)-3(b-c)^2=0,$$

and solving for X^2, we obtain the values stated in the question.

Another solution. Equations (**A**) are equivalent to

$$(x+y+z)^2=a+b+c=A \text{ (say)},$$

$$(x+\omega^2y+\omega z)^2=a+\omega b+\omega^2c=B,$$

$$(x+\omega y+\omega^2z)^2=a+\omega^2b+\omega c=C,$$

where ω is an imaginary cube root of 1.

Therefore the eight solutions are given by

$$x+y+z=\surd A,\quad x+\omega^2y+\omega z=\surd B,\quad x+\omega y+\omega^2z=\surd C,\ \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{B})$$

where $\surd A$ is either square root of A and so for $\surd B$, $\surd C$.

From equations (**B**), by addition,

$$3x=\surd A+X,$$

where $X=\surd B+\surd C$, and therefore

$$X^2=B+C+2\surd B\,.\,\surd C$$

$$=2a-b-c\pm\surd(a^2+b^2+c^2-bc-ca-ab).$$

Also, from the second and third of equations (**B**),

$$2x-(y+z)=X,\quad \therefore\ y+z=\tfrac{1}{3}(2\surd A-X),$$

and

$$(\omega^2-\omega)(y-z)=\surd C-\surd B=\frac{C-B}{X}=\frac{(\omega^2-\omega)(c-b)}{X}\,;$$

$$\therefore\ y-z=-(b-c)/X.$$

These results agree with the preceding if we change the sign of X, which is permissible.

Ex. 7. *Show that, if the equations*

$$by^2+cz^2=2fyz,\quad cz^2+ax^2=2gzx,\quad ax^2+by^2=2hxy\ \ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

are consistent, then

$$abc+2fgh-af^2-bg^2-ch^2=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{B})$$

From equations (**A**), by multiplication,

$$ax^2(b^2y^4+c^2z^4)+\ldots+\ldots+2abcx^2y^2z^2=8fghx^2y^2z^2.\ \ldots\ldots\ldots\ldots\ldots(\mathbf{C})$$

Also

$$b^2y^4+c^2z^4=(4f^2-2bc)\,y^2z^2,$$

with two similar equations. Substituting in (**C**) and dividing by $x^2y^2z^2$ (assumed not to be zero), the result (**B**) follows.

Special cases. Considering the original equations it will be seen that

(i) If $x=0$, $y\neq0$, $z\neq0$, then $cz^2=0$, and $c=0$;

hence the equations become, $by^2=2fyz$, $by^2=0$, and we have $b=0$, and $f=0$.

(ii) If $x=0$, $y=0$ and $z\neq0$, the equations reduce to the single equation $cz^2=0$; and thus do not constitute a system of equations.

Thus, if the equations (**A**) form a system of equations with a solution other than (0, 0, 0), then the condition (**B**) holds.

Ex. 8. *Prove that, if*

$$ax^2+by^2+cz^2-(b+c)yz-(c+a)zx-(a+b)xy=0$$

and

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

then, if no two of a, b, c *are equal,*

$$\frac{x}{(b-c)^2}=\frac{y}{(c-a)^2}=\frac{z}{(a-b)^2}=\frac{xyz}{a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy}.$$

The given equations can be written

$$x(-ax+by+cz)+y(ax-by+cz)+z(ax+by-cz)=0,$$

$$x(-x+y+z)+y(x-y+z)+z(x+y-z)=0.$$

Multiplying these by $(x+y-z)$, $(ax+by-cz)$ respectively and subtracting, we find that

$$xy\{y(a-b)+z(c-a)\}=xy\{z(b-c)+x(a-b)\};$$

and two similar equations can be obtained. Hence, *assuming that none of the three,* z, y, z, *is zero,* we may write

$$y(a-b)+z(c-a)=k,\quad z(b-c)+x(a-b)=k,\quad x(c-a)+y(b-c)=k.$$

Hence, *if no two of* a, b, c *are equal,* $x=-\dfrac{k(b-c)}{(c-a)(a-b)}$,

$$\therefore\ \frac{x}{(b-c)^2}=\frac{y}{(c-a)^2}=\frac{z}{(a-b)^2}=\lambda\ (\text{say});$$

whence also $\lambda=\dfrac{-ax+by+cz}{-a(b-c)^2+b(c-a)^2+c(a-b)^2}=\dfrac{ax-by-cz}{(b+c)(c-a)(a-b)}$;

$$\therefore\ \Sigma ax(ax-by-cz)=\lambda^2(b-c)(c-a)(a-b)\,.\,\Sigma a(b^2-c^2)$$

$$=\lambda^2(b-c)^2(c-a)^2(a-b)^2=xyz/\lambda,$$

and the result in question follows.

The consideration of special cases is left to the reader.

EXERCISE XVIII

1. If (α, β), (α', β') are the roots of

$$u=ax^2+bx+c=0,\quad u'=a'x^2+b'x+c'=0,$$

prove that

$$a^2a'^2(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta')=(ac'-a'c)^2-(ab'-a'b)(bc'-b'c),$$

and consequently *the necessary and sufficient condition that the equations have a common root is* $R=0$, *where*

$$R=(ac'-a'c)^2-(ab'-a'b)(bc'-b'c).$$

2. Eliminate x, y from the equations

$$ax^2+bx+c=0,\quad a'y^2+b'y+c'=0,\quad xy=z,$$

and show that the roots of the resulting equation in z are $\alpha\alpha'$, $\alpha\beta'$, $\beta\alpha'$, $\beta\beta'$, where (α, β), (α', β') are the roots of

$$ax^2+bx+c=0,\quad a'x^2+b'x+c'=0.$$

Show that this equation is

$$a^2a'^2z^4-aa'bb'z^3+(b^2a'c'+b'^2ac-2aca'c')z^2-bcb'c'z+c^2c'^2=0.$$

Prove also that the equation whose roots are α/α', α/β', β/α', β/β' is obtained from the last equation by interchanging a' and c'.

Solve the equations in Exx. 3-28.

3. $x^3+y^3=8, \quad xy+(x+y)=2.$ [Put $x+y=\lambda$, $xy=\mu$.]

4. $x^2y+xy^2+x+y=3, \quad 2(x-1)(y-1)=-1.$

5. $x(1-y^2)=2y, \quad y(1-x^2)=2x.$

6. $x+y+z=9, \quad x^2+y^2+z^2=29, \quad x^{-1}+y^{-1}+z^{-1}=\frac{13}{12}.$

7. $(y+z)(z+x)(x+y)=\frac{45}{4}, \quad (x+1)(y+1)(z+1)=9, \quad xyz=1.$

8. $\dfrac{x^2}{3}+\dfrac{y^2}{5}=\dfrac{9}{x}+\dfrac{25}{y}=8.$

9. $yz+2y+4z=2zx+4z+x=4xy+x+2y=7.$

10. $\dfrac{x^2}{y}=16-xy, \quad \dfrac{y^2}{x}=9-xy.$

11. $\dfrac{x^4+3}{y^2}=\dfrac{x^2y^2-6}{xy}=\dfrac{y^4+3}{x^2}.$

12. $\dfrac{x^2}{a-y}+\dfrac{y^2}{a-x}=a, \quad \dfrac{x}{a^2-y^2}+\dfrac{y}{a^2-x^2}=\dfrac{1}{a}.$

13. $(1-x)^2+(1-y)^2=(1-x)(1-y)(1-xy), \quad ay(1-x)^2+bx(1-y)^2=0.$
[Put $x=1-\lambda$, $y=1-\mu$ and prove that $a\lambda^2-\lambda(a-b)+a-b=0$ and $b\mu^2-\mu(b-a)+b-a=0.$]

14. $y^2+z^2=a+y+z, \quad z^2+x^2=b+z+x, \quad x^2+y^2=c+x+y.$

15. $yz+zx+xy=a^2-x^2=b^2-y^2=c^2-z^2.$
[$(x+y)(x+z)=a^2$, etc.]

16. $x(y+z-x)=a^2, \quad y(z+x-y)=b^2, \quad z(x+y-z)=c^2.$

17. $x-\dfrac{1}{y}=a-\dfrac{1}{b}, \quad y-\dfrac{1}{z}=b-\dfrac{1}{c}, \quad z-\dfrac{1}{x}=c-\dfrac{1}{a}.$
[Put $x=a+\lambda$, $y=b+\mu$, $z=c+\nu$.]

18. $ayz+by+cz=bzx+cz+ax=cxy+ax+by=a+b+c.$

19. $x^2+y+z=y^2+z+x=z^2+x+y=3.$

20. $\dfrac{x}{y+z+1}=\dfrac{y}{z+x}=\dfrac{z}{x+y-1}=x+y+z.$

21. $x^2+(y-z)^2=a^2, \quad y^2+(z-x)^2=b^2, \quad z^2+(x-y)^2=c^2.$

22. $x(bc-xy)=y(xy-ac),$
$xy(ay+bx-xy)=abc(x+y-c).$

23. $(x-y)(x-z)=ax, \quad (y-z)(y-x)=by, \quad (z-x)(z-y)=cz.$

24. $yz(a-x)=ax^2, \quad zx(b-y)=by^2, \quad xy(c-z)=cz^2.$

25. $3(x+yz)=4(y+zx)=5(z+xy), \quad 8x-9y=1.$

26. $u+v=2, \quad ux+vy=-1, \quad ux^2+vy^2=11, \quad ux^3+vy^3=5.$

27. $x+y=x'+y'=2a, \quad \dfrac{1}{x}+\dfrac{1}{y'}=\dfrac{2}{y}, \quad \dfrac{1}{y}+\dfrac{1}{x'}=\dfrac{2}{x}.$

28. $\sqrt{x+1}+\sqrt{y-2}=\sqrt{x-1}+\sqrt{y+2}=3.$

29. If $\dfrac{3x+1}{5x-3}=\dfrac{2y+1}{5y-2}=\dfrac{x+y+5}{8}$, prove that $y+1=\pm x$, and solve the equations.

30. If $\dfrac{x^2}{a}+\dfrac{y^2}{b}=\dfrac{a^2}{x}+\dfrac{b^2}{y}=a+b$, show that either $\dfrac{x}{a}+\dfrac{y}{b}+1=0$ or $x+y=a+b$.

[Put $x=aX$, $y=bY$, and show $(X+Y+1)(aX+bY-a-b)=0$.]

31. Eliminate x, y, z from

$$\frac{x}{a}(y+z-x)=\frac{y}{b}(z+x-y)=\frac{z}{c}(x+y-z) \text{ and } ax+by+cz=0.$$

32. If $x^3-y^4=y^3-x^4=a^4$ and $x+y=-a$, and x, y are unequal, prove that
(i) $xy=a^2(a-1)/(2a-1)$, (ii) $3a^3-3a^2+3a-1=0$, (iii) $3a^2(x^2+y^2)=-1$,
(iv) $3a(x^3+y^3)=2-a$, and (v) $3a(x^4+y^4+1)=2+4a^2$.

33. If
$$y^2z^2+(y+z)^2+\lambda yz=1+\lambda,$$
$$z^2x^2+(z+x)^2+\lambda zx=1+\lambda \text{ and } x\neq y,$$
then will
$$x^2y^2+(x+y)^2+\lambda xy=1+\lambda.$$

34. If x, y, z satisfy the equations

$$\frac{1}{x}+\frac{1}{y+z}=\frac{1}{a},\quad \frac{1}{y}+\frac{1}{z+x}=\frac{1}{b},\quad \frac{1}{z}+\frac{1}{x+y}=\frac{1}{c},$$

prove that $x(b+c-a)=y(c+a-b)=z(a+b-c)$: and hence solve the equations.

35. If $ax^2+by^2+cz^2=0$, $ayz+bzx+cxy=0$ and $x^3+y^3+z^3+\lambda xyz=0$, prove that $a^3+b^3+c^3+\lambda abc=0$.

36. If $a+b+c=0$ and $x+y+z=0$, prove that

$$\Sigma a^2x^2-\Sigma bcyz=\tfrac{1}{4}\Sigma a^2\,.\,\Sigma x^2.$$

[Use the identities $(by+cz-ax)^2=(bz+cy)^2$, etc.]

37. Given that

$$ax(y+z-x)+by(z+x-y)+cz(x+y-z)=0$$
and
$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0,$$

show that
$$\frac{x}{a(b-c)^2}=\frac{y}{b(c-a)^2}=\frac{z}{c(a-b)^2}.$$

38. Eliminate x, y, z from

$$(z+x-y)(x+y-z)=ayz,$$
$$(x+y-z)(y+z-x)=bzx,$$
$$(y+z-x)(z+x-y)=cxy,$$

showing that the result is

$$abc=(a+b+c-4)^2.$$

39. Eliminate x, y, z from

$$x+y-z=a,\quad x^2+y^2-z^2=b^2,\quad x^3+y^3-z^3=c^3,\quad xyz=d^3.$$

40. Eliminate x and y from

$$x^2+xy=a^2,\quad y^2+xy=b^2,\quad x^2+y^2=c^2.$$

41. If $x^2+y^2+z^2=1$, $x+2y+3z=3$, $3x+y+z=3$, prove that

$$x-8y+5z=\pm 3.$$

42. Show that the equations

$$x+y+z=0, \quad \frac{x^2}{32}-\frac{y^2}{25}-\frac{z^2}{63}=0, \quad 32yz-25zx-63xy=0$$

are consistent, and find $x : y : z$.

43. If x, y, z are not all zero and

$$x^2-ayz=y^2-bzx=z^2-cxy=\tfrac{1}{2}(x^2+y^2+z^2),$$

prove that $a^2+b^2+c^2=2abc+1$ and $x^2/(1-a^2)=y^2/(1-b^2)=z^2/(1-c^2)$.

44. Show that if $x=a, y=b, z=c$ is a solution of

$$px^3+qy^3+rz^3=sxyz,$$

then

$$x=a(qb^3-rc^3), \quad y=b(rc^3-pa^3), \quad z=c(pa^3-qb^3)$$

is another solution.

45. If $\frac{x}{l}(y+z-x)=\frac{y}{m}(z+x-y)=\frac{z}{n}(x+y-z)$ and $l+m+n=0$, show that

$$x/l^2=y/m^2=z/n^2.$$

46. Eliminate x, y from $(b-x)(c-y)=a^2$, $(c-x)(a-y)=b^2$, $(a-x)(b-y)=c^2$.

47. If the four expressions

$$x^2+y^2+z^2+m(yz+zx+xy), \qquad y^2+z^2+u^2+m(zu+uy+yz),$$
$$z^2+u^2+x^2+m(ux+xz+zu), \qquad u^2+x^2+y^2+m(xy+yu+ux),$$

are equal, prove that each is equal to $\frac{1}{2}(x^2+y^2+z^2+u^2)$.

48. If $x_1, x_2, \ldots x_5$ are all different and

$$\frac{x_1^2+x_2^2}{x_1x_2}=\frac{x_2^2+x_3^2}{x_2x_3}=\frac{x_3^2+x_4^2}{x_3x_4}=\frac{x_4^2+x_5^2}{x_4x_5}=k,$$

prove that $x_1^2+x_5^2=(k^4-4k^2+2)x_1x_5$.

49. (i) Eliminate x, y, z, t from the equations

$$ax+by+cz+dt=0, \qquad a'x+b'y+c'z+d't=0,$$
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{t}=0, \qquad \frac{a'}{x}+\frac{b'}{y}+\frac{c'}{z}+\frac{d'}{t}=0,$$

showing that the result is

$$LMN+Lmn+Mnl+Nlm=0,$$

where

$$l=(bc'), \quad m=(ca'), \quad n=(ab'),$$
$$L=(ad'), \quad M=(bd'), \quad N=(cd').$$

(ii) Use this to prove Exercise XVI, 25, (iii).

[(i) Prove that $\frac{l}{(x^2-t^2)yz}=\ldots=\ldots=\frac{L}{(y^2-z^2)xt}=\ldots=\ldots,$

the unwritten fractions being obtained by the cyclic substitutions (xyz), (lmn), (LMN). Now put x^2, y^2, z^2, t^2 for x, y, z, t in Ex. XVI, 25, (i).

(ii) Let $x=\cos\alpha+\iota\sin\alpha$, $y=\cos\beta+\iota\sin\beta$, etc., where $\alpha, \beta, \gamma, \delta$ are the angles which the sides a, b, c, d make with a fixed line. By projecting along and perpendicular to this line, prove that the four given equations hold.]

CHAPTER XI

RECIPROCAL AND BINOMIAL EQUATIONS

1. Reciprocal Equations. A *reciprocal equation* is an equation which possesses the following property: *If α is any root, then $1/\alpha$ is also a root.*

Let $\alpha, \beta, \ldots$ be the roots of

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

then $1/\alpha, 1/\beta, \ldots$ are the roots of

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0.$$

Hence (**A**) is a reciprocal equation if and only if

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \ldots = \frac{a_{n-1}}{a_1} = \frac{a_n}{a_0}.$$

Denoting each of these fractions by k, we have

$$k^2 = \frac{a_0}{a_n} \cdot \frac{a_n}{a_0} = 1,$$

so that $k = \pm 1$.

We shall say that *Reciprocal Equations are of the first or second types according as $k = +1$ or $k = -1$, that is according as $a_r = a_{n-r}$ or $a_r = -a_{n-r}$, for $r = 0, 1, 2, \ldots n$.*

Theorem 1. *The solution of any reciprocal equation depends on that of a reciprocal equation of the first type and of even degree.*

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ be a reciprocal equation. *If this is of the first type,* $x^n f\left(\frac{1}{x}\right) = f(x)$.

Let n be odd and equal to $2m+1$. Then the equation may be written

$$f(x) = a_0(x^{2m+1} + 1) + a_1x(x^{2m-1} + 1) + \ldots + a_mx^m(x+1) = 0.$$

Therefore $f(x)$ is divisible by $x+1$, and if $\phi(x)$ is the quotient,

$$x^{2m}\phi\left(\frac{1}{x}\right) = x^{2m}\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x}+1} = x^{2m+1} \cdot \frac{f\left(\frac{1}{x}\right)}{1+x} = \frac{f(x)}{x+1} = \phi(x);$$

hence $\phi(x) = 0$ is a reciprocal equation of the first type of degree $2m$.

If $f(x)=0$ *is of the second type,* $x^n f\left(\frac{1}{x}\right)=-f(x)$.

Let $n=2m+1$. Grouping the terms as before, we can show that $f(x)$ is divisible by $x-1$, and that, if $\phi(x)$ is the quotient, then $x^{2m}\phi\left(\frac{1}{x}\right)=\phi(x)$: with the same result as before.

Again, if $n=2m$, since $a_r=-a_{2m-r}$, it follows that $a_m=-a_m$, and therefore $a_m=0$. Hence the equation may be written

$$f(x)=a_0(x^{2m}-1)+a_1x(x^{2m-2}-1)+\ldots+a_{m-1}x^{m-1}(x^2-1)=0.$$

Therefore $f(x)$ is divisible by x^2-1, and if $\phi(x)$ is the quotient,

$$x^{2m-2}\phi\left(\frac{1}{x}\right)=x^{2m-2}\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x^2}-1}=-x^{2m}\frac{f\left(\frac{1}{x}\right)}{x^2-1}=\frac{f(x)}{x^2-1}=\phi(x).$$

Thus in every case the solution of $f(x)=0$ depends on that of a reciprocal equation $\phi(x)=0$ of the first type and of even degree.

We say that a reciprocal equation is of the *standard form* when it is of the first type and of even degree.

Theorem 2. *The solution of a reciprocal equation of the first type and of degree* $2m$ *depends on that of an equation of degree* m.

Let the equation be

$$a_0x^{2m}+a_1x^{2m-1}+\ldots+a_1x+a_0=0.$$

Dividing by x^m, this may be written

$$a_0\left(x^m+\frac{1}{x^m}\right)+a_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\ldots+a_m=0. \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

Let $x+\frac{1}{x}=z$ and $x^r+\frac{1}{x^r}=u_r$, so that $u_1=z$; then

$$u_2=\left(x+\frac{1}{x}\right)^2-2=z^2-2.$$

Also $$\left(x+\frac{1}{x}\right)\left(x^{r-1}+\frac{1}{x^{r-1}}\right)=\left(x^r+\frac{1}{x^r}\right)+\left(x^{r-2}+\frac{1}{x^{r-2}}\right),$$

therefore $$u_r=zu_{r-1}-u_{r-2}.$$

Putting $r=3, 4, \ldots$ in succession,

$$u_3=z(z^2-2)-z=z^3-3z,$$

$$u_4=z(z^3-3z)-(z^2-2)=z^4-4z^2+2,$$

and so on.

Thus equation (**A**) can be expressed as an equation of degree m in z, and if z_1 is one of its roots, the corresponding values of x are given by $x^2-z_1x+1=0$.

The reader may verify that

$$u_5 = z^5 - 5z^3 + 5z, \qquad u_6 = z^6 - 6z^4 + 9z^2 - 2,$$
$$u_7 = z^7 - 7z^5 + 14z^3 - 7z, \qquad u_8 = z^8 - 8z^6 + 20z^4 - 16z^2 + 2.$$

Ex. 1. *Solve* $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$.

One root is -1. Dividing by $x+1$, we have $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

Dividing by x^2 and grouping the terms,

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$$

If $z = x + \frac{1}{x}$, this becomes $6(z^2 - 2) + 5z - 38 = 0$, that is $6z^2 + 5z - 50 = 0$;

$$\therefore\ z = \tfrac{5}{2} \text{ or } -\tfrac{10}{3}, \quad \text{and } x \text{ is given by } x + \frac{1}{x} = \tfrac{5}{2} \text{ or } -\tfrac{10}{3}.$$

Whence we find $x = 2, \frac{1}{2}, -3$ or $-\frac{1}{3}$. Thus the roots are $-1, 2, \frac{1}{2}, -3, -\frac{1}{3}$.

2. The Binomial Equation $x^n - 1 = 0$.

(1) *If* α *is a root of* $x^n - 1 = 0$, *so also is* α^r, *where* r *is any integer.*

For $(\alpha^r)^n = (\alpha^n)^r = 1$.

(2) *If* m *is prime to* n, *the equations* $x^m - 1 = 0$ *and* $x^n - 1 = 0$ *have no common root except* 1.

For if α is a common root, then $\alpha^{pm} = 1$ and $\alpha^{qn} = 1$, where p, q are any positive integers. Therefore $\alpha^{pm - qn} = 1$, and since m is prime to n, p and q can be found so that $pm - qn = \pm 1$. (See Ch. I, 4, (3).) Hence $\alpha = 1$.

(3) *If* n *is a prime number and* α *is any imaginary root of* $x^n - 1 = 0$, *the* n *roots are* $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$.

For every one of this set is a root and no two of them are equal. For suppose that the two α^a and α^b $(a > b)$ are equal. Then $\alpha^{a-b} = 1$, and the equations $x^n - 1 = 0$ and $x^{a-b} = 1$ have a common imaginary root. This is impossible; for n is a prime and $1 \leqslant a - b < n$, thus n is prime to $a - b$.

(4) *If* $n = pqr \ldots$ *where* $p, q, r, \ldots$ *are primes or powers of primes, the roots of* $x^n - 1 = 0$ *are the* n *terms of the product*

$$(1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1})(1 + \beta + \beta^2 + \ldots \beta^{q-1})(1 + \gamma + \gamma^2 + \ldots + \gamma^{r-1}) \ldots,$$

where α *is a root of* $x^p - 1 = 0$, β *of* $x^q - 1 = 0$, γ *of* $x^r - 1 = 0$, *etc.*

Take the case of three factors p, q, r; similar reasoning applying in all cases.

Any term of the product, for instance $\alpha^a \beta^b \gamma^c$, is a root. For

$$(\alpha^a)^n = (\alpha^p)^{aqr} = 1,$$

and similarly $(\beta^b)^n = 1$ and $(\gamma^c)^n = 1$.

If any two terms are equal, for instance, if $a^a\beta^b\gamma^c = \alpha^{a'}\beta^{b'}\gamma^{c'}$, then $\beta^{b-b'}\gamma^{c-c'} = \alpha^{a'-a}$, which is impossible, for $\beta^{b-b'}\gamma^{c-c'}$ is a root of $x^{qr}-1=0$ and $\alpha^{a'-a}$ is a root of $x^p-1=0$; and since p is prime to qr, these equations have no common root except 1.

(5) As explained in Ch. V, 17, the roots of $x^n-1=0$ are

$$\cos\frac{2r\pi}{n}+\iota\sin\frac{2r\pi}{n}, \quad \text{where} \quad r=0, 1, 2, \ldots n-1.$$

The imaginary roots are therefore

$$\cos\frac{2r\pi}{n}\pm\iota\sin\frac{2r\pi}{n}, \quad \text{where} \quad r=1, 2, \ldots \frac{n-1}{2} \quad \text{or} \quad \frac{n-2}{2},$$

according as n is odd or even.

If x^n-1 is divided by $x-1$ or x^2-1 according as n is odd or even and $\phi(x)$ is the quotient, then $\phi(x)=0$ is a reciprocal equation, and if this is transformed by the substitution $z=x+\frac{1}{x}$, the values of z are

$$2\cos\frac{2r\pi}{n} \quad \text{where} \quad r=1, 2, \ldots \frac{n-1}{2} \quad \text{or} \quad \frac{n-2}{2}.$$

3. Special Roots of $x^n-1=0$.

(1) If r is less than n and prime to n, then $\cos\frac{2r\pi}{n}+\iota\sin\frac{2r\pi}{n}$ is a root of $x^n-1=0$, but *it is not a root of any equation of lower degree, and of the same type.*

For, if $$\left(\cos\frac{2r\pi}{n}+\iota\sin\frac{2r\pi}{n}\right)^m=1, \text{ where } m<n,$$

then $$\cos\frac{2rm\pi}{n}+\iota\sin\frac{2rm\pi}{n}=1.$$

Therefore rm is divisible by n, and since n is prime to r, m must be divisible by n. This is impossible, for $m<n$.

Definition. Any root of $x^n-1=0$ which is not a root of an equation of the same type and of lower degree is called a *special* root of the equation.

The special roots of $x^n-1=0$ are $\cos\frac{2r\pi}{n}+\iota\sin\frac{2r\pi}{n}$, where $r<n$ and prime to n. Thus, $x^n-1=0$ has $\phi(n)$ special roots, where $\phi(n)$ is the number of numbers less than n and prime to n, including unity.

If $r<n$ and prime to n then $n-r$ is prime to n, and therefore

$$\cos\frac{2r\pi}{n}\pm\iota\sin\frac{2r\pi}{n}$$

are special roots. These can be arranged in pairs, $\left(\alpha, \frac{1}{\alpha}\right)$, $\left(\beta, \frac{1}{\beta}\right)$, etc., so that they are the roots of a reciprocal equation.

(2) *If α is a special root of $x^n - 1 = 0$, the n roots are $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$.*

For every one of the set is a root. And if two of them are equal, for instance, if $\alpha^a = \alpha^b$, then $\alpha^{a-b} = 1$, so that α is a root of $x^{a-b} = 1$. This is impossible, for the last equation is of lower degree than $x^n - 1 = 0$.

(3) *If α is any special root of $x^n - 1 = 0$, the complete set of special roots is $\alpha^a, \alpha^b, \alpha^c, \ldots$ where $a, b, c, \ldots$ are the numbers less than n and prime to it, including unity.*

For if a is any number less than n and prime to it, the remainders when $a, 2a, 3a \ldots (n-1)a$ are divided by n are the numbers $1, 2, 3, \ldots n-1$, taken in some order or other (Ch. I, 10). And since $\alpha^n = 1$, it follows that every number in the set $\alpha^a, \alpha^{2a}, \alpha^{3a}, \ldots \alpha^{(n-1)a}$, occurs somewhere in the set $\alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$. Therefore α^a is a special root.

4. The Equation $x^n - A = 0$.

(1) Let $A = a(\cos\alpha + \iota \sin\alpha)$, where $a = |A|$, then the roots of $x^n - A = 0$ are $\sqrt[n]{a} \cdot \left\{\cos\frac{\alpha + 2r\pi}{n} + \iota \sin\frac{\alpha + 2r\pi}{n}\right\}$, where $r = 0, 1, 2, \ldots n-1$.

For by De Moivre's theorem, this expression is a root for every r, and its n values are all different.

Since $\cos\frac{\alpha + 2r\pi}{n} + \iota \sin\frac{\alpha + 2r\pi}{n} = \left(\cos\frac{\alpha}{n} + \iota \sin\frac{\alpha}{n}\right)\left(\cos\frac{2r\pi}{n} + \iota \sin\frac{2r\pi}{n}\right)$,

it follows that *the n values of $\sqrt[n]{A}$ may be found by multiplying any value of $\sqrt[n]{A}$ by the n-th roots of unity.*

(2) The roots of $x^n + 1 = 0$ are

$$\cos\frac{(2r+1)\pi}{n} + \iota \sin\frac{(2r+1)\pi}{n}, \quad \text{where} \quad r = 0, 1, 2, \ldots n-1.$$

This is obtained by putting $a = 1, \alpha = \pi$ in (1).

Ex. 1. *Solve $x^5 - 1 = 0$. Deduce the values of cos 36° and cos 72°.*

The real root is 1. Dividing by $x - 1$,

$$x^4 + x^3 + x^2 + x + 1 = 0, \text{ that is, } x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$$

Let $z = x + \frac{1}{x}$ then $z^2 + z - 1 = 0$, giving $z = \frac{1}{2}(-1 \pm \sqrt{5})$.

The values of x are $\cos\frac{2r\pi}{5} \pm \iota \sin\frac{2r\pi}{5}$ $(r = 1, 2)$.

Therefore the values of z are $2\cos\frac{2\pi}{5}$, and $2\cos\frac{4\pi}{5}$. Now $\cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}$;

$$\therefore \cos\frac{\pi}{5} = \cos 36° = \frac{\sqrt{5}+1}{4}, \quad \text{and} \quad \cos\frac{2\pi}{5} = \cos 72° = \frac{\sqrt{5}-1}{4}.$$

Ex. 2. What are the special roots of $x^{15}-1=0$. *Find the equation of which these are the roots. Show that this can be reduced to an equation of the fourth degree. What are its roots?*

The numbers <15 and prime to 15 are

$$1,\ 2,\ 4,\ 7,\ 15-7,\ 15-4,\ 15-2,\ 15-1.$$

The special roots are $\cos\frac{2r\pi}{15}\pm\iota\sin\frac{2r\pi}{15}$ $(r=1, 2, 4, 7)$.

The non-special roots are the roots of $x^3-1=0$ and $x^5-1=0$.

The L.C.M. of x^3-1 and x^5-1 is $(x^2+x+1)(x^5-1)$ and

$$\frac{x^{15}-1}{(x^2+x+1)(x^5-1)}=\frac{x^{10}+x^5+1}{x^2+x+1}=x^8-x^7+x^5-x^4+x^3-x+1.$$

Therefore the equation whose roots are the special roots is

$$x^8-x^7+x^5-x^4+x^3-x+1=0,$$

that is,
$$\left(x^4+\frac{1}{x^4}\right)-\left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)-1=0.$$

If $z=x+\frac{1}{x}$, this becomes

$$(z^4-4z^2+2)-(z^3-3z)+z-1=0,$$

that is,
$$z^4-z^3-4z^2+4z+1=0,$$

of which the roots are $2\cos\frac{2r\pi}{15}$ $(r=1, 2, 4, 7)$.

5. The Equation $x^{17}-1=0$. Gauss discovered that *the solution of this equation depends on that of four quadratic equations.* This can be proved as follows.

The imaginary roots of $x^{17}-1=0$ are

$$\cos\frac{2r\pi}{17}\pm\iota\sin\frac{2r\pi}{17},\quad \text{where}\quad r=1, 2, 3, \ldots 8.$$

Let $y_r=\cos\frac{2r\pi}{17}$, then we have

$$y_{17-r}=y_r \quad\text{and}\quad 2y_ry_s=y_{r+s}+y_{r-s}. \qquad \text{.....(A)}$$

Also the sum of the imaginary roots is -1, therefore

$$y_1+y_2+y_3+\ldots+y_8=-\tfrac{1}{2}. \qquad \text{.....(B)}$$

Equations (A) lead to an arrangement in pairs of $y_1, y_2, \ldots y_8$, which has a remarkable cyclic property. Starting with the pair y_1y_4, we have

$$\left.\begin{aligned} 2y_1y_4&=y_3+y_5, & 2y_3y_5&=y_2+y_8\\ 2y_2y_8&=y_6+y_7, & 2y_6y_7&=y_1+y_4\end{aligned}\right\}, \qquad \text{.....(C)}$$

On trial, it will appear that no y except y_4, taken with y_1, will lead to a like result.

Let $\quad y_1+y_4=2\alpha,\quad y_3+y_5=2\beta,\quad y_2+y_8=2\gamma,\quad y_6+y_7=2\delta$
then $\quad y_1y_4=\beta,\quad y_3y_5=\gamma,\quad y_2y_8=\delta,\quad y_6y_7=\alpha$ } .....(D)

It follows that *any relation connecting* $\alpha, \beta, \gamma, \delta$ *continues to hold after the cyclic substitution* $(\alpha\beta\gamma\delta)$.

Such relations are the following.

By (B) $$\alpha+\beta+\gamma+\delta=-\tfrac{1}{4}. \quad \text{.....(E)}$$

Again, using equations (A),

$$\alpha\gamma=\tfrac{1}{4}(y_1y_2+y_1y_8+y_4y_2+y_4y_8)=\tfrac{1}{8}(y_1+y_3+y_7+y_8+y_2+y_6+y_4+y_5),$$

therefore $$\alpha\gamma=-\tfrac{1}{16}. \quad \text{.....(F)}$$

Consequently $$\beta\delta=-\tfrac{1}{16}. \quad \text{.....(G)}$$

Similarly we can show that

$$16\alpha\beta=-1+4\alpha-4\beta, \quad \text{.....(H)}$$

$$4\alpha^2=1+2\beta+\gamma. \quad \text{.....(I)}$$

Finally let $$\alpha+\gamma=2\lambda, \quad \beta+\delta=2\mu. \quad \text{.....(J)}$$

Then by (E) $$\lambda+\mu=-\tfrac{1}{8}.$$

Also $$4\lambda\mu=\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta.$$

Hence by (H) and three equations derived from this by the cyclic substitution $(\alpha\beta\gamma\delta)$, we find that

$$\lambda\mu=-\tfrac{1}{16}.$$

Thus

y_1, y_4	are the roots of	$y^2-2\alpha y+\beta=0$	
y_3, y_5	,, ,,	$y^2-2\beta y+\gamma=0$	
y_2, y_8	,, ,,	$y^2-2\gamma y+\delta=0$	
y_6, y_7	,, ,,	$y^2-2\delta y+\alpha=0$	
α, γ	,, ,,	$y^2-2\lambda y-\tfrac{1}{16}=0$	
β, δ	,, ,,	$y^2-2\mu y-\tfrac{1}{16}=0$	
λ, μ	,, ,,	$y^2+\tfrac{1}{8}y-\tfrac{1}{16}=0$	

. .....(K)

Since $\cos\theta$ decreases as θ increases from 0 to π, *the greater root of each quadratic is that which is stated first.*

Thus, starting with the last equation and working upwards, the values of $y_1, y_2, \ldots y_8$ can be found, and then the roots of $x^{17}-1=0$ are given by $x^2-2y_1x+1=0$, etc.

Ex. 1. *Express* $\cos\dfrac{2\pi}{17}$ *as a surd.*

By (K), $$\lambda=-\tfrac{1}{16}+\tfrac{1}{16}\sqrt{17}, \quad \mu=-\tfrac{1}{16}-\tfrac{1}{16}\sqrt{17},$$

and $$\alpha=\lambda+\sqrt{p}, \quad \gamma=\lambda-\sqrt{p}, \quad \beta=\mu+\sqrt{q},$$

where $$p=\frac{1}{16^2}(34-2\sqrt{17}), \quad q=\frac{1}{16^2}(34+2\sqrt{17}).$$

Also $$\cos\frac{2\pi}{17}=y_1=\alpha+\sqrt{(\alpha^2-\beta)};$$

and by (I), $$\alpha^2-\beta=\tfrac{1}{4}(1+\gamma-2\beta)=\tfrac{1}{4}(1+\lambda-2\mu-\sqrt{p}-2\sqrt{q});$$

therefore,
$$\cos\frac{2\pi}{17}=\frac{1}{16}\{-1+\sqrt{17}+\sqrt{(34-2\sqrt{17})}$$
$$+\tfrac{1}{8}\sqrt{\{17+3\sqrt{17}-\sqrt{(34-2\sqrt{17})}-2\sqrt{(34+2\sqrt{17})}\}}.$$

NOTE. (i) The cosines of $\frac{4\pi}{17}, \frac{6\pi}{17}, \ldots \frac{16\pi}{17}$ can be expressed in a similar form; the surd values of the sines of these angles involve one more radical sign. (Ex. XIX, 22.)

(ii) Any length determined by a quadratic equation can be found geometrically. It is therefore possible, *by the use of ruler and compasses only*, to effect the following construction.

6. To inscribe a regular 17-sided Polygon in a Circle. Take the radius of the circle as unit of length. Let O be the centre. Draw $x'Ox$, $y'Oy$ at right angles, cutting the circle in A, A' and B, B'. Let the vertices of a regular inscribed 17-gon be marked 0, 1, 2, ... 17, the point O coinciding with B. The ordinates $y_1, y_2, \ldots y_8$ of the points 1, 2, ... 8 are given by the equations of Art. 5. To solve these geometrically, observe that the roots of $y^2-2ay+b=0$ are the ordinates of the points where the axis $x=0$ cuts the circle $x^2+y^2-2ay+b=0$

Thus to find the points $(0, y_1)$, $(0, y_2)$, ... $(0, y_8)$ (two of which are marked y_3, y_5 in Fig. 27), we have to draw the circles in the following table.

Circle	Centre	Ends of chord $y=0$	Ends of diameter $x=0$	Name of circle
$x^2+y^2+\frac{1}{8}y-\frac{1}{16}=0$	$(0, -\frac{1}{16})$	$(\pm\frac{1}{4}, 0)$	$(0, \lambda), (0, \mu)$	ω
$x^2+y^2-2\lambda y-\frac{1}{16}=0$	$(0, \lambda)$	$(\pm\frac{1}{4}, 0)$	$(0, \alpha), (0, \gamma)$	λ
$x^2+y^2-2\mu y-\frac{1}{16}=0$	$(0, \mu)$	$(\pm\frac{1}{4}, 0)$	$(0, \beta), (0, \delta)$	μ
$x^2+y^2-2\alpha y+\beta=0$	$(0, \alpha)$	$(\pm\sqrt{-\beta}, 0)$	$(0, y_1), (0, y_4)$	α
$x^2+y^2-2\beta y+\gamma=0$	$(0, \beta)$	$(\pm\sqrt{-\gamma}, 0)$	$(0, y_3), (0, y_5)$	β
$x^2+y^2-2\gamma y+\delta=0$	$(0, \gamma)$	$(\pm\sqrt{-\delta}, 0)$	$(0, y_2), (0, y_8)$	γ
$x^2+y^2-2\delta y+\alpha=0$	$(0, \delta)$	$(\pm\sqrt{-\alpha}, 0)$	$(0, y_6), (0, y_7)$	δ

The centres of the circles are marked ω, λ, μ, α, etc., in Fig. 27. The circles ω, λ, μ are easily drawn, giving the points α, β, γ, δ. To draw the last four circles notice that:

(i) *The circles α, β, γ, δ meet $y=0$ at the same points as the circles on $B\beta$, $B\gamma$, $B\delta$, $B\alpha$ as diameters, respectively.*

For, since the equation to the circle on $B\beta$ as diameter is
$$x^2+\left(y-\frac{1+\beta}{2}\right)^2=\left(\frac{1-\beta}{2}\right)^2,$$
which is satisfied by $x=\pm\sqrt{-\beta}$, $y=0$, we can draw the circles β and γ.

(ii) *The diameters of the circles* α, β, γ, δ *are equal to the tangents from* B *to the circles* β, γ, δ, α, *respectively.*

For (diameter of $\odot\ \alpha)^2 = 4(\alpha^2 - \beta) = (2\beta + \gamma + 1) - 4\beta$, and the square of tangent from B to $\odot\ \beta = 1 - 2\beta + \gamma$. Hence we can draw the circles α, δ.

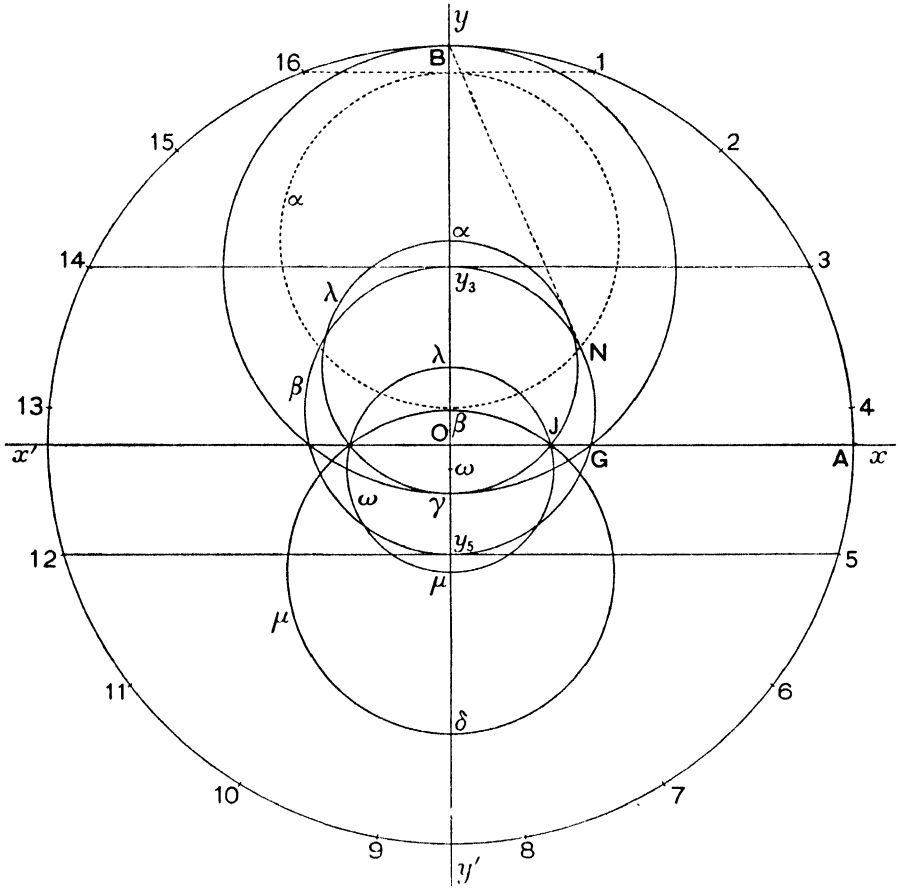


FIG. 27.

Construction. Along Ox take $OJ = \frac{1}{4}OA$. Along Oy' take $O\omega = \frac{1}{4}OJ$. With centre ω and radius ωJ draw a circle cutting yOy' in λ, μ. With centre λ and radius λJ draw a circle cutting yOy' in α, γ. With centre μ and radius μJ draw a circle cutting yOy' in β, δ. Draw the circle on $B\gamma$ as diameter cutting Ox in G. With centre β and radius βG draw a circle cutting yOy' in y_3, y_5.

Through y_3, y_5 draw parallels to $x'Ox$. These lines meet the given circle in the vertices 3, 14 and 5, 12 of the required 17-gon, which can be completed by bisecting the arc 35 at 4 and setting off arcs equal to 34.

Or we may draw the circles α, γ, δ as explained above.

EXERCISE XIX

RECIPROCAL AND BINOMIAL EQUATIONS

Solve the equations in Exx. 1-9 :

1. $x^4-8x^3+17x^2-8x+1=0$.
2. $x^4+6x^3-5x^2+6x+1=0$.
3. $2x^4+3x^3+5x^2+3x+2=0$.
4. $2x^4+x^3-17x^2+x+2=0$.
5. $2x^5+5x^4-5x-2=0$.
6. $3x^5-10x^4-3x^3-3x^2-10x+3=0$.
7. $2x^6-x^5-2x^3-x+2=0$.
8. $2x^7-x^6-3x^4-3x^3-x+2=0$.
9. $x^8+1+(x+1)^8=2(x^2+x+1)^4$.

[Write this $x^4+x^{-4}+(x+x^{-1}+2)^4=2(x+x^{-1}+1)^4$, and show that this reduces to $(x+x^{-1}+1)(x+x^{-1}+2)=0$.]

10. If $s_n=1+u_1+u_2+\ldots+u_n$ where $u_n=x^n+x^{-n}$ and $z=x+x^{-1}$, prove that

$$s_n=\frac{u_{n+1}-u_n}{z-2} \quad \text{and} \quad s_n=zs_{n-1}-s_{n-2}.$$

Hence show that

$$s_0=1, \quad s_1=z+1, \quad s_2=z^2+z-1,$$
$$s_3=z^3+z^2-2z-1, \quad s_4=z^4+z^3-3z^2-2z+1,$$
$$s_5=z^5+z^4-4z^3-3z^2+3z+1,$$
$$s_6=z^6+z^5-5z^4-4z^3+6z^2+3z-1.$$

Note that $s_n=0$ is the equation on which the solution of $x^{2n+1}-1=0$ depends.

$\left[1+s_n=\frac{1-x^n}{1-x}+\frac{1-x^{-n}}{1-x^{-1}} \text{ and } (1-x)(1-x^{-1})=2-z.\right]$

11. If α is an imaginary root of $x^7-1=0$, the equation whose roots are $\alpha+\alpha^6$, $\alpha^2+\alpha^5$, $\alpha^3+\alpha^4$ is $s_3=z^3+z^2-2z-1=0$.

12. If n is a prime, the special roots of $x^{2n}-1=0$ are the roots of $x^n+1=0$.

13. The special roots of $x^9-1=0$ are the roots of $x^6+x^3+1=0$, and their values are $\cos\frac{2r\pi}{9}\pm\iota\sin\frac{2r\pi}{9}$, where $r=1, 2, 4$.

14. The special roots of $x^{12}-1=0$ are the roots of $x^4-x^2+1=0$, and their values are $\pm\left(\cos\frac{\pi}{6}\pm\iota\sin\frac{\pi}{6}\right)$, that is, $\pm\frac{1}{2}(\sqrt{3}\pm\iota)$.

[To find the equation, divide $x^{12}-1$ by the L.C.M. of x^2-1, x^3-1, x^4-1, x^6-1, i.e. by the L.C.M. of x^4-1, x^6-1.]

15. If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, prove that the equation whose roots are $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$, $\gamma+\frac{1}{\gamma}$ is

$$adz^3+(ac+bd)z^2+(ab+bc+cd-3ad)z+(a-c)^2+(b-d)^2=0.$$

[Prove that $ad\Pi\{x^2-x(\alpha+\alpha^{-1})+1\}=\{(ax^3+bx^2+cx+d)(dx^3+cx^2+bx+a)\}$, and write $(x^2+1)/x=z$.]

16. Show that there is a value of k for which $x^6-15x^3-8x^2+2$ is divisible by x^3+kx+1, and find this value.

[Show that $x^6-15x^3-8x^2+2=0$ has two roots α, β such that $\alpha\beta=1$. Then $k=-(\alpha+\beta)$.]

17. For the equation $x^4+px^3+qx^2+px+1=0$, prove that

(i) the roots are all real if, and only if,

$$4(q-2)<p^2<(\tfrac{1}{2}q+1)^2 \quad \text{and} \quad 2q+4<p^2;$$

(ii) the roots are all imaginary if, and only if, $p^2<(\frac{1}{2}q+1)^2$ and at least one of the two inequalities $p^2<4(q-2)$, $p^2<2q+4$ holds;

(iii) two roots are real and two imaginary if $(\frac{1}{2}q+1)^2<p^2$.

[If $z=x+x^{-1}$, then $z^2+pz+q-2=0$. Let z_1, z_2 be the values of z, then

$$x=\tfrac{1}{2}(z_1\pm\sqrt{z_1^2-4}) \quad \text{or} \quad \tfrac{1}{2}(z_2\pm\sqrt{z_2^2-4}).$$

If *any* value of x is real, one and therefore both values of z must be real, and so $p^2>4(q-2)$. The conditions in question can now be obtained by considering the equation whose roots are z_1^2-4, z_2^2-4.]

18. *If in the last example all the roots of the given equation are imaginary and the values of z are real, prove that* $|p|<4$ *and* $q<6$.

[For then $4(q-2)<p^2<2q+4$, etc.]

19. If $(x-1)^5=a(x^5-1)$ and $x\neq 1$, prove that

$$x+\frac{1}{x}=\frac{4+a\pm\sqrt{5a(4+a)}}{2(1-a)}.$$

Also if $0<a>16$, the given equation has three real and two imaginary roots, but if $a<0$ or $a>16$, the only real root is 1.

20. Show that, if r is any integer, the roots of

$$1-nx-\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\ldots+(-1)^{\frac{1}{2}n(n+1)}x^n=0$$

are the values of $\tan\dfrac{(4r+1)\pi}{4n}$.

21. If $y_r=\cos\dfrac{2r\pi}{5}$, show that

$$2y_1y_2=y_1+y_2=-\tfrac{1}{2},$$

and deduce the construction indicated in the figure for inscribing a regular pentagon in a circle.

In the figure $OH=\frac{1}{2}OA$, and $OK=\frac{1}{2}OH$.

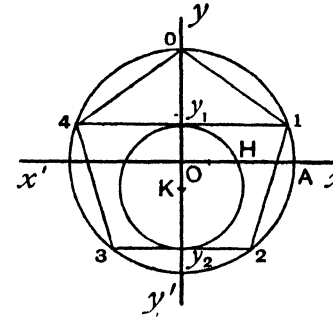


FIG. 28.

22. Prove that the values of $\sin\dfrac{2\pi}{17}$ and $\cos\dfrac{\pi}{17}$ are respectively

$$\sqrt{[\tfrac{1}{32}\{17-\sqrt{17}+\sqrt{(34-2\sqrt{17})}\}-\tfrac{1}{16}\sqrt{\{17+3\sqrt{17}+\sqrt{(34-2\sqrt{17})}+2\sqrt{(34+2\sqrt{17})}\}}]},$$

and $\frac{1}{16}\{1-\sqrt{17}+\sqrt{(34-2\sqrt{17})}\}+\frac{1}{8}\sqrt{\{17+3\sqrt{17}+\sqrt{(34-2\sqrt{17})}+2\sqrt{(34+2\sqrt{17})}\}}$

[Referring to Art. 5, Ex. 1, $\sin\dfrac{2\pi}{17}=\sqrt{(\frac{1}{2}-\frac{1}{2}y_2)}$ and

$$\delta=\mu-\sqrt{q}, \quad y_2=\gamma+\sqrt{(\gamma^2-\delta)}=\gamma+\tfrac{1}{2}\sqrt{(1+\alpha-2\delta)};$$

$$\therefore\ y_2=\gamma+\tfrac{1}{2}\sqrt{\{1+\lambda-2\mu+\sqrt{p+2\sqrt{q}}\}}.]$$

CHAPTER XII

CUBIC AND BIQUADRATIC EQUATIONS

1. The Cubic Equation. The standard form of the cubic equation is

$$u = ax^3 + 3bx^2 + 3cx + d = 0. \quad \text{.....(A)}$$

If $x = y - b/a$ this equation becomes

$$y^3 + \frac{3H}{a^2}y + \frac{G}{a^3} = 0, \quad \text{.....(B)}$$

where $$H = ac - b^2, \quad G = a^2d - 3abc + 2b^3.$$

If $z = ay = ax + b$, the equation is

$$z^3 + 3Hz + G = 0. \quad \text{.....(C)}$$

If α, β, γ are the roots of (A), those of (B) are $\alpha + b/a$, $\beta + b/a$, $\gamma + b/a$, those of (C) are $a\alpha + b$, $a\beta + b$, $a\gamma + b$.

2. Equation whose Roots are the Squares of the Differences of the Roots. The differences of the roots are the same for equations (A) and (B). Hence the equation whose roots are

$$(\beta - \gamma)^2, \quad (\gamma - \alpha)^2, \quad (\alpha - \beta)^2,$$

is obtained by putting $q = 3H/a^2$, $r = G/a^3$ in Ch. VI, 17.

The resulting equation is

$$a^6z^3 + 18a^4Hz^2 + 81a^2H^2z + 27(G^2 + 4H^3) = 0, \quad \text{.....(D)}$$

whence the important equality

$$a^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27(G^2 + 4H^3). \quad \text{.....(E)}$$

It is easily verified that

$$G^2 + 4H^3 = a^2\Delta, \quad \text{.....(F)}$$

where $$\Delta = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2, \quad \text{.....(G)}$$

so that $$a^4(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27\Delta. \quad \text{.....(H)}$$

The function Δ is called the *discriminant* of the cubic u, and its vanishing is the necessary and sufficient condition that the equation $u = 0$ should have two equal roots.

3. Character of the Roots. Excluding the case in which two roots are equal, and remembering that imaginary roots occur in pairs, the possibilities are

(i) *All the roots may be real*, and then by (H), $\Delta<0$.

(ii) *One root may be real and two imaginary*; and, denoting the roots by α, $\lambda \pm \iota\mu$, we find that the product of the squares of the differences of the roots is equal to $-4\mu^2\{(\alpha-\lambda)^2+\mu^2\}^2$, so that $\Delta>0$.

Hence *if* $\Delta<0$, *all the roots are real and unequal; if* $\Delta>0$, *one root is real and two are imaginary.*

Also it follows from equation (D) that $H=0$, $G=0$ are the necessary and sufficient conditions for *three equal roots.*

4. Cardan's Solution. If $z=ax+b$, the equation

$$ax^3+3bx^2+3cx+d=0 \quad \text{.....(A)}$$

becomes

$$z^3+3Hz+G=0. \quad \text{.....(B)}$$

Let $z=m^{\frac{1}{3}}+n^{\frac{1}{3}}$; then $z^3-3m^{\frac{1}{3}}n^{\frac{1}{3}}z-(m+n)=0.$ .....(C)

Comparing (B) and (C), $m^{\frac{1}{3}}n^{\frac{1}{3}}=-H$, $m+n=-G$.

Therefore m and n are the roots of $t^2+Gt-H^3=0$, ...(D)

and we may take $m=\frac{1}{2}(-G+\sqrt{G^2+4H^3})$.

If Q denotes any one of the three values of

$$\sqrt[3]{\{\tfrac{1}{2}(-G+\sqrt{G^2+4H^3})\}},$$

the three values of $m^{\frac{1}{3}}$ are Q, ωQ, $\omega^2 Q$, where ω is an imaginary cube root of unity.

Also, because $m^{\frac{1}{3}}n^{\frac{1}{3}}=-H$, the corresponding values of $n^{\frac{1}{3}}$ are

$$-H/Q, \quad -\omega^2 H/Q, \quad -\omega H/Q.$$

Hence the values of z, that is of $ax+b$, are

$$Q-H/Q, \quad \omega Q-\omega^2 H/Q, \quad \omega^2 Q-\omega H/Q.$$

NOTE. If $G^2+4H^3<0$, all the roots of the cubic are real, but Cardan's solution gives them in an imaginary form, which is very unsuitable for practical purposes.* In this case a solution can be easily obtained as follows.

5. Trigonometrical Solution when $G^2+4H^3<0$. Taking the equation

$$z^3+3Hz+G=0, \quad \text{.....(A)}$$

when $G^2+4H^3<0$, a solution can be obtained by using the equation

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

* This is sometimes called the *irreducible case.*

If $\cos 3\theta$ is known, $\cos\theta$ is determined by

$$\cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta = 0. \quad \text{.....(B)}$$

Let $z = q\cos\theta$, so that

$$\cos^3\theta + \frac{3H}{q^2}\cos\theta + \frac{G}{q^3} = 0. \quad \text{.....(C)}$$

Equations (B) and (C) are identical if

$$q = 2\sqrt{-H} \quad \text{and} \quad \cos 3\theta = -\frac{4G}{q^3} = -\frac{G}{2\sqrt{-H^3}}. \quad \text{.....(D)}$$

Now $G^2 < -4H^3$, and so a real value of θ can be found to satisfy the last equation. If α is any such value, the roots of (A) are

$$q\cos\alpha, \quad q\cos\left(\frac{2\pi}{3}+\alpha\right), \quad q\cos\left(\frac{2\pi}{3}-\alpha\right).$$

Ex. 1. *If all the roots of* $ax^3+3bx^2+3cx+d=0$ *are real, show that the equation can be reduced to*

$$t^3 - t + \mu = 0, \quad \text{where} \quad 27\mu^2 < 4,$$

by a substitution of the form $x = p + qt$, *where* p *and* q *are real.*

If $z = ax+b$, the equation becomes $z^3 - hz + G = 0$,

where $h = -3H = 3(b^2-ac)$. Since all the roots are real, $G^2+4H^3<0$, $\therefore$ $H<0$ and $h>0$.

Writing $z = \sqrt{h}\,.\,t$, the equation becomes $t^3 - t + \mu = 0$ where $\mu = G/\sqrt{h^3}$, and the substitution is

$$x = -\frac{b}{a} + \frac{\sqrt{h}}{a}\,.\,t.$$

NOTE. The roots of the cubic can be expressed as convergent series by using this reduction. (See Ch. XXIX, 3.)

6. Two Important Functions of the Roots.

Let $$L = \alpha + \omega\beta + \omega^2\gamma, \quad M = \alpha + \omega^2\beta + \omega\gamma,$$

where ω is an imaginary cube root of unity, and observe that

$$\omega^3 = 1, \quad 1 + \omega + \omega^2 = 0, \quad \omega - \omega^2 = \pm\iota\sqrt{3}.$$

(1) *The interchange of any two of the letters* α, β, γ *transforms either of the functions* L^3, M^3 *into the other.*

Thus the transposition (α, β) changes L^3 into

$$(\beta + \omega\alpha + \omega^2\gamma)^3 = (\omega\alpha + \omega^3\beta + \omega^2\gamma)^3 = \omega^3 M^3 = M^3.$$

It is the existence of functions possessing this property which makes the solution of a cubic equation depend on that of a quadratic.

(2) L *and* M *are functions of the differences of* α, β, γ. For since

$$1 + \omega + \omega^2 = 0,$$

they are unaltered by writing $\alpha+h$, $\beta+h$, $\gamma+h$ for α, β, γ.

(3) We have the following equalities:

$$L+M=2\alpha-\beta-\gamma, \qquad L-M=(\omega-\omega^2)(\beta-\gamma),$$
$$\omega^2L+\omega M=2\beta-\gamma-\alpha, \quad \omega^2L-\omega M=(\omega-\omega^2)(\gamma-\alpha),$$
$$\omega L+\omega^2M=2\gamma-\alpha-\beta, \quad \omega L-\omega^2M=(\omega-\omega^2)(\alpha-\beta).$$

Also, by Ex. XII, 9,

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=-27G/a^3.$$

Therefore
$$L^3+M^3=-27G/a^3, \quad \text{.....(A)}$$
$$L^3-M^3=-3(\omega-\omega^2)(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta). \quad \text{.....(B)}$$

Moreover,
$$LM=\Sigma\alpha^2-\Sigma\beta\gamma=-9H/a^2. \quad \text{.....(C)}$$

(4) From equations **(A)** and **(C)** it follows that $(\frac{1}{3}aL)^3$, $(\frac{1}{3}aM)^3$ *are the roots of*

$$t^2+Gt-H^3=0,$$

which is the auxiliary quadratic in Cardan's solution.

7. The Cubic as the Sum of Two Cubes.

If $u\equiv ax^3+3bx^2+3cx+d$, *then constants* A, B, λ, μ *can be found such that*

$$u\equiv A(x-\lambda)^3+B(x-\mu)^3, \quad \text{.....(A)}$$

provided that u *has no square factor. Also* λ, μ *are the roots of*

$$\mathbf{H}\equiv(ac-b^2)x^2+(ad-bc)x+(bd-c^2)=0.$$

By expanding and equating coefficients it will be seen that the identity **(A)** holds if

$$\left.\begin{array}{ll} A+B=a & \lambda A+\mu B=-b \\ \lambda^2A+\mu^2B=c & \lambda^3A+\mu^3B=-d \end{array}\right\} \quad \text{.....(B)}$$

Eliminating A, B from the first three and from the last three of these equations,

$$a(\lambda^2\mu-\lambda\mu^2)+b(\lambda^2-\mu^2)+c(\lambda-\mu)=0,$$
$$b(\lambda^3\mu^2-\lambda^2\mu^3)+c(\lambda^3\mu-\lambda\mu^3)+d(\lambda^2\mu-\lambda\mu^2)=0.$$

Assume that neither λ nor μ is zero and that $\lambda\neq\mu$, and divide the first of these equations by $\lambda-\mu$ and the second by $\lambda\mu(\lambda-\mu)$, then

$$a\lambda\mu+b(\lambda+\mu)+c=0, \quad \text{and} \quad b\lambda\mu+c(\lambda+\mu)+d=0.$$

Eliminating μ,

$$(a\lambda+b)(c\lambda+d)-(b\lambda+c)^2=0.$$

Therefore λ and, by symmetry, μ are the roots of

$$(ax+b)(cx+d)-(bx+c)^2=0,$$

which is the same as

$$\mathbf{H}\equiv(ac-b^2)x^2+(ad-bc)x+(bd-c^2)=0. \quad \text{.....(C)}$$

Also A, B are given by $A+B=a$, $\lambda A+\mu B=-b$.

If $\lambda=0$, then by equations (**B**),

$$bd-c^2=0 \quad \text{and} \quad \mu=-\frac{c}{b}=-\frac{d}{c}=-\frac{ad-bc}{ac-b^2},$$

so that also in this case λ and μ are the roots of $\mathbf{H}=0$.

If $\lambda=\mu$, the reasoning fails. This case will arise if, and only if, $u=0$ has two equal roots. For using equation (**G**) of Art. 2, we find that

$$(ad-bc)^2-4(ac-b^2)(bd-c^2)=\Delta.$$

Hence, if $u=0$ has two equal roots, so also has $\mathbf{H}=0$ and *vice versa*.

Ex. 1. *Use the preceding to solve the equation* $2x^3+3x^2-21x+19=0$.

Let $$2x^3+3x^2-21x+19=A(x-\lambda)^3+B(x-\mu)^3. \quad \text{.....(A)}$$

Then λ, μ are the roots of $(2x+1)(-7x+19)-(x-7)^2=0$, giving $\lambda=1$, $\mu=2$; and, equating the coefficients of x^3 and x^2 in (**A**),

$$A+B=2, \quad A+2B=-1, \quad \text{so that} \quad A=5, \ B=-3.$$

Hence the given equation may be written $5(x-1)^3=3(x-2)^3$, and the roots are given by $\sqrt[3]{5}(x-1)=k.\sqrt[3]{3}(x-2)$, where $k=1, \omega, \omega^2$.

Since $k^3=1$, $$x=\frac{5^{\frac{1}{3}}-2k3^{\frac{1}{3}}}{5^{\frac{1}{3}}-k3^{\frac{1}{3}}}=\frac{(5^{\frac{1}{3}}-2k3^{\frac{1}{3}})(5^{\frac{2}{3}}+k5^{\frac{1}{3}}.3^{\frac{1}{3}}+k^2 3^{\frac{2}{3}})}{2}.$$

Hence the roots are

$$\tfrac{1}{2}(-1-\sqrt[3]{75}-\sqrt[3]{45}), \quad \tfrac{1}{2}(-1-\omega\sqrt[3]{75}-\omega^2\sqrt[3]{45}), \quad \tfrac{1}{2}(-1-\omega^2\sqrt[3]{75}-\omega\sqrt[3]{45}).$$

8. The Hessian. (1) The function **H** which occurs in the last article is called the *Hessian*, and is of great importance in the theory of the cubic. The roots of $\mathbf{H}=0$ will be denoted by λ, μ, and we shall write

$$\mathbf{H}=h_0x^2+h_1x+h_2,$$

where $$h_0=H=ac-b^2, \quad h_1=ad-bc, \quad h_2=bd-c^2,$$

so that $$h_1^2-4h_0h_2=\Delta. \quad \text{.....(A)}$$

(2) Using the identities of Exercise XII, 4, 5, 6, we find that

$$-18\mathbf{H}=a^2\{(x-\alpha)^2(\beta-\gamma)^2+(x-\beta)^2(\gamma-\alpha)^2+(x-\gamma)^2(\alpha-\beta)^2\}. \quad \text{.....(B)}$$

(3) We can find the values of λ, μ in terms of α, β, γ as follows. Let

$$L=\alpha+\omega\beta+\omega^2\gamma, \qquad M=\alpha+\omega^2\beta+\omega\gamma,$$
$$L'=\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta, \quad M'=\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta.$$

Then, if $P=(x-\alpha)(\beta-\gamma)$, $Q=(x-\beta)(\gamma-\alpha)$, $R=(x-\gamma)(\alpha-\beta)$, we have

$$P+Q+R=0, \quad \text{therefore} \quad \Sigma P^2=-2\Sigma QR$$

and $$\Sigma P^2=\tfrac{2}{3}(\Sigma P^2-\Sigma QR)=\tfrac{2}{3}(P+\omega Q+\omega^2R)(P+\omega^2Q+\omega R).$$

It follows from **(B)** that one root λ of $\mathbf{H}=0$ is given by

$$(\lambda-\alpha)(\beta-\gamma)+\omega(\lambda-\beta)(\gamma-\alpha)+\omega^2(\lambda-\gamma)(\alpha-\beta)=0,$$

that is,
$$\lambda(\omega^2-\omega)L+(\omega^2-\omega)L'=0,$$

so that
$$\lambda=-L'/L \text{ and similarly } \mu=-M'/M. \quad \text{.....(C)}$$

(4) It is easy to verify that the substitution

$$t=H(ax+b)$$

reduces the equation $h_0x^2+h_1x+h_2=0$ to

$$t^2+Gt-H^3=0;$$

which is the auxiliary quadratic in Cardan's solution.
Thus the roots of the auxiliary quadratic are

$$H(a\lambda+b), \quad H(a\mu+b).$$

EXERCISE XX

THE CUBIC

1. Find to five places of decimals the real root of
(i) $x^3+29x-97=0$, (ii) $x^3+6x^2+27x-26=0$, (iii) $x^3-3x^2+5x-43=0$.

2. Find to five places of decimals the real roots of
(i) $x^3-3x+1=0$, (ii) $2x^3+11x^2-10x+1=0$.

3. If α, β, γ are the roots of $x^3-3qx+r=0$ and $k=\pm\sqrt{3(4q^3-r^2)}$, show that
(i) $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=3k$.
(ii) $(\beta-\gamma)(\gamma-\alpha)+(\gamma-\alpha)(\alpha-\beta)+(\alpha-\beta)(\beta-\gamma)=-9q$.
(iii) The equation whose roots are $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ is
$$x^3-9qx+3k=0.$$
(iv) $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=\frac{3}{2}(r+k)$, $\alpha^2\gamma+\beta^2\alpha+\gamma^2\beta=\frac{3}{2}(r-k)$.
(v) $\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha=\alpha^3\gamma+\beta^3\alpha+\gamma^3\beta=-9q^2$.
(vi) If the roots are all real and $\alpha>\beta>\gamma$, the difference between any two of the roots cannot exceed $2\sqrt{(3q)}$, and the difference between the greatest and least must exceed $3\sqrt{q}$.

4. If α, β, γ are the roots of $ax^3+3bx^2+3cx+d=0$,
(i) the equation whose roots are $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ is
$$a^3x^3+9Hax\pm\sqrt{-27(G^2+4H^3)}=0;$$
(ii) the equation whose roots are $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha$, $\alpha^2\gamma+\beta^2\alpha+\gamma^2\beta$ is
$$a^4x^2-3a^2(ad-3bc)x+9(a^2d^2-6abcd+3ac^3+3b^3d)=0.$$

In the following examples the letters L, M, L', M', h_0, h_1, h_2 have the meanings assigned in Art. 8, and unless otherwise stated α, β, γ are the roots of

$$u\equiv ax^3+3bx^2+3cx+d=0.$$

5. Show that $L^2+\frac{3b}{a}M=3M'$, and $M^2+\frac{3b}{a}L=3L'$.

6. If $u=A(x-\lambda)^3+B(x-\mu)^3$ and α, β, γ are the roots of $u=0$, assume that

$$A^{\frac{1}{3}}(\alpha-\lambda)+B^{\frac{1}{3}}(\alpha-\mu)=\omega A^{\frac{1}{3}}(\beta-\lambda)+B^{\frac{1}{3}}(\beta-\mu)=\omega^2 A^{\frac{1}{3}}(\gamma-\lambda)+B^{\frac{1}{3}}(\gamma-\mu)=0. \quad \textbf{(A)}$$

Hence prove that $A/B=-L^3/M^3$, $\lambda=-L'/L$, $\mu=-M'/M$.

[Multiply equations (A) by (1, 1, 1), (1, ω, ω^2), (1, ω^2, ω), add and use Ex. 5.]

7. If $k=\pm\sqrt{(-\frac{1}{3}\Delta)}$, prove that

$$\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha+3\frac{d}{a}=\frac{9}{2a^2}(h_1+k), \quad \text{and} \quad \alpha^2\gamma+\beta^2\alpha+\gamma^2\beta+3\frac{d}{a}=\frac{9}{2a^2}(h_1-k).$$

8. Find the values of l, m, n such that

$$\beta\gamma+l\beta+m\gamma+n=0, \quad \gamma\alpha+l\gamma+m\alpha+n=0, \quad \alpha\beta+l\alpha+m\beta+n=0;$$

proving that $2h_0 l=h_1+k$, $2h_0 m=h_1-k$, $h_0 n=h_2$, where $k=\pm\sqrt{(-\frac{1}{3}\Delta)}$.

That is to say, *any two roots* (α, β) *of the cubic equation* $u=0$ *are connected by the* '*homographic*' *relation*

$$2h_0\alpha\beta+(h_1+k)\alpha+(h_1-k)\beta+2h_2=0.$$

9. Except when $\Delta=0$, there are two substitutions of the form

$$x=(ly+m)/(l'y+m'),$$

which will transform the cubic equation $u=0$ into itself, and these are

$$x=-\frac{(h_1+k)y+2h_2}{2h_0y+h_1-k},$$

where $3k^2=4h_0h_2-h_1^2$ and h_0, h_1, h_2 are the coefficients of the Hessian.

Explain why this fails if $\Delta=0$.

[This is merely another way of stating the last part of Ex. 8.]

10. Find substitutions of the form $x=(ly+m)/(l'y+m')$ which will transform the equation $x^3-3x^2+3=0$ into itself, showing that these are

$$x=(2y-3)/(y-1) \quad \text{and} \quad x=(y-3)/(y-2).$$

11. Find substitutions of the form $x=(ly+m)/(l'y+m')$ which will transform the equation $x^3+3x^2-6x+1=0$ into itself, showing that these are

$$x=(y-1)/y \quad \text{and} \quad x=1/(1-y).$$

12. Show that any pair of roots α, β of the equation $x^3-3x^2+3x-2=0$ are connected by a relation of the form $p\alpha+q\beta+r=0$ where p, q, r are the same numbers whatever pair is chosen, proving that the relation is

$$(\sqrt{3}-\iota)\alpha+(\sqrt{3}+\iota)\beta-2\sqrt{3}=0.$$

13. If α, β, γ are the roots of $x^3-3qx+r=0$, find l, m, n such that

$$l\alpha^2+m\alpha+n=\beta, \quad l\beta^2+m\beta+n=\gamma, \quad l\gamma^2+m\gamma+n=\alpha;$$

proving that, if α, β, γ are real and $\alpha>\beta>\gamma$, then

$$\frac{l}{q}=\frac{2m}{r-k}=\frac{n}{-2q^2}=\frac{1}{k},$$

where $k=\sqrt{\frac{1}{3}(4q^3-r^2)}$.

Hence show that with these values of l, m, n the substitution

$$y=lx^2+mx+n$$

transforms the equation $x^3-3qx+r=0$ into itself.

14. If α, β, γ are the roots of $x^3-21x+35=0$, show that $\alpha^2+2\alpha-14$ is equal to either β or γ. (In other words, the substitution $y=x^2+2x-14$ transforms the equation into itself.)

15. If α, β, γ are the roots of the cubic $u=0$ and

$$\alpha^3+p\alpha+q=k, \quad \beta^3+p\beta+q=\omega k, \quad \gamma^3+p\gamma+q=\omega^2 k,$$

prove that

$$p=\frac{3b}{a}+\lambda, \quad q=\frac{2c}{a}+\lambda\frac{b}{a},$$

where λ is a root of the Hessian $\mathbf{H}=0$.

Further, if $\lambda=-L'/L$, $\mu=-M'/M$, then $3k=M(\lambda-\mu)=-\dfrac{a^2M}{27H}(L^3-M^3)$.

[Multiply by $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, and add.]

16. Prove that the equation $u=0$ is transformed into $y^3=k^3$ by the substitution * $y=x^2+px+q$,

if

$$p=\frac{3b}{a}+\lambda, \quad q=\frac{2c}{a}+\lambda\frac{b}{a},$$

where λ is a root of $\mathbf{H}=0$ and k is determined as in Ex. 15.

17. Find substitutions of the form $y=x^2+px+q$ which will transform

$$u\equiv x^3+3x^2+15x+13=0$$

into the form $y^3=k^3$, giving the value of k in each case.

[λ, μ are the roots of $x^2+2x-3=0$ and $(\frac{1}{3}L)^3$, $(\frac{1}{3}M)^3$ are the roots of

$$t^2+Gt-H^3=0,$$

i.e. of $t^2=4^3$. If we take $\lambda=1$, $\mu=-3$, we must take $L=-6$, $M=6$. It is thus found that each of the substitutions, $y=x^2+4x+11$ and $y=x^2+7$, will reduce $u=0$ to $y^3=8^3$.]

18. In Ex. 17, if $y=x^2+4x+11$, we may have $y=8$, giving $x^2+4x+3=0$. Explain why it is not to be expected that both roots of this equation satisfy $u=0$.

9. The Standard Form of the Biquadratic Equation is

$$u=ax^4+4bx^3+6cx^2+4dx+e=0. \quad \text{.....(A)}$$

If $x=y-b/a$, this becomes

$$ay^4+\frac{6H}{a}y^2+\frac{4G}{a^2}y+\frac{K}{a^3}=0, \quad \text{.....(B)}$$

where $H=ac-b^2$, $G=a^2d-3abc+2b^3$, $K=a^3e-4a^2bd+6ab^2c-3b^4$.

If $z=ay=ax+b$, the equation becomes

$$z^4+6Hz^2+4Gz+K=0. \quad \text{.....(C)}$$

If α, β, γ, δ are the roots of (A), those of (B) are $\alpha+b/a$, $\beta+b/a$, $\gamma+b/a$, $\delta+b/a$, and those of (C) are $a\alpha+b$, $a\beta+b$, $a\gamma+b$, $a\delta+b$.

* This is a case of '*Tschirnhausen's Transformation.*' (See Art. 17.)

10. Some Important Functions of the Roots. Let

$$\lambda=\beta\gamma+\alpha\delta,\quad \mu=\gamma\alpha+\beta\delta,\quad \nu=\alpha\beta+\gamma\delta.$$

(1) Any rearrangement of the letters α, β, γ, δ transforms any one of the functions λ, μ, ν into itself or into one of the other two.

It will be found that the existence of functions having this property makes the solution of (**A**) depend on that of a cubic equation.

(2) The cyclic substitution $(\alpha\beta\gamma)$ changes λ to μ, μ to ν, ν to λ.

We have

$$\left.\begin{aligned}(\beta-\gamma)(\alpha-\delta)&=\nu-\mu\\(\gamma-\alpha)(\beta-\delta)&=\lambda-\nu\\(\alpha-\beta)(\gamma-\delta)&=\mu-\lambda\end{aligned}\right\},\quad \text{.....(D)}$$

Hence if two of α, β, γ, δ are equal, then two of λ, μ, ν are equal, and *vice versa*.

Also if three of α, β, γ, δ are equal, then $\lambda=\mu=\nu$, and *vice versa*.

(3) *The functions λ, μ, ν are the roots of*

$$a^3y^3-6a^2cy^2+4a(4bd-ae)y-8(2ad^2+2eb^2-3ace)=0. \quad \text{.....(E)}$$

For

$$\Sigma\lambda=\Sigma\alpha\beta=6c/a,$$

$$\Sigma\mu\nu=\Sigma\alpha^2\beta\gamma=\Sigma\alpha\,.\,\Sigma\alpha\beta\gamma-4\alpha\beta\gamma\delta=4(4bd-ae)/a^2,$$

$$\lambda\mu\nu=\Sigma\alpha^2\beta^2\gamma^2+\alpha\beta\gamma\delta\Sigma\alpha^2$$

$$=\frac{4}{a^2}(4d^2-3ce)+\frac{4e}{a^3}(4b^2-3ac)$$

$$=\frac{8}{a^3}(2ad^2+2eb^2-3ace).$$

The second term of this equation can be removed by the substitution

$$y=(4t+2c)/a, \quad \text{.....(F)}$$

and the equation becomes

$$4t^3-It+J=0, \quad \text{.....(G)}$$

where

$$I=ae-4bd+3c^2, \quad \text{.....(H)}$$

$$J=ace+2bcd-ad^2-c^3-eb^2. \quad \text{.....(I)}$$

Expressed as a determinant

$$J=\begin{vmatrix}a&b&c\\b&c&d\\c&d&e\end{vmatrix}. \quad \text{.....(J)}$$

It will be shown that the solution of (**A**) depends on equation (**G**), which is called *the reducing cubic*.

The functions I and J are of great importance in the theory of the biquadratic.

(4) **Roots of the Reducing Cubic.** Let t_1, t_2, t_3 be the roots of equation (G) corresponding to λ, μ, ν respectively. Then by (F)

$$\frac{4t_1}{a} = \lambda - \frac{2c}{a} = \lambda - \tfrac{1}{3}(\lambda+\mu+\nu) = \tfrac{1}{3}\{\lambda - \mu - (\nu - \lambda)\}; \quad \text{.....(K)}$$

whence, by (A), $t_1 = \frac{1}{12}a\{(\gamma-\alpha)(\beta-\delta) - (\alpha-\beta)(\gamma-\delta)\}$, ...(L)
with similar values for t_2, t_3.

If $\alpha, \beta, \gamma, \delta$ are all real, or if they are all imaginary, then t_1, t_2, t_3 are all real, and conversely.

If two of the roots $\alpha, \beta, \gamma, \delta$ are real and two are imaginary, then only one of the three t_1, t_2, t_3 is real, and conversely.

For λ, μ, ν, and therefore also t_1, t_2, t_3, are all real if $\alpha, \beta, \gamma, \delta$ are all real, or if $\alpha, \beta, \gamma, \delta$ are of the forms $l \pm \iota m$, $l' \pm \iota m'$.

Also if α, β are real and $\gamma = l + \iota m$, $\delta = l - \iota m$, then ν is real and λ, μ are imaginary. Moreover, one of these cases must arise, and so the converse statements are true.

(5) **The Functions I, J.** Since t_1, t_2, t_3 are functions of the *differences* of $\alpha, \beta, \gamma, \delta$, *so also are* I, J.

It follows that I and J are the same for equations (A) and (B), and therefore in equations (H) and (I) we may put

$$b = 0, \quad c = H/a, \quad d = G/a^2, \quad e = K/a^3,$$

giving $$a^2 I = K + 3H^2 \quad \text{and} \quad a^3 J = HK - G^2 - H^3,$$

whence the important identities

$$K = a^2 I - 3H^2, \quad G^2 + 4H^3 = a^2(HI - aJ). \quad \text{.....(M)}$$

Another important equality is

$$(\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2 = 24I/a^2. \quad \text{.....(N)}$$

For $$\Sigma(\beta-\gamma)^2(\alpha-\delta)^2 = \Sigma(\nu-\mu)^2 = 2(\Sigma\lambda^2 - \Sigma\mu\nu),$$

and
$$\Sigma\lambda^2 - \Sigma\mu\nu = (\Sigma\lambda)^2 - 3\Sigma\mu\nu$$
$$= \left(\frac{6c}{a}\right)^2 - 3\,\frac{4(4bd-ae)}{a^2} = \frac{12}{a^2}(ae - 4bd + 3c^2) = 12I/a^2.$$

(6) **The Discriminant.** By equations (D),

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2$$
$$= a^6(\mu-\nu)^2(\nu-\lambda)^2(\lambda-\mu)^2$$
$$= 4^6(t_2-t_3)^2(t_3-t_1)^2(t_1-t_2)^2 \text{ by equation (K)}$$
$$= 256\Delta \text{ by Art. 2,} \quad \text{.....(O)}$$

where $$\Delta = I^3 - 27J^2. \quad \text{.....(P)}$$

The function Δ is called the *discriminant* of the quartic u, and its vanishing is the necessary and sufficient condition that the equation $u=0$ may have two equal roots.

11. Character of the Roots. If two roots are equal, then $\Delta=0$. Excluding this case and remembering that if there are any imaginary roots, then these occur in pairs, the possible cases are :

(1) *All the roots may be real*, and then $\Delta>0$.

(2) *Two roots may be real and two imaginary.* Denoting the roots by α, β, $l\pm\iota m$, it is easily shown that the product of the squares of the differences of the roots is $-4m^2(\alpha-\beta)^2\{(\alpha-l)^2+m^2\}^2\{(\beta-l)^2+m^2\}^2$, and so $\Delta<0$.

(3) *All the roots may be imaginary.* Denoting them by $l\pm\iota m$, $l'\pm\iota m'$, we find that the product of the squares of the differences of the roots is $16m^2m'^2\{(l-l')^2+(m+m')^2\}^2\{(l+l')^2+(m-m')^2\}^2$, and so $\Delta>0$.

Hence (4) *If* $\Delta<0$, *two roots are real and two imaginary.*

(5) If $\Delta>0$, *the roots are all real or all imaginary.*

For the criterion distinguishing the two cases of (5), see Art. 15.

12. Ferrari's Solution of the Biquadratic. Writing the equation

$$u=ax^4+4bx^3+6cx^2+4dx+e=0, \quad \text{.....(A)}$$

we assume that

$$au=(ax^2+2bx+s)^2-(2mx+n)^2. \quad \text{.....(B)}$$

Expanding and equating coefficients, we have

$$2m^2=as+2b^2-3ac, \quad mn=bs-ad, \quad n^2=s^2-ae. \quad \text{.....(C)}$$

Eliminating m, n,

$$(s^2-ae)(as+2b^2-3ac)=2(bs-ad)^2,$$

which reduces to

$$s^3-3cs^2+(4bd-ae)s+(3ace-2ad^2-2eb^2)=0. \quad \text{.....(D)}$$

The second term can be removed by the substitution

$$s=2t+c \quad \text{.....(E)}$$

and equation (D) becomes

$$4t^3-It+J=0, \quad \text{.....(F)}$$

which is the ' reducing cubic.' (See Art. 10.) Equations (C) become

$$\left.\begin{aligned} m^2&=at+b^2-ac=at-H\\ mn&=2bt+bc-ad\\ n^2&=(2t+c)^2-ae \end{aligned}\right\}. \quad \text{.....(G)}$$

Thus, if t_1 is a root of (F) *and*

$$m_1 = \sqrt{(at_1 + b^2 - ac)}, \quad n_1 = (2bt_1 + bc - ad)/m_1,$$

the equation $u = 0$ *can be put into the form*

$$(ax^2 + 2bx + c + 2t_1)^2 - (2m_1x + n_1)^2 = 0,$$

and its roots are the roots of the quadratics

$$ax^2 + 2bx + c + 2t_1 = \pm(2m_1x + n_1).$$

It should be noticed that the three roots of (F) correspond to the three ways of expressing u as the product of two quadratic factors.

Ex. 1. *Solve* $u = x^4 + 3x^3 + x^2 - 2 = 0$.

Let $$u = (x^2 + px + s)^2 - (mx + n)^2.$$

Expanding and equating coefficients, we have

$$p = \tfrac{3}{2}, \quad m^2 = 2s + \tfrac{5}{4}, \quad mn = \tfrac{3}{2}s, \quad n^2 = s^2 + 2;$$

$$\therefore \ 4s^3 - 2s^2 + 8s + 5 = 0.$$

The last equation is satisfied if $s = -\frac{1}{2}$, and then $m^2 = -1 + \frac{5}{4}$, $mn = -\frac{3}{4}$. Thus we may take

$$s = -\tfrac{1}{2}, \quad m = \tfrac{1}{2}, \quad n = -\tfrac{3}{2},$$

and

$$u = \{x^2 + (p + m)x + s + n\}\{x^2 + (p - m)x + s - n\}$$
$$= (x^2 + 2x - 2)(x^2 + x + 1).$$

Therefore the roots are $-1 \pm \sqrt{3}$, ω, ω^2, where ω is an imaginary cube root of 1.

13. Deductions from Ferrari's Solution.

Let β, γ be the roots of $ax^2 + 2bx + c + 2t_1 + (2m_1x + n_1) = 0$, ...(H)

then α, δ are the roots of $ax^2 + 2bx + c + 2t_1 - (2m_1x + n_1) = 0$. .....(I)

Further, let (γ, α), (α, β) be the roots of the equations obtained from (H) by changing the suffix 1 into 2 and 3 respectively.

Now $$a\beta\gamma = c + 2t_1 + n_1, \qquad a\alpha\delta = c + 2t_1 - n_1,$$

$$a(\beta + \gamma) = -2(b + m_1), \quad a(\alpha + \delta) = -2(b - m_1),$$

therefore $$a(\beta\gamma + \alpha\delta) = 2c + 4t_1,$$

which agrees with equation (F) of Art. 10.

Also $$a(\beta\gamma - \alpha\delta) = 2n_1, \quad a(\beta + \gamma - \alpha - \delta) = -4m_1, \ \text{.....(J)}$$

with similar equations given by the cyclic substitutions (α, β, γ) and (1, 2, 3) of the suffixes.

Hence (i) *the values of m are*

$$-\tfrac{1}{4}a(\beta + \gamma - \alpha - \delta), \quad -\tfrac{1}{4}a(\gamma + \alpha - \beta - \delta), \quad -\tfrac{1}{4}a(\alpha + \beta - \gamma - \delta).$$

In connection with these functions it should be noted that by Ch. VI, 16, Ex. 3, (iii), p. 96,

$$(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)(\alpha + \beta - \gamma - \delta) = 32G/a^3. \ \text{.....(K)}$$

Also, by (F) and (G), *the equation whose roots are*

$$\tfrac{1}{16}a^2(\beta+\gamma-\alpha-\delta)^2,\quad \tfrac{1}{16}a^2(\gamma+\alpha-\beta-\delta)^2,\quad \tfrac{1}{16}a^2(\alpha+\beta-\gamma-\delta)^2$$

is

$$4(y+H)^3-a^2I(y+H)+a^3J=0. \qquad \text{.....(L)}$$

This is an important equation, obtained in another way in Art. 15.

(ii) *The values of n are* $\tfrac{1}{2}a(\beta\gamma-\alpha\delta)$, $\tfrac{1}{2}a(\gamma\alpha-\beta\delta)$, $\tfrac{1}{2}a(\alpha\beta-\gamma\delta)$.

(iii) The values of $-\dfrac{n_1}{2m_1}$, $-\dfrac{n_2}{2m_2}$, $-\dfrac{n_3}{2m_3}$ are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\quad \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\quad \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta},$$

and from (F), (G) it follows that *these functions are the values of z found by eliminating t from*

$$z=-\frac{1}{2}\cdot\frac{2bt+bc-ad}{at+b^2-ac} \quad \text{and} \quad 4t^3-It+J=0. \qquad \text{.....(M)}$$

14. Biquadratic as the Product of Quadratic Factors.

Descartes' method. Let $u=ax^4+4bx^3+6cx^2+4dx+e$, and assume that

$$u=a(x^2+2lx+m)(x^2+2l'x+m').$$

Expanding and equating coefficients, we have

$$l+l'=2\frac{b}{a},\quad m+m'=6\frac{c}{a}-4ll',\quad lm'+l'm=2\frac{d}{a},\quad mm'=\frac{e}{a}.$$

Now

$$\begin{vmatrix} 1 & 1 & 0 \\ l & l' & 0 \\ m & m' & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \\ l' & l & 0 \\ m' & m & 0 \end{vmatrix}=\begin{vmatrix} 2 & l+l' & m+m' \\ l+l' & 2ll' & lm'+l'm \\ m+m' & lm'+l'm & 2mm' \end{vmatrix}=0.$$

Substituting the above values for $l+l'$, $m+m'$, etc., and multiplying each row by $a/2$, we have

$$\begin{vmatrix} a & b & 3c-2all' \\ b & all' & d \\ 3c-2all' & d & e \end{vmatrix}=0.$$

If we write

$$t=c-all',$$

the equation becomes

$$\begin{vmatrix} a & b & c+2t \\ b & c-t & d \\ c+2t & d & e \end{vmatrix}=0,$$

which when expanded is

$$4t^3-It+J=0.$$

Corresponding to any root t_1 of this equation, we can now find values of l, m, l', m' which will satisfy the conditions.

Again, if the equation is given in the form $u \equiv x^4 + px^2 + qx + r = 0$, we proceed as in Art. 15, using l instead of $2l$, to obtain the cubic in the form $l^2(l^2+p)^2 - 4rl^2 - q^2 = 0$, which reduces to an equation lacking the second term on substituting $z - 2p/3$ for l^2.

NOTE. In numerical work, unless this cubic has rational roots, the solution becomes very laborious by the methods described in this chapter; the student can convince himself of this by verifying the steps of Cardan's method for solving the equation, $z^3 - 11z - 15 = 0$, which is obtained as above for the biquadratic, $x^4 - 3x^2 - x + 2 = 0$.* Later, by one of the methods described in Ch. XXVII, it will be found that one root can be found approximately with very little trouble. This is all that is required, since it is only necessary to have one way of breaking up u into quadratic factors.

15. Four Real Roots. In Art. 11, it has been shown that $\Delta > 0$ is a *necessary* condition that the four roots should be real; also, by De Gua's rule (Ch. VI, 11, Ex. 1), if $H > 0$, at least two of the roots must be imaginary.

Hence, $\Delta > 0$, $H < 0$ are *necessary* conditions that all the roots should be real; but these two alone are not *sufficient*.

The complete set of conditions may be found by using Sturm's theorem, given later. This, the usual course adopted in text-books, involves rather tedious reckoning; the conditions can be found much more easily by the use of either Ferrari's or Descartes' solution of the biquadratic.

Theorem. *The necessary and sufficient conditions that the roots of*

$$u \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

are all real is (i) $\Delta > 0$, $H < 0$, *and* $12H^2 > a^2 I$,

or their equivalents (ii) $\Delta > 0$, $H < 0$, *and* $2HI > 3aJ$.

Let $z = ax + b$; then by Art. 9, (**C**), and Art. 10, (**M**), z is given by

$$v \equiv z^4 + 6Hz^2 + 4Gz + K = 0, \quad \text{where} \quad K = a^2 I - 3H^2.$$

Now, $v = 0$ has the same number of real roots as $u = 0$.

For $v = 0$, using the method of Descartes, and supposing that

$$v \equiv (z^2 + 2lz + m)(z^2 - 2lz + m'),$$

we get, by equating coefficients,

$$-4l^2 + m + m' = 6H, \quad -l(m - m') = 2G, \quad mm' = K.$$

Eliminating m and m' from these equations, $(2l^3 + 3Hl)^2 - G^2 = l^2 K$, or

$$4l^6 + 12Hl^4 + (9H^2 - K)l^2 - G^2 = 0.$$

Hence, since $K = a^2 I - 3H^2$, the values of l^2 are the roots of

$$4y^3 + 12Hy^2 + (12H^2 - a^2 I)y - G^2 = 0; \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

which, with the substitution, $y + H = at$, reduces to $4t^3 - It + J = 0$.

* Thus, $u^3 + v^3 = 15$, $uv = 11/3$, $u^3 - v^3 = \pm 5{\cdot}27397$, $u = 2{\cdot}16423$, $v = 1{\cdot}6943$, $z = u + v = 3{\cdot}85845$, from which $u \equiv (x^2 + 2{\cdot}42042x + 1{\cdot}63580)(x^2 - 2{\cdot}42042x + 1{\cdot}22265) = 0$.

Now the three values of l^2 correspond to the three ways of expressing v as the product of two quadratic factors. Hence,

(i) If the roots of $v=0$ are all real, the three values of l^2 must be all real and positive, i.e. the three roots of (**A**) are all positive.

(ii) If the roots of $v=0$ are all imaginary, then, since their sum is zero, we may denote them by $\lambda \pm \iota\mu$, $-\lambda \pm \iota\mu'$, and thus the values of l^2 are λ^2, $-\frac{1}{4}(\mu+\mu')^2$, $-\frac{1}{4}(\mu-\mu')^2$, i.e. the roots of (**A**) are all real, but only one is positive ; and conversely.

(iii) If two of the roots of $v=0$ are real and two imaginary, we may denote them by $\lambda, \mu, \lambda' \pm \iota\mu'$, where $\lambda+\mu+2\lambda'=0$; and then the values of l^2, i.e. the roots of (**A**), are λ'^2, $\frac{1}{4}(\lambda+\lambda' \pm \iota\mu')^2$, of which one is real and the other two imaginary ; and conversely.

It follows that the roots of $v=0$, and therefore those of $u=0$, are all real, if, and only if, the roots of (**A**) are all real and positive.

Now, the roots of (**A**) are real if those of $4t^3 - It + J = 0$ are real ; and this is the case if, and only if, $\Delta = I^3 - 27J^2 > 0$; also, the roots of (**A**) are, by Descartes' rule (Ch. VI, 9, 10), all real and positive, if, and only if, H is negative and $12H^2 - a^2I$ is positive ; which proves the first part of the theorem.

To prove the equivalence of the second set of conditions, suppose first that

$$\Delta > 0, \quad H < 0, \quad \text{and} \quad 12H^2 > a^2I.$$

Then, since $\Delta = I^3 - 27J^2 > 0$, we have $I > 0$; hence, since $H < 0$, it follows that

$$HI < 0.$$

Also, $12H^2I^2 > a^2I^3 > 27a^2J^2$, so that $4H^2I^2 > 9a^2J^2$, i.e. $|\, 2HI \,| > |\, 3aJ \,|$; and, since HI is negative, therefore $2HI < 3aJ$.

Next, suppose that $\Delta > 0$, $H < 0$, and $2HI < 3aJ$.

First observe that, if p and q are any real numbers, positive or negative, $p^2 + pq + q^2 = (p + \frac{1}{2}q)^2 + \frac{3}{4}q^2 > 0$; hence, since $p^3 - q^3 = (p-q)(p^2+pq+q^2)$, it follows that $p^3 \gtrless q^3$ according as $p \gtrless q$.*

Then, since $2HI < 3aJ$, and $\Delta > 0$; we have $8H^3I^3 < 27a^3J^3 < a^3I^3J$; but I is positive, and therefore $8H^3 < a^3J$. Now, $G^2 + 4H^3 = a^2(HI - aJ)$; hence, by addition, $G^2 + 12H^3 < a^2HI$;

therefore $12H^3 < a^2HI$, and dividing by H, which is negative, $12H^2 > a^2I$.

Thus, the two sets of conditions are equivalent.

* In general, if n is *odd*, $p^n - q^n = (p-q)(p^{n-1} + p^{n-2}q + \ldots + q^{n-1})$, where the last factor is the product of pairs of complex conjugate factors, and is therefore positive ; hence $p^n \gtrless q^n$ according as $p \gtrless q$.

But, if n is even, $p^n - q^n = (p-q)(p+q)$ (a positive factor), hence $p^n \gtrless q^n$ as $|p| \gtrless |q|$, and not as $p \gtrless q$.

Alternatively, using Ferrari's method, the equation whose roots are

$$\tfrac{1}{16}a^2(\beta+\gamma-\alpha-\delta)^2, \quad \tfrac{1}{16}a^2(\gamma+\alpha-\beta-\delta)^2, \quad \tfrac{1}{16}a^2(\alpha+\beta-\gamma-\delta)^2$$

is $$4(y+H)^3-a^2I(y+H)+a^3J=0,$$

which, when expanded, is equation (**A**) on p. 192.

There are three possibilities.

(i) All the roots of $u=0$ may be real; in that case the roots of (**A**) are *all three real and positive.*

(ii) All the roots of $u=0$ may be imaginary; in that case we may take

$$\alpha=\lambda+\iota\mu, \quad \beta=\lambda-\iota\mu, \quad \gamma=\lambda'+\iota\mu', \quad \delta=\lambda'-\iota\mu';$$

and the roots of (**A**) are $-\tfrac{1}{4}a^2(\mu-\mu')^2$, $-\tfrac{1}{4}a^2(\mu+\mu')^2$, $\tfrac{1}{4}a^2(\lambda-\lambda')^2$, which are all three *real*, but *one only* is *positive.*

(iii) Two of the roots of $u=0$ may be real and two imaginary; we may take α, β to be the real roots and $\gamma=\lambda+\iota\mu$, $\delta=\lambda-\iota\mu$; and then

$$\tfrac{1}{16}a^2(\beta-\alpha+2\iota\mu)^2, \quad \tfrac{1}{16}a^2(\alpha-\beta+2\iota\mu)^2, \quad \tfrac{1}{16}a^2(\alpha+\beta-2\lambda)^2$$

are the roots of (**A**), of which *one only is real.*

The proof proceeds as before.

Another method of solving the biquadratic, due to *Euler*, is given in Exercise XXI, 17.

16. Transformation into the Reciprocal Form.

The substitution $x=py+q$ transforms the equation $u=0$ into

$$ap^4y^4+4Bp^3y^3+6Cp^2y^2+4Dpy+E=0, \quad \text{.....(A)}$$

where $$B=aq+b, \quad C=aq^2+2bq+c,$$

$$D=(a, b, c, d\,\|\,q, 1)^3, \quad E=(a, b, c, d, e\,\|\,q, 1)^4.$$

This will be a reciprocal equation if $ap^4=E$, $Bp^3=Dp$,

that is, if q, p are given by $aD^2=B^2E$, $p^2=D/B$. .....(**B**)

Suppose that the conditions (**B**) are satisfied, and that the roots of (**A**) are y_1, y_2, y_3, y_4 where $y_2y_3=y_1y_4=1$. Let α, β, γ, δ be the corresponding roots of $u=0$, so that

$$\alpha=py_1+q, \quad \beta=py_2+q, \quad \gamma=py_3+q, \quad \delta=py_4+q.$$

It follows that

$$p^2=(\beta-q)(\gamma-q)=(\alpha-q)(\delta-q), \quad \text{.....(C)}$$

and therefore

$$q=\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}, \quad p^2=\frac{(\beta-\alpha)(\beta-\delta)(\gamma-\alpha)(\gamma-\delta)}{(\beta+\gamma-\alpha-\delta)^2}. \quad \text{.....(D)}$$

Hence by Art. 13, (iii), *the values of q are the roots of the equation in z found by eliminating t from*

$$z=-\frac{1}{2}\cdot\frac{2bt+bc-ad}{at+b^2-ac} \quad \textit{and} \quad 4t^3-It+J=0. \quad \text{.....(E)}$$

NOTE. The points corresponding to q and $q \pm p$, as given by (C), are respectively the centre C and the foci F, F' of the involution determined by the points $\alpha, \beta, \gamma, \delta$, in which (β, γ) and (α, δ) are corresponding pairs of points. That is to say, the line segments are such that $C\beta \,.\, C\gamma = C\alpha \,.\, C\delta = CF^2 = F'C^2$.

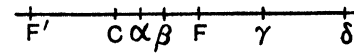


FIG. 29.

Ex. 1. *Find a substitution of the form* $x = py + q$ *which will change the equation*

$$x^4 + x^3 - 2x - 1 = 0 \quad \ldots\ldots\ldots\

Case of the biquadratic. Let α, β, γ, δ be the roots of

$$u \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

We shall prove that *in general three sets of values of p and q can be found such that if x is eliminated between*

$$u = 0 \quad \textit{and} \quad y = x^2 + px + q,$$

the resulting equation is of the form

$$y^4 + fy^2 + g = 0. \quad \ldots\ldots\ldots\

As will be seen in the next example, *each value of y gives one and only one value of x which satisfies the equation* $u=0$.

Ex. 1. *Find a substitution of the form* $y=x^2+px+q$ *which will reduce the equation*

$$x^4+12x-5=0 \quad \text{.....(A)}$$

to the form $y^4+fy^2+g=0$. *Find the values of f, g, and use the result to solve the equation.*

Here $a=1, b=0, c=0, d=3, e=-5$, so that $I=-5, J=-9$; and using Art. 13, (iii), we have

$$z=\frac{3}{2t} \quad \text{where} \quad 4t^3+5t-9=0.$$

One value of t is 1, giving $z=\frac{3}{2}$, and then $p=3$, $q=0$. One such substitution is therefore

$$y=x^2+3x. \quad \text{.....(B)}$$

To eliminate x from (A) and (B), we have *

$$y(x^2-3x)=x^4-9x^2=-9x^2-12x+5, \text{ from (A)};$$

$$\therefore\ (y+9)x^2-3(y-4)x-5=0. \quad \text{.....(C)}$$

But $$x^2+43x-y=0;$$

$$\therefore\ \frac{x^2}{y^2-4y+5}=\frac{3x}{y^2+9y-5}=\frac{1}{2y+5};$$

$$\therefore\ (y^2+9y-5)^2=9(y^2-4y+5)(2y+5),$$

which reduces to

$$y^4+98y^2-200=0.$$

The values of y are $\pm\sqrt{2}$, $\pm 10\iota$, and the corresponding values of x are given by

$$x=\tfrac{1}{3}(y^2+9y-5)/(2y+5).$$

Hence we find that the roots of (A) are

$$-1\pm\sqrt{2}, \quad 1\pm 2\iota.$$

EXERCISE XXI

THE BIQUADRATIC

Unless otherwise stated $\alpha, \beta, \gamma, \delta$ *are the roots of*

$$u=ax^4+4bx^3+6cx^2+4dx+e=0.$$

1. If $\beta+\gamma=\alpha+\delta$, show that $a^2d+2b^3-3abc=0$.

2. If $(\beta+\gamma)\alpha\delta=(\alpha+\delta)\beta\gamma$, show that $e^2b+2d^3-3edc=0$.

[Deduce from Ex. 1.]

3. If $\beta\gamma=\alpha\delta$, show that $ad^2=b^2e$.

4. If
$$L=\beta\gamma+\alpha\delta+\omega(\gamma\alpha+\beta\delta)+\omega^2(\alpha\beta+\gamma\delta),$$
$$M=\beta\gamma+\alpha\delta+\omega^2(\gamma\alpha+\beta\delta)+\omega(\alpha\beta+\gamma\delta),$$

show that the interchange of any two of the four $\alpha, \beta, \gamma, \delta$ changes L^3 into M^3 and M^3 into L^3.

* In the case of the general equation, $u=0$, and the substitution, $y=x^2+px+q$, the method for obtaining the second quadratic (C) is

$$y[ax^2+(4b-ap)x]=(x^2+px+q)[ax^2+(4b-ap)x]=ax^4+4bx^3+mx^2+nx,$$

and thus, $y[ax^2+(4b-ap)x]+6cx^2+4dx+e=mx^2+nx$.

5. Show that the equation whose roots are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}, \quad \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta}, \quad \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}$$

is $$aD^2=B^2E$$

where $B=az+b$, $D=(a, b, c, d)(z, 1)^3$, $E=(a, b, c, d, e)(z, 1)^4$.

[Diminish the roots of $u=0$ by z and use Ex. 3.]

6. If α, β, γ, δ are the roots of $u\equiv x^4+qx^2+rx+s=0$, show that the equation whose roots are

$$\beta\gamma+\alpha\delta, \quad \gamma\alpha+\beta\delta, \quad \alpha\beta+\gamma\delta$$

is $$z^3-qz^2-4sz+4qs-r^2=0,$$

and that if z_1 is a root of this equation, then

$$u\equiv\left(x^2-\frac{rx}{\lambda_1-\lambda_2}+\lambda_1\right)\left(x^2+\frac{rx}{\lambda_1-\lambda_2}+\lambda_2\right),$$

where λ_1, λ_2 are the roots of

$$\lambda^2-\lambda z_1+s=0.$$

7. Show that the equation whose roots are

$$(\beta-\gamma)(\alpha-\delta), \quad (\gamma-\alpha)(\beta-\delta), \quad (\alpha-\beta)(\gamma-\delta)$$

is $$a^3z^3-12aIz\pm16\sqrt{\Delta}=0.$$

[Use equations (D), (G) of Art. 10, and Exercise XX, 3, (iii).]

8. Solve $x^4-2x^3+a(2x-1)=0$ by putting it in the form

$$(x^2-x+s)^2=(2s+1)x^2-2(a+s)x+a+s^2,$$

and choosing s so that the right-hand side is a perfect square. Hence show that the equation has two and only two real roots unless $a=1$ or 0.

Solve by Ferrari's method:

9. $x^4+12x-5=0$.

10. $x^4+x^3-2x-1=0$.

11. $x^4-3x^2-4x-3=0$.

12. $x^4-4x^3+5x+2=0$.

13. $x^4+12x^3+54x^2+96x+40=0$.

14. Express $x^4-4x^3+7x^2-6x+3$ as the product of quadratic factors in three different ways.

15. If α, β, γ, δ are the roots of $x^4+3x^3+x^2-2=0$, prove that the equation whose roots are

$$(\beta+\gamma-\alpha-\delta)^2, \quad (\gamma+\alpha-\beta-\delta)^2, \quad (\alpha+\beta-\gamma-\delta)^2$$

is $$z^3-19z^2+243z-225=0.$$

[Use equation (L) of Art. 13.]

16. Show that the equation whose roots are

$$\tfrac{1}{16}a^2\{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)\}^2$$

and two similar expressions is

$$4(x+\mathbf{L})^3-e^2I(x+\mathbf{L})+e^3J=0$$

where $\mathbf{L}=ce-d^2$.

[This follows from equation (L) of Art. 13 by changing α, β, γ, δ into their reciprocals.]

17. *Euler's Solution.* If $\surd l, \surd m, \surd n$ denote either of the square roots of l, m, n and $z = \surd l + \surd m + \surd n$, prove that

(i) $z^4 - 2\Sigma l \,.\, z^2 - 8\surd l \surd m \surd n \,.\, z + \Sigma l^2 - 2\Sigma mn = 0.$

(ii) If this equation is identical with $z^4 + 6Hz^2 + 4Gz + K = 0,$

then l, m, n are the values of s given by $4s^3 + 12Hs^2 + (9H^2 - K)s - G^2 = 0,$

which is the same as $4(s+H)^3 - a^2 I(s+H) + a^3 J = 0.$

(iii) The complete solution of the biquadratic $u = 0$ is given by

$$a\alpha + b = -\surd l + \surd m + \surd n, \quad a\beta + b = \surd l - \surd m + \surd n,$$
$$a\gamma + b = \surd l + \surd m - \surd n, \quad a\delta + b = -\surd l - \surd m - \surd n,$$

the square roots being chosen so that $\surd l \,.\, \surd m \,.\, \surd n$ has the same sign as G.

Observe that the values of l, m, n are $\frac{1}{16}a^2(\beta + \gamma - \alpha - \delta)^2$, etc., see equations (**L**) and (**K**) of Art. 13.

18. Find a substitution of the form $x = py + q$ which will transform the equation

$$x^4 - 14x^2 - 40x - 11 = 0$$

into the reciprocal form. Show that one such substitution is $x = 2y - 1$. Use this to solve the equation.

19. In Ex. 1 of Art. 17, if each value of y which satisfies the equation

$$y^4 + 98y^2 - 200 = 0$$

is substituted for y in $y = x^2 + 3x$, we get eight values of x, four of which satisfy

$$x^4 + 12x - 5 = 0.$$

Without solving any equations, show that the other four values of x are the roots of

$$x^4 + 12x^3 + 54x^2 + 96x + 40 = 0. \qquad (Cf.\ Ex.\ 13.)$$

[The left-hand side is

$$\{(x^2+3x)^4 + 12(x^2+3x) - 5\} \div (x^4 + 12x - 5).]$$

20. Find a substitution of the form $y = x^2 + px + q$ which will reduce the equation $x^4 + x^3 - 2x - 1 = 0$ to the form $y^4 + fy^2 + g = 0.$

Show that one such substitution is $y = x^2 - x - \frac{1}{2}$, this reducing the equation to $(2y)^4 + 26(2y)^2 - 11 = 0$. Use this to solve the equation. (*Cf. Ex.* 10.)

21. Find a substitution of the form $y = x^2 + px + q$ which will reduce the equation

$$2x^4 + 8x^3 + 4x + 13 = 0$$

to the form $y^4 + fy^2 + g = 0$, showing that one such substitution is

$$y = x^2 + 3x - 1,$$

this reducing the equation to

$$4\left(\frac{y}{3}\right)^4 + 4\left(\frac{y}{3}\right)^2 - 9 = 0.$$

CHAPTER XIII

THEORY OF IRRATIONALS

1. Sections of the System of Rational Numbers. Suppose the rational numbers to be represented by points in a straight line, and let the line be cut at any point P.

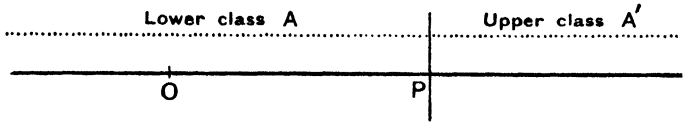


FIG. 30.

The system of rationals is thus divided into two parts, which will be called *the lower class* A and *the upper class* A'. The class A contains all the rationals to the left of P, and the class A' contains all those to the right of P. Thus, any number in A is less than any number in A'. Two distinct cases arise, according as the point P does, or does not, represent a rational number.

First case. Suppose P to represent some rational, 3 for example. The class A contains every rational less than 3, and the class A' contains every rational greater than 3. *The number* 3 *may be assigned to either class.* If 3 belongs to A, it is the greatest number of this class, and there is no least number in A'.

For if a' is supposed to be the least number in A', rationals exist which are less than a' and greater than 3, and which therefore belong to the class A'.

Similarly if 3 is assigned to A', it is the least number of this class, and the class A has no greatest number.

Thus *corresponding to any rational*, the classification is such that ***either the class A has a greatest number or the class A' has a least number.***

Second case. Suppose that the point P does not represent a rational number. For example, let OP be equal to the side of a square whose area is 7 square units.

Classify the rationals according to the following rule :

The lower class A is to contain all the negative numbers, zero and every positive number whose square is less than 7.

The upper class A' is to contain every positive number whose square is greater than 7.

The classification is such that

(i) *Every rational is included in one or other of two classes A, A'.* For no rational exists whose square is equal to 7.

(ii) *Any number in A is less than any number in A'.* For if a is any positive number in A and a' is any number in A', we have $a^2<7<a'^2$, and so $a<a'$.

(iii) *The class A has no greatest number and the class A' has no least number.* For, suppose that a is the greatest number of A, then $a^2<7$, and we can find a rational b such that $b>a$ and $b^2<7$. To do this, we have to find b (greater than a), so that $b^2-a^2<7-a^2$. This will be the case if

$$b-a<\frac{7-a^2}{b+a}<\frac{7-a^2}{2a}.$$

Thus, b belongs to the class A, and a is not the greatest number in A. Similarly it can be shown that A' has no least number.

This classification is said to *define* the *irrational number* which we denote by $\sqrt{7}$, and we assign it a place on the scale with rationals as follows : $\sqrt{7}$ is to follow or to precede any positive rational a, according as $a^2<7$ or $a^2>7$.

2. Dedekind's Definition. Suppose that a certain rule enables us to divide the whole system of rationals into two classes, a lower class A and an upper class A', so that any number in A is less than any number in A'. Two cases arise :

(i) If A has a greatest number or if A' has a least number, the classification defines a *rational number,* namely the greatest number in A or the least number in A'.

(ii) If there is no greatest number in A and no least number in A', the classification defines an *irrational number* which is to follow all the numbers in A and to precede all those in A'.

Any rational or irrational number is called a *real number,* and the aggregate of rationals and irrationals is known as the *system of real numbers.*

The classification just described will be called the *classification* or the *section* (A, A'), and the real number defined by it may be known as the number (A, A').

The *real number zero* is defined by the section (A, A'), where A contains all the negative rationals and A' contains all the positive rationals.

If A contains some positive rationals, the section (A, A') defines a *positive real number.*

If A' contains some negative rationals, (A, A') defines a *negative real number.*

We are not justified in regarding a ' real number ' as a ' number ' until we have given fresh definitions of equality, inequality and the fundamental operations of arithmetic. Moreover, these definitions must be in agreement with those already given for the system of rationals.

3. Equality and Inequality. Two real numbers are said to be *equal* when they are defined by the same classification of rationals.

If α and β are real numbers, and some of the rationals which follow α precede β, we say that α is *less than* β and that β is *greater than* α.

If α and β are unequal real numbers, infinitely many rationals lie between them.

For let $\alpha<\beta$, where α and β are defined by the classifications (A, A'), (B, B') respectively. Since these are different classifications, at least one rational r lies between α and β. Then r follows α and precedes β, therefore r belongs to each of the classes A' and B. If r is the *only* rational between

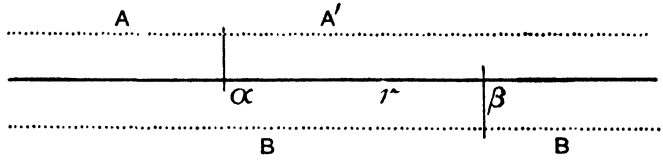


FIG. 31.

α and β, then r is the least number in A' and the greatest in B. Hence, by Art. 2, (i), each of the classifications (A, A'), (B, B') defines the same number r, and we should have $\alpha=r=\beta$, which is impossible, for $\alpha<\beta$. Thus at least two, and therefore infinitely many, rationals lie between α and β. (See Ch. II, 10.)

4. Theorem. *Let A and A' be two classes of rationals such that*

(i) *Each class contains at least one number.*

(ii) *Any number in A is less than any number in A'.*

(iii) *Two numbers a and a' can be chosen from A and A' respectively, so that*

$$a' - a < \epsilon,$$

where ϵ is any positive number we may choose, however small.

Then there is one and only one real number α such that $a \leqslant \alpha \leqslant a'$ where a is any number in A and a' is any number in A'.

Proof. Divide the system of rationals into two classes according to the following rule : The lower class is to contain every rational less than any number a' in A'. The upper class is to contain every rational greater than any number a in A.

It will be proved that not more than one rational can escape classification. For if two rationals l and l' $(l<l')$ so escape, then for every a and a'

$$a < l < l' < a'.$$

This contradicts the hypothesis, for we can choose a and a' so that

$$a' - a < l' - l,$$

so that either a or a' lies between l and l'.

If there is one rational which escapes classification, then it is to be assigned to either the upper or the lower class. Thus, by Dedekind's definition, the classification defines a real number α which satisfies the conditions stated above.

This theorem may also be stated as follows :

If (a_n) and (a_n') are sequences of rationals such that

$$\text{(i)}\ a_1 \leqslant a_2 \leqslant a_3 \ldots \leqslant a_n \ldots \leqslant a_n' \ldots \leqslant a_3' \leqslant a_2' \leqslant a_1',$$

and (ii) *it is possible to find n such that*

$$a_n' - a_n < \epsilon,$$

where ϵ is any positive number we may choose, however small, then one real number α, and only one, exists such that $a_n \leqslant \alpha \leqslant a_n'$ for every n.

5. Endless Decimals. Using the ordinary notation, let

$$a_0 \cdot a_1 a_2 \ldots a_n \ldots$$

be an endless decimal, where the successive figures are formed according to some definite rule.

Let $\quad d_n = a_0 \cdot a_1 a_2 \ldots a_n \quad$ and $\quad d_n' = d_n + 1/10^n.$

The sequences (d_n) and (d_n') obviously satisfy the conditions of the last theorem, and therefore a real number δ is defined by

$$d_0 \leqslant d_1 \leqslant d_2 \ldots \leqslant \delta \ldots \leqslant d_2' \leqslant d_1' \leqslant d_0'.$$

We say that the decimal *represents* or *is equal to* this real number δ.

Conversely, *any real number* δ *can be represented by a decimal.* For, using the same notation as in the preceding, if δ is known, then for any suffix n, we can find d_n so that $d_n \leqslant \delta \leqslant d_n'$. That is to say, we can calculate to any number of figures a decimal which represents δ.

Of course, a terminating decimal may be regarded as an endless decimal in which, after a certain stage, all the figures are zeros.

Since any rational can be represented by a decimal which terminates or recurs, it follows that *a non-recurring endless decimal represents an irrational number, and conversely.*

6. The fundamental Operations of Arithmetic. In what follows, a Greek letter denotes a *real* number, a small Roman letter represents a *rational.* Real numbers will be defined by using the theorem of Art. 4, and when we say that a real number α is defined by $a \leqslant \alpha \leqslant a'$, it is presumed that a, a' are any numbers in classes A, A' of rationals which satisfy the conditions of the theorem.

(1) *Addition.* If α, β are real numbers defined by

$$a \leqslant \alpha \leqslant a' \quad \text{and} \quad b \leqslant \beta \leqslant b',$$

then $\alpha + \beta$ is defined by $a + b \leqslant \alpha + \beta \leqslant a' + b'$.

This classification defines a real number, for

(i) There is at least one $a + b$ and one $a' + b'$.

(ii) Any $a + b \leqslant$ any $a' + b'$.

(iii) We can choose a, a', b, b' so that

$$a' - a < \tfrac{1}{2}\epsilon \quad \text{and} \quad b' - b < \tfrac{1}{2}\epsilon,$$

where ϵ is any positive number, however small, and then

$$(a' + b') - (a + b) = (a' - a) + (b' - b) < \epsilon.$$

We define $\alpha + \beta + \gamma$ as meaning $(\alpha + \beta) + \gamma$, and it is easy to show that the commutative and associative laws hold good.

Ex. 1. *Show that* $\alpha + \beta = \beta + \alpha$.

It is obvious that $\alpha + \beta$ and $\beta + \alpha$ are defined by the same classification of rationals.

Ex. 2. *Prove that* $\alpha + 0 = \alpha$.

Zero is defined by $b < 0 < b'$, where b is any negative rational and b' is any positive rational.

Let α be defined by $a < \alpha < a'$, then $\alpha + 0$ is defined by

$$a + b < \alpha + 0 < a' + b',$$

and since every b is negative and every b' positive,

$$a + b < \alpha < a' + b'.$$

Thus $\alpha + 0$ and α are defined by the same classification of rationals and are equal.

Ex. 3. *Prove that* $\alpha+(-\alpha)=0$.

If α is defined by $a<\alpha<a'$, then $\alpha+(-\alpha)$ is defined by

$$a-a'<\alpha+(-\alpha)<a'-a.$$

Now every $a<$ any a', therefore $a-a'<0<a'-a$.

Hence $\alpha+(-\alpha)$ and 0 are defined by the same classification, and are equal.

(2) *Subtraction.* We define $\alpha-\beta$ as $\alpha+(-\beta)$. Whence it follows that $(\alpha-\beta)+\beta=\alpha$, for

$$\{\alpha+(-\beta)\}+\beta=\alpha+\{\beta+(-\beta)\}=\alpha+0=\alpha.$$

(3) *Multiplication.* If α, β are positive real numbers defined by

$$a\leqslant\alpha\leqslant a' \quad \text{and} \quad b\leqslant\beta\leqslant b',$$

then $\alpha\beta$ is defined by $ab\leqslant\alpha\beta\leqslant a'b'$. These conditions define a real number, for

(i) There is at least one ab and one $a'b'$.

(ii) Every $ab\leqslant$ any $a'b'$.

(iii) We can choose a, b, a', b' so that $a'b'-ab<\epsilon$, where ϵ is any assigned positive number. For $a'b'-ab=a'(b'-b)+b(a'-a)$.

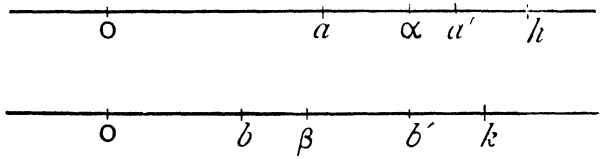


FIG. 32.

Choose rationals h, k, so that $h>\alpha$ and $k>\beta$, then if $a'<h$,

$$a'b'-ab<h(b'-b)+k(a'-a).$$

We can now choose a, b, a', b', so that

$$b'-b<\epsilon/2h \quad \text{and} \quad a'-a<\epsilon/2k,$$

and then $a'b'-ab<\epsilon$.

For zero and negative factors, the definitions are

$$\alpha\,.\,0=0=0\,.\,\alpha,$$

$$\alpha(-\beta)=-\alpha\beta=\beta(-\alpha) \quad \text{and} \quad (-\alpha)(-\beta)=\alpha\beta.$$

Further, $\alpha\beta\gamma$ is taken to mean $(\alpha\beta)\gamma$. It is easy now to show that the commutative, distributive and associative laws hold good.

Ex. 4. *Show that* $\sqrt{7}\,.\,\sqrt{7}=7$.

$\sqrt{7}$ is defined by $a<\sqrt{7}<a'$, where a, a' are any positive rationals such that

$$a^2<7<a'^2.$$

Hence $\sqrt{7}\,.\,\sqrt{7}$ is defined by $a^2<7<a'^2$, and this classification also defines the number 7.

(4) *Division.* If α is a positive real number defined by $a \leqslant \alpha \leqslant a'$, then $1/\alpha$ is defined by $1/a' \leqslant 1/\alpha \leqslant 1/a$.

These conditions define a real number, for

(i) There is at least one $1/a'$ and one $1/a$. (ii) Every $1/a' \leqslant$ any $1/a$.

(iii) We can choose a, a' so that $1/a - 1/a' < \epsilon$, where ϵ is any assigned positive number.

To prove this, we have to show that a, a' can be chosen so that

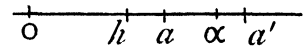


FIG. 33.

$$a' - a < aa'\epsilon.$$

First choose h so that $0 < h < \alpha$. Next choose a, a' so that $a > h$ and $a' - a < h^2\epsilon$.

Since $h < a < a'$, we have $h^2 < aa'$ and $a' - a < aa'\epsilon$.

NOTE. This reasoning depends on the existence of positive rationals less than α. Consequently *no meaning is assigned to* $1/0$.

Further definitions are $1/(-\alpha) = -1/\alpha$ and $\alpha/\beta = \alpha \,.\, 1/\beta$;
whence it follows that $(\alpha/\beta) \,.\, \beta = \alpha$; for $(\alpha \,.\, 1/\beta) \,.\, \beta = \alpha(1/\beta \,.\, \beta) = \alpha$.

Ex. 5. If α, β are positive real numbers defined by $a < \alpha < a'$ and $b < \beta < b'$, then α/β is defined by $a/b' < \alpha/\beta < a'/b$.

This follows from the definition of $1/\beta$ and multiplication.

7. Powers and Roots. If n is a positive integer and α a real number, the nth power of α (written α^n) is defined by $\alpha^n = \alpha \,.\, \alpha \,.\, \alpha \ldots$ to n factors. If m, n are positive integers, it follows that $\alpha^m \,.\, \alpha^n = \alpha^{m+n}$ and $(\alpha^m)^n = \alpha^{mn}$.

That these formulae may hold for zero and negative values of m and n, we must have $\alpha^0 = 1$ and $\alpha^{-n} = 1/\alpha^n$.

Theorem 1. (i) *If α is a positive real number, n a positive integer and x, x' positive rationals such that $x^n < \alpha < x'^n$, then positive rationals x_1, x_1' exist such that*

$$x^n < x_1^n < \alpha < x_1'^n < x'^n.$$

For if $x < x_1 < x'$, then

$$x_1^n - x^n = (x_1 - x)(x_1^{n-1} + x_1^{n-2}x + \ldots + x^{n-1}) < (x_1 - x) \,.\, nx'^{n-1}.$$

We can choose $x_1 > x$, so that $x_1 - x < (\alpha - x^n)/nx'^{n-1}$: and then

$$x_1^n - x^n < \alpha - x^n, \quad \text{so that} \quad x_1^n < \alpha.$$

The existence of a number x_1' such that $\alpha < x_1'^n < x'^n$ can be proved in a similar way.

(ii) *If ϵ is any positive number, however small, then $x_1'^n - x_1^n < \epsilon$, provided that $x_1' - x_1 < \epsilon/nx'^{n-1}$.*

For as in the preceding, $x_1'^n - x_1^n < (x_1' - x_1) \,.\, nx'^{n-1}$.

Theorem 2. *If α is a positive real number, there is one positive real number ξ, and one only, whose n-th power is equal to α.*

If $\alpha = x^n$ where x is a positive rational, there is only one such number x, for the nth powers of unequal positive rationals are unequal.

If no such x exists, divide the system of rationals into a lower class A and an upper class A', according to the following rule :

The class A is to contain every negative number, zero, and every positive number x such that $x^n < \alpha$.

The class A' is to contain every positive number x' such that $x'^n > \alpha$.

Then (i) no rational escapes classification; for no rational x exists such that $x^n = \alpha$.

(ii) Every $x <$ every x', for $x^n < \alpha < x'^n$.

(iii) There is no greatest x and no least x'. This follows from the last theorem.

Therefore the classification defines a real number ξ.

Again, for every x and every x',

$$x^n < \alpha < x'^n \quad \text{and} \quad x^n < \xi^n < x'^n.$$

Also every rational is an x or an x', therefore these may be chosen so that $x' - x$ is as small as we like. Hence, as in Theorem 1, they may be chosen so that $x'^n - x^n < \epsilon$, where ϵ is any positive number, however small.

Thus both α and ξ^n are defined by the same classification of rationals, a classification which satisfies the conditions of the theorem of Art. 4 and therefore $\xi^n = \alpha$.

Definitions. If α is a positive real number, the *principal n-th root* of α is defined as the positive real number whose nth power is equal to α. This is written $\sqrt[n]{\alpha}$, and is generally called the *n-th root of* α.

Thus $$(\sqrt[n]{\alpha})^n = \alpha.$$

It follows that

$$\sqrt[n]{\alpha} \cdot \sqrt[n]{\beta} = \sqrt[n]{(\alpha\beta)} \quad \text{and} \quad \sqrt[m]{(\sqrt[n]{\alpha})} = \sqrt[mn]{\alpha} = \sqrt[n]{(\sqrt[m]{\alpha})}.$$

If n is odd, we have

$$(-\sqrt[n]{\alpha})^n = (-1)^n (\sqrt[n]{\alpha})^n = -\alpha.$$

When n is an odd integer, we therefore define the *principal n-th root*, or simply the *n-th root*, of $-\alpha$ as $-\sqrt[n]{\alpha}$; thus

$$\sqrt[n]{(-\alpha)} = -\sqrt[n]{\alpha}.$$

If n is even, the nth power of every real number is positive, and therefore no real number exists which is the nth root of a negative number.

Indices which are Rational Fractions. If p, q are positive integers, $a^{\frac{p}{q}}$ is *defined* as $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$.

This definition assigns no meaning to a^x when x is *irrational*. This case is considered in Art. 9.

8. Surds. If a is not a perfect nth power, $\sqrt[n]{a}$ is called a *surd* of the nth order.

Two surds of the same order are *like* surds when their quotient is rational: otherwise they are *unlike* surds.

The following theorems and examples depend on the fact that a rational number cannot be equal to an irrational.

Theorem 1. *If $x+\sqrt{y}=a+\sqrt{b}$ where x, y, a, b are rationals, then $x=a$ and $y=b$, or else y and b are the squares of rationals.*

For suppose that $x\neq a$ and let $x=a+z$, then $z+\sqrt{y}=\sqrt{b}$, and by squaring,

$$2z\sqrt{y}=b-y-z^2.$$

Therefore $\sqrt{y}$ is rational, and from the given equation $\sqrt{b}$ is rational. On the other hand, if $x=a$, then $y=b$.

Ex. 1. *If $a+b\sqrt{p}+c\sqrt{q}=0$ where a, b, c are rationals and $\sqrt{p}$, $\sqrt{q}$ are unlike surds, then $a=0$, $b=0$, $c=0$.*

By transposing and squaring we can show that

$$2ab\sqrt{p}=c^2q-a^2-b^2p.$$

If $ab\neq 0$, the left-hand side would be irrational and the right-hand side would be rational; which is impossible.

Therefore $ab=0$, and consequently $a=0$ or $b=0$.

If $a=0$, then $b\sqrt{p}+c\sqrt{q}=0$; and, if $b\neq 0$, then $\sqrt{p}/\sqrt{q}=-c/b$; so that $\sqrt{p}$ and $\sqrt{q}$ would be like surds; which is not the case.

Therefore $b=0$ and $c=0$.

If $b=0$ then $a+c\sqrt{q}=0$, and therefore $a=0$ and $c=0$.

Ex. 2. *If $a+b\sqrt[3]{p}+c\sqrt[3]{p^2}=0$ where a, b, c, p are rationals and p is not a perfect cube, then a, b, c are all zero.*

Multiplying the given equation by $\sqrt[3]{p}$ we have

$$cp+a\sqrt[3]{p}+b\sqrt[3]{p^2}=0,$$

and eliminating the terms containing $\sqrt[3]{p^2}$,

$$(b^2-ac)\sqrt[3]{p}=c^2p-ab.$$

Since $\sqrt[3]{p}$ is irrational, it follows that

$$b^2=ac \quad \text{and} \quad c^2p=ab,$$

therefore

$$c^4p^2=a^2b^2=a^3c.$$

If $c\neq 0$ we should have $p^2=\left(\frac{a}{c}\right)^3$, so that $\sqrt[3]{p^2}$ would be rational, which is not the case.

Hence $c=0$, and therefore also $a=0$ and $b=0$.

Theorem 2. *Suppose that p, q are rationals and that q is not a perfect square, then if $p+\sqrt{q}$ is a root of $f(x)=0$ where $f(x)$ is a polynomial with rational coefficients, $p-\sqrt{q}$ is also a root of $f(x)=0$.*

For we have $\{x-(p+\sqrt{q})\}\{x-(p-\sqrt{q})\}=(x-p)^2-q$, and if $f(x)$ is divided by $(x-p)^2-q$, we obtain an identity of the form

$$f(x)=Q\{(x-p)^2-q\}+Rx+S,$$

where R, S are rationals. Putting $x=p+\sqrt{q}$, we have

$$0=R(p+\sqrt{q})+S.$$

Since $\sqrt{q}$ is irrational it follows that $Rp+S=0$ and $R=0$. Consequently $S=0$ and $f(x)=Q\{(x-p)^2-q\}$, therefore $p-\sqrt{q}$ is a root of $f(x)=0$.

9. Irrational Indices. The theorems of Art. 7 hold when a is a positive *real* number, the indices being *rationals*, and lead to the following:

Definition. Let a be a real number greater than unity, and let ξ be an irrational defined by $x<\xi<x'$ where x, x' belong to classes of rationals satisfying the conditions stated in Art. 4. Then a^ξ is defined by

$$a^x<a^\xi<a^{x'}.$$

These conditions define a real number, for

(i) there is at least one a^x and one $a^{x'}$, (ii) every $a^x<$every $a^{x'}$,

(iii) we can choose x and x' so that $a^{x'}-a^x<\epsilon$ where ϵ is any assigned positive number, however small.

To prove this, choose a number $h>\xi$, then $a^x<a^h$ and

$$a^{x'}-a^x=a^x(a^{x'-x}-1)<a^h(a^{x'-x}-1).$$

Next, by Ch. II, 18, (7), we can find a positive integer n such that

$$a^{\frac{1}{n}}-1<\epsilon/a^h.$$

Finally, choose x, x' so that $x'-x<1/n$, and then

$$a^{x'}-a^x<a^h(a^{\frac{1}{n}}-1)<\epsilon.$$

NOTE. The real positive value of a^ξ, defined in this way, is called its *principal value*, to distinguish it from other values which will be found later. At present a^ξ will stand for its principal value.

If $0<a<1$, a^ξ is defined by the classification $a^{x'}<a^\xi<a^x$. Or it may be defined as $(1/b)^\xi$, where $b=1/a$. We define 1^ξ as 1. At present, we are not able to assign a meaning to a^ξ when a is negative.

It is easy to show that if $a>0$, the index laws

$$a^x.a^y=a^{x+y} \quad \text{and} \quad (a^x)^y=a^{xy}$$

hold for all real values of x and y.

10. Logarithms. Theorem. *If a and N are positive real numbers and $a \neq 1$, there exists a single real number ξ such that $a^\xi = N$.*

Proof. Suppose that no rational x exists such that $a^x = N$ and first let $a > 1$. Then, by Art. 7, rationals x, x' exist such that $a^x < N < a^{x'}$.

Divide the system of rationals into a lower class A and an upper class A' by the following rule :

The class A is to contain every rational x such that $a^x < N$.

The class A' is to contain every rational x' such that $a^{x'} > N$.

Then (i) no rational escapes classification; for it is assumed that no rational x exists such that $a^x = N$. (ii) Every $x <$ every x'; for $a^x < a^{x'}$.

(iii) There is no greatest x; for if x_1 is supposed to be the greatest, as in the last article, we can find $x_2 > x_1$, so that $a^{x_2} - a^{x_1} < N - a^{x_1}$, i.e. so that $a^{x_2} < N$. Similarly it can be shown that there is no least x'.

The classification therefore defines a real number ξ.

Again, for every x and every x',

$$a^x < N < a^{x'} \quad \text{and} \quad a^x < a^\xi < a^{x'};$$

and since every rational is an x or an x', these may be chosen so that $x' - x$ is as small as we like. Hence, as in Art. 9, they may be chosen so that $a^{x'} - a^x < \epsilon$, where ϵ is any positive number, however small.

Thus both N and a^ξ are defined by the same classification of rationals, a classification which satisfies the conditions of Art. 4, and so $a^\xi = N$.

If $a < 1$, we find ξ so that $(1/a)^\xi = 1/N$, and then $a^\xi = N$.

Definition. If any positive real number a is chosen as ' base ' and N is any positive real number, the number ξ given by the relation $a^\xi = N$ is called the *logarithm* of N to the base a, and we write $\xi = \log_a N$.

11. Definitions. We say that (i) the set of real numbers (x) which lie between a and b forms the (*open*) *interval* (a, b) or the *range* $a < x < b$.

(ii) The real values of x such that $a \leqslant x \leqslant b$ form the *closed interval* (a, b) or the *range* $a \leqslant x \leqslant b$.

(iii) If $a < x \leqslant b$, the real numbers x form an interval (a, b) *open at a and closed at b*, or the *range* $(a < x \leqslant b)$.

' Interval ' generally means ' open interval.'

Any number x in the interval (a, b) is often called a *point* in the interval.

(iv) If x_1, x_2 are any numbers in the interval (a, b) and $f(x)$ is a function of x such that $f(x_1) \leqslant f(x_2)$ when $x_1 < x_2$, we say that $f(x)$ *increases steadily* through the interval, unless the sign is always that of equality, in which case $f(x)$ is constant in the interval.

If $f(x_1)<f(x_2)$ when $x_1<x_2$, then $f(x)$ is said to *increase steadily in the stricter sense.*

If $f(x_1)\geqslant f(x_2)$ when $x_1<x_2$, $f(x)$ *decreases steadily*; and if $f(x_1)>f(x_2)$ when $x_1<x_2$, $f(x)$ *decreases steadily in the stricter sense.*

(v) If a is a real number, the '*neighbourhood of* a' is the set of real numbers in the interval $(a-\epsilon, a+\epsilon)$, where ϵ is as small as we like.

12. Sections of the System of Real Numbers.

(1) **Dedekind's Theorem.** *If the system of real numbers is divided into two classes A and A' such that* (i) *each class contains at least one number,* (ii) *every real number belongs to one or other of the classes,* (iii) *any number in A is less than any number in A', then there is a single real number α, such that all numbers less than α belong to A and all numbers greater than α belong to A'. The number α may be regarded as belonging to either of the classes.*

Proof. Consider the *rationals* in A and A'. These form two classes A_1 and A_1', which form a section of the system of rationals and define a real number α. Two cases arise.

(i) If α is rational, it is the greatest number in A_1 or the least in A_1'. If α is the greatest number in A_1, it is also the greatest number in A. For suppose that there is a number β in A greater than α. Between α and β there are rationals greater than α which therefore belong to A_1' and also to A'. Hence β must belong to A', which is not the case.

Similarly if α is the least number in A_1', it is also the least in A'.

(ii) If α is irrational, it is greater than all the numbers in A_1 and less than all those in A_1'. Also, α must belong to A or to A', and just as in the preceding, we can show that it is the greatest number in A or the least in A'

The theorem just proved is of great importance. It shows that *any section of the system of real numbers defines a real number.* Thus the consideration of sections of this kind leads to no further generalisation of our idea of number. All this is sometimes expressed by saying that *the system of real numbers is closed,* or *the aggregate of real numbers is perfect.*

On the other hand, the rational numbers do not form a closed system, or in other words, the aggregate of rational numbers is not perfect, for a section of the system of rationals does not always define a rational.

(2) *If the numbers in the interval (a, b) are divided into two classes A and A', as in* (1), *the section determines a real number α.*

For if we take with A all the real numbers x such that $x\leqslant a$ and with A' all the real numbers x' such that $x'\geqslant b$, we obtain a section of the system of real numbers.

13. The Continuum. It follows from Art. 1 that if we take any point O in a straight line as origin and any length we may choose as unit of length, then to any point P in the line there corresponds a real number p.

By choosing suitable axioms regarding the characteristics of a straight line we can ensure the truth of the converse of this statement, namely, that to every real number p there corresponds a point P in the line. Thus the correspondence between the points of a straight line and the real numbers is complete.

If p is the real number corresponding to the point P of the line, we say that p is the measure of the length of OP. Hence the system of real numbers is adequate for the measurement of the *length* of a straight line. This is an instance of what is known as a *continuous magnitude.*

The aggregate of real numbers is called the *arithmetic continuum*, and the aggregate of the points of a straight line is called the *linear continuum.*

Ratio and Proportion

NOTE. *In Arts.* 14-16 *capital letters will denote concrete magnitudes and not numbers; small letters will denote positive integers.*

14. Equality and Inequality of Concrete Magnitudes. In considering how a magnitude or a quantity of any kind is to be measured, at the outset we must *define* what is meant by saying that 'A is equal to B,' 'A is greater than or less than B,' where A and B are quantities of the kind. The exact meaning of these statements depends on *the particular kind* of quantity we are considering. For instance, let A and B denote segments of straight lines; if A can be made to coincide with B, we say $A=B$; if A can be made to coincide with a part of B, we say $A<B$ and $B>A$.

The student should reflect on the meaning of equal angles, equal velocities, equal forces, equal quantities of heat, etc.

15. Ratio. We assume that a magnitude B can be divided into any number (n) of equal parts. Denoting any one of these parts by C, B is said to contain C (exactly) n times, and C is called the nth part of B. This is expressed shortly by writing $B=nC$ or $C=\frac{1}{n}B$. If A contains the nth part of B exactly m times, we denote this by writing $A=\frac{m}{n}B$.

Definition. If A and B are magnitudes of the same kind, and if integers a and b exist such that A contains the bth part of B exactly a times $\left(\text{that is, if } A=\frac{a}{b}B\right)$, the magnitudes A and B are said to be *commensurable*, and the *ratio of A to B* (written $A:B$) is defined as the number a/b.

If no integers a and b exist such that $A=\frac{a}{b}B$, the magnitudes A and B are said to be *incommensurable.*

Axiom of Archimedes. If A and B are magnitudes of the same kind, no matter how small A may be compared with B, an integer m can always be found such that $mA>B$ or $\frac{1}{m}B<A$.

16. Ratio of Two Incommensurables. If A and B are incommensurable magnitudes, the ratio of A to B (written $A:B$) is defined as the irrational number determined by the following classification of the entire system of rationals :

the Lower Class is to contain every rational a/b for which $bA>aB$,

the Upper Class is to contain every rational a'/b' for which $b'A<a'B$.

This classification defines an irrational number ; because

(1) every rational falls into one or other of the two classes ; for, by hypothesis, there is no rational a/b for which $bA=aB$:

(2) every a/b is less than any a'/b' ; for $\frac{a}{b}B<A<\frac{a'}{b'}B$:

(3) there is no greatest a/b ; for suppose a_1/b_1 to be the greatest, then by hypothesis $b_1A>a_1B$.

Now by the axiom of Archimedes, a multiple of a magnitude (however small the magnitude may be) can be found which exceeds any magnitude of the same kind, however great. Therefore an integer r can be found such that $r(b_1A-a_1B)>a_1B$, and therefore $rb_1A>(r+1)a_1B$. Let $rb_1=b_2$ and $(r+1)a_1=a_2$, then $b_2A>a_2B$ and $a_2/b_2=a_1(r+1)/rb_1>a_1/b_1$, so that a_1/b_1 is not the greatest a/b. Similarly it can be shown that there is no least a'/b' Hence the classification defines an irrational number.

Theorem. If A, B, C, D are four magnitudes of the same kind, the ratio of A to B is equal to that of C to D if $mA>=\text{or}<nB$ according as $mC>=\text{or}<nD$, for all positive integral values of m and n.*

Proof. If rationals m, n exist such that $mA=nB$, then by hypothesis $mC=nD$ and $A/B=C/D=n/m$. If no such rationals m, n exist, then A, B and also C, D are incommensurables, and the ratios A/B, C/D are irrationals defined by the same classification and are therefore equal.

If A, B, C, D are magnitudes of the same kind and $A:B=C:D$, then A, B, C, D are said to be *in proportion.*

* This is Euclid's definition of the equality of two ratios.

EXERCISE XXII

IRRATIONALS

1. Show that no rational exists whose nth power is equal to a/b, where a/b is a positive fraction in its lowest terms unless a and b are perfect nth powers.

[Suppose that $\left(\frac{x}{y}\right)^n = \frac{a}{b}$ where x/y is a positive fraction in its lowest terms. Then $bx^n = ay^n$. Since y is prime to x, y^n must be a divisor of b. Let $b = ky^n$, where k is a whole number. Then $a = kx^n$. Hence $k = 1$, for a is prime to b. Therefore $a = x^n$ and $b = y^n$, that is to say, a and b are perfect nth powers.]

2. If $\sqrt{x} + \sqrt{y} = \sqrt{a} + \sqrt{b}$, then either $x = a$, $y = b$ *or* $x = b$, $y = a$, *or* $\sqrt{x}$, $\sqrt{y}$, $\sqrt{a}$, $\sqrt{b}$ are all rational or all like surds.

[Squaring, $x + y = a + b$, and $\sqrt{xy} = \sqrt{ab}$, unless $\sqrt{xy}$, $\sqrt{ab}$ are rational.]

3. Find the condition that rationals x, y may exist such that

$$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y},$$

where a, b are given rationals and b is not a perfect square. Also find x and y in terms of a and b.

[Proceed as in Ex. 2, and show that $x = \frac{1}{2}(a \pm \sqrt{a^2 - b})$.]

4. If $a + b\sqrt{p} + c\sqrt{q} + d\sqrt{pq} = 0$ where $\sqrt{p}$, $\sqrt{q}$ are unlike surds, then a, b, c, d are all zero.

[We have $$a + b\sqrt{p} + (c + d\sqrt{p})\sqrt{q} = 0;$$
$$\therefore\ (a + b\sqrt{p})(c - d\sqrt{p}) + (c^2 - d^2 p)\sqrt{q} = 0.$$

Hence by Art. 8, Ex. 1,

$$ac = bdp, \quad bc = ad, \quad c^2 = dp,$$

and consequently, if $c \neq 0$ and $a \neq 0$, then $p = c^2$.]

5. If $x + \sqrt[3]{y} = a + \sqrt[3]{b}$ where x, y, a, b are rational and y, b are not perfect cubes, then $x = a$ and $y = b$.

[Put $z = x - a$. Show that $z^3 + y - b + 3z\sqrt[3]{by} = 0$; $\therefore\ \sqrt[3]{y} = m\sqrt[3]{b^2}$ where m is rational. Now use Art. 8, Ex. 2.]

6. If $a\sqrt[3]{p^2} + b\sqrt[3]{pq} + c\sqrt[3]{q^2} = 0$ and neither of p^2q, pq^2 is a perfect cube, show that a, b, c are all zero.

[Show that $c^3q^2 - a^3p^2 - b^3pq = 3ab(ap\sqrt[3]{pq^2} + bq\sqrt[3]{p^2q})$, and that the expression in brackets is not zero, since p/q is not a cube.]

7. Show that $\sqrt{2}$ and $\sqrt{3}$ are cubic functions of $\sqrt{2} + \sqrt{3}$ with rational coefficients, and that $\sqrt{2} - \sqrt{6} + 3$ is the ratio of two linear functions of $\sqrt{2} + \sqrt{3}$.

[For the first part, let $y = \sqrt{2} + \sqrt{3}$; find y^3, and eliminate $\sqrt{3}$ and $\sqrt{2}$ in turn: for the second part, find the value of the product $(\sqrt{2} - \sqrt{6} + 3)(\sqrt{2} + \sqrt{3} + x)$, and show that this is equal to $y + 5$ when $x = 1$.]

8. Find the equation of lowest degree with rational coefficients of which one root is

(i) $\sqrt{2} + \sqrt{3}$; (ii) $\sqrt[3]{2} + 3\sqrt[3]{4}$; (iii) $\sqrt[3]{2} + \sqrt[3]{3}$.

[(i) If $x = \sqrt{2} + \sqrt{3}$, $x^2 - 1 = 2\sqrt{2}x$, etc.;
(ii) if $y = \sqrt[3]{2} + 3\sqrt[3]{4}$, $y^3 = 110 + 18(\sqrt[3]{2} + 3\sqrt[3]{4})$;
(iii) if $z = \sqrt[3]{2} + \sqrt[3]{3}$, $z^3 = 5 + 3\sqrt[3]{6} \cdot z$, and $(z^3 - 5)^3 = 162z^3$.]

9. If all the solutions of

$$ax^2+2hxy+by^2=1, \quad a'x^2+2h'xy+b'y^2=1$$

are rational and a, h, b, a', h', b' are rational, then $(h-h')^2-(a-a')(b-b')$ and $(ab'-a'b)^2-4(ah'-a'h)(hb'-h'b)$ are the squares of rationals.

[Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (-\alpha_1, -\beta_1)(-\alpha_2, -\beta_2)$ be the solutions. Prove that $\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}$ are the roots of

$$(a-a')t^2+2(h-h')t+b-b'=0.$$

Also, show that $\alpha_1\beta_1, \alpha_2\beta_2$ are the roots of

$$z^2\{(ab')^2-4(ah')(hb')\}+2z\{(a-a')(hb')-(b-b')(ah')\}+(a-a')(b-b')=0,$$

where $(ab')=ab'-a'b$. Hence show that

$$\alpha_1^2\alpha_2^2=\frac{(b-b')^2}{(ab')^2-4(ah')(hb')},$$

whence the second result follows.]

10. If a, b, x, y are rationals such that

$$(ay-bx)^2+4(a-x)(b-y)=0,$$

prove that *either* $x=a$, $y=b$, *or* $1-ab$ and $1-xy$ are the squares of rationals.

[Put $x-a=X$, $y-b=Y$; $\therefore\ (aY-bX)^2+4XY=0$.

If X and Y are not zero, solve for $\frac{X}{Y}$, and show that $1-ab$ is a perfect square.

Next observe that the given equation is unaltered by the interchange of x and a, y and b.]

CHAPTER XIV

INEQUALITIES

In this chapter, various methods of dealing with inequalities are explained. Many of the results are of fundamental importance.

1. Weierstrass' Inequalities. *If $a_1, a_2, \ldots a_n$ are positive numbers less than 1 whose sum is denoted by s_n, then*

$$1-s_n<(1-a_1)(1-a_2)\ldots(1-a_n)<1/(1+s_n),$$
$$1+s_n<(1+a_1)(1+a_2)\ldots(1+a_n)<1/(1-s_n),$$

where, in the last inequality, it is supposed that $s_n<1$.

For $$(1-a_1)(1-a_2)=1-(a_1+a_2)+a_1a_2>1-(a_1+a_2),$$
$$(1-a_1)(1-a_2)(1-a_3)>\{1-(a_1+a_2)\}(1-a_3)>1-(a_1+a_2+a_3),$$

and continuing thus, we can prove that

$$(1-a_1)(1-a_2)\ldots(1-a_n)>1-s_n.$$

In the same way

$$(1+a_1)(1+a_2)\ldots(1+a_n)>1+s_n.$$

Again, $1-a_1<1/(1+a_1)$, and $1+a_1<1/(1-a_1)$, for $0<a_1<1$. Using these and similar inequalities, we have

$$(1-a_1)(1-a_2)\ldots(1-a_n)<1/(1+a_1)(1+a_2)\ldots(1+a_n)<1/(1+s_n),$$

and if $s_n<1$,

$$(1+a_1)(1+a_2)\ldots(1+a_n)<1/(1-a_1)(1-a_2)\ldots(1-a_n)<1/(1-s_n).$$

2. *Many inequalities depend on the fact that the square of a real number is positive.*

Ex. 1. *If $a>0$ and $b>0$, then $\frac{1}{2}(a+b)>\sqrt{ab}$.*
For $(\sqrt{a}-\sqrt{b})^2>0$.

Ex. 2. *If a, b, c are positive and not all equal, then*

$$(a+b+c)(bc+ca+ab)>9abc.$$

For $$(a+b+c)(bc+ca+ab)-9abc$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-6abc$$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2>0.$$

3. *A special arrangement of terms or factors is sometimes useful.*

Ex. 1. *Show that* $\underline{|n} > n^{\frac{n}{2}}$.

We have $(\underline{|n})^2 = 1 \,.\, n \,.\, 2(n-1) \,.\, 3(n-2) \ldots r(n-r+1) \ldots n \,.\, 1,$

and $r(n-r+1) > n$ if $r^2 - r(n+1) + n < 0$; that is if $(r-1)(r-n) < 0$ or if $1 < r < n$.

Therefore $(\underline{|n})^2 > n^n$.

Ex. 2. *If a, b, c are positive numbers any two of which are together greater than the third, then*

$$1/(b+c-a) + 1/(c+a-b) + 1/(a+b-c) > 1/a + 1/b + 1/c.$$

Now, since $a(c+a-b)(a+b-c) > 0$, we have $1/(c+a-b) + 1/(a+b-c) > 2/a$,

if $a(a+b-c+c+a-b) > 2(c+a-b)(a+b-c)$ or $a^2 > a^2 - (b-c)^2$;

which is the case. Similarly, $1/(a+b-c) + 1/(b+c-a)$, $1/(b+c-a) + 1/(c+a-b)$, are greater than $2/b$, $2/c$ respectively; and the result is obtained by addition.

4. *If a, b, x are positive, then* $(a+x)/(b+x) \gtrless a/b$, *according as* $a \lesseqgtr b$.

For since $b(b+x)$ is positive, $(a+x)/(b+x) \gtrless a/b$

according as $ab + bx \gtrless ab + ax$, that is, according as $ax \lesseqgtr bx$ or $a \lesseqgtr b$.

Ex. 1. *Prove that* $\dfrac{1}{2\sqrt{(n+1)}} < \dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{2n-1}{2n} < \dfrac{1}{\sqrt{(2n+1)}}$.

Let $u_n = \dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{(2n-1)}{2n}$, then $u_n < \dfrac{2}{3} \cdot \dfrac{4}{5} \cdots \dfrac{2n}{2n+1}$; $\therefore\ u_n^2 < \dfrac{1}{2n+1}$.

Also $(2n+1)u_n = \dfrac{3}{2} \cdot \dfrac{5}{4} \cdots \dfrac{2n+1}{2n} > \dfrac{4}{3} \cdot \dfrac{6}{5} \cdots \dfrac{2n+2}{2n+1}$; $\therefore\ (2n+1)^2 u_n^2 > n+1$.

Therefore $u_n < \dfrac{1}{\sqrt{2n+1}}$ and $u_n > \dfrac{\sqrt{(n+1)}}{2n+1} > \dfrac{1}{2\sqrt{(n+1)}}$.

5. *If both sides of an inequality are symmetric functions of a, b, c, ... h, k, there is no loss of generality in assuming that* $a \geqslant b \geqslant c \ldots \geqslant h \geqslant k$.

Ex. 1. *If a, b, c are all positive and* $n \geqslant 0$ *or* $n \leqslant -1$, *then*

$$a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) \geqslant 0.$$

First let $n > 0$. On account of symmetry, we may assume that $a \geqslant b \geqslant c$. Hence,

$a^n \geqslant b^n \geqslant c^n$ and $a^n(a-b)(a-c) \geqslant b^n(a-b)(b-c)$, also $c^n(c-a)(c-b) \geqslant 0$;

therefore $a^n(a-b)(a-c) + c^n(c-a)(c-b) \geqslant b^n(a-b)(b-c)$,

whence the result in question.

If $n = 0$, $\Sigma a^n(a-b)(a-c) = \Sigma(a-b)(a-c) = \Sigma a^2 - \Sigma bc$

and $\Sigma a^2 - \Sigma bc = \frac{1}{2}\Sigma(a-b)^2 > 0$.

If $n \leqslant -1$, let $n = -m-1$, so that $m \geqslant 0$. Let $a = 1/\alpha$, $b = 1/\beta$, $c = 1/\gamma$, then it is easily seen that $\Sigma a^n(a-b)(a-c) = \dfrac{1}{\alpha\beta\gamma}\Sigma\alpha^m(\alpha-\beta)(\alpha-\gamma) \geqslant 0$ by the preceding.

6. *If $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are two sets of real numbers, then*

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2),$$

the sign of equality occurring only when $a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n$.

For let $A = \Sigma a_r^2$, $B = \Sigma a_r b_r$, $C = \Sigma b_r^2$, then for all values of λ,

$$\Sigma(a_r + \lambda b_r)^2 = A + 2\lambda B + \lambda^2 C.$$

Therefore $A + 2\lambda B + \lambda^2 C \geqslant 0$. If the sign of equality occurs, we must have $a_r + \lambda b_r = 0$ for every r, and so

$$-\lambda = a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n.$$

Hence the roots of $A + 2\lambda B + \lambda^2 C = 0$ are equal and $B^2 = AC$.

Otherwise $A + 2\lambda B + \lambda^2 C > 0$ for all values of λ, and therefore $B^2 < AC$.

The theorem also follows from the identity

$$\Sigma a_r^2 \,.\, \Sigma b_r^2 = (\Sigma a_r b_r)^2 + S,$$

where S is the sum of the $\frac{1}{2}n(n-1)$ squares of the form $(a_1b_2 - a_2b_1)^2$.

7. *If $a_1, a_2, \ldots a_n$ are n positive numbers, not all equal to one another, then*

$$n\Sigma a_r^{x+y} \gtrless \Sigma a_r^x \,.\, \Sigma a_r^y,$$

according as x and y have the same or opposite signs.

First suppose that x and y have the same sign. Let r, s be any two of the numbers 1, 2, ... n, then $a_r^x - a_s^x$ and $a_r^y - a_s^y$ are both positive, both negative or zero. Hence $(a_r^x - a_s^x)(a_r^y - a_s^y) \geqslant 0$, and consequently

$$a_r^{x+y} + a_s^{x+y} \geqslant a_r^x a_s^y + a_s^x a_r^y.$$

There are $\frac{1}{2}n(n-1)$ relations of this sort, *not all* equalities. Also a_r^{x+y} occurs in $n-1$ of them and $a_r^x a_s^y$ in only one.

Hence by addition

$$(n-1)\Sigma a_r^{x+y} > \Sigma a_r^x a_s^y \text{ where } s \neq r.$$

Therefore $$n\Sigma a_r^{x+y} > \Sigma a_r^{x+y} + \Sigma a_r^x a_s^y = \Sigma a_r^x \,.\, \Sigma a_r^y.$$

If x and y have opposite signs, then $(a_r^x - a_s^x)(a_r^y - a_s^y) \leqslant 0$, and $<$ must be substituted for $>$ in the preceding.

8. The inequality in Art. 7 can be written

$$\frac{\Sigma a_r^{x+y}}{n} \gtrless \frac{\Sigma a_r^x}{n} \cdot \frac{\Sigma a_r^y}{n}.$$

Hence it follows that *if $a_1, a_2, \ldots a_n$ are n positive numbers, not all equal to one another, and $x, y, z, \ldots$ are all positive or all negative, then*

$$\frac{\Sigma a_r^{x+y+z+\ldots}}{n} > \frac{\Sigma a_r^x}{n} \cdot \frac{\Sigma a_r^y}{n} \cdot \frac{\Sigma a_r^z}{n} \ldots .$$

9. *If a is any positive number except* 1, *and x, y are positive rationals,* then*

$$(a^x-1)/x>(a^y-1)/y \quad \text{if} \quad x>y.$$

First suppose that $x=p$, $y=q$ where p, q are positive integers. If $p>1$, we have

$$\begin{aligned}&(a^p-1)/p-(a^{p-1}-1)/(p-1)\\&=\{p(a^p-a^{p-1})-(a^p-1)\}/p(p-1)\\&=(a-1)\{pa^{p-1}-(a^{p-1}+a^{p-2}+\ldots+1)\}/p(p-1).\end{aligned}$$

Now, the expression in the large brackets can be written

$$\{(a^{p-1}-a^{p-2})+(a^{p-1}-a^{p-3})+\ldots+(a^{p-1}-1)\}$$

and each term of this is positive or negative according as $a\gtrless 1$.

Hence, in either case, $(a^p-1)/p-(a^{p-1}-1)/(p-1)>0$,

that is, $(a^p-1)/p$ decreases as p decreases, and therefore

$$(a^p-1)/p>(a^q-1)/q \quad \text{if} \quad p>q.$$

If one or both of x, y are fractions, we may take $x=p/d$, $y=q/d$ where p, q, d are positive integers and $p>q$. We have to show that

$$\frac{a^{p/d}-1}{p/d}>\frac{a^{q/d}-1}{q/d}, \quad \text{that is} \quad \frac{b^p-1}{p}>\frac{b^q-1}{q},$$

where $b=a^{\frac{1}{d}}$. This follows from the preceding, for $b>0$ and $\neq 1$.

NOTE. It has been shown that if $a>0$ and $\neq 1$, as x passes through positive rational values, $(a^x-1)/x$ *increases with* x. Putting $1/a$ for a, it follows that $(a^{-x}-1)/x$ increases, and consequently $(1-a^{-x})/x$ *decreases as* x *increases*. Further, putting $1/x$ for x, we conclude that $x(a^{\frac{1}{x}}-1)$ *decreases and* $x(1-a^{-\frac{1}{x}})$ *increases as* x *increases*.

Ex. 1. *If $a>1$ and $n\geqslant 1$, then* $n(a^{\frac{1}{n}}-1)>1-\frac{1}{a}$.

If $n>1$, $n(a^{\frac{1}{n}}-1)=a^{\frac{1}{n}}.n(1-a^{-\frac{1}{n}})>a^{\frac{1}{n}}(1-a^{-1})>1-a^{-1}$.

If $n=1$, the inequality becomes $a-1>1-\frac{1}{a}$, or $a(a-1)>a-1$, i.e. $(a-1)^2>0$.

10. *If a and b are positive and unequal and x is any rational number except* 1, *then*

$$xa^{x-1}(a-b)>a^x-b^x>xb^{x-1}(a-b),$$

unless $0<x<1$, *in which case* $xa^{x-1}(a-b)<a^x-b^x<xb^{x-1}(a-b)$.

Proof. First observe that, in either of the cases, the truth of one of the inequalities involves that of the other. Thus, if for all positive and

* It can be shown that this is true when x, y are positive real numbers. (See Ex. XXIII, 37-39.)

unequal values of a and b, $xa^{x-1}(a-b)>a^x-b^x$, we may interchange a and b, so that $xb^{x-1}(b-a)>b^x-a^x$, and therefore $a^x-b^x>xb^{x-1}(a-b)$.

Thus it is only necessary to prove one inequality in each case.

Let $a/b=k$, then

(i) if x is positive $a^x-b^x \gtrless xb^{x-1}(a-b)$ according as $k^x-1 \gtrless x(k-1)$, or according as $(k^x-1)/x \gtrless (k-1)/1$, that is according as $x \gtrless 1$; (See Art. 9.)

(ii) if x is negative and equal to $-y$, then $xa^{x-1}(a-b) \gtrless a^x-b^x$ according as $-yk^{-y-1}(k-1) \gtrless k^{-y}-1$, that is according as $-y(k-1) \gtrless k-k^{y+1}$, or according as $k^{y+1}-1 \gtrless k(y+1)-y-1$, or according as

$$(k^{y+1}-1)/(y+1) \gtrless (k-1)/1;$$

that is (by Art. 9) according as $y+1 \gtrless 1$, or $y \gtrless 0$. Now $y>0$, and so

$$xa^{x-1}(a-b)>a^x-b^x.$$

This completes the proof of the theorem.

11. When $b=1$, the second inequalities in Art. 10 become

$$a^x-1>x(a-1) \text{ if } x<0 \text{ or } >1,$$
$$a^x-1<x(a-1) \text{ if } 0<x<1.$$

Writing n for x and $1+x$ for a, it follows that, *if n is any rational except* 1, *and* $x>-1$ *and* $\neq 0$, *then*

$$(1+x)^n>1+nx \text{ if } n<0 \text{ or } >1,$$
$$(1+x)^n<1+nx \text{ if } 0<n<1.$$

Or again, changing the sign of x, *if* $x<1$ *and* $\neq 0$,

$$(1-x)^n>1-nx \text{ if } n<0 \text{ or } >1,$$
$$(1-x)^n<1-nx \text{ if } 0<n<1.$$

Ex. 1. *If m, n are positive and $m<n$, it is possible to find positive values of x such that*

$$(1+x)^m<1+nx.$$

For this inequality holds if $x>0$ and

$$(1+x)^{-m}>1/(1+nx).$$

Now $(1+x)^{-m}>1-mx$, hence the inequality holds if $1-mx>1/(1+nx)$, or if $(1+nx)(1-mx)>1$, or if $x\{(n-m)-mnx\}>0$, or if $0<x<(n-m)/mn$.

Ex. 2. *If $s_n=a_1+a_2+\ldots+a_n$ where $a_1, a_2, \ldots a_n$ are all positive,* then

$$1+s_n+\frac{s_n^2}{\underline{|2}}+\ldots+\frac{s_n^n}{\underline{|n}}-(1+a_1)(1+a_2)\ldots(1+a_n)\geqslant 0.$$

Denote the left-hand side by u_n, then if $n>1$,

$$u_n-u_{n-1}>\sum_{r=1}^{r=n}\frac{s_n^r-s_{n-1}^r}{\underline{|r}}-a_n(1+a_1)(1+a_2)\ldots(1+a_{n-1}).$$

Now $s_n^r - s_{n-1}^r > r a_n s_{n-1}^{r-1}$, when $r>1$ (Art. 10), and when $r=1$, this becomes an equality, therefore

$$\frac{u_n - u_{n-1}}{a_n} > \sum_{r=1}^{r=n} \frac{s_{n-1}^{r-1}}{\lfloor r-1} - (1+a_1)(1+a_2)\ldots(1+a_{n-1}) > u_{n-1};$$

therefore $u_n - u_{n-1} > a_n u_{n-1}$ and $u_n > (1+a_n)u_{n-1} > u_{n-1}$.

Now $u_1 = 0$, therefore $u_n > 0$ for $n = 2, 3, 4$, etc.

12. Arithmetic and Geometric Means. (1) If $a, b, c, \ldots k$ are n numbers and

$$A = \frac{1}{n}(a+b+c+\ldots+k) \quad \text{and} \quad G = \sqrt[n]{(abc\ldots k)},$$

then A is called the *arithmetic mean* and G the *geometric mean* of $a, b, c, \ldots k$.

13. Theorem. *The arithmetic mean of n positive numbers, which are not all equal to one another, is greater than their geometric mean.*

For let A be the arithmetic mean and G the geometric mean of the n positive numbers

$$a, b, c, d, \ldots k. \quad \ldots\ldots\ldots\

Alternative proof. Taking the set (**A**), we have

$$ab \leqslant \{\tfrac{1}{2}(a+b)\}^2, \quad cd \leqslant \{\tfrac{1}{2}(c+d)\}^2,$$
$$abcd \leqslant \{\tfrac{1}{2}(a+b) \,.\, \tfrac{1}{2}(c+d)\}^2 \leqslant \{\tfrac{1}{4}(a+b+c+d)\}^4.$$

Proceeding in this way, we can show that if n is a power of 2, then

$$abc \ldots k \leqslant \left\{\frac{1}{n}(a+b+\ldots k)\right\}^n,$$

so that $G \leqslant A$.

If n is not a power of 2, consider the set $a, b, c, \ldots k, A, A, A, \ldots$, where A occurs r times and $n+r$ is a power of 2. By the preceding,

$$abc \ldots k \,.\, A^r \leqslant \left\{\frac{a+b+c+\ldots k+rA}{n+r}\right\}^{n+r}$$

The right-hand side is equal to A^{n+r}, and therefore $G^n A^r \leqslant A^{n+r}$, so that $G \leqslant A$, the sign of equality only occurring when all the numbers $a, b, \ldots k$ are equal.

Ex. 1. *Show that* $n^n > 1 \,.\, 3 \,.\, 5 \ldots (2n-1)$.

We have $\{1+3+5+\ldots+(2n-1)\}/n > \sqrt[n]{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}$.

Now $1+3+5+\ldots+2n-1 = \frac{1}{2}n\{1+(2n-1)\} = n^2$;

$\therefore \; n > \sqrt[n]{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}$ and $n^n > 1 \,.\, 3 \,.\, 5 \ldots (2n-1)$.

Ex. 2. *If* $a_1, a_2, a_3 \ldots a_n$ *are positive and* $(n-1)s = a_1 + a_2 + \ldots + a_n$, *then*

$$a_1 a_2 a_3 \ldots a_n \geqslant (n-1)^n (s-a_1)(s-a_2) \ldots (s-a_n).$$

For $(s-a_2)+(s-a_3)+\ldots+(s-a_n) = a_1$,

therefore $a_1/(n-1) \geqslant \{(s-a_2)(s-a_3) \ldots (s-a_n)\}^{1/(n-1)}$.

There are n inequalities (or equalities) of this kind, and the result follows by multiplication.

14. Theorem. *If* $a, b, \ldots k$ *and* $x, y, \ldots w$ *are two sets of* n *positive numbers, those in the second set being rational, then*

$$\left(\frac{ax+by+\ldots+kw}{x+y+\ldots+w}\right)^{x+y+\ldots+w} \geqslant a^x b^y \ldots k^w.$$

For in Art. 13 we may take $x = x'/g$, $y = y'/g$, $\ldots w = w'/g$, where x', y', $\ldots w'$ and g are positive integers, and then it is sufficient to prove that

$$\frac{ax'+by'+\ldots kw'}{x'+y'+\ldots+w'} \geqslant (a^{x'} b^{y'} \ldots k^{w'})^{1/(x'+y'+\ldots+w')}.$$

Now the left-hand side is the arithmetic mean and the right-hand side is the geometric mean of $a, a, \ldots b, b, \ldots k, k \ldots$, where a occurs x' times, b occurs y' times, ... and k occurs w' times, so that the result follows by the last theorem.

15. Theorem. *If $a, b, c, \ldots k$ are n positive numbers which are not all equal to one another, and m is any rational except 0 or 1, then*

$$\frac{a^m+b^m+c^m+\ldots+k^m}{n} \gtrless \left(\frac{a+b+c+\ldots+k}{n}\right)^m,$$

according as m does not or does lie between 0 and 1.

For let $$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\ldots=\frac{k}{w}=\frac{a+b+c+\ldots+k}{n},$$

then $$x+y+z+\ldots+w=n,$$

and $$\frac{1}{n}(a^m+b^m+c^m+\ldots+k^m) \gtrless \left\{\frac{1}{n}(a+b+c+\ldots+k)\right\}^m,$$

according as $$x^m+y^m+z^m+\ldots+w^m \gtrless n.$$

If m does not lie between 0 and 1, by Art. 11,

$$x^m-1>m(x-1), \quad y^m-1>m(y-1), \ldots w^m-1>m(w-1).$$

Hence $$x^m+y^m+\ldots+w^m-n>m(x+y+\ldots+w-n)=0,$$

and therefore $$x^m+y^m+\ldots+w^m>n.$$

If $0<m<1$, by Art. 11, the sign $>$ must be replaced by $<$ in all these inequalities, so that $$x^m+y^m+\ldots+w^n<n.$$

This completes the proof of the theorem.

16. Theorem. *If $a, b, \ldots k$ and $x, y, \ldots w$ are two sets of n rationals and those of the first set are not all equal to one another, then, m being any rational,*

$$\frac{xa^m+yb^m+\ldots+wk^m}{x+y+\ldots+w} \gtrless \left(\frac{xa+yb+\ldots+wk}{x+y+\ldots+w}\right)^m,$$

according as m does not or does lie between 0 and 1.

This depends on Art. 15 in the same way that Art. 14 depends on Art. 13.

17. Application to Questions of Maxima and Minima Values. In the theorems of Arts. 13-15 the inequalities become equalities when all the numbers $a, b, c, \ldots k$ are equal. Hence we draw the following conclusions:

Suppose that $x, y, z, \ldots w$ are n positive variables and that c is a constant, then:

(1) If $x+y+z+\ldots+w=c$, the value of $xyz \ldots w$ is greatest when

$$x=y=\ldots=w=c/n,$$

so that the greatest value of $xyz \ldots w$ is $(c/n)^n$.

(2) If $xyz \ldots w=c$, the value of $x+y+\ldots w$ is least when $x=y=\ldots=w$, so that the least value of $x+y+\ldots+w$ is $nc^{\frac{1}{n}}$.

Again, if m is any rational except 0 or 1, from Art. 3 we conclude that:

(3) If $x+y+z+\ldots+w=c$ then, according as m does not or does lie between 0 and 1, the least or the greatest value of $x^m+y^m+\ldots+w^m$ occurs when $x=y=\ldots=w$, the value in question being $n^{1-m}.c^m$.

(4) If $x^m+y^m+\ldots+w^m=c$, then according as m does not or does lie between 0 and 1, the greatest or the least value of $x+y+\ldots+w$ occurs when $x=y=\ldots=w$, the value in question being $n^{1-1/m}.c^{1/m}$.

Ex. 1. *Find the greatest value of xyz for positive values of x, y, z, subject to the condition*

$$yz+zx+xy=12.$$

Since $yz+zx+xy=12$, the value of $(yz)(zx)(xy)$ is greatest when $yz=zx=xy$, that is when $x=y=z=2$.

Hence the greatest value of $(yz)(zx)(xy)$ is 2^6, and the greatest value of xyz is 8.

Ex. 2. *If the sum of the sides of a triangle is given, prove that the area is greatest when the triangle is equilateral.*

Let a, b, c be the sides of the triangle, and let $a+b+c=2s$.

If Δ is the area, then $\Delta^2=s(s-a)(s-b)(s-c)$.

Now $(s-a)+(s-b)+(s-c)=s=$ a constant,

Hence the value of $(s-a)(s-b)(s-c)$ is greatest when $s-a=s-b=s-c$.

Therefore the value of Δ is greatest when $a=b=c$.

Ex. 3. *Find the least value of $3x+4y$ for positive values of x and y, subject to the condition $x^2y^3=6$.*

Since $x^2y^3=6$, if λ, μ are any constants, we have $(\lambda x)(\lambda x)(\mu y)(\mu y)(\mu y)=6\lambda^2\mu^3$.

Therefore $\lambda x+\lambda x+\mu y+\mu y+\mu y$ is least when $\lambda x=\mu y=(6\lambda^2\mu^3)^{\frac{1}{5}}$.

Hence the least value of $2\lambda x+3\mu y$ is $5(6\lambda^2\mu^3)^{\frac{1}{5}}$.

Putting $2\lambda=3$ and $3\mu=4$, it follows that the least value of $3x+4y$ is

$$5\left(6\cdot\frac{3^2}{2^2}\cdot\frac{4^3}{3^3}\right)^{\frac{1}{5}}=10.$$

Ex. 4. *Find the least value of $x^{-1}+y^{-1}+z^{-1}$ for positive values of x, y, z which satisfy the condition $x+y+z=c$.*

In Art. 17, (3), putting $m=-1$, $n=3$, since m does not lie between 0 and 1, the least value of $x^{-1}+y^{-1}+z^{-1}$ is 3^2c^{-1} or $9/c$.

EXERCISE XXIII

*In the inequalities marked with an *, all the numbers concerned are supposed to be positive.*

1. If $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are two sets of numbers and all those in the second set are positive, then $(a_1+a_2+\ldots+a_n)/(b_1+b_2+\ldots+b_n)$ lies between the greatest and least of a_1/b_1, a_2/b_2, ..., a_n/b_n.

2. If a, b, c are any real numbers, show that

$$(b+c-a)^2+(c+a-b)^2+(a+b-c)^2\geqslant bc+ca+ab.$$

3. If $l^2+m^2+n^2=1$ and $l'^2+m'^2+n'^2=1$, then $ll'+mm'+nn'\leqslant 1$.
[$\Sigma(mn'-m'n)^2+(ll'+mm'+nn')^2=\Sigma l^2\,.\,\Sigma l'^2$.]

4. Show that $a(x+y)^2+bz^2+cz(x-y)$ cannot be positive for all real values of the quantities involved.

5. If any two of a, b, c are together greater than the third and $x+y+z=0$, show that
$$a^2yz+b^2zx+c^2xy\leqslant 0.$$

6. If $a>0$ and $x>y$, show that $a^x+a^{-x}>a^y+a^{-y}$.

7.* If a, b, c, d are in harmonical progression, then $a+d>b+c$.
[Let $p-3q$, $p-q$, $p+q$, $p+3q$ be the reciprocals of a, b, c, d.]

8.* If $x\neq y$, then $(ax^n+by^n)/(ax^{n-1}+by^{n-1})$ increases as n increases through integral values.

9. If a, b, c, d are all >1, then $8(abcd+1)>(a+1)(b+1)(c+1)(d+1)$.
[$2(ab+1)>(a+1)(b+1)$, etc.]

10.* $\dfrac{b^2+c^2}{b+c}+\dfrac{c^2+a^2}{c+a}+\dfrac{a^2+b^2}{a+b}\geqslant a+b+c$.
[For, $(b^2+c^2)/(b+c)\geqslant\frac{1}{2}(b+c)$, etc.]

11. If $x>0$ or <-1, then $x+x^2+x^3+\ldots+x^{2n}\leqslant n(1+x^{2n+1})$.
[Show that $x^{r+1}+x^{2n-r}\leqslant 1+x^{2n+1}$.]

12.* (i) $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\geqslant 0$,
(ii) $a^3+b^3+c^3+3abc\geqslant a^2(b+c)+b^2(c+a)+c^2(a+b)$.
[The second inequality is merely another form of the first.]

13. If a, b, c ... k are n positive numbers, then
(i) $\Sigma a\,.\,\Sigma 1/a\geqslant n^2$; (ii) $\Sigma a^3\,.\,\Sigma 1/a\geqslant n\Sigma a^2$.

14. If a, b, c are positive and not all equal, then
$$6abc<a^2(b+c)+b^2(c+a)+c^2(a+b)<2(a^3+b^3+c^3).$$
[$3\Sigma a^3>\Sigma a\,.\,\Sigma a^2$ and $\Sigma a(b-c)^2>0$.]

15. If $a\neq b$ and x, y are any two positive numbers such that $x+y=1$, then
$$a^xb^y<ax+by.$$
[If $a>b$, then $\left(\dfrac{a}{b}\right)^x-1<x\left(\dfrac{a}{b}-1\right)$.]

16. If $n>0$ and $x>1$, then
$$\frac{x^{n+1}-x^{-(n+1)}}{x^n-x^{-n}}>\frac{n+1}{nx}.$$ [Use Art. 9.]

17. If $p>1$ and r is any rational except zero, then
$$(p+1)^{r+1}-p^{r+1}\gtrless(r+1)p^r\gtrless p^{r+1}-(p-1)^{r+1},$$
where the upper or lower signs are to be taken according as r does not, or does, lie between 0 and -1.

Hence show that $(1^r+2^r+3^r+\ldots+n^r)/n^{r+1}$ lies between $1/(r+1)$ and
$$\{(1+1/n)^{r+1}-1/n^{r+1}\}/(r+1).$$
[In Art. 10, let $x=r+1$. First put $a=p+1$, $b=p$, and next $a=p$, $b=p-1$. Finally let $p=1, 2, 3, \ldots n$ in succession, and add.]

18.* $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\ldots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\geqslant n.$

19.* $a^2b+b^2c+c^2a\geqslant 3abc.$ **20.** $(n+1)^n>2^n\,\underline{|n}$.

21. If $x>1$ and n is a positive integer, $(x^n-1)/(x-1)>nx^{\frac{1}{2}(n-1)}$.

22. If any two of a, b, c are together greater than the third, then

$$(a+b+c)^3>27(b+c-a)(c+a-b)(a+b-c).$$

23. If all the factors are positive, then

(i) $abc\geqslant \Pi(b+c-a)$; (ii) $abcd\geqslant \Pi(b+c+d-2a)$.

24. The greatest value of $xyz(d-ax-by-cz)$ is $d^4/4^4abc$, provided that all the factors are positive, and a, b, c, d are given positive numbers.

25. If a, b, c are positive rationals and x, y, z are positive variables such that $x+y+z$ is constant, show that $x^ay^bz^c$ has its greatest value when $x/a=y/b=z/c$.

26. If x, y, z are positive and $x+y+z=1$, then $8xyz<(1-x)(1-y)(1-z)<\frac{8}{27}$. [Observe that $\Sigma(1-x)=2$.]

27. If $s_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ and $n>2$, show that

$$n(n+1)^{\frac{1}{n}}-n<s_n<n-(n-1)n^{-\frac{1}{n-1}}.$$

[Consider the sequences $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\frac{n-1}{n}$ and $\frac{2}{1},\frac{3}{2},\frac{4}{3},\ldots\frac{n+1}{n}$.]

28. Let $a_1, a_2, \ldots a_n$ be a sequence of positive numbers and let A, G, H respectively be the arithmetic, geometric and harmonic means of a_1 and a_n, then

(i) If $a_1, a_2, \ldots a_n$ is an arithmetical progression, $G^n<a_1a_2\ldots a_n<A^n$.

(ii) If $a_1, a_2, \ldots a_n$ is a harmonical progression,

$$nH<a_1+a_2+\ldots+a_n<nA \quad\text{and}\quad H^n<a_1a_2\ldots a_n<G^n.$$

29. If $x_1x_2x_3\ldots x_n=a^n$ where a is a constant, then

(i) The sum of the products of every r of the x's $>C_r^n a^r$.

(ii) The least value of $(x_1+k)(x_2+k)\ldots(x_n+k)$ is $(a+k)^n$.

30. If a is any positive number except 1, and x, y, z are rationals no two of which are equal and which may be positive or negative, then

$$a^x(y-z)+a^y(z-x)+a^z(x-y)>0.$$

[First let x, y, z be integers and let $x>y>z$ (Art. 5). Consider the A.M. and the G.M. of the set of numbers $a^x, a^x, a^x, \ldots, a^z, a^z, a^z, \ldots$ where a^x occurs $(y-z)$ times and a^z occurs $(x-y)$ times.

If x, y, z are fractions, denote them by p/d, q/d, r/d where p, q, r, d are integers and d is positive. This case can then be at once deduced from the preceding.]

31. Deduce the theorem of Art. 9 from Ex. 30.

32. If $a_1, a_2, \ldots, a_n$ are positive and $p>q$ then

$$(a_1{}^q+a_2{}^q+\ldots+a_n{}^q)<n^{p-q}(a_1{}^p+a_2{}^p+\ldots+a_n{}^p).$$

[In Art. 15 put $m=\frac{q}{p}$, $a=a_1{}^p$, etc.]

33.* $\dfrac{3}{b+c+d}+\dfrac{3}{c+d+a}+\dfrac{3}{d+a+b}+\dfrac{3}{a+b+c}\geqslant\dfrac{16}{a+b+c+d}$.

[In Art. 15, put $m=-1$, $n=4$, $\frac{1}{3}(b+c+d)$ for a, etc.

34.* Use Art. 16 to show that

$$\frac{a^{-1}b+b^{-1}c+c^{-1}a}{a+b+c}\geqslant\frac{a+b+c}{bc+ca+ab}.$$

35.* $3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)\geqslant abc(a+b+c)^3$.
[Use Ex. 34.]

36. If $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are two sets of positive numbers arranged in descending order, then

$$(a_1+a_2+\ldots+a_n)(b_1+b_2+\ldots b_n)\leqslant n(a_1b_1+a_2b_2+\ldots+a_nb_n).$$

[Let $u_n=n\Sigma a_rb_r-\Sigma a_r\,.\,\Sigma b_r$. Use the relation $(a_r-a_n)(b_r-b_n)\geqslant 0$ to show that $u_n>u_{n-1}$.]

37. *If $a>1$ and p, q are positive integers, show that*

$$\frac{a^p-1}{p}-\frac{a^q-1}{q}>\frac{p-q}{2}(a-1)^2 \text{ when } p>q.$$

[Let $u_p=(a^p-1)/p-(a^{p-1}-1)/(p-1)$, then by Art. 9,

$$u_p=\frac{(a-1)^2}{p(p-1)}\{a^{p-2}+a^{p-3}(a+1)+a^{p-4}(a^2+a+1)+\ldots+(a^{p-2}+a^{p-3}+\ldots+1)\};$$

$$\therefore\ u_p>\frac{(a-1)^2}{p(p-1)}\{1+2+3+\ldots+(p-1)\},\quad \therefore\ u_p>\tfrac{1}{2}(a-1)^2$$

and $$\frac{a^p-1}{p}-\frac{a^q-1}{q}=u_p+u_{p-1}+\ldots+u_{q+1}>\frac{p-q}{2}(a-1)^2.]$$

38. *If $a>1$ and x, y are positive rationals, then*

$$\frac{a^x-1}{x}-\frac{a^y-1}{y}>\frac{x-y}{2}\cdot\left(\frac{a-1}{a}\right)^2 \text{ if } x>y.$$

[Put $x=p/d$, $y=q/d$ where p, q, d are positive integers. Let $a^{1/d}=b$; then, using Ex. 37, we find that

$$\frac{a^x-1}{x}-\frac{a^y-1}{y}>d\,.\,\frac{p-q}{2}\,.\,(b-1)^2=\frac{x-y}{2}\{d(a^{\frac{1}{d}}-1)\}^2,$$

and by Art. 9, Ex. 1, $d(a^{\frac{1}{d}}-1)>1-a^{-1}$.]

39. *If $a>1$ and x, y are positive real numbers, then*

$$(a^x-1)/x>(a^y-1)/y \text{ if } x>y.$$

[Let x', y' be rational approximations to x, y such that $x'>x$, $y'<y$, and therefore $x'-y'>x-y$. Let $f(x)=(a^x-1)/x$, and suppose that

$$f(x)=f(x')+\epsilon,\quad f(y)=f(y')+\epsilon',$$

then $$f(x)-f(y)=f(x')-f(y')+\epsilon-\epsilon'>\tfrac{1}{2}(x'-y')(1-a^{-1})^2+\epsilon-\epsilon';$$

$$\therefore\ f(x)-f(y)>\tfrac{1}{2}(x-y)(1-a^{-1})^2+\epsilon-\epsilon'.$$

We can choose x', y' so that $|\epsilon-\epsilon'|<\frac{1}{2}(x-y)(1-a^{-1})^2$, and so $f(x)-f(y)>0$, when $x>y$. See Ch. XIII, 7.]

CHAPTER XV

SEQUENCES AND LIMITS

(*Continued from Art.* 11 *on p.* 15)

1 Limit of a Function of the Positive Integral Variable n.

When n is large, $1+\frac{1}{n}$ is nearly equal to 1; further, by making n large enough, we can make $1+\frac{1}{n}$ as nearly equal to 1 as we like.

This is roughly what is meant when we say that 1 is the limit of $1+\frac{1}{n}$ as n tends to infinity. In this explanation, the meaning of 'large enough,' 'as nearly equal to 1 as we like,' is by no means clear.

If u_n is a function of the positive integral variable n, the 'limit of u_n as n tends to infinity' is usually defined as follows:

Definitions. (1) If corresponding to any positive number ϵ that we may choose, no matter how small, there is a positive integer m such that for every integer n greater than or equal to m

$$|u_n - l| < \epsilon,$$

where l is a fixed number, then l is called the *limit of u_n as n tends to infinity.*

This is expressed by writing

$$\lim_{n\to\infty} u_n = l, \quad \text{or simply} \quad \lim u_n = l.$$

We also say that u_n *tends to l as n tends to infinity*; and express this by

$$u_n \to l \quad \text{as} \quad n \to \infty.$$

(2) If, corresponding to any positive number M that we may choose, no matter how great, there is a positive integer m, such that for every integer n greater than or equal to m,

$$u_n > M,$$

then u_n is said to *tend to infinity as n tends to infinity*, and we write

$$u_n \to \infty \quad \text{or} \quad \lim u_n = \infty.$$

If $u_n \to \infty$ and $v_n = -u_n$, we say that v_n *tends to minus infinity* $(-\infty)$.

If $u_n \to l$ we say that the sequence (u_n) is *convergent*, and that it converges to the limit l. If $u_n \to \infty$ or $-\infty$, the sequence (u_n) is said to be *divergent*.

If u_n does not tend to a limit, nor to $+\infty$, nor to $-\infty$, it is said to *oscillate* and the sequence (u_n) is called *oscillatory*.

If u_n oscillates and a positive number M can be found such that, for every positive integer n,

$$|u_n| < M,$$

then u_n is said to *oscillate finitely*. Otherwise, u_n *oscillates infinitely*.

Ex. 1. *How does the function a^n behave as $n \to \infty$?*

Referring to Ch. II, 18, (5), it will be seen that if $a > 1$, $a^n \to \infty$; if $a = 1$, $a^n = 1$ for every n; if $|a| < 1$, $a^n \to 0$; if $a = -1$, $a^n = \pm 1$ and a^n oscillates finitely; if $a < -1$, $a^n \to \infty$ or $-\infty$ according as n is even or odd, and so a^n oscillates infinitely.

It is often useful to represent the first few terms of a sequence (u_n) *graphically*. This is done by plotting points whose coordinates are $(1, u_1)$, $(2, u_2)$, $(3, u_3)$, ... It is usual to join points representing consecutive terms.

Ex. 2. *If $u_n = (-1)^n\left(1 + \frac{1}{n}\right)$, represent graphically the first few terms of the sequence (u_n), and examine the character of the sequence as $n \to \infty$.*

As $n \to \infty$, $u_{2n} \to 1$ and $u_{2n+1} \to -1$.
Thus u_n oscillates finitely.

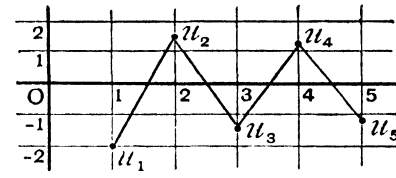


FIG. 34.

2. Fundamental Theorems on Limits.

(1) *If $u_n \to l$ and $v_n \to l'$, then*

(i) $u_n + v_n \to l + l'$, (ii) $u_n - v_n \to l - l'$, (iii) $u_n v_n \to ll'$,

(iv) $1/u_n \to 1/l$, *unless* $l = 0$, (v) $u_n/v_n \to l/l'$, *unless* $l' = 0$.

Proof. Choose a positive number ϵ, no matter how small.

(i) and (ii). By the definition of a limit we can choose m_1 and m_2 so that

$$|u_n - l| < \tfrac{1}{2}\epsilon \quad \text{for} \quad n \geqslant m_1$$

and

$$|v_n - l'| < \tfrac{1}{2}\epsilon \quad \text{for} \quad n \geqslant m_2.$$

If we denote the greater of m_1, m_2 by m, then each of these inequalities will hold for $n \geqslant m$, and for such values of n we have

$$|u_n + v_n - (l + l')| \leqslant |u_n - l| + |v_n - l'| < \epsilon$$

and

$$|u_n - v_n - (l - l')| \leqslant |u_n - l| + |v_n - l'| < \epsilon.$$

Hence, by the definition of a limit,

$$u_n + v_n \to l + l' \quad \text{and} \quad u_n - v_n \to l - l'.$$

(iii) Let $u_n = l + \alpha$ and $v_n = l' + \beta$. Then

$$u_n v_n - ll' = \beta l + \alpha(l' + \beta).$$

Choose any positive number L greater than $|\, l' + \beta \,|$, then

$$|\, u_n v_n - ll' \,| < |\, \beta l \,| + |\, \alpha \,| \,.\, L.$$

By the definition of a limit we can choose m so that

$$|\, \alpha \,| = |\, u_n - l \,| < \tfrac{1}{2}\frac{\epsilon}{L} \quad \text{and} \quad |\, \beta \,| = |\, v_n - l' \,| < \tfrac{1}{2}\frac{\epsilon}{|\, l \,|},$$

provided only that $n \geqslant m$. Hence also $|\, u_n v_n - ll' \,| < \epsilon$.

Therefore $$u_n v_n \to ll'.$$

(iv) Let $u_n = l + \alpha$, then

$$\frac{1}{u_n} - \frac{1}{l} = -\frac{\alpha}{l(l+\alpha)}.$$

If l is not zero we can choose a positive number L less than $|\, l + \alpha \,|$, and then

$$\left|\, \frac{1}{u_n} - \frac{1}{l} \,\right| < \frac{|\, \alpha \,|}{|\, l \,|\, L}.$$

Now by the definition of a limit we can choose m so that

$$|\, \alpha \,| = |\, u_n - l \,| < |\, l \,| \,.\, L \,.\, \epsilon,$$

provided only that $n \geqslant m$. Hence also

$$\left|\, \frac{1}{u_n} - \frac{1}{l} \,\right| < \epsilon, \text{ and therefore } \frac{1}{u_n} \to \frac{1}{l}.$$

(v) This follows from (iii) and (iv), for

$$\frac{u_n}{v_n} = u_n \cdot \frac{1}{v_n} \to l \cdot \frac{1}{l'} \text{ when } l' \neq 0.$$

(2) *If $u_n \to l$ and $v_n - u_n \to 0$, then $v_n \to l$.*

For $v_n - l = (v_n - u_n) + (u_n - l)$, therefore $|\, v_n - l \,| \leqslant |\, v_n - u_n \,| + |\, u_n - l \,|$.

Now we can choose m so that for $n \geqslant m$ both

$$|\, u_n - l \,| < \tfrac{1}{2}\epsilon \quad \text{and} \quad |\, v_n - u_n \,| < \tfrac{1}{2}\epsilon.$$

Hence for such values of n,

$$|\, v_n - l \,| < \epsilon, \quad \text{and therefore} \quad v_n \to l.$$

3. Monotone Sequences. If $u_{n+1} \geqslant u_n$ for all values of n, u_n is said to *increase steadily*, and (u_n) is called a *monotone increasing sequence.*

If $u_{n+1} \leqslant u_n$ for all values of n, u_n is said to *decrease steadily*, and (u_n) is called a *monotone decreasing sequence.*

Theorem 1. *If every term of a monotone increasing sequence* (u_n) *is less than a fixed number* k, *then the sequence converges to a limit equal to, or less than,* k.

Proof. Divide the system of rational numbers into two classes as follows :

The *lower class* is to contain every rational a such that, for some value of n (and therefore for all greater values), $u_n \geqslant a$.

The *upper class* is to contain every rational a' such that $u_n < a'$ for all values of n.

No rational escapes classification, and every a is less than any a'. Hence the classification defines a real number A.

We shall prove that the sequence converges to A as a limit.

First observe that u_n cannot exceed A. For if $u_n > A$, u_n would exceed every rational which lies between u_n and A. This is impossible, for such a rational would belong to the upper class.

Next, choose a positive number ϵ, no matter how small, and let a be a rational such that

$$A - \epsilon < a < A.$$

Then a belongs to the lower class, and is exceeded by some term u_m of the sequence and by all succeeding terms;

therefore $$0 < A - u_n < \epsilon \quad \text{if} \quad n \geqslant m.$$

Hence $$\lim u_n = A.$$

Again $k \geqslant A$, for otherwise k would be exceeded by some term of the sequence, thus

$$\lim u_n \leqslant k.$$

It follows that if u_n increases steadily as $n \to \infty$, then u_n tends to a limit or to $+\infty$.

For if u_n does not tend to $+\infty$, a positive number k exists such that $u_n < k$ for all values of n.

Theorem 2. *If every term of a monotone decreasing sequence* (u_n) *is greater than a fixed number* k, *then the sequence converges to a limit equal to or greater than* k.

This can be proved by an argument similar to the preceding, or thus : Let $v_n = -u_n$, then (v_n) is an increasing sequence of which every term is less than $-k$. Therefore (v_n) converges to a limit less than $-k$. Hence (u_n) converges to a limit greater than k.

Hence also *if* u_n *decreases steadily as* $n \to \infty$, then u_n tends to a limit or to $-\infty$.

Ex. 1. *If* $u_n = n(x^{\frac{1}{n}} - 1)$ *where* $x>0$, *show that* u_n *tends to a limit as* $n\to\infty$.

If $x=1$, $u_n=0$ for every n, and so $\lim u_n=0$.

If $x>1$, $u_n>0$, and by XIV, 9, u_n decreases as n increases. Hence u_n tends to a limit, which we shall denote by $f(x)$, as $n\to\infty$.

If $0<x<1$, let $x=1/y$, then $y>1$ and

$$u_n = n(y^{-\frac{1}{n}} - 1) = -y^{-\frac{1}{n}} \cdot n(y^{\frac{1}{n}} - 1) \to -f(y).$$

4. Theorems. (1) *If for all values of* n,* u_n *is positive and* $u_{n+1}>ku_n$ *where* k *is a constant greater than unity, then* $u_n\to\infty$.

For $u_n>ku_{n-1}>k^2u_{n-2}\ldots>k^{n-1}u_1$. Now $k>1$, therefore $u_n\to\infty$.

(2) *If for all values of* n, u_n *is positive and* $u_{n+1}<ku_n$ *where* k *is a positive constant less than unity, then* $u_n\to 0$.

The proof is similar to that of Theorem (1).

(3) *If* u_n *is positive and* $\lim u_{n+1}/u_n=l$, *then* $u_n\to\infty$ *if* $l>1$, *and* $u_n\to 0$ *if* $l<1$.

For if $l>1$ we can choose k so that $l>k>1$, and then by the definition of a limit we can find m such that $u_{n+1}/u_n>k$ for $n\geqslant m$. Hence by Theorem (1), $u_n\to\infty$.

If $l<1$, we can choose k so that $l<k<1$, and then we can find m so that $u_{n+1}/u_n<k$, for $n\geqslant m$. Hence, by Theorem (2), $u_n\to 0$.

Ex. 1. *If* p *is a given integer, positive or negative, show that*

(i) $\dfrac{x^n}{n^p}\to\infty$ if $x>1$; (ii) $\dfrac{x^n}{n^p}\to 0$ if $|x|<1$.

If $u_n=\dfrac{x^n}{n^p}$ then $\dfrac{u_{n+1}}{u_n}=\left(\dfrac{n}{n+1}\right)^p . x\to x$ as $n\to\infty$,

and the results follow from Theorem (3).

5. Exponential Inequalities and Limits.

(1) *If* n *is a positive integer,* $\left(1+\dfrac{1}{n}\right)^n<3$.

For, by the Binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{\underline{2}}\left(1-\frac{1}{n}\right)+\frac{1}{\underline{3}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots \text{ to } n+1 \text{ terms}$$

$$<1+1+\frac{1}{\underline{2}}+\frac{1}{\underline{3}}+\ldots+\frac{1}{\underline{n}}$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}<1+\frac{1}{1-\frac{1}{2}}<3.$$

* Or, if $n\geqslant m$ where m is a fixed number, i.e. after a certain stage.

(2) *If m and n are positive integers and $m>n$, then*

$$\left(1+\frac{1}{m}\right)^m>\left(1+\frac{1}{n}\right)^n \quad \textit{and} \quad \left(1-\frac{1}{m}\right)^{-m}<\left(1-\frac{1}{n}\right)^{-n}.$$

For since $m/n>1$, by XIV, 11,

$$\left(1+\frac{1}{m}\right)^{\frac{m}{n}}>1+\frac{m}{n}\cdot\frac{1}{m}, \quad \text{and therefore} \quad \left(1+\frac{1}{m}\right)^m>\left(1+\frac{1}{n}\right)^n.$$

Moreover, $0<n/m<1$, therefore by XIV, 11, $\left(1-\frac{1}{n}\right)^{\frac{n}{m}}<1-\frac{n}{m}\cdot\frac{1}{n}$.

Hence $\left(1-\frac{1}{n}\right)^n<\left(1-\frac{1}{m}\right)^m$ and $\left(1-\frac{1}{m}\right)^{-m}<\left(1-\frac{1}{n}\right)^{-n}$.

(3) **The number e.** *If $u_n=\left(1+\frac{1}{n}\right)^n$ and $u_n'=\left(1-\frac{1}{n}\right)^{-n}$, then as $n\to\infty$ through positive integral values, u_n tends to a finite limit, so also does u_n'; and the limit is the same for both. This limit is denoted by e, and is one of the most important numbers in mathematics.*

For (i) it has been shown that u_n increases with n and that $u_n<3$ for every n. Therefore $u_n\to e$ as $n\to\infty$, where e is a fixed number less than 3.

(ii) Again, $$u_n'-u_n=u_n\left(\frac{u_n'}{u_n}-1\right)=u_n\left\{\left(1-\frac{1}{n^2}\right)^{-n}-1\right\}.$$

Now if $n>1$, $$1<\left(1-\frac{1}{n^2}\right)^{-n}<1+n\cdot\frac{1}{n^2} \quad \text{(by XIV, 11)},$$

therefore $$0<u_n'-u_n<\frac{u_n}{n}<\frac{3}{n}. \quad \text{Now } \frac{3}{n}\to 0,$$

therefore $$u_n'-u_n\to 0 \quad \text{and} \quad u_n'\to e.$$

It should be noticed that

$$u_n<e<u_n'.$$

It can be shown that $e=2{\cdot}7182818284\ldots$.

EXERCISE XXIV

1. State how the following functions behave as $n\to\infty$:

(i) $2+(-1)^n$. (ii) $n+(-1)^n$. (iii) $\frac{1}{n}+2(-1)^n$.

(iv) $2+(-1)^n\frac{1}{n}$. (v) $n^2+(-1)^n n$.

2. Prove that $\frac{x^n}{\lfloor n}\to 0$ as $n\to\infty$.

3. Represent graphically the first few terms of the following sequences, and say how each behaves as $n \to \infty$.

(i) $\frac{1}{2}, \frac{2}{3}, \ldots \frac{n}{n+1}, \ldots$. (ii) $-1+1, 1+\frac{1}{2}, \ldots (-1)^n + \frac{1}{n}, \ldots$.

4. If $|x|<1$, then $\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} x^n \to 0$ as $r \to \infty$.

5. Find $\lim_{n\to\infty} \frac{x^n+n}{x^n-n}$. [The limit is 1, if $|x|>1$, and -1 otherwise.]

6. If $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$, prove that $\frac{1}{2} < u_n < 1$, and that u_n tends to a limit as $n \to \infty$.

7. Show that $\frac{1}{2}n < \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n} < n$.

Hence show that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \to \infty$ as $n \to \infty$.

[Group the terms thus: $\frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \ldots$.]

8. If $\lim_{n\to\infty} n(x^{\frac{1}{n}} - 1) = f(x)$, prove that if $x>0$ and $y>0$,

(i) $f(1)=0$; (ii) $f\left(\frac{1}{x}\right) = -f(x)$; (iii) $f(xy) = f(x) + f(y)$.

[See Art. 3, Ex. 1. For the last part we have $f(xy) = \lim n(x^{\frac{1}{n}} y^{\frac{1}{n}} - 1)$. Now

$x^{\frac{1}{n}} y^{\frac{1}{n}} - 1 = y^{\frac{1}{n}}(x^{\frac{1}{n}} - 1) + (y^{\frac{1}{n}} - 1)$, and $\lim y^{\frac{1}{n}} = 1$;

$\therefore f(xy) = \lim n(x^{\frac{1}{n}} - 1) + \lim n(y^{\frac{1}{n}} - 1) = f(x) + f(y)$.]

9. If $u_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}$, $v_n = \frac{3 \cdot 5 \cdot 7 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots 2n}$, prove that

$$u_n \to 0, \quad v_n \to \infty \quad \text{and} \quad \tfrac{1}{2} < u_n v_n < 1.$$

[If $y>0$, then $\frac{x}{y} \gtrless \frac{x+1}{y+1}$, according as $x \gtrless y$, hence

$$u_n < \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1}, \quad \therefore u_n^2 < \frac{1}{2n+1} \quad \text{and} \quad u_n \to 0;$$

$$v_n > \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n+2}{2n+1}, \quad \therefore v_n^2 > \frac{2n+2}{2} \quad \text{and} \quad v_n \to \infty;$$

$u_n v_n < \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1} \cdot v_n = 1$ and $v_n u_n > \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n+2}{2n+1} u_n = \frac{1}{2} \cdot \frac{2n+2}{2n+1} > \frac{1}{2}$.]

10. If $w_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \sqrt{2n+1}$, prove that w_n tends to a limit which lies between $1/\sqrt{2}$ and 1 as $n \to \infty$.

[Show that $w_{n+1} < w_n$, then by Ex. 9, $w_n^2 = u_n v_n$ and $\frac{1}{\sqrt{2}} < w_n < 1$. Wallis has shown that this limit $= \sqrt{\frac{2}{\pi}}$.]

11. If $n \geqslant 3$, prove that $\sqrt[n+1]{(n+1)} < \sqrt[n]{n}$. Hence show that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

[For the first part use the inequality $\left(1+\frac{1}{n}\right)^n < 3$. Thus $\sqrt[n]{n}$ decreases steadily; also $\sqrt[n]{n} > 1$, therefore $\sqrt[n]{n} \to l$ where $l \geqslant 1$. If $l > 1$, for $n > 3$ we should have $\sqrt[n]{n} > l$ and $l^n/n < 1$. But $l^n/n \to \infty$ when $l > 1$ (XV, 4, Ex. 1).]

12. If r is any rational except zero and $u_n = (1^r + 2^r + 3^r + \ldots + n^r)/n^{r+1}$, show that (i) $(r+1)u_n$ lies between 1 and $\left(1+\frac{1}{n}\right)^{r+1} - \frac{1}{n^{r+1}}$.

(ii) If $r+1 > 0$, $\lim_{n\to\infty} u_n = 1/(r+1)$.

[In Exercise XXIII, 17, put 1, 2, 3, ..., n in succession for p, and add. The necessity for the condition $r+1>0$ in (ii) should be explained.]

13. Show that $\lim_{n\to\infty} \{(a+1)^r + (a+2)^r + \ldots + (a+n)^r\}/n^{r+1} = 1/(r+1)$.

6. General Principle of Convergence.

The necessary and sufficient condition for the convergence of the sequence (u_n) is as follows.

Corresponding to any positive number ϵ that we may choose, no matter how small, it must be possible to find m so that, for every positive integer p,

$$|u_{m+p} - u_m| < \epsilon. \qquad \text{.....(A)}$$

That is to say, the difference between u_m and any subsequent term must be numerically less than ϵ, so that if the terms of the sequence are represented graphically, all the terms beginning with u_m lie in a strip of width 2ϵ.

This statement, called the *general principle of convergence*, is of the greatest importance, and may be proved as follows.

(i) *The condition is necessary.* For if $u_n \to l$, we can find m so that

$$|u_m - l| < \tfrac{1}{2}\epsilon \quad \text{and} \quad |u_{m+p} - l| < \tfrac{1}{2}\epsilon$$

for every positive integer p. Now $u_{m+p} - u_m = (u_{m+p} - l) + (l - u_m)$.

Hence $|u_{m+p} - u_m| \leqslant |u_{m+p} - l| + |u_m - l|$, and $|u_{m+p} - u_m| < \epsilon$.

(ii) *The condition is sufficient.* Represent the terms $u_1, u_2, \ldots$ by the points $u_1, u_2, \ldots$ on the x-axis. The condition asserts that m can be found so that all the points,

$$u_{m+1}, \quad u_{m+2}, \quad u_{m+3}, \ldots,$$

lie on a segment of length 2ϵ with its middle point at u_m.

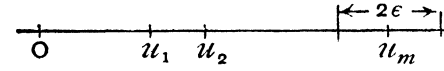


FIG. 35.

By choosing ϵ small enough, we can make this segment as short as we like. Hence it *seems reasonable* to conclude that as $n \to \infty$, the point u_n tends to a limiting position somewhere near to u_m. If the student is satisfied with this reasoning, he may omit the proof on the next page.

Given that, for every positive ϵ, *no matter how small, there exists a positive integer* m *such that*

$$|u_n - u_m| < \epsilon$$

for every integer n *greater than* m, *it is required to prove that the sequence* (u_n) *is convergent.*

Proof. Using the given condition, we can determine a positive integer m_1 so that, for every integer $m_2 > m_1$, u_{m_2} is within the interval

$$(u_{m_1} - \epsilon,\ u_{m_1} + \epsilon).$$

Denote this interval by (a_1, b_1).

Again, we can determine m_2 so that, for every integer $m_3 > m_2$, u_{m_3} is within the interval $(u_{m_2} - \frac{1}{2}\epsilon,\ u_{m_2} + \frac{1}{2}\epsilon)$. Now u_{m_3} is within (a_1, b_1), for $m_3 > m_2$. Thus the whole or part of $(u_{m_2} - \frac{1}{2}\epsilon,\ u_{m_2} + \frac{1}{2}\epsilon)$ is within (a_1, b_1).

Denote the common part of these intervals by (a_2, b_2).

This is of length $\frac{1}{2}\epsilon$, lies entirely within (a_1, b_1), and contains u_n for $n > m_2$. The figure illustrates the case in which the intervals overlap.

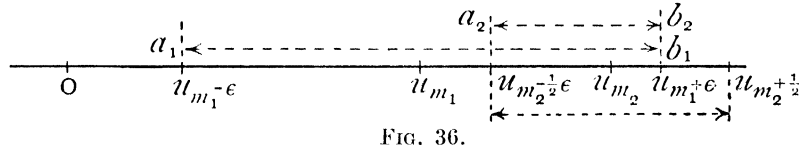


FIG. 36.

By repeating the process, we can find a succession of intervals (a_1, b_1), (a_2, b_2), ... (a_r, b_r) ..., such that (i) each contains all that follow;

(ii) for sufficiently large values of n, u_n is within each of the intervals;

(iii) $b_n - a_n < \dfrac{1}{2^{n-2}}\epsilon \to 0$ as $n \to \infty$.

Hence by Ch. XIII, 4, there is one real number l and one only, such that $a_n < l < b_n$ for every n, and this is the limit of the sequence (u_n).

NOTE. The necessary and sufficient condition for the convergence of the sequence (u_n) is usually stated (less simply) as follows.

Corresponding to any positive number ϵ *that we may choose, no matter how small, it must be possible to find* m *so that*

$$|u_{n+p} - u_n| < \epsilon \quad \text{.....(B)}$$

if $n \geqslant m$, *for every positive integer* p.

The condition **(B)**, though apparently more general than **(A)**, is equivalent to it. To prove this, we have only to show that if **(A)** is possible, then **(B)** is possible when $n > m$.

By the condition (**A**), we can find m so that

$$| u_{m+p} - u_m | < \tfrac{1}{2}\epsilon,$$

and then all the terms u_{m+1}, u_{m+2}, u_{m+3}, ... lie in the interval

$$(u_m - \tfrac{1}{2}\epsilon, \quad u_m + \tfrac{1}{2}\epsilon).$$

Therefore the difference between any two terms which follow u_m is less than ϵ. Hence we can find m so that

$$| u_{n+p} - u_n | < \epsilon \quad \text{if} \quad n > m.$$

Therefore the conditions (**A**) and (**B**) are one and the same.

7. Bounds of a Sequence. (1) If there exists a number M such that, for all values of n,

$$u_n \leqslant M,$$

the sequence (u_n), and also the function u_n, are said to be *bounded above* (or *on the right*).

If there exists a number N such that, for all values of n,

$$u_n \geqslant N,$$

then (u_n), and also u_n, are said to be *bounded below* (or *on the left*).

A sequence (u_n), or a function u_n, which is bounded above and below, is said to be *bounded*.

Examples.

1. *Let* $u_n = (-1)^n\left(1 + \frac{1}{n}\right)$.

Here (u_n) has a *greatest* term, namely $u_2 = \frac{3}{2}$: it has also a *least* term, namely $u_1 = -2$. These are called *the upper and lower bounds* of (u_n), or of u_n (Fig. 37).

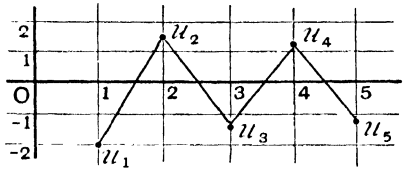


Fig. 37.

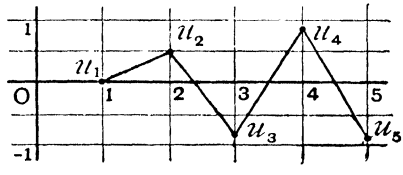


Fig. 38.

2. *Let* $u_n = (-1)^n\left(1 - \frac{1}{n}\right)$.

Although there is no greatest term, we say that 1 is the *upper bound* of (u_n), or of u_n, for the following reasons :

(i) No term of (u_n) exceeds 1.

(ii) Infinitely many terms exceed any number less than 1, for $u_{2n} \to 1$.

For similar reasons, we say that -1 is the *lower bound* of (u_n) of u_n (Fig. 38).

Theorem. *If u_n is bounded above there exists a number h such that*

(i) *no term of (u_n) exceeds h ;*

(ii) *at least one term of (u_n) exceeds any number less than h.*

This number h is called the *upper bound* of the sequence (u_n) or of the function u_n.

Proof. Divide the system of rationals into two classes.

The lower class is to contain every rational a such that for some value of n

$$u_n \geqslant a.$$

The upper class is to contain every rational a' such that for all values of n

$$u_n < a'.$$

Since u_n is bounded above, both classes exist. No rational escapes classification, and every a is less than any a'. Hence the classification defines a real number h.

Moreover, no term of (u_n) exceeds h, for in that case it would exceed a rational a' greater than h. This is impossible, for a' belongs to the upper class.

Also if ϵ is any positive number, no matter how small, we can find a rational a such that

$$h - \epsilon < a < h.$$

Now a belongs to the lower class and is exceeded by some term of (u_n). Hence at least one term of (u_n) exceeds $h - \epsilon$.

Hence the number h satisfies the conditions stated above.

This theorem is often stated as follows : *If u_n is bounded above, then it has an upper bound.*

In a similar way we can show that : *If u_n is bounded below there exists a number l such that*

(i) *no term of (u_n) is less than l ;*

(ii) *at least one term of (u_n) is less than any number greater than l.*

The number l is called the *lower bound* of the sequence (u_n) or of the function u_n.

NOTE. *If (u_n) has no greatest term, infinitely many terms of the sequence lie between h and $h - \epsilon$.*

For every term is less than h, and at least one term u_m is greater than $h - \epsilon$. Since there is no greatest term, infinitely many terms are greater than u_m, and therefore also greater than $h - \epsilon$.

Similarly, *if (u_n) has no least term, infinitely many terms of the sequence lie between l and $l + \epsilon$.*

Terms selected from $u_1, u_2, u_3, \ldots$ according to some definite rule are said to form a *subsequence.*

Ex. 1. *Explain how to select a subsequence which will converge to the upper bound h of (u_n).*

If $u_1 \neq h$, let a_2 be the first term of (u_n) greater than u_1. If $a_2 \neq h$, let a_3 be the first term of (u_n) greater than a_2. Continuing in this way we form the required subsequence $u_1, a_2, a_3, \ldots$. The formal proof is left to the student.

8. Upper and Lower Limits (of Indetermination) of a Sequence.

(1) Let (u_n) be a bounded sequence and let $h_1, h_2, h_3, \ldots$ and $l_1, l_2, l_3, \ldots$ be the upper and lower bounds of the sequences $u_1, u_2, u_3 \ldots, u_2, u_3, u_4 \ldots$, $u_3, u_4, u_5 \ldots$, etc., respectively.

If $h_1 = u_1$, then u_1 is not less than any term of $u_2, u_3, u_4, \ldots$; therefore $h_1 \geqslant h_2$. If $h_1 \neq u_1$, then h_1 is the upper bound of $u_2, u_3, u_4, \ldots$, and so $h_1 = h_2$. Thus, in any case, $h_1 \geqslant h_2$. Similarly $h_2 \geqslant h_3$, $h_3 \geqslant h_4$, and so on.

Hence $h_1, h_2, h_3, \ldots$ is a steadily decreasing sequence and every term is greater than l_1. Therefore this sequence tends to a limit H greater than or equal to l_1.

Similarly $l_1, l_2, l_3, \ldots$ is an increasing sequence and every term is less than h_1. Therefore this sequence tends to a limit L less than or equal to h_1.

The numbers H and L are called the *upper and lower limits (of indetermination)* of the sequence (u_n).

It is usual to write $\overline{\lim}\, u_n = H$ and $\underline{\lim}\, u_n = L$.

(2) *Examples.*

(i) *Let* $u_n = (-1)^n\left(1 - \frac{1}{n}\right)$. Here $h_1 = h_2 = h_3 = \ldots = H$. (Fig. 39.)

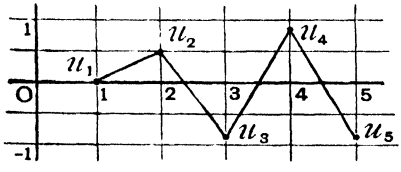


FIG. 39.

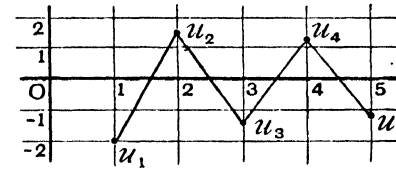


FIG. 40.

(ii) *Let* $u_n = (-1)^n\left(1 + \frac{1}{n}\right)$.

Here $h_1 = h_2 = u_2$; $h_3 = h_4 = u_4$; etc. $\therefore H = \lim u_{2n} = 1$. (Fig. 40.)

Also $l_1 = u_1$; $l_2 = l_3 = u_3$; etc. $\therefore L = \lim u_{2n+1} = -1$.

(3) It is easy to see that, if $h_n > h_{n+1}$, then $h_n = u_n$ and $h_1, h_2, \ldots h_{n-1}$ are all terms of $u_1, u_2, \ldots, u_{n-1}$. As $n \to \infty$ two cases arise :

(i) After a certain stage, all the h's are equal, that is to say, m exists so that

$$h_m = h_{m+1} = h_{m+2} = \ldots = H.$$

In this case, (u_n) has no greatest term. None of the terms from u_m onwards can exceed H, but infinitely many terms exceed $H - \epsilon$.

(ii) There is no such stage as that described in (i). That is, no matter how great m may be, we can find n, greater than m, so that $h_n > h_{n+1}$. Therefore every h is a term of (u_n), and in all cases H is such that, after a certain stage, every term of (u_n) is less than $H + \epsilon$.

Hence *in all cases, infinitely many terms of* (u_n) *are greater than* $H - \epsilon$ *; also, after a certain stage all the terms of* (u_n) *are less than* $H + \epsilon$, *in other words, only a* finite *number of terms exceed* $H + \epsilon$.

In a similar way we can show that *infinitely many terms of* u_n *are less than* $L + \epsilon$ *and, after a certain stage, all the terms are greater than* $L - \epsilon$*; in other words, only a finite number of terms are less than* $L - \epsilon$.

It follows that, *if* $H = L = l$, *then the sequence converges to* l *as a limit.*

For, after a certain stage, all the terms of (u_n) lie between $L - \epsilon$ and $H + \epsilon$, that is between $l - \epsilon$ and $l + \epsilon$.

(4) *General Principle of Convergence.* We can at once deduce a proof that the condition in Art. 6 is *sufficient* for convergence.

Given that m *can be found so that* $|u_n - u_m| < \epsilon$ *for* $n > m$, *it is required to prove that* (u_n) *is convergent.*

First observe that u_n cannot tend to $+\infty$ or to $-\infty$. Therefore (u_n) has upper and lower limits H and L. We can therefore choose m and n $(n > m)$ so that $|H - u_n| < \frac{1}{3}\epsilon$, $|u_m - L| < \frac{1}{3}\epsilon$, $|u_n - u_m| < \frac{1}{3}\epsilon$; and since $H - L = H - u_n + u_m - L + u_n - u_m$,

$$|H - L| \leqslant |H - u_n| + |u_n - L| + |u_n - u_m| < \epsilon.$$

That is to say, the difference between the *fixed* numbers H and L is less than any positive number ϵ, however small. Therefore $H = L$ and (u_n) is convergent.

9. Theorems. (1) *If* (u_n) *is an increasing sequence and if, after a certain stage,*

$$u_n - u_{n-1} < k(u_{n-1} - u_{n-2}),$$

where k *is a positive constant less than unity, then the sequence is convergent. But if* $u_n - u_{n-1} \geqslant u_{n-1} - u_{n-2}$, *the sequence is divergent.*

In the first case, suppose that the conditions hold for $n \geqslant s + 2$,

Then if $m \geqslant s$, $u_{m+1} - u_m < k(u_m - u_{m-1}) \ldots < k^{m-s}(u_{s+1} - u_s)$,

therefore $u_{m+1} - u_m < k^m c$ where $c = (u_{s+1} - u_s)/k^s$.

Also
$$u_{m+2} - u_{m+1} < k^{m+1} c,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_{m+n} - u_{m+n-1} < k^{m+n-1} c.$$

Hence by addition

$$u_{m+n} - u_m < k^m c(1 + k + k^2 + \ldots k^{n-1}) < k^m c/(1-k).$$

Choose any positive number ϵ, no matter how small; then, since $k<1$, we can find m so that $k^m < \epsilon(1-k)/c$, and since $u_{m+n} > u_m$, it follows that

$$| u_{m+n} - u_m | < \epsilon \text{ for every } n.$$

Therefore the sequence (u_n) converges.

In the second case, if $n \geqslant s$, $u_{n+1} - u_n \geqslant u_{s+1} - u_s$.

Now $u_{s+1} - u_s$ is a fixed positive quantity, therefore obviously $u_n \to \infty$.

(2) *If* $u_n > 0$ *for all values of* n *and* $u_{n+1}/u_n \to l > 0$ *where* l *is a fixed number, then* $u_n^{\frac{1}{n}} \to l$.

Because $l>0$, we can choose a positive number $\epsilon < l$ and as small as we like. Then we can find m so that

$$l - \epsilon < u_{m+1}/u_m < l + \epsilon,$$
$$l - \epsilon < u_{m+2}/u_{m+1} < l + \epsilon,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$l - \epsilon < u_n/u_{n-1} < l + \epsilon,$$

and by multiplication,

$$(l-\epsilon)^{n-m} < u_n/u_m < (l+\epsilon)^{n-m}.$$

Since $l - \epsilon > 0$, it follows that

$$(l-\epsilon)^n < \frac{u_n}{u_m}(l-\epsilon)^m < \frac{u_n}{u_m} \cdot l^m, \quad \text{and} \quad (l+\epsilon)^n > \frac{u_n}{u_m}(l+\epsilon)^m > \frac{u_n}{u_m} \cdot l^m;$$

therefore
$$(l-\epsilon)^n < \frac{u_n}{u_m} \cdot l^m < (l+\epsilon)^n$$

and
$$\left(\frac{u_m}{l^m}\right)^{\frac{1}{n}} \cdot (l-\epsilon) < u_n^{\frac{1}{n}} < \left(\frac{u_m}{l^m}\right)^{\frac{1}{n}} (l+\epsilon).$$

Now u_m/l^m is a fixed number, therefore $(u_m/l^m)^{\frac{1}{n}} \to 1$ and $u_n^{\frac{1}{n}} \to l$.

NOTE. *The converse of this theorem is not true.* That is to say, if $u_n^{\frac{1}{n}} \to l$, it does *not* follow that $u_{n+1}/u_n \to l$. For consider as a special case the sequence 1, 2, 1, 2, ... in which the terms are alternately 1 and 2. Here $u_n^{\frac{1}{n}} \to 1$, but u_{n+1}/u_n does not tend to a limit.

(3) *If* $s_n = u_1 + u_2 + \ldots + u_n$ *and* $\lim_{n\to\infty} u_n = l$, *then* $\lim_{n\to\infty} \frac{1}{n} s_n = l$.

We can find m so that for any positive ϵ,

$$l-\epsilon < u_{m+1} < l+\epsilon,$$
$$l-\epsilon < u_{m+2} < l+\epsilon,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$l-\epsilon < u_n < l+\epsilon.$$

Hence by addition,

$$(n-m)(l-\epsilon) < s_n - s_m < (n-m)(l+\epsilon) \quad \text{for} \quad n>m$$

and

$$\left(1-\frac{m}{n}\right)(l-\epsilon) < \frac{s_n - s_m}{n} < \left(1-\frac{m}{n}\right)(l+\epsilon). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Now let m and n tend to ∞ in such a way that $n/m \to \infty$. (For example, we may suppose that $m\to\infty$ and $n>m^2$.)

We shall show that under these circumstances $s_m/n \to 0$.

Since $u_n \to l$ we can find a fixed number k such that $|u_r| < k$ for every r, and then

$$|s_m| \leqq |u_1| + |u_2| + \ldots + |u_m| < mk,$$

so that

$$\frac{1}{n}|s_m| < \frac{m}{n}k \quad \text{and} \quad s_m/n \to 0.$$

Thus in the inequality (A), m/n and s_m/n tend to zero and ϵ can be taken as small as we like, therefore $s_n/n \to l$.

(4) *If* $P_n = a_n b_1 + a_{n-1}b_2 + a_{n-2}b_3 + \ldots + a_1 b_n$ *and* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$,

then

$$\lim_{n\to\infty} \frac{1}{n} P_n = ab.$$

Let $a_n = a + \alpha_n$ and $b_n = b + \beta_n$, so that $\alpha_n \to 0$, $\beta_n \to 0$, then

$$\frac{1}{n}P_n = ab + \frac{a}{n}(\beta_1 + \beta_2 + \ldots + \beta_n) + \frac{b}{n}(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + Q_n,$$

where

$$Q_n = \frac{1}{n}(\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \ldots + \alpha_1\beta_n).$$

Now by (3), $\frac{1}{n}(\alpha_1 + \alpha_2 + \ldots + \alpha_n) \to 0$ and $\frac{1}{n}(\beta_1 + \beta_2 + \ldots + \beta_n) \to 0$,

and we shall prove that $Q_n \to 0$.

We have

$$|Q_n| \leqq \frac{1}{n}\sum_{r=1}^{r=n} |\alpha_r \beta_{n-r+1}|,$$

and since $\alpha_n \to 0$ and $\beta_n \to 0$ we can find a fixed number k such that for every s

$$|\alpha_s| < k \quad \text{and} \quad |\beta_s| < k.$$

We can also find m so that

$$|\alpha_s| < \epsilon/k \quad \text{and} \quad |\beta_s| < \epsilon/k \quad \text{for} \quad s>m.$$

Hence $|\alpha_r\beta_{n-r+1}|<k|\beta_{n-r+1}|<\epsilon$ for $n-r+1>m$, ...(A)

and $|\alpha_r\beta_{n-r+1}|<k|\alpha_r|<\epsilon$ for $r>m$. .....(B)

If $n>2m$, the inequality (A) will hold for $r<m+1$, i.e. for $r\leqslant m$, so that, with this proviso, $|\alpha_r\beta_{n-r+1}|<\epsilon$ for $1\leqslant r\leqslant n$.

Therefore $|Q_n|<\epsilon$: consequently $Q_n\to 0$ and $\frac{1}{n}P_n\to ab$.

10. Complex Sequences. (1) If $z_n=x_n+\iota y_n$ where x_n and y_n are functions of a positive integral variable n, the sequence (z_n) is said to be *complex*.

If corresponding to any positive ϵ, however small, there is a positive integer m such that

$$|z_n-z|<\epsilon \quad \text{for} \quad n\geqslant m,$$

where z is a fixed number, then z is called the *limit of* z_n as $n\to\infty$, and we write $\lim_{n\to\infty} z_n=z$ or $\lim z_n=z$ or $z_n\to z$.

Under these circumstances the sequence (z_n) is said to converge to the limit z.

In order that $\lim z_n=z=x+\iota y$ *it is necessary and sufficient that*

$$\lim x_n=x \quad \textit{and} \quad \lim y_n=y.$$

For $$|z_n-z|=\sqrt{\{(x_n-x)^2+(y_n-y)^2\}} \qquad \text{.....(A)}$$

Therefore $|z_n-z|\geqslant|x_n-x|$ and $|z_n-z|\geqslant|y_n-y|$.

First suppose that $z_n\to z$. By definition,

$$|z_n-z|\to 0, \quad \text{therefore} \quad |x_n-x| \quad \text{and} \quad |y_n-y|$$

tend to zero, so that $x_n\to x$ and $y_n\to y$.

Again, if $x_n\to x$ *and* $y_n\to y$, from equation (A) it follows that

$$|z_n-z|\to 0, \quad \text{therefore} \quad z_n\to z.$$

All this is evident geometrically, for $|z_n-z|$ is the distance between the points z_n and z.

(2) The theorems of Ch. XV, 2, regarding the limits of the sum, difference, product and quotient of two functions of n, continue to hold when the functions are complex.

(3) *Limit of* z^n *as* $n\to\infty$ *when* z *is complex. If* $|z|<1$, *then* $z^n\to 0$. *Otherwise* z^n *does not tend to a limit except when* $z=1$, *and then* $\lim z^n=1$.

For let $z=r(\cos\theta+\iota\sin\theta)$, then $z^n=r^n(\cos n\theta+\iota\sin n\theta)$, and

(i) if $|z|=r<1$, then $r^n\to 0$ and $z^n\to 0$;

(ii) if $z=1$ then $z^n=1$, and if $|z|>1$ then $r^n\to\infty$ and z^n does not tend to a limit.

11. General Principle of Convergence. The necessary and sufficient condition for the convergence of a complex sequence (z_n) is as follows:

Corresponding to any positive ϵ, however small, it must be possible to find m so that $|z_{m+p}-z_m|<\epsilon$ for every positive integer p.

Proof. *If (z_n) is convergent,* then (x_n) and (y_n) are convergent sequences; and by Art. 6, we can therefore choose m so that for every positive integer p,

$$|x_{m+p}-x_m|<\tfrac{1}{2}\epsilon \quad \text{and} \quad |y_{m+p}-y_m|<\tfrac{1}{2}\epsilon.$$

Now $$|z_{m+p}-z_m|\leqslant|x_{m+p}-x_m|+|y_{m+p}-y_m|,$$

therefore $$|z_{m+p}-z_m|<\epsilon.$$

Hence the condition is *necessary.*

Next suppose that $|z_{m+p}-z_m|<\epsilon$, then $|x_{m+p}-x_m|\leqslant|z_{m+p}-z_m|<\epsilon$.

Therefore (x_m) is convergent (Art. 6), and similarly it can be shown that (y_n) is convergent. Hence (z_n) is convergent, and the condition is *sufficient.*

NOTE. As in Art. 6, the condition may be stated thus. *Corresponding to any positive ϵ, however small, it must be possible to find m so that*

$$|z_{n+p}-z_n|<\epsilon,$$

if $n\geqslant m$, for every positive integer p.

EXERCISE XXV

1. If u_n is a decreasing function of n, then, after a certain stage, all the terms of u_n have the same sign.

[Either u_n tends to a limit or to $-\infty$. First let $u_n\to l$. If $l\geqslant 0$, u_n is positive for all values of n, for u_n tends to l *from above.* If $l<0$, we can choose m so that

$$l<u_n<0 \quad \text{for} \quad n\geqslant m.$$

Therefore the terms of (u_n) beginning with u_m are negative.

If $u_n\to-\infty$, we can find m so that $u_n<-k$ for $n\geqslant m$.]

2. If u_n is an increasing function, the statement in Ex. 1 is true.

3. Show that the positive proper fractions can be arranged in a definite order, so as to form a sequence. Represent the first few terms graphically; indicate terms which form subsequences converging to the upper and lower bounds as a limit.

[The sequence, omitting such fractions as $\frac{2}{4}$, is

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{3}{4},\ \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{4}{5},\ \tfrac{1}{6},\ \tfrac{5}{6},\ \cdots.$$

The required subsequences are indicated by the dotted lines.]

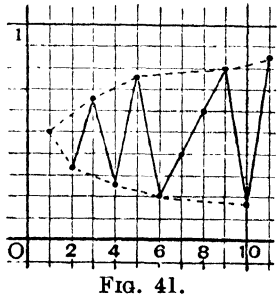


FIG. 41.

4. If $u_n>0$ for all values of n and $u_{n+1}/u_n \to l$, then $(nu_n)^{\frac{1}{n}} \to l$.
[See Ex. XXIV, 11, and Art. 9, (2).]

5. Prove that $\lim \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)=0$. [Art. 9, (3).]

6. If $u_n=\frac{(n+1)^n}{\lfloor n}$, show that $\frac{u_n}{u_{n-1}} \to e$, and consequently $(n+1)/(\lfloor n)^{\frac{1}{n}} \to e$.

7. If $u_n=\frac{1}{1\,.\,n}+\frac{1}{2(n-1)}+\frac{1}{3(n-2)}+\ldots+\frac{1}{n\,.\,1}$, prove that

$$u_n=\frac{2}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\to 0.$$

[We have $(n+1)u_n=\left(1+\frac{1}{n}\right)+\left(\frac{1}{2}+\frac{1}{n-1}\right)+\left(\frac{1}{3}+\frac{1}{n-2}\right)+\ldots$.]

8. If $u_n=\frac{1}{1^p\,.\,n^p}+\frac{1}{2^p\,.\,(n-1)^p}+\frac{1}{3^p(n-2)^p}+\ldots+\frac{1}{n^p\,.\,1^p}$, where $0<p<\frac{1}{2}$, prove that $u_n \to \infty$.

[Show that $r(n-r+1)\leqslant \frac{1}{4}(n+1)^2$ for $r=1, 2, 3, \ldots n$. Hence show that $u_n>n\,.\,4^p/(n+1)^{2p}\to \infty$.]

9. If $k>0$ and α, $-\beta$ are the positive and negative roots of $x^2-x-k=0$ prove that (i) if $u_n=\sqrt{(k+u_{n-1})}$ and $u_1>0$, then $u_n \to \alpha$;
(ii) if $v_n=\sqrt{(k-v_{n-1})}$ and $v_1<k$, then $v_n \to \beta$.
Illustrate geometrically.

[(i) Here $u_n^2=k+u_{n-1}$, $u_{n-1}^2=k+u_{n-2}$, $\therefore\ u_n^2-u_{n-1}^2=u_{n-1}-u_{n-2}$;
hence $u_n \gtrless u_{n-1}$, according as $u_{n-1} \gtrless u_{n-2}$; therefore (u_n) is monotone.
First suppose that $u_1>\alpha$, then $u_1^2>u_1+k$ for $u_1^2-u_1-k=(u_1-\alpha)(u_1+\beta)$, $\therefore\ u_2^2=u_1+k<u_1^2$, $\therefore\ u_2<u_1$ and (u_n) is a decreasing sequence.
Again, $u_2^2=u_1+k>u_2+k$, $\therefore\ u_2>\alpha$. Similarly $u_n>\alpha$ for $n=3, 4, \ldots$. Hence $u_n \to l \geqslant \alpha$. Thus $u_n-u_{n-1}\to 0$ and it follows that

$$(u_n-\alpha)(u_n+\beta)=(u_n^2-u_n-k)-(u_n^2-u_{n-1}-k)\to 0,\quad \therefore\ u_n\to\alpha.$$

The case in which $u_1<\alpha$ can be treated in the same way.

(ii) Draw graphs of $y=x$ and $y=\sqrt{(k+x)}$, meeting at A. In Fig. 42, we have

$ON_r=N_rp_r=u_r$, and $N_rP_r=\sqrt{(k+u_r)}=u_{r+1}$,

and thus $P_rp_r=u_r-u_{r+1}$.

The figure illustrates the case in which

$$u_1=ON_1=N_1p_1>\alpha,$$

and shows that (u_n) is a decreasing sequence.
Thus $N_rP_r>N_rp_r$ for every r, and so N_r is always to the right of N. Thus $u_n \to l \geqslant \alpha$ and

$$u_n-u_{n+1}=P_np_n\to 0.$$

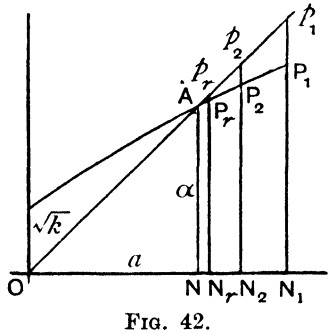


FIG. 42.

Hence N_r tends to N as a limiting position and $u_n \to \alpha$.]

10. If $u_n = f(u_{n-1})$ where $f(x)$ is such that (i) $f(x) > 0$ if $x > 0$, (ii) $f(x)$ increases with x, and (iii) the equation $x = f(x)$ has a single positive root α, prove that $u_n \to \alpha$.

[It is advisable to draw a figure.]

11. Show that as the number of roots tends to infinity, the expressions

$$\sqrt{7+\sqrt{7+\sqrt{7+}}}\ldots, \quad -\sqrt{7-\sqrt{7-\sqrt{7-}}}\ldots$$

tend to the roots of $x^2 - x - 7 = 0$ as limits.

12. If $x = \sqrt{7-\sqrt{7+y}}$, prove that

$$y = (x^2-7)^2 - 7.$$

Draw a graph of the last equation, and prove that if

$$u_n = \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7+}}}}\ldots \text{ to } 2n \text{ roots},$$

then $u_1, u_3, u_5 \ldots$ is a decreasing sequence,

$u_2, u_4, u_6 \ldots$ is an increasing sequence,

and that both sequences converge to 2 as a limit.

Prove also that

$$\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7-}}}}\ldots \text{ to } \infty = 3.$$

13. If u_1, v_1 are given unequal numbers and

$$u_n = \tfrac{1}{2}(u_{n-1} + v_{n-1}), \quad v_n = \sqrt{(u_{n-1}v_{n-1})} \text{ for } n \geqslant 2,$$

prove that (i) u_n decreases, and v_n increases as n increases. (ii) Each of the functions u_n and v_n tends to a limit, which is the same for both. This limit is called the *arithmetico-geometric mean* of u_1 and v_1 (*Gauss*).

[$u_n - v_n = \tfrac{1}{2}(\sqrt{u_{n-1}} - \sqrt{v_{n-1}})^2 > 0$, $\therefore\ u_n > v_n$ for every n;

$\therefore\ u_n = \tfrac{1}{2}(u_{n-1} + v_{n-1}) < u_{n-1}$ and $v_n = \sqrt{(u_{n-1}v_{n-1})} > v_{n-1}$.

Thus $v_1 < v_2 \ldots < v_n \ldots < u_n \ldots < u_2 < u_1$.

Hence u_n tends to a limit, and so does v_n.

Also $\lim v_n = \lim v_{n-1} = \lim (2u_n - u_{n-1}) = \lim u_n$.]

CHAPTER XVI

CONVERGENCE OF SERIES (1)

1. Definitions. Let u_n be a function of n which has a definite value for all positive integral values of n. An expression of the form

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots ,$$

in which every term is followed by another term, is called an *infinite series*. This series will be denoted by $\Sigma_{n=1}^{\infty} u_n$, or by Σu_n, and the sum of its first n terms by s_n.

The sum of the p terms immediately following the nth term is denoted by $R_{n,\,p}$, so that

$$R_{n,\,p} = u_{n+1} + u_{n+2} + \ldots + u_{n+p} = s_{n+p} - s_n.$$

As n tends to infinity, there are four distinct possibilities : s_n may tend to a finite limit, to infinity or to minus infinity, or it may do none of these things.

If s_n tends to a finite limit s, the series is said to be *convergent* and s is called its *sum to infinity*. Thus s is defined by

$$\lim_{n \to \infty} s_n = s, \quad \text{or briefly,} \quad \lim s_n = s.$$

This is also expressed by writing

$$u_1 + u_2 + u_3 + \ldots \text{ to } \infty = s \quad \text{or} \quad \Sigma_1^{\infty} u_n = s, \quad \text{or} \quad \Sigma u_n = s.$$

If s_n tends to ∞ or to $-\infty$, the series is said to be *divergent*.

If s_n tends to no limit, whether finite or infinite, the series is said to *oscillate*. We say that the series oscillates *finitely* or *infinitely*, according as s_n oscillates between finite limits, or between $+\infty$ and $-\infty$.

Series which diverge or oscillate are often said to be *non-convergent*.

The sum (to infinity) of a convergent series is essentially a *limit*, and is not a sum in the sense described in the definition of addition. We are therefore not justified in assuming that the sum of an infinite series is unaltered by changing the order of the terms or by the introduction or removal of brackets.

In fact, changes of this kind *may* alter the sum, or they may transform a convergent series into one which diverges or oscillates.

Thus the series $(1-1)+(1-1)+(1-1)+\ldots$ is convergent and its sum is zero, but the series $1-1+1-1+\ldots$ oscillates.

2. The Geometric Series. *The series* $1+x+x^2+\ldots+x^{n-1}+\ldots$ *converges if* $|x|<1$ *and its sum is* $1/(1-x)$. *It diverges if* $x\geqslant 1$, *oscillates finitely if* $x=-1$, *and oscillates infinitely if* $x<-1$.

For $s_n=(1-x^n)/(1-x)$ if $x\neq 1$. If $|x|<1$, $x^n\to 0$ and $s_n\to 1/(1-x)$. If $x\geqslant 1$, $s_n\to\infty$. If $x=-1$, $s_n=1$ or 0, according as n is odd or even. If $x<-1$, $s_n\to\infty$ or $-\infty$ according as n is odd or even.

3. Theorems. (1) *If m is a given number, the series*

$$u_1+u_2+u_3+\ldots \quad \textit{and} \quad u_{m+1}+u_{m+2}+u_{m+3}+\ldots$$

both converge, both diverge or both oscillate.

For let $s_n=u_1+u_2+\ldots+u_n$, $t_n=u_{m+1}+u_{m+2}+\ldots+u_{m+n}$.

Then $t_n=s_{m+n}-s_m$ and s_m is a fixed number, hence

(i) if s_{m+n} tends to a finite limit or to $\pm\infty$, so also will t_n;

(ii) if s_{m+n} tends to no limit, whether finite or infinite, neither will t_n.

(2) *If k is a fixed number and Σu_n converges to a sum s, then $\Sigma k u_n$ is convergent and its sum is ks. Also if the first series diverges or oscillates, so does the second, unless $k=0$.*

For if $s_n\to s$, then $ks_n\to ks$. If $k\neq 0$ and $s_n\to\pm\infty$, so does ks_n. If s_n tends to no fixed limit, finite or infinite, neither does ks_n.

4. Addition and Subtraction of Series. *If the series*

$$u_1+u_2+\ldots \quad \textit{and} \quad v_1+v_2+\ldots$$

converge and their sums are s and t respectively, then

(i) $(u_1+v_1)+(u_2+v_2)+\ldots$ *converges, and its sum is* $s+t$;

(ii) $(u_1-v_1)+(u_2-v_2)+\ldots$ *converges, and its sum is* $s-t$.

(i) For $\lim\{(u_1+v_1)+(u_2+v_2)+\ldots+(u_n+v_n)\}$

$=\lim\{(u_1+u_2+\ldots+u_n)+(v_1+v_2+\ldots+v_n)\}$

$=\lim(u_1+u_2+\ldots+u_n)+\lim(v_1+v_2+\ldots+v_n)=s+t.$

NOTE. In the proof, all the series in brackets are *finite*.

(ii) The proof is similar to the preceding.

SERIES OF POSITIVE TERMS

We first consider series in which *all the terms are positive*. Under this heading we include series in which all the terms are positive *beginning with some particular term*.

5. Introduction and Removal of Brackets. Let Σu_n be a series of positive terms, and let the terms be arranged in groups without altering

their order. Denote the sum of the terms in the nth group by v_n, then

(i) *If Σu_n converges to a sum s, so does Σv_n.*

For let
$$s_n = u_1 + u_2 + \ldots + u_n,$$
$$t_n = v_1 + v_2 + \ldots + v_n.$$

Then for every n we can find m and p, so that
$$s_m \leqslant t_n < s_{m+p},$$
where m tends to infinity with n.

Now $\lim s_m = \lim s_{m+p} = s$, therefore $\lim t_n = s$.

(ii) *If Σv_n converges to a sum s, so does Σu_n.*

For corresponding to every n we can find m, so that
$$t_m \leqslant s_n < t_{m+1}.$$
But $\lim t_m = \lim t_{m+1} = s$, therefore $\lim s_n = s$.

6. Changing the Order of Terms. *If the terms of a convergent series of positive terms are rearranged, the series remains convergent and its sum is unaltered.*

Proof. Let Σu_n be convergent, and let the terms be rearranged in any manner. Denote the new series by Σv_n, so that every u is a v and every v is a u. Let
$$s_n = u_1 + u_2 + \ldots + u_n,$$
$$t_n = v_1 + v_2 + \ldots + v_n.$$

Suppose that the first n terms of Σu_n are among the first m terms of Σv_n. Further, suppose that the first m terms of Σv_n are among the first $(n+p)$ terms of Σu_n.

Thus
$$n \leqslant m \leqslant n + p,$$
and m tends to infinity with n.

Also
$$s_n \leqslant t_m \leqslant s_{n+p}.$$

Now $\lim s_n = \lim s_{n+p} = s$, therefore $\lim t_m = s$.

Hence Σv_n is convergent and has the same sum as Σu_n.

NOTE. The argument fails for such a derangement as
$$u_1 + u_3 + u_5 + \ldots + u_2 + u_4 + u_6 + \ldots,$$
where Σu_n is broken up into two (or any finite number of) infi

For here we cannot find m so that the first n terms of Σu_n m terms of Σv_n. Thus u_2 does not occur till infinitely many of t have been placed.

The rule for the addition of series applies in cases of this kind, but Σu_n is broken up into *infinitely many* infinite series.

Derangements of the last sort are considered in another volume.

7. Theorem. *A series of positive terms cannot oscillate, and if s_n is always less than some fixed number k, the series converges and its sum is less than k.*

For s_n increases with n, and therefore tends to a finite limit or to ∞ (Ch. XV, 3). Hence the series cannot oscillate. Further, if $s_n<k$ for every n, then $s_n \to s$, where s is a fixed number less than k (Ch. XV, 3).

8. Comparison Tests. (1) *If Σu_n and Σa_n are two series of positive terms and Σa_n is known to be convergent, then Σu_n will be convergent if*

(i) * *$u_n \leqslant a_n$ for all values of n ;*

(ii) * *or if u_n/a_n is always less than some fixed positive number k ;*

(iii) *or if u_n/a_n tends to a finite limit.*

Proof. (i) If $a_1+a_2+\ldots$ to $\infty = t$, we have

$$u_1+u_2+\ldots+u_n \leqslant a_1+a_2+\ldots a_n<t\ ;$$

and since t is a fixed number, it follows that Σu_n is convergent.

(ii) For all values of n, we have $u_n<ka_n$. Now Σka_n is convergent, therefore Σu_n is convergent.

(iii) This follows at once from (ii), for if $\lim u_n/a_n$ is finite, we can find a positive constant k, such that $u_n/a_n<k$ for all values of n.

(2) *If Σu_n and Σa_n are two series of positive terms and Σa_n is known to be divergent, then Σu_n will be divergent if*

(i) * *$u_n \geqslant a_n$ for all values of n ;*

(ii) * *or if u_n/a_n is always greater than some fixed positive number k ;*

(iii) *or if u_n/a_n tends to a limit greater than zero.*

Proof. (i) Let N be a positive number, no matter how great. Since Σa_n is divergent, m can be found such that

$$a_1+a_2+\ldots+a_n>N \quad \text{provided that} \quad n>m\ ;$$

$$\therefore\ u_1+u_2+\ldots+u_n>N \quad \text{provided that} \quad n>m,$$

is divergent.

values of n, $u_n>ka_n$. Now Σa_n is divergent; therefore t, therefore Σu_n is divergent.

at once from (ii), for if $\lim u_n/a_n$ is finite, a positive ound such that $u_n/a_n>k$ for all values of n.

y these tests, we require certain series which are known or divergent. The geometric series and the series of the hose which are most generally useful as *test series.*

ficient that these conditions should hold *for all values of n greater than some*

9. Theorem. *The series* $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$ *is convergent if* $p>1$ *and divergent if* $p\leqslant 1$.

Denote the given series by S. Since S is a series of positive terms, its convergence or divergence is not affected by grouping the terms in brackets in any way we please.

First let $p>1$. Group the terms as follows:

$$1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

where the first terms in the brackets are $1/2^p$, $1/2^{2p}$, $1/2^{3p}$, etc., and the brackets contain 2, 4, 8, ... terms, respectively.

Each term in (A) after the first is less than the corresponding term in

$$1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\ldots,$$

which is the same as $1+\frac{2}{2^p}+\frac{4}{4^p}+\ldots$, or $1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\ldots$. ...(B)

Now (B) is a geometric series whose common ratio $1/2^{p-1}$ is less than unity, for $p>1$. Hence (B) is convergent, and therefore S is convergent.

Next, if $p=1$, the series S becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$. Group the terms thus: $1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots$, ...(C)

where the last terms in the brackets are $1/2^2$, $1/2^3$, $1/2^4$, etc., and the brackets contain 2, 4, 8, ... terms, respectively.

Each term of (C) is greater than the corresponding term of

$$1+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4})+(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8})+\ldots,$$

which is the same as $1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\ldots$, or $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$. .....(D)

Now (D) is divergent, and therefore S is divergent.

Lastly, if $p<1$, each term of S after the first is greater than the corresponding term of $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$; but the last series is divergent, therefore S is divergent.

10. Examples on the Comparison Tests.

Ex. 1. *Is the series* $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ *convergent or divergent?*

Here $u_n=\frac{2n-1}{n(n+1)(n+2)}$; let $a_n=\frac{1}{n^2}$.* Then we have

$$\lim\frac{u_n}{a_n}=\lim\frac{n^2(2n-1)}{n(n+1)(n+2)}=2.$$

Now Σa_n is convergent; therefore Σu_n is convergent.

* The nth term a_n of a suitable test series is found thus: Keeping only the highest powers of n in the numerator and denominator of u_n, we obtain $2n/n^3$ or $2/n^2$. Disregarding the numerical factor 2, we take $1/n^2$ for a_n.

Ex. 2. *Test for convergence or divergence the series*

$$\frac{1}{1+x}+\frac{x}{1+x^2}+\frac{x^2}{1+x^3}+\ldots+\frac{x^{n-1}}{1+x^n}+\ldots, \text{ where } x>0.$$

If $x<1$, $x^n\to 0$ and $u_n/x^{n-1}\to 1$. Now Σx^{n-1} is convergent, therefore Σu_n is convergent.

If $x>1$, $x^{-n}\to 0$ and $u_n=\frac{1}{x}\cdot\frac{x^n}{1+x^n}=\frac{1}{x}\cdot\frac{1}{x^{-n}+1}\to\frac{1}{x}$. Hence Σu_n is divergent.

If $x=1$, $u_n=\frac{1}{2}$, and the series is divergent.

Tests derived from the Geometric Series

11. D'Alembert's Test. *A series Σu_n of positive terms is convergent if $u_{n+1}/u_n<k<1$, where k is a fixed number and n may have any value.*

The series is divergent if $u_{n+1}/u_n\geqslant 1$ for all values of n.

For if $u_{n+1}/u_n<k<1$ for every n, then

$$\frac{u_n}{u_1}=\frac{u_n}{u_{n-1}}\cdot\frac{u_{n-1}}{u_{n-2}}\cdots\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}<k^{n-1}.$$

Thus $u_n<u_1\,.\,k^{n-1}$ and $\Sigma u_1\,.\,k^{n-1}$ is convergent, therefore Σu_n is convergent.

Again, if $u_{n+1}/u_n\geqslant 1$ for all values of n, then

$$u_n\geqslant u_{n-1}\geqslant u_{n-2}\cdots\geqslant u_1,$$

therefore $u_1+u_2+\ldots+u_n\geqslant nu_1\to\infty$. Hence Σu_n is divergent.

In particular, *if u_{n+1}/u_n tends to a limit l, then Σu_n is convergent when $l<1$ and divergent when $l>1$.*

For if $l<1$, choose k so that $l<k<1$. Then by the definition of a limit $u_{n+1}/u_n<k$ for $n>m$, where m is a fixed number. Hence

$$u_{m+1}+u_{m+2}+\ldots$$

is convergent, and therefore Σu_n is convergent.

Again, if $l>1$, then m can be found so that $u_{n+1}>u_n$ for $n>m$. Hence $u_{m+1}+u_{m+2}+\ldots$ is divergent, and therefore Σu_n is divergent.

Note. (i) It is sufficient that the conditions should hold *for all values of n greater than some fixed value.*

(ii) *Nothing is said as to the case in which* $\lim u_{n+1}/u_n=1$. In this case the series may be convergent or it may be divergent, and we say that the *test fails.*

For example, consider the series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots \quad\text{and}\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.$$

The first is divergent, the second is convergent, and in both we have $\lim u_{n+1}/u_n=1$

If, however, u_{n+1}/u_n tends to 1 as a limit *from above*, so that always $u_{n+1}>u_n$ the series is divergent. The series $1+2+3+4+\ldots$ is an instance of this.

12. Cauchy's Test. *A series Σu_n of positive terms is convergent if $u_n^{\frac{1}{n}} < k < 1$ where k is a fixed number and n can have any value.*

The series is divergent if $u_n^{\frac{1}{n}} \geqslant 1$ for infinitely many values of n.

For if $u_n^{\frac{1}{n}} < k < 1$, then $u_n < k^n$, and since $k < 1$ the series Σk^n is convergent, and consequently Σu_n is convergent.

If $u_n^{\frac{1}{n}} \geqslant 1$, and consequently $u_n \geqslant 1$, for infinitely many values of n, then Σu_n is obviously divergent.

In particular, *if $u_n^{\frac{1}{n}}$ tends to a limit l, then Σu_n is convergent when $l < 1$ and divergent when $l > 1$.*

The proof is similar to that in the last article.

It is sufficient that the conditions should hold for $n > m$, where m is a fixed number. Also nothing is said as to the case in which $\lim u_n^{\frac{1}{n}} = 1$, but if the limit is approached *from above*, Σu_n is divergent.

13. Comparing these tests, that of D'Alembert is more generally *useful* than Cauchy's. For in most of the series with which we are concerned, u_{n+1}/u_n is a simpler function than u_n.

On the other hand, Cauchy's test is more *general* than D'Alembert's. That this is the case will be seen by noting that

(i) Cauchy's condition for divergence is wider than D'Alembert's. For in D'Alembert's test the condition has to be satisfied by *all* values of n greater than a fixed value, and this is not the case in Cauchy's.

(ii) It has been shown in Ch. XV, 9, (2), that if $u_{n+1}/u_n \to l$ then $u_n^{\frac{1}{n}} \to l$. But if $u_n^{\frac{1}{n}} \to l$, it does *not* follow that u_{n+1}/u_n tends to any limit.

So we may expect that Cauchy's test will sometimes succeed when D'Alembert's fails, as in Ex. 3 below.

Ex. 1. *Show that the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$ is convergent if $0 < x < 1$ and divergent if $x \geqslant 1$.*

Here $u_n = \frac{x^n}{n}$, $u_{n+1} = \frac{x^{n+1}}{n+1}$; $\therefore \lim \frac{u_{n+1}}{u_n} = \lim \frac{n}{n+1} x = x$.

Hence, if $0 < x < 1$, the series is convergent, and if $x > 1$, it is divergent.

If $x = 1$, D'Alembert's test fails, but in this case the series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots,$$

and is divergent.

Ex. 2. *Show that the series* $1+x+x^2/\lfloor 2+x^3/\lfloor 3+\ldots$ *is convergent for all positive values of* x.

Here $u_n=\dfrac{x^{n-1}}{\lfloor n-1}$, $u_{n+1}=\dfrac{x^n}{\lfloor n}$; $\therefore \lim \dfrac{u_{n+1}}{u_n}=\lim \dfrac{x}{n}=0$.

Hence the series is convergent.

Ex. 3. *Test for convergence the series* $a+b+a^2+b^2+a^3+b^3+\ldots$.

Here $$\frac{u_{2n}}{u_{2n-1}}=\left(\frac{b}{a}\right)^n, \qquad \frac{u_{2n+1}}{u_{2n}}=a.\left(\frac{a}{b}\right)^n,$$

and D'Alembert's test is inapplicable. Using Cauchy's test, since $u_n^{\frac{1}{n}}\to a^{\frac{1}{2}}$ or $b^{\frac{1}{2}}$ according as n is odd or even, the series is convergent if $0<a<1$ and $0<b<1$, and divergent if $a\geqslant 1$, or if $b\geqslant 1$.

EXERCISE XXVI

1. State the character of the series

(i) $(1-\frac{1}{2})+(1-\frac{3}{4})+(1-\frac{7}{8})+\ldots$, (ii) $1-\frac{1}{2}+1-\frac{3}{4}+1-\frac{7}{8}+\ldots$.

showing that for the second series $\lim s_{2n}=1$ and $\lim s_{2n+1}=2$.

2. Show that the series $x+x^{-1}+x^2+x^{-2}+x^3+x^{-3}+\ldots$ is never convergent.

3. Show that, if $a_1+a_2+a_3+\ldots$ is convergent, so also is

$$\tfrac{1}{2}(a_1+a_2)+\tfrac{1}{2}(a_2+a_3)+\tfrac{1}{2}(a_3+a_4)+\ldots.$$

Prove also that the converse is true if $a_n>0$, for $n>$ some fixed number m. [Consider the sum to n terms of the second series.]

4. Criticise the following:

$$2(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\tfrac{1}{8}+\ldots)=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots=(1+\tfrac{1}{3}+\tfrac{1}{5}+\ldots)+(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\ldots).$$

$$\therefore \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\ldots=1+\tfrac{1}{3}+\tfrac{1}{5}+\ldots,$$

which is absurd, for $\frac{1}{2}<1$, $\frac{1}{4}<\frac{1}{3}$, and so on.

5. If n is a positive integer, show that

(i) $\dfrac{1}{2}<\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+\ldots+\dfrac{1}{2n}<1$. (ii) $\dfrac{1}{2}n<\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\ldots+\dfrac{1}{2^p}<n$.

(iii) Given that $2^{20}>10^6+1$, show that the sum of the first million terms of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ is less than 20.

6. Show that

(i) $\dfrac{1}{(m+1)(m+2)}+\dfrac{1}{(m+2)(m+3)}+\dfrac{1}{(m+3)(m+4)}+\ldots \text{ to } \infty=\dfrac{1}{m+1}$.

(ii) The sum to infinity of the series $\dfrac{1}{(m+1)^2}+\dfrac{1}{(m+2)^2}+\dfrac{1}{(m+3)^2}+\ldots$ lies between $\dfrac{1}{m+1}$ and $\dfrac{1}{m+1}+\dfrac{1}{(m+1)^2}$.

(iii) Hence show that, in order to find the sum to infinity of the series $\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\ldots$, with an error less than 0·001, at least 1000 terms must be taken.

Determine whether the series in Exx. 7-14 are convergent or divergent.

7. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots$.

8. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots$.

9. $\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\dots$.

10. $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\dots$.

11. $\frac{1\,.\,2}{3^2\,.\,4^2}+\frac{3\,.\,4}{5^2\,.\,6^2}+\frac{5\,.\,6}{7^2\,.\,8^2}+\dots$.

12. $\frac{1}{\lfloor 1}+\frac{3}{\lfloor 2}+\frac{5}{\lfloor 3}+\frac{7}{\lfloor 4}+\dots$.

13. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\dots$.

14. $\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\dots$.

Test for convergence or divergence the series whose nth terms are

15. $\frac{\sqrt{n}}{n^2+1}$.

16. $\frac{n}{(a+nb)^2}$.

17. $\sqrt{\frac{2^n-1}{3^n-1}}$.

18. $\frac{n^p}{(n+1)^q}$.

19. $\frac{n^3+a}{2^n+a}$.

20. $\frac{n^3}{\lfloor n-1}$.

21. $\sqrt{n^2+1}-n$.

22. $\sqrt{n^3+1}-\sqrt{n^3}$.

For what positive values of x are the series in Exx. 23-31 convergent, and for what values are they divergent ?

23. $1+2x+3x^2+4x^3+\dots$.

24. $1+x/2^2+x^2/3^2+x^3/4^2+\dots$.

25. $\frac{1}{1\,.\,2\,.\,3}+\frac{x}{4\,.\,5\,.\,6}+\frac{x^2}{7\,.\,8\,.\,9}+\dots$.

26. $\frac{x}{a+\sqrt{1}}+\frac{x^2}{a+\sqrt{2}}+\frac{x^3}{a+\sqrt{3}}+\dots$.

27. $\frac{1}{2^{a+x}}+\frac{2^a}{3^{a+x}}+\frac{3^a}{4^{a+x}}+\dots$.

28. $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\dots$.

29. $\Sigma\frac{a^n}{a^n+x^n}$.

30. $\Sigma\frac{1}{x^n+x^{-n}}$.

31. $\Sigma\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$.

32. Explain why D'Alembert's test cannot be stated thus : A series of positive terms is convergent if the ratio of each term to the preceding is less than unity.

Exx. 33-37 *refer to series of positive terms.*

33. If Σu_n is convergent and $\frac{v_{n+1}}{v_n}\leqslant\frac{u_{n+1}}{u_n}$ for $n\geqslant m$, then Σv_n is convergent.

34. If Σu_n is divergent and $\frac{v_{n+1}}{v_n}\geqslant\frac{u_{n+1}}{u_n}$, then Σv_n is divergent.

35. If Σu_n is convergent, so is $\Sigma {u_n}^2$.

[After a certain stage $u_n<1$, for $u_n\to 0$; $\therefore\ {u_n}^2<u_n$.]

36. If Σu_n is convergent, so is $\Sigma u_n/(1+u_n)$.

37. If $\Sigma {u_n}^2$ is convergent, so is $\Sigma u_n/n$. [For $u_n/n\leqslant\frac{1}{2}({u_n}^2+1/n^2)$.]

Find the sum to infinity of the series in Exx. 38-41.

38. $\frac{1}{1\,.\,3\,.\,5}+\frac{1}{3\,.\,5\,.\,7}+\frac{1}{5\,.\,7\,.\,9}+\dots$.

39. $\frac{1}{1\,.\,4}+\frac{1}{2\,.\,5}+\frac{1}{3\,.\,6}\dots$.

40. $\frac{2}{1\,.\,4\,.\,5}+\frac{3}{2\,.\,5\,.\,6}+\frac{4}{3\,.\,6\,.\,7}+\dots$.

41. $\frac{1}{1\,.\,3\,.\,7}+\frac{2}{3\,.\,5\,.\,9}+\frac{3}{5\,.\,7\,.\,11}+\dots$.

Terms Alternately Positive and Negative

14. Theorem. *The series* $u_1 - u_2 + u_3 - \dots$, *in which the terms are alternately positive and negative, is convergent if each term is numerically less than the preceding and* $\lim u_n = 0$.

Proof. We have

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}), \quad \dots\dots\dots\dots\dots\dots\textbf{(A)}$$

and this can be written in the form

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - u_{2n}. \quad \dots\dots\dots\dots\dots\dots\textbf{(B)}$$

Now $u_1 - u_2$, $u_2 - u_3$, $u_3 - u_4$, ... are all positive, for

$$u_1 > u_2 > u_3 > \dots .$$

From (**A**) it therefore follows that s_{2n} is positive and increases with n, and from (**B**) it is seen that s_{2n} is always less than the fixed number u_1.

Hence, s_{2n} tends to a limit which is less than u_1 (XV, 3).

Again, $s_{2n+1} = s_{2n} + u_{2n+1}$ and $\lim u_{2n+1} = 0$; therefore s_{2n+1} tends to the same limit as s_{2n}.

The series is therefore convergent, its sum is positive and less than u_1.

Ex. 1. *Show that the series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ *is convergent.*

Here the terms are alternately positive and negative, each term is numerically less than the preceding and $\lim 1/n = 0$; hence the series is convergent.

Series with Terms either Positive or Negative

15. Absolutely Convergent Series. The series Σu_n, containing positive and negative terms, is said to be *absolutely convergent* if the series $\Sigma |u_n|$ is convergent.

If Σu_n is convergent and $\Sigma |u_n|$ divergent, then Σu_n is said to be *conditionally convergent* or *semi-convergent.*

Thus the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$ is absolutely convergent, because the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is convergent. But, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is conditionally convergent, for it is convergent and $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent.

Theorem. *An absolutely convergent series is convergent.*

Let Σu_n be absolutely convergent; then by definition $\Sigma |u_n|$ is convergent.

Now $u_n + |u_n| = 2u_n$ or 0, according as u_n is positive or negative.

Therefore every term of the series $\Sigma(u_n + |u_n|)$ is $\geqslant 0$ and is $\leqslant$ the corresponding term of the convergent series $\Sigma 2|u_n|$.

Hence $\Sigma(u_n + |u_n|)$ is convergent and consequently, by Art. 4, Σu_n is convergent.

NOTE. If the reader fails to see the point in the statement of this theorem, he should note that, if we say that Σu_n is absolutely convergent, we assert the convergence of *another* series, $\Sigma|u_n|$, and *not* that of Σu_n itself.

From Arts. 11 and 12, it follows that *the series Σu_n is absolutely convergent if, after a certain stage,*

$$\left|\frac{u_{n+1}}{u_n}\right| < k \quad \text{or if} \quad |u_n|^{\frac{1}{n}} < k,$$

where k is a fixed positive number less than unity.

16. General Condition for Convergence. Since s_n is a function of the positive integral variable n, by Ch. XV, 6, (B), *the necessary and sufficient condition for the convergence of Σu_n is as follows*:

Corresponding to any positive number ϵ that we may choose, however small, it must be possible to find m so that, if $n \geqslant m$, for every positive integer p,

$$|s_{n+p} - s_n| < \epsilon,$$

that is,

$$|R_{n,p}| = |u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \epsilon.$$

Thus if Σu_n is convergent and p is any integer,

$$R_{n,p} = u_{n+1} + u_{n+2} + \ldots + u_{n+p} \to 0 \quad \text{as} \quad n \to \infty \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

and in particular $u_n \to 0$.

But *the condition* (**A**) *is not sufficient to ensure the convergence of Σu_n unless p is allowed to tend to ∞ in any way whatever.*

Ex. 1. *Apply the general condition to the series*

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots .$$

Here

$$R_{n,p} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+p} > \frac{p}{n+p}.$$

For any *fixed* value of p, $p/(n+p) \to 0$ as $n \to \infty$. But if $p = n^2$, then $p/(n+p) \to 1$; hence, $R_{n,p}$ cannot tend to zero, and so the series is not convergent.

17. Pringsheim's Theorem. *If Σu_n is a convergent series of positive decreasing terms, then $nu_n \to 0$.*

For if p is any positive integer

$$pu_{m+p} < u_{m+1} + u_{m+2} + \ldots + u_{m+p} \to 0 \quad \text{as} \quad m \to \infty .$$

Let $p = m$, therefore $mu_{2m} \to 0$ and $2mu_{2m} \to 0$; i.e. $nu_n \to 0$, when $n = 2m$.

Let $p = m+1$, therefore $(m+1)u_{2m+1} \to 0$; and

$(2m+1)u_{2m+1} = 2(m+1)u_{2m+1} - u_{2m+1} \to 0$; i.e. $nu_n \to 0$, when $n = 2m+1$.

This proves the theorem, whether n is odd or even.

Ex. 1. *Use this theorem to show that the series $\Sigma \frac{1}{n}$ is divergent.*

Here $nu_n = n \cdot \frac{1}{n} = 1$, and does not tend to zero. Hence $\Sigma \frac{1}{n}$ cannot be convergent; and, since it is a series of positive terms, it must be divergent.

18. Introduction and Removal of Brackets. Let the terms of Σu_n be arranged in groups without altering their order. Denote the sum of the terms of the nth group by v_n, then

(1) *If Σu_n is convergent, so also is Σv_n, and these series have the same sum.*

For let $\quad s_n = u_1 + u_2 + \ldots + u_n, \quad t_n = v_1 + v_2 + \ldots + v_n.$

Then for any value of n we can find m so that either

$$s_n = t_m \quad \text{or} \quad s_n = t_m + (u_{m+1} + u_{m+2} + \ldots u_{m+p}),$$

where the terms in the bracket are included in the group v_{m+1}.

If Σu_n is convergent,

$$R_{n,p} = u_{n+1} + u_{n+2} + \ldots u_{n+p} \to 0 \quad \text{as} \quad n \to \infty,$$

also $\quad m \to \infty \quad$ as $\quad n \to \infty$, $\quad$ therefore $\quad s_n - t_m \to 0.$

Hence Σv_n converges to the same sum as Σu_n.

(2) *If Σv_n is convergent, Σu_n is not necessarily so.*

For, given that Σv_n is convergent, we cannot conclude that $R_{n,p} \to 0$ for all values of p.

(3) Denote $|\, u_n \,|$ by u_n', and suppose that the terms of $\Sigma u_n'$ are grouped as in § (1). Let v_n' be the sum of the terms of the nth group. Then, *if $\Sigma v_n'$ is convergent*, so also is u_n' (Art. 5); and, as $n \to \infty$,

$$|\, R_{n,p} \,| = |\, u_{n+1} + u_{n+2} + \ldots + u_{n+p} \,| \leqslant u'_{n+1} + u'_{n+2} + \ldots + u'_{n+p} \to 0.$$

Therefore, as in § (1), *Σv_n converges to the same sum as Σu_n.*

19. Rearrangement of Terms. The following example shows that a rearrangement of the terms of a *semi-convergent* series may alter its sum, and a rearrangement of the signs may change the series into one which is divergent.

Ex. 1. *Let s be the sum of the semi-convergent series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad \ldots\ldots\ldots\

that is, $$\sigma_{3n}=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left($$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right.$$

$$\therefore \lim \sigma_{3n}=\frac{1}{2}\lim s_{2n}=\frac{1}{2}s. \quad \text{Also}$$

Hence, the second series converges and

(ii) Let $$t_n=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$$

then $$t_{3n}>\left(\frac{1}{3}+\frac{1}{3}-\frac{1}{3}\right)+\left(\frac{1}{6}+\frac{1}{6}-\frac{1}{6}\right)+($$

$$\therefore\ t_{3n}>\frac{1}{3}\left(1+\frac{1}{2}\right.$$

Now the series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is divergent; t_{3n+2} all tend to ∞. Hence the series (C) is

Theorem. *If the terms of an absolu the series remains convergent and its sur*

Denote the new series by Σv_n, so tha

We have $u_n+|u_n|=2u_n$ or 0, accor And since $\Sigma|u_n|$ is a convergent se $\Sigma(u_n+|u_n|)$, for its terms are $\leqslant$ the c

Let $\Sigma|u_n|=s$ and $\Sigma(u_n+|u_n|)=$ $\Sigma|u_n|$ and $\Sigma(u_n+|u_n|)$ are series o unaltered by any rearrangement of tern

$$\Sigma|v_n|=s, \quad \Sigma(v_n+|v_n|)=s',$$

Ex. 1. *Find a range of values of x for whicl*

$$1+(x-x^2)+(x-x^2)^2-$$

can be arranged as a convergent series of ascendi

Expanding the terms of (A) and removing b

$$1+x-x^2+x^2-2x^3+x^4$$

Now, if $|x-x^2|<1$, the series (A) is co *justify* either the removal of brackets or a dera

However, by Arts. 15, 18 (1), 19, each of these *convergent*; and, by Art. 5, this will be the cas

$$1+(x'+x'^2)+(x'+$$

where $x'=|x|$, is convergent; that is, if $x'+$

$$x'<\tfrac{1}{2}(\sqrt{5}-1), \quad \text{or} \quad -\tfrac{1}{2}(\sqrt{5}-$$

which is the required range.

NOTE. What has been said does not preclud can be made over a *wider range.*

earching for the sum of an infinite
ontent with an approximate value.

$. + u_n + \ldots$, ...(A)

$\llcorner u_{n+3} + \ldots$ . .....(B)

and its sum, denoted by R_n, is called
ries (A). Thus we have

$_n + R_n$,

approximate value of s. It should be
s.

er limit to the numerical value of the
o find a number which $|R_n|$ cannot
t series

$+ \ldots + (-1)^n u_n + \ldots$,

ive terms. Here

$_{n+1} - u_{n+2} + u_{n+3} - \ldots\}$.

rackets is positive and less than u_{n+1}

Hence *the error in taking* s_n *as an*
ly less than u_{n+1}.

ror in taking $x - x^2/2$ as the sum of the
ss than $x^3/3$.

n which all the terms are positive or in
erms are all of the same sign. This is
n often proceed as follows.

$^3 + \ldots + \frac{x^n}{n} + \ldots$ *where* $0 < x < 1$. The series is

$$\frac{}{+1}(1 + x + x^2 + \ldots) < \frac{x^{n+1}}{(n+1)(1-x)}.$$

ue of s is numerically less than $x^{n+1}/(n+1)(1-x)$.

gence or Divergence. For a con-
lue of R_n for a given value of n, *the more*
ge.

alue of r, the more rapid is the convergence
$r^2 + \ldots$.

$/4^2 + \ldots$ is convergent, but in order to find
mals, at least 1000 terms must be taken
fore say that the series *converges slowly*.

Next, consider the series $1+\frac{1}{2}+$. of a million terms is less than 20 (s therefore said to *diverge slowly*. *In prac* the reckoning so that we have to deal w

For instance, the series, $s=1-\frac{1}{2}$ -

By grouping the terms, we find

$$s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+$$

This series converges more rapidly find that $s=2[\frac{1}{3}+\frac{1}{5}(\frac{1}{3})^3+\frac{1}{5}(\frac{1}{3})^5+$. rapidly than a geometrical series

22. Series of Complex

that the series Σx_n and Σy_n c s_n, σ_n, τ_n be the sums to n te Ch. XV, 10, $\lim s_n=$ lii and we say that Σz_n converges t

Thus Σz_n *is convergent if* Σx_n *a*

(2) By Ch. XV, 11, *the neces of* Σz_n is as follows : *Corres it must be possible to find*

$|s_{n+p}-s_n|<\epsilon$, tha

(3) **Definition of A** terms and $\Sigma|z_n|$ is

As for real se *convergent.*

For $|z_n|=\sqrt{(}$ then $\Sigma|z_n|$ is c

Consequently

(4) It is impo. Σx_n *and* Σy_n, *an*

The first part c

Hence if $\Sigma|x_n|$

(5) The theorems c of brackets and the complex series. This

, is complex, the series

$$\ldots + z^{n-1} + \ldots$$

and when convergent, its sum is $1/(1-z)$.

$$+ z^{n-1} = (1-z^n)/(1-z).$$

' if $|z|>1$, z^n does not converge to
), (3), $z^n \to 0$ and $s_n \to 1/(1-z)$,
s sum is $1/(1-z)$.

re $0<r<1$, we have

$$\text{s } 2\theta + \iota \sin 2\theta) + \ldots \text{ to } \infty$$

$$\frac{-r\cos\theta + \iota r \sin\theta}{-2r\cos\theta + r^2}.$$

$$\text{+} \ldots = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2},$$

$$+ \ldots \quad = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

hat these results are also true
<1.

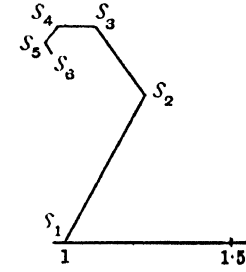


FIG. 43.

then $u_n \to 0$.

. then we can find

$$\ldots \left(1 + \frac{k}{n-1}\right)$$

$$\ldots + \frac{1}{n-1}\Big).$$

Now the series $\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\dots$ is divergent, therefore

$$u_m/u_n\to\infty \quad \text{and} \quad u_n\to 0.$$

25. The Binomial Series. For any given value of n, the series

$$1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\dots+\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}x^r+\dots$$

is called the *Binomial Series*. When n is a positive integer, the series terminates and its sum is $(1+x)^n$. In all other cases, it is an infinite series, which we denote by $u_0+u_1+u_2+\dots+u_r+\dots$.

(1) **Ultimate Signs of the Terms.** *From the beginning, or after a certain stage, the terms are alternately positive and negative, or they all have the same sign according as* $x\gtrless 0$.

For $$u_{r+1}/u_r=(n-r)x/(r+1).$$

Hence if m is the least of the numbers 0, 1, 2, ... which is greater than n, the terms u_m, u_{m+1}, u_{m+2}, ... are alternately positive and negative, or all have the same sign according as $x\gtrless 0$.

(2) **Convergence, etc.** We shall prove that *the series is absolutely convergent if* $|x|<1$; *and when* $x=1$, *the series converges if* $n>-1$, *oscillates finitely if* $n=-1$ *and oscillates infinitely if* $n<-1$.

Proof. Denote the series by $u_0+u_1+u_2+\dots$, then

$$\frac{u_{r+1}}{u_r}=\frac{n-r}{r+1}x\to -x, \text{ as } r\to\infty\ ; \text{ therefore } \lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right|=|x|,$$

and the series is absolutely convergent if $|x|<1$.

Next let $x=1$. If $r>n$, u_{r+1} and u_r have opposite signs, so that, after a certain stage, the terms are alternately positive and negative. Also

$$\left|\frac{u_r}{u_{r+1}}\right|=\frac{r+1}{r-n}=1+\frac{a_r}{r} \quad \text{where} \quad a_r=(n+1)\Big/\left(1-\frac{n}{r}\right).$$

(i) *If* $n>-1$, as $r\to\infty$, $a_r\to n+1>0$. Hence, by Art. 24, $u_r\to 0$. Also $|u_{r+1}|<|u_r|$. Hence, by Art. 14, the series is convergent.

(ii) *If* $n=-1$ the series is $1-1+1-1+\dots$ and oscillates finitely.

(iii) *If* $n<-1$, let $n+1=-m$ so that $m>0$, then

$$|u_r|=(1+m)\left(1+\frac{m}{2}\right)\dots\left(1+\frac{m}{r}\right)>1+m\left(1+\frac{1}{2}+\dots+\frac{1}{r}\right).$$

Hence $|u_r|\to\infty$ and the series oscillates infinitely.
The case in which $x=-1$ is considered in Ch. XX, 6.

EXERCISE XXVII

RANGE OF CONVERGENCE

Show that the series in Exx. 1-5 are convergent.

1. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$.

2. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$.

3. $\frac{1}{a+1}-\frac{1}{a+2}+\frac{1}{a+3}-\frac{1}{a+4}+\dots$.

4. $1-\frac{1}{2}+\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6}+\dots$.

5. $1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\dots$.

6. Prove that the series $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\dots$ is convergent if $0<x<1$.

7. Show that the series $1+x+x^2/\underline{|2}+x^3/\underline{|3}+\dots$ is convergent for all values of x, real or complex.

8. Show that $\frac{1}{a}+\frac{1}{a+1}-\frac{1}{a+2}+\frac{1}{a+3}+\frac{1}{a+4}-\frac{1}{a+5}+\dots$ is a divergent series.

9. Show that the series

$$\frac{\mu}{2}+\frac{3}{\underline{|2}}\left(\frac{\mu}{2}\right)^2+\frac{4.6}{\underline{|3}}\left(\frac{\mu}{2}\right)^3+\dots+\frac{(n+1)(n+3)(n+5)\dots(3n-3)}{\underline{|n}}\left(\frac{\mu}{2}\right)^n+\dots$$

is convergent if $\mu^2<4/27$.

[Consider separately the sum of the odd and the sum of the even terms.]

10. (i) Find a range of values of x for which the series

$$1+(\tfrac{5}{2}x-x^2)+(\tfrac{5}{2}x-x^2)^2+\dots$$

is convergent.

(ii) Find a range for which the series can be arranged as a convergent series of ascending powers of x.

11. If $0<x\leqslant 0{\cdot}4$, show that the series

$$1+(2x\cos\theta-x^2)+(2x\cos\theta-x^2)^2+\dots$$

can be arranged as a convergent series of ascending powers of x.

[It is enough to show that with the given condition $|2x\cos\theta|+x^2<1$.]

12. Let the terms of Σu_n be arranged in groups, without altering their order, and denote the sum of the terms of the nth group by v_n.

Given that Σv_n is convergent, prove that Σu_n converges to the same sum as Σv_n in the following cases:

(i) If the number of terms in every bracket is finite and $u_n\to 0$.

(ii) If all the terms in each bracket have the same sign.

[See Art. 18. In case (i), $R_{n,p}\to 0$. In case (ii), $v_n\to 0$; hence, every term in v_n tends to zero, and case (i) applies.]

13. Show that the series $1+z+z^2+\dots$ is absolutely convergent when $z=1/(1+\iota)$, and find its sum.

14. If $z_n=\dfrac{(-1)^n}{n}+\dfrac{1}{n^2}\iota$, show that Σz_n is convergent, but not absolutely convergent.

Prove the results in Exx. 15-17, where R_n is the remainder after n terms in the given series.

15. For the series $1+\dfrac{1}{\lfloor 1}+\dfrac{1}{\lfloor 2}+\dfrac{1}{\lfloor 3}+\dots$, $R_n<\dfrac{n+1}{n\lfloor n}$.

16. For the series $1-\dfrac{x}{\lfloor 1}+\dfrac{x^2}{\lfloor 2}-\dfrac{x^3}{\lfloor 3}+\dots$, $|R_n|<\dfrac{x^n}{\lfloor n}$ provided that $0<x<1$.

17. For the series $1+\dfrac{x}{\lfloor 1}+\dfrac{x^2}{\lfloor 2}+\dfrac{x^3}{\lfloor 3}+\dots$, $R_n<\dfrac{x^n}{\lfloor n}\cdot\dfrac{n+1}{n+1-x}$ provided that $0<x<n+1$.

Prove the results stated in Exx. 18-23 :

18. $\Sigma_1^\infty \dfrac{1}{4n^2-1}=\dfrac{1}{2}$.

19. $\Sigma_1^\infty \dfrac{n-1}{\lfloor n}=1$.

20. If $|x|>1$, $\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1-x^4}+\dots \text{ to } \infty=\dfrac{1}{x-1}$.

21. $\dfrac{x}{1-x^2}+\dfrac{x^2}{1-x^4}+\dfrac{x^4}{1-x^8}+\dots \text{ to } \infty=\dfrac{x}{1-x}$ or $\dfrac{1}{1-x}$, according as $|x|\lesseqgtr 1$.

22. If $b-1>a>0$, then

$$1+\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\dots \text{ to } \infty=\frac{b-1}{b-a-1}.$$

23. If $b-2>a>0$, then

$$\frac{a}{b}+2\frac{a(a+1)}{b(b+1)}+3\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\dots \text{ to } \infty=\frac{a(b-1)}{(b-a-1)(b-a-2)}.$$

24. Show that the sum to n terms of the series

$$\cos\theta+\cos 2\theta+\cos 3\theta+\dots$$

is $\quad \frac{1}{2}\{\sin(n+\frac{1}{2}\theta)\operatorname{cosec}\frac{1}{2}\theta-1\}$.

Hence show that the series oscillates finitely except when θ is an even multiple of π, in which case the series diverges.

$[2\cos r\theta\sin\dfrac{\theta}{2}=\sin(r+\frac{1}{2}\theta)-\sin(r-\frac{1}{2}\theta).]$

25. Show that the series $\Sigma\sin n\theta$ oscillates finitely except when θ is a multiple of π, in which case it converges to zero.

26. Show that the series $z+\frac{1}{2}z^2+\frac{1}{3}z^3+\dots$ is convergent, but not absolutely convergent, when $z=\cos\dfrac{\pi}{4}+\iota\sin\dfrac{\pi}{4}$. Illustrate the slowness of the convergence by plotting the points $s_1, s_2, \dots s_7$ on an Argand diagram.

CHAPTER XVII

CONTINUOUS VARIABLE

1. Definitions. We say that x *varies continuously from* a *to* b when it passes *once* through all real values between a and b. Under these circumstances x is called a continuous real variable.

Limit of $f(x)$. Consider the function $f(x)=\{(1+x)^3-1\}/x$.

When $x=0$, $f(x)$ has no definite value, for it then takes the form 0/0. In other words the function $f(x)$ is *undefined* for the value zero of x. For every value of x except zero, we have $f(x)=(3x+3x^2+x^3)/x=3+3x+x^2$.

Thus, *if* x *is nearly equal to zero,* $f(x)$ *is nearly equal to* 3*: further, by making* x *small enough, we can make the difference between* $f(x)$ *and* 3 *as small as we please.* This is *roughly* what is meant by saying that 3 *is the limit of* $f(x)$ *as* x *tends to zero.*

The matter may be put precisely as follows: Choose any positive number ϵ, no matter how small. If $|x|<1$, then

$$|f(x)-3|=|3x+x^2|<4|x|,$$

therefore $\quad |f(x)-3|<\epsilon \quad$ provided that $\quad |x|<\frac{1}{4}\epsilon.$

Thus, *as* x *tends to zero,* $|f(x)-3|$ *becomes and remains less than any positive number* ϵ *that we may choose, no matter how small.* This is *precisely* what is meant by saying that 3 is the limit of $f(x)$ as x tends to zero.

It should be observed that $f(x)$ never *attains* the value 3. The usual definitions are as follows:

(1) *If, corresponding to any positive number* ϵ *which we may choose, no matter how small, there exists a number* η *such that*

$$|f(x)-l|<\epsilon \quad \textit{provided only that} \quad |x-a|<\eta,$$

where l *and* a *are fixed numbers, then* l *is called the limit of* $f(x)$ *as* x *tends to* a.

This is expressed by writing $\lim_{x\to a} f(x)=l$.

Or we may say that $f(x)\to l$ as $x\to a$, where it is understood *that* x *may approach* a *from either side, i.e.* x *may tend to* $a+0$ *or* $a-0$.

Observe that nothing is said as to the value of $f(x)$ when $x=a$. Such a value may or may not exist. In any case, we are not concerned with it.

(2) *If, corresponding to any positive number ϵ, no matter how small, there exists a positive number N such that*

$$|f(x)-l|<\epsilon, \quad \textit{provided only that} \quad x>N,$$

where l is a fixed number, then l is called the limit of $f(x)$ as x tends to infinity.

This is expressed by writing $\lim_{x\to\infty} f(x)=l$ or $f(x)\to l$ as $x\to\infty$.

(3) *If to any positive number M, no matter how great, there corresponds a positive number N such that*

$$f(x)>M, \quad \textit{provided only that} \quad x>N,$$

we say that $f(x)$ tends to infinity with x, and we write $f(x)\to\infty$ as $x\to\infty$ or (less correctly) $\lim_{x\to\infty} f(x)=\infty$.

The reader should be able to frame for himself corresponding definitions for the meaning of '$f(x)$ tends to minus infinity as $x\to\infty$.'

2. Fundamental Theorems. Let u, v be functions of x which tend to the limits l, l' as x tends to a or as x tends to infinity, then

(i) $\lim(u+v)=l+l'$, (ii) $\lim(u-v)=l-l'$, (iii) $\lim uv=ll'$,

(iv) $\lim 1/u=1/l$ unless $l=0$, (v) $\lim u/v=l/l'$ unless $l'=0$.

Apart from some verbal changes which the reader can easily make for himself, the proofs are the same as those in Ch. XV, 2.

3. Two Theorems on Polynomials. What is said in this article holds good, no matter whether the variable x and the coefficients are real or complex.

Let $f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$, then

(1) *If ϵ is any positive number, no matter how small, we can find a positive number η so that*

$$|f(x)-a_0|<\epsilon \quad \textit{provided only that} \quad |x|<\eta.$$

Proof. For shortness, let $|x|=x'$ and let k be the greatest of the numbers

$$|a_1|, \quad |a_2|, \ldots, |a_n|;$$

then
$$\begin{aligned}|f(x)-a_0| &\leqslant |a_1|x'+|a_2|x'^2+\ldots+|a_n|x'^n\\ &<kx'(1+x'+x'^2+\ldots+x'^n)<kx'/(1-x').\end{aligned}$$

If $x'<1$, then $kx'/(1-x')<\epsilon$, provided that

$$kx'<\epsilon(1-x'), \quad \text{that is if} \quad x'<\epsilon/(k+\epsilon).$$

Therefore $|f(x)-a_0|<\epsilon$, if $|x|<\epsilon/(k+\epsilon)$;

and so $\epsilon/(k+\epsilon)$ is a suitable value of η.

(2) *If M is a positive number, no matter how great, we can find a positive number m so that*

$$|f(x)|>M, \quad \textit{provided only that} \quad |x|\geqslant m.$$

The letters having the same meaning as in (1), we have

$$|f(x)| = x'^n \cdot \left| a_n + a_{n-1}\cdot\frac{1}{x} + a_{n-2}\cdot\frac{1}{x^2} + \ldots + a_0\cdot\frac{1}{x^n}\right|$$

$$\geqslant x'^n \cdot \left\{ |a_n| - \left| a_{n-1}\cdot\frac{1}{x} + a_{n-2}\cdot\frac{1}{x^2} + \ldots + a_0\cdot\frac{1}{x^n}\right|\right\}.$$

Choose any number ϵ between 0 and $|a_n|$. Then by the last theorem

$$\left| a_{n-1}\cdot\frac{1}{x} + a_{n-2}\cdot\frac{1}{x^2} + \ldots + a_0\cdot\frac{1}{x^n}\right| < \epsilon$$

if $$\frac{1}{x'} < \frac{\epsilon}{k+\epsilon}, \quad \text{that is if} \quad x' > \frac{k+\epsilon}{\epsilon}.$$

For such values of x we have $|f(x)| > x'^n\{|a_n| - \epsilon\}$.

In particular, put $\epsilon = \frac{1}{2}|a_n|$, then

$$|f(x)| > \tfrac{1}{2}|a_n|\, x'^n \quad \text{if} \quad x' > \frac{2k+|a_n|}{|a_n|}.$$

Hence if m is the greater of the two numbers

$$\sqrt[n]{\frac{2M}{|a_n|}} \quad \text{and} \quad \frac{2k+|a_n|}{|a_n|},$$

then $$f(x) > M \quad \text{if} \quad |x| > m.$$

From Theorem (1) *it follows that* $\lim_{x\to 0} f(x) = a_0$.*

Again, if $x = y + a$, then $y \to 0$ as $x \to a$ and $f(a)$ is the term independent of y in the polynomial $f(y+a)$. Hence

$$\lim_{x\to a} f(x) = \lim_{y\to 0} f(y+a) = f(a).$$

Ex. 1. *If* $f(x) = \dfrac{2x^2+x-3}{3x^2+2x-5}$, *find* $\lim f(x)$; (i) *as* $x \to 0$, (ii) *as* $x \to \infty$, (iii) *as* $x \to 1$.

(i) $$\lim_{x\to 0} f(x) = \frac{\lim(2x^2+x-3)}{\lim(3x^2+2x-5)} = \frac{-3}{-5} = \frac{3}{5}.$$

(ii) Let $x = 1/y$, then $y \to 0$ as $x \to \infty$ and

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0} \frac{3y^2-y-2}{5y^2-2y-3} = \frac{\lim(3y^2-y-2)}{\lim(5y^2-2y-3)} = \frac{2}{3}.$$

(iii) Let $x = 1+y$, then $y \to 0$ as $x \to 1$ and

$$\lim_{x\to 1} f(x) = \lim_{y\to 0} \frac{2y^2+5y}{3y^2+8y} = \lim_{y\to 0} \frac{2y+5}{3y+8} = \frac{5}{8}.$$

* The reader must understand that this is *not* a roundabout way of saying that $f(x) = a_0$ when $x = 0$.

4. Continuous and Discontinuous Functions. We confine ourselves to one-valued functions of the real variable x.

We may say that $f(x)$ is a continuous function of x in the interval (a, b) if the curve whose equation is $y=f(x)$ is continuous between the points where $x=a$ and $x=b$.

This supposes that we know what is meant by a 'continuous curve.' A representation of the curve in Fig. 44 can be drawn without allowing the pencil to leave the paper, and we say that it is *continuous*. In Fig. 45, the curve cannot be so drawn near the points where $x=x_1$ and $x=x_2$, and we say that the curve is *discontinuous* at these points.

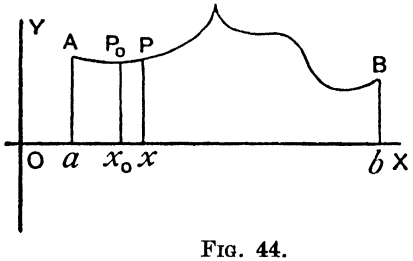


FIG. 44.

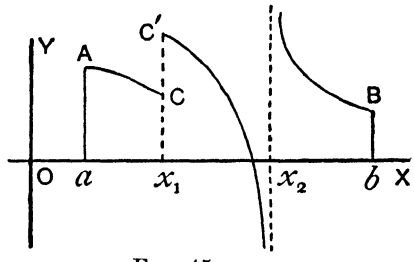


FIG. 45.

We choose the following as the distinctive feature of a continuous curve : *As the point P (x, y) passes along the curve, any small change in the value of x is accompanied by a* small *change (or by no change at all) in the value of y.*

(This is true everywhere in Fig. 44, but is not true near the points where $x=x_1$ and $x=x_2$ in Fig. 45.)

More precisely thus : let $P_0(x_0, y_0)$ be any point on the curve and $P(x, y)$ any other point on it *on either side of* P_0. Choose a positive number ϵ, no matter how small. Then if $|y-y_0|<\epsilon$ for all sufficiently small values of $|x-x_0|$, the curve is said to be *continuous at* P_0.

Further, we say that a curve is *continuous* if it is continuous at every point on it.

Thus the curve AB is continuous in Fig. 44, but in Fig. 45 it is discontinuous at the points where $x=x_1$ and $x=x_2$.

Definitions. (1) *The single-valued function $f(x)$ is continuous at $x=x_0$ (or at the point x_0, or for x_0), if for any positive number ϵ that we may choose, however small, a number η exists such that $|f(x)-f(x_0)|<\epsilon$, provided that $|x-x_0|<\eta$.*

This is equivalent to the following : *$f(x)$ is continuous at x_0 if $f(x)\to f(x_0)$ as $x\to x_0$, where x may tend to x_0 either from the left or from the right.*

It is important that the student should fully realise all that is implied in the definition just given. If $f(x)$ is continuous at x_0, then

(i) $f(x)$ has a *single definite* value when $x = x_0$;

(ii) if h tends to zero through *positive* values, then $f(x_0+h) \to$ a finite number L_1, and $f(x_0-h)$, a finite number L_2;

(iii) and we must have $L_1 = f(x_0) = L_2$.

Ex. 1. A simple case, in which $f(x_0)$, L_1, L_2 all exist, but are not all equal, is given by Goursat.

If $f(x) = x^2 + \frac{x^2}{1+x^2} + \ldots + \frac{x^2}{(1+x^2)^n} + \ldots$ to ∞, then, when $x=0$, every term of $f(x)$ is zero, and so $f(x)=0$; but, *for all values of x except zero*, $f(x) = 1+x^2$.

Thus, $f(0)=0$, and $\lim_{x \to 0} f(x) = 1$, so that $f(x)$ is discontinuous at $x=0$.

(2) *The function $f(x)$ is continuous in any interval if it is continuous at every point of the interval.*

If the interval (a, b) is *closed* $(a<b)$, so that the end points are included, $f(x)$ *is continuous at* a *if* $f(x) \to f(a)$ *as* $x \to a+0$ *and at* b *if* $f(x) \to f(b)$ *as* $x \to b-0$.

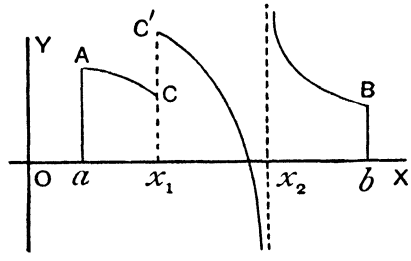


FIG. 45.

Types of Discontinuities. (i) In Fig. 45 let y_1, y_1' be the ordinates of C, C', then $y \to y_1$ as $x \to x_1$ from the left, i.e. as $x \to x_1 - 0$ and $y \to y_2$ as $x \to x_1$ from the right, i.e. as $x \to x_1 + 0$. This is expressed by writing

$$\lim_{x \to x_1 - 0} y = y_1 \quad \text{and} \quad \lim_{x \to x_1 + 0} y = y_2.$$

(ii) At the point where $x = x_2$ in Fig. 45, it is supposed that y is infinite. Thus y is a discontinuous function of x at $x = x_1$, and at $x = x_2$. For example, $1/x$ and $1/x^2$ are discontinuous at $x=0$.

(iii) It may happen that $f(x)$ has real values only on one side of the point $x = x_0$. In such a case the curve $y = f(x)$ stops abruptly and $f(x)$ is discontinuous at x_0.

Taking (as usual) $\sqrt{x}$ to denote the positive square root of the number x, we see that $\sqrt{x_0 - x}$ is discontinuous at x_0.

5. Theorems on Continuous Functions.

(1) It follows from the definition that *if $f(x)$ and $\phi(x)$ are continuous at $x=x_0$, so also are the sum, difference and product of these functions.*

The same is true for the quotient $f(x)/\phi(x)$, except when $\phi(x_0)=0$.

(2) **Function of a Function.** If $y=\phi(u)$ where $u=f(x)$, then y is called a *function of a function* of x, and is denoted by $\phi\{f(x)\}$.

Theorem. *If $f(x)$ tends to a finite limit l as $x\to a$, and $\phi(x)$ is continuous at $x=l$, then* $\lim_{x\to a}\phi\{f(x)\}=\phi\{\lim_{x\to a}f(x)\}$.

For if $\lim f(x)=l$, then $f(x)=l+\eta$, where $\eta\to 0$ as $x\to a$; also $\phi(x)$ is continuous at $x=l$, therefore

$$\lim_{x\to a}\phi\{f(x)\}=\lim_{\eta\to 0}\phi(l+\eta)=\phi(l)=\phi\{\lim_{x\to a}f(x)\}.$$

6. Continuity of Rational Functions. If $f(x)$ is a polynomial, then, by Art. 3, $\lim_{x\to a}f(x)=f(a)$, where a is any real number.

Therefore *the polynomial is continuous for all values of x.*

Hence also *any rational function of x (i.e. a polynomial or the quotient of two polynomials) is continuous for all values of x except for such as make the denominator vanish.*

7. The Function x^n. *If n is rational, x^n is continuous for all values of x for which x^n has definite real values.*

(i) *Suppose that $x>0$.* If x_0 is any positive number, we can choose a, b so that $0<a<x_0<b$.

Let x be any number other than x_0 in the interval (a, b); then by Ch. XIV, 10, $x^n-x_0^n$ lies between $nx^{n-1}(x-x_0)$ and $nx_0^{n-1}(x-x_0)$.

Now both x^{n-1} and x_0^{n-1} lie between a^{n-1} and b^{n-1}, hence if k is the greater of the last two numbers, $|x^n-x_0^n|<k\,.\,|n|\,.\,|x-x_0|$.

If then ϵ is any assigned positive number, however small,

$$|x^n-x_0^n|<\epsilon, \quad \text{provided that} \quad |x-x_0|<\frac{\epsilon}{k\,|n|}.$$

Hence x^n is continuous at $x=x_0$.

(ii) *If $x<0$,* let $x=-y$, so that $x^n=(-1)^n y^n$ where it is supposed that x^n has a real value. Now y^n is continuous for positive values of y, therefore x^n is continuous for negative values of x.

(iii) *At the point $x=0$.* If n is positive, $\lim_{x\to 0}x^n=0$. If n is negative, $x^n\to\infty$ or to $-\infty$, according as $x\to+0$ or to -0. Hence x^n is continuous at $x=0$ for positive, but not for negative values of n.

8. Fundamental Theorems. (1) *If $f(x)$ is continuous at $x=a$ and $f(a)\neq 0$, then $f(x)$ has the same sign as $f(a)$ for all values of x in the neighbourhood of a; that is to say, if $a-\eta<x<a+\eta$ where η is arbitrarily small.*

For since $f(a)\neq 0$, we can choose ϵ so that $f(a)-\epsilon$ and $f(a)+\epsilon$ have the same sign as $f(a)$: and since $f(x)$ is continuous at $x=a$, we can find η so that

$$f(a)-\epsilon<f(x)<f(a)+\epsilon, \quad \text{provided that} \quad |x-a|<\eta.$$

For such values of x, $f(x)$ has the same sign as $f(a)$.

(2) *If $f(x)$ is continuous for the range $a\leqslant x\leqslant b$, then as x passes from the value a to the value b, $f(x)$ assumes at least once every value between $f(a)$ and $f(b)$.*

From the graphical point of view the truth of this statement is obvious.

Let A, B be the points on the curve $y=f(x)$ whose abscissae are a and b, and let k be any number between $f(a)$ and $f(b)$.

We assume that any straight line which passes between the points A, B cuts the curve at least once.

It follows that the line $y=k$ cuts the curve at least once. Hence as x varies from a to b, $f(x)$ takes the value k at least once.

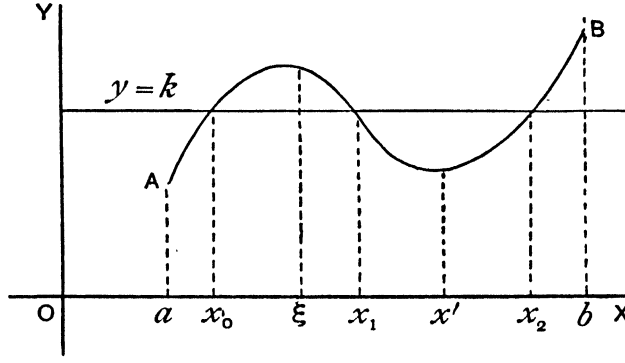


FIG. 46.

Proof. Suppose that $a<b$ and $f(a)<f(b)$. Let k be any number between $f(a)$ and $f(b)$. Because $f(x)$ is continuous at $x=a$ and $f(a)<k$, there are values of x in the interval (a, b) for which $f(x)<k$: and because $f(x)$ is continuous at $x=b$ and $f(b)>k$, there are values of x in (a, b) for which $f(x)>k$.

Divide the real numbers in the interval (a, b) into two classes as follows:

The lower class A is to contain every number x such that $f(\xi)<k$ where ξ is any number in the range $a<\xi\leqslant x$.

The upper class A' is to contain every number x' not included in A.

(Thus in Fig. 46, class A consists of the range $(a<x<x_0)$. The range $(x_1\leqslant x'\leqslant x_2)$ belongs to class A', for although $f(x')<k$, yet $f(\xi)>k$ for some values of ξ in the range $a<\xi\leqslant x'$.)

It has been shown that both classes exist.

Also every number in the interval (a, b) is included, and every x is less than any x'.

Therefore the classification defines a real number x_0, which is either the greatest number in A or the least number in A'.

It remains to prove that $f(x_0)=k$.

If $f(x_0)-k<0$, by reason of the continuity of $f(x)$ at x_0, we can find ϵ so that $f(x)-k<0$ if $x_0-\epsilon\leqslant x\leqslant x_0+\epsilon$. Thus $f(\xi)<k$ when $\xi\leqslant x_0+\epsilon$. Hence $x_0+\epsilon$ belongs to class A, which is impossible, as it is greater than x_0.

If $f(x_0)-k>0$, we can find ϵ so that $f(x)-k>0$ if $x_0-\epsilon\leqslant x\leqslant x_0+\epsilon$. Thus $x_0-\epsilon$ belongs to class A', which is impossible, for it is less than x_0.

Hence it follows that $f(x_0)=k$. It is easy to modify the proof to suit the case when $f(a)>f(b)$.

NOTE. If $f(x)=k$ for more than one value of x in the interval (a, b), then x_0, determined as above, is the least of these values.

9. Derivatives. If $f(x)$ is a function of x such that

$$\{f(x+h)-f(x)\}/h$$

tends to a limit as h tends to zero, this limit is called the *derivative of* $f(x)$, *and is denoted by* $f'(x)$. Thus $f'(x)$ is defined by the equation

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

The function $f'(x)$ is also called the *differential coefficient* of $f(x)$.

It is possible that for some values of x, this limit does not exist. If x_0 is such a value, $f(x)$ *has no derivative at* $x=x_0$. *This is certainly the case when* $f(x)$ *is discontinuous at* $x=x_0$. For in order that $\{f(x_0+h)-f(x_0)\}/h$ may tend to a limit as $h\to 0$, $f(x_0+h)$ must tend to $f(x_0)$.

Ex. 1. *If* $f(x)=x^n$, *where* n *is a positive integer, prove that* $f'(x)=nx^{n-1}$.

We have $$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\left(nx^{n-1}+\frac{n(n-1)}{\underline{|2}}hx^{n-2}+\ldots+h^{n-1}\right)=nx^{n-1}.$$

10. Tangent to a Curve. Let P be any point on a curve and let Q be a point which is supposed to move along the curve, so as to approach P from either side. It is supposed that the curve is continuous near P, so that Q may be as near to P as we like.

If a straight line $T'PT$ exists such that, as Q approaches P from either side, the angle which PQ makes with $T'PT$ becomes and (as Q continues to approach P) remains less than any angle we may choose, however small, then $T'PT$ is called the *tangent* to the curve at P.

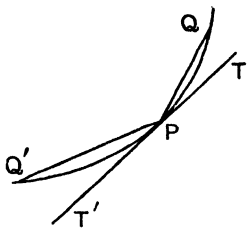


FIG. 47.

This is sometimes expressed by saying that *the tangent to a curve at P is the limiting position of the chord PQ as Q tends to coincidence with P.*

Notice that (i) *Q is not supposed to reach P, for if Q were to coincide with P, PQ would cease to be a chord.*

(ii) The curve in Fig. 48 is supposed to represent the path of a ball which is thrown from A, strikes the ground at B and proceeds to C. Just as the ball hits the ground, it is moving in the direction TB and, just after, in the direction BT'. We may, if we choose, say that the curve has *two* tangents at B, namely BT and BT'. Strictly speaking, according to the definition, the curve has no tangent at B.

This is an instance of *a continuous curve which has no definite tangent at one point.*

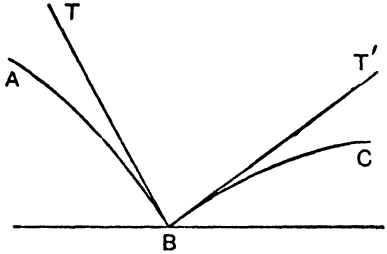


FIG. 48.

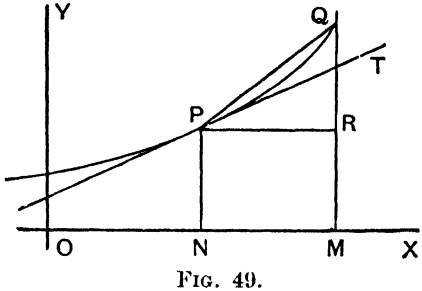


FIG. 49.

11. Gradient of Tangent. Let (x, y) be the coordinates of a point P on the curve whose equation is $y=f(x)$. We shall suppose that the curve has a definite tangent PT at P, and that this tangent is not parallel to the y-axis, as in Fig. 49.

Let Q be the point on the curve whose coordinates are $(x+h, y+k)$.

Draw the ordinates PN, QM, and draw PR parallel to OX to meet QM in R. Then

$$PR = OM - ON = (x+h) - x = h,$$
$$RQ = MQ - NP = (y+k) - y = k,$$

and since P and Q are on the curve, we have

$$y = f(x) \quad \text{and} \quad y+k = f(x+h); \quad \therefore\ k = f(x+h) - f(x);$$
$$\therefore \text{ gradient of chord } PQ = \frac{k}{h} = \frac{f(x+h)-f(x)}{h}.$$

Now, as Q tends to coincidence with P, PQ tends to the limiting position PT and h tends to zero;

$$\therefore \text{ gradient of } PT = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x).$$

Hence $f'(x)$ *is the gradient of the tangent to the curve* $y = f(x)$ *at the point* (x, y).

If PT is nearly parallel to OY, $f'(x)$ will be large, and if P tends to coincide with a point where the tangent is parallel to the y-axis, we may expect that $f'(x)$ will tend to infinity.

Ex. 1. *Find the equation of the tangent to the curve, whose equation is* $y = x^3$, *at the point whose abscissa is* a.

If $f(x) = x^3$, we have $f'(x) = 3x^2$; $\therefore$ the gradient of the tangent at the point (a, a^3) is $3a^2$, and the equation to the tangent is

$$y - a^3 = 3a^2(x-a) \quad \text{or} \quad y = 3a^2x - 2a^3.$$

12. Notation of the Differential Calculus. In the differential calculus the derivative of $f(x)$ is given a different name and a different notation is used.

Any number nearly equal to x is denoted by $x + \delta x$; and, if $y = f(x)$, the value of y corresponding to $x + \delta x$ is denoted by $y + \delta y$; thus we have

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

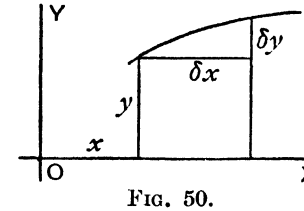


FIG. 50.

This limit is denoted by $\frac{dy}{dx}$, and is called *the differential coefficient of* y *(with regard to* x*)*. In finding $\frac{dy}{dx}$ we are said to *differentiate* y *with regard to* x, and the process is called *differentiation.*

We often write $\frac{df(x)}{dx}$ in the form $\frac{d}{dx}f(x)$, where $\frac{d}{dx}$ denotes the *operation of differentiating* $f(x)$: it is also written shortly as $\frac{df}{dx}$.

13. Rules of Differentiation.

(1) If u, v are functions of x, then

$$\text{(i)}\ \frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}, \qquad \text{(ii)}\ \frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx},$$

$$\text{(iii)}\ \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}, \qquad \text{(iv)}\ \frac{d}{dx}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dx},$$

$$\text{(v)}\ \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{1}{v^2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right).$$

It is assumed that $\frac{du}{dx}$ and $\frac{dv}{dx}$ have definite values, also that in (iv) $u\neq 0$ and in (v) $v\neq 0$.

Proof. Let δu, δv, $\delta(uv)$, etc., be the increments in u, v, uv, etc., corresponding to an increment δx in x.

Because $\frac{du}{dx}$, $\frac{dv}{dx}$ are not infinite, δu and δv tend to zero as $\delta x\to 0$.

(i) and (ii). These are left to the student.

(iii) $\delta(uv)=(u+\delta u)(v+\delta v)-uv=u\,\delta v+(v+\delta v)\delta u$;

$$\therefore\ \frac{d}{dx}(uv)=\lim\left(u\frac{\delta v}{\delta x}\right)+\lim\,(v+\delta v)\frac{\delta u}{\delta x}$$

$$=u\lim\frac{\delta v}{\delta x}+\lim(v+\delta v)\lim\frac{\delta u}{\delta x}$$

$$=u\frac{dv}{dx}+v\frac{du}{dx}.$$

(iv) $\delta\left(\frac{1}{u}\right)=\frac{1}{u+\delta u}-\frac{1}{u}=-\frac{\delta u}{u(u+\delta u)}$;

$$\therefore\ \frac{d}{dx}\left(\frac{1}{u}\right)=-\lim\frac{\frac{\delta u}{\delta x}}{u(u+\delta u)}=-\frac{\lim\frac{du}{dx}}{u\lim(u+\delta u)}=-\frac{1}{u^2}\frac{du}{dx}.$$

(v) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{d}{dx}\left(\frac{1}{v}\cdot u\right)=\frac{1}{v}\frac{du}{dx}+u\frac{d}{dx}\left(\frac{1}{v}\right)$;

$$\therefore\ \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}=\frac{1}{v^2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right).$$

Ex. 1. *Find* $\frac{dy}{dx}$, (i) *if* $y=\frac{2x-1}{3x+1}$; (ii) $y=\frac{x^3+3x}{x-1}$.

We have (i) $\frac{dy}{dx}=\frac{(3x+1)2-(2x-1)3}{(3x+1)^2}=\frac{5}{(3x+1)^2}$.

(ii) $y=x^2+x+4+\frac{4}{x-1}$; $\therefore\ \frac{dy}{dx}=2x+1-\frac{4}{(x-1)^2}$.

Ex. 2. *Show that the rules for the differentiation of a product and a quotient can be written in the forms*

$$\text{(i)}\ \frac{1}{p}\frac{dp}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}\quad \textit{and}\quad \text{(ii)}\ \frac{1}{q}\frac{dq}{dx}=\frac{1}{u}\frac{du}{dx}-\frac{1}{v}\frac{dv}{dx},$$

where $p=uv$ *and* $q=u/v$.

(iii) *Deduce that if* $z=(u_1u_2\ldots u_n)/(v_1v_2\ldots v_m)$, *where* $u_1, u_2, \ldots, v_1, v_2, \ldots$ *are functions of* x; *then*

$$\frac{1}{z}\frac{dz}{dx}=\Sigma_{r=1}^{r=n}\left(\frac{1}{u_r}\cdot\frac{du_r}{dx}\right)-\Sigma_{s=1}^{s=m}\left(\frac{1}{v_s}\frac{dv_s}{dx}\right).$$

(i) If $p=uv$, then $\dfrac{dp}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$; and, if each side is divided by $p(=uv)$, the result follows.

(ii) This follows in a similar manner.

(iii) Let $U=u_1u_2\ldots u_n$, $V=v_1v_2\ldots v_m$; then

$$z=U/V\quad\text{and}\quad\frac{1}{z}\frac{dz}{dx}=\frac{1}{U}\frac{dU}{dx}-\frac{1}{V}\frac{dV}{dx};$$

and by repeated application of the rule for a product,

$$\frac{1}{U}\frac{dU}{dx}=\frac{1}{u_1}\frac{du_1}{dx}+\frac{1}{u_2u_3\ldots u_n}\frac{d(u_2u_3\ldots u_n)}{dx}$$

$$=\frac{1}{u_1}\frac{du_1}{dx}+\frac{1}{u_2}\frac{du_2}{dx}+\frac{1}{u_3u_4\ldots u_n}\frac{d(u_3u_4\ldots u_n)}{dx}=\Sigma_{r=1}^{r=n}\left(\frac{1}{u_r}\cdot\frac{du_r}{dx}\right).$$

Similarly $\dfrac{1}{V}\dfrac{dV}{dx}=\Sigma_{s=1}^{s=m}\left(\dfrac{1}{v_s}\dfrac{dv_s}{dx}\right)$; and the result follows.

Ex. 3. *Prove that, if* y *is a function of* x, *and* $\dfrac{dy}{dx}$ *exists, then for all rational values of* n,

$$\frac{d}{dx}(y^n)=ny^{n-1}\frac{dy}{dx}.$$

Let $w=y^n$; and (i) let n be a positive integer; then $w=y\,.\,y\,.\,y\,.\ldots y$ (n factors);

$$\therefore\ \frac{1}{w}\frac{dw}{dx}=\frac{1}{y}\frac{dy}{dx}+\frac{1}{y}\frac{dy}{dx}+\ldots+\frac{1}{y}\frac{dy}{dx}\ (n\text{ terms})=\frac{n}{y}\frac{dy}{dx}.$$

(ii) Let $n=-m$, where m is a positive integer; then $w=y^{-m}$,

$$\therefore\ wy^m=1\quad\text{and}\quad\frac{1}{w}\frac{dw}{dx}+\frac{1}{y^m}\frac{dy^m}{dx}=0.$$

Hence, by (i),
$$\frac{1}{w}\frac{dw}{dx}+\frac{m}{y}\frac{dy}{dx}=0\quad\text{or}\quad\frac{1}{w}\frac{dw}{dx}=\frac{n}{y}\frac{dy}{dx}.$$

(iii) Let $n=p/q$ where p and q are integers; then $w=y^{\frac{p}{q}}$ and $w^q=y^p=z$, say.

Hence, by (i) or (ii),
$$\frac{q}{w}\frac{dw}{dx}=\frac{1}{z}\frac{dz}{dx}=\frac{p}{y}\frac{dy}{dx},\quad\text{and}\quad\frac{1}{w}\frac{dw}{dx}=\frac{n}{y}\frac{dy}{dx}.$$

That is, in all cases
$$\frac{1}{w}\frac{dw}{dx}=\frac{n}{y}\frac{dy}{dx};\ \text{and multiplying each side by } w(=y^n),$$

we have
$$\frac{dw}{dx}=\frac{d}{dx}(y^n)=ny^{n-1}\cdot\frac{dy}{dx}.$$

NOTE. It should be carefully observed that there is always a factor $\dfrac{dy}{dx}$, unless $y=x$.

14. Function of a Function.

(1) *If* $u=f(x)$ *is a single-valued function of* x, *continuous for* $a\leqslant x\leqslant b$, *and* $y=\phi(u)$ *is a single-valued function of* u, *continuous for the corresponding range of values of* u, *then* y *is a continuous function of* x *for* $a\leqslant x\leqslant b$.

Let x_0 be any value of x in the given range, and u_0, y_0 the corresponding values of u and y.

Because $y=\phi(u)$ is continuous at u_0, for any assigned positive number ϵ, we can find a number η such that

$$|y-y_0|<\epsilon, \quad \text{provided that} \quad |u-u_0|<\eta,$$

and because $f(x)$ is continuous at x_0, we can find η' so that

$$|u-u_0|<\eta, \quad \text{and consequently} \quad |y-y_0|<\epsilon \quad \text{if} \quad |x-x_0|<\eta'.$$

This proves the theorem.

(2) *If the derivatives* $f'(x)$ *and* $\phi'(u)$ *exist for* $a<x<b$ *and for corresponding values of* u, *and if* $F(x)=\phi(u)=\phi\{f(x)\}$, *then*

$$F'(x)=\phi'(u)\,.f'(x), \quad \text{that is,} \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Let δu, δy be the increments u and y corresponding to an increment δx in x, then if $\delta u\neq 0$,

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}. \qquad \text{.....(A)}$$

As $\delta x\to 0$, $\dfrac{\delta u}{\delta x}\to\dfrac{du}{dx}$ and since $\dfrac{du}{dx}$ is finite, $\delta u\to 0$

and therefore $$\frac{\delta y}{\delta u}\to\frac{dy}{du}.$$

Consequently

$$\frac{dy}{dx}=\lim\frac{\delta y}{\delta u}\cdot\lim\frac{\delta u}{\delta x}=\frac{dy}{du}\cdot\frac{du}{dx}. \qquad \text{.....(B)}$$

It may happen that $\delta u=0$ for some value of δx, and then equation (A) does not hold.

In this case we must have $\delta y=0$, for otherwise $\dfrac{dy}{du}$ would not be finite: hence $\dfrac{dy}{dx}=0$. Also $\dfrac{du}{dx}=0$, for $\delta u=0$. Thus the equation

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

is true also in this case.

Ex. 1. *Find* $\dfrac{dy}{dx}$, (i) when $y=(ax+b)^n$, (ii) when $y=\dfrac{1}{a+bx^2}$.

(i) Let $u=ax+b$, then $y=u^n$, and $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=nu^{n-1}\,.\,a=na(ax+b)^{n-1}$.

(ii) Let $u=a+bx^2$, then $y=\dfrac{1}{u}$, and $\dfrac{dy}{dx}=\left(-\dfrac{1}{u^2}\right).2bx=-\dfrac{2bx}{(a+bx^2)^2}$.

Ex. 2. *If* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *then* $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$.

We regard y as a function of x, and differentiate both sides of the given equation with regard to x, thus $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$.

15. Derivative of x^n. *If* $y=x^n$ *where* n *is any rational, then*

$$\frac{dy}{dx}=nx^{n-1}.$$

Proof. If x is positive, $x+h$ is positive for all sufficiently small values of h, and by Ch. XIV, 10, $(x+h)^n-x^n$ lies between $nh(x+h)^{n-1}$ and nhx^{n-1}. Also x^{n-1} is a continuous function of x. Therefore

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=nx^{n-1}.$$

If x is negative, let $x=-u$, then $y=(-1)^nu^n$ and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=(-1)^n\,.\,nu^{n-1}\,.\,(-1)=nx^{n-1}.$$

*Or,** having proved that the derivative of x^n is nx^{n-1} when n is a positive integer, we can prove that this is also true when $n=p/q$ as follows.

Let $y=x^{\frac{p}{q}}$, therefore $y^q=x^p$. Now $\frac{d}{dx}y^q=qy^{q-1}\frac{dy}{dx}$,

therefore $qy^{q-1}\frac{dy}{dx}=px^{p-1}$ and $\frac{dy}{dx}=\frac{p}{q}\frac{x^{p-1}}{y^{q-1}}=\frac{p}{q}\frac{x^{p-1}}{x^{\frac{p}{q}(q-1)}}=\frac{p}{q}x^{\frac{p}{q}-1}$.

16. Meaning of the Sign of $f'(x)$. *If* $f'(x_0)>0$, *then for all values of* x *in the neighbourhood of* x_0, $f(x)\gtrless f(x_0)$ *according as* $x\gtrless x_0$.

For $\{f(x_0+h)-f(x_0)\}/h$ tends to a positive limit as $h\to 0$. Hence for sufficiently small values of h, $f(x_0+h)-f(x_0)$ has the same sign as h.

Similarly *if* $f'(x_0)<0$, *for every* x *in the neighbourhood of* x_0,

$$f(x)\lesseqgtr f(x_0) \quad as \quad x\gtrless x_0.$$

NOTE. If $f'(x)>0$ only for the *single* value x_0 of x, it does *not* follow that $f(x)$ increases steadily as x increases through x_0.

For if $x_1<x_2<x_0$ where x_1, x_2 are in the neighbourhood of x_0, it has only been proved that $f(x_1)<f(x_0)$ and $f(x_2)<f(x_0)$, and we cannot conclude that $f(x_1)<f(x_2)$.

17. Complex Functions of a Real Variable x. If $y=f(x)+\iota\phi(x)$, then y is called a *complex function* of x and its derivative is $f'(x)+\iota\phi'(x)$. The rules of Art. 13 obviously hold for such functions.

* See also Art. 13, Ex. 3.

18. Higher Derivatives. If the derivative of $f'(x)$ exists, it is called the second derivative or the second differential coefficient of $f(x)$.

If $y=f(x)$, this is written in any of the forms

$$f''(x), \quad \frac{d^2y}{dx^2}, \quad \left(\frac{d}{dx}\right)^2 f(x).$$

So by n differentiations (when possible) we obtain the nth derivative or the nth differential coefficient of $f(x)$, which is written in any of the forms

$$f^{(n)}(x), \quad \frac{d^ny}{dx^n}, \quad \left(\frac{d}{dx}\right)^n f(x).$$

The following are important instances.

(i) If $$y=x^n, \quad \frac{dy}{dx}=nx^{n-1}, \quad \frac{d^2y}{dx^2}=n(n-1)x^{n-2},$$

$$\frac{d^ry}{dx^r}=n(n-1)(n-2)\ldots(n-r+1)x^{n-r}.$$

(ii) If $$f(x)=a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{\lfloor 2}a_2x^{n-2}+\ldots+a_n,$$

then $$f'(x)=n\left\{a_0x^{n-1}+(n-1)a_1x^{n-2}+\frac{(n-1)(n-2)}{\lfloor 2}a_2x^{n-3}+\ldots+a_{n-1}\right\}.$$

Or, using the notation of Ch. III, 1, if $f(x)=(a_0, a_1, a_2, \ldots a_n\rangle x, 1)^n$, then $$f'(x)=n(a_0, a_1, a_2, \ldots a_{n-1}\rangle x, 1)^{n-1},$$

and by successive differentiation

$$f''(x)=n(n-1)(a_0, a_1, \ldots a_{n-2}\rangle x, 1)^{n-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f^{(n-1)}(x)=n(n-1)\ldots 3\,.\,2\,.\,(a_0x+a_1),$$

$$f^{(n)}(x)=\lfloor n\, a_0.$$

19. Maxima and Minima. We say that $f(x)$ has a *maximum value* at $x=x_0$ when $f(x_0)>f(x)$ for all values of x in the neighbourhood of x_0. In other words, if a number ϵ exists such that $f(x_0)>f(x)$ when

$$x_0-\epsilon<x<x_0+\epsilon,$$

then $f(x_0)$ is a maximum value of $f(x)$. If in the preceding, the sign $>$ is replaced by $<$, then $f(x_0)$ is a *minimum* value of (fx).

Maxima and minima values of a function are sometimes called *stationary* or *turning* values.

A *necessary* condition that $f(x_0)$ may be a maximum or a minimum value of $f(x)$ is that $f'(x_0)=0$.

This follows from Art. 16; but the condition is not *sufficient*.

If $f(x_0)$ is a maximum value, then for sufficiently small values of h, $f(x_0+h)<f(x_0)$. Hence by Art. 16, as x increases through x_0, the sign of $f'(x)$ changes from + to −.

Suppose now that $f''(x_0)$ exists and is not zero. Then since $f'(x)$ decreases as x increases through x_0, $f''(x_0)$ *must be negative.*

Similarly if $f(x_0)$ is a minimum value, as x increases through x_0, the sign of $f'(x)$ must change from − to +, and if $f''(x_0)$ is not zero, it must be *positive.*

If the sign of $f'(x)$ does not change as x passes through x_0, $f(x)$ is neither a maximum nor a minimum. The case in which $f''(x_0)=0$ will be considered later. In this case $f(x_0)$ may be a maximum, a minimum or neither of these.

To illustrate this geometrically, in Fig. 51, the tangents at A, B, C to the curve $y=f(x)$ are parallel to OX.

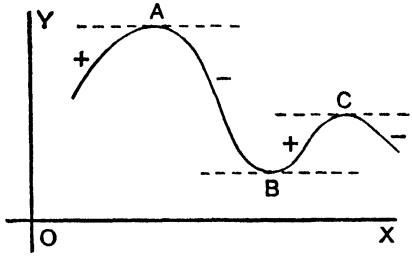


FIG. 51.

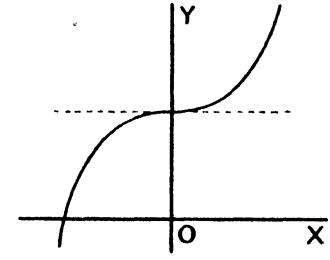


FIG. 52.

The sign of $f'(x)$ for different parts of the curve is as indicated, and $f(x)$ has maximum values at A, C and a minimum value at B.

Fig. 52 represents the graph of $y=x^3+1$. Here $\frac{dy}{dx}=3x^2$, so that $\frac{dy}{dx}=0$ when $x=0$, but as x passes through O, the sign of $\frac{dy}{dx}$ does not change. Thus y is neither a maximum nor a minimum when $x=0$.

Ex. 1. *Search for maxima and minima values of*

$$f(x)=2x^3-3x^2-12x+6.$$

Here $$f'(x)=6x^2-6x-12=6(x+1)(x-2).$$

Thus $$f'(x)=0 \text{ if } x=-1 \text{ or } 2.$$

As x increases through -1, the sign of $f'(x)$ changes from + to −; and as x increases through 2, the sign of $f'(x)$ changes from − to +.

Thus $f(-1)$ is a maximum and $f(2)$ a minimum value.

Or thus, $$f''(x)=12x-6=6(2x-1),$$

so that $f''(-1)<0$ and $f''(2)>0$, leading to the same results as before.

20. Points of Inflexion. In Fig. 53 APB is supposed to be part of a curve represented by $y=f(x)$.

If a point moves along the curve from A to B, the gradient of the tangent increases along the arc AP and then decreases.

Thus $\frac{dy}{dx}$ has a maximum value at P, and therefore $\frac{d^2y}{dx^2}=0$ at P.

If Q is a point on the curve near P, the secant QP cuts the curve at another point Q' near P, and we may regard the tangent TPT' as the limiting position of the line QPQ', when the three points Q, P, and Q' are very close together.

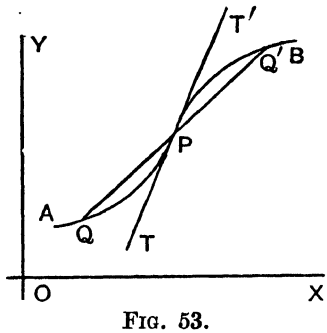


FIG. 53.

Definition. A point (x, y) on a curve at which $\frac{dy}{dx}$ is a maximum or a minimum is called a *point of inflexion.*

At such a point, $\frac{d^2y}{dx^2}=0$, and the curve crosses the tangent.

EXERCISE XXVIII

Find the limits of the functions in Exx. 1-3 (i) as x tends to zero; (ii) as x tends to ∞.

1. $\frac{1+2x+3x^2}{2+3x+4x^2}$.

2. $\frac{2x^2-3x^3+4x^4}{3x+4x^2-5x^3}$.

3. $\frac{(1+x)^2-(1-x)^2}{(1+x)^3-(1-x)^3}$.

Find the values of

4. $\lim_{x\to 1}\frac{3x^2-7x+4}{2x^2-5x+3}$.

5. $\lim_{x\to 2}\frac{x^3-x^2-x-2}{x^3-3x^2+3x-2}$.

6. $\lim_{x\to a}\frac{x^3-a^3}{x^4-2x^2a^2+a^4}$.

7. $\lim_{h\to 0}\frac{(x+h)^{\frac{3}{2}}-x^{\frac{3}{2}}}{h}$.

8. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

9. Prove that (i) $\lim_{x\to 0}\cos x=1$; (ii) $\lim_{x\to 0}\frac{\sin x}{x}=1$.

10. Prove that $\sin x$ and $\cos x$ are continuous functions of x.
[$\sin(x+h)-\sin x=2\sin\frac{1}{2}h\cos(x+\frac{1}{2}h)$.]

11. For what values of x is $\tan x$ discontinuous?

12. Prove that $\frac{d}{dx}\sin x=\cos x$, $\frac{d}{dx}\cos x=-\sin x$, $\frac{d}{dx}\tan x=\sec^2 x$.
[$\frac{\sin(x+h)-\sin x}{h}=\frac{2\sin\frac{1}{2}h}{h}\cdot\cos(x+\frac{1}{2}h)\to\cos x$ as $h\to 0$.]

13. If $f(x)=(x-a)^m(x-b)^n$, prove that

$$f'(x)=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-mb-na].$$

Hence, if m, n are positive, $f'(x)$ vanishes for a value of x between a and b.

14. Prove the following:

(i) $\left(\frac{d}{dx}\right)^r(ax+b)^n=n(n-1)\dots(n-r+1)a^r(ax+b)^{n-r}$.

(ii) $\left(\frac{d}{dx}\right)^r\frac{1}{(x-a)^n}=(-1)^r\frac{n(n+1)\dots(n+r-1)}{(x-a)^{n+r}}$.

(iii) $\left(\frac{d}{dx}\right)^n\frac{1}{1-x^2}=\frac{1}{2}\lfloor n\left\{\frac{1}{(1-x)^{n+1}}+(-1)^n\cdot\frac{1}{(1+x)^{n+1}}\right\}$.

15. Find the derivatives of

(i) $\frac{ax^2+2bx+c}{a'x^2+2b'x+c'}$. (ii) $\sqrt{\left(\frac{1-x}{1+x}\right)}$. (iii) $\frac{x}{\sqrt{(a^2+x^2)^3}}$.

(iv) $\sin^n x$. (v) $\cos^n x$. (vi) $\tan^n x$.

16. Find $\frac{dy}{dx}$ if (i) $ax^2+2hxy+by^2+2gx+2fy+c=0$, (ii) if $x^3+y^3=3axy$.

17. Prove that $\frac{d^2x}{dy^2}=-\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^3$.

18. If $y=A\cos\mu x+B\sin\mu x$, then $\frac{d^2y}{dx^2}=-\mu^2y$.

19. If $y=\cos x+\iota\sin x$, then $\frac{dy}{dx}=\iota y$.

20. Prove that if $\theta=\tan^{-1}a/x$, then

$$\left(\frac{d}{dx}\right)^n\frac{a}{x^2+a^2}=(-1)^n\lfloor n\cdot\frac{\sin(n+1)\theta}{(x^2+a^2)^{-(n+1)/2}},$$

$$\left(\frac{d}{dx}\right)^n\frac{x}{x^2+a^2}=(-1)^n\lfloor n\cdot\frac{\cos(n+1)\theta}{(x^2+a^2)^{-(n+1)/2}}.$$

[Consider the values of $1/(x-\iota a)\pm1/(x+\iota a)$.]

21. Find the equation to the tangent at the origin to the curve whose equation is $y=ax+bx^2+cx^3$.

22. If P, Q, Q' are points on the parabola $y=px^2+qx+r$ whose abscissae are a, $a-h$, $a+h$ respectively, prove that the tangent at P is parallel to the chord QQ'.

23. Prove that the function $2x^3-3x^2+6x-5$ always increases with x.

24. Sketch roughly the graph of $y=x(x-1)^2$, and find the greatest value of $x(x-1)^2$ for values of x between 0 and 1.

25. (i) If $f(x)=(x-1)^3(x-2)$, for what values of x does $f'(x)$ vanish?

(ii) Sketch roughly the graph of $y=f(x)$ from $x=0$ to $x=2$, and show that one of the values of x found in (i) gives a minimum value of $f(x)$ and that the other gives neither a maximum nor a minimum.

26. Prove that $x=-b/2a$ gives a maximum or a minimum value of ax^2+bx+c according as $a\lesseqgtr0$.

27. (i) When does the function $3x^4-4x^3-36x^2-1$ increase with x?

(ii) Find the turning values of the function, and state the character of each.

28. Show that the turning values of $(x-1)(x-2)(x-3)$ are approximately $\pm 0{\cdot}38$.

29. Four equal squares are cut off from the corners of a rectangular sheet of tin, 8 inches long and 5 inches wide. The rest of the sheet is bent so as to form an open box on a rectangular base. Find the volume of the box of greatest capacity which can be formed in this way.

30. The regulations for Parcel Post require that the sum of the length and girth of a parcel must not exceed 6 feet. Prove that the right circular cylinder of greatest volume which can be sent is 2 feet long and 4 feet in girth.

31. Show that the height of the right circular cylinder of greatest volume which can be cut out from a spherical ball of radius r is $2r/\sqrt{3}$.

21. Inverse Functions. Let $f(x)$ be a single-valued function of x, continuous for the range $a \leqslant x \leqslant b$. Further, suppose that throughout the range, either $f(x)$ increases with x, or decreases as x increases. We shall prove that

(1) *The equation $y=f(x)$ determines x as a single-valued continuous function of y.*

This function is denoted by $f^{-1}(y)$, so that the statements $y=f(x)$ and $x=f^{-1}(y)$ mean the same thing. Also the function f^{-1} is called the *inverse* of f.

Proof. Since $f(x)$ is continuous, as x varies continuously from a to b, $f(x)$ takes every value from $f(a)$ to $f(b)$ at least once. Also $f(x)$ takes every value once only, for $f(x_1)=f(x_2)$ only when $x_1=x_2$. Thus, corresponding to any value y_0 of y between $f(a)$ and $f(b)$, there is a single value x_0 of x such that $y_0=f(x_0)$.

Again, $f^{-1}(y)$ is continuous. For, taking the case in which y increases with x, let y_0 be any value of y between $f(a)$ and $f(b)$ and x_0 the corresponding value of x, so that

$$a<x_0<b.$$

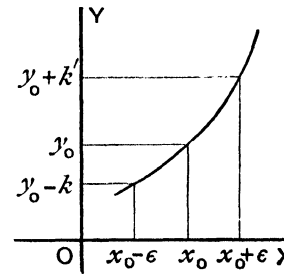


FIG. 54.

For sufficiently small values of ϵ, $x_0-\epsilon$ and $x_0+\epsilon$ are within the interval (a, b). Let y_0-k and y_0+k' be the corresponding values of y, and let η be the smaller of the two k, k'.

As y varies from $y_0-\eta$ to $y_0+\eta$, x lies between $x_0-\epsilon$ and $x_0+\epsilon$, for x increases with y.

Hence $\quad |x-x_0|<\epsilon, \quad$ provided that $\quad |y-y_0|<\eta$;

and therefore $x=f^{-1}(y)$ is continuous at y_0.

If y decreases as x increases, we take y_0+k and y_0-k' as the values of y corresponding to $x_0-\epsilon$ and $x_0+\epsilon$. To show that $f^{-1}(y)$ is continuous at $y=f(a)$ and $y=f(b)$, we consider the variation of y for the ranges

$$a<x<a+\epsilon \quad \text{and} \quad b-\epsilon<x<b.$$

(2) *If* $\dfrac{dy}{dx}$ *exists and is never zero for the range* $a<x<b$, *then* $\dfrac{dx}{dy}=1\Big/\dfrac{dy}{dx}$.

For if δx and δy are corresponding increments of x and y, then, since $\dfrac{dy}{dx}$ is finite, $\delta y\to 0$ as $\delta x\to 0$, and *vice versa*. Therefore

$$\frac{dx}{dy}=\lim\frac{\delta x}{\delta y}=\lim 1\Big/\frac{\delta y}{\delta x}=1\Big/\lim\frac{\delta y}{\delta x}=1\Big/\frac{dy}{dx}.$$

22. The Inverse Circular Functions. The numerically smallest value of y which has the same sign as x and is such that $\sin y=x$ is called the *principal value of* $\sin^{-1}x$. A similar definition applies to $\tan^{-1}x$.

The smallest positive value of y such that $\cos y=x$ is called the *principal value of* $\cos^{-1}x$.

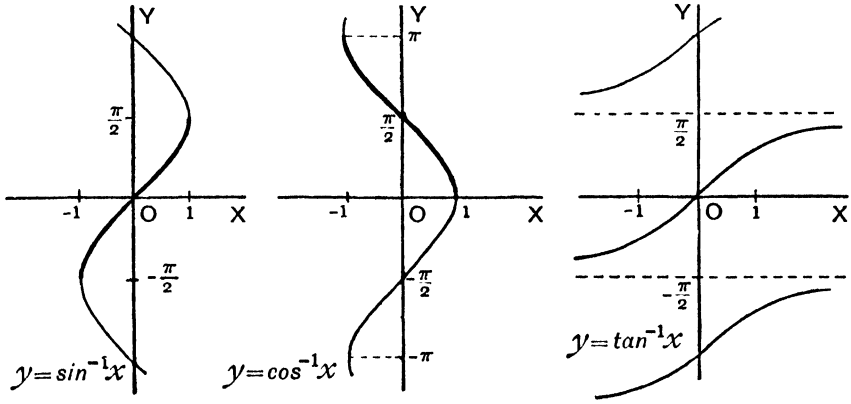


FIG. 55.

Thus the principal values of $\sin^{-1}x$ and $\tan^{-1}x$ lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, and that of $\cos^{-1}x$ lies between 0 and π.

Unless otherwise stated, $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ will be used to represent the principal values of the functions; for these, the derivatives of $\sin^{-1}x$ and $\cos^{-1}x$ are positive and negative respectively, as can be seen from Fig. 55.

Ex. 1. *Show that* $\dfrac{d}{dx}\tan^{-1}x=\dfrac{1}{1+x^2}$.

Let $y=\tan^{-1}x$, then $x=\tan y$ and $\dfrac{dx}{dy}=\sec^2 y=1+x^2$; $\therefore\ \dfrac{dy}{dx}=\dfrac{1}{1+x^2}$.

23. Bounds of a Function. Let $f(x)$ be a function which has a definite value for every value of x in the interval (a, b).

If a number M exists such that $f(x) \leqslant M$ for every x in (a, b), then $f(x)$ is said to be *bounded above.*

If a number N exists such that $f(x) \geqslant N$ for every x in (a, b), $f(x)$ is said to be *bounded below.*

A function which is bounded above and below is said to be *bounded.*

Suppose that $f(x)$ is bounded above in (a, b).

Let S denote the aggregate of the values of $f(x)$ as x varies continuously from a to b.

By applying Dedekind's theorem (Ch. XIII, 12) to the set S, we can show that *a number h exists such that*

(i) *no value of $f(x)$ exceeds h;*

(ii) *at least one value of $f(x)$ exceeds any number less than h.*

This number h is called the *upper bound* of $f(x)$.

Similarly, *if $f(x)$ is bounded below, a number l exists such that*

(i) *no value of $f(x)$ is less than l;*

(ii) *at least one value of $f(x)$ is less than any number greater than l.*

This number l is called the *lower bound* of $f(x)$.

Ex. 1. Consider the function $f(x) = \lim_{n \to \infty} \frac{nx}{1+nx^2}$. If $x=0$, $f(x)=0$. If $x \neq 0$. $f(x) = 1/x$. Thus $f(x)$ has a definite value for every value of x. But $f(x)$ is not bounded in any interval including zero, for by making x small enough, we can make $1/x$ exceed any number we may choose.

Theorem 1. *If $f(x)$ is continuous in the closed interval (a, b), it is bounded in (a, b).*

Suppose that $f(x)$ is not bounded in (a, b). Because $f(x)$ is continuous at $x=a$, for sufficiently small values of $x-a$, $f(x)$ lies between $f(a)-\epsilon$ and $f(a)+\epsilon$, where ϵ is any small positive number. Hence there are values of x in (a, b) such that $f(x)$ is bounded in (a, x) and (if our supposition is possible) not bounded in (x, b).

Divide the real numbers in (a, b) into a lower class A and an upper class A' as follows: The number x is to be placed in A or in A' according as $f(x)$ is or is not bounded in (a, x).

According to the supposition, both classes exist and every number in A is less than every number in A'. The classification therefore defines a real number α, which is the greatest number in A or the least in A'.

Now $f(x)$ is continuous at $x=\alpha$, therefore for sufficiently small values of $x-\alpha$, $f(x)$ lies between $f(\alpha-\epsilon)$ and $f(\alpha+\epsilon)$, where ϵ is any small

positive number. Hence $f(x)$ is bounded in $(a, \alpha+\epsilon)$, and so $\alpha+\epsilon$ belongs to A, which is impossible, for $\alpha+\epsilon>\alpha$. This proves the theorem.

Theorem 2.* *If $f(x)$ is continuous in (a, b) and h, l are its upper and lower bounds, $f(x)$ takes the values h and l at least once as x varies continuously from a to b.*

For if ϵ is an assigned positive number, however small, there is at least one value of x in (a, b) for which $f(x)>h-\epsilon$, so that

$$h-f(x)<\epsilon \quad \text{and} \quad 1/\{h-f(x)\}>1/\epsilon.$$

Hence $1/\{h-f(x)\}$ is not bounded. But if $h-f(x)$ does not vanish for some value of x in the interval, $1/\{h-f(x)\}$ is continuous in (a, b).

Therefore $h=f(x)$ for some value of x in (a, b).

Similarly there is a value of x for which $f(x)=l$.

24. Rolle's Theorem.

Suppose that $f(x)$ is continuous in the closed interval (a, b) and has a derivative $f'(x)$ for every x such that $a<x<b$.

*If $f(a)=0$ and $f(b)=0$, then $f'(x)=0$ for at least one value of x between a and b.***

This theorem is of the greatest importance. A strict proof (as given on the next page) depends on the rather difficult considerations of the last article, but from the graphical point of view its truth is obvious.

Let the curve $y=f(x)$ cut the x-axis at A, B. Suppose that it is continuous from A to B, and that at every point it has a definite and finite gradient. The theorem asserts that *there is at least one point on the curve between A and B where the tangent is parallel to the x-axis.*

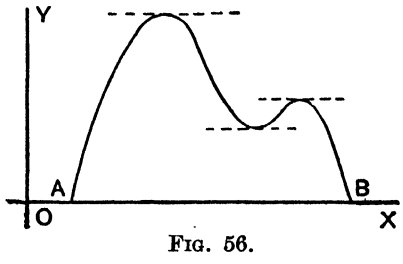


FIG. 56.

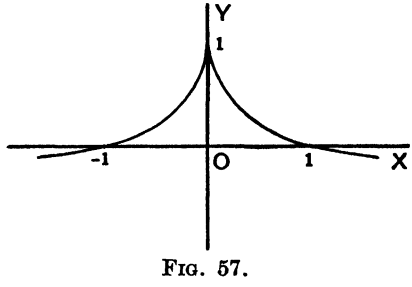


FIG. 57.

Fig. 57 represents the curve $y=1-x^{\frac{2}{3}}$. Here $f'(x)=-\infty$ at $x=0$ and $f'(x)$ is not zero for any value of x between -1 and $+1$.

* This proof is taken from Hardy's *Pure Mathematics*.

** If $f'(x)$ exists for $a<x<b$, $f(x)$ must be continuous in the open interval (a, b), but not necessarily in the *closed* interval.

Proof. Because $f(x)$ is continuous in (a, b), it is bounded in (a, b) and attains at least once its upper bound h and its lower bound l.

Either (i) $f(x)=0$ for all values of x in (a, b), in which case $f'(x)=0$ for every x; or (ii) some values of $f(x)$ are positive or negative.

If positive values of $f(x)$ exist, h is positive and unequal to $f(a)$ or $f(b)$. Moreover, $f'(h)=0$, for otherwise there would be values of x in the neighbourhood of h for which $f(x)>h$, which is impossible.

If there are negative values of $f(x)$, then l is negative, unequal to $f(a)$ or $f(b)$ and $f'(l)=0$. For if $f'(l)\neq 0$, there would be values of x near l for which $f(x)<l$, which is impossible. This proves the theorem.

A simpler proof, for the case in which $f(x)$ is a polynomial, is given in the next chapter.

25. The Mean-Value Theorem. *If $f(x)$ is continuous for $a\leqslant x\leqslant b$ and $f'(x)$ exists for $a<x<b$, then*

$$f(b)-f(a)=(b-a)f'(\xi), \quad \text{.....(A)}$$

where ξ is some number between a and b.

Proof. Let $$\phi(x)=f(b)-f(x)-\frac{b-x}{b-a}\{f(b)-f(a)\},$$

then $$\phi'(x)=-f'(x)+\frac{1}{b-a}\{f(b)-f(a)\}.$$

Also $\phi(a)=0$ and $\phi(b)=0$, hence by Rolle's theorem $\phi'(x)=0$ for some value ξ of x between a and b, which proves the result in question.

Geometrically. Let A, B be the points on the curve $y=f(x)$ where $x=a, b$ respectively. The theorem asserts that there is a point C on the curve between A and B where the tangent is parallel to AB. If $P(x, y)$ is any point on the curve, and the ordinate NP (or NP produced) meets AB in Q, it is easily shown that

$$\phi(x)=NQ-NP=PQ.$$

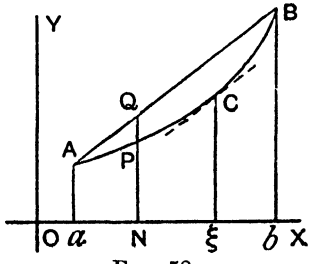


FIG. 58.

Corollary. *If, throughout any given interval,*

(i) *$f'(x)$ is always positive, then $f(x)$ increases with x;*

(ii) *if $f'(x)$ is always negative, $f(x)$ decreases as x increases;*

(iii) *if $f'(x)$ is always zero, then $f(x)$ is constant throughout the interval.*

All this follows from equation (A), where it is supposed that a, b are any two values of x in the interval.

NOTE. It has not been assumed that $f'(x)$ is continuous.

26. Integration. Suppose that $f(x)$ is a given function of x, and that we try to find a function $\phi(x)$ such that $\phi'(x)=f(x)$. Such a function is called an *integral of* $f(x)$ *(with regard to* x*)*, and is denoted by $\int f(x)\,dx$.

If $\phi(x)$ *and* $\psi(x)$ *are any two functions such that* $\phi'(x)=\psi'(x)$, *then* $\phi(x)$ *and* $\psi(x)$ *differ by a constant.*

For $$\frac{d}{dx}\{\phi(x)-\psi(x)\}=\phi'(x)-\psi'(x)=0\,;$$

therefore, by Art. 25, Cor. (iii), $\phi(x)-\psi(x)=$ a constant C.

Hence if *any* value of $\phi(x)$ is denoted by $\int f(x)\,dx$, the *general* value is $\int f(x)\,dx+C$. Thus $\frac{d}{dx}\phi(x)=f(x)$, and $\phi(x)=\int f(x)\,dx+C$, are two equations which mean the same thing.

Ex. 1. Since $\frac{d}{dx}x^{n+1}=(n+1)x^n$, therefore $\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$.

NOTE. The symbol $\int \ldots dx$ is to be regarded as denoting the *operation* of integration, just as $\frac{d}{dx}$ denotes that of differentiation. These are *inverse* operations.

'Integration' can be applied to find the *area of a curve* as in the next example.

Ex. 2. *If* PN *is the ordinate of any point* $P(x, y)$ *on the curve represented by* $y=kx^n$ *where* $n>0$, *prove that the area* ONP *bounded by* ON, NP *and the arc* OP *is equal to* $1/(n+1)$ *of the rectangle contained by* ON, NP.

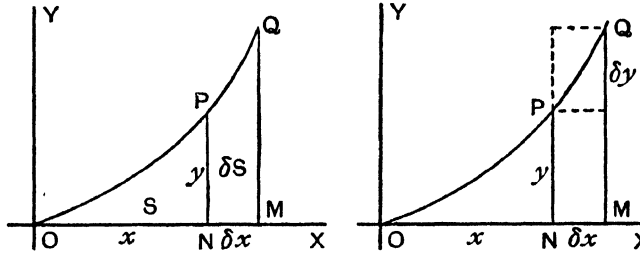


FIG. 59.

Denote the area by S and let x, y be the coordinates of P. Let δy, δS be the increments in y, S corresponding to the increment δx in x. If Q is the point $(x+\delta x, y+\delta y)$ and QM its ordinate, then rect. $PM<\delta S<$ rect. QN, i.e. $y\delta x<\delta S<(y+\delta y)\,\delta x$;

therefore $$y<\frac{\delta S}{\delta x}<y+\delta y\,;\quad \text{and since } \delta y\to 0,\quad \therefore\ y=\frac{dS}{dx}.$$

Hence $$S=\int y\,dx=\int kx^n\,dx=\frac{kx^{n+1}}{n+1}+C,\ \text{where } C \text{ is a constant.}$$

Now $S=0$ when $x=0$; $\therefore\ C=0$ and $S=kx^{n+1}/(n+1)=xy/(n+1)$.

27. Taylor's Theorem, also known as the **General Mean Value Theorem.**

If $f(x)$ is continuous in the closed interval (a, b) and the first n derivatives $f'(x)$, $f''(x) \ldots f^{(n)}(x)$ exist for all values of x between a and b, then

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{\lfloor 2}(b-a)^2 f''(a) + \ldots + \frac{1}{\lfloor n-1}(b-a)^{n-1}f^{(n-1)}(a) + \frac{1}{\lfloor n}(b-a)^n f^{(n)}(\xi), \quad \ldots\ldots(\mathbf{A})$$

where ξ is some number between a and b.

The proof is similar to that of Art. 25. Let

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{1}{\lfloor 2}(b-x)^2 f''(x) - \ldots - \frac{1}{\lfloor n-1}(b-x)^{n-1}f^{(n-1)}(x),$$

and

$$\phi(x) = F(x) - \left(\frac{b-x}{b-a}\right)^n F(a),$$

then

$$F'(x) = -\frac{1}{\lfloor n-1}(b-x)^{n-1}f^{(n)}(x),$$

and

$$\phi'(x) = -\frac{1}{\lfloor n-1}(b-x)^{n-1}f^{(n)}(x) + n\frac{(b-x)^{n-1}}{(b-a)^n}F(a)$$

$$= \frac{n(b-x)^{n-1}}{(b-a)^n}\left\{F(a) - \frac{1}{\lfloor n}(b-a)^n f^{(n)}(x)\right\}.$$

Now $\phi(a) = 0$ and $\phi(b) = 0$, therefore by Rolle's theorem $\phi'(x) = 0$ for some value ξ of x between a and b, hence

$$F(a) = \frac{1}{\lfloor n}(b-a)^n f^{(n)}(\xi),$$

which is equivalent to equation (**A**).

Let $b = a + h$, then equation (**A**) *may be written*

$$f(a+h) = f(a) + hf'(a) + \frac{1}{\lfloor 2}h^2 f''(a) + \ldots + \frac{1}{\lfloor n-1}h^{n-1}f^{(n-1)}(a) + R_n, \quad \ldots(\mathbf{B})$$

where

$$R_n = \frac{1}{\lfloor n}h^n f^{(n)}(a+\theta h) \quad \textit{and} \quad 0 < \theta < 1.$$

For any number between a and $a+h$ may be written in the form $a+\theta h$ where $0<\theta<1$. In general, *the value of θ depends on n.*

Taylor's Series. *Suppose that $f(x)$ is continuous in the closed interval $(a-c, a+c)$, and the n derivatives $f'(x)$, $f''(x), \ldots f^{(n)}(x)$ exist when $a-c<x<a+c$ and that $|h|<c$.* Then equation (**B**) holds.

Further suppose that $R_n = \frac{1}{\lfloor n} h^n f^{(n)}(a+\theta h) \to 0$ *as* $n \to \infty$, *then*

$$f(a+h) = f(a) + hf'(a) + \frac{1}{\lfloor 2} h^2 f''(a) + \ldots + \frac{1}{\lfloor n} h^n f^{(n)}(a) + \ldots \text{ to } \infty . \quad \ldots(\mathbf{C})$$

For if s_n is the sum to n terms of the series,

$$f(a+h) - s_n = R_n \to 0 \quad \text{so that} \quad s_n \to f(a+h).$$

The series in (**C**) is called *Taylor's Series* and R_n is *Lagrange's* form of the remainder after n terms. When $a=0$, equation (**A**) becomes

$$f(h) = f(0) + hf'(0) + \frac{1}{\lfloor 2} h^2 f''(0) + \ldots \text{ to } \infty . \quad \ldots\ldots\ldots\ldots\ldots(\mathbf{D})$$

This expansion is known as *Maclaurin's Series.*

The Complex Variable

28. The Quantity $x + \iota y$, which we denote by z, is called the complex variable. Here x and y are real variables, independent of one another and each capable of assuming any values from $-\infty$ to $+\infty$.

As z passes from a value z_0 to a value z_1, we suppose the variation to be *continuous.* By this we mean that the point z is to move along a continuous curve from the point z_0 to the point z_1.

Thus the variation of z is continuous if the variations of x and y are continuous.

29. Function of a Complex Variable z. If $Z = X + \iota Y$, where X and Y are functions of x and y, we say that Z is a *complex function of the real variables x and y.*

If z is known, so also are x and y. Consequently X, Y and Z are known. Hence it would be perfectly reasonable to call Z a function of z.

But it is usual to restrict the meaning of the term 'function of z' as follows.

We use the equation $Z = f(z)$ to indicate that Z is the result of definite operations performed on z, the variables x and y not occurring explicitly, and we say that Z is a function of z.

Thus $2x + 3\iota y$ is a function of z in the first sense, but not in the restricted sense.

For if z is known, so is $2x + 3\iota y$; but $2x + 3\iota y$ cannot be expressed in terms of z alone, i.e. without the explicit use of x or y.

It follows that if

$$Z = X + \iota Y = f(z) = f(x + \iota y),$$

one series of operations determines X and Y in terms of x and y.

30. Definitions relating to Limits and Continuity.

(1) To say that z *tends to zero* ($z \to 0$) is to say that z varies in such a way that $|z|$ becomes and remains less than any positive ϵ we may choose, however small.

Thus, after a certain stage, the point z (on an Argand diagram) is within a circle with centre O and radius ϵ, which may be as small as we like.

To say that z tends to a ($z \to a$), where a is a fixed number, is to say that $z - a \to 0$. After a certain stage, the point z is within a circle with centre a and radius ϵ, which may be as small as we like.

(2) We say that $f(z)$ tends to a limit l as z tends to a if, for any positive ϵ that we may choose, however small, η can be found so that

$$|f(z) - l| < \epsilon, \quad \text{provided only that} \quad |z - a| < \eta.$$

In other words, as z approaches a *by any path whatever* the distance of the point $f(z)$ from the fixed point l becomes and remains less than any length ϵ that we may choose, however small.

(3) The function $f(z)$ is continuous for $z = z_0$ or at (the point) z_0 if for any positive ϵ that we may choose, however small, we can find η so that

$$|f(z) - f(z_0)| < \epsilon, \quad \text{provided only that} \quad |z - z_0| < \eta.$$

This is the same as saying that as z tends to z_0 along any path whatever, $f(z)$ tends to $f(z_0)$ as a limit.

It is assumed that $f(z)$ has a definite value at every point in the neighbourhood of the point z_0.

(4) The function $f(z)$ is continuous in a region A if it is continuous at every point of the region.

Hence if z describes a continuous curve in the region of continuity of the function $f(z)$, the point $f(z)$ will also describe a continuous curve.

31. Continuity of Rational Functions of z. All that has been said in Arts. 2 and 3, with regard to the real variable, holds for the complex variable. In particular :

(i) $\lim_{z \to 0} (a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n) = a_0$.

(ii) A polynomial in z is continuous for all values of z.

(iii) A rational function of z is continuous for all values of z except for those which make the denominator zero.

EXERCISE XXIX

1. Prove that (i) $\sin x - x$ always decreases with x.

(ii) $\tan x - x$ increases with x in the intervals
$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, ...

(iii) $\dfrac{\sin x}{x}$ decreases as x increases from 0 to $\frac{1}{2}\pi$.

2. If a, $ax+b$, $ax^2+2bx+c$, $ax^3+3bx^2+3cx+d$ and $ax^4+4bx^3+bcx^2+4dx+e$ are denoted by u_0, u_1, u_2, u_3 respectively, show that, if $U=u_0^2u_3-3u_0u_1u_2+2u_1^3$, $V=u_0u_4-4u_1u_3+3u_2^2$, then $\dfrac{dU}{dx}=0$ and $\dfrac{dV}{dx}=0$, so that U and V are independent of x.

3. If
$$u_n=\cos x-\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\ldots+(-1)^n\frac{x^{2n}}{\lfloor 2n}\right)$$
and
$$v_n=\sin x-\left(x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\ldots+(-1)^n\frac{x^{2n+1}}{\lfloor 2n+1}\right),$$
prove that
$$\frac{du_n}{dx}=-v_{n-1},\quad \frac{dv_n}{dx}=u_n.$$

Hence show that u_n and v_n are positive or negative according as n is odd or even.

[If $x=0$, then $u_n=0$ and $v_n=0$; also $u_0=\cos x-1<0$, $v_0=\sin x-x<0$.

Now $\dfrac{du_1}{dx}=-v_0>0$, $\therefore$ u_1 is an increasing function, $\therefore$ $u_1>0$;

and $\dfrac{dv_1}{dx}=u_1>0$, $\therefore$ v_1 is an increasing function, $\therefore$ $v_1>0$;

and so on, in succession for (u_2, v_2), (u_3, v_3), ...]

4. Prove that for all values of x

(i) $\cos x=1-\dfrac{x^2}{\lfloor 2}+\dfrac{x^4}{\lfloor 4}-\ldots$ to ∞;

(ii) $\sin x=x-\dfrac{x^3}{\lfloor 3}+\dfrac{x^5}{\lfloor 5}-\ldots$ to ∞.

[(i) Let t_r be the rth term of the series and s_r the sum to r terms. Then by Ex. 3, for every x, $s_{2n}<\cos x<s_{2n+1}$ and $s_{2n+1}-s_{2n}=t_{2n+1}\to 0$ as $n\to\infty$. Therefore $s_r\to\cos x$ as $r\to\infty$.]

5. Use Taylor's theorem to prove that

(i) $\sin(x+h)=\sin x+h\cos x-\dfrac{h^2}{\lfloor 2}\sin x-\dfrac{h^3}{\lfloor 3}\cos x+\ldots$;

(ii) $\cos(x+h)=\cos x-h\sin x-\dfrac{h^2}{\lfloor 2}\cos x+\dfrac{h^3}{\lfloor 3}\sin x+\ldots$.

[(i) If $f(x)=\sin x$, $|R_n|<\dfrac{1}{\lfloor n}\cdot|h^n|\to 0$ as $n\to\infty$.]

6. Deduce the expansions of $\sin x$ and $\cos x$ in Ex. 4 from Taylor's theorem.

7. Prove, by means of Art. 22, Ex. 1, that, if $0<x\leqslant 1$, then

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{ to } \infty .$$

[Let $S_r = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{r-1}\frac{x^{2r-1}}{2r-1}$, and $f(x) = \tan^{-1}x - S_r$; then $f(0)=0$, and $f'(x) = 1/(1+x^2) - (1 - x^2 + x^4 - \ldots + (-1)^{r-1}x^{2r-2}$, so that

$$(1+x^2)f'(x) = (-1)^r x^{2r},$$

which is positive or negative according as r is even or odd.

Hence, $\tan^{-1}x$ lies between S_r and S_{r+1}, and $|S_r - S_{r+1}| \to 0$, as $r \to \infty$, so long as x is positive and not greater than unity.]

8. If $f(x)$ is a function whose first derivative is $1/x$, show that, if $0<x\leqslant 1$,

$$f(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \text{ to } \infty .$$

[Proceed as in Ex. 7.]

9. If $0<\theta\leqslant \pi/6$, prove that $\tan\theta$ lies between $\theta + \frac{1}{3}\theta^3$ and $\theta + \frac{2}{9}\theta^3$.

[For the second inequality, let $f(\theta) = \theta + a\theta^3 - \tan\theta$; show that $f(0)$, $f'(0)$, $f''(0)$, are all zero, and that $f'''(\theta) \geqslant 0$ throughout the interval $0<\theta\leqslant\pi/6$ if $9a \geqslant 8$; hence, in succession, $f''(\theta)$, $f'(\theta)$, $f(\theta)$, are all positive.]

The Binomial Theorem

10. If n is a positive integer, show that the expansion of $(a+h)^n$ by the Binomial theorem is given by Taylor's series.

11. If n is any rational number and $-1<x<1$, (i) show that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}x^{r-1} + R_r$$

where $$R_r = \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r(1+\theta x)^{n-r} \text{ and } 0<\theta<1.$$

(ii) If $0<x<1$, prove that $R_r \to 0$ as $r \to \infty$, so that in this case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \ldots \text{ to } \infty . \quad \ldots\ldots\ldots\

CHAPTER XVIII

THEORY OF EQUATIONS (2), POLYNOMIALS (2), RATIONAL FRACTIONS (1)

1. Multiple Roots. *If α is an r-multiple root of $f(x)=0$, where $f(x)$ is a polynomial, then $(x-\alpha)^{r-1}$ is a common factor of $f(x)$ and $f'(x)$.*

For $f(x)=(x-\alpha)^r \cdot \phi(x)$ where $\phi(x)$ is a polynomial and

$$f'(x)=r(x-\alpha)^{r-1} \cdot \phi(x)+(x-\alpha)^r \cdot \phi'(x).$$

2. Rolle's Theorem for Polynomials. *If $f(x)$ is a polynomial, at least one real root of $f'(x)=0$ lies between any two real roots of $f(x)=0$.*

Proof. Let α, β be consecutive real roots of $f(x)=0$, so that

$$f(x)=(x-\alpha)^m(x-\beta)^n \cdot \phi(x)$$

where m, n are positive integers, and $\phi(x)$ is not divisible by $x-\alpha$ or $x-\beta$. Then $\phi(x)$ is of invariable sign for $\alpha \leqslant x \leqslant \beta$ and

$$f'(x)=(x-\alpha)^{m-1}(x-\beta)^{n-1} \cdot \psi(x),$$

where

$$\psi(x)=\{m(x-\beta)+n(x-\alpha)\} \cdot \phi(x)+(x-\alpha)(x-\beta)\phi'(x).$$

Now $\psi(\alpha)=m(\alpha-\beta)\phi(\alpha)$ and $\psi(\beta)=n(\beta-\alpha)\phi(\beta)$.

Therefore $\psi(\alpha)$ and $\psi(\beta)$ have unlike signs; moreover, $\psi(x)$ is everywhere continuous.

Hence $\psi(x)$ and consequently $f'(x)$ is equal to zero for some value of x between α and β.

3. Deductions from Rolle's Theorem.

(1) *If all the roots of $f(x)=0$ are real, so also are those of $f'(x)=0$, and the roots of the latter equation separate those of the former.*

For if $f(x)$ is of degree n, $f'(x)$ is of degree $n-1$, and a root of $f'(x)=0$ exists in each of the $n-1$ intervals between the n roots of $f(x)=0$.

(2) *If all the roots of $f(x)=0$ are real, so also are those of $f'(x)=0$, $f''(x)=0$, $f'''(x)=0$, ..., and the roots of any one of these equations separate those of the preceding equation.*

This follows from (1).

(3) *Not more than one root of* $f(x)=0$ *can*

(i) *lie between two consecutive roots of* $f'(x)=0$, *or*

(ii) *be less than the least of these, or*

(iii) *be greater than the greatest of them.*

For let $\beta_1, \beta_2, \ldots \beta_r$ be the real roots of $f'(x)=0$, any of which may be multiple roots, and suppose that $\beta_1<\beta_2<\ldots<\beta_r$. Let α_1, α_2 be real roots of $f(x)=0$.

If $\alpha_1=\alpha_2$, then α_1 is one of the set $\beta_1, \beta_2, = \beta_r$. (Art. 1.)

If $\alpha_1\neq\alpha_2$, by Rolle's theorem, a root of $f'(x)$ lies between α_1 and α_2.

Hence (i) if $\beta_1<\alpha_1<\alpha_2<\beta_2$, then β_1 and β_2 cannot be consecutive roots; (ii) if $\alpha_1<\alpha_2<\beta_1$, then β_1 cannot be the least root of $f'(x)=0$; (iii) if $\beta_r<\alpha_1<\alpha_2$, then β_r cannot be the greatest.

Thus, *not more than one root of* $f(x)=0$ *can lie in any one of the open intervals*

$$(-\infty, \beta_1),\ (\beta_1, \beta_2),\ \ldots\ (\beta_{r-1}, \beta_r),\ (\beta_r, \infty).$$

(4) *If* $f'(x)=0$ *has* r *real roots, then* $f(x)=0$ *cannot have more than* $r+1$ *real roots.*

If $f(x)=0$ has no multiple roots, none of the roots of $f'(x)=0$ is a root of $f(x)=0$, and the theorem follows from (3).

If $f(x)=0$ has an m-multiple root, we regard this as the limiting case in which m roots tend to equality. Thus the theorem is true in all cases.

(5) *If* $f^{(r)}(x)$ *is any derivative of* $f(x)$ *and the equation* $f^{(r)}(x)=0$ *has imaginary roots, then* $f(x)=0$ *has at least as many.*

It follows from (4) that $f(x)=0$ has at least as many imaginary roots as $f'(x)=0$.

(6) If all the real roots $\beta_1, \beta_2, \ldots$ of $f'(x)=0$ are known, we can find the number of real roots of $f(x)=0$ by considering the signs of $f(\beta_1), f(\beta_2), \ldots$. A single root of $f(x)=0$, or no root, lies between β_1 and β_2, according as $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, or the same sign.

Ex. 1. *Find the character of the roots of* $f(x)\equiv 3x^4-8x^3-6x^2+24x+7=0$.

Here $f'(x)=12(x^2-1)(x-2)$. The roots of $f'(x)=0$ are $-1, 1, 2$.

x	$-\infty$	-1	1	2	∞
$f(x)$	$+$	$-$	$+$	$+$	$+$

and $f(x)=0$ has a real root <-1, one between -1 and 1, and two imaginary roots.

Ex. 2. *If* $b^2-ac<0$, *then the equation* $f(x)=(a, b, c, \ldots k)(x, 1)^n=0$ *has at least two imaginary roots.* See Ch. VI, 11, Ex. 1.

For $$f^{(n-2)}(x)=n(n-1)\ldots 3\,.\,(ax^2+2bx+c)$$

and the equation $ax^2+2bx+c=0$ has two imaginary roots.

4. Sums of Powers of the Roots of an Equation—Newton's Theorem.

Let $\alpha, \beta, \ldots \kappa$ be the roots of

$$f(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0, \quad \ldots\ldots\ldots\

Ex. 1. *For equation* (**A**) *of the last article, find the value of* $\Sigma(\alpha-\beta)^r$, *where* r *is an even number, in terms of* $s_0, s_1, s_2, \ldots$.

If $c_1, c_2, \ldots$ are the coefficients of $x, x^2, \ldots$ in the expansion of $(1+x)^r$, then

$$(\alpha-x)^r=\alpha^r-c_1x\alpha^{r-1}+c_2x^2\alpha^{r-2}-\ldots+x^r.$$

In this equation, substitute $\alpha, \beta, \gamma, \ldots \kappa$ in succession for x and add, therefore,

$$(\alpha-\alpha)^r+(\alpha-\beta)^r+(\alpha-\gamma)^r+\ldots+(\alpha-\kappa)^r$$
$$=s_0\alpha^r-c_1s_1\alpha^{r-1}+c_2s_2\alpha^{r-2}-\ldots+s^r.$$

Similar equations can be deduced from the expansions of $(\beta-x)^r$, $(\gamma-x)^r$, etc.: hence, by addition, we have

$$2\,\Sigma(\alpha-\beta)^r=s_0s_r-c_1s_1s_{r-1}+c_2s_2s_{r-2}-\ldots+s_rs_0.$$

5. Order and Weight of Symmetric Functions. Let $\alpha, \beta, \gamma \ldots$ be the roots of

$$x_1^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

Consider any function of the coefficients; for example, let

$$u=p_1^2p_2-2p_2^2-p_1p_3+4p_4.$$

Let u be expressed in terms of the roots by means of the equalities

$$p_1=-\Sigma\alpha, \quad p_2=\Sigma\alpha\beta, \quad p_3=-\Sigma\alpha\beta\gamma, \ldots .$$

Consider *the highest power of any root* α *which occurs in this expression for* u, and denote it by α^s.

Each of $p_1, p_2, \ldots$ involves α to the first degree only, therefore α^s arises from the highest term $p_1^2p_2$ and *from this alone*: also the index s is the degree of this term, namely 3.

The *order* of a symmetric function of the roots of an equation is defined as the index of the highest power of any root which occurs in the function.

Thus the *order* of $\Sigma\alpha^3\beta$ is 3, and that of $\Sigma\alpha\beta\gamma$ is 1.

From what has been said it will be clear that *the order of a symmetric function of the roots is the degree of the function when expressed in terms of the coefficients* $p_1, p_2, \ldots p_n$.

If u *is a symmetric function of order* s *of the roots of*

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{B})$$

then a_0^su *is a homogeneous function of* $a_0, a_1, a_2, \ldots$ *of degree* s.

For if the equation is written in the form (**A**), u is a function of $p_1, p_2, \ldots$ in which the highest term is of degree s and $p_1=a_1/a_0$, $p_2=a_2/a_0$, etc.

The *weight* of a symmetric function (u) of the roots $\alpha, \beta, \ldots$ is its *degree in* $\alpha, \beta, \ldots$.

Thus the *weight* of $\Sigma\alpha^3\beta$ is 4, and that of $\Sigma\alpha\beta\gamma$ is 3.

Theorem. *If a symmetric function* (u) *of the roots of equation* (**A**) *is expressed in terms of the coefficients* $p_1, p_2, \ldots$ *the sum of the suffixes of every term, when written at full length,* is equal to the weight* (w) *of* u.

Proof. Denote any term in u by $p_1^h p_2^k p_3^l \ldots$; and in equation (**A**) write $x = y/\lambda$. The equation becomes $y^n + \lambda p_1 y^{n-1} + \lambda^2 p_2 y^{n-2} + \ldots + \lambda^n p_n \ldots 0$, and the roots of this are $\lambda\alpha, \lambda\beta, \lambda\gamma, \ldots$.

If then we multiply every root by λ, the effect of this is

(i) to multiply u by λ^w ;

(ii) to multiply p_1, p_2, p_3 by $\lambda, \lambda^2, \lambda^3, \ldots$ respectively ;

(iii) to multiply $p_1^h p_2^k p_3^l \ldots$ by $\lambda^{h+2k+3l+\ldots}$;

$$\therefore \quad w = h + 2k + 3l + \ldots ,$$

which proves the theorem.

Ex. 1. *Write down the literal part of* $\Sigma\alpha^2\beta^2$, *expressed as a function of the coefficients.*

$\Sigma\alpha^2\beta^2$ is of order 2 and weight 4. The only terms of weight 4 are such as contain $p_1^4, p_1^2 p_2, p_1 p_3, p_2^2, p_4$. The first two of these are of too high an order.

Thus we have $\Sigma\alpha^2\beta^2 = a p_1 p_3 + b p_2^2 + c p_4$,

where a, b, c are independent of p_1, p_2, p_3.

6. Partial Derivatives. Suppose that $u \equiv f(x, y, z, \ldots)$ is a function of the independent variables $x, y, z, \ldots$. The operation of differentiating u with respect to x, on the supposition that x *alone varies*, is denoted by the symbol $\frac{\partial}{\partial x}$, and $\frac{\partial u}{\partial x}$ is called the first *partial* derivative of u with respect to x. Sometimes the ordinary symbol $\frac{d}{dx}$ is used in the same sense, the context implying that the differentiation is partial. We also write

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x},$$

with similar notation for higher derivatives.

For example, if $u = x^m y^n$,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(n x^m y^{n-1}) = mn x^{m-1} y^{n-1},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(m x^{m-1} y^n) = mn x^{m-1} y^{n-1}.$$

Therefore

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

It follows that *equation* (**A**) *holds when* u *is a polynomial in* x, y.

* Thus the term $p_1^2 p_2^3$ is written $p_1 p_1 p_2 p_2 p_2$.

It is proved in books on the ' Calculus ' that the commutative property represented by (**A**) holds when the functions in question are continuous. A strict proof is difficult, and here we are only concerned with polynomials.

Again, we often regard $h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y}$ as an operator, and we write

$$h\frac{\partial u}{\partial x}+k\frac{\partial u}{\partial y}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)u=Du,$$

$$D^2u=D(Du)=D\left(h\frac{\partial u}{\partial x}+k\frac{\partial u}{\partial y}\right)=h\left(h\frac{\partial^2 u}{\partial x^2}+k\frac{\partial^2 u}{\partial x\,\partial y}\right)+k\left(h\frac{\partial^2 u}{\partial x\,\partial y}+k\frac{\partial^2 u}{\partial y^2}\right)$$
$$=h^2\frac{\partial^2 u}{\partial x^2}+2hk\frac{\partial^2 u}{\partial x\,\partial y}+k^2\frac{\partial^2 u}{\partial y^2},$$

if h, k are constants; a result which we may express as follows:

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2u=\left\{h^2\left(\frac{\partial}{\partial x}\right)^2+2hk\frac{\partial}{\partial x}\frac{\partial}{\partial y}+k^2\left(\frac{\partial}{\partial y}\right)^2\right\}u.$$

The value of D^nu can be written down in the same way by the Binomial theorem.

7. Taylor's Theorem for Polynomials. (1) *If $f(x)$ is a polynomial of degree n in x, then*

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\ldots+\frac{h^n}{\lfloor n}f^{(n)}(x). \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

Proof. This follows from Ch. XVII, or thus: we may assume that

$$f(x+h)=a_0+ha_1+\frac{h^2a_2}{\lfloor 2}+\ldots+\frac{h^n}{\lfloor u}a_n$$

where $a_0, a_1, \ldots a_n$ are independent of h, and if $X=x+h$ we have

$$\frac{d}{dh}f(x+h)=\frac{dX}{dh}\cdot\frac{d}{dX}f(X)=f'(x+h).$$

Hence, by successive differentiation with regard to h,

$$f'(x+h)=a_1+ha_2+\frac{h^2}{\lfloor 2}a_3+\ldots+\frac{h^{n-1}}{\lfloor n-1}a_n,$$

$$f''(x+h)=a_2+ha_3+\frac{h^2}{\lfloor 2}a_4+\ldots+\frac{h^{n-2}}{\lfloor n-2}a_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$f^{(n)}(x+h)=a_n.$$

The result follows by putting $h=0$ in these equations; for we find that

$$a_1=f'(x),\quad a_2=f''(x),\quad \ldots,\quad a_n=f^{(n)}(x).$$

(2) *If* $u \equiv f(x, y)$ *is a polynomial of degree* n *in* x, y, *then*

$$f(x+h, y+k) = u + Du + \frac{1}{\lfloor 2} D^2u + \ldots + \frac{1}{\lfloor n} D^n u. \quad \ldots\ldots\ldots\ldots (\mathbf{B})^*$$

For we may assume that

$$f(x+h, y+k) = a_0 + (b_0h + b_1k) + \Sigma_{m=2}^{n} \frac{1}{\lfloor m} (p_0, p_1, \ldots p_m \S h, k)^m, \quad \ldots (\mathbf{C})$$

where $a_0, b_0, \ldots$ are independent of h, k.

Putting $h=0$, $k=0$ we have $a_0 = u$; and differentiating $m-r$ times with regard to h and r times with regard to k, we find that

$$\left(\frac{\partial}{\partial h}\right)^{m-r} \left(\frac{\partial}{\partial k}\right)^{r} f(x+h, y+k) = p_r + \text{terms involving } h, k. \quad \ldots\ldots\ldots (\mathbf{D})$$

Then, if $X = x+h$, $Y = y+k$, the left-hand side of (**D**) is equal to

$$\left(\frac{\partial}{\partial X}\right)^{m-r} \left(\frac{\partial}{\partial Y}\right)^{r} f(X, Y) = f_{m-r, r}(X, Y), \text{ say, i.e. } f_{m-r, r}(x+h, y+k);$$

and putting $h=0$, $k=0$, we have $p_r = f_{m-r, r}(x, y) = \dfrac{\partial^m u}{\partial x^{m-r} \partial y^r}$, which proves the theorem.

(3) *If* $u \equiv f(x, y, z, \ldots)$ *is a polynomial of degree* n *in* $x, y, z, \ldots$ *then*

$$f(x+h, y+k, z+l, \ldots) = u + Du + \frac{1}{\lfloor 2} D^2u + \ldots + \frac{1}{\lfloor n} D^n u, \quad \ldots\ldots\ldots (\mathbf{E})$$

where D *denotes the operator* $h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y} + l\dfrac{\partial}{\partial z} + \ldots$.

The proof is similar to the above, using the Multinomial theorem.

8. Euler's Theorem. (1) *Let* u *be a polynomial homogeneous in* x, y *and of degree* n, *then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu, \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u,$$

and, in general, $\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^r u = n(n-1)(n-2)\ldots(n-r+1)u.$

Proof. Let $u = f(x, y)$, then by Taylor's theorem,

$$f(x+\lambda x, y+\lambda y) = u + \lambda\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \frac{\lambda^2}{\lfloor 2}\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \ldots \quad \ldots (\mathbf{A})$$

and since u is homogeneous and of degree n in x, y,

$$f(x+\lambda x, y+\lambda y) = (1+\lambda)^n u. \quad \ldots\ldots\ldots\ldots (\mathbf{B})$$

The coefficients of the powers of λ in the expressions on the right of (**A**) and (**B**) must be equal, whence the results follow.

* See end of Art. 6.

(2) *If* $u=f(x, y, z, \ldots)$ *is a polynomial homogeneous in* $x, y, z, \ldots$ *and of degree* n, *then*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+\ldots=nu,$$

$$D^r u=n(n-1)(n-2)\ldots(n-r+1)u,$$

where D^r *stands for* $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\ldots\right)^r$ *in its expanded form.*

Putting $h=\lambda x$, $k=\lambda y$, $l=\lambda z$, ... in equation (**E**) of Art. 7, the proof is similar to the preceding.

9. Partial Fractions. Let $f(x)/\phi(x)$ be a rational proper fraction which is to be expressed in partial fractions. *If* $x-a$ *is a non-repeated factor of* $\phi(x)$, *the fraction corresponding to* $x-a$ *is*

$$\frac{f(a)}{\phi'(a)}\cdot\frac{1}{x-a}.$$

For let $\phi(x)=(x-a)\psi(x)$ and $\dfrac{f(x)}{\phi(x)}=\dfrac{A}{x-a}+\dfrac{F(x)}{\psi(x)}$.

From the second equation

$$f(x)=A\psi(x)+(x-a)F(x),$$

and from the first, by differentiation,

$$\phi'(x)=(x-a)\psi'(x)+\psi(x).$$

Putting $x=a$ and observing that $\psi(a)\neq 0$, we have

$$f(a)=A\psi(a) \quad \text{and} \quad \phi'(a)=\psi(a), \quad \text{therefore} \quad A=f(a)/\phi'(a).$$

Ex. 1. *Express* $x^{m-1}/(x^n-1)$ *in partial fractions, where* m, n *are positive integers and* $m\leqslant n$.

If $x-a$ is any factor of x^n-1, then $a^n=1$ and

$$\frac{x^{m-1}}{x^n-1}=\Sigma\frac{a^{m-1}}{na^{n-1}}\cdot\frac{1}{x-a}=\frac{1}{n}\Sigma\frac{a^m}{x-a}.$$

The imaginary roots of $x^n-1=0$ are

$$\cos\frac{2r\pi}{n}\pm\iota\sin\frac{2r\pi}{n} \text{ where } r=1, 2, \ldots \tfrac{1}{2}(n-1) \text{ or } \tfrac{1}{2}(n-2),$$

according as n is odd or even.

Let $\quad a=\cos\dfrac{2r\pi}{n}+\iota\sin\dfrac{2r\pi}{u}$, then $a^{-1}=\cos\dfrac{2r\pi}{n}-\iota\sin\dfrac{2r\pi}{n}$;

and denote the sum of the fractions corresponding to $x-a$ and $x-a^{-1}$ by u_r, then by the preceding,

$$u_r=\frac{1}{n}\left(\frac{a^m}{x-a}+\frac{a^{-m}}{x-a^{-1}}\right)=\frac{1}{n}\frac{x(a^m+a^{-m})-(a^{m-1}+a^{-(m-1)})}{x^2-x(a+a^{-1})+1};$$

$$\therefore\ u_r=\frac{2}{n}\cdot\frac{x\cos rm\alpha-\cos r(m-1)\alpha}{x^2-2x\cos r\alpha+1} \text{ where } \alpha=\frac{2\pi}{n}.$$

Hence, according as n is odd, or even, $x^{n-1}/(x^n-1)$ *is equal to*

$$\frac{1}{n}\cdot\frac{1}{x-1}+\Sigma_{r=1}^{r=\frac{1}{2}(n-1)}u_r, \quad \text{or,} \quad \frac{1}{n}\cdot\frac{1}{x-1}+\frac{(-1)^m}{n}\cdot\frac{1}{x+1}+\Sigma_{r=1}^{r=\frac{1}{2}(n-2)}u_r.$$

10. Theorem. The following theorem is often useful.

If $\alpha, \beta, \ldots$ *are the roots of* $u\equiv(a_0, a_1, \ldots a_n)(x, 1)^n=0$ *and*

$$v\equiv\phi(a_0, a_1, \ldots a_n)$$

is a symmetric function of $\alpha, \beta, \ldots$, *involving only differences of the roots, then v satisfies the equation*

$$a_0\frac{\partial v}{\partial a_1}+2a_1\frac{\partial v}{\partial a_2}+\ldots+na_{n-1}\frac{\partial v}{\partial a_n}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

For the substitution $x=X+h$ transforms $u=0$ into

$$U\equiv(a_0, A_1, A_2, \ldots A_n)(X, 1)^n,$$

where $A_1=a_1+a_0h$, $A_2=a_2+2a_1h+a_0h^2$, etc., and since the roots of $U=0$ are $\alpha-h, \beta-h, \ldots$, we have

$$v=\phi(a_0, a_1, \ldots a_n)=\phi(a_0, A_1, \ldots A_n). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

If the right-hand side of (**B**) is expanded in powers of h, every coefficient must vanish, and by the extension of Taylor's theorem the coefficient of h is the same as that in

$$a_0h\frac{\partial v}{\partial a_1}+(2a_1h+a_0h^2)\frac{\partial v}{\partial a_2}+(3a_2h+3a_1h^2+a_0h^3)\frac{\partial v}{\partial a_3}+\ldots;$$

$$\therefore\ a_0\frac{\partial v}{\partial a_1}+2a_1\frac{\partial v}{\partial a_2}+3a_2\frac{\partial v}{\partial a_3}+\ldots=0.$$

Ex. 1. *For the biquadratic* $(a_0, a_1, a_2, a_3, a_4)(x, 1)^4=0$, *find in terms of the coefficients the value of*

$$v=(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta).$$

This function is of order 3 and weight 3. We may therefore assume that

$$a_0^3v=pa_0^2a_3+qa_0a_1a_2+ra_1^3,$$

where p, q, r are independent of $a_0, a_1, \ldots$. And since v is a function of the differences of $\alpha, \beta, \ldots$, we have

$$a_0\frac{\partial v}{\partial a_1}+2a_1\frac{\partial v}{\partial a_2}+3a_2\frac{\partial v}{\partial a_3}+4a_3\frac{\partial v}{\partial a_4}=0;$$

$$\therefore\ a_0(qa_0a_2+3ra_1^2)+2a_1qa_0a_1+3a_2pa_0^2=0;$$

$$\therefore\ a_0a_1^2(3r+2q)+a_0^2a_2(3p+q)=0.$$

This is true for all values of a_0, a_1, a_2, therefore

$$3r+2q=0, \quad 3p+q=0, \quad \text{giving} \quad q=-3p, \quad r=2p;$$

$$\therefore\ a_0^3v=p(a_0^2a_3-3a_0a_1a_2+2a_1^3).$$

The value of p may be found by giving special values to $\alpha, \beta, \gamma, \delta$. Thus by taking $\alpha=0$, $\beta=1$, $\gamma=-1$, $\delta=4$, we find that $p=32$, which agrees with Ex. 3 of VI, 16.

EXERCISE XXX

1. The equation $f(x) \equiv x^3 + 3qx + r = 0$ has two equal roots if $r^2 + 4q^3 = 0$. Prove this (i) by applying the H.C.F. process to $f(x)$ and $f'(x)$; (ii) by eliminating x between $f(x) = 0$ and $f'(x) = 0$.

2. The condition that the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may have two equal roots is

$$(ad - bc)^2 - 4(ac - b^2)(bd - c^2) = 0.$$

Show that this may be obtained by eliminating x from

$$ax^2 + 2bx + c = 0 \quad \text{and} \quad bx^2 + 2cx + d = 0.$$

3. Find the values of a for which $ax^3 - 9x^2 + 12x - 5 = 0$ has equal roots, and solve the equation in one case.

4. The equation $x^n - qx^{n-m} + r = 0$ has two equal roots if

$$\left\{\frac{q}{n}(n-m)\right\}^n = \left\{\frac{r}{m}(n-m)\right\}^m.$$

5. All the roots of $x^n - \frac{n \,.\, n-1}{\lfloor 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} x^{n-4} - \ldots = 0$ are real.

[Use Ex. VII, 14.]

6. Show that the equation $x^3 - 7x + 7 = 0$ has two roots between 1 and 2 and one between -3 and -4.

7. If the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ has three roots, each equal to α, show that α is a root of

$$n^2 x^{n-1} + (n-1)^2 p_1 x^{n-2} + (n-2)^2 p_2 x^{n-3} + \ldots + p_{n-1} = 0.$$

8. Show that the equation $x^4 - 4x^3 + 10x^2 + 7x - 5 = 0$ has one positive, one negative and two imaginary roots.

9. The equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ has four real roots if

$$-13 < r < -8,$$

two real roots if $-8 < r < 19$, and no real root if $r > 19$.

[The roots of $f'(x) = 0$ are -2, -1, 1.]

10. The equation $f(x) = (x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ has one real and two imaginary roots.

11. The equation $x^5 - 5ax + 4b = 0$ has three real roots or only one, according as $a^5 \gtrless b^4$.

12. The equation $x^5 + 5ax^3 + b = 0$ has one and only one real root in the following cases: (i) if $a > 0$; (ii) if $a < 0$ and $b^2 + 108a^5 > 0$.

13. The equation $x^5 + 5ax^2 + b = 0$ has one and only one real root in the following cases: (i) if a, b have the same sign; (ii) if a, b have opposite signs and $b(b^3 + 108a^5) > 0$.

[Deduce from Ex. 12 by putting $1/x$ for x.]

14. Show that if $3b^2 < a^2$ there are real values of x, y, z which satisfy

$$x + y + z = a, \quad yz + zx + xy = b^2,$$

and that these values are all positive if $a > 0$ and $a^2 < 4b^2$.

[x, y, z are the roots of an equation of the form $\theta^3 - a\theta^2 + b^2\theta + \kappa = 0$ and if $\theta = \phi + a/3$, this reduces to one of the form $\phi^3 + \phi(b^2 - \frac{1}{3}a^2) + k' = 0$.]

15. If $\lambda_1, \lambda_2, \lambda_3$ are the values of λ given by the equation

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$

prove that these are all real; also, if a, b, c and λ_1, λ_2, λ_3 are in descending order, then

$$a^2>\lambda_1>b^2>\lambda_2>c^2>\lambda_3$$

and

$$x^2=\frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(a^2-b^2)(a^2-c^2)}.$$

16. If $\alpha, \beta, \gamma, \ldots$ are the roots of the equation

$$x^n+\frac{x^{n-1}}{\lfloor 1}+\frac{x^{n-2}}{\lfloor 2}+\ldots+\frac{1}{\lfloor n}=0,$$

show that

$$\Sigma\alpha^r=0 \quad \text{for} \quad r=2, 3, 4, \ldots n.$$

17. If $f(x)=0$ is a cubic equation whose roots are α, β, γ, and α is the harmonic mean of the roots of $f'(x)=0$, prove that $\alpha^2=\beta\gamma$.

18. The sum of the ninth powers of the roots of $x^3+3x+9=0$ is zero.

19. If s_n is the sum of the nth powers of the roots of $x^4-x^2+1=0$, show that

$$s_{2n-1}=0, \quad s_{2n}=4\cos\frac{n\pi}{3}.$$

20. If α, β, ... are the roots of $(a_0, a_1, a_2, \ldots a_n\not\downarrow t, 1)^n=0$, show that the equation whose roots are $\alpha y-x$, $\beta y-x$, etc., is $(a_0, u_1, u_2, \ldots u_n\not\downarrow t, 1)^n=0$, where $u_1=a_0x+a_1y$, $u_2=a_0x^2+2a_1xy+a_2y^2$, etc.

21. If m and n are positive integers and $m\leqslant n$, prove that

$$\frac{x^{m-1}}{x^n+1}=\frac{(-1)^{n-m}}{n(x+1)}+\frac{2}{n}\Sigma_{r=0}^{r=\frac{1}{2}(n-3)}u_r \quad \text{or} \quad \frac{2}{n}\Sigma_{r=0}^{r=\frac{1}{2}(n-2)}u_r,$$

according as n is odd or even, where

$$u_r=-\frac{x\cos(2r+1)m\alpha-\cos(2r+1)(m-1)\alpha}{x^2-2x\cos(2r+1)\alpha+1} \quad \text{and} \quad \alpha=\frac{\pi}{n}.$$

22. Let k be given by the equation

$$\Delta'=\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix}=0.$$

If this equation has a repeated root k_1, prove that

$$k_1=a-gh/f=b-hf/g=c-fg/h,$$

provided that none of f, g, h is zero. If one of these (g) is zero, so also is another (h), and then

$$k_1=a \quad \text{and} \quad (a-b)(a-c)=f^2.$$

Under these circumstances, show that

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-k_1(x^2+y^2+z^2)$$

is a perfect square.

[Denoting the cofactors of the elements of Δ' by A', F', etc., the equations $A'=0$, $B'=0$, $C'=0$ are quadratics in k, and the roots of any one of them separate those of $\Delta'=0$. Hence if $k=k_1$, then A', B', C' and consequently F', G', H' are all zero. See Ch. IX, 21, (4).]

CHAPTER XIX

EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND SERIES

1. Continuity of a^x and $\log_a x$. If $a>0$, the function a^x is continuous for all real values of x. This follows from Ch. XIII, 9.

Hence the inverse function $\log_a x$ is continuous for positive values of x. (Ch. XVII, 21.) The reader is reminded that $\log x$ stands for $\log_e x$.

2. Exponential Inequalities and Limits. (Continued from Ch. XV, 5.)

(1) *As $x\to\infty$ through real values,*

$$\left(1+\frac{1}{x}\right)^x\to e \quad \textit{and} \quad \left(1-\frac{1}{x}\right)^{-x}\to e.$$

For let $m<x<m+1$, where m is a positive integer, then

$$1+\frac{1}{m+1}<1+\frac{1}{x}<1+\frac{1}{m};$$

therefore
$$\left(1+\frac{1}{m+1}\right)^m<\left(1+\frac{1}{x}\right)^x<\left(1+\frac{1}{m}\right)^{m+1}.$$

If $x\to\infty$, then $m\to\infty$ and by Ch. XV, 5, (3),

$$\left(1+\frac{1}{m+1}\right)^m=\left(1+\frac{1}{m+1}\right)^{m+1}\div\left(1+\frac{1}{m+1}\right)\to e,$$

$$\left(1+\frac{1}{m}\right)^{m+1}=\left(1+\frac{1}{m}\right)^m\cdot\left(1+\frac{1}{m}\right)\to e;$$

therefore
$$\left(1+\frac{1}{x}\right)^x\to e.$$

Next let $x=y+1$, then as $x\to\infty$, $y\to\infty$ and

$$\left(1-\frac{1}{x}\right)^{-x}=\left(\frac{y}{y+1}\right)^{-(y+1)}=\left(1+\frac{1}{y}\right)^y\cdot\left(1+\frac{1}{y}\right)\to e.$$

It follows that if $x\to\infty$ or $-\infty$,

$$\lim\left(1+\frac{1}{x}\right)^x=e,$$

and consequently
$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$$

(2) *If z is any real number,*

$$\lim_{x\to\infty}\left(1+\frac{z}{x}\right)^{x}=e^{z}.$$

This is obviously true if $z=0$. If $z\neq 0$, let $x=yz$, then as $x\to\infty$, $y\to\infty$ or $-\infty$, according as $z\gtrless 0$. In either case, by the preceding,

$$\left(1+\frac{z}{x}\right)^{x}=\left(1+\frac{1}{y}\right)^{yz}=\left\{\left(1+\frac{1}{y}\right)^{y}\right\}^{z}\to e^{z}.$$

(3) *If* $a>0$, *then*

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\log a.$$

This is obviously true if $a=1$. If $a\neq 1$, let $a^{x}=1+y$, then as $x\to 0$, $y\to 0$ and

$$\lim_{x\to 0}\frac{x}{a^{x}-1}=\lim_{y\to 0}\frac{\log_a(1+y)}{y}=\lim_{y\to 0}\log_a(1+y)^{\frac{1}{y}}.$$

Now by Ch. XVII, 5, (2),

$$\lim_{y\to 0}\log_a(1+y)^{\frac{1}{y}}=\log_a\{\lim_{y\to 0}(1+y)^{\frac{1}{y}}\}=\log_a e=\frac{1}{\log a};$$

therefore

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\log a.$$

Here x may tend to zero through positive or negative values, so that *each of the functions*

$$\frac{a^{x}-1}{x} \quad \textit{and} \quad \frac{1-a^{-x}}{x}$$

tends to log a as a limit as $x\to 0$.

It has been shown in Ch. XIV, 9, that the first of these functions increases and the second decreases as x increases from zero. Hence

$$\textit{if } x>0, \textit{ then } \frac{1-a^{-x}}{x}<\log a<\frac{a^{x}-1}{x}.$$

3. The Exponential Theorem. *For all real values of x,*

$$e^{x}=1+x+\frac{x^{2}}{\lfloor 2}+\frac{x^{3}}{\lfloor 3}+\ldots+\frac{x^{n}}{\lfloor n}+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textbf{(A)}$$

Proof. The series (**A**) is convergent for all values of x, and its sum is a function of x which we shall denote by $E(x)$. If n is a positive integer,

$$\left(1+\frac{x}{n}\right)^{n}=1+n\cdot\frac{x}{n}+\frac{n(n-1)}{\lfloor 2}\left(\frac{x}{n}\right)^{2}+\frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{x}{n}\right)^{3}+\ldots \text{ to } n+1 \text{ terms}$$

$$=1+x+\frac{x^{2}}{\lfloor 2}p_1+\frac{x^{3}}{\lfloor 3}p_2+\ldots+\frac{x^{n}}{\lfloor n}p_{n-1},$$

where

$$p_r=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{r}{n}\right).$$

Now $p_r<1$, and since $1/n, 2/n, \ldots (n-1)/n$ are positive numbers less than unity, for the values $2, 3, \ldots, (n-1)$ of r we have, by Ch. XIV, 1,

$$p_r>1-\frac{1}{n}(1+2+3+\ldots+r)=1-\frac{r(r+1)}{2n};$$

$$\therefore\ 0<1-p_r<r(r+1)/2n.$$

Let s_n be the sum to n terms of (**A**), and let $|x|=x_1$; then

$$\left|s_{n+1}-\left(1+\frac{x}{n}\right)^n\right|\leqslant\frac{x_1^2}{\lfloor\underline{2}}(1-p_1)+\frac{x_1^3}{\lfloor\underline{3}}(1-p_2)+\ldots+\frac{x_1^n}{\lfloor\underline{n}}(1-p_{n-1})$$

$$<\frac{x_1^2}{2n}\left\{\frac{1.2}{\lfloor\underline{2}}+\frac{2.3}{\lfloor\underline{3}}x_1+\frac{3.4}{\lfloor\underline{4}}x_1^2+\ldots+\frac{(n-1)n}{\lfloor\underline{n}}x_1^{n-2}\right\}$$

$$<\frac{x_1^2}{2n}\left\{1+x_1+\frac{x_1^2}{\lfloor\underline{2}}+\frac{x_1^3}{\lfloor\underline{3}}+\ldots\text{to }\infty\right\}<\frac{x_1^2}{2n}\cdot E(x_1).$$

Now $E(x_1)$ is finite for all values of x_1, therefore

$$\lim_{n\to\infty}s_{n+1}=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x;$$

but
$$\lim_{n\to\infty}s_{n+1}=E(x);\quad\therefore\ e^x=E(x).$$

This result is called the *Exponential Theorem.*

In particular, when $x=1$ we have

$$e=1+1+\frac{1}{\lfloor\underline{2}}+\frac{1}{\lfloor\underline{3}}+\ldots+\frac{1}{\lfloor\underline{n}}+\ldots,$$

from which it can be shown that $e=2{\cdot}7182818284\ldots.$

Ex. 1. *Show that e is irrational.*

Suppose that $e=p/q$, where p and q are integers; then

$$\frac{p}{q}=1+1+\frac{1}{\lfloor\underline{2}}+\ldots+\frac{1}{\lfloor\underline{q}}+R, \qquad \ldots\ldots\ldots\

5. An Irrational Index. *When $a>0$ and x is irrational, we take the equation $a^x = E(x \log a)$ as defining the meaning of a^x.*

This agrees with what has been said in Ch. XIII, 9, for it will be shown later that the sum of the series denoted by $E(x)$ is a continuous function.

6. Derivatives of a^x, $\log x$ and x^n.

(1) *Suppose that* $a>0$, then by Art. 2, (3),

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} a^x \cdot \frac{a^h - 1}{h} = a^x \log a.$$

In particular, $$\frac{d}{dx}e^x = e^x.$$

NOTE. It should be observed that, if $a<0$, a^x has not been defined for all real values of x.

(2) If $y = \log x$, then $x = e^y$ and $\dfrac{dx}{dy} = e^y = x$,

therefore $\dfrac{dy}{dx} = \dfrac{1}{x}$; that is, $\dfrac{d}{dx}\log x = \dfrac{1}{x}$.

(3) It has been shown that when n is rational, $\dfrac{d}{dx}x^n = nx^{n-1}$. To prove that this is true for all real values of n, we proceed thus:

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\log x} = \frac{n}{x}e^{n\log x} = \frac{n}{x}x^n = nx^{n-1}.$$

NOTE. Since it has been shown in (1) that, for the curve $y=a^x$, the gradient dy/dx is $y \log a$, it follows that, if the tangent at a point P on the graph meets OX in T, and PN is the ordinate of P, then $TN = 1/\log a$: i.e. the *subtangent is constant*. This fact gives a rapid construction for the graph of $y=a^x$, with considerable accuracy.

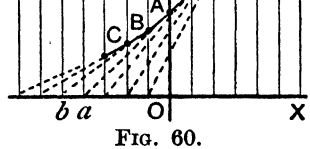


FIG. 60.

For instance, for $y=e^x$, where $TN=1$, take a sheet of ruled exercise paper and draw a line perpendicular to the ruling as the axis of x; and mark one of the rulings as the axis of y. On this take OA, equal to, say, 4 intervals, as the unit; and on OX, to the left of O, take $Oa=OA$. Draw Aa until it meets the first ruling to the left of OY at B. Without removing the pencil-point from B, rotate the ruler until it passes through b, the foot of the ruling next to the left of a, and draw bB until it meets the ruling next to the left of A; and so on. Find points on the curve to the right of OY in the same way, and replace the broken line by a 'fair curve.'

If, in Fig. 60, OX and OY are interchanged, the curve is then the graph of $y = \log x$.

Also, if the paper is turned through 90° in its own plane, and then turned over, the curve as seen through the back of the paper is the graph of $y = \log x$, with the axes in the usual position.

7. Inequalities and Limits (continued).

(1) *For all real values of* x *other than zero* $e^x > 1+x$.

For let $f(x) = e^x - 1 - x$, then $f(0) = 0$ and $f'(x) = e^x - 1$,

so that $f'(x) \gtrless 0$ according as $x \gtrless 0$. Hence, if x increases or decreases from zero, in either case $f(x)$ increases from zero, and is therefore positive.

(2) *If* $x > -1$ *and* $\neq 0$, *then* $x > \log(1+x)$.

This follows from (1) by taking logarithms.

(3) *If* $x > 0$, *then* $\frac{1}{2}x^2/(1+x) < x - \log(1+x) < \frac{1}{2}x^2$,

and if $-1 < x < 0$, *then* $\frac{1}{2}x^2 < x - \log(1+x) < \frac{1}{2}x^2/(1+x)$.

For let $f(x) = x - \log(1+x) - \frac{1}{2}x^2$, then

$$f'(x) = 1 - \frac{1}{1+x} - x = -\frac{x^2}{1+x} < 0.$$

Hence $f(x)$ decreases as x increases from -1, and since $f(0) = 0$, it follows that $f(x) \gtrless 0$ according as $x \lesseqgtr 0$.

Again, if $\phi(x) = x - \log(1+x) - \frac{1}{2}\dfrac{x^2}{1+x}$, then

$$\phi'(x) = 1 - \frac{1}{1+x} - \frac{x}{1+x} + \frac{1}{2}\frac{x^2}{(1+x)^2} = \frac{1}{2}\left(\frac{x}{1+x}\right)^2 > 0.$$

Hence $\phi(x)$ increases as x increases from -1, and since $\phi(0) = 0$, it follows that $\phi(x) \lesseqgtr 0$ according as $x \lesseqgtr 0$.

This proves all the inequalities in question.

(4) *If* $x > -1$ *and* $\neq 0$ *and* l, h *are respectively the smaller and the greater of the numbers* 1 *and* $1+x$, *then*

$$\frac{1}{2}\frac{x^2}{h} < x - \log(1+x) < \frac{1}{2}\frac{x^2}{l}.$$

This is merely another way of stating the inequalities in (3).

(5) *For any positive value of* n, (i) $\lim_{x\to\infty} \dfrac{\log x}{x^n} = 0$, (ii) $\lim_{x\to 0} (x^n \log x) = 0$.

(i) For all positive values of m we have by Art. 2, (3),

$$\frac{\log x}{x^n} < \frac{1}{x^n}\frac{x^m - 1}{m} < \frac{1}{mx^{n-m}}.$$

Choose m so that $0 < m < n$, then *as* $x \to \infty$,

$$x^{n-m} \to \infty \quad \text{and} \quad (\log x)/x^n \to 0.$$

(ii) Put $x = 1/y$, then as $x \to 0$, $y \to \infty$ and

$$x^n \log x = -(\log y)/y^n \to 0.$$

(6) *For all values of* r, $\lim_{x\to\infty} x^r/e^x = 0$.

This is obvious when $r<0$. If $r>0$, let $e^x=y$, then, putting $\frac{1}{r}=n$,

$$\frac{x^r}{e^x}=\frac{(\log y)^r}{y}=\left(\frac{\log y}{y^n}\right)^r.$$

Now as $x\to\infty$, $y\to\infty$; and, by (5), since $n=\frac{1}{r}>0$, $(\log y)/y^n\to 0$. Hence it follows that $x^r/e^x\to 0$.

8. The Manner in which e^x and $\log x$ tend to Infinity. Let n be any positive number.

First suppose that $x\to\infty$, then $e^x\to\infty$ and $\log x\to\infty$.

Now, by Art. 7, (6), $x^n/e^x\to 0$, therefore e^x *tends to infinity faster than* x^n, *however great* n *may be.*

Again, by Art. 7, (5), $(\log x)/x^n\to 0$, therefore $\log x$ *tends to infinity more slowly than* x^n, *no matter how small* n *may be.*

If $x\to 0$, then $-\log x\to\infty$, and by Art. 7, (5), $(\log x)/x^{-n}\to 0$; hence $-\log x$ *tends to* ∞ *more slowly than* $1/x^n$, *no matter how small* n *may be.*

9. Theorem. *If* $u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$, *then as* $n\to\infty$, $u_n\to\gamma$ *where* γ *is a fixed number lying between* 0 *and* 1.

Proof. By Art. 7, (2),

$$u_n-u_{n-1}=\frac{1}{n}-\log\frac{n}{n-1}=\frac{1}{n}+\log\left(1-\frac{1}{n}\right)<0,$$

so that u_n decreases as n increases. Again, by Art. 7, (2),

$$\log\left(1+\frac{1}{n}\right)<\frac{1}{n}<-\log\left(1-\frac{1}{n}\right), \quad \text{that is,} \quad \log\frac{n+1}{n}<\frac{1}{n}<\log\frac{n}{n-1}.$$

Substituting $n-1, n-2, \ldots 2$, in succession for n, we have

$$\log\frac{n}{n-1}<\frac{1}{n-1}<\log\frac{n-1}{n-2}, \quad \ldots, \quad \log\tfrac{3}{2}<\tfrac{1}{2}<\log\tfrac{2}{1}.$$

Also $\log\frac{2}{1}<1$; and by addition, it follows that

$$\log(n+1)<1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}<1+\log n; \text{ i.e., } \log(n+1)-\log n<u_n<1.$$

But $\log(n+1)-\log n>0$, hence $0<u_n<1$; and, from above, $u_n<u_{n-1}$.

Thus u_n is a decreasing function which is always >0. Consequently $u_n\to\gamma$ where γ is a fixed number >0; and, since $u_n<1$, $\gamma<1$.

NOTE. The number γ is known as *Euler's Constant.* It can be shown that $\gamma=0{\cdot}577215\ldots$. It is often convenient to state the theorem as follows:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}=\gamma+\log n+\epsilon_n, \quad \textit{where } \epsilon_n\to 0 \textit{ as } n\to\infty.$$

10. Series for log 2. Let s_r be the sum to r terms of the series,

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots+(-1)^{n-1}\frac{1}{n}+\ldots \text{ to } \infty .$$

then
$$s_{2n}=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$$
$$=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$
$$=(\gamma+\log 2n+\epsilon_{2n})-(\gamma+\log n+\epsilon_n)=\log 2+\epsilon_{2n}-\epsilon_n.$$

Therefore $s_{2n}\to\log 2$. Also $s_{2n+1}-s_{2n}\to 0$, hence $s_{2n+1}\to\log 2$.

Thus,
$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots+(-1)^{n-1}\frac{1}{n}+\ldots \text{ to } \infty .$$

11. The Exponential Function. (1) For all values of z, real or complex, the series $1+z+\dfrac{z^2}{\lfloor 2}+\dfrac{z^3}{\lfloor 3}+\ldots$ is convergent. Its sum is called the *Exponential Function*, and is denoted by $E(z)$, or by $\exp z$.

(2) **Theorem.** *If $z=x+\iota y$ where x and y are real,*

$$E(z)=\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^x(\cos y+\iota\sin y).$$

Proof. It has been shown in Art. 3 that the first of these equalities holds when z is real, and exactly the same proof holds when z is complex.

Let $1+\dfrac{z}{n}=1+\dfrac{x}{n}+\iota\dfrac{y}{n}=\rho(\cos\phi+\iota\sin\phi)$, where $\rho>0$ and $-\pi<\phi\leqslant\pi$.

For sufficiently large values of n, $1+\dfrac{x}{n}>0$; hence, since $\rho>0$, and $\rho\cos\phi=1+\dfrac{x}{n}$, we have $\cos\phi>0$, and $-\tfrac{1}{2}\pi<\phi<\tfrac{1}{2}\pi$.

Also $\tan\phi=y/(n+x)\to 0$, therefore $\phi\to 0$, and $\phi/\tan\phi\to 1$; hence
$$\lim n\phi=\lim(n\tan\phi)=\lim\{ny/(n+x)\}=y.$$

Now $\rho^2=\left(1+\dfrac{x}{n}\right)^2+\dfrac{y^2}{n^2}=\left(1+\dfrac{x}{n}\right)^2\sec^2\phi$; therefore $\rho^n=\left(1+\dfrac{x}{n}\right)^n\sec^n\phi$;

and, since $\lim\sec^n\phi=\lim(1+\tan^2\phi)^{\frac{1}{2}n}=\lim\left(1+\dfrac{y^2}{n^2}\right)^{\frac{1}{2}n}=\lim(e^{y^2})^{\frac{1}{2n}}=1$,

we have $\lim\rho^n=e^x$; hence, $E(z)=\lim\rho^n(\cos n\phi+\iota\sin n\phi)$

and
$$E(z)=e^x(\cos y+\iota\sin y).$$

(3) It follows that $E(z)$ *cannot vanish for any value of z,* for neither e^x nor $\cos y+\iota\sin y$ can vanish.

(4) *The function $E(z)$ is periodic, and its period is $2\iota\pi$.*

For if k is any integer, positive or negative, the values of $\cos y$ and $\sin y$ are unaltered by adding $2k\pi$ to y. Therefore $E(z)$ is unaltered by adding $2\iota k\pi$ to z, thus $E(z+2\iota k\pi)=E(z)$.

12. A Complex Index. When z is real, e^z is a multiple-valued function, except when z is an integer. The real positive value of e^z is called its principal value, and it has been shown that

$$E(z)=\text{the principal value of } e^z.$$

We take this equation as defining the principal value of e^z when z is complex.

This implies that for imaginary (as for real) values of z, e^z is to be regarded as a multiple-valued function.

For the present, we take e^z as denoting its principal value defined by $E(z)$.

Thus if $z=x+\iota y$, and $z'=x'+\iota y'$, we write

$$e^z=E(z)=e^x(\cos y+\iota\sin y), \quad \text{and} \quad e^{z'}=e^{x'}(\cos y'+\iota\sin y');$$

thus,
$$\begin{aligned} e^z\,.\,e^{z'} &= e^x\,.\,e^{x'}\,(\cos y+\iota\sin y)(\cos y'+\iota\sin y') \\ &= e^{x+x'}\{\cos(y+y')+\iota\sin(y+y')\}=e^{z+z'}. \end{aligned}$$

Hence the same index laws hold for imaginary as for real indices.

Again, if a is real and positive and z is real, $a^z=e^{z\log a}=E(z\log a)$.

In agreement with this, *if $a>0$, we define the principal value of a^z as $E(z\log a)$ and, at present, we take a^z as representing this value.*

13. Series for Sine and Cosine. For all real values of x and y, it has been shown that $E(x+\iota y)=e^x(\cos y+\iota\sin y)$. When $x=0$, this gives

$$\begin{aligned} \cos y+\iota\sin y &= E(\iota y) \\ &= 1+\iota y+\frac{\iota^2y^2}{\lfloor 2}+\frac{\iota^3y^3}{\lfloor 3}+\dots \\ &= \left(1-\frac{y^2}{\lfloor 2}+\frac{y^4}{\lfloor 4}-\dots\right)+\iota\left(y-\frac{y^3}{\lfloor 3}+\frac{y^5}{\lfloor 5}-\dots\right). \end{aligned}$$

Equating real and imaginary parts, we have, for all real values of y,

$$\cos y=1-\frac{y^2}{\lfloor 2}+\frac{y^4}{\lfloor 4}-\dots, \quad \sin y=y-\frac{y^3}{\lfloor 3}+\frac{y^5}{\lfloor 5}-\dots.$$

14. Exponential Values of Sine and Cosine. Using e^z to mean exactly the same as $E(z)$, by the preceding,

$$\cos y+\iota\sin y=E(\iota y)=e^{\iota y}, \quad \cos y-\iota\sin y=E(-\iota y)=e^{-\iota y};$$

therefore
$$\cos y=\tfrac{1}{2}(e^{\iota y}+e^{-\iota y}), \quad \sin y=\frac{1}{2\iota}(e^{\iota y}-e^{-\iota y}).$$

15. Series which can be summed by the Exponential Theorem.

(1) *The series* $\Sigma u_n \cdot \frac{x^n}{\lfloor n}$, *where* u_n *is a polynomial in* n.

We can find $a_0, a_1, \ldots a_r$, independent of n, so that

$$u_n = a_0 + a_1 n + a_2 n(n-1) + \ldots + a_r n(n-1)\ldots(n-r+1),$$

and then

$$\Sigma_{n=0}^{\infty} u_n \cdot \frac{x^n}{\lfloor n} = a_0 \Sigma_{n=0}^{\infty} \frac{x^n}{\lfloor n} + a_1 x \Sigma_{n=1}^{\infty} \frac{x^{n-1}}{\lfloor n-1} + a_2 x^2 \Sigma_{n=2}^{\infty} \frac{x^{n-2}}{\lfloor n-2}$$

$$+ \ldots + a_r x^r \Sigma_{n=r}^{\infty} \frac{x^{n-r}}{\lfloor n-r}$$

$$= (a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r) e^x.$$

Ex. 1. *Find the value of* $\Sigma(n^3+5)/\lfloor n$.

Here $n^3+5 = n(n-1)(n-2) + 3n(n-1) + n + 5$; hence, putting $a_0 = 5$, $a_1 = 1$, $a_2 = 3$, $a_3 = 1$, and $x = 1$ in the result above, we have

$$\Sigma_{n=0}^{\infty} (n^3+5) = (5+1+3+1)e = 10e.$$

(2) *The series* $\Sigma u_n x^n / (n+a)(n+b)\ldots(n+k) \lfloor n$ *where* u_n *is a polynomial in* n *and* $a, b, \ldots k$ *are unequal positive integers.*

Series of this kind can be summed as in the following example.

Ex. 2. *Find the value of*

$$\Sigma_{n=1}^{n=\infty} \frac{n^2+1}{n+2} \cdot \frac{x^n}{\lfloor n}.$$

Denote the sum by S, then

$$S = \Sigma_{n=1}^{n=\infty} \frac{(n^2+1)(n+1)}{\lfloor n+2} \cdot x^n.$$

Now $(n^2+1)(n+1) = n^3 + n^2 + n + 1$

$$= a + b(n+2) + c(n+2)(n+1) + d(n+2)(n+1)n,$$

where a, b, c, d can be found by division, their values being $a = -5$, $b = 5$, $c = -2$, $d = 1$. Thus

$$S = -5\Sigma_1^{\infty} \frac{x^n}{\lfloor n+2} + 5\Sigma_1^{\infty} \frac{x^n}{\lfloor n+1} - 2\Sigma \frac{x^n}{\lfloor n} + \Sigma \frac{x^n}{\lfloor n-1}.$$

Now $$\Sigma_1^{\infty} \frac{x^n}{\lfloor n+2} = \frac{1}{x^2}\left(e^x - 1 - x - \frac{x^2}{2}\right),$$

$$\Sigma_1^{\infty} \frac{x^n}{\lfloor n+1} = \frac{1}{x}(e^x - 1 - x), \quad \Sigma_1^{\infty} \frac{x^n}{\lfloor n} = e^x - 1,$$

$$\Sigma_1^{\infty} \frac{x^n}{\lfloor n-1} = x e^x$$

$$\therefore\ S = \frac{1}{2x^2}(10 - x^2) - \frac{e^x}{x^2}(5 - 5x + 2x^2 - x^3).$$

EXERCISE XXXI

1. Show that $\lim_{x\to\infty}\left(1-\frac{k}{x^2}\right)^x=1$.

Find the sum to infinity of the series in Exx. 2-9:

2. $1+\frac{2}{\lfloor 1}+\frac{3}{\lfloor 2}+\frac{4}{\lfloor 3}+\dots$.

3. $\frac{3}{\lfloor 2}+\frac{5}{\lfloor 3}+\frac{7}{\lfloor 4}+\frac{9}{\lfloor 5}+\dots$.

4. $\frac{3}{\lfloor 2}+\frac{5}{\lfloor 4}+\frac{7}{\lfloor 6}+\frac{9}{\lfloor 8}+\dots$.

5. $1+\frac{1+2}{\lfloor 2}+\frac{1+2+3}{\lfloor 3}+\dots$.

6. $\frac{1}{2}-\frac{1}{3\lfloor 1}+\frac{1}{4\lfloor 2}-\frac{1}{5\lfloor 3}+\dots$.

7. $1^3+\frac{2^3}{\lfloor 1}+\frac{3^3}{\lfloor 2}+\frac{4^3}{\lfloor 3}+\dots$.

8. $1^3-\frac{2^3}{\lfloor 1}+\frac{3^3}{\lfloor 2}-\frac{4^3}{\lfloor 3}+\dots$.

9. $1+\frac{x}{\lfloor 2}+\frac{2x^2}{\lfloor 3}+\frac{3x^3}{\lfloor 4}+\dots$.

Prove that:

10. $\Sigma_1^\infty\frac{n-1}{(n+2)\lfloor n}x^n=\frac{1}{x^2}\{(x^2-3x+3)e^x+\frac{1}{2}x^2-3\}$.

11. $\Sigma_1^\infty\frac{2n-1}{(n+3)\lfloor n}=\frac{1}{3}(43-15e)$.

12. $\Sigma_0^\infty\frac{(n+1)^3}{\lfloor n}x^n=(1+7x+6x^2+x^3)e^x$.

13. If $e^{ax}\cos bx=u_0+u_1x+\dots+u_nx^n+\dots$, then $u_n=(a^2+b^2)^{\frac{n}{2}}\frac{\cos n\theta}{\lfloor n}$, where $\theta=\tan^{-1}\frac{b}{a}$.

16. Theorem. *If* $-1<x\leqslant 1$, *then*

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n-1}\frac{x^n}{n}+\dots.$$

Proof. (i) *Suppose that* $0<x<1$. Let s_n be the sum to n terms of the series, then

$$\frac{ds_n}{dx}=1-x+x^2-\dots+(-1)^{n-1}x^{n-1}=\frac{1-(-x)^n}{1+x}.$$

Let $y_n=\log(1+x)-s_n$; then $y_n=0$ when $x=0$, and

$$\frac{dy_n}{dx}=\frac{1}{1+x}-\frac{1-(-x)^n}{1+x}=\frac{(-x)^n}{1+x}.$$

Therefore $y_n\gtrless 0$, according as n is even or odd.

Hence $\log(1+x)$ lies between s_n and s_{n+1} for every n.

But $s_{n+1}-s_n=(-1)^nx^{n+1}/(n+1)\to 0$ as $n\to\infty$,

therefore $s_n\to\log(1+x)$ as $n\to\infty$,

which proves this case of the theorem. (*Cf.* Ex. XXIX, 8.)

(ii) The case in which $x=1$ has been dealt with in Art. 10.

(iii) *When* $x<0$, changing the sign of x, it has to be shown that

$$\textit{if } 0<x<1, \textit{ then } \quad -\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots .$$

Let $y_n=-\log(1-x)-s_n$, where s_n is the sum to n terms of this series; then $y_n=0$ when $x=0$, and $\dfrac{dy_n}{dx}=\dfrac{1}{1-x}-\dfrac{1-x^n}{1-x}=\dfrac{x^n}{1-x}>0$; therefore $y_n>0$ for every n.

Let $z_n=y_n-\dfrac{x^{n+1}}{n+1}\cdot\dfrac{1}{1-x}$; then $z_n=0$ when $x=0$, and

$$\frac{dz_n}{dx}=\frac{x^n}{1-x}-\left\{\frac{x^n}{1-x}+\frac{1}{n+1}\cdot\frac{x^{n+1}}{(1-x)^2}\right\}=-\frac{1}{n+1}\cdot\frac{x^{n+1}}{(1-x)^2}<0;$$

therefore $z_n<0$. Hence $0<y_n<\dfrac{1}{n+1}\cdot\dfrac{x^{n+1}}{1-x}$; and, since $0<x<1$, the last expression $\to 0$ as $n\to\infty$; thus $y_n\to 0$ and $s_n\to -\log(1-x)$, which proves the theorem in this case.

17. Logarithmic Series. If $-1<x<1$, we have

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots . \quad \ldots\ldots\ldots\ldots\textbf{(A)}$$

Changing the sign of x,

$$\log(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots\right), \quad \ldots\ldots\ldots\ldots\textbf{(B)}$$

and since $\log\dfrac{1+x}{1-x}=\log(1+x)-\log(1-x)$, it follows that

$$\log\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right). \quad \ldots\ldots\ldots\ldots\textbf{(C)}$$

Writing $\dfrac{1+x}{1-x}=\dfrac{n+1}{n}$, so that $x=\dfrac{1}{2n+1}$, we have

$$\log(n+1)-\log n=2\left(\frac{1}{2n+1}+\frac{1}{3}\cdot\frac{1}{(2n+1)^3}+\frac{1}{5}\cdot\frac{1}{(2n+1)^5}+\ldots.\right). \quad \ldots\textbf{(D)}$$

Ex. 1. *If* $0<x<1$ *and* $s_n=x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots$ *to* n *terms, show that the error in taking* $2s_n$ *as the value of* $\log\dfrac{1+x}{1-x}$ *is less than* $\dfrac{2x^{2n+1}}{2n+1}\cdot\dfrac{1}{1-x^2}$.

If R_n is the remainder after n terms of the series,

$$R_n=\frac{x^{2n+1}}{2n+1}+\frac{x^{2n+3}}{2n+3}+\frac{x^{2n+5}}{2n+5}+\ldots<\frac{x^{2n+1}}{2n+1}(1+x^2+x^4+\ldots);$$

$$\therefore \text{ the error}=2R_n<\frac{2x^{2n+1}}{2n+1}\cdot\frac{1}{1-x^2}.$$

18. Series which can be summed by the Logarithmic Series.

Ex. 1. *If k is a positive integer and $|x|<1$, then*

$$\Sigma_{n=1}^{n=\infty}\frac{x^n}{n+k}=-\frac{1}{x^k}\left\{x+\frac{x^2}{2}+\ldots+\frac{x^k}{k}+\log(1-x)\right\}.$$

Ex. 2. *If $|x|<1$, find the value of* $\Sigma_{n=1}^{n=\infty}\frac{n^3+n^2+1}{n(n+2)}x^n$.

The series is convergent when $|x|<1$. Let S be the sum, then since

$$\frac{n^3+n^2+1}{n(n+2)}=n-1+\frac{1}{2}\cdot\frac{1}{n}+\frac{3}{2}\cdot\frac{1}{n+2},$$

we have

$$S=\Sigma_1^\infty(n-1)x^n+\frac{1}{2}\Sigma_1^\infty\frac{x^n}{n}+\frac{3}{2}\frac{x^n}{n+2};$$

$$\therefore\ S=\frac{x^2}{(1-x)^2}-\frac{1}{2}\log(1-x)-\frac{3}{2x^2}\left\{x+\frac{x^2}{2}+\log(1-x)\right\}.$$

The method of Ex. 2 *can be applied to sum any series of which the general term is*

$$Px^n/(n+a)(n+b)\ldots(n+k),$$

where P is a polynomial in n and $a, b, \ldots k$ are unequal positive integers.

19. Calculation of Napierian Logarithms.

The number e is chosen as the base of the system of logarithms used in theoretical work. Such logarithms are called *Napierian*, after Napier, the inventor of logarithms. In theoretical work, $\log N$ means $\log_e N$; just as, in practical reckoning, $\log N$ means $\log_{10} N$.

The method of applying the equations of Art. 17 to the calculation of Napierian logarithms is exhibited in the following example.

Ex. 1. *Calculate $\log_e 2$ to seven places of decimals.*

Let $\frac{1+x}{1-x}=2$; $\therefore\ x=\frac{1}{3}$; $\therefore\ \log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\ldots\right\}$.

Carrying the reckoning to nine places, we have

$1/3$	$=0{\cdot}333$	333	333	$1/3$	$=0{\cdot}333$	333	333
$1/3^3$	$=0{\cdot}037$	037	037	$1/(3\,.\,3^3)$	$=0{\cdot}012$	345	679
$1/3^5$	$=0{\cdot}004$	115	226	$1/(5\,.\,3^5)$	$=0{\cdot}000$	823	045
$1/3^7$	$=0{\cdot}000$	457	247	$1/(7\,.\,3^7)$	$=0{\cdot}000$	065	321
$1/3^9$	$=0{\cdot}000$	050	805	$1/(9\,.\,3^9)$	$=0{\cdot}000$	005	645
$1/3^{11}$	$=0{\cdot}000$	005	645	$1/(11\,.\,3^{11})$	$=0{\cdot}000$	000	513
$1/3^{13}$	$=0{\cdot}000$	000	627	$1/(13\,.\,3^{13})$	$=0{\cdot}000$	000	048
$1/3^{15}$	$=0{\cdot}000$	000	070	$1/(15\,.\,3^{15})$	$=0{\cdot}000$	000	005
					$=0{\cdot}346$	573	589
							2
					$0{\cdot}693$	147	178

$\therefore\ \log_e 2=0{\cdot}693147178$ *nearly.*

The error in the value obtained for $\log_e 2$ is due to two causes:

(i) We have taken $2S_8$ for $\log_e 2$. By Ex. 1 of Art. 17, the error on this account is less than

$$\frac{2(\frac{1}{3})^{17}}{17} \cdot \frac{1}{1-\frac{1}{9}} < 0{\cdot}000\ 000\ 002.$$

(ii) The error in the *calculated* value of $2S_8$ is numerically less than

$$2 \times 8 \times 0{\cdot}000\ 000\ 000\ 5 = 0{\cdot}000\ 000\ 008.$$

Adding the results of (i) and (ii), we see that 0·693 147 178 is an approximate value of $\log_e 2$ with an error numerically less than 0·000 000 01.

$$\therefore\ 0{\cdot}693\ 147\ 168 < \log_e 2 < 0{\cdot}693\ 147\ 188.$$

Hence, to seven figures, $\log_e 2 = 0{\cdot}6931472$.

Having found $\log_e 2$, the Napierian logarithms of the natural numbers can be found in succession by using formula of (D) Art. 17.

The logarithms of successive prime numbers may be calculated by the following method, due to Borda, in which the series employed converge very rapidly. Using the identities

$$x^3 - 3x + 2 = (x-1)^2(x+2) \quad \text{and} \quad x^3 - 3x - 2 = (x+1)^2(x-2)$$

it will be seen that

$$\frac{(x-1)^2(x+2)}{(x+1)^2(x-2)} = \frac{x^3-3x+2}{x^3-3x-2} = \frac{1+2/(x^3-3x)}{1-2/(x^3-3x)};$$

$$\therefore\ \log(x-1) + \tfrac{1}{2}\log(x+2) - \log(x+1) - \tfrac{1}{2}\log(x-2)$$
$$= \frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5 + \dots.$$

Putting $x = 5, 6, 7, 8$, it will be found that

$$\log 2 - \tfrac{3}{2}\log 3 + \tfrac{1}{2}\log 7 = 0{\cdot}0181838220\ \dots,$$
$$\tfrac{1}{2}\log 2 + \log 5 - \log 7 = 0{\cdot}0101013536\ \dots,$$
$$-2\log 2 + 2\log 3 - \tfrac{1}{2}\log 5 = 0{\cdot}0062112599\ \dots,$$
$$-\tfrac{5}{2}\log 3 + \tfrac{1}{2}\log 5 + \log 7 = 0{\cdot}0040983836\ \dots.$$

Hence it can be shown that

$$\log 2 = 0{\cdot}693147180\ \dots \quad \log 3 = 1{\cdot}098612288\ \dots,$$
$$\log 5 = 1{\cdot}609437912\ \dots \quad \log 7 = 1{\cdot}945910149\ \dots.$$

The logarithms of successive primes 11, 13, etc., can now be obtained by putting $x+2 = 11$, 13, etc., in the formula.

The method of the example which follows, in which logarithms to a base 10 are calculated without the use of the logarithmic series, is of historical interest, being one of the methods given by Briggs.

Ex. 2. *Obtain the logarithm of* 7 *to the base* 10, *by the use of square-roots only, to* 5 *decimal places.*

In the reckoning below, when a square-root has been obtained which is greater than 7, the square-root of the product of it and the last square-root obtained that was less than 7 is found next; and *vice versa*. Thus we have

$$\log\sqrt{10} = \log 3{\cdot}162277 \ldots = \log A = 0{\cdot}5,$$
$$\log\sqrt{10A} = \log 5{\cdot}623413 \ldots = \log B = 0{\cdot}75,$$
$$\log\sqrt{10B} = \log 7{\cdot}498942 \ldots = \log C = 0{\cdot}875,$$
$$\log\sqrt{BC} = \log 6{\cdot}493816 \ldots = \log D = 0{\cdot}8125,$$
$$\log\sqrt{CD} = \log 6{\cdot}978305 \ldots = \log E = 0{\cdot}84375,$$
$$\log\sqrt{CE} = \log 7{\cdot}233941 \ldots = \log F = 0{\cdot}859375,$$

and so on; until we arrive at

$$\log 7{\cdot}000032 = 0{\cdot}845100 \ldots,$$
$$\log 6{\cdot}999968 = 0{\cdot}845096 \ldots;$$

thus, $\log_{10} 7 = 0{\cdot}84510$ to five places, and probably is equal to 0·845098 to six places.

20. Calculation of Common Logarithms.

We have

$$\log_{10} N = \log_e N / \log_e 10.$$

The expression $1/(\log_e 10)$ is called the *modulus* of the common system of logarithms and is denoted by μ. It can be shown that, to fifteen places of decimals,

$$\mu = 0{\cdot}434294481903251.$$

Thus $\log_{10} N = \mu \log_e N$, *where* μ *has the above value.*

21. The Hyperbolic Functions.

(1) The functions known as the hyperbolic cosine, sine and tangent of x, and denoted by $\cosh x$, $\sinh x$ and $\tanh x$, are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The reciprocals of $\cosh x$, $\sinh x$, $\tanh x$ are called the hyperbolic secant, cosecant and cotangent of x, and we write

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}, \quad \coth x = \frac{1}{\tanh x}.$$

(2) Hence the following equations:

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x,$$

$$\frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

Also, $\cosh(-x)=\cosh x$, $\sinh(-x)=-\sinh x$, $\tanh(-x)=-\tanh x$, so that $\cosh x$ is an *even* function, and $\sinh x$, $\tanh x$ are odd functions of x. Graphs of these functions are shown below : rough sketches of the first two can be constructed quickly from the curves $y=e^x$, $y=e^{-x}$ (see p. 309), by observing that, in Fig. 61, C is the middle point of AB, and $ED=BC$. The ordinate of $y=\tanh x$ is equal to the ratio ED/EC; this can be found numerically or by a geometrical construction.

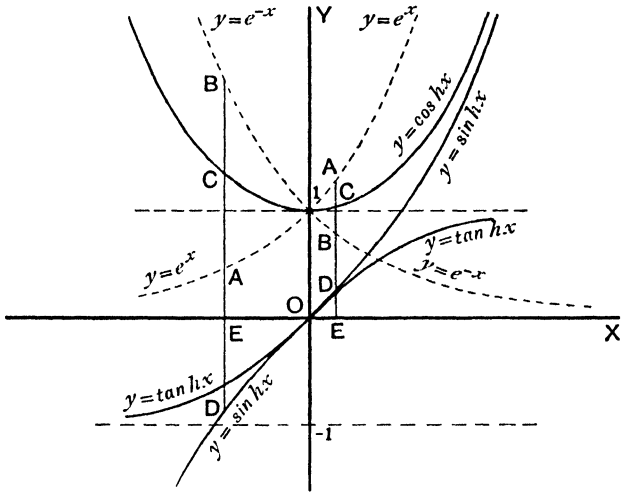


FIG. 61.

(3) *Addition Formulae.* We have

$$\begin{aligned}2\cosh x\cosh y &= \tfrac{1}{2}(e^x+e^{-x})(e^y+e^{-y})\\ &= \tfrac{1}{2}(e^{x+y}+e^{-(x+y)}+e^{x-y}+e^{-(x-y)})\\ &= \cosh(x+y)+\cosh(x-y).\end{aligned}$$

Similarly,

$$2\sinh x\sinh y=\cosh(x+y)-\cosh(x-y),$$
$$2\sinh x\cosh y=\sinh(x+y)+\sinh(x-y),$$
$$2\cosh x\sinh y=\sinh(x+y)-\sinh(x-y).$$

Therefore

$$\cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y,$$
$$\sinh(x\pm y)=\sinh x\cosh y\pm\cosh x\sinh y,$$

and consequently

$$\tanh(x\pm y)=\frac{\tanh x\pm\tanh y}{1\pm\tanh x\tanh y},$$

where in each equation the upper or lower sign is to be taken throughout.

(4) Using the exponential values of $\sin x$ and $\cos x$ (Art. 14), we have

$$\cos x = \frac{e^{\iota x} + e^{-\iota x}}{2} = \cosh \iota x,$$

$$\sin x = \frac{e^{\iota x} - e^{-\iota x}}{2\iota} = \frac{1}{\iota} \sinh \iota x,$$

$$\tan x = \frac{1}{\iota} \tanh \iota x,$$

and $\cosh x = \cos \iota x$, $\sinh x = \iota \sin \iota x$, $\tanh x = \iota \tan \iota x$.

22. Inverse Hyperbolic Functions. Let $y = \cosh x$, then

$$e^x + e^{-x} = 2y \quad \text{and} \quad e^{2x} - 2e^x y + 1 = 0,$$

therefore $e^x = y \pm \sqrt{(y^2 - 1)}$ and $x = \log \{y \pm \sqrt{(y^2 - 1)}\}$.

Thus if $y > 1$, there are two real values of x, equal in magnitude and of opposite signs, which correspond to any given value of y.

The *positive* value is denoted by $\cosh^{-1} y$, so that

$$\cosh^{-1} y = \log \{y + \sqrt{(y^2 - 1)}\}.$$

Next, let $y = \sinh x$, then $e^{2x} - 2e^x y - 1 = 0$, and since $e^x > 0$ for real values of x, we have $e^x = y + \sqrt{(y^2 + 1)}$, so that $x = \log \{y + \sqrt{(y^2 + 1)}\}$, and we write

$$\sinh^{-1} y = \log \{y + \sqrt{y^2 + 1}\}.$$

Again, if $$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

then $$e^{2x} = \frac{1 + y}{1 - y} \quad \text{and} \quad x = \tfrac{1}{2} \log \frac{1 + y}{1 - y}.$$

Hence we write

$$\tanh^{-1} y = \tfrac{1}{2} \log \frac{1 + y}{1 - y}.$$

Thus $\cosh^{-1} y$ is a single-valued and continuous function of y (see Ch. XVII, 21) for all values of $y > 1$; $\sinh^{-1} y$ is a single-valued and continuous function of y for all values of y; $\tanh^{-1} y$ is a single-valued and continuous function of y over the range $-1 < y < 1$.

EXERCISE XXXII

The notation of Art. 9 is used in Exx. 2 and 3.

1. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} \to \log 2$ as $n \to \infty$.

2. Prove that $1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} = \log 2 + \frac{1}{2}\gamma + \frac{1}{2} \log n + \epsilon_{2n} - \epsilon_n$, so that $1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} - \frac{1}{2} \log n \to \log 2 + \frac{1}{2}\gamma$.

3. Let the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ be deranged as follows. Write down the first p positive terms followed by the first q negative terms, then the next p positive terms followed by the next q negative terms, and so on.

Prove that the series so deranged is convergent, and that its sum is

$$\log 2+\tfrac{1}{2}\log p/q.$$

[Let σ_r be the sum of the first r groups of p positive followed by q negative terms, then

$$\sigma_r=\left(1+\frac{1}{3}+\ldots+\frac{1}{2pr-1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{rq}\right)$$

$$=(\log 2+\tfrac{1}{2}\gamma+\tfrac{1}{2}\log pr+\epsilon_{2pr}-\epsilon_{pr})-\tfrac{1}{2}(\gamma+\log qr+\epsilon_{qr})\to\log 2+\tfrac{1}{2}\log p/q.$$

Also, if s_n is the sum to n terms of the deranged series, r can be found so that all these terms are contained in σ_r, but not in σ_{r-1}. Thus σ_r-s_n is the sum of a finite number of terms each of which $\to 0$. Also $r\to\infty$ as $n\to\infty$, and consequently $s_n\to\log 2+\frac{1}{2}\log p/q$.]

4. Use the series for $\log(1+x)$ to show that:

(i) if $0<x<1$, then $0<x-\log(1+x)<\frac{1}{2}x^2$;

(ii) if $-1<x<0$, then $0<x-\log(1+x)<\frac{1}{2}x^2/(1+x)$.

[For (ii) let $x=-y$, then $0<y<1$ and

$$-\log(1-y)=y+\tfrac{1}{2}y^2+\tfrac{1}{3}y^3+\ldots<y+\tfrac{1}{2}y^2(1+y+y^2+\ldots),\text{ etc.}]$$

5. (i) Show that $\log 50-\log 49=\mu\left\{\frac{1}{50}+\frac{1}{2}\cdot\frac{1}{50^2}+\frac{1}{3}\cdot\frac{1}{50^3}+\ldots\right\}$.

(ii) Hence calculate $\log_{10}7$ to five places of decimals, given that

$$\log_{10}2=0{\cdot}301030,\quad \mu=0{\cdot}434294.$$

6. (i) Show that $\log_{10}1{\cdot}1=2\mu\left\{\frac{1}{21}+\frac{1}{3}\cdot\frac{1}{21^3}+\frac{1}{5}\cdot\frac{1}{21^5}+\ldots\right\}$.

(ii) Given $\mu=0{\cdot}434294$, find $\log_{10}11$ to five places of decimals.

7. (i) Show that $\log_{10}\frac{13}{12}=2\mu\left\{\frac{1}{25}+\frac{1}{3}\cdot\frac{1}{25^3}+\frac{1}{5}\cdot\frac{1}{25^5}+\ldots\right\}$.

(ii) Given that $\log_{10}2=0{\cdot}301030$, $\log_{10}3=0{\cdot}477121$, $\mu=0{\cdot}434294$, find $\log_{10}13$ to five places of decimals.

8. (i) Show that $\log_e(1+x+x^2)=\log_e(1-x^3)-\log_e(1-x)$.

(ii) Expand $\log_e(1+x+x^2)$ to six terms, and show that the coefficient of x^n in the expansion is $-2/n$ or $1/n$ according as n is or is not a multiple of 3.

9. Find the coefficients of x^{4m} and x^{4m+1} in the expansion of

$$\log_e(1+x+x^2+x^3).$$

Sum to infinity the series in Exx. 10-14.

10. $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots$.

11. $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\ldots$.

12. $\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\ldots$.

13. $\frac{1}{1.2}+\frac{x}{3.4}+\frac{x^2}{5.6}+\frac{x^3}{7.8}+\ldots$.

14. $1+\frac{x^2}{3}+\frac{x^4}{15}+\ldots+\frac{x^{2n}}{4n^2-1}+\ldots$.

15. In the expansion of $(1+bx+cx^2)\log_e(1+x)$,

(i) Find the coefficients of x^3 and x^4, and determine the values of b and c so that each of these coefficients may vanish.

(ii) Giving to b and c the values so determined, expand the given expression as far as the term containing x^5.

(iii) Hence find an expression of the form $\dfrac{x+ax^2}{1+bx+cx^2}$ which will give the best possible approximate values of $\log_e(1+x)$ when x is small.

16. (i) Prove that $(6+6x+x^2)\log_e(1+x)-(6x+3x^2)$

$$=x^5\left\{\frac{1\cdot2}{3\cdot4\cdot5}-\frac{2\cdot3}{4\cdot5\cdot6}x+\frac{3\cdot4}{5\cdot6\cdot7}x^2-\dots\right\},$$

(ii) Denoting the series in brackets by $u_1-u_2+u_3-\dots$, prove that

$$\frac{u_n}{u_{n-1}}=\frac{(n+1)^2}{(n-1)(n+4)}x.$$

(iii) If $0<x<\frac{2}{3}$, show that $u_n<u_{n-1}$.

(iv) Hence show that *if* $0<x<\dfrac{2}{3}$, *then* $\dfrac{6x+3x^2}{6+6x+x^2}$ *is an approximate value of* $\log_e(1+x)$, *with an error in defect less than* $x^5/180$.

17. Use Ex. 16 (iv) to show that $\dfrac{759}{32003}$ is an approximate value of $\log_e 1{\cdot}024$, with an error in defect less than $5/10^{11}$. Hence find $\log_{10}1{\cdot}024$ and $\log_{10}2$, each to ten places of decimals.

18. If $a-b=h$, where h is small, show that

(i) $$\frac{a-b}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{h}{a}+\frac{h^2}{2a^2}+\frac{h^3}{2a^3}+\dots.$$

(ii) $$2\left(\frac{a-b}{a+b}\right)+\frac{2}{3}\left(\frac{a-b}{a+b}\right)^3=\frac{h}{a}+\frac{h^2}{2a^2}+\frac{h^3}{3a^3}+\dots.$$

(iii)* Hence show that, if a is nearly equal to b,

$$\log_e\frac{a}{b}=\frac{a-b}{2}\left(\frac{1}{a}+\frac{1}{b}\right),\ \textit{nearly},$$

the error being approximately equal to $(a-b)^3/6a^3$.

Sum the series in Exx. 19, 20:

19. $\Sigma_{n=1}^{n=\infty}\dfrac{n^2+1}{n(n+2)}x^n$. **20.** $\Sigma_{n=1}^{n=\infty}\dfrac{(n+1)^3}{n(n+3)}x^n$.

21. If s_n is the sum to n terms of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{5\cdot6\cdot7}+\frac{1}{9\cdot10\cdot11}+\frac{1}{13\cdot14\cdot15}+\dots,$$

prove that

$$2s_n=\frac{1}{2n+1}+\frac{1}{2n+3}+\frac{1}{2n+5}+\dots+\frac{1}{4n-1}.$$

Hence show that the sum to infinity of the series is $\frac{1}{4}\log 2$.

* This is Napier's formula.

22. If s_n is the sum to n terms of the series

$$\frac{1}{2.3.4}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\ldots+\frac{1}{(3n-1)3n(3n+1)}+\ldots,$$

prove that

$$2s_n=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{3n+1}-1.$$

Hence show that the sum to infinity of the series is $\frac{1}{2}(\log 3-1)$.

23. Prove that

(i) $\dfrac{1}{1.2.3}+\dfrac{1}{3.4.5}+\dfrac{1}{5.6.7}+\ldots$ to $\infty=\log 2-\frac{1}{2}$.

(ii) $\dfrac{1}{1.2.3.4}+\dfrac{1}{3.4.5.6}+\dfrac{1}{5.6.7.8}+\ldots$ to $\infty=\frac{2}{3}\log 2-\frac{5}{12}$.

24. By expanding $\log(1+x+x^2)$ in various ways show that

(i) $1-n+\dfrac{n(n-3)}{\lfloor 2}-\dfrac{n(n-4)(n-5)}{\lfloor 3}+\ldots=(-1)^n.2$ or $(-1)^{n-1}$ according as n is of the form $3r$ or $3r\pm1$.

(ii) $1-\dfrac{n^2}{\lfloor 2}+\dfrac{n^2(n^2-1^2)}{\lfloor 4}-\dfrac{n^2(n^2-1^2)(n^2-2^2)}{\lfloor 6}+\ldots=(-1)^n$ or $(-\frac{1}{2})^{n-1}$ according as n is of the form $3r$ or $3r\pm1$.

(iii) $1-\dfrac{n(n+1)}{\lfloor 3}+\dfrac{(n-1)n(n+1)(n+2)}{\lfloor 5}-\dfrac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{\lfloor 7}+\ldots=\dfrac{(-1)^n}{2n+1}$ or $\dfrac{(-2)^{n-1}}{2n+1}$

according as n is of the form $3r$ or $3r\pm1$.

(iv) $C_1^n-\frac{1}{2}C_3^{n+1}+\frac{1}{3}C_5^{n+2}-\frac{1}{4}C_7^{n+3}+\ldots=0$, $\dfrac{1}{n}$, $\dfrac{3}{n}$ or $\dfrac{4}{n}$ according as n is of the form $6r$, $6r\pm1$, $6r\pm2$, $6r-3$.

[These are obtained from the identities:

(i) $\log(1-x^3)-\log(1-x)=\log\{1+x(1+x)\}$;

(ii) and (iii) $\log(1-x^3)-\log(1-x)-\log(1+x^2)=\log\left\{1+\dfrac{x}{1+x^2}\right\}$;

(iv) $\log(1-x^3)-2\log(1-x^2)+\log(1-x)=\log\left\{1-\dfrac{x}{(1+x)^2}\right\}$.]

25. Use the identities

$x^4-25x^2+144=(x+3)(x-3)(x+4)(x-4)$, $x^4-25x^2=x^2(x+5)(x-5)$

to show that

$$\log(x+3)+\log(x-3)+\log(x+4)+\log(x-4)-2\log x-\log(x+5)-\log(x-5)$$
$$=2\left\{\frac{72}{x^4-25x^2+72}+\frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3+\ldots\right\}.$$

This formula is useful in finding the logarithms of large primes.

CHAPTER XX

CONVERGENCE (2)

SERIES OF POSITIVE TERMS (*Continued*)

1. Cauchy's Condensation Test. *If $f(n)$ is a (positive*) decreasing function of n, and a is any positive integer >1, then the two series, $\Sigma f(n)$ and $\Sigma a^n f(a^n)$, are both convergent or both divergent.*

Group the terms of $\Sigma f(n)$ as follows :

$$\begin{aligned}&\{f(1) \quad +f(2) \quad +\ldots+f(a)\},\\ &+\{f(a+1) \ +f(a+2) \ +\ldots+f(a^2)\},\\ &+\{f(a^2+1)+f(a^2+2)+\ldots+f(a^3)\}+\ldots,\end{aligned}$$

where, if v_n denotes the sum of the terms of the nth group,

$$v_n=f(a^{n-1}+1)+f(a^{n-1}+2)+\ldots+f(a^n).$$

The number of these terms is a^n-a^{n-1}, and since $f(n)$ is a decreasing function,

$$(a^n-a^{n-1})f(a^n)\leqslant v_n\leqslant(a^n-a^{n-1})f(a^{n-1}) ;$$

$$\therefore \ \frac{1}{a}(a-1)a^n f(a^n)\leqslant v_n\leqslant(a-1)a^{n-1}f(a^{n-1}).$$

Now, if $\Sigma a^n f(a^n)$ is convergent, so also is Σv_n; and therefore $\Sigma f(n)$ converges (Ch. XVI, 5). If $\Sigma a^n f(a^n)$ is divergent, so also is Σv_n; and therefore $\Sigma f(n)$ diverges. But a series of positive terms must converge or diverge, therefore $\Sigma f(n)$ and $a^n\Sigma f(a^n)$ are both convergent or both divergent.

2. An important Test Series. *The series $\Sigma\dfrac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $p\leqslant 1$.*

If a is any positive integer, this series is convergent or divergent according as the series Σv_n is convergent or divergent, where

$$v_n=\frac{a^n}{a^n(\log a^n)^p}=\frac{1}{(\log a)^p}\cdot\frac{1}{n^p}.$$

Since $(\log a)^p$ is constant, the given series is convergent or divergent according as $\Sigma 1/n^p$ is convergent or divergent, that is according as $p>1$ or $p\leqslant 1$.

* In any case, the terms of the decreasing function $f(n)$ are ultimately of the same sign (Exercise XXV, 1).

3. Kummer's Test. *Let Σu_n and $\Sigma 1/d_n$ be two series of positive terms and suppose that $\Sigma 1/d_n$ is divergent; also suppose that $v_n = d_n u_n/u_{n+1} - d_{n+1}$; then*

(i) *if a fixed number k can be found so that after a certain stage, say for $n \geqslant m$, $v_n > k > 0$, the series Σu_n is convergent;*

(ii) *if $v_n < 0$ for $n \geqslant m$, Σu_n is divergent.*

Proof. (i) Putting $m, m+1, \ldots, n-1$ in succession for n in the given condition, we have

$$d_m u_m - d_{m+1} u_{m+1} > k u_{m+1},$$
$$d_{m+1} u_{m+1} - d_{m+2} u_{m+2} > k u_{m+2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$d_{n-1} u_{n-1} - d_n u_n > k u_n;$$

whence $$u_{m+1} + u_{m+2} + \ldots + u_n < \frac{1}{k}(d_m u_m - d_n u_n) < \frac{1}{k} d_m u_m.$$

Thus if s_r is the sum to r terms of Σu_n,

$$s_n < s_m + \frac{1}{k} d_m u_m, \text{ a fixed number};$$

therefore Σu_n is convergent (Ch. XVI, 7).

(ii) Since $d_n \dfrac{u_n}{u_{n+1}} - d_{n+1} < 0$ for $n \geqslant m$, we have

$$d_m u_m < d_{m+1} u_{m+1} < \ldots < d_n u_n,$$

therefore $u_n > d_m u_m \cdot \dfrac{1}{d_n}$. Now $d_m u_m$ is a fixed number and $\Sigma 1/d_n$ is divergent, therefore Σu_n is divergent.

NOTE. If $v_n \to 0$ we cannot find k so that $v_n > k > 0$ for $n \geqslant m$, and the test fails. In part (i) of the test, $\Sigma 1/d_n$ need not diverge; in fact, d_n may be any positive function of n.

4. Raabe's Test. *Let Σu_n be a series of positive terms, then*

(i) *if after a certain stage, say for $n \geqslant m$,*

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > k > 1,$$

where k is a fixed number, then Σu_n is convergent.

(ii) *If* $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) < 1$, *then Σu_n is divergent.*

This follows at once from Kummer's test by taking $d_n = n$, giving

$$v_n = n\left(\frac{u_n}{u_{n+1}} - 1\right).$$

In particular, if $\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=l$, *then* Σu_n *converges when* $l>1$ *and diverges when* $l<1$: *but if* $l=1$, *the test fails.*

This test may be tried when D'Alembert's test fails.

Ex. 1. *Consider the convergence or divergence of* $x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\dots$.

Here $$\frac{u_n}{u_{n+1}}=\frac{2n(2n+1)}{(2n-1)^2}\cdot\frac{1}{x^2};\quad\therefore\ \lim\frac{u_n}{u_{n+1}}=\frac{1}{x^2}.$$

By D'Alembert's test, Σu_n converges if $|x|<1$ and diverges if $|x|>1$. If $|x|=1$, the test fails; with Raabe's test, however, we have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim\frac{6n^2-n}{(2n-1)^2}=\frac{3}{2}>1;$$

therefore Σu_n is convergent.

NOTE. This series is the expansion of $\sin^{-1}x$, and was discovered by Newton. From it, he deduced the series for $\sin x$.

5. Gauss's Test. *Let* Σu_n *be a series of positive terms and suppose that* u_n/u_{n+1} *can be expressed in the form*

$$\frac{u_n}{u_{n+1}}=1+\frac{a}{n}+\frac{b_n}{n^p},$$

where $p>1$ *and* $|b_n|<$ *a fixed number* k *or (in particular)* b_n *tends to a finite limit as* $n\to\infty$, *then* Σu_n *converges if* $a>1$ *and diverges if* $a\leqslant 1$.

Proof. First suppose that $a\neq 1$, then

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=a+\frac{b_n}{n^{p-1}}\to a\quad\text{as}\quad n\to\infty,$$

for $|b_n|<k$ and $p>1$. Therefore, by Raabe's test, Σu_n converges if $a>1$ and diverges if $a<1$.

If $a=1$, Raabe's test fails and we apply Kummer's. Taking $d_n=n\log n$ in Art. 3 we have

$$\begin{aligned}v_n&=n\log n\cdot\frac{u_n}{u_{n+1}}-(n+1)\log(n+1)\\&=n\left(1+\frac{1}{n}+\frac{b_n}{n^p}\right)\log n-(n+1)\log(n+1)\\&=(n+1)\log\frac{n}{n+1}+\frac{\log n}{n^{p-1}}\cdot b_n.\end{aligned}$$

Now $$(n+1)\log\frac{n}{n+1}=(n+1)\log\left(1-\frac{1}{n+1}\right)\to -1\ \text{ as }\ n\to\infty$$

and $$\frac{\log n}{n^{p-1}}\cdot b_n\to 0,\quad\text{for}\quad p>1\ \text{(Ch. XIX, 7, (5))},\quad\text{and}\quad |b_n|<k.$$

Therefore $v_n\to -1$ and, after a certain stage, $v_n<0$. Hence Σu_n *is* divergent.

In many cases u_n/u_{n+1} can be expanded in a series of powers of $1/n$, and we have the following rule :

If $$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b}{n^2} + \dots,$$

the series being convergent for sufficiently large values of n, *then* Σu_n *is convergent if* $a > 1$ *and divergent if* $a \leqslant 1$.

For in this case b_n is the sum of a convergent series.

This is the most generally useful test for series of positive terms, and in cases where it applies, it is better to use it at once instead of beginning with D'Alembert's and Raabe's tests.

Ex. 1. *Discuss the convergence or divergence of* $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots$.

By division we can show that $$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 = 1 + \frac{1}{n} + \frac{b_n}{n^2},$$

where $$b_n - = \frac{1}{4}\left(1 + \frac{1}{n}\right) \Big/ \left(1 + \frac{1}{2n}\right)^2 \to -\frac{1}{4} \quad \text{as} \quad n \to \infty.$$

Therefore by Gauss's test the series is divergent. Here the tests of D'Alembert and Raabe both fail.

CONTINUED FROM CH. XVI, 25

6. Binomial Series. *When* $x = -1$, *the series*

$$1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \dots$$

converges if $n > 0$ *and diverges if* $n < 0$.

When $x = -1$, after a certain stage all the terms have the same sign. Denoting the series by $u_0 + u_1 + u_2 + \dots$, we can show by division that

$$\frac{u_r}{u_{r+1}} = -\frac{r+1}{n-r} = 1 + \frac{n+1}{r} + \frac{b_r}{r^2},$$

where $$b_r = n(n+1) \Big/ \left(1 - \frac{n}{r}\right) \to n(n+1) \quad \text{as} \quad r \to \infty.$$

Hence, by Gauss's test, the series converges or diverges according as $n + 1 \gtrless 1$, that is according as $n \gtrless 0$.

Apparently the test says that the series diverges when $n = 0$, but here the reasoning fails because u_r/u_{r+1} is of the form $0/0$.

7. The Hyper-geometric Series. The series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots$$

is known as the *hyper-geometric series*, and is of great importance.

If none of the constants α, β, γ is a negative integer, the series is endless, and we shall prove that

(i) *the series is absolutely convergent if* $|x|<1$, *and is divergent if* $|x|>1$;

(ii) *when* $x=1$, *the series converges if* $\gamma>\alpha+\beta$, *and not otherwise;*

(iii) *when* $x=-1$, *the series converges if* $\gamma+1>\alpha+\beta$, *and not otherwise.*

Proof. Denoting the series by $u_0+u_1+\ldots+u_n+\ldots$ where

$$u_n=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{1\cdot 2\ldots n\cdot\gamma(\gamma+1)\ldots(\gamma+n-1)}x^n,$$

we have

$$\frac{u_n}{u_{n+1}}=\frac{(1+n)(\gamma+n)}{(\alpha+n)(\beta+n)}x=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}x\to x.$$

(i) Hence the series converges absolutely if $|x|<1$, and diverges if $|x|>1$.

(ii) *When* $x=1$, $u_n/u_{n+1}\to 1$. Hence, after a certain stage, the terms have the same sign.

Applying Gauss's test, we can show by division that

$$\frac{u_n}{u_{n+1}}=1+(\gamma+1-\alpha-\beta)\frac{1}{n}+\frac{b_n}{n^2},$$

where

$$b_n=\left(A+\frac{B}{n}\right)\Big/\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)$$

and A, B are fixed numbers. Thus $b_n\to A$ and Σu_n converges or diverges according as $\gamma+1-\alpha-\beta>1$ or $\leqslant 1$, that is according as

$$\gamma>\alpha+\beta \quad \text{or} \quad \gamma\leqslant\alpha+\beta.$$

(iii) *When* $x=-1$, after a certain stage, the terms are alternately positive and negative.

Also $|u_{n+1}|<|u_n|$ if $(\alpha+n)(\beta+n)<(1+n)(\gamma+n)$; that is, if

$$\alpha\beta+n(\alpha+\beta)<\gamma+n(1+\gamma),$$

or if

$$1+\gamma-\alpha-\beta>(\alpha\beta-\gamma)\frac{1}{n}.$$

This is true for sufficiently large values of n if $\gamma+1>\alpha+\beta$.

Again, as in the preceding, $-\dfrac{u_n}{u_{n+1}}=1+\dfrac{a_n}{n}$ where $a_n\to\gamma+1-\alpha-\beta$ as $n\to\infty$. Hence by Ch. XVI, 24, $u_n\to 0$ if $\gamma+1>\alpha+\beta$, but not otherwise. Therefore by Ch. XVI, 14, the series converges if $\gamma+1>\alpha+\beta$, but not otherwise.

8. De Morgan's and Bertrand's Test. *Let Σu_n be a series of positive terms, and suppose that u_n/u_{n+1} can be expressed in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{a_n}{n \log n},$$

then (i) *if after a certain stage $a_n > 1+k$ where k is a fixed positive number, Σu_n is convergent.*

(ii) *If $a_n \leqslant 1$, Σu_n is divergent.*

Using Kummer's test, we take $d_n = n \log n$. Then, as in Art. 5,

$$v_n = n \log n \,.\, \frac{u_n}{u_{n+1}} - (n+1)\log(n+1)$$
$$= (n+1)\log\left(1 - \frac{1}{n+1}\right) + a_n.$$

Therefore $v_n \to a_n - 1$ as $n \to \infty$. Hence Σu_n converges if $a_n > 1+k$ and diverges if $a_n < 1$.

If $a_n = 1$, $v_n < 0$, for $\log\left(1 - \frac{1}{n+1}\right) < -\frac{1}{n+1}$, therefore Σu_n is divergent.

Series with Terms Positive or Negative

9. Theorem. *If Σu_n is a convergent series of positive terms and $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, positive or negative, and all numerically less than some fixed number k, then $\Sigma a_n u_n$ is absolutely convergent.*

For
$$|a_{m+1}u_{m+1}| + |a_{m+2}u_{m+2}| + \ldots + |a_{m+p}u_{m+p}|$$
$$= |a_{m+1}|\,u_{m+1} + |a_{m+2}|\,u_{m+2} + \ldots + |a_{m+p}|\,u_{m+p}$$
$$< k(u_{m+1} + u_{m+2} + \ldots + u_{m+p}).$$

Since Σu_n is convergent, we can find m such that for all values of p,

$$u_{m+1} + u_{m+2} + \ldots + u_{m+p} < \epsilon/k,$$

where ϵ is any positive number, as small as we choose. Therefore

$$|a_{m+1}u_{m+1}| + |a_{m+2}u_{m+2}| + \ldots + |a_{m+p}u_{m+p}| < \epsilon.$$

Hence $\Sigma a_n u_n$ is absolutely convergent.

Ex. 1. *If Σu_n is a convergent series of positive terms, $\Sigma u_n \cos n\theta$ and $\Sigma u_n \sin n\theta$ are absolutely convergent for all values of θ.*

10. Abel's Inequality. *If $u_1, u_2, u_3, \ldots$ is a sequence of real numbers such that* $l < u_1 + u_2 + u_3 + \ldots + u_r < h$

for the values 1, 2, 3, ... n of r, and if $a_1, a_2, a_3, \ldots$ is a decreasing sequence of positive terms, then $a_1 l < a_1u_1 + a_2u_2 + \ldots + a_nu_n < a_1 h$.

For let $$s_n = u_1 + u_2 + \ldots + u_n,$$

and $$t_n = a_1u_1 + a_2u_2 + \ldots + a_nu_n.$$

Then $u_1 = s_1$, $u_2 = s_2 - s_1$, $u_3 = s_3 - s_2$, etc., therefore

$$\begin{aligned} t_n &= a_1s_1 + a_2(s_2 - s_1) + \ldots + a_n(s_n - s_{n-1}) \\ &= (a_1 - a_2)s_1 + (a_2 - a_3)s_2 + \ldots + (a_{n-1} - a_n)s_{n-1} + a_ns_n. \end{aligned}$$

Now the factors $a_1 - a_2$, $a_2 - a_3$, $\ldots$ $a_{n-1} - a_n$, a_n are positive or zero, and their sum is a_1.

Also s_1, s_2, $\ldots$ s_n are all greater than l and less than h.

Therefore $$a_1l < t_n < a_1h.$$

11. Dirichlet's Test. *If Σu_n converges or oscillates between finite limits and a_1, a_2, a_3, $\ldots$ is a decreasing sequence of positive terms which tends to zero as a limit, then Σa_nu_n is convergent.*

Let $$s_n = u_1 + u_2 + \ldots + u_n.$$

Since Σu_n converges or oscillates finitely, we can find numbers h, l so that, for all values of m and p,

$$l < s_{m+p} - s_m < h,$$

that is to say,

$$l < u_{m+1} + u_{m+2} + \ldots + u_{m+p} < h.$$

Therefore, by Abel's inequality

$$a_{m+1}l < a_{m+1}u_{m+1} + a_{m+2}u_{m+2} + \ldots + a_{m+p}u_{m+p} < a_{m+1}h.$$

Now $a_{m+1} \to 0$ as $m \to \infty$, therefore for any ϵ we can find m so that

$$| a_{m+1}u_{m+1} + a_{m+2}u_{m+2} + \ldots + a_{m+p}u_{m+p} | < \epsilon ;$$

hence Σa_nu_n is convergent.

Ex. 1. *If (a_n) is a decreasing sequence of positive terms which tends to zero as a limit, both the series $\Sigma a_n \cos n\theta$ and $\Sigma a_n \sin n\theta$ are convergent unless $\theta = 0$ or $2k\pi$, in which case $\Sigma a_n \cos n\theta$ diverges.*

For if θ is not a multiple of 2π, the series $\Sigma \cos n\theta$ and $\Sigma \sin n\theta$ oscillate between finite limits (Exercise XXVII, 24, 25).

Hence the result follows by Dirichlet's test.

12. Abel's Test. *If Σu_n converges and a_1, a_2, a_3, $\ldots$ is a decreasing sequence of positive terms, then Σa_nu_n is convergent.*

For as n tends to infinity, a_n tends to some limit l, therefore

$$a_n - l \to 0.$$

Hence by Dirichlet's test, $\Sigma(a_n - l)u_n$ is convergent; that is to say, $\Sigma a_nu_n - l\Sigma u_n$ is convergent.

But Σu_n is convergent, therefore Σa_nu_n is convergent.

Power Series

A series of the type $\Sigma a_n x^n$ is called a *Power Series*. In Arts. 13-15, x may be real or complex.

13. *If $\Sigma a_n x^n$ converges (absolutely or conditionally) for $x = x_1$, it converges absolutely for all values of x such that $|x| < |x_1|$.*

For since $\Sigma a_n x_1^n$ is convergent, $a_n x_1^n \to 0$. Therefore a positive number k exists such that $|a_n x_1^n| < k$ for every n. Hence if $|x| < |x_1|$,

$$|a_n x^n| = |a_n x_1^n| \cdot \left|\frac{x}{x_1}\right|^n < k \left|\frac{x}{x_1}\right|^n.$$

Therefore every term of $\Sigma |a_n x^n|$ is less than the corresponding term of the convergent series

$$k + k\left|\frac{x}{x_1}\right| + k\left|\frac{x}{x_1}\right|^2 + \dots .$$

Therefore $\Sigma |a_n x^n|$ is convergent and $\Sigma a_n x^n$ is absolutely convergent.

14. *If $\Sigma a_n x^n$ is non-convergent for $x = x_1$, it is non-convergent for every x such that $|x| > |x_1|$.*

For if the series converge for $x = x_2$ where $|x_2| > |x_1|$, by the last theorem it converges for $x = x_1$, which is contrary to the hypothesis.

15. *With regard to the series $\Sigma a_n x^n$, either* (i) *it converges for $x = 0$ and for no other value of x, or* (ii) *it converges absolutely for all values of x, or* (iii) *a positive number R exists such that $\Sigma a_n x^n$ converges absolutely when $|x| < R$ and is non-convergent when $|x| > R$.*

Proof. Suppose that there is a value of x other than zero for which $\Sigma a_n x^n$ converges. Also, suppose that there is a value x' of x for which it is non-convergent. Choose a positive number a greater than $|x'|$, then, by Art. 14, the series is non-convergent for $x = a$.

Divide the real numbers in the interval $(0, a)$ into two classes.

The lower class is to contain every real number r such that $\Sigma a_n x^n$ converges if $|x| = r$. *The upper class* is to contain every real number r', such that $\Sigma a_n x^n$ does not converge if $|x| = r'$. Both classes exist, and it follows from Arts. 13 and 14 that every r is less than any r'.

Therefore the classification defines a real positive number R which separates the classes and may be assigned to either class. This number R is such that $\Sigma a_n x^n$ converges absolutely if $|x| < R$, and is non-convergent if $|x| > R$. Nothing is said as to what happens when $|x| = R$.

By writing $R=0$, $R=\infty$ in the exceptional cases (i) and (ii) respectively, these may be included in (iii).

All that has been said applies to any power series, real or complex.

16. For a real series $\Sigma a_n x^n$, the interval $(-R, R)$ is called the *interval of convergence.* The series converges absolutely or is non-convergent according as x is within or outside the interval. At either end point $\Sigma a_n x^n$ may converge, diverge or oscillate.

For a complex series $\Sigma a_n z^n$, R is called the *radius of convergence.* The circle (O, R) with centre at the origin and radius R is called the *circle of convergence.* The series converges absolutely or is non-convergent according as the point z is inside or outside the circle. If z is on the circumference, the character of the series is not determined.

17. Let $\Sigma a_n z^n$ be any series, real or complex, and suppose that

$$\lim |a_{n+1}/a_n| = l.$$

Then

$$\lim \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = l|z|,$$

and it follows from D'Alembert's test that $R=1/l$.

Again, if $\lim |a_n|^{\frac{1}{n}} = l$, then $\lim |a_n z^n|^{\frac{1}{n}} = l|z|$, and by Cauchy's test $R=1/l$.

18. In considering the behaviour of a complex series *at points on its circle of convergence,* the following theorem is often useful.

If (a_n) *is a decreasing sequence of positive numbers such that* $a_{n+1}/a_n \to 1$ *and* $a_n \to 0$, *then,* (i) *if* Σa_n *converges,* $\Sigma a_n z^n$ *converges absolutely at every point on its circle of convergence,* (ii) *if* Σa_n *diverges,* $\Sigma a_n z^n$ *converges* (*though not absolutely*) *at every point on the circle, except at the point* $z=1$, *where it diverges,* (iii) *the same statements hold for the series* $\Sigma(-1)^n a_n z^n$, *except that it diverges at the point* $z=-1$, *when* $\Sigma(-1)^n a_n$ *is divergent.*

For $1/R = \lim a_{n+1}/a_n = 1$, and at a point on the circle of convergence

$$a_n z^n = a_n(\cos\theta + \iota \sin\theta)^n = a_n(\cos n\theta + \iota \sin n\theta).$$

Moreover, (i) if Σa_n converges, both $\Sigma a_n \cos n\theta$ and $\Sigma a_n \sin n\theta$ are absolutely convergent (Art. 9, Ex. 1); therefore $\Sigma a_n z^n$ is absolutely convergent when $|z|=1$. (ii) If Σa_n diverges, both $\Sigma a_n \cos n\theta$ and $\Sigma a_n \sin n\theta$ converge unless $\theta=0$ or $2k\pi$; in which case $\Sigma a_n \cos n\theta$ diverges (Art. 11, Ex. 1). Hence the statement (ii) follows.

(iii) This follows from (i) and (ii) on substituting $-z$ for z.

19. If $\Sigma a_n z^n$ converges when $z=z_1$, by Art. 13, it is absolutely convergent when $|z|<|z_1|$. Denoting its sum by $f(z)$, we shall prove that

$$f(z)=a_0+a_1z+a_2z^2+\ldots+(a_n+\eta)z^n, \quad \text{where } |\eta|\to 0 \text{ as } |z|\to 0.$$

We have $$\eta z^n=a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots,$$

and $$|\eta z^n|\leqslant|a_{n+1}z^{n+1}|+|a_{n+2}z^{n+2}|+\ldots.$$

As in Art 13, we can find a fixed number k, such that for every r, $|a_rz_1^r|<k$; and therefore $|a_rz^r|<k\left|\frac{z}{z_1}\right|^r$; and thus, if $|z|<|z_1|$,

$$|\eta z^n|<k\left\{\left|\frac{z}{z_1}\right|^{n+1}+\left|\frac{z}{z_1}\right|^{n+2}+\ldots\right\}=k\cdot\left|\frac{z}{z_1}\right|^{n+1}\cdot\frac{|z_1|}{|z_1|-|z|}.$$

Hence if $z\neq 0$, $$\eta<\frac{k}{|z_1|^n}\cdot\frac{|z|}{|z_1|-|z|}.$$

Therefore $$|\eta|\to 0 \text{ as } |z|\to 0.$$

20. Criterion for the Identity of Power Series. *If* $\Sigma a_nz^n=\Sigma b_nz^n$ *for every value of* z *whose modulus is less than some number* μ, *then* $a_n=b_n$ *for every value of* n.

For by Art. 19, for every n,

$$a_0+a_1z+a_2z^2+\ldots+(a_n+\eta)z^n=b_0+b_1z+b_2z^2+\ldots+(b_n+\eta')z^n,$$

where η and η' tend to zero as $|z|\to 0$.

By making $|z|\to 0$, it will be seen that $|a_0-b_0|<$any positive number, however small, and since a_0, b_0 are constants, it follows that $a_0=b_0$.

Since the above equation holds for values of z other than zero, we can divide by z, therefore

$$a_1+a_2z+\ldots+(a_n+\eta)z^{n-1}=b_1+b_2z+\ldots+(b_n+\eta)z^{n-1}.$$

Making $z\to 0$ as before, we have $a_1=b_1$, and by continuing the process, $a_n=b_n$ for every n.

21. Binomial Series. We consider the convergence of the series

$$1+nz+\frac{n(n-1)}{\lfloor 2}z^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}z^r+\ldots,$$

where n is real and z complex.

Denoting the series by $u_0+u_1z+u_2z^2+\ldots$, we have

$$R=\lim_{r\to\infty}\left|\frac{u_r}{u_{r+1}}\right|=\lim_{r\to\infty}\left|\frac{r+1}{n-r}\right|=1.$$

Therefore, it follows from Art. 17 that

(i) *The series converges absolutely if* $|z|<1$, *and is non-convergent if* $z>1$.

When $|z|=1$, the point z is on the circle of convergence, and we shall prove that

(ii) *If* $n>0$, *the series converges absolutely at every point on its circle of convergence.*

(iii) *If* $-1<n<0$, *the series converges (though not absolutely) at every point on the circle, except at the point* $z=-1$, *where it diverges.*

Let accents indicate absolute values, so that $u_r'=|u_r|$, then if $r>n$,

$$\frac{u_{r+1}'}{u_r'}=\frac{r-n}{r+1}.$$

Therefore $u_{r+1}' \gtreqless u_r'$, according as $r-n \gtreqless r+1$, that is according as $n \lesseqgtr -1$.

If $n\leqslant -1$, u_r' increases with r, or is constant and *the series cannot be convergent.*

If $n>-1$, we apply the theorem of Art. 18. After a certain stage, u_r' is a decreasing sequence, $u_{r+1}'/u_r' \to 1$ and, as shown in Ch. XVI, 25, $u_r' \to 0$.

These are the conditions required in Art. 18, to apply the theorem of this article, and we have to consider the convergence of $\Sigma u_r'$.

After a certain stage, the terms of Σu_r are alternately positive and negative, so that $\Sigma u_r'$ is convergent or divergent, according as $\Sigma(-1)^r u_r$ is convergent or divergent, that is according as $n \gtrless 0$. (See Art. 6.)

Hence the statements (ii) and (iii) follow immediately from Art. 18.

Multiplication of Series

22. *If* $u_1+u_2+u_3+\ldots$ *and* $v_1+v_2+v_3+\ldots$ *are convergent series of positive terms, or if they are absolutely convergent real or complex series, their sums being* s *and* t *respectively, then the series*

$$u_1v_1+(u_1v_2+u_2v_1)+(u_1v_3+u_2v_2+u_3v_1)+\ldots$$

is absolutely convergent and its sum is st.

Multiply the terms of $u_1+u_2+u_3+\ldots$ by $v_1, v_2, v_3, \ldots$, and arrange the products as below.

$$\left.\begin{array}{llll} u_1v_1, & u_2v_1, & u_3v_1, & \ldots \\ u_1v_2, & u_2v_2, & u_3v_2, & \ldots \\ u_1v_3, & u_2v_3, & u_3v_3, & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(A)}$$

This array extends to infinity on the right and below, and we shall consider two ways of arranging the terms so as to form simply infinite series.

Let s_n, t_n be the sums to n terms of Σu_n, Σv_n respectively; then if σ_n is the sum of the first n terms of the first n rows of (**A**), we have

$$\sigma_n = s_n t_n \to st.$$

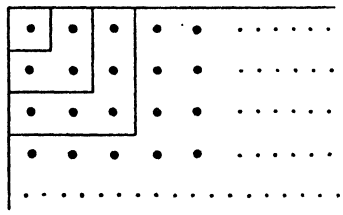


FIG. 62.

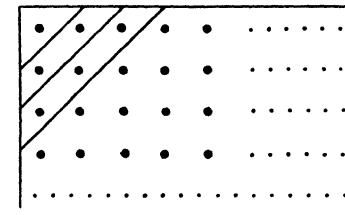


FIG. 63.

Draw a set of squares, as in Fig. 62, where the terms are represented by dots and the nth square just contains the terms in σ_n. Then the terms in (**A**) can be arranged to form the infinite series

$$u_1v_1 + (u_1v_2 + u_2v_2 + u_2v_1) + (u_1v_3 + u_2v_3 + u_3v_3 + u_3v_2 + u_3v_1) + \dots, \quad \dots\dots (\mathbf{B})$$

where the nth term is the sum of the terms of (**A**) between the nth and the $(n-1)$th squares.

The sum to n terms of this series is σ_n, and its sum to infinity is st.

Again, the terms of (**A**) can be arranged to form the infinite series

$$u_1v_1 + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + \dots, \quad \dots\dots\dots\dots (\mathbf{C})$$

where the nth term is the sum of the terms between the nth and the $(n-1)$th diagonals, drawn as in Fig. 63.

If the brackets are removed in (**B**) and (**C**), we have two series which differ only in the arrangement of terms. The first series is absolutely convergent and its sum is st; this is therefore also true for the second series (Chap. XVI, 5, 6 and 19).

Ex. 1. *If* $E(x) = 1 + x + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \dots$, *prove that, for all values of* x *and* y,

$$E(x) \,.\, E(y) = E(x+y).$$

The series equivalent to $E(x)$ and $E(y)$ are absolutely convergent, therefore

$$E(x) \,.\, E(y) = 1 + d_1 + d_2 + \dots + d_n + \dots,$$

where
$$d_n = \frac{x^n}{\lfloor n} + \frac{x^{n-1}y}{\lfloor n-1 \rfloor 1} + \frac{x^{n-2}y^2}{\lfloor n-2 \rfloor 2} + \dots + \frac{y^n}{\lfloor n} = \frac{(x+y)^n}{\lfloor n};$$

$$\therefore \; E(x) \,.\, E(y) = E(x+y).$$

In Arts. 23 and 24, the series (**C**) will be denoted by Σd_n, where

$$d_n = u_1v_n + u_2v_{n-1} + u_3v_{n-2} + \dots + u_nv_1;$$

it should be noted that, in every term of d_n, the sum of the suffixes is $n+1$.

23. *If Σu_n and Σv_n are convergent with sums s and t respectively and* one *of these series, say Σu_n, is absolutely convergent, then Σd_n is convergent and its sum is .st.* (Mertens.)

Let s_n, t_n, D_n be the sums to n terms of Σu_n, Σv_n, Σd_n respectively. Considering the array (**A**) in Art. 22, it will be seen that

$$s_n t_n - D_n = u_2 w_1 + u_3 w_2 + u_4 w_3 + \ldots + u_n w_{n-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{B})$$

where $w_1 = v_n, \quad w_2 = v_{n-1} + v_n, \quad w_3 = v_{n-2} + v_{n-1} + v_n,$

and generally $w_m = v_{n-m+1} + v_{n-m+2} + \ldots + v_n ;$

lastly $w_{n-1} = v_2 + v_3 + \ldots + v_n.$

The series Σv_n is convergent; and hence (i) we can find a positive number μ such that $|v_r + v_{r+1} + v_{r+2} + \ldots + v_n| < \epsilon$ for $r > \mu$, where ϵ is an arbitrarily small positive number; and therefore

$|w_1|, \quad |w_2|, \ldots |w_m|,$ are all $< \epsilon,$ provided that $n - m + 1 > \mu ; \ldots (\mathbf{C})$

(ii) the sequence (t_n) is bounded; hence we can find a positive number k such that $|t_n| < k$ for every n, and consequently

$$|w_{m+1}|, \quad |w_{m+2}|, \ldots |w_{n-1}| \quad \text{are all} \quad < k.$$

We therefore consider separately the sum of the first m terms of the series in (**B**), and the sum of the remaining terms, where m is to be properly chosen.* Let

$$P = u_2 w_1 + u_3 w_2 + \ldots + u_{m+1} w_m,$$

$$Q = u_{m+2} w_{m+1} + u_{m+3} w_{m+2} + \ldots + u_n w_{n-1} ;$$

then
$$\begin{aligned} |P| &\leqslant |u_2|\,|w_1| + |u_3|\,|w_2| + \ldots + |u_{m+1}|\,|w_m| \\ &< \{|u_2| + |u_3| + \ldots + |u_{m+1}|\}\epsilon, \end{aligned}$$

with the condition (**C**). Hence, if s' is the sum of the convergent series $\Sigma |u_n|$, then $|P| < s'\epsilon.$

Also
$$\begin{aligned} |Q| &\leqslant |u_{m+2}|\,|w_{m+1}| + |u_{m+3}|\,|w_{m+2}| + \ldots + |u_n|\,|w_{n-1}| \\ &< \{|u_{m+2}| + |u_{m+3}| + \ldots + |u_n|\}k, \end{aligned}$$

and since $\Sigma |u_n|$ is convergent, we can find a positive number μ' such that $|u_{m+2}| + |u_{m+3}| + \ldots + |u_n| < \epsilon,$ provided that $m + 2 > \mu', \ldots (\mathbf{D})$

and then $|Q| < k\epsilon.$

The conditions (**C**) and (**D**) are satisfied if $n > 2m$ and $m >$ either of the two, $\mu - 1$ and $\mu' - 2$. Hence, as m, and consequently n, tend to infinity,

$$|P| \to 0, \quad |Q| \to 0 \quad \text{and} \quad s_n t_n - D_n \to 0,$$

which proves the theorem.

* This artifice is often useful.

24. *If Σu_n and Σv_n are convergent with sums s and t respectively, and if Σd_n is convergent, then its sum is st.* (Abel.)

Let s_n, t_n, D_n be the sums to n terms of Σu_n, Σv_n, Σd_n respectively, then

$$D_1 = u_1 t_1, \quad D_2 = u_2 t_1 + u_1 t_2, \quad D_3 = u_3 t_1 + u_2 t_2 + u_1 t_3, \quad \text{etc.}$$

Hence $$D_1 + D_2 + D_3 + \ldots + D_n = s_n t_1 + s_{n-1} t_2 + s_{n-2} t_3 + \ldots + s_1 t_n.$$

Now $s_n \to s$ and $t_n \to t$, therefore by Ch. XV, 9, (4),

$$\frac{1}{n}(s_n t_1 + s_{n-1} t_2 + \ldots + s_1 t_n) \to st.$$

Also, if D is the sum of Σd_n, which is convergent, by Ch. XV, 9, (3),

$$\frac{1}{n}(D_1 + D_2 + \ldots + D_n) \to D.$$

Therefore $D = st$.

Ex. 1. *If* $-1 < x \leqslant 1$, *prove that*

$$\{\log(1+x)\}^2 = d_1 x^2 - d_2 x^3 + d_3 x^4 - \ldots,$$

where $$d_n = \frac{2}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

Under suitable conditions, the rule for the multiplication of series gives

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)^2 = d_1 x^2 - d_2 x^3 + d_3 x^4 - \ldots,$$

where $$d_n = \frac{1}{n \,.\, 1} + \frac{1}{(n-1)\,2} + \frac{1}{(n-2)\,3} + \ldots + \frac{1}{1 \,.\, n}$$

$$= \frac{2}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

If $-1 < x < 1$, the series for $\log(1+x)$ is absolutely convergent and the result in question holds by Art. 22.

If $x = 1$, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent; and so is $d_1 - d_2 + d_3 - \ldots$, for it is easily shown that $d_n < d_{n-1}$, and $d_n \to 0$ by Ch. XV, 9, (3). Hence Art. 24 applies, and the statement is true in this case also.

EXERCISE XXXIII

Prove that the series:

1. $1 + \frac{1}{2} + \frac{1 \,.\, 3}{2 \,.\, 4} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} + \ldots$ is divergent.

2. $1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$ is convergent if $b - 1 > a > 0$, its sum being $\frac{b-1}{b-a-1}$.

3. $1 + \frac{1}{2} \cdot \frac{a}{b} + \frac{1 \,.\, 3}{2 \,.\, 4} \cdot \frac{a(a+1)}{b(b+1)} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \cdot \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$ is convergent if $2(b-a) > 1$.

4. If $\dfrac{u_n}{u_{n+1}} = \dfrac{n^p + an^{p-1} + bn^{p-2} + \ldots + k}{n^p + a'n^{p-1} + b'n^{p-2} + \ldots + k'}$, show that Σu_n is convergent if $a - a' > 1$ and divergent if $a - a' \leqslant 1$.

5. The series $\Sigma x^n \cos n\theta$ and $\Sigma x^n \sin n\theta$ are absolutely convergent if $|x| < 1$.

6. The series $\Sigma \dfrac{1}{n^2} \cos n\theta$ and $\Sigma \dfrac{1}{n^2} \sin n\theta$ are absolutely convergent.

7. The series $\Sigma \dfrac{1}{n} \cos n\theta$ and $\Sigma \dfrac{1}{n} \sin n\theta$ are convergent excepting that the first series diverges when θ is an even multiple of π.

8. The series $\Sigma \dfrac{1}{n} \cos n\theta$ and $\Sigma \dfrac{1}{n} \sin n\theta$ are not absolutely convergent excepting the last series, when θ is a multiple of π.

[Use the following:

$|\cos n\theta| \geqslant \cos^2 n\theta \geqslant \frac{1}{2}(1 + \cos 2n\theta)$ and $|\sin n\theta| \geqslant \sin^2 n\theta \geqslant \frac{1}{2}(1 - \cos 2n\theta)$.]

9. If $u_1, u_2, u_3, \ldots$ is a decreasing sequence of positive terms tending to zero as a limit, the series

$$u_1 - \tfrac{1}{2}(u_1 + u_2) + \tfrac{1}{3}(u_1 + u_2 + u_3) - \ldots$$

is convergent.

[Denote the series by $v_1 - v_2 + v_3 - \ldots$. Using Ch. XV, 9, (3), show that $v_1, v_2, v_3, \ldots$ is a decreasing sequence tending to zero as a limit.]

10. Discuss the convergence of Σu_n where

$$u_n = \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right) \sin n\theta.$$

11. Show that, if $0 < p < \frac{1}{2}$, the rule for multiplication fails to obtain a convergent series for

$$\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots\right)^2$$

[Show that the series is $1 - d_1 + d_2 - \ldots$ where

$$d_n = \Sigma_{r=1}^{r=n} \frac{1}{r^p (n - r + 1)^p}$$

and

$$r(n - r + 1) \leqslant \tfrac{1}{4}(n+1)^2.]$$

12. If $a_n = \left(1 + \dfrac{x}{n}\right) e^{-\frac{n}{x}}$, show that the series whose nth term is $a_n - 1$ is absolutely convergent for all values of x.

[$a_n - 1 = -\dfrac{1}{2}\dfrac{x^2}{n^2} + \dfrac{1}{3}\dfrac{x^3}{n^3} - \ldots$; therefore $n^2 . |a_n - 1| \to \dfrac{1}{2}|x|^2$ as $n \to \infty$. Also $\Sigma 1/n^2$ is convergent.]

13. If $F(\alpha, \beta, \gamma, x) = 1 + \dfrac{\alpha\beta}{1 . \gamma} x + \dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2 + \ldots$ and $|x| < 1$, prove that

$$\gamma F(\alpha, \beta, \gamma, x) - (\gamma - \alpha) F(\alpha, \beta + 1, \gamma + 1, x) = \alpha(1 - x) F(\alpha + 1, \beta + 1, \gamma + 1, x).$$

Under what circumstances is this true (i) when $x = 1$, and (ii) when $x = -1$?

CHAPTER XXI

BINOMIAL AND MULTINOMIAL THEOREMS

1. A General Statement. The Binomial theorem asserts that the sum of the series

$$1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots,$$

when convergent, is one of the values of $(1+x)^n$. (Of course, in a complete statement, this value must be specified.)

The series terminates if, and only if, n is a positive integer; in which case the sum is $(1+x)^n$.

This leads to the following theorem, which enables us to deal with the case in which n is any rational number.

2. Vandermonde's Theorem. The following notation is often used; we write

$$x_r=x(x-1)(x-2)\ldots(x-r+1),$$

where x is any number whatever. Vandermonde's theorem asserts that *if m and n are any numbers whatever, then*

$$(m+n)_r=m_r+C_1^r m_{r-1}n_1+C_2^r m_{r-2}n_2+\ldots+n_r. \quad \ldots\ldots\ldots\ldots\text{(A)}$$

Proof. First suppose that m and n are positive integers; then

$$(1+x)^m=1+\frac{m_1}{\lfloor 1}x+\frac{m_2}{\lfloor 2}x^2+\ldots+\frac{m_r}{\lfloor r}x^r+\ldots,$$

$$(1+x)^n=1+\frac{n_1}{\lfloor 1}x+\frac{n_2}{\lfloor 2}x^2+\ldots+\frac{n_r}{\lfloor r}x^r+\ldots.$$

The coefficient of x^r in the product of these series is therefore equal to the coefficient of x^r in $(1+x)^{m+n}$, that is to say,

$$\frac{(m+n)_r}{\lfloor r}=\frac{m_r}{\lfloor r}+\frac{m_{r-1}n_1}{\lfloor r-1\ \lfloor 1}+\frac{m_{r-2}n_2}{\lfloor r-2\ \lfloor 2}+\ldots+\frac{n_r}{\lfloor r}.$$

Multiplying by $\lfloor r$, we obtain equation (A).

This equation holds for all positive integral values of m and n; and since each side is a *polynomial* in m, n it is true for *all* values of m, n.

3. Case of a Rational Index. *If n is any rational number and $-1<x<1$, the sum of the series*

$$1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots$$

is the real positive value of $(1+x)^n$.

We first give what is essentially *Euler's* proof, although in the last step we have to assume a property of infinite series which is not proved till later.

Euler's proof. The series is absolutely convergent if $|x|<1$; denote its sum by $f(n)$, and let

$$n_r=n(n-1)(n-2)\ldots(n-r+1),$$

so that

$$f(n)=1+n_1x+\frac{n_2}{\lfloor 2}x^2+\ldots+\frac{n_r}{\lfloor r}x^r+\ldots.$$

For all values of m and n, the series corresponding to $f(m)$, $f(n)$ are absolutely convergent, therefore by the rule for the multiplication of series,

$$f(m)\,.\,f(n)=1+d_1x+d_2x^2+\ldots+d_rx^r+\ldots,$$

where

$$d_r=\frac{m_r}{\lfloor r}+\frac{m_{r-1}n_1}{\lfloor r-1\,\lfloor 1}+\frac{m_{r-2}n_2}{\lfloor r-2\,\lfloor 2}+\ldots+\frac{n_r}{\lfloor r}.$$

Hence, by Vandermonde's theorem,

$$d_r=\frac{(m+n)_r}{\lfloor r},$$

and

$$f(m)\,.\,f(n)=1+(m+n)_1x+\frac{(m+n)_2}{\lfloor 2}x^2+\ldots,$$

so that

$$f(m)\,.\,f(n)=f(m+n). \qquad \ldots\ldots\ldots\

Now $f(n)$ does not vanish for any value of x in the interval, and consequently its sign is the same throughout the interval.

But when $x=0$, $f(n)=1$, therefore the sign is positive and $f(n)$ is equal to the real positive value of $(1+x)^n$.

Negative index. Let n be a positive rational. Putting $m=-n$ in equation (**B**),

$$f(n)\,.\,f(-n)=f(0)=1\;; \qquad \text{.....(C)}$$

therefore
$$f(-n)=\frac{1}{(1+x)^n}=(1+x)^{-n}.$$

NOTE. (i) Equations (**A**)-(**C**) hold when x is complex, provided that $|x|<1$. Hence *if x is complex, $|x|<1$ and n rational, $f(x)$* is one of the values of $(1+x)^n$. In particular, if n is a positive integer, $f(-n)=(1+x)^{-n}$.

(ii) *If n and x are real and $|x|<1$,* it can be shown that $f(n)$ is continuous for all values of n. Hence if $(1+x)^n$ is defined as in Ch. XIII, 7, when n is irrational, we have $(1+x)^n=f(n)$.

4. Second Proof for the Case of a Real Index. The following proof depends on very elementary considerations and, although it is unsuitable for reproduction *in toto*, the reader will find it a very useful exercise to work through the details. We shall prove that *if n and x are real and $|x|<1$, then*

$$(1+x)^n=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r+\ldots,$$

where $(1+x)^n$ has its real positive value.

Proof. Let $t_{n,\,r}$ be the rth term and $s_{n,\,r}$ the sum to r terms of the series. Also let

$$y_{n,\,r}=(1+x)^n-s_{n,\,r}.$$

First suppose that $0<x<1$, then

$$y_{n,\,1}=(1+x)^n-1\gtrless 0 \quad \text{according as} \quad n\gtrless 0. \qquad \text{.....(A)}$$

Also
$$\frac{dy_{n,\,r}}{dx}=ny_{n-1,\,r-1} \text{ and } y_{n,\,r}=0 \text{ when } x=0. \qquad \text{.....(B)}$$

If for any assigned values of n and r it can be shown that $y_{n-1,\,r-1}$ is of invariable sign for $0<x<1$, then by (**B**), $y_{n,\,r}$ must have the same sign as $ny_{n-1,\,r-1}$.

Repeating this reasoning when $n-1$, $n-2$, ... are substituted for n and observing that, by (**A**), $y_{n-r+1,\,1}$ has the same sign as $n-r+1$, it follows that the three expressions

$$y_{n,\,r}, \quad n(n-1)\ldots(n-r+2)y_{n-r+1,\,1}, \quad \text{and} \quad n(n-1)\ldots(n-r+1)$$

have the same sign. Therefore $y_{n,\,r}$ *has the same sign as* $t_{n,\,r+1}$.

After a certain stage, say for $r>$ some fixed number k, the signs of

$t_{n,\,r+1}$, and consequently those of $y_{n,\,r}$ are alternately + and −, as r increases. Therefore $(1+x)^n$ lies between $s_{n,\,r}$ and $s_{n,\,r+1}$ for $r>k$.

But since the series is convergent, $s_{n,\,r+1}-s_{n,\,r}\to 0$ as $r\to\infty$.

Therefore $$s_{n,\,r}\to(1+x)^n \quad \text{as} \quad r\to\infty,$$

which proves this case of the theorem.

Next let $x<0$. Changing the sign of x, it has to be shown that *if* $0<x<1$, *then*

$$(1-x)^n=1-nx+\frac{n(n-1)}{\lfloor 2}x^2-\ldots+(-1)^r\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots.$$

Here $$y_{n,\,r}=(1-x)^n-s_{n,\,r},$$

so that $$y_{n,\,1}=(1-x)^n-1\gtrless 0 \quad \text{according as} \quad n\lesseqgtr 0; \qquad \ldots\ldots\ldots\ldots\text{(C)}$$

also $$\frac{dy_{n,\,r}}{dx}=-ny_{n-1,\,r-1} \quad \text{and} \quad y_{n,\,r}=0 \quad \text{when} \quad x=0. \qquad \ldots\ldots\ldots\ldots\text{(D)}$$

Reasoning as in the previous case and observing that by (C), $y_{n-r+1,\,1}$ and $n-r+1$ have *opposite* signs, it follows that the three expressions

$$y_{n,\,r}, \quad (-1)^{r-1}n(n-1)\ldots(n-r+2)y_{n-r+1,\,1}$$

and $$(-1)^r n(n-1)\ldots(n-r+1)$$

have the same sign. Hence, as before,

$$y_{n,\,r} \quad \textit{has the same sign as} \quad t_{n,\,r+1}. \qquad \ldots\ldots\ldots\ldots\text{(E)}$$

After a certain stage, say for $r>k$, the sign of $t_{n,\,r+1}$ is invariable, and we proceed thus:

Let $$z_{n,\,r}=y_{n,\,r}-t_{n,\,r+1}(1-x)^{-1}.$$

For every r, we have

$$t_{n,\,r+1}-xt_{n,\,r}=t_{n,\,r+1}\left(1+\frac{r}{n-r+1}\right)=t_{n,\,r+1}\frac{n+1}{n-r+1}=t_{n+1,\,r+1}.$$

Therefore

$$(1-x)s_{n,\,r}=s_{n+1,\,r}-xt_{n,\,r} \quad \text{and} \quad (1-x)y_{n,\,r}=y_{n+1,\,r}+xt_{n,\,r}. \qquad \ldots\ldots\text{(F)}$$

Hence
$$\begin{aligned}(1-x)z_{n,\,r}&=(1-x)y_{n,\,r}-t_{n,\,r+1}\\&=y_{n+1,\,r}+xt_{n,\,r}-t_{n,\,r+1}\\&=y_{n+1,\,r}-t_{n+1,\,r+1}\\&=y_{n+1,\,r+1}. \qquad \ldots\ldots\ldots\ldots\text{(G)}\end{aligned}$$

Hence by (E), $z_{n,\,r}$ has the same sign as $t_{n+1,\,r+2}$. Now

$$t_{n+1,\,r+2}=-\frac{n+1}{r+1}t_{n,\,r+1},$$

so that by (E)

$$z_{n,\,r} \quad \textit{has the same sign as} \quad -(n+1)y_{n,\,r}. \qquad \ldots\ldots\ldots\ldots\text{(H)}$$

(i) *If* $n > -1$ it follows that $y_{n,r}$ and $z_{n,r}$ have opposite signs. Therefore $y_{n,r}$ lies between 0 and $t_{n,r+1}.(1-x)^{-1}$. But $t_{n,r+1} \to 0$ as $r \to \infty$, therefore $y_{n,r} \to 0$ and $s_{n,r} \to (1-x)^n$.

(ii) *If* $n < -1$, then $-m-1 < n \leqslant -m$ where m is a positive integer. If $y_{n+1,r} \to 0$ as $r \to \infty$, since $t_{n,r} \to 0$ it follows from (**F**) that $y_{n,r} \to 0$.

Now $n+m > -1$, and therefore, by case (i), $y_{n+m,r} \to 0$. Hence in succession it follows that $y_{n+m-1,r}$, $y_{n+m-2,r}, \dots y_{n,r}$ all tend to zero.

Hence $s_{n,r} \to (1-x)^n$, and the theorem is proved in all cases.

5. We shall prove that *if* n *is a negative integer,* z *complex and* $|z|<1$, *then*

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{\underline{|2}} z^2 + \dots .$$

Using the same notation as before, we see that the series is convergent (Chap. XX, 21) and $t_{n,r} \to 0$ as $r \to \infty$. Also $(1+z)y_{n,r} = y_{n+1,r} - zt_{n,r}$ (compare equation (**F**)).

Hence as before, if $y_{n+1,r} \to 0$, then $y_{n,r} \to 0$.

Now when $n = -1$,

$$y_{n,r} = \frac{1}{1+z} - (1 - z + z^2 - \dots + (-1)^{r-1} z^{r-1}) = (-1)^r \frac{z^r}{1+z} \to 0.$$

Hence in succession $y_{n,r} \to 0$ for $n = -2, -3, -4, \dots$. So that

$$s_{n,r} \to (1+z)^n$$

when n is a negative integer.

The theorem is obviously true when n is a positive integer. The case in which n is a fraction is considered later.

6. The reader should be familiar with the following particular instances; in each case it is assumed that $|x|<1$:

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{\underline{|2}}(-x)^2 + \dots = 1 + x + x^2 + x^3 + \dots ,$$

and in a similar way

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots ,$$

$$(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \frac{3.4.5.6}{1.2.3.4}x^4 + \dots$$

$$= \tfrac{1}{2}[1.2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + \dots],$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\underline{|2}}x^2 + \frac{n(n+1)(n+2)}{\underline{|3}}x^3 + \dots ,$$

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots .$$

Ex. 1. *Expand* $1/(3+2x)^2$ *to three terms* (i) *in ascending powers of* x; (ii) *in descending powers of* x; (iii) *say for what values of* x *each expansion is valid.*

(i) $$\frac{1}{(3+2x)^2}=\frac{1}{9}\left(1+\frac{2}{3}x\right)^{-2}=\frac{1}{9}\left\{1-2\left(\frac{2}{3}x\right)+3\left(\frac{2}{3}x\right)^2-\dots\right\};$$

$$\therefore\ 1/(3+2x)^2=\frac{1}{9}-\frac{4}{27}x+\frac{4}{27}x^2 \text{ to three terms.}$$

(ii) $$\frac{1}{(2x+3)^2}=\frac{1}{4x^2}\left(1+\frac{3}{2x}\right)^{-2}=\frac{1}{4x^2}\left\{1-2\left(\frac{3}{2x}\right)+3\left(\frac{3}{2x}\right)^2-\dots\right\};$$

$$\therefore\ \frac{1}{(2x+3)^2}=\frac{1}{4x^2}-\frac{3}{4x^3}+\frac{27}{16x^4} \text{ to three terms.}$$

(iii) The first expansion is valid if $\left|\frac{2}{3}x\right|<1$, that is if $|x|<\frac{3}{2}$.

The second is valid if $\left|\frac{3}{2x}\right|<1$, that is if $|x|>\frac{3}{2}$.

Ex. 2. *Sum the series* $1+\frac{5}{8}+\frac{5.8}{8.12}+\frac{5.8.11}{8.12.16}+\dots$ *to* ∞.

To see if this can be done by the Binomial theorem, take some particular term, say the fourth, and proceed thus:

$$\frac{5.8.11}{8.12.16}=\frac{\frac{5}{3}\cdot\frac{8}{3}\cdot\frac{11}{3}}{2.3.4}\left(\tfrac{3}{4}\right)^3=2\cdot\frac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdot\frac{11}{3}}{1.2.3.4}\cdot\left(\tfrac{3}{4}\right)^4,$$

which is equal to $2n(n+1)(n+2)(n+3)x^4/\underline{|4}$, where $n=\frac{2}{3}$ and $x=\frac{3}{4}$. Let s be the sum of the series; treating every term as above, we have

$$s=2\left\{\frac{\frac{2}{3}}{1}\cdot\frac{3}{4}+\frac{\frac{2}{3}\cdot\frac{5}{3}}{1.2}\left(\tfrac{3}{4}\right)^2+\frac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}}{1.2.3}\left(\tfrac{3}{4}\right)^3+\dots\right\}$$

$$=2[(1-\tfrac{3}{4})^{-\frac{2}{3}}-1]=4\sqrt[3]{2}-2.$$

7. Numerically Greatest Term in the Expansion of $(1+x)^n$ where $|x|<1$.

Let u_r be the rth term, and let accents indicate numerical values so that $x'=|x|$ and $u_r'=|u_r|$; *if* n *is positive,*

$$\frac{u_{r+1}'}{u_r'}=\frac{|n-r+1|}{r}\cdot x'.$$

Let k be the positive integer such that $k<n+1<k+1$.

If $r\geqslant k+1$, $$\frac{u_{r+1}'}{u_r'}=\frac{r-n-1}{r}x'<1;$$

therefore u_{k+1}', u_{k+2}', ... is a decreasing sequence, and the numerically greatest term is among the first $k+1$ terms.

If $r\leqslant k$, then $u_{r+1}'\gtreqless u_r'$ according as $(n+1-r)x'\gtreqless r$,

that is according as $r\gtreqless\frac{(n+1)x'}{1+x'}$.

(i) *If* $\frac{(n+1)x'}{1+x'} = i+f$, where i is a positive integer and f is a positive proper fraction, the $(i+1)$th term is numerically the greatest term.

(ii) *If* $\frac{(n+1)x'}{1+x'} = i$, where i is a positive integer, then, numerically, the ith term is equal to the $(i+1)$th, and each is greater than any other term.

Next, let n be negative and equal to $-m$, then

$$\frac{u_{r+1}'}{u_r'} = \frac{m+r-1}{r} \cdot x' ;$$

therefore $u_{r+1}' \gtreqless u_r'$ according as $(m+r-1)x' \gtreqless r$,

that is according as $r \gtreqless \frac{(m-1)x'}{1-x'}$.

It follows that: (i) *If* $m \leqslant 1$, the first term is the greatest.

(ii) *If* $\frac{(m-1)x'}{1-x'} = i+f$, where i is a positive integer or zero and f is a positive proper fraction, the $(i+1)$th term is the greatest.

(iii) *If* $\frac{(m-1)x'}{1-x'} = i$, where i is a positive integer, the ith term is equal to the $(i+1)$th, and each of these is greater than any other.

We have thus proved the following important property of a binomial series: *If* $|x|<1$, *the terms in the expansion of* $(1+x)^n$ *either decrease numerically from the beginning of the series or else they increase numerically up to a greatest term (or to two equal terms) and then decrease numerically.*

Ex. 1. *Which is the numerically greatest term (or terms) in the expansion of* $(1+x)^{\frac{25}{2}}$ *when* $x=\frac{4}{5}$ *?*

Let u_r denote the rth term;

$$\frac{u_{r+1}}{u_r} = \frac{\frac{25}{2}-r+1}{r} \cdot \frac{4}{5} = \frac{54-4r}{5r} . \quad \text{.....(A)}$$

First consider the terms for which u_{r+1}/u_r is negative. This will be the case if $4r>54$ or if $r \geqslant 14$. For such values of r,

$$\left|\frac{u_{r+1}}{u_r}\right| = \frac{4r-54}{5r} < 1 ; \quad \therefore \ |u_{r+1}|<|u_r| ;$$

Hence the greatest term or terms must be included among the first 14 terms of the expansion.

If $r \leqslant 13$, u_{r+1}/u_r is positive, and from (A) it follows that

$$|u_{r+1}| \gtreqless |u_r| \text{ according as } 54-4r \gtreqless 5r,$$

$$\text{that is according as } r \lesseqgtr 6.$$

Hence, the sixth term is equal to the seventh, and each of these is greater than any other term.

8. Approximate Values. The Binomial series can be used to obtain approximate values, as follows :

Ex. 1. *Calculate* $\sqrt[3]{2}$ *to six places of decimals.*

The perfect cubes are 1, 8, 27, 64, 125, ... Search for two of these such that one is roughly twice the other. We have

$$2\times 64 = 125+3\,; \quad \therefore\ 2 = \frac{125}{64}\left(1+\frac{3}{125}\right); \quad \therefore\ \sqrt[3]{2} = \frac{5}{4}\left(1+\frac{3}{5^3}\right)^{\frac{1}{3}};$$

$$\therefore\ \sqrt[3]{2} = \frac{5}{4}\left\{1+\frac{1}{3}\cdot\frac{3}{5^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\,.\,2}\cdot\frac{3^2}{5^6}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\,.\,2\,.\,3}\cdot\frac{3^3}{5^9}+\ldots\right\}$$

$$= \frac{5}{4}+\frac{1}{4\,.\,5^2}-\frac{1}{4\,.\,5^5}+\frac{1}{12\,.\,5^7}-\frac{1}{6\,.\,5^{10}}+\ldots \quad \text{.....(A)}$$

Denoting this series by $u_1+u_2-u_3+u_4-u_5+\ldots$, we have

$$u_1 = \frac{5}{4} = 1{\cdot}25,$$

$$u_2 = \frac{1}{10^2} = 0{\cdot}01, \qquad u_3 = \frac{u_2}{5^3} = 0{\cdot}000\ \ 08$$

$$u_4 = \frac{u_3}{3\,.\,5^2} = \underline{0{\cdot}000\ \ 001\ \ 07}, \qquad u_5 = \frac{2u_4}{5^3} = \underline{0{\cdot}000\ \ 000\ \ 02}.$$
$$1{\cdot}260\ \ 001\ \ 07 \qquad\qquad 0{\cdot}000\ \ 080\ \ 02$$

Therefore, $\sqrt[3]{2} = 1{\cdot}260\ \ 001\ \ 07 - 0{\cdot}000\ \ 080\ \ 02$ nearly

$= 1{\cdot}259\ \ 921\ \ 05$ nearly. .....(B)

To estimate the possible error in this result, let s_n be the sum to n terms of the series (A) and R_n the remainder after n terms. The error arises from two causes :

(i) *The error caused by taking* s_5 *as the value of* $\sqrt[3]{2}$ is R_5. Now, in the series (A), the terms beginning with the second are alternately positive and negative, also each is numerically less than the preceding ; therefore R_5 is positive and less than u_6.

Now $$u_6 = -\frac{\frac{1}{3}-4}{5}\cdot\frac{3}{5^3}u_5 = \frac{11}{5^4}u_5 < 0{\cdot}000\ 000\ 000\ 4\,;$$

$\therefore$ the error is less than 0·000 000 000 4.

(ii) *The error in the calculated value of* s_5 arises from the errors in the calculated values of u_4 and u_5. These cannot exceed 0·000 000 005, hence the error in the calculated value of s_5 cannot exceed 0·000 000 01.

Adding the results of (i) *and* (ii), we see that the error in the result (B) is numerically less than 0·000 000 010 4. Thus, to six places of decimals,

$$\sqrt[3]{2} = 1{\cdot}259921.$$

Ex. 2. *If* x *is small and* $n>1$, *obtain an approximation to* $(1+x)^{\frac{1}{n}}$ *in the form* $\dfrac{p+qx}{p'+q'x}$ *where* p, q, p', q' *are independent of* x.

We have

$$(1+x)^{\frac{1}{n}} = 1+\frac{x}{n}-\frac{n-1}{\underline{2}}\left(\frac{x}{n}\right)^2+\frac{(n-1)(2n-1)}{\underline{3}}\left(\frac{x}{n}\right)^3-\ldots, \quad \text{.....(A)}$$

and $$\frac{1+ax}{1+bx} = 1+\frac{(a-b)x}{1+bx} = 1+(a-b)x-(a-b)bx^2+(a-b)b^2x^3-\ldots, \quad \text{.....(B)}$$

Choose a and b so that

$$a-b=1/n \quad \text{and} \quad (a-b)b=(n-1)/2n^2;$$
$$\therefore\ b=(n-1)/2n \quad \text{and} \quad a=(n+1)/2n.$$

Substituting these values for a and b in (**B**), we have

$$\frac{2n+(n+1)x}{2n+(n-1)x}=1+\frac{x}{n}-\frac{n-1}{2}\cdot\left(\frac{x}{n}\right)^2+\left(\frac{n-1}{2}\right)^2\left(\frac{x}{n}\right)^3-\ldots\ldots \qquad \text{(C)}$$

If x is small, the series (**A**) and (**C**) are both convergent, and since they are identical as far as the terms involving x^2, it follows that

$$(1+x)^{\frac{1}{n}}=\frac{2n+(n+1)x}{2n+(n-1)x} \quad \textit{nearly.}$$

9. Theorem. *If* $n>1$ *and* $0<x<1$, *then* $\dfrac{2n+(n+1)x}{2n+(n-1)x}$ *is an approximate value of* $(1+x)^{\frac{1}{n}}$, *with an error in defect less than* $\dfrac{n^2-1}{12n^3}x^3$.

By multiplying each side of equation (**A**) in the last example by

$$2n+(n-1)x,$$

it can be shown that

$$(2n+\overline{n-1}\,.\,x)(1+x)^{\frac{1}{n}}-(2n+\overline{n+1}\,.\,x)$$
$$=\frac{n^2-1}{6n^2}x^3\left\{1-\frac{2n-1}{4n}\cdot 2x+\frac{(2n-1)(3n-1)}{4n\,.\,5n}\cdot 3x^2-\ldots\right\}.$$

Denoting the series in brackets by

$$u_1-u_2+u_3-\ldots+(-1)^r u_r+\ldots,$$

we have

$$\frac{u_r}{u_{r-1}}=\frac{rn-1}{(r+2)n}\cdot\frac{r}{r-1}\cdot x.$$

Now $(rn-1)r<(r+2)(r-1)n$ provided that $r\geqslant 2$, for

$$(rn-1)r-(r+2)(r-1)n=-r-(r-2)n<0;$$

and since $0<x<1$, it follows that $u_r<u_{r-1}$.

The sum of the series is therefore positive and less than unity;

$$\therefore\ 0<(1+x)^{\frac{1}{n}}-\frac{2n+(n+1)x}{2n+(n-1)x}<\frac{n^2-1}{6n^2}\cdot\frac{x^3}{2n+(n-1)x};$$

and, since $n>1$ and x is positive, the last expression is less than

$$(n^2-1)x^3/12n^3,$$

whence the result follows.

Ex. 1. *Show that* 635/504 *is an approximation to* $\sqrt[3]{2}$ *with an error in defect less than* 0·0000005.

Here $\sqrt[3]{2}=\frac{5}{4}\left(1+\frac{3}{125}\right)^{\frac{1}{3}}$; hence, an approximation is

$$\frac{5}{4}\cdot\frac{750+12}{750+6},\quad \text{or}\quad \frac{635}{504},$$

with an error in defect $<\frac{5}{4}\cdot\frac{8}{12\,.\,27}\cdot\frac{3^3}{125^3}<\frac{43}{10^8}<\frac{5}{10^7}$.

EXERCISE XXXIV

1. If x is positive, which is the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$?

2. If x is positive, the terms in the expansion of $(1-x)^{\frac{19}{2}}$, beginning with a certain term, are all positive. Prove this, and find which is the first of this series of positive terms.

Which is the numerically greatest term or terms in the expansions of

3. $(1+x)^{\frac{35}{2}}$ when $x=\frac{3}{4}$?

4. $(1-x)^{-\frac{3}{2}}$ when $x=\frac{9}{10}$?

5. $(1-x)^{-13}$ when $x=\frac{1}{3}$?

6. $(1+x)^{-\frac{5}{3}}$ when $x=\frac{15}{17}$?

7. Between what numbers must x lie in order that, in the expansion of $(1-x)^{-\frac{3}{2}}$, the third term may be the greatest?

8. If $x>0$, which is the first negative term in the expansion of

$$(11-13x)/(1-x)^3?$$

9. If $0<x<1$, show that (i) $\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2$ *nearly*, the error being in defect and less than $\frac{1}{16}x^3$.

(ii) $\sqrt{1-x}=1-\frac{1}{2}x-\frac{1}{8}x^2$ *nearly*, the error being in excess and numerically less than $\frac{1}{16}x^3/(1-x)$.

10. If x is large and positive, show that

$$\sqrt[3]{x^3+1}=x+\frac{1}{3}\cdot\frac{1}{x^2}-\frac{1}{9}\cdot\frac{1}{x^5} \text{ nearly},$$

the error being in defect and less than $5/81x^8$.

11. If $0<x<1$ and R_n is the remainder after n terms in the expansion of $1/(1-x)^2$, show that

$$R_n<(n+1)x^n/(1-x)^2.$$

12. If $0<x<0{\cdot}01$, show that (i) $1+2x+3x^2$ is an approximate value of $1/(1-x)^2$ with an error in defect less than $0{\cdot}000005$.

(ii) $1-2x+3x^2$ is an approximate value of $1/(1+x)^2$ with an error in excess less than $0{\cdot}000004$.

13. If $0<x<1$, and R_n is the remainder after n terms in the expansion of $1/(1-x)^3$, show that

$$R_n<\tfrac{1}{2}(n+1)(n+2)x^n/(1-x)^3.$$

14. If $0<x<0{\cdot}01$, show that (i) $1+3x+6x^2$ is an approximate value of $1/(1-x)^3$ with an error in defect less than $0{\cdot}000011$.

(ii) $1-3x+6x^2$ is an approximate value of $1/(1+x)^3$ with an error in excess less than $0\,00001$.

15. If ϵ is small, show that, neglecting cubes and higher powers of ϵ,

(i) $\frac{1}{2}\left(\frac{x}{\sqrt{x}-\epsilon}+\sqrt{x}-\epsilon\right)=\sqrt{x}+\frac{\epsilon^2}{2\sqrt{x}}$.

(ii) $\frac{1}{3}\left\{\frac{x}{(\sqrt[3]{x}-\epsilon)^2}+2(\sqrt[3]{x}-\epsilon)\right\}=\sqrt[3]{x}+\frac{\epsilon^2}{\sqrt[3]{x}}$.

16. *If we have tables giving the square roots and cube roots of numbers to f significant figures, show that the roots can be found* by division only, *to 2f significant figures as follows.*

If $\sqrt{x}=a$ *to f significant figures, then* $\sqrt{x}=\frac{1}{2}\left(\frac{x}{a}+a\right)$ *to 2f significant figures.*

If $\sqrt[3]{x}=a$ *to f significant figures, then* $\sqrt[3]{x}=\frac{1}{3}\left(\frac{x}{a^2}+2a\right)$ *to 2f significant figures.*

Show also that in each case the error is in excess.

[Put $a=\sqrt{x}-\epsilon$, and use Ex. 15.]

17. Given $\sqrt{2}=1{\cdot}41421$ (approx.), show, by division, that

$$\sqrt{2}=1{\cdot}41421356237 \text{ (approx.)}.$$

18. Given $\sqrt[3]{10}=2{\cdot}1544$ (approx.), show, by division, that

$$\sqrt[3]{10}=2{\cdot}154434690 \text{ (approx.)}.$$

19. Show that (i) if x is small, $\sqrt{(x^2+4)}-\sqrt{(x^2+1)}=1-\frac{1}{4}x^2+\frac{7}{64}x^4$ *nearly*;

(ii) if x is large, $\sqrt{(x^2+4)}-\sqrt{(x^2+1)}=\frac{3}{2x}\left(1-\frac{5}{4x^2}+\frac{21}{8x^4}\right)$ *nearly.*

20. If x is small, show that approximately

(i) $\sqrt{(4-3x)}\,.\,\sqrt[3]{(8+3x)}\div(9-\frac{2}{3}x)^{\frac{3}{2}}=\frac{4}{27}(1-\frac{5}{36}x)$;

(ii) $\dfrac{(1-2x)^{-\frac{1}{2}}-(1+2x)^{\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}-(1+x)^{\frac{1}{2}}}=4+2x$;

(iii) $(1+4x+10x^2)^{\frac{1}{2}}=1+2x+3x^2-6x^3$;

(iv) $(1+4x+10x^2)^{-\frac{1}{2}}=1-2x+x^2+10x^3$;

(v) $(1+x+x^2+x^3)^{\frac{1}{5}}=1+\frac{1}{5}x+\frac{3}{25}x^2+\frac{11}{125}x^3$.

In each of the examples 21-26, (i) prove the given equality; (ii) by expanding the right-hand side to four terms, deduce an approximate value of the given surd; (iii) find an upper limit to the numerical value of the error.

21. $\sqrt[3]{1001}=10(1+0{\cdot}001)^{\frac{1}{3}}$.

22. $\sqrt{2}=\frac{7}{5}(1-0{\cdot}02)^{-\frac{1}{2}}$.

23. $\sqrt[3]{4}=\frac{8}{5}(1+0{\cdot}024)^{-\frac{1}{3}}$.

24. $\sqrt[3]{5}=\frac{5}{3}(1+0{\cdot}08)^{\frac{1}{3}}$.

25. $\sqrt[5]{3}=\frac{5}{4}(1-0{\cdot}01696)^{\frac{1}{5}}$.

26. $\sqrt[5]{4}=\frac{4}{3}(1-\frac{13}{256})^{\frac{1}{5}}$.

27. If a is nearly equal to b, show that

$$\sqrt[n]{\frac{a}{b}}=\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b} \text{ nearly},$$

and that if $a>b$, the error is in defect and is less than

$$\frac{n^2-1}{12n^3}\left(\frac{a-b}{b}\right)^3.$$

[Assume the result of Art. 9, and put $x=(a-b)/b$.]

28. Show that $\sqrt[5]{\frac{543}{540}}=\frac{2709}{2706}$ *nearly*, the error being in defect and less than $3/10^9$.

29. If $a-b=h$, where h is small, show that

(i) $\sqrt{\frac{a}{b}}=1+\frac{1}{2}\frac{h}{b}-\frac{1}{8}\frac{h^2}{b^2}+\frac{1}{16}\frac{h^3}{b^3}-\frac{5}{128}\frac{h^4}{b^4}+\dots$.

(ii) $\frac{a}{a+b}=\frac{1}{2}+\frac{1}{4}\frac{h}{b}-\frac{1}{8}\frac{h^2}{b^2}+\frac{1}{16}\frac{h^3}{b^3}-\frac{1}{32}\frac{h^4}{b^4}+\dots$.

(iii) Hence show that if a is nearly equal to b, then

$$\sqrt{\frac{a}{b}}=\frac{a}{a+b}+\frac{1}{4}\frac{a+b}{b}, \text{ nearly},$$

the error being approximately equal to $(a-b)^4/128b^4$.

10. Application of the Binomial Theorem to the Summation of Series.

(1) Many series may be summed by expanding some function of x as a convergent power series in two different ways and equating coefficients.

Ex. 1.* If $u_r=r(r+1)(r+2)\dots(r+q-1)$

and $v_r=(n-r)(n-r+1)(n-r+2)\dots(n-r+p-1)$,

prove that

$$u_1v_1+u_2v_2+u_3v_3+\dots+u_{n-1}v_{n-1}=\frac{\lfloor p\,\lfloor q}{\lfloor p+q+1}\cdot\frac{\lfloor p+q+n-1}{\lfloor n-2}.$$

It will be easily seen that

$$u_1=\lfloor q,\quad u_2=\lfloor q\,.\,(q+1),\quad u_3=\lfloor q\,.\,\frac{(q+1)(q+2)}{\lfloor 2},\quad \dots\dots\dots\dots,\quad u_{n-1}=\lfloor q\,.\,\frac{(q+1)(q+2)\dots(q+n-2)}{\lfloor n-2}.$$

$$v_{n-1}=\lfloor p,\quad v_{n-2}=\lfloor p(p+1),\quad v_{n-3}=\lfloor p\,.\,\frac{(p+1)(p+2)}{\lfloor 2},\quad \dots\dots\dots\dots,\quad v_1=\lfloor p\,.\,\frac{(p+1)(p+2)\dots(p+n-2)}{\lfloor n-2}.$$

therefore $u_1+u_2x+u_3x^2+\dots+u_{n-1}x^{n-2}+\dots=\lfloor q\,(1-x)^{-(q+1)}$,

and $v_{n-1}+v_{n-2}x+v_{n-3}x^2+\dots+v_1x^{n-2}+\dots=\lfloor p\,(1-x)^{-(p+1)}$.

Hence if S is the sum of the series in question,

$$S/\lfloor p\,\lfloor q=\text{coefft. of } x^{n-2} \text{ in } (1-x)^{-(p+1)}.(1-x)^{-(q+1)}$$
$$=\text{coefft. of } x^{n-2} \text{ in } (1-x)^{-(p+q+2)}$$
$$=\lfloor p+q+n-1\,/\,(\lfloor p+q+1\,\lfloor n-2).$$

(2) **Theorem.** *If* $f(x)=a_0+a_1x+a_2x^2+a_3x^2+\dots+a_nx^n+\dots$ *for* $|x|<k$, *where* k *is any positive number, then* $a_0+a_1+a_2+\dots+a_n$ *is equal to the coefficient of* x^n *in the expansion of* $f(x)/(1-x)$.

For $$\frac{f(x)}{1-x}=(a_0+a_1x+a_2x^2+\dots)(1+x+x^2+\dots),$$

and the series are absolutely convergent for sufficiently small values of x.

* This equality is required in the Theory of Probabilities.

(3) In what follows, we suppose that $|x|<1$ except when n is a positive integer, so that all the series which occur are convergent.

We shall write $(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_rx^r + \ldots$;

then,
$$n(1+x)^{n-1} = n + \frac{n(n-1)}{\lfloor 1}x + \frac{n(n-1)(n-2)}{\lfloor 2}x^2 + \ldots$$
$$= c_1 + 2c_2x + 3c_3x^2 + \ldots + rc_rx^{r-1} + \ldots,$$

and, by a succession of similar steps,*

$$n(n-1)(1+x)^{n-2} = 1\,.\,2c_2 + 2\,.\,3c_3x + \ldots + r(r-1)c_rx^{r-2} + \ldots,$$
$$n(n-1)(n-2)(1+x)^{n-3} = 1\,.\,2\,.\,3c_3 + \ldots + r(r-1)(r-2)c_rx^{r-3} + \ldots,$$

and so on, as far as required.

Hence *we can sum the series* $\Sigma_{r=0}^{\infty} u_rc_rx^r$ *where* u_r *is a polynomial in* r.

For, dividing u_r and successive quotients by r, $r-1$, $r-2$, ... in this order we can find constants A_0, A_1, A_2, A_3, ... such that

$$u_r = A_0 + A_1r + A_2r(r-1) + A_3r(r-1)(r-2) + \ldots;$$

then, $\Sigma_{r=0}^{\infty} u_rc_rx^r = A_0(1+x)^n + A_1nx(1+x)^{n-1} + A_2n(n-1)x^2(1+x)^{n-2}$
$$+ A_3n(n-1)(n-2)x^3(1+x)^{n-3} + \ldots.$$

Ex. 2. *Find* $\Sigma_{r=0}^{n}(r+1)^3c_rx^r$.

Denoting the sum by S and proceeding as above, we find that

$$(r+1)^3 = 1 + 7r + 6r(r-1) + r(r-1)(r-2);$$
$$\therefore\ S = (1+x)^n + 7nx(1+x)^{n-1} + 6n(n-1)x^2(1+x)^{n-2} + n(n-1)(n-2)x^3(1+x)^{n-3}$$
$$= (1+x)^{n-3}\{1 + (7n+3)x + (6n^2+8n+3)x^2 + (n+1)^3x^3\}.$$

(4) Again, by expanding $(1+x)^{n+1}$, we can show that

$$\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + c_0x + \frac{c_1}{2}x^2 + \ldots + \frac{c_r}{r+1}x^{r+1} + \ldots,$$

and, by similar steps,

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} + \frac{1}{n+1}x + \frac{c_0}{1\,.\,2}x^2 + \ldots + \frac{c_r}{(r+1)(r+2)}x^{r+2} + \ldots,$$

$$\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)} = \frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+1)(n+2)}x + \frac{1}{2(n+1)}x^2$$
$$+ \frac{c_0}{1\,.\,2\,.\,3}x^3 + \ldots + \frac{c_r}{(r+1)(r+2)(r+3)}x^{r+3} + \ldots;$$

and so on. These results can also be obtained by integration.

* These results can also be obtained by differentiating with regard to x, although so far the process has not been justified in case of an infinite series.

Hence, *if u_r is a polynomial in r, we can sum the series*

$$\Sigma_{r=0}^{\infty} \frac{u_r}{(r+1)(r+2)(r+3)\dots(r+k)} c_r x^r,$$

For, dividing u_r and successive quotients by $r+k, r+k-1, \dots r+2, r+1$ in this order, we can find an equation of the form

$$\frac{u_r}{(r+1)(r+2)\dots(r+k)} = v_r + \frac{A_1}{r+1} + \frac{A_2}{(r+1)(r+2)} + \dots \frac{A_k}{(r+1)(r+2)\dots(r+k)},$$

where $A_k, A_{k-1}, \dots A_1$ are the remainders in the successive divisions and v_r is the final quotient, and then the sum of the series is

$$\Sigma v_r c_r x^r + \frac{A_1}{x} \Sigma \frac{c_r}{r+1} x^{r+1} + \frac{A_2}{x^2} \Sigma \frac{c_r}{(r+1)(r+2)} x^{r+2} + \dots .$$

The value of $\Sigma v_r c_r x^r$ is found by § (3), and the other sums are given by the above equations.

Further, *we can sum the series*

$$\Sigma_{r=0}^{\infty} \frac{u_r}{(r+a)(r+b)\dots(r+k)} c_r x^r,$$

where $a, b, \dots k$ are unequal positive integers of which k is the greatest.

For this can be reduced to the preceding by multiplying numerator and denominator of the fraction by such factors as will convert the denominator into $(r+1)(r+2)(r+3)\dots(r+k)$.

Ex. 2. *Find* $S = \Sigma_{r=0}^{\infty} \frac{1}{r+3} c_r x^r$.

We have $\frac{1}{r+3} = \frac{r^2+3r+2}{(r+1)(r+2)(r+3)} = \frac{1}{r+1} - \frac{2}{(r+1)(r+2)} + \frac{2}{(r+1)(r+2)(r+3)}$;

hence, $S = \frac{1}{x} \Sigma \frac{c_r}{r+1} x^{r+1} - \frac{2}{x^2} \Sigma \frac{c_r}{(r+1)(r+2)} x^{r+2} + \frac{2}{x^3} \frac{c_r}{(r+1)(r+2)(r+3)} x^{r+3}$

$$= \frac{1}{x} \cdot \frac{(1+x)^{n+1}-1}{n+1} - \frac{2}{x^2} \left\{ \frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} \right\}$$

$$+ \frac{2}{x^3} \left\{ \frac{(1+x)^{n+3}-1}{(n+1)(n+2)(n+3)} - \frac{x}{(n+1)(n+2)} - \frac{x^2}{2(n+1)} \right\}.$$

$$= \frac{(1+x)^{n+1}}{(n+1)x} \left\{ 1 - \frac{2}{n+2} \cdot \frac{1+x}{x} + \frac{2}{(n+2)(n+3)} \cdot \left(\frac{1+x}{x} \right)^2 \right\}$$

$$- \frac{1}{x} \cdot \frac{1}{n+1} + \frac{2}{x^2} \left\{ \frac{1}{(n+1)(n+2)} + \frac{x}{n+1} \right\}$$

$$- \frac{2}{x^3} \left\{ \frac{1}{(n+1)(n+2)(n+3)} + \frac{x}{(n+1)(n+2)} + \frac{x^2}{2(n+1)} \right\},$$

$$\therefore \quad S = \frac{(1+x)^{n+1}}{(n+1)(n+2)(n+3)x} \left\{ n^2+3n+2 - \frac{2(n+1)}{x} + \frac{2}{x^2} \right\} - \frac{2}{(n+1)(n+2)(n+3)x^3}.$$

Ex. 3. *Find* $\Sigma_{r=0}^{\infty} \frac{r^2+1}{r+3} c_r x^r$.

We have $$\frac{r^2+1}{r+3} = r - 3 + \frac{10}{r+3}$$

and $$\Sigma(r-3)c_r x^r = nx(1+x)^{n-1} - 3(1+x)^n ;$$

$\therefore$ the required sum $= (1+x)^{n-1}\{(n-3)x - 3\} + 10S$, where S is the same as in Ex. 2.

11. Multinomial Theorem for any Real Index.

If $b, c, \ldots k$ are any m numbers, n is real, and

$$f(x) = (1 + bx + cx^2 + \ldots + kx^m)^n,$$

then (i) $f(x)$ *can be expanded in the form*

$$1 + u_1 x + u_2 x^2 + \ldots + u_r x^r + \ldots ,$$

where the series terminates if n *is a positive integer, and in other cases is absolutely convergent for sufficiently small values of* x.

(ii) *The coefficient* u_r *of* x^r *is the sum of all terms of the form*

$$\frac{n(n-1)(n-2)\ldots(\alpha+1)}{\lfloor\beta \lfloor\gamma \cdots \lfloor\kappa} b^\beta c^\gamma \ldots k^\kappa,$$

where $\beta, \gamma, \ldots \kappa$ *have any positive integral or zero values such that*

$$\beta + 2\gamma + \ldots + m\kappa = r,$$

and the corresponding values of α *are given by* $\alpha + \beta + \gamma + \ldots + \kappa = n$.

For let $y = bx + cx^2 + \ldots + kx^m$, then by the Binomial theorem

$$f(x) = (1+y)^n = 1 + \Sigma_{s=1}^{s=\infty} \frac{n(n-1)\ldots(n-s+1)}{\lfloor s} y^s, \quad \ldots\ldots\ldots\text{(A)}$$

provided that $|y| < 1$, and by Ch. XVII, 3, (1), we can find a positive number ρ such that $|y| < 1$ when $|x| < \rho$.

By substituting $bx + cx^2 + \ldots$ for y in (**A**) and expanding the terms, we obtain an expansion of $f(x)$ in powers of x which is convergent if $|x| < \rho$.

In order that the terms of this series may be arranged in ascending powers of x, it is sufficient that the series is convergent when all the terms are made positive. This will be the case if $b'x' + c'x'^2 + \ldots + k'x'^m < 1$, where $x', b', c', \ldots$ stand for $|x|, |b|, |c|, \ldots$; and we can find a positive number σ such that this is the case if $|x| < \sigma$.

This concludes the proof of the statement (i).

Again, s being any positive integer, by Ch. III, 16, the coefficient of x^r in $(bx + cx^2 + \ldots + kx^m)^s$ is the sum of all terms of the form

$$\frac{\lfloor s}{\lfloor\beta \lfloor\gamma \cdots \lfloor\kappa} b^\beta c^\gamma \ldots k^\kappa,$$

where $\beta, \gamma, \ldots \kappa$ have any positive integral or zero values such that

$$\beta + 2\gamma + \ldots + m\kappa = r, \quad \ldots\ldots\ldots\

5. If n is a positive integer, show that

$$1+3n+\frac{3\cdot 4}{1\cdot 2}\frac{n(n-1)}{\lfloor 2}+\frac{4\cdot 5}{1\cdot 2}\frac{n(n-1)(n-2)}{\lfloor 3}+\ldots+\frac{(n+1)(n+2)}{2}$$
$$=\text{coefficient of } x^n \text{ in } \{2-(1-x)\}^n/(1-x)^3=(n^2+7n+8)2^{n-3}.$$

6. If n is a positive integer, show that

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{\lfloor 3}2^{n-6}+\ldots=n+1.$$

[From the expansion of $\{1-x(2-x)\}^{-1}=(1-x)^{-2}$.]

7. Prove that if n is a positive integer,

$$1-n\cdot\frac{1+x}{1+nx}+\frac{n(n-1)}{\lfloor 2}\cdot\frac{1+2x}{(1+nx)^2}+\frac{n(n-1)(n-2)}{\lfloor 3}\cdot\frac{1+3x}{(1+nx)^3}+\ldots=0.$$

8. Show that, for all values of m and n,

$$C_r^n-C_{r-1}^nC_1^m+C_{r-2}^nC_2^{m+1}-C_{r-3}^nC_3^{m+2}-\ldots+(-1)^rC_r^{m+r-1}$$
$$=\text{the coefficient of } x^r \text{ in the expansion of } (1+x)^{n-m}.$$

9. If $p+q=1$ and n, r are positive integers $(n>r)$, show that

$$p^n+C_1^np^{n-1}q+C_2^np^{n-2}q^2+\ldots+C_{n-r}^np^rq^{n-r}$$
$$=p^r\left\{1+rq+\frac{r(r+1)}{\lfloor 2}q^2+\ldots+\frac{r(r+1)\ldots(n-1)}{\lfloor n-r}q^{n-r}\right\}.$$

[Divide by p^r, put $1-q$ for p, and show that the coefficient of q^k on the left-hand side when expanded is

$$C_k^n-C_{k-1}^nC_2^{n-r-k+1}+C_{k-2}^nC_2^{n-r-k+2}-\ldots+(-1)^kC_k^{n-r}$$
$$=\text{coefficient of } x^k \text{ in } (1+x)^n\times(1+x)^{-(n-r-k+1)}.]$$

In Exx. 10-13, if $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_rx^r+\ldots$ where n may have any value, find the sum of the series, $\Sigma_{r=0}^{\infty}mc_rx^r$, when m is

10.* $(r+1)^2$; **11.** $\frac{r+1}{r+2}$; **12.** $\frac{1}{(r+1)(r+3)}$; **13.*** $\frac{r^2-r+1}{(r+1)(r+3)}$.

14. If $u_r=A_0+A_1r+A_2r(r-1)+\ldots+A_hr(r-1)\ldots(r-h+1)$ and
$\phi(x)=A_0x^h+A_1nx^{h-1}+A_2n(n-1)x^{h-2}+\ldots+A_hn(n-1)\ldots(n-h+1)$,
prove that

$$\Sigma u_rc_rx^r=(1+x)^{n-h}[x^h\phi(1)+\frac{x^{h-1}}{\lfloor 1}\phi'(1)+\frac{x^{h-2}}{\lfloor 2}\phi''(1)+\ldots].$$

15. Prove that $C_r^rC_{13-r}^{52-r}+C_r^{r+1}C_{13-r}^{51-r}+C_r^{r+2}C_{13-r}^{50-r}+\ldots$ to 40 terms $=C_{24}^{53}$.
[Consider the product $(1-x)^{-(r+1)}\times(1-x)^{-(14-r)}$.]

16. Show that

$$\Sigma\frac{1}{\lfloor 2p+1\,\lfloor 2q+1\,\lfloor 2r+1}=\frac{3^{2n+3}-3}{4\lfloor 2n+3},$$

the summation being taken for all positive integral values, including zero, of p, q, r such that $p+q+r=n$.
[Consider the expansion of

$$(a+b+c)^{2n+3}-(b+c-a)^{2n+3}-(c+a-b)^{2n+3}-(a+b-c)^{2n+3}.]$$

* Check the answer by putting $n=1$.

CHAPTER XXII

RATIONAL FRACTIONS (2), RECURRING SERIES AND DIFFERENCE EQUATIONS

1. Expansion of a Rational Fraction. Any rational proper fraction in x, whose denominator does not contain x as a factor, may be written in the form

$$\frac{P}{Q}=\frac{a_0+a_1x+a_2x^2+\ldots+a_mx^m}{1+p_1x+p_2x^2+\ldots+p_rx^r} \quad \text{where } m<r.$$

Various ways of expanding such a fraction in a series of the form $\Sigma u_n x^n$ are given below.

If it can be shown that the series obtained by any process is convergent for all values of x in any interval including zero, then by Ch. XX, 20, *any other method of expansion yields identically the same series.*

2. Theorem. *The rational fraction P/Q can be expanded in a convergent series of the form*

$$u_0+u_1x+u_2x^2+\ldots+u_nx^n+\ldots,$$

if and only if $|x|<|\lambda|$, where λ is the root of $Q=0$ with the least modulus.

For, by the method of partial fractions, P/Q can be expressed as the sum of several terms of the form $A(x-\alpha)^{-r}$, where A is a constant, r a positive integer and α is any root, real or imaginary, of the equation $Q=0$.

All of these terms can be expanded in a convergent series of ascending powers of x if, and only if, $|x|<|\lambda|$, where λ is the root of $Q=0$ with the least modulus.

When this is the case, the expansion of P/Q can be obtained by adding the various series.

Ex. 1. *For what values of x can the fraction*

$$1/(x^2+2x-1)(x^2-x+5)$$

be expanded in a series of ascending powers of x?

Equating the denominator to zero, the roots are

$$-1\pm\sqrt{2} \quad \text{and} \quad \tfrac{1}{2}(1\pm i\sqrt{19})$$

and their moduli are $\sqrt{2}\pm1$, $\sqrt{5}$, $\sqrt{5}$.

Hence the expansion is possible if and only if $|x|<\sqrt{2}-1$.

3. Methods of Expansion. (1) Suppose that

$$P/Q = u_0 + u_1 x + u_2 x^2 + \dots$$

where the series is assumed to be convergent and P, Q are as described in Art. 1. Multiplying by Q,

$$a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m = (1 + p_1 x + p_2 x^2 + \dots + p_r x^r)(u_0 + u_1 x + u_2 x^2 + \dots).$$

Expanding the right-hand side and equating coefficients, the values of $u_0, u_1, u_2, \dots$ are determined by the equations

$$u_0 = a_0, \quad u_1 + p_1 u_0 = a_1, \quad u_2 + p_1 u_1 + p_2 u_0 = a_2, \quad \dots\dots\dots,$$

$$u_{r-1} + p_1 u_{r-2} + p_2 u_{r-3} + \dots + p_{r-1} u_0 = a_{r-1},$$

and for $n \geqslant r$, $\qquad u_n + p_1 u_{n-1} + p_2 u_{n-2} + \dots + p_r u_{n-r} = 0.$

Hence, after a certain stage, any r successive coefficients are connected by a linear relation. A series having this property is called a *Recurring Series.*

(2) The process just described is equivalent to *synthetic division* carried out as in the next example. (See Ch. III, 3.)

Ex. 1. *Show that, for sufficiently small values of* x,

$$\frac{1 + x + x^2 + x^3}{1 - 2x + 3x^2 - 4x^3} = 1 + 3x + 4x^2 + 4x^3 + 8x^4 + 20x^5 + \dots .$$

The reckoning is as follows. $1 - (2 - 3 + 4)$

$1 + 1 + 1 + 1$	
$+ 4 + 12 + 16 + 16 + 32$	(a)
$- 3 - 9 - 12 - 12 - 24$	(b)
$+ 2 + 6 + 8 + 8 + 16$	(c)
$1 + 3 + 4 + 4 + 8 + 20$	(d)

The first term in the quotient (d) is 1; $1(2 - 3 + 4) = 2 - 3 + 4$; put 2, -3, $+4$ diagonally in (c), (b), (a); $2 + 1 = 3$, the next term in (d); $3(2 - 3 + 4) = 6 - 9 + 12$; put 6, -9, $+12$ diagonally in (c), (b), (a); $6 - 3 + 1 = 4$, the next term in (d).

If the reader will do this example by the method in Art. 3, (1), he will see that the reckoning is essentially the same as that just described.

(3) *The method of Partial Fractions.*

Ex. 2. *If* $1/(1 - 2x + 5x^2) = u_0 + u_1 x + \dots + u_n x^n + \dots$ *find the value of* u_n.

We have
$$\frac{1}{1 - 2x + 5x^2} = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \frac{1}{\alpha - \beta}\left\{\frac{\alpha}{1 - \alpha x} - \frac{\beta}{1 - \beta x}\right\}$$

where α, β are the roots of $x^2 - 2x + 5 = 0$, so that we may take $\alpha = 1 + 2\iota$, $\beta = 1 - 2\iota$.

Hence
$$\frac{1}{1 - 2x + 5x^2} = \frac{1}{4\iota}\{\alpha(1 + \alpha x + \alpha^2 x^2 + \dots) - \beta(1 + \beta x + \beta^2 x^2 + \dots)\};$$

$$\therefore \; u_n = \frac{1}{4\iota}(\alpha^{n+1} - \beta^{n+1}).$$

Now $\alpha = \sqrt{5}(\cos\theta + \iota \sin\theta)$, $\beta = \sqrt{5}(\cos\theta - \iota \sin\theta)$, where $\theta = \tan^{-1} 2$,

$$\therefore \; \alpha^{n+1} - \beta^{n+1} = 5^{\frac{1}{2}(n+1)} \cdot 2\iota \sin(n+1)\theta;$$

hence
$$u_n = \tfrac{1}{2} \cdot 5^{\frac{1}{2}(n+1)} \sin(n+1)\theta, \text{ where } \theta = \tan^{-1} 2.$$

(4) A rational fraction can also be expanded by the *Multinomial Theorem*. Using this method and that of Partial Fractions and equating coefficients, we can obtain many important results, as in the following exercise.

EXERCISE XXXVI

1. Use synthetic division to expand

$$\text{(i)}\ \frac{1}{(1-x)^2(1-2x)}; \quad \text{(ii)}\ \frac{1-4x+x^2}{1-6x+11x^2-6x^3},$$

as far as the terms containing x^5. Find also the coefficient of x^n in each expansion.

2. Show that, if p and q are real numbers, the condition that $1/(1+px+qx^2)$ can be expanded in a convergent series of ascending powers of x is as follows:

(i) if $p^2>4q$, then $|x|<|(p'-\sqrt{p^2-4q})/2q|$, where $p'=|p|$;

(ii) if $p^2<4q$, then $|x|<1/\sqrt{q}$.

3. If $|x|$ lies between $|\alpha|$ and $|\beta|$, show that the fraction $1/(x-\alpha)(x-\beta)$ can be expanded in the form $\ldots+b_2x^{-2}+b_1x^{-1}+a_0+a_1x+a_2x^2+\ldots$, where the series extends to ∞ on the right and left.

[Let $|\alpha|<|\beta|$. The fraction $=(\alpha-\beta)\{1/(x-\alpha)-1/(x-\beta)\}$.

If $|\alpha|<|x|<|\beta|$, prove that

$$1/(x-\alpha)=x^{-1}(1+\alpha x^{-1}+\alpha^2x^{-2}+\ldots),$$

and

$$1/(x-\beta)=-\beta^{-1}(1+\beta^{-1}x+\beta^{-2}x^2+\ldots).]$$

4. If H_n is the sum of the homogeneous products of n dimensions which can be formed with a, b, c, show that

$$a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)=-H_n(b-c)(c-a)(a-b).$$

[H_n is the coefficient of x^n in the product

$$(1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)(1+cx+c^2x^2+\ldots),$$

i.e. in the expansion of $1/(1-ax)(1-bx)(1-cx)$. The result is obtained by the method of partial fractions.]

5. Prove that, for sufficiently small values of x,

$$\frac{1}{1-px+qx^2}=1+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$

where

$$a_n=p^n-(n-1)p^{n-2}q+\frac{(n-2)(n-3)}{\lfloor 2}p^{n-4}q^2-\ldots,$$

the $(r+1)$th term being $(-1)^rC_r^{n-r}p^{n-2r}q^r$.

6. If $\alpha+\beta=p$, $\alpha\beta=q$, use the identity $\dfrac{1}{1-\alpha x}+\dfrac{1}{1-\beta x}=\dfrac{2-px}{1-px+qx^2}$

to prove that $\alpha^n+\beta^n=p^n-\dfrac{n}{\lfloor 1}p^{n-2}q+\dfrac{n(n-3)}{\lfloor 2}p^{n-4}q^2-\ldots,$

the $(r+1)$th term being $(-1)^r\dfrac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{\lfloor r}p^{n-2r}q^r.$

[Expand both sides, equate coefficients and use Ex. 5.]

7. If $\alpha+\beta=p$, $\alpha\beta=q$, use the identity $\dfrac{1}{1-\alpha x}-\dfrac{1}{1-\beta x}=\dfrac{(\alpha-\beta)x}{1-px+qx^2}$

to prove that $\dfrac{\alpha^n-\beta^n}{\alpha-\beta}=p^{n-1}-\dfrac{n-2}{\lfloor 1}p^{n-3}q+\dfrac{(n-3)(n-4)}{\lfloor 2}p^{n-5}q^2-\dots,$

the $(r+1)$th term being $(-1)^r\dfrac{(n-r-1)(n-r-2)\dots(n-2r)}{\lfloor r}p^{n-2r-1}q^r.$

[Expand both sides and use Ex. 5.]

8. If n is a positive integer, prove that

(i) $2\cos n\theta=(2\cos\theta)^n-\dfrac{n}{\lfloor 1}(2\cos\theta)^{n-2}+\dfrac{n(n-3)}{\lfloor 2}(2\cos\theta)^{n-4}-\dots,$

the $(r+1)$th term being $(-1)^r\dfrac{n(n-r-1)\dots(n-2r+1)}{\lfloor r}(2\cos\theta)^{n-2r}.$

(ii) $\dfrac{\sin n\theta}{\sin\theta}=(2\cos\theta)^{n-1}-\dfrac{n-2}{\lfloor 1}(2\cos\theta)^{n-3}+\dfrac{(n-3)(n-4)}{\lfloor 2}(2\cos\theta)^{n-5}-\dots,$

the $(r+1)$th term being $(-1)^r\dfrac{(n-r-1)(n-r-2)\dots(n-2r)}{\lfloor r}(2\cos\theta)^{n-2r-1}.$

[In Exx. 6, 7 put $\alpha=z$, $\beta=z^{-1}$ where $z=\cos\theta+\iota\sin\theta$.]

9. Show that (i) the coefficient of x^{3n} in the expansion of $\dfrac{1+x}{(1+x+x^2)^3}$ is $\frac{1}{2}(n+1)(3n+2)$.

(ii) The coefficient of x^{4n} in the expansion of $\dfrac{1}{(1-x)(1-x^2)(1-x^4)}$ is $(n+1)^2$.

4. Recurring Series. (1) Let $u_0+u_1x+u_2x^2+\dots$ be a series in which any $r+1$ successive coefficients are connected by the equation

$$u_n+p_1u_{n-1}+p_2u_{n-2}+\dots+p_ru_{n-r}=0, \quad \dots\dots\dots\dots\dots\dots(\mathbf{A})$$

where r is a fixed number and $p_1, p_2, \dots p_r$ are constants. Such a series is called a *Recurring Series of the r-th order.*

Some authors call equation (**A**) the *Scale of Relation* of the series; others use this term to denote the polynomial $1+p_1x+p_2x^2+\dots+p_rx^r$. We shall take it to mean *either* of these things.

If the coefficients $u_0, u_1, \dots u_{r-1}$ are known, the subsequent coefficients can be found in succession by equation (**A**). Thus a recurring series of the rth order depends on $2r$ constants.

Hence if the first $2r$ coefficients are given, in general the series can be continued as a recurring series of the rth order, and in one way only. Also it can be continued as a recurring series of the $(r+1)$th order in a doubly infinite number of ways. For to do this we can give u_r and u_{r+1} any values.

It may happen that the first $2r$ terms belong to a recurring series of order $r-1$. Two conditions are necessary that this may be so, and then the series can be continued as a recurring series of the $(r-1)$th order.

Ex. 1. *Discuss the question of continuing* $\Sigma u_n x^n$ *as a recurring series when the first six terms are known.*

Let the scale of relation be $u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$.

Then p, q, r are given by

$$u_3 + pu_2 + qu_1 + ru_0 = 0, \quad u_4 + pu_3 + qu_2 + ru_1 = 0, \quad u_5 + pu_4 + qu_3 + ru_2 = 0.$$

Now $\Sigma u_n x^n$ can be continued as a recurring series of the third order if p, q, r have definite values and $r \neq 0$. This will be the case if

$$\begin{vmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \\ u_3 & u_4 & u_5 \end{vmatrix} \neq 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

In case either determinant is zero, suppose that the scale is $u_n + pu_{n-1} + qu_{n-2} = 0$; then

$$u_2 + pu_1 + qu_0 = 0, \qquad u_3 + pu_2 + qu_1 = 0,$$
$$u_4 + pu_3 + qu_2 = 0, \qquad u_5 + pu_4 + qu_3 = 0.$$

That these equations may be consistent, each of the determinants in (A) must vanish, so that *two* conditions are necessary that $\Sigma u_n x^n$ may be a recurring series of the second order.

If it is one of the first order, the scale being $u_n + pu_{n-1} = 0$, then

$$u_1 + pu_0 = 0, \quad u_2 + pu_1 = 0, \ldots \quad u_5 + pu_4 = 0,$$

so that the *four* conditions $u_1^2 - u_0 u_2 = 0$, $u_2^2 - u_1 u_3 = 0$, $u_3^2 - u_2 u_4 = 0$, $u_4^2 - u_3 u_5 = 0$ must be satisfied.

(2) *Any recurring series* $\Sigma u_n x^n$ *is convergent for sufficiently small values of* x.

For let the scale be

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_r u_{n-r} = 0, \quad \ldots\ldots\ldots\ldots\ldots\text{(A)}$$

and let $|u_0|, |u_1|, \ldots |p_1|, |p_2|, \ldots$ be denoted by $u_0', u_1', \ldots p_1', p_2', \ldots$. Then by (A),

$$\begin{aligned} |u_n| &= |p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_r u_{n-r}| \\ &\leqslant p_1' u_{n-1}' + p_2' u_{n-2}' + \ldots + p_r' u_{n-r}' \\ &\leqslant g u_{n-s}', \end{aligned}$$

where g is the greater of the numbers $p_1' + p_2' + \ldots + p_r'$ and 1, and u_{n-s}' is the greatest of the set $u_{n-1}', u_{n-2}', \ldots u_0'$. In the same way

$$|u_{n-s}| \leqslant g u_{n-t}',$$

where u_{n-t}' is the greatest of $u_{n-s-1}', u_{n-s-2}', \ldots u_0'$.

Continuing thus, by multiplication we can show that $|u_n| \leqslant g^n A$, where A is the greatest of $u_{n-1}', u_{n-2}', \ldots u_0'$.

Thus $$|u_n x^n| \leqslant |gx|^n . A,$$

and $\Sigma u_n x^n$ is convergent if $|x| < 1/g$, for then $|u_n x^n|$ is less than the $(n+1)$th term of a convergent series of positive terms.

(3) *The sum to infinity of any recurring series* $\Sigma u_n x^n$, *when convergent, is a rational proper fraction, which is called the* generating function *of the series.*

Let the scale be

$$u_n + p_1 u_{n-1} + p_2 u_{n-2} + \ldots + p_r u_{n-r} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

and let

$$s = u_0 + u_1 x + u_2 x^2 + \ldots \text{ to } \infty . \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{B})$$

Multiplying (**B**) by $p_1 x$, $p_2 x^2$, ... $p_r x^r$ and adding, we find that

$$s(1 + p_1 x + p_2 x^2 + \ldots + p_r x^r) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{r-1} x^{r-1}, \quad \ldots\ldots (\mathbf{C})$$

where $a_0 = u_0$, $a_1 = u_1 + p_1 u_0$, $a_2 = u_2 + p_1 u_1 + p_2 u_0$, ... ,

$$a_{r-1} = u_{r-1} + p_1 u_{r-2} + \ldots p_{r-1} u_0,$$

the coefficients of powers of x higher than x^{r-1} vanishing on account of equation (**A**).

Thus s is equal to a rational fraction, of which the denominator is

$$Q = 1 + p_1 x + p_2 x^2 + \ldots + p_r x^r.$$

It follows from Art. 2 that *the recurring series is convergent if* $|x| < |\lambda|$, *where* λ *is the root of* $Q = 0$ *which has the least modulus.*

(4) *If* u_n *is a positive integral function of* n *of degree* r, *then* $\Sigma u_n x^n$ *is a recurring series of which the scale is* $(1-x)^{r+1}$.

To prove this, we can proceed as in Ch. VIII, 7. Let

$$s = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n \ldots ,$$

then

$$s(1-x) = u_0 + v_1 x + v_2 x^2 + \ldots + v_n x^n + \ldots ,$$

where $v_n = u_n - u_{n-1}$, so that v_n is a polynomial in n of degree $r-1$. By r steps of this kind, we find that

$$s(1-x)^r = u_0 + k(x + x^2 + x^3 + \ldots),$$

where k is independent of x, and therefore

$$s(1-x)^{r+1} = u_0 + (k - u_0)x,$$

which proves the theorem,

Ex. 2. *Find the sum to infinity of*

$$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots , \text{ when } |x| < 1.$$

This is a convergent recurring series, and the scale is $(1-x)^3$. Denoting the sum by S,

$$\begin{aligned} S &= 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots , \\ -3xS &= -3 \cdot 1^2 x - 3 \cdot 2^2 x^2 - 3 \cdot 3^2 x^3 - \ldots , \\ 3x^2 S &= 3 \cdot 1^2 x^2 + 3 \cdot 2^2 x^3 + \ldots , \\ -x^3 S &= - 1^2 x^3 - \ldots ; \end{aligned}$$

$$\therefore \ S(1-x)^3 = 1 + x \quad \text{and} \quad S = \frac{1+x}{(1-x)^3}.$$

(5) *The sum to n terms* of a recurring series can be found by a process similar to that in § (3), or in the last example.

Ex. 3. *Find the sum to n terms of* $u_0+u_1x+u_2x^2+\dots$, *where* $u_n+pu_{n-1}+qu_{n-2}=0$ *for* $n>1$.

Let $s_n=u_0+u_1x+u_2x^2+\dots+u_{n-1}x^{n-1}$

then $pxs_n=pu_0x+pu_1x^2+\dots+pu_{n-2}x^{n-1}+pu_{n-1}x^n$,

and $qx^2s_n=qu_0x^2+\dots+qu_{n-3}x^{n-1}+qu_{n-2}x^n+qu_{n-1}x^{n+1}$.

Since $u_r+pu_{r-1}+qu_{r-2}=0$ for $r>1$, by addition,

$$s_n(1+px+qx^2)=u_0+(u_1+pu_0)x-u_nx^n+qu_{n-1}x^{n+1},$$

which determines the value of s_n, unless x is a root α of the equation $1+px+qx^2=0$. In this case the sum may be deduced from the general result by supposing that $x\to\alpha$.

(6) Given the scale $u_n+p_1u_{n-1}+\dots+p_ru_{n-r}=0$, and the values of u_0, u_1, $\dots$ u_{r-1}, we can find the value of u_n by (i) finding the generating function; (ii) expressing this as the sum of partial fractions of the form $A(1-\alpha x)^{-m}$, where α is a root of $x^r+p_1x^{r-1}+p_2x^{r-2}+\dots+p_r=0$; and (iii) expanding the partial fractions and collecting the coefficients of x^n.

Examples are given in the next article.

5. Examples of Finite Difference Equations.

Suppose that any $r+1$ consecutive terms of the sequence (u_n) are connected by the equation

$$u_n+p_1u_{n-1}+p_2u_{n-2}+\dots+p_ru_{n-r}=0, \quad \dots\dots\dots\dots\dots\dots(\mathbf{A})$$

where p_1, p_2, $\dots$ p_r are constants. From this point of view, equation (**A**) is called a *Linear Finite Difference Equation with constant coefficients*.

So far as equation (**A**) is concerned, the first r terms of the sequence may have any values whatever, that is to say they are *arbitrary constants*.

Thus the process described in the last section enables us to find the general value of u_n which satisfies equation (**A**), and shows that this value involves r arbitrary constants. The general value of u_n is called the *general solution*, and the process of finding it is called *solving the equation*.

The process leads to the following results.

(i) *The general solution of* $u_n+pu_{n-1}+qu_{n-2}=0$ *is* $u_n=A\alpha^n+B\beta^n$ *or* $(A+nB)\alpha^n$, *where* α, β *are the roots of* $x^2+px+q=0$ *and* A, B *are arbitrary constants, the first or second form being taken according as* $\alpha\neq\beta$ *or* $\alpha=\beta$.

(ii) *The general solution of* $u_n+pu_{n-1}+qu_{n-2}+ru_{n-3}=0$ *is*

$$u_n=A\alpha^n+B\beta^n+C\gamma^n,$$

where α, β, γ *are the roots of* $x^3+px^2+qx+r=0$ *and* A, B, C *are arbitrary constants, provided that no two of the roots are equal.*

If $\alpha=\beta\neq\gamma$, *the solution is* $u_n=(A+nB)\alpha^n+C\gamma^n$.

If $\alpha=\beta=\gamma$, *then* $u_n=(A+nB+n^2C)\alpha^n$.

(iii) *For any other value of r the general solution of equation* (**A**) *has a similar form.*

We give the proof of (ii).

If u_0, u_1, u_2 have any values, then

$$u_0+u_1x+u_2x^2+\ldots+u_nx^n+\ldots=\frac{a+bx+cx^2}{1+px+qx^2+rx^3}=\frac{P}{Q}\text{ (say)},$$

where a, b, c are functions of u_0, u_1, u_2. Also

$$1+px+qx^2+rx^3=(1-\alpha x)(1-\beta x)(1-\gamma x),$$

where α, β, γ are the roots of

$$x^3+px^2+qx+r=0.$$

(i) If no two of α, β, γ are equal,

$$\Sigma u_nx^n=\frac{P}{Q}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}+\frac{C}{1-\gamma x},$$

where A, B, C are arbitrary constants.

Expanding the fractions on the right and equating coefficients,

$$u_n=A\alpha^n+B\beta^n+C\gamma^n.$$

(ii) If $\alpha=\beta\neq\gamma$, then

$$\frac{P}{Q}=\frac{A}{1-\alpha x}+\frac{B}{(1-\alpha x)^2}+\frac{C}{1-\gamma x}$$

and

$$u_n=(A+nB)\alpha^n+C\gamma^n.$$

(iii) If $\alpha=\beta=\gamma$, then

$$\frac{P}{Q}=\frac{A}{1-\alpha x}+\frac{B}{(1-\alpha x)^2}+\frac{C}{(1-\alpha x)^3}$$

and

$$\begin{aligned}u_n&=A\alpha^n+Bn\alpha^n+C\,.\,\tfrac{1}{2}(n+1)(n+2)\alpha^n\\&=(A+nB'+n^2C')\alpha^n,\end{aligned}$$

where B', C' are arbitrary constants.

Ex. 1. *The first term of a sequence is* 1, *the second is* 2, *and every other term is the sum of the two preceding terms. Find the n-th term.*

Denoting the sequence by $u_1, u_2, \ldots$ we have

$$u_n-u_{n-1}-u_{n-2}=0 \quad\text{and}\quad u_1=1,\ u_2=2.$$

Hence $u_n=A\alpha^n+B\beta^n$, $2=A\alpha^2+B\beta^2$, $1=A\alpha+B\beta$, where α, β are the roots of $x^2-x-1=0$.

Eliminating A, B from these equations, we find that

$$(\alpha-\beta)u_n=(2-\beta)\alpha^{n-1}-(2-\alpha)\beta^{n-1}.$$

Now $\alpha+\beta=1$, $\alpha\beta=-1$ and if $\alpha>\beta$, $\alpha-\beta=\sqrt{5}$, hence $2-\beta=1+\alpha=\alpha^2$ (for $\alpha^2-\alpha-1=0$), and similarly $2-\alpha=\beta^2$. Therefore

$$u_n=\frac{1}{\sqrt{5}}(\alpha^{n+1}-\beta^{n+1})=\frac{1}{2^{n+1}\sqrt{5}}\{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}\}.$$

Ex. 2. *Find the n-th term of a recurring series of which the first four terms are*

$$1+2x+7x^2+20x^3.$$

Find also the sum of the first n terms when $x=-1$.

Denote the series by $u_0+u_1x+u_2x^2+\dots$, and let the scale be

$$u_n+pu_{n-1}+qu_{n-2}=0.$$

Then $7+2p+q=0$ and $20+7p+2q=0$, giving $p=-2$, $q=-3$, and the scale is $u_n-2u_{n-1}-3u_{n-2}=0$.

The roots of $x^2-2x-3=0$ are 3, -1, and therefore

$$u_n=A\,.\,3^n+B(-1)^n,$$

where $A+B=1$, $3A-B=2$. Hence $A=\frac{3}{4}$, $B=\frac{1}{4}$.

Thus the nth term $=u_{n-1}x^{n-1}=\frac{1}{4}[3^n+(-1)^{n-1}]x^{n-1}$;

$$\therefore \text{ sum to } n \text{ terms} = \frac{3}{4}\cdot\frac{1-3^nx^n}{1-3x}+\frac{1}{4}\cdot\frac{1-(-x)^n}{1+x}.$$

Now $$\lim_{x\to-1}\frac{1-(-x)^n}{1+x}=\lim_{y\to1}\frac{1-y^n}{1-y}=n;$$

$\therefore$ the sum to n terms of the series $1-2+7-20+\dots$

$$=\frac{3}{4}\cdot\frac{1-(-3)^n}{4}+\frac{n}{4}=\frac{1}{16}\{4n+3+(-3)^{n+1}\}.$$

Ex. 3. *Find the general solution of* $u_n-u_{n-1}+u_{n-2}=0$ *in a real form.*

The solution is $u_n=C\alpha^n+D\beta^n$ where α, β are the roots of $x^2-x+1=0$ and C, D are arbitrary constants. Changing the constants, this may be written

$$u_n=A(\alpha^n+\beta^n)-\iota B(\alpha^n-\beta^n).$$

The values of α, β are

$$\frac{1}{2}\pm\iota\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}\pm\iota\sin\frac{\pi}{3}.$$

Hence the values of α^n, β^n are $\cos\frac{n\pi}{3}\pm\iota\sin\frac{n\pi}{3}$, and so

$$\alpha^n+\beta^n=2\cos\frac{n\pi}{3},\quad \alpha^n-\beta^n=2\iota\sin\frac{n\pi}{3},$$

and the solution is $$u_n=A\cos\frac{n\pi}{3}+B\sin\frac{n\pi}{3}.$$

EXERCISE XXXVII

1. Find the nth term and the sum to n terms of the recurring series:

(i) $1+2+5+14+\dots$; (ii) $1+2+5+12+\dots$.

2. Show that the recurring series

$$2+\tfrac{7}{5}+1+\tfrac{9}{125}+\dots$$

is convergent, and that the sum to infinity is $7\frac{1}{2}$.

3. Show that, if $u_n-2u_{n-1}+4u_{n-2}-3u_{n-3}=0$, the series Σu_nx^n is convergent if $|x|<\frac{1}{\sqrt{3}}$; and if the first three terms are $1+2x+3x^2$, the sum to infinity is

$$(1+2x^2)/(1-2x+4x^2-3x^3).$$

[The moduli of the roots of $x^3-2x^2+4x-3=0$ are 1, $1/\sqrt{3}$, $1/\sqrt{3}$.]

4. If $3u_n - 7u_{n-1} + 5u_{n-2} - u_{n-3} = 0$ and $u_1 = 1$, $u_2 = 8$, $u_3 = 17$, find the values of u_n and $u_1 + u_2 + u_3 + \ldots + u_n$.

5. Find the nth term of the recurring series $1 + 2 + 3 + 5 + 7 + 9 + \ldots$.

6. If u_n is the nth term of the recurring series $1 + 2 + 3 + 8 + 13 + 30 + \ldots$, show that $u_n = \frac{1}{9}\{(3n-4)(-1)^n + 2^{n+2}\}$, and find the sum to infinity of the series

$$u_1 + u_2/\underline{|1} + u_3/\underline{|2} + u_4/\underline{|3} + \ldots .$$

7. If $u_n - pu_{n-1} + qu_{n-2} = 0$, where $p^2 < 4q$, and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, show that

$$u_n \sin\theta = u_2 q^{\frac{n-2}{2}} \sin(n-1)\theta - u_1 q^{\frac{n-1}{2}} \sin(n-2)\theta, \quad \text{where} \quad \tan\theta = \frac{\sqrt{(4q-p^2)}}{p}.$$

8. If $\Sigma u_n x^n$, $\Sigma v_n x^n$ are recurring series of which the scales are $1 + px + qx^2$ and $1 + p'x + q'x^2$, then $\Sigma(u_n + v_n)x^n$ is a recurring series whose scale is

$$(1 + px + qx^2)(1 + p'x + q'x^2).$$

9. If Σu_n, Σv_n are recurring series of which the scales are

$$u_n + pu_{n-1} + qu_{n-2} = 0 \quad \text{and} \quad v_n + p'u_{n-1} + q'u_{n-2} = 0,$$

where $p^2 \neq 4q$ and $p'^2 \neq 4q'$, then $\Sigma u_n v_n$ is a recurring series whose scale is

$$u_n - pp'u_{n-1} + (p^2q' + p'^2q - 2qq')u_{n-2} - pp'qq'u_{n-3} + q^2q'^2u_{n-4} = 0.$$

[If α, β and α', β' are the roots of $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ respectively, then $u_n v_n = A\alpha^n\alpha'^n + B\alpha^n\beta'^n + C\alpha'^n\beta^n + D\alpha'^n\beta'^n$,

where A, B, C, D are constants. Hence the scale of $\Sigma u_n v_n x^n$ is

$$(1 - \alpha\alpha'x)(1 - \alpha\beta'x)(1 - \alpha'\beta x)(1 - \alpha'\beta'x).]$$

10. If $\Sigma u_n x^n$, $\Sigma v_n x^n$ are recurring series whose scales are $1 + px + qx^2$ and $1 - 2kx + k^2x^2$, then $\Sigma u_n v_n x^n$ is a recurring series whose scale is

$$(1 + pkx + qk^2x^2)^2.$$

Hence show that $\Sigma n u_n x^n$ is a recurring series whose scale is $(1 + px + qx^2)^2$.

11. If $\Sigma u_n x^n$ is a recurring series whose scale is $1 + px + qx^2$, then $\Sigma u_n^2 x^n$ is a recurring series whose scale is $[1 - (p^2 - 2q)x + q^2x^2](1 - qx)$.

[If α, β are the roots of $x^2 + px + q = 0$, then $u_n^2 = A(\alpha^2)^n + B(\beta^2)^n + Cq^n$, where A, B, C are constants. Hence the scale of $\Sigma u_n^2 x^n$ is

$$(1 - \alpha^2x)(1 - \beta^2x)(1 - qx).]$$

12. If $\dfrac{1}{1 + px + qx^2} = \Sigma_{n=0}^{\infty} u_n x^n$, use Ex. 11 to show that

$$\Sigma u_n^2 x^n = \frac{1 + qx}{[1 - (p^2 - 2q)x + q^2x^2](1 - qx)},$$

13. Prove that each of the series

$$\cos\alpha + x\cos(\theta + \alpha) + x^2\cos(2\theta + \alpha) + \ldots,$$
$$\sin\alpha + x\sin(\theta + \alpha) + x^2\sin(2\theta + \alpha) + \ldots,$$

is a recurring series, the scale being $1 - 2x\cos\theta + x^2$ in each case. Show also that the sum to n terms of the first series is

$$\frac{\cos\alpha - x\cos(\alpha - \theta) - x^n\cos(\alpha + n\theta) + x^{n+1}\cos(\alpha + \overline{n-1}\theta)}{1 - 2x\cos\theta + x^2},$$

and that the sum to n terms of the second series may be obtained from this expression by changing cos to sin in the numerator.

6. Finite Difference Equations. Let (u_n) be a sequence in which some or all of the terms $u_n, u_{n-1}, u_{n-2}, \dots u_{n-r}$ are connected by an equation which holds for all values of $n \geqslant r$, where r is a fixed number. Such an equation is called a *Finite Difference Equation.*

An important class of these equations has been considered in Art. 5. Here we give methods which apply to various cases which occur in ordinary algebra. The general theory belongs to the Calculus of Finite Differences.

To *solve* a difference equation is to express u_n as a function of n in the most general form.

The result is called the *general solution,* and from what has been said in Art. 5 it is clear that this solution must involve one or more arbitrary constants.

A *particular solution* is obtained by giving these constants special values.

If u_n is a given function of n containing one or more arbitrary constants, a difference equation may be obtained as in the following examples.

Ex. 1. *If* $u_n = Cn - 2$ *where* C *is an arbitrary constant, obtain the corresponding difference equation.*

The required equation is found by eliminating C from

$$u_n = Cn - 2 \quad \text{and} \quad u_{n-1} = C(n-1) - 2,$$

giving

$$(n-1)u_n - nu_{n-1} = 2.$$

Ex. 2. *If* $u_n = A\alpha^n + B\beta^n$ *where* A, B *are arbitrary constants and* α, β *are given unequal numbers neither of which is zero, prove that*

$$u_n - (\alpha + \beta)u_{n-1} + \alpha\beta u_{n-2} = 0.$$

The result is found by eliminating A, B from

$$u_n = A\alpha^n + B\beta^n, \quad u_{n-1} = A\alpha^{n-1} + B\beta^{n-1}, \quad u_{n-2} = A\alpha^{n-2} + B\beta^{n-2}.$$

7. Solution of some Elementary Types.

(1) *If* $u_n = au_{n-1}$, *then* $u_n = Ca^n$ *where* C *is an arbitrary constant.*

For $u_n/u_{n-1} = u_{n-1}/u_{n-2} = \dots = u_2/u_1 = a$, and u_1 may have any value.

(2) *If* $u_n = a_n u_{n-1}$ *where* a_n *is a given function of* n, *then* $u_n = Ca_1a_2 \dots a_n$ *where* C *is an arbitrary constant.*

For

$$\frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \dots \frac{u_2}{u_1} = a_n a_{n-1} \dots a_2,$$

therefore

$$u_n = \frac{u_1}{a_1} \cdot a_1 a_2 \dots a_n,$$

and u_1/a_1 may have any value we choose.

(3) *If* $u_n - a_n u_{n-1} = b_n$ *where* a_n *and* b_n *are given functions of* n, *the general solution is*

$$u_n = a_1 a_2 \dots a_n \left\{ \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \dots + \frac{b_n}{a_1 a_2 \dots a_n} + C \right\},$$

where C *is an arbitrary constant.*

Let $v_n = a_1 a_2 \dots a_n$, and divide each side of the given equation by v_n. The result is

$$u_n/v_n - u_{n-1}/v_{n-1} = b_n/v_n.$$

In this, writing $n-1, n-2, \dots 2$ in succession for n, we have

$$u_{n-1}/v_{n-1} - u_{n-2}/v_{n-2} = b_{n-1}/v_{n-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$u_2/v_2 - u_1/v_1 = b_2/v_2.$$

Whence by addition,

$$u_n/v_n - u_1/v_1 = b_1/v_1 + b_2/v_2 + \dots + b_n/v_n - b_1/v_1.$$

This gives the result in question, for the value of $u_1/v_1 - b_1/v_1$ is arbitrary.

In examples of this type, it is better to apply the method just described rather than to use the actual result.

Ex. 1. *Solve the equation* $(n-1)u_n - nu_{n-1} = 2$.

Dividing by $n(n-1)$, we have

$$u_n/n - u_{n-1}/(n-1) = 2/n(n-1).$$

Writing $n-1, n-2, \dots 2$ in succession for n and adding,

$$\frac{u_n}{n} - \frac{u_1}{1} = 2\left\{ \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{(n-1)n} \right\} = 2\left(1 - \frac{1}{n}\right).$$

Hence $u_n = Cn - 2$ where $C = 2 + u_1$, and since u_1 is an arbitrary constant, so also is C. (See Art. 6, Ex. 1.)

Ex. 2. *Find the general solution of* $u_n - nu_{n-1} = \lfloor n$.

Dividing by $\lfloor n$, the equation becomes $u_n/\lfloor n - u_{n-1}/\lfloor n-1 = 1$, and proceeding as before, we find that $u_n = \lfloor n(n+C)$ where C is an arbitrary constant.

(4) *The general solution of* $u_n - \alpha u_{n-1} = c\beta^n$ *is*

$$A\alpha^n + c\beta^{n+1}/(\beta - \alpha) \quad \text{or} \quad (A + cn)\alpha^n,$$

according as α *and* β *are unequal or equal.*

For dividing by α^n, we have

$$\frac{u_n}{\alpha^n} - \frac{u_{n-1}}{\alpha^{n-1}} = c\left(\frac{\beta}{\alpha}\right)^n.$$

Writing $n-1, n-2, \dots 2$ in succession for n and adding,

$$\frac{u_n}{\alpha^n} - \frac{u_1}{\alpha} = c\left\{ \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2} + \dots + \frac{\beta^n}{\alpha^n} - \frac{\beta}{\alpha} \right\}.$$

Therefore $$u_n = c\alpha^n \left\{\frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2} + \ldots + \frac{\beta^n}{\alpha^n}\right\} + B\alpha^n,$$

where $B(=u_1/\alpha - c\beta/\alpha)$ is an arbitrary constant.

Hence, if $\alpha \neq \beta$, $$u_n = c\beta \cdot \frac{\beta^n - \alpha^n}{\beta - \alpha} + B\alpha^n,$$

which may be written $$u_n = A\alpha^n + c\beta^{n+1}/(\beta - \alpha),$$

where $A(=B - c\beta/(\beta - \alpha))$ is an arbitrary constant.

If $\alpha = \beta$, then $u_n = cn\alpha^n + B\alpha^n$.

(5) *The general solution of* $u_n - (\alpha + \beta)u_{n-1} + \alpha\beta u_{n-2} = 0$ *is*

$$u_n = A\alpha^n + B\beta^n \quad \text{or} \quad u_n = (A + nB)\alpha^n,$$

according as α *and* β *are unequal or equal, where* A, B *are arbitrary constants.*

For the equation may be written

$$u_n - \alpha u_{n-1} = \beta(u_{n-1} - \alpha u_{n-2}).$$

Hence by § (1), $$u_n - \alpha u_{n-1} = C\beta^n,$$

where C is an arbitrary constant, and therefore, by § (4),

$$u_n = A\alpha^n + C\beta^{n+1}/(\beta - \alpha) \quad \text{or} \quad (A + Cn)\alpha^n,$$

according as $\alpha \neq \beta$ or $\alpha = \beta$. This establishes the result in question, for $C\beta/(\beta - \alpha)$ is an arbitrary constant if $\alpha \neq \beta$. (See also Art. 5.)

8. Linear Difference Equations. (1) An equation of the form

$$au_n + bu_{n-1} + cu_{n-2} + \ldots + ku_{n-r} = l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

is called a *linear difference equation.* Here r is a fixed number, and $a, b, \ldots k, l$ are given functions of n or constants. All the equations in Art. 7 are of this type.

With regard to equation (**A**), it should be noticed that

(i) *The general solution involves* r *arbitrary constants,* for $u_{r+1}, u_{r+2}, \ldots$ can be found in succession in terms of $u_1, u_2, \ldots u_r$, which may have any values.

(ii) *If* v_n *is the general solution of*

$$au_n + bu_{n-1} + cu_{n-2} + \ldots + ku_{n-r} = 0,$$

and w_n *is any particular solution of equation* (**A**), *then* $v_n + w_n$ *is the general solution of* (**A**). For it is obviously a solution, also it is the general solution, because v_n involves r arbitrary constants.

(iii) The method of dealing with such equations as

$$au_n + bu_{n-1} + \ldots + ku_{n-r} = l_n,$$

where $a, b, \ldots k, r$ are constants and l_n is a given function of n, is illustrated in the next example.

Ex. 1. *Solve the equation* $u_n - 5u_{n-1} + 6u_{n-2} = n^2 + 5^n + 2^n$.

The general solution of $u_n - 5u_{n-1} + 6u_{n-2} = 0$ is

$$u_n = A \cdot 2^n + B \cdot 3^n. \quad \text{.....(A)}$$

Next search for a particular solution of

$$u_n - 5u_{n-1} + 6u_{n-2} = n^2.$$

Assume as a trial solution $u_n = a + bn + cn^2$, then for all values of n,

$$a + bn + cn^2 - 5\{a + b(n-1) + c(n-1)^2\} + 6\{a + b(n-2) + c(n-2)^2\} = n^2.$$

Equating coefficients of powers of n, we find that

$$2c = 1, \quad 2b - 14c = 0, \quad 2a - 7b + 19c = 0,$$

giving $c = \frac{1}{2}$, $b = \frac{7}{2}$, $a = \frac{15}{2}$. and so a solution is

$$u_n = \tfrac{1}{2}(n^2 + 7n + 15). \quad \text{.....(B)}$$

Again, take the equation

$$u_n - 5u_{n-1} + 6u_{n-2} = 5^n,$$

and assume as a trial solution $u_n = a \cdot 5^n$, then

$$a(5^n - 5 \cdot 5^{n-1} + 6 \cdot 5^{n-2}) = 5^n, \text{ giving } a = \tfrac{25}{6}.$$

Thus a solution is

$$u_n = \tfrac{1}{6} \cdot 5^{n+2}. \quad \text{.....(C)}$$

Finally, take the equation

$$u_n - 5u_{n-1} + 6u_{n-2} = 2^n.$$

Since 2 is a root of $x^2 - 5x + 6 = 0$, it is clear that $a \cdot 2^n$ is not a solution. Trying $u_n = an \cdot 2^n$, we have

$$a\{n \cdot 2^n - 5(n-1)2^{n-1} + 6(n-2)2^{n-2}\} = 2^n,$$

giving $a = -2$. Thus a solution is

$$u_n = -n \cdot 2^{n+1}. \quad \text{.....(D)}$$

The general solution of the given equation is obtained by adding the right-hand sides of equations (**A**)–(**D**), and is

$$u_n = \tfrac{1}{2}(n^2 + 7n + 15) + \tfrac{1}{6} \cdot 5^{n+2} + (A - 2n) \cdot 2^n + B \cdot 3^n.$$

EXERCISE XXXVIII

1. The first term of a sequence is 1, and every other term is the sum of all the preceding terms. Find the nth term.

2. If $2u_n - u_{n-1} = na$, show that

$$u_n = \frac{u_1}{2^{n-1}} + a\left\{n - 1 + \frac{1}{2^{n-2}}\right\}.$$

3. The first term of a sequence is a, the second is b, and every other term is the arithmetic mean of the two preceding terms. Show that the nth term is

$$\tfrac{1}{3}(a + 2b) - \tfrac{1}{3}(a - b)(-\tfrac{1}{2})^{n-2},$$

thus proving that the nth term tends to $\frac{1}{3}(a + 2b)$ as a limit as $n \to \infty$.

4. The first term of a sequence is 1, the second is a, and every other term is the geometric mean of the two preceding terms.

Show that the nth term is a^x where $x = \frac{1}{3}\{2 + (-\frac{1}{2})^{n-2}\}$.

5. If $u_n = pu_{n-1} + qu_{n-2}$ and u_n tends to a limit other than zero as $n \to \infty$, show that $|q| < 1$, and the roots of $x^2 - px - q = 0$ are 1 and $-q$; also show that

$$\lim u_n = \frac{u_2 + qu_1}{1+q}.$$

6. Show that the equation

$$u_n - (\alpha+\beta+\gamma)u_{n-1} + (\beta\gamma + \gamma\alpha + \alpha\beta)u_{n-2} - \alpha\beta\gamma u_{n-3} = 0$$

can be written in the form

$$u_n - \alpha u_{n-1} - (\beta+\gamma)(u_{n-1} - \alpha u_{n-2}) + \beta\gamma(u_{n-2} - \alpha u_{n-3}) = 0.$$

Hence prove that

$$u_n - \alpha u_{n-1} = B\beta^n + C\gamma^n \quad \text{or} \quad (B+Cn)\beta^n,$$

according as $\beta \neq \gamma$ or $\beta = \gamma$, where B, C are arbitrary constants. Deduce the general solution of the equation as given in Art. 5, (ii).

7. If $u_n = nu_{n-1} + (-1)^n$ and $u_0 = 1$, show that $u_n/\lfloor\underline{n} \to e^{-1}$ as $n \to \infty$.
[$u_n/\lfloor\underline{n} = u_{n-1}/\lfloor\underline{n-1} + (-1)^n/\lfloor\underline{n}$.]

8. If $u_n = n(u_{n-1} + u_{n-2})$, find u_n in terms of u_0 and u_1, and show that

$$u_n/\lfloor\underline{n} + 1 \to u_0 + (u_1 - 2u_0)e^{-1} \quad \text{as} \quad n \to \infty.$$

[Write the equation $u_n - (n+1)u_{n-1} = -(u_{n-1} - nu_{n-2})$.]

9. If $u_n = (n-1)(u_{n-1} + u_{n-2})$ and $u_1 = 0$, $u_2 = 1$, prove that

$$u_n = \left\{ \frac{1}{\lfloor\underline{2}} - \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} - \ldots + \frac{(-1)^n}{\lfloor\underline{n}} \right\}.$$

[Write the equation in the form

$$u_n - nu_{n-1} = -\{u_{n-1} - (n-1)u_{n-2}\}.]$$

10. Obtain a particular solution of

$$u_n - 5u_{n-1} + 6u_{n-2} = n$$

by assuming $u_n = an + b$.

Hence find the general solution of the equation.

11. Find the general solution of $u_n - 5u_{n-1} + 6u_{n-2} = 7^n$.
[Assume $u_n = a \cdot 7^n$ as a trial solution.]

12. Find the general solution of

$$u_n - u_{n-1} - u_{n-2} = n^2.$$

[Take as a trial solution $u_n = an^2 + bn + c$.]

13. Find a particular solution of

$$u_n + u_{n-1} = \sin n\alpha$$

by assuming $u_n = a\cos n\alpha + b \sin n\alpha$.

Hence show that the general solution is

$$u_n = C(-1)^n + \frac{\sin(n+1)\alpha + \sin n\alpha}{2(1+\cos\alpha)}.$$

14. If $u_n u_{n-1} = \frac{a}{n}$, find u_n in terms of u_1.

[$nu_n = (n-1)u_{n-2}$; note that there are two forms.]

15. Show that the general solution of the equation

$$u_n u_{n-1} + a u_n + b u_{n-1} + c = 0$$

can be written in the form

$$u_n = \frac{A\alpha^n + \beta^n}{A\alpha^{n-1} + \beta^{n-1}} - a,$$

where A is an arbitrary constant and α, β are the roots of

$$x^2 - (a-b)x + c - ab = 0;$$

unless $\alpha = \beta$, in which case

$$u_n = \frac{a-b}{2(A+n)} - \tfrac{1}{2}(a+b).$$

[Write the equation as $u_n(u_{n-1}+a) + b u_{n-1} + c = 0$. Put $u_{n-1} + a = \dfrac{v_{n-1}}{v_{n-2}}$, so that $u_n + a = \dfrac{v_n}{v_{n-1}}$. The equation then reduces to

$$v_n + (b-a)v_{n-1} + (c-ab)v_{n-2} = 0,$$

unless $v_{n-1} = 0$.]

16. If $u_n u_{n-1} + 3u_n - 4u_{n-1} - 2 = 0$, prove that

$$u_n = \frac{A \cdot 2^n + 5^n}{A \cdot 2^{n-1} + 5^{n-1}} - 3.$$

17. If $u_n u_{n-1} + 5u_n + u_{n-1} + 9 = 0$, then

$$u_n = \frac{2}{A+n} - 3.$$

18. If $u_{n+1} = \dfrac{n(n+1)}{2n - u_n}$ and $u_1 = a$, prove that

$$u_{n+1} = (n+1) \cdot \frac{a + n(1-a)}{a + (n+1)(1-a)}$$

and

$$\lim_{n\to\infty} \frac{u_1 u_2 \ldots u_{n+1}}{\lfloor n} = \frac{a}{1-a}.$$

[Put $u_n = n v_n$.]

CHAPTER XXIII

THE OPERATORS Δ, E, D. INTERPOLATION

1. The Operator Δ. Let u_n be a function of the positive integral variable n, and consider the sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$.

(i) The meaning of the symbol Δ is defined by

$$\Delta u_n = u_{n+1} - u_n,$$

and Δ is regarded as denoting the *operation* which when applied to u_n produces the difference $u_{n+1} - u_n$. Here n may have any value, so that

$$\Delta u_1 = u_2 - u_1, \quad \Delta u_2 = u_3 - u_2, \text{ etc.}$$

(ii) *If* $\Delta u_n = v_n$, *then* $v_1 + v_2 + v_3 + \ldots + v_n = u_{n+1} - u_1$.

For $v_n = u_{n+1} - u_n$, $v_{n-1} = u_n - u_{n-1}, \ldots \quad v_1 = u_2 - u_1$,

whence the result follows by addition.

(iii) *If* u_n, v_n *are functions of* n, *then* $\Delta(u_n + v_n) = \Delta u_n + \Delta v_n$.

For
$$\begin{aligned}\Delta(u_n + v_n) &= (u_{n+1} + v_{n+1}) - (u_n + v_n) \\ &= (u_{n+1} - u_n) + (v_{n+1} - v_n) = \Delta u_n + \Delta v_n.\end{aligned}$$

In this connection it should be noted that if

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n,$$

then Δs_n *is not equal to* $\Delta u_1 + \Delta u_2 + \ldots + \Delta u_n$ *unless* $u_1 = 0$, for

$$\Delta s_n = (u_1 + u_2 + \ldots + u_n + u_{n+1}) - (u_1 + u_2 + \ldots + u_n) = u_{n+1},$$

and
$$\begin{aligned}\Delta u_1 + \Delta u_2 + \ldots + \Delta u_n &= (u_2 - u_1) + (u_3 - u_2) + \ldots + (u_{n+1} - u_n) \\ &= u_{n+1} - u_1.\end{aligned}$$

(iv) Following the index notation of Algebra, the symbol Δ^2 denotes that the operation Δ is to be repeated, and we write

$\Delta^2 u_n = \Delta(\Delta u_n)$, $\Delta^3 u_n = \Delta(\Delta^2 u_n)$, and so on ; thus

$$\begin{aligned}\Delta^2 u_n &= \Delta u_{n+1} - \Delta u_n \\ &= (u_{n+2} - u_{n+1}) - (u_{n+1} - u_n) = u_{n+2} - 2u_{n+1} + u_n, \\ \Delta^3 u_n &= \Delta^2 u_{n+1} - \Delta^2 u_n \\ &= (u_{n+3} - 2u_{n+2} + u_{n+1}) - (u_{n+2} - 2u_{n+1} + u_n) \\ &= u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n.\end{aligned}$$

This process may be continued to any extent and, from the way in which the successive differences are formed, it is obvious that the numerical coefficients in the expressions for $\Delta^2 u_n, \Delta^3 u_n, \ldots$ are the same as those in the expansions of $(1-x)^2, (1-x)^3, \ldots$. Hence we conclude that

$$\Delta^r u_n = u_{n+r} - C_1^r u_{n+r-1} + C_2^r u_{n+r-2} - \ldots + (-1)^n u_n, \quad \ldots\ldots\ldots\ldots(\mathbf{A})$$

a result which is proved more concisely in Art. 4.

For the sequence $u_0, u_1, u_2, \ldots$ *a table of differences* is conveniently written thus:

$$\begin{array}{ccccccc} u_0 & & u_1 & & u_2 & & u_3 \ldots \\ & \Delta u_0 & & \Delta u_1 & & \Delta u_2 \ldots & \\ & & \Delta^2 u_0 & & \Delta^2 u_1 \ldots & & \\ & & \ldots\ldots\ldots & & & & \end{array}$$

where $\Delta u_0 = u_1 - u_0$, $\Delta u_1 = u_2 - u_1$, $\Delta^2 u_0 = \Delta u_1 - \Delta u_0$, etc.

Thus, for the sequence $0^2, 1^2, 2^2, 3^2, \ldots$, where $u_n = n^2$, we write

u	0		1		4		9		16 ...
Δu		1		3		5		7	
$\Delta^2 u$			2		2		2		
$\Delta^3 u$				0		0			

and $\Delta 0^2 = 1$, $\Delta^2 0^2 = 2$, $\Delta^3 0^2 = 0$.

(v) If u_x is a function of *any* variable x, the meaning of the operation Δ applied to u_x is *defined* by $\Delta u_x = u_{x+1} - u_x$.

All that has been said about the symbol Δ still holds, for in finding the successive differences of u_x we are merely concerned with the sequence $u_x, u_{x+1}, u_{x+2}, \ldots$.

Illustrations. (i) If $u_n = n(n-1)(n-2)$, then

$$\Delta u_n = (n+1)n(n-1) - n(n-1)(n-2) = 3n(n-1),$$

$$\Delta^2 u_n = 3(n+1)n - 3n(n-1) = 6n, \quad \Delta^3 u_n = 6(n+1) - 6n = 6$$

and $$\Delta^r u_n = 0 \quad \text{for} \quad r > 3.$$

(ii) If $u_n = n^3$, since $n^3 = n(n-1)(n-2) + 3n(n-1) + n$, by the preceding, $\Delta u_n = 3n(n-1) + 6n + 1 = 3n^2 + 3n + 1$, $\Delta^2 u_n = 6n + 6$, $\Delta^3 u_n = 6$, $\Delta^4 u_n = 0$.

2. Special Cases. (i) The following notation is often used; we write

$$x_r = x(x-1)(x-2)\ldots(x-r+1) \quad \text{and} \quad x_{-r} = \frac{1}{x(x+1)(x+2)\ldots(x+r-1)}.$$

If x is variable and Δ applies to x, then

$$\Delta x_r = (x+1)x(x-1)\ldots(x-r+2) - x(x-1)(x-2)\ldots(x-r+1)$$

$$= x(x-1)\ldots(x-r+2)\{(x+1) - (x-r+1)\} = rx(x-1)\ldots(x-r+2),$$

that is $$\Delta x_r = r x_{r-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

Similarly we can show that $\Delta x_{-r} = -rx_{-(r+1)}$, ...(B)

it being assumed that *no denominator vanishes.*

These formulae may be compared with

$$\frac{d}{dx}x^r = rx^{r-1} \quad \text{and} \quad \frac{d}{dx}x^{-r} = -rx^{-(r+1)}.$$

Ex. 1. *Show that*

$$n(n-1)+(n-1)(n-2)+\ldots+2\,.\,1 = \tfrac{1}{3}(n+1)n(n-1),$$
$$n(n-1)(n-2)+(n-1)(n-2)(n-3)+\ldots+3\,.\,2\,.\,1 = \tfrac{1}{4}(n+1)n(n-1)(n-2),$$

and so on.

If S is the sum of the first series,

$$S = n_2 + (n-1)_2 + (n-2)_2 + \ldots ;$$

and since $\Delta n_3 = 3n_2$, by Art. 1, (ii),

$$S = \tfrac{1}{3}(n+1)_3 + C \text{ where } C \text{ is a constant.}$$

Putting $n=2$, we find that $C=0$. Hence the result. Similarly for the second series. Of course, the rule in Ch. VIII, 3, might have been applied.

(ii) **A Polynomial.** If u_x is a polynomial in x of degree r, dividing by x, $x-1$, $x-2$, ... in succession, we can find a_0, a_1, ... a_r, independent of x, such that

$$u_x = a_0 + a_1x_1 + a_2x_2 + \ldots + a_rx_r,$$

and then
$$\Delta u_x = a_1 + 2a_2x_1 + 3a_3x_2 + \ldots + ra_rx_{r-1},$$
$$\Delta^2 u_x = 2.1\,a_2 + 3.2\,a_3x_1 + \ldots + r(r-1)\,a_rx_{r-2},$$

and so on. Finally

$$\Delta^r u_x = a_r\,.\,\lfloor r \quad \text{and} \quad \Delta^{r+1}u_x = 0.$$

Ex. 2. *If* $m<r$, *then* $\Delta^m u_0 = \lfloor m\,.\,a_m$.

For $\Delta^m u_x = \lfloor m\,.\,a_m$ + terms containing x as a factor.

3. The Operator E. The effect of the operation denoted by E, when applied to u_n, is to increase n by unity. Thus for all values of n,

$$Eu_n = u_{n+1}.$$

If a is a constant, the following equations define the meaning of $E+a$, $a+E$, Ea and aE, regarded as operators.

$$(E+a)\,u_n = u_{n+1} + au_n = (a+E)\,u_n,$$
$$(Ea)\,u_n = E(au_n) = au_{n+1} = (aE)\,u_n.$$

Further, we write

$$E^2u_n = E(Eu_n) = Eu_{n+1} = u_{n+2},$$
$$E^3u_n = E(E^2u_n) = Eu_{n+2} = u_{n+3},$$

and so on ; thus, if r is a positive integer,

$$E^ru_n = u_{n+r}.$$

Hence, if p, q are positive integers,

$$E^pE^qu_n = E^p(E^qu_n) = E^pu_{n+q} = u_{n+p+q},$$

so that $$E^pE^qu_n = E^{p+q}u_n = E^qE^pu_n.$$

Again, if a, b are constants,

$$\begin{aligned}(E+a)(E+b)u_n &= (E+a)(u_{n+1}+bu_n)\\ &= u_{n+2}+bu_{n+1}+au_{n+1}+abu_n\\ &= \{E^2+(a+b)E+ab\}u_n.\end{aligned}$$

Thus, *so far as addition (subtraction) and multiplication are concerned, the operator E combines with itself and with constants according to the laws of Algebra.*

The same is true for Δ, for

$$\Delta u_n = u_{n+1}-u_n = (E-1)\,u_n.$$

More generally, if u_x is a function of *any* variable x, the operation E is defined by

$$Eu_x = u_{x+1}.$$

4. Fundamental Theorems. *If u_x is a function of the variable x, and r is a positive integer, then*

(i) $\Delta^r u_x = u_{x+r} - C_1^r u_{x+r-1} + C_2^r u_{x+r-2} - \ldots + (-1)^r u_x$, ...(A)

(ii) $u_{x+r} = u_x + C_1^r \Delta u_x + C_2^r \Delta^2 u_x + \ldots + \Delta^r u_x$. .....(B)

Proofs. The first theorem follows from the reasoning in Art. 1, (iv), on writing x for n.

More easily thus:

$$\begin{aligned}(\text{i})\ \Delta^r u_x &= (E-1)^r u_x\\ &= \{E^r - C_1^r E^{r-1} + C_2^r E^{r-2} - \ldots + (-1)^r\}u_x\\ &= E^r u_x - C_1^r E^{r-1}u_x + C_2^r E^{r-2}u_x - \ldots + (-1)^r u_x\\ &= u_{x+r} - C_1^r u_{x+r-1} + C_2^r u_{x+r-2} - \ldots + (-1)^r u_x.\end{aligned}$$

Again, (ii) $u_{x+r} = E^r u_x = (1+\Delta)^r u_x$, and therefore

$$\begin{aligned}u_{x+r} &= (1 + C_1^r\Delta + C_2^r\Delta^2 + \ldots + \Delta^r)\,u_x\\ &= u_x + C_1^r\Delta u_x + C_2^r\Delta^2 u_x + \ldots + \Delta^r u_x.\end{aligned}$$

(iii) *If u_x is a polynomial in x of degree $r-1$, then*

$$u_{x+r} - C_1^r u_{x+r-1} + C_2^r u_{x+r-2} - \ldots + (-1)^r u_x = 0. \quad \ldots\ldots\ldots\ldots(\text{C})$$

For the left-hand side $= \Delta^r u_x = 0$ (Art. 2, (ii)).

(iv) *If s_n is the sum to n terms of the series $u_1 + u_2 + u_3 + \ldots$, then*

$$s_n = nu_1 + \frac{n(n-1)}{\lfloor 2}\Delta u_1 + \frac{n(n-1)(n-2)}{\lfloor 3}\Delta^2 u_1 + \ldots + \Delta^{n-1}u_1. \quad \ldots\ldots(\text{D})$$

For $u_n = E^{n-1}u_1 = (1+\Delta)^{n-1}u_1$, therefore

$$u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{\lfloor 2}\Delta^2 u_1 + \ldots + \Delta^{n-1}u_1.$$

Similarly

$$u_{n-1} = u_1 + (n-2)\Delta u_1 + \frac{(n-2)(n-3)}{\lfloor 2}\Delta^2 u_1 + \ldots + \Delta^{n-2}u_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$u_2 = u_1 + \Delta u_1,$$

$$u_1 = u_1.$$

Adding and using the results of Art. 2, Ex. 1, the result follows at once.

Ex. 1. *Find a cubic function of* n *which has the values* 1, -3, -1, 13 *when* $n = 1, 2, 3, 4$ *respectively.*

Denoting the function by u_n, we have

n	1		2		3		4
u_n	1		-3		-1		13
Δu_n		-4		2		14	
$\Delta^2 u_n$			6		12		
$\Delta^3 u_n$				6			

and $\Delta^r u_n = 0$ for $r > 3$, since u_n is a cubic function.

Also $$u_n = E^{n-1}u_1 = (1+\Delta)^{n-1}u_1;$$

$$\therefore\ u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{\lfloor 2}\Delta^2 u_1 + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}\Delta^3 u_1$$

$$= 1 + (-4)(n-1) + 6\frac{n^2-3n+2}{\lfloor 2} + 6\frac{n^3-6n^2+11n-6}{\lfloor 3}$$

$$= 5 - 2n - 3n^2 + n^3.$$

Ex. 2. *Sum the series*

$$2 \cdot 3 + 3 \cdot 6 + 4 \cdot 11 + \ldots + (n+1)(n^2+2).$$

Here $u_n = (n+1)(n^2+2)$. This is a cubic function of n, so that $\Delta^r u_n = 0$ for $n > 3$. Writing down the first four terms, we find, as on the right, that $u_1 = 6$, $\Delta u_1 = 12$, $\Delta^2 u_1 = 14$, $\Delta^3 u_1 = 6$.

u_n	6		18		44		90
Δu_n		12		26		46	
$\Delta^2 u_n$			14		20		
$\Delta^3 u_n$				6			

Hence by (iv),

$$s_n = 6n + 12 \cdot \frac{n(n-1)}{\lfloor 2} + 14\frac{n(n-1)(n-2)}{\lfloor 3} + 6\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}$$

$$= \tfrac{1}{12}n(3n^3 + 10n^2 + 21n + 38). \quad \text{(Cf. Ch. VIII, 4, Ex. 1.)}$$

Ex. 3. *If* $S = x^n - C_1^r(x+1)^n + C_2^r(x+2)^n - \ldots + (-1)^r(x+r)^n$ *where* r, n *are positive integers, then* $S = 0$ *if* $r > n$ *and* $S = (-1)^n \lfloor n$ *if* $r = n$.

For if Δ applies to x, $S = (1-E)^r x^n = (-1)^r \Delta^r x^n$, and the result follows by Art. 2, (ii).

Ex. 4.* (*Bernoulli's Numbers.*) *If* B_r *is defined as in* Ch. VIII, 8, (5), *then*

$$(-1)^r B_r = \left\{1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots + (-1)^r \frac{\Delta^r}{r+1}\right\} 0^r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

For $n^r = E^n 0^r = (1+\Delta)^n 0^r$ and $\Delta^m 0^r = 0$ if $m > r$;

$$\therefore\ n^r = \left\{n\Delta + \frac{n(n-1)}{\lfloor 2} \Delta^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} \Delta^r\right\} 0^r. \quad \ldots\ldots\ldots\ldots\text{(B)}$$

Hence if $S_r = 1^r + 2^r + \ldots + n^r$, we have

$$S_r = \left\{\frac{(n+1)n}{\lfloor 2} \Delta + \frac{(n+1)n(n-1)}{\lfloor 3} \Delta^2 + \ldots + \frac{(n+1)n(n-1)\ldots(n-r+1)}{\lfloor r+1} \Delta^r\right\} 0^r. \quad \ldots\text{(C)}$$

Now, by definition, B_r = coefficient of n in S_r, hence

$$\therefore\ B_r = \left\{\frac{\Delta}{\lfloor 2} - \frac{\lfloor 1\ \Delta^2}{\lfloor 3} + \frac{\lfloor 2\ \Delta^3}{\lfloor 4} - \ldots + (-1)^{r-1} \frac{\lfloor r-1}{\lfloor r+1} \Delta^r\right\} 0^r;$$

$$\therefore\ B_r = \left\{\Delta\left(1 - \frac{1}{2}\right) - \Delta^2\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + (-1)^{r-1}\left(\frac{1}{r} - \frac{1}{r+1}\right)\Delta^r\right\} 0^r. \quad \ldots\ldots\ldots\text{(D)}$$

Again, equating the coefficients of n in (B),

$$\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots + (-1)^{r-1} \frac{\Delta^r}{r}\right\} 0^r = 0 \quad \text{if} \quad r > 1. \quad \ldots\ldots\ldots\ldots\text{(E)}$$

Hence if $r > 1$,

$$B_r = \left\{1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots + (-1)^r \frac{\Delta^r}{r+1}\right\} 0^r. \quad \ldots\ldots\ldots\ldots\ldots\text{(F)}$$

Moreover, $\Delta 0^r = 1$, $B_1 = \frac{1}{2}$ and $(-1)^r B_r = B_r$ for $r > 1$ (Ch. VIII, 8, (5)).
Hence equation (A) holds for $r > 0$.

Ex. 5. Apply equation (A) of Ex. 4 to show that $B_2 = \frac{1}{6}$.

$$B_2 = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3}\right) 0^2 = 0 - \tfrac{1}{2} + \tfrac{2}{3} = \tfrac{1}{6}.$$

5. Operators in a Fractional Form. Many writers give the following 'proof' of the theorem in Art. 4, (iv):

$$\begin{aligned} s_n &= u_1 + u_2 + \ldots + u_n \\ &= (1 + E + E^2 + \ldots + E^{n-1})\, u_1 \\ &= \frac{E^n - 1}{E - 1}\, u_1 = \frac{(1+\Delta)^n - 1}{\Delta}\, u_1 \\ &= \left(n + \frac{n(n-1)}{\lfloor 2} \Delta + \frac{n(n-1)(n-2)}{\lfloor 3} \Delta^2 + \ldots + \Delta^{n-1}\right) u_1 \\ &= n u_1 + \frac{n(n-1)}{\lfloor 2} \Delta u_1 + \frac{n(n-1)(n-2)}{\lfloor 3} \Delta^2 u_1 + \ldots + \Delta^{n-1} u_1. \end{aligned}$$

Here the operator is written in a fractional form in order to transform a polynomial in E into a polynomial in Δ.

* This result was obtained by Cayley by a much more difficult process (Vol. IX, pp. 259-262 of his collected works).

The result, when obtained, can be verified by multiplication and addition; this fact justifies the process.

But at every stage *the operator must be regarded as a whole.* Thus the equation

$$s_n = \frac{E^n - 1}{E - 1} u_1$$

does not imply that $(E-1)s_n = (E^n - 1)u_1$.

In fact the last statement is not true, for

$$(E-1)s_n = s_{n+1} - s_n = u_{n+1} \quad \text{and} \quad (E^n - 1)u_1 = u_{n+1} - u_1.$$

We conclude that, in such transformations, *the operator can be put in the form $f(\Delta)/\phi(\Delta)$, where $f(\Delta)$ and $\phi(\Delta)$ are polynomials in Δ, provided that $f(\Delta)$ is divisible by $\phi(\Delta)$, and $f(\Delta)$, $\phi(\Delta)$ are not regarded as separate operators.*

6. Interpolation. Suppose that y is a function of x whose value has been determined by experiment, or otherwise, for $x = a, b, c, \ldots$.

The problem of interpolation is to find approximate values of y for intermediate values of x.

The graphical method is to plot the points (x, y) for $x = a, b, c, \ldots$, and draw a 'smooth' curve through the points. We assume that this curve represents as nearly as possible the graph of the function y.

Let us suppose that the equation to the curve so drawn is $y = f(x)$. It is natural to take for $f(x)$ the simplest function of x which satisfies the conditions. If y is known for n values of x, we assume that

$$y = p_0 + p_1 x + p_2 x^2 + \ldots + p_{n-1} x^{n-1},$$

where the n constants $p_0, p_1, \ldots$ are to be determined by making the curve go through the n points.

If only two values of y are known, the curve is the straight line joining the corresponding points, and we have the rule of *proportional parts* as used in working with tables of logarithms and trigonometric functions.

7. Lagrange's Formula. If $y = y_1, y_2, y_3$ for $x = a, b, c$ respectively, we assume that

$$y = y_1 \frac{(x-b)(x-c)}{(a-b)(a-c)} + y_2 \frac{(x-a)(x-c)}{(b-a)(b-c)} + y_3 \frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

For y is a quadratic function of x of which the values are y_1, y_2, y_3 when $x = a, b, c$.

Similarly if $y = y_1, y_2, y_3, y_4$ when $x = a, b, c, d$ we take

$$y = y_1 \frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + \text{three similar terms};$$

with similar formulae when more than four values of y are given.

8. If the values of y are known for values of x *at equal intervals*, instead of applying Lagrange's formulae, we use the operators E and Δ.

For example, suppose that $y=y_0,\ y_1,\ y_2,\ y_3$ for $x=x_0,\ x_1,\ x_2,\ x_3$. Assume that y is a cubic function of x; then $\Delta^r y_0=0$ for $r\geqslant 4$, and if n is a positive integer,

$$y_n=E^n y_0=(1+\Delta)^n y_0,$$

therefore

$$y_n=y_0+n\Delta y_0+\frac{n(n-1)}{\lfloor 2}\Delta^2 y_0+\frac{n(n-1)(n-2)}{\lfloor 3}\Delta^3 y_0.$$

Hence the equation

$$y=y_0+x\Delta y_0+\frac{x(x-1)}{\lfloor 2}\Delta^2 y_0+\frac{x(x-1)(x-2)}{\lfloor 3}\Delta^3 y_0$$

represents a cubic curve which passes through the four points (x_0, y_0) (x_1, y_1), ... and gives an approximate value of y corresponding to any value of x within the specified range.

Ex. 1. *Given that*

$$\sin 45^\circ=0{\cdot}7071,\quad \sin 50^\circ=0{\cdot}7660,\quad \sin 55^\circ=0{\cdot}8192,\quad \sin 60^\circ=0{\cdot}8660,$$

find $\sin 52^\circ$.

Let $u_x=10^4 \sin(45+5x)^\circ$. Construct a table of differences as below.

u	Δu	$\Delta^2 u$	$\Delta^3 u$
$u_0=7071$			
	589		
$u_1=7660$		-57	
	532		-7
$u_2=8192$		-64	
	468		
$u_3=8660$			

$$\therefore\ \Delta u_0=589,\quad \Delta^2 u_0=-57,\quad \Delta^3 u_0=-7.$$

Assuming that $\Delta^r u_0=0$ for $r>3$, approximate values of u_x are given by

$$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{\lfloor 2}\Delta^2 u_0+\frac{x(x-1)(x-2)}{\lfloor 3}\Delta^3 u_0.$$

Let $52=45+5x$, so that $x=1{\cdot}4$ and

$$\begin{aligned}
u_0 &= 7071\\
x\Delta u_0=1{\cdot}4\times 589 &= 824{\cdot}6\\
\frac{x(x-1)}{\lfloor 2}\Delta^2 u_0=-0{\cdot}7\times 0{\cdot}4\times 57 &= -15{\cdot}96\\
\frac{x(x-1)(x-2)}{\lfloor 3}\Delta^3 u_0=1{\cdot}4\times 0{\cdot}4\times 0{\cdot}7 &= 0{\cdot}392\\
&\overline{7895{\cdot}992}\\
&15{\cdot}96\\
u_x &\overline{7880{\cdot}032}
\end{aligned}$$

$\therefore$ $\sin 52^\circ=0{\cdot}7880$ to four places.

The correct seven-place value is 0·7880108.

If we only use the values of sin 50 and sin 55, the rule of proportional parts would give

$$\sin 52^\circ = 0{\cdot}7660 + \tfrac{2}{5} \times 0{\cdot}0532 = 0{\cdot}7873.$$

If u_0 and several other terms of the sequence $u_0, u_1, u_2, \ldots$ are known and approximate values of the missing terms are required, we may proceed as follows.

Ex. 2. *If* $u_0 = 4{\cdot}3315$, $u_1 = 4{\cdot}7046$, $u_3 = 5{\cdot}6713$, $u_5 = 7{\cdot}1154$, *find approximate values of* u_2 *and* u_4.

Four points on the graph of u_x are known. We therefore assume that it can be represented by a cubic function of x. This is the same as assuming that $\Delta^4 u_x = 0$. Hence we have approximately

$$\Delta^4 u_0 = u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0,$$

$$\Delta^4 u_1 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0\ ;$$

$$\therefore\ u_4 + 6u_2 = 4(u_1 + u_3) - u_0 = 37{\cdot}1721,$$

$$u_4 + u_2 = \tfrac{1}{4}(u_5 + 6u_3 + u_1) = 11{\cdot}4619.$$

$$\therefore\ u_2 = 5{\cdot}1420,\quad u_4 = 6{\cdot}3199 \text{ (approx.)}.$$

NOTE. In this example $u_0 = \tan 77^\circ$, $u_1 = \tan 78^\circ$, $u_3 = \tan 80^\circ$, $u_5 = \tan 82^\circ$, so we have found that $\tan 79^\circ = 5{\cdot}1420$, $\tan 81^\circ = 6{\cdot}3199$ (approx.).

The correct values are 5·1446 and 6·3138.

It should be noticed that the approximations just found are much closer than those given by

$$u_2 = \tfrac{1}{2}(u_1 + u_3) = 5{\cdot}1879,\quad u_4 = \tfrac{1}{2}(u_3 + u_5) = 6{\cdot}3933.$$

9. Bessel's Formula. In using Mathematical Tables, the effect of second differences may be easily and accurately allowed for by using a formula due to Bessel.

Let u_{-1}, u_0, u_1, u_2 be four consecutive values of a function u_x, as given in the tables, and suppose that values of u_x are required for values of x between 0 and 1.

Denoting first and second differences by a, b, c and d, e, as shown on the right, Bessel's formula is

u_{-1}			
	a		
u_0		d	
	b		f
u_1		e	
	c		
u_2			

$$u_x = u_0 + xb + \frac{x(x-1)}{2} \cdot \frac{d+e}{2}, \quad \text{.....(A)}$$

which is the same as

$$u_x = u_0 + xb + \frac{x(x-1)}{4}(c-a). \quad \text{.....(B)}$$

The values of u_0, a, b, c can be taken from the tables, so that very little labour is required in using the formula.

To show that this gives a good approximation, and to estimate the error, if we neglect differences of the fourth and higher orders, we have

$$u_x=(1+\Delta)^{x+1}u_{-1}=u_{-1}+(x+1)a+\frac{(x+1)x}{2}d+\frac{(x+1)x(x-1)}{6}f.$$

Now $u_0-u_{-1}=a$, $b-a=d$, $e-d=f$; and therefore

$$u_x=u_0+xb+\frac{x(x-1)}{2}d+\frac{(x+1)x(x-1)}{6}f$$

$$=u_0+xb+\frac{x(x-1)}{2}\left(\frac{d+e}{2}-\frac{e-d}{2}\right)+\frac{(x+1)x(x-1)}{6}f,$$

so that if v_x is the approximation given by (**A**),

$$u_x=v_x-\frac{x(x-1)}{2}\cdot\frac{f}{2}+\frac{(x+1)x(x-1)}{6}f$$

$$=v_x+\frac{x(x-1)(2x-1)}{12}f.$$

Hence the error is approximately $\frac{1}{12}x(x-1)(2x-1)f$.

Ex. 1. *Use tables of square roots to find* $\sqrt{12344}$.

Using Barlow's tables, we take

$u_{-1}=\sqrt{12330}=\ldots\ldots\ldots$

$\cdot045020=a$

$u_0=\sqrt{12340}=111\cdot085553$

$\cdot045001=b$

$u_1=\sqrt{12350}=\ldots\ldots\ldots\ldots$

$\cdot044983=c.$

$u_2=\sqrt{12360}=\ldots\ldots\ldots\ldots.$

(The differences a, b, c are given in the tables.) For $\sqrt{12344}$, we have

$x=\cdot4$, $\frac{1}{4}x(x-1)=-\cdot06$, $c-a=-\cdot000037$.

$$\begin{aligned} u_0&=111\cdot085553\\ xb&=\quad\cdot0180004\\ \tfrac{1}{4}x(x-1)(c-a)&=\quad\cdot00000224\\ \sqrt{12344}&=111\cdot103556\end{aligned}$$

The result is correct to the last figure.

10. The Operator D. When we write

$$\frac{du}{dx}=\frac{d}{dx}u,\quad \frac{d^nu}{dx^n}=\left(\frac{d}{dx}\right)^n u,$$

the symbol $\frac{d}{dx}$ is regarded as an *operator*, which is sometimes denoted by D. Thus

$$D^mD^nu=D^{m+n}u=D^nD^mu.$$

If a is a constant, the following equations define the meaning of $D+a$, $a+D$, Da and aD, regarded as operators.

$$(D+a)u=Du+au=(a+D)u,\quad (Da)u=D(au)=aDu=(aD)u.$$

Hence if a, b are constants, $(D+a)(D+b)u=(D+a)(Du+bu)$,

and $(D+a)(D+b)u=D^2u+bDu+aDu+abu=\{D^2+(a+b)D+ab\}u$.

Thus, so far as addition, subtraction and multiplication are concerned, the operator D combines with itself and with *constants* according to the laws of Algebra.

11. The n-th Derivative of a Product. If u, v are functions of x,

$$\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}.$$

Let the symbols D_1, D_2 denote differentiation with regard to x, and suppose that D_1 only operates on u and its differential coefficients, and that D_2 operates only on v and its differential coefficients, so that

$$D_1(uv)=\frac{du}{dx}v,\quad D_2(uv)=u\frac{dv}{dx};\ \text{and hence}\ \frac{d}{dx}(uv)=(D_1+D_2)uv.$$

As D_1 and D_2 are independent of each other, they obey the laws of addition and multiplication, therefore

$$\left(\frac{d}{dx}\right)^n(uv)=(D_1+D_2)^n(uv)=(D_1{}^n+C_1^nD_1{}^{n-1}D_2+C_2^nD_1{}^{n-2}D_2{}^2+\ldots+D_2{}^n)(uv);$$

that is, $$\frac{d^n}{dx^n}(uv)=\frac{d^nu}{dx^n}v+C_1^n\frac{d^{n-1}u}{dx^{n-1}}v+C_2^n\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}+\ldots+u\frac{d^nv}{d^nx}.$$

This result is known as *Leibniz' Theorem*, and is often written in the form $$(uv)_n=u_nv+C_1^nu_{n-1}v_1+C_2^nu_{n-2}v_2+\ldots+uv_n,$$

where the suffixes indicate differentiation with regard to x.

Ex. 1. *If* $y=u+v$ *where* $u^m=x+(x^2-k)^{\frac{1}{2}}$, $v^m=x-(x^2-k)^{\frac{1}{2}}$, *then*

$$(x^2-k)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-\frac{1}{m^2}\right)y_n=0,\ \textit{for}\ n=0,\ 1,\ 2,\ \ldots.$$

Differentiating with regard to x, we have

$$mu^{m-1}u_1=1+x(x^2-k)^{-\frac{1}{2}}=u^m(x^2-k)^{-\frac{1}{2}};\ \text{hence}\ (x^2-k)^{\frac{1}{2}}u_1=u/m.$$

Differentiating again,

$$(x^2-k)^{\frac{1}{2}}u_2+x(x^2-k)^{-\frac{1}{2}}u_1=u_1/m=(x^2-k)^{-\frac{1}{2}}.u/m^2;$$

$$\therefore\ (x^2-k)u_2+xu_1-u/m^2=0.$$

Similarly $$(x^2-k)v_2+xv_1-v/m^2=0,$$

whence by addition, $$(x^2-k)y_2+xy_1-y/m^2=0.$$

Differentiating n times by Leibniz' theorem,

$$\begin{aligned}(x^2-k)y_{n+2}+n.2x.y_{n+1}+\frac{n(n-1)}{2}\cdot 2y_n\\ +\quad xy_{n+1}+\quad ny_n\\ -y_n/m^2=0;\end{aligned}$$

$$\therefore\ (x^2-k)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-\frac{1}{m^2}\right)y_n=0.$$

This example leads to a remarkable theorem on cubic equations, given in Ch. XXIX.

EXERCISE XXXIX

1. Find the polynomial of the third degree in n which has the values 2, 11, 32, 71, when $n=1, 2, 3, 4$ respectively. (Use Art. 4.)

2. Use the method of Art. 4, Ex. 2 to find the sum to n terms of the series

(i) $1^3+3^3+5^3+\dots$; (ii) $1^2\,.\,2^2+2^2\,.\,3^2+3^2\,.\,4^2+\dots$.

3. If $u_n=x^n$ and Δ applies to n, show that

(i) $\Delta^r u_n=x^n(x-1)^r$;

(ii) $1+x+x^2+\dots+x^{n-1}=n+\dfrac{n(n-1)}{\lfloor 2}(x-1)+\dfrac{n(n-1)(n-2)}{\lfloor 3}(x-1)^2$
$+\dots+(x-1)^{n-1}$.

4. Show that

(i) $x^n-C_1^n(x-y)^n+C_2^n(x-2y)^n-\dots+(-1)^n(x-ny)^n=y^n\,\lfloor n$.

(ii) $xy-C_1^n(x-1)(y-1)+C_2^n(x-2)(y-2)-\dots+(-1)^n(x-n)(y-n)$
$=x+y-1$, 2 or 0, according as $n=1$, $n=2$ or $n>2$.

(iii) $(n-1)^2C_1^n+(n-3)^2C_3^n+(n-5)^2C_5^n+\dots=2^{n-3}n(n+1)$, if $n>2$.

[In each case let S be the sum of the series.

(i) Let $u_r=(x-ry)^n$, then $S=(1-E)^nu_0=(-1)^n\Delta^nu_0$.
Also $\Delta^nu_r=(-y)^n\,\lfloor n$, etc.

(ii) Let $u_r=(x-r)(y-r)$, then $S=(-1)^n\Delta^nu_0$. Also
$\Delta u_0=(x-1)(y-1)-xy=-x-y+1$, $\Delta^2u_0=2$, $\Delta^nu_r=0$ if $n>2$.

(iii) Let $u_r=r^2$, then $S=\frac{1}{2}\{(E+1)^n-(E-1)^n\}u_0$;
$\therefore\ S=\frac{1}{2}\{(2+\Delta)^n-\Delta^n\}u_0$. Also $u_0=0$, $\Delta u_0=1$, $\Delta^2u_0=2$, and $\Delta^su_r=0$ if $s>2$.]

5. If $x_r=x(x-1)(x-2)\dots(x-r+1)$, show that
$$(x+n)_r=x_r+C_1^nrx_{r-1}+C_2^nr(r-1)x_{r-2}+\dots\,,$$
the last term being $r(r-1)\dots(r-n+1)x_{r-n}$ if $r>n$ and $C_r^n\,\lfloor r$ if $r\leqslant n$.

Verify that, if $n=3$ and $r=2$, the identity becomes
$$(x+3)(x+2)=x(x-1)+6x+6.$$

6. If $x^r=x_r+ax_{r-1}+bx_{r-2}+\dots$, prove that
$$a=\tfrac{1}{2}r(r-1),\quad b=\tfrac{1}{24}r(r-1)(r-2)(3r-5).$$
[Equate to 0 the coefficients of x^{r-1} and x^{r-2} on the right.]

7. Show that

(i) $\Delta^rx^{r+1}=(x+\frac{1}{2}r)\,\lfloor r+1$;

(ii) $\Delta^rx^{r+2}=\frac{1}{2}[x^2+rx+\frac{1}{12}r(3r+1)]\,.\,\lfloor r+2$.

[Use Ex. 6.]

8. Prove that
$$x^n-C_1^r(x+1)^n+C_2^r(x+2)^n-\dots+(-1)^r(x+r)^n$$
$$=(-1)^r(x+\tfrac{1}{2}r)\,\lfloor r+1 \quad \text{when} \quad n=r+1,$$
and $$=(-1)^r\tfrac{1}{2}\{x^2+rx+\tfrac{1}{12}r(3r+1)\}\,\lfloor r+2 \text{ when } n=r+2.$$
[Use Ex. 7.]

9. If none of the denominators vanish, prove that

$$\text{(i)}\ \Delta^n \frac{1}{x} = \frac{(-1)^n \lfloor n}{x(x+1)(x+2)\dots(x+n)}.$$

Deduce that

$$\text{(ii)}\ \frac{\lfloor n}{x(x+1)(x+2)\dots(x+n)} = \frac{1}{x} - C_1^n \frac{1}{x+1} + C_2^n \frac{1}{x+2} - \dots + (-1)^n \frac{1}{x+n};$$

$$\text{(iii)}\ \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+n} = n\frac{1}{x+1} - \frac{n(n-1)}{2}\frac{1}{(x+1)(x+2)} + \frac{n(n-1)(n-2)}{3}\frac{1}{(x+1)(x+2)(x+3)} - \dots.$$

10. Prove that

$$\frac{1}{m} + n\frac{2}{m+1} + \frac{n(n-1)}{1\,.\,2}\frac{2^2}{m+2} + \dots + \frac{2^n}{m+n} = \frac{3^n}{m} - \frac{n\,.\,2\,.\,3^{n-1}}{m(m+1)} + \frac{n(n-1)\,.\,2^2\,.\,3^{n-2}}{m(m+1)(m+2)} - \dots + (-1)^n \frac{n(n-1)\dots 1\,.\,2^n}{m(m+1)\dots(m+n)}.$$

11. If $u_n = 1/(an+b)$, show that

$$\text{(i)}\ \Delta^r u_n = (-1)^r \lfloor r \frac{a^r}{(an+b)(a\,.\,n+1+b)\dots(a\,.\,n+r+b)};$$

$$\text{(ii)}\ \frac{1}{b} + \frac{1}{a+b} + \frac{1}{2a+b} + \dots + \frac{1}{(n-1)a+b} = \frac{n}{b} - \frac{n(n-1)}{b(a+b)}\cdot\frac{a}{2} + \frac{n(n-1)(n-2)}{b(a+b)(2a+b)}\cdot\frac{a^2}{3} - \dots.$$

12. If $x_{-r} = 1/x(x+1)(x+2)\dots(x+r-1)$, show that

$$(x+n)_{-r} = x_{-r} - C_1^n r\, x_{-(r+1)} + C_2^n r(r+1) x_{-(r+2)} - \dots \text{ to } n+1 \text{ terms.}$$

13. If n is a positive integer and y is any number such that no denominator vanishes, prove that

$$\text{(i)}\ 1 + \frac{n}{y} + \frac{n(n-1)}{y(y-1)} + \frac{n(n-1)(n-2)}{y(y-1)(y-2)} + \dots = \frac{y+1}{y-n+1};$$

$$\text{(ii)}\ 1 + 2\frac{n}{y} + 3\frac{n(n-1)}{y(y-1)} + 4\frac{n(n-1)(n-2)}{y(y-1)(y-2)} + \dots = \frac{(y+1)(y+2)}{(y-n+1)(y-n+2)};$$

$$\text{(iii)}\ 1 + 3\frac{n}{y} + 6\frac{n(n-1)}{y(y-1)} + 10\frac{n(n-1)(n-2)}{y(y-1)(y-2)} + \dots = \frac{(y+1)(y+2)(y+3)}{(y-n+1)(y-n+2)(y-n+3)};$$

each series being continued to $n+1$ terms.

[These results follow from Ex. 12 by putting successively $r=1$, $x=-(y+1)$; $r=2$, $x=-(y+2)$; $r=3$, $x=-(y+3)$.]

14. Show that

$$\text{(i)}\ \frac{1}{x(x+1)}-C_1^n\frac{1}{(x+1)(x+2)}+C_2^n\frac{1}{(x+2)(x+3)}-\dots$$
$$+(-1)^n\frac{1}{(x+n)(x+n+1)}=\frac{\lfloor n+1}{x(x+1)(x+2)\dots(x+n+1)};$$

$$\text{(ii)}\ \frac{1}{x(x+1)(x+2)}-C_1^n\frac{1}{(x+1)(x+2)(x+3)}+C_2^n\frac{1}{(x+2)(x+3)(x+4)}$$
$$-\dots\text{ to }n+1\text{ terms}=\frac{\lfloor n+2}{x(x+1)(x+2)\dots(x+n+2)}.$$

15. If $u_n=\dfrac{a(a-1)(a-2)\dots(a-n+1)}{b(b-1)(b-2)\dots(b-n+1)}$, prove that

$$\Delta u_n=\frac{a-b}{b}\cdot\frac{a(a-1)\dots(a-n+1)}{(b-1)(b-2)\dots(b-n)},$$

$$\Delta^2u_n=\frac{a-b}{b}\cdot\frac{a-b+1}{b-1}\cdot\frac{a(a-1)\dots(a-n+1)}{(b-2)(b-3)\dots(b-n-1)},$$

$$\Delta^ru_n=\frac{a-b}{b}\cdot\frac{a-b+1}{b-1}\dots\frac{a-b+r-1}{b-r+1}\cdot\frac{a(a-1)\dots(a-n+1)}{(b-r)(b-r-1)\dots(b-r-n+1)}.$$

16. If $u_0=1$, $u_1=\dfrac{a}{b}$, $u_2=\dfrac{a(a-1)}{b(b-1)}$, etc., as in Ex. 15, prove that

$$\Delta^ru_0=\frac{a-b}{b}\cdot\frac{a-b+1}{b-1}\cdot\frac{a-b+2}{b-2}\cdot\dots\cdot\frac{a-b+r-1}{b-r+1}.$$

[Putting $n=1$ in Ex. 15, find Δu_0, Δ^2u_0, etc., in succession from the equations

$$\Delta u_0=u_1-u_0,\quad \Delta^2u_0=\Delta u_1-\Delta u_0,\quad \Delta^3u_0=\Delta^2u_1-\Delta^2u_0,\quad\text{etc.}]$$

17. Prove that

$$\left(1-\frac{a}{b}\right)\left(1-\frac{a}{b-1}\right)\left(1-\frac{a}{b-2}\right)\dots\left(1-\frac{a}{b-n+1}\right)$$
$$=1-n\frac{a}{b}+\frac{n(n-1)}{\lfloor 2}\frac{a(a-1)}{b(b-1)}-\frac{n(n-1)(n-2)}{\lfloor 3}\frac{a(a-1)(a-2)}{b(b-1)(b-2)}+\dots$$

to $n+1$ terms.

[Using Ex. 16, the left-hand side $=(-1)^n\Delta^nu_0=(1-E)^nu_0$.]

18. Show that the sum of the products two together of

$$\frac{1}{x},\ \frac{1}{x+1},\ \frac{1}{x+2},\dots,\frac{1}{x+n-1},$$

is equal to

$$\frac{n(n-1)}{2x(x+1)}\left\{1-\frac{n-2}{\lfloor 1}\frac{1}{x+2}+\frac{(n-2)(n-3)}{\lfloor 2}\cdot\frac{1}{(x+2)(x+3)}-\dots\text{ to }n-1\text{ terms}\right\}.$$

[Equate the coefficients of a^2 in Ex. 17 and put $x=-b$.]

19. In Art. 4, Ex. 4, it has been shown that

$$(-1)^r B_r = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots + (-1)^r \frac{\Delta^r}{r+1}\right) 0^r.$$

Prove also that

$$B_r = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots + (-1)^r \frac{\Delta^r}{r+1}\right) 1^r,$$

and use each of these formulae to show that $B_4 = -\frac{1}{30}$.

[Equation (D) of Ex. 4 can be written

$$B_r = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots\right)(1+\Delta)0^r \quad \text{and} \quad (1+\Delta)0^r = E0^r = 1^r.]$$

20. Given that

$$\sqrt{12500} = 111{\cdot}803399,$$
$$\sqrt{12510} = 111{\cdot}848111,$$
$$\sqrt{12520} = 111{\cdot}892806,$$
$$\sqrt{12530} = 111{\cdot}937483,$$

show that $\sqrt{12516} = 111{\cdot}874929$.

21. Given that

$$\sec 89^\circ\, 4' = 61{\cdot}3911,$$
$$\sec 89^\circ\, 5' = 62{\cdot}5072,$$
$$\sec 89^\circ\, 6' = 63{\cdot}6646,$$
$$\sec 89^\circ\, 7' = 64{\cdot}8657,$$

show that Bessel's formula gives

$$\sec 89^\circ\, 5'\, 40'' = 63{\cdot}2741.$$

22. If $S_n = 1^n + \frac{2^n}{\lfloor 1} + \frac{3^n}{\lfloor 2} + \ldots + \frac{(m+1)^n}{\lfloor m} + \ldots$ to ∞, prove that

(i) $\Delta^r S_n = \frac{2^n \cdot 1^r}{\lfloor 1} + \frac{3^n \cdot 2^r}{\lfloor 2} + \frac{4^n \cdot 3^r}{\lfloor 3} + \ldots$.

(ii) $\Delta^r S_0 = 1^{r-1} + \frac{2^{r-1}}{\lfloor 1} + \frac{3^{r-1}}{\lfloor 2} + \ldots = S_{r-1}$.

(iii) $S_n = S_{n-1} + C_1^n S_{n-2} + C_2^n S_{n-3} + \ldots + C_{n-1}^n S_0 + S_0$.

(iv) $S_n = S_{n-1} + C_1^n S_{n-1} - C_2^n S_{n-2} - \ldots + (-1)^{n-1} S_0$.

(v) Use (iii) or (iv) to find S_1, S_2, ... S_7, showing that for

$$n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,$$
$$S_n/e = 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140.$$

[(iii) and (iv), we have $S_n = (\Delta+1)^n S_0$ and $S_{n-1} = (E-1)^n S_0$.]

23. If $T_n = 1^n - \frac{2^n}{\lfloor 1} + \frac{3^n}{\lfloor 2} - \ldots + (-1)^m \frac{(m+1)^n}{\lfloor m} + \ldots$ to ∞, prove that

(i) $\Delta^r T_0 = -T_{r-1}$.

(ii) $-T_n = T_{n-1} + C_1^n T_{n-2} + C_2^n T_{n-3} + \ldots + C_{n-1}^n T_0 - T_0$.

(iii) Using (ii), show that for

$$n = 1,\ 2,\ 3,\ 4,\ 5,$$
$$T_n \cdot e = 0,\ -1,\ -1,\ 2,\ 9.$$

CHAPTER XXIV

CONTINUED FRACTIONS (1)

1. Definitions. Any expression of the form

$$a_1 \pm \cfrac{b_2}{a_2 \pm \cfrac{b_3}{a_3 \pm \dots}} \qquad \text{.....(A)}$$

is called a *continued fraction*, and is written $a_1 \pm \dfrac{b_2}{a_2 \pm} \dfrac{b_3}{a_3 \pm} \dots$ .....(B)

The signs $\pm$ may be supposed to be attached to the b's, and so any continued fraction may be expressed in the form

$$F = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \dots \frac{b_n}{a_n +} \dots, \qquad \text{.....(C)}$$

where the a's and b's may be positive or negative numbers. The quantities a_1, b_2/a_2, ... b_n/a_n ... are called the *elements* of F, and the fraction obtained by stopping at any particular stage is called a *convergent*. Thus the first, second, third, ..., convergents are $\dfrac{a_1}{1}$, $a_1 + \dfrac{b_2}{a_2}$, $a_1 + \dfrac{b_2}{a_2 +} \dfrac{b_3}{a_3}$, ...

It is obvious that the fraction is unaltered if b_n, a_n, b_{n+1} $(n = 2, 3, \dots)$ are all multiplied by any number k or if the signs of b_n, a_n and b_{n+1} are all changed. Thus any continued fraction may be written in the form (C) where a_2, a_3, ... are all positive.

Ex. 1. *Prove that, if each fraction contains* n *elements,*

$$\frac{2}{4+} \frac{2}{4+} \frac{2}{4+} \dots = \frac{1}{2+} \frac{1}{4+} \frac{1}{2+} \frac{1}{4+} \dots .$$

The fraction on the left is equal to $\dfrac{1}{2+} \dfrac{1}{4+} \dfrac{2}{4+} \dfrac{2}{4+} \dots = \dfrac{1}{2+} \dfrac{1}{4+} \dfrac{1}{2+} \dfrac{1}{4+} \dots .$

2. Formation of Convergents. (1) Let u_n denote the nth convergent of the continued fraction,

$$F = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \dots \frac{b_n}{a_n +} \dots, \quad \text{so that} \quad u_1 = \frac{a_1}{1}, \quad u_2 = a_1 + \frac{b_2}{a_2} = \frac{a_2 a_1 + b_2}{a_2}.$$

We shall write $u_1 = p_1/q_1$, $u_2 = p_2/q_2$ where

$$p_1 = a_1, \quad p_2 = a_2 a_1 + b_2, \quad q_1 = 1, \quad q_2 = a_2.$$

Observing that u_3 may be obtained from u_2 by changing a_2 into a_2+b_3/a_3, we have

$$u_3=\frac{(a_2+b_3/a_3)a_1+b_2}{a_2+b_3/a_3}=\frac{a_3(a_2a_1+b_2)+b_3a_1}{a_3a_2+b_3}=\frac{a_3p_2+b_3p_1}{a_3q_2+b_3q_1},$$

and so $u_3=p_3/q_3$, where $p_3=a_3p_2+b_3p_1$ and $q_3=a_3q_2+b_3q_1$.

Proceeding thus we can show that, if p_n and q_n are defined for successive values of n by the equations

$$p_n=a_np_{n-1}+b_np_{n-2},\quad q_n=a_nq_{n-1}+b_nq_{n-2},\quad \text{.....(A)}$$

then p_n/q_n is the nth convergent of F.

Equations (**A**) are called the *recurrence formulae.* It will be found convenient to write

$$p_0=1\quad\text{and}\quad q_0=0,\quad \text{.....(B)}$$

and it will be seen that $p_2=a_2p_1+b_2p_0$ and $q_2=a_2q_1+b_2q_0$, so that equations (**A**) hold for $n\geqslant 2$.

(2) It is important to state all this rather more precisely :

Definition. *For the continued fraction*

$$F=a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots\frac{b_n}{a_n+}\cdots$$

the quantities p_n and q_n are defined by the equations

$$p_n=a_np_{n-1}+b_np_{n-2},\quad q_n=a_nq_{n-1}+b_nq_{n-2},\quad \text{.....(A)}$$

with the initial values $p_0=1$, $q_0=0$, $p_1=a_1$, $q_1=1$.

Theorem. *If p_n and q_n are defined in this way, then p_n/q_n is the n-th convergent of F.*

Proof. Assume that this holds for the values 2, 3, ... r of n, and denote the nth convergent by u_n, then

$$u_r=\frac{p_r}{q_r}=\frac{a_rp_{r-1}+b_rp_{r-2}}{a_rq_{r-1}+b_rq_{r-2}}.$$

Now p_{r-1}, q_{r-1}, p_{r-2}, q_{r-2} are independent of a_r and b_r ; also u_r may be transformed into u_{r+1} by writing a_r+b_{r+1}/a_{r+1} for a_r, therefore

$$u_{r+1}=\frac{\left(a_r+\dfrac{b_{r+1}}{a_{r+1}}\right)p_{r-1}+b_rp_{r-2}}{\left(a_r+\dfrac{b_{r+1}}{a_{r+1}}\right)q_{r-1}+b_rq_{r-2}}=\frac{a_{r+1}(a_rp_{r-1}+b_rp_{r-2})+b_{r+1}p_{r-1}}{a_{r+1}(a_rq_{r-1}+b_rq_{r-2})+b_{r+1}q_{r-1}}.$$

Hence

$$u_{r+1}=\frac{a_{r+1}p_r+b_{r+1}p_{r-1}}{a_{r+1}q_r+b_{r+1}q_{r-1}}=\frac{p_{r+1}}{q_{r+1}}.$$

Thus the theorem holds for $n=r+1$; but it holds for $n=2$, and therefore for $n=3$, 4, 5, ... in succession.

(3) For the fraction $$\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots,$$

p_n and q_n are defined by equations (**A**) with the initial values

$$p_0=0,\quad q_0=1,\quad p_1=b_1,\quad q_1=a_1.$$

Again, the fraction $$a_1-\frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\cdots\frac{b_n}{a_n-}\cdots$$

is obtained from the fraction in § (2) by changing the signs of the b's, so that the recurrence formulae are

$$p_n=a_np_{n-1}-b_np_{n-2},\quad q_n=a_nq_{n-1}-b_nq_{n-2}.$$

In the following articles we shall call the reader's attention to certain types of continued fractions which will be considered in detail later.

Ex. 1. *Prove that* $(1-x)^{-2}=1+\dfrac{2x}{1-}\,\dfrac{3x}{2+}\,\dfrac{x}{3-}\,\dfrac{2x}{1}$.

Calculating the convergents of the continued fraction, we have

$$\frac{p_1}{q_1}=\frac{1}{1},\quad \frac{p_2}{q_2}=\frac{1+2x}{1},\quad \frac{p_3}{q_3}=\frac{2(1+2x)-3x\,.\,1}{2\,.\,1-3x\,.\,1}=\frac{2+x}{2-3x};$$

$$\frac{p_4}{q_4}=\frac{3(2+x)+x(1+2x)}{3(2-3x)+x\,.\,1}=\frac{6+4x+2x^2}{6-8x};$$

$$\frac{p_5}{q_5}=\frac{1\,.\,(6+4x+2x^2)-2x(2+x)}{1\,.\,(6-8x)-2x(2-3x)}=\frac{1}{(1-x)^2},$$

which proves the statement in question. *Notice that no fractions are reduced to lower terms, until the final stage.*

3. Infinite Continued Fractions.

With regard to the fraction

$$F=a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots\frac{b_n}{a_n+}\cdots,$$

if the number of elements is finite, F is called a *terminating* continued fraction.

We may, however, suppose the b's and a's to be determined by a rule of such a kind that there is no last element. In this case F stands for an *infinite* continued fraction, *which is to be regarded as defining the sequence of convergents*

$$p_1/q_1,\quad p_2/q_2,\ \cdots,\ p_n/q_n,\ \cdots,$$

and we say that *the continued fraction is convergent, divergent or oscillatory according as this sequence converges, diverges or oscillates.*

If the sequence converges, the *value* of the fraction F is defined as $\lim p_n/q_n$, and we write

$$F=a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots=\lim p_n/q_n.$$

Illustrations. Let x stand for $\dfrac{3}{2-}\,\dfrac{3}{2-}\,\dfrac{3}{2-}\ldots$ to ∞, where every element after the first is $-3/2$. *If we assume that x may be treated as a number,* we have $x=3/(2-x)$ leading to $x=1\pm\sqrt{2\iota}$, a result which is clearly meaningless. Again, if x stands for

$$\frac{3}{2+}\,\frac{3}{2+}\,\frac{3}{2+}\ldots \text{ to } \infty,$$

and we assume that x is a definite number, we have

$$x=3/(2+x);\quad \therefore\ x^2+2x-3=0;\quad \therefore\ x=1 \quad \text{or} \quad -3.$$

We have thus proved that if *the fraction has a value,* its value must be unity, but *we have not shown that a value exists.*

In neither of these examples can the first step be justified without knowing that the continued fraction is convergent.

4. Simple Continued Fractions.

An expression of the form

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\cdots,$$

where every a is a positive integer, except that a_1 may be zero, is called a *simple continued fraction.*

Theorem. *Any rational number can be expressed as a simple terminating continued fraction which can be arranged so as to have either an odd or an even number of quotients.*

Let A/B be the number, A and B being positive integers. In the process of finding the G.C.M. of A and B, let $a_1, a_2, \ldots$ be the quotients and $r_1, r_2, \ldots$ the remainders, so that

$$A=a_1B+r_1,\quad B=r_1a_2+r_2,\quad r_1=r_2a_3+r_3,\quad \text{etc.},$$

and therefore

$$\frac{A}{B}=a_1+\frac{1}{B/r_1},\quad \frac{B}{r_1}=a_2+\frac{1}{r_1/r_2},\quad \frac{r_1}{r_2}=a_3+\frac{1}{r_2/r_3},\ \text{etc.}$$

This process terminates, and if a_n is the last quotient,

$$\frac{A}{B}=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n}.$$

Again, if $a_n>1$, we can write

$$\frac{1}{a_n}=\frac{1}{a_n-1+\dfrac{1}{1}},$$

and if $a_n=1$,

$$\frac{1}{a_{n-1}+}\,\frac{1}{a_n}=\frac{1}{a_{n-1}+1}.$$

Thus matters can always be arranged so that the continued fraction has either an odd, or an even, number of quotients, as required.

Ex. 1. *Express* $\frac{157}{68}$ *as a simple continued fraction with* (i) *an even and* (ii) *an odd number of quotients.*

	68 | 157	$2=a_1$	
$a_2=3$	63 | 136		
	5 | 21	$4=a_3$	
$a_4=5$	5 | 20		
	0 | 1		

i.e. $\dfrac{157}{68}=2+\dfrac{1}{3+}\dfrac{1}{4+}\dfrac{1}{5}$

or $2+\dfrac{1}{3+}\dfrac{1}{4+}\dfrac{1}{4+}\dfrac{1}{1}$.

NOTE. The fact that the number of quotients can be made odd or even as required is of importance. For instance, the penultimate convergents to 157/68 in the two forms above are 30/13 and 127/55; and it will be found that (127, 55) and (30, 13) are the *least solutions* in positive integers of

$$68x-157y=+1 \quad \text{and} \quad 68x-157y=-1$$

respectively. (See Art. 9 and Ch. XXV, 3.)

Ex. 2. *Use the* H.C.F. *process to express* $(1-x)^{-2}$ *as a continued fraction.*

The H.C.F. process is

$1/2x$	$1-2x+x^2$ | $1-x/2$	1 | $1-2x+x^2$	1
$-9/2x$	$-x\,.\,3/2+x^2$ | $-x\,.\,3/2$	$2x-x^2$ | $2x-x^2\,.\,4/3$	$-4/3$
	x^2	$x^2\,.\,1/3$	$1/3$

giving

$$\frac{1}{(1-x)^2}=1+\frac{1}{1/2x+}\frac{1}{-4/3+}\frac{1}{-9/2x+}\frac{1}{1/3}$$

$$=1+\frac{2x}{1+}\frac{2x}{-4/3+}\frac{1}{-9/2x+}\frac{1}{1/3}$$

$$=1+\frac{2x}{1-}\frac{3x}{2+}\frac{-3/2}{-9/2x+1/3}\frac{1}{}$$

$$=1+\frac{2x}{1-}\frac{3x}{2+}\frac{x}{3-}\frac{2x}{1}.$$

In the last three steps, the statement at the end of Art. 1 has been used.

5. Recurring Continued Fractions. If, after a certain stage, the elements recur in the same order, we have a '*recurring*' continued fraction.

The recurring elements form the '*recurring period*' or the '*cycle*,' and the non-recurring elements, if such exist, form the '*acyclic part*' of the fraction.

The cycle is usually denoted by putting asterisks under the first and last of the recurring elements, thus

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{3+}\frac{1}{4+}\cdots \text{ is denoted by } \frac{1}{1+}\frac{1}{2+}\frac{1}{\underset{*}{3+}}\frac{1}{\underset{*}{4}}.$$

Here $\dfrac{1}{3+}\dfrac{1}{4+}$ is the cycle and $\dfrac{1}{1+}\dfrac{1}{2}$ the acyclic part.

Ex. 1. (i) *If* α, β *are the roots of* $x^2 - ax + b = 0$, *prove that the n-th convergent of*

$$a - \frac{b}{a-} \frac{b}{a-} \cdots \frac{b}{a-} \cdots$$

is equal to $(\alpha^{n+1} - \beta^{n+1})/(\alpha^n - \beta^n)$ *or* $(1 + 1/n)\alpha$ *according as* $\alpha \neq \beta$ *or* $\alpha = \beta$.

(ii) *Hence show that the continued fraction is convergent if* $a^2 \geqslant 4b$, *and that its value is the numerically greater root of* $x^2 - ax + b = 0$.

(i) Here p_n and q_n are given by

$$p_n = ap_{n-1} - bp_{n-2}, \quad q_n = aq_{n-1} - bq_{n-2},$$

together with $p_0 = 1$, $p_1 = a$, $q_0 = 0$, $q_1 = 1$.

First suppose that $\alpha \neq \beta$, then by Ch. XXII, 5, (i),

$$p_n = A\alpha^n + B\beta^n, \quad q_n = A'\alpha^n + B'\beta^n,$$

where A, B, A', B' are constants. Therefore

$$p_1 = A\alpha + B\beta = a = \alpha + \beta, \qquad q_1 = A'\alpha + B'\beta = 1,$$
$$p_0 = A + B = 1, \qquad q_0 = A' + B' = 0,$$

giving $$A/\alpha = -B/\beta = A' = -B' = 1/(\alpha - \beta);$$

thus $$p_n = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) \quad \text{and} \quad q_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

leading to the first result.

Next let $\alpha = \beta$, so that $a = 2\alpha$ and $b = \beta^2$, then by Ch. XXII, 5, (i),

$$p_n = (A + nB)\alpha^n, \quad q_n = (A' + nB')\alpha^n.$$

Proceeding as before, we find that $A = B = 1$, $A' = 0$, $B' = 1/\alpha$, hence

$$p_n = (n+1)\alpha^n, \quad q_n = n\alpha^{n-1}, \quad p_n/q_n = (1 + 1/n)\alpha.$$

(ii) If $a^2 > 4b$, then α, β are real. Let $|\alpha| > |\beta|$, then $\beta^n/\alpha^n \to 0$ as $n \to \infty$ and

$$\frac{p_n}{q_n} = \frac{\alpha^{n+1}}{\alpha^n} \cdot \frac{1 - (\beta/\alpha)^{n+1}}{1 - (\beta/\alpha)^n} \to \alpha.$$

If $a^2 = 4b$, then $\alpha = \beta$ and $p_n/q_n = (1 + 1/n)\alpha \to \alpha$.

Hence, in both cases, the continued fraction is convergent and its value is α.

Ex. 2. *If* p_n/q_n *is the n-th convergent of the recurring fraction*

$$\frac{1}{a+} \frac{1}{b+} \frac{1}{a+} \frac{1}{b+} \cdots,$$

prove that $$p_n - (ab+2)p_{n-2} + p_{n-4} = 0$$

and $$q_n - (ab+2)q_{n-2} + q_{n-4} = 0.$$

If n is *even*, the recurrence formulae give

$$p_n = bp_{n-1} + p_{n-2},$$
$$p_{n-1} = ap_{n-2} + p_{n-3},$$
$$p_{n-2} = bp_{n-3} + p_{n-4},$$

whence eliminating p_{n-1} and p_{n-3}, we have

$$p_n - (ab+2)p_{n-2} + p_{n-4} = 0.$$

If n is odd, we have the same equations as before, except that a and b are interchanged, leading to the same result. And so for the q's.

Ex. 3. *If* p_n/q_n *is the n-th convergent of*

$$\frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots\frac{(2n-3)^2}{2+}\cdots$$

prove that

$$p_n/q_n = 1-\frac{1}{3}+\frac{1}{5}-\ldots+(-1)^{n-1}\frac{1}{2n-1}.$$

Hence show that the fraction is convergent and that its value is $\pi/4$.

If v_n denotes either p_n or q_n, we have

$$v_n = 2v_{n-1}+(2n-3)^2 v_{n-2};$$

$$\therefore\ v_n-(2n-1)v_{n-1} = -(2n-3)\{v_{n-1}-(2n-3)v_{n-2}\},$$

and this holds if we write $n-1,\ n-2,\ \ldots 2$ in succession for n, therefore

$$v_n-(2n-1)v_{n-1} = (-1)^{n-1}(2n-3)(2n-5)\ldots 3\,.\,1(v_1-v_0).$$

Replacing the v's by p's and observing that $p_0=0$ and $p_1=1$, we have

$$p_n-(2n-1)p_{n-1} = (-1)^{n-1}1\,.\,3\,.\,5\ldots(2n-3);$$

$$\therefore\ \frac{p_n}{1\,.\,3\ldots(2n-1)}-\frac{p_{n-1}}{1\,.\,3\ldots(2n-3)} = (-1)^{n-1}\frac{1}{2n-1};$$

$$\therefore\ \frac{p_n}{1\,.\,3\ldots(2n-1)} = A+\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\ldots+(-1)^{n-1}\frac{1}{2n-1},$$

where A is independent of n. Putting $n=1$, we find that $A=0$.

Next, replacing the v's by q's and observing that $q_0=1$ and $q_1=1$, we have

$$q_n-(2n-1)q_{n-1}=0,$$

whence we find that $q_n = 1\,.\,3\,.\,5\ldots(2n-1)$, leading to the first result.

Again, the series $1-1/3+1/5-\ldots$ is convergent and its sum is $\pi/4$; therefore the fraction converges and its value is $\pi/4$.

EXERCISE XL

1. Find the first four convergents (unreduced) of

$$\text{(i)}\ \frac{1}{2+}\frac{2}{3+}\frac{3}{4+}\frac{4}{5};\quad \text{(ii)}\ \frac{1}{1-}\frac{2}{2-}\frac{3}{3-}\frac{4}{4}.$$

2. Without calculating the convergents, prove that

$$\frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5} = \frac{1}{1+}\frac{1}{2+}\frac{2}{3+}\frac{3}{4}.$$

3. Express $\frac{233}{151}$ as a simple continued fraction having (i) an even and (ii) an odd number of quotients.

4. The reckoning on the right shows that

$$\frac{85}{52} = 2-\frac{1}{3-}\frac{1}{4-}\frac{1}{5}.$$

Explain this.

3	52	85	2
	57	104	
5	5	19	4
	5	20	
	0	1	

5. Express $\frac{203}{85}$ in the form $a_1-\dfrac{1}{a_2-}\dfrac{1}{a_3-}\ldots$, where the a's are integers greater than unity.

6. Express $(2a^2-a-1)/(2a^2-3a)$ in the form $1+\dfrac{1}{f_1+}\dfrac{1}{f_2+}\ldots$, and verify by calculating the convergents.

7. Show that the nth convergent of

$$2-\frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\cdots$$

is $(n+1)/n$, and that the value of the infinite fraction is unity.

8. Show that the nth convergent of

$$\frac{1}{1-}\frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\cdots$$

is n, and that the infinite fraction is divergent.

9. There being $(n+1)$ elements in all, show that

$$2-\frac{1}{2-}\frac{1}{2-}\cdots\frac{1}{2-}\frac{1}{x}=\frac{(n+1)x-n}{nx-n+1}.$$

10. If $a^2>4b$, the numerically smaller root of $x^2-ax+b=0$ is equal to

$$\frac{b}{a-}\frac{b}{a-}\frac{b}{a-}\cdots\frac{b}{a-}\cdots.$$

[This follows from Art. 5, Ex. 1.]

11. For the fraction

$$\frac{b}{a+}\frac{b}{a+}\frac{b}{a+}\cdots+\frac{b}{a+}\cdots,$$

prove that $p_n=-\alpha\beta(\alpha^n-\beta^n)/(\alpha-\beta)$, $q_n=(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta)$ where α, β are the roots of $x^2-ax-b=0$. Hence show that if a, b are positive, the fraction is convergent and equal to the positive root of $x=b/(a+x)$.

12. If p_n/q_n is the (unreduced) nth convergent of

$$a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots\frac{b_n}{a_n+}\cdots,$$

and the *number of terms* in p_n, q_n respectively are denoted by u_n, v_n respectively, then u_n/v_n is the nth convergent of

$$1+\frac{1}{1+}\frac{1}{1+}\cdots.$$

Hence show that

$$u_{n-1}=v_n=\frac{1}{2^{n-1}}(C_1^n+5C_3^n+5^2C_5^n+\ldots).$$

13. Prove that

(i) $$\frac{1}{1-}\frac{3x}{1+}\frac{x}{1-}\frac{2x}{3+}\frac{x}{2-}\frac{x}{1}=\frac{1}{(1-x)^3}.$$

(ii) $$\frac{1}{1+}\frac{3x}{1-}\frac{2x}{1+}\frac{x}{3-}\frac{5x}{2+}\frac{x}{5-}\frac{3x}{1}=(1-x)^3.$$

14. Prove that, each fraction containing n elements,

$$\frac{1}{1-}\frac{1}{4-}\frac{1}{1-}\frac{1}{4-}\cdots=2\left(\frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\cdots\right).$$

Hence show the nth convergent of the first fraction reduces to $2n/(n+1)$.

15. Prove that

$$\frac{1}{1-}\frac{1^2}{3-}\frac{2^2}{5-}\frac{3^2}{7-}\cdots\frac{(n-1)^2}{2n-1}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n},$$

showing that the infinite continued fraction is divergent.

[$p_n=(2n-1)p_{n-1}-(n-1)^2p_{n-2}$; $\therefore\ p_n-np_{n-1}=(n-1)\{p_{n-1}-(n-1)p_{n-2}\}$.]

16. Prove that

$$\frac{a^2}{2a+1-}\ \frac{(a+1)^2}{2a+3-}\ \frac{(a+2)^2}{2a+5-}\ \cdots\ \frac{(a+n-1)^2}{2a+2n-1-}\ \cdots = a,$$

showing incidentally that, if $k_n = a(a+1)(a+2)\ldots(a+n)$, then

$$\frac{q_n}{k_n} = \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \ldots \frac{1}{a+n} \quad \text{and} \quad p_n/q_n = a - k_n/q_n.$$

Use this to show that the fraction in Ex. 15 is divergent.
[Here $q_n - (a+n)q_{n-1} = (a+n-1)\{q_{n-1} - (a+n-1)q_{n-2}\}$.]

17. Putting $a = -m$, in Ex. 16, where m is a positive integer, show that

$$\frac{m}{2m-1-}\ \frac{(m-1)^2}{2m-3-}\ \frac{(m-2)^2}{2m-5-}\ \cdots\ \frac{1^2}{1} = 1.$$

Also prove this result by induction.

18. For the fraction $\dfrac{1}{1+}\ \dfrac{2}{2+}\ \dfrac{3}{3+}\ \cdots\ \dfrac{n}{n+}\ \cdots$, prove that

$$\frac{p_n}{\lfloor n+1} = 1 - \frac{q_n}{\lfloor n+1} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots \text{ to } n \text{ terms.}$$

Hence show that the fraction is convergent and equal to $1/(e-1)$.

19. If n is a positive integer, prove by induction that

$$\frac{n}{n+}\ \frac{n-1}{n-1+}\ \frac{n-2}{n-2+}\ \cdots\ \frac{2}{2+}\ \frac{1}{1+}\ \frac{1}{2} = \frac{n+1}{n+2}.$$

20. For the fraction

$$\frac{1}{1+}\ \frac{1^2}{1+}\ \frac{2^2}{1+}\ \frac{3^2}{1+}\ \cdots\ \frac{(n-1)^2}{1+}\ \cdots,$$

prove that $\dfrac{p_n}{q_n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \ldots + (-1)^{n-1}\dfrac{1}{n}$.

Hence show that the fraction is convergent and equal to $\log 2$.

6. Simple Continued Fractions are of the form

$$x_1 = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+}\ \cdots\ \frac{1}{a_n+}\ \cdots,$$

where the a's are positive integers, except that a_1 may be zero.

Here $a_1, a_2, \ldots$ are called the (*partial*) *quotients* and if

$$x_2 = a_2 + \frac{1}{a_3+}\ \frac{1}{a_4+}\ \cdots, \quad x_3 = a_3 + \frac{1}{a_4+}\ \frac{1}{a_5+}\ \cdots, \text{ etc.,}$$

then $x_2, x_3, \ldots$ are called the *complete quotients*.

Thus we have

$$x_1 = a_1 + \frac{1}{a_2+}\ \cdots\ \frac{1}{a_{n-1}+}\ \frac{1}{x_n};$$

that is to say, *the continued fraction x_1 is obtained from its n-th convergent by substituting the complete quotient x_n for the quotient a_n.*

Properties of the Convergents

7. As in Art. 2, p_n and q_n are defined by the equations

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}, \qquad \text{.....(A)}$$

with the initial values $p_0 = 1, \quad p_1 = a_1, \quad q_0 = 0, \quad q_1 = 1.$

The convergents are $\left(\frac{1}{0}\right), \ \frac{a_1}{1}, \ \frac{a_1 a_2 + 1}{a_2}, \ \frac{a_1 a_2 a_3 + a_3 + a_1}{a_2 a_3 + 1}$, etc., and it will be seen that q_n *is the same function of* $a_2, a_3, \ldots a_n$ *as* p_{n-1} *is of* $a_1, a_2, \ldots a_{n-1}$.

8. We have $p_2 > p_1$, and if $n \geqslant 3$, $p_n - p_{n-1} = (a_n - 1) p_{n-1} + p_{n-2} \geqslant p_{n-2} \geqslant 1$; and so for the q's, since $q_2 \geqslant q_1$, thus p_n *and* q_n *increase with* n, *and in the case of an infinite continued fraction, they tend to infinity with* n.

9. From the recurrence formulae (**A**) we have

$$\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1} (a_n q_{n-1} + q_{n-2}) \\ &= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}). \end{aligned}$$

This holds if we write $n-1, n-2, \ldots 2$ in succession for n, also

$$p_1 q_0 - p_0 q_1 = -1,$$

therefore

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n. \qquad \text{.....(B)}$$

Again, we have

$$\begin{aligned} p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}) \\ &= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}); \end{aligned}$$

therefore

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n. \qquad \text{.....(C)}$$

From (**B**) and (**C**) we obtain the formulae

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^n \frac{1}{q_n q_{n-1}} \quad \text{and} \quad \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^{n-1} \frac{a_n}{q_n q_{n-2}}. \qquad \text{....(D)}$$

10. *Every convergent* p_n/q_n *is in its lowest terms, also* p_n *is prime to* p_{n-1} *and* q_n *to* q_{n-1}.

For if the numbers belonging to any of the pairs (p_n, q_n), (p_n, p_{n-1}), (q_n, q_{n-1}) have a common factor g, then on account of equation (**B**), g would divide ± 1.

11. From the recurrence formulae (**A**), we have

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{p_{n-1}/p_{n-2}} \quad \text{and} \quad \frac{q_n}{q_{n-1}} = a_n + \frac{1}{q_{n-1}/q_{n-2}}.$$

The first of these holds if we write $n-1, n-2, \ldots 2$ for n.

The second of these holds if we write $n-1, n-2, \ldots 3$ for n.

Also $p_1/p_0=a_1$ and $q_2/q_1=a_2$; therefore

$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}, \quad \text{.....(E)}$$

$$\frac{q_n}{q_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\frac{1}{a_2}. \quad \text{.....(F)}$$

It follows that *if* p_r/q_r, p_r'/q_r' *are respectively the r-th convergents of*

$$a_1+\frac{1}{a_2+}\cdots\frac{1}{a_n} \quad \textit{and} \quad a_n+\frac{1}{a_{n-1}+}\cdots\frac{1}{a_1},$$

where in the second the quotients occur in the reverse order, then

$$p_n'=p_n, \quad q_n'=p_{n-1}, \quad p_{n-1}'=q_n, \quad q_{n-1}'=q_{n-1}. \quad \text{.....(G)}$$

For by the preceding, $p_n'/q_n'=p_n/p_{n-1}$ and $p_{n-1}'/q_{n-1}'=q_n/q_{n-1}$, and by Art. 10 these fractions are in their lowest terms.

12. It has been shown that if p'/q', p/q are consecutive convergents of a simple continued fraction, then $pq'-p'q=\pm1$.

Conversely, let $pq'-p'q=\pm1$ where $q'\leqslant q$, and suppose that p/q is expressed as a simple continued fraction with an even or an odd number of quotients according as the arbitrary sign is + or −. We can prove that p'/q' *is the convergent immediately preceding* p/q.

For if p''/q'' is the last convergent but one, by hypothesis and equation (**B**) of Art. 9,

$$pq'-p'q=pq''-p''q, \quad \text{and therefore} \quad p(q'-q'')=q(p'-p'').$$

Now p is prime to q, for $pq'-p'q=\pm1$, therefore q divides $q'-q''$ or else $q'=q''$.

Also $q'\leqslant q$ and $q''\leqslant q$, therefore q cannot divide $q'-q''$, and consequently $q'=q''$ and $p'=p''$, which proves the theorem.

13. Sequences of Odd and Even Convergents. Let $u_n=p_n/q_n$, then, from equations (**D**) of Art. 9, it follows that

$$u_n-u_{n-1} \quad \text{and} \quad u_n-u_{n-2}$$

are of opposite sign. Therefore u_n lies between u_{n-1} and u_{n-2}, that is to say, *every convergent lies between the two preceding convergents.*

Now $u_1<u_2$ and consequently $u_1<u_3<u_2$; $u_3<u_4<u_2$; $u_3<u_5<u_4$, and so on, therefore

$$u_1<u_3<u_5\ldots<u_6<u_4<u_2.$$

Thus the odd convergents form an increasing sequence, the even convergents form a decreasing sequence, and every odd convergent is less than any even convergent.

Consequently, as n tends to infinity, both u_{2n-1} and u_{2n} tend to finite limits, and these limits are the same, for by equation (D) of Art. 9,

$$u_{2n} - u_{2n-1} = 1/q_{2n}q_{2n-1} \to 0.$$

Hence u_n tends to a limit x such that for every n,

$$u_{2n-1} < x < u_{2n}.$$

In other words, (i) *every infinite simple continued fraction is convergent, and* (ii) *its value is greater than any odd convergent and* (iii) *less than any even convergent.*

14. *Each convergent is a closer approximation to the value of the continued fraction than the preceding.*

For let $z = a_1 + \dfrac{1}{a_2 +} \cdots \dfrac{1}{a_n +} \dfrac{1}{a_{n+1} +} \cdots$ and $z' = a_{n+1} + \dfrac{1}{a_{n+2} +} \cdots$,

so that z' is the $(n+1)$th complete quotient, then

$$p_{n+1}/q_{n+1} = (a_{n+1}p_n + p_{n-1})/(a_{n+1}q_n + q_{n-1}) ;$$

and therefore $$z = (z'p_n + p_{n-1})/(z'q_n + q_{n-1}).$$

Consequently $$z'(zq_n - p_n) = -(zq_{n-1} - p_{n-1})$$

and $$z - \frac{p_n}{q_n} = -\frac{q_{n-1}}{z'q_n}\left(z - \frac{p_{n-1}}{q_{n-1}}\right).$$

Now $q_n > q_{n-1}$ and $z' \geqslant 1$, therefore

$$\left| z - \frac{p_n}{q_n} \right| < \left| z - \frac{p_{n-1}}{q_{n-1}} \right|,$$

that is to say, z is nearer to p_n/q_n than to p_{n-1}/q_{n-1}.

15. *Any convergent p/q of a simple continued fraction is a closer approximation to the value of the fraction than any rational fraction r/s with a denominator less than q.*

For let $z = a_1 + \dfrac{1}{a_2 +} \cdots$, and let p'/q' be the convergent immediately preceding p/q. By Art. 14, if r/s is nearer to z than p/q, it is also nearer to z than p'/q'. Moreover, z lies between p/q and p'/q'; therefore r/s lies between p/q and p'/q', and

$$\left| \frac{r}{s} - \frac{p'}{q'} \right| < \left| \frac{p}{q} - \frac{p'}{q'} \right|, \quad \text{i.e.} \quad \left| \frac{r}{s} - \frac{p'}{q'} \right| < \frac{1}{qq'}.$$

Therefore $q \,.\, |\, rq' - sp' \,| < s$; and, since $rq' - sp'$ is an integer, it follows that

$$s > q.$$

16. *If ϵ is the numerical value of the error, in representing the value z of the continued fraction, by p_n/q_n, then*

$$a_{n+2}/q_nq_{n+2}<\epsilon<1/q_nq_{n+1}, \quad \text{.....(H)}$$

and within wider limits

$$1/q_n(q_n+q_{n+1})<\epsilon<1/q_n{}^2. \quad \text{.....(I)}$$

For if $u_n=p_n/q_n$, then by Art. 13,

$$u_n, \quad u_{n+2}, \quad z, \quad u_{n+1},$$

is an increasing or a decreasing sequence, therefore

$$|u_n-u_{n+2}|<|u_n-z|<|u_n-u_{n+1}|.$$

Also by equations (**D**),

$$|u_n-u_{n+2}|=a_{n+2}/q_nq_{n+2} \quad \text{and} \quad |u_n-u_{n+1}|=1/q_nq_{n+1},$$

whence we have the inequalities (**H**).

Again, $q_{n+2}=a_{n+2}q_{n+1}+q_n$, therefore

$$q_{n+2}/a_{n+2}=q_{n+1}+q_n/a_{n+2}\leqslant q_{n+1}+q_n$$

and

$$1/q_n(q_n+q_{n+1})\leqslant a_{n+2}/q_nq_{n+2}.$$

Also $q_{n+1}>q_n$, and therefore $1/q_nq_{n+1}<1/q_n{}^2$.

Hence the inequalities (**I**) follow from (**H**).

17. *Any positive irrational z can be expressed as a simple continued fraction, and that in only one way.*

Let $z=a_1+\dfrac{1}{z_2}$ where a_1 is the integral part of z (which may be zero), then $1/z_2<1$ and $z_2>1$. Continuing the process, let

$$z_2=a_2+1/z_3, \quad z_3=a_3+1/z_4, \quad \ldots \quad z_n=a_n+1/z_{n+1},$$

where a_2, a_3, ... are integers such that, for every r,

$$a_r<z_r<a_r+1,$$

since $z_2>1$, it follows in succession that a_2, a_3, ... are *positive* integers and we have

$$z=a_1+\cfrac{1}{a_2+}\cdots\cfrac{1}{a_n+}\cfrac{1}{z_{n+1}}.$$

The process can be continued indefinitely, and leads to the infinite continued fraction

$$F=a_1+\cfrac{1}{a_2+}\cdots\cfrac{1}{a_n+}\cdots.$$

Again, we have $a_2<z_2$, $a_3<z_3$, etc., therefore

$$a_1+\frac{1}{a_2}>z, \quad a_2+\frac{1}{a_3}>z_2, \quad \text{etc.},$$

and

$$a_1+\cfrac{1}{a_2+}\cfrac{1}{a_3}<z, \quad a_2+\cfrac{1}{a_3+}\cfrac{1}{a_4}<z_2, \quad \text{etc.}$$

Continuing thus, it can be shown that if u_r is the rth convergent of F, then

$$u_{2n-1} < z < u_{2n}.$$

Now as n tends to infinity, u_{2n-1} and u_{2n} tend to the same limit, which must therefore be equal to z. That is to say, the fraction F converges to z as a limit, and we write

$$z = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n +} \cdots .$$

It remains to show that this mode of expression is unique. Suppose that

$$z = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots = b_1 + \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots .$$

The integral parts of these continued fractions are equal, therefore

$$a_1 = b_1 \quad \text{and} \quad a_2 + \frac{1}{a_3 +} \cdots = b_2 + \frac{1}{b_3 +} \cdots .$$

Hence, by similar reasoning,

$$a_2 = b_2 \quad \text{and} \quad a_3 + \frac{1}{a_4 +} \cdots = b_3 + \frac{1}{b_4 +} \cdots ,$$

and so on indefinitely. Thus the continued fractions are identical.

When z is expressed in this way, any convergent of the continued fraction will be called a *convergent to z*.

18. *The value of any infinite simple continued fraction is an irrational number.*

For any rational number can be expressed as a finite simple continued fraction.

Ex. 1. *Express* $(\sqrt{37}+8)/9$ *as a simple continued fraction.*

The integral part of this surd is 1, and

$$\frac{\sqrt{37}+8}{9} = 1 + \frac{\sqrt{37}-1}{9} = 1 + \frac{4}{\sqrt{37}+1}, \quad \frac{\sqrt{37}+1}{4} = 1 + \frac{\sqrt{37}-3}{4} = 1 + \frac{7}{\sqrt{37}+3},$$

$$\frac{\sqrt{37}+3}{7} = 1 + \frac{\sqrt{37}-4}{7} = 1 + \frac{3}{\sqrt{37}+4}, \quad \frac{\sqrt{37}+4}{3} = 3 + \frac{\sqrt{37}-5}{3} = 3 + \frac{4}{\sqrt{37}+5},$$

$$\frac{\sqrt{37}+5}{4} = 2 + \frac{\sqrt{37}-3}{4} = 2 + \frac{7}{\sqrt{37}+3}, \quad \frac{\sqrt{37}+3}{7} = 1 + \text{etc.}$$

The complete quotient $(\sqrt{37}+3)/7$ has occurred before, and the subsequent work consists of repetitions of the last three steps, thus, with the notation of Art. 5 :

$$\frac{\sqrt{37}+8}{9} = 1 + \frac{1}{1+} \underset{*}{\frac{1}{1+}} \frac{1}{3+} \underset{*}{\frac{1}{2}},$$

the continued fraction being a recurring fraction.

19. Approximations. (1) If a given number x is expressed as a simple continued fraction of which p_r/q_r is the rth convergent, then p_n/q_n is an approximate value of x in defect or excess, according as n is odd or even, and the error is numerically less than $1/q_nq_{n+1}$, and *a fortiori* less than $1/q_n^2$. Now $q_{n+1}=a_{n+1}q_n+q_{n-1}$, hence *if* a_{n+1} *is large,* p_n/q_n *is a specially good approximation.*

Or again, if we require *an approximation with an error numerically less than a given fraction* $1/a$, we calculate the convergents until we find one p_n/q_n such that $q_nq_{n+1}\geqslant a$. This is the required approximation. Of course, the conditions are satisfied if $q_n\geqslant\sqrt{a}$.

Ex. 1. *It is required to find good approximations to the value of* π *in the form of vulgar fractions, given that*

$$\pi=3+\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

The convergents are

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \cdots.$$

The denominators 106, 33102 are large compared with the preceding ones, therefore 22/7 and 355/113 are specially good approximations, and they are both in excess.

For 355/113, *the error is less than*

$$\frac{1}{113\times 33102}<\frac{1}{3740000}<0{\cdot}00000027.$$

20. The problem may be of such a kind that no suitable approximation can be found among the convergents (as in Art 22, Ex. 2).

We may then proceed as follows: Choose two convergents p/q, p'/q', one an odd and the other an even convergent. If m, m' are positive integers, the fraction

$$\frac{P}{Q}=\frac{mp+m'p'}{mq+m'q'}$$

lies between p/q and p'/q'. Also x lies between these convergents, therefore P/Q is a closer approximation than p/q, and perhaps also than p'/q'; and it may be possible to choose m, m' to suit the conditions of the problem in question.

21. The most important fractions of this kind are the so-called *intermediate convergents.* Let p_n/q_n be the nth convergent of

$$x=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

Consider the sequence

$$\frac{p_n}{q_n},\ \frac{p_n+p_{n+1}}{q_n+q_{n+1}},\ \frac{p_n+2p_{n+1}}{q_n+2q_{n+1}},\ \cdots\ \frac{p_n+a_{n+2}p_{n+1}}{q_n+a_{n+2}q_{n+1}}\left(=\frac{p_{n+2}}{q_{n+2}}\right),$$

the $(r+1)$th term being P_r/Q_r where $P_r=p_n+rp_{n+1}$, $Q_r=q_n+rq_{n+1}$.

The terms of this sequence (excluding the first and last) are called the *intermediate convergents* between p_n/q_n and p_{n+2}/q_{n+2}. The sequence possesses the following properties :

(i) It is an increasing or a decreasing sequence according as n is odd or even, and the denominators form an increasing sequence.

(ii) Every fraction P_r/Q_r is in its lowest terms.

(iii) Any fraction which lies between P_r/Q_r and P_{r+1}/Q_{r+1} has a denominator greater than either Q_r or Q_{r+1}.

(iv) Limits to the numerical value of the error in taking P_r/Q_r as the value of x are given by

$$\frac{a_{n+2}-r}{q_{n+2}Q_r} < \left| x - \frac{P_r}{Q_r} \right| < \frac{1}{q_{n+1}Q_r}$$

For we have $P_{r+1}Q_r - P_rQ_{r+1} = p_{n+1}q_n - p_nq_{n+1} = (-1)^{n+1}$,

whence the statements (i)-(iii) follow : and because

$$P_r/Q_r, \quad p_{n+2}/q_{n+2}, \quad x, \quad p_{n+1}/q_{n+1},$$

is either an increasing or a decreasing sequence, therefore

$$\left| \frac{p_{n+2}}{q_{n+2}} - \frac{P_r}{Q_r} \right| < \left| x - \frac{P_r}{Q_r} \right| < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{P_r}{Q_r} \right|.$$

Since $r < a_{n+2}$, this proves the statement.

Now consider the sequence formed by writing down the odd convergents

$$p_1/q_1, \ \dots, \ p_3/q_3, \ \dots, \ p_5/q_5 \ \dots, \ \text{.....(A)}$$

and inserting all the intermediate convergents, then

(i) This is an *increasing* sequence of fractions in their lowest terms, and each fraction is a closer approximation in defect to x than the preceding.

(ii) The denominators form an increasing sequence.

(iii) If P/Q, P'/Q' are successive terms, any fraction which lies between them has a denominator greater than Q or Q'.

Again, the sequence

$$p_2/q_2, \ \dots, \ p_4/q_4, \ \dots, \ p_6/q_6, \ \dots, \ \text{.....(B)}$$

formed by the even convergents and their intermediates, has the same properties as the preceding, except that it is a *decreasing* sequence in which each term is a closer approximation *in excess* to x than the preceding.

22. Problem. *It is required to find the fraction which, of all those with denominators less than a given number a, is the closest approximation (in defect or in excess) to a given number x.*

To do this we express x as a simple continued fraction, and calculate the convergents p_1/q_1, p_2/q_2, ... until we find p_n/q_n such that

$$q_n \leqslant a < q_{n+1}.$$

If $q_n = a$, then p_n/q_n is the required fraction.

If $q_n < a$, we first consider approximations in defect. Taking the sequence (**A**) formed by the odd convergents and their intermediates, we find successive terms P/Q, P'/Q' such that $Q \leqslant a < Q'$; then P/Q is the closest approximation in defect of all fractions with denominators less than a.

For if h/k is nearer to x than P/Q, then h/k must coincide with a term to the right of P/Q in the sequence, or else it must lie between two such terms, and in either case $k \geqslant Q' > a$.

In the same way, by considering the sequence (**B**) formed by the even convergents and their intermediates, we can find the fraction with denominator less than a which is the closest approximation in excess. Of these two fractions, the one which is nearer to x is the number required.

Ex. 1. *Find the fraction with denominator less than* 500 *which is nearest to* $\sqrt{14}$.

As in Art 18, Ex. 1, we can show that $\sqrt{14} = 3 + \underset{*}{\frac{1}{1}} + \frac{1}{2} + \frac{1}{1} + \underset{*}{\frac{1}{6}}$,

and we find that $\dfrac{p_7}{q_7} = \dfrac{333}{89}$, $\dfrac{p_8}{q_8} = \dfrac{449}{120}$, $\dfrac{p_9}{q_9} = \dfrac{3027}{809}$.

For the approximation in defect, we insert intermediate convergents between p_7/q_7 and p_9/q_9. These are of the form $P_r/Q_r = (333 + 449r)/(89 + 120r)$.

The conditions $Q_r \leqslant 500 < Q_{r+1}$ give $r = 3$, and $P_3/Q_3 = 1680/449$.

For the approximation in excess, we consider the convergents p_8/q_8, p_{10}/q_{10}. Since $a_{10} = 1$, there are no intermediate convergents and the approximation in excess is $p_8/q_8 = 449/120$.

Finally, we can show that $14 - (1680/449)^2 < (449/120)^2 - 14$,
so that the required approximation is 1680/449.

Ex. 2. *The mean tropical year is* 365·2422642 *days. It is required to find a* practically convenient *method of correcting the calendar, the year being taken as* 365 *days.*

We have $0{\cdot}2422642 = 0 + \dfrac{1}{4} + \dfrac{1}{7} + \dfrac{1}{1} + \dfrac{1}{4} + \dfrac{1}{1} + \dfrac{1}{6} + \dfrac{1}{11} \cdots$,

and the convergents are

$$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}, \frac{47}{194}, \frac{321}{1325} \cdots.$$

Take 1/4 as a first approximation. This amounts to adding 1 day every 4 years, and is an over-correction.

The fraction 7/29 is inconvenient; but 8/33 suggests an addition of 24 days in every 99 years, i.e. make every fourth year a leap-year, except at the end of a century. This is also an over-correction.

None of the other convergents give convenient approximations; and the same is true for the intermediate convergents in the first few intervals. We therefore proceed as in Art. 20. We choose the convergents 1/4 and 39/161, the first an even and the second an odd convergent; then, if m, m' are positive integers,

$$\frac{1}{4} < \frac{m \cdot 1 + m' \cdot 39}{m \cdot 4 + m' \cdot 161} < \frac{39}{161}.$$

The values $m = 19$, $m' = 2$ give the approximation

$$\frac{19 \cdot 1 + 2 \cdot 39}{19 \cdot 4 + 2 \cdot 161} = \frac{97}{400}.$$

This gives one more day to be added in 400 years than the fraction 8/33; i.e. it makes every fourth year a leap-year, except those at the end of a century, with the further correction that every fourth century is to be counted as a leap-year.

The fact that this approximation, 97/400, can be obtained from the convergents 1/4 and 47/194, which are both in excess, shows that this also is an over-correction.

The error is less than a day in 4000 years.

Miscellaneous Theorems

23. Let p/q be a fraction in its lowest terms, which is an approximate value of some quantity z. It may be important to know *under what circumstances p/q is a convergent to z.* The answer to this is contained in the following:

Let p/q be expressed as a simple continued fraction with an even or an odd number of quotients according as $p/q \gtrless z$, and let p'/q' be the last convergent but one.

If $\left| z - \frac{p}{q} \right| < \frac{1}{q(q+q')}$, *or a fortiori if* $\left| z - \frac{p}{q} \right| < \frac{1}{2q^2}$,

then p/q is a convergent to z.

For suppose that

$$\frac{p}{q} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n},$$

where n is even or odd according as $p/q \gtrless z$, and let z' be determined by

$$z = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n + \frac{1}{z'}}.$$

It is sufficient to prove that $z' > 1$, for then z' can be expressed as a simple continued fraction in which the first quotient is not zero. We have

$$z = \frac{pz' + p'}{qz' + q'} \quad \text{and therefore} \quad \frac{p}{q} - z = \frac{pq' - p'q}{q(qz' + q')},$$

Now $pq' - p'q = 1$ or -1, according as $p/q \gtrless z$; therefore in both cases $q(qz' + q')$ is *positive* and

$$\frac{1}{q(qz' + q')} = \left| \frac{p}{q} - z \right| < \frac{1}{q(q+q')}.$$

Hence $qz' + q' > q + q'$ and $z' > 1$, which proves the theorem.

24. *If p_r/q_r is the r-th convergent of the simple continued fraction of which the value is z, then $z^2 \gtrless p_n p_{n-1}/q_n q_{n-1}$, according as n is even or odd.*

For if k is the complete $(n+1)$th quotient,

$$z = (kp_n + p_{n-1})/(kq_n + q_{n-1}).$$

Therefore $z^2 \gtrless p_n p_{n-1}/q_n q_{n-1}$, according as

$$q_n q_{n-1}(kp_n + p_{n-1})^2 - p_n p_{n-1}(kq_n + q_{n-1})^2 \gtrless 0.$$

The expression on the left is equal to

$$(k^2 p_n q_n - p_{n-1}q_{n-1})(p_n q_{n-1} - p_{n-1}q_n) = (-1)^n (k^2 p_n q_n - p_{n-1}q_{n-1}),$$

and this is positive or negative according as n is even or odd, for $k \geqslant 1$, $p_n > p_{n-1}$, and $q_n > q_{n-1}$, which proves the theorem.

25. *If p/q is an even convergent, and X/Y an odd convergent, of a simple continued fraction whose value is z, then*

$$pX \gtrless z^2 qY,$$

according as p/q precedes or follows X/Y in the sequence of convergents.

(i) *If p/q precedes X/Y,* let X'/Y' be the convergent which *immediately* precedes X/Y. Then X'/Y' is an even convergent, and it occurs later than p/q or else coincides with it, therefore

$$\frac{X}{Y} < z < \frac{X'}{Y'} \leqslant \frac{p}{q};$$

and since X/Y is an odd convergent immediately preceded by X'/Y',

$$z^2 < \frac{XX'}{YY'} \quad \text{and} \quad z^2 < \frac{pX}{qY}.$$

(ii) *If p/q follows X/Y,* let p'/q' be the convergent which *immediately* precedes p/q. Then p'/q' is an odd convergent, and it occurs later than X/Y or else coincides with it, therefore

$$\frac{X}{Y} \leqslant \frac{p'}{q'} < z < \frac{p}{q};$$

and since p/q is an even convergent immediately preceded by p'/q',

$$z^2 > \frac{pp'}{qq'} \quad \text{and} \quad z^2 > \frac{pX}{qY}.$$

26. Symmetric Continued Fractions. A simple finite continued fraction, in which the quotients equidistant from the beginning and end are equal, is said to be *symmetric*. For instance, the fractions,

$$\frac{247}{77} = 3 + \frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3}, \quad \frac{425}{132} = 3 + \frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3},$$

are symmetric, the former having an odd number of quotients, and the latter an even number.

The following is an important property of a fraction of this kind with an even number of quotients.

(1) *Let* $\frac{p}{q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r}$ *and* $\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{a_r+}\frac{1}{a_{r-1}+}\cdots\frac{1}{a_1}$, *the fractions* p/q, P/Q *being in their lowest terms, and let* p'/q', P'/Q' *be the convergents immediately preceding* p/q *and* P/Q, *then*

$$P=p^2+p'^2,\quad Q'=q^2+q'^2\quad\text{and}\quad Q^2+1=PQ'.$$

For by Art. 11, $P/P'=P/Q$, therefore $P'=Q$, also

$$\frac{p}{p'}=a_r+\frac{1}{a_{r-1}+}\cdots\frac{1}{a_1}\quad\text{and}\quad\frac{q}{q'}=a_r+\frac{1}{a_{r-1}+}\cdots\frac{1}{a_2};$$

therefore

$$\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{p'}{p}=\frac{p^2+p'^2}{pq+p'q'},$$

$$\frac{P'}{Q'}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{q'}{q}=\frac{pq+p'q'}{q^2+q'^2}.$$

If $X=p^2+p'^2$ and $Y=pq+p'q'$, we have

$$Yp-Xq=(-1)^r p'\quad\text{and}\quad Xq'-Yp'=(-1)^r p;$$

therefore any common factor of X and Y divides p and p', which are prime to one another, therefore X/Y is in its lowest terms. So also is P/Q, therefore $P=X$ and $Q=Y$, and consequently

$$P=p^2+p'^2\quad\text{and}\quad Q'=q^2+q'^2.$$

Moreover, $PQ'-P'Q=(-1)^{2r}$ and $P'=Q$, therefore $Q^2+1=PQ'$.

Ex. 1. *If* p_r/q_r *is the r-th convergent of the continued fraction*

$$\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{a_r+}\cdots\frac{1}{a_1},$$

prove that the remainders in the process of finding the G.C.M. *of* Q, P *are*

$$p_{2r-2},\ p_{2r-3},\ \ldots,\ p_1,\ 1.$$

For $P=p_{2r}$, $Q=p_{2r-1}$ and $p_{2r}=a_1p_{2r-1}+p_{2r-2}$ with similar equations, the last being $p_2=a_1p_1+1$. Hence the G.C.M. process is that shown on the right.

a_2	p_{2r-1}	p_{2r}	a_1
	a_2p_{2r-2}	a_1p_{2r-1}	
	p_{2r-3}	p_{2r-2}	a_2
	.....	.....	
	p_1	p_2	a_1
		a_1p_1	
		1	

Ex. 2. *Given that* $218^2+1=25\,.\,1901$, *express* 1901 *as the sum of two squares.*

The reckoning shows that

$$\frac{1901}{218}=8+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{8}.$$

1	218	1901	8
1	61	157	2
2	26	35	1
8	8	9	1
		1	

Hence, if p_r/q_r is the rth convergent,

$1901=p_3^2+p_4^2=26^2+35^2$. (See Ex. 1 and Art. 26.)

27. Application to the Theory of Numbers. *If Q is any positive integer, any factor of Q^2+1 can be expressed as the sum of two squares.*

For let $Q^2+1=PR$ where $P>R$, then obviously $P>Q$ and P is prime to Q. Let

$$\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_{2r}}; \quad \text{.....(A)}$$

then if P'/Q' is the convergent immediately preceding P/Q,

$$PQ'-P'Q=1=PR-Q^2.$$

Therefore $P(R-Q')=Q(Q-P')$ and, since P is prime to Q, it is a divisor of $Q-P'$ or else $Q=P'$. The first alternative is impossible, for $P>Q$ and $P>P'$, therefore $Q=P'$ and $R=Q'$.

Consequently $\frac{P}{Q}=\frac{P}{P'}=a_{2r}+\frac{1}{a_{2r-1}+}\cdots\frac{1}{a_2+}\frac{1}{a_1}$, and therefore $a_{2r}=a_1$, $a_{2r-1}=a_2$, etc., and the continued fraction is symmetric.

Denoting the mth convergent of (A) by p_m/q_m, we have, by Art. 26,

$$P=p_r^2+p_{r-1}^2, \quad R=Q'=q_r^2+q_{r-1}^2,$$

which proves the theorem.

28. *Conversely, if $P=x^2+y^2$ where x is prime to y and $x>y$, then an integer Q can be found so that Q^2+1 is divisible by P.*

Let $\frac{x}{y}=a_r+\frac{1}{a_{r-1}+}\cdots\frac{1}{a_1}$, and $\frac{X}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{a_r+}\frac{1}{a_{r-1}+}\cdots\frac{1}{a_1}$;

then, if p_m/q_m is the mth convergent to the last continued fraction,

$$X=p_r^2+p_{r-1}^2=x^2+y^2=P \quad \text{and} \quad Q=p_{2r-1}.$$

Also $Xq_{2r-1}-Qp_{2r-1}=1$, therefore $Q^2+1=Pq_{2r-1}$.

Ex. 1. *Given that* $3637=46^2+39^2$, *find a solution in positive integers of the equation*

$$x^2+1=3637y.$$

Using Art. 28, we have $\frac{46}{39}=1+\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3}$, and writing down the continued fraction

$$3+\frac{1}{1+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3},$$

of which the last two convergents are $\frac{1027}{290}$, $\frac{3637}{1027}$, we have $3637\times290-1027^2=1$, and the required solution therefore is (1027, 290).

Or thus: the last two convergents of $3+\frac{1}{1+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1}$ are $\frac{39}{11}$ and $\frac{46}{13}$. Therefore

$$y=11^2+13^2=290 \quad \text{and} \quad x=\sqrt{(3637\,.\,290-1)}=1027.$$

29. *If x is prime to y, any factor of x^2+y^2 is the sum of two squares.*

For we can find an integer Q such that x^2+y^2 is a factor of Q^2+1 and *every* factor of Q^2+1 is the sum of two squares.

30. Simple Recurring Continued Fractions. These may be grouped as follows :

(i) Fractions which have *no* acyclic part.

(ii) Those with an acyclic part consisting of *a single* quotient.

(iii) Those with an acyclic part consisting of *more than one* quotient.

The reasons for this classification will appear later. The conclusions are of fundamental importance, and we proceed to consider these types in order.

31. No Acyclic Part.

If
$$\alpha = b_1 + \frac{1}{b_2+}\,\frac{1}{b_3+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots, \qquad \text{.....(A)}$$

the fraction having *no acyclic part, then α is a root of a quadratic equation with integral coefficients. Moreover, if β is the second root of the equation, then*

$$-\frac{1}{\beta} = b_n + \frac{1}{b_{n-1}+}\,\frac{1}{b_{n-2}+}\cdots\frac{1}{b_1+}\,\frac{1}{b_n+}\,\frac{1}{b_{n-1}+}\cdots, \qquad \text{.....(B)}$$

where the order of the quotients is reversed, whence it follows that

$$-1<\beta<0.$$

Denoting the rth convergents of the fractions (A), (B) by p_r/q_r and p_r'/q_r' respectively, we have

$$\alpha = b_1 + \frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{\alpha} = \frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}},$$

therefore α is the positive root of

$$q_n x^2 - (p_n - q_{n-1})x - p_{n-1} = 0\,; \qquad \text{.....(C)}$$

and in the same way the fraction (B) is the positive root of

$$q_n' x^2 - (p_n' - q_{n-1}')x - p_{n-1}' = 0,$$

which by Art. 11 is the same equation as

$$p_{n-1}x^2 - (p_n - q_{n-1})x - q_n = 0. \qquad \text{.....(D)}$$

Now equation (D) can be transformed into equation (C) by writing $-1/x$ for x. Therefore $-1/\beta$ is the positive root of (D), and is consequently equal to the fraction (B).

Also, from (B), β is negative, and $-\frac{1}{\beta}>1$; from which it follows that $\beta>-1$.

32. A single Acyclic Quotient.

If $$\alpha = a_1 + \frac{1}{b_1+}\,\frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots, \quad \text{.....(A)}$$

where the single quotient a_1 does not recur, then α is a root of a quadratic equation whose second root cannot lie between 0 and -1.

Let $x = a_1 + \frac{1}{b_1+}\,\frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\cdots$ and $y = b_1 + \frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\cdots$,

then $x - a_1 = 1/y$, and if p_r/q_r is the rth convergent of the second fraction, as in Art. 31,

$$q_n y^2 - (p_n - q_{n-1})\,y - p_{n-1} = 0, \quad \text{.....(B)}$$

and therefore $$p_{n-1}(x-a_1)^2 + (p_n - q_{n-1})(x-a_1) - q_n = 0. \quad \text{.....(C)}$$

Denoting the roots of (B) by α, β where α is positive, and the corresponding roots of (C) by α', β', we have

$$\alpha' = a_1 + 1/\alpha, \quad \beta' = a_1 + 1/\beta.$$

The root α' is the value of the fraction (A), and by Art. 31 we have

$$\beta' = a_1 - \left(b_n + \frac{1}{b_{n-1}+}\,\frac{1}{b_{n-2}+}\cdots\right) = a_1 - b_n - f,$$

where f is a positive proper fraction.

Now a_1 and b_n are unequal, for if $a_1 = b_n$ then a_1 would belong to the cycle, which is contrary to the hypothesis. Consequently $\beta' = I - f$ or $-I - f$ where I is a positive integer, and therefore β' cannot lie between 0 and -1.

33. More than one Quotient Acyclic.

If $$\alpha = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_m+}\,\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots, \quad \text{.....(A)}$$

in which at least two quotients do not recur, then α is a root of a quadratic equation with rational coefficients whose second root is positive.

Let $x = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_m+}\,\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots$ and $y = b_1 + \frac{1}{b_2+}\cdots\frac{1}{b_n+}\cdots$,

and let P_r/Q_r and p_r/q_r be the rth convergents of the first and second fractions respectively, then

$$x = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_m+}\,\frac{1}{y} = \frac{yP_m + P_{m-1}}{yQ_m + Q_{m-1}}, \quad \text{.....(B)}$$

and $$q_n y^2 - (p_n - q_{n-1})\,y - p_{n-1} = 0. \quad \text{.....(C)}$$

If α, β are the roots of (C), α being positive, then by Art. 31,

$$\alpha = b_1 + \frac{1}{b_2+}\cdots\frac{1}{b_n+}\,\frac{1}{b_1+}\cdots \quad \text{and} \quad -\frac{1}{\beta} = b_n + \frac{1}{b_{n-1}+}\cdots\frac{1}{b_1+}\,\frac{1}{b_n+}\cdots \quad \text{...(D)}$$

Also if we eliminate y from equations (**B**), (**C**) we obtain a quadratic in x with rational coefficients, whose roots α', β' are found by substituting α, β respectively for y in (**B**). Thus α' is equal to the fraction (**A**) and

$$\beta' = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{m-1} +} \frac{1}{a_m + 1/\beta}.$$

From equation (**D**) it follows that $a_m + 1/\beta = a_m - b_n - f$, where f is a positive proper fraction. Now a_m is not equal to b_n, for if $a_m = b_n$ then a_m would belong to the cycle, which is contrary to the hypothesis. It follows that

$$a_m + 1/\beta = I - f \quad \text{or} \quad -I - f,$$

where I is a positive integer. Thus

$$a_{m-1} + \frac{1}{a_m + 1/\beta} = a_{m-1} + \frac{1}{I - f} \quad \text{or} \quad a_{m-1} - \frac{1}{I + f},$$

and since a_{m-1} is a positive integer, the expression on the left, and therefore also β', is positive in all cases.

NOTE. The argument depends on the existence of a_{m-1}, and fails if there is not more than one quotient in the acyclic part.

34. Summary. It has been proved that *any simple recurring continued fraction is a root of a quadratic equation with rational coefficients. Also the second root β of the equation is restricted as follows:*

(i) *If there is no acyclic part, then* $-1 < \beta < 0$.

(ii) *If the acyclic part consists of a single quotient, then* $\beta < -1$ *or* $\beta > 0$.

(iii) *If the acyclic part contains at least two quotients, then* $\beta > 0$.

Ex. 1. *Find the value of* $1 + \frac{1}{1+} \overset{*}{\frac{1}{1+}} \frac{1}{3+} \overset{*}{\frac{1}{2}}$.

Denoting the value of the fraction by x, we have

$$x = 1 + \frac{1}{1+} \frac{1}{y} \quad \text{where} \quad y = 1 + \frac{1}{3+} \frac{1}{2+} \frac{1}{y};$$

$$\therefore \; x = (2y+1)/(y+1) \quad \text{and} \quad y = (9y+4)/(7y+3),$$

giving $$y = (\sqrt{37}+3)/7 \quad \text{and} \quad x = (\sqrt{37}+8)/9.$$

EXERCISE XLI

1. For the fraction $\frac{1}{a+} \frac{1}{a+} \frac{1}{a+} \ldots$, prove that $p_n = q_{n-1}$ and

$$1/(1 - ax - x^2) = 1 + q_1 x + q_2 x^2 + \ldots + q_n x^n + \ldots,$$

provided that $|x| < \frac{1}{2}(-a + \sqrt{a^2+4})$. (See Ch. XXII, 2.)

2. For the fraction $\cfrac{1}{a_1+}\ \cfrac{1}{a_2+}\ \cdots\ \cfrac{1}{a_n+}\ \cdots$, prove that

(i) $p_nq_{n-3}-q_np_{n-3}=(-1)^{n-1}(a_na_{n-1}+1)$,

(ii) $p_nq_{n-4}-q_np_{n-4}=(-1)^{n-1}(a_na_{n-1}a_{n-2}+a_n+a_{n-2})$.

3. Prove that

$$a+\cfrac{1}{b+}\ \cfrac{1}{2a+}\ \cfrac{1}{b+}\ \cfrac{1}{2a+}\ \cdots=\sqrt{\frac{a}{b}(ab+2)}.$$

Deduce the following:

$$\sqrt{a(a+1)}=a+\cfrac{1}{2+}\ \cfrac{1}{2a+}\ \cfrac{1}{2+}\ \cfrac{1}{2a+}\ \cdots,\qquad \sqrt{a(a+2)}=a+\cfrac{1}{1+}\ \cfrac{1}{2a+}\ \cfrac{1}{1+}\ \cfrac{1}{2a+}\ \cdots,$$

$$\sqrt{a^2-1}=a-1+\underset{*}{\cfrac{1}{1+}}\ \underset{*}{\cfrac{1}{2(a-1)}}\cdots,\qquad \sqrt{a^2+2}=a+\underset{*}{\cfrac{1}{a+}}\ \underset{*}{\cfrac{1}{2a+}}\cdots.$$

4. Prove that

$$\sqrt{a^2+1}=a+\cfrac{1}{2a+}\ \cfrac{1}{2a+}\ \cfrac{1}{2a+}\ \cdots,$$

$$\sqrt{a^2-2}=a-1+\underset{*}{\cfrac{1}{1+}}\ \cfrac{1}{a-2+}\ \cfrac{1}{1+}\ \underset{*}{\cfrac{1}{2(a-1)}}.$$

5. Prove that

(i) $$a+\underset{*}{\cfrac{1}{b+}}\ \underset{*}{\cfrac{1}{c}}=\frac{abc+c+a-b+\sqrt{\{(abc+a+b+c)^2+4\}}}{2(bc+1)}.$$

(ii) $$a+\underset{*}{\cfrac{1}{b+}}\ \cfrac{1}{c+}\ \cfrac{1}{b+}\ \underset{*}{\cfrac{1}{2a}}=\sqrt{\left(a+\frac{1}{b}\right)\left(a+\frac{c}{bc+2}\right)}.$$

6. Show that 449/120 differs from $\sqrt{14}$ by less than 1/90,000. (See Art. 22, Ex. 1.)

7. A straight ruler is graduated in inches and centimetres, the ratio of an inch to a centimetre being N. If N is expressed as a simple continued fraction and p/q is any convergent, show that the distance between the graduations 'p centimetres' and 'q inches' is less than the distance between any two preceding graduations corresponding to a convergent of the fraction.

8. Find the fractions which, of all those with denominators less than 500, are the closest approximations (i) in defect and (ii) in excess to 0·2422642. (See Art. 22, Ex. 2.)

9. Given that 1 metre = 3·2809 ft., obtain the two approximations, 8 kilometres = 5 miles, 103 kilometres = 64 miles; and find an upper limit of error in each case.

10. Given that 1 kilogramme = 2·2046 pounds, show that 44 kilogrammes is slightly greater than 97 pounds, the error being less than half a grain.
Obtain also the usual approximation, 7 grammes = 108 grains.

11. If p/q, p'/q', p''/q'' are consecutive convergents to a simple continued fraction F, prove that F lies between $\sqrt{pp'/qq'}$ and $\sqrt{p'p''/q'q''}$.

12. For the fraction $F=\dfrac{1}{a+}\dfrac{1}{b+}\dfrac{1}{a+}\dfrac{1}{b+}\dots$, prove that

$$\text{(i) } q_n=p_{n-1}; \quad \text{(ii) } F=\frac{1}{2a}\{-ab+\sqrt{ab(ab+4)}\}.$$

Also if α, β are the roots of $x^2-(ab+2)x+1=0$ and $d_n=\alpha^n-\beta^n$, then

$$\text{(iii) } ap_{2n}=bq_{2n-1}=abd_n/d_1; \qquad \text{(iv) } p_{2n+1}=q_{2n}=(d_{n+1}-d_n)/d_1;$$

(v) if $c=ab+2$, and $x^2<\frac{1}{2}(-c+\sqrt{c^2+4})$, then

$$\frac{1+bx-x^2}{1-cx^2+x^4}=p_1+p_2x+\dots+p_{n+1}x^n+\dots,$$

$$\frac{1+ax-x^2}{1-cx^2+x^4}=1+q_1x+q_2x^2+\dots+q_nx^n+\dots.$$

[If u_n stands for p_{2n} or for p_{2n-1}, then, by Art. 5, Ex. 2,

$$u_n-cu_{n-1}+u_{n-2}=0, \quad \therefore\ u_n=A\alpha^n+B\beta^n.$$

If $u_n=p_{2n}$, we have $A+B=p_0=0$, $A\alpha+B\beta=p_2=b$,

giving $$A=-B=b/d_1, \quad \therefore\ p_{2n}=bd_n/d_1.$$

If $u=p_{2n-1}$, since $c=\alpha+\beta$, then $A\alpha+B\beta=p_1=1$, $A\alpha^2+B\beta^2=p_3=c-1$,

giving $A=(1-\alpha^{-1})/d_1$, $B=-(1-\beta^{-1})/d_1$, $\therefore\ p_{2n-1}=(d_n-d_{n-1})/d_1$.

Finally, it is easy to show that

$$1/(1-cx^2+x^4)=(d_1+d_2x^2+\dots d_{n+1}x^{2n}+\dots)/d_1,$$

and then the results in (v) can be obtained by equating coefficients.]

13. If p_n/q_n is the nth convergent to $\dfrac{1}{a_1+}\dfrac{1}{a_2+}\dots\dfrac{1}{a_n}$, then

$$p_{n-1}q_{s-1}-q_{n-1}p_{s-1}$$

is the numerator of the $(n-s)$th convergent to $\dfrac{1}{a_n+}\dfrac{1}{a_{n-1}+}\dfrac{1}{a_{n-2}+}\dots$.

14. Prove that the fraction $a-\dfrac{1}{a-}\dfrac{1}{a-}\dots\dfrac{1}{a-}\dfrac{1}{x}$, in which a is equal to -1 and is repeated any number of times, must have one of three values, and that if x satisfies the equation $2x^3+3x^2-3x-2=0$, the fraction satisfies this equation.

15. If $a_1+\dfrac{1}{a_2+}\dots\dfrac{1}{a_{r-1}+}\dfrac{1}{a_r+}\dfrac{1}{a_{r-1}+}\dots\dfrac{1}{a_2+}\dfrac{1}{a_1}=\dfrac{P}{Q}$ and P/Q is in its lowest terms, prove that Q^2-1 is divisible by P.

16. If P, Q, R are positive integers such that $Q^2-1=PR$ and $P>R$, then P/Q can be expressed in the form

$$\frac{P}{Q}=a_1+\frac{1}{a_2+}\dots\frac{1}{a_{r-1}+}\frac{1}{a_r+}\frac{1}{a_{r-1}+}\dots\frac{1}{a_2+}\frac{1}{a_1},$$

and if p_m/q_m is the mth convergent, then

$$P=p_r(p_r+p_{r-2}), \quad Q=p_rq_{r-1}+p_{r-1}q_{r-2}, \quad R=q_{2r}=q_{r-1}(q_r+q_{r-2}).$$

Verify by taking $P=33$, $Q=23$.

CHAPTER XXV

INDETERMINATE EQUATIONS OF THE FIRST DEGREE

In this chapter a, b, c, ... *stand for positive integers.*

1. Under this heading we consider integral solutions, and more particularly positive integral solutions, of a single linear equation in two or more variables or of a system of m linear equations in n variables ($m<n$).

2. A Single Equation of the Form $ax \pm by = \pm c$. It is obvious that we need only consider the forms

$$ax - by = c \quad \text{and} \quad ax + by = c.$$

If either of these equations is satisfied by integral values of x, y, then any common divisor of a, b must divide c. Hence *if* a, b *have a common divisor which does not divide* c, *no solution in integers exists.*

If a, b have a common divisor which divides c, this can be removed by division. Hence *there is no loss of generality in supposing that* a *is prime to* b, *and we shall assume this to be the case.*

3. The Equation $ax - by = 1$ (a prime to b).

(i) *To find a solution in positive integers*, express a/b as a simple continued fraction with an *even* number of quotients (Ch. XXIV, 4). The last convergent is a/b. Let p/q be the convergent immediately preceding a/b. Then since a/b is an even convergent, $aq - bp = 1$ (Ch. XXIV, 9), and therefore (q, p) is a solution.

(ii) *To find the general solution in integers*, suppose that (x, y) is *any* solution in integers, then

$$ax - by = 1 = aq - bp.$$

Therefore $a(x-q) = b(y-p)$. Hence $a(x-q)$ is divisible by b, and since a is prime to b, $x-q$ must be divisible by b. Consequently

$$(x-q)/b = (y-p)/a = t,$$

where t is an integer or zero. Hence

$$x = q + bt, \quad y = p + at \quad \text{where} \quad t = 0, \quad \pm 1, \quad \pm 2, \quad \text{etc.}$$

This is called the *general solution in integers.*

(iii) Since p/q is a convergent which precedes a/b, it follows that $p<a$ and $q<b$, and from the form of the general solution we conclude that (q, p) *is the only solution in positive integers such that* $p<a$ *and* $q<b$. We shall call this *the 'least' solution in positive integers.*

(iv) Since $a(b-q)-b(a-p)=-1$, the 'least' solution in positive integers of $ax-by=-1$ is $(b-q, a-p)$.

4. The Equation ax − by = c (a prime to b).

(i) *To find a solution in positive integers,* express a/b as a simple continued fraction with an even number of quotients, and let p/q be the convergent immediately preceding a/b. Then

$$aq-bp=1 \quad \text{and} \quad aqc-bpc=c.$$

Hence (qc, pc) is a solution.

(ii) *To find the general solution in integers.* The equation may be written $a(x-qc)=b(y-pc)$, and since a is prime to b, $x-qc$ must be divisible by b. Therefore $(x-qc)/b=(y-pc)/a=t$, where t is an integer or zero. Hence the general solution is

$$x=qc+bt, \quad y=pc+at \quad (t=0, \pm1, \pm2, \ldots).$$

(iii) Let m be the integral part of the smaller of pc/a, qc/b, then the *least solution in positive integers* is (α, β) where $\alpha=qc-bm$, $\beta=pc-am$, and *the general solution in positive integers is*

$$x=\alpha+bt, \quad y=\beta+at \quad (t=0, 1, 2, \ldots).$$

Thus *the equation has infinitely many positive integral solutions.*

Ex. 1. *Find the general solution in positive integers of*

(i) $13x-17y=5$: (ii) $13x-17y=-5$; (iii) $13x-17y=996$.

We find that

$$\frac{13}{17}=0+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4}}}=0+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{1}}}}.$$

The convergents of the continued fractions are

$$\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{13}{17}; \quad \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{10}{13}, \frac{13}{17},$$

and

$$13 \cdot 4-17 \cdot 3=1, \quad 13 \cdot 13-17 \cdot 10=-1.$$

Hence *for the first equation* a solution is (20, 15). The general solution in integers is

$$x=20+17t, \quad y=15+13t \quad (t=0, \pm1, \pm2, \ldots).$$

The least solution in positive integers (given by $t=-1$) is (3, 2), and the general solution in positive integers is $x=3+17t$, $y=2+13t$ $(t=0, 1, 2, \ldots)$.

Similarly *for the second equation* a solution is given by $x=13 \cdot 5=65$, $y=10 \cdot 5=50$, and the general solution in positive integers is

$$x=14+17t, \quad y=11+13t \quad (t=0, 1, 2, \ldots).$$

In case of *the third equation*, the reckoning may be shortened thus: Dividing 996 by 13 (the smaller of the coefficients), we find that $996 = 13 \cdot 76 + 8 = 13 \cdot 77 - 5$.

Hence the equation may be written $13(x-77) - 17y = -5$, and by the preceding the least positive integral solution is given by $x - 77 = 14$, $y = 11$.

Therefore the general solution in positive integers is

$$x = 91 + 17t, \quad y = 11 + 13t \quad (t = 0, 1, 2, \ldots).$$

NOTE. In this way the solution of $ax - by = \pm c$ can be made to depend on that of an equation of the same form in which c is less than or equal to the smaller of $\frac{1}{2}a$, $\frac{1}{2}b$.

5. The Equation $ax + by = c$ (a prime to b).

(i) *General solution in integers.* Let a/b be expressed as a simple continued fraction with an even number of quotients, and let p/q be the last convergent but one. Then $aq - bp = 1$, and the equation may be written

$$ax + by = c(aq - bp) \quad \text{or} \quad a(cq - x) = b(y + cp).$$

Since a is prime to b, $cq - x$ must be divisible by b, and therefore

$$(qc - x)/b = (y + pc)/a = t,$$

where t is an integer or zero.

Hence the general solution in integers is

$$x = qc - bt, \quad y = at - pc \quad (t = 0, \pm 1, \pm 2, \text{ etc.}).$$

(ii) *Positive integral solutions.* The values of t (if such exist) for which x and y are positive are given by the conditions, $pc/a < t < qc/b$.

Let m, n have positive integral or zero values such that

$$pc/a = m + f, \quad qc/b = n + g, \quad \text{where} \quad 0 \leqslant f < 1 \quad \text{and} \quad 0 < g \leqslant 1;$$

then x, y will be positive integers if $t = m + 1, \; m + 2, \ldots \; n$.

Hence if N is the number of solutions in positive integers,

$$N = n - m = \frac{qc}{b} - \frac{pc}{a} + f - g = \frac{c}{ab} + f - g.$$

Now $-1 \leqslant f - g < 1$, and therefore

$$\frac{c}{ab} - 1 \leqslant N < \frac{c}{ab} + 1.$$

Hence, *if c is not divisible by ab*, the number (N) of solutions in positive integers is one of the integers next to c/ab, and is greater than or less than c/ab according as $f \gtrless g$.

If c is divisible by ab, then pc/a and qc/b are integers; therefore

$$f = 0, \; g = 1 \quad \text{and} \quad N = \frac{c}{ab} - 1.$$

Ex. 1. *Find the positive integral solutions of* $13x+17y=3000$.

Use the rule just given, to show that there are 13 *such solutions.*

As in Art. 4, Ex. 1, we have $13 \cdot 4 - 17 \cdot 3 = 1$, and the equation may be written

$$13x+17y=3000(13 \cdot 4 - 17 \cdot 3) \quad \text{or} \quad 13(12000-x)=17(y+9000):$$

and the integral solutions are given by $x=12000-17t$, $y=13t-9000$.

Moreover, $$\frac{12000}{17}=705\frac{15}{17}, \quad \frac{9000}{13}=692\frac{4}{13};$$

therefore the positive integral solutions are given by $t=693, 694, \ldots 705$. These solutions are (15, 165), (32, 152), ... (219, 9).

Now, $13<\frac{3000}{13 \cdot 17}<14$; also $f=\frac{4}{13}$, $g=\frac{15}{17}$; so that $f<g$; hence the number of solutions is 13. Since $705-693+1=13$, this result agrees with the preceding.

6. Two Equations with Three Unknowns.

(1) The general solution in integers, and all the positive integral solutions, if any exist, may be found as in the next example.

Ex. 1. *Find all the positive integral solutions of the pair of equations,*

$$2x-2y+3z=7, \quad -4x+7y+3z=19.$$

Eliminating one of the unknowns (z), we have $2x-3y=-4$; and the general solution in integers of this equation is $x=3m-2$, $y=2m$, $(m=0, \pm1, \pm2 \ldots)$.

Substituting these values of x, y in one of the given equations, say in the first, we find that $2m+3z=11$; and the general solution in integers of this equation is

$$m=4-3t, \quad z=1+2t, \text{ and therefore } x=3m-2=10-9t, \quad y=2m=8-6t.$$

Hence the general solution in integers is $x=10-9t$, $y=8-6t$, $z=1+2t$.

The only values of t for which x, y, z are positive are 0 and 1, therefore the only positive integral solutions are (10, 8, 1) and (1, 2, 3).

(2) *If* (α, β, γ) *is a solution in integers of the two equations*

$$ax+by+cz=d, \quad a'x+b'y+c'z=d',$$

it is required to find the general solution in integers.

(i) *If none of the three* (bc'), (ca'), (ab') *is zero*, by Ch. X, 6, (4), the equations are equivalent to

$$\frac{x-\alpha}{(bc')}=\frac{y-\beta}{(ca')}=\frac{z-\gamma}{(ab')}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

Denote each member of (**A**) by t/f where t may be any integer or zero and f is an integer to be chosen later, then

$$x-\alpha=(bc')t/f, \quad y-\beta=(ca')t/f, \quad z-\gamma=(ab')t/f.$$

Thus x, y, z will be integers for *all* integral values of t if and only if f is a common factor of (bc'), (ca'), (ab').

Now every common factor of these numbers is a factor of their G.C.M., hence *all integral solutions are given by*

$$\frac{x-\alpha}{(bc')}=\frac{y-\beta}{(ca')}=\frac{z-\gamma}{(ab')}=\frac{t}{g},$$

where g *is the* G.C.M. *of* (bc'), (ca'), (ab') *and* $t=0, \pm1, \pm2, \ldots$.

(ii) *If* $(bc')=0$, the equations are equivalent to

$$x=\alpha, \quad \frac{y-\beta}{(ca')}=\frac{z-\gamma}{(ab')},$$

and it is easily seen that *the general solution in integers is*

$$x=\alpha, \quad \frac{y-\beta}{(ca')}=\frac{z-\gamma}{(ab')}=\frac{t}{g_1},$$

where g_1 *is the* G.C.M. *of* (ca'), (ab').

7. A Single Equation of the Form $ax+by+cz+\ldots=d$.

(1) If $a, b, c, \ldots$ are all positive, then all the positive integral solutions (if any exist) may be found as follows:

Ex. 1. *Find the positive integral solutions of* $7x+11y+26z=123$.

The greatest coefficient is 26, and $7+11+26z\leqslant123$, hence $z\leqslant4$. Therefore the only possibilities are

$z=1$, $7x+11y=97$, of which there is one solution (6, 5),
$z=2$, $7x+11y=71$, „ „ „ one „ (7, 2),
$z=3$, $7x+11y=45$, „ „ „ no „ ,
$z=4$, $7x+11y=19$, „ „ „ no „ .

Thus there are two solutions in positive integers, namely (6, 5, 1) and (7, 2, 2).

Ex. 2. *Find all the positive integral solutions of* $3x_1+4x_2+7x_3+11x_4=67$.

Here $11x_4\leqslant67-(3+4+7)$, therefore $x_4<5$, and we may have

$x_4=4$, $x_3=1$, $3x_1+4x_2=16$, $x_1=4$, $x_2=1$,
$x_3=2$, $3x_1+4x_2=9$, no solution,
$x_4=3$, $x_3=1$, $3x_1+4x_2=27$, $x_1=5, 1$, $x_2=3, 6$,
$x_3=2$, $3x_1+4x_2=20$, $x_1=4$, $x_2=2$,
$x_3=3$, $3x_1+4x_2=13$, $x_1=3$, $x_2=1$,
$x_3=4$, $3x_1+4x_2=6$, no solution.

The other possibilities, namely $x_4=2, 1$, may be treated in the same way. There are 28 solutions in all.

(2) If (α, β, γ) is an integral solution of $ax+by+cz=d$ (where a, b, c are positive or negative integers), *other solutions are given by*

$$x=\alpha+bw-cv, \quad y=\beta+cu-aw, \quad z=\gamma+av-bu,$$

where u, v, w *have any integral or zero values.*

This is obviously true. Further, we can show that *if any two of the three a, b, c are prime to each other, then every integral solution can be so expressed.*

For if $x-\alpha$, $y-\beta$, $z-\gamma$ are denoted by X, Y, Z respectively, the equation becomes

$$aX+bY+cZ=0. \quad \text{.....(A)}$$

Let a be prime to b, then integers u, v can be found so that, whatever integral value Z may have,

$$av-bu=Z.$$

If, then, X, Y, Z are integers which satisfy (A),

$$a(X+cv)+b(Y-cu)=aX+bY+cZ=0\,;$$

and since a is prime to b, we must have

$$X+cv=bw, \quad Y-cu=-aw,$$

where w is some integer or zero. This proves the statement in question.

EXERCISE XLII

1. Find the least solution in positive integers of

(i) $68x-157y=1$; (ii) $68x-157y=-1$.

In Exx. 2-7, find the general solution in integers and the least positive integral values of x, y.

2. $8x-27y=125$. 3. $27x-8y=125$. 4. $29x-13y=1$.

5. $17x-41y=1$. 6. $12x-7y=211$. 7. $199x-225y=20$.

Find all the positive integral solutions of equations 8-10.

8. $7x+8y=50$. 9. $11x+7y=151$. 10. $12x+29y=700$.

11. Express 68/77 as the sum of two proper fractions whose denominators are 7 and 11.

12. Show that $1/(13\times17)$ can be expressed in two ways as the difference between two proper fractions whose denominators are 13 and 17 ; and express the given fraction in these ways.

13. AB and CD are two rods, each one foot in length. AB is scaled off into 13 equal parts, and CD into 17 equal parts. If the two rods are placed side by side, with the scales in contact, and the ends of the rods in alignment, show that the distance between one scale division of AB and one of CD can never be less than $1/(13\times17)$ of a foot, and that there are two cases in which the distance has this value. Which are the pairs of divisions in these cases ?

14. The sum of two positive integers is 100. If one is divided by 7 the remainder is 1, and if the other is divided by 9 the remainder is 7. Find the numbers.

15. If c is increased by kab, then the number of positive integral solutions of $ax+by=c$ is increased by k. [For c/ab is increased by k and f, g are unaltered.]

16. The least value of c for which the equation $ax+by=c$ has N solutions is given by
$$c=a+b+(N-1)ab.$$

[This follows from Ex. 15, coupled with the fact that $a+b$ is the least value of c for which the equation has one solution.]

17. Find the least positive integer, a, for which the equation $3x+4y=a$ has six positive integral solutions.

18. Find the least value of c in order that the equation $13x+17y=c$ may have 13 solutions in positive integers, showing that the required value is 2682. Verify by finding the number of solutions for $c=2682$, 2681.

19. The number of solutions in positive integers of the equation $x+by=c$ is the integral part of c/b.

Find all the positive integral solutions of equations 20-23.

20. $\left.\begin{aligned} x+3y-4z&=8,\\ 2x+y+3z&=39.\end{aligned}\right\}$

21. $\left.\begin{aligned} 7x+11y+13z&=201,\\ 11x+13y+7z&=209.\end{aligned}\right\}$

22. $9x+16y+25z=109.$

23. $17x+25y+23z=270.$

24. Find the least number which when divided by 7, 11, 20, leaves the remainders 1, 5, 9, respectively. Find also the general form of all such numbers.

25. A certain set of stamps is made up of four different sorts: a collector has one of each sort and some duplicates. The values of the stamps are 2d., 3d., 9d., 1s., respectively; and the lot are worth 3s. 9d. Find all the different ways in which the batch could have been made up.

26. A man has in his pocket three half-crowns, seven other silver coins which are either shillings or sixpences, and eightpence in coppers; he paid a bill of 9s. 7d. Find the possible number of different ways of doing so.

27. Find the sum of money which is expressed in farthings by the same digits written in the same order as those which are required to express it in pounds, shillings and pence.

How many solutions are there to this problem ?

28. If (p_1, p_2, p_3, p_4) is an integral solution of $ax_1+bx_2+cx_3+dx_4=e$, show that other solutions are given by
$$\begin{aligned} x_1&=p_1+bw-cv-du',\\ x_2&=p_2+cu-dv'-aw,\\ x_3&=p_3+dw'+av-bu,\\ x_4&=p_4+au'+bv'-cw', \end{aligned}$$
where u, v, w, u', v', w' have any integral or zero values.

29. If $ax+by+c=0$, $a'x+b'y+c'=0$ and x is an integer, prove that y is an integer, unless both b and b' are divisible by $ab'-a'b$.

[For $a(bc')+b(ca')+c(ab')=0$ and $a'(bc')+b'(ca')+c'(ab')=0$.]

CHAPTER XXVI

THEORY OF NUMBERS (2)

1. Congruence. (1) Let n be any positive whole number, which we shall call the *modulus*. If a, b are any numbers, positive or negative, such that $a-b$ is divisible by n, then a and b are said to be *congruent* with respect to the modulus n, and each is said to be a *residue* of the other.

This is expressed by writing $a \equiv b \pmod{n}$ or $a-b \equiv 0 \pmod{n}$, where the symbol $\equiv$ is an abbreviation for '*is congruent with.*' When no doubt exists as to the modulus in question, we simply write $a \equiv b$. Any statement of this kind is called *a congruence.*

Every residue of a to the modulus n is of the form $a+qn$, where q may be positive or negative. If r is the remainder when a is divided by n, we have $a \equiv r \pmod{n}$ and $0 \leqslant r < n$;

thus r is the *least positive residue* of a.

Again, $a \equiv r \equiv -(n-r) \pmod{n}$, and either r or $n-r \leqslant \frac{1}{2}n$. Hence we can always find r' so that $a \equiv r' \pmod{n}$ and $|r'| \leqslant \frac{1}{2}n$, where r' may be positive or negative. This number r' is called the *absolute least residue* of a.

For example, if 5 is the modulus, $34 \equiv 4 \equiv -1$ and $-34 \equiv -4 \equiv 1$. Thus the least positive residues of 34 and -34 are 4 and 1, and the absolute least residues are -1 and 1 respectively.

(2) **Fundamental Theorems.** *If* $a \equiv b \pmod{n}$ *and* $a' \equiv b' \pmod{n}$, *then* (i) $a+a' \equiv b+b'$. (ii) $a-b \equiv a'-b'$. (iii) $aa' \equiv bb'$, *and in particular* $ac \equiv bc$.

(iv) *If* a *and* b *are divisible by* d, *a number prime to* n, *then* $a/d \equiv b/d$.

For by hypothesis $a-b$ and $a'-b'$ are divisible by n, hence

$$(a+a')-(b+b') \quad \text{and} \quad (a-a')-(b-b') \quad \text{are divisible by } n.$$

Again, $$aa'-bb' = (a-b)a' + (a'-b')b,$$

therefore $aa'-bb'$ is divisible by n. This proves the theorems (i)-(iii).

Lastly, $a-b=qn$, where q is a whole number, then since a and b are divisible by d, so also is qn. Now d is prime to n, therefore q/d is a whole number, and since $a/d - b/d = q/d \,.\, n$, it follows that $a/d \equiv b/d \pmod{n}$.

In particular, *if* $a \equiv b \pmod{p}$ *where* p *is a prime and if* d *is any common divisor of* a *and* b, *then* $a/d \equiv b/d \pmod{p}$, *except when* $d \equiv 0 \pmod{p}$.

Thus a close analogy exists between 'congruences to a prime modulus' and 'equalities.'

(3) The following are immediate consequences of the theorems in § (2):

(i) *If to any modulus* $a \equiv a'$, $b \equiv b'$, ... $k \equiv k'$, *then to the same modulus* $ab \ldots k \equiv a'b' \ldots k'$, *and in particular* $a^m \equiv a'^m$.

(ii) *If* $f(x, y, z, \ldots)$ *is a polynomial in* $x, y, z, \ldots$ *with integral coefficients and* $x \equiv x'$, $y \equiv y'$, $z \equiv z'$, ... *to any modulus, then to the same modulus* $f(x, y, z, \ldots) \equiv f(x', y', z', \ldots)$.

Ex. 1. *Give a test as to the divisibility of a number by* 7, 11 *or* 13.

Any number N may be written in the form

$$N = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \quad \text{where} \quad t = 1000 \quad \text{and} \quad 0 \leqslant a_r < t.$$

Let $s = a_0 - a_1 + a_2 - \ldots + (-1)^n a_n$, then we have $t + 1 = 1001 = 7 \cdot 11 \cdot 13$; hence $t \equiv -1$ and $t^r \equiv (-1)^r$, with regard to each of the moduli 7, 11 and 13. Consequently $N \equiv s$ (mod 7, 11 or 13).

Thus if $N = 12345671$, then $s = 671 - 345 + 12 = 338$. Now 338 is divisible by 13 but not by 7 or 11; therefore 12345671 is divisible by 13 but not by 7 or 11.

2. The Numbers less than a Given Number and prime to it.

If n is any number greater than 1, the number of positive integers less than n and prime to it is denoted by $\varphi(n)$: moreover, we *define* the value of $\varphi(1)$ as 1. Here, it should be noted that 1 is to be regarded as prime to every other number. Thus the numbers less than 12 and prime to it are 1, 5, 7, 11, and $\varphi(12) = 4$.

(1) *If* a *is prime to* n, *the number of terms of the arithmetical progression*

$$x, \quad x + a, \quad x + 2a, \ldots \quad x + (n-1)a$$

which are prime to n *is* $\varphi(n)$.

For if these numbers are divided by n, the remainders are 0, 1, 2, ... $n-1$, taken in a certain order (Ch. I, 10), and if a number is prime to n, so also is its remainder. Consequently, as many terms in the progression are prime to n as there are numbers less than n and prime to it.

(2) *If* m *is prime to* n, *then* $\varphi(mn) = \varphi(m) \cdot \varphi(n)$.

To enumerate the numbers less than mn and prime to it, we arrange the numbers 1, 2, 3, ... mn in n rows and m columns as below:

1	...	2	...	k	...	m
$m+1$	...	$m+2$	...	$m+k$	...	$2m$
$2m+1$	...	$2m+2$	...	$2m+k$	...	$3m$
.....						
$(n-1)m+1$		$(n-1)m+2$ ...		$(n-1)m+k$ ...		nm.

Since m is prime to n, the numbers in question are those which are prime to both m and n. Now every member of the kth column is or is not prime to m, according as k is or is not prime to m.

Thus the numbers in question, being prime to m, are members of the $\varphi(m)$ columns headed by numbers prime to m.

Moreover, every column contains $\varphi(n)$ numbers which are also prime to n, for the numbers in each column form an arithmetical progression of n terms with a common difference m which is prime to n.

It follows that there are $\varphi(m) \cdot \varphi(n)$ numbers less than mn and prime to both m and n, that is to mn, and so $\varphi(mn) = \varphi(m) \cdot \varphi(n)$.

(3) *If* $m_1, m_2, m_3 \ldots m_r$ *are prime to one another, then*

$$\varphi(m_1 m_2 m_3 \ldots m_r) = \varphi(m_1) \cdot \varphi(m_2) \cdot \varphi(m_3) \ldots \varphi(m_r).$$

For by (2), $\varphi(m_1 m_2) = \varphi(m_1) \cdot \varphi(m_2)$. Also m_3 is prime to both m_1 and m_2, therefore it is prime to $m_1 m_2$. Hence

$$\varphi(m_1 m_2 m_3) = \varphi(m_1 m_2) \cdot \varphi(m_3) = \varphi(m_1) \cdot \varphi(m_2) \cdot \varphi(m_3),$$

and so on, for any number of steps.

(4) *If* p *is a prime, then* $\varphi(p^r) = p^r(1 - 1/p)$.

The proposition is true when $r = 1$, for $\varphi(p) = p - 1$.

If $r > 1$, since p is a prime, of the numbers 1, 2, 3, ... p^r, those which are not prime to p^r are p, $2p$, $3p$, ... p^r, and their number is p^{r-1}.

All the rest are prime to p^r, and their number is $p^r - p^{r-1}$, which proves the theorem.

(5) *If* $n = p^a \cdot q^b \cdot r^c \ldots$ *where* $p, q, r, \ldots$ *are the prime factors of* n, *then*

$$\varphi(n) = n\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right) \ldots .$$

For $p^a, q^b, r^c \ldots$ are prime to one another, and the result follows from (3) and (4).

Ex. 1. *Find the number of integers less than* n *and prime to it, when*

$$n = 1024, 1025, 1026.$$

We have $1024 = 2^{10}$, hence $\varphi(1024) = 1024\,(1 - \frac{1}{2}) = 512$.

$1025 = 5^2 \cdot 41$ and $\varphi(1025) = 1025\,(1 - \frac{1}{5})\,(1 - \frac{1}{41}) = 800$.

$1026 = 2 \cdot 3^3 \cdot 19$ and $\varphi(1026) = 1026\,(1 - \frac{1}{2})\,(1 - \frac{1}{3})\,(1 - \frac{1}{19}) = 324$.

This illustrates the irregularity in the variation of the function $\varphi(n)$.

Ex. 2. *If* $n \geqslant 2$, *the sum of the integers less than* n *and prime to it is* $\frac{1}{2} n \varphi(n)$.

This is obviously true when $n = 2$. If $n > 2$ and x is any number less than n and prime to it, so also is $n - x$.

Hence the numbers less than n and prime to it can be arranged in pairs such that each pair has a sum n.

Thus $\varphi(n)$ must be even, the number of pairs is $\frac{1}{2}\varphi(n)$ and the sum of the numbers in question is $\frac{1}{2} n \varphi(n)$.

(6) **Theorem.** *If* $d_1(=1)$, d_2, d_3, ... $d_r(=n)$ *are the divisors of any number* n, *then*

$$\varphi(d_1)+\varphi(d_2)+\varphi(d_3)+\ldots+\varphi(d_r)=n.$$

Let $n=p^aq^br^c\ldots$ where $p, q, r, \ldots$ are primes and let

$$S=\varphi(d_1)+\varphi(d_2)+\ldots\ \varphi(d_r),$$

then every divisor of n is of the form $p^xq^yr^z\ldots$ where $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$, $0\leqslant z\leqslant c$, etc., so that

$$S=\Sigma\varphi(p^xq^yr^z\ldots)=\Sigma\{\varphi(p^x)\,.\,\varphi(q^y)\,.\,\varphi(r^z)\ldots\},$$

where $x, y, z, \ldots$ have all values subject to the above conditions. Hence S is equal to the product

$$[1+\varphi(p)+\varphi(p^2)+\ldots+\varphi(p^a)][1+\varphi(q)+\ldots+\varphi(q^b)]$$
$$[1+\varphi(r)+\ldots+\varphi(r^c)]\ldots.$$

Now $1+\varphi(p)+\varphi(p^2)+\ldots+\varphi(p^a)$
$$=1+(p-1)+(p^2-p)+\ldots+(p^a-p^{a-1})=p^a,$$

and similarly for the other series in the brackets. Therefore

$$S=p^a\,.\,q^b\,.\,r^c\ldots=n.$$

For instance, the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24; and

$$\varphi(1)+\varphi(2)+\varphi(3)+\varphi(4)+\varphi(6)+\varphi(8)+\varphi(12)+\varphi(24)$$
$$=1+1+2+2+2+4+4+8=24.$$

3. Fermat's Theorem. *If* p *is a prime and* a *is any number prime to* p, *then* $a^{p-1}-1$ *is divisible by* p.

For since a is prime to p, if a, $2a$, $3a$, ... $(p-1)a$ are divided by p, the remainders are 1, 2, 3, ... $p-1$ taken in a certain order.

Also the product of the numbers a, $2a$, $3a$, ... is congruent (mod p) with the product of the remainders, therefore $a^{p-1}\lfloor p-1\equiv\lfloor p-1$; and since $\lfloor p-1$ is prime to p, therefore $a^{p-1}\equiv 1\,(\mathrm{mod}\,p)$, which proves the theorem.

Corollary 1. *If* p *is a prime and* a *is any number whatever, then* a^p-a *is divisible by* p.

For $a^p-a=a(a^{p-1}-1)$, and if a is prime to p, $a^{p-1}-1$ is divisible by p, which is a prime number. Hence in all cases a^p-a is divisible by p.

Corollary 2. *If* p *is an odd prime and* a *is prime to* p, *then*

$$a^{\frac{1}{2}(p-1)}\equiv\pm 1\,(\mathrm{mod}\,p).$$

For $a^{p-1}-1=(a^{\frac{1}{2}(p-1)}-1)(a^{\frac{1}{2}(p-1)}+1)$, and $a^{p-1}-1$ is divisible by p, therefore p is a divisor of $a^{\frac{1}{2}(p-1)}-1$ or of $a^{\frac{1}{2}(p-1)}+1$, since p is a prime number.

Ex. 1. *The fifth power of any number N has the same right-hand digit as N.*
For $N^5 - N$ is divisible by 5 and by 2.

Ex. 2. *The ninth power of any number N is of one of the forms* $19m$, $19m \pm 1$.
If N is prime to 19, since $\frac{1}{2}(19-1) = 9$, by Cor. 2 we have $N^9 \equiv \pm 1 \pmod{19}$. Otherwise we must have $N = 19m$.

4. Euler's Extension of Fermat's Theorem. *If n is any number and a is prime to n, then* $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Let $a_1 (= 1)$, a_2, a_3, ... $a_{\varphi(n)}$ be the numbers, in ascending order, less than n and prime to it, and consider the products

$$aa_1, \quad aa_2, \quad aa_3, \ldots \quad aa_{\varphi(n)}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

If these are divided by n, the remainders are all different. For if we suppose that two products as aa_r, aa_s $(r > s)$ give the same remainder, then $a(a_r - a_s)$ would be divisible by n. This is impossible, for a is prime to n and $a_r - a_s < n$.

Also the remainders are all prime to n, for the factors of any product are both prime to n.

Hence the remainders are the numbers a_1, a_2, ... $a_{\varphi(n)}$ taken in some order or other. Now the product of the numbers in the set (**A**) is congruent (mod n) with the product of the remainders, therefore

$$a^{\varphi(n)} \cdot a_1 a_2 \ldots a_{\varphi(n)} \equiv a_1 a_2 \ldots a_{\varphi(n)} \pmod{n},$$

and dividing by $a_1 a_2 \ldots a_{\varphi(n)}$, which is prime to n, we have the result in question.

5. Wilson's Theorem. *If p is a prime number, then* $\underline{|p-1} + 1$ *is divisible by p.*

For if a is any one of the numbers 1, 2, 3, ... $p-1$, and if the products $1 \cdot a$, $2a$, $3a$, ... $(p-1)a$ are divided by p, the remainders are the numbers 1, 2, 3, ... $p-1$, taken in some order or other. Hence, for every a, there is one number a' and one only, such that $aa' \equiv 1 \pmod{p}$.

If $a' = a$, then $a^2 - 1$ must be divisible by p, and since p is a prime and $a < p$, it follows that either $a + 1 = p$ or $a - 1 = 0$.

Thus 1 and $p-1$ are the only values of a for which a' can be equal to a.

If then a is any one of the $p-3$ numbers 2, 3, ... $p-2$, then a' is not equal to a. Consequently, these numbers can be arranged in $\frac{1}{2}(p-3)$ pairs, such that the product of each pair is congruent with 1.

Therefore $2 \cdot 3 \ldots (p-2) \equiv 1 \pmod{p}$ and $\underline{|p-1} \equiv p - 1 \pmod{p}$.

Hence $\underline{|p-1} + 1 \equiv 0 \pmod{p}$, which is the result in question.

NOTE. The numbers a, a' are called *associated residues*, each being the *associate* of the other. (Euler.)

Conversely, if $\lfloor p-1+1$ *is divisible by* p, *then* p *is a prime number.* For otherwise p would have a factor q which would divide $\lfloor p-1$ and $\lfloor p-1+1$.

Ex. 1. *Prove that* $\lfloor 28+233$ *is divisible by* 899.

We have $899=29\,.\,31$, also $233\equiv 1 \pmod{29}$ and $233\equiv 16 \pmod{31}$.

By Wilson's theorem, $\lfloor 28+1\equiv 0 \pmod{29}$, therefore $\lfloor 28+233\equiv 0 \pmod{29}$.

Again, to the modulus 31, we have $\lfloor 30+1\equiv 0$, therefore $30\,.\,29\,.\,\lfloor 28+1\equiv 0$, and so $(-1)(-2)\,\lfloor 28+32\equiv 0$. Since 2 is prime to 31, it follows that $\lfloor 28+16=0$, and therefore $\lfloor 28+233\equiv 0$. Hence $\lfloor 28+233$ is divisible by 29 and 31, and therefore by 899.

6. Lagrange's Theorem. *If* p *is a prime and* r *is any number less than* $p-1$, *the sum of the products of the numbers* $1, 2, 3, \ldots, p-1$ *taken* r *together is divisible by* p.

Let $f(x)=(x+1)(x+2)\ldots(x+p-1)$; then if a_r denotes the sum of the products of $1, 2, 3, \ldots, p-1$ taken r together,

$$f(x)=x^{p-1}+a_1x^{p-2}+a_2x^{p-3}+\ldots+a_{p-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Also we have the identity $(x+p)f(x)=(x+1)f(x+1)$, that is,

$$(x+p)(x^{p-1}+a_1x^{p-2}+a_2x^{p-3}+\ldots+a_{p-1})$$
$$=(x+1)^p+a_1(x+1)^{p-1}+a_2(x+1)^{p-2}+\ldots+a_{p-1}(x+1). \quad \text{(B)}$$

Equating the coefficients of x^{p-2}, x^{p-3}, $\ldots$ x, we find that

$$pa_1=C_2^p+C_1^{p-1}\,.\,a_1,$$
$$pa_2=C_3^p+C_2^{p-1}a_1+C_1^{p-2}a_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$pa_{p-2}=C_{p-1}^p+C_{p-2}^{p-1}a_1+\ldots\,C_1^2a_{p-2}.$$

Now, since p is a prime, C_r^p is divisible by p, for $r<p$; and C_r^{p-1}, C_r^{p-2}, etc., are all prime to p. It follows in succession that $a_1, a_2, \ldots a_{p-2}$ are all divisible by p, which is the theorem in question. For another proof, see Art. 11, (6).

NOTE. We can immediately deduce the theorems of Wilson and Fermat, thus:

(i) Equating the terms independent of x in the identity (B),

$$pa_{p-1}=1+a_1+a_2+\ldots+a_{p-1},$$

and since $a_1, a_2, \ldots, a_{p-2}$ are divisible by p, so also is $a_{p-1}+1$, that is $\lfloor p-1+1$ is divisible by p, which is Wilson's theorem.

(ii) If x is any number prime to p, one of the numbers $x+1, x+2, \ldots, x+p-1$ must be divisible by p, hence $f(x)\equiv 0 \pmod{p}$.

Also $f(x)\equiv x^{p-1}+a_{p-1}\equiv x^{p-1}-1 \pmod{p}$, therefore $x^{p-1}-1\equiv 0 \pmod{p}$, which is Fermat's theorem.

Ex. 1. *If p is a prime, greater than* 3, *prove that*

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{p-1}\right)\cdot \lfloor\underline{p-1} \equiv 0 \pmod{p^2}.$$

If, in the identity (**A**), we put x equal to p and $-2p$ in succession, we have

$$(p+1)(p+2)\ldots(2p-1)=a_{p-1}+a_{p-2}\cdot p+a_{p-3}\cdot p^2+\ldots+p^{p-1},$$

$$(2p-1)(2p-2)\ldots(p+1)=a_{p-1}-a_{p-2}\cdot 2p+a_{p-3}\cdot 4p^2+\ldots+(2p)^{p-1};$$

and since the left-hand sides are identical, we have by subtraction,

$$3p\cdot a_{p-2}-3p^2\cdot a_{p-3}+9p^3\cdot a_{p-4}-\ldots-(2^{p-1}-1)p^{p-1}=0.$$

Now, since p is a prime greater than 3, a_{p-3} exists; and, by Lagrange's theorem, it is divisible by p.

Hence, $3p\cdot a_{p-2}$ is divisible by p^3; and, since 3 is prime to p, it follows that

$$a_{p-2} \text{ is divisible by } p^2.$$

Ex. 2. *If p is a prime, greater than* 3, *prove that, for all integral values of m,*

$$\lfloor\underline{mp}-\lfloor\underline{m}\cdot(\lfloor\underline{p})^m$$

is divisible by p^{m+3}.

Substituting rp for x in the identity (**A**), we have, as in Ex. 1,

$$(rp+1)(rp+2)\ldots(rp+p-1)\equiv a_{p-1} \pmod{p^3},$$

where $r=1, 2, 3, \ldots, m-1$, and p is prime and greater than 3.

Hence, $$\lfloor\underline{(r+1)p}\,/\,(\lfloor\underline{rp}\cdot\lfloor\underline{p})\equiv r+1 \pmod{p^3};$$

and, by giving r the values $1, 2, \ldots, m-1$, it follows that

$$\lfloor\underline{mp}\,/\,(\lfloor\underline{p})^m\equiv\lfloor\underline{m} \pmod{p^3};$$

hence, $$\lfloor\underline{mp}-\lfloor\underline{m}(\lfloor\underline{p})^m\equiv 0 \pmod{p^{m+3}}.$$

EXERCISE XLIII

1. If $N=a_0+a_1t+a_2t^2+\ldots+a_nt^n$, and $S=a_0-a_1+a_2-\ldots+(-1)^nt^n$, where $t=10000$ and $0\leqslant a_r<t$, prove that $N\equiv S$(mod 73 or 137). Hence state a rule for the divisibility of a number by 73 or 137. Apply it to 30414 and 81103.

2. If p is a prime and $x_1, x_2, x_3, \ldots x_a$ are any numbers, show that

$$(x_1+x_2+\ldots+x_a)^p\equiv x_1^p+x_2^p+\ldots+x_a^p \pmod{p},$$

and in particular $(ax)^p\equiv ax^p \pmod{p}$, where x is any number.

Consequently, $a^p\equiv a \pmod{p}$, and if a is not divisible by p, $a^{p-1}\equiv 1 \pmod{p}$, which is Fermat's theorem.

3. Prove that, if n is a prime greater than 7, then n^6-1 is divisible by 504.

4. Prove that, if n is a prime greater than 13, then $n^{12}-1$ is divisible by 65520.

5. If p and q are different primes, then $p^{q-1}+q^{p-1}-1$ is divisible by pq.

6. If p is a prime of the form $4m+1$, show that $\lfloor\underline{\frac{1}{2}(p-1)}$ is a solution of the congruence, $x^2+1\equiv 0 \pmod{p}$.

[Show that $1\cdot 2\cdot 3\cdot\ldots\cdot 2m\equiv(4m)\cdot(4m-1)\cdot(4m-2)\ldots(2m+1) \pmod{p}$.]

7. If p is a prime of the form $4m-1$, show that $\lfloor \frac{1}{2}(p-1)$ is a solution of the congruence, $x^2-1\equiv 0 \pmod p$.

8. If the congruence, $x^2\equiv -1 \pmod p$, has no solution, show that

(i) $\lfloor p-1\equiv(-1)^{\frac{1}{2}(p-1)}$; and (ii) that p is of the form $4m-1$.

9. Show that $\lfloor 18+1$ is divisible by 437.

10. If p is a prime, $2\lfloor p-3+1$ is divisible by p.

11. If n is any odd number, show that

$$\lfloor n-1\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right]\equiv 0 \pmod n.$$

12. Arrange the numbers 2, 3, 4, ... , 15 in pairs, a and a', such that for each pair, $aa'\equiv 1 \pmod{17}$.

13. Let $n=a^pb^qc^r\ldots$, where $a, b, c, \ldots$ are different primes, and consider the groups

$$\left.\begin{array}{l} a,\ 2a,\ 3a,\ \ldots\ n/a\ .\ a \\ b,\ 2b,\ 3b,\ \ldots\ n/b\ .\ b \\ \ldots\ldots\ldots\ldots \end{array}\right\},\ \ldots\ldots\ldots(\mathbf{A}) \qquad \left.\begin{array}{l} ab,\ 2ab,\ 3ab,\ \ldots\ n/ab\ .\ ab \\ bc,\ 2bc,\ 3bc,\ \ldots\ n/bc\ .\ bc \\ \ldots\ldots\ldots\ldots \end{array}\right\},\ \ldots\ldots\ldots(\mathbf{B})$$

together with groups (C), (D), etc., formed in a similar way from the combinations of $a, b, c, \ldots$ taken 3, 4, ... together. If $s_1, s_2, s_3, \ldots$ denote the sums of the rth powers of the numbers in the groups (A), (B), (C), ... respectively, show that the sum of the rth powers of the numbers not greater than n and not prime to it is $s_1-s_2+s_3-s_4+\ldots$.

14. If S_2 is the sum of the squares, and S_3 the sum of the cubes, of the numbers less than n and prime to it, show that

$$\text{(i)}\quad S_2=\frac{1}{3}n^3\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots+\frac{1}{6}n(1-a)(1-b)(1-c)\ldots,$$

$$\text{(ii)}\quad S_3=\frac{1}{4}n^4\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots+\frac{1}{4}n^2(1-a)(1-b)(1-c)\ldots,$$

where $a, b, c, \ldots$ are the different prime factors of n.

[Use the last example to show that the sum of the squares of the numbers not greater than n and not prime to it is

$$\frac{1}{3}n^3\left(\Sigma\frac{1}{a}-\Sigma\frac{1}{ab}+\ldots\right)+\frac{1}{2}n^2+\frac{1}{6}n(\Sigma a-\Sigma ab+\ldots).]$$

7. Roots of a Congruence. If $f(x)$ is a polynomial of the rth degree in x, with integral coefficients, any value of x which satisfies the condition $f(x)\equiv 0 \pmod n$ is called a *solution* or *root* of the congruence $f(x)\equiv 0$, which is said to be of the rth degree.

If x_0 is a root of the congruence $f(x)\equiv 0 \pmod n$, so also is any number congruent with x_0 to the modulus n.

Two roots will be regarded as *distinct* only when they are incongruent with respect to the modulus, and when we say that a congruence has r roots we mean that it has r distinct roots.

Ex. 1. It follows from Fermat's theorem that, *if p is a prime, the roots of the congruence* $x^{p-1}-1\equiv 0 \pmod p$ *are* $1, 2, 3, \ldots p-1$.

Ex. 2. Every square number is of the form $5n$ or $5n\pm 1$, thus no square can be congruent with 2 to the modulus 5. In other words, *the congruence* $x^2\equiv 2 \pmod 5$ *has no solution.*

NOTE. It is obvious that, in dealing with the congruence $f(x)\equiv 0 \pmod n$, we may replace any coefficient by any number congruent with it to the modulus n. Thus the congruence $13x\equiv 17 \pmod 5$ is identical with $3x\equiv 2$ or with $3x\equiv -3$ or with $x\equiv -1 \pmod 5$.

8. The Linear Congruence. (1) *If a is prime to n, the congruence* $ax\equiv b \pmod n$ *has just one distinct root.*

For if the terms of the arithmetical progression $0, a, 2a, \ldots (n-1)a$ are divided by n, since a is prime to n, the remainders are the numbers $0, 1, 2, \ldots n-1$, in some order or other. Hence there is just one value of x such that $ax\equiv b \pmod n$ and $0\leqslant x<n$.

To solve $ax\equiv b \pmod n$ *when a is prime to n,* let a/n be expressed as a simple continued fraction with an even number of quotients, and let y_0/x_0 be the last convergent but one.

Then $ax_0-ny_0=1$, so that $ax_0\equiv 1 \pmod n$. Therefore $ax\equiv b\equiv abx_0$, and since a is prime to n, $x\equiv bx_0$, which is the required solution.

If a is small, or the product of small primes, it is generally easier to proceed as in Exx. 1-3 below.

(2) *If a is not prime to n, and g is the greatest common divisor of these numbers, there are g incongruent solutions of* $ax\equiv b \pmod n$, *provided that b is divisible by g. If b is not divisible by g, there is no solution.*

For let $a=ga'$, $n=gn'$, so that a' is prime to n'. We require values of x such that $ga'x-b$ is divisible by gn'. No such value exists unless b is divisible by g. If this condition is satisfied and $b=gb'$, then $a'x-b'$ is divisible by n' and the given congruence is equivalent to $a'x\equiv b' \pmod{n'}$.

Since a' is prime to n', the last congruence has one solution $\alpha<n'$, and g solutions $<n$, namely, α, $\alpha+n'$, $\alpha+2n'$, $\ldots$ $\alpha+(g-1)n'$, which are the g distinct roots of $ax\equiv b \pmod n$.

(3) The expression $b/a \pmod n$ is used to denote *any* solution of

$$ax\equiv b \pmod n.$$

If a and n have a common divisor which does not divide b, then $b/a \pmod n$ has no meaning: in every other case it has infinitely many values. If a is prime to n, these values are all congruent (mod n): if a is not prime to n, they are all congruent (mod n/g) where g is the greatest common divisor of a and n.

Such expressions possess many of the properties of ordinary fractions, and it is easy to prove the following:

(i) If $a \equiv a'$ and $b \equiv b' (\mathrm{mod}\, n)$, the expressions $a/b (\mathrm{mod}\, n)$ and $a'/b' (\mathrm{mod}\, n)$ are equivalent.

(ii) $am/bm (\mathrm{mod}\, mn)$ is equivalent to $a/b (\mathrm{mod}\, n)$.

(iii) $ak/bk (\mathrm{mod}\, n)$ is equivalent to $a/b (\mathrm{mod}\, n)$, if k is prime to n.

Ex. 1. *Solve* (i) $5x \equiv 2$ (*mod* 7); (ii) $15x \equiv 6$ (*mod* 21).

(i) Here $5x \equiv 2 - 7 \equiv -5 (\mathrm{mod}\, 7)$, therefore $x \equiv -1 \equiv 6 (\mathrm{mod}\, 7)$.

(ii) Since 3 is a divisor of 15, 6, 21, the congruence is equivalent to $5x \equiv 2 (\mathrm{mod}\, 7)$, so that $x \equiv 6 (\mathrm{mod}\, 7)$ and $x \equiv 6, 13, 20 (\mathrm{mod}\, 21)$.

Ex. 2. *Solve* $36x \equiv 7$ (*mod* 157).

We have $12x \equiv \frac{7}{3} \equiv \frac{-150}{3} \equiv -50$, therefore $x \equiv \frac{-50}{12} \equiv \frac{-25}{6} \equiv \frac{132}{6} \equiv 22 (\mathrm{mod}\, 157)$.

Ex. 3. *Solve* $17x \equiv 11$ (*mod* 43).

We have $\frac{43}{17} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8}}}$ and the third convergent is 5/2, thus

$$1 = 43 \,.\, 2 - 17 \,.\, 5 \equiv -17 \,.\, 5 (\mathrm{mod}\, 43) \quad \text{and} \quad \tfrac{1}{17} (\mathrm{mod}\, 43) \equiv -5.$$

Therefore $x \equiv \frac{11}{17} (\mathrm{mod}\, 43) \equiv -55 \equiv 31 (\mathrm{mod}\, 43)$.

(4) **Simultaneous Congruences.** (i) *It is required to find x so that*

$$x \equiv a (\mathit{mod}\, \alpha) \quad \mathit{and} \quad x \equiv b (\mathit{mod}\, \beta).$$

We have $x = a + \alpha y$ where y is given by

$$a + \alpha y \equiv b (\mathrm{mod}\, \beta) \quad \text{or} \quad \alpha y \equiv b - a (\mathrm{mod}\, \beta).$$

Let g be the greatest common divisor of α and β. If $b - a$ is not divisible by g there is no solution. Otherwise, there is just one value y_1 of y less than β/g, which satisfies the last congruence, and the general value of y is given by $y = y_1 + \beta/g \,.\, t$, so that the general solution is

$$x = x_1 + \alpha\beta/g \,.\, t \quad \text{where} \quad x_1 = a + \alpha y_1.$$

Thus a solution x_1 of the given congruences exists if, and only if, $b - a$ is divisible by g, the greatest common divisor of α, β, and then the congruences are equivalent to the single congruence $x \equiv x_1 (\mathrm{mod}\, l)$, where l is the L.C.M. of α and β.

It should be observed that solutions always exist when α is prime to β.

(ii) *To find x so as to satisfy a number of relations of the form*

$$x \equiv a (\mathit{mod}\, \alpha), \quad x \equiv b (\mathit{mod}\, \beta), \quad x \equiv c (\mathit{mod}\, \gamma), \ldots$$

we replace the first two by $x \equiv x_1 (\mathrm{mod}\, l)$ as in the last section. Taking this with the third, the first three congruences are equivalent to

$$x \equiv x_2 (\mathrm{mod}\, m),$$

where x_2 is any solution (found as before) and m is the L.C.M. of l and γ,

and therefore of α, β, γ. Continuing thus, it will be seen that *the given congruences are together equivalent to* $x \equiv \xi \pmod{L}$ *where* ξ *is any solution and* L *is the* L.C.M. *of* $\alpha, \beta, \gamma, \ldots$. *Moreover, a solution always exists if* $\alpha, \beta, \gamma, \ldots$ *are prime to one another.*

Ex. 1. *Given that* $35x \equiv 4 \pmod{9}$ *and* $55x \equiv 2 \pmod{12}$, *find the general value of* x.

We have $x = \frac{4}{35} \pmod{9} \equiv \frac{4}{-1} \equiv 5 \pmod{9}$, and $x = \frac{2}{55} \pmod{12} \equiv \frac{14}{7} \equiv 2 \pmod{12}$.

Hence $x = 5 + 9y \equiv 2 \pmod{12}$, giving $y = \frac{-3}{9} \pmod{12} \equiv \frac{-1}{3} \pmod{4} \equiv 1 \pmod{4}$.

Thus $y = 1 + 4t$ and $x = 5 + 9(1 + 4t) = 14 + 36t$, which is the general solution.

(5) We can apply these methods to solve a linear congruence when the modulus is a composite number, as in the next example.

Ex. 2. *Solve* $19x \equiv 1 \pmod{140}$.

We have $140 = 4 \cdot 5 \cdot 7$, and therefore $19x \equiv 1$ for the moduli 4, 5 and 7: and since these numbers are prime to one another, any value of x which satisfies these conditions is a solution. Thus we have

$$x \equiv \frac{1}{19} \pmod{4} \equiv \frac{1}{-1} \equiv -1 \pmod{4}, \quad x \equiv \frac{1}{19} \pmod{5} \equiv -1 \pmod{5},$$

and

$$x \equiv \frac{1}{19} \pmod{7} \equiv \frac{8}{-2} \equiv -4 \pmod{7}.$$

Hence $x = -1 + 4y \equiv -1 \pmod{5}$, so that $y = 5z$.

Therefore $x = -1 + 20z \equiv -4 \pmod{7}$ and $z \equiv \frac{-3}{20} \pmod{7} \equiv 3 \pmod{7}$. Thus $z = 3 + 7t$, $y = 15 + 35t$ and $x = 59 + 140t$, which is the general solution.

In this case the solution can be found more easily by the method of § (3), Ex. 3, thus $\frac{140}{19} = 7 + \frac{1}{2+} \frac{1}{1+} \frac{1}{2+} \frac{1}{2}$. The fourth convergent is $\frac{59}{8}$, and $59 \cdot 19 - 140 \cdot 8 = 1$.

Therefore $x \equiv \frac{1}{19} \pmod{140} \equiv 59 \pmod{140}$.

9. A Theorem on Fractions. *If* n *is the product of factors* $a, b, c, d, \ldots l$, *which are prime to one another, and* m *is prime to* n, *the fraction* m/n *can be expressed uniquely in the form*

$$\frac{m}{n} = \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} + \ldots + \frac{\lambda}{l} \pm k, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{A})$$

where $\alpha, \beta, \ldots \lambda, k$ *are positive integers and* $\alpha < a$, $\beta < b$, *etc.*

Let $a' = bcd \ldots l$, then a is prime to a', and therefore integers x, x' can be found so that $a'x + ax' = m$. Moreover, x' is prime to a', for any common factor of these two would divide m, which is prime to a'. Thus we have

$$\frac{m}{aa'} = \frac{x}{a} + \frac{x'}{a'} \quad \text{and} \quad \frac{m}{n} = \frac{x}{a} + \frac{x'}{bcd \ldots l}.$$

Proceeding in this way, we can obtain a relation of the form

$$\frac{m}{n}=\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\ldots\frac{w}{l},$$

where x, y, z, w are integers, and then m/n can be put in the required form by writing $x=\alpha+k_1a$, $y=\beta+k_2b$, etc., where $\alpha, \beta, \ldots$ are the least positive residues of $x, y, \ldots$ with regard to the moduli $a, b, \ldots$ respectively.

Again, if we multiply each side of the equality (A) by n, it will be seen that $\alpha \,.\, bc \ldots l \equiv m \pmod{\alpha}$, $\beta \,.\, ac \ldots l \equiv m \pmod{b}$, etc. Since $bc \ldots l$ is prime to a, α has just one positive value less than a. Similarly β has just one value less than b, and so on. Hence the expression found for m/n is unique.

Ex. 1. *Express* $\frac{101}{1540}$ *in the form* $\frac{x}{4}+\frac{y}{5}+\frac{z}{7}+\frac{u}{11}+k$, *the fractions being positive proper fractions and* k *an integer.*

Here x, y, z, u are determined by

$$\begin{aligned}
&5 \,.\, 7 \,.\, 11x \equiv 101 \pmod 4, && 1 \,.\, (-1)(-1)x \equiv 1, && x \equiv 1 \pmod 4;\\
&4 \,.\, 7 \,.\, 11y \equiv 101 \pmod 5, && (-1) \,.\, 2 \,.\, 1 \,.\, y \equiv 1 \equiv -4, && y \equiv 2 \pmod 5;\\
&4 \,.\, 5 \,.\, 11z \equiv 101 \pmod 7, && (-3)(-2) \,.\, 4z \equiv 3, && z \equiv 1 \pmod 7;\\
&4 \,.\, 5 \,.\, 7u \equiv 101 \pmod{11}, && 4u \equiv 1 \equiv 12, && u \equiv 3 \pmod{11}.
\end{aligned}$$

With $x=1, y=2, z=1, u=3$, we find that $\frac{1}{4}+\frac{2}{5}+\frac{1}{7}+\frac{3}{11}=1\frac{101}{1540}$; so that $k=-1$.

10. The General Congruence. In this article $f(x)$, $\varphi(x)$, ... will denote polynomials with integral coefficients.

(1) If every coefficient of a polynomial $f(x)$ is divisible by n, then $f(x)$ is divisible by n for all values of x, and we say that $f(x)$ is *identically congruent* with zero to the modulus n. This is expressed by writing

$$f(x) \equiv\!\!\!\equiv 0 \pmod n.$$

Again, $f(x)$ is said to be identically congruent with $\varphi(x) \pmod n$ if $f(x)-\varphi(x) \equiv\!\!\!\equiv 0 \pmod n$, which is also written as $f(x) \equiv\!\!\!\equiv \varphi(x) \pmod n$.

(2) **Division (mod n).** If $f(x)$ is divided by $\varphi(x)$, a polynomial of lower degree than $f(x)$, we obtain an identical relation of the form

$$f(x)=Q'(x) \,.\, \varphi(x)+R'(x),$$

where the remainder $R'(x)$ is of lower degree than $\varphi(x)$.

Suppose that any coefficient which occurs in the division is replaced by any number congruent with it (mod n). The process modified in this way is called *division (mod n)*, and leads to an identical congruence of the form

$$f(x) \equiv Q(x) \,.\, \varphi(x)+R(x) \pmod n,$$

where $R(x)$ is of lower degree than $\varphi(x)$.

If $R(x) \equiv 0 \pmod n$, then $f(x)$ is said to be *divisible (mod n)* by $\varphi(x)$.

If $f(x)$ is divisible (mod n) by $(x-\alpha)^r$, the congruence $f(x)\equiv 0 \pmod n$ is said to have r roots equal to α.

Ex. 1. *Prove that* 2, 4, 6 *are roots of* $5x^3+3x^2-4x-2\equiv 0 \pmod 7$.

Division (mod 7) by $x-2$ and $x-4$ in succession is shown on the right. In the third line 13 is replaced by -1 and -6 by $+1$. In the last line 19 is replaced by 5 and 21 by 0. The reckoning shows that

$$5x^3+3x^2-4x-2 \equiv 5(x-2)(x-4)(x+1) \pmod 7,$$

so that the congruence holds when $x=2$, 4 or 6.

$1-2$	$5+\ 3-\ 4-2$
	$\quad 10-\ 2+2$
$1-4$	$5-\ 1+\ 1+0$
	$\quad 20+20$
	$5+\ 5+\ 0$

11. Congruences to a Prime Modulus. The analogy between congruences and equalities is carried a step further by the following theorems, in which the modulus p is a prime.

(1) *If $MN\equiv 0 \pmod p$ where p is a prime, then either $M\equiv 0 \pmod p$, or $N\equiv 0 \pmod p$.* For since MN is divisible by the prime p, either M or N is so divisible.

(2) *If α is a root of $f(x)\equiv 0 \pmod p$, then $f(x)$ is divisible (mod p) by $x-\alpha$.* For by division (mod p) we have $f(x)\equiv Q(x).(x-\alpha)+R$, where R is independent of x. Putting $x=\alpha$ we have $R\equiv f(\alpha)\equiv 0 \pmod p$; therefore $f(x)$ is divisible (mod p) by $x-\alpha$.

(3) *If $f(x)$ is divisible (mod p) by $(x-\alpha)^r$ and by $(x-\beta)^s$ where $\alpha\neq\beta$, then it is divisible (mod p) by $(x-\alpha)^r(x-\beta)^s$.* For any common divisor of $x-\alpha$ and $x-\beta$ is a divisor of $\alpha-\beta$ which is independent of x.

(4) *If $f(x)$ is a polynomial of degree n, the congruence $f(x)\equiv 0 \pmod p$ cannot have more than n roots, any root which occurs r times being counted as r distinct roots.* This follows at once from the preceding theorems.

(5) *If $f(x)\equiv 0 \pmod p$ is a congruence of the n-th degree with n roots and $\varphi(x)$ is a factor of $f(x)$ of degree r, then the congruence $\varphi(x)\equiv 0 \pmod p$ has r roots.* This is an immediate consequence of (4).

(6) *If p is a prime, it follows from Fermat's theorem that the roots of the congruence $x^{p-1}-1\equiv 0 \pmod p$ are* 1, 2, 3, ... , $p-1$, *and therefore*

$$x^{p-1}-1 \equiv (x-1)(x-2)\dots(x-p+1) \pmod p.$$

Hence if a_r denotes the sum of the products r together of 1, 2, 3, ... $p-1$, it follows that $a_1x^{p-2}-a_2x^{p-3}+\dots+a_{p-2}x-\{\lfloor p-1+1\}\equiv 0 \pmod p$.

Hence $a_r\equiv 0 \pmod p$ for $r<p-1$, which is Lagrange's theorem (Art. 6); also $\lfloor p-1+1\equiv 0 \pmod p$, which is Wilson's theorem.

Ex. 1. *Solve* $x^2+x+1\equiv 0 \pmod 7$.

Here $4x^2+4x+4\equiv 0$, or $(2x+1)^2\equiv -3\equiv 4$; and $2x+1\equiv \pm 2$. Hence $x\equiv 2$ or $4 \pmod 7$.

In treatises on the Theory of Numbers the notion of congruence is extended to complex numbers, thus completing the analogy referred to at the head of this article.

12. Reduction of a Congruence. Let $f(x) \equiv 0 \pmod{p}$ be a congruence of the nth degree to a prime modulus p. Let the roots of the congruence be $\alpha_1, \alpha_2, \ldots \alpha_r$, where some of these may be multiple roots.

By a modified H.C.F. *process we can find a congruence* $R(x) \equiv 0 \pmod{p}$ *of degree r of which the roots are* $\alpha_1, \alpha_2, \ldots \alpha_r$.

For the roots of $x^{p-1} - 1 \equiv 0 \pmod{p}$ are $1, 2, 3, \ldots p-1$, hence every solution of $f(x) \equiv 0 \pmod{p}$ is a solution of $x^{p-1} - 1 \equiv 0 \pmod{p}$.

Let the H.C.F. process be carried out for the functions $f(x)$ and $x^{p-1} - 1$ with the modification that any coefficient may be replaced by any number congruent with it (mod p); then the last divisor $R(x)$ is the highest common divisor (mod p) of $f(x)$ and $x^{p-1} - 1$.

The truth of this statement depends on the following. Let u, v be any two consecutive remainders in the process. If w is the next remainder, $w = au + bv$ where neither a nor b is divisible by p. Hence the common divisors (mod p) of u and v are the same as those of v and w.

It follows that $R(x) \equiv 0 \pmod{p}$ is the congruence referred to above.

Ex. 1. *Show that the congruence* $3x^4 - 2x^3 - 4x^2 - x - 1 \equiv 0 \pmod{7}$ *has only one distinct root and solve it.*

Carrying out the modified H.C.F. process for the left-hand side and $x^6 - 1$, it is found that the last divisor is $x + 2$, showing that -2 is the *only* distinct root.

By division (mod 7) we find that $3x^4 - 2x^3 - 4x^2 - x - 1 \equiv 3(x+2)^2(x^2 - 3) \pmod{7}$.

Now $x^2 \equiv 3 \pmod{7}$ has no solution, hence -2 is the only distinct root.

EXERCISE XLIV

1. Find the form of x, if

(i) $29x \equiv 1 \pmod{13}$; (ii) $2x \equiv 3 \pmod{7}$ and $3x \equiv 5 \pmod{11}$;

(iii) $8x \equiv 1 \pmod{7}$, $3x \equiv 4 \pmod{11}$, and $7x \equiv 3 \pmod{20}$.

2. Solve (i) $78x \equiv 1 \pmod{179}$; (ii) $78x \equiv 13 \pmod{179}$.

3. Solve (i) $16x \equiv 31 \pmod{1217}$; (ii) $30x \equiv 31 \pmod{1861}$;
(iii) $15x \equiv 28 \pmod{1009}$; (iv) $26x \equiv 35 \pmod{1901}$.

4. If $x \equiv a \pmod{22}$ and $x \equiv b \pmod{40}$, show that $a - b$ is even and $x \equiv 100a - 99b \pmod{440}$.

5. If $x \equiv a \pmod{16} \equiv b \pmod{5} \equiv c \pmod{11}$, prove that
$$x \equiv 385a + 176b - 560c \pmod{880}.$$

6. If $x \equiv a \pmod{7} \equiv b \pmod{11} \equiv c \pmod{13}$, prove that
$$x \equiv -286a + 364b - 77c \pmod{1001}.$$

7. If $x \equiv a \pmod 3 \equiv b \pmod 5 \equiv c \pmod 7$, prove that

$$x \equiv -35a + 21b + 15c \pmod{105}.$$

8. Express $\frac{1}{1760}$ in the form $\frac{x}{32} + \frac{y}{5} + \frac{z}{11} \pm k$, where x, y, z, k are positive integers and $x < 32$, $y < 5$, $z < 11$.

9. Express $\frac{1}{1001}$ in the form $\frac{x}{7} + \frac{y}{11} + \frac{z}{13} \pm k$, where x, y, z, k are positive integers and $x < 7$, $y < 11$, $z < 13$.

10. Find the form of numbers, of which the first, second and third powers are of the forms $3n+1$, $4n+1$ and $5n+1$ respectively.

11. If $n^2 - 2$ is divisible by 7, and $n^2 - 3$ is divisible by 11, find the form of n.

12. If $x^2 + 4x + 2$ is divisible by 23, find the form of x.

13. If n^2 and $(n+1)^2$ are of the forms $11m+4$ and $12m+4$, find the form of n.

14. Find four consecutive numbers divisible by 5, 7, 9, 11 respectively.

15. Prove that the congruence $3x^4 - 4x^3 - x^2 + 3x - 4 \equiv 0 \pmod 7$ has only two incongruent solutions, and find them.

16. Prove that the congruence $4x^4 \equiv 5x^3 + x^2 + 2x + 5 \pmod{11}$ has four incongruent solutions, and find them.

17. Prove that the congruence $11x^9 + 1 \equiv 0 \pmod{29}$ has only one distinct solution, and find it.

18. *Bicycle gear as a revolution counter.*

A bicycle with no free wheel has

19 teeth in the back chain-wheel,
62 teeth in the front chain-wheel,
121 links in the chain,

and the circumference of the back wheel of the bicycle is known to be almost exactly 7 feet.

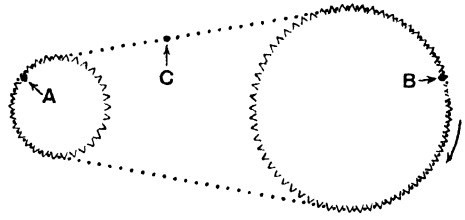


FIG. 64.

Prove the following rule to find the number of revolutions (R) of the back wheel (and so the distance travelled) in any journey of not more than 10 miles.

Mark a tooth A in the back chain-wheel, a tooth B in the front chain-wheel, and a link C in the chain. At the start count the teeth or links from A to B and from A to C in the direction of turning. At the finish count these again.

Take the increase from A to B or 62 minus the decrease, and denote it by a.

Take the increase from A to C or 121 minus the decrease, and denote it by b.

Then the number R is the remainder in the division $(3025a + 2108b) \div 7502$.

[If x is the number of revolutions of the back wheel,

$$19x \equiv a \pmod{62} \quad \text{and} \quad 19x \equiv b \pmod{121}.$$

Hence show that $x \equiv 49a \pmod{62}$ and $x \equiv 51b \pmod{121}$,

and therefore $x \equiv 3025a + 2108b \pmod{7502}$.]

CHAPTER XXVII

RESIDUES OF POWERS OF A NUMBER, RECURRING DECIMALS

Residues of Powers of a Number.

1. *If a and g are prime to any modulus n, the least positive residues of successive terms of the geometrical progression $a, ag, ag^2, \ldots$ recur, and if t is the number of terms in the recurring period, $t \leqslant \varphi(n)$ and is independent of a.*

Denote the residues by r_0, r_1, r_2, etc. Each of these is one of the $\varphi(n)$ numbers $<n$ and prime to it, for since a and g are prime to n, so is every residue. Hence, of the numbers $r_0, r_1, r_2, \ldots r_{\varphi(n)}$, at least two, say r_s and r_{s+t}, are equal. Therefore $ag^s \equiv ag^{s+t} \pmod{n}$; and, since a is prime to n, hence $g^t \equiv 1 \pmod{n}$. It follows that $ag^t \equiv a$, $ag^{t+1} \equiv ag$, $ag^{t+2} \equiv ag^2$, etc., and so $r_t = r_0$, $r_{t+1} = r_1$, $r_{t+2} = r_2$, etc., where $t \leqslant \varphi(n)$.

Moreover, *the number of terms in the period is the least index for which* $g^t \equiv 1 \pmod{n}$.

It should be noticed that: (i) No two residues in the period are equal. (ii) If $r_\mu = 1$, then $\mu \equiv 0 \pmod{t}$, and conversely. (iii) If $r_\mu = r_\nu$, then $\mu \equiv \nu \pmod{t}$, and conversely. (iv) $r_{\mu+1} \equiv g r_\mu \pmod{n}$. (v) If $n-1$ occurs as a residue, the subsequent residues are $n - r_1$, $n - r_2$, $n - r_3$, etc.; for they are congruent (mod n) with $-g$, $-g^2$, $-g^3$, etc.

Thus *for the modulus* 19, the residues of $2^0, 2^1, 2^2, \ldots$ are

Index.	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 ... .
Residue.	1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1 ... .

Having found $r_4 = 16$, we have $r_5 \equiv 16 \times 2 \equiv 13 \pmod{19}$. Again, $r^9 = 18 = 19 - 1$, and the subsequent residues are $19 - 2$, $19 - 4$, etc.

Here $t = 18 = \varphi(19)$, and *every* number not divisible by 19 is congruent (mod 19) with some power of 2.

For the modulus 13, the residues of $5^0, 5^1, 5^2, \ldots$ are

Index.	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
Residue.	1, 5, 12, 8, 1, 5, 12, 8, 1, 5, 12, 8, 1.

Here $t = 4$, which is a divisor of $\varphi(13) = 12$.

2. *If g is prime to n and t is the least index other than zero for which $g^t \equiv 1 \pmod{n}$, then $t = \varphi(n)$ or is a divisor of $\varphi(n)$.*

For $g^{\varphi(n)} \equiv 1 \pmod{n}$, so that t may be equal to $\varphi(n)$, and by the preceding $t \leqslant \varphi(n)$. If $t < \varphi(n)$, let $\varphi(n) = at + b$, where $0 \leqslant b < t$, then

$$g^{\varphi(n)} = g^{at+b} = (g^t)^a \cdot g^b \equiv g^b \pmod{n}.$$

Therefore $g^b \equiv 1 \pmod{n}$, and consequently $b = 0$, for $b < t$. Hence, if $t < \varphi(n)$, it is a divisor of $\varphi(n)$.

Second proof. Let the least positive residues of $g^0, g^1, g^2, \ldots g^{t-1}$ be

$$a_0, a_1, a_2, \ldots a_{t-1}. \quad \ldots\ldots\ldots\

3. An Odd Prime Modulus: Primitive Roots. Let p be an odd prime, g any number not divisible by p, and t the least index for which $g^t \equiv 1 \pmod{p}$, then g is said to *belong to the index t.*

Since $\varphi(p) = p-1$, t is equal to or is a divisor of $p-1$. If $t = p-1$, so that the period of residues includes every number less than p, the period is said to be *complete* and g is called a *primitive root of the modulus p.* If t is a divisor of $p-1$, the period is *incomplete* and g may be called a *subordinate root.*

Hence *if g is a primitive root of p, every number prime to p is congruent with some power of g.*

It can be proved that, for any odd prime modulus, primitive roots exist; for the present we assume this to be the case. When one has been found, the others can be found as in the following illustration.

Illustration. In the margin, for the modulus 13 the least positive residues of $2^0, 2^1, 2^2, \ldots 2^{11}$ are written down round the circumference of a circle.

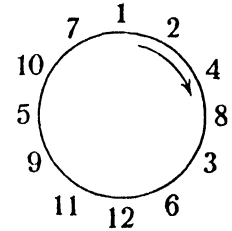


FIG. 65.

Every number less than 13 occurs, and so 2 is a primitive root of 13.

Hence, without further reckoning, we can write down the residues (mod 13) of powers of *any* number.

Thus for powers of 6, since $6 \equiv 2^5$, $6^2 \equiv 2^{10}$, etc., the residues of $6^0, 6^1, 6^2, \ldots$ are found by starting with 1 and taking every *fifth* number. The residues are

1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11.

Every number less than 13 appears, the reason being that 5 is prime to 12.

Hence 6 is a primitive root; and, in the same way, if μ is any number less than 12 and prime to it, 2^μ is congruent (mod 13) with a primitive root.

Since $\varphi(12) = 4$, there are *four* primitive roots of 13; they are congruent with $2^1, 2^5, 2^7, 2^{11}$, and are equal to 2, 6, 11, 7, the first, fifth, seventh and eleventh numbers in order round the circle, *counting from* 1.

Again, we have $5 \equiv 2^9$, and the residues of $5^0, 5^1, 5^2, \ldots$ are obtained by counting every ninth number, or every third number in the reverse order.

Since 3 is the greatest common divisor of 9 and 12 and $12 = 3 \cdot 4$, after four counts we reach the number, 5, from which we started.

Thus 5 is a subordinate root; it belongs to the index 4, and the period is 1, 5, 12, 8.

4. If p is an odd prime, g any number >1 and not divisible by p, t the least index for which $g^t \equiv 1 \pmod p$ and $r_0, r_1, r_2, \ldots$ the least positive residues of $g^0, g^1, g^2, \ldots$, then

$$\begin{aligned} r_0 + r_1 + r_2 + \ldots + r_{t-1} &\equiv 1 + g^1 + g^2 + \ldots + g^{t-1} \\ &\equiv (g^t - 1)/(g-1) \bmod p \\ &\equiv 0 \pmod p, \end{aligned}$$

that is to say, *the sum of the residues which form the period is divisible by* p.

Thus if $p=13$ and $g=5$, then $t=4$ and

$$r_0 + r_1 + r_2 + r_3 = 1 + 5 + 12 + 8 = 13 \times 2.$$

Again, suppose that e is a factor of t (not t itself) and $t=ef$. Starting with any residue r_m and taking every eth residue, we have the sequence

$$r_m, \quad r_{m+e}, \quad r_{m+2e}, \ \ldots .$$

These residues recur, and they form a sub-period containing f *residues;* for $g^x \equiv 1 \pmod p$ if $x=ef$, and for no smaller value of x. Also

$$\begin{aligned} r_m + r_{m+e} + r_{m+2e} + \ldots + r_{m+(f-1)e} &\equiv g^m (1 + g^e + g^{2e} + \ldots + g^{(f-1)e}) \pmod p \\ &\equiv g^m (g^{ef} - 1)/(g^e - 1) \pmod p \\ &\equiv 0 \pmod p. \end{aligned}$$

Hence *the sum of the residues forming the sub-period is divisible by* p.

Thus if $p=19$ and $g=2$ (See Art. 1, (v)), then $t=18$ and

$$r_0 + r_2 + r_4 + \ldots + r_{16} = 1 + 4 + 16 + 7 + 9 + 17 + 11 + 6 + 5 = 4 \times 19,$$

$$r_1 + r_4 + r_7 + r_{10} + r_{13} + r_{16} = 2 + 16 + 14 + 17 + 3 + 5 = 3 \times 19.$$

5. Recurring Decimals. Let $m<n$ and prime to n, then

(1) *If* n *is prime to* 10, m/n *is equal to a pure recurring decimal with a period of* t *figures, where* t *is the least index other than zero for which* $10^t \equiv 1 \pmod n$; *so that* $t = \varphi(n)$ *or is a divisor of* $\varphi(n)$.

For the remainders $r_1, r_2, r_3, \ldots$, in the process of expressing m/n as a decimal, are the least positive residues (mod n) of m, $10m$, $10^2m, \ldots$. Hence, by Arts. 1 and 2, the remainders recur, with a period of t figures.

Also if $m/n = 0{\cdot}\dot{a}_1 a_2 \ldots \dot{a}_t$, then $r_s/n = 0{\cdot}\dot{a}_{s+1} a_{s+2} \ldots a_t a_1 \ldots \dot{a}_s$, so that all the fractions $m/n, r_1/n, r_2/n, \ldots$ have the same period. It follows that

(i) If $t=\varphi(n)$ and m is any number $<n$ and prime to it, all the fractions m/n have the same period.

(ii) If $\varphi(n) = et$ where $e>1$, the fractions m/n can be arranged in e groups of t fractions, such that the periods are the same for fractions in the same group and different for those in different groups.

This follows from the reasoning in the last part of Art. 2.

(iii) *If* $n=p$, *any prime except* 2 *or* 5, *then* $t=p-1$, *or* t *is a divisor of* $p-1$, *according as* 10 *is or is not a primitive root of* p.*

If $n=p^r$, then $\varphi(n)=p^{r-1}(p-1)$ and t is equal to or is a divisor of $p^{r-1}(p-1)$. The case in which $n=p^2$ deserves further consideration.

(iv) *If* p *is a prime, not* 2 *or* 5, *and* $1/p$ *has a period of* t *figures, then* $1/p^2$ *has a period of* t *or* pt *figures.*

For let T be the number of figures in the period of $1/p^2$, and let

$$F(t)=1+10^t+10^{2t}+\ldots+10^{(p-1)t}.$$

Then $F(t)\equiv 0 \pmod p$, for $10^t\equiv 1 \pmod p$, and therefore

$$10^{pt}-1=(10^t-1)F(t)\equiv 0 \pmod{p^2}.$$

Hence T is pt, or a divisor of pt, and, since p is a prime, $T=t$ or $p\tau$, where τ is t, or a divisor of t.

If $\tau<t$, we have

$$1+10^\tau+10^{2\tau}+\ldots+10^{(p-2)\tau}=(10^{(p-1)\tau}-1)/(10^\tau-1)\equiv 0 \pmod p,$$

for 10^τ is not $\equiv 1$. Hence $F(\tau)\equiv 10^{(p-1)\tau}\equiv 1 \pmod p$.

Now $$10^{p\tau}-1=(10^\tau-1)F(\tau),$$

and therefore $10^{p\tau}-1$ is not $\equiv 0 \pmod{p^2}$.

Consequently τ is not $<t$ and $T=t$ or pt.

NOTE. The only known cases in which $1/p^2$ has a period of t figures are when $p=3$ or 487.

(2) *If* $F=\dfrac{m}{2^\mu 5^\nu}<1$, *where* m *is prime to* 10, *and* k *is the greater of the two* μ, ν, *then* F *is a terminating decimal with* k *figures.*

For 10^kF is an integer with not more than k digits, the last of which is not zero.

(3) *If* $F=\dfrac{m}{2^\mu 5^\nu n}<1$, *where* m, n *and* 10 *are prime to one another, then* F *is a mixed recurring decimal, with a period of* t *figures, where* t *is the number of figures in the period of* $1/n$.

For let k be the greater of the two μ, ν, then $10^kF=m'/n$, where m' is prime to n and less than 10^k. Therefore $10^kF=I+f$, where I is an integer of not more than k digits and f is a pure recurring decimal with a period of t figures.

(4) *If* $F=\dfrac{m}{p^a q^b r^c \ldots}$ *where* p, q, r, $\ldots$ *are different primes other than* 2 *and* 5 *and* m *is prime to* p, q, r, $\ldots$, *then if* t, t', t'' $\ldots$ *are the numbers of figures in the periods of the decimal equivalents of* F, $1/p^a$, $1/q^b$, *etc., respectively,* t *is the* L.C.M. *of* t', t'', t''', *etc.*

* The primes less than 100 of which 10 is a primitive root are 7, 17, 19, 23, 29, 47, 59, 61, 97.

For let $n = p^a q^b \dots$, then $10^t \equiv 1 \pmod{n}$, therefore $10^t \equiv 1 \pmod{p^a}$ and, since t' is the least index such that $10^{t'} \equiv 1 \pmod{p^a}$, t must be a multiple of t'; similarly, t is a multiple of t'', t''', etc.

Therefore t is a common multiple of t', t'', etc.

Again, if τ is *any* common multiple of t', t'', ... we have $10^\tau \equiv 1$ for each of the moduli p^a, q^b, ..., and since p^a, q^b, ... are prime to one another, $10^\tau \equiv 1 \pmod{n}$.

It follows that t is the *least* common multiple of t', t'', etc.

Given a table showing the periods of decimals equivalent to fractions $1/n$, where n is a prime or a power of a prime, we can easily express *any* fraction as a decimal by using the theorem of Ch. XXVI, 9.

5. In practice, the following theorem is useful.

Let $m/n = 0 \cdot a_1 a_2 a_3 \dots$ where n is prime to 10 and m prime to n. If, after r steps of division, the remainder $n - m$ occurs, the subsequent figures of the decimal are $9 - a_1$, $9 - a_2$, $9 - a_3$, ..., and the period contains $2r$ figures.

For let $m, m_1, m_2, \dots$ be the least positive residues of $m, 10m, 10^2m, \dots$, then

$$m_r = n - m \equiv -m \pmod{n}.$$

Hence the subsequent residues are congruent with $-m_1$, $-m_2$, etc., and are equal to $n - m_1$, $n - m_2$, etc.

Moreover, a_{r+1} is the greatest integer in $10(n-m)/n$;

hence $$a_{r+1} = I(10 - a_1 + f) = 9 - a_1,$$

where f is a positive proper fraction. Similarly $a_{r+2} = 9 - a_2$, etc.

Ex. 1. To express 1/73 as a decimal, the division need not be carried beyond the stage shown in the reckoning, for since the remainder $72 (= 73 - 1)$ occurs, the subsequent remainders are $73 - 10$, $73 - 27$, etc., the corresponding figures in the decimal being $9 - 0$, $9 - 1$, etc. Thus

$$\begin{array}{r|l} 73 & 100 \,(\cdot 0136 \\ & \underline{270} \\ & \underline{510} \\ & \underline{720} \end{array}$$

$$1/73 = 0 \cdot \dot{0}1369\dot{8}6\dot{3}.$$

Again, the number of figures in the period is 8 and $\varphi(73) = 72$. Hence for values of m less than 73, the fractions $m/73$ can be arranged in $72/8 = 9$ groups such that the period is the same for those in the same group. By the preceding, the complete set of remainders for 1/73 is

1, 10, 27, 51, 72, 63, 46, 22, ...(A)

and one group is obtained by giving m these values. To get a second group, choose any number less than 73 not included in the set (A), say 2. The remainders for 2/73 are congruent (mod 73) with 2×1, 2×10, 2×27, etc., and are therefore

2, 20, 54, 29, 71, 53, 19, 44. .....(B)

A second group is obtained by giving m these values. A third group can be found in the same way by choosing a number less than 73 and not included in the sets (A), (B)—and so on.

EXERCISE XLV

NOTE. The prime factors of $f(t)=10^t-1$ for $t=1, 2, 3, \ldots 9$ are given in the following table:

$f(1)=3^2$, $f(2)=3^2 \cdot 11$, $f(3)=3^3 \cdot 37$,
$f(4)=3^2 \cdot 11 \cdot 101$, $f(5)=3^2 \cdot 41 \cdot 271$, $f(6)=3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$,
$f(7)=3^2 \cdot 239 \cdot 4649$, $f(8)=3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137$, $f(9)=3^4 \cdot 37 \cdot 333667$.

1. For the modulus 19, given the least positive residues of 2^0, 2^1, 2^2, etc. (Art. 1), write down those of 10^0, 10^1, 10^2, etc., and 7^0, 7^1, 7^2, etc. What are the primitive roots of 19?

2. For the modulus 17, find the least positive residues of 3^0, 3^1, 3^2, etc. What are the primitive roots of 17?

3. If p is an odd prime, g belongs to the index t, and r_0, r_1, r_2, etc., are the least positive residues of g^0, g^1, g^2, etc., prove that

(i) $p-1$ occurs as a residue if, and only if, t is even.

(ii) $r_0 r_1 r_2 \ldots r_{t-1} \equiv -1$ or $+1 \pmod p$ according as t is even or odd. Verify when $p=13$, $g=2, 3, 5$.

(iii) If g is a primitive root of p, show that theorem (ii) becomes Wilson's theorem.

[(i) If t is even, $g^{t/2} \equiv -1 \pmod p$.

(ii) $r_0 r_1 r_2 \ldots r_{t-1} \equiv g^s \pmod p$, where $s=0+1+2+\ldots+(t-1)$.]

RECURRING DECIMALS

Here n is prime to 10, p is a prime not 2 or 5, t is the number of figures in the period of $1/n$ or $1/p$; r_0, r_1, r_2, etc., are the least positive residues of 10^0, 10^1, 10^2, etc. (mod n or p), and we write $\frac{1}{n}$ or $\frac{1}{p}=0 \cdot a_1 a_2 a_3 \ldots$.

4. Prove that $0 \cdot \dot{a}_1 a_2 \ldots \dot{a}_t = a_1 a_2 \ldots a_t / (10^t - 1)$ where $a_1 a_2 \ldots a_t$ is the number of which the digits are $a_1, a_2, \ldots$.

[The decimal $= a_1 a_2 \ldots a_t \left(\frac{1}{10^t} + \frac{1}{10^{2t}} + \frac{1}{10^{3t}} + \ldots \right)$.]

5. If $\frac{1}{n}=0 \cdot a_1 a_2 \ldots a_r \ldots$ and $\frac{n-1}{n}=0 \cdot b_1 b_2 \ldots b_r \ldots$, prove that

$$a_1+b_1=a_2+b_2=\ldots a_r+b_r=\ldots .$$

6. If $1/n=0 \cdot \dot{a}_1 a_2 \ldots \dot{a}_t$ and r is the remainder immediately preceding the remainder 1 in the division, then r and a_t are the least positive integers x, y such that

$$10x-ny=1$$

and

$$a_1 a_2 \ldots a_t \times r = a_t a_1 a_2 \ldots a_{t-1},$$

where the a's are digits of the numbers indicated. Having found r and a_t, all the figures in the period may be found as in Ex. 8.

[$1/n=0 \cdot a_1 a_2 \ldots a_t a_1 \ldots$, $r/n=0 \cdot a_t a_1 \ldots a_{t-1} \ldots$; explain why no figure has to be carried when a_1 is multiplied by r.]

7. If p is a prime and $1/p = 0\cdot\dot{a}_1a_2 \ldots a_r a_{r+1} \ldots \dot{a}_{2r}$, prove that

$$a_{r+1} + a_1 = a_{r+2} + a_2 = \ldots = a_{2r} + a_r = 9,$$

and give an example to show that such relations do not hold if p is a composite number.

8. *Express* $\frac{1}{13}$ *as a decimal by using the equality* $10 \times 4 - 13 \times 3 = 1$.

(i) Put down 3 (the last figure of the period) and multiply by 4 as follows:

$$\begin{array}{r} \cdot \quad 076923 \\ 4 \end{array}$$

$4 \times 3 = 12$, put 2 to the left of 3 and carry 1.
$4 \times 2 + 1 = 9$, put 9 to the left of 2.
$4 \times 9 = 36$, put 6 to the left of 9 and carry 3.
$4 \times 6 + 3 = 27$, put 7 to the left of 6 and carry 2.
$4 \times 7 + 2 = 30$, put 0 to the left of 7.

If the process is continued, the figures recur and $\frac{1}{13} = 0\cdot\dot{0}76923\dot{}$.

(ii) Or thus—in Ex. 7, $a_t a_1 a_2 \ldots a_{t-1}/r = a_1 a_2 \ldots a_t$.
Hence the following: Put down 3 and divide by 4 as below:

$$4\,)\underline{3}$$
$$076923.$$

4 into 3 is 0, which is the first figure of the period.
4 into 30 is 7 and 2 over. Put 7 to the right of 0.
4 into 27 is 6 and 3 over. Put 6 to the right of 7.
4 into 36 is 9. Put 9 to the right of 6.
4 into 9 is 2 and 1 over. Put 2 to the right of 9.
4 into 12 is 3. Put 3 to the right of 2.
4 into 3 is 0, and the figures recur.

9. Cases in which $1/p$ can be quickly expressed as a decimal by the methods of Ex. 8 are $p = 17, 19, 23, 29, 43, 59, 79, 89$.

10. Justify the following:

If
$$\begin{array}{l} A = 0588 \\ B = \quad 2352 \\ C = \quad\quad 9408 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$
then
$$\frac{1}{17} = \cdot 0588235294 \ldots$$

$$\begin{array}{r} 17\,)\underline{100}\,(\cdot 0588 \\ \underline{150} \\ \underline{140} \\ 4 \end{array}$$

where $B = 4A$, $C = 4B$, etc.

[The division shows that $10^4 = 588 \times 17 + 4$;

$$\therefore \frac{1}{17} = \frac{588}{10^4}\left(1 - \frac{4}{10^4}\right)^{-1} = 0\cdot 0588\left(1 + \frac{4}{10^4} + \frac{4^2}{10^8} + \ldots\right).]$$

11. By the method of Ex. 10 show that

$$\frac{1}{31} = 0\cdot\dot{0}32258064516129\dot{}.$$

[After 6 steps of division, the remainder 2 occurs.]

12. Show that for values of $m<41$ the fractions $m/41$ can be arranged in 8 groups such that all the fractions in any group have the same period. What are the values of m for the group to which 2/41 belongs ?

13. Prove that $\frac{1}{81}=0\cdot\dot{0}12345679\dot{0}$, and show that the values of m for which $m/81$ has this period are given by $m\equiv 1 \pmod 9$.

14. If n is any number prime to 10, prove that a number n' can be found such that every digit in the product nn' is 1.

Find n' when $n=41$ and when $n=123$.

[Express $\frac{1}{n}$ or $\frac{1}{9n}$ as a decimal, according as n is or is not prime to 3.]

15. If n is a prime or a power of a prime, prove that the values of n for which the decimal equivalent of $1/n$ has a period of t figures are as follows :

t	1	2	3	4	5	6	7	8	9
n	3, 9	11	27, 37	101	41, 271	7, 13	239, 4649	73, 137	333667

[The values of n for which $1/n$ has a period of t figures are included among the divisors of 10^t-1. See note at the head of this Exercise.]

16. Show that the only prime p for which the decimal equivalent of $1/p$ has (i) a period of 10 figures is 9091, and (ii) a period of 12 figures is 9901.

Also express 1/9091 and 1/9901 as decimals, explaining why only 5 and 6 steps of division respectively are necessary.

6. Primitive Roots (continued). The process of finding a primitive root is one of trial. The orthodox methods of shortening the reckoning are given in another volume ; but in the example below we give a method, founded on the assumption of Art. 3, which is usually successful. For convenience in solving the congruences in Ex. XLVI, we here give *the least primitive root, g, for any prime modulus, p, less than* 100.

$g=2$, $p=3,\ 5,\ 11,\ 13,\ 19,\ 29,\ 37,\ 53,\ 59,\ 61,\ 67,\ 83$;

$g=3$, $p=7,\ 17,\ 31,\ 43,\ 79,\ 89$;

$g=5$, $p=23,\ 47,\ 73,\ 79$; $g=6$, $p=41$; $g=7$, $p=71$.

Ex. 1. *Find the primitive roots of* 67.

If 1/67 is converted into a decimal fraction, the reckoning shows that 10 is a subordinate root of modulus 67, with a period of 33 figures. Art. 3 suggests that this period consists of every alternate figure of the period of *some* primitive root, g ; and, consequently, that $g^2\equiv 10 \pmod{67}$ will lead to a primitive root.

We find that a value of g is ± 12 ; and, converting 12/67 into a decimal fraction, we obtain, as remainders in the process, the residues of the odd powers of 12. When these are interpolated between the remainders in the reckoning for 1/67, we find that all the numbers less than 67 are included ; thus 12 is a primitive root, and its period is

1, *12*, 10, *53*, 33, *61*, 62, *7*, 17, *3*, 36, *30*, 25, *32*, 49, *15*, 21, etc.

From this, as in Art. 3, all the other primitive roots can be immediately written down, together with their periods ; thus we find that other primitive roots are 61, 7, 32, 51, 41, 13, 2, etc., these being the residues of 12^k, where k is prime to 66.

7. The Congruence $x^n \equiv a \pmod{p}$. It is supposed that p is an odd prime.

(1) *If n is prime to $p-1$, there is a single solution.*

For, if $x^n \equiv a$, then $x^{n\lambda} \equiv a^\lambda$; and, since n is prime to $p-1$, there is, by Chap. XXV, 3, a single value of λ, less than $p-1$, such that

$$\lambda n \equiv 1 \pmod{p-1}.$$

If λ has this value, $x^{n\lambda} = x \,.\, (x^{p-1})^k \equiv x \pmod{p}$; and thus, $x \equiv a^\lambda \pmod{p}$, which is the only solution.

Ex. 1. *Solve the congruence,* $x^{13} \equiv 21$ (*mod* 23).

Here $x^{13\lambda} \equiv 21^\lambda \pmod{23}$; and $13 \,.\, 17 \equiv 1 \pmod{22}$; hence we have

$$x \equiv 21^{17} \equiv (-2)^{17} \equiv (-2) \,.\, (256)^2 \equiv (-2) \,.\, (3)^2 \equiv 5 \pmod{23}.$$

(2) *If n is not prime to $p-1$, there are several solutions, or there is no solution.*

Let d be the highest common divisor of n and $p-1$; and let g be a primitive root of p. Then, since the period of g contains all the numbers less than p, we can find r, such that $g^r \equiv a \pmod{p}$.

Let $x \equiv g^s \pmod{p}$; then $g^{ns} \equiv a \equiv g^r \pmod{p}$, and therefore

$$ns \equiv r \pmod{p-1}. \quad \text{.....(A)}$$

The congruence (**A**), and therefore also the congruence $x^n \equiv a$, has d solutions, or no solution, according as r is, or is not, divisible by d; for d is a factor of n and $p-1$ (Chap. XXVI, 8). Also, the solutions of $x^n \equiv a \pmod{p}$ are given by $x \equiv g^s \pmod{p}$, where s has any value which satisfies (**A**).

Ex. 2. *Solve the congruence,* $x^{15} \equiv 31$ (*mod* 37).

A primitive root of 37 is 2; and we find that $2^9 \equiv 31$. Hence, if $x \equiv 2^s$, then $2^{15s} \equiv 2^9$, and $15s \equiv 9 \pmod{36}$; hence $s \equiv 3, 15, 27 \pmod{36}$, and

$$x = 2^3,\ 2^{15},\ 2^{27}, \text{ i.e. } 8,\ 23,\ 6.$$

EXERCISE XLVI

Solve the congruences in Exx. 1-6:

1. $x^3 \equiv 1 \pmod{19}$.
2. $x^5 \equiv 2 \pmod{17}$.
3. $x^7 \equiv 8 \pmod{13}$.
4. $x^{11} \equiv 5 \pmod{31}$.
5. $x^6 \equiv 31 \pmod{41}$.
6. $x^{21} \equiv 2 \pmod{31}$.
7. Find the general solution in integers of $x^2 = 19y + 5$.
8. Find three solutions in integers of $x^3 = 73y + 3$.
9. Show, without calculation, that a primitive root of 237 will satisfy one at least of the congruences, $g \equiv 10$, $g^2 \equiv 10$, $g^4 \equiv 10$.

CHAPTER XXVIII

NUMERICAL SOLUTION OF EQUATIONS

1. The Problem under consideration is *to find the actual or approximate values of the real roots of any equation with numerical coefficients.*

This question is quite distinct from that of finding an algebraical solution. In fact, no algebraical solution of the general equation of the fifth or higher degree has been discovered.

If any rational or multiple roots exist they can be found and removed from the equation (Ch. XVIII, 1, and VI, 14). It may be convenient not to remove the multiple roots.

Thus we are only concerned with irrational roots. The usual procedure is as follows :

(i) We find an interval which contains *all* the roots. Ways of doing this have been given in Ch. VI, 12. A method due to Newton which involves more calculation, but which yields closer limits, is given in the next article.

(ii) We *separate* the roots, that is to say we find intervals each of which contains a single root or a multiple root.

(iii) Taking any interval which contains a single root, by a process of approximation, we find smaller and smaller intervals which contain the root.

2. Newton's Method of finding an Upper Limit to the Roots. This depends on the following theorem :

If h is a number such that $f(x)$ and all its derivatives are positive when $x=h$, then h is an upper limit to the roots of $f(x)=0$.

For if n is the degree of $f(x)$, then

$$f(x+h)=f(h)+xf'(h)+\ldots+\frac{1}{\lfloor n}x^n f^{(n)}(h).$$

Hence if $f(h)$, $f'(h)$, ... $f^{(n)}(h)$ are all positive, $f(x+h)>0$, for $x\geqslant 0$, that is to say $f(x)>0$ for $x\geqslant h$, which proves the theorem.

NOTE. In applying this theorem, we suppose that the coefficient of the highest power of x in $f(x)$ is positive, so that $f^{(n)}(x)>0$.

We then find the least integer h_1 such that $f^{(n-1)}(x)>0$ when $x=h_1$. If this value of x does not make $f^{(n-2)}(x)>0$, we find by trial a greater integer h_2 such that

$$f^{(n-2)}(h_2)>0.$$

Continuing thus, we can find the least integer h which makes $f(x)$ and all its derivatives positive.

It should be observed that if we have found a number h_r such that

$$f^{(n)}(x),\ f^{(n-1)}(x),\ \dots\ f^{(r)}(x)$$

are all positive, then these functions are all positive for $x>h_r$. This follows from the above theorem, for the derivatives of $f^{(r)}(x)$ are $f^{(r+1)}(x),\ \dots\ f^{(n-1)}(x),\ f^{(n)}(x)$.

Ex. 1. *Find the integral part of the greatest root of*

$$f(x)=x^3-10x^2-11x-100=0. \quad \text{(Cf. Ex. 1, p. 91.)}$$

Here $$f'(x)=3x^2-20x-11,\quad f''(x)=2(3x-10),\quad f'''(x)=6.$$

The least integral values of x for which $f''(x)$, $f'(x)$ and $f(x)$ are positive are respectively 4, 8 and 12. Thus 12 is the least integer which exceeds the greatest root, the integral part of which is therefore 11.

EXERCISE XLVII

In Exx. 1-5 find an upper limit to the roots by Newton's method.

1. $x^3-20x^2-31x+1609=0$.
2. $x^3-2x^2-51x-110=0$.
3. $x^4-4x^3-3x+23=0$.
4. $x^5+4x^4-2x^3+10x^2-2x=962$.
5. $x^5+6x^4-10x^3-112x^2-207x=110$.

3. Separation of the Roots. Rules for this purpose were given by Fourier and by Sturm. Fourier's rule, though often convenient, is incomplete, while that of Sturm definitely separates the roots, but its application may be very laborious. We require the following theorems.

Subsidiary Theorems. (1) If α is a root of $f(x)=0$, as the variable x increases through the value α, the functions $f(x)$ and $f'(x)$ have opposite signs just before $x=\alpha$ and the same signs just after $x=\alpha$.

For since $f(\alpha)=0$, we have

$$f(\alpha+h)=hf'(\alpha)+\frac{h^2}{\lfloor 2}f''(\alpha)+\dots+\frac{h^r}{\lfloor r}f^{(r)}(\alpha)+\dots, \quad \text{.....(A)}$$

$$f'(\alpha+h)=f'(\alpha)+hf''(\alpha)+\dots+\frac{h^{r-1}}{\lfloor r-1}f^{(r)}(\alpha)+\dots \quad \text{.....(B)}$$

Suppose that $f^{(r)}(\alpha)$ is the first term of the sequence $f'(\alpha)$, $f''(\alpha)$, ... which is not zero. For sufficiently small values of h, the signs of $f(\alpha+h)$ and $f'(\alpha+h)$ are respectively the signs of the first terms in (**A**) and (**B**) which do not vanish; they are therefore the same as the signs of $h^rf^{(r)}(\alpha)$ and $h^{r-1}f^{(r)}(\alpha)$. Hence $f(\alpha+h)$ and $f'(\alpha+h)$ have the same or opposite signs according as $h\gtrless 0$, which proves the theorem.

(2) *If α is an r-multiple root of $f(x)=0$, then, as the variable x increases through the value α, the signs of the functions*

$$f(x),\quad f'(x),\quad f''(x),\ \dots f^{(r-1)}x,\quad f^{(r)}(x) \qquad \text{.....(C)}$$

are alternately + and −, or − and +, just before $x=\alpha$, and all the functions have the same sign just after $x=\alpha$.

For, when $x=\alpha$, all the terms of the sequence (C) except the last vanish, and the result follows by applying theorem (1) to every two consecutive terms. It follows that *as x increases through the value α, r changes of sign are lost in the sequence* (C).

It is important to observe that since $f^{(r)}(\alpha)\neq 0$ and $f^{(r)}(x)$ is continuous at $x=\alpha$, this function has the same sign just before and just after $x=\alpha$.

4. Sturm's Functions. Let $f(x)$ be a polynomial and $f_1(x)$ its first derivative. *Let the operation of finding the* H.C.F. *of $f(x)$ and $f_1(x)$ be performed with this alteration:*

The sign of each remainder is to be changed before it is used as a divisor: the sign of the last remainder is also to be changed.

Denote the *modified* remainders by $f_2(x), f_3(x), \dots f_r(x)$; then $f(x)$, $f_1(x), f_2(x), \dots f_r(x)$ are called *Sturm's functions* and $f_1(x), f_2(x), \dots f_r(x)$ are known as the *auxiliary functions.*

In the ordinary H.C.F. process we can multiply (or divide) any remainder by any constant. *In the modified process it is essential that such multipliers should be positive.*

Sturm's functions are connected by the equations

$$\left.\begin{aligned} f(x) &= q_1 f_1(x) - f_2(x), \\ f_1(x) &= q_2 f_2(x) - f_3(x), \\ &\dots\dots\dots\dots\dots\dots \\ f_{r-2}(x) &= q_{r-1} f_{r-1}(x) - f_r(x), \end{aligned}\right\} \qquad \text{.....(A)}$$

where $q_1, q_2, \dots$ are the quotients in the process just described, these quotients being functions of x.

It is important to notice that the relation between $f(x)$ and $f_1(x)$ is essentially different from that connecting $f_1(x)$ and $f_2(x)$; $f_2(x), f_3(x)$; etc.

5. Sturm's Theorem. *If $f(x)$ is a polynomial and a, b are any real numbers $(a<b)$, the number of distinct roots of $f(x)=0$ which lie between a and b (any multiple root which may exist being counted once only) is equal to the excess of the number of changes of sign in the sequence of Sturm's functions*

$$f(x),\quad f_1(x),\quad f_2(x), \dots f_r(x), \qquad \text{.....(S)}$$

when $x=a$ over the number of changes of sign in the sequence when $x=b$.

Proof for the case in which $f(x)=0$ has no repeated root.

Equations (**A**) lead to the following conclusions.

(i) Since $f(x)=0$ has no repeated root, $f(x)$ and $f_1(x)$ have no common factor, and consequently $f_r(x)$ is independent of x.

(ii) No two consecutive terms of the sequence (**S**) can vanish for the same value of x, for, if this happened, all the subsequent terms including $f_r(x)$ would vanish.

(iii) If any term of (**S**) except the first is zero, the terms which precede and follow it have opposite signs. Thus if $f_s(x)=0$, then

$$f_{s-1}(x)=-f_{s+1}(x).$$

A similar remark applies if one of the q's is zero. Thus if $q_s=0$, then $f_{s-1}(x)=-f_{s+1}(x)$.

Suppose now that x increases continuously from $x=a$ to $x=b$. As x varies, no one of Sturm's functions can change its sign unless x passes through a value which makes that function vanish, for these functions are polynomials.

Suppose that x passes through a value α which makes *just one* of Sturm's functions vanish.

(i) If α is a root of $f(x)=0$, one change of sign is lost in the sequence (**S**). For $f(x)$ and $f_1(x)$ have opposite signs just before $x=\alpha$, and they have the same signs just after $x=\alpha$.

(ii) If α is a root of $f_m(x)$ where $m=1, 2, \ldots$ or $r-1$, no change of sign is gained or lost. For $f_m(\alpha)=0$, and therefore $f_{m-1}(\alpha)=-f_{m+1}(\alpha)$.

If $m=1$, $f_0(\alpha)$ stands for $f(\alpha)$. Moreover, $f_{m-1}(x)$ and $f_{m+1}(x)$ are continuous at $x=\alpha$, so that each of these is of invariable sign near $x=\alpha$.

Hence, just before and also just after $x=\alpha$, the signs of $f_{m-1}(x), f_m(x), f_{m+1}(x)$ are either $+\,+\,-$ or $+\,-\,-$ or $-\,+\,+$ or $-\,-\,+$, showing that no change of sign is lost or gained.

If x passes through a value α which makes more than one of Sturm's functions vanish, no two of them can be consecutive functions.

If $f(x)$ is one of them, then by the preceding, one change of sign is lost between $f(x)$ and $f_1(x)$, and no change is gained or lost in the sequence $f_1(x), f_2(x), \ldots f_r(x)$. Thus one change is lost in the sequence (**S**).

If $f(x)$ is not one of them, no change of sign is gained or lost.

Thus, as x increases, a change of sign in the sequence (**S**) is lost whenever x passes through a root of $f(x)=0$, and under no other circumstance is a change gained or lost. This proves the theorem.

Remarks. In applying Sturm's theorem, labour may be saved by the following considerations.

(i) If there is no repeated root, the last function $f_r(x)$ is independent of x and its *sign only* is required.

Let x have a value such that $f_{r-1}(x)=0$, then $f_r(x)=-f_{r-2}(x)$. Thus, if the last three functions are $a'x^2+b'x+c'$, $a''x+b''$, a''',

then $$a'''=-\left(a'\cdot\frac{b''^2}{a''^2}-b'\cdot\frac{b''}{a''}+c'\right);$$

and from this the sign of a''' may often be found without much calculation.

(ii) If, at any stage, we arrive at a function $f_s(x)$ such that all the roots of $f_s(x)=0$ are imaginary, the H.C.F. process need not be continued, and we can use $f(x)$, $f_1(x)$, ... $f_s(x)$, instead of the complete set of functions.

For the essential property of the last function is that it should remain of invariable sign for all real values of x, and $f_s(x)$ has this property.

(iii) Suppose that one of Sturm's functions as $f_m(x)$ vanishes when $x=a$; then, in counting the number of changes of sign in the sequence $f(a), f_1(a), \ldots f_r(a)$, we may regard the sign of $f_m(a)$ as either + or −. For if $f_m(a)=0$, then $f_{m-1}(a)$ and $f_{m+1}(a)$ have opposite signs.

Ex. 1. *Find the number and position, relative to numbers on the scale of integers, of the real roots of*

$$f(x)=x^4-3x^3-2x^2+7x+3=0.$$

Here $f_1(x)=4x^3-9x^2-4x+7$, and the modified H.C.F. process is as follows:

	$f_1(x)=$	$4-\quad 9-\quad 4+\quad 7$	$4-12-\quad 8+\quad 28+\quad 12=f(x)$	1
			$4-\quad 9-\quad 4+\quad 7$	
			$-\quad 3-\quad 4+\quad 21+\quad 12$	
			4	
			$-12-16+\quad 84+\quad 48$	-3
		43	$-12+27+\quad 12-\quad 21$	
4		$172-\quad 387-\quad 172+\quad 301$	$43-\quad 72-\quad 69=f_2(x)$	
		$172-\quad 288-\quad 276$	83	
		$-\quad 99+\quad 104+\quad 301$	$3569-5976-\quad 5727$	43
		43	$3569-8213$	
-99		$-4257+4472+12943$	$2237-\quad 5727$	
		$-4257+7128+\quad 6831$	83	
		$32)\ 2656-\quad 6112$	$185671-475341$	2237
	$f_3(x)=$	$83-\quad 191$	$185671+427267$	
			$f_4(x)=\quad +\quad 48074$	

Thus $f_2(x)=43x^2-72x-69$, $f_3(x)=83x-191$, $f_4(x)=+48074$.

The sign of $f_4(x)$ is found more easily by Remark (i). If $f_3(x)=0$, then $f_4(x)=-f_2(x)$. When $x=\frac{191}{83}$, $f_2(x)$ is negative. Therefore $f_4(x)$ is positive.

The number of changes of sign in the sequence of Sturm's functions for various values of x are shown below.

x	$-\infty$	-2	-1	0	2	3	∞
$f(x)$	+	+	−	+	+	+	+
$f_1(x)$	−	−	−	+	−	+	+
$f_2(x)$	+	+	+	−	−	+	+
$f_3(x)$	−	−	−	−	−	+	+
$f_4(x)$	+	+	+	+	+	+	+
Number of changes of sign	4	4	3	2	2	0	0

Hence the equation $f(x)=0$ has one root between -2 and -1, one between -1 and 0 and two between 2 and 3.

Ex. 2. *Find the number and position of the real roots of*

$$\text{(i) } x^5-x-1=0\,; \quad \text{(ii) } x^6-2x^2+3x-4=0.$$

(i) Taking $f(x)=x^5-x-1$, we have $f_1(x)=5x^4-1$, and the modified H.C.F. process is

$$\begin{array}{r|l} 5x^4-1 & 5x^5-5x-5 \\ & \underline{5x^5-\ x} \\ & f_2(x)=4x+5 \end{array} \Big| \; x.$$

Also, when $f_2(x)=0$, $f_3(x)=-f_1(x)$, therefore

$$f_3(x)=-\{5(-\tfrac{5}{4})^4-1\}<0.$$

For various values of x, the signs of Sturm's functions are :

x	$-\infty$	0	1	2	∞
$f(x)$	−	−	−	+	+
$f_1(x)$	+	−	+	+	+
$f_2(x)$	−	+	+	+	+
$f_3(x)$	−	−	−	−	−
Number of changes of sign	2	2	2	1	1

showing that there is one real root between 1 and 2, and that the other roots are imaginary.

(ii) Here $f(x)=x^6-2x^2+3x-4$, $f_1(x)=6x^5-4x+3$.

$$\begin{array}{r|l} 6x^5-4x+3 & 6x^6-12x^2+18x-24 \\ & \underline{6x^6-\ 4x^2+\ 3x} \\ & f_2(x)=\ 8x^2-15x+24 \end{array} \Big| \; x.$$

The roots of $f_2(x)=0$ are imaginary, for $15^2<4\,.\,8\,.\,24$, and we need not proceed further.

x	$-\infty$	-2	-1	1	2	∞
$f(x)$	+	+	−	−	+	+
$f_1(x)$	−	−	+	+	+	+
$f_2(x)$	+	+	+	+	+	+

Thus there is one real root between -2 and -1, another between 1 and 2, the other roots being imaginary.

6. Sturm's Theorem (Multiple Roots). It remains to consider the case in which the equation $f(x)=0$ has repeated roots. Suppose that

$$f(x)=(x-\alpha)^p(x-\beta)^q(x-\gamma)^r \dots$$

where $p, q, r, \dots$ are positive integers and $\alpha, \beta, \gamma, \dots$ are all different (real or imaginary), then

$$f_1(x)=u\,.\,\{p(x-\beta)(x-\gamma)\dots+q(x-\alpha)(x-\gamma)\dots+r(x-\alpha)(x-\beta)\dots+\dots\},$$

where $$u=(x-\alpha)^{p-1}(x-\beta)^{q-1}(x-\gamma)^{r-1}\dots .$$

Hence u is the H.C.F. of $f(x)$ and $f_1(x)$, and all of Sturm's functions,

$$f(x),\quad f_1(x),\quad f_2(x),\dots f_r(x),\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{S})$$

are divisible by u. Denote the quotients by

$$\phi(x),\quad \psi_1(x),\quad \psi_2(x),\dots \psi_r(x),\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{S}')$$

so that $$\phi(x)=(x-\alpha)(x-\beta)(x-\gamma)\dots ,$$

and $\psi_r(x)$ is independent of x. Also for any value of x, the number of changes of sign in the sequence (**S**) is the same as that in the sequence (**S'**).

Dividing equations (**A**) of Art. 4 by u, we have the identities

$$\phi(x)=q_1\psi_1(x)-\psi_2(x),\quad \psi_1(x)=q_2\psi_2(x)-\psi_3(x),\quad \text{etc.}$$

Whence, as in Art. 5, it follows that no change of sign is gained or lost as x passes through a value which makes one of the set $\psi_1(x), \psi_2(x), \dots \psi_r(x)$ vanish.

Again, $$\psi_1(\alpha)=p(\alpha-\beta)(\alpha-\gamma)\dots=p\phi_1(\alpha),$$

therefore $\psi_1(x)$ and $\phi_1(x)$ have the same sign when $x=\alpha$, and since these functions are continuous, they have the same signs for values of x near $x=\alpha$. Hence, by Art. 5, as x increases through any root α of $\phi(x)$ a change of sign is lost in the sequence (**S'**).

It follows that the number of real roots of $\phi(x)=0$ in the interval (a, b), where $a<b$ is equal to the excess of the number of changes of sign in the sequence (**S'**) when $x=a$ over the number of changes of sign when $x=b$. That is, the number of *distinct* roots of $f(x)$ in the interval (a, b) is equal to the excess of the number of changes of sign in the sequence (**S**) when $x=a$ over the number of changes of sign when $x=b$.

Conditions for the Reality of all the Roots. Let $f(x)$ be a polynomial of degree n with its leading coefficient positive. In general, the number of Sturm's functions is $n+1$.

The necessary and sufficient conditions that the roots of $f(x)=0$ may be all real and different are: (i) *the number of Sturm's functions must be $n+1$. and* (ii) *the leading coefficients of all these functions must be positive.*

For, as x passes from $-\infty$ to $+\infty$, n changes of sign will be lost in the sequence of Sturm's functions if and only if both of these conditions hold.

7. Application of Sturm's Theorem. For the biquadratic,

$$f(z)=z^4+6Hz^2+4Gz+K,$$

where $K=a^2I-3H^2$ (Ch. XII, 9; 10 (6)), Sturm's functions are $f(z)$ and

$$f_1(z)=z^3+3Hz+G,$$
$$f_2(z)=-3Hz^2-3Gz-K,$$
$$f_3(z)=-(2HI-3aJ)z-GI,$$
$$f_4(z)=I^3-27J^2=\Delta.$$

In proving this we require the identity (Ch. XII, 10, (6))

$$G^2+4H^3=a^2(HI-aJ). \quad \text{.....(A)}$$

Sturm's process is

	z^3+ $3Hz+G$		$z^4+6Hz^2+4Gz+K$	z
		$3H^2$	z^4+3Hz^2+Gz	
$-Hz$	$3H^2z^3+9H^3z$ $+3GH^2$		$f_2(z)=-3Hz^2-3Gz-K$	
	$3H^2z^3+3GHz^2$ $+HKz$			
G	$-3GHz^2+(12H^3-a^2HI)z+3H^2G$			
	$-3GHz^2-3G^2z$ $-KG$			
	$a^2)$ $-a^2(2HI-3aJ)z-a^2GI$		*	

$$f_3(z)=-Lz-GI \quad \text{where} \quad L=2HI-3aJ.$$

Again, if $f_3(z)=0$, then $f_4(z)=-f_2(z)$, therefore

$$\begin{aligned}L^2f_4(z)&=3H(GI)^2-3G(GI)L+KL^2\\&=(a^2I-3H^2)(3aJ-2HI)^2+3G^2I(3aJ-HI)\\&=9a^4IJ^2-12a^3HI^2J+a^2H^2(4I^3-27J^2)\\&\qquad+3I(G^2+4H^3)(3aJ-HI)\ *\\&=a^2H^2(4I^3-27J^2)-3a^2H^2I^3\\&=a^2H^2(I^3-27J^2).\end{aligned}$$

Disregarding square factors, we may therefore take

$$f_4(z)=I^3-27J^2.$$

Hence if Δ is positive and both H and $2HI-3aJ$ are negative, all the roots are real. If Δ is positive and at least one of the two, H and $2HI-3aJ$, is positive, all the roots are imaginary.

For, as x passes from $-\infty$ to $+\infty$, in the first case four changes of sign are lost in the sequence of Sturm's functions, and in the second case no change of sign is lost.

If Δ is negative, two roots are real and two imaginary.

All this has been proved more easily in Ch. XII.

* The identity (A) is used in these steps.

8. Fourier's Theorem. *Let $f(x)$ be a polynomial of the n-th degree and let $f_1(x), f_2(x), \ldots f_n(x)$ be its successive derivatives. Let R be the number of real roots of $f(x)=0$ which lie between a and b, where a, b are any real numbers, of which a is less than b, and an r-multiple root is counted r times. Let N, N' respectively denote the number of changes of sign in the sequence*

$$f(x),\ f_1(x),\ f_2(x), \ldots f_n(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{F})$$

when $x=a$ and when $x=b$, then

$$N \geqslant N' \quad \text{and} \quad R \leqslant N - N';$$

also $(N-N')-R$ is an even number or zero.

In particular if $N-N'=1$, there is just one real root in the interval, and if $N-N'=0$ no real root lies between a and b.

In this connection $f(x)$ and its derivatives are called Fourier's functions.

Proof. Let x increase continuously from $x=a$ to $x=b$. As x varies, no change of sign can be gained or lost in the sequence (**F**) unless x passes through a value which makes one of its terms vanish.

(1) Suppose that x passes through a root α of $f(x)=0$.

(i) If α is an unrepeated root, a change of sign is lost between $f(x)$ and $f_1(x)$. This has been proved in Art. 3.

(ii) If α is an r-multiple root, r changes of sign are lost in the sequence $f(x), f_1(x), \ldots f_r(x)$. (See Art. 3.)

(2) Suppose that x passes through a value α' which makes one or more of the functions $f_1(x), f_2(x), \ldots f_n(x)$ vanish.

(i) If $f_m(\alpha')=0$, but neither $f_{m-1}(\alpha')$ nor $f_{m+1}(\alpha')$ is zero, there is no gain or loss of changes of sign when $f_{m-1}(\alpha')$ and $f_{m+1}(\alpha')$ have opposite signs: but if these have the same sign, two changes are lost.

For $f_m(x)$ and $f_{m+1}(x)$ have opposite signs just before $x=\alpha'$, and they have the same sign just after $x=\alpha'$.

(ii) Suppose that when $x=\alpha'$ the r functions

$$f_m(x),\ f_{m+1}(x), \ldots f_{m+r-1}(x)$$

all vanish, but neither $f_{m-1}(x)$ nor $f_{m+r}(x)$ is zero.

Then if r is odd there is a loss of $r-1$ or $r+1$ changes of sign according as $f_{m-1}(\alpha')$ and $f_{m+r}(\alpha')$ have opposite signs or the same sign, while if r is even there is a loss of r changes of sign in either case.

Thus, under this heading, no change of sign is ever gained, and if any are lost, their number is even.

Ex. 1. *Apply Fourier's method to separate the roots of*

$$f(x)=2x^5+7x^4-40x^3-23x^2+38x-4=0.$$

Here
$$f_1(x)=2(5x^4+14x^3-60x^2-23x+19),$$
$$f_2(x)=2(20x^3+42x^2-120x-23),$$
$$f_3(x)=8(15x^2+21x-30),$$
$$f_4(x)=8(30x+21),$$
$$f_5(x)=240.$$

By Newton's method we find that all the roots lie in the interval $(-7, 4)$. For various values of x the signs of Fourier's functions are as below.

x	-7	-6	-2	-1	0	1	3	4
$f(x)$	−	+	+	−	−	−	−	+
$f_1(x)$	+	+	−	−	+	−	+	+
$f_2(x)$	−	−	+	+	−	−	+	+
$f_3(x)$	+	+	−	−	−	+	+	+
$f_4(x)$	−	−	−	−	+	+	+	+
$f_5(x)$	+	+	+	+	+	+	+	+
Number of changes of sign	5	4	4	3	3	1	1	0

These results show that there is one root in each of the intervals $(-7, -6)$, $(-2, -1)$, $(3, 4)$, and either two roots or none at all in the interval $(0, 1)$.

Subdividing the last interval, we find that when $x=0{\cdot}5$, $f(x)$ is positive. Hence there is a root in each of the intervals $(0, 0{\cdot}5)$ and $(0{\cdot}5, 1)$.

It should be noticed that we need not find the signs of $f_1(x)$, $f_2(x)$, etc., when $x=0{\cdot}5$, for it has been shown that either two roots, or none at all, lie between 0 and 1.

Remarks. (1) In applying Fourier's theorem, as stated above, the question arises as to *what sign is to be attached to one of Fourier's functions which may happen to vanish when* $x=a$ *or* b.

This difficulty is met by observing that, for sufficiently small values of h, the number of roots between a and b $(a<b)$ is the same as the number of roots between $a+h$ and $b-h$. We may therefore put $a+h$ instead of a and $b-h$ instead of b, where $h\to 0$.

(2) **Disadvantage of Fourier's Method.** Suppose that (as in the last example) all the roots of an equation are accounted for except two, which, if real, must lie in a certain interval (a, b). If these roots are nearly equal, it may require a large number of trials to separate them. If we fail to do this the possibilities are:

(i) The roots may be very nearly equal, in which case they can be separated by further trials.

(ii) The roots may be equal, and if we begin by testing for equal roots it would be better to use Sturm's process.

(iii) The roots may be imaginary, and no matter how many trials are made, we may not be able to discover whether this is or is not the case.

(3) If the roots of $f(x)$ are diminished by a, the transformed equation is

$$f(a)+xf_1(a)+\frac{x^2}{\lfloor 2}f_2(a)+\ldots+\frac{x^n}{\lfloor n}f_n(a)=0,$$

so that N *is the number of changes of sign in this equation. Similarly* N' *is the number of changes of sign in the equation obtained by diminishing the roots of* $f(x)=0$ *by* b. If N and N' are *defined* in this way, we obtain *Budan's* statement of Fourier's theorem.

Imaginary Roots. As x increases from $-\infty$ to $+\infty$, n changes of sign are lost in the sequence of Fourier's functions. If it is known that a certain interval contains no root of $f(x)=0$, and that, as x increases through the interval, s changes are lost, then the equation cannot have more than $n-s$ real roots, and must therefore have at least s imaginary roots.

EXERCISE XLVIII

1. From Fourier's theorem deduce (i) Descartes' rule of signs; (ii) Newton's method of finding an upper limit to the roots.

In Exx. 2-11, find intervals (a, b) where a, b are consecutive integers which contain the real roots of the following equations.

In Exx. 2-4 *use Fourier's method; and in Exx.* 5-11 *use Sturm's method.*

2. $x^3-3x^2-4x+11=0$.
3. $x^3-20x^2-31x+1609=0$.
4. $2x^3+11x^2-10x+1=0$.
5. $x^3+2x^2-51x+110=0$.
6. $2x^3-15x^2-8x+166=0$.
7. $x^4-4x^3-3x+23=0$.
8. $3x^4-10x^2-6x+16=0$.
9. $x^4-x^3-4x^2+4x+1=0$.
10. $x^5-3x^4-24x^3+95x^2-46x-101=0$.
11. $x^5+3x^4+2x^3-3x^2-2x-2=0$.

12. Use Sturm's theorem to show that the equation $x^3+3Hx+G=0$ has three real and distinct roots if and only if $G^2+4H^3<0$.

13. If $e<0{\cdot}55$, prove that the equation $x^4+4x^3+6x^2-4x+e=0$ has two real and two imaginary roots. $[I=e+7,\ J=-4.]$

9. Newton's Method of approximating to a Root. Suppose that the equation $f(x)=0$ has a single unrepeated root in a small interval (α, β), so that $f'(x)\neq 0$ within the interval. Let $\alpha+h$ be the actual value of the root, then by the second mean value theorem,

$$f(\alpha+h)=f(\alpha)+hf'(\alpha)+\tfrac{1}{2}h^2f''(a+\theta h)=0,$$

where $0<\theta<1$. Hence

$$h=-\frac{f(\alpha)}{f'(\alpha)}+\epsilon \quad \text{where} \quad \epsilon=-\tfrac{1}{2}h^2\frac{f''(\alpha+\theta h)}{f'(\alpha)}. \qquad \ldots\ldots\ldots\ldots\text{(A)}$$

It follows, that if k is the greatest value of $f''(x)$ in the interval, and

$$\alpha_1 = \alpha - f(\alpha)/f'(\alpha),$$

the value of the root is $\alpha_1 + \epsilon$ where

$$|\epsilon| < \tfrac{1}{2}h^2 \frac{k}{|f'(\alpha)|} < \tfrac{1}{2}(\beta-\alpha)^2 \cdot \frac{k}{|f'(\alpha)|}. \quad \text{.....(B)}$$

Thus, if $\beta - \alpha$ is a small number of the first order, the error in taking α_1 as the value of the root is of the second order of smallness, unless $k/f'(\alpha)$ is large. By repeating the process, the approximation can be carried to any required degree of accuracy.

NOTE. In the preceding, $f(x)$ may be *any* function of x such that $f(x)$, $f'(x)$ and $f''(x)$ are continuous in the interval (α, β).

Ex. 1. *Find an approximate value of a, such that* $a^{\sqrt{a}} = 3$.

Put $a = x^2$; then $f(x) = x^{2x} - 3 = 0$; and $f'(x) = x^{2x}(2 + 2\log_e x)$.

Also $f(x) = -2$ when $x = 1$, and $f(x) = +0{\cdot}375$ when $x = 1{\cdot}5$.

Starting with $\alpha = 1{\cdot}5$, the work proceeds as follows.

$f(\alpha) = 0{\cdot}375$, $f'(\alpha) = 9{\cdot}487$, $f(\alpha)/f'(\alpha) = 0{\cdot}04$, and $\alpha_1 = 1{\cdot}46$;

$f(\alpha_1) = 0{\cdot}0193$, $f'(\alpha_1) = 8{\cdot}324$, $f(\alpha_1)/f'(\alpha_1) = 0{\cdot}0023$, and $\alpha_2 = 1{\cdot}4577$;

$f(\alpha_2) = 0{\cdot}000255$, $f'(\alpha_2) = 8{\cdot}262$, $f(\alpha_2)/f'(\alpha_2) = 0{\cdot}000031$, and $\alpha_3 = 1{\cdot}457669$;

hence, the value of a required is $(1{\cdot}457669)^2 = 2{\cdot}12480$.

This method is often employed in cases where the interval (α, β) is not very small, and then it is important to know under what circumstances α_1 is a closer approximation to the root than α.

Fourier's Rule. We shall assume that neither $f'(x)$ nor $f''(x)$ is zero for any value of x in the interval (α, β), so that each of these functions retains the same sign for all values of x with which we are concerned. Now

$$f(\alpha + h) = f(\alpha) + hf'(\alpha + \theta_1 h) = 0,$$

where $0 < \theta_1 < 1$, so that

$$h = -f(\alpha)/f'(\alpha + \theta_1 h).$$

Hence if $t = -f(\alpha)/f'(\alpha)$, it follows that t has the same sign as h.

Now α_1 is certainly a better approximation than α if α_1 lies between α and $\alpha + h$, and since $\alpha_1 = \alpha + t$, this is the case if $|t| < |h|$, that is if

$$f'(\alpha) > f'(\alpha + \theta_1 h).$$

This condition is satisfied if $f(\alpha)$ and $f''(\alpha)$ have the same sign. For $f'(x)$ increases with x or decreases as x increases according as $f''(x) \gtrless 0$, and $h \gtrless 0$ according as $f(\alpha)$ and $f'(\alpha)$ have different signs or the same sign.

Thus we have Fourier's rule, which is as follows: *If neither* $f'(x)$ *nor* $f''(x)$ *is zero near the point* α, *and* $f(\alpha)$ *and* $f''(\alpha)$ *have the same sign, then* α_1 *is a better approximation than* α.

The various cases which can arise are illustrated in Fig. 66, (i)-(iv). It is to be noted that the tangent to the graph of $f(x)$ at the point where $x=\alpha$ cuts the x-axis at the point α_1.

In the figures on the left, $f(\alpha)$ and $f''(\alpha)$ have the same sign; in those on the right they have opposite signs and Fourier's rule does not apply.

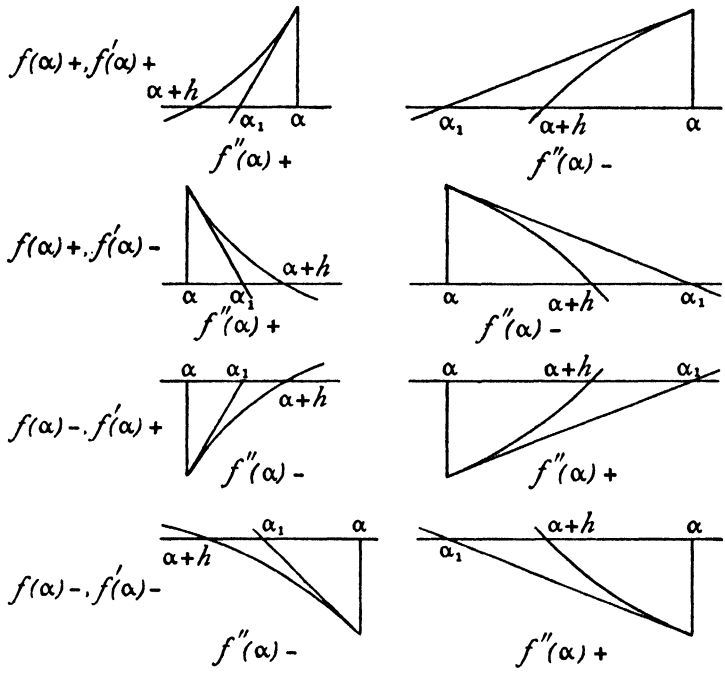


FIG. 66.

We shall now show that α_1 is a better approximation than α, *provided that the interval is sufficiently small.*

The true value of the root $=\alpha+h=\alpha_1+\epsilon$, and therefore α_1 is a better approximation than α if $|\epsilon|<|h|$, that is if

$$\tfrac{1}{2}h^2\left|\frac{f''(\alpha+\theta h)}{f'(\alpha)}\right|<|h| \quad \text{or if} \quad |h|<2\left|\frac{f'(\alpha)}{f''(\alpha+\theta h)}\right|.$$

If $f''(x)\neq 0$ in the interval (α,β), we can find a positive number l such that $|l|<|f''(\alpha+\theta h)|$, and we draw the following conclusion:

If neither $f'(x)$ nor $f''(x)$ is zero in the interval (α,β) and

$$|h|<\frac{2|f'(\alpha)|}{l},$$

then α_1 is a better approximation than α.

This is certainly the case if $|\beta-\alpha| < \dfrac{2|f'(\alpha)|}{l}$.

Moreover, in cases where Fourier's rule does not apply, $|h|<|t|$, and α_1 is a closer approximation than α if

$$|t|<\frac{2|f'(\alpha)|}{l}, \quad \text{that is, if} \quad l|f(\alpha)|<2\{f'(\alpha)\}^2. \ldots\ldots\ldots\ldots\text{(C)}$$

In this case, by (A),

$$|\epsilon|<\frac{1}{2}t^2\cdot\frac{k}{|f'(\alpha)|}, \quad \text{that is} \quad |\epsilon|<\frac{1}{2}\frac{\{f(\alpha)\}^2}{|f'(\alpha)|^3}\cdot k. \ldots\ldots\ldots\ldots\text{(D)}$$

Ex. 2. Find the real root of $x^3=2x+5$ to nine significant figures.

Let $f(x)=x^3-2x-5$, then $f'(x)=3x^2-2$, $f''(x)=6x$.

Since $f(2)<0<f(3)$, the root lies between 2 and 3. Also $f(2)=-1$, $f'(2)=10$, $f''(2)=12$.

Taking $\alpha=2$ and applying Newton's method, $\alpha_1=2-\dfrac{-1}{10}=2{\cdot}1$.

Here $f(2)$ and $f''(2)$ have opposite signs, and Fourier's rule does not apply. But $h<0{\cdot}1$, and the least value of $|f''(x)|$ in the interval (2, 3) is 12. Hence

$$|h|<\frac{2|f''(\alpha)|}{l},$$

and α_1 is a better approximation than α.

Moreover, $f''(x)$ increases with x, and therefore

$$f''(\alpha+\theta h)<f''(\alpha+h)=f''(2{\cdot}1)=12{\cdot}6,$$

so that
$$|\epsilon|=\frac{1}{2}h^2\frac{f''(\alpha+\theta h)}{f'(\alpha)}<\frac{1}{2}(0{\cdot}1)^2\cdot\frac{12{\cdot}6}{10}<0{\cdot}007.$$

Next, taking $\alpha=2{\cdot}1$,* we have $f(2{\cdot}1)=0{\cdot}061$, $f'(2{\cdot}1)=11{\cdot}23$, $f''(2{\cdot}1)=12{\cdot}6$.

Hence
$$\alpha_1=2{\cdot}1-\frac{0{\cdot}061}{11{\cdot}23}=2{\cdot}094569\ldots.$$

Also the value of h is equal to that of ϵ in the first approximation, so that

$$|h|<0{\cdot}007,$$

and consequently
$$|\epsilon|<\frac{1}{2}(0{\cdot}007)^2\frac{f''(2{\cdot}1)}{f'(2{\cdot}1)}<0{\cdot}00003.$$

Moreover, ϵ is negative, and so the root lies between 2·09457 and 2·09453. Hence the error in taking 2·09455 as the root is numerically less than 0·00002.

Again, taking $\alpha=2{\cdot}09455$, we find that

$$f(\alpha)=-0{\cdot}000016557\ldots \quad \text{and} \quad f'(\alpha)=11{\cdot}11607,$$

giving
$$\alpha_1=\alpha-f(\alpha)/f'(\alpha)=2{\cdot}094551483\ldots.$$

Also
$$|\epsilon|<\frac{1}{2}(0{\cdot}00002)^2\cdot\frac{12{\cdot}6}{11}<10^{-11}.24.$$

Moreover, ϵ is negative, and so the value of the root to nine significant figures is **2·09455148.**

* This approximation might have been obtained graphically.

Ex. 3. *A donkey is tethered to a point A in the circumference of a circular field of radius a by a rope of length l. Find the value of l in order that he may be able to graze over exactly half of the field.*

In Fig. 67 AB is the stretched rope, $\angle AOB = x$, $\angle OAB = x'$ measured in radians.

Thus $2x' + x = \pi$, $l = 2a \sin \frac{x}{2}$.

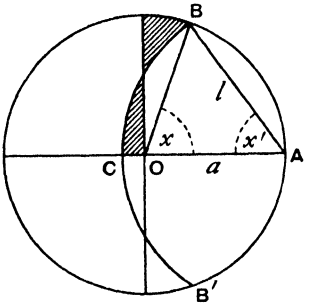


FIG. 67.

Area donkey can graze over

$$= \text{segment } ABB' + \text{segment } CBB'$$
$$= a^2(x - \tfrac{1}{2}\sin 2x) + l^2(x' - \tfrac{1}{2}\sin 2x').$$

Equating this to $\frac{1}{2}a^2\pi$, it will be found that x is given by $f(x) = x - \tan x + \pi(\frac{1}{2}\sec x - 1) = 0$.

Hence, $f'(x) = \tan x\left(\frac{\pi}{2}\sec x - \tan x\right)$,

and $f''(x) \equiv \frac{\pi}{2}\sec x(1 + 2\tan^2 x) - 2\tan x \sec x$.

In a question of this kind it is important to start with a really good approximation, to be found graphically or otherwise. We draw the figure so that the shaded areas are nearly equal, and then by measurement

$$\angle AOB = 71° = 1{\cdot}239 \text{ radians (nearly)}.$$

Taking $\alpha = 1{\cdot}239$, we find that, approximately, $f(\alpha) = 0{\cdot}018$, $f'(\alpha) = 5{\cdot}578$, and the next approximation is

$$\alpha_1 = \alpha - f(\alpha)/f'(\alpha) = 1{\cdot}239 - 0{\cdot}003 = 1{\cdot}236.$$

If x increases from 0 to $\pi/2$, $f''(x)$ increases from $\pi/2$ to ∞; therefore $f''(\alpha)$ is positive. Hence $f(\alpha)$ and $f''(\alpha)$ have the same sign and, by Fourier's rule, α_1 is a closer approximation than α. Further, we can show that $f''(\alpha) < 34$; therefore $f''(x) < 34$ if $x < \alpha$. Hence, if $\alpha_1 + \epsilon$ is the true value of x, by Art. 9 (**B**),

$$|\epsilon| < \tfrac{1}{2}(0{\cdot}003)^2 \cdot \tfrac{34}{5} < 0{\cdot}00004.$$

Thus the following results are correct to the last figure:

$$\angle AOB = 1{\cdot}236 \text{ radians} = 70°\ 50', \quad \text{and} \quad l/a = 2\sin 35°\ 25' = 1{\cdot}16.$$

Hence $l = 7a/6$, very nearly.

10. Nearly equal Roots. Suppose that the equation $f(x) = 0$ has two nearly equal roots within the small interval (α, β), and let $\alpha + h$ be the true value of one of them, then

$$f(\alpha + h) = a + bh + ch^2 + dh^3 + \ldots = 0,$$

where $a = f(\alpha)$, $b = f'(\alpha)$, $c = \frac{1}{2}f''(\alpha)$, etc.

Neglecting squares of h, we have $a + bh = 0$, giving Newton's rule. But the roots are separated by a root of $f'(x) = 0$, so that $f'(x)$ is zero for some value of x in the interval. Consequently $f'(\alpha)$ is small, and the term ch^2 cannot be neglected in comparison with bh, so that Newton's rule does not apply. Neglecting cubes of h, we have $a + bh + ch^2 = 0$.

This equation has nearly equal roots, and is therefore approximately the same as $(a+\frac{1}{2}bh)^2=0$. Hence we have

$$h=-\frac{2a}{b}=-\frac{b}{2c} \text{ (approx.)}.$$

This result is used in Horner's method (Art. 11, Ex. 3) to *suggest* a trial value of h. It is only applicable when $2a/b$ and $b/2c$ are nearly equal.

In the same way, if the equation has three nearly equal roots in the interval (α, β), the value of one of them being $\alpha+h$, neglecting h^4, we have $a+bh+ch^2+dh^3=0$; which is nearly the same as $(a+\frac{1}{3}bh)^3=0$. Hence,

$$h=-\frac{3a}{b}=-\frac{b}{c}=-\frac{c}{3d} \text{ (nearly)}.$$

11. Horner's Method. The process described in this article, due to Horner, is the most convenient way of finding approximate values of the irrational roots of equations of the type $f(x)=0$ where $f(x)$ is a polynomial. Any rational roots which may exist can also be found in the same way. One great merit of the method is the very concise and orderly way in which the reckoning can be arranged. The root is evolved as a decimal, the figures of which are obtained in succession.

In what follows, we are only concerned with positive roots. To find the negative roots of $f(x)=0$, we find the positive roots of $f(-x)=0$.

Consider the equation $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$, ...(**A**)

and suppose that this has a single root α in the interval $(p, p+1)$, where p is a positive integer. (The case of two nearly equal roots will be considered later.) Let $\alpha=p\cdot qrs\ldots$ where $q, r, s\ldots$ are the figures in the decimal part of the root: then p can be found as in Ch. XXVIII, 2, or preferably figure by figure as in Ex. 2 below.

Decrease the roots of (**A**) by p, and let the transformed equation be

$$a_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n=0. \quad \text{.....(B)}$$

This has a single root between 0 and 1, namely $0\cdot qrs\ldots$. In finding the value of q, we avoid the use of decimals by multiplying the roots of (**B**) by 10, obtaining the equation

$$\phi(x)=a_0x^n+10b_1x^{n-1}+\ldots+10^{n-1}b_1x+10^nb_n=0. \quad \text{.....(C)}$$

This has a single root between 0 and 10, namely $q\cdot rs\ldots$. Moreover, *q is the greatest number such that if the roots of* (**C**) *are decreased by q, the sign of the last term of the transformed equation is the same as that of the last term of* (**C**). This follows from the fact that $\phi(x)=0$ has a root between q and $q+1$, and there is no root between 0 and q, so that $\phi(0)$, $\phi(q)$ and $-\phi(q+1)$ have the same signs.

In practice *the value of q is suggested and, after a certain stage, is definitely given by Newton's method.*

For, regarding p as an approximate value of α, we have

$$0{\cdot}qrs \ldots = -f(p)/f'(p) = -b_n/b_{n-1} \text{ (approximately).}$$

Having found the correct value of q, we decrease the roots of (**C**) by q, multiply the roots of the resulting equation by 10, so obtaining an equation of which $r{\cdot}s \ldots$ is a root.

Continuing as above, we can find as many figures as we choose in the decimal representation of α.

When a certain number of figures have been found, about as many more can be found by a contracted process explained in the next example.

Ex. 1. *Find to ten significant figures the real root of*

$$f(x) = x^3 - 111 = 0. \quad \ldots\ldots\ldots\

NOTE. Denoting the root by α, the successive steps show that

$$4<\alpha<5, \quad 4{\cdot}8<\alpha<4{\cdot}9, \quad 4{\cdot}80<\alpha<4{\cdot}81, \quad \text{etc.}$$

Also that for the values 4, 4·8, 4·805 ... 4·8058955338 of x,

$$f(x) = -47, \quad -0{\cdot}408, \quad -0{\cdot}062 \ldots, \quad \ldots -0{\cdot}000000004 \ldots .$$

EXPLANATION OF THE SUCCESSIVE STEPS

(*a*) Decreasing the roots by 4, the resulting equation is

$$x^3 + 12x^2 + 48x - 47 = 0. \quad \ldots\ldots\ldots\

Ex. 2. *Find to nine significant figures the real root of* $x^3 - x^2 - x - 2000 = 0$.

It is easily seen that the root lies between 10 and 20, and the first step is to decrease the roots by 10. In practice, the work is arranged thus :

	1	−1	−1	−2000	10
		9	89	−1110	2
		19	279	−428000	9
$\frac{11}{3} = 3\cdot\ldots$		29	341	−32621000	6
		31	40700	−4139264	8
		33	43931	−321124	6
$\frac{43}{41} = 1\cdot\ldots$		350	4724300	−34568	
		359	4746956		
		368	476964,8		
$\frac{33}{47} = 0{\cdot}7\ldots$		3770	477267,8		
		3776	477571		
		3782	47759,4		
$\frac{41}{48} = 0{\cdot}8\ldots$		37,88	4776,2) 34568 (7237		
		,38	1135		
			180		
			37		

The required root is 12·9686724, to nine significant figures.

NOTE. It will be seen that Newton's approximation gives 1 more than the correct figure until we come to the 8.

Ex. 3. *Find approximate values of the two roots of*

$$3x^4 - 61x^3 + 127x^2 + 220x - 520 = 0$$

which lie between 2 *and* 3.

The reckoning is as follows, and the explanation is on the opposite page.

3 − 61 + 127 + 220 − 520 (2 .....(*a*)
− 55 + 17 + 254 − 12
− 49 − 81 + 92
− 43 − 167
− 37

3 − 370 − 16700 + 92000 − 120000 (2 .....(*b*)
− 364 − 17428 + 57144 − 5712
− 358 − 18144 + 20856
− 352 − 18848
− 346

3 − 3460 − 1884800 + 20856000 − 57120000 (5 .....(*c*)
− 3445 − 1902025 + 11345875 − 390625
− 3430 − 1919175 + 1750000
− 3415 − 1936250
− 3400

− 3,4 − 19362,5 + 175000 − 390625 (4 .....(*d*)
− 19376 + 97496 − 641
− 19390 + 19936
− 19404

For the smaller root :

$$\begin{array}{rrrl} -194{,}04 & +1993{,}6 & -641 & (\ 0 \ \ldots\ldots (e) \\ -1{,}94 & +199{,}3 & -641 & (\ 3 \ \ldots\ldots (f) \\ & +193 & -\ 62 & \\ & +187 & & \\ & 18{,}7 & -\ 62 & (\ 3 \ \ldots\ldots (g) \\ & 1{,}8 & -\ 6 & (\ 3 \ \ldots\ldots (h) \end{array}$$

For the greater root :

$$\begin{array}{rrrl} -194{,}0 & +1993{,}6 & -\ 641 & (\ 9 \ \ldots\ldots (e') \\ & +247 & +1582 & \\ & -1499 & & \\ -1{,}9 & -149{,}9 & +1582 & (\ 9 \ \ldots. (f') \\ & -167 & +\ 79 & \\ & -184 & & \\ & -18{,}4 & 79 & (\ 4 \ \ldots. (h') \\ & & 5 & \\ & -1{,}8 & 5 & (\ 2 \end{array}$$

Thus the roots are 2·2540333 ... and 2·2549942 ... , where the last figure in each case is not to be relied upon.

Explanation of the Successive Steps

(*a*) Decreasing the roots by 2, the resulting equation is

$$3x^4 - 37x^3 - 167x^2 + 92x - 12 = 0.$$

By the question, this has two roots between 0 and 1.

(*b*) Suspecting two nearly equal roots, we apply the rule of Art. 10, and the figure 2 is suggested by

$$-\frac{2(-12)}{92} = 0{\cdot}2 \ldots \quad \text{and} \quad -\frac{92}{2(-167)} = 0{\cdot}2 \ldots .$$

Decreasing the roots by 2, the sign of the last term of the resulting equation is −. If the roots are decreased by 3, it will be found that again the sign of the last term is −. So far, then, we have no assurance that 2 is the proper figure.

But when the roots are decreased by 2, the signs of the terms are + − − + −, and when they are decreased by 3, the signs are + − − − −, so that two changes of sign are lost. Hence, by Art. 8, the two roots lie between 2·2 and 2·3.

(*c*) and (*d*). Similar remarks apply to these steps. The figure 5 in (*c*) is suggested by

$$-\frac{2(-5712)}{20856} = 0{\cdot}5 \quad \text{and} \quad \frac{20856}{2(-18848)} = 0{\cdot}5 \ldots .$$

Thus the roots lie between 2·254 and 2·255.

(*e*) and (*e'*). Here the rule of Art. 10 fails, and the separation of the roots is a matter of trial. If at (*e*) we decrease the roots by 1 (instead of by 0), the signs of the terms are + − − + +, one change of sign being lost. The same is true if, as at (*e'*), we decrease the roots by 9. Thus one root lies between 2·2540 and 2·2541, and the other between 2·2549 and 2·255 ; and the roots are separated.

Immediately after the roots have been separated, i.e. beginning at the stages (*f*), (*f'*), the proper figure is suggested by Newton's rule.

EXERCISE XLIX

Find by Newton's method to five significant figures:

1. The two positive roots of $x^3 - 7x + 7 = 0$.
2. The positive root of $2x^3 - 3x - 6 = 0$.

Find the roots indicated below correct to seven places of decimals by Horner's method.

3. The real root of $x^3 + 29x - 97 = 0$.
4. The cube root of 4129.
5. The three real roots of $x^3 - 3x + 1 = 0$.
6. The two roots of $2x^3 + 11x^2 - 10x + 1 = 0$ in the interval (0, 1).
7. The two roots of $x^3 - 20x^2 - 31x + 1609 = 0$ in the interval (13, 15).
8. The two roots of $2x^3 - 15x^2 - 8x + 166 = 0$ in the interval (5, 6).
9. The greatest and least roots of

$$3x^4 - 61x^3 + 127x^2 + 220x - 520 = 0.$$

10. The two roots of $3x^4 - 10x^2 - 6x + 16 = 0$ in the interval (1, 2).
11. The two roots of $3x^4 - 2x^2 - 34x + 40 = 0$ in the interval (1, 2).
12. The two roots of $7x^4 - 11x^2 - 32x + 40 = 0$ in the interval (1, 2).
13. The two roots of $x^4 - 2x^3 - 13x^2 - 11x + 133 = 0$ in the interval (3, 4).
14. The two roots of $3x^4 - 4x^3 - 12x^2 + 12x + 9 = 0$ in the interval (1, 2).
15. The two roots of $2x^4 - 16x^3 - 17x^2 + 392x - 782 = 0$ in the interval (4, 5).
16. The two roots of $x^4 + 58x^2 - 388x + 571 = 0$ in the interval (2, 3).
17. The two roots of $2x^4 - 27x^3 + 25x^2 + 179x - 275 = 0$ in the interval (2, 3).
18. The positive root of $2x^4 + 5x^3 + 4x^2 + 3x = 8002$.
19. The greatest root of the following:

(i) $x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962$.
(ii) $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x = 110$.
(iii) $x^5 + 12x^4 + 59x^3 + 150x^2 + 201x = 207$.

20. If $f(x) = e^x - 1 - 2x$, show that the equation $f(x) = 0$ has just one solution other than zero, the solution being $x = 1{\cdot}260 \ldots$.
Show also that $f(x) < 0$ if $0 < x < 1{\cdot}260$.

21. If $f(x) = e^x - e^{-x} - 4x$, show that the equation $f(x) = 0$ has just one positive solution given by $x = 2{\cdot}177 \ldots$, and that $f(x) < 0$ if $0 < x < 2{\cdot}177$.

22. Find an approximate solution of $x^x = 10$, showing that $x = 2{\cdot}50616$ (nearly), where the last figure is not to be relied on. Verify that for this value of x, $x^x = 9{\cdot}9995$ (approx.).
[Here x is given by $f(x) = x^{-1} - \log_{10} x = 0$. By drawing the curves $y = x^{-1}$ and $y = \log_{10} x$, it will be found that $x = 2{\cdot}5$ is a good approximation. Five applications of Newton's process give $x = 2{\cdot}50616$.]

CHAPTER XXIX

IMPLICIT FUNCTIONS, CURVE TRACING

1. Implicit Functions. If two variables x and y are connected by an equation represented by $f(x, y)=0$, we say that y is an *implicit function* of x.

In particular, if $f(x, y)$ is a polynomial in x, y and $f(x, y)=0$, then y is called an *algebraic function* of x.

Taking as a simple case the equation

$$y^2-2xy+2x^2=1,$$

this gives $y=x\pm\sqrt{(1-x^2)}$, and so defines y as a two-valued function of x.

But if, as generally happens, we are unable to express y explicitly in terms of x, we are not justified in assuming without further enquiry that y is really a function of x in the sense hitherto understood.

The general question involves difficult analysis, and cannot be considered here. Some exercise in tracing curves from their equations will, however, convince the student that, at any rate in a large number of cases, the equation $f(x, y)=0$ *does* define y as a single- or multiple-valued function of x.

The general form of a curve represented by an algebraic equation can generally be found by pure algebra, and affords excellent practice in finding approximate values.

2. A Rule for Approximations. The following process is of great practical use.

Suppose that $$y=a+xf(y)+x^2\phi(y)+x^3\psi(y)+\dots, \quad \text{.....(A)}$$

where x is small and $f(y)$, $\phi(y)$, etc., do not involve x. *It is required to approximate to the value of y, in ascending powers of x.*

Let $$\left.\begin{aligned} f(y+h)&=f(y)+hf_1(y)+h^2f_2(y)+\dots\\ \phi(y+h)&=\phi(y)+h\phi_1(y)+h^2\phi_2(y)+\dots\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}\right\}. \quad \text{.....(B)}$$

The first approximation is obtained by putting $x=0$, giving

$$y=a. \quad \text{.....(C)}$$

For the second approximation, let $y=a+\alpha$ where α is $O(x)$,* then

$$\begin{aligned} y &= a+xf(a+\alpha)+x^2\phi(a+\alpha)+\dots \\ &= a+x\{f(a)+\alpha f_1(a)+\dots\}+x^2\{\phi(a)+\dots\}+\dots , \end{aligned}$$

and neglecting small quantities of the second order, i.e. terms containing $x\alpha$, x^2, etc., *the second approximation is*

$$y=a+xf(a), \quad \dots\dots\dots\

To do this, write equation (A) in the form

$$\sin y = y - \tfrac{1}{6}\sin^3 y - \tfrac{3}{40}\sin^5 y - \tfrac{5}{112}\sin^7 y - \dots \quad \dots\dots\dots\

3. Cubic Equation with Three Real Roots. If all the roots of $ax^3+3bx^2+3cx+d=0$ are real, the equation can be reduced to

$$y^3-y+\mu=0 \quad \text{where} \quad \mu^2<4/27 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

by a transformation of the form $x=p+qy$, where p, q are real numbers (Ch. XII, 5, Ex. 1). The last example suggests the possibility of expressing the roots of equation (A) as power series in μ.

Theorem. *If* $\mu^2<4/27$, *the roots of* $y^3-y+\mu=0$ *are*

$$F(\mu)-F(-\mu), \quad 1-F(\mu), \quad -1+F(-\mu),$$

where

$$F(\mu)=\frac{\mu}{2}+\frac{3}{\lfloor 2}\left(\frac{\mu}{2}\right)^2+\frac{4\,.\,6}{\lfloor 3}\left(\frac{\mu}{2}\right)^3+\frac{5\,.\,7\,.\,9}{\lfloor 4}\left(\frac{\mu}{2}\right)^4+\frac{6\,.\,8\,.\,10\,.\,12}{\lfloor 5}\left(\frac{\mu}{2}\right)^5$$

$$+\ldots+\frac{(n+1)(n+3)(n+5)\ldots(3n-3)}{\lfloor n}\left(\frac{\mu}{2}\right)^n+\ldots\,.$$

Write the equation in the form $y^3-y=2x$, ……(A)

where $x=-\mu/2$ and $x^2<1/27$.

Assuming that the values of y can be expressed as power series in x, by Taylor's theorem, these values are given by

$$y=(y)_0+x(y_1)_0+\frac{x^2}{\lfloor 2}(y_2)_0+\ldots+\frac{x^n}{\lfloor n}(y_n)_0+\ldots\,, \quad \ldots\ldots\ldots\text{(B)}$$

where $y_n=\dfrac{d^n y}{dx_n}$ and $(y)_0$, $(y_1)_0$, $(y_2)_0$, … are the values of y, y_1, y_2, … when $x=0$. (Of course, x is to be made zero *after* differentiation.)

The assumption will be justified if the series is convergent.

Let $y=u+v$, then $y^3-3uvy=u^3+v^3$. This equation will be identical with (A) if $\qquad u^3+v^3=2x \quad \text{and} \quad 3uv=1.$

Thus we may take $u^3=x+(x^2-k)^{\frac{1}{2}}$, $v^3=x-(x^2-k)^{\frac{1}{2}}$, where $k=1/27$.

Writing $u_n=\dfrac{d^n u}{dx^n}$, $v_n=\dfrac{d^n v}{dx^n}$, and differentiating with regard to x,

we have $\qquad 3u^2u_1=1+x(x^2-k)^{-\frac{1}{2}}=u^3(x^2-k)^{-\frac{1}{2}}\,;$

therefore $(x^2-k)^{\frac{1}{2}}u_1=u/3\,;$ and similarly $(x^2-k)^{\frac{1}{2}}v_1=-v/3.$

Differentiating again, we have $(x^2-k)^{\frac{1}{2}}u_2+xu_1(x^2-k)^{-\frac{1}{2}}=u_1/3,$

giving $\qquad (x^2-k)\,u_2+xu_1=(x^2-k)^{\frac{1}{2}}\,.\,u_1/3=u/9.$

Similarly, $\qquad (x^2-k)\,v_2+xv_1=v/9.$

Since $y=u+v$, it follows that $(x^2-k)\,y_2+xy_1=y/9.$ ……(C)*

* An alternative proof of this result is indicated in Ex. 6 of the next Exercise.

Differentiating n times by Leibniz' theorem (Ch. XXIII, 11),

$$(x^2-k)y_{n+2}+(2n+1)xy_{n+1}+(n^2-\tfrac{1}{9})y_n=0. \quad \text{.....(D)}$$

Putting $x=0$ and $k=1/27$, we have for $n=0, 1, 2, \ldots$

$$(y_{n+2})_0=3(3n+1)(3n-1)(y_n)_0. \quad \text{.....(E)}$$

The possible values of $(y)_0$ are $0, \pm 1$.

(i) *If* $(y)_0=-1$, since $y_1=2/(3y^2-1)$ we have $(y_1)_0=1$.

Hence from (**E**) in succession,

$$(y_2)_0=3\,.\,1\,.\,(-1)(y)_0=3, \qquad (y_3)_0=\frac{6\,.\,4\,.\,2}{2}(y_1)_0=6\,.\,4,$$

$$(y_4)_0=\frac{9\,.\,7\,.\,5}{3}(y_2)_0=9\,.\,7\,.\,5, \qquad (y_5)_0=\frac{12\,.\,10\,.\,8}{2}(y_3)_0=12\,.\,10\,.\,8\,.\,6;$$

and if $n\geqslant 2$, $\qquad (y_n)_0=(3n-3)(3n-5)\ldots(n+3)(n+1)$.

Thus $y=-1+F(x)$ where

$$F(x)=x+\frac{3}{\lfloor 2}x^2+\frac{4\,.\,6}{\lfloor 3}x^3+\ldots+\frac{(n+1)(n+3)\ldots(3n-3)}{\lfloor n}x^n+\ldots,$$

for such values of x as make the series convergent.

Denoting the series by $u_1+u_2+u_3+\ldots$, we have

$$\frac{u_{n+2}}{u_n}=\frac{3(3n+1)(3n-1)}{(n+2)(n+1)}x^2\to 27x^2 \text{ as } n\to\infty.$$

Hence the series Σu_{2n}, Σu_{2n+1}, and consequently also Σu_n, are convergent if $x^2<1/27$.

(ii) *If* $(y)_0=1$, then $(y_1)_0=1$. Hence the numerical value of $(y_n)_0$ is the same as before, the sign being + or − according as n is odd or even. Therefore

$$y=1-F(-x).$$

(iii) Since the sum of the roots of (**A**) is zero, the root corresponding to $(y)_0=0$ is given by $y=F(-x)-F(x)$.

Putting $x=-\mu/2$, we have the results stated above.

A more general theorem, which includes this, is given in Ex. L, 7.

EXERCISE L

1. Having given that

$$x=\tan x-\tfrac{1}{3}\tan^3 x+\tfrac{1}{5}\tan^5 x-\tfrac{1}{7}\tan^7 x+\ldots,$$

prove that

$$\tan x=x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\tfrac{17}{315}x^7+\ldots.$$

2. If $y=x+ay^n$ where x, y are both small, prove that approximately

$$y=x+ax^n+na^2x^{2n-1}+\tfrac{1}{2}n(3n-1)a^3x^{3n-2}.$$

3. If $y=x+ay^2+by^3$ and x, y are both small, then approximately

$$y=x+ax^2+(b+2a^2)x^3+5a(b+a^2)x^4.$$

4. If $y=1+xa^y$ and x is small, then approximately

$$y=1+xa+x^2a^2\log a+\tfrac{3}{2}x^3a^3(\log a)^2.$$

5. If s, t are positive integers $(s>t)$ and λ is small, the equation

$$y^s-y^t+(s-t)\lambda=0$$

has a root nearly equal to

$$1-\lambda-(-1+s+t)\frac{\lambda^2}{\lfloor 2}-(-1+2s+t)(-1+s+2t)\frac{\lambda^3}{\lfloor 3}.$$

6. If $y^3-y=2x$, show that

(i) $4(27x^2-1)=(3y^2-1)^3-3(3y^2-1)^2$;

(ii) $y_2=-\dfrac{24y}{(3y^2-1)^3}=-\dfrac{12(y-3xy_1)}{(3y^2-1)^2}$,

where $y_1=\dfrac{dy}{dx}$, $y_2=\dfrac{d^2y}{dx^2}$. Deduce that $(x^2-\frac{1}{27})y_2+xy_1=\frac{1}{9}y$.

7. If $y=u+v$ where $u^m=x+\sqrt{(x^2-k)}$, $v^m=x-\sqrt{(x^2-k)}$, m being a positive integer, then

(i) x and y are connected by the equation

$$2x=y^m-\frac{m}{\lfloor 1}y^{m-2}k^{\frac{1}{m}}+\frac{m(m-3)}{\lfloor 2}y^{m-4}k^{\frac{2}{m}}-\frac{m(m-4)(m-5)}{\lfloor 3}y^{m-6}k^{\frac{3}{m}}+\dots. \quad \text{.....(A)}$$

(ii) If x is sufficiently small,

$$y=(y)_0+x(y_1)_0+\frac{x^2}{\lfloor 2}(y_2)_0+\dots, \quad \text{.....(B)}$$

where $(y)_0$ is a value of y obtained by putting $x=0$ in (**A**), and $(y_1)_0$, $(y_2)_0$, ... are the corresponding values of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, ...

(iii) Using Ch. XXIII, 11, Ex. 1, show that

$$k(y_{n+2})_0=\left(n^2-\frac{1}{m^2}\right)(y_n)_0.$$

Hence prove that equation (**B**) holds if $x^2<k$.

(iv) If m is odd, one value of $(y)_0$ is 0, and then

$$y=(-1)^{\frac{m-1}{2}}\cdot\frac{2}{mk^{\frac{m-1}{2m}}}\left\{x+\frac{x^3}{\lfloor 3}\cdot\frac{1}{k}\left(1^2-\frac{1}{m^2}\right)+\frac{x^5}{\lfloor 5}\cdot\frac{1}{k^2}\left(1^2-\frac{1}{m^2}\right)\left(2^2-\frac{1}{m^2}\right)+\dots\right\}.$$

(v) If $m=5$ and $k^{\frac{1}{5}}=\frac{1}{5}$, equation (**A**) becomes

$$y^5-y^3+\tfrac{1}{5}y=2x,$$

and the values of $(y)_0$ are 0, $\pm\sqrt{\left(\frac{1}{2}\pm\frac{1}{2\sqrt{5}}\right)}$.

Thus all the roots are real, and they can be expressed as power series in x if $x^2<\dfrac{1}{5^5}$.

(vi) If $m=7$ and $k^{\frac{1}{7}}=\frac{1}{7}$, the equation is $y^7-y^5+\frac{2}{7}y^3-\frac{1}{49}y=2x$.

4. Tangents and Asymptotes. If P is a point on the curve and Q a point on it near P, the tangent at P is defined as the limiting position of the secant PQ as Q moves up to P. Thus as Q moves to P, the distance of Q from the tangent at P becomes indefinitely small, so that near P the curve is indefinitely close to the tangent.

If P moves to an infinite distance along a branch of the curve, and the tangent at P tends to a limiting position, this limiting position is called an *asymptote* to the curve. It may happen that an infinite branch has no asymptote. We then find the simplest equation which nearly represents the branch at ∞, and say that this is the equation to a *curvilinear asymptote*.

In the examples which follow, tangents and asymptotes are found by a process of approximation.

5. Intersections of a Straight Line and Curve. If $u=0$ is the equation to a curve of the nth degree and $y=\mu x+\nu$ that of any straight line, the abscissae of the points of intersection are given by an equation of the form

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{A})$$

obtained by substituting $\mu x+\nu$ for y in $u=0$.

(i) If this equation has two equal roots, the line meets the curve in 'two coincident points' and, except in special cases (see (iii)), touches the curve. If three roots are equal, the line meets the curve in three coincident points. Except in special cases (see (iii)), the point of contact is a *point of inflexion* (Ch. XVII, 20). At such a point, the curve *crosses the tangent.*

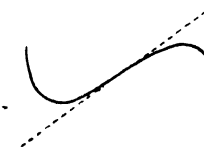


FIG. 68.

(ii) If $a_0=0$, one root of equation (**A**) is infinite, and the line meets the curve in *one point at infinity.*

If $a_0=0$ and $a_1=0$, the line meets the curve in *two points at infinity* and, except in special cases (see (iv)), is an asymptote.

(iii) Any point (D) where a curve crosses itself is called a *double point*, or *node.* At such a point there are two tangents, each meeting the curve in *three* consecutive points. Any other straight line through D meets the curve in *two* consecutive points. (Fig. 69.)

If the double point is at infinity, the tangents are parallel straight lines. So in this case the curve has two parallel asymptotes, each meeting the curve in three points at infinity. *Any* straight line parallel to the asymptotes meets the curve in two points at infinity.

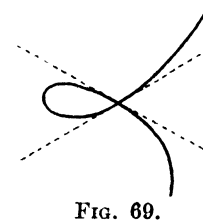


FIG. 69.

If the tangents at a double point coincide, the point is called a *cusp*. There are two kinds of cusp, as shown in Fig. 70.

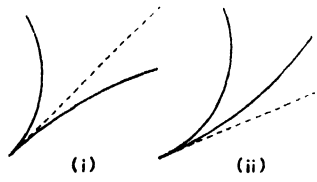


FIG. 70.

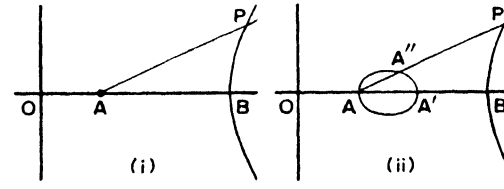


FIG. 71.

In tracing a curve, it is important to be able to account for *all the n points* in which any straight line meets the curve, remembering that imaginary points occur in pairs.

(iv) Fig. 71, (i), is a sketch of the curve whose equation is

$$y^2 = (x-a)^2(x-b).$$

The axis of x meets the curve in $A(a, 0)$ and $B(b, 0)$. The equation to any straight line AP through A is $y = m(x-a)$; and, on eliminating y, we have $(x-a)^2(x-b-m^2) = 0$; hence, AP meets the curve in *two coincident points at* A, and at a point P, given by $x = b + m^2$. Therefore, A is the *only point* on the curve to the left of the line $y = b$; and thus A may be regarded as *a double point at which the tangents are imaginary*.

A point on a curve, such as A, is called a *conjugate point*, if no other point in its immediate neighbourhood is on the curve.

The conjugate point at A, in Fig. 71, (i), may be considered as the limiting form, when $a = a'$, of the oval in Fig. 71, (ii), which is a sketch of the curve whose equation is

$$y^2 = (x-a)(x-a')(x-b),$$

6. Curve Tracing. To find the general shape of a curve represented by a given equation, proceed as follows. (i) Consider any symmetry which the curve may possess, e.g. symmetry with regard to either axis, or in opposite quadrants. (ii) Mark any points on the curve which can be found easily, e.g. the points (if any) where it cuts the axes. (iii) Find the approximate form of the curve near some of these points.

Near (0, 0) (if on the curve) x, y are small, *but not necessarily of the same order*. To find the form near any other point, move the origin to that point.

The quickest way of finding the tangent at any point is to find the value of $\frac{dy}{dx}$. But, without further differentiation, we cannot say on which side of the tangent the curve lies, or discover any peculiarities which the point may possess. So, for a *few* points, it is better to use a process of approximation, as in Ex. 1.

(iv) Find any regions in which there is no part of the curve. (Exx. 1, 2.)

(v) Find the form of the curve near infinity, i.e. where one or both of x and y are large, but not necessarily of the same order.

In doing this, we find the (linear) asymptotes (if there are any), and the approximate form of any ' parabolic ' branches.

The reader should be able to draw the ' parabolas ' represented by $y^m = x^n$, where m and n are positive integers. For example,

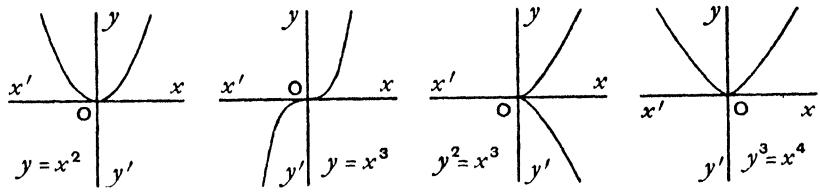


FIG. 72.

(vi) By considering the value of $\frac{dy}{dx}$, find the tangents (if any exist) which are parallel to the axes. Notice that, at a point of inflexion, $\frac{d^2y}{dx^2} = 0$.

(vii) It may be possible to find y explicitly in terms of x or x in terms of y; or again, we may be able to express x and y as functions of some variable t. We can then find any number of points on the curve and the gradients at these points.*

7. Examples on Curve Tracing.

Ex. 1. *Trace the curve* $(y^2 - 1)y = (x^2 - 4)x$. .....(A)

The equation is unaltered by writing $-x$, $-y$ for x, y. Hence there is symmetry in opposite quadrants. The points $(0, 0)$, $(0, \pm 1)$, $(\pm 2, 0)$, $(\pm 2, \pm 1)$ are on the curve.

Write the equation $y^3 - x^3 - y + 4x = 0$. .....(A)

Near $(0, 0)$, x and y are small and of the same order.

1st approx., $y = 4x$.

This is the equation to the tangent at $(0, 0)$.

2nd approx., $y = 4x + 63x^3$,

obtained by putting $(4x)^3$ for y^3 in (A), showing that the curve is above the tangent in the first quadrant and below it in the third quadrant. Thus $(0, 0)$ is a *point of* inflexion.

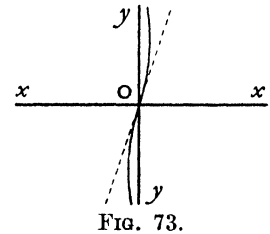


FIG. 73.

* The examples given in Art. 7 and in Exercise LI have been chosen so that one of these methods can be used. Thus in each case the curve can be ' plotted ' to any degree of accuracy required; so that the student can verify the conclusions arrived at by the methods just described, and gain confidence in using them. However, when only the *characteristic form* of the curve is required, the advantages of the above method over the method of plotting, even when the latter can be done, will be obvious. [See Ex. 22 of Exercise LI.]

Near (0, 1), putting $Y = y - 1$, equation (**A**) becomes $2Y + 3Y^2 + Y^3 = (x^2 - 4)x$.

1st approx., $2Y = -4x$ or $y = 1 - 2x$,

which is the equation to the tangent at (0, 1).

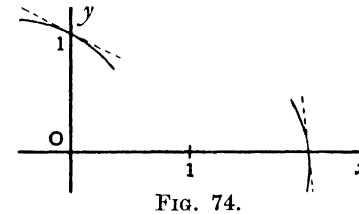


FIG. 74.

2nd approx., $2Y + 3(-2x)^2 = -4x$, giving $y = 1 - 2x - 6x^2$, which shows that the curve is below the tangent.

Near (2, 0), putting $X = x - 2$, we find that $y = -8x - 6x^2$ (approx.), showing that the equation to the tangent is $y = -8(x - 2)$ and that the curve is below it.

Again, $dy/dx = (3x^2 - 4)/(3y^2 - 1)$. Therefore, at each of the points $(\pm 2, \pm 1)$, the gradient of the tangent is 4. The tangent is parallel to Ox where $x = \pm 2/\sqrt{3} = \pm 1{\cdot}16$; and the tangent is parallel to Oy, where $y = \pm 1/\sqrt{3} = \pm 0{\cdot}58$.

Near (∞, ∞), x and y are both large and of the same order; and approximately

$$y^3 - x^3 = 0, \quad \text{giving} \quad y - x = 0.$$

2nd approx. From (**A**), $$y - x = \frac{y - 4x}{x^2 + xy + y^2} = \frac{-3x}{3x^2} = -\frac{1}{x}.$$

This shows that, as $x \to \infty$, the curve becomes indefinitely close to the line $y - x = 0$, which is therefore an asymptote.

Again, on the curve $y = x - \frac{1}{x}$ (approx.), and on the asymptote $y = x$; showing that the curve is *below* the asymptote on the right and *above* it on the left.

Thus the curve is as given in Fig. 75.

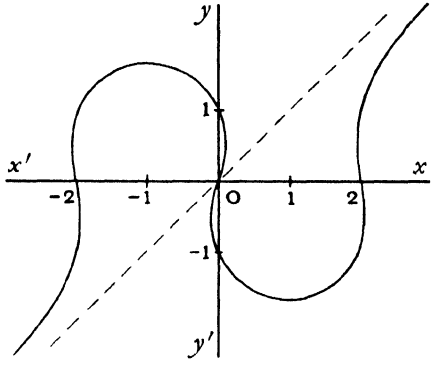


FIG. 75.

Putting $y = tx$ in equation (**A**), it will be seen that any number of points on the curve can be found by giving any values to t in the equations $y = tx$, $x^2 = (4 - t)/(1 - t^3)$.

If $1 < t < 4$, the corresponding point is *imaginary*; thus no part of the curve lies in the space swept out by a line turning round O from the position $y = x$ to the position $y = 4x$.

Ex. 2. *Trace the curve* $(x-1)(x-2)y^2=2x^2$. .....(A)

The equation is unaltered by writing $-y$ for y; hence the curve is symmetrical with regard to Ox. Also y is imaginary if $1<x<2$; hence no part of the curve lies between the lines $x=1$, $x=2$. Write the equation in the form

$$x^2y^2-3xy^2+2(y^2-x^2)=0. \quad \text{.....(A)}$$

Near (0, 0), the curve is approximately given by $y^2-x^2=0$ or $y=\pm x$.

Taking $y=x$ as a first approximation, from (A)

$$2(y-x)=(3xy^2-x^2y^2)/(y+x);$$

giving as a second approximation

$$y-x=\frac{1}{2}\cdot\frac{3x^3}{2x}=\frac{3}{4}x^2,$$

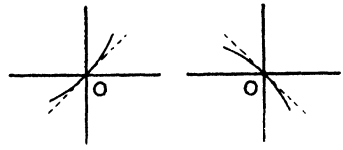


FIG. 76.

showing that $y-x=0$ is a tangent at (0, 0) and that the curve is above the tangent. Similarly $y+x=0$ is a tangent at (0, 0) which is a *double* point.

Near (1, ∞), $x-1=\frac{2x^2}{(x-2)y^2}=\frac{2.1^2}{(1-2)y^2}=-\frac{2}{y^2}$.

Near (2, ∞), $x-2=\frac{2x^2}{(x-1)y^2}=\frac{2.2^2}{(2-1)y^2}=\frac{8}{y^2}$,

showing that $x-1=0$ and $x-2=0$ are asymptotes, the curve approaching the first from the left and the second from the right.

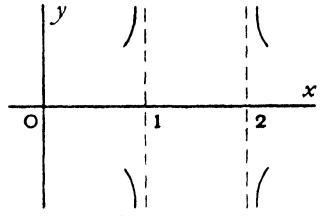


FIG. 77.

Again, writing equation (A) in the form $x^2(y^2-2)=3xy^2-2y^2$, it will be seen that near $(\infty, \sqrt{2})$, $y-\sqrt{2}=\frac{3}{\sqrt{2}.x}$; and near $(\infty, -\sqrt{2})$, $y+\sqrt{2}=-\frac{3}{\sqrt{2}.x}$.

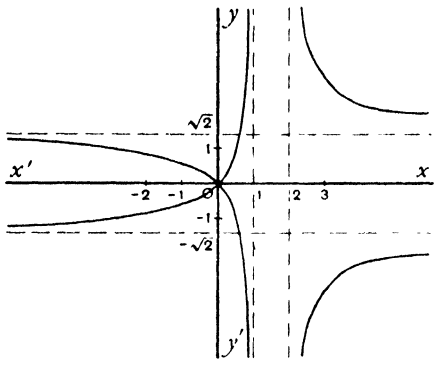


FIG. 78.

We can now sketch the curve, as in Fig. 78.

NOTE. The asymptotes parallel to Ox can be found *by equating to zero the coefficient of the highest power of* y *in* (A): similarly for those parallel to Oy. Also, for all values of k except ± 1, the line $y=k$ meets the curve in two finite points and in two points at ∞. The same is true for the line $x=l$, where $l<1$ or >2.

8. In the next example we consider the possibility of finding the approximate form of a curve near the origin, or near infinity, *by retaining only certain terms in its equation.*

Ex. 1. *Trace the curve* $x^3+y^3-3axy=0 \quad (a>0)$. .....(A)

Omitting the first term, we have $y^3-3axy=0$.

Disregarding the factor y, we consider the equation $y^2-3ax=0$. .....(B)

If y is small, x is small and $O(y^2)$; so that x^3 is $O(y^6)$, and can therefore be neglected in comparison with the other terms in (A). Hence (B) approximately represents the curve or part of the curve near (0, 0).

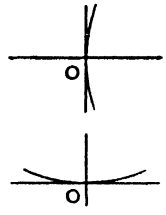


FIG. 79.

Similarly, by omitting the second term, we find that part of the curve near (0, 0) is represented by $x^2-3ay=0$.

Omitting the third term, we get $x^3+y^3=0$.

This represents the curve near (∞, ∞); for if y is $O(x)$, then $3axy$ is $O(x^2)$, and can be neglected in comparison with the other terms of (A). Thus near (∞, ∞), as a first approximation $y+x=0$.

For a second approximation, $y+x=3a\cdot\dfrac{xy}{x^2-xy+y^2}=3a\cdot\dfrac{-x^2}{3x^2}=-a.$

A third approximation is obtained by substituting $-x-a$ for y, thus

$$y+x=3a\frac{xy}{x^2-xy+y^2}=3a\cdot\frac{-x(x+a)}{x^2+x(x+a)+(x+a)^2}=-a\left(1-\frac{a^2}{3x^2}+\dots\right),$$

(the last step being obtained by long division). Hence the third approximation is $y+x=-a+\dfrac{a^3}{3x^2}$, showing that $y+x=-a$ is an asymptote, and that the curve approaches it from above at both ends.

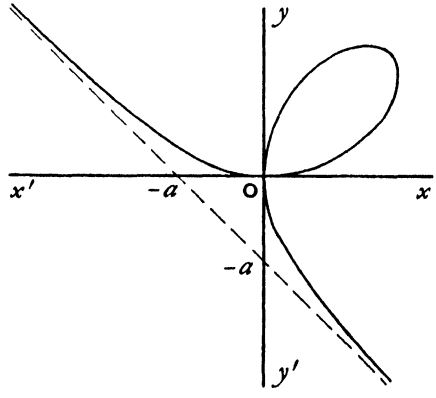


FIG. 80.

The curve is symmetrical with regard to the line $y=x$, for its equation is unaltered if x and y are interchanged. Further, if $y=tx$, then $x=3at/(1+t^3)$, so that any number of points can be found by giving different values to t; e.g. $t=1$, $x=y=3a/2$.

9. Newton's Parallelogram. Consider the curve represented by $u=0$ where u is a polynomial in x, y.

Let v be the expression obtained by retaining only certain terms of u. A rule which goes by the name of '*Newton's Parallelogram*' enables us to choose these terms so that $v=0$ may represent the curve, or part of it, near the origin or near infinity, when such parts exist.

Let ax^my^n, bx^py^q, cx^ry^s, ... be any terms of u. Represent these by points $A, B, C, \ldots$ whose coordinates referred to any axes are (m, n), (p, q), (r, s), ...

The figure so formed is '*Newton's Diagram.*'

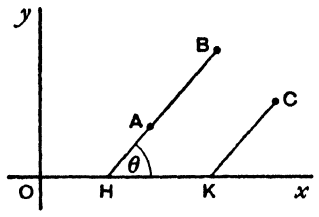


FIG. 81.

Suppose that ax^my^n, bx^py^q are small or large numbers of the same order; then y^{n-q} is $O(x^{p-m})$, and therefore y is $O(x^{(p-m)/(n-q)})$. Join AB, cutting Ox in H, and let the angle $xHA=\theta$; then y is $O(x^{-\cot\theta})$ and ax^my^n is $O(x^{m-n\cot\theta})$. Also, $m-n\cot\theta=OH$; therefore the terms represented by A, B are $O(x^{OH})$.

Draw CK parallel to AB to meet Ox in K; then, for similar reasons, the term cx^ry^s represented by C is $O(x^{OK})$. Thus the term C is of an order lower or higher than the terms A, B according as C is on the same side of AB as O, or on the opposite side. Hence the following rule.

Rule. Mark on Newton's diagram points $A, B, C, \ldots$ representing all the terms of the polynomial u. Choose any two of these, say A, B.

Let v be the expression whose terms are those represented by A, B and any points which may happen to lie on the line AB.

If all the other points lie on the side of AB remote from the origin O, then $v=0$ approximately represents the curve $u=0$ (or part of it) near O (if on the curve).

If all the other points lie on the same side of AB as O, then $v=0$ represents the curve $u=0$ (or part of it) near infinity.

NOTE. If the straight line AB passes through 0, the rule fails; for the corresponding terms cannot be of the same order.

In the following examples the points representing the 1st, 2nd, 3rd, ... terms of the equation are marked 1, 2, 3, ... in the diagram.

Ex. 1. $x^3+y^3-3axy=0$.

The terms 1, 3 and 2, 3 give the form of the curve near the origin.

The terms 1, 2 give the form near infinity. (See Ex. 1, Art. 8.)

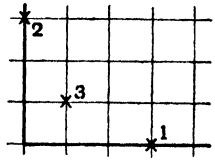


FIG. 82.

Ex. 2. $(ax-by)^2=ax^2y+y^4$.

Write the equation in the form,

$$a^2x^2-2abxy+b^2y^2-ax^2y-y^4=0.$$

The form near the origin is given by terms 1, 2, 3, and near infinity by the terms 1, 4 and 4, 5.

Thus *near* (0, 0), $(ax-by)^2=0$.

Near infinity, $a^2x^2-ax^2y=0$ and $ax^2y+y^4=0$, giving $y-a=0$ and $y^3+ax^2=0$.

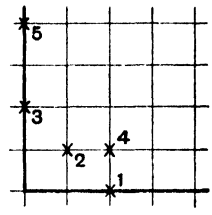


FIG. 83.

EXERCISE LI

Trace the curves represented by the following equations and work out the accompanying details. (For hints for *plotting*, see Ex. 22.)

1. $y^3-y+x=0$. [Fig. 84, (i).]

Near (0, 0), $y=x+x^3$. Near (0, 1), $y=1-\frac{1}{2}x-\frac{3}{8}x^2$.

Near (∞,∞), $y=-x^{\frac{1}{3}}-\frac{1}{3}x^{-\frac{1}{3}}$.

The tangent is perpendicular to Ox at the points $\pm\frac{2}{3\sqrt{3}}$, $\pm\frac{1}{\sqrt{3}}$.

2. $y^2=x^2(x+1)$. [Fig. 84, (ii)]. Near (0, 0), $y=\pm(x+\frac{1}{2}x^2)$; near ∞, $y^2=x^3$.

3. $x^3+y^3=a^2x$. [Fig. 84, (iii)]. Near (0, 0), $y^3=a^2x$; near (∞,∞), $y=-x+\frac{a^2}{3x}$.

4. $x^3+y^3=3ax^2$. [Fig. 84, (iv)].

Near (0, 0), $y^3=3ax^2$; near (∞,∞), $y=-x+a+\frac{a^2}{x}$.

5. $(x-a)y^2=a^2x$. [Fig. 84, (v)]. Near (0, 0), $y^2+ax=0$;

near (a,∞), $x=a+\frac{a^3}{y^2}$; near $(\infty,\pm a)$, $y=\pm\left(a+\frac{a^2}{2x}\right)$.

6. $y^2(x-a)=x^2(x+a)$. [Fig. 84, (vi)]. O is on the curve, but the curve passes through no real point near O: consequently O is a '*conjugate point*.'

Near (∞,∞), $y=\pm\left(x+a+\frac{a^2}{2x}\right)$. If $\frac{dy}{dx}=0$, $x=\frac{1}{2}a(\sqrt{5}+1)=1{\cdot}6$ approx.

7. $x(y-x)^2=a^2y$. [Fig. 84, (vii)]. Near (0, 0), $x^3=a^2y$. Near $(0,\infty)$, $x=\frac{a^2}{y}$.

Near (∞,∞), $y=x\pm a$, closer approximations being $y=x\pm a+\frac{a^2}{2x}$.

Tangents are parallel to Ox where $y=3x$.

8. $x(y-x)^2=ay^2$. [Fig. 84, (viii)]. Near (0, 0), $x^3-ay^2=0$.

Near (a,∞), $x=a+\frac{2a^2}{y}$. Near (∞,∞), 1st approx., $y=x\pm\sqrt{ax}$.

2nd approx., $y=x\pm\sqrt{ax}+a$. (**A**). 3rd approx., $y=x\pm\sqrt{ax}+a\pm\sqrt{\frac{a^3}{x}}$.

This shows that the curve becomes indefinitely close to that represented by (**A**), i.e. to the parabola $(y-x-a)^2=ax$. This is therefore a parabolic asymptote.

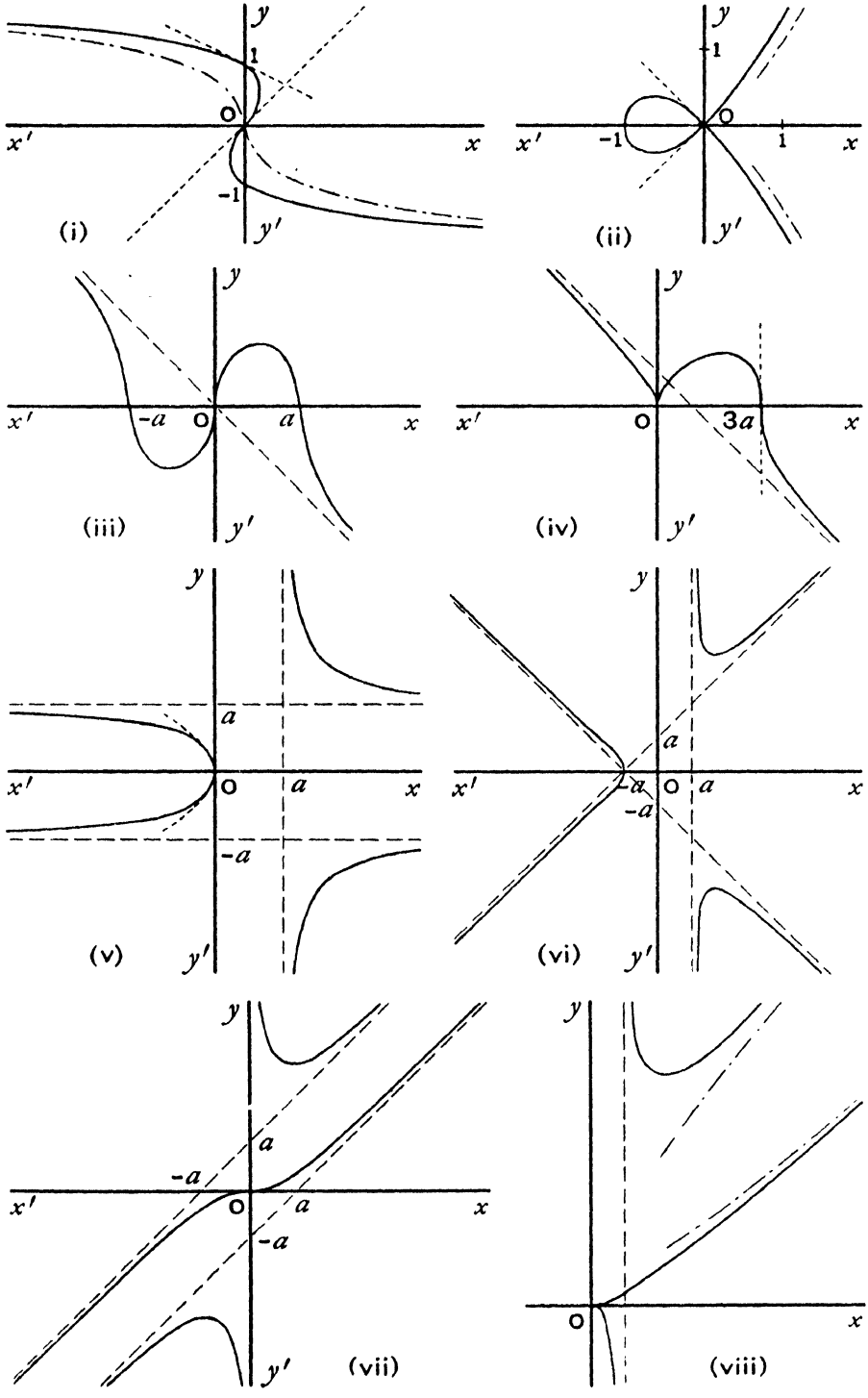

y
1
O
x'
x
-1
y'
(i)
y
1
O
x'
-1
1
x
y'
(ii)
y
x'
-a
O
a
x
(iii)
y'
y
x'
O
3a
x
(iv)
y'
y
a
x'
O
a
x
-a
(v)
y'
y
a
x'
-a
O
a
x
-a
(vi)
y'
y
a
-a
x'
O
a
x
-a
y'
(vii)
y
O
x
(viii)

FIG. 84.

9. $x^4 - 3xy^2 + 2y^3 = 0$. [Fig. 85, (i).]

Near (0, 0), $x^3 - 3y^2 = 0$ and $y = \frac{3}{2}x - \frac{2}{9}x^2$.

Near (∞, ∞), 1st approx., $x^4 + 2y^3 = 0$, i.e. $y = -\dfrac{x^{\frac{4}{3}}}{2^{\frac{1}{3}}}$. .....(A)

2nd approx., $y = -\dfrac{x^{\frac{4}{3}}}{2^{\frac{1}{3}}} + \dfrac{x}{2}$. .....(B)

Equation (A) is represented by —·—·— in the diagram; it gives the general trend of the curve, but (B) shows that it is not a parabolic asymptote.

If $y = tx$, then $x = t^2(3 - 2t)$. Mark points on the curve corresponding to $t = 1, 2, -\frac{1}{2}, -1$.

10. $ax(y - x)^2 = y^4$. [Fig. 85, (ii).] Near (0, 0), $y - x = 0$ and $ax = y^2$.

With $y = x$ as 1st approx., the 2nd is $y = x \pm \dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$. Near (∞, ∞), $ax^3 - y^4 = 0$.

11. $x^4 - a^2xy - b^2y^2 = 0$. [Fig 85, (iii).] Near (0, 0), $y = -\dfrac{a^2}{b^2}x - \dfrac{x^3}{a^2}$ and $x^3 - a^2y = 0$. Near (∞, ∞), by solving for y and expanding in descending powers of x, show that approximately

$$y = \pm\frac{x^2}{b} - \frac{a^2x}{2b^2} \mp \frac{a^4}{8b^3}.$$

The equations $y = \pm\dfrac{x^2}{b}$ are represented by —·—·— in the diagram.

12. $(3x - 2y)^2 - 3x^2y - y^4 = 0$. [Fig. 85, (iv).] .....(A)

The points (0, 0) $(0, \pm 2)$ are on the curve. Near (0, 0), $3x - 2y = 0$.

2nd approx., $y = \dfrac{3x}{2} \pm \dfrac{3}{2\sqrt{2}}x^{\frac{3}{2}}$, showing that the point O is a cusp, the tangent being $3x - 2y = 0$, which meets the curve at *one* other point, viz. $(-\frac{8}{9}, -\frac{4}{3})$.

Near $(\infty, 3)$, $y = 3 - \dfrac{12}{x}$. The asymptote $y = 3$ cuts the curve again at $(-\frac{5}{4}, 3)$.

Near (∞, ∞), $3x^2 + y^3 = 0$ (represented by —·—·—). .....(B)

2nd approx., $y = -3^{\frac{1}{3}}x^{\frac{2}{3}} - 1$, showing that the curve (A) is below (B) and that $(y + 1)^3 + 3x^2 = 0$ is a parabolic asymptote.

If $3x^2 + y^3 = 0$, then $(3x - 2y)^2 = 0$, showing that the curves (A), (B) *touch* at the point $(-\frac{8}{9}, -\frac{4}{3})$.

By solving for x, show that $y < -1$ or else $0 < y < 4$. Also show that the lines $y = -1$ and $y = 4$ touch the curve.

Fig. 85, (v), (vi), (vii), (viii), are curves whose equations are given in Exx. 13-16. These are of the form $xy^2 - 2ky = px^3 + qx^2 + rx + s$.* (See Exx. 17, 18.)

13. $xy^2 - 12y = x^3 - 15x^2 + 63x - 85$.

14. $xy^2 - 16y = x^3 - 14x^2 + 60x - 104$.

15. $xy^2 - 24y = x^3 - 10x^2 - 5x + 150$.

16. $xy^2 - 8y = x^3 - 8x^2 - 3x - 8$.

17. Show that every curve whose equation is $xy^2 - 2ky = px^3 + qx^2 + rx + s$ has three asymptotes, or only one, according as p is positive or negative.

* Curves having an equation of this form belong to the first of four classes into which Newton divided cubic curves (*Enumeratio Linearum tertii ordinis*, 1706).

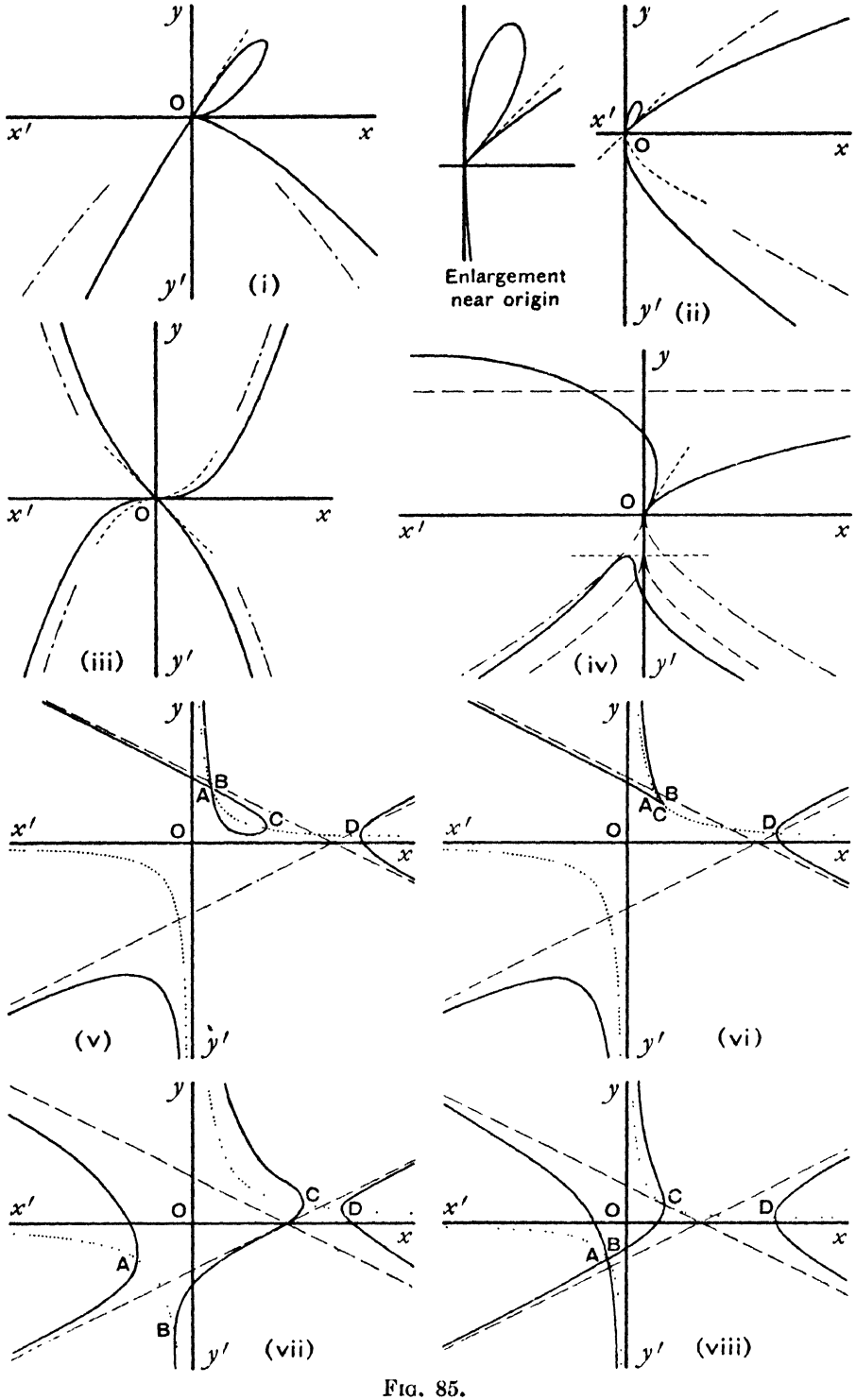


FIG. 85.

18. (i) Show that, if $xy^2 - 2ky = px^3 + qx^2 + rx + s$, where $p \neq 0$, then, for values of x other than zero, y is given by an equation of the form

$$(xy - k)^2 = p(x - a)(x - b)(x - c)(x - d).$$

(ii) In Exx. 13-16 employ the methods of the previous examples to find first and second approximations to the curve at infinity; and, by the help of the points, A, B, C, D, which correspond to $x = a, b, c, d$, and of the points where the curves cut the asymptotes, trace the curves.

NOTE. Diagrams (v) and (vi) show that a cusp is of a higher order of singularity than a node; while (vii) and (viii) show the 'change of partners' that generally occurs when two of the points, A, B, C, D, approach one another, coincide, and then separate again, owing to a change in the coefficients, p, q, r, s.

Points of Inflexion

19. Show that the points of inflexion of the cubic curve

$$y(a'x^2 + 2b'x + c') = ax^3 + 3bx^2 + 3cx + d \quad \text{.....(A)}$$

lie on the straight line whose equation is

$$\tfrac{4}{3}y(a'c' - b'^2) = (ac' - 2bb' + ca')x + (bc' - 2cb' + da').$$

[From (i), by differentiation,

$$\tfrac{1}{3}\frac{dy}{dx}(a'x^2 + 2b'x + c') + \tfrac{2}{3}y(a'x + b') = ax^2 + 2bx + c. \quad \text{.....(B)}$$

Differentiating again, and putting $\dfrac{d^2y}{dx^2} = 0$, we have

$$\tfrac{2}{3}\frac{dy}{dx}(a'x + b') + \tfrac{1}{3}ya' = ax + b. \quad \text{.....(C)}$$

The coordinates of a point of inflexion satisfy equations (A), (B), and (C). Multiplying equations (A), (B), (C) respectively by

$$a', \quad -2(a'x + b'), \quad a'x^2 + 2b'x + c',$$

the result is obtained by addition.

20. Show that the curve (p. 475), whose equation is $(y^2 - 1)y = (x^2 - 4)x$, has three points of inflexion which very nearly lie on the line $x = 64y$.

[If $p = dy/dx$, show that, at a point of inflexion, $x^2 = (4p^6 - p^4)/(p^6 - 1)$, and $y^2 = (4p^2 - 1)/(p^6 - 1)$; so that $p^4 + 8p^3 - 2p - 4 = 0$.]

21. Prove that the points of inflexion on the curve $y^2 = x^2(x^2 + 2ax + b)$ are determined by the equation $2x^3 + 6ax^2 + 3(a^2 + b)x + 2ab = 0$.

[At a point of inflexion $y\dfrac{dy}{dx} = 2x^3 + 3ax^2 + bx$, and $\left(\dfrac{dy}{dx}\right)^2 = 6x^2 + 6ax + b$.]

22. Plot some of the curves in Examples 1-12, as follows:

For Ex. 1, plot $x = -y^3$, and $x = y$, and add corresponding abscissae.
For Exx. 2, 5, 6, 7, 8, 11, express y in terms of x.
For Exx. 3, 4, 9, 10, use $y = tx$, or $x = ty$.
For Ex. 12, express x in terms of y.

CHAPTER XXX

INFINITE PRODUCTS

1. If $a_1, a_2, \ldots a_n \ldots$ is a sequence of numbers, real or complex, the product $a_1 a_2 \ldots a_n$ will be denoted by P_n or by $\Pi_{r=1}^{r=n} a_r$. Thus we write

$$P_n = \Pi_{r=1}^{r=n} a_r = a_1 a_2 \ldots a_n.$$

Definition. If P_n tends to a finite limit P, *different from zero*, as $n \to \infty$, except when a factor of P_n is zero, we say that the infinite product $a_1 a_2 a_3 \ldots$ is *convergent and that its value is* P, or that it *converges to* P, and we write

$$P = a_1 a_2 a_3 \ldots \text{ to } \infty = \Pi_{n=1}^{n=\infty} a_n, \text{ or simply } = \Pi a_n.$$

If $P_n \to \infty$ or $-\infty$, or if $P_n \to 0$ when no factor of P_n is zero, the infinite product is said to *diverge* or to be *divergent.*

If P_n oscillates, the infinite product is said to *oscillate.*

The reader may ask why we say that the product *diverges* (to zero) when $P_n \to 0$. The reason is as follows. According to the definition, if $a_1, a_2, \ldots$ are real and positive, *the infinite product* Πa_n *and the infinite series* $\Sigma \log a_n$ both converge, both diverge or both oscillate.

For $\log P_n = \log a_1 + \log a_2 + \ldots + \log a_n$, and P_n tends to a finite limit, to ∞, to zero or oscillates according as $\log P_n$ tends to a finite limit, to ∞, to $-\infty$ or oscillates.

If $P_n \to$ a finite limit, then $P_{n-1} \to$ the same limit, and if this limit is not zero $a_n = P_n / P_{n-1} \to 1$. Thus *if* Πa_n *is convergent, then* $a_n \to 1$.

It is generally convenient to write an infinite product in the form

$$P = (1+u_1)(1+u_2) \ldots (1+u_n) \ldots .$$

A necessary (but not sufficient) condition for convergence is that $u_n \to 0$, and so, after a certain stage, $|u_n| < 1$.

The character of P, as regards convergence, will not be affected by removing any number of factors from the beginning of the product. Consequently there will be no loss of generality in supposing that $|u_n| < 1$ for every n.

The simplest and at the same time the most important case is when all the u's have the same sign.

2. Theorem. *If $u_1, u_2, u_3, \ldots$ are positive and less than unity, each of the infinite products*

$$(1+u_1)(1+u_2)(1+u_3)\ldots, \quad (1-u_1)(1-u_2)(1-u_3)\ldots,$$

is convergent or divergent according as the series Σu_n is convergent or divergent.

Proof. The series Σu_n is convergent or divergent.

Suppose that Σu_n is convergent and that its sum is s; then m can be found so that

$$u_m + u_{m+1} + u_{m+2} + \ldots \text{ to } \infty < 1.$$

Also the omission of the first $m-1$ factors of a product does not affect its convergency.

Hence there is no loss of generality in assuming that $s<1$ and consequently $s_n<s<1$ for every n.

By Ch. XIV, 1, we have

$$P_n = (1+u_1)(1+u_2)\ldots(1+u_n) < \frac{1}{1-s_n} < \frac{1}{1-s},$$

$$Q_n = (1-u_1)(1-u_2)\ldots(1-u_n) > 1-s_n > 1-s.$$

Also P_n increases and Q_n decreases as n increases. Therefore P_n and Q_n tend to positive limits as $n\to\infty$, and both of the products $\Pi(1+u_n)$ and $\Pi(1-u_n)$ are convergent.

Next suppose that Σu_n is divergent. By Ch. XIV, 1,

$$P_n = (1+u_1)(1+u_2)\ldots(1+u_n) > 1+s_n \to \infty,$$

$$Q_n = (1-u_1)(1-u_2)\ldots(1-u_n) < \frac{1}{1+s_n} \to 0.$$

Moreover, $Q_n>0$. Therefore $P_n\to\infty$ and $Q_n\to 0$, so that $\Pi(1+u_n)$ diverges to ∞ and $\Pi(1-u_n)$ diverges to zero.

3. General Condition for Convergence. Let (a_n) be a sequence of numbers, real or complex, none of which is zero. The necessary and sufficient condition for the convergence of the infinite product

$$P = a_1 a_2 a_3 \ldots$$

is as follows: *Corresponding to any positive number ϵ, however small, an integer m must exist such that for all values of p*

$$|\, a_{m+1}a_{m+2}\ldots a_{m+p} - 1 \,| < \epsilon,$$

that is to say,

$$\left| \frac{P_{m+p}}{P_m} - 1 \right| < \epsilon \quad \textit{for} \quad p = 1, 2, 3, \ldots, \ldots\ldots\ldots\

Also, by Ch. XV, 6, we can find m such that

$$|P_{m+p}-P_m|<k\epsilon \quad \text{for} \quad p=1, 2, 3, \ldots .$$

Therefore
$$\left|\frac{P_{m+p}}{P_m}-1\right|<\frac{k\epsilon}{|P_m|}<\frac{k\epsilon}{k}=\epsilon,$$

so that condition (**A**) holds.

Conversely, if condition (**A**) holds, then

$$-\epsilon<\frac{P_{m+p}}{P_m}-1<\epsilon \quad \text{and} \quad 1-\epsilon<\frac{P_{m+p}}{P_m}<1+\epsilon. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{B})$$

Taking $\epsilon<1$, we have $|P_{m+p}|>(1-\epsilon).|P_m|$, so that $|P_n|>$some fixed positive number when $n>m$. Consequently P_n cannot tend to zero.

Also by (**B**), P_{m+p} and P_m have the same sign, and therefore

$$|P_{m+p}-P_m|=|P_{m+p}|-|P_m|<\epsilon.|P_m|.$$

Hence $|P_{m+p}-P_m|<\epsilon'$, where ϵ' is positive and as small as we choose. Hence $P_n\to$ a limit different from zero and the product P is convergent. (See Ch. XV, 6.)

4. Absolute Convergence. If $u_1, u_2, u_3, \ldots$ are real or complex numbers whose absolute values, or moduli, are denoted by $u_1', u_2', u_3', \ldots$, the infinite product

$$P=(1+u_1)(1+u_2)(1+u_3)\ldots$$

is said to be *absolutely convergent* when the product

$$P'=(1+u_1')(1+u_2')(1+u_3')\ldots$$

is convergent.

Hence the product $\Pi(1+u_n)$ is absolutely convergent if, and only if, the series Σu_n is absolutely convergent.

Theorem. *An absolutely convergent infinite product* (P) *is convergent.*

Let P and P' be defined as above and suppose that P' is convergent. Then, by Art. 3, we can find m so that

$$|(1+u'_{m+1})(1+u'_{m+2})\ldots(1+u'_{m+p})-1|<\epsilon \quad \text{for} \quad p=1, 2, 3, \ldots .$$

Now
$$|(1+u_{m+1})(1+u_{m+2})\ldots(1+u_{m+p})-1|$$
$$\leqslant|(1+u'_{m+1})(1+u'_{m+2})\ldots(1+u'_{m+p})-1|.$$

This follows on expanding each side and remembering that for any numbers $a, b, c, \ldots$

$$|a+b+c+\ldots|\leqslant|a|+|b|+|c|+\ldots .$$

Therefore

$$|(1+u_{m+1})(1+u_{m+2})\ldots(1+u_{m+p})-1|<\epsilon \quad \text{for} \quad p=1, 2, 3, \ldots .$$

Hence, by Art. 3, P is convergent.

5. Derangement of Factors. We shall prove that *if the factors of an absolutely convergent infinite product* (P) *are rearranged in any way, the product remains absolutely convergent and its value is unaltered.*

Consider the infinite products

$$P=(1+u_1)(1+u_2)\ldots, \quad Q=(1+v_1)(1+v_2)\ldots,$$

where P is absolutely convergent and Q is formed by rearranging the factors of P, so that every v is a u and every u is a v.

Since P is absolutely convergent, so is the series Σu_n. This series remains absolutely convergent when its terms are rearranged. Therefore Σv_n is absolutely convergent, and consequently the product Q is absolutely convergent.

It remains to be shown that $P=Q$. Let $a_r=1+u_r$, $b_r=1+v_r$.

For any suffix m, it is possible to find n so that all the factors of

$$P_m=a_1a_2\ldots a_m \quad \text{are included among the factors of} \quad Q_n=b_1b_2\ldots b_n,$$

and therefore

$$Q_n/P_m=a_pa_q\ldots a_s,$$

where $p, q, \ldots s$ are all $>m$. Also, as m tends to ∞, so does n, and since P is convergent, it follows from Art. 3 that

$$|\, a_pa_q\ldots a_s-1\,|\to 0, \quad \text{and therefore} \quad Q_n/P_m\to 1.$$

Thus P_m and Q_n tend to the same limit, that is to say, $P=Q$.

6. Expansion of an Infinite Product as a Series. Consider the infinite product

$$P=(1+xu_1)(1+xu_2)(1+xu_3)\ldots,$$

where $u_1, u_2, u_3, \ldots$ and x are any numbers, real or complex.

Let p_r be the sum of the products of $u_1, u_2, \ldots u_n$ taken r together, so that

$$P_n=(1+xu_1)(1+xu_2)\ldots(1+xu_n)=1+p_1x+p_2x^2+\ldots+p_nx^n.$$

We shall prove that *if the series* Σu_n *is absolutely convergent, then*

(i) *As* $n\to\infty$, $p_r\to$ *a finite limit* l_r *for* $r=1, 2, \ldots n$.

(ii) *For all values of* x,

$$P=(1+xu_1)(1+xu_2)(1+xu_3)\ldots \text{ to } \infty = 1+l_1x+l_2x^2+\ldots \text{ to } \infty.$$

Let $u_r'=|\,u_r\,|$, $x'=|\,x\,|$, and

$$P_n'=(1+x'u_1')(1+x'u_2')\ldots(1+x'u_n')=1+p_1'x'+p_2'x'^2+\ldots+p_n'x'^n,$$

so that p_r' is the sum of the products of $u_1', u_2', \ldots u_n'$, taken r together.

Each of these products multiplied by $\lfloor r$ occurs in the expansion of

$$(u_1'+u_2'+\ldots+u_n')^r,$$

therefore $p_r' \,.\, \underline{|r}$ occurs in the expansion, and it follows that

$$p_r' < (u_1' + u_2' + \ldots + u_n')^r / \underline{|r}.$$

Since Σu_n is absolutely convergent, $\Sigma u_n'$ converges to a sum s', and therefore

$$p_r' < s'^r / \underline{|r}.$$

Also p_r' increases with n, therefore, as $n \to \infty$, $p_r' \to$ a limit l_r' such that

$$0 < l_r' < s'^r / \underline{|r}.$$

If $r \to \infty$, then $s'^r / \underline{|r} \to 0$, and consequently $l_r' \to 0$, so that l_r' is finite, however great r may be.

Thus as $n \to \infty$, p_r' becomes an infinite series, which is convergent; and p_r becomes an absolutely convergent series. Hence $p_r \to$ a finite limit l_r, and we may write $l_r = (1 + \epsilon_r) p_r$ where $\epsilon_r \to 0$.

Let $Q_n = 1 + l_1 x + l_2 x^2 + \ldots + l_n x^n$, then

$$\begin{aligned} |\, Q_n - P_n \,| &= |\, \epsilon_1 p_1 x + \epsilon_2 p_2 x^2 + \ldots + \epsilon_n p_n x^n \,| \\ &\leqslant |\, \epsilon_m \,| \,.\, (p_1' x' + p_2' x'^2 + \ldots + p_n' x'^n) \\ &\leqslant |\, \epsilon_m \,| \,.\, (P_n' - 1), \end{aligned}$$

where ϵ_m is the greatest of $|\, \epsilon_1 \,|$, $|\, \epsilon_2 \,|$, etc.

Therefore $Q_n - P_n \to 0$, and since $P_n \to P$, it follows that, for all values of x,

$$P = 1 + l_1 x + l_2 x^2 + \ldots \text{ to } \infty .$$

Ex. 1. *Find the coefficient of* z^r *in the expansion of*

$$(1 + zx)(1 + zx^2)(1 + zx^3) \ldots (1 + zx^n).$$

Assume that

$$(1 + zx)(1 + zx^2) \ldots (1 + zx^n) = 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n.$$

In this identity put zx for z, therefore

$$(1 + zx^2)(1 + zx^3) \ldots (1 + zx^{n+1}) = 1 + p_1 zx + p_2 z^2 x^2 + \ldots + p_n z^n x^n ;$$

therefore

$$\begin{aligned} &(1 + zx^{n+1})(1 + p_1 z + p_2 z^2 + \ldots + p_n z^n) \\ &\quad = (1 + zx)(1 + p_1 zx + p_2 z^2 x^2 + \ldots + p_n z^n x^n). \end{aligned}$$

Equating the coefficients of z^r,

$$p_r + p_{r-1} x^{n+1} = p_r x^r + p_{r-1} x^r ;$$

$$\therefore \; p_r = \frac{1 - x^{n-r+1}}{1 - x^r} \cdot x^r p_{r-1}.$$

Putting $r - 1$, $r - 2$, \ldots 2, 1 in succession for r,

$$p_{r-1} = \frac{1 - x^{n-r+2}}{1 - x^{r-1}} \cdot x^{r-1} p_{r-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$p_1 = \frac{1 - x^n}{1 - x} \cdot x.$$

Whence by multiplication

$$p_r = \frac{(1 - x^n)(1 - x^{n-1})(1 - x^{n-2}) \ldots (1 - x^{n-r+1})}{(1 - x)(1 - x^2)(1 - x^3) \ldots (1 - x^r)} \cdot x^{\frac{1}{2} r(r+1)}.$$

Ex. 2. *If* $|x|<1$, *show that for all values of* z,

$$(1+zx)(1+zx^2)(1+zx^3)\ldots \text{ to } \infty = 1+l_1z+l_2z^2+\ldots+l_rz^r+\ldots,$$

where

$$l_r = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)}\cdot x^{\frac{1}{2}r(r+1)}.$$

The series $x+x^2+x^3\ldots$ is absolutely convergent, therefore the infinite product is convergent, and denoting its value by P, by the theorem just proved,

$$P = 1+l_1z+l_2z^2+\ldots+l_rz^r+\ldots,$$

where $l_r = \lim_{n\to\infty} p_r$ and p_r is the same as in Ex. 1. Also x^n, x^{n-1}, $\ldots x^{n-r+1}$ all tend to zero as $n\to\infty$, therefore l_r has the value stated above.

7. Another Method. When the factors are real and positive, the theory of infinite products can be made to depend on that of infinite series by the use of logarithms. Let $P_n = (1+u_1)(1+u_2)\ldots(1+u_n)$, where $u_1, u_2, \ldots$ are real and $-1<u_r<1$ for every r. As explained in Art. 1, the last assumption involves no loss of generality. Then

$$\log P_n = \log(1+u_1)+\log(1+u_2)+\ldots+\log(1+u_n).$$

Hence the infinite product P and the infinite series $\Sigma\log(1+u_n)$ both converge, both diverge or both oscillate. By Ch. XIX, 7, (4),

$$\text{if } x>-1 \text{ and } \neq 0, \text{ then } \frac{1}{2}\frac{x^2}{h}<x-\log(1+x)<\frac{1}{2}\frac{x^2}{l}, \quad \ldots\ldots\ldots(\mathbf{A})$$

where l is the smaller and h the greater of the numbers 1 and $1+x$.

Let

$$u_{m+1}+u_{m+2}+\ldots+u_n = r_{m,n},$$
$$u_{m+1}^2+u_{m+2}^2+\ldots+u_n^2 = s_{m,n},$$
$$\log(1+u_{m+1})+\log(1+u_{m+2})+\ldots+\log(1+u_n) = t_{m,n}.$$

Putting $u_{m+1}, u_{m+2}, \ldots u_n$ for x in (**A**), by addition we find that

$$\frac{1}{2}\frac{s_{m,n}}{H}<r_{m,n}-t_{m,n}<\frac{1}{2}\frac{s_{m,n}}{L},$$

where L is the least and H the greatest of the numbers 1, $1+u_1$, $1+u_2$, $\ldots$, $1+u_n$.

(i) *Suppose that* Σu_n^2 *is convergent*, then by the general principle of convergence we can find m so that for all values of n, $s_{m,n}<2L\epsilon$, and therefore

$$0<r_{m,n}-t_{m,n}<\epsilon,$$

where ϵ is positive and as small as we please.

Hence by the general principle of convergence, P *converges, diverges to* ∞, *diverges to* 0, *or oscillates, as* Σu_n *converges, diverges to* ∞, *diverges to* $-\infty$, *or oscillates.*

(ii) *Suppose that* Σu_n^2 *diverges*, then since $r_{m,n} - t_{m,n} > s_{m,n}/2H$, it follows that, for sufficiently large values of n,

$$r_{m,n} - t_{m,n} > N$$

for any positive value of N, however great. Hence

(*a*) *If* Σu_n *converges or oscillates between finite limits, then* P *diverges to* 0.

(*b*) *If* Σu_n *diverges to* $+\infty$ *or oscillates so that its upper limit is* $+\infty$, *then* P may *converge.*

This includes all possible cases, and when the logarithm of a complex number has been defined and its properties investigated, the same method can be applied when the factors of P are complex numbers.

EXERCISE LII

Obtain the results in Exx. 1-5 by considering the value of P_n in each case.

1. $(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{4})\ldots \to \infty$. $[P_n = \frac{1}{2}(n+1).]$

2. $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})\ldots \to 0$. $\left[P_n = \frac{1}{n}.\right]$

3. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots \to \frac{1}{2}$. $\left[P_n = \frac{1}{2}.\frac{n+1}{n}.\right]$

4. $(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{5})\ldots \to \frac{1}{2}$. $[P_{2n+1} = \frac{1}{2}$ and $P_n/P_{n-1} \to 1.]$

5. $(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\ldots \to 1$. $[P_{2n+1} = 1.]$

6. Show that

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}.$$

Hence if $|x| < 1$,

$$(1+x)(1+x^2)(1+x^4)\ldots \text{ to } \infty = \frac{1}{1-x}.$$

7. Show that

$$\sin x = 2^n \sin\frac{x}{2^n}\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^n}.$$

Remembering that $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, deduce that

$$\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots \text{ to } \infty = \frac{\sin x}{x}.$$

8. Show that $(1+x)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\ldots$ diverges to ∞ or to 0 according as $x \gtrless 0$.

9. Show that the infinite products

$$\left(1+\frac{x}{1^2}\right)\left(1+\frac{x}{2^2}\right)\left(1+\frac{x}{3^2}\right)\left(1+\frac{x}{4^2}\right)\ldots \text{ and } \left(1+\frac{x}{1^2}\right)\left(1-\frac{x}{2^2}\right)\left(1+\frac{x}{3^2}\right)\left(1-\frac{x}{4^2}\right)\ldots$$

are convergent for all values of x.

10. If $|x|<1$ and $|zx|<1$, show that

$$\frac{1}{(1-zx)(1-zx^2)\dots(1-zx^n)}=1+q_1z+q_2z^2+\dots+q_rz^r+\dots,$$

where $$q_r=\frac{(1-x^n)(1-x^{n+1})\dots(1-x^{n+r-1})}{(1-x)(1-x^2)\dots(1-x^r)}\cdot x^r.$$

[Proceed as in Art. 6, Ex. 1.]

11. If n, r are positive integers, show that

$$(1-x^n)(1-x^{n+1})\dots(1-x^{n+r-1})$$

is exactly divisible by $(1-x)(1-x^2)\dots(1-x^r)$.

12. If $|x|<1$ and $|zx|<1$, show that

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\dots \text{ to } \infty}=1+t_1z+t_2z^2+\dots+t_rz^r+\dots \text{ to } \infty,$$

where $$t_r=x^r/(1-x)(1-x^2)\dots(1-x^r).$$

[Having explained why such an expansion is possible, put zx for z and show that

$$(1-zx)(1+t_1z+t_2z^2+\dots)=1+t_1zx+t_2z^2x^2+\dots.$$

Hence by equating the coefficients of z^r,

$$(1-x^r)t_r=x\,.\,t_{r-1}.]$$

13. If $(1+zx)(1+zx^3)\dots(1+zx^{2n-1})=1+p_1z+p_2z^2+\dots+p_nz^n$, show that

$$p_r=\frac{(1-x^{2n})(1-x^{2(n-1)})\dots(1-x^{2(n-r+1)})}{(1-x^2)(1-x^4)\dots(1-x^{2r})}\cdot x^{r^2}.$$

14. If $|x|<1$, show that for all values of z,

$$(1+zx)(1+zx^3)(1+zx^5)\dots=1+l_1z+l_2z^2+\dots+l_rz^r+\dots,$$

where $$l_r=\frac{1}{(1-x^2)(1-x^4)\dots(1-x^{2r})}\cdot x^{r^2}.$$

15. If $|x|<1$ and $|zx|<1$, show that

(i) $$\frac{1}{(1-zx)(1-zx^3)\dots(1-zx^{2n-1})}=1+q_1z+q_2z^2+\dots,$$

where $$q_r=\frac{(1-x^{2n})(1-x^{2(n+1)})\dots(1-x^{2(n+r-1)})}{(1-x^2)(1-x^4)\dots(1-x^{2r})}\cdot x^r.$$

(ii) $$\frac{1}{(1-zx)(1-zx^3)(1-zx^5)\dots \text{ to } \infty}=1+t_1z+t_2z^2+\dots \text{to } \infty,$$

where $$t_r=x^r/(1-x^2)(1-x^4)\dots(1-x^{2r}).$$

CHAPTER XXXI

PERMUTATIONS, COMBINATIONS AND DISTRIBUTIONS

1. Combinations with Repetitions. The number of combinations of n things r together, when each may be taken as often as we please, is denoted by H_r^n, and is the same as the number of homogeneous products of degree r which can be formed with n letters $a, b, c, \ldots k$.

The *sum* of these products is the coefficient of x^r in the expansion of

$$(1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)\ \ldots\ (1+kx+k^2x^2+\ldots).$$

When $a, b, \ldots k$ are all equal to unity, we obtain the *number* of the products. Thus

$$H_r^n = \text{coefft. of } x^r \text{ in } (1+x+x^2+\ldots)^n$$
$$= \text{coefft. of } x^r \text{ in } (1-x)^{-n};$$

therefore
$$H_r^n = \frac{n(n+1)\ldots(n+r-1)}{\lfloor r} = C_r^{n+r-1}.$$

2. Combinations and Permutations of n Things, not all different. Let C be the number of combinations, and P the number of permutations, of n letters $aa\ldots bb\ldots cc\ldots$ taken k together, then

(1) *C is equal to the coefficient of x^k in the product*

$$(1-x)^{-f}(1-x^{p+1})(1-x^{q+1})(1-x^{r+1})\ \ldots,$$

where f is the number of different letters, p is the number of a's, q the number of b's, r the number of c's, and so on.

For the *sum* of the combinations in question is the coefficient of x^k in the product

$$(1+ax+a^2x^2+\ldots+a^px^p)(1+bx+b^2x^2+\ldots+b^qx^q)\ldots$$
$$\ldots(1+cx+c^2x^2+\ldots+c^rx^r)\ \ldots \text{ to } f \text{ factors};$$

and their *number* is the coefficient of x^k in the expression

$$(1+x+x^2+\ldots+x^p)(1+x+x^2+\ldots+x^q)(1+x+x^2+\ldots+x^r)\ \ldots,$$

which is obtained from the preceding by putting $a, b, c, \ldots$ equal to unity. Whence the result follows.

(2) *P is $\lfloor k$ times the coefficient of x^k in the product*

$$\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^p}{\lfloor p}\right)\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^q}{\lfloor q}\right)$$
$$\cdot\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^r}{\lfloor r}\right) \ldots \text{ to } f \text{ factors.}$$

For the number of permutations involving λa's, μb's, νc's, ... is

$$\lfloor \lambda+\mu+\nu+\ldots \,/\, \lfloor \lambda \, \lfloor \mu \, \lfloor \nu \ldots .$$

Hence
$$\frac{P}{\lfloor k}=\Sigma\frac{1}{\lfloor \lambda \, \lfloor \mu \, \lfloor \nu \ldots},$$

where the summation is to include every set of values of $\lambda, \mu, \nu, \ldots$ chosen (with repetitions) from 0, 1, 2, 3, ... such that

$$\lambda+\mu+\nu+\ldots=k.$$

Therefore $P/\lfloor k$ is the coefficient of x^k in the above product.

Ex. 1. *Find the number of combinations (C) and the number of permutations (P) of the letters of the word parallel taken four together.*

We have by the preceding,

$$C=\text{coefficient of } x^4 \text{ in } (1-x)^{-5}(1-x^3)(1-x^4)(1-x^2)^3,$$
$$P/\lfloor 4=\text{coefficient of } x^4 \text{ in } \left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}\right)\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}\right)\left(1+\frac{x}{\lfloor 1}\right)^3,$$

whence we find that $C=22$, $P=286$.

3. Distributions. If n things are divided into r lots or classes, the result is called a *distribution*. A number of different cases arise, thus :

(i) Some of the things may be alike, or they may be all different.

(ii) The *order* of the things in each class may, or may not, have to be considered. In the first case the classes are called *groups* ; in the second, they are called *parcels*.

(iii) When a distribution has been made, it may, or may not, be necessary to consider the order in which the classes stand.

In the first case the *classes are said to be different* (meaning that they have to be distinguished from one another) ; in the second case they are said to be *indifferent*.

(iv) Blank lots may, or may not, be admissible, a ' blank lot ' being a ' lot ' which contains none of the things.

For example, consider what is implied in the following questions.

Ex. 1. *How many different signals can be made by flying five flags a, b, c, d, e on three masts ?*

Denoting the masts by 1, 2, 3, the following arrangements give different signals.

	1	2	3
(A)	*a*	*d*	*e*
	b		
	c		

	1	2	3
(B)	*b*	*d*	*e*
	a		
	c		

	1	2	3
(C)	*d*	*a*	*e*
		b	
		c	

Because such arrangements as (A) and (B) are different, it is a case of distribution in *groups*.

Because (A), (C) are different, the *groups are said to be different.*

Further *blank lots are admissible*, for all the flags may be flown on only one or only two masts. (See Art. 4, Ex. 1.)

Ex. 2. *In how many ways can five books a, b, c, d, e be divided among three people denoted by* 1, 2, 3 *?*

Such distributions as (A), (B) in Ex. 1 are identical, so it is a case of distribution in *parcels.* Also the distributions (A), (C) are different, and so the *parcels are said to be different.*

Further, it is implied that each person gets at least one book, and so blank lots are not allowed. (See Art. 5, Ex. 1.)

Ex. 3. *In how many ways can five books be tied up in three bundles ?*

Here the order of the books in a bundle does not matter, so the distribution is in *parcels.* Also the bundles are not to be distinguished from one another, and so *the parcels are indifferent.* (See Art. 5, Ex. 2.)

4. Arrangement in Groups. *The number of ways in which n different things can be arranged in r different groups is*

$$r(r+1)(r+2)\dots(r+n-1) \quad \textit{or} \quad \lfloor n\, C^{n-1}_{r-1},$$

according as blank groups are or are not admissible.

Proof. (i) Let n letters $a, b, \dots k$ be written in a row in any order. All the arrangements of the letters in r groups, blank groups being admissible, can be obtained thus : place among the letters $r-1$ marks of partition, and arrange the $n+r-1$ things (consisting of letters and marks) in all possible orders.* Since $r-1$ of the things are alike, the number of different arrangements is $\lfloor n+r-1 / \lfloor r-1 = r(r+1)(r+2)\dots(r+n-1)$.

(ii) All the arrangements of the letters in r groups, none of the groups being blank, can be obtained as follows : (*a*) Arrange the letters in all possible orders. This can be done in $\lfloor n$ ways. (*b*) In every such arrangement, place $(r-1)$ marks of partition in $(r-1)$ out of the $(n-1)$ spaces between the letters. This can be done in C^{n-1}_{r-1} ways.

Hence the required number is $\lfloor n\, C^{n-1}_{r-1}$.

Ex. 1. *How many different signals can be made by flying five different flags on three masts ?*

The required number $= 3 . 4 . 5 . 6 . 7 = 2520$.

* Thus one arrangement of 7 letters in 5 groups, the second blank, is indicated by $a \,||\, bcd \,|\, ef \,|\, g$.

5. Distribution in Parcels.

(1) *The number of ways in which n different things can be distributed into r different parcels, blank lots being admissible, is* r^n.

For each of the n things can be assigned to any one of the r parcels.

(2) *The number of ways in which n different things can be distributed into r different parcels, there being no blank lots, is*

$$r^n - C_1^r(r-1)^n + C_2^r(r-2)^n - \ldots + (-1)^{r-1}C_{r-1}^r,$$

which is $\lfloor n$ *times the coefficient of* x^n *in the expansion of* $(e^x-1)^r$.

Proof. In any distribution, denote the parcels by $\alpha_1, \alpha_2, \ldots \alpha_r$, and consider the distributions in which blanks are allowed.

The total number of these is r^n.

The number in which α_1 is blank is $(r-1)^n$.

Therefore the number in which α_1 is not blank is $\Big\}\ r^n-(r-1)^n$.

Of these last, the number in which α_2 is blank is $\Big\}\ (r-1)^n-(r-2)^n$.

Therefore the number in which α_1, α_2 are not blank is $\Big\}\ r^n-2(r-1)^n+(r-2)^n$.

Of these last, the number in which α_3 is blank is $\Big\}\ (r-1)^n-2(r-2)^n+(r-3)^n$.

Therefore the number in which α_1, α_2, α_3 are not blank is $\Big\}\ r^n-3(r-1)^n+3(r-2)^n-(r-3)^n$.

This process can be continued as far as we like, and it is obvious that the coefficients are formed as in a binomial expansion.

Hence *the number of distributions in which no one of x assigned parcels is blank is* $r^n - C_1^x(r-1)^n + C_2^x(r-2)^n - \ldots + (-1)^x(r-x)^n$.

When $x=r$, this gives the result in question.

A *complete* proof is as follows :

Let u_x denote the number of distributions in which no one of the parcels $\alpha_1, \alpha_2, \ldots \alpha_x$ is blank. Of these, the number in which α_{x+1} is blank is obtained by changing r into $r-1$ in u_x (which is a function of r). This number is represented by $E^{-1}u_x$, where E^{-1} denotes the operation of changing r into $r-1$.

Therefore the number of distributions in which no one of $\alpha_1, \alpha_2, \ldots \alpha_{x+1}$ is blank is $u_x - E^{-1}u_x$, that is to say, $u_{x+1} = (1-E^{-1})u_x$.

Also we have $u_1 = r^n - (r-1)^n = (1-E^{-1})r^n$.

Hence $$u_x = (1 - E^{-1})^x r^n$$
$$= (1 - C_1^x E^{-1} + C_2^x E^{-2} - \ldots + (-1)^x E^{-x}) r^n$$
$$= r^n - C_1^x (r-1)^n + C_2^x (r-2)^n - \ldots + (-1)^x (r-x)^n,$$

and when $x = r$, this gives the result in question.

Ex. 1. *In how many ways can five different books be distributed among three persons, if each person is to have at least one book?*

The required number $= 3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$.

Ex. 2. *In how many ways can five different books be tied up in three bundles?*

The required number $= \frac{1}{\lfloor 3} (3^5 - 3 \cdot 2^5 + 3 \cdot 1^5) = \frac{150}{6} = 25$.

(3) *The number of ways in which n things of the same sort can be distributed into r different parcels is* C_{r-1}^{n+r-1} *or* C_{r-1}^{n-1}, *according as blank lots are or are not admissible.*

Proof. If there are no blank lots, any such distribution can be effected as follows: place the n things in a row and put marks of partition in a selection of $r-1$ out of the $n-1$ spaces between them. This can be done in C_{r-1}^{n-1} ways, which is therefore the number of such distributions.

If blank lots are allowed, the number of distributions is the same as that of $n+r$ things of the same sort into r parcels with no blank lots. For such a distribution can be effected thus: put one of the $n+r$ things into each of the r parcels, and distribute the remaining n things into r parcels, blank lots being allowed. Hence the number in question is C_{r-1}^{n+r-1}.

These theorems can also be proved as in Ex. 2 *of the next section.*

(4) *The number of ways in which n things of the same sort can be distributed into r different parcels, where no parcel is to contain more than k things and there are no blank lots, is the coefficient of* x^n *in the expansion of*

$$(x + x^2 + x^3 + \ldots x^k)^r \quad \textit{or of} \quad (1 - x^{k+1})^r (1 - x)^{-r}.$$

For the product of r factors, each of which is $x + x^2 + x^3 + \ldots + x^r$, may be denoted by $\Sigma x^\lambda x^\mu x^\nu \ldots$, where x^λ, x^μ, x^ν, ... are any terms selected from x, x^2, ... x^k, the summation to include every such selection.

The coefficient of x^n in the product is therefore the number of ways in which r numbers $\lambda, \mu, \ldots$ can be chosen from 1, 2, 3, ... k, repetitions being allowed, such that their sum may be n, and this is the required number.

Distributions of things of the same sort into *indifferent parcels* are considered in Art. 8.

Ex. 1. *In how many ways can three persons, each throwing a single die once, make a total score of* 11 ?

The required number is equal to the co. of x^{11} in $(x+x^2+x^3+\ldots+x^6)^3$

$$=\text{co. of } x^{11} \text{ in } x^3(1-x^6)^3(1-x)^{-3}$$

$$=\text{co. of } x^8 \text{ in } (1-x)^{-3}(1-3x^6)$$

$$=\text{co. of } x^8 \text{ in } \left(\frac{3.4}{2}x^2+\frac{9.10}{2}x^8\right)(1-3x^6)$$

$$=45-18=27.$$

Ex. 2. *Apply this method to prove the theorems of the last section.*

The number of distributions of n like things into r parcels is the coefficient of x^n in $(1+x+x^2+\ldots \text{ to } \infty)^r$ or in $(x+x^2+x^3+\ldots \text{ to } \infty)^r$, according as blank lots are or are not allowed.

These expressions are respectively equal to $(1-x)^{-r}$ and $x^r(1-x)^{-r}$, etc.

6. Derangements. Any change in the order of the things in a group is called a *derangement.*

If n things are arranged in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$$\lfloor n \left\{1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\ldots+(-1)^n\cdot\frac{1}{n}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Proof. Denote the group of things by $a_1, a_2, \ldots a_n$. The total number of arrangements is $\lfloor n$; and of these

The number in which a_1 stands first, i.e. is *in situ*, is	$\lfloor n-1.$
Therefore the number in which a_1 is not *in situ* is	$\lfloor n-\lfloor n-1.$
Of these last, the number in which a_2 is *in situ* is	$\lfloor n-1-\lfloor n-2.$
Therefore the number in which neither a_1 nor a_2 is *in situ* is	$\lfloor n-2\,\lfloor n-1+\lfloor n-2.$
Of these last, the number in which a_3 is *in situ* is	$\lfloor n-1-2\,\lfloor n-2+\lfloor n-3.$
Therefore the number in which no one of a_1, a_2, a_3 is *in situ* is	$\lfloor n-3\,\lfloor n-1+3\,\lfloor n-2-\lfloor n-3.$

This process can be continued as far as we like, and it is obvious that the coefficients are formed as in a binomial expansion.

Hence *the number of arrangements in which no one of x assigned letters is* in situ *is*

$$\lfloor n-C_1^x\,\lfloor n-1+C_2^x\,\lfloor n-2-\ldots+(-1)^x\,\lfloor n-x.$$

If $x=n$, the last term is $(-1)^n$, and we obtain the result in question.

A *complete* proof is as follows :

Let v_x denote the number of arrangements in which no one of a_1, a_2, ... a_x is *in situ*.

Of these, the number in which a_{x+1} is *in situ* is obtained by changing n into $n-1$ in v_x (which is a function of n).

This number is represented by $E^{-1}v_x$, where E^{-1} denotes the operation of changing n into $n-1$.

Therefore the number of arrangements in which no one of a_1, a_2, ... a_{x+1} is *in situ* is $v_x - E^{-1}v_x$, that is to say,

$$v_{x+1} = (1 - E^{-1})v_x.$$

Also, as in the preceding, we have

$$v_1 = \underline{|n} - \underline{|n-1} = (1 - E^{-1})\underline{|n}.$$

Hence it follows that

$$\begin{aligned} v_x &= (1 - E^{-1})^x \underline{|n} \\ &= (1 - C_1^x E^{-1} + C_2^x E^{-2} - \ldots + (-1)^x E^{-x}) \underline{|n} \\ &= \underline{|n} - C_1^x \underline{|n-1} + C_2^x \underline{|n-3} - \ldots + (-1)^x \underline{|n-x}. \end{aligned}$$

If $x = n$, the last term is $(-1)^n$, and we obtain the result in question.

NOTE. (i) The expression (**A**) is sometimes called *subfactorial* n, and is denoted by $\underline{||}n$. It is easily shown that $\underline{||}n$ is the integer nearest to $\underline{|n}/e$.

(ii) The values of $\underline{||}n$ for $n = 1, 2, 3, \ldots$ can be calculated thus : Begin with 1 and multiply successively by 1, 2, 3, ... , increasing any even product by 1 and decreasing every odd product by 1 before the next multiplication. Thus

$$\underline{||}1 = 1.1 - 1 = 0, \qquad \underline{||}2 = 0.2 + 1 = 1,$$
$$\underline{||}3 = 3.1 - 1 = 2, \qquad \underline{||}4 = 4.2 + 1 = 9.$$

7. A General Theorem. In Arts. 5 and 6 we have instances of a theorem, stated by Whitworth * as follows :

If there are N sets of letters and, if out of r assigned letters a, b, c, ... , each letter occurs in N_1 sets, each combination of two letters occurs in N_2 sets, each combination of three letters occurs in N_3 sets, and so on ; and finally all the r letters occur in N_r sets, then the number of sets free from all the letters is

$$N - C_1^r N_1 + C_2^r N_2 - C_3^r N_3 + \ldots + (-1)^r N_r.$$

Or more generally : * *If there are N events and r possible conditions such that every single condition is satisfied in N_1 of the events, every two of the conditions are simultaneously satisfied in N_2 of the events, ... and finally all the r conditions are simultaneously satisfied in N_r of the events, then the number of events free from all the conditions is as stated above.*

The proof is exactly similar to that in the last article.

* Whitworth, *Choice and Chance*, p. 73.

8. Partition of Numbers. (1) In this section we consider a number as formed by the addition of other numbers.

Any selection of the numbers 1, 2, 3, 4, ... (with repetitions), of which the sum is n, is called a *partition of* n ; if the selection contains p numbers, it is called a *partition of* n *into* p *parts* or, shortly, a *p-partition of* n.

Thus $7=3+2+1+1$, and we say that 3211 is a 4-partition of 7.

Ex. 1. *Supposing that all the partitions of* 1, 2, 3, 4, 5 *have been written down, explain how to find all the partitions of* 6.

Take first 6 ; then 5 followed by 1 ; then 4, followed by all the partitions of 2 (that is, 2, 11) ; then 3, followed by all the partitions of 3 (that is, 3, 21, 111) ; then 2, followed by the partitions of 4 *which contain no part greater than* 2 (that is, 22,211, 1111) ; lastly, 111111. The complete set is

6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111.

(2) *If* Π_p^n *is the number of* p*-partitions of* n,* *then*

$$\Pi_p^n = \Pi_{p-1}^{n-1} + \Pi_p^{n-p}. \quad \text{.....(A)}$$

To make the p-partitions of n is to distribute n elements into p indifferent parcels, blanks being inadmissible.

This can be done by placing one of the elements in each of the p parcels and then distributing the remaining $n-p$ elements into 1 or 2 or 3 ... or p parcels. Therefore

$$\Pi_p^n = \Pi_1^{n-p} + \Pi_2^{n-p} + \Pi_3^{n-p} + \ldots + \Pi_p^{n-p}. \quad \text{.....(B)}$$

Changing n into $n-1$ and p into $p-1$, we have

$$\Pi_{p-1}^{n-1} = \Pi_1^{n-p} + \Pi_2^{n-p} + \Pi_3^{n-p} + \ldots + \Pi_{p-1}^{n-p},$$

whence the result (A) follows.

If $p > \frac{1}{2}n$, *then* $\Pi_p^n = \Pi_{p-1}^{n-1}$.

For in this case $p > n-p$, and therefore $\Pi_p^{n-p} = 0$.

(3) **Construction of a table of values of** Π_p^n. We have

$$\Pi_1^n = 1, \quad \Pi_2^n = \tfrac{1}{2}(n-1) \quad \text{or} \quad \tfrac{1}{2}n,$$

according as n is odd or even. Also $\Pi_n^n = 1$, $\Pi_{n-1}^n = 1$.

In the table on page 501, denote the nth space of the mth row by (n, m): call the set of spaces $(n, 1)$, $(n+1, 2)$, $(n+2, 3)$, ... *the n-th diagonal.*

We can fill up the first and second rows and the first and second diagonals, and using § (2), we have

$$\Pi_1^{n+1} = \Pi_1^n, \quad \Pi_2^{n+2} = \Pi_1^n + \Pi_2^n, \quad \Pi_3^{n+3} = \Pi_1^n + \Pi_2^n + \Pi_3^n, \text{ etc.}$$

Hence the numbers in the $(n+1)$th diagonal are the sums of one, two, three, ... terms of the nth column, starting from the top.

* Π_p^n is subsequently denoted by $P(n, p, *)$.

The second column is 1, 1, 0, 0, ... , therefore the third diagonal is 1, 2, 2, 2, ... The third column is 1, 1, 1, 0, 0, ... , and the fourth diagonal 1, 2, 3, 3, 3, ... ; and so on to any extent.

TABLE OF p-PARTITIONS OF n (i.e. VALUES OF $P(n, p, *)$)

$n=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$p=1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3			1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33
4				1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64
5					1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84
6						1	1	2	3	5	7	11	14	20	26	35	44	58	71	90
7							1	1	2	3	5	7	11	15	21	28	38	49	65	82
8								1	1	2	3	5	7	11	15	22	29	40	52	70
9									1	1	2	3	5	7	11	15	22	30	41	54
10										1	1	2	3	5	7	11	15	22	30	42

(4) **Various classes of partitions of n** may be considered: the number of those in any particular class is denoted by a symbol of the form $P(n, \ , \ )$ where the number and the nature of the parts are respectively indicated in the first and second spaces following n.

In the first space, p means that there are p parts; and $\leqslant p$ means that the number of parts does not exceed p.

In the second space, q means that the greatest of the parts is q; and $\leqslant q$ means that no part exceeds q.

An *asterisk* means that no restriction is placed on the number or the nature of the parts, as the case may be.

If the parts are to be unequal, then Q is used instead of P. Thus:

$P(n, p, q)$ is the number of partitions of n into p parts, the greatest of which is q.

$P(n, p, *)$ is the number of p-partitions of n.

$P(n, \leqslant p, *)$ is the number of partitions of n into p or any smaller number of parts.

$Q(n, *, \leqslant q)$ is the number of partitions of n into unequal parts, none of which exceeds q.

(5) The values of $P(n, \leqslant p, *)$ can be found from the table on p. 501. For instance, the number of partitions of 10 into not more than 4 parts is the sum of the first four numbers in the 10th column, and is therefore the 4th number in the 11th diagonal, that is 23.

That is to say that

$$P(10, \leqslant 4, *) = P(14, 4, *),$$

and, in general, we have

$$P(n, \leqslant p, *) = P(n+p, p, *). \quad \text{.....(C)}$$

(6) We have the following relations :

$$P(n, p, q) = P(n-q, p-1, \leqslant q)$$

and

$$Q(n, p, q) = Q(n-q, p-1, \leqslant q). \quad \text{.....(D)}$$

For consider any partition of n into p parts, of which the greatest is q. If we remove the part q, we have a partition of $n-q$ into $p-1$ parts, none of which exceeds q. Also from every partition of the latter kind we can derive one of the former.

(7) **Conjugate Partitions.** Consider a partition of 8 into 3 parts of which the greatest is 4, say 4 3 1.

This may be written as in the margin, and if we sum the columns instead of the rows, we have 3 2 2 1, a partition of 8 into 4 parts of which 3 is the greatest.

```
1 1 1 1
1 1 1
1
```

Clearly 4 3 1 can be derived from 3 2 2 1 by the same process, and these are called *conjugate partitions.*

Thus *to every partition of n into p parts of which the greatest is q, there corresponds a partition of n into q parts of which the greatest is p.*

Consequently, we have

$$P(n, p, q) = P(n, q, p), \quad \text{.....(E)}$$

$$P(n, p, *) = P(n, *, p),$$

$$P(n, \leqslant p, q) = P(n, q, \leqslant p), \quad \text{.....(F)}$$

$$P(n, \leqslant p, *) = P(n, *, \leqslant p). \quad \text{.....(G)}$$

Thus by § 5,

$$P(n, *, \leqslant p) = P(n+p, p, *). \quad \text{.....(H)}$$

(8) **Euler's use of Series in the Enumeration of Partitions.**

(i) $Q(n, *, \leqslant q)$ *is the coefficient of x^n in the expansion of*

$$(1+x)(1+x^2)(1+x^3)\ldots(1+x^q).$$

For this coefficient is the number of ways in which n can be obtained by adding a selection of the numbers 1, 2, 3, ... q.

(ii) $P(n, *, \leqslant q)$ *is the coefficient of* x^n *in the expansion of*

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots(1-x^q)^{-1}.$$

For consider the product

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots(1+x^q+x^{2q}+\dots).$$

Any terms out of the first, second, ... qth factors may be represented by $x^\alpha, x^{2\beta}, x^{3\gamma}\dots x^{q\theta}$ where $\alpha, \beta, \gamma \dots \theta$ are any of the numbers $0, 1, 2, \dots q$. If the product of these terms is x^n, we have

$$\alpha + 2\beta + 3\gamma + \dots + q\theta = n.$$

Hence the coefficient of x^n in the product is the number of ways in which n can be made up by adding any selection of the numbers 1, 2, 3, ... q, *repetitions being allowed.* Hence the result follows.

(iii) $Q(n, p, \leqslant q)$ *is the coefficient of* $z^p x^n$ *in the expansion of*

$$(1+zx)(1+zx^2)(1+zx^3)\dots(1+zx^q),$$

and is therefore the coefficient of $x^{n-\frac{1}{2}p(p+1)}$ *in the expansion of*

$$\frac{(1-x^q)(1-x^{q-1})\dots(1-x^{q-p+1})}{(1-x)(1-x^2)\dots(1-x^p)}.$$

For the coefficient of $z^p x^n$ in the first product is the number of ways in which n can be obtained by adding a selection of p of the numbers 1, 2, 3, ...q.

The rest follows from Ch. XXX, 6, Ex. 1.

(iv) It follows that $Q(n, p, *)$ is the coefficient of $x^{n-\frac{1}{2}p(p+1)}$ in the expansion of $(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots(1-x^p)^{-1}$.

Hence by (ii), $Q(n, p, *) = P(n - \frac{1}{2}p(p+1), *, \leqslant p)$,

whence by § (5), $Q(n, p, *) = P(n - \frac{1}{2}p(p-1), p, *)$;

thus the number of partitions of n *into* p *unequal parts can be found from the table of values of* $P(n, p, *)$ *in* (3).

Ex. 1. *In how many ways can* 17 *be made up by adding four of the numbers* 1, 2, 3, ... , 9 ?

We have $Q(17, 4, *) = P(17 - \frac{1}{2}.4.3, 4, *) = P(11, 4, *) = 11.$

Thus the total number of ways of forming 17 by adding four of the numbers 1, 2, 3, ... , is 11.

From this number we must subtract the number of partitions with a part greater than 9.

These partitions are 10, 4, 2, 1 ; 11, 3, 2, 1 ; and their number is 2.

Or thus, $Q(17, 4, 10) = Q(7, 3, \leqslant 10) = Q(7, 3, *) = P(4, 3, *) = 1,$

and similarly $Q(17, 4, 11) = 1$. Hence the required number is $11 - 2 = 9$.

(v) The reader should have no difficulty in verifying the following :

$$(1-zx)^{-1}(1-zx^2)^{-1}\ldots(1-zx^q)^{-1}=1+\Sigma P(n,\ p\leqslant q)z^px^n,$$

$$(1-zx)^{-1}(1-zx^2)^{-1}(1-zx^3)^{-1}\ldots \text{ to } \infty =1+\Sigma P(n,\ p,\ *)z^px^n,$$

$$(1-z)^{-1}(1-zx)^{-1}\ldots(1-zx^q)^{-1}=1+\Sigma P(n,\leqslant p,\leqslant q)z^px^n,$$

$$(1-z)^{-1}(1-zx)^{-1}(1-zx^2)^{-1}\ldots \text{ to } \infty =1+\Sigma P(n,\leqslant p,\ *)z^px^n.$$

EXERCISE LIII

1. Find the number of ways of displaying 5 flags on 4 masts if all the flags are to be used, and (i) one or more of the masts need not be used ; (ii) all the masts are to be used.

2. Find the number of ways in which n things, of which r are alike, can be arranged in circular order.

3. Show that a selection of 10 balls can be made from an unlimited number of red, white, blue and green balls in 286 different ways, and that 84 of these contain balls of all four colours.

4. Show that the number of different selections of 5 letters which can be made from five a's, four b's, three c's, two d's and one e is 71.

5. Show that a selection of $(n-1)$ things can be made from $3n$ things, of which n are alike of one sort, n are alike of another sort and n are alike of a third sort, in $n\,.\,2^{n-1}$ different ways.

6. In the expansion of $(a_1+a_2+\ldots+a_p)^n$ where n is any positive integer not greater than p, prove that the coefficient of any term in which none of the numbers $a_1, a_2, \ldots a_p$ occurs more than once is $\lfloor n$.

7. In how many ways can a white balls, b black balls and c red balls be put in m different bags, if one or more of the bags may remain empty ?

8. In an examination, the maximum mark for each of three papers is n ; for the fourth paper it is $2n$. Prove that the number of ways in which a candidate can get $3n$ marks is

$$\tfrac{1}{6}(n+1)(5n^2+10n+6).$$

[Number = coefficient of x^{3n} in $(1-x^{n+1})^3(1-x^{2n+1})(1-x)^{-4}$.]

9. In how many ways can 5 rings be worn on the 4 fingers of one hand ?

10. In how many ways can an examiner assign 30 marks to 8 questions, giving not less than 2 marks to any question ?

11. The number of ways in which $2n$ things of one sort, $2n$ of another sort and $2n$ of a third sort can be divided between two persons so that each may have $3n$ things is $3n^2+3n+1$.

[Number of selections of $3n$ things = coefficient of x^{3n} in $(1-x^{2n+1})^3(1-x)^{-3}$.]

12. The total number of permutations of n things taken 1, 2, 3, ... or n together is the integer nearest to $e\lfloor n-1$.

$$\left[\text{Total number} = e\lfloor n-1-\left(\frac{1}{n+1}+\frac{1}{n+1\,.\,n+2}+\ldots\right).\right]$$

13. There are 5 letters and 5 directed envelopes, (i) In how many ways can all the letters be put into the wrong envelopes ? (ii) In how many ways can 2 letters be rightly placed and 3 letters wrongly placed ?

14. The number of arrangements of the n terms $a_1, a_2, \ldots a_n$ in which no one of r sequences $a_1a_2, a_2a_3, \ldots a_ra_{r+1}$ occurs is

$$\underline{|n} - C_1^r \underline{|n-1} + C_2^r \underline{|n-2} - \ldots + (-1)^r C_r^r \underline{|n-r}.$$

[For the total number of arrangements is $\underline{|n}$. Any one of the sequences occurs in $\underline{|n-1}$ of these ; any two in $\underline{|n-2}$; and so on.]

15. Show that the number of arrangements of the letters *abcd* which involve none of the sequences *ab, bc, cd* is 11. Verify by writing down the arrangements.

16. The number of arrangements of the letters *abcd* in which neither a, b nor c, d come together is

$$\underline{|4} - C_1^2 . 2 \underline{|3} + C_2^2 . 2^2 \underline{|2} = 8.$$

Explain this, and write down the arrangements in question.

17. A party of 10 consists of 2 Englishmen, 2 Scotsmen, 2 Welshmen and 4 men of other nationalities (all different). In how many ways can they stand in a row so that no two men of the same nationality are next one another ?

In how many ways can they sit at a round table ?

18. The number of combinations n together of $3n$ letters of which n are a and n are b and the rest unlike is $(n+2)2^{n-1}$.

[Number = coefficient of x^n in $(1-x^{n+1})^2(1+x)^n(1-x)^{-2}$, or in $(2-1-x)^n(1-x)^{-2}$
= coefficient of x^n in $2^n(1-x)^{-2} - n \cdot 2^{n-1}(1-x)^{-1}$, etc.]

19. Show that if $a_1, a_2, \ldots a_m$ are distinct prime numbers other than unity, the number of solutions in integers (excluding unity) of the equation

$$x_1x_2x_3 \ldots x_n = a_1a_2a_3 \ldots a_n$$

is n^m. Show also that the number of solutions in which at least one x is unity is

$$C_1^n(n-1)^m - C_2^n(n-2)^m + \ldots + (-1)^{n-1}C_{n-1}^n \cdot 1^m.$$

[Solution follows from the general theorem of Art. 7.]

20. Given n pairs of gloves, in how many ways can each of n persons take a right-handed and a left-handed glove without taking a pair ?

21. (i) If n things are arranged in circular order, the number of ways of selecting three of the things no two of which are next each other is

$$\tfrac{1}{6}n(n-4)(n-5).$$

(ii) If the n things are arranged in a row, the number of such sets of three is

$$\tfrac{1}{6}(n-2)(n-3)(n-4).$$

22. Find the number of positive integral solutions of $x+y+z+w=20$ under the following conditions :

(i) Zero values of x, y, z, w are included.

(ii) Zero values are excluded.

(iii) No variable may exceed 10 ; zero values excluded.

(iv) Each variable is an odd number.

(v) x, y, z, w have different values (zero excluded).

23. Prove that the number of positive integral solutions (zero values excluded) of the equation

$$x+2y+3z=n$$

is $$\tfrac{1}{12}(n-1)(n-5)+\tfrac{1}{8}(-1)^n+c,$$

where $c=\frac{11}{24}$ or $\frac{1}{8}$ according as n is or is not a multiple of 3. Verify when $n=9$, 10.

[The required number is co. of x^n in

$$(x+x^2+x^3+\ldots)(x^2+x^4+x^6+\ldots)(x^3+x^6+x^9+\ldots).]$$

24. The number of independent solutions in unequal positive integers, zero included, of

$$x+y+z+u=2p$$

is the integer nearest to $\frac{1}{96}p^2(2p-3)$.

25. Show that $\Pi_3^{6n}=3n^2$.

[$\Pi_3^{6n}-\Pi_3^{6n-3}=3n-1$, $\Pi_3^{6n-3}-\Pi_3^{6n-6}=3n-2$, ... $\Pi_3^6-\Pi_3^3=2$, $\Pi_3^3=1$.]

26. A necklace is made up of 3 beads of one sort and $6n$ of another, those of each sort being similar. Show that the total number of possible arrangements of the beads is $3n^2+3n+1$.

[Having put on the three beads, the number of arrangements of the $6n$ beads in the spaces between the three (which are indifferent) is

$$\Pi_3^{6n}+\Pi_2^{6n}+\Pi_1^{6n}=3n^2+3n+1.]$$

27. Find the values of (i) $P(18, 5\ *)$, (ii) $P(12, \leqslant 5, *)$, (iii) $P(12, *, 4)$.

28. In how many ways can 20 be expressed as the sum of four unequal positive integers ?

29. Show that the total number of partitions of n is $P(2n, n, *)$. Find the total number of partitions of 10.

30. Show that $P(2n+r, n+r, *)=P(2n, n, *)$. Hence use the table of Art. 8, (3), to find $P(25, 18, *)$.

31. Show that $P(n, p, \leqslant q)$ is equal to the coefficient of x^{n-p} in the expansion of

$$\frac{(1-x^q)(1-x^{q+1})\ldots(1-x^{q+p-1})}{(1-x)(1-x^2)\ldots(1-x^p)}.$$

Hence show that

$$Q(n, p, \leqslant q)=P\{n-\tfrac{1}{2}p(p-1),\ p,\ \leqslant(q-p+1)\}.$$

32. Show that

$$\Pi_p^n=\text{co. of } x^{n-p} \text{ in } \frac{1}{(1-x)(1-x^2)\ldots(1-x^p)}.$$

EXERCISE LIV *

1. There are n points in a plane, no three being collinear except p of them, which are collinear. How many triangles can be drawn with their vertices at three of the points ?

2. There are n points in a plane, no three being collinear. Show that the number of n-sided polygons with their vertices at these points is $\frac{1}{2}\underline{|n-1}$. Verify if $n=4$.

* The results in Exx. 9-12 were given by Dudeney

3. If m points in a straight line are joined in all possible ways to n points in another straight line, then, excluding the $(m+n)$ points, the number of points of intersection is

$$\tfrac{1}{2}mn(m-1)(n-1).$$

4. If n straight lines of indefinite length are drawn in a plane, no two of them being parallel and no three meeting in a point, then

(i) the number of points of intersection is $\frac{1}{2}n(n-1)$;

(ii) the plane is divided into $\frac{1}{2}(n^2+n+2)$ parts.

[(ii) If u_n is the number of parts, show that $u_n = u_{n-1} + n$.]

5. Show that, in general, n planes divide space into $\frac{1}{6}(n^3+5n+6)$ regions. What are the exceptional cases?

[If no three planes have a common line of intersection and no four meet in a point, then $v_n = v_{n-1} + u_{n-1}$ where v_n is the required number and u_n is the same as in Ex. 4.]

6. The number of squares formed on a piece of squared paper by m horizontal lines and n vertical lines $(m<n)$ is

$$\tfrac{1}{6}m(m-1)(3n-m-1).$$

7. (i) If m, n are positive integers such that $m \leqslant n$ and $m+n=a$, where a is constant, prove that the greatest value of $m(m-1)(3n-m-1)$ is

$$\tfrac{1}{4}(a+1)(a-1)(a-3) \quad \text{or} \quad \tfrac{1}{4}a(a-1)(a-2),$$

according as a is odd or even.

(ii) If a straight rods of indefinite length are placed so as to form the greatest possible number of squares, prove that this number is

$$\tfrac{1}{24}(a+1)(a-1)(a-3) \quad \text{or} \quad \tfrac{1}{24}a(a-1)(a-2),$$

according as a is odd or even.

8. Show that the number of ways in which three numbers in arithmetical progression can be selected from $1, 2, 3, \ldots n$ is $\frac{1}{4}(n-1)^2$ or $\frac{1}{4}n(n-2)$, according as n is odd or even.

9. The sides of a triangle are a, b, c inches where a, b, c are integers and $a \leqslant b \leqslant c$.

If c is given, show that the number of different triangles is $\frac{1}{4}(c+1)^2$ or $\frac{1}{4}c(c+2)$, according as c is odd or even.

10. Of the triangles in the last question, the number of those which are isosceles or equilateral is $\frac{1}{2}(3c-1)$ if c is odd and $\frac{1}{2}(3c-2)$ if c is even.

11. Each side of a triangle is an integral number of inches, no side exceeding c inches. Prove that the number of different (i.e. non-congruent) triangles which can be so formed is

$$\tfrac{1}{24}(c+1)(c+3)(2c+1) \quad \text{or} \quad \tfrac{1}{24}c(c+2)(2c+5),$$

according as c is odd or even.

12. Of the triangles in the last question, the number of those which are isosceles or equilateral is $\frac{1}{4}(3c^2+1)$ or $\frac{3}{4}c^2$, according as c is odd or even.

CHAPTER XXXII

PROBABILITY

1. First Principles. In speaking of the probability of an event, doubt is implied as to whether it will or will not happen. The degree of doubt depends on our knowledge of the controlling conditions. A complete knowledge of these enables us to say that the event is certain to happen or certain to fail.

Measurement of Probabilities. We are constantly forming rough estimates of probabilities, saying that some event is unlikely, likely, very likely or certain to happen. This implies that (at any rate in certain cases) probability can be measured and can be compared with *certainty*, which must be regarded as a degree of probability. Thus there is nothing to prevent us from choosing *certainty* as the *unit of probability*, and this is always done.

In what follows the words *probability* and *chance* are synonymous, both of them being used to denote *measure of probability.*

Suppose that n balls $A, B, \ldots K$, all of the same sort, are placed in a bag, and that one of them is drawn. There is nothing to favour the drawing of one ball more than another, in other words it is *equally likely* that $A, B, \ldots$ or K will be drawn.

Thus if x is the chance that A is drawn, the chance that any other ball, say B, is drawn is also x.

Our notions with regard to probabilities therefore lead us to assert that the chance that *either A or B* is drawn is $2x$, and the chance that one of the n balls is drawn is nx. But it is certain that one ball will be drawn, and so $nx=1$ and $x=1/n$. Thus the probability of drawing A is $1/n$.

The following definition is a statement of *Laplace's First Principle.*

Definition. If an event can happen in a ways and fail in b ways, and there is nothing to lead us to believe that one of these ways should occur rather than another, that is to say, all the ways are equally likely, then

the chance that the event happens is	$a/(a+b)$,
the chance that the event fails is	$b/(a+b)$,
the odds in favour of the event are as	$a : b$,
the odds against the event are as	$b : a$.

If $b=0$, the event is certain to happen, and its probability is 1; *if* $a=0$, it is certain to fail, and the probability of its happening is 0.

If $a=b$, the event is as likely to happen as to fail, and we say that there is an *even chance* of its happening.

If p, q are the respective chances of the happening and failing of an event, then

$$p+q=1.$$

Events of a type to which this definition does not directly apply will be considered later.

Examples. (i) *If a coin is tossed,* the chance of 'heads' is $\frac{1}{2}$.

(ii) *If a six-faced die is thrown,* the chance that 'ace' turns up is $\frac{1}{6}$.

(iii) *If a ball is drawn from a bag containing* 3 *white and* 2 *black balls,* the chance of drawing white is $\frac{3}{5}$; the odds against black are as 3 : 2.

(iv) *If a card is drawn from a well-shuffled pack,*

the chance that the card is ace of spades is $\frac{1}{52}$;
the chance that the card is a spade is $\frac{13}{52}$;
the chance that the card is an ace is $\frac{4}{52}$.

It is easy to make a mistake as to the meaning of '*equally likely.*'

Ex. 1. A man of much practical experience in the laying of odds said: '*If you toss three pennies simultaneously, it is an even chance that they will fall all alike, i.e. all heads or all tails.*

'For they must fall all heads, or all tails, or 2 heads and a tail, or 2 tails and a head. So they fall all alike in 2 out of 4 possible cases.'

As a matter of fact, *the odds are* 3 *to* 1 *against their falling alike.* For, if the pennies are denoted by a, b, c, then a can fall head or tail, and so for b and c. Hence there are 8 *equally likely* cases of which 2 are favourable.

2. Exclusive Events. Events are said to be *mutually exclusive* when the happening of any one of them precludes the happening of any other. In the case of mutually exclusive events, *the chance that one or other of them occurs is the sum of the chances of the separate events.*

This statement is known as *Laplace's second principle.* It can also be stated as follows: *If an event can happen in several different ways, one only of which can occur on the same occasion, the chance of its happening is the sum of the chances of its happening in the several different ways.*

For consider n events of which the chances are a_1/g, a_2/g, ..., a_n/g respectively, where the letters denote integers. Out of g equally likely ways, the events can happen in a_1, a_2, ... a_n ways respectively, and since the events cannot concur, one or other of them can happen in $a_1+a_2+\ldots+a_n$ ways. The chance of this is therefore $(a_1+a_2+\ldots+a_n)/g$, which is the sum of the chances of the events.

Ex. 1. *In a single cast with two dice, what is the chance of throwing* (i) *two aces,* (ii) *doublets,* (iii) *five-six* (*i.e. one die turns up five and the other six*) *?*

Any face of either die may turn up, so there are 6×6 equally likely cases.

These are made up of the six throws (1, 1), (2, 2), ... (6, 6), together with the thirty throws (1, 2), (2, 1), ... (5, 6), (6, 5).

We count (5, 6), (6, 5) as the same throw, and call it *five-six*. Thus

$$\text{the chance of throwing two aces} = \tfrac{1}{36};$$
$$\text{the chance of throwing doublets} = \tfrac{6}{36};$$
$$\text{the chance of throwing five-six} = \tfrac{2}{36}.$$

Ex. 2. *In a single cast with two dice, what are the odds against throwing* 7, *i.e. against two numbers of which the sum is* 7 *?*

Out of 36 possible cases, those in favour of the event are (6, 1), (1, 6), (5, 2), (2, 5), (4, 3), (3, 4), and the number of these is 6, therefore

$$\text{the chance of throwing 7 is } \tfrac{6}{36} = \tfrac{1}{6},$$

and the odds against throwing 7 are as 5 : 1.

Ex. 3. *In a single cast with three dice, what is the chance of throwing* (i) *four-five-six,* (ii) *eleven,* (iii) *less than eleven,* (iv) *more than ten ?*

The number of possible cases is 6^3.

(i) Among these *four-five-six* occurs $\lfloor 3$ times;

$$\therefore \text{ chance of throwing four-five-six} = \frac{\lfloor 3}{6^3} = \frac{1}{36}.$$

(ii) Let s be the number of cases in which 11 is thrown, then

$$s = \text{coefft. of } x^{11} \text{ in } (x + x^2 + x^3 + \ldots + x^6)^3 = 27;$$

$$\therefore \text{ chance of throwing 11 is } \frac{27}{6^3} = \frac{1}{8}.$$

(iii) Let t be the number of cases in which less than 11 is thrown. These cases are those in which exactly 3, 4, 5, ... 10 are thrown. Therefore t is the sum of the coefficients of x^3, x^4, x^5, ... x^{10} in the expansion of

$$(x + x^2 + x^3 + \ldots + x^6)^3;$$

hence,

$$\begin{aligned} t &= \text{coefficient of } x^{10} \text{ in } x^3(1 - x^6)^3(1 - x)^{-4} \\ &= \text{coefficient of } x^7 \text{ in } (1 - x)^{-4}(1 - 3x^6) \\ &= \text{coefficient of } x^7 \text{ in } \left(4x + \frac{8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3} x^7\right)(1 - 3x^6) \\ &= 120 - 12 = 108; \end{aligned}$$

$$\therefore \text{ chance of throwing less than 11 is } \frac{108}{6^3} = \frac{1}{2}.$$

(iv) The total number of possible cases is made up of those in which more than 10 are thrown and those in which less than 11 are thrown;

$$\therefore \text{ chance of throwing more than } 10 = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

(In the old game of *passe-dix*, a player stakes on throwing more than 10.)

Ex. 4. *A party of ten take their seats at a round table. What are the odds against two specified persons* (*A*, *B*) *sitting together ?*

A having taken his place, *B* has a choice of 9 places, 2 of which are next to *A*, hence the odds against *B* sitting next to *A* are as 7 : 2.

Ex. 5. Four cards are drawn from a pack of 52 cards. What is the chance that one of each suit is drawn?

Four cards can be selected from 52 in C_4^{52} ways; and this is the number of possible cases. One of each suit can be selected in 13^4 ways; and this is the number of favourable cases;

$$\therefore \text{ required chance} = \frac{13^4}{C_4^{52}} = \frac{28561}{270725}.$$

Ex. 6. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn?

Five balls can be selected out of ten in C_5^{10} ways. This is the number of possible cases. Also three white balls and two black balls can be selected in $C_3^6 . C_2^4$ ways, which is the number of favourable cases, therefore

$$\text{the required chance} = \frac{C_3^6 . C_2^4}{C_5^{10}} = \frac{10}{21}.$$

3. Independent Events. Events are said to be *independent* when the happening of any one of them does not affect the happening of any of the others.

Laplace's third principle. The probability of the concurrence of several independent events is the product of their separate probabilities.

It is sufficient to consider two independent events A_1 and A_2, of which the probabilities are p_1 and p_2 respectively. Suppose that A_1 can happen in a_1 ways and fail in b_1 ways, all of which are equally likely. Also suppose that A_2 can happen in a_2 ways and fail in b_2 ways, all equally likely.

Then $p_1 = a_1/(a_1+b_1)$, $p_2 = a_2/(a_2+b_2)$, and there are the following possibilities:

A_1 and A_2 may both happen, and this may occur in a_1a_2 ways.
A_1 may happen and A_2 fail, and this may occur in a_1b_2 ways.
A_2 may happen and A_1 fail, and this may occur in a_2b_1 ways.
A_1 and A_2 may both fail, and this may occur in b_1b_2 ways.

The total number of ways in which both A_1 and A_2 are concerned is $(a_1+b_1)(a_2+b_2)$, and these are all equally likely. Hence the chance that both A_1 and A_2 occur is $a_1a_2/(a_1+b_1)(a_2+b_2)$, which is equal to p_1p_2.

It should also be noticed that

the chance that A_1 happens and A_2 fails is $p_1(1-p_2)$;
the chance that A_2 happens and A_1 fails is $p_2(1-p_1)$;
the chance that both A_1 and A_2 fail is $(1-p_1)(1-p_2)$.

Hence also the chance that *one and only one* of the events happens is

$$p_1(1-p_2)+p_2(1-p_1) = p_1+p_2-2p_1p_2.$$

Also the chance that *at least one* of the events happens (i.e. both do not fail) is $1-(1-p_1)(1-p_2) = p_1+p_2-p_1p_2$.

Ex. 1. *What is the chance of throwing ace with a single die in two trials?*

The chance of not throwing ace in one trial is $\frac{5}{6}$,

the chance of not throwing ace in two trials is $(\frac{5}{6})^2$;

$\therefore$ the chance of throwing at least one ace $= 1 - (\frac{5}{6})^2 = \frac{11}{36}$.

Or thus,

Chance of success in both trials $= \frac{1}{36}$.

Chance of success in the first trial and failure in the second $= \frac{1}{6} \cdot \frac{5}{6}$.

Chance of failure in the first and success in the second $= \frac{5}{6} \cdot \frac{1}{6}$.

Chance of at least one success $= \frac{1}{36} + 2 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{11}{36}$.

Ex. 2. *Referring to the last example, explain why the following reasoning is incorrect: The chance of throwing ace in the first trial is $\frac{1}{6}$ and the chance of ace in the second trial is $\frac{1}{6}$, therefore the chance of ace in two trials is $\frac{2}{6}$.*

The throwing of ace in the first trial does not exclude the possibility of throwing ace in the second trial. Thus the events are not mutually exclusive and Art. 2 does not apply. It should be noticed that, if the reasoning were correct, it would follow that the chance of throwing ace in 6 trials would be $\frac{6}{6}$, i.e. that ace would *certainly* occur in 6 trials; which is obviously false.

Ex. 3. *In how many throws with a single die will it be an even chance that ace turns up at least once?*

The chance against ace turning up in n throws is $(\frac{5}{6})^n$, and the chance that it turns up at least once is $1 - (\frac{5}{6})^n$. If then n is the required number, we have

$$1 - (\tfrac{5}{6})^n = \tfrac{1}{2}; \quad \therefore\ (\tfrac{5}{6})^n = \tfrac{1}{2}; \quad \therefore\ n = 3{\cdot}8 \text{ nearly.}$$

Hence it is less than an even chance that ace turns up once in 3 trials, and more than an even chance that it turns up in 4 trials.

4. Interdependent Events. If two events A_1, A_2 are such that p_1 is the probability of A_1, and p_2 is the probability of A_2 *on the supposition that A_1 has happened*, then the probability that both A_1 and A_2 happen is $p_1 p_2$.

Also if p_3 is the probability of a third event A_3 *after A_1 and A_2 have happened*, then the probability that A_1, A_2 and A_3 will happen is $p_1 p_2 p_3$, and so on for any number of dependent events.

For the reasoning of the last article holds good under these conditions.

Ex. 1. *One bag contains* 3 *white balls and* 2 *black balls; another contains* 5 *white and* 3 *black balls. If a bag is chosen at random and a ball is drawn from it, what is the chance that it is white?*

The chance that the first bag is chosen is $\frac{1}{2}$, and the chance of drawing a white ball from it is $\frac{3}{5}$;

$\therefore$ the chance of choosing the first bag and drawing a white is $\frac{1}{2} \cdot \frac{3}{5}$.

Similarly the chance of choosing the second bag and drawing a white is $\frac{1}{2} \cdot \frac{5}{8}$;

$\therefore$ the chance of drawing a white from a bag chosen at random is

$$\tfrac{1}{2} \cdot \tfrac{3}{5} + \tfrac{1}{2} \cdot \tfrac{5}{8} = \tfrac{49}{80}.$$

NOTE. If all the balls were put into one bag and a ball were drawn, the chance of its being white is $\frac{8}{13}$, which is not the same as before.

Ex. 2. *A person draws a card from a pack, replaces it, and shuffles the pack. He continues doing this until he draws a spade. What is the chance that he will have to make* (i) *at least three trials,* (ii) *exactly three trials?*

The chance of success at any particular trial is $\frac{1}{4}$. *In the first case,* he will have to fail at the first and second trials;

$$\therefore \text{ required chance} = (\tfrac{3}{4})^2 = \tfrac{9}{16}.$$

In the second case, he must fail at the first and second trials and succeed at the third;

$$\therefore \text{ required chance} = (\tfrac{3}{4})^2 \cdot \tfrac{1}{4} = \tfrac{9}{64}.$$

Ex. 3. *Five balls are drawn, one by one, from a bag containing* 6 *white and* 4 *black balls. What is the chance that* 3 *white and* 2 *black balls are drawn?*

First method. It makes no difference to the result if we suppose the 5 balls drawn *simultaneously.* Hence, as in Art. 2,

$$\text{the required chance} = C_3^6 \cdot C_2^4 / C_5^{10} = \tfrac{10}{21}.$$

Second method. Suppose the balls drawn in the order indicated by *wwwbb.* The chance that this occurs is easily seen to be

$$\tfrac{6}{10} \cdot \tfrac{5}{9} \cdot \tfrac{4}{8} \cdot \tfrac{4}{7} \cdot \tfrac{3}{6}.$$

Also the chance that 3 white and 2 black balls are drawn in any other particular order is obtained from the preceding by merely altering the order of the numerators.

The number of orders in which this can occur is the number of permutations of the letters *wwwbb,* and is therefore $\lfloor 5 / \lfloor 3 \lfloor 2 = 10$;

$$\therefore \text{ the required chance} = 10 \cdot \tfrac{6}{10} \cdot \tfrac{5}{9} \cdot \tfrac{4}{8} \cdot \tfrac{4}{7} \cdot \tfrac{3}{6},$$

which is equal to the result previously obtained.

5. Another Way of estimating Probabilities. It is obvious that the definition in Art. 1 only applies to events of a restricted type; in particular, to games of chance. There are other cases in which probability is estimated by considering the frequency with which the event occurs, or is believed to occur, 'in the long run.' For example:

(i) It is an observed fact that about 51 per cent. of the children born in Europe are boys. Hence we say that the probability that a child, about to be born, will be a boy is about 0·51.

(ii) According to 'tables of mortality,' such as are used by Insurance Companies, out of 81,188 men 50 years old, 70,552 live to the age of 60. We therefore say that the chance that a man of age 50 will live another 10 years is $70552/81188 = 0{\cdot}869$.

In general, if an event has happened in pN trials out of a total of N trials, where N is a large number, we say that p is the probability that the event will happen on a fresh trial.

This notion of probability is not at variance with that contained in Laplace's first principle (Art. 1).

If a six-faced die is thrown a large number of times, we expect one face to turn up about as often as another, and this has been verified experimentally. Thus in a large number of throws we may expect ace to turn up in about $\frac{1}{6}$ of the total number of trials, and from this point of view we say that the chance of throwing ace is $\frac{1}{6}$.

6. Expectation. Suppose that a person (A) has a ticket in a lottery which gives him a chance p of a prize of £a.

If the lottery were held N times, where N is a large number, we may expect him to get the prize about pN times, receiving £pNa. Thus we may say that *on an average* he receives £pa for a single lottery, and this is called his *expectation*.

Next suppose that A's ticket gives him a chance p_1 of receiving £a_1, a chance p_2 of receiving £a_2, and so on.

If the lottery were held N times, where N is large, we may expect him to receive £a_1 on about p_1N occasions, £a_2 on about p_2N occasions, and so on. Altogether he may be expected to get about £$N(p_1a_1+p_2a_2+\ldots)$.

Thus we may say that *on an average* he gets £$(p_1a_1+p_2a_2+\ldots)$ for a single lottery. This sum is the sum of the expectations arising from his chances of securing the separate sums, and is called his *expectation.*

Definitions. The *average value* of a quantity P, subject to risk, is the average value which P assumes in the long run. This value is also called the *expected value* of P or the *expectation* with regard to P.

If a quantity P can assume the values $P_1, P_2, P_3, \ldots$ and the chances that it has these values are respectively $p_1, p_2, p_3, \ldots$, the *average* or the *expected* value of P is $p_1P_1+p_2P_2+p_3P_3+\ldots$.

This is merely a generalisation of what has been said in the case of a lottery.

It should be observed that every p denotes the chance that P has the corresponding value, so that *events need not be independent.*

Ex. 1. A man's *expectation of life* is usually taken to mean the average number of years which men of his age survive.

If p_r is the chance that he will survive r years, dying before the end of the next year, his expectation of life is roughly $p_1 \cdot 1+p_2 \cdot 2+p_3 \cdot 3+\ldots$. A closer estimate will be given in another volume.

Ex. 2. *A person draws* 2 *balls from a bag containing* 3 *white and* 5 *black balls. If he is to receive* 10*s. for every white ball which he draws and* 1*s. for every black ball, what is his expectation?*

The number of ways in which 2 balls can be drawn is $C_2^8=28$.

Of these the number of ways in which

2 white balls can be drawn is $C_2^3 = 3$,

1 white and 1 black ball can be drawn is $3 \cdot 5 = 15$,

2 black balls can be drawn is $C_2^5 = 10$.

Hence his chances of receiving 20s., 11s. and 2s. are respectively 3/28, 15/28, 10/28;

$$\therefore \text{ expectation} = \left(\tfrac{3}{28} \cdot 20 + \tfrac{15}{28} \cdot 11 + \tfrac{10}{28} \cdot 2\right) = 8\text{s. } 9\text{d.}$$

7. Successive Events. Let p, q be the respective chances of the happening and of the failing of an event at a single trial, then :

(1) *The chance of its happening exactly r times in n trials is* $C_r^n p^r q^{n-r}$.

For the chance of its happening r times and failing $n-r$ times *in a specified order is* $p^r q^{n-r}$.

Now the number of different orders in which these things can occur is the number of different arrangements of r things of one sort and $n-r$ things of another sort. This number is $\underline{|n} / \underline{|r} \, \underline{|n-r}$, which is equal to C_r^n.

Thus the event in question can happen in C_r^n ways, which are mutually exclusive, and the chance that it happens in any one of them is $p^r q^{n-r}$; therefore the required chance is $C_r^n p^r q^{n-r}$.

(2) We have $C_r^n = C_{n-r}^n$, therefore *the chances that the event happens exactly n times, exactly $(n-1)$ times, exactly $(n-2)$ times, ... are respectively the first, second, third, ... terms in the expansion*

$$(p+q)^n = p^n + C_1^n p^{n-1} q + C_2^n p^{n-2} q^2 + \dots .$$

(3) The greatest term in the expansion is that in which r is the integral part of $\dfrac{(n+1)q/p}{1+q/p}$, that is of $(n+1)q$. Hence *the most probable case* is that of $n-r$ successes and r failures where r is the integral part of $(n+1)q$.

If nq is an integer, then $r = nq$ and $n-r = np$, so that *in the most probable case the ratio of the number of successes to the number of failures is* $p : q$.

(4) It may be useful to state an important theorem due to Bernoulli : If the chance of an event happening on a single trial is p and if a number of independent trials are made, the probability that the ratio of the number of successes to the number of trials differs from p by less than any assigned number, however small, can be made as near to certainty as we choose, by making the number of trials sufficiently great.

The proof of this is beyond our present scope, and requires a use of approximate values of large factorials. For instance, we may use Stirling's theorem, which states that

$$\underline{|n} = \sqrt{2\pi n} \cdot n^n e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \dots\right).$$

(5) *If $u_{n,r}$ is the chance that the event happens* at least r times *in n trials, we shall prove that*

$$\text{(i) } u_{n,r} = p^n + C_1^n p^{n-1}q + C_2^n p^{n-2}q^2 + \ldots + C_{n-r}^n p^r q^{n-r},$$

$$\text{(ii) } u_{n,r} = p^r\left\{1 + rq + \frac{r(r+1)}{\lfloor 2}q^2 + \ldots + \frac{r(r+1)\ldots(n-1)}{\lfloor n-r}q^{n-r}\right\}.$$

For if the event happens *at least r times*, then (i) it must happen *exactly* n times, or *exactly* $(n-1)$ times, or *exactly* $(n-2)$ times, ... or *exactly* r times, and the chances that these things occur are the successive terms of the first expression.

Or, (ii) it must happen *exactly* r times in *just* r trials, or in *just* $(r+1)$ trials, or in *just* $(r+2)$ trials, ... or in *just* n trials. Now that it may happen *exactly* r times in *just* $(r+s)$ trials, it must happen in the last of these and also in $(r-1)$ out of the preceding $(r+s-1)$ trials. The chance of this is $p\,C_{r-1}^{r+s-1}p^{r-1}q^s = C_{r-1}^{r+s-1}p^r q^s$; and therefore

$$\begin{aligned} u_{n,r} &= p^r + C_{r-1}^r p^r q + C_{r-1}^{r+1}p^r q^2 + \ldots + C_{r-1}^{n-1}p^r q^{n-r} \\ &= p^r\{1 + C_1^r q + C_2^{r+1}q^2 + \ldots + C_{n-r}^{n-1}q^{n-r}\}, \end{aligned}$$

which is the second of the expressions in question. That these two expressions are equal may be proved as in Exercise XXXV, Ex. 9.

Ex. 1. *If a die is thrown* 5 *times, what is the chance that ace turns up* (i) *exactly* 3 *times,* (ii) *at least* 3 *times?*

Let x and y be the required chances, then $x = C_3^5\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{250}{6^5}$,

and $$y = \left(\frac{1}{6}\right)^5 + C_1^5\left(\frac{1}{6}\right)^4\cdot\frac{5}{6} + C_1^5\left(\frac{1}{6}\right)^3\cdot\left(\frac{5}{6}\right)^2 = \frac{46}{6^4},$$

or by the second formula,

$$y = \left(\frac{1}{6}\right)^3\left\{1 + 3\cdot\frac{5}{6} + \frac{3.4}{1.2}\left(\frac{5}{6}\right)^2\right\} = \frac{46}{6^4}.$$

Ex. 2. *A and B throw alternately with a single die, A having the first throw. The person who first throws ace is to receive* £1. *What are their expectations?*

The chances that the stake is won at the first, second, third, ... throws are

$$\frac{1}{6},\ \frac{5}{6}\cdot\frac{1}{6},\ \left(\frac{5}{6}\right)^2\cdot\frac{1}{6},\ \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}, \ldots;$$

$\therefore$ A's chance $= \frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^4\cdot\frac{1}{6} + \ldots = \frac{1}{6}\cdot\frac{1}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$; B's chance $= 1 - \frac{6}{11} = \frac{5}{11}$.

Hence, the expectations of A and B are £$\frac{6}{11}$ and £$\frac{5}{11}$ respectively.

Ex. 3. **'The Problem of Points.'** *A and B play a series of games which cannot be drawn, and p, q are their respective chances of winning a single game. What is the chance that A wins m games before B wins n games?*

Here A must win at least m games out of $m+n-1$, and the required chance is obtained by substituting these numbers for r and n in either of the formulae in (5).

(6) **Several Alternatives.** So far, at each trial there have been two alternatives—the event may succeed or it may fail.

Consider the case of an experiment which at any trial must produce one of three results denoted by A, B, C. Let the chances that A, B, C occur at any trial be p, q, r respectively. One of the three results has to occur, therefore $p+q+r=1$. Also the chance that in n trials A occurs α times, B occurs β times and C occurs γ times (where $\alpha+\beta+\gamma=n$) is

$$\frac{\lfloor n}{\lfloor\alpha\,\lfloor\beta\,\lfloor\gamma}\,p^\alpha q^\beta r^\gamma,$$

that is to say, the chance is the term containing $p^\alpha q^\beta r^\gamma$ in the expansion of

$$(p+q+r)^n.$$

For the chance that A may happen α times, B β times and C γ times *in a specified order* is $p^\alpha q^\beta r^\gamma$.

Now the number of orders in which these things can occur is the number of arrangements of α things of one sort (A), β things of another sort (B), and γ things of a third sort (C), placed in a row.

This number is $\lfloor n / \lfloor\alpha\,\lfloor\beta\,\lfloor\gamma$. Hence the result follows.

Ex. 4. *A card is drawn from a pack, the card is replaced and the pack shuffled. If this is done six times, what is the chance that the cards drawn are* 2 *hearts,* 2 *diamonds and* 2 *black cards ?*

At any trial, the chance that a heart or a diamond is drawn is $\frac{1}{5}\frac{3}{2}=\frac{1}{4}$; also the chance that a black card is drawn is $\frac{1}{2}$. Hence by the preceding,

$$\text{required chance}=\frac{\lfloor 6}{\lfloor 2\,\lfloor 2\,\lfloor 2}\cdot\left(\frac{1}{4}\right)^2\cdot\left(\frac{1}{4}\right)^2\cdot\left(\frac{1}{2}\right)^2=\frac{45}{512}.$$

Ex. 5. *A 'hand' of six cards is dealt from a pack in the ordinary way. Find the chance that it consists of* 2 *hearts,* 2 *diamonds and* 2 *black cards.*

The chance that the

first	card dealt is a heart	is $\frac{1}{5}\frac{3}{2}$,
second	card dealt is a heart	is $\frac{1}{5}\frac{2}{1}$,
third	card dealt is a diamond	is $\frac{1}{5}\frac{3}{0}$,
fourth	card dealt is a diamond	is $\frac{1}{4}\frac{2}{9}$,
fifth	card dealt is a black	is $\frac{2}{4}\frac{6}{8}$,
sixth	card dealt is a black	is $\frac{2}{4}\frac{5}{7}$.

Hence the chance that 2 hearts, 2 diamonds and 2 black cards are dealt *in this or in any specified order* is

$$\tfrac{1}{5}\tfrac{3}{2}\cdot\tfrac{1}{5}\tfrac{2}{1}\cdot\tfrac{1}{5}\tfrac{3}{0}\cdot\tfrac{1}{4}\tfrac{2}{9}\cdot\tfrac{2}{4}\tfrac{6}{8}\cdot\tfrac{2}{4}\tfrac{5}{7}.$$

The required chance is obtained by multiplying this number by $\lfloor 6/\lfloor 2\,\lfloor 2\,\lfloor 2$; and is therefore equal to $\frac{7605}{78302}$.

Ex. 6. *Find the chance that in a game of whist the dealer (A) has exactly two honours.*

The card turned up is, or is not, an honour; and the chances of these possibilities are respectively $\frac{4}{13}$ and $\frac{9}{13}$.

In the first case A certainly has one honour. Of the remaining 51 cards, 12 belong to A and 39 to the other players.

Let p be the chance that, of the 3 remaining honours, 1 is among the 12 cards and 2 are among the 39, then

$$p = C_1^3 \cdot \tfrac{12}{51} \cdot \tfrac{39}{50} \cdot \tfrac{38}{49}.$$

In the second case, where the card turned up is not an honour, if p' is the chance that 2 honours belong to A and 2 to the other players, we have

$$p' = C_2^4 \cdot \tfrac{12}{51} \cdot \tfrac{11}{50} \cdot \tfrac{39}{49} \cdot \tfrac{38}{48};$$

$$\therefore \text{ the required chance} = \tfrac{4}{13} p + \tfrac{9}{13} p' = \tfrac{2232}{8330}.$$

8. Probability of Causes. (1) Suppose that an event has happened which must have arisen from one of a certain number of causes C_1, C_2, ... *What is the probability that a specified cause,* C_1, *actually led to the event?*

This question is said to be one of *inverse probability.*

Ex. 1. *Each of three bags A, B, C contains white and black balls, the numbers of which are as follows:*

	A	B	C
white - -	a_1	a_2	a_3
black - -	b_1	b_2	b_3

A bag is chosen at random, a ball is drawn from it and is found to be white.

It is required to find the probabilities Q_1, Q_2, Q_3 *that the ball came from A, B, C respectively.*

If the numbers of the balls are altered as below, the probabilities in question remain unchanged:

	A	B	C
white - -	a_1x	a_2y	a_3z
black - -	b_1x	b_2y	b_3z

when x, y, z are any numbers whatever. Choose these so that

$$x(a_1+b_1) = y(a_2+b_2) = z(a_3+b_3).$$

The three bags now contain the same number of balls, therefore any one of the $(a_1x+a_2y+a_3z)$ white balls is as likely to be drawn as another.

If the ball which was drawn came from A, then it belonged to the group a_1x of white balls, and so for the other possibilities.

Therefore $$Q_1 : Q_2 : Q_3 = a_1x : a_2y : a_3z = p_1 : p_2 : p_3,$$

where $$p_1 = a_1/(a_1+b_1), \quad p_2 = a_2/(a_2+b_2), \quad p_3 = a_3/(a_3+b_3).$$

Now a white ball *is* drawn, therefore

$$Q_1+Q_2+Q_3 = 1 \quad \text{and} \quad Q_1 = p_1/(p_1+p_2+p_3),$$

with similar values for Q_2, Q_3.

It is to be noticed that p_1 *is the probability that the event will occur on the supposition that the ball comes from A, and so for* p_2, p_3.

Ex. 2. *The same as the preceding, except that there are m_1 bags such as A containing a_1 white and b_1 black balls, m_2 such as B, and m_3 such as C, as in Ex.* 1. *Also Q_1, Q_2, Q_3 are the chances that the ball came from an A, B, C bag respectively.*

Alter the numbers of the balls as in the preceding. Then any one of the

$$m_1a_1x + m_2a_2y + m_3a_3z$$

white balls is as likely to be drawn as another, and Q_1 is the chance that it comes from one of the m_1 groups of a_1x balls, so also for Q_2, Q_3, therefore

$$Q_1 : Q_2 : Q_3 = m_1a_1x : m_2a_2y : m_3a_3z = m_1p_1 : m_2p_2 : m_3p_3,$$

and as before, $$Q_1 + Q_2 + Q_3 = 1,$$

therefore $$Q_1 = m_1p_1/(m_1p_1 + m_2p_2 + m_3p_3),$$

with similar values for Q_2, Q_3.

NOTE. If P_1 is the chance estimated *before* the evont that an A bag will be chosen and P_2, P_3 have similar meanings, we have

$$P_1 : P_2 : P_3 = m_1 : m_2 : m_3 ;$$

therefore $$Q_1 : Q_2 : Q_3 = P_1p_1 : P_2p_2 : P_3p_3$$

and $$Q_1 = P_1p_1/(P_1p_1 + P_2p_2 + P_3p_3), \quad \text{etc.}$$

Observe that P_1p_1 is the chance that a white ball is drawn, and that from an A bag.

(2) **A General Statement.** Suppose that an event has occurred which must have been due to one of the causes, C_1, C_2, ... C_n.

Let P_r be the probability of the existence of the cause C_r, estimated *before* the event took place.

Let p_r be the probability of the event on the assumption that the cause C_r exists.

Then the probability Q_r of the existence of the cause C_r, estimated after *the event has occurred, is given by*

$$Q_r = P_rp_r/(P_1p_1 + P_2p_2 + \dots P_np_n).$$

For an argument similar to that in the last two examples shows that

$$Q_1 : Q_2 : Q_3 : \dots = P_1p_1 : P_2p_2 : P_3p_3 : \dots ,$$

and since the event *has* happened,

$$Q_1 + Q_2 + \dots + Q_n = 1,$$

whence the result follows.

NOTE. It is usual to call P_1, P_2, ... P_n the *a priori* probabilities of the existence of the causes and Q_1, Q_2, ... Q_n the *a posteriori* probabilities.

The product P_rp_r is the *antecedent probability* that the event will occur, and that from the rth cause.

The argument depends on the assumption (1) that Q_1, Q_2, ... *are proportional to* P_1p_1, P_2p_2, ... , which is justified in (1)

In particular, if an event is due to one of *two causes*, the odds in favour of its having occurred from the first cause are as $P_1p_1 : P_2p_2$.

The way in which these principles are applied to determine the *Probability of Future Events* is illustrated in Ex. 4.

Ex. 1. *A bag contains* 5 *balls, and of these it is equally likely that* 0, 1, 2, 3, 4, 5 *are white. A ball is drawn and is found to be white. What is the chance that this is the only white ball?*

There are 6 possible hypotheses; the number of white balls may be 0, 1, 2, 3, 4, 5.

Denoting these possibilities by $C_0, C_1, \dots C_5$, and using a notation similar to the above, we have $P_0 = P_1 = \dots = P_5 = \frac{1}{6}$; and $p_0 = 0, \quad p_1 = \frac{1}{5}, \quad p_2 = \frac{2}{5}, \dots, \quad p_5 = \frac{5}{5}$;

$$\therefore \; P_0p_0 + P_1p_1 + \dots + P_5p_5 = \frac{1}{6} \cdot \frac{1+2+3+4+5}{5} = \frac{1}{2};$$

$$\therefore \text{ the required chance} = Q_1 = P_1p_1 \div \tfrac{1}{2} = \tfrac{1}{15}.$$

Ex. 2. *If in the last example the ball which was drawn is replaced, what is the chance that a second drawing will give a white ball?*

$$Q_0p_0 + Q_1p_1 + \dots + Q_5p_5 = \tfrac{2}{6}(p_0^2 + p_1^2 + \dots + p_5^2) = \frac{2}{6 \cdot 5^2}(1^2 + 2^2 + 3^2 + 4^2 + 5^2).$$

Hence the required chance $= \tfrac{11}{15}$.

Ex. 3. *A pack of cards is counted, face downwards, and it is found that one card is missing. Two cards are drawn and are found to be spades. What are the odds against the missing card being a spade?*

The 'event' is that two spades are drawn. There are two possible hypotheses:

(C_1) The missing card is a spade.

(C_2) The missing card is not a spade.

The *a priori* probabilities P_1, P_2 of C_1, C_2 are $P_1 = \frac{1}{4}, \quad P_2 = \frac{3}{4}$.

The chances p_1, p_2 of the event under the hypotheses C_1, C_2 are

$$p_1 = \tfrac{12}{51} \cdot \tfrac{11}{50}, \quad p_2 = \tfrac{13}{51} \cdot \tfrac{12}{50}.$$

The odds against the missing card being a spade are as

$$P_2p_2 : P_1p_1 = \tfrac{3}{4} \cdot \tfrac{13}{51} \cdot \tfrac{12}{50} : \tfrac{1}{4} \cdot \tfrac{12}{51} \cdot \tfrac{11}{50} = 39 : 11.$$

Ex. 4. *A bag contains m balls which are either white or black, all possible numbers being equally likely. If p white and q black balls have been drawn in $p+q$ successive trials without replacement, the chance that another drawing will give a white ball is*

$$(p+1)/(p+q+2).$$

The possible hypotheses, all equally likely, are that the number of white balls are

$$m-q, \quad m-q-1, \quad \dots \quad m-q-r+1, \quad \dots \quad p.$$

Denote these by $C_1, C_2, \dots C_r, \dots C_{m-q-p+1}$.

If $p_1, p_2, \dots$ are the antecedent probabilities of the event under these hypotheses, we have $p_r = C_p^{m-q-r+1} \cdot C_q^{q+r-1} / C_{p+q}^m = A u_r v_r,$

where $u_r = r(r+1)(r+2) \dots (r+q-1),$

$$v_r = (n-r)(n-r+1)(n-r+2) \dots (n-r+p-1),$$

$$n = m-q-p+2,$$

and A is independent of r.

If $Q_1, Q_2, \dots$ are the *a posteriori* probabilities of $C_1, C_2, \dots$, then $Q_r = u_r v_r / S,$

where $S = \sum_{r=1}^{r=n-1} u_r v_r = \dfrac{\lfloor p \rfloor q}{\lfloor p+q+1}} \cdot \dfrac{\lfloor m+1}{\lfloor n-2}}$ (by Ch. XXI, 10, Ex. 1.)

If p_r' is the chance that, with hypothesis C_r, another drawing gives a white ball,

$$p_r' = \frac{m-q-p-r+1}{m-q-p} = \frac{n-1-r}{m-q-p};$$

thus $p_{n-1}=0$, and the chance that another drawing gives a white ball is

$\Sigma_{r=1}^{r=n-2} Q_r p_r' = S'/(m-q-p)S$, where $S' = \Sigma_{r=1}^{r=n-2} u_r(n-1-r)(n-r)\ldots(n-r+p)$.

Thus S' can be obtained from S by writing $n-1$ for n and p for $p-1$; and hence

$$S' = \frac{\lfloor p+q+2}{\lfloor p+1 \lfloor q} \cdot \frac{\lfloor m+1}{\lfloor n-3} \quad \text{and} \quad \frac{S'}{S} = \frac{(p+1)(n-2)}{p+q+2};$$ the chance $=(p+1)/(p+q+2)$.

9. Value of Testimony. The theory of probability has been used to estimate the value of the testimony of witnesses. Such an application is open to adverse criticism. It rests on two assumptions which can hardly be justified, namely: (i) that to each witness there pertains a constant p (his *credibility*), which measures the average frequency with which he speaks the truth; (ii) that the statements of witnesses are independent of one another in the sense required in the theory of probability.

If we are prepared to make these assumptions, the procedure is as follows.

Ex. 1. *If p is the probability that a statement made by A is true and p' has a similar meaning for B, what are the odds in favour of the truth of a statement which A and B concur in making?*

The 'event' is the agreement of A and B in making a certain statement. The possibilities are: (i) the statement is true; (ii) it is false.

If it is true, the chance that they both say it is true is pp'.

If it is false, the chance that they both say it is true is $(1-p)(1-p')$.

Thus the *antecedent probabilities* of the event on the hypotheses (i), (ii) are

$$pp' \quad \text{and} \quad (1-p)(1-p'),$$

and the odds in favour of the truth of the statement are as $pp' : (1-p)(1-p')$.

Ex. 2. *A bag contains n balls, one of which is white. The probabilities that A and B speak the truth are p, p' respectively. A ball is drawn from the bag, and A and B both assert that it is white. What are the odds in favour of its being white?*

The 'event' is the agreement of A and B in making a statement. The possibilities are (i) it is true, (ii) it is false.

The *a priori* probabilities P_1, P_2, of (i), (ii) are $P_1 = 1/n$, $P_2 = (n-1)/n$.

If p_1, p_2 are the chances of the event under (i) and (ii), we have $p_1 = pp'$.

In the case of (ii), $(n-1)$ balls remain in the bag, and one of these is white. The chance that A should choose this ball and wrongly assert that it was drawn from the bag is $(1-p)(n-1)$. The chance that B should do the same thing is $(1-p')(n-1)$; hence

$$p_2 = (1-p)(1-p')(n-1)^2,$$

and the odds in favour of a white ball having been drawn are as

$$P_1p_1 : P_2p_2 = \frac{1}{n}pp' : \frac{n-1}{n}\cdot(1-p)(1-p')\cdot\frac{1}{(n-1)^2}$$
$$= (n-1)pp' : (1-p)(1-p').$$

10. Geometrical Applications. The following statements are axiomatic :

(i) If a point is taken at random on a given straight line AB, the chance that it falls on a particular segment PQ of the line is PQ/AB.

Or we may say that the total number of cases is represented by AB, and the number of favourable cases by PQ.

(ii) If a point is taken at random on an area S which includes an area σ, the chance that the point falls on σ is σ/S.

Ex. 1. *Given that* $x+y=2a$ *where* a *is constant and that all values of* x *between* 0 *and* $2a$ *are equally likely, show that it is an even chance that* $xy>\frac{3}{4}a^2$.

Let AB be a diameter of a circle with centre O and radius a. Take a point P at random in AB.

Let $AP=x$, $PB=y$, then $x+y=2a$, and all values of x between 0 and $2a$ are equally likely.

Draw the ordinate PQ, then $PQ^2=AP\,.\,PB=xy$.

If A', B' are the mid-points of OA, OB, the ordinates at these points are equal to $a\,.\,\sqrt{\frac{3}{4}}$.

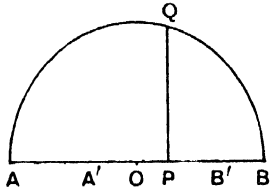


FIG. 86.

Hence $PQ>a\sqrt{\frac{3}{4}}$ if, and only if, P lies in $A'B'$.

Hence the chance that $xy>\frac{3}{4}a^2$ is $A'B'/AB$, that is $\frac{1}{2}$.

Ex. 2. *Two points* P, Q *are taken at random on a straight line* OA *of length* a. *Show that the chance that* $PQ>b$, *where* $b<a$, *is* $(a-b)^2/a^2$.

The points are as likely to fall in the order O, P, Q, A as in the order O, Q, P, A.

We may therefore suppose that Q is to the right of P.

Draw OA' at right angles to OA and equal to it.

Complete the figure as in the diagram, where

$$OL=PR=b.$$

If δx is small, the number of cases in which the distance of P from O lies between x and $x+\delta x$ and Q is in PA, is represented by $\delta x\,.\,PA$, i.e. by the area of the shaded rectangle.

Of these, the favourable cases are those in which Q lies in RA, and their number is represented by the upper part of the shaded rectangle cut off by LM.

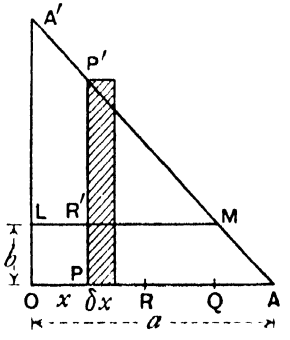


FIG. 87.

Hence the total number of cases is represented by area of the triangle OAA', and the total number of favourable cases by the area of the triangle LMA';

$$\therefore \text{ the required chance}=\frac{\triangle LMA'}{\triangle OAA'}=\left(\frac{a-b}{a}\right)^2.$$

EXERCISE LV

1. If two balls are drawn from a bag containing 2 white, 4 red and 5 black balls, what is the chance that

(i) they are both red ?

(ii) one is red and the other black ?

2. A bag contains m white and n black balls. If $p+q$ balls are drawn without replacement, the chance that these consist of p white and q black balls is $C_p^m \cdot C_q^n / C_{p+q}^{m+n}$.

3. If four balls are drawn from a bag containing 3 white, 4 red and 5 black balls, what is the chance that *exactly* two of them are black ? (i.e. two are black and two are white or red.)

4. In the last question what is the chance that *at least* three of the four balls drawn from the bag are black ?

5. Out of 20 consecutive numbers, two are chosen at random ; prove that the chance that their sum is odd is $\frac{10}{19}$

6. In a throw with three dice what is the chance that the dice fall (i) all alike, (ii) two alike and the third different, (iii) all different ?

Explain why the sum of these probabilities is unity, and verify the truth of this statement.

[(ii) Any particular throw of this sort, as five-five-six, occurs in 3 cases, and there are $6 \cdot 5$ throws of this kind.]

7. What is the chance of throwing 10 in a single cast with three dice ?

8. In a single cast with four dice what is the chance of throwing (i) two doublets, (ii) exactly 12, (iii) less than 12 ?

9. How many times must a pair of dice be thrown that it may be at least an even chance that double six may occur at least once ?

10. If four cards are drawn from a pack, what is the chance that they are ace, king, queen, knave (of the same or of different suits) ?

11. A letter is chosen at random out of ' assinine ' and one is chosen at random out of ' assassin.' Show that the chance that the same letter is chosen on both occasions is $\frac{1}{4}$.

[The number of possible cases is $7 \cdot 8$. Of these 14 are favourable.]

12. If the letters of ' attempt ' are written down at random, find the chance that (i) all the t's are together, (ii) no two t's are together.

13. A party of n men of whom A, B are two, form single rank. What is the chance that (i) A, B are next one another, (ii) that exactly m men are between them, (iii) that not more than m men are between them ?

14. From a bag containing a balls, m balls are drawn simultaneously and replaced ; then n balls are drawn. Show that the chance that exactly r balls are common to the two drawings is $C_r^m C_{n-r}^{a-m} / C_n^a$. Prove also that this expression is unaltered by interchanging m and n.

[Consider the second drawing. There are C_n^a possible cases. The favourable cases consist of selections of r balls out of the m already drawn, with selections of $n-r$ out of $a-m$ not already drawn.]

15. A squad of $4n$ men form fours. Find the chance that two specified men are next one another in the same four.

16. If two squares are chosen at random on a chess board, show that

(i) the chance that they have a side common is $\frac{1}{18}$;

(ii) the chance that they have contact at a corner is $\frac{7}{144}$.

[(i) Out of 64×63 possible cases, the number of favourable cases is

$$4 \cdot 2 + 24 \cdot 3 + 36 \cdot 4.]$$

17. One bag contains a white balls and b black balls; another contains a' white and b' black balls.

Let p be the chance of drawing a white ball from a bag chosen at random, and let p' be the chance of drawing a white ball from a bag containing $(a+a')$ white and $(b+b')$ black balls.

Find p, p', and show that $p \gtrless p'$ according as $a+b-a'-b'$ and $ab'-a'b$ have the same or opposite signs.

18. If A's chance of winning a single game against B is $\frac{2}{3}$, find A's chance of winning (i) two games (at least) out of the first three; (ii) four games (at least) out of the first six.

19. A draws a card from a pack of n cards marked 1, 2, 3, ... n; the card is replaced in the pack and then B draws a card. Find the chance that A draws (i) the same card as B; (ii) a higher card than B; (iii) a lower card than B. Verify that the sum of these chances is unity.

[For (ii), chance $= \frac{1}{n}\left\{\frac{1}{n} + \frac{2}{n} + \ldots + \frac{n-1}{n}\right\}$.]

20. A bag contains n cards marked 1, 2, 3, ... n. If A is to draw all the cards in succession and is to receive 1 shilling for every card which comes out in its proper order, prove that his expectation is 1 shilling.

[The expectation arising from each ball is $\frac{1}{n}$ of a shilling, etc. The fact that the probabilities, on which the several expectations depend, are not independent does not affect the argument.]

21. A bag contains n tickets numbered 1, 2, 3, ... n. A person draws two tickets at once, and is to receive a number of shillings equal to the product of the numbers drawn. What is his expectation?

22. What is the chance that a hand of five cards contains (i) exactly two aces; (ii) at least two aces?

[For (ii), number of favourable cases $= C_2^4 \cdot C_3^{48} + C_3^4 \cdot C_2^{48} + 48$.]

23. If p_1, p_2, p_3, p_4 are the probabilities of four independent events, find the chance that (i) two and not more than two of the events happen; (ii) at least two of the events happen.

24. Two persons each make a single throw with a pair of dice. What is the chance that the throws are equal?

[If $(x+x^2+\ldots+x^6)^2 = a_2x^2+\ldots+a_{12}x^{12}$, then the chance $= \frac{1}{36^2}(a_2^2+\ldots+a_{12}^2)$: and this is equal to {coefficient of x^{10} in $\left(\frac{1-x^6}{1-x}\right)^4$}$/36^2$.]

25. The decimal part of the logarithms of two numbers taken at random are found to seven places. What is the chance that the second can be subtracted from the first without ' borrowing ' ?

26. In a game of whist, find the chance that each of the four players should have an honour :

(i) if the card turned up is an honour ;

(ii) if this card is not an honour ;

(iii) before the last card is turned up.

27. In a game of whist, find the chance that each party has two honours :

(i) if the card turned up is an honour ;

(ii) if this card is not an honour ;

(iii) before the last card dealt is turned up.

28. Out of n persons sitting at a round table, three, A, B, C, are chosen at random. Prove that the chance that no two of these three are sitting next one another is $(n-4)(n-5)/(n-1)(n-2)$.

[Relative to A, B can sit in $n-3$ places. For 2 of them, C can sit in $n-5$ places ; for $n-5$ of them, C can sit in $n-6$ places. Hence, the number of favourable cases $=2(n-5)+(n-5)(n-6)$.]

29. Three cards are drawn from a bag containing n cards marked 1, 2, 3, ... n. Find the chance that (i) the three cards form a sequence ; (ii) they contain a sequence of two ; (iii) they involve no sequence.

30. In a set of tennis between A and B, the chance that the server wins a game is always p, the chance that he loses is q. The score is five all, they proceed to play deuce and vantage games and A has the service. Denoting A's chance of winning the set by x, show that $x=pq+(p^2+q^2)x$, whence it follows that A and B have equal chances of winning the set.

31. A and B play with two dice on the condition that A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Show that A's chance is to B's chance as 30 : 31. (Huyghens.)

32. Three players A, B, C of equal skill engage in a match consisting of a series of games in which A plays B, B plays C, C plays A, A plays B, and so on, in rotation. The player who first wins two consecutive games wins the match. Prove that the chances which A, B, C respectively have of winning the match are as 12 : 20 : 17.

[First show that if A loses the first game, his chance of winning the match is $\frac{1}{7}$: if he wins the first game, his chance is $\frac{17}{49}$.]

33. A bag contains 3 balls, and it is equally likely that 1, 2 or all of them are white. A ball is drawn and is found to be white.

(i) What is the chance that this is the only white ball ?

(ii) What is the chance that another drawing will give a white ball ?

34. The same question as the last, except that each ball is as likely as not to be white.

[Proceeding as in Ex. 1, p. 520, we see that P_0, P_1, P_2, P_3 are the terms in the expansion of $(\frac{1}{2}+\frac{1}{2})^3$. The rest proceeds as in Ex. 2, p. 520.]

35. A bag contains 5 balls, and it is not known how many of these are white. Two balls are drawn, and these are white. What is the chance that all are white ?

36. The chances that A, B, C speak the truth are respectively p, p', p''. What are the odds in favour of an event actually having happened which

(i) all three assert to have happened ?

(ii) A, B assert to have happened and C denies ?

Geometrical

37. The sides of a rectangle are chosen at random, each less than a given length a, all such lengths being equally likely. Show that the chance that the diagonal is less than a is $\pi/4$.

[Draw a square, $OACB$, whose side is a ; and the arc AB. If $ONPM$ is one of the rectangles, the favourable cases are when P lies in the quadrant OAB.]

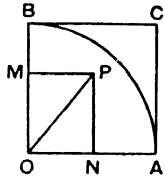


FIG. 88.

38. A floor is paved with rectangular bricks, each of length a and breadth b. A circular disc of diameter c is thrown on the floor. Show that the chance that it falls entirely on one brick is $(a-c)(b-c)/ab$.

39. A point P is taken at random in a straight line AB. Show that the chance that the greater of the parts AP, PB is at least k times the smaller is $2/(k+1)$.

40. If two points are taken at random on the circumference of a circle, the chance that their distance apart is greater than the radius of the circle is $\frac{2}{3}$.

[The unfavourable cases are represented by the thick arc in Fig. 89.]

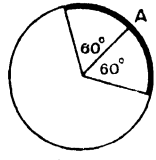


FIG. 89.

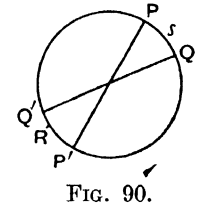


FIG. 90.

41. If three points, P, Q, R, are taken at random on the circumference of a circle, the chance that they do not lie on the same semicircle is $\frac{1}{4}$.

[Choose P. Choose Q on one semicircle on PP' ; R must lie on $P'Q'$. (Fig. 90.)

If $PQ=s$, number of favourable cases $=\int_0^{\frac{1}{2}c} s\,ds=\frac{1}{8}c^2$; total number $=\frac{1}{2}c\,.\,c$, etc.]

42. On a straight line AB of length $a+b$, two segments PQ, $P'Q'$, of lengths a, b respectively, are measured at random. (i) If $a>b$, the chance that $P'Q'$ lies entirely within PQ is $(a-b)/a$. (ii) If c is less than a or b, the chance that the common part of PQ, $P'Q'$ is less than c is c^2/ab.

[(ii) Let $AP=x$. If $LQ=c$, $MQ=x$, and $P'Q'$ overlaps PQ towards B, P' must lie in LM. Hence, the number of favourable cases $=\int_0^c (c-x)dx=\frac{1}{2}c^2$;

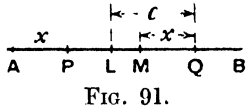


FIG. 91.

.. *total* number of favourable cases $=c^2$. Total number of cases $=ab$, etc.]

CHAPTER XXXIII

CONTINUED FRACTIONS (2)

EXPRESSION OF A QUADRATIC SURD AS A SIMPLE CONTINUED FRACTION

1. Surds of the Form $\pm(\sqrt{N}\pm b_1)/r_1$. From Ch. XXIV, 33, it follows that *any simple recurring continued fraction is equal to a quadratic surd.* In other words, its value is of the form $\pm(\sqrt{N}\pm b_1)/r_1$, where N, b_1, r_1 are positive integers, except that b_1 may be zero, and N is not a perfect square.

We shall prove that conversely *any positive number of the form*

$$\pm(\sqrt{N}\pm b_1)/r_1$$

can be expressed as a simple recurring continued fraction.

In considering this theorem, it is to be observed that:

(i) *There is no loss of generality in assuming that $N-b_1^2$ is divisible by r_1.*

For $(\sqrt{N}\pm b_1)/r_1=(\sqrt{Nr_1^2}\pm b_1r_1)/r_1^2$ and $Nr_1^2-(b_1r_1)^2$ is divisible by r_1^2, hence $(\sqrt{N}\pm b_1)/r_1$ can always be replaced by an expression of the same form for which the above condition holds.

(ii) We need only consider surds which are greater than unity.

For if $N-b_1^2=\pm r_1r_2$, where r_2 is a positive integer, then

$$\frac{\sqrt{N}\pm b_1}{r_1}=1\div\frac{\sqrt{N}\mp b_1}{r_2}.$$

It follows that every positive quadratic surd or its reciprocal is of one of four types considered in the next article.

2. Types of Quadratic Surds. In every case it is assumed that

(i) *The surd in question is greater than unity.*

(ii) *$N-b_1^2$ is divisible by r_1.*

(A) *The type $(\sqrt{N}+b_1)/r_1$ where $b_1^2<N$.* This will be called the *normal* type, and is of special importance in the theory: it includes the forms $\sqrt{A/B}$ and $\sqrt{N}$.

To express $(\sqrt{N}+b_1)/r_1$ *as a simple continued fraction,* we form the equations

$$\frac{\sqrt{N}+b_1}{r_1}=a_1+\frac{\sqrt{N}-b_2}{r_1}=a_1+\frac{r_2}{\sqrt{N}+b_2}, \quad \text{.....(A)}$$

where a_1 is the integral part of $(\sqrt{N}+b_1)/r_1$ and

$$b_2=a_1r_1-b_1, \quad r_1r_2=N-b_2^2. \quad \text{.....(B)}$$

We shall first show that b_2, r_2 *are positive integers.*

Since a_1 is the integral part of $(\sqrt{N}+b_1)/r_1$,

$$a_1r_1<\sqrt{N}+b_1<a_1r_1+r_1, \quad \text{.....(C)}$$

and therefore

$$b_2<\sqrt{N}<b_2+r_1. \quad \text{.....(D)}$$

If we suppose that $b_2\leqslant 0$, it follows that $\sqrt{N}<r_1$, but $b_1<\sqrt{N}$, therefore $b_1<r_1$ and consequently $b_1<a_1r_1$. This is contrary to the supposition that $b_2\leqslant 0$. Hence b_2 is positive, and it is obviously an integer.

Again, b_2 is a positive number less than $\sqrt{N}$, therefore $N-b_2^2>0$, and consequently r_2 is positive. Further, we have

$$N-b_2^2=N-(a_1r_1-b_1)^2=N-b_1^2-r_1(a_1^2r_1-2a_1b_1),$$

and since $N-b_1^2$ is divisible by r_1, so also is $N-b_2^2$. Hence r_2 is a positive integer. Continuing the process, we form the equations

$$\frac{\sqrt{N}+b_n}{r_n}=a_n+\frac{\sqrt{N}-b_{n+1}}{r_n}=a_n+\frac{r_{n+1}}{\sqrt{N}+b_{n+1}}, \quad \text{.....(E)}$$

for $n=2, 3, \ldots$ where a_n is an integer such that

$$a_n<(\sqrt{N}+b_n)/r_n<a_n+1, \quad \text{.....(F)}$$

and

$$b_{n+1}=a_nr_n-b_n, \quad r_nr_{n+1}=N-b_{n+1}^2, \quad \text{.....(G)}$$

leading to

$$\frac{\sqrt{N}+b_1}{r_1}=a_1+\frac{1}{a_2+}\ \ldots\ \frac{1}{a_n+}\ \ldots\ . \quad \text{.....(H)}$$

By steps similar to the above, we can show that *every a, b, r is a positive integer.* For taking $n=2$, by the preceding $(\sqrt{N}+b_2)/r_2>1$, therefore a_2 is a positive integer. Also $N-b_2^2$ is divisible by r_2. Reasoning as before, it will be seen that b_3 and r_3 are positive integers, and so for $n=3, 4, 5, \ldots$.

Finally, *the fraction is periodic,* for the nth complete quotient is

$$(\sqrt{N}+b_n)/r_n,$$

and for every n, $b_n<\sqrt{N}$, therefore $r_n=(b_n+b_{n+1})/a_n<2\sqrt{N}$.

Hence the fraction $(\sqrt{N}+b_n)/r_n$ cannot have more than $2N$ distinct values, and one of the complete quotients must occur again. From this stage, all the succeeding a's recur and in the same order: the same is true for the b's and r's. Thus *the continued fraction is periodic.*

It will now be shown that in the process of expressing a positive quadratic surd of any other type as a simple continued fraction, *a stage must occur at which the complete quotient is a surd of normal type. Whence it will follow that in every case the continued fraction is periodic.*

(B) *The type* $(\sqrt{N}-b_1)/r_1$ *where* $b_1^2<N$. Here the second complete quotient is a surd of normal type.

For instance, $$\frac{\sqrt{57}-7}{2}=1\Big/\frac{2}{\sqrt{57}-7}=1\Big/\frac{\sqrt{57}+7}{4}=\underset{*}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{\underset{*}{7}}$$

and $$\frac{\sqrt{57}-3}{2}=2+\frac{\sqrt{57}-7}{2}=2+1\Big/\frac{\sqrt{57}+7}{4}.$$

(C) *The form* $(b_1+\sqrt{N})/r_1$ *where* $b_1^2>N$. We form the equations

$$\frac{b_1+\sqrt{N}}{r_1}=a_1+\frac{b_2+\sqrt{N}}{r_1}=a_1+\frac{r_2}{b_2-\sqrt{N}},$$

$$\frac{b_2-\sqrt{N}}{r_2}=a_2+\frac{b_3-\sqrt{N}}{r_2}=a_2+\frac{r_3}{b_3+\sqrt{N}},\quad \text{etc.},$$

where a_1, a_2, ... are the integral parts of the fractions on the left and, for every n,

$$b_{n+1}=b_n-a_nr_n,\quad r_nr_{n+1}=b_{n+1}^2-N.$$

As in (A), every r is an integer and every b is an integer or zero, and by hypothesis $b_1>\sqrt{N}$.

Suppose that b_1, b_2, ... b_n are all greater than $\sqrt{N}$; then r_1, r_2, ... r_n are all positive, and b_1, b_2, ... b_{n+1} is a decreasing sequence of integers. Hence a stage must occur at which

$$b_n>\sqrt{N}>b_{n+1}.$$

This being so, *n cannot be even*; for in that case we should have

$$b_n-\sqrt{N}>a_nr_n,$$

and therefore $b_{n+1}>\sqrt{N}$.

Hence *n is odd*, and $b_n+\sqrt{N}>a_nr_n$. Consequently $\sqrt{N}+b_{n+1}>0$; and since also $\sqrt{N}-b_{n+1}>0$, it follows that $N-b_{n+1}^2>0$, and therefore r_{n+1} is negative.

Now $(n+1)$ being even, the $(n+1)$th complete quotient may be written

$$(\sqrt{N}-b_{n+1})/(-r_{n+1});$$

and since $b_{n+1}^2<N$, this or the next quotient is a surd of normal type according as $b_{n+1}\gtrless 0$.

(D) *The type* $(b_1-\sqrt{N})/r_1$ *where* $b_1^2>N$. Here the second complete quotient is a surd of type (C).

3. Theorem. *Let the surd* $(\sqrt{N}+b)/r$ *be expressed as a simple recurring fraction, then* (i) *if* $b<\sqrt{N}<b+r$, *the fraction has no acyclic part;* (ii) *if* $\sqrt{N}>b+r$, *it has an acyclic part consisting of a single quotient;* (iii) *if* $b>\sqrt{N}$, *it has an acyclic part of one or more quotients.*

For if $x=(\sqrt{N}+b)/r$, then $(rx-b)^2=N$ and the second root of this quadratic is $(-\sqrt{N}+b)/r$; hence, from Chap. XXIV, 33, it follows that (i) *if* $-1<(-\sqrt{N}+b)/r<0$, the fraction has no acyclic part and these conditions are equivalent to $b<\sqrt{N}<b+r$; (ii) *if* $(-\sqrt{N}+b)/r<-1$, that is, if $\sqrt{N}>b+r$, the acyclic part consists of a single quotient; and (iii) *if* $(-\sqrt{N}+b)/r>0$, that is, if $b>\sqrt{N}$, the acyclic part contains one or more quotients.

In particular, *if* $A>B$, *the simple continued fraction equivalent to* $\sqrt{A/B}$ *has a single non-recurring quotient.*

4. Method of Reckoning. In practice we replace the written work involving surds by an easy mental process, as in the next example.

Ex. 1. *Express* $(\sqrt{37}+8)/9$ *as a simple continued fraction.*

Here a_1 = integral part of the surd = 1, also $b_1=8$, $r_1=9$. The various quantities are now found in succession from the equations

$$b_n=a_{n-1}r_{n-1}-b_{n-1}, \quad r_n=(N-b_n^2)/r_{n-1}, \quad a_n=\text{integral part of } (\sqrt{N}+b_n)/r_n,$$

giving the table:

n	1	2	3	4	5	6
b	8	1	3	4	5	3
r	9	4	7	3	4	7
a	1	1	1	3	2	1

The reckoning, if continued, is a repetition of the part between the dotted lines, and

$$\frac{\sqrt{37}+8}{9}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+}}}}\cdots.$$

* *

NOTE. The third complete quotient, $(\sqrt{37}+3)/7$, marks the beginning of the recurring period, for $3<\sqrt{37}<3+7$. It will be shown in the last article of this chapter that, *from this stage, the above reckoning can be replaced by a* G.C.M. *process.*

5. $\sqrt{A/B}$ as a Continued Fraction.

(1) It is supposed that A, B are positive integers and that $A>B$. We have $\sqrt{\dfrac{A}{B}}=\dfrac{\sqrt{AB}}{B}=\dfrac{\sqrt{N}+b_1}{r_1}$, where $N=AB$, $b_1=0$, $r_1=B$. The surd is therefore of the type discussed in Art. 3, and it is expressed as a simple continued fraction by constructing equations of the type

$$\frac{\sqrt{N}+b_n}{r_n}=a_n+\frac{\sqrt{N}-b_{n+1}}{r_n}=a_n+\frac{r_{n+1}}{\sqrt{N}+b_{n+1}}, \text{ for values of } n=1, 2, 3, \ldots;$$

where a_n is an integer such that $a_n < (\sqrt{N}+b_n)/r_n < a_n+1$,

and $$b_{n+1} = a_n r_n - b_n, \quad r_n r_{n+1} = N - b_{n+1}^2,$$

leading to $$\sqrt{\frac{A}{B}} = \frac{\sqrt{N}}{r_1} = a_1 + \frac{1}{a_2+} \cdots \frac{1}{a_n+} \cdots .$$

As in Art. 2, every a, b and r is a positive integer and the continued fraction is periodic. Also, it has been shown in Art. 3 that a_1 is the only non-recurring quotient.

(2) **The following inequalities** are required. Every complete quotient is greater than 1, therefore $\sqrt{N}+b_n > r_n$. .....**(A)**

Also $r_{n-1} = (N-b_n^2)/r_n = (\sqrt{N}-b_n)(\sqrt{N}+b_n)/r_n > \sqrt{N}-b_n$, therefore

$$\sqrt{N} < b_n + r_{n-1}. \quad \text{.....(B)}$$

Again, if $n>1$, the continued fraction equivalent to $(\sqrt{N}+b_n)/r_n$ has no acyclic part, therefore, as in Art. 3, $b_n < \sqrt{N} < b_n + r_n$ $(n \geqslant 2)$.....**(C)**

For any suffix m, $b_m < \sqrt{N}$, hence from **(B)** and **(C)**,

$$b_m - b_n < r_{n-1} \quad \text{and} \quad b_m - b_n < r_n \quad (n \geqslant 2). \quad \text{.....(D)}$$

(3) **The Cycle of Quotients.** Let c be the number of elements in the cycle, then

$$\frac{\sqrt{N}}{r_1} - a_1 = \frac{1}{\overset{*}{a_2}+} \frac{1}{a_3+} \cdots \frac{1}{\overset{*}{a_{c+1}}}.$$

Now $\sqrt{N}/r_1 - a_1$ is a root of $r_1^2 (x+a_1)^2 = N$, the other root of this equation being $-\sqrt{N}/r_1 - a_1$. Therefore by Ch. XXIV, 31,

$$\frac{\sqrt{N}}{r_1} + a_1 = a_{c+1} + \frac{1}{a_c+} \frac{1}{a_{c-1}+} \cdots \frac{1}{\overset{*}{a_2}}.$$

Hence $$a_{c+1} + \frac{1}{a_c+} \cdots \frac{1}{a_2+} \frac{1}{\overset{*}{a_{c+1}}+} \cdots = 2a_1 + \frac{1}{a_2+} \cdots \frac{1}{a_{c+1}+} \frac{1}{a_2+} \cdots,$$

and therefore $$a_{c+1} = 2a_1, \quad a_c = a_2, \quad a_{c-1} = a_3, \text{ etc.}$$

Summary. It has been shown that $\sqrt{A/B}$ can be expressed as a simple recurring fraction in which (i) *there is a single non-recurring element,* a_1 *;* (ii) *the last partial quotient of the cycle is* $2a_1$ *;* (iii) *for the rest of the cycle, the partial quotients equidistant from the beginning and end are equal.*

All this may be expressed by writing

$$\sqrt{\frac{A}{B}} = \frac{\sqrt{N}}{r_1} = a_1 + \frac{1}{\overset{*}{a_2}+} \frac{1}{a_3+} \cdots \frac{1}{a_3+} \frac{1}{a_2+} \frac{1}{\overset{*}{2a_1}+} \cdots ;$$

and we shall call the sequence, $a_2, a_3, \ldots a_3, a_2$, the *reciprocal part* of the cycle of quotients.

(4) **The b and r Cycles.** Let c be the number of elements in the cycle, so that

$$\frac{\sqrt{N}}{r_1} = a_1 + \underset{*}{\frac{1}{a_2 +}} \cdots \underset{*}{\frac{1}{a_{c+1}}}.$$

The recurrence is due to the fact that the second complete quotient appears again as the $(c+2)$th. From this stage all the complete quotients recur in the same order; hence, $(\sqrt{N}+b_{c+m})/r_{c+m} = (\sqrt{N}+b_m)/r_m$, and therefore $b_{c+m} = b_m$ and $r_{c+m} = r_m$ for $m \geqslant 2$.

Remembering that $a_{c+1} = 2a_1$, $a_{c-m} = a_{2+m}$ for $m = 0, 1, 2, \ldots$, we have

$$r_{c+1} = (N - b_{c+2}^2)/r_{c+2} = (N - b_2^2)/r_2 = r_1, \quad r_c = (N - b_{c+1}^2)/r_{c+1} = (N - b_2^2)/r_1 = r_2,$$

$$b_{c+1} = a_{c+1}r_{c+1} - b_{c+2} = 2a_1r_1 - b_2 = b_2, \quad b_c = a_c r_c - b_{c+1} = a_2 r_2 - b_2 = b_3,$$

and so on.

Hence, (i) *the cycle of b's is* $b_2, b_3, \ldots b_{c+1}$ *and is reciprocal, i.e.* $b_{c+1} = b_2$, $b_c = b_3$, *etc.*; (ii) *the cycle of r's is* $r_1, r_2, \ldots r_c$, *and is reciprocal after the first term, i.e.* $r_c = r_2$, $r_{c-1} = r_3$, *etc.*

The character of the recurrence and the reciprocity of the three sequences is exhibited below, where the recurring periods are enclosed in brackets.

1	2	3		c	$c+1$	$c+2$,
b_1	$[b_2$	b_3	$\ldots$	b_3	$b_2]$	b_2,
$[r_1$	r_2	r_3	$\ldots$	$r_2]$	r_1	r_2,
a_1	$[a_2$	a_3	$\ldots$	a_2	$2a_1]$	a_2.

It should be noticed that the a and b cycles correspond, and the *reciprocal* parts of the a and r cycles correspond.

(5) **Calculation of the Quotients.** In finding the values of b_n, r_n, a_n, *if we can tell when the middle of the b cycle has been reached,* no further calculation is necessary. This matter is settled by the following theorem.

If b_m, r_m, a_m *are respectively equal to* b_{n+1}, r_n, a_n, *then* b_m *and* b_{n+1} *are equidistant from the ends of the b cycle.*

For $$r_{n+1} = (N - b_{n+1}^2)/r_n = (N - b_m^2)/r_m = r_{m-1}.$$

Also $$a_{m-1}r_{m-1} = b_{m-1} + b_m \quad \text{and} \quad a_{n+1}r_{n+1} = b_{n+2} + b_{n+1}.$$

Subtracting and using the hypothesis, we find that

$$a_{m-1} - a_{n+1} = (b_{m-1} - b_{n+2})/r_{m-1}. \quad \ldots\ldots\ldots\

Putting $m=n$ and $m=n+1$ in succession, we have the following:

(i) *If two consecutive b's, namely b_m and b_{m+1}, are equal, then r_m and a_m are the mid-terms of the reciprocal parts of the r and a cycles. In this case c is even.*

(ii) *If $r_n=r_{n+1}$ and $a_n=a_{n+1}$ (so that two consecutive r's are equal and likewise the corresponding a's), then r_n, r_{n+1} and a_n, a_{n+1} are the mid-terms of the reciprocal parts of the r and a cycles, and b_{n+1} is the mid-term of the b cycle. In this case c is odd.*

Ex. 1. *Express $\sqrt{\frac{17}{11}}$ and $\sqrt{61}$ as simple continued fractions.*

(i) $\sqrt{\frac{17}{11}}=\frac{\sqrt{187}}{11}$. Hence $N=187$, $r_1=11$, $a_1=1$, and proceeding as in Art. 4, we construct the following table by a mental process:

b	0	[11	13	11	11		
r	[11	6	3	22			
a	1	[4	8	1	8	4	2].

Because $b_4=b_5=11$, therefore a_4 is the mid-term of the reciprocal part of the a cycle. This cycle can therefore be completed, for its last term is $2a_1=2$, thus

$$\sqrt{\frac{17}{11}}=1+\underset{*}{\frac{1}{4+}}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{8+}\,\underset{*}{\frac{1}{4+}}\,\frac{1}{2}.$$

(ii) For $\sqrt{61}$ we have $r_1=1$, $a_1=7$ and the b, r, a, table is

b	0	[7	5	7	5	4	6					
r	[1	12	3	4	9	5	5					
a	7	[1	4	3	1	2	2	1	3	4	1	14].

Seeing that $a_6=a_7$ and $r_6=r_7$, we conclude that a_6, a_7 are the mid-terms of the reciprocal part of the a cycle. Also the last quotient of the cycle is 14.

Therefore $$\sqrt{61}=7+\underset{*}{\frac{1}{1+}}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\underset{*}{\frac{1}{14}}.$$

(6) **Relations connecting the p's and q's.** Suppose that

$$\sqrt{\frac{A}{B}}=\frac{\sqrt{N}}{r_1}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_n+}\cdots,$$

where $N=AB$, $r_1=B$; and let p_r/q_r be the rth convergent, then

$$\frac{\sqrt{N}}{r_1}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_n+}\frac{1}{z}, \quad \text{where} \quad z=\frac{\sqrt{N}+b_{n+1}}{r_{n+1}};$$

therefore
$$\frac{\sqrt{N}}{r_1}=\frac{zp_n+p_{n-1}}{zq_n+q_{n-1}}=\frac{(\sqrt{N}+b_{n+1})p_n+r_{n+1}p_{n-1}}{(\sqrt{N}+b_{n+1})q_n+r_{n+1}q_{n-1}}$$

and
$$\sqrt{N}(\sqrt{N}q_n+b_{n+1}q_n+r_{n+1}q_{n-1})=r_1(\sqrt{N}p_n+b_{n+1}p_n+r_{n+1}p_{n-1}).$$

Equating rational and irrational parts,

$$Nq_n=r_1(b_{n+1}p_n+r_{n+1}p_{n-1}), \qquad r_1p_n=b_{n+1}q_n+r_{n+1}q_{n-1}.$$

Solving for b_{n+1} and r_{n+1} and putting $N=AB$ and $r_1=B$, we find that

$$Aq_nq_{n-1}-Bp_np_{n-1}=(-1)^nb_{n+1}, \quad \text{.....(A)}$$

$$Bp_n^2-Aq_n^2=(-1)^nr_{n+1}. \quad \text{.....(B)}$$

(7) **Integral Solutions of $Bx^2-Ay^2=M$.** If for some value of n it happens that $(-1)^nr_{n+1}=M$, then, by equation (**B**), (p_n, q_n) is a solution of $Bx^2-Ay^2=M$.

Ex. 1. *Find two positive integral solutions of* $11x^2-17y^2=3$.

Expressing $\sqrt{\frac{17}{11}}$ as a continued fraction (§ (5), Ex. 1), we find that $r_3=r_5=3$. Also $p_2/q_2=5/4$, $p_4/q_4=46/37$, and therefore (5, 4), (46, 37) are solutions.

6. $\sqrt{N}$ as a Continued Fraction. (1) Here $r_1=1$ and all the conclusions of the last article hold. Additional theorems are the following.

(i) *In the cycle of r's, one member and only one is equal to* 1, *namely* r_1.

For suppose that $r_m=1$, then, remembering that $b_2=a_1$ and using the equations

$$a_mr_m=b_m+b_{m+1}, \quad r_mr_{m+1}=N-b_{m+1}^2,$$

we have $a_m=$integral part of $(\sqrt{N}+b_m)/r_m=a_1+b_m=b_2+b_m$.

Also $a_m=b_m+b_{m+1}$, therefore $b_{m+1}=b_2$ and

$$r_{m+1}=N-b_{m+1}^2=N-b_2^2=r_1r_2=r_2.$$

Therefore $(\sqrt{N}+b_{m+1})/r_{m+1}=(\sqrt{N}+b_2)/r_2$, and $r_m=1$ marks the beginning of a new cycle of r's.

(ii) *If* $r_n\geqslant 2$, *then* $a_n\leqslant b_n$. For in this case $n>1$ and $\sqrt{N}<b_n+r_n$, so that

$$a_n<(\sqrt{N}+b_n)/r_n<(2b_n+r_n)/r_n<b_n+1\leqslant b_n.$$

(iii) *If* $r_n=2$, *the number* (c) *of elements in the cycle belonging to* $\sqrt{N}$ *is even. Also* a_n *is the mid-term of the reciprocal part of the 'a' cycle, and* $a_n=a_1$ *or* a_1-1.

For a_1 is the integral part of $\sqrt{N}$ and $\sqrt{N}>b_n$, therefore $a_1\geqslant b_n$. Also a_n is the integral part of $(a_1+b_n)/r_n$, and so if $r_n=2$ we have

$$a_n>(a_1+b_n)/2-1>b_n-1\geqslant b_n.$$

But by the preceding, $a_n\leqslant b_n$, therefore $a_n=b_n$. Also $2a_n=b_n+b_{n+1}$, therefore $b_n=b_{n+1}$.

Hence by § (5), (i), c is even and a_n is the mid-term of the reciprocal part of the a cycle. Finally a_n is the integral part of $(a_1+a_n)/2$, so that $a_n=(a_1+a_n)/2$ or $(a_1+a_n-1)/2$, and therefore $a_n=a_1$ or a_1-1.

(2) Equations (**A**) and (**B**) at the top of the page become

$$Nq_nq_{n-1}-p_np_{n-1}=(-1)^nb_{n+1}, \quad \text{.....(A)}$$

$$p_n^2-Nq_n^2=(-1)^nr_{n+1}. \quad \text{.....(B)}$$

(3) **Integral Solutions of $x^2 - Ny^2 = M$.** If, for some value of n, it is found that $(-1)^n r_{n+1} = M$, then (p_n, q_n) is a solution of $x^2 - Ny^2 = M$.

In particular, if c is the number of elements in the cycle belonging to $\sqrt{N}$, the necessary and sufficient condition that r_{n+1} may be equal to 1 is that $n = tc$ where $t = 1, 2, 3, \ldots$; hence, it follows that

(i) *for the equation* $x^2 - Ny^2 = 1$, solutions are (p_{tc}, q_{tc}) when c is even and (p_{2tc}, q_{2tc}) when c is odd;

(ii) *for the equation* $x^2 - Ny^2 = -1$, solutions are $(p_{(2t-1)c}, q_{(2t-1)c})$ when c is odd; the method giving no solution when c is even.

Ex. 1. *Find a positive integral solution of* $x^2 - 13y^2 = -1$.

We find that $\sqrt{13} = 3 + \underset{*}{\cfrac{1}{1+}}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \underset{*}{\cfrac{1}{6}}$,

so that $c = 5$. Also $p_5 = 18$, $q_5 = 5$ and $(18, 5)$ is a solution.

(4) **Calculation of Convergents.** The following formulae are useful in this connection. If we multiply equations (**A**), (**B**) firstly by q_n and q_{n-1}, and secondly by p_n and p_{n-1}, we obtain by addition

$$p_n = r_{n+1}q_{n-1} + b_{n+1}q_n \quad \text{and} \quad Nq_n = r_{n+1}p_{n-1} + b_{n+1}p_n. \quad \ldots\ldots\ldots(\mathbf{C})$$

Again, changing n into $n-1$ in equation (**B**) and using equation (**A**), we find that $p_{n-1} = r_n q_n - b_{n+1}q_{n-1}$ and $Nq_{n-1} = r_n p_n - b_{n+1}p_{n-1}$. ....(**D**)

In general, the p's are considerably larger than the q's, and to find p_n/q_n we need only calculate the q's and then use the first of equations (**C**).

(5) **The Convergent p_c/q_c.** (i) *If* $m + n = c$, *where* c *is the number of elements in the cycle belonging to* $\sqrt{N}$, *then*

$$p_c = p_m q_{n+1} + p_{m-1}q_n \quad \textit{and} \quad q_c = q_m q_{n+1} + q_{m-1}q_n. \quad \ldots\ldots\ldots(\mathbf{E})$$

For $$\frac{p_c}{q_c} = a_1 + \cfrac{1}{a_2+} \cdots \cfrac{1}{a_m + \cfrac{1}{f}},$$

where $$f = a_{m+1} + \cfrac{1}{a_{m+2}+} \cdots \cfrac{1}{a_c} = a_{n+1} + \cfrac{1}{a_n+}\ \cfrac{1}{a_{n-1}+} \cdots \cfrac{1}{a_2} = \frac{q_{n+1}}{q_n};$$

therefore $$\frac{p_c}{q_c} = \frac{fp_m + p_{m-1}}{fq_m + q_{m-1}} = \frac{p_m q_{n+1} + p_{m-1}q_n}{q_m q_{n+1} + q_{m-1}q_n}.$$

If X is the numerator and Y the denominator of the last fraction,

$$Xq_m - Yp_m = (-1)^{m-1}q_n \quad \text{and} \quad Xq_{m-1} - Yp_{m-1} = (-1)^m q_{n+1}.$$

Hence any common factor of X and Y is a factor of q_n and of q_{n+1}, and since these numbers are prime to one another, so also are X and Y. Thus the fractions p_c/q_c and X/Y are in their lowest terms, and therefore $p_c = X$ and $q_c = Y$, which are the results in question.

(ii) In particular, *if c is odd and* $n=\frac{1}{2}(c-1)$,

$$p_c=p_nq_n+p_{n+1}q_{n+1} \quad \text{and} \quad q_c=q_n{}^2+q_{n+1}^2. \quad \text{.....(F)}$$

If c is even and $n=\frac{1}{2}c$,

$$p_c=p_nq_{n+1}+p_{n-1}q_n \quad \text{and} \quad q_c=q_n(q_{n+1}+q_{n-1}). \quad \text{.....(G)}$$

(iii) *If* $m+n=c$, *then*

$$p_n+\sqrt{N}q_n=(-1)^m(p_m-\sqrt{N}q_m)(p_c+\sqrt{N}q_c), \quad \text{.....(H)}$$

which is equivalent to the two equations

$$p_mp_c-Nq_mq_c=(-1)^mp_n, \quad p_mq_c-q_mp_c=(-1)^mq_n. \quad \text{.....(I)}$$

For using equations (**E**), $p_mq_c-q_mp_c=(p_mq_{m-1}-q_mp_{m-1})q_n=(-1)^mq_n$.

Also we have the identity

$$(p_mp_c-Nq_mq_c)^2-N(p_mq_c-q_mp_c)^2=(p_m{}^2-Nq_m{}^2)(p_c{}^2-Nq_c{}^2).$$

Now $\quad p_m{}^2-Nq_m{}^2=(-1)^mr_{m+1}, \quad p_n{}^2-Nq_n{}^2=(-1)^nr_{n+1}$;

and, since $m+n=c$, we have $r_{m+1}=r_{n+1}$; and therefore

$$p_n{}^2-Nq_n{}^2=(-1)^c(p_m{}^2-Nq_m{}^2)=(p_m{}^2-Nq_m{}^2)(p_c{}^2-Nq_c{}^2).$$

Hence $\quad (p_mp_c-Nq_mq_c)^2=p_n{}^2.$

It remains to consider the sign of $p_mp_c-Nq_mq_c$.

The convergent p_m/q_m precedes p_c/q_c. First suppose that c is even. If m is even, $\sqrt{N}<p_c/q_c<p_m/q_m$ and $p_mp_c>Nq_mq_c$. If m is odd, it follows from Ch. XXIV, 25, that $p_mp_c<Nq_mq_c$. The same results follow in a similar way when c is odd, and thus $p_mp_c-Nq_mq_c=(-1)^mp_n$.

It should be noticed that *these results hold if c is replaced by tc.*

(iv) *If N is a prime and c is even and* $n=\frac{1}{2}c$, *then*

$$p_c+\sqrt{N}q_c=\tfrac{1}{2}(p_n+\sqrt{N}q_n)^2, \quad \text{.....(J)}$$

which is equivalent to $\quad p_c=\frac{1}{2}(p_n{}^2+Nq_n{}^2) \quad$ *and* $\quad q_c=p_nq_n$.

Also $p_n{}^2-Nq_n{}^2=(-1)^n\,.\,2$, *and the mid-term of the reciprocal part of the a cycle is* a_1 *or* a_1-1.

For by equations (**I**), $p_np_c-Nq_nq_c=(-1)^np_n$ and $p_nq_c-q_np_c=(-1)^nq_n$, therefore

$$p_c/q_c=(p_n{}^2+Nq_n{}^2)/2p_nq_n \quad \text{.....(K)}$$

and $\quad \left(\dfrac{p_n{}^2-Nq_n{}^2}{2p_nq_n}\right)^2=\left(\dfrac{p_c}{q_c}\right)^2-N=\dfrac{1}{q_c{}^2}; \quad$ hence $\quad \dfrac{(p_n{}^2-Nq_n{}^2)}{p_nq_n}=(-1)^n\dfrac{2}{q_c},$

the sign being determined by the fact that $p_n\gtrless\sqrt{N}q_n$, according as n is even or odd. Since N is a prime and p_n is prime to q_n, the last fraction is in its lowest terms. Hence $p_n{}^2-Nq_n{}^2=(-1)^n\,.\,2$ or $(-1)^n$, and the latter alternative is impossible.

Therefore $p_n^2 - Nq_n^2 = (-1)^n \cdot 2$ and $q_c = p_n q_n$

Hence, $r_{n+1} = 2$; and the rest follows by § (1), (iii), p. 534.

NOTE. *When N is not a prime, p_c and q_c may be found by using equation* (K) *and noting that p_c/q_c is in its lowest terms.*

Thus we find that $$\sqrt{21} = 4 + \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{8},$$
$$\qquad\qquad\qquad

(ii) Since $\sqrt{N}$ is irrational, equations (**L**) are *equivalent* to the single relation

$$p_{n+tc}+\sqrt{N}q_{n+tc}=(p_n+\sqrt{N}q_n)(p_{tc}+\sqrt{N}q_{tc}).$$

Putting $t=1$ and $n=c,\ 2c,\ 3c,\ \ldots$ in succession, we find that

$$p_{2c}+\sqrt{N}q_{2c}=(p_c+\sqrt{N}q_c)^2,\quad p_{3c}+\sqrt{N}q_{3c}=(p_c+\sqrt{N}q_c)^3\ \ldots,$$

and generally, for every positive t, $p_{tc}+\sqrt{N}q_{tc}=(p_c+\sqrt{N}q_c)^t$; .....(**P**)

and therefore $p_{n+tc}+\sqrt{N}q_{n+tc}=(p_n+\sqrt{N}q_n)(p_c+\sqrt{N}q_c)^t$. .....(**Q**)

Ex. 1.* *Find a solution in positive integers of* $x^2-61y^2=1$.

In Ex. 1, p. 533, $\sqrt{61}$ is expressed as a continued fraction, the number of quotients in the cycle being 11. Hence the smallest solution of the equation in question is $(p_{22},\ q_{22})$. The mid-elements of the cycle are the 5th and 6th, and $p_5=164$, $q_5=21$, $p_6=453$, $q_6=58$. Hence by equations (**F**),

$$p_{11}=p_5q_6+p_6q_6=164\,.\,21+453\,.\,58=29718,\quad q_{11}=q_5^2+q_6^2=21^2+58^2=3805.$$

Using equations (**O**) and observing that $p_{11}^2-61q_{11}^2=-1$,

$$p_{22}=p_{11}^2+61q_{11}^2=2p_{11}^2+1=1766319049,\qquad q_{22}=2p_{11}q_{11}=226153980.$$

Thus the least solution is (1766319049, 226153980).

7. The Cycle belonging to $z=(\sqrt{N}+b)/r$. Suppose that

$$z>1,\quad b<\sqrt{N}<b+r,\quad N-b^2=rr',$$

where r' is a positive integer, then the simple continued fraction corresponding to z has no acyclic part (Art. 3), and the cycle can be calculated as follows.

Rule. *Let* (x, y) *be a solution in positive integers of any one of the equations*

$$x^2-Ny^2=\pm1\quad or\quad \pm4, \qquad .....(\mathbf{A})$$

the latter pair being used only when r, r' *are both even and* N, b *are odd. Substitute* x/y *for* $\sqrt{N}$ *in the given surd. Express the result as a simple continued fraction with an even or an odd number of quotients according as the sign on the right of the chosen equation is* + *or* −. *This fraction consists of one or more of the cycles belonging to* $(\sqrt{N}+b)/r$.

Proof. Since $N=b^2+rr'$, the surd $(\sqrt{N}+b)/r$ is the positive root of

$$rz^2-2bz-r'=0. \qquad .....(\mathbf{B})$$

(1) We shall prove that this equation can be written in the form

$$z=(pz+p')/(qz+q'),\ \text{where } p,\ q,\ p',\ q' \text{ are positive integers} \qquad .....(\mathbf{C})$$

such that $pq'-p'q=\pm1$ and $q'\leqslant q$. .....(**D**)

For equation (**C**) is the same as $qz^2-(p-q')z-p'=0$, which is identical with (**B**) if

$$q=ry,\quad p-q'=2by,\quad p'=r'y, \qquad .....(\mathbf{E})$$

where y may be any rational.

* This problem was set by Fermat as a challenge to the English mathematicians of his time.

From (D), (E) we have $\frac{1}{4}(p+q')^2 = Ny^2 \pm 1 = x^2$, ...(F)

where x is rational. All possible cases are included in the following :

(A) x, y are positive integers such that $x^2 - Ny^2 = \pm 1$.

(B) $x = \xi/2$, $y = \eta/2$ where ξ, η are odd integers such that $\xi^2 - N\eta^2 = \pm 4$.

In case (B), r, r' are both even, for $q = ry$, $p' = r'y$. Hence $N - b^2$ is divisible by 4, and if N is odd so also is b.

In both cases, from (E) and (F), we find that

$$p = x + by, \quad q' = x - by, \quad \text{.....(G)}$$

and

$$p/q = (x/y + b)/r. \quad \text{.....(H)}$$

It remains to be shown that $q' \leqslant q$. This will be so if $x/y \leqslant b + r$.

Three cases must be considered :

(i) *If* $x^2 - Ny^2 = -1$ *or* $\xi^2 - N\eta^2 = -4$, then

$$x/y < \sqrt{N} < b + r \quad \text{and} \quad q' < q.$$

(ii) *If* $x^2 - Ny^2 = 1$, the least value of y is 1, and

$$x^2/y^2 = N + 1/y^2 \leqslant N + 1.$$

Now $N < (b+r)^2$, therefore $N \leqslant (b+r)^2 - 1$. Hence

$$x/y \leqslant \sqrt{N+1} \leqslant b + r \quad \text{and} \quad q' \leqslant q.$$

(iii) *If* $x = \xi/2$, $y = \eta/2$ *where* $\xi^2 - N\eta^2 = 4$, the least value of η is 1 and

$$x/y \leqslant \sqrt{N+4}.$$

Let $N = (b+r)^2 - k$ so that k is a positive integer, then $rr' = r(2b+r) - k$ where r, r' are even. Therefore k is divisible by 4, and 4 is its least value. Hence

$$x/y \leqslant \sqrt{N+4} \leqslant b + r \quad \text{and} \quad q' \leqslant q.$$

(2) It follows that if

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n}}, \quad \text{.....(I)}$$

where n is even or odd according as $pq' - p'q = 1$ or -1, then p'/q' is the convergent immediately preceding p/q. (Ch. XXIV, 12.) Now let z' be determined by

$$z = a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n + \cfrac{1}{z'}}},$$

then

$$z = \frac{pz' + p'}{qz' + q'} \quad \text{and} \quad \frac{pz' + p'}{qz' + q'} = \frac{pz + p'}{qz + q'}.$$

Hence $z' = z$ and *the fraction* (I) *consists of one or more of the cycles belonging to* z.

The rule can be used (*as in Ex.* 2) *to express any quadratic surd as a continued fraction, beginning at the stage where recurrence commences.*

Ex. 1. *Express* $(\sqrt{621}+21)/18$ *as a simple continued fraction.*

Since $21<\sqrt{621}<21+18$, there is no acyclic part. Also $18r'=621-21^2=180$, giving $r'=10$.

Thus r, r' are even and b is odd, also

$$25^2-621\,.\,1^2=4.$$

Using the rule, we substitute $25/1$ for $\sqrt{621}$ in the given surd, and express the result as a simple continued fraction with an *even* number of quotients, thus

$$\frac{25+21}{18}=\frac{23}{9}=2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4},$$

and

$$\frac{\sqrt{621}+21}{18}=2+\underset{*}{\frac{1}{1+}}\,\frac{1}{1+}\,\underset{*}{\frac{1}{4}}.$$

Ex. 2. *Express* $(\sqrt{13}+7)/4$ *as a simple continued fraction.*

Since $7>\sqrt{13}$, there is an acyclic part. The reckoning on the right shows that the second complete quotient is $(\sqrt{13}+1)/3$, and since $1<\sqrt{13}<1+3$, we can apply the rule. We have

b	7,	1
r	4,	3
a	2,	1

$$3^2-13\,.\,1^2=-4 \quad \text{and} \quad 18^2-13\,.\,5^2=-1.$$

Since $r=3$, an odd number, the first equation is useless. Taking the second, we substitute $18/5$ for $\sqrt{13}$ in $(\sqrt{13}+1)/3$, and express the result as a simple continued fraction with an *odd* number of quotients, thus

$$\left(\frac{18}{5}+1\right)\frac{1}{3}=\frac{23}{15}=1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1}.$$

This is the cycle belonging to the given surd, and

$$\frac{\sqrt{13}+7}{4}=2+\underset{*}{\frac{1}{1+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\underset{*}{\frac{1}{1}}.$$

EXERCISE LVI

1. Verify some of the following results, where the values of b, r, a are given as far as the middle of the first cycle:

(i) $\sqrt{13}$
$b\,.\,0\,.\,3\,.\,1\,.\,2$
$r\,.\,1\,.\,4\,.\,3\,.\,3$
$a\,.\,3\,.\,1\,.\,1\,.\,1$

(ii) $\sqrt{19}$
$b\,.\,0\,.\,4\,.\,2\,.\,3\,.\,3$
$r\,.\,1\,.\,3\,.\,5\,.\,2\,.\,5$
$a\,.\,4\,.\,2\,.\,1\,.\,3\,.\,1$

(iii) $\sqrt{31}$
$b\,.\,0\,.\,5\,.\,1\,.\,4\,.\,5\,.\,5$
$r\,.\,1\,.\,6\,.\,5\,.\,3\,.\,2\,.\,3$
$a\,.\,5\,.\,1\,.\,1\,.\,3\,.\,5\,.\,3$

(iv) $\sqrt{43}$
$b\,.\,0\,.\,6\,.\,1\,.\,5\,.\,4\,.\,5\,.\,5$
$r\,.\,1\,.\,7\,.\,6\,.\,3\,.\,9\,.\,2\,.\,9$
$a\,.\,6\,.\,1\,.\,1\,.\,3\,.\,1\,.\,5\,.\,1$

(v) $\sqrt{91}$
$b\,.\,0\,.\;\;9\,.\,1\,.\,8\,.\;\;7\,.\,7$
$r\,.\,1\,.\,10\,.\,9\,.\,3\,.\,14\,.\,3$
$a\,.\,9\,.\;\;1\,.\,1\,.\,5\,.\;\;1\,.\,5$

(vi) $\sqrt{109}$
$b\,.\;\;0\,.\,10\,.\,8\,.\;\;7\,.\,5\,.\,9\,.\;\;7\,.\,8\,.\,10$
$r\,.\;\;1\,.\;\;9\,.\,5\,.\,12\,.\,7\,.\,4\,.\,15\,.\,3\,.\;\;3$
$a\,.\,10\,.\;\;2\,.\,3\,.\;\;1\,.\,2\,.\,4\,.\;\;1\,.\,6\,.\;\;6$

(vii) $\sqrt{a^2+1}\begin{cases} b\,.\,0\,.\,a\,.\,a \\ r\,.\,1\,.\,1\,.\,1 \\ a\,.\,a\,.\,2a\,.\,2a \end{cases}$ (viii) $\sqrt{a^2-1}\begin{cases} b\,.\,0\,.\,a-1\,.\,a-1 \\ r\,.\,1\,.\,2(a-1)\,.\,1 \\ a\,.\,a-1\,.\,1\,.\,2(a-1) \end{cases}$

(ix) $\sqrt{a^2+2}\begin{cases} b\,.\,0\,.\,a\,.\,a \\ r\,.\,1\,.\,2\,.\,1 \\ a\,.\,a\,.\,a\,.\,2a \end{cases}$ (x) $\sqrt{a^2-2}\begin{cases} b\,.\,0\,.\,a-1\,.\,a-2\,.\,a-2 \\ r\,.\,1\,.\,2a-3\,.\,2\,.\,2a-3 \\ a\,.\,a-1\,.\,1\,.\,a-2\,.\,1 \end{cases}$ $(a>2)$

(xi) $\sqrt{a(a+1)}\begin{cases} b\,.\,0\,.\,a\,.\,a \\ r\,.\,1\,.\,a\,.\,1 \\ a\,.\,a\,.\,2\,.\,2a \end{cases}$ (xii) $\sqrt{a^2+4}\begin{cases} b\,.\,0\,.\,a\,.\,a-2\,.\,2 \\ r\,.\,1\,.\,4\,.\,a\,.\,a \\ a\,.\,a\,.\,\frac{1}{2}(a-1)\,.\,1\,.\,1 \end{cases}$ $(a>1$ and odd$)$

(xiii) $\sqrt{a(a+4)}\begin{cases} b\,.\,0\,.\,a+1\,.\,a-2\,.\,a\,.\,a \\ r\,.\,1\,.\,2a-1\,.\,4\,.\,a\,.\,4 \\ a\,.\,a+1\,.\,1\,.\,\frac{1}{2}(a-1)\,.\,2\,.\,\frac{1}{2}(a-1) \end{cases}$ $(a>1$ and odd$)$

2. In certain cases, a solution of one of the equations $x^2-Ny^2=\pm 1$ or ± 4 can be written down at once; thus

if $N=a^2\pm 1$, then $a^2-N\,1^2=\mp 1$;
if $N=a^2+2$, then $(a^2\pm 1)^2-Na^2=1$;
if $N=a(a+1)$, then $(2a+1)^2-N\,.\,2^2=1$;
if $N=a^2\pm 4$, then $a^2-N\,.\,1^2=\mp 4$.

Apply this to write down the integral solutions of $x^2-Ny^2=1$ when

$$N=51,\ 47,\ 56.$$

3. Find a solution of $x^2-109y^2=-4$, and apply the H.C.F. process to show that

$$\frac{\sqrt{109}+7}{10}=\underset{*}{1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9+}\,\frac{1}{1+}\,\frac{1}{2}}.$$

4. Obtain the following:

(i) $\dfrac{\sqrt{17}+15}{8}=\underset{*}{2+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3}}$; (ii) $\dfrac{\sqrt{61}+7}{6}=\underset{*}{2+\frac{1}{2+}\,\frac{1}{7}}$;

(iii) $\sqrt{\dfrac{13}{5}}=\underset{*}{1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}}$; (iv) $\dfrac{\sqrt{19}+1}{6}=\underset{*}{1+\frac{1}{8+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2}}$;

(v) $\sqrt{\dfrac{17}{3}}=\underset{*}{2+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4}}$.

5. Find two positive integral solutions of

(i) $5x^2-13y^2=7$, (ii) $5x^2-13y^2=-7$, (iii) $3x^2-17y^2=7$.

6. Find positive integral solutions of $x^2-621y^2=4$ and of $x^2-621y^2=1$. Referring to Art. 7, verify that if 623/25 and 7775/312 are substituted for $\sqrt{621}$ in the fraction $(\sqrt{621}+21)/18$, the first gives two cycles and the second three cycles of the fraction.

7. If $N=(2m+3)^2-4$, prove that

$$\frac{\sqrt{N}+2m-1}{4}=m+\underset{*}{\frac{1}{2+}}\ \underset{}{\frac{1}{m+}}\ \underset{}{\frac{1}{1+}}\ \underset{}{\frac{1}{4(m+1)+}}\ \underset{*}{\frac{1}{1}}.$$

8. If a is odd, find a positive integral solution of

(i) $x^2-(a^2+4)y^2=a$; (ii) $x^2-(a^2+4)y^2=-a$.

9. Prove that $r_n\leqslant 2a_1$, and that if $r_n=2a_1$, then $a_n=1$ and a_n is the mid-term of the reciprocal part of the a cycle.

Show also that if N is an odd prime and $r_n=2a_1$, then N must be equal to 3.

[If $r_n=2a_1$, $a_n<(\sqrt{N}+b_n)/r_n\leqslant 1$, therefore $b_n+b_{n+1}=a_nr_n=2a_1$, so that $b_n=b_{n+1}$. Hence a_n is the mid-term of the reciprocal part of the a cycle. If N is an odd prime, we must have $a_n=a_1$ or a_1-1.

Hence $a_1=1$ or 2, and $N=3, 5, 7$. On trial it is found that $N=3$.]

10. If $r_n=r_{n+1}=a_1$, then $a_n=a_{n+1}=1$, so that a_n, a_{n+1} are the mid-terms of the reciprocal part of the a cycle.

[$a_n=I(a_1+b_n)/a_1=1$ or 2, according as $b_n\neq a_1$ or $b_n=a_1$. If $b_n=a_1$, $b_{n+1}=2a_1-b_n=a_1$, so that $b_n=b_{n+1}$. This is impossible. Hence $a_n=1$ and $a_{n+1}=I(a_1+b_{n+1})/a_1=1$.]

11. If a_1 is an odd prime, or a power of such a number, and $r_n=a_1$, then $a_n=1$.

[For $r_nr_{n+1}=N-b_{n+1}^2$, $r_{n-1}r_n=N-b_n^2$, therefore b_n and b_{n+1} are solutions of $x^2\equiv N(\mathrm{mod}\ a_1)$, and since each $\leqslant a_1$, and a_1 is an odd prime, $b_n+b_{n+1}=a_1$, therefore $a_n=(b_n+b_{n+1})/r_n=a_1/a_1=1$.]

12. If, after a certain stage, the simple continued fractions which are equivalent to two irrationals z and z' are identical, prove that

$$z'=(pz+p')/(qz+q') \quad \text{where} \quad pq'-p'q=\pm 1$$

where p, p', q, q' are integers. Show also that the converse is true.

13. If the number (c) of elements in the cycle belonging to $\sqrt{N}$ is even, (i) prove that $p_{c/2}/q_{c/2}=(p_c\pm 1)/q_c$, according as c is or is not divisible by 4.

(ii) By taking $N=21$, illustrate the fact that equations (J) of Art. 6, (5), (iv), are not necessarily true unless N is a prime, and verify the equations for $N=19$.

14. If the number (c) of elements in the cycle belonging to $\sqrt{N}$ is odd:

(i) Prove that N is the sum of two squares which are prime to one another. Also, by taking $N=205$, show that the converse of this is not generally true.

(ii) If $m=\frac{1}{2}(c-1)$ and $n=\frac{1}{2}(c+1)$, prove that

$$p_m^2+p_n^2=N(q_m^2+q_n^2)=Nq_c,$$
$$p_m^2+Nq_m^2=(2p_mq_mp_c+1)/q_c,$$
$$p_m^2-Nq_m^2=(p_mq_n+p_nq_m)/q_c,$$

and verify when $N=29$.

[(i) N is a factor of p_c^2+1.]

MISCELLANEOUS EXERCISES (A)

1. Show that the product of any two numbers, as 19 and 35, can be found as on the right.* In the left-hand column, each number is the quotient when the preceding number is divided by 2. Any remainders are disregarded. In the right-hand column, each number is twice the preceding.

19	35
9	70
4	~~140~~
2	~~280~~
1	560

$19 \times 35 = 665$

The numbers 140 and 280 which are opposite *even* numbers on the left are crossed out, and the sum of the other numbers on the right is 19×35.

[The reckoning shows that

$$19 \times 35 = (1 + 2 + 2^2 \cdot 0 + 2^3 \cdot 0 + 2^4)35 = 35 + 70 + 560.]$$

2. Show that n^3 is the sum of n consecutive odd numbers, and find the middle or the two middle numbers.

3. Take any sum of money less than £12, say £9 8s. 7d. Reverse the order of pence and pounds, obtaining £7 8s. 9d. The difference between these sums is £1 19s. 10d. Reversing the order of pence and pounds, we obtain £10 19s. 1d. Adding the last two sums, we have £12 18s. 11d. Prove that the result will be the same whatever sum less than £12 is chosen.

4. If a/b is a proper fraction and $q_1, q_2, \ldots q_n$ are the quotients and $r_1, r_2, \ldots r_n$ the remainders when b is divided by $a, r_1, r_2, \ldots r_{n-1}$, respectively, show that

(i) $$\frac{a}{b} = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \ldots + (-1)^{n-1} \frac{1}{q_1 q_2 \cdots q_n} + R_n,$$

where $$R_n = (-1)^n \frac{1}{q_1 q_2 \cdots q_n} \cdot \frac{r_n}{b}.$$

(ii) For some value of n, $R_n = 0$, so that a/b can always be expressed in the form

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{r} - \frac{1}{s} + \ldots,$$

where $p, q, \ldots$ are positive integers, their number being finite.

(iii) Express in this way $\frac{3}{5}\frac{2}{1}$ and $\frac{2}{5}\frac{5}{3}$.

5. Prove that $a+b+c+d$ and $a+b-c-d$ are factors of

$$2(a^4+b^4+c^4+d^4) - (a^2+b^2+c^2+d^2)^2 + 8abcd,$$

and find the remaining factors.

6. Solve $$\left(\frac{x-1}{x+2}\right)^3 = \frac{x-4}{x+5}.$$

7. If x, y, z are real, prove that

$$(x^2+y^2+z^2)/(yz+zx+xy)$$

cannot lie between 1 and -2.

8. (i) Find the simplest equation with integral coefficients which has

$$-\sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{3}{2}} \quad \text{and} \quad -\sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{3}{2}}$$

among its roots. What are the other roots of the equation?

(ii) Prove that $4(x^2+x+1)^3 - 27x^2(x+1)^2 = (x-1)^2(2x+1)^2(x+2)^2$.

*Known as the Russian peasants' method of multiplication.

9. If $x+y+z=x^2+y^2+z^2=2$, show that

$$x(1-x)^2=y(1-y)^2=z(1-z)^2.$$

10. If $a+b+c+d=0$ and $x+y+z+t=0$, prove that the results of rationalising the equations

$$\sqrt{ax}+\sqrt{by}+\sqrt{cz}+\sqrt{dt}=0, \qquad \sqrt{bx}+\sqrt{ay}+\sqrt{dz}+\sqrt{ct}=0,$$

are equivalent. State the two other equations which lead to equivalent results.

11. Find the equation whose roots are given by

$$y_i=x_i-x_i^2+(x_1^2+x_2^2+x_3^2), \quad (i=1, 2, 3),$$

where x_1, x_2, x_3 are the roots of $x^3-x^2+4=0$.

12. Solve the equation

$$6x^4-3x^3+8x^2-x+2=0,$$

being given that it has a pair of roots whose sum is zero.

13. If $\dfrac{a}{x-md}+\dfrac{b}{x-mc}+\dfrac{c}{x+mb}+\dfrac{d}{x+ma}=0$ and $a+b+c+d=0$, prove that the only finite value of x is $m(ac+bd)/(a+b)$.

14. If $(ca'-ac')^2<4(ab'-a'b)(bc'-b'c)$, prove that $b^2>ac$ and $b'^2>a'c'$.

15. If $a<b<c<d$, and $a-c\leqslant b-d$, the roots of

$$(x-a)(x-c)=k(x-b)(x-d)$$

are real for all values of k.

16. If $l+m+n=0$, show that the expression

$$a^2l^2+b^2m^2+c^2n^2-2bcmn-2canl-2ablm$$

is positive if $abc(a+b+c)$ is positive.

17. Show that the expression $(cx-az)^2-(ay-bx)(bz-cy)$ is the product of two linear functions of x, y, z, and that, when a, b, c are real, the coefficients in the linear functions are real if $b^2\geqslant 4ac$.

18. If the cubic derived from the equation

$$\frac{a}{x+a}+\frac{b}{x+b}=\frac{c}{x+c}+\frac{d}{x+d}$$

has a pair of equal roots, then either one of the numbers a, b is equal to one of the numbers c, d or

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}.$$

19. Express in factors

$$(bc-a^2)(ca-b^2)+(ca-b^2)(ab-c^2)+(ab-c^2)(bc-a^2).$$

20. Show that if a, b, c are real, the roots of

$$1/(x+a)+1/(x+b)+1/(x+c)=3/x$$

are real.

21. Eliminate x, y, z from the equations:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a, \qquad \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b,$$

$$\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=c.$$

22. Prove that $10^n-(5+\sqrt{17})^n-(5-\sqrt{17})^n$ is divisible by 2^{n+1}.

23. (*a*) Solve $(x-1)^3+(x-2)^3+(x-3)^3+\ldots+(x-n)^3=0$.

(*b*) If x, y, z are unequal and

$$(x-1)(y-2)=(y-1)(z-2)=(z-1)(x-2),$$

show that each product is equal to -1 and that

$$yz+zx+xy-3(x+y+z)+9=0$$

and

$$xyz-2(x+y+z)+9=0.$$

24. By the method of partial fractions, prove that, if $a, b, c, \ldots k$ are n numbers, then the sum of n fractions of the type

$$\frac{a^m}{(a-b)(a-c)(a-d)\ldots(a-k)}$$

is zero if m is an integer less than $n-1$ and is unity if $m=n-1$.

25. If n is a positive integer,

$$\frac{2}{\lfloor n+1 \rfloor \lfloor n-1}\cdot\frac{1}{a+1}-\frac{2}{\lfloor n+2 \rfloor \lfloor n-2}\cdot\frac{4}{a+4}+\frac{2}{\lfloor n+3 \rfloor \lfloor n-3}\cdot\frac{9}{a+9}-\ldots \text{ to } n \text{ terms}$$

$$=\frac{1}{(a+1)(a+4)(a+9)\ldots(a+n^2)}.$$

26. Under what conditions are the following equations consistent?

$$x+y+z=1, \quad 2x-y+z=c_1, \quad (4a+b)y+az=c_2, \quad -bx+(a+3b)y=c_3?$$

27. Show that, if the roots of $x^3-ax^2+bx-c=0$ are not in A.P., then there are in general three transformations of the form $x=y+\lambda$, such that the transformed cubic in y has its roots in G.P.

28. (*a*) Prove algebraically that, if a, b, α, β are real numbers, then

$$\sqrt{(a+\alpha)^2+(b+\beta)^2}\leqslant\sqrt{a^2+b^2}+\sqrt{\alpha^2+\beta^2},$$

the positive root being indicated by the radical sign.

(*b*) If a and b are positive and $a+b=1$, show that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geqslant 12\tfrac{1}{2}.$$

29. Examine, for different positive or negative values of x and p, the convergence or divergence of

$$\text{(i)}\ \Sigma\frac{x^n}{n^p}; \qquad \text{(ii)}\ \Sigma\frac{\sqrt{(n+1)}-\sqrt{n}}{n^p}.$$

30. Show how to find the nth term and the sum of n terms of a recurring series of which the scale of relation is of the form $u_n+pu_{n-1}+qu_{n-2}=0$ and of which the first four terms are known.

Apply the method to the series

(i) 1, 2, 5, 14 ..., $\quad (p=-4, \quad q=3)$

(ii) 1, 2, 5, 12 .... $\quad (p=-2, \quad q=-1)$

31. (*a*) Show that $2\Sigma b^2c^2(b^2-c^2)(a^4-b^2c^2)$

$$=\Sigma(a^2b^2+a^2c^2-a^4)[(b^4-c^2a^2)(a^2-b^2)+(c^4-a^2b^2)(c^2-a^2)].$$

(*b*) Eliminate x, y, z from

$$x^2y+x^2z=a^3, \quad y^2z+y^2x=b^3, \quad z^2x+z^2y=c^3, \quad xyz=d^3.$$

32. (*a*) Show that any root of the number n can be expressed in the form

$$\sqrt[3]{a\sqrt[3]{b\sqrt[3]{c\ldots}}},$$

where each of the quantities $a, b, c, \ldots$ is n, 1 or n^{-1}.

(*b*) If x is less than $\sqrt{N}$, but greater than $\frac{1}{2}\sqrt{N}$, show that $\dfrac{3N+x^2}{N+3x^2}\cdot x$ is nearer to $N^{\frac{1}{2}}$ than is either $\dfrac{N+x^2}{2x}$ or x.

33. Show that

$$\begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix} = \pm\frac{a^2\Pi(a_i-a_j)}{\Pi a_i \Pi(a-a_i)^2}.$$

Write out the terms of the product in the numerator, and give the resulting expression its correct sign.

34. Discuss the reality of the roots of

$$x^4+4x^3-2x^2-12x+a=0,$$

for all real values of a.

[Denoting the equation by $f(x)=0$, consider the roots of $f'(x)=0$.]

35. Solve the equation $46x^3+72x^2+18x-11=0$, by expressing the left-hand side as the sum of two cubes.

36. If $(1+x)^n=1+c_1x+c_2x^2+\ldots+c_nx^n$, show that

$$c_1(1-x)-\tfrac{1}{2}c_2(1-x)^2+\tfrac{1}{3}c_3(1-x)^3-\ldots+(-1)^{n-1}\frac{1}{n}c_n(1-x)^n$$

$$=1-x+\tfrac{1}{2}(1-x^2)+\tfrac{1}{3}(1-x^3)+\ldots+\frac{1}{n}(1-x^n).$$

[Denoting the expressions by u and v, show that $\dfrac{du}{dx}=\dfrac{dv}{dx}$, etc.]

37. Prove that there is only one set of real values of x, y, z which satisfy the equation

$$(1-x)^2+(x-y)^2+(y-z)^2+z^2=\tfrac{1}{4},$$

and find them.

[Writing a, b, c, d, for $(1-x)$, $(x-y)$, $(y-z)$, z, we have $a+b+c+d=1$ and $a^2+b^2+c^2+d^2=\frac{1}{4}$; hence $\Sigma(a-b)^2=0$.]

38. If ω is a fifth root of 2 and $x=\omega+\omega^2$, prove that

$$x^5=10x^2+10x+6.$$

39. Prove that, if $r^2=p^2s$, the product of a pair of roots of the equation

$$x^4+px^3+qx^2+rx+s=0$$

is equal to the product of the other pair.

Solve $$x^4-x^3-16x^2-2x+4=0.$$

40. Obtain the roots of

$$x^4-6x^3+13x^2-6x+1=0$$

in the form $(\sqrt{3}\pm\sqrt{2})\,e^{\pm\frac{\iota\pi}{6}}$.

41. Prove that $x^{4n+1}+\iota=(x+\iota)\cdot\Pi_{r=1}^{2n}(x^2+2\iota x\cos r\alpha-1)$, where n is a positive integer and $(4n+1)\alpha=2\pi$.

42. If three real numbers satisfy the equations

$$x+y+z=5 \quad \text{and} \quad yz+zx+xy=8,$$

prove that none of them is less than 1 nor greater than $2\frac{1}{3}$.

43. Find a general formula for all the positive integers which, when divided by 5, 6, 7, will leave remainders 1, 2, 3 respectively, and show that 206 is the least of them.

44. Show that a determinant of order n can be expressed as one of order $n-1$ as follows.

If $\Delta = \begin{vmatrix} a_1b_1c_1 & \dots & l_1 \\ a_2b_2c_2 & \dots & l_2 \\ \dots\dots & & \\ a_nb_nc_n & \dots & l_n \end{vmatrix}$, then $\Delta a_1^{n-2} = \begin{vmatrix} (a_1b_2)(a_1b_3)(a_1b_4) & \dots & (a_1b_n) \\ (a_1c_2)(a_1c_3)(a_1c_4) & \dots & (a_1c_n) \\ \dots\dots & & \\ (a_1l_2)(a_1l_3)(a_1l_4) & \dots & (a_1l_n) \end{vmatrix}$.

[Multiply Δ by $\begin{vmatrix} a_1 & 0 & 0 & \dots & 0 \\ -b_1 & a_1 & 0 & \dots & 0 \\ -c_1 & 0 & a_1 & \dots & 0 \\ \dots\dots & & & & \\ -l_1 & 0 & 0 & \dots & a_1 \end{vmatrix}$.]

45. Show that, if n is a positive integer, the number of solutions of the equation

$$n=2n_1+3n_2,$$

where n_1 and n_2 are positive integers or zero, is equal to N or $N+1$ according as r is or is not equal to 1, where N denotes the quotient and r the remainder when n is divided by 6.

46. Prove that $\dfrac{a^3b^3}{(a+b)^4} \leqslant \dfrac{3^3a^2}{4^4}$ or $\dfrac{3^3b^2}{4^4}$, a, b being positive.

47. If n is a positive integer,

$$1+3n+\frac{3\cdot4}{1\cdot2}\cdot\frac{n(n-1)}{\lfloor 2}+\frac{4\cdot5}{1\cdot2}\cdot\frac{n(n-1)(n-2)}{\lfloor 3} + \dots + \frac{(n+1)(n+2)}{1\cdot2} = (n^2+7n+8)2^{n-3}.$$

48. If a, b, c, d are real and

$$\frac{b^2-ac}{ad-bc} = \frac{ad-bc}{c^2-bd},$$

show that each ratio $= \left(\frac{a}{d}\right)^{\frac{1}{3}}$ and that $a^{\frac{1}{3}}c+bd^{\frac{1}{3}}+a^{\frac{2}{3}}d^{\frac{2}{3}}=0$, the real values of the roots being taken in all cases.

49. Prove that the condition that

$$(ax+by+cz)(bx+cy+az)(cx+ay+bz)$$

can be expressed in the form P^3+Q^3 where P and Q are linear functions of x, y, z is

$$a^3+b^3+c^3=3abc;$$

and find values of P and Q.

50. If $x_1, x_2, \dots x_n$ are positive numbers such that $\Sigma x=n$, prove that $\Sigma x^m \geqslant n$ if $m>1$.

51. If r is an odd integer and $q_r = \dfrac{1 \cdot 3 \cdot 5 \ldots (2r-1)}{2 \cdot 4 \cdot 6 \ldots 2r}$, prove that

$$q_r + q_1 q_{r-1} + q_2 q_{r-2} + \cdots$$

to $\frac{1}{2}(r+1)$ terms is equal to $\frac{1}{2}$.

52. If $\left(\dfrac{1+x}{1-x}\right)^n = 1 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots$, show that

$$(r+1)a_{r+1} - 2na_r + (r-1)a_{r-1} = 0.$$

Also show that

$$a_r = 2^n C_r^{n+r-1} - 2^{n-1} C_1^n C_r^{n+r-2} + 2^{n-2} C_2^n C_r^{n+r-3} - \ldots + (-1)^{n-1} \cdot 2 C_1^n.$$

53. Sum the series $\Sigma_{n=0}^{\infty} \dfrac{3n+2}{n^2+3n+2} \cdot x^n$.

54. Examine the convergence of the series whose nth term is

$$x^n/(x^{2n} + x^n + 1)$$

for any value of x.

55. If m is a positive integer, show that

$$1 - m^2 + \frac{m^2(m^2-1)}{1^2 \cdot 2^2} + \ldots + (-1)^m \frac{m^2(m^2-1^2) \ldots \{(m^2-(m-1)^2\}}{(\lfloor m)^2} = 0.$$

56. Prove that, as the real variable x changes steadily from $-\infty$ to $+\infty$, the function

$$y = \frac{(x-a)^2}{x-b},$$

where a and b are real and $a > b$, assumes twice over all values except those in an interval of length $4(b-a)$; and locate this range of values precisely.

57. Prove that, if

$$\begin{vmatrix} a & a^3 & a^4 - 1 \\ b & b^3 & b^4 - 1 \\ c & c^3 & c^4 - 1 \end{vmatrix} = 0,$$

and a, b, c are all different, then $abc(ab+bc+ca) = a+b+c$.

58. Prove that, in the expansion of $(1+x)^n(1-x)^{-2}$ in ascending powers of x, where n is a positive integer, the sum of the first $n+r$ coefficients is

$$2^{n-3}\{n^2 + n(4r+3) + 4r(r+1)\}.$$

59. The graph of $y = \dfrac{ax+b}{(x-1)(x-4)}$ has a turning point at $P(2, -1)$. Find the values of a and b, and show that y is a maximum at P. Sketch the curve.

60. Prove that, if $x \geqslant 100$, then $x^{\frac{1}{3}} - \frac{1}{2}\{(x+1)^{\frac{1}{3}} + (x-1)^{\frac{1}{3}}\} = \frac{1}{9}x^{-\frac{5}{3}}$ (nearly), the approximation being correct to at least eight decimal places.

61. If $f'(x)$ is positive, show that $f(x)$ is increasing.

Prove that $2x + x\cos x - 3\sin x > 0$ if $0 < x < \dfrac{\pi}{2}$.

62. Prove that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \ldots \text{ to infinity} = \log 2 - \tfrac{1}{2},$$

and that

$$\frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} + \ldots \text{ to infinity} = 1.$$

63. Solve the equations:

$$\left.\begin{aligned} y^2+z^2-x(y+z)&=a\\ z^2+x^2-y(z+x)&=b\\ x^2+y^2-z(x+y)&=c \end{aligned}\right\}.$$

[Show that $b+c-a=2(x^2-yz)$, etc.]

64. By expanding $(1-2x-3x^2)^{-1}$ in two different ways, prove that

$$2^n+(n-1)\,.\,3\,.\,2^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}\,3^2\,.\,2^{n-4}$$
$$+\frac{(n-3)(n-4)(n-5)}{\lfloor 3}\,3^3\,.\,2^{n-6}+\ldots=\tfrac{1}{4}\{3^{n+1}+(-1)^n\}.$$

65. If $ax+by+cz=1$ and a, b, c are positive, show that the values of x, y, z, for which $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is stationary, are given by

$$ax^2=by^2=cz^2.$$

Show that this is a true maximum or minimum, if $xyz>0$.

66. Show that the equation $\tan x=1+x$

has an infinite number of real roots, and find graphically the approximate value of the smallest positive root.

67. Assuming that $x\{\log(1+x)\}^{-1}$ can be expanded in ascending powers of x, find the first four terms of the expansion.

By help of this result prove that a capital sum accumulating at compound interest at r per cent. per annum will be increased ten-fold after

$$\left(\frac{230\cdot 26}{r}+1\cdot 15\right)\text{ years approximately.}$$

68. If $x=y-y^2+2y^3-y^4$, and it is assumed that y can be expanded in ascending powers of x, show that

$$y=x+x^2-4x^4+\ldots.$$

69. Draw graphs of

$$\text{(i) } y=\frac{2(x^2-6x)}{x^2+x-6}\,;\quad \text{(ii) } y^2=\frac{2(x^2-6x)}{x^2+x-6}\,.$$

70. By expanding

$$\left(2+x+\frac{1}{x}\right)^{2m+n}$$

in two ways, prove that

$$\lfloor 2m+n\ \sum_{r=0}^{r=m}\frac{2^{2m-2r}}{\lfloor r\ \lfloor 2m-2r\ \lfloor n+r}=\frac{\lfloor 4m+2n}{\lfloor 2m+2n\ \lfloor 2m}\,.$$

71. Show that $(a-\frac{1}{a}-x)(4-3x^2)$, where a is a positive constant, has one maximum value and one minimum value. Find these values and show that the difference between them is $\frac{4}{9}\left(a+\frac{1}{a}\right)^3$. What is the least value of this difference for various values of a?

72. Prove that $x^4-18x^2+4dx+9=0$

has four real roots, if $d^4\leqslant 1728$.

73. If $r^2x = aR + bR^3$, where $R \geqslant r$,
and the quantities denoted by all the letters are real and positive, prove that

$$x \geqslant 2\sqrt{ab}.$$

74. Prove that, if ϵ is small, one root of

$$x^3 + x^2 - (1 + 2\epsilon^2)x - 1 = 0$$

is approximately $1 + \frac{1}{2}\epsilon^2$, and find corresponding approximations to the other roots.

75. If α, β, γ are the roots of

$$x^3 - ax^2 + bx - c = 0,$$

the area of the triangle, of which the lengths of the sides are α, β, γ, is

$$\tfrac{1}{4}\{a(4ab - a^3 - 8c)\}^{\frac{1}{2}}.$$

If the triangle is right-angled, show that

$$a(4ab - a^3 - 8c)(a^2 - 2b) = 8c^2.$$

76. If n is a positive integer, prove that

(i) $\dfrac{2^n}{n} < \dfrac{n^n}{\lfloor n} < 3^{n-1}$ when $n > 3$;

(ii) $1^1 \cdot 2^2 \cdot 3^3 \ldots n^n < \left(\dfrac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$ when $n > 1$.

77. If y is positive, show that $\log y$ lies between $2\left(\dfrac{y-1}{y+1}\right)$ and $\dfrac{y^2-1}{y}$.

78. Prove that all the roots of the equation

$$x^4 - 14x^2 + 24x = k$$

are real if $8 < k < 11$.

79. Find a function $(ax^2 + 2bx + c)/(a'x^2 + 2b'x + c')$ which has turning values 3 and 4 when $x = 2$ and -2 respectively, and has the value 6 when $x = 0$.

80. If a, b are unequal positive quantities, show that

$$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}$$

81. Show that two pencils of straight lines in one plane, containing p and q lines respectively, no two lines being coincident or parallel, divide the plane into $pq + 2p + 2q - 1$ parts.

82. Show that the number 30 may be divided into three unequal parts in 61 ways, while if equal parts are allowed, there are 75 ways.

83. In how many ways can a batsman make 14 runs in 6 balls, not scoring more than 4 off any ball ?

84. Prove that the equation

$$x^4 + 4rx + 3s = 0$$

has no real roots if $r^4 < s^3$.

85. Find the minimum values of

$$\frac{a^m + x^m}{2} - \left(\frac{a+x}{2}\right)^m \quad \text{and} \quad \frac{a^m + b^m + x^m}{3} - \left(\frac{a+b+x}{3}\right)^m$$

for positive values of x, where a, b, m are positive and $m > 1$.

86. Find the condition that $ax+b/x$ can take any real value for real values of x.

Express $\xi=(x-a)(x-b)/(x-c)(x-d)$ in terms of y where $y=(x-d)/(x-c)$; and hence, or otherwise, show that ξ can take all real values if

$$(c-a)(c-b)(d-a)(d-b)$$

is negative.

87. Find the maximum and minimum values of

$$y=(x+1)^3(x+3)^2(x+2),$$

and draw a rough graph of the curve.

88. Prove that $7^{2n}-48n-1$ is divisible by 2304.

89. If

$$(x_1-x_2)^2+(y_1-y_2)^2=a^2,$$
$$(x_2-x_3)^2+(y_2-y_3)^2=b^2,$$
$$(x_3-x_1)^2+(y_3-y_1)^2=c^2;$$

then
$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

90. Show that an approximate solution of $x\log x+x-1=\epsilon$, where ϵ is small, is

$$x=1+\epsilon/2-\epsilon^2/16.$$

91. Find the limits of $\dfrac{x^3+y^3}{x-y}$, as x, y tend to zero along the curves

(i) $y=x-x^2$, (ii) $y=x-x^3$, (iii) $y=x-x^6$.

92. If a, b, c are real, prove that square root of $a^2+b^2+c^2-bc-ca-ab$ is greater than $\frac{1}{2}(2a-b-c)$ or than $\dfrac{\sqrt{3}}{2}(a-c)$, where $a>b>c$.

93. Prove that if

$$a_r=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\ldots+\frac{1}{\lfloor r},$$

then
$$1+\frac{1}{2}a_1+\frac{1}{2^2}a_2+\ldots \text{ to } \infty=2\sqrt{e}.$$

94. Show that a sum of n pence can be made up of pennies, halfpennies and farthings in $(n+1)^2$ different ways.

95. If $x>2$, prove that

$$1+\frac{n}{x}+\frac{n(n+1)}{\lfloor 2\,x^2}+\ldots=1+\frac{n}{x-1}+\frac{n(n-1)}{\lfloor 2(x-1)^2}+\ldots$$

and deduce that

$$n+\frac{n(n-1)}{\lfloor 2}\,\frac{(r-1)}{\lfloor 1}+\frac{n(n-1)(n-2)}{\lfloor 3}\,\frac{(r-1)(r-2)}{\lfloor 2}+\ldots=\frac{n(n+1)\ldots(n+r-1)}{\lfloor r}.$$

96. Find approximations to the roots of

$$(x-1)(x-2)(x-3)=\alpha,$$

where α is small.

97. If $3u_n-7u_{n-1}+5u_{n-2}-u_{n-3}=0$ and $u_0=1$, $u_1=8$, $u_2=17$, prove that

$$2u_n=20n-7+3^{2-n}.$$

98. Expand $\{2x+\sqrt{(9+3x^2)}\}/(3-x)$ in ascending powers of x, as far as the fifth power of x, and show that for small values of x it leads to a good approximation for e^x.

Deduce that $e^{\frac{1}{4}}=1{\cdot}2840\ldots$.

99. If a, b are positive, and p and q are positive rationals such that $1/p+1/q=1$, prove that
$$ab\leqslant a^p/p+b^q/q.$$

100. By means of the expansions of e^x and $\log(1+x)$, prove that, when n is large,
$$\left(1+\frac{1}{n}\right)^n=e\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\ldots\right).$$

Show that e is given approximately by
$$2e=(1{\cdot}1)^{10}+(0{\cdot}9)^{-10},$$
with an error of about 0·46 per cent.

101. If $r>p>q$, show that the number of combinations of $p+q$ things r at a time, p things being of one sort and q of another, is $p+q-r+1$.

102. Show that one root of the equation
$$x^3=100(x-1)$$
is approximately 1·0103, and find the other roots correct to two places of decimals.

103. Prove that, if n is a positive integer, the coefficient of x^n in $(1+x+x^2)^n$ is
$$1+\frac{n(n-1)}{(\lfloor 1)^2}+\frac{n(n-1)(n-2)(n-3)}{(\lfloor 2)^2}+\ldots.$$

104. The number of ways of selecting r pairs from n different things is
$$\frac{n(n-1)\ldots(n-2r+1)}{2^r\lfloor r}=\frac{\lfloor n}{2^r\lfloor r\,\lfloor n-2r}.$$

105. Having given that
$$x+y+z=1,\qquad x^2+y^2+z^2=2,\qquad x^3+y^3+z^3=3,$$
prove that
$$x^4+y^4+z^4=4\tfrac{1}{6}.$$

106. If a, b, c are real and
$$a^2+b^2+c^2+2abc=1,$$
then a^2, b^2, c^2 are all greater than 1 or all less than 1.

107. If $x+y+z=xyz$ and $x^2=yz$, then y and z may have any real values and $x^2\geqslant 3$.

108. Solve
$$x^{\frac{1}{4}}+(1552-x)^{\frac{1}{4}}=10.$$

109. Show that the result of eliminating x and y from the three equations
$$\frac{1}{x-a}+\frac{1}{y-a}=\frac{1}{a},\qquad \frac{1}{x-b}+\frac{1}{y-b}=\frac{1}{b},$$
and
$$x^2+y^2=2(a^2+b^2),$$
is
$$a^2+b^2-6ab=0.$$

110. Prove that $\log_e\{\log_e(1+x)^{\frac{1}{x}}\}=-\frac{1}{2}x+\frac{5}{24}x^2-\frac{1}{8}x^3-\ldots$.

Find, without using tables, the value of $\log_e(\log_e 1{\cdot}01)$ correct to five places of decimals, having given $\log_e 10=2{\cdot}302585$.

111. Show that the coefficient of x^{3n+1} in the expansion of $\dfrac{8-2x}{(x+2)(x^3+8)}$ in a series of ascending powers of x is

$$(-1)^{n+r}\frac{3n+3}{2^{3n+3}}.$$

112. Find to five places of decimals the real root of $x^3+x+1=0$.

113. If $x^{3-\mu}=a^3$ where μ is small, show that an approximate root of the equation is given by $x=a\{1+\frac{1}{3}\mu\log_e a+\frac{1}{9}\mu^2\log_e a(1+\frac{1}{2}\log_e a)\}$.

114. Prove that $(1+x)^{\frac{1}{1-x}}=1+x+x^2+\frac{3}{2}x^3+\frac{11}{6}x^4+\dots$.

115. The difference between the A.M. and the G.M. of a set of nearly equal numbers is to a first approximation equal to the quotient of half the difference between the square of their A.M. and the A.M. of their squares by any of the numbers.

116. Find the sum of the terms after the nth in the expansion of $(1+x)/(1-x)^2$ in ascending powers of x.

Prove that the ratio of this sum to the sum of the corresponding terms in the expansion of $(1-x)^{-1}$ can be made equal to any given number λ, which is greater than $2n$, by suitable choice of x. Explain clearly why the restriction upon λ is necessary.

117. An examination consists of three papers, to each of which m marks are assigned as a maximum. One candidate obtains a total of $2m$ marks on the three papers. Show that there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur.

118. Show that the $2n$-th convergent of $\dfrac{3}{4-}\dfrac{4}{3+}\dfrac{3}{4-}\dfrac{4}{3+}\dots$ differs from the value of the continued fraction by $13/\{9\,.\,12^n+4(-1)^n\}$.

[Prove that (i) $p_{2n}=11p_{2n-2}+12p_{2n-4}$, with a like result for the q's; (ii) $q_{2n}=p_{2n}+(-1)^n$ and $q_{2n-1}/4=(q_{2n}-p_{2n-1})/5=p_{2n}/9$.]

119. If the infinite product

$$(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\left(1+\frac{x}{2^3}\right)^2\dots$$

is expressed in the form $a_0+a_1x+a_2x^2+\dots+a_rx^r+\dots$,

then $$(2^{r+1}-1)a_{r+1}=4(a_r+a_{r-1}).$$

120. Find the coefficient of z^n in the expansion in ascending powers of z of $\dfrac{1+xz}{1-z-xz^2}$, and show that

$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^{n+2}-\left(\frac{1-\sqrt{4x+1}}{2}\right)^{n+2}\right\}$$
$$=1+nx+\frac{(n-1)(n-2)}{\lfloor 2}x^2+\frac{(n-2)(n-3)(n-4)}{\lfloor 3}x^3+\dots.$$

121. Prove that

$$mx^{m+n}-(m+n)x^m+n,$$

where m and n are positive integers, is divisible by $(x-1)^2$, and that the quotient is

$$mx^{m+n-2}+\dots+p_rx^r+\dots+n$$

where $p_r=m(m+n-r-1)\quad(r\geqslant m)$, $\qquad p_r=(r+1)n\quad(r<m)$.

122. Prove that the number of ways in which n prizes may be distributed among q people so that everybody may have one *at least* is

$$q^n - q(q-1)^n + \frac{q(q-1)}{\lfloor 2} (q-2)^n - \dots .$$

123. Prove that when x is sufficiently small

$$\frac{1}{\log(1+x)} = \frac{1}{x} + \frac{1}{2} - \frac{1}{12}x + \frac{1}{24}x^2 - \dots .$$

Also, if $n > 1$,

$$\frac{1}{\log\left(1 + x + \frac{x^2}{\lfloor 2} + \dots + \frac{x^n}{\lfloor n}\right)} = \frac{1}{x} + \frac{x^{n-1}}{\lfloor n+1} - \frac{x^n}{(n+2)\lfloor n} + \dots .$$

124. A man owes a sum of £a, and repays £b at the end of each year, partly to pay the interest at $100n$ per cent. and partly to repay capital. Show that the sum will be repaid in λ years, where λ is the integer equal to or next greater than

$$-\log\left(1 - \frac{an}{b}\right) \Big/ \log(1+n).$$

125. Show that the number of combinations taken n together of $3n$ letters of which n are a, n are b, and the rest unlike is

$$C_n^n + 2C_{n-1}^n + 3C_{n-2}^n + \dots + nC_1^n + n + 1,$$

and that this sum is equal to $2^{n-1}(n+2)$.

126. If a, b are positive integers, the probability that $\frac{1}{5}(a^2+b^2)$ is a positive integer is $\frac{9}{25}$.

127. If

$$x + (b+c)y + bcz = b - c,$$
$$x + (c+a)y + caz = c - a,$$
$$x + (a+b)y + abz = a - b,$$

where $a \neq b \neq c$, prove that $zx - y^2 = 3$.

128. Ladders 15 ft. and 10 ft. long are placed in a passage, each with its top against a wall and its foot against the opposite wall. If the point where the ladders cross is 5 ft. from the ground, find the width of the passage. (*Sunday Times.*)

[If the width of the passage is x ft. and the tops of the ladders are a ft. and b ft. from the ground, then

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{5}, \text{ and } a^2 - b^2 = 125.$$

Hence if $b = 5(y+1)$, show that $0 < y < 1$, and

$$y^4 + 2y^3 + 5y^2 - 2y - 1 = 0,$$

giving $y = 0{\cdot}5761287$, $b = 7{\cdot}880644$, $x = \sqrt{(100 - b^2)} = 6{\cdot}155929$.]

MISCELLANEOUS EXERCISES (B)

1. A party consisting of 7 robbers and a monkey had stolen a number N of nuts, which were to be divided equally among the robbers. In the night a robber woke and decided to take his share. He found that, if he gave one to the monkey, the remaining nuts could be divided into 7 equal lots. He gave one to the monkey and took his share. The other robbers woke in succession, found that the same thing was possible, and each gave one to the monkey and took one-seventh of the remaining nuts. In the morning they divided what were left, and they found that, if they gave one to the monkey, the nuts could be divided into 7 equal lots. Prove that the least value of N is $7^8 - 6 = 5764795$.

[The value -6 of N satisfies the numerical conditions; for $-6 - 1 = -7$, $-7 \div 7 = -1$, $-7 - (-1) = -6$. Thus -6 is unaltered by the set of operations corresponding to what each robber did. Hence the values of N are given by $-6 + k \cdot 7^8$.]

2. Let a/b be a proper fraction and $c_1, c_2, c_3, \ldots$ an infinite sequence of positive integers. If $q_1, q_2, q_3, \ldots q_n$ are the quotients and $r_1, r_2, r_3, \ldots r_n$ the remainders when $ac_1, r_1c_2, r_2c_3, \ldots r_{n-1}c_n$ respectively, are divided by b, show that

(i) $$\frac{a}{b} = \frac{q_1}{c_1} + \frac{q_2}{c_1c_2} + \frac{q_3}{c_1c_2c_3} + \ldots + \frac{q_n}{c_1c_2 \ldots c_n} + R_n,$$

where $$R_n = \frac{1}{c_1c_2 \ldots c_n} \cdot \frac{r_n}{b};$$

(ii) $q_1 < c_1$, $q_2 < c_2$, $\ldots$ $q_n < r_n$.

(iii) $r_n \equiv ac_1c_2 \ldots c_n \pmod{b}$;

(iv) If a is a prime to b and $a < b$, then

$$\frac{a}{b} = \frac{q_1}{c_1} + \frac{q_2}{c_1c_2} + \frac{q_3}{c_1c_2c_3} + \ldots,$$

where the series terminates if $c_1c_2 \ldots c_n$ is divisible by b, for some value of n: otherwise the series is convergent.

3. Express the following fractions in the form:

$$\frac{q_1}{3} + \frac{q_2}{3 \cdot 5} + \frac{q_3}{3 \cdot 5 \cdot 7} + \ldots + \frac{q_n}{3 \cdot 5 \ldots (2n+1)} + \ldots;$$

(i) $\frac{7}{9}$; (ii) $\frac{1}{2}$; (iii) $\frac{1}{4}$.

In the last two cases, find the value of q_n.

4. Show that

$$\frac{1}{8} = \frac{1}{3^2} + \frac{3}{3^2 \cdot 5^2} + \frac{6}{3^2 \cdot 5^2 \cdot 7^2} + \ldots + \frac{\frac{1}{2}n(n+1)}{3^2 \cdot 5^2 \ldots (2n+1)^2} + \ldots.$$

5. Use Ex. 2, (iv), to show that e is irrational.

6. Sum the series

$$n^2 + 2(n-1)^2 + 3(n-2)^2 + \ldots,$$

where n is a positive integer.

Prove that $$S_1 + \frac{S_2}{\lfloor 2} + \ldots + \frac{S_n}{\lfloor n} + \ldots \text{ to infinity} = \frac{17e}{6},$$

where S_n is the sum of the squares of the first n positive integers.

7. Prove that, if x, y, z are positive and $x+y+z=1$, then

$$\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right)\geqslant 8.$$

8. From a cask containing x apples, a man sells half the contents of the cask and half an apple to one customer, half the remainder and half an apple to a second, and so on, always selling half of what are left and half an apple. In no case is an apple divided. After n sellings, a apples are left. Show that

$$x=2^n(a+1)-1.$$

9. Show that the planes $ny-mz=\lambda$, $lz-nx=\mu$, $mx-ly=\nu$, have a common line if, and only if, $l\lambda+m\mu+n\nu=0$.

10. If $\beta=\dfrac{a\alpha+b}{c\alpha+d}$ $\gamma=\dfrac{a\beta+b}{c\beta+d}$, where $ad-bc\neq 0$,

and a, b, c, d are connected by the equation $a^2+d^2+ad+bc=0$,

prove that $$\alpha=\frac{a\gamma+b}{c\gamma+d}.$$

Show also that α, β, γ are the roots of

$$x\cdot\frac{ax+b}{cx+d}\cdot\frac{-dx+b}{cx-a}=\alpha\cdot\frac{a\alpha+b}{c\alpha+d}\cdot\frac{-d\alpha+b}{c\alpha-a}.$$

11. If $k_r=\dfrac{\lfloor 2r-2}{\lfloor r\ \lfloor r-1}$, show that $k_r=k_1k_{r-1}+\ldots+k_pk_{r-p}+\ldots+k_{r-1}k_1$,

where r is a positive integer greater than unity, and $p<r$.

12. Show that $\dfrac{a^m+b^m+c^m}{3}>\dfrac{a^p+b^p+c^p}{3}\cdot\dfrac{a^q+b^q+c^q}{3}\cdot\dfrac{a^r+b^r+c^r}{3}$,

where p, q, r are positive quantities such that $m=p+q+r$.

13. If $x_r=x(x-1)(x-2)\ldots(x-r+1)$, prove that

$$(2n-1)_{n-1}\cdot(2n-3)_{n-3}\cdot(2n-5)_{n-5}\ldots>n_n\cdot n_{n-3}\cdot n_{n-5}\cdot n_{n-7}\ldots,$$

each product being continued as long as the suffixes are greater than zero.

14. If l, m, l', m', l'' and m'' are integers, and if α/β is not rational, and if

$$l\alpha+m\beta=l'\alpha+m'\beta,$$

show that $l=l'$, and $m=m'$.

Also show that no two of the numbers

$$(2l+5m)\alpha+l\beta,\quad (2l'+5m'+1)\alpha+l'\beta,\quad (2l''+5m'')\alpha+(l''+1)\beta$$

can be equal.

15. One root of the equation $x^6-9x^5+18x^4+9x^3+27x^2-54x-36=0$ is $3-3^{\frac{1}{3}}-3^{\frac{2}{3}}$; find all the roots.

16. If

$$a_1yz+b_1(y+z)+c_1=0,$$
$$a_2zx+b_2(z+x)+c_2=0,$$
$$a_3xy+b_3(x+y)+c_3=0,$$

are true for an infinite series of values of x, y, z, prove that, in general,

$$a_2c_3+a_3c_2=2b_2b_3,\quad a_3c_1+a_1c_3=2b_1b_3,\quad a_1c_2+a_2c_1=2b_1b_2.$$

17. If a, b, c are unequal, show that

$$ax^{b-c}+bx^{c-a}+cx^{a-b}\geqq a+b+c.$$

18. If

$$a(x+y+b)+x^2y^2+bxy(x+y)=0,$$
$$a(z+x+b)+z^2x^2+bzx(z+x)=0,$$

where y and z are unequal, prove that

$$a(y+z+b)+y^2z^2+byz(y+z)=0.$$

19. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

is equal to zero.

20. Prove that

$$\frac{nx}{(1+x)^n-(1-x)^n}=\Sigma\frac{A_r}{x^2+\tan^2\frac{r\pi}{n}},$$

where $A_r=(-1)^{r-1}\sin^2\frac{r\pi}{n}\cos^{n-4}\frac{r\pi}{n}$ and $r=1, 2, 3, \ldots \frac{1}{2}(n-1)$ or $\frac{1}{2}n-1$, according as n is odd or even.

21. Determine the condition that $(x^{n+1}-x^n+1)$ shall be divisible by (x^2-x+1).

22. Show that the first four terms in the expansions of

$$(1+x)^n+a\left(1+\frac{x}{p}\right)^{pn} \quad \text{and} \quad b\left(1+\frac{x}{q}\right)^{qn}+c\left(1+\frac{x}{r}\right)^{rn}$$

are identical if

$$1+a=b+c, \quad 1+\frac{a}{p}=\frac{b}{q}+\frac{c}{r}, \quad 1+\frac{a}{p^2}=\frac{b}{q^2}+\frac{c}{r^2},$$

and find the values of a, b, c in terms of p, q, r.

23. Find the limiting values of

$$(\cos x)^{\frac{1}{x}}, \quad (\cos x)^{\frac{1}{x^2}}, \quad (\cos x)^{\frac{1}{x^3}},$$

as x tends to zero through positive or negative values.

24. Show that, if l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are real quantities satisfying the six relations,

$$l_1^2+m_1^2+n_1^2=l_2^2+m_2^2+n_2^2=l_3^2+m_3^2+n_3^2=1,$$
$$l_2l_3+m_2m_3+n_2n_3=l_3l_1+m_3m_1+n_3n_1=l_1l_2+m_1m_2+n_1n_2=0,$$

then

$$l_1^2+l_2^2+l_3^2=m_1^2+m_2^2+m_3^2=n_1^2+n_2^2+n_3^2=1,$$
$$m_1n_1+m_2n_2+m_3n_3=n_1l_1+n_2l_2+n_3l_3=l_1m_1+l_2m_2+l_3m_3=0$$

and

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=\pm 1.$$

25. If $z^3+3hz+g=\frac{1}{\mu-\nu}[\mu(z+\nu)^3-\nu(z+\mu)^3]$ for all values of z, show that

$$2\mu=\frac{g}{h}+\frac{\sqrt{\Delta}}{h} \quad \text{and} \quad 2\nu=\frac{g}{h}-\frac{\sqrt{\Delta}}{h},$$

where $$\Delta=g^2+4h^3.$$

Show that the cubic $4z^3-27a^2(z+a)=0$ has two equal roots.

26. If $u=(b-c)^n+(c-a)^n+(a-b)^n$, where n is a positive integer, prove that

(i) If u is divisible by $\Sigma a^2-\Sigma bc$, then n is of the form $3k\pm1$, where k is an integer.

(ii) If u is divisible by $(\Sigma a^2-\Sigma bc)^2$, then u is of the form $3k+1$.

27. If all the roots of the equation

$$x^n+p_1x^{n-1}+\ldots+p_n=0$$

are rational and negative, show that

$$\text{(i)} \quad \left(\frac{p_1}{n}\right)^n \geqslant p_n,$$

$$\text{(ii)} \quad \left(\frac{p_1}{n}\right)^r \geqslant p_r\frac{\lfloor r \lfloor n-r}{\lfloor n}.$$

28. Prove that, if n is a positive integer, and $\theta_r=(\alpha+2r\pi)/n$, then

$$\frac{1}{x^{2n}-2x^n\cos\alpha+1}=\frac{1}{n\sin\alpha}\sum_{r=0}^{n-1}\frac{\sin\alpha-x\sin(\alpha-\theta_r)}{x^2-2x\cos\theta_r+1}.$$

29. Investigate the maximum and minimum values of $(x+1)^5/(x^5+1)$, and trace its graph.

Prove that the equation $(x+1)^5=m(x^5+1)$ has three real roots if $0<m<16$ and only one real root (-1) if $m<0$ or $m>16$.

30. If n and $n+2$ are both prime numbers, prove that $(n-2)\lfloor n-1-2$ is divisible by $n(n+2)$.

31. Prove that, if ϵ is small, the equation

$$x^2-3x+2=\epsilon\,.\,x^3$$

has a root nearly equal to $1-\epsilon+4\epsilon^2$, and find approximations to the other two roots.

32. If $a, b, c, \ldots k, l$ are positive numbers arranged in descending order, and

$$a+b+c+\ldots+k+l=\frac{\mu a-l}{\mu-1},$$

show that μ lies between the greatest and least of the numbers

$$a/b,\ b/c,\ c/d,\ \ldots,\ k/l.$$

33. Discuss the convergence of the series whose nth terms are

$$\frac{(\lfloor n)^2}{\lfloor 2n}x^n, \quad (-1)^n\frac{n}{n+1}.$$

34. Find the seventh roots of unity and show their positions in the z plane where $z=x+iy$.

If z is any one of the imaginary roots, find the equation whose roots are

$$z+z^4+z^2 \quad \text{and} \quad z^6+z^3+z^5.$$

35. If d denotes the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix},$$

and D denotes the determinant obtained by changing a_r into A_r where

$$A_1 = a_1^2 + 2a_2a_5 + 2a_3a_4, \quad A_2 = a_4^2 + 2a_1a_2 + 2a_3a_5, \quad A_3 = a_2^2 + 2a_1a_3 + 2a_4a_5,$$
$$A_4 = a_5^2 + 2a_1a_4 + 2a_2a_3, \quad A_5 = a_3^2 + 2a_1a_5 + 2a_2a_4,$$

show that $d^2 = D$.

36. If m, n, p, q are positive integers and if x, y, z all tend to a, prove that

$$\lim \frac{x^m(y^n - z^n) + y^m(z^n - x^n) + z^m(x^n - y^n)}{x^p(y^q - z^q) + y^p(z^q - x^q) + z^p(x^q - y^q)} = \frac{mn(m-n)}{pq(p-q)} a^{m+n-p-q}.$$

37. Prove that, in the expansion of

$$(1+x)^m + (1-x)^m,$$

where $-1 < x < 1$, the terms are either all positive or, after a certain stage, become and remain negative, and that in the latter case the last positive term exceeds numerically the sum of the infinite series of negative terms.

Deduce or otherwise prove that

$$(1+x)^m + (1-x)^m - 2$$

vanishes with or has the same sign as $m(m-1)x^2$.

38. Show that the number of distinct sets of three positive integers (none zero) whose sum is the odd integer $2n+1$, is given by the least integer containing $\frac{1}{3}(n^2+n)$.

39. Show that the number of distinct sets of three positive integers (none zero) whose sum is an even integer $2n$ is the integral part of $\frac{1}{3}n^2$.

40. Prove that

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - (1+x)e^{-x}$$

is positive for positive values of x, and increases as x increases.

41. Find to 6 places of decimals the root of

$$x^3 + 3x^2 - 42x - 41 = 0$$

which lies between 5 and 6.

42. Prove that the coefficient of x^{3n} in the expansion of $\dfrac{1}{(1-x)(1-x^3)(1-x^6)}$ in powers of x is

$$\frac{1}{4}\left\{(n+2)^2 - \frac{1-(-1)^n}{2}\right\}.$$

43. If the volume of a right circular cone is 20 cubic feet, prove that, when the cone is such that the curved surface is a minimum, the radius of the base is approximately 2·381 feet.

44. Arrange the following numbers in order, so that as $x \to \infty$, the ratio of each number to the preceding may tend to infinity:

$$x^2, \quad 2^x, \quad x^x, \quad e^x, \quad x^{\log x}, \quad (\log x)^2, \quad 2^{\log x}.$$

45. Prove that

$$\frac{(x-1)(x-2)\dots(x-n)}{x(x+1)(x+2)\dots(x+n)} = \Sigma_{r=0}^{n}(-1)^{n-r}\frac{\lfloor n+r}{(\lfloor r)^2\,\lfloor n-r}\cdot\frac{1}{x+r};$$

and deduce, or otherwise prove, that

$$-\frac{\lfloor n+1}{\lfloor n-1}+\frac{\lfloor n+2}{\lfloor 1\,\lfloor 2\,\lfloor n-2}-\frac{\lfloor n+3}{\lfloor 2\,\lfloor 3\,\lfloor n-3}+\frac{\lfloor n+4}{\lfloor 3\,\lfloor 4\,\lfloor n-4}-\dots$$
$$+(-)^n\frac{\lfloor 2n}{\lfloor n-1\,\lfloor n} = (-)^n n(n+1).$$

46. Prove that, if

$$\frac{a}{l^2}+\frac{b}{m^2}+\frac{c}{n^2}=\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=\frac{al}{x^3}+\frac{bm}{y^3}+\frac{cn}{z^3}=0,$$

where a, b, c are not zero and no two of $\frac{x}{l}, \frac{y}{m}, \frac{z}{n}$ are equal, then

$$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0.$$

47. If $a_1a_2a_3\dots a_n=\lambda^n$, prove that the sum of the products r together of the a's is greater than

$$\lfloor n\,.\,\lambda^r/\{\lfloor r\,\lfloor n-r\}.$$

48. Solve the equations:

$$x+y+z+w=1,$$
$$ax+by+cz+dw=\lambda,$$
$$a^2x+b^2y+c^2z+d^2w=\lambda^2,$$
$$a^3x+b^3y+c^3z+d^3w=\lambda^3.$$

Prove that, if a, b, c, d, λ are all real and unequal, at least two and not more than three of x, y, z, w are positive.

49. Prove that if

$$y^2+yz+z^2=a^2,\quad z^2+zx+x^2=b^2,\quad x^2+xy+y^2=c^2,\quad yz+zx+xy=0,$$

then
$$a\pm b\pm c=0.$$

50. Prove that $\dfrac{\lfloor 72}{(\lfloor 36)^2}-1$ is divisible by 73.

51. Prove that if m is prime and $p<m$,

$$\lfloor p-1\,\lfloor m-p+(-1)^{p-1}\equiv 0 \pmod m.$$

52. If $\alpha^2+\beta^2=\kappa\alpha\beta$, $\beta^2+\gamma^2=\kappa\beta\gamma$, $\gamma^2+\delta^2=\kappa\gamma\delta$, $\delta^2+\epsilon^2=\kappa\delta\epsilon$, and if $\alpha, \beta, \gamma, \delta, \epsilon$ are all different, then

$$\alpha^2+\epsilon^2=(\kappa^4-4\kappa^2+2)\alpha\epsilon.$$

53. Prove that

$$\frac{1}{\lfloor r}+\frac{1}{2^2}\frac{n(n-1)}{\lfloor 1\,\lfloor r+1}+\frac{1}{2^4}\cdot\frac{n(n-1)(n-2)(n-3)}{\lfloor 2\,\lfloor r+2}+\dots=\frac{\lfloor 2n+2r}{2^n\,\lfloor n+r\,\lfloor n+2r}.$$

54. If
$$(1+px+x^2)^n=1+a_1x+a_2x^2+\dots+a_{2n}x^{2n},$$
prove that
$$a_r=a_{2n-r};$$
and that
$$1+3a_1+5a_2+\dots+(4n+1)a_{2n}=(2n+1)(2+p)^n.$$

55. If of $3n$ letters there are n a's, n b's and n c's, the number of combinations of these r together is equal to the number of combinations $3n-r$ together.

Also, if $n>r>2n+1$, the number of combinations is

$$\tfrac{1}{2}(n+1)(n+2)+(r-n)(2n-r)$$

and the maximum number of combinations is

$$\tfrac{1}{4}\{3(n+1)^2+1\} \quad \text{or} \quad \tfrac{3}{4}(n+1)^2,$$

according as n is even or odd.

56. Assuming that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$ tends to a limit ν as $n\to\infty$, prove that

$$\text{(i)}\ \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\ldots-\frac{1}{n^2}\right)=\gamma;$$

$$\text{(ii)}\ \lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{1}+\frac{n-1}{2}+\frac{n-2}{3}+\ldots+\frac{1}{n}-\log\lfloor n\right)=\gamma.$$

57. Expand $\log(1-x-x^2)$ as far as the term containing x^4, and if

$$\log(1-x-x^2)=-u_1x-\tfrac{1}{2}u_2x^2-\tfrac{1}{3}u_3x^3-\ldots,$$

obtain an expression for u_n, and prove that

$$u_n=u_{n-1}+u_{n-2}.$$

58. Find $\lim_{x\to1}\dfrac{1-x+\log x}{1-(2x-x^2)^{\frac{1}{2}}}$.

59. Find $\lim_{x\to0}\dfrac{a^x-b^x}{c^x-d^x}$, where a, b, c, d are positive and $c\neq d$.

60. If
$$x^2+y^2+z^2=\xi^2+\eta^2+\zeta^2=1,$$
and
$$x+y+z=\xi+\eta+\zeta=x\xi+y\eta+z\zeta=0,$$
show that
$$3x^2=(\eta-\zeta)^2 \quad \text{and} \quad x^2+\xi^2=\tfrac{2}{3}.$$

61. If n points are taken in space so that not more than 3 of them lie in any plane and not more than 2 in any straight line, show that the points define $\frac{1}{2}n(n-1)$ straight lines and $\frac{1}{2}\lfloor n-1$ polygons of n sides.

62. Prove that the continued fraction $a-\dfrac{1}{a-}\,\dfrac{1}{a-}\cdots\dfrac{1}{a-}\,\dfrac{1}{x}$ in which a is equal to -1 and is repeated any number of times, must have one of three values, and that if x satisfies the equation $2x^3+3x^2-3x-2=0$, the fraction satisfies this equation.

63. If $(1+x)^n=c_0+c_1x+\ldots c_nx^n$, where n is a positive integer, prove that

$$\frac{c_0}{1^2}-\frac{c_1}{2^2}+\frac{c_2}{3^2}\ldots+\frac{(-1)^nc_n}{(n+1)^2}=\frac{1}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right).$$

64. If a, b, c are real and l, m, n are integers, find the values of x for which $(x-a)^l(x-b)^m(x-c)^n$ has maxima or minima.

Determine which of these values give maxima and which minima, (i) when l, m, n are all even, (ii) when l, m, n are all odd.

65. Solve the equations:

$$y+z=a(1-yz),\quad z+x=b(1-zx),\quad x+y+z-xyz=c(1-xy-yz-zx).$$

66. Sum the series $1+\dfrac{2}{\lfloor1}+\dfrac{3}{\lfloor2}+\dfrac{8}{\lfloor3}+\dfrac{13}{\lfloor4}+\dfrac{30}{\lfloor5}+\dfrac{55}{\lfloor6}+\ldots$ to ∞.

67. Discuss the convergence of the series

$$\Sigma \frac{\lfloor 2n}{(\lfloor n)^2} z^n,$$

for all values of z, real or complex, distinguishing cases of absolute and conditional convergence.

68. The equations of four lines are

$$(1)\ y=a,\ z=a'; \qquad (2)\ z=b,\ x=b';$$
$$(3)\ x=c,\ y=c'; \qquad (4)\ x=y=z.$$

Find the equations of the line drawn from the point (k, k, k) to cut the lines (1) and (2).

Prove that two lines can be drawn to cut all the four given lines, and that these lines cut the line (4) in two points determined by $x=y=z=k$, where

$$(k-a)(k-b)(k-c)=(k-a')(k-b')(k-c').$$

69. Discuss the reality and equality of the roots of

$$x^4-2\lambda(3x^2-4x+3)+3\lambda^2=0$$

for different real positive values of λ.

70. A bag contains 21 balls of 6 different colours, there being 6 of the first colour, 5 of the second, 4 of the third, and so on. Show that the number of different selections of 6 balls which can be made from the bag is 259.

71. If $\Delta f(a)=f(a+w)-f(a)$, $\Delta^2 f(a)=\Delta f(a+w)-\Delta f(a)$, etc.

prove that $$f(a+xw)=f(a)+x\Delta f(a)+\frac{x(x-1)}{\lfloor 2}\Delta^2 f(a)+\dots.$$

Use this to find $\log 7{\cdot}063$, given that

	Δ log	Δ^2 log	Δ^3 log
log 7 = 0·845098			
	12234		
log 7·2 = 0·857332		−334	
	11900		16
log 7·4 = 0·869232		−318	
	11582		
log 7·6 = 0·880814			

72. A series is such that the sum of the rth term and the $(r+1)$th is always r^4. Prove that

(i) the rth term is $\frac{1}{2}r(r-1)(r^2-r-1)-(-1)^r c$;

(ii) the sum of r terms is $\dfrac{r(r^2-1)(3r^2-7)}{30}+\dfrac{c}{2}\{1-(-1)^r\}$.

73. If n is a positive integer,

$$\frac{1}{2}<\frac{1}{3n+1}+\frac{1}{3n+2}+\dots+\frac{1}{5n}+\frac{1}{5n+1}<\frac{2}{3}.$$

74. Prove that, if $\dfrac{p}{q}$, $\dfrac{r}{s}$ are fractions such that $qr-ps=1$, then the denominator of any fraction whose value lies between $\dfrac{p}{q}$ and $\dfrac{r}{s}$ is not less than $q+s$.

Prove that there are two and only two fractions with denominators less than 19 which lie between $\frac{10}{13}$ and $\frac{4}{5}$.

75. Determine ranges of x for which the function $(\log x)/x$ (i) increases and (ii) decreases as x increases.

Hence prove the following theorems, wherein n is a given positive number and only positive values of x are considered:

(i) The equation $n^x = x$ has two roots, one root or none according to the value of n.

(ii) The inequality $n^x > x^n$ is a consequence of either $x > n$ or $x < n$, according to the value of n.

State the critical values of n.

76. From a bag containing 9 red and 9 blue balls, 9 are drawn at random, the balls being replaced. Show that the probability that 4 balls of each colour will be included is a little less than $\frac{1}{2}$.

77. For the continued fraction $\cfrac{1}{a_1+}\ \cfrac{1}{a_2+}\ \ldots$, prove that $p_{n-1}+q_n$ is not altered if $a_1, a_2, \ldots a_n$ are permuted cyclically.

78. An approximate value of ϕ, measured in radians, is $3\sin\phi/(2+\cos\phi)$ provided that $\phi < \frac{1}{2}\pi$. Establish this result when ϕ is small and show that the error is approximately $\phi^5/180$.

Hence express approximately the acute angles of a right-angled triangle in terms of the sides and deduce that, if $c^2 = a^2 + b^2$, then

$$\frac{6(a+b)c+3c^2}{ab+2(a+b)c+4c^2},$$

is nearly equal to $\frac{1}{2}\pi$ for all positive values of a and b.

79. Given a_1 things of one kind, a_2 things of another kind, etc., a_r things of an rth kind, show that the number of different groups which can be formed from one or more of the above things is

$$(a_1+1)(a_2+1)\ldots(a_r+1)-1,$$

and show that if $r=4$ and $a_1 < n < a_2 < a_3 < a_4$, the number of groups having n members is

$$\tfrac{1}{6}[(n+1)(n+2)(n+3)-(n-a_1)(n-a_1+1)(n-a_1+2)].$$

80. When $n \geqslant 2$, if

$$x_n = a_n x_{n-1} + x_{n-2},$$
$$y_n = a_n y_{n-1} + y_{n-2},$$

and if

$$x_0 = 1, \quad x_1 = a_1, \quad y_0 = 0, \quad y_1 = 1,$$

verify by induction that $x_{n+1}y_n - x_n y_{n+1} = (-1)^{n+1}$, and deduce that

$$\frac{x_n}{y_n} = a_1 + \frac{1}{y_1 y_2} - \frac{1}{y_2 y_3} + \ldots + (-1)^n \frac{1}{y_{n-1} y_n}.$$

81. Find the equation determining the values of x for which $\sin mx/\sin x$ is stationary.

Hence, or otherwise, show that, if m is an integer, $\sin^2 mx/\sin^2 x$ never exceeds m^2.

82. A quadratic function of x takes the values y_1, y_2, y_3, corresponding to three equidistant values of x. Prove that if $y_1 + y_3 > 2y_2$, the minimum value of the function is

$$y_2 - \frac{(y_3-y_1)^2}{8(y_1+y_3-2y_2)}.$$

83. Show that

$$(n+1)^{n-1}(n+2)^n > 3^n(\lfloor n)^2.$$

84. The digits of a number are 1, 2, 3, 4, 5, 6, 7, 8, 9, written at random in any order. Show that the odds are 115 to 11 against the number being divisible by 11.

85. If m, n are positive, show that

$$\left(\frac{mn+1}{m+1}\right)^{m+1} > n^m.$$

86. The coefficient of x^{2n} in the expansion of $(1+x^2)^n/(1-x)^3$ is

$$2^{n-1}(n^2+4n+2).$$

87. If $ax_1^2+by_1^2+cz_1^2=ax_2^2+by_2^2+cz_2^2=ax_3^2+by_3^2+cz_3^2=d,$

$$ax_2x_3+by_2y_3+cz_2z_3=ax_3x_1+by_3y_1+cz_3z_1=ax_1x_2+by_1y_2+cz_1z_2=f,$$

prove that $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\{(d+2f)/abc\}^{\frac{1}{2}}.$

88. Prove that

$$\begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix} = -4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\,.\,\Sigma\sin(\alpha+\beta).$$

$$\begin{vmatrix} \sin^3\alpha & \sin^3\beta & \sin^3\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix} = -\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin(\alpha+\beta+\gamma).$$

89. Denoting the number of combinations of n letters taken r together, all the letters being unlike, by ${}_nC_r$, show that if r of the letters are alike, the rest being unlike, the number of combinations is

$${}_{n-r}C_r+{}_{n-r}C_{r-1}+\ldots+{}_{n-r}C_1+1. \quad (n\geqslant 2r)$$

Show that the number of combinations in groups of n which can be formed from $3n$ letters, of which n are a and n are b and the rest unlike, is

$${}_nC_n+2{}_nC_{n-1}+3{}_nC_{n-2}+\ldots+n{}_nC_1+(n+1),$$

and show that this sum is $2^{n-1}(n+2)$.

90. Prove that if n is a positive integer,

$$1-\frac{n}{1^2}x+\frac{n(n-1)}{1^2\,.\,2^2}x^2-\frac{n(n-1)(n-2)}{1^2\,.\,2^2\,.\,3^2}x^3+\ldots$$

$$=e^x\{1-\frac{n+1}{1^2}x+\frac{(n+1)(n+2)}{1^2\,.\,2^2}x^2-\ldots\}.$$

91. Investigate the convergence of the series whose nth term is

$$e^{-2\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)}.$$

92. Prove that

$$\frac{1}{n(n+1)}-\frac{C_1^n}{(n+1)(n+2)}+\frac{C_2^n}{(n+2)(n+3)}-\ldots+(-1)^n\frac{C_n^n}{2n(2n+1)}=\frac{\lfloor n-1\rfloor n+1}{\lfloor 2n+1}.$$

93. If t^2+at+b and t^2+at-b both have rational factors, then

$$a=\lambda(\mu^2+\nu^2), \quad b=\lambda^2\mu\nu(\mu^2-\nu^2),$$

where λ, μ, ν are integers and μ is prime to ν.

[For instance, t^2+5t+6 and t^2+5t-6.]

94. For the fraction $\dfrac{1^2}{1+}\dfrac{2^2}{1+}\dfrac{3^2}{1+}\cdots$, prove that $p_n+q_n=\underline{|n+1}$.

95. Discuss the reality of the roots of

$$16x^4+24x^2+16kx+9=0,$$

for all real values of k, and solve the equation when $k=\sqrt{2}$.

96. If $ax^2+2hx+b>0$ for all real values of x, show that

$$a(x^2+y^2)+b(z^2+1)+h\{xz+y+\sqrt{3}(z-yz)\}>0$$

for all real values of x, y, z, the coefficients being real.

97. 2^n players of equal skill enter for a tournament. They are drawn in pairs, and the winners of each round are drawn again for the next. Find the chance that two given competitors will play against each other in the course of the tournament. Also show that if $n=7$ the chance that a given competitor either wins or is beaten by the actual winner is $\frac{1}{16}$.

98. If there are four relations,

$$A_ia_j+B_ib_j+C_ic_j+D_id_j=0 \quad \text{for} \quad i=1,2\,;\ j=1,2\,;$$

show that $\dfrac{(B,C)}{(a,d)}=\dfrac{(C,A)}{(b,d)}=\dfrac{(A,B)}{(c,d)}=\dfrac{(A,D)}{(b,c)}=\dfrac{(B,D)}{(c,a)}=\dfrac{(C,D)}{(a,b)}$.

99. If $\sin x=xy^2$ where x^2 and $y-1$ are both small, show that

(i) $y=1-\dfrac{x^2}{12}+\dfrac{x^4}{1440}$ approx.;

(ii) $x^2=-12(y-1)+\frac{6}{5}(y-1)^2$ approx.

100. In the game of *Craps*, each player in turn acts as 'banker.' The banker throws with two dice numbered 1 to 6. If he throws 7 or 11 he wins. If he throws 2, 3 or 12, he loses. If he throws any other number, he throws again and continues to throw until either the number he threw first or 7 turns up. In the first case he wins, in the second he loses.

Show that the odds *against* the bank are 251 to 244.

101. If $u_nu_{n-1}+au_n+bu_{n-1}+c=0$, show that

$$\frac{u_n-\alpha}{u_n-\beta}=\frac{\beta+a}{\alpha+a}\cdot\frac{u_{n-1}-\alpha}{u_{n-1}-\beta},$$

where α, β are the roots of

$$x^2+(a+b)x+c=0,$$

unless $\alpha=\beta$, in which case

$$\frac{1}{u_n-\alpha}-\frac{1}{u_{n-1}-\alpha}=\frac{2}{a-b}.$$

Hence obtain the general solution of the first equation, and show that the results so obtained agree with those in Exercise XXXVIII, 15, p. 372.

102. If $\begin{vmatrix} 1 & \cos x & 0 & 0 \\ \cos x & 1 & \cos\alpha & \cos\beta \\ 0 & \cos\alpha & 1 & \cos\gamma \\ 0 & \cos\beta & \cos\gamma & 1 \end{vmatrix}=0,$

prove that $\sin^2\gamma\sin^2x=\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\gamma$.

103. If $\dfrac{ab}{b+x}-\dfrac{cd}{d+y}=\dfrac{bc}{x}-\dfrac{ad}{y}=z$, then either $\dfrac{x}{b}+\dfrac{y}{d}+1=0$ or $z=a-c$.

104. If

$$a^2x^2+b^2y^2+c^2z^2=0,$$

$$a^2x^3+b^2y^3+c^2z^3=0,$$

and

$$\frac{1}{x}-a^2=\frac{1}{y}-b^2=\frac{1}{z}-c^2,$$

prove that

$$a^4x^3+b^4y^3+c^4z^3=0,$$

and

$$a^6x^3+b^6y^3+c^6z^3=a^4x^2+b^4y^2+c^4z^2.$$

105. X and Y are two places on a road, 4 miles apart. A man A, starting from X at any time between one and four o'clock, walks to Y at 4 miles an hour. A man B, starting from Y at any time between one and four o'clock, walks to X at 4 miles an hour. Show that the odds in favour of the men meeting on the way are as 5 : 4, all times of starting being equally likely.

[Suppose that A and B start at x hrs. and y hrs. past one, respectively. Taking axes Ox, Oy at right angles, set off $OL=3$ units along Ox and draw a square on OL. The total number of cases is represented by the area of the square, and the number of favourable cases by the part of the square between the lines $x-y=\pm 1$.]

106. Show that, in a game of whist, the chance that a hand is void of a suit is about $\frac{1}{20}$. Also the chance that in 25 deals a certain player has a hand void of a suit *at least twice* is about 0·358.

[Chance that a hand is void of a suit $=4\,.\,C_{13}^{39}\div C_{13}^{52}$.]

107. If $u\equiv ax^4+4bx^3+6cx^2+4dx+e$ and t is any root of

$$4(x+H)^3-a^2I(x+H)+a^3J=0,$$

show that the roots of $u=0$ are given by

$$(ax+b\pm\sqrt{t})^2+t+3H\mp G/\sqrt{t}=0.$$

ANSWERS

EXERCISE I

PAGE

10. 7. 28^2-27^2; 8^2-3^2. 10. 212, 213, ..., 219, 220. 22. 20, 4836.

11. 23. 6, 28, 496. 27. 120. 28. 360. 30. (i) $\frac{(p^r-1)}{(p-1)}$; (ii) $\frac{(p^r-1)}{(p-1)}-r$

EXERCISE II

22. 19. (i) 5·394, 0·006; (ii) 2·594; 0·002; (iii) 2·276; 0·002.

EXERCISE III

31. 1. $5x^3-20x^2+80x-314$; 1258. 2. $x^4-3x^3-6x^2-5x-10$; -21.
3. $\frac{1}{3}x^3-\frac{10}{9}x^2+\frac{34}{27}x-\frac{40}{81}$; $\frac{1}{81}$. 4. $\frac{1}{2}x^4-\frac{1}{4}x^3+\frac{1}{8}x^2-\frac{1}{16}x+\frac{1}{32}$; $-\frac{33}{32}$.
5. $3x^2+x-3$; $-3x-7$. 6. $3x^3+x-3$; $5x+1$. 7. 13.
8. (i) $(x+5)^3-13(x+5)^2+56(x+5)$; (ii) $(x+1)^3-(x+1)+80$;
(iii) $(x+\frac{1}{3})^3+(x+\frac{1}{3})^2+\frac{2156}{27}$.
9. (i) y^3+5y^2-9; (ii) $y^3-75y+7$; (iii) y^3-15y^2+257.
10. $-n+7n(n+1)-6n(n+1)(n+2)+n(n+1)(n+2)(n+3)$.
11. $-\frac{55}{4}$; 0, $\frac{11}{3}$. 12. $(x+1)(x+2)(x-2)(x-3)$.
13. $(x-3y)(2x+y)(x^2+xy+y^2)$. 14. $3x^3+4x^2-5x-2$.
15. $7x^2-11x-6$. 16. $3x^2-7xy+4y^2$.
18. $(x+1)(x^2-x+1)(x^4-x^3+x^2-x+1)(x^8+x^7-x^5-x^4-x^3+x+1)$.
20. $a(ca'-c'a)^3+b(ab'-a'b)^2(ca'-c'a)+c(ab'-a'b)^3=0$.
21. $a=3$, $b=-3$, $c=1$.

32. 22. $p^2-3q^2+3r^2-s^2=0$. 23. $x^2-3y^2+3z^2-w^2=0$.
24. $a=5$, $b=-3$, $c=1$. 25. $a=1$, $b=2$.
26. $a=8$, $b=0$. 27. $(3x+4y-1)(x-y+3)$.
28. $(3x-4y+2)(x+2y-3)$. 29. x^2-2x+4.
30. $(x-1)^2(x+5)$; $(x+3)^2(x-3)$. 31. (i) $3(p+a^2)x+q-2a^3$.
33. $\lambda=1$, $(2x+y-2)(x-y+1)$; $\lambda=-7/8$, $(x-7y+4)(x-y-4)/8$.
34. $\lambda=-1$, $(y-1)(y-2)$; $\lambda=-2$, $-(x+y)(x-y)$;
$\lambda=-\frac{10}{9}$, $-(x+3y+4)(x-3y-4)/9$.

EXERCISE IV

39. 16. 120. 17. 60. 18. 15. 20. (i) 2^{2n}.
21. (i) $x^{\frac{1}{2}n(n+1)}$; (ii) $\underline{|n+1}$.
23. $(-1)^m C^n_m$, if $n=2m$; $(-1)^m(n+1)C^n_m$, if $n=2m+1$.

EXERCISE V

PAGE

43. 1. $(x-1)^2$. 2. $2x+1$. 3. x^2+x-3. 4. x^2+x-1.
5. $(x-1)^2$. 6. $3x-4$. 7. $X=\frac{1}{4}(x+2)$, $Y=\frac{1}{4}(x-2)$.
8. $X=x^3+1$, $Y=x$. 9. $X=1+x^2$, $Y=1-x+x^2$.
10. $\lambda=-2\mu=1$; $A=-\frac{1}{2}$, $B=-1$, $C=\frac{3}{2}$, $D=-\frac{1}{2}$.

EXERCISE VI

50. 3. $-2\Sigma x^3+3\Sigma x^2y-12xyz$. 4. $2\Sigma x^3+6\Sigma x^2y-12xyz$.
17. $(a-b)^2+(b-c)^2+(c-a)^2$.
18. $p=a+f-g-h$, $q=b+g-h-f$, $r=c+h-f-g$.

51. 24. 6. 25. 1. 26. -2. 27. $\dfrac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}$.
28. $-abc\dfrac{(b-c)(c-d)(a-b)}{(1+bc)(1+ca)(1+ab)}$.
29. $-(b+c-2a)(c+a-2b)(a+b-2c)$. 30. $-9(b-c)(c-a)(a-b)$.
31. $-4(b-c)(c-a)(a-b)(a+b+c)$.
32. $3(b+c+2a)(c+a+2b)(a+b+2c)$. 33. $24abc$.
34. $-(b-c)(c-a)(a-b)(\Sigma a^2+3\Sigma bc)$.
35. $(1-abc)(1-a^2-b^2-c^2+2abc)$.
36. (16)(25)(34); (12)(24)(36)(65); (16)(65)(53)(24).

EXERCISE VII

60. 1. $2, \pi/6$; $2, -\pi/6$; $2, \dfrac{5\pi}{6}$; $2, -\dfrac{5\pi}{6}$. 3. $(18+\iota)/25$.
4. (i) $(5-\iota)/13$; (ii) $(-2+2\iota)/8$; (iii) $\frac{1}{2}\left(1+\iota\cot\dfrac{\theta}{2}\right)$.

61. 5. (i) $(x^2+y^2)^{\frac{n}{2}}\left[\cos n\left(\tan^{-1}\dfrac{y}{x}\right)+\iota\sin n\left(\tan^{-1}\dfrac{y}{x}\right)\right]$;
(ii) $(x-\iota y)(x^2+y^2)$; (iii) $(x^2-y^2-1-2\iota xy)/(x^2+y^2-2x+1)$.
6. (i) 1625; (ii) 13/125. 7. $-\pi/3$. 13. $1-2\iota$; $\frac{1}{2}(1\pm\iota\sqrt{3})$.

EXERCISE VIII

67. 7. $2-\iota$, $-4-3\iota$.

EXERCISE IX

78. 3. If $X+\iota Y=(1+x+\iota y)/(1-x-\iota y)$, then $X=(1-x^2-y^2)/[(1-x)^2+y^2]$, and $Y=2y/[(1-x)^2+y^2]$; $X=0$, if $x^2+y^2=1$. Geometrically, take any line AOB, so that $BO=OA$, and draw AL, BM, parallel to Ox to meet a line through 1, parallel to BOA, in L and M: construct OMN similar to $OL1$, with $\angle$s $1ON$, LOM described in opposite directions. If A represents z, then L, M, N represent $1+z$, $1-z$ and $(1+z)/(1-z)$. If $|z|=1$, $OAL1$, $BO1M$ are rhombi; hence $\angle OM$ is a right angle; and N lies on Oy.

5. If $a=p+\iota q$, $px+qy=p^2+q^2$, which passes through (p, q) and is perpendicular to $qx=py$.

EXERCISE X

PAGE

86. 1. (i) $4x^3+2x^2-3x-1=0$; (ii) $2x^3+9x^2+11x+3=0$; (iii) $4x^3-11x^2+9x-2=0$; (iv) $2x^3-9x^2+11x-3=0$; (v) $4x^3-13x^2+17x+1=0$.

2. (i) $4x^3-8x^2+8x-3=0$; (ii) $x^3-4x^2+8x-6=0$.

3. $\frac{1}{2}(3\pm\sqrt{5})$; $\frac{1}{2}(-1\pm\sqrt{5})$. 4. $\pm 3/2$, $\frac{1}{2}(1\pm\sqrt{5})$.

5. 2, 9/2, −1/2. 6. 2, 1/3, −1/2. 7. 5, 3, 1, −1.

8. 4, 2, 1, $\frac{1}{2}$. 9. −2, 3, 6. 10. 3, −2, 1/2, −1/2.

11. (i) $x^3-8x-15$; (ii) $x^4-23x^2+59x-52=0$.

12. (i) $x^3-14x^2+11x^3-75=0$; (ii) $x^4-25x^3+375x^2-1260x-11700=0$.

EXERCISE XI

94. 3. 4, −2. 4. 11, −2. 5. 4, −6. 6. 2, 0.
7. 7, −1. 8. 3·1, −3. 9. 7, −6. 13. 4.
14. 6. 15. 3, 6, −4. 16. 2/3. 17. 3/2, 2/3.
18. 2, 2, 2/3, −1/2. 19. 3/2, 2/3.

EXERCISE XII

99. 17. $-a$, $-b$, $-\frac{1}{2}(a+b\pm\iota\sqrt{3a^2+2ab+3b^2})$.

EXERCISE XIII

104. 1. $\frac{1}{x-2}-\frac{1}{x-1}-\frac{1}{(x-1)^2}$. 2. $\frac{1}{x+3}-\frac{1}{x}+\frac{3}{x^2}-\frac{6}{x^3}$.

3. $\frac{10}{x-3}-\frac{10}{x-2}-\frac{9}{(x-2)^3}-\frac{5}{(x-2)^3}$. 4. $\frac{1}{x-1}+\frac{x}{x^2+x+1}$.

5. $\frac{2}{x-2}-\frac{x}{x^2+x+2}$. 6. $\frac{3x+2}{x^2+3}-\frac{2x+1}{x^2+2}$.

7. $\frac{x+2}{x^2-x+1}+\frac{x-3}{(x^2-x+1)^2}-\frac{x-2}{(x^2-x+1)^3}$.

8. $\frac{2}{3(x-1)}+\frac{1}{(x-1)^2}+\frac{x+2}{3(x^2+x+1)}$.

9. $\frac{x-2}{x^2+1}-\frac{1}{x+2}+\frac{1}{(x+2)^2}+\frac{3}{(x+2)^3}$.

10. $\frac{4}{x^2+1}-\frac{3}{(x^2+1)^2}+\frac{1}{(x^2+1)^3}-\frac{4}{x^2+2}$.

11. $\frac{x-1}{x^2+1}+\frac{1}{(x^2+1)^2}-\frac{x-1}{x^2+2}$ 12. $\left[\frac{4}{(x+1)^2}+\frac{5}{x^2+x+1}+\frac{3x}{(x^2+x+1)^2}\right]/9$.

13. $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)^2}+\frac{1}{4(x+1)^2}+\frac{1}{2(x-1)^3}-\frac{1}{2(x+1)^3}$.

14. $\frac{1}{2\sqrt{2}}\left[\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$.

PAGE

104. 15. $-1+\dfrac{1}{1-ax}+\dfrac{1}{1-bx}$; $(a^{n+1}+b^{n+1})x^{n+1}-ab(a^n+b^n)x^{n+2}$.

16. $1+\Sigma\dfrac{2a(a+b)(a+c)}{(a-b)(a-c)}\cdot\dfrac{1}{x-a}$. 17. $1+\Sigma\dfrac{a^3}{(a-b)(a-c)}\cdot\dfrac{1}{x-a}$.

18. Put $x=d$ in Ex. 17.

105. 19. $x+a+b+c+\Sigma\dfrac{a^4}{(a-b)(a-c)}\cdot\dfrac{1}{x-a}$; put $x=d$.

EXERCISE XIV (A)

116. 1. $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$.

2. $\frac{1}{12}(3n-1)(3n+2)(3n+5)(3n+8)+\frac{20}{3}$.

3. $\frac{1}{3}n(2n+1)(2n-1)$. 4. $\frac{1}{6}n(n+1)(2n+13)$.

5. $\frac{1}{12}n(n+1)(n+2)(3n+1)$. 6. $\frac{1}{12}n(n+1)(n+2)(3n+5)$.

7. $\frac{1}{12}n(n+1)(n+2)(3n+13)$. 8. $\frac{3}{4}n(n+1)(3n+1)(3n-2)$.

9. $\frac{1}{2}n(4n^2+n-1)$. 10. 2926. 11. 286. 12. 50336. 13. 4300.

14. $\frac{1}{15}n(n+1)(n+2)(3n^2+6n+1)$. 15. $\frac{1}{6}n(n+1)(n+2)$.

16. $\frac{1}{12}n(n+1)^2(n+2)$.

19. (i) $\frac{1}{2}n(6n^2-3n-1)$; (ii) $\frac{1}{8}n(n-1)(9n^2-9n-2)$, where $n\geqslant 2$.

20. 220. 21. 344. 23. 715.

EXERCISE XIV (B)

117. 1. $\dfrac{n}{n+1}$. 2. $\dfrac{n}{2n+1}$. 3. $\dfrac{n(n+3)}{4(n+1)(n+2)}$.

4. $\dfrac{n(3n+7)}{20(3n+2)(3n+5)}$. 5. $\dfrac{n(n^2+6n+11)}{18(n+1)(n+2)(n+3)}$.

6. $\dfrac{n(3n+1)}{4(n+1)(n+2)}$. 7. $\dfrac{n(4n+5)}{3(2n+1)(2n+3)}$.

8. $\dfrac{n(29n^2+138n+157)}{36(n+1)(n+2)(n+3)}$. 9. $\dfrac{1}{2}-\dfrac{1.3.5.\ldots.(2n+1)}{2.4.6.\ldots.(2n+2)}$.

10. $\dfrac{4.7.\ldots.(3n+4)}{2.5.\ldots.(3n+2)}-2$. 11. $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

12. $6-\dfrac{2n+3}{2^{n-1}}$. 13. $\dfrac{a-\{a+(n-1)d\}r^n}{1-r}+\dfrac{dr(1-r^{n-1})}{(1-r)^2}$.

14. $34-(4n^2+12n+17)/2^{n-1}$. 15. $n/(1+x)\{1+(n+1)x\}$.

16. $\lfloor n+1-1$. 17. $\left\{1-\dfrac{1}{1.3.5.\ldots.(2n+1)}\right\}\Big/2$. 18. $\dfrac{n(n+2)}{(n+1)^2}$.

19. $\dfrac{1}{140}-\dfrac{1}{4(2n+7)(2n+9)}-\dfrac{2}{3(2n+5)(2n+7)(2n+9)}$

$$-\frac{1}{(2n+3)(2n+5)(2n+7)(2n+9)}$$

or $\dfrac{1}{140}-\dfrac{1}{48}\left[\dfrac{1}{2n+3}+\dfrac{1}{2n+5}+\dfrac{1}{2n+7}-\dfrac{3}{2n+9}\right]$.

PAGE

117. 20. $1-\dfrac{x^n}{(x+1)(x+2)\dots(x+n)}$. $\dfrac{1-x^n}{(1-x)^3}-\dfrac{nx^n}{(1-x)^2}-\dfrac{n(n+1)x^n}{2(1-x)}$.

118. 22. $\dfrac{n(n+1)}{4(n+2)}$. 23. $\dfrac{2^n}{n+2}-\dfrac{1}{2}$. 24. $\dfrac{1}{2}-\dfrac{1}{n+2}\cdot\dfrac{1}{n}$.

26. $\dfrac{1}{x-1}-\dfrac{k}{x^k-1}$, where $k=2^n$. 27. $\dfrac{1}{1-x}-\dfrac{1}{1-x^k}$, where $k=2^n$.

28. $1-\dfrac{1}{(1+a_1)(1+a_2)\dots(1+a_n)}$.

EXERCISE XV

129. 8. (i) a^3+b^3; (ii) $-(a^3+b^3)^2$; (iii) 36.

131. 20. (i) $(\beta-\gamma)(\gamma-\alpha)(\gamma-\beta)$; (ii) $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$; (iii) $(\mu-\nu)(\nu-\lambda)(\lambda-\mu)$.

EXERCISE XVI

143. 1. $0, \pm\sqrt{\tfrac{3}{2}(a^2+b^2+c^2)}$.

146. 19. $39^2=30^2+20^2+14^2+5^2$.

EXERCISE XVII

158. 1. $\frac{11}{8}, \frac{21}{16}, -\frac{1}{16}$; (ii) 1, −2, −1, 3. 2. 3 : −2 : 10. 3. −3/2.

EXERCISE XVIII

165. 3. $(2, 0), (0, 2), (-\frac{1}{2}\pm\frac{11}{2}\iota, -\frac{1}{2}\mp\frac{1}{2}\iota), (-2\pm\iota\sqrt{2}, -2\mp\iota\sqrt{2})$.

4. $(1\pm\frac{1}{2}\sqrt{2}, 1\mp\frac{1}{2}\sqrt{2}), (-\frac{3}{4}\pm\frac{1}{4}\sqrt{57}, -\frac{3}{4}\mp\frac{1}{4}\sqrt{57})$.

5. $(0, 0), (\pm\sqrt{3}, \mp\sqrt{3}), (\pm\iota, \mp\iota)$. 6. 2, 3, 4, in any order.

7. 2, 1, $\frac{1}{2}$, in any order. 8. $(3, 5), (-\frac{9}{2}, \frac{5}{2}), (\frac{3}{4}, -\frac{25}{4})$.

9. $(1, 1, 1), (-\frac{7}{5}, -\frac{7}{3}, 7)$.

10. $x, \omega x, \omega^2 x=4$ or $-\sqrt[3]{36}$; $y, \omega^2 y, \omega y=2$ or $-\sqrt[3]{162}$.

11. $(\frac{1}{2}\sqrt{5}\pm\frac{1}{2}\iota\sqrt{3}, \frac{1}{2}\sqrt{5}\mp\frac{1}{2}\iota\sqrt{3}), (\frac{1}{2}\iota\sqrt{5}\pm\frac{1}{2}\sqrt{3}, \frac{1}{2}\iota\sqrt{5}\mp\frac{1}{2}\sqrt{3})$ or $x=-y=$ a fourth root of $\frac{3}{2}$.

[Use the identity, $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$.]

12. The only solutions are $(0, a), (a, 0)$. [Remove fractions and show that $xy=0$ or a^2, and that the latter alternative is impossible.]

13. $(1, 1), \left(\dfrac{a+b\pm\sqrt{(a-b)(a+3b)}}{2a}, \dfrac{a+b\pm\sqrt{(b-a)(b+3a)}}{2b}\right)$ with any arrangement of signs.

14. $x=\frac{1}{2}\{1\pm\sqrt{(1+2b+2c-2a)}\}$, etc., with any arrangement of signs.

15. $x=\pm(a^2b^2+a^2c^2-b^2c^2)/abc$, etc., the upper or lower sign to be taken throughout.

16. $kx=a^2(b^2+c^2-a^2)$, etc., where $k^2=(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$.

17. (a, b, c) or $x=(ab-ab^2c-1)/(ab^2c^2-b^2c+b)$, etc. [Put $x=a+\lambda$, $y=b+\mu$, $z=c+\nu$.]

PAGE

165. 18. $(1, 1, 1)$ or $x=(a+b+c)/(a-b-c)$, etc.

19. $(1, 1, 1)$, $(-3, -3, -3)$, $(\pm\sqrt{2}, \pm\sqrt{2}, 1\mp\sqrt{2})$, in any order.

20. $(\frac{1}{2}, \frac{1}{6}, -\frac{1}{6})$.

21. If $A=\pm\sqrt{(b^2+c^2-a^2)}$, $B=\pm\sqrt{(c^2+a^2-b^2)}$, $C=\pm\sqrt{(a^2+b^2-c^2)}$, then $2x=B+C$, $2y=C+A$, $2z=A+B$, giving eight solutions. [Show that $2x(y-z)=b^2-c^2$.]

22. $(\pm\sqrt{ac}, \pm\sqrt{bc})$, $(\frac{1}{2}(a-b+c)\pm\sqrt{Q}, \frac{1}{2}(b+c-a)\mp\sqrt{Q})$ where
$$Q=\Sigma a^2-2\Sigma bc.$$

23. If $\Sigma bc\neq 0$, the only solutions are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. If $\Sigma bc=0$, the equations are not independent. [Show that
$$x(y-z)/bc=\ldots=\ldots.]$$

24. $(0, 0, 0)$ or $x=\pm\dfrac{b^3c^3+c^3a^3+a^3b^3-3a^2b^2c^2}{a(ca-b^2)(ab-c^2)}$, etc.

25. $(-1, -1, -1)$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $(\frac{4}{5}, \frac{3}{5}, 0)$.

26. The values of x, y, u, v are $(3, -2, \frac{1}{5}, \frac{7}{5})$ or $(-2, 3, \frac{7}{5}, \frac{3}{5})$.

27. $x=y=x'=y'=a$. 28. $(\frac{5}{4}, \frac{17}{4})$.

166. 29. $(2, 1)$, $(-1, -2)$, $(-5, 4)$.

31. $a^3+b^3+c^3=a^2(b+c)+b^2(c+a)+c^2(a+b)$.

32. (i) Subtract the second equation from the first, and divide by $x-y$; (ii) prove $(x^3+y^3)(1-xy)=2a^4-a^5$; then, from the fact that x, y are roots of $(2a-1)t^2+a(2a-1)t+a^2(a-1)=0$, obtain $(2a-1)(x^2+y^2)=a^2$, $(2a-1)(x^3+y^3)=a^3(a-2)$, and thus
$$xy=2a^2-a+1=a^2(a-1)/(2a-1)$$
from (i); for (iii), (iv), (v), multiply the result (ii) by $a+1$, and obtain $3a^4=-2a-1$, and the results follow.

34. $2x(b+c-a)=\ldots=\ldots-\Sigma a^2+2\Sigma bc$.

39. $(a^3-3ab^2+2c^3)(a^4-3a^2b^2+3b^4-4ac^3)+16d^3(a^2-b^2)^2=0$.

40. $a^4+b^4=c^2(a^2+b^2)$.

167. 42. $16:5:-21$. 46. $(bc+ca+ab)(a^2+b^2+c^2-bc-ca-ab)=0$.

EXERCISE XIX

177. 1. $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(5\pm\sqrt{21})$. 2. $(-7\pm3\sqrt{5})$, $\frac{1}{2}(1\pm\iota\sqrt{3})$.

3. $\frac{1}{2}(-1\pm\iota\sqrt{3})$, $\frac{1}{4}(-1\pm\iota\sqrt{15})$. 4. $\frac{1}{2}(\pm\sqrt{5})$, $\frac{1}{4}(-7\pm\sqrt{33})$.

5. 1, $\frac{1}{2}(-3\pm\sqrt{15})$, $\frac{1}{4}(-1\pm\iota\sqrt{15})$. 6. -1, $2\pm\sqrt{3}$, $\frac{1}{2}(1\pm\iota\sqrt{35})$.

7. -1, -1, $\pm\iota$, $\frac{1}{2}(3+\iota\sqrt{7})$.

8. -1, $\pm\iota$, $\frac{1}{8}[3+\sqrt{33}\pm\sqrt{(6\sqrt{33}-42)}]$, $\frac{1}{8}[3-\sqrt{33}\pm\iota\sqrt{(6\sqrt{33}+42)}]$.

9. -1, -1, $\frac{1}{2}(-1+\iota\sqrt{3})$. 16. $k=-3$.

EXERCISE XX

184. 1. (i) 2·68061; (ii) 0·80125; (ii) 4·22524.

2. (i) $2\cos 40°$ or 1·53209, 0·34730, −1·87939.
(ii) 0·69073, 0·11479, −6·30553.

EXERCISE XXI

PAGE

198. 9. $-1\pm\sqrt{2}$, $1\pm2\iota$.

10. $\frac{1}{4}(-1+\sqrt{5}\pm\sqrt{-2+6\sqrt{5}})$, $\frac{1}{4}(-1+\sqrt{5}\pm\iota\sqrt{2+6\sqrt{5}})$.

11. $\frac{1}{2}(1\pm\sqrt{13})$, $\frac{1}{2}(1\pm\iota\sqrt{3})$. 12. $\frac{1}{2}(1\pm\sqrt{5})$, $\frac{1}{2}(3\pm\sqrt{17})$.

13. $-2\pm\sqrt{2}$, $-4\pm2\iota$.

14. $(x^2-x+1)(x^2-3x+3)$, $\{x^2-x(2-\iota\sqrt{3})-\iota\sqrt{3}\}\{x^2-x(2+\iota\sqrt{3})+\iota\sqrt{3}\}$, $\{x^2-2x+\frac{1}{2}(3+\iota\sqrt{3})\}\{x^2-2x+\frac{1}{2}(3-\iota\sqrt{3})\}$.

199. 18. $\sqrt{5}\pm\sqrt{2+2\sqrt{5}}$, $-\sqrt{5}\pm\sqrt{2-2\sqrt{5}}$; the other substitutions are given by $q^2+2q+5=0$.

20. Roots as in Ex. 10; the other substitutions are given by $16t^2+4t-3=0$.

EXERCISE XXII

214. 7. $\sqrt{2}=\frac{1}{2}(y^3-9y)$; $\sqrt{3}=-\frac{1}{2}(y^3-11y)$; $\sqrt{2}-\sqrt{6}+3=(y+5)/(y+1)$.

8. $x^4-10x^2+1=0$; $y^3-18y-110=0$; $z^9-15z^6-117z^3-125=0$.

EXERCISE XXIV

233. 1. (i) Oscillates between 1 and 3; (ii) diverges; (iii) oscillates between -2 and $+2$; (iv) converges to 2; (v) diverges.

EXERCISE XXVI

254. 1. (i) Convergent; (ii) oscillates finitely.

255. 7. Divergent. 8. Divergent. 9. Divergent. 10. Divergent.
11. Convergent. 12. Convergent. 13. Convergent. 14. Divergent.
15. Convergent. 16. Divergent. 17. Convergent.
18. Divergent if $p-q+1\geqslant0$, convergent if $p-q+1<0$.
19. Convergent. 20. Convergent. 21. Divergent. 22. Convergent.
23. $x<1$. 24. $x\leqslant1$. 25. $x\leqslant1$. 26. $x<1$. 27. $x>1$. 28. $x<1$.
29. $x>a$. 30. $x\neq1$. 31. $x<1$. 38. $\frac{1}{12}$.
39. $\frac{11}{18}$. 40. $\frac{49}{144}$. 41. $\frac{7}{24}$.

EXERCISE XXVII

264. 10. (i) Between $\frac{1}{4}(5-\sqrt{41})$ and $\frac{1}{4}$, or between 2 and $\frac{1}{4}(5+\sqrt{41})$;
(ii) if $|x|<\frac{1}{4}(\sqrt{41}-5)$.

265. 13. $1-\iota$.

EXERCISE XXVIII

282. 1. $\frac{1}{4}$, $\frac{3}{4}$. 2. 0, $-\infty$. 3. $\frac{2}{3}$, 0. 4. 1. 5. 7/3. 6. ∞. 7. $\frac{3}{2}x^{\frac{1}{2}}$. 8. 1.

283. 15. (i) $2\{(ab'-a'b)x^2+(ac'-a'c)x+bc'-b'c\}/(a'x^2+2b'x+c')^2$;
(ii) $-(1-x)^{-\frac{1}{2}}(1+x)^{\frac{3}{2}}$; (iii) $(a^2-2x^2)/(a^2+x^2)^{\frac{5}{2}}$; (iv) $n\sin^{n-1}x\cos x$;
(v) $-n\cos^{n-1}x\sin x$; (vi) $n\tan^{n-1}x\sec^2x$.

PAGE

283. 16. (i) $-(hx+by+f)/(ax+hy+g)$; (ii) $(x^2-ay)/(ax-y^2)$.

21. $y=ax$. 24. $4/27$, when $x=1/3$.

25. 1, 7/4; the graph has a point of inflexion at (1, 0), where the gradient is zero; another at $(3/2, -1/16)$, where the gradient is $-\frac{1}{4}$; a minimum at $(7/4, -27/256)$.

284. 27. Between $x=-2$ and $x=0$, and for $x>3$; max. at $(0, -1)$; min. at $(-2, -65)$; min. at $(3, -190)$.

29. 18 cubic inches.

EXERCISE XXX

304. 3. $a=2$, $x=1, 1, 5/2$; or $a=\frac{5}{2}\frac{4}{5}$, $x=3/5, 3/5, 5/6$.

305. 18. Use $\Sigma x^9=-27\Sigma(x+3)^3$, and substitute the values of Σx^3, Σx^2, Σx.

EXERCISE XXXI

315. 2. $2e$ 3. e. 4. $e-1$. 5. $3e/2$. 6. $1-2/e$. 7. $15e$. 8. $-1/e$.

9. $\dfrac{1}{x}[(x-1)e^x+x+1]$.

EXERCISE XXXII

322. 5. (ii) 0·84510. 6. (ii) 1·04139. 7. (ii) 1·11394.

9. $-3/4m$; $+1/(4m+1)$. 10. $\log_e 2$. 11. $2\log_e 2-1$.

12. $2-\log_e 2$. 13. $\dfrac{1}{2\sqrt{x}}\log\dfrac{1+\sqrt{x}}{1-\sqrt{x}}+\dfrac{1}{2x}\log(1-x)$.

14. $\dfrac{1}{4}\left[6+\left(x-\dfrac{1}{x}\right)\log\dfrac{1+x}{1-x}\right]$.

323. 15. $c-\dfrac{b}{2}+\dfrac{1}{3}$; $-\dfrac{c}{2}+\dfrac{b}{3}-\dfrac{1}{4}$; $b=1$, $c=\dfrac{1}{6}$; $a=\dfrac{1}{2}$.

17. $1{\cdot}010299956\frac{6}{7}$; 0·3010299957.

19. $\dfrac{5}{4x}(x+2)+\dfrac{x}{1-x}+\dfrac{1}{2x^2}(5-x^2)\log(1-x)$.

20. $\dfrac{4}{9x^2}(6+3x+2x^2)+\dfrac{2x-x^2}{(1-x)^2}+\dfrac{1}{3x^3}(8-x^3)\log(1-x)$.

EXERCISE XXXIII

339. 10. Convergent for all values of θ (see Art. 11, Ex. 1).

13. (i) If $\gamma>\alpha+\beta+1$; (ii) if $\gamma>\alpha+\beta$.

EXERCISE XXXIV

349. 1. 8th term. 2. 11th term. 3. 8th term. 4. 5th term.

5. 6th and 7th terms. 6. 5th and 6th terms.

7. $\frac{4}{5}<x<\frac{6}{7}$. 8. 11th term.

350. 21. 10·0033322284; 5.10^{-13}. 22. 1·414214; 5.10^{-7}.

23. 1·5860098; 5.10^{-8}. 24. 1·70998; 3.10^{-6}.

25. 1·2431626; 4.10^{-8}. 26. 1·319508; 3.10^{-7}.

EXERCISE XXXV

PAGE

356. 10. $(1+x)^{n-2}[1+(3n+2)x+(n+1)^2x^2]$.

11. $\dfrac{(1+x)^n}{(n+1)(n+2)}\left[(n+1)^2-\dfrac{n}{x}+\dfrac{1}{x^2}\right]-\dfrac{1}{(n+1)(n+2)x^2}$.

12. $P-Q$, where $P=\dfrac{(1+x)^n}{(n+1)(n+2)(n+3)}\left[n+2+\dfrac{2n+3}{x}+\dfrac{n}{x^2}-\dfrac{1}{x^3}\right]$ and

$$Q=\frac{1}{2(n+1)x}-\frac{1}{(n+1)(n+2)(n+3)x^3}.$$

13. $P-Q$, where $Q=\dfrac{3}{2(n+1)x}-\dfrac{13}{(n+1)(n+2)(n+3)x^3}$ and

$$P=\frac{(1+x)^n}{(n+1)(n+2)(n+3)}\left[n^2+n^2-n+2-\frac{5n^2-n-9}{x}+\frac{13n}{x^2}-\frac{13}{x^3}\right].$$

EXERCISE XXXVI

359. 1. (i) $1+4x+11x^2+26x^3+57x^4+120x^5$; $2^{n+2}-n-3$.

(ii) $1+2x+2x^2-4x^3-34x^4-148x^5$; $-1+3\,.\,2^n-3^n$.

EXERCISE XXXVII

365. 1. (i) $\frac{1}{2}(1+3^{n-1})$, $\frac{1}{4}(2n-1+3^n)$;

(ii) $\dfrac{1}{2\sqrt{2}}[(1+\sqrt{2})^n-(1-\sqrt{2})^n]$, $\frac{1}{4}[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}-2]$.

366. 4. $\frac{1}{2}(20n-27+3^{3-n})$, $\frac{1}{4}(20n^2-34n+27-3^{3-n})$.

5. $\frac{1}{9}[(23-3n)2^{n-2}+(-1)^n]$. 6. $-\frac{1}{3}+\dfrac{4}{9e}+\dfrac{8e^2}{9}$.

EXERCISE XXXVIII

370. 1. 2^{n-2}.

371. 8. $u_n=\underline{\lfloor n+1}\left[u_0+u_1-2u_0\left\{\dfrac{1}{\underline{\lfloor 2}}-\dfrac{1}{\underline{\lfloor 3}}+\ldots+(-1)^{n+1}\dfrac{1}{\underline{\lfloor n+1}}\right\}\right]$

10. $A\,.\,2^n+B\,.\,3^n+\frac{1}{4}(4n+7)$. 11. $A\,.\,2^n+B\,.\,3^n+\frac{1}{20}\,.\,7^{n+2}$.

12. $A\left(\dfrac{1+\sqrt{5}}{2}\right)^n+B\left(\dfrac{1-\sqrt{5}}{2}\right)^n-(n^2+6n+13)$.

14. $u_{2n}=\dfrac{2n-1}{2n}\cdot\dfrac{2n-3}{2n-2}\cdot\ldots\cdot\dfrac{1}{2}\cdot\dfrac{a}{u_1}$; $u_{2n+1}=\dfrac{2n}{2n+1}\cdot\dfrac{2n-2}{2n-1}\cdot\ldots\cdot\dfrac{2}{3}\cdot u_1$.

EXERCISE XXXIX

384. 1. x^3+2x-1. 2. (i) $\frac{1}{3}n(4n^2-1)$; (ii) $\dfrac{n}{15}(3n^4+15n^3+25n^2+15n+2)$

EXERCISE XL

PAGE

394. 1. (i) $\frac{1}{2}, \frac{3}{8}, \frac{15}{38}, \frac{87}{222}$; (ii) $1, \frac{+2}{0}, \frac{+3}{-3}, \frac{+4}{-12}$.

3. $1+\cfrac{1}{1+}\,\cfrac{1}{1+}\,\cfrac{1}{5+}\,\cfrac{1}{3+}\,\cfrac{1}{4}$. 5. $3-\cfrac{1}{2-}\,\cfrac{1}{3-}\,\cfrac{1}{4-}\,\cfrac{1}{5}$.

6. $1+\cfrac{1}{a-2+}\,\cfrac{1}{1+}\,\cfrac{1}{2(a-1)}$; $\frac{1}{1}, \frac{a-1}{a-2}, \frac{a}{a-1}, \frac{2a^2-a-1}{2a^2-3a}$.

EXERCISE XLI

412. 8. $\frac{86}{355}=0{\cdot}242253\ldots$, $\frac{47}{194}=0{\cdot}242268\ldots$.

10. $12\frac{1}{2}$ yards; 3 inches.

EXERCISE XLII

419. 1. (i) 127, 55; (ii) 30, 13. 2. $19+27t$, $1+8t$.

3. $7+8t$, $8+27t$. 4. $9+13t$, $20+29t$.

5. $29+41t$, $12+17t$. 6. $24+7t$, $11+12t$.

7. $155+225t$, $137+199t$. 8. 6, 1. 9. 1, 20; 8, 9.

10. 39, 8; 10, 20. 11. $\frac{3}{7}+\frac{5}{11}$. 12. $\frac{4}{17}-\frac{3}{13}$; $\frac{10}{13}-\frac{13}{17}$.

13. 3, 4; 13, 10. 14. 57, 43.

420. 17. 67. 18. 13, $x=205-17t$; 12, $x=201-17t$.

20. 14, 2, 3; 1, 13, 8. 21. 11, 3, 7.

22. 2, 1, 3; 3, 2, 2; 4, 3, 1. 23. 2 . 3 . 7.

24. 1149; $1149+1540t$.

25.

2d.	3d.	9d.	1d.
3	2	1	2
3	3	2	1
3	6	1	1
6	4	1	1
6	1	2	1
9	2	1	1

26.

2s. 6d.	1s.	6d.	1d.
2	2	5	1
2	1	6	7
1	6	1	7
1	7	0	1

27. One; £12 12s. 8d.

EXERCISE XLIII

428. 12. 2, 9; 3, 6; 4, 13; 5, 7; 8, 15; 10, 12; 11, 14.

EXERCISE XLIV

434. 1. (i) $9+13t$; (ii) $75+77t$; (iii) $379+770t$.

2. (i) $149+179t$; (ii) $30+179t$.

3. (i) $78+1217t$; (ii) $1800+1861t$; (iii) $540+1009t$; (iv) $1683+1901t$.

435. 8. $\frac{7}{32}+\frac{3}{8}+\frac{2}{11}-1$. 9. $\frac{5}{7}+\frac{4}{11}+\frac{12}{13}-2$. 10. $30k+1$.

11. $\pm 17+77k$. 12. $3+23t$, $-7+23t$.

PAGE

435. 13. $\pm 13+66t$, $\pm 9+66t$. 14. $1735+3465t$, etc.
15. 2, 4; $x^2+1\equiv(\text{mod } 7)$ has no solution.
16. ± 2, -3, -4. 17. 3.

EXERCISE XLV

442. 1. 1, 10, 5, 12, etc., i.e. those of 19 in reverse order; 1, 7, 11, a period of **3**, since 7 is the 6th number after 1, and 6 is a factor of $(19-1)$; 2^1, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, or 2, 13, 14, 15, 3, 10.

2. 1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6; 3^{2n+1}, i.e. every other number in the preceding, since $2n+1$ is prime to $17-1$, or 3, 10, 5, 11, 14, 7, 12, 6.

444. 12. 99999 is divisible by 41, hence recurring periods contain 5 digits; 2, 20, 36, 32, 37, the remainders in the division of 2 by 41.

14. If $n=41$, $n'=271$; $n=123$, $n'=903342366757$.

16. Because $p-1$ occurs as a remainder.

EXERCISE XLVI

445. 1. 1, 7, 11. 2. 15. 3. 5. 4. 25.
5. 8, 33. 6. 2, 10, 19. 7. $19k\pm 9$, $19k^2\pm 18k+4$.
8. (25, 214), (54, 2157), (67, 4120).

EXERCISE XLVII

447. 1. 15. 2. 10. 3. 4. 4. 4. 5. 5.

EXERCISE XLVIII

456. 2. $(-2, -1)$, (1, 2), (3, 4). 3. $(-8, -9)$, (13, 14), (14, 15).
4. $(-6, -7)$ and two in (0, 1). 5. $(-9, -10)$ and two in (3, 4).
6. $(-3, -4)$ and two in (5, 6). 7. (2, 3), (3, 4).
8. Two in (1, 2). 9. $(-2, -1)$, (0, 1) and two in (1, 2).
10. $(-6, -5)$, $(-1, 0)$, (4, 5). 11. One root in (1, 2).

EXERCISE XLIX

466. 1. 1·3569, 1·6920. 2. 1·7838. 3. 2·6306166.
4. 16·0428539. 5. 0·3472964, 1·5320862, -1·8793826.
6. 0·1147994, 0·6907309. 7. 13·8440609, 14·2895592.
8. 5·2100150, 5·2973245. 9. 17·7459667, -1·9216606.
10. 1·2783089, 1·5511616. 11. 1·4007219, 1·5823564.
12. 1·0156820, 1·52858586. 13. 3·4334634, 3·6617081.
14. 1·7320508, 1·8666161. 15. 4·4707625, 4·7814825.
16. 2·59861938, 2·76190631. 17. 2·1010205, 2·1084952.
18. 7·3355540.
19. (i) 3·3548487; (ii) 4·4641016; (iii) 0·6386058.

EXERCISE LIII

PAGE

504. 1. (i) 6720; (ii) 480. 2. $\underline{|n-1}/\underline{|r}$. 7. $C_a^{m+a-1} \cdot C_b^{m+b-1} \cdot C_c^{m+c-1}$.
9. 1024. 10. 116280.

505. 13. (i) 44; (ii) 20. 17. (i) 1895040; (ii) 145680.
20. $\underline{|n} \cdot \underline{|n}$. 22. (i) 1771; (ii) 969; (iii) 885; (iv) 165; (v) 552.

506. 27. (i) 57; (ii) 47; (iii) 15. 28. 23. 29. 42. 30. 15.

EXERCISE LIV

506. 1. $C_3^n - C_3^p$.

EXERCISE LV

523. 1. (i) 6/55; (ii) 4/11. 3. 14/33. 4. 5/33.
6. (i) 1/36; (ii) 5/12; (iii) 5/9. 7. 1/8.
8. (i) 1/12; (ii) 125/1296; (iii) 155/648. 9. 25.
10. 236/270725. 12. (i) 1/7; (ii) 2/7.
13. (i) $2/n$; (ii) $2(n-m-1)/n(n-1)$; (iii) $(m+1)(2n-m-2)/n(n-1)$.

524. 15. $1/2n$. 18. (i) 20/27, (ii) 496/729.
19. (i) $1/n$; (ii) $(n-1)/2n$; (iii) $(n-1)2n$. 21. $(n+1)(3n+2)$ pence.
22. (i) 2162/54145; (ii) 2257/54145.
23. (i) $\Sigma p_1p_2(1-p_3)(1-p_4) = \Sigma p_1p_2 - 3\Sigma p_1p_2p_3 + 6p_1p_2p_3p_4$;
(ii) $\Sigma p_1p_2 - 2\Sigma p_1p_2p_3 + 3p_1p_2p_3p_4$.
24. 73/648.

525. 25. $(0{\cdot}55)^7$.
26. (i) $p = \underline{|3} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49} = \frac{2197}{20825}$; (ii) $p' = \underline{|4} \cdot \frac{12}{51} \cdot \frac{13}{50} \cdot \frac{13}{49} \cdot \frac{13}{48} = p$;
(iii) $\frac{4}{13}p + \frac{9}{13}p' = p$.
27. (i) $p = 3 \cdot \frac{25}{51} \cdot \frac{26}{50} \cdot \frac{25}{49} = \frac{325}{833}$; (ii) $p' = 6 \cdot \frac{25}{51} \cdot \frac{24}{50} \cdot \frac{26}{49} \cdot \frac{25}{48} = p$;
(iii) $\frac{4}{13}p + \frac{9}{13}p' = p$.
29. (i) $6/n(n-1)$; (ii) $6(n-3)/n(n-1)$; (iii) $(n-3)(n-4)/n(n-1)$.
33. (i) 1/6; (ii) 7/9. 34. (i) 1/4; (ii) 2/3.

526. 35. 1/2.
36. (i) $pp'p''$ to $(1-p)(1-p')(1-p'')$; (ii) $pp'(1-p'')$ to $(1-p)(1-p')p''$.

EXERCISE LVI

541. 5. (i) (2, 1), (50, 31); (ii) (3, 2), (29, 18); (iii) (5, 2), (12, 5).
6. (i) (25, 1), (623, 25), ...; (7775, 312), ...

542. 8. (i) $[\frac{1}{2}a(a-1)+1, \frac{1}{2}(a-1)]$; (ii) $[\frac{1}{2}a(a+1)+1, \frac{1}{2}(a+1)]$.

MISCELLANEOUS EXERCISES (A)

2. n^2 or n^2-1 and n^2+1. 4. (iii) $\frac{1}{1}-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{408}$; $\frac{1}{2}-\frac{1}{34}+\frac{1}{884}-\frac{1}{46852}$.

5. $a+c-b-d$, $a+d-b-c$. 6. $x=-\frac{1}{2}$. 8. $36x^4-156x^2+25=0$, $\sqrt{\frac{2}{3}}\pm\sqrt{\frac{3}{2}}$.

10. $\sqrt{cx}+\sqrt{dy}+\sqrt{az}+\sqrt{bt}=0$, $\sqrt{dx}+\sqrt{cy}+\sqrt{bz}+\sqrt{at}=0$.

11. $y^3-3y^2-y+19=0$. 12. $\pm\frac{\iota}{\sqrt{3}}$, $\frac{1\pm\sqrt{-15}}{4}$. 19. $\Sigma ab(\Sigma ab-\Sigma a^2)$.

21. $ab=c+1$. 23. $x=\frac{1}{2}(n+1)$ or $\frac{1}{2}\{n+1\pm\sqrt{(1-n^2)}\}$.

26. $a+b\neq 0$ and $c_1(a-b)+c_2-c_3=2a-b$.

29. (i) Convergent if $|x|<1$ or if $x=-1$. Divergent if $|x|>1$. When $x=1$ the series is convergent if $p>1$, divergent if $p\leqslant 1$.
(ii) Convergent if $p>\frac{1}{2}$, divergent if $p\leqslant\frac{1}{2}$.

30. (i) $\frac{1}{2}(1+3^{n-1})$; (ii) $\frac{\sqrt{2}}{4}\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\}$.

31. $d^6(a^3+b^3+c^3+2d^3)=a^3b^3c^3$.

33. $-\dfrac{a^2(a_2-a_3)(a_3-a_1)(a_1-a_2)}{\Pi a_i \Pi(a-a_i)^2}$.

34. If $a>9$, none real, $-7\leqslant a\leqslant 9$, four real, $a<-7$ two real.

35. $46x=-24+15\,.\,4^{\frac{1}{3}}k+10\,.\,2^{\frac{1}{3}}k^2$ where $k=1$, ω or ω^2.

37. $x=\frac{3}{4}$, $y=\frac{1}{2}$, $z=\frac{1}{4}$.

39. $x=-2\pm\sqrt{2}$ or $\frac{1}{2}(5\pm\sqrt{17})$. 43. $206+210n$.

49. Equate the factors to $P+Q$, $P+\omega Q$, $P+\omega^2 Q$.

53. $\dfrac{x-4}{x^2}\log(1-x)-\dfrac{4}{x}$. 54. Convergent unless $x=\pm 1$. 56. $0<y<4(b-a)$.

59. $a=1$, $b=0$. [$y=0$, $x=1$, $x=4$, are asymptotes; y is a minimum at $(-2, -\frac{1}{9})$.]

63. $x=\dfrac{b^2+c^2-ab-ac}{2\sqrt{(\Sigma a^3-3abc)}}$, etc. 66. 64° 50′ nearly. 67. $1+\dfrac{x}{2}-\dfrac{x^2}{12}+\dfrac{x^3}{24}$.

71. 32/9. 74. $-1+\epsilon-\dfrac{\epsilon^2}{4}$, $-1-\epsilon-\dfrac{\epsilon^2}{4}$.

76. (i) Since $\left(1+\dfrac{1}{m}\right)^m<e<3$, $\therefore\ \dfrac{2}{1}\cdot\left(\dfrac{3}{2}\right)^2\cdot\left(\dfrac{4}{3}\right)^3\cdots\left(\dfrac{n}{n-1}\right)^{n-1}<3^{n-1}$.

(ii) Consider the arithmetic and geometric means of the factors of

$$1^1\,.\,2^2\,.\,3^3\,.\,\ldots\,.\,n^n$$

when written at full length thus $1\,.\,2\,.\,2\,.\,3\,.\,3\,.\,3\,\ldots$.

79. $\dfrac{6(x^2-12x+4)}{x^2-20x+4}$. 83. 31.

85. Minimum values when (i) $x=a$, (ii) $x=\frac{1}{2}(a+b)$.

87. $\frac{27}{64}$ is a maximum value; 0 and $-\frac{128}{729}$ are minimum values.

91. 0, 2, ∞. 96. $1+\frac{\alpha}{2}+\frac{3\alpha^2}{8}$, $2-\alpha$, $3+\frac{\alpha}{2}-\frac{3\alpha^2}{8}$.

99. Put $p=1/m$, $q=1/n$. 102. 9·46, $-10\cdot47$. 108. 4^4, 6^4, $(5\pm\sqrt{-151})^4$.

110. $-4\cdot61015$. 112. $-0\cdot68233$.

123. Take $1+x+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^n}{\lfloor n}=e^x-\left(\frac{x^{n+1}}{\lfloor n+1}+\ldots\right)$.

MISCELLANEOUS EXERCISES (B)

3. (i) $\frac{2}{3}+\frac{1}{3.5}+\frac{4}{3.5.7}+\frac{6}{3.5.7.9}$; (ii) $q_n=n$;
(iii) $q_1=0$, $q_{4n+2}=3+6n$, $q_{4n+3}=5+6n$,
$q_{4n+4}=q_{4n+5}=2+2n$, for $n=0, 1, 2, \ldots$.

6. $\frac{1}{12}n(n+1)^2(n+2)$.

15. $3-3^{\frac{1}{3}}k-3^{\frac{2}{3}}k^2$ and $3^{\frac{1}{3}}k$, where k has the values 1, ω, ω^2.

21. n is of the form $6k+1$.

22. The values of $-a$, b, c are $p^2\frac{(q-1)(r-1)}{(p-q)(p-r)}$ and similar expressions.

23. 1, $e^{-\frac{1}{2}}$, 0. 29. 16 and 0.

31. $2+8\epsilon+32\epsilon^2$ and $\frac{1}{\epsilon}-3-7\epsilon$.

33. (i) Convergent if $|x|<4$, divergent if $x\geqslant 4$, oscillates if $x=-4$.
(ii) Oscillates between finite limits.

35. $D=\Pi(a_1+a_2\omega+a_3\omega^2+a_4\omega^3+a_5\omega^4)^2$, where $\omega^5=1$.

36. Put $x=a+h$, $y=a+k$, $z=a+l$, where h, k and $l\to 0$.

41. 5·674619. 44. $(\log x)^2$, $2^{\log x}$, x^2, $x^{\log x}$, 2^x, e^x, x^x.

48. $x=\frac{(\lambda-b)(\lambda-c)(\lambda-d)}{(a-b)(a-c)(a-d)}$, etc.

58. -1. 59. $(\log a-\log b)/(\log c-\log d)$.

64. Let α, β, $(\alpha<\beta)$ be the roots of $\Sigma l(x-b)(x-c)=0$. (i) If l, m, n are odd, $x=\alpha$ gives a max. and $x=\beta$ a min. value. (ii) If l, m, n are even, $x=\alpha$ and $x=\beta$ give max. values, and $x=a$, b, c give min. values, each zero.

65. $x=\frac{c-a}{1+ca}$, $y=\frac{c-b}{1+cb}$, $z=\frac{a+b-c+abc}{1+bc+ca-ab}$.
Put $x=\tan X$, $y=\tan Y$, $z=\tan Z$.

66. $\frac{1}{9}(8e^2+4e^{-1})$. The sum $=\Sigma_1^\infty\frac{2^{n+2}+(-1)^n(3n-4)}{9\lfloor n-1}$.

67. Converges absolutely if $|z|<\frac{1}{4}$, diverges if $|z|>\frac{1}{4}$. If $|z|=\frac{1}{4}$, it converges, but not absolutely.

68. $\dfrac{x-k}{(a'-k)(b'-k)}=\dfrac{y-k}{(a-k)(b-k)}=\dfrac{z-k}{(a'-k)(b-k)}$.

69. If $\lambda>2$, the roots are all real; if $0<\lambda<2$, two are real and two imaginary; if $\lambda<0$, all are imaginary; equal roots if $\lambda=1$ or 2.

71. 0·8489892.

73. For the first part, use Chap. XIV, 15, p. 223, putting $m=-1$. For the second part, show that the sum of the series $<\dfrac{n}{3n+1}+\dfrac{n+1}{4n+1}<\dfrac{2}{3}$.

75. $(\log x)/x$ increases as x increases from 0 to e and decreases as x increases from e to ∞.

(i) One root if $0<n<1$; two roots if $1<n<e^{\frac{1}{e}}$; no root if $n>e^{\frac{1}{e}}$.

(ii) $n^x>x^n$ if $x<n<e$, or if $e<n<x$.

A rough graph of $(\log x)/x$ should be drawn.

84. Consider the number of ways in which 4 digits can be chosen so that their sum is 17 or 28.

85. This is so if $n\left(1-\dfrac{n-1}{n(m+1)}\right)^{m+1}>1$; then use Chap. XIV, 11.

91. Convergent; use Euler's constant, p. 311, and comparison test (iii), p. 250.

95. If $k^2<4$, all the roots are imaginary; if $k^2\geqslant 4$ two are real and two imaginary.

If $$k=\sqrt{2},\quad x=-\frac{\iota}{2}\pm\frac{3^{\frac{1}{4}}}{\sqrt{2}}\left(\iota\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\text{ or}$$

$$\frac{\iota}{2}\pm\frac{3^{\frac{1}{4}}}{\sqrt{2}}\left(\iota\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\text{ where }\tan\theta=\sqrt{2}.$$

INDEX

The numbers refer to pages.

Abel's inequality, 330
  test for convergence, 331
  theorem on multiplication of series, 338
Absolute error, 15
  least residues, 421
  value, 13
Absolutely convergent series, 256, 261
Aggregate, 14, 211
Alternating functions, 45
Amplitude of complex number, 56
Argand diagram, 56, 77
  product and quotient, 69, 70
Arithmetic mean, 221
Arithmetical progression, residues of terms, 8
Approximations, $\sqrt[n]{A}$, 15-17
  to roots of equations, 456-461
  to sums of series, 260
Approximate value as a fraction of $\sqrt[n]{(1+x)}$ (Newton), 347, 348
  of $\log_e(1+x)$, 323
  of $\log_e(a/b)$ (Napier), 323
Archimedes' axiom (Eudoxus), 13
Associated residues, 425
Associative law, 13
Asymptote, 473

Bernoulli's numbers, 113, 114, 378
  theorem, 115
Bessel's interpolation formula, 381
Bicycle-gear as revolution counter, 435
Bilinear substitution, 80
Binomial equations, 170, 171
  solution of $x^{17}-1$, 173
Binomial series, 263, 328
  for complex variable, 334
Binomial theorem, 34, 294
  general statement, 340
  elementary proof, 342
  Euler's proof, 341
Biquadratic equation, 186-199
  condition for four real roots, 192, 453
  Euler's solution, 190
  Ferrari's and Descartes' solutions, 189-192
  functions I, J, and the discriminant Δ, 187, 188
  reducing cubic, 187-191
  transformation into reciprocal form, 194
  Tschirnhausen's transformation, 195
Bordered determinant, 138
Bounds of a function, 286
  of a sequence, 237
Brackets, introduction and removal of, 258

Cardan's solution of cubic, 180
Cauchy's condensation test, 325
  test for convergence, 253
Change of order of terms, 249, 258
Cofactors of determinant, 123
Combinations and permutations, 493
Commutative law, 13
Comparison tests for convergence, 250
Complex functions of a real variable, 279
Complex numbers, 52 *et seq.*
  modulus and amplitude, 56
  product and quotient, 69
  use as operator, 76
Complex sequences, 243
Complex variable, 291, 334
  in binomial series, 328
  limits and continuity, 292
  roots of equations, 66
Composite numbers, 4
Conjugate numbers, 60
  partitions, 402
Congruences, linear, 428
  reduction of, 433
  $x^a \equiv b \pmod{p}$, 445
Continued fractions, 389
  approximations by, 402
  equivalent surds, 401, 527
  calculation of convergents, 535
Continued product, 33
Continuous functions, 269-271
  variable, 266
Continuum, 212
Convergence, general principle of, 235, 240, 244, 257
Convergence of infinite products, 486
Convergence of sequences, 231, 235, 244, 257
Convergence of series, 247 *et seq.*
  binomial series, 263, 328, 334

comparison tests, 250
hypergeometric series, 328
(for other tests see under the respective proper names)
Cube roots of unity, 65
Cubic as sum of two cubes, 182
Cubic equation, 85
auxiliary quadratic, 182, 184
Cardan's solution, 180
character of roots, 180
functions G, H, and the discriminant Δ, 179
irreducible case, 180
roots as power series, 469, 472
Curve tracing, 474 *et seq.*
Cusp, 474
Cyclic expressions, 46

D'Alembert's test for convergence, 252
Dedekind, definition of irrational, 201
theorem on section of system of real numbers, 211
De Gua's rule, 90
De Moivre's theorem, 59, 63
De Morgan's and Bertrand's test for convergence, 330
Derivatives of polynomials, 273
of a^x, $\log x$, x^n, 309
Descartes' rule of signs, 88
Determinants, 119 *et seq.*
bordered, 138
expansion, 132, 137
minors and cofactors, 193
product of two, 134
reciprocal, 136
symmetric, 138-140
use of remainder theorem, 126
Differences, method of, 106
Differentiation, 273, 276
Dirichlet's test for convergence, 331
Discontinuous functions, 269
Discriminant of biquadratic, 188
of cubic, 179
of quadratic in x, y, 30
Displacements and vectors, 70
Distributions, 494, 496
Distributive law, 13
Division (mod n), 432
Division transformation, 24
Divisors, number and sum of, 6
Double points on curves, 473
Double roots of equations, 42

Elements of set, 14
Endless decimals, 16, 203
Equation of a plane, 154
Equations, cubic, biquadratic, 85
equivalent systems, 149
numerical, 447
rational roots, 92
symmetric functions of the roots, 95
Errors in excess and defect, 15
Exclusive events, 509
Expansion of $f(x+h)$, 35, 290
Exponential curve, 309
function and series, 19, 306, 312
inequalities and limits, 232, 306, 310
values of sine and cosine, 313
Exponential theorem, 307
Expectation, 514
Eudoxus, theorem of, 13
Euler's constant, 311
solution of biquadratic, 199
theorem on polynomials, 301

Factorial n, powers of primes in, 9
Factors of $\cos m\phi - \cos n\theta$, 68
of $x^n \pm 1$, 64
of $x^{2n} - 2x^n \cos\theta + 1$, 68
Fermat's theorem, 424, 433
Euler's extension, 425, 437
Ferrari's solution of biquadratic, 189
Finite difference equations, 363, 367
Fourier's theorem on position of roots of an equation, 454
rule in connection with Newton's method, 457
Function of a function, 271, 278
Fundamental laws of arithmetic, 13, 53, 54
of order, 12

Gauss's test of convergence, 327
General principle of convergence, 235, 240, 244, 257
Generating function, 362
Geometric series, 248, 262
Gradient of tangent, 274
Greatest term in binomial expansion, 345

Hessian of cubic, 183
Higher derivatives, 280
Highest common factor, 40
Horner's method, 461
Hyperbolic functions, 319
Hypergeometric series, 328

Imaginary roots of equations, 65, 82
number of, 89, 90
Implicit functions, 467
Independent events, 511
Indeterminate equations, 415
Indetermination, limits of, 239
Induction, method of, 9
Inequalities, fundamental, 216, 219
A.M. $\geq$ G.M., 221
exponential, 239
logarithmic, 306, 310
Infinite continued fractions, 390
Infinite products, 485
derangement of factors, 488
expansion as series, 488

Infinite sequences, 15
Inflexion, points of, 282, 473
  line of inflexions, 484
Integration, 289
Interdependent events, 512
Intermediate convergents, 402
Interpolation, Lagrange's formula, 379
Interval, 210
  of convergence, 333
Inverse functions in general, 83, 284
  circular functions, 285
  hyperbolic functions, 321
Inverse probability, 518, 528
Inversions, 48
Irrational numbers, 12, 17
  Dedekind's definition, 201
Irrational indices, 209

Kummer's test for convergence, 306

Lagrange's interpolation formula, 379
  theorem on divisibility, 426, 433
Least positive residues, 421
Leibniz' theorem on differentiation of product, 383
Limits, exponential and logarithmic, 232, 306, 310
  of a function, 228, 229
  of indetermination, 239
  of sequence, 15, 16
  to roots of equation, 90
Line at infinity, 152
Linear congruences, 429
Linear difference equations, 369
Logarithmic curve, 309
  function and series, 306, 310-316
Logarithms, 210, 317

Maclaurin's theorem, 291
Magnification, 80
Matrices, 135
Maxima and minima, by differentiation, 280
  by inequalities, 223
Mean value theorem, 288
Minors of determinants, 123, 132
Modulus of complex number, 56-59
Monotone sequences, 230
Multinomial theorem, positive integral index, 35-37
  real index, 354
Multiple roots of equations, 82
Multiplication of series, 335-338
  Abel's theorem, 338
  $E(x) \times E(y) = E(x+y)$, 336
  Merten's theorem, 337

Newton's approximation to $\sqrt[n]{1+x}$, 348
  approximation to roots of an equation, 456
  method of divisors, 92
  parallelogram, 479
  sums of powers of roots of an equation, 297
  upper limits to roots, 446
Node, 473

Operators Δ, E, D, 373, 378, 382
  $\cos\theta + \iota \sin\theta$, or $E(\iota\theta)$, 75
Order and weight of functions, 298

Partial derivatives, 299
Partial fractions, 100, 302
Partition of numbers, 500
Perfect numbers, 11
Pfaffian, 141
Polygon of five sides, 178
  of seventeen sides, 175
Power-series, 332
  criterion for identity, 334
Prime and composite functions, 42
Primitive roots, 438, 444
Principal value of amplitude, 56
  of inverse circular functions, 285
  of inverse hyperbolic functions, 321
Principal nth root, 207
Pringsheim's theorem on convergence, 257
Probability, 508
  of causes, 518
  geometrical methods, 522, 526
Product of consecutive integers, 7
  Lagrange's theorem, 426, 433

Quadratic function of x, y, 29

Raabe's test for convergence, 326
Radius of convergence, 333
Ratio and proportion, 212
Rational fractions, 100, 357
  integral functions, 23
  roots of equations, 92
Real numbers, 18, 201
Reciprocal equations, 168
  determinants, 136
Rectangular arrays, 135
Recurring continued fractions, 392, 409
  decimals, 439
  series, 360
Relative error, 15
Remainder theorem, 26
  use in determinants, 126
Residues, associated, 425
  of terms of A.P., 8
  of terms of G.P., 436
Reversion of series, 467
Rolle's theorem, 287
  deductions from, 295
Roots of equations, number of, 81
  position and character of, 87, 448, 454
Roots, of complex numbers, 62
  of ± 1, 64
  of congruences, 428
Rule of Sarrus, 128

Scalar quantities, 71
Scale of relation, 360
Section of rational system, 200
  of real system, 211
Sequence, definition of, 15
  bounds, upper and lower limits, 237, 239
  general principle of convergence, 235, 244
  monotone, 230
Series, approximation to sum, 260
  of complex terms, 261
  summation of, 106
  Taylor's, Maclaurin's, 290
Series for $\cos\theta$, $\sin\theta$, 313
  for $\cos n\theta$, $\sin n\theta$, 61, 360
  for $\cos^n\theta$, $\sin^n\theta$, 62
  for $\log_e 2$, 312
  for $\tan^{-1}x$, 294
Series summable by the binomial theorem, 351
  exponential series, 314
  logarithmic series, 317
Σn^r, in terms of σ, σ' where $\sigma = n(n+1)$, $\sigma' = 2n+1$, 110-112
$\Sigma u_n x^n$, where u_n is a polynomial in n, 110
$\Sigma(a+nb)(a+\overline{n-1}b)\ldots(a+\overline{n+r-1}b)$, 106
$\Sigma 1/(a+nb)\ldots(a+n+r-1b)$, 106
$\Sigma \dfrac{a(a+1)\ldots(a+r-1)}{b(b+1)\ldots(b+r-1)}$, 110
Simple continued fractions, 391
Smallness, orders of, 14
Skew-symmetric determinants, 138, 140
Special roots of $x^n - 1$, 171
Stirling's approximation to $\lfloor\underline{n}$, 515
Sturm's theorem, 448, 452
Substitutions, 47
Successive events, 515
Sums of powers of roots, 297
  of two squares, 407, 408
Surds, 208
  expressed as continued fractions, 401, 427, 435
Symbols, $I(x/y)$, $\lfloor\underline{n}$, P_r^n, C_r^n, 7, 8
  $O(x)$, 14
  ι as an operator, 75
  $f^{-1}(x)$, 83
Symmetric continued fractions, 406
  determinants, 138
Symmetric functions, 44
  of roots of equations, 95
  order and weight of, 298
Synthetic division, 24, 84
Systems of equations, linear, 149
  of any degree, 160

Tangent to a curve, 274
Tangents and asymptotes, 473
Taylor's theorem and series, 290
  for polynomials, 300
Testimony, value of, 521
Test-series for convergence $\Sigma 1/n^p$, 251; $\Sigma 1/n(\log n)^p$, 325
Tetrahedron, volume of, 159
Three planes, 156
Transformation, $Z = (az+b)/(cz+d)$, 80
Transformation of equations, 82-84
Transpositions, 44
Tschirnhausen's transformation for cubic, 186
  for biquadratic, 195

Undetermined coefficients, 28

Vandermonde's theorem, 340
Vectors, 70-76

Weierstrass' inequalities, 216
Whitworth's theorem, 499
Wilson's theorem, 425, 433