UNIVERSAL
LIBRARY

OU_166091

UNIVERSAL
LIBRARY

OSMANIA UNIVERSITY LIBRARY

Call No 510 C73M. Accession No 22483

Author Commission on Secondary School Education

Title Mathematics in General Educati

This book should be returned on or before the date last marked below.

PROGRESSIVE EDUCATION ASSOCIATION PUBLICATIONS

COMMISSION ON SECONDARY SCHOOL CURRICULUM

MATHEMATICS IN GENERAL EDUCATION

COPYRIGHT, 1940, BY
D. APPLETON-CENTURY COMPANY, INC.

All rights reserved. This book, or parts thereof, must not be reproduced in any form without permission of the publisher.

450

Copyright, 1938 by the Progressive Education Association

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

Mathematics in General Education is the Report of the Committee on the Function of Mathematics in General Education of the Commission on Secondary School Curriculum. This Commission was established by the Executive Board of the Progressive Education Association in 1932, and charged with the task of examining the fundamental problems of general education at the secondary level (junior and senior high school and junior college).

The proposals of the Commission are designed to apply to the education of all adolescents (ranging in age, approximately, from twelve to twenty) whether or not they plan to go to college. Although they provide a sound basis of preparation for college and for professional school, they are intended to contribute primarily to the purpose of helping the student achieve a socially adequate and personally satisfying life.

On the assumption that the processes and goals of such education must be relevant to the needs of the learner as he interacts with his social medium, the Commission established, first, a Study of Adolescents to provide essential information on the problems, interests, and inclinations of young people. The insight into adolescent development contributed by this Study has profoundly influenced the work of the Commission as a whole.

Second, a series of Committees was established in each of a number of areas of instruction in the secondary school—art, English, science, social studies.[1] Originally the Committee on the Function of Science in General Education included teachers of mathematics, but later a separate Committee on the Function of Mathematics in General Education was organized. All of these Committees assumed the responsibility of exploring the contribution of their

[1] For a discussion of the reasons why attention was focused on the resources of the usual subject-matter areas see: V. T. Thayer, Caroline B. Zachry, and Ruth Kotinsky, *Reorganizing Secondary Education* (New York, D. Appleton-Century Co., 1939), pp. 421–428. This volume may be consulted for a more detailed discussion of the origin and work of the Commission as a whole, and a statement of its fundamental educational position.

particular fields to meeting the needs of young people in the democratic society of America today. In addition, each Committee undertook to implement its point of view with a series of suggestions to teachers.

Mathematics in General Education thus constitutes one of a series of publications to result from the work of the Commission, its Committees in the various areas of instruction, and the Study of Adolescents. It examines the study and teaching of mathematics for their values in relation to the whole process of general education. To this end the personnel of the Committee included students of educational theory and practice as well as teachers of mathematics in secondary schools and colleges. Though complete agreement on all specifics among the members of such a group is perhaps impossible to achieve, especially when their common task involves ploughing new ground, all the members of this Committee are in complete accord on the basic principles and fundamental lines of suggestion presented in the Report. Their differences have been of a kind to lend a balance and perspective which could not otherwise have been attainable.

Prior to issuing this final Report, the Committee submitted tentative formulations for criticisms and suggestions to numerous groups of mathematics teachers drawn from different geographical regions and working under widely varying conditions. It is particularly indebted for the criticism, suggestions, and contributions submitted by mathematics teachers attending the Workshops of the Progressive Education Association.

It is difficult if not impossible adequately to express appreciation to all individuals who have contributed to the Report. The Committee is particularly indebted to Dr. Peter Blos, of the Study of Adolescents, who prepared the chapter on "Understanding the Student." Dr. Blos gave generously of his time and attention to make explicit for the Committee much of the background which influenced their deliberations.

Special acknowledgment is also due to George A. Boyce, formerly of Bronxville, N. Y., Public Schools; Arnold Dresden, of Swarthmore College; J. A. Lauwerys, of the Institute of Education in London; Julius H. Hlavaty, of the High School of Science, New York City; I. A. Richards, of Harvard University; M. F. Rosskopf, of John Burroughs School, St. Louis, Missouri; Vera Sanford, of

the Oneonta, N. Y., State Normal School; Raleigh Schorling of the University of Michigan; Caroline B. Zachry, of the Study of Adolescents, Commission on Secondary School Curriculum, Progressive Education Association; and Louis Zahner, of Groton School. In expressing its indebtedness, the Committee in no way wishes to imply that these persons are in complete accord with the proposals of this Report; these proposals are the responsibility of the undersigned.

ALBERT A. BENNETT, *Professor of Mathematics, Brown University.*

CUTHBERT DANIEL, *Editorial Consultant, Radio Research Project, Princeton University.*

HAROLD FAWCETT, *Associate Director and Teacher of Mathematics, Ohio State University School.*

MAURICE L. HARTUNG, *Assistant Professor of the Teaching of Mathematics, University of Chicago.*

ROBERT J. HAVIGHURST, *Director for General Education, General Education Board.*

JOSEPH JABLONOWER, *Member Board of Examiners, New York City, and formerly Head of the Department of Mathematics, Fieldston School.*

RUTH KOTINSKY, *Secretary of the Commission on Secondary School Curriculum, Progressive Education Association.*

V. T. THAYER, *Chairman of the Commission on Secondary School Curriculum, Progressive Education Association, and Educational Director of the Ethical Culture Schools.*

CONTENTS

PAGE

PREFACE v

PART I

THE TEACHING OF MATHEMATICS IN RELATION TO GENERAL EDUCATION

CHAPTER

I. INTRODUCTION 3
Reasons for a Reëxamination of the Aims and Content of Mathematics Instruction 3
The Persistence of Disciplinary, Practical, and Cultural Aims: An Historical Sketch 3
Changing Social and Economic Conditions Affecting Secondary Education 6
New Psychological Theories Affecting Secondary Education 8
A Loss of Confidence in the Educational Values of Mathematics 9
A Brief Description of the Report 12
To Whom This Report Is Addressed 14

II. MATHEMATICS IN RELATION TO THE PURPOSES OF GENERAL EDUCATION 16
Needs as the Basis of Educational Planning 17
The Concept of Educational Needs 17
Classification of Educational Needs 20
Personal Living 20
Immediate Personal-Social Relationships 21
Social-Civic Relationships 22
Economic Relationships. 22
The Major Ideals of Democracy as the Basis of Educational Values 24
Optimum Development of Personality 27
Reciprocal Individual and Group Responsibility for Promoting Common Concerns 28
The Free Play of Intelligence 29
Personal Characteristics Essential to Democratic Living . . 31

CHAPTER PAGE

Social Sensitivity 32
Esthetic Appreciation 33
Tolerance 35
Coöperativeness 36
Self-Direction 36
Creativeness 37
The Disposition and Ability to Use Reflective Thinking in the Analysis and Solution of Problem Situations . . 39
Summary Formulation of the Purpose of General Education 42
The Rôle of Mathematics in Achieving the Purpose of General Education 43
The Rôle of Mathematics in Meeting the Needs of Students 44
The Rôle of Mathematics in Developing Personal Characteristics Essential to Democratic Living 48
Social Sensitivity 48
Esthetic Appreciation 49
Tolerance, Coöperativeness, Self-Direction, and Creativeness 50
The Disposition and Ability to Use Reflective Thinking in the Solution of Problems 52

PART II

MAJOR UNDERSTANDINGS GROWING OUT OF MATHEMATICAL EXPERIENCE

III. Introduction to Part II: Some Concepts Basic to Problem-Solving, Crucial in Democracy, and Pervasive in Mathematics 59
The Relation of These Concepts to the Problem-Solving Process 59
The Relation of These Concepts to Democracy 63
The Relation of These Concepts to the Development of an Understanding of Mathematics as a Unified Field . . . 69
The Relation of These Concepts to the Organization of the Curriculum 71

IV. Formulation and Solution 75
Focusing Students' Attention on the Process and Nature of Formulation 77
Restricting and Idealizing the Problem 77
The Statement of the Problem: The Choice of Helpful Concepts 82
Explicitness and Clarity of the Premises 82

CHAPTER PAGE

The Relation of the Formulation to the Possibility of Solution 83
Focusing Students' Attention on the Nature of Acceptable Solutions 84
Types of Solutions and the Possibility of Solution . . . 84
Criteria for the Acceptability of Solutions 87

V. DATA 90
Building Students' Understanding of the Nature of Data . . 91
Characteristics of Acceptable Data 92
The Variety of Types of Data 95
Focusing Students' Attention on the Processes of Collecting and Recording Data 96
The Question of Accessibility 96
The Choice of Units 97
The Use of Measuring and Recording Devices 100
The Choice and Use of Sources 101
Focusing Students' Attention on the Processes of Organizing Data 102
The Construction and Use of Tables 103
The Construction and Use of Graphs 109

VI. APPROXIMATION 112
Building Students' Understanding of Approximation in Measurement and Computation 115
Approximations Originating in Measurement 115
Approximations Originating in Mathematical Theory . . 119
Building Students' Understanding of Statistical concepts . . 121
Measures of Central Tendency. 122
Measures of Dispersion 127
Approximate Relations Among Associated Variables . . 130

VII. FUNCTION 139
Developing Students' Understanding of Mathematical Functions of one Variable 141
Fundamental Notations Related to the Function Concept . 141
Common Types of Quantitative Functions 148
Developing Students' Understanding of Functions of Several Independent Variables 159
Developing Students' Understanding of the Generality of the Function Concept 161
Generality as to Arguments 162
Propositional Functions 162
Relations 163

CHAPTER PAGE
VIII. OPERATIONS 168
Building Students' Understanding of Concepts Basic to Operations 170
Concepts Basic to Counting and the Fundamental Operations 170
Concepts Basic to Comparisons: Difference, Ratio, Proportionality 177
Concepts Basic to Other Operations 180
Helping Students Develop the Necessary Skills and Techniques 181
A Point of View on Drill 181
A Point of View on Correctness in Computation 183

IX. PROOF 187
Developing Students' Understanding of the Nature of Proof in the Broad Sense 190
Cases in Which Proof of Any Kind Is Unnecessary, Impossible, or Premature 191
Inductive Proof 192
Developing Students' Understanding of the Nature of Proof in the Narrower Sense 195
The If-Then Principle 195
The Interplay of Deduction and Induction 203
Logical Deduction, Truth, and Fact 204
Four Stages of Abstraction with Regard to Definitions and Assumptions 205

X. SYMBOLISM 214
Improving Students' Understanding of Symbols Through Study of Mathematical Usage and Comparison with Non-Mathematical Usage 216
Some Uses of Symbols in Mathematics 217
Distinctions Between M and NM Uses of Symbols . . . 225
The Social Origins of Symbols 232
Some Broader Aspects of Symbolism 234

PART III

THE DEVELOPMENT AND NATURE OF MATHEMATICS

XI. THE DEVELOPMENT AND NATURE OF MATHEMATICS 241
Understandings Related to the Development of Mathematics in Response to the Needs of the Times 242
Mathematics as a Direct Response to the Spirit and Needs of the Times 242

CHAPTER PAGE

The Insufficiency of the Needs and Spirit of the Times as an Explanation of All Mathematical Development . . 245
Understandings Related to the Development of Mathematics as a Science 248
The Cumulative Character of Mathematical Development . 248
Some Non-Cumulative Aspects of Mathematical Development 254
Light Thrown on the Nature of Mathematics by Certain Aspects of Its Development 256
Approaches to the Development of These Understandings . 258
Some Connections in Which Historical Materials May Be Introduced 258
Adaptation to the Maturity of Students 261
Adaptation to the Special Interests of Individual Students 262

PART IV

UNDERSTANDING THE STUDENT AND EVALUATING HIS GROWTH

XII. Understanding the Student 269
The Adolescent as the Teacher Observes Him 273
The Adolescent as the Teacher Understands Him 275
The Strains and Conflicts of Adolescence 275
Typical Adolescent Behavior in Response to Strain . . . 280
The Influence of Early Experience in the Family . . . 286
The Study of Personality 292
The Developmental Approach 292
An Illustrative Case: Paul 293
Introduction 293
Identifying Data: Family and School 294
As the School Sees Paul 295
As the Parents See Paul 303
As Paul Sees Himself 305
Reconstructing Paul's Social and Intellectual Development 314
The Rôle of Mathematics in Paul's Personality . . . 324
The Significance of the Individual Case 325
The Implications for Teaching 329

XIII. The Evaluation of Student Achievement 338
General Characteristics and Methods of Evaluation . . . 340
Some Characteristics of a Satisfactory Evaluation Program . 340
Major Steps in the Process of Evaluation 345

CHAPTER PAGE

Sample Evaluation Techniques 354
The Ability to Recognize Quantitative Factors in Problem-Situations 354
Understanding of Some of the Qualities of Good Data . . 355
Arrangement of Data in Tabular Form 357
The Proper Use of Approximate Numbers 357
Choosing a Form of Graphical Representation and Drawing Graphs 359
The Interpretation of Data. 360
The Ability to Apply Several Rather General Principles of Variation 366
The Nature of Proof 368
Evaluation of Qualities of Personality Other than Reflective Thinking 372
Sample Interpretations of Test Data 375
Interpretations Based on a Single Test 375
Interpretations Based on Several Tests 378

APPENDICES

I. A Source Unit on Normal Variability 385

II. Illustrative Short Activities 403

Index 413

Part I

THE TEACHING OF MATHEMATICS IN RELATION TO GENERAL EDUCATION

I

INTRODUCTION

The purposes of this Introduction are, first, to explain the factors that have led to a reconsideration of the aims and content of mathematical instruction and, second, to provide a descriptive survey of the Report as a whole.

REASONS FOR A REËXAMINATION OF THE AIMS AND CONTENT OF MATHEMATICS INSTRUCTION

Attention is usually focused on the modification or fundamental reconstruction of education when school practices and procedures, appropriate to the past, are carried into a period of changed social conditions and revised theories of learning. This introductory discussion makes clear that the reconsideration of mathematics instruction represented in this Report was undertaken because of precisely such a situation.

The Persistence of Disciplinary, Practical, and Cultural Aims: An Historical Sketch

Until very recent times, mathematics instruction was accepted almost without question by most educators as an essential part of a secondary education. This was reflected both in college-entrance requirements and in state legislation. A Massachusetts law [1] of 1827, for instance, required the teaching of algebra, geometry, and surveying in every high school in towns of 500 families or more. Similar legislation was passed in other states and remained in force for decades.

One of the reasons for the compulsory inclusion of mathe-

[1] *An Act of the Commonwealth of Massachusetts, A.D., 1827, To Provide for the Instruction of Youth* (Boston, Christian Register Office, 1828), pp. 1–2.

matics in the curriculum during this period was the disciplinary effect it was presumed to have upon the adolescent mind. When called upon to defend mathematics beyond this, educators cited, on the one hand, its traditional standing as an element of a cultural education, and, on the other, its usefulness in the practical affairs of life. Since 1890 mathematical education has been subjected to repeated scrutiny both by individual reformers and by committees representing various national organizations. Yet these three conceptions of the purpose and value of the study of mathematics—disciplinary, utilitarian, and cultural—continue to influence practice and content to the present day.

The first comprehensive survey of the program and purpose of secondary education in the United States, made by the Committee of Ten [2] in 1893, assumed that mathematics has a general disciplinary value. This assumption freed the Committee from the necessity of examining critically the traditional content of courses or considering the degree to which this content might be helpful to young people in solving their problems. Its recommendations [3] for the mathematics curriculum dealt primarily with the placement of traditional subject-matter. The Committee suggested, for example, that some algebra and what is now called intuitional geometry should be taught much earlier than had been previously customary.

As an outgrowth of the work of the Committee of Ten, the College Entrance Examination Board [4] was established and began to function in 1900. The requirements set by this Board (and by its counterparts in regional accrediting agen-

[2] The official title of this Committee was "The Committee on Secondary School Studies of the National Education Association."

[3] *Report of the Committee of Ten on Secondary School Studies* (New York, Published for the National Education Association by the American Book Co., 1894), pp. 104–116.

[4] The report of the Committee of Ten led in 1895 to the appointment of the joint Committee of the Secondary Department and the Department of Higher Education of the National Education Association. This Committee evolved into the Committee on College Entrance Requirements, which, after four years of work with associations of teachers in secondary schools and colleges, formulated plans for the College Entrance Examination Board.

cies) have continued to exert a profound influence on the content of courses of study, and have reflected, in modified though clearly recognizable form, the faith in the disciplinary aim that characterized the nineteenth century.

The first stimulating criticism of mathematical work being done at the secondary-school level came from influential teachers of mathematics. At the turn of the century Felix Klein in Germany,[5] John Perry in England,[6] and E. H. Moore in the United States [7] suggested certain changes in methods and in the organization of materials. These proposed changes were designed to make the subject function more effectively in the thinking of students. The recommendations of these men are noteworthy in that they made little mention of disciplinary aims and brought practical aims to the fore. The organization they proposed foreshadowed what is now sometimes called an integrated course. The integration, however, was to take place upon the basis of the logical interconnections among a number of subjects, rather than primarily with reference to students' needs and interests. Essentially similar goals and the "fusion" of various divisions of elementary mathematics—for example, algebra and trigonometry—were also recommended in 1914 by T. Percy Nunn,[8] a leader in mathematical education in England.

In recent years, the Report of the National Committee on Mathematical Requirements of the Mathematical Association of America,[9] published in 1923, has been widely recog-

[5] Felix Klein, *Elementary Mathematics from an Advanced Standpoint,* translated from the 3rd German ed. by E. H. Hedrick and C. A. Noble (New York, The Macmillan Co., 1932), *passim.*

[6] John Perry, Editor, *Discussion on the Teaching of Mathematics,* British Association Meeting, Glasgow (New York, The Macmillan Co., 1901).

[7] Eliakim Hastings Moore, "On the Foundations of Mathematics," *Bulletin* of the American Mathematical Society, Vol. 9, May, 1903, pp. 402–424; reprinted in *A General Survey of Progress in the Last Twenty-five Years, The First Yearbook* of the National Council of Teachers of Mathematics, 1926, pp. 32–57.

[8] T. Percy Nunn, *The Teaching of Algebra (Including Trigonometry)* (New York, Longmans, Green and Co., 1914), pp. 16–21.

[9] *The Reorganization of Mathematics in Secondary Education,* A Report by the National Committee on Mathematical Requirements under the Auspices of The Mathematical Association of America, Inc., 1923.

nized as authoritative. Accepting the customary classification of aims into practical or utilitarian, disciplinary, and cultural, the Committee gave a more satisfactory statement of them than had previously been available. The influence of this report on point of view and practice of teachers can hardly be overestimated. Authors of mathematics texts have almost invariably claimed that their books conform with its recommendations. Teachers in training have studied it as they have studied no other pronouncement in the field, and writers on methods of teaching continue to devote many of their pages to a discussion of its views.

This historical sketch indicates that successive recommendations in regard to the aims, content, and method of mathematics instruction have not deviated significantly from traditional doctrine. Thus the question is raised as to whether or not purposes and procedures characteristic of the past have not been carried into the present without sufficiently fundamental reconstruction. In order to answer this question it is necessary to consider certain changes in the social structure and in psychological theory as these affect secondary education.

Changing Social and Economic Conditions Affecting Secondary Education

The economic readjustments and social changes of recent years call for a reconsideration of the aims and purposes of secondary education as a whole. Changing social and economic conditions have, in the first place, brought about a gradual postponement of gainful occupation until now most young people of secondary-school age are virtually excluded from employment. This situation has resulted in part from a general reduction in economic opportunity for all, and in part from the fact that modern business and industry employ relatively fewer youthful workers than formerly. Reduction in employment opportunities for youth has contributed to an unprecedented increase in secondary-school enrolments, and

has set certain new problems in the education of adolescents.

Whereas in 1900 only 11.4 per cent of the appropriate age group were enrolled in the secondary schools, 64 per cent were enrolled in 1934. This change in size of enrolments entailed changes in the characteristics of the secondary-school population. The student body now includes a far wider range of academic ability and of types of interests, talents, and life goals than ever before. Furthermore, a generation ago almost all high-school students came seeking preparation either for skilled jobs or for college—which, it was assumed, would eventually lead to professional employment. The schools served them by helping them meet college-entrance requirements or by equipping them with requisite vocational skills. With the growing limitations on individual economic opportunity and the increasing recognition of the unethical quality of mere "getting ahead" as an ideal, preparation solely for successful vocational performance is seriously to be questioned. Instead, adolescents must be helped to profit from a prolonged period of non-participation in the world's work by learning to manage the problems with which the conditions of their lives now confront them—much as younger children profit from childhood by living fully and richly on their own level.

Economic and social changes are bringing about profound changes in the adolescent's immediate social relationships in the family and in other face-to-face groups. They are making his effective participation in wider social and economic life more and more difficult. And they call for a new system of values by which he may endow his personal life with worth. Problems in these areas perplex all adolescents, and it is in these areas that social readjustment is now necessary.

Thus social and economic changes have given rise to the necessity that aims and methods of secondary education designed to equip a fairly homogeneous and relatively restricted group for individual success be reconstructed so as to equip all adolescents for creative participation in a wider social life.

But there is an additional way in which these changes are giving a new turn to educational planning. A period of profound economic readjustment and uncertainty has been followed by mounting threats to the democratic way of life. Adolescents not only need help in reformulating their personal, social, and economic relationships in response to the new conditions that influence them; it is increasingly recognized that they must also be helped to do this in ways which harmonize with democratic ideals and conserve democratic values.

New Psychological Theories Affecting Secondary Education

Though many teachers continue to justify the study of mathematics in terms of its disciplinary values, methods of instruction have been based more generally upon the theory of specific-habit responses than upon the training of supposed "faculties of the mind." But just as the specific-habit theory of learning superseded faculty psychology, more recent theories are now superseding habit formation as the key to learning.

Evidence is accumulating to the effect that the individual responds as a whole to whole situations which confront him, and is to some degree remade as a person in the course of his experience. This means that in planning educational experience for students, the school must take into account the whole personality, emotional as well as intellectual—what it is now, how it changes, and what it would best become. Furthermore, it means that such factors as teacher, classmates, method, and school atmosphere constitute a part of the learning situation; they influence what the student learns—how he changes as a person. Methods themselves must be such that the student deals with whole situations rather than practices upon specific skills.

When education overlooks the importance of the wholeness of the personality of the learner and the wholeness to which he responds, any specific habit learned in school may

fail to function when it is called for in out-of-school situations, or when it does function, may be put to questionable uses. The ability to compute a percentage, for example, may fail to come into play when it would facilitate making a personal budget, or it may be used to mislead the public in a newspaper article.

The inadequacy and inappropriateness of preëstablished specific habits are emphasized by the unprecedented complexity and novelty of the problems which a scientific age and its social and economic consequences have brought in their wake. The newer conception of the individual and the way he learns is particularly important from a social point of view: it gives hope of personalities capable of dealing constructively with ever new environmental conditions—changing them in desirable directions, and being changed themselves in the process.

A Loss of Confidence in the Educational Values of Mathematics

Changes in mathematics instruction have not kept pace with the changing interests and concerns of the student body or with emerging conceptions of the proper aims and purposes of secondary education. The teacher has been made increasingly aware of the inappropriateness of traditional courses by the indifference of many students to the subject, or their outspoken dislike for it. He has also been disturbed by criticism of the mathematics curriculum voiced by specialists in education, many of whom are known to understand mathematics and to prize it for what it has meant to them.

Loss of confidence in the educational values of mathematics is strikingly evidenced by a decrease in the per cent of the total high-school population enrolled in mathematics courses. The great increase in the total number of enrolled students has partially concealed this fact, since the schools are filled and many teachers are overwhelmed with large classes. The situation is made clear, however, by the follow-

ing table showing the total registrations in the secondary schools of ten states across the country for 1928 and 1934, and the per cents of these students who were registered during the same years in certain mathematics courses.

THE NUMBER OF STUDENTS IN SECONDARY SCHOOLS AND THE PER CENT REGISTERED IN CERTAIN MATHEMATICS COURSES FOR TEN REPRESENTATIVE STATES * IN 1928 AND 1934

Region	*Years*	*Total No. of Pupils in Secondary Schools (to nearest thousand)*	*% Registered in General Mathematics*	*% Registered in Elementary Algebra*	*% Registered in Plane Geometry*
United States ..	1928	2,897,000	5.4	27	18
	1934	5,402,000	2.4	19	12
1. Massachusetts ..	1928	119,000	4.8	19	15
	1934	246,000	4.1	13	9
2. New York	1928	363,000	1.1	26	18
	1934	531,000	1.8	12	8
3. Ohio	1928	177,000	5.8	26	14
	1934	412,000	3.1	18	10
4. Illinois	1928	194,000	3.3	29	20
	1934	340,000	0.6	24	18
5. Tennessee	1928	31,000	6.0	34	20
	1934	62,000	0.9	29	16
6. Wisconsin	1928	77,000	3.4	28	19
	1934	149,000	0.6	20	17
7. Missouri	1928	82,000	4.9	27	18
	1934	112,000	1.7	23	15
8. Oklahoma	1928	50,000	5.8	29	22
	1934	106,000	1.3	24	18
9. Colorado	1928	33,000	3.0	28	20
	1934	59,000	1.8	18	12
10. California	1928	190,000	3.6	21	14
	1934	347,000	3.1	12	9

* See Carl A. Jessen, "Registrations in Mathematics," *School Life*, Vol. 22, No. 7, pp. 211–212.

In each of the states listed the per cent of the school population registered in both elementary algebra and plane geometry was less in 1934 than it was in 1928. The situation in the State of New York, where about 10 per cent of the high-school students of the country were enrolled in 1934, deserves special attention. Not only did the per cent of students registered in mathematics decrease, but the actual number of them thus registered also declined.[10] Yet during this short period the total school population in New York State increased about 46 per cent. These data reveal in quantitative terms a situation which many mathematics teachers have sensed but have not fully realized.

Some of the underlying causes for a decline in the per cent of students enrolled in mathematics courses have already been suggested. As the total number of failures in these courses mounted owing to changes in the interests and capacities of the growing student body, educational administrators, many of whom had small confidence in mathematics as taught, sought to meet the situation by making these courses elective. Under the new conditions many students availed themselves of the privilege of omitting mathematics from their programs, and their course advisers acquiesced, seeing no reason to urge them to study the subject except for the sake of meeting college entrance requirements. Eventually colleges were pressed to admit students who had not taken these courses. As more and more colleges have yielded, even this compulsion upon secondary-school students to elect algebra and geometry is losing its force. As a result, the education of a large portion of the high-school population involves no more than incidental experience with mathematics.

Yet mathematicians and teachers of mathematics are convinced of the value of their field in the education of youth. The Committee shares the conviction that, divested of much

[10] In 1928 New York State had about 96,000 students studying elementary algebra and about 64,000 studying plane geometry; in 1934 the figures were 65,000 and 43,000 respectively. This is a decrease of about one-third in each case.

of its conventional content and formal organization, mathematics as a mode of thought and an instrument of analysis has an indispensable function in general education. It has much to contribute in meeting the needs of students—both as these are felt by students themselves and as they are defined by educators. This Report therefore attempts to present an acceptable statement of the aims and purposes of general education under modern conditions, to interpret the function of mathematics in serving these purposes, and to suggest appropriate content and method.

A BRIEF DESCRIPTION OF THE REPORT

Part I of this Report outlines the basic educational philosophy that has guided the thinking of the Committee and includes a discussion of the rôle of the teacher of mathematics in the achievement of the aims of general education.

Part II is devoted to a discussion of certain broad concepts and related abilities that are involved in problem-solving: Formulation and Solution, Data, Approximation, Function, Operation, Proof, and Symbolism. The discussion of these concepts centers on the special contribution the mathematics teacher can make in their development.

Part III consists of a chapter on helping students appreciate the development and nature of mathematics.

Part IV contains two chapters that are intended to help the teacher clarify his views concerning, first, the student as a human being whose development is influenced by a variety of potent factors, including his family, social background, friends, and the school; and second, the problem of evaluating the growth of the student toward the objectives of general education.

An Appendix contains an illustrative "source unit" and some activities that should help the teacher in the effort to apply the general principles discussed in the other parts of the Report.

This outline indicates that the Report discusses mathemat-

ics teaching within the context of general education. By definition, general education is not primarily concerned with preparation for specific vocations or for college. It is not specialized or restricted to any particular group. It emphasizes meeting the educational needs of each student, and the group with which it is concerned is the secondary-school population as a whole.

In such a program the curriculum for a given school or group must, in the last analysis, be determined in the light of the needs of the particular individuals who make up the group to be taught. Since students differ widely as to their needs, capacities, and interests, it would not be consistent with the purpose of this Report to outline a detailed course of study to be followed by all, or even to propose a number of alternative courses. Furthermore, teachers also differ in their capacities and interests, and in general do more effective work under conditions that allow them some freedom in planning their own programs. For these reasons, instead of recommending a single more or less formal course of study, the report outlines a set of fundamental concepts and guiding principles designed to serve as a basis upon which teachers may so organize their own work as to make it appropriate to the possibilities and limitations of individual schools, or classes, or, ideally, individual students.

In seeking to meet the needs of the large group who constitute the bulk of the school population, the more specialized needs of certain students must not be neglected. Some of the concepts and skills mentioned in this Report should doubtless be developed only with those secondary-school students who show special aptitude, or whose definite vocational interest calls for more mathematics than may reasonably be given to all. Prospective engineers and scientists fall into these classes. The range of topics discussed thus becomes large in order to make possible the selection necessary to care for the wide range of needs, abilities, and interests to be served.

In formulating the outlines of a program through which mathematical education may advance during the next few years, the Committee had of necessity to be idealistic. To make the proposed program effective and to supply innumerable details of possible content and organization will require experimentation, both extensive and intensive, over a period of years.[11] In the light of such experimentation certain suggestions of this Report will almost certainly need to be modified or even rejected. But without making such recommendations the Committee could hardly hope to outline a forward-looking program, and without such a program mathematics for general education is not likely to become consonant with the needs of times.

TO WHOM THIS REPORT IS ADDRESSED

This report is addressed primarily to the growing group of well-trained teachers who are dissatisfied with the mathematics curriculum in their schools and are seeking a basis for a fundamental reconstruction consistent with modern educational theory. Convinced that the curriculum should be reorganized, some of them are already moving forward along lines proposed here. It is hoped that in this discussion they will find helpful suggestions and from it gain greater confidence when they discover ideas of their own coinciding with those which have survived the gauntlet of Committee deliberation.

It is also hoped that the Report will be helpful to administrators and curriculum experts who are interested in bringing about changes in the curriculum and are seeking clues

[11] It is with respect to this point that the task undertaken by this Committee differed from that of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics. In preparing its report on *The Place of Mathematics in Secondary Education*, the Joint Commission, after discussing the general aims of education, sought to outline a program of the sort being offered at the moment by some schools in advance of the great majority. Most of its suggestions have been tested to some extent in practice, and the Joint Commission took a practical rather than experimental point of view.

as to the proper rôle of mathematics in the program of secondary education.

The Report is also addressed to teachers in training and those who guide them. Although the literature appropriate to courses on the teaching of mathematics is becoming extensive, much of it is devoted to special methods of teaching particular topics. By focusing attention upon broader aspects of teaching mathematics, the discussion in this volume is intended to help future teachers see such details in proper perspective.

Parts of the Report will be of interest to all teachers of mathematics at the secondary level. Those who, for one reason or another, find it impossible to embark upon the major reorganization recommended may nevertheless become acquainted with the point of view and find certain suggestions they can put into effect. Every effort of this sort, if successful, will in the long run serve to promote the purpose of the Committee—to help teachers of mathematics better meet the needs of boys and girls.

III

MATHEMATICS IN RELATION TO THE PURPOSES OF GENERAL EDUCATION

The fundamental assumption of this Report is that the mathematical curriculum should be designed to contribute as directly as possible to the achievement of the aims of general education. Some statement of these aims, acceptable both to students of broad educational problems and to teachers of mathematics, is therefore in order.

In the past, teachers of mathematics have often evinced impatience with discussions of educational theory, but more and more they are recognizing that some assumptions, some basic principles, some philosophy are inherent in any educational program. In the majority of cases, however, this philosophy is implicit rather than explicit. Under these conditions it is neither readily subject to corrective criticism nor does it serve as an adequate guide to the selection of content and methods of instruction.

The task of developing an acceptable philosophy of general education and exploring its implications for the teaching of mathematics calls for coöperation between mathematicians and teachers of mathematics on the one hand, and experts on educational problems on the other [1]—an undertaking in which teachers of mathematics first participate in the formulation of the objectives of general education, and then make explicit the precise ways in which the study and teaching of mathematics can contribute to their realization.

For these reasons this chapter is devoted to a discussion of the basic educational assumptions and philosophy which

[1] See Robert J. Havighurst, "Can Mathematicians and Educationists Cooperate?" *The Mathematics Teacher*, Vol. 30, May, 1937, pp. 211–213.

have guided the preparation of this Report—a formulation that represents the combined thinking of mathematicians, teachers of mathematics, and students of educational theory.

NEEDS AS THE BASIS OF EDUCATIONAL PLANNING

The Concept of Educational Needs

The Committee subscribes to the view that the needs of the student constitute a key conception in the educative process. But the word *need* has several meanings, and different groups of educators have interpreted it in different ways. Hence the sense in which it is used here must be set forth.

The word *needs* is sometimes used to refer to the desires, wishes, inclinations, and urges that impel an individual at any given moment; their expression and fulfilment give him satisfaction. This conception of needs, crucially important though it is for the purposes of education, is likely to prove inadequate when taken by itself; it has frequently led to the assumption that education results when the individual is merely permitted to act on the basis of his present impelling motives, without reference to the way these motives may change in the course of experience.

In undertaking to relate educational procedures to the desires and inclinations of the individual it is important to bear in mind that the environment creates needs in this sense as well as satisfies them: [2]

> . . . development is not something that takes place within the organism independent of where a person is or what he is doing. Rather, development is due to the interaction of intrinsic factors (factors within the organism, such as hormones, glandular secre-

[2] *Preliminary Exploration of the Field of Immediate Social Relations of Boys and Girls in Secondary Schools*, a tentative report of the Committee on Immediate Social Relations of Adolescents of the Commission on Secondary School Curriculum (New York, Progressive Education Association, 1939), p. 1. Mimeographed.

tions, metabolism, etc.) and extrinsic factors (factors without the organism, such as temperature, food, education, culture). What a child becomes as he proceeds through life depends upon the results of the interaction of these intrinsic and extrinsic factors.

The human organism at birth has need for certain conditions of living, if the basic processes necessary for life are to be maintained. As these conditions are provided, the infant absorbs them, makes them over into himself and creates new needs for maintenance of life. But the very conditions with which he is surrounded in themselves create needs within him, if life is to continue: to handle certain foods, to get along with certain people, to behave acceptably to his community. Thus, as the extrinsic factors play upon the developing child, they create needs peculiar to those factors.

The adolescent's desires, inclinations, and tendencies to react in certain ways, then, result from his past experience in a social medium. They are both personal and social in origin. As he continues to live, they undergo change. The way they change—the particular quality they take on—depends upon the conditions that surround him and the way he reacts to them. But they are constantly under revision as life goes on.

Education cannot neglect the quests and urges of the individual since they determine the situations that he seeks and avoids, and profoundly affect what he learns. If they are too severely thwarted or overly indulged, the personality may be so distorted as to be incapable of effective functioning in life.

But neither can education merely accept them and give them more or less free play, precisely because they are always undergoing change. Education must determine the direction in which this process of change should go—the kinds of inclinations and tendencies to act that should desirably characterize the individual, the type of activity which should desirably bring him satisfaction. It must, in other words, modify the first sense in which *needs* were defined. For educational purposes the concept of needs must take into account

the difference between a present state or condition of things and a more desirable one foreseen as possible in the future, as when it is said, "He needs to become more considerate." It is not enough to know what the personality of the student now is, or what he now wants; in addition it is necessary to try to decide and define what he should become.

Emphasis upon this aspect of *needs* in isolation from the other is also undesirable. All too frequently it has led to a prescription of what is to be learned, as when it is said that students need to know French, or how to operate a typewriter, or how to solve an equation, or any number of similar things. Such prescription, without reference to the desires and tendencies to act characteristic of the student's level of development, tends to thwart the achievement of present satisfactions, and so runs the danger of distorting the personality by giving rise to indifference and resentment. Furthermore, it frequently defeats its own purposes because learning remote from current desires and inclinations may be rejected, mal-assimilated, or misused.

How, then, shall the educational needs of the learner be defined? In this Report they are taken to be the resultant of his present inclinations and quests, on the one hand, and the demands of desirable social living on the other. Thus [3] "the need of the student to select and use goods and services wisely" refers to a want (biological tension) or a desire on the one hand, and the requirements, demands, standards of social living on the other. To speak of a need without including both its personal and social aspects is to leave out an

[3] From this point to page 43 the Committee has adopted, with certain omissions and amendments, the statement presented in: Committee on the Function of Science in General Education of the Commission on Secondary School Curriculum (Progressive Education Association), *Science in General Education* (New York, D. Appleton-Century Co., 1938), pp. 25–46. The membership of the Committee on the Function of Science in General Education and that of the Committee on the Function of Mathematics in General Education worked in close contact while defining their basic positions. At the outset it was assumed that the same group could explore both fields, and the mathematics committee as a separate group was appointed only after the need for distinctive treatment of problems in each field became evident.

indispensable element. Merely to say that John wants something, or that Teacher X believes John needs a particular piece of knowledge, is to leave out the element of interaction between the two necessary components.

CLASSIFICATION OF EDUCATIONAL NEEDS

The diversity and range of possible needs is so great that some method of classification is almost essential. The Committee accepts the categories used for this purpose by the Commission as a whole.[4] They are phrased in terms of four "basic aspects of living" as follows:

1. Personal Living
2. Immediate Personal-Social Relationships
3. Social-Civic Relationships
4. Economic Relationships

Clearly these categories are not mutually exclusive, for complex human relationships do not lend themselves to neat compartmentalization. They are intended to serve as convenient centers of reference for identifying worthy interests and needs and for selecting and organizing appropriate learning experiences.

Personal Living

In proposing this category, the Committee by no means intends to consider the individual apart from his environment. But as the individual interacts with his environment he steadily builds unique characteristics that mark him off from others—he develops as a person.

Observation of the behavior of the adolescent in this as-

[4] For a discussion of the factors that led to this classification, see: V. T. Thayer, Caroline B. Zachry, and Ruth Kotinsky, for the Commission on Secondary School Curriculum (Progressive Education Association), *Reorganizing Secondary Education* (New York, D. Appleton-Century Co., 1939), Ch. II, "The Needs of Adolescents as a Basis of Educational Reorganization."

pect of living reveals at once some of his needs as a growing, developing personality. He needs to attain a maximum of physical and mental health and to be able to maintain it. This involves assurance that he is developing normally. He needs to increase his ability to guide his own conduct and to feel himself moving toward recognized adult status. He inquires into the nature of the world and attempts to build a world picture and find his place in it; these are indications of a need for understanding of the external world as it affects his philosophy of life. For his own satisfaction he needs to develop a variety of interests and to increase his esthetic appreciation of many types of experience.

Immediate Personal-Social Relationships

By the time the individual reaches adolescence, he is involved in a multitude of relationships with persons in his immediate environment—his parents, his brothers and sisters, his schoolmates and chums. He has close contacts with persons of both sexes and with individuals both younger and older than himself. He also has many adjustments to make because of his relationships with culture groups whose manners and customs differ from his own. He is called upon to assume responsibilities within the family and within other face-to-face social groups—athletic teams, school clubs and societies, and the like. He develops an interest in the opposite sex, with all of its attendant individual disturbances. A new world of contacts and immediate relationships with persons is opening to him, and he seeks to establish himself in it on a self-reliant basis, a process that involves a gradual emancipation from home and family ties.

Among the primary needs of the adolescent in this aspect of living are to develop maturity as a participant in home and family life, maturity in relationships with age mates of both sexes, satisfying relationships with adults outside the home, and ideals or standards to guide the conduct of all such relationships.

Social-Civic Relationships

While the adolescent grows as a person and grows in the range, depth, and meaningfulness of his immediate personal-social relationships, he is simultaneously involved in relationships of a broader social character. There is, of course, no clear-cut line of demarcation between his contacts with the family group and those with the wider community and society, but a distinction between immediate personal relationships and wider social relationships nevertheless proves valuable for the purposes of education.

This distinction may be made in two ways. First, a line may be drawn at the point where the adolescent comes into contact with institutions other than the home. His participation in the activities of school, club, church, local community, and government may be conceived to be primarily social-civic in character, for though he may have immediate personal relationships within these institutions, through them he also acts with others for wide common purposes. Second, social-civic relationships require much greater use of general principles and conceptualization than do immediate personal-social relationships—a fact well illustrated by a comparison of the problems of getting along with people in face-to-face relationships and those encountered in working on such social-civic problems as housing, community health and recreation, community planning, crime, divorce, and government.

In social-civic relationships the adolescent needs, primarily, an extension and deepening of his areas of social concern, responsible participation in social movements and the activities of social institutions, and an increasing awareness of the social implications of his own activities.

Economic Relationships

Again it is pertinent to point out the close relationship between this aspect of living and those previously discussed. Even the immediate social life of the individual in the home

has definite economic implications, and economic ideals and practices profoundly affect both the adolescent's personal philosophy and his development as a person. Social life and economic life are of course very closely related in modern industrial society. However, economic life in a narrower sense may be thought of as a separable area, and it is in this delimited sense that the term is here used.

This aspect of living includes the activities of the individual in the production and distribution of goods and services. Social changes have tended to remove the adolescent from contact with productive economic life. They have delayed his entrance into economic activity and surrounded it with difficulty and uncertainty; as a result, the vocational orientation of young people has become an educational problem. The adolescent has more responsibilities than ever before for the selection and purchase of goods; the secondary school must help him learn to use his intelligence as a consumer both now and in the future. Also, the adolescent is entering into his rôle as a citizen in a society characterized by growing industrialization, by associational activity on an increasing scale, and by more and more disposition toward the solution of economic problems by governmental means. The applications of the sciences, including mathematics, to the use and control of energy and materials lie at the bottom of many of the processes of social change. The secondary school must help the adolescent to see the rôle of mathematics in analyzing the problems of economic life and in making intelligent judgments about such economic institutions as capitalism and other forms of economic organization, monopolies, collective bargaining, public ownership, money and banking, and domestic and international commerce.

The principal needs of the adolescent in this aspect of living are emotional assurance of progress toward adult status through participation in adult economic affairs, vocational orientation, wise selection and use of goods, and participation as a citizen in the solution of basic economic problems.

In conclusion, it is important to note that the needs mentioned above are illustrative rather than definitive, and are by no means intended to comprise an exhaustive list.[5] The identification of the specific needs of the students in any given school requires study of both the local community and of the students themselves. It is also necessary to determine carefully which of the particular needs identified most urgently demand attention, and to select appropriate curricular materials and methods. These are tasks which call for the active participation of all the teachers in the school.

THE MAJOR IDEALS OF DEMOCRACY AS THE BASIS OF EDUCATIONAL VALUES

The concept of meeting educational needs does not of itself indicate the directions of growth to be fostered in the process—that is, provide the sense of direction necessary for planning and guiding educational procedure. Needs are defined as the resultant of the present personality of the student and what he should desirably become from a social point of view. But what is desirable from a social point of view? What directions of development are socially valuable?

The Committee believes that the major values to be fostered are those associated with democracy as a way of life. This is a broader concept than that involved in political democracy because it refers to all the relationships of life. But in a democratic society, ideals and patterns of belief and action evolve from the will of the people, and consequently they are always in process of reconstruction and reinterpretation as the needs and conditions of the people change. This fact makes it impossible to set forth definitive formulations of ideal social arrangements; reinterpretation of both the ideals of a democracy and their embodiment in social institu-

[5] For a more extended discussion of needs, see Thayer, Zachry, and Kotinsky, *op. cit.*, Chs. V–VIII inclusive. For another recent statement of educational objectives, see The Educational Policies Commission, *The Purposes of Education in American Democracy* (Washington, D. C., 1938).

tions must come from the people as a whole. This being true, it follows that the program of the school cannot be set up with completeness or finality; it must be kept flexible in order to meet the changing personal-social needs of individuals and the changing demands of the social order. Furthermore, since democracy rests ultimately upon the intelligence of the common man, it follows that the student must be given increasing responsibility for and aid toward reconstructing his own beliefs, attitudes, and plans of action upon the basis of his maturing intelligence. This implies that he must have a responsible share in planning the program of the school.

These considerations complicate enormously the problem of defining the purpose of the school in a democratic society and of formulating an appropriate and effective program. Yet the schools have an inescapable responsibility to make it possible for young people to contribute effectively to the progressive refinement of the democratic way of life. Unless curriculum-making groups are willing to seek earnestly for the deeper meanings and implications of democracy, there is little hope that the school will have much direct influence in bringing about progressive refinement of democratic ideals and practices.

Acting upon this belief, the Committee presents its view as to the meaning of democracy and indicates some implications both for the school in general and for the mathematics program.

In a recent article, Dewey presents succinctly the broad idea of democracy which the Committee believes to be basic.[6]

> Democracy . . . means voluntary choice, based on an intelligence that is the outcome of free association and communication with others. It means a way of living together in which mutual and free consultation rule instead of force, and in which coöperation instead of brutal competition is the law of life; a social order in which all the forces that make for friendship, beauty, and knowl-

[6] John Dewey, "Education and Social Change," *The Social Frontier*, Vol. 3, May, 1937, p. 238.

edge are cherished in order that each individual may become what he, and he alone, is capable of becoming.

Otto puts the matter in a slightly different way: [7]

> . . . Democracy is not a mere association of individuals whose purposes or acts are individualistic in the *laissez faire* sense. It is not even primarily a form of government. *It is an intelligent use of cooperative means for the progressive attainment of significant personalities.* Significant personalities cannot be unfolded from within; they must be acquired by individuals in union with other individuals intent upon a similar quest.

Analysis of the above quotations reveals at least three basic ideals of democratic living which, though intimately interrelated, are capable of isolation for the purpose of discussion and clarification: first of all, the fostering and development of distinctive personalities; second, "mutual and free consultation" in associated living, or the "use of coöperative means," instead of ruthless competition; and third, intelligence growing out of "free association and communication" as the basis of choice and action.

Obviously each one of these three aspects of democracy could be expanded to include the other two. For example, significant personalities can be developed only through free consultation and association with others and through the use of intelligence in the continuous reconstruction of experience. Free association in carrying out common ends and purposes implies a respect for the personality of those who are associated, and satisfactory outcomes of such association can be secured only when intelligence is placed above caprice, whim, or selfishness. And, finally, reliance on collective intelligence would imply action leading toward consequences satisfactory to all concerned: personality would have to be respected, and there would of necessity be free association

[7] Max C. Otto, "John Dewey's Philosophy," *The Social Frontier,* Vol. 3, June, 1937, p. 266. (Italics in original.)

and consultation. To consider any one of the aspects of democracy out of relationship to the others is to deal with a sterile abstraction.

On the other hand, to extend the meaning of any one to include everything which is meant by democratic living leads to vagueness. Consequently, the Committee has chosen to discuss each aspect in turn.

Optimum Development of Personality

High regard for the individual is probably the most distinctive and pervasive characteristic of democratic living. In democracies, personalities are held to be precious, unique, and not capable of duplication. The optimum development of each individual, irrespective of birth, economic status, race, creed, or color, is to be encouraged and nurtured both because of the enhanced enjoyment of individual living which comes through full development of distinctive qualities and because of contributions which such distinctive qualities make to the common life.

To assert that human personality grows as individuals share in promoting common ends is merely to express the faith that democracy is the form of social organization best adapted to the full realization of the worth of the human personality. But the development of human personality as a goal must not be confused with individualistic action as a method for attaining the good life. Unrestrained individualism cannot be tolerated in a democratic program of living, for the living of each must be such as to allow others the realization of their potentialities. The disastrous effects of an outworn concept of individualism in a society which has become fundamentally interdependent have been exemplified as America has shifted from an agrarian to an industrial economy. This is not to argue for abandonment of the basic respect for personality so inherent in the American tradition, but rather that conditions for its realization must be con-

tinuously reinterpreted in the light of the changing situation. This position is well stated in the following quotation: [8]

> . . . While the philosophy of individualism which emphasizes the worth of the individual and his right to an opportunity for the full development and free expression of his personality is as true as ever, and is still the goal of American life, individualism as a *method* of achieving the good life is no longer practical. . . . If we are to realize the goal of securing for the individual those conditions of life to which Americans have always aspired, we must abandon the philosophy and the practice of trying to achieve that goal only by individualistic means. Social responsibility and collective action for decent incomes, for security, for health, for recreation, for education, constitute the only practical road to the goal of individualism.

The concept of individualism is undergoing marked revision in the light of far-reaching developments in technology and their attendant effects upon human living. To some these profound changes mean that the individual is to be subordinated to society as a whole, as has happened elsewhere in the world. To others the changes mean that the "American Dream" of ever richer development of human personality has a fair chance of realization, if only the fruits of technology can be made available to all men. The Committee holds, as has been stated, that the ideal of maximum development of the individual should still be cherished as a goal, and that the schools should help to interpret its meaning in the light of rapidly changing social, political, and economic conditions.

Reciprocal Individual and Group Responsibility for Promoting Common Concerns

The development of significant personalities can be achieved only through mutual sharing of interests and purposes; democratic participation in group life has its own distinctive characteristics and values which require special

[8] *Democracy in Transition*, by a group of social scientists in The Ohio State University (New York, D. Appleton-Century Co., 1937), pp. 29–30.

consideration. As Bode points out: [9] "All . . . social organizations are means to an end, and this end lies inside the process of living together and working together; it is not located on a far-off mountain top created by an iridescent dream. The kingdom of heaven is within us, within the everyday lives of a toiling, sweating humanity."

To insure a satisfying associated life, the group, on the one hand, must accept furtherance of the concerns of its members as its highest value and must recognize that such an end necessitates the participation of the individual in determining policies that affect him, and the individual, for his part, must recognize that his own purposes cannot be considered apart from the needs and purposes of the group. Increasingly the conditions of American life call for refinement of the concept of associated living: for seeing the significance of working together for the common good, for seeing one's own acts with a full sense of their bearing on the welfare of all, for seeing how both advantage and responsibility are inherent in the relationship of individual and group, and for seeing how reciprocal examination and criticism of action are thus entailed.

The Free Play of Intelligence

A third dominant ideal of the democratic society, fundamental to the refinement of democracy as a way of life, is reliance upon the free play of intelligence in solving problems of human concern—in contrast with the making of decisions on the basis of traditional beliefs, uncritical acceptance of authority, or blind impulse, or on the basis of one set of prescribed values uncriticized by comparison with the values of others.

In a democracy, institutions, policies, and programs are constantly in process of making. They are set up by the people themselves, rather than by some external authority

[9] Boyd H. Bode, *Democracy as a Way of Life* (New York, The Macmillan Co., 1937), pp. 49–50. Quoted by permission of the publishers.

or by some small group, and are subject to modification or rejection in accordance with the will of the people as a whole. It follows then that there must be no barriers to the free play of criticism and evaluation from every quarter; otherwise institutions and policies would no longer be subjected to critical scrutiny in the light of the values of the entire group. The alternative to reliance on collective intelligence in solving common problems is resort either to a philosophy of drift or to authoritarianism.

In its broadest aspects this ideal implies a disposition to abide by the results of collective thinking; to forego personal bias and interest when these are ruled out by the facts; to go where the evidence points, irrespective of its origin, or of authority, convention, or prejudice. Democracy rests upon faith in the intelligence of common men. Once this faith is destroyed, the alternative is some form of dictatorship, which means the destruction of democracy itself. The schools have never yet consciously and deliberately organized their programs in a way to encourage and promote intelligent action on the part of all people. Until this has been done, it is impossible to say that faith in the intelligence of the common man is unjustified.

The application of these ideals to the secondary-school program would cut very deeply. Each student would be studied to discover his potentialities and nurtured in whatever unique way would build him to his highest stature as an individual. It would mean that students would be encouraged to work together at the problems facing them, of whatever kind; to carry this study wherever significant bearings appeared; to bring to bear the best that combined study could find; and to reach a conclusion that seemed to them justified in the light of the evidence. It would mean that no problem, no creed, no system of belief, could be walled off and kept isolated from thorough study, analysis, and evaluation. Within the limits of their maturity, students would be encouraged to formulate their own outlook on life, scrutinize

it in the light of the best that is known, and reconstruct it in the light of new evidence.

Out of the welter of conflicting ideals and practices that beset adolescents on every hand, they would seek to weave unity and consistency into their lives. From continued study of problems and continued generalization of results, they would increasingly learn methods of intelligent search, standards of evaluation, and dissatisfaction with shoddy procedures; they would come to build reliance on intelligence. And the school would be committed to the task of helping them in the process rather than determining their beliefs and patterns of conduct for them.

The acceptance of this point of view would mean that the organization and administration of the school would be directed toward the fullest possible exemplification of these three interrelated ideals: the school would be organized in a way to promote the maximum growth of personality of teachers, students, and all members of the school community; to provide for the fullest possible sharing of interests and concerns, with group responsibility for their satisfactory realization; and to promote the ever-widening use of intelligence in all of its activities and the free play of opinion to this end. In application, the point of view would affect student-teacher relationships in the classroom and the face-to-face relationships of students. Not only would it have important bearings upon method and procedure; it would provide, in addition, direction for the curriculum in all areas. Later parts of this report indicate how, in particular, it would affect the mathematics program of the school.

PERSONAL CHARACTERISTICS ESSENTIAL TO DEMOCRATIC LIVING

Up to this point the Committee has been mainly concerned with outlining the major ideals of democracy as an indication of the general direction it believes the secondary school

should take. But in order to promote a democratic group life it is necessary to have individuals with characteristics appropriate to its pursuit; it is necessary to decide, at least in a general way, the kind of behavior to be fostered in adolescents as they grow toward mature participation in the common life. To this end the Committee proposes to set forth some of the characteristics that seem to be desirable from the point of view of development of personality and at the same time essential to democratic living.[10]

Social Sensitivity

Social sensitivity as a characteristic of behavior has been described as "an awareness of, and responsiveness to, social and human phenomena." [11] But in order to be helpful, the concept must be broken down into the more specific attitudes and abilities that characterize the socially sensitive person. These are perhaps best indicated by an analytical description of how such a person functions.

a. He enters appreciatively and sympathetically into the lives of others. His concern should extend beyond the boundaries of the particular social group with which he is primarily identified and should extend, as his contacts broaden, to ever-widening areas. The ideal, of course, is to develop individuals who recognize their kinship with the human race as a whole, so that human suffering in China or Ethiopia becomes their active concern and so that they cherish human achievement, even remote in space or time, as a contribution to present-day living.

[10] It may be remarked, parenthetically, that the ideals of a democratic society and the characteristics of personality implied by them might well be called needs. However, for purposes of clarity the Committee considers it wise to distinguish these from needs as used in the above discussion, for two reasons: first, it helps to identify the common needs of adolescents existing as going concerns in the lives of boys and girls, irrespective of the direction the meeting of these needs should take if they are to foster democracy as a way of life; and second, it avoids the danger of listing as needs certain ideals and values of adult democratic society which at any given time may not be identifiable as biological tensions in boys and girls.

[11] Hilda Taba, "Social Sensitivity," Progressive Education Association, Evaluation in the Eight Year Study, *Bulletin* No. 6, University of Chicago, December, 1936 (originally published at Ohio State University).

b. He is sensitive to the effects of his own actions as a citizen, producer, consumer, upon the welfare of others. This necessarily involves the disposition to consider his own acts in terms of all their foreseeable consequences to other people. Again this is a question of continuing growth, but the socially sensitive person has disciplined himself to see as far as possible and always to widen the area of his vision.

c. He is willing to take measures counter to his own immediate comfort or the comfort of the group with which he is immediately identified, for the sake of a broader social good.

d. He looks upon social maladjustments as problems to be solved through the application of intelligence, rather than as evidence of the perversity of an unalterable human nature. Sensitivity here is to the malleable elements in situations and to evidences of the way people respond with better adaptations when the resulting values can be made apparent. Wide historical perspective should aid here.

e. He evaluates all measures, including the economic and political, upon the basis of all the human values involved in the situation. For example, it has been shown that the emerging concept of democratic living requires reconstruction of the concept of individualism. Retention of the older conception as a basis of operation in society—whether it be embodied in law or otherwise maintained—is antithetical to the highest good of all. The person who is socially sensitive recognizes this and acts accordingly.

The school can do much to promote social sensitivity on the part of students. It can organize its activities in such a way as to provide constant opportunity for effective social participation in the life of the school. It can make the community a laboratory for the study of social problems, and to a limited extent students can participate in community activities. The school can encourage and help the student to study the effects on others both of his own actions and of the actions of policy-making groups inside and outside the school.

Esthetic Appreciation

Appreciations are characterized by enjoyment. To appreciate deeply means to cherish, to esteem, to prize. The pres-

ence of appreciation in experience marks the difference between mere understanding and a recognition of qualities which are appealing and enjoyable. The enjoyment may be a very simple response to something like a landscape or a game, or it may be one that relies upon a body of experience or knowledge for its value, as in the study of painting, geometry, geology, and architecture. Enjoyments may emerge from reading a poem, seeing a piece of sculpture, watching a machine, working through a demonstration in mathematics, making an experiment in the laboratory, solving a problem in human welfare, or observing a sunset.

In addition to the quality of enjoyment, esthetic appreciation also carries with it the idea of appraisal. Some enjoyable things are of more value socially and esthetically than others, and thus the cultivation of appreciation involves not only enjoyment but also the development of personal standards, tastes, and judgments concerning the value of a given experience. This is but another way of saying that appreciations have both intrinsic and instrumental values.

The behavior of a person who is growing in his ability to enjoy a variety of things and to appreciate them deeply is characterized by an increasing zest for living and by higher standards of value, both of which contribute to the richness of experience, thereby making for deeper and keener enjoyments.

The school can do much to help its students develop and enhance their enjoyments. If teachers are sensitive to what is significant to the student, they can help him see meanings in life about him that will add greatly to the richness of his day-to-day experience. Many enthusiasms and enjoyments the student will "catch" from his teachers as they open up areas meaningful and truly satisfying to him. In addition teachers should, wherever possible, cultivate whatever shreds of interest they find in their students and help these interests to develop into something finer and more satisfying.

Tolerance

Clearly akin to social sensitivity is tolerance—the increasing disposition and ability to appraise sympathetically, in terms of their unique contributions to the enrichment of the common life, those points of view, customs, and traditions that differ from one's own, or from those of the group with which one is identified.

This characteristic of personality is distinctly democratic in character, for only among peoples dedicated to the full and free development of personality through associated living is the individual encouraged to respect viewpoints that differ materially from his own. In a really democratic group life the contributions of different racial or national groups are cherished, out of the principle of respect for personality as well as for their contribution to the progressive refinement of individual and group living. "Elasticity of the emotions and the intellect" not only makes possible a sympathetic understanding and appreciation of honest differences, but also the utilization of these differences in enhancing human values. The individual who is achieving such elasticity is in process of achieving a genuine understanding of the meaning of democracy, is growing toward maturity of participation in democratic living.

Exercise of tolerance does not, however, mean a *laissez-faire* attitude toward any and all ways of believing; it does not mean absence of standards, preferences, considerations. Tolerance is rooted in the promotion of qualities and conditions that add to the variety and richness of human living and that make possible the cultivation of ideas for its improvement. On this basis, it would be impossible to condone tolerance of traits or practices that militate against this criticized value—ruthlessness, dogmatism, cruelty, injustice, lying, suppression of just civil liberties, exploitation of human lives. Exercise of tolerance has, therefore, not merely the

negative aspect of non-interference but also a positive aspect that imposes an obligation to maintain actual means of discernment, criticism, and thoughtful variation. Stated from a different angle, tolerance is by no means an "absolute value" —like all other values, it must be weighed in connection with other claims that deserve consideration.

Coöperativeness

As members of an industrial culture, the American people are undoubtedly becoming more and more interdependent. This, as has been pointed out, calls for rather drastic modification of the prevailing concept of individualism. The trend toward a growing collectivism means that, if the democratic tradition is to be maintained, new and more effective means of coöperation must be devised. According to Day,[12] one of the basic ideals of democracy has always been the settlement of controversies between groups or classes of individuals by peaceful means, through resort to discussion, persuasion, the ballot, acceptance, and appraisal. But cooperation means much more than the settling of controversies; it means a vital sharing of interests, an active concern with promoting the values of others, and a working together to promote more wholesome human relationships. It is the business of the school to develop individuals who have not only the disposition but also the understanding and ability to put this ideal into more general practice.

Self-Direction

A task of education which has an important bearing upon the development of the student is the setting up of an environment conducive to growth in self-reliance—one promoting in the student the ability to direct his own activities with a minimum of adult control and guidance. This does not mean that a student should be free to do as he pleases, but

[12] Edmund E. Day, "Basic Responsibilities of General Education in America," *The Educational Record,* Vol. 17, Supplement No. 10, October, 1936, p. 18.

rather that he should have an opportunity to attain independence through the exercise of intelligence. A sane and workable concept of freedom involves both the opportunity and the disposition to assume full responsibility for the consequences of one's acts, no less than the opportunity for independent volition. The individual must discipline himself to act through intelligence rather than through caprice, or blind habit. This characteristic of intelligent self-direction is marked by the following abilities: to carry out independent interests apart from required work of the school; to direct wisely one's own activities without the coercion of others; to utilize effectively the resources of one's environment; and to reach independent conclusions without undue reliance upon the judgment of others.

The development of this characteristic has important implications for school practice. The school environment must be organized in such way as to give the student increasing control over his experience. In all of his life situations that involve the making of choices, he must be encouraged to take responsibility and helped to make his choices wisely. As he grows more mature emotionally, socially, and intellectually, these situations will necessarily involve more complex relationships. The teacher will frequently be called upon to study the emotional blocks that prevent the use of intelligence and to help the student to overcome the inhibitions and fears which are interfering with his growth. Careful guidance of the student's experience is called for, instead of the constant imposition of adult plans and points of view.

Creativeness

Creativeness [18] means the ability of the individual to behave in ways new to him, and to bring new things into being. It involves synthesizing previously unrelated elements of

[18] This term among others is intended to be used in the sense made vivid by John Dewey in *How We Think* (Boston, D. C. Heath and Co., New edition, 1933).

See also Harold Rugg, *Culture and Education in America* (New York, Har-

experience. In many cases—painting, musical composition, or delineation of a character in a story—there is conscious arrangement of the elements to produce a certain effect. In other cases, there seems to be a factor of discovery—the individual suddenly becomes aware of a previously obscure relation among the elements. Anecdotes associated with the discovery of certain physical principles—Archimedes and the crown of Hiero, Galileo watching the swinging chandeliers in the cathedral, and Newton observing the fall of an apple —are based upon this type of creativeness.[14] Whatever may be the psychological nature of this phenomenon of sudden awareness of relations, experiences of this sort are not confined to men of genius. Whoever discovers simple numerical or geometrical relations, or formulates an hypothesis, is acting creatively.

Creativeness viewed in this wide sense is manifested in various ways by different individuals. One may be concerned with finding new applications of principles; the inventor is often of this type. Another looks for and finds new relationships among data first met as isolated items, or aspires to discover generalizations in terms of which to subsume many diverse principles. Another seeks further causes, reasons, and proofs. But whatever be the direction of discovery, insofar as a novel situation is mastered, creativeness is functioning. For the purposes of education the value of creative behavior

---

court, Brace and Co., 1931), Chapter XIX, "Self-Cultivation and the Creative Act: Issues and Criteria."

For a discussion of certain aspects of creativeness, see June E. Downey, *Creative Imagination: Studies in the Psychology of Literature* (New York, Harcourt, Brace and Co., 1929).

[14] These stories may be apocryphal. For relatively modern and completely authentic instances of such creative work in science, see "Mathematical Creation," Chapter III of *Science and Method* in *The Foundations of Science* by H. Poincaré, translated by George Bruce Halstead (New York, The Science Press, 1913). Karl Pearson, *The Grammar of Science* (New York, The Macmillan Co., 3rd edition revised and enlarged, 1911), Part I. Physical, "Science and the Imagination," pp. 30–32. Eric Temple Bell, *The Search for Truth* (New York, Reynal and Hitchcock, New Edition, 1935). H. Poincaré, *Science and Hypothesis* (New York, The Science Press, 1905).

is not to be judged in terms of the actual newness of the result in the story of human thought, but in terms of the effect upon the individual of producing a thing new to him.

To exhibit creativeness, the power to master a strange situation, often, but not always, presupposes a spirit of inquiry, a cultivated curiosity. Persons of placid negative temperament remain passive in situations that call forth creativeness on the part of those who are imaginative, curious, original. What, then, can the school do to help the adolescent develop his creative possibilities? Primarily, it must try to understand the drives that are, at the moment, giving character to his behavior. It must, in other words, take stock of his present needs and interests and encourage him to achieve their satisfaction in ways that are as full, as varied, as significant as possible. For example, if he shows unusual interest and ability along certain lines, the school can help him to engage in activities that reach out beyond present experience into uncharted seas, rather than hold him to routine. It must cultivate an atmosphere in which the devising of new and better ideas and procedures is encouraged, rather than one which conveys the feeling that it is somehow not quite right to differ from accepted and established ways and that variations interfere with smooth working arrangements. Above all the school must refrain from doing the student's thinking for him, but must instead stimulate him to the most fruitful endeavor along lines that grip him.

The Disposition and Ability to Use Reflective Thinking in the Analysis and Solution of Problem Situations

In a democracy, where each person is expected to take part in policy-making and to direct his own life, the disposition and ability to analyze problem situations is peculiarly necessary.

The youth of today must contend with problems that call for a high degree of effective intelligence. They live in a transition period heavily charged with both threat and promise, and their

social participation must be creative rather than adaptive merely. The society in which they are to play a responsible rôle becomes increasingly complex. Without instrumentalities of skilful analysis it is impossible even to disentangle the intricacies of its organization; control over the dynamic factors involved requires facility in understanding intellectual abstractions and adeptness in using them to formulate effective plans of action. Furthermore, under modern conditions this latter process is dependent for its success upon adroitness in the communication and clarification of thought, on the one hand, and sensitivity to speciousness in special pleading on the other.

Perhaps never in history has there been so crucial a case for emphasis upon the development of intelligence—if intelligence be taken to mean insight into the ramified complexities of modern life and the inclination and ability to bring order and value out of them. It is essential that experience in the secondary school eventuate in methods of thought adequate for creative social participation, and that students, as persons, become increasingly capable of putting these methods to work in the interest of democratic values.[15]

Reflective thinking originates when an individual feels a sense of perplexity, which leads to the identification and formulation of a problem. Further progress depends upon the development of a tentative hypothesis (or hypotheses) based upon the data at hand, and this in turn often suggests the need for additional data. If the data tend on analysis to support the original hypothesis, this hypothesis is taken as a conclusion, and the problem is considered solved. On the other hand, the data may eventually be found to weaken the hypothesis or make it wholly untenable, which leads to a similar examination of alternative hypotheses until a satisfactory solution is reached.

None of these phases into which reflective thinking may be analyzed is really distinct in the sense that it occurs as a definite step in a sequence. They all go on more or less simultaneously, each doing something to perfect the others. Thus

[15] Thayer, Zachry, and Kotinsky, *op. cit.*, Chapter I, "Reasons for a Reëxamination of Secondary Education."

the analysis of an hypothesis may give rise to new perplexities and new or modified hypotheses requiring new data. The final conclusion is but the culmination of an interacting process that has been going on since the inception of the problem.

The acceptance of responsibility by the teacher for the cultivation of reflective thinking obviously has important implications for both content and method. It calls for an emphasis upon the problem approach—and the problems must be genuine for the student, rather than important only to the teacher. It means that the student should have continuous opportunity to arrive at understandings through thinking, rather than through memorizing, and opportunity and encouragement to apply such understandings to new situations. The teacher should seize every opportunity to help the student see how reflective thinking, as practised in the school-room, is applicable elsewhere.

In stressing reflective thinking as "the method of intelligent learning" no gratuitous assumptions are here made concerning mental discipline or the transfer of reflective thinking from situations in which it is learned and practised to situations that are quite different. Reflective thinking must, for the great majority of people, be learned and practised in connection with the solution of social problems if it is to be applied by them to such problems at a later time. These considerations suggest that through school experience the student should come to appreciate the value of reflective thinking, to see how the method has enabled man to banish many unreasonable fears and superstitions, better to control his life and better to attain his ideals. It is hoped that through the development of such understandings the student may develop both ability and determination to apply reflective thinking to his problems in every pertinent area.

This list of characteristics of personality is by no means complete. One must also keep in mind that these characteris-

tics are not mere "abstract absolutes," valuable as ends in themselves. "Coöperation" may be practised with a band of racketeers or "reflective thinking" used in planning a bank robbery, and the more the individual cultivates these characteristics with reference merely to such isolated aims the greater menace he is to society. But as the Committee sees it, none of these characteristics is justifiably to be sought in isolation; always they are relative to the constituent ideals of shared democratic living. Thus "coöperativeness" is only to be encouraged when it furthers respect for personality, shared responsibility for common concerns, and the like. It must be always interpreted in the light of the complete personal-social situation in which it functions.

The interrelatedness of these characteristics must also be stressed. They do not exist as independent entities which can be singled out and cultivated separately; they are distinguishable aspects of the total personality of the individual, and each enters into and remakes the others. As the individual seeks to achieve unity and consistency of behavior, these qualities give character to his pattern of living. Moreover, there is no single pattern of life which fits all individuals. Young people differ in their capacities, interests, and needs, and it is neither feasible nor desirable to expect all to develop the same types of personality. On the contrary, there are an indefinite number of individual patterns, and to all of them the broad purpose of general education can be definitely related, though with differing application to each individual.

SUMMARY FORMULATION OF THE PURPOSE OF GENERAL EDUCATION

A brief statement of the purpose of general education may now be formulated. General education is designed to help the individual meet his needs in ways that are consistent with and promote social welfare. A fourfold classification of the basic aspects of living facilitates the identification of these

needs and of the experiences through which they may be met. In this process education should strive to achieve the values of the democratic way of life and to develop related desirable qualities of personality. The purpose of general education may, then, be summarized as follows:

THE PURPOSE OF GENERAL EDUCATION IS TO PROVIDE RICH AND SIGNIFICANT EXPERIENCES IN THE MAJOR ASPECTS OF LIVING, SO DIRECTED AS TO PROMOTE THE FULLEST POSSIBLE REALIZATION OF PERSONAL POTENTIALITIES, AND THE MOST EFFECTIVE PARTICIPATION IN A DEMOCRATIC SOCIETY.

THE RÔLE OF MATHEMATICS IN ACHIEVING THE PURPOSE OF GENERAL EDUCATION

The teacher of mathematics can make certain indispensable and distinctive contributions to the attainment of the broad purposes of general education. He can also make certain contributions that are similar in nature to those made by teachers in other fields. This section seeks to define both of these types of contribution in the large, and so to foreshadow detailed discussion of the special resources of mathematics in subsequent chapters.

Any field of study deserves a place in the curriculum only insofar as it has a unique rôle to play in meeting the educational needs of students. Although teachers in all departments of the school share the major purposes of education and can unite in discussing their common objectives and common difficulties, the teacher who by taste and training is especially well equipped along some one line may be expected to have his own particular contribution to make.

But in investigating the rôle of a particular field of study it is necessary to examine the tracery of its interconnections with other fields. In the past this necessity has escaped the attention of many, largely because of their continuing faith in pure discipline. As a result, the resources peculiar to each field have seldom been properly focused upon attaining com-

mon aims, and students have neither recognized the mutually reënforcing rôles of the concepts, methods, and techniques of the various areas of human knowledge nor have they profited from this reënforcement in meeting their needs.

Furthermore, the experience of the student in the school has all too seldom been designed as experience in democratic living. The teacher of each subject, preoccupied with imparting his own particular knowledges and skills, has failed to devise classroom methods with this larger aim in view. The development of desirable characteristics of personality has been largely left to chance through lack both of insight on the part of teachers in the different fields and of coöperative planning and action on the part of the staff as a whole.

The generality of the educational aims formulated in the first part of this chapter makes further discussion necessary in order to clarify the common and differential provinces of the various areas of instruction. The present discussion therefore attempts to make explicit the interconnections of mathematics with other fields in meeting the educational needs of students and developing personalities capable of creative participation in the democratic way of life.

The Rôle of Mathematics in Meeting the Needs of Students

Adolescents encounter certain problems [16] as they strive to meet their needs in the basic aspects of living. The "wise selection and use of goods," a need in the area of economic relationships, for example, may find a specific instance in the problem of purchasing a new suit of clothes. The function of the school is obviously not to select and buy the suit for the student. But school experience should enable him to make his own selection more wisely—not only in this instance but in others. Helping him to solve this problem is one way of helping him to meet his needs.

[16] "Problem" is here to be interpreted not as an exercise of the traditional sort assigned for solution in mathematics classes, but as a difficulty appreciated by the student and awakening in him a desire for its solution.

But the wise selection of any article involves a number of considerations. What style is appropriate for the individual and the occasions on which he intends to wear the suit? What are the wearing qualities of various fabrics? What determines their relative warmth or coolness? What factors are to be weighed in deciding how much to pay for a suit? Should one buy only union-made clothes?

These questions illustrate the fact that a need cannot be adequately met without the coöperation of teachers of home economics, art, natural and social science, as well as teachers of mathematics. On the one hand, some essential consideration in the wise selection of goods would be lacking without the pooling of their separate and distinct points of view, sensitivities, techniques, methods, concepts, and backgrounds of knowledge. And on the other hand, all fields of human study and knowledge rely upon reflective thinking, and all teachers should be concerned with improving the student's ability in problem-solving, as well as in developing other desirable characteristics of the personality. Only by coöperative work and planning can they reënforce one another's efforts to these ends.

The rôle of the mathematics teacher in this process is analogous to that of any other teacher. Just as the art teacher may be concerned with the problem of taste, and the teacher of natural science brings his specialized knowledges and techniques to bear in analyzing and solving some particular aspects of a problem, the mathematics teacher makes his special contribution whenever quantitative data and relationships or the facts and relationships of space and form are encountered. The highly effective special symbolism and methods of mathematics have been developed in order to treat just such aspects of experience, and the actual problems of living involve them to an extent that should not be underestimated. The teacher of mathematics bears the responsibility of equipping students to solve such problems with the aid of mathematical concepts and methods as they seek to meet their needs

throughout life. In this process he also has the responsibility of throwing light on the nature of problem-solving.

The mathematics teacher also has a rôle to play in meeting certain needs that do not involve problems of quantitative and spatial relationships. Like other teachers, he is the representative in the school of a special field of human activity of inestimable social significance to all. For some persons in the world outside the school—engineers, scientists, actuaries, statisticians—mathematical activities are a means of livelihood, and to some they are a source of intense personal satisfaction. Adolescents need to feel the social bearings and import of their activities as students, and the teacher of mathematics may acquaint them with persons, groups, and institutions using mathematics in furthering the advance of human knowledge or in solving community and social problems. In this same process he helps them meet their need to understand the vocational reference of their school interests. And he may help them to estimate the value of mathematics as a personal interest on a basis sounder than either its promise of pecuniary rewards or the public esteem in which it is held. In meeting needs of this kind the teacher of mathematics uses his field precisely as the teacher of art, or English, or science, or any other field uses his, but it is his own special field, and he alone is fully sensitive to its personal and social values.

The importance of mathematics in contemporary culture raises the question of the extent to which students in general should be urged to study advanced mathematics. The professional mathematician not engaged in research at a university, or in teaching, has not many lines of occupation open to him. But some writers, enthusiastic over the value of mathematical information and discipline, would seek to show that practically every worth-while activity calls for advanced training in mathematics either directly or indirectly. Nor is supposedly expert evidence lacking. For example, a jurist perhaps attributes most of his critical acumen to early train-

ing along mathematical lines. Some surgeon may believe his powers of analytical diagnosis due to algebraic training, and credit the study of geometry for much of his essential spatial intuition and sense of form. A minister may find in mathematics evidence of the Infinite Mind of God and see in the invariance of mathematical relations a promise of an everlasting future of perfect harmony. Despite the sincerity of such views, one may wonder whether those who enjoy mathematics may not attribute to thinking in this subject what may more properly be said to characterize thinking in general. Similar encomiums indeed have been voiced for classical training, for a godly mother, and for the incentive due to a childhood of privation. Where one algebraist finds the finger of the Lord in an algebraic identity, another no less eminent admits that as for the existence of God, he has found no need to make such an hypothesis.

In brief, the citing of selected examples as contrasted with the technique of statistical sampling and the use of controls is poor reasoning even if used in a noble cause. One could by such methods make out a strong case for the advantage to one's future career of being born, say, on a Thursday. An impressive list of distinguished persons may be cited who have been born on any desired preassigned day of the week—namely, approximately one-seventh of all famous people whose birthday is known. And the chief warning to teachers to be content with more modest claims for the values of mathematical training lies in the number of those who dislike mathematics and still have found success in some chosen line of work.

Yet it seems true that to a rapidly increasing extent specialists are being called for in all lines of adult enterprise, and with specialization comes a steadily increasing mathematization of all fields. For example, business administration, psychology, education, and biology all invoke mathematical statistics of an advanced sort where but a few years ago mathematics would have been regarded as an impertinent

intrusion. Furthermore, in an enlightened community it might be reasonable to expect, for every practising specialist, many intelligent alert citizens who understand the nature of the problems being tackled by the specialists who serve them.

The rôle of the mathematics teacher in meeting needs that involve problems in quantitative and spatial relationships is illustrated in the Appendix of this report. His rôle in developing desirable characteristics of personality is discussed in the section below, and the whole of Part II is devoted to an exploration of his very special contribution in the common task of developing a high order of ability in reflective thinking and problem-solving.

THE RÔLE OF MATHEMATICS IN DEVELOPING PERSONAL CHARACTERISTICS ESSENTIAL TO DEMOCRATIC LIVING

In stating objectives for mathematics instruction it has not been customary to stress the development of such qualities of personality as tolerance, coöperativeness, self-direction, creativeness, social sensitivity, and esthetic appreciativeness. Yet teachers of mathematics can share with other teachers in this task, in some instances through the choice of problems on which students work, in others through the special resources of mathematics, and in still others through the way they guide the conduct of the class as a social group.

Social Sensitivity

Social sensitivity is a quality of personality that may be increased through the proper choice of problems to be studied. If these problems are socially significant, students may not only learn proper techniques for presenting and interpreting data, but may also at the same time become increasingly familiar with and sensitive to important social facts and concepts. For maximum effectiveness in this respect

teachers of mathematics should coöperate with teachers of the social studies and other fields in choosing the problems and related data to be analyzed and studied. It should also be noted that without mathematical concepts and methods, chiefly statistical in nature, it is impossible to be fully sensitive to many dynamic social factors and their interplay. Thus the teacher of mathematics has a unique contribution to make in equipping the students with methods for understanding wider social problems and relationships.

Esthetic Appreciation

The teacher of mathematics may contribute to growth in sensitivity to the esthetic quality of experience through appropriate use of the unique content and methods of his own field. Many people respond favorably to quantitative facts and relations. They enjoy statistical comparisons, economic estimates, and mathematical formulations of physical principles. Arithmetic and algebra are for them natural and highly prized means for interpreting their environment; through them they may find ways of manifesting their personality. The recognition of familiar elements in new contexts, which contributes to the satisfaction of the successful student in mathematical courses, also influences his appreciation of geometric form as seen in the world around him. Mathematical instruction may enhance appreciation of geometric form as it occurs in nature, art, industry, and architecture. In the application of geometric construction to artistic design, the student has an opportunity to exhibit his esthetic taste and to create new combinations more or less expressive of some aspect of his personality.

Mathematical exposition, which involves the choice and arrangement of words as well as of mathematical symbols, has esthetic overtones. On reading an artistic piece of mathematical exposition, many mathematicians feel pleasure akin to that experienced on reading a selection of good poetry or on seeing a well-painted picture. Under superior teaching second-

ary-school students may have similar experiences at their own level of appreciation. A taste for strict logical deduction, a hearty respect for the power of reasoning, a confident faith in the value of sound inference, all these aspects of an appreciation for logical thinking are particularly appropriate objectives of mathematical study. Finally, an understanding of the rôle that mathematics has played in the drama of civilization has appreciational qualities which appeal to many students.

Tolerance, Coöperativeness, Self-Direction, and Creativeness

Such qualities of the personality as tolerance, coöperativeness, self-direction, and creativeness may be developed no matter what may be the content of the problems to which students are giving their attention, provided only that it is a problem real to them. Furthermore, the development of these characteristics is not dependent upon the peculiar methods and concepts of mathematics. But the mathematics class, like any other class, is a social group, and its experiences or activities may provide experience in either desirable or undesirable social living.

If the mathematics teacher is fully aware of the meaning and desirability of tolerance and coöperativeness, he can foster their development by consciously adjusting classroom activities to this end. This is especially true when the class undertakes work on a comprehensive problem which requires coöperative activity in the collection of data, for example, or even in the formulation of the problem. Again, individuals or small groups, taking their departure from tenable but different basic assumptions, may on occasion present conflicting conclusions on the same problem. In a case like this properly guided class discussion may contribute to growth in intelligent tolerance. These examples serve to show how certain types of class activity offer a means of developing characteristics of personality appropriate to democratic group life.

In addition, mathematics, like other fields, can contribute to an appreciation of the values of tolerance and coöperation through developing insight into its own history and development as a science. The following quotation clarifies this type of contribution.[17]

> The . . . teacher can use his subject-matter in several ways to foster attitudes of coöperativeness in his students. . . . He can show them how the sciences themselves offer striking evidence of coöperation. A science, they can see, is seldom nationalistic or race conscious; it is the product of the collective activity of men who are widely separated in space and time, linked in many cases only by their common interests and attempts to understand and control nature. Even the geniuses of science have depended upon their fellow-workers—Newton himself said, "If I have seen a little farther than others, it is because I have stood on the shoulders of giants." The increasingly rapid progress of science in modern times may be attributed to coöperative attack by many men upon more difficult and complicated problems; to no small extent it has been due to the fact that scientists publish their work widely and look for help from other scientists throughout the world.

In the same way the general classroom methods used in teaching mathematics may lead to growth in intelligent self-direction and creativeness, or they may retard such growth. Responsibility for planning and carrying through individual projects and for evaluating one's performance in them is useful in this connection. Teachers of mathematics can encourage young people to "reach out beyond present experience into uncharted seas," rather than confine them to routine activities. A simple illustration relates to geometric exercises of the traditional sort. Here the opportunities for creative activity could be greatly increased if, instead of giving both the assumptions about the figure and the conclusion, the teacher returned at least occasionally to the practice of an earlier day and presented the figure only,[18] suggesting to students that they attempt to discover all the facts and rela-

[17] *Science in General Education*, p. 284.

[18] It is said that when an ancient geometer had discovered a new theorem, he presented only the figure and the word "Behold!" to his colleagues.

tionships which appear to hold true about it. If the figure is not too simple this constitutes a stimulating creative exercise.[19] Creativeness may also be encouraged in discovering and formulating problems, in devising methods of attack, in recognizing relationships among data, in discovering methods of proof, and in presenting conclusions in expositional or other forms. But if mathematics is to be a field for creative activity, the approach to problems must involve a type of investigational experience which is an adventure into the unknown—it must provide constant opportunity for discovery.

Mere moralistic precepts are of course futile in the development of any of these characteristics. All aspects of classroom activity have to be intelligently guided with reference to their effects upon the personalities of students if growth in this respect is to be achieved. The personality of the teacher, the pattern of student-teacher and group relationships, and the atmosphere of the school are also highly influential. The mathematics teacher must have insight into dynamic factors in the personality development of adolescents if his part in their education is to be fully effective.[20]

The Disposition and Ability to Use Reflective Thinking in the Solution of Problems

The development of intelligence in analyzing problem situations, otherwise referred to as reflective thinking, although but a part of the purpose of general education, is so essential a part as to be given a major place in this Report. But reflective thinking is not the special province of any single subject-matter course. Rather, after one fashion or another, it is the concern of all departments of the school, for it stands at the heart of the method of scientific inquiry—not only

[19] Perhaps one reason why the Theorem of Pythagoras is so famous is that it has been a rich field for creative activity of this kind. No less than 167 different proofs are known, one of these having been discovered by President James A. Garfield!

[20] See Chapter XII, "Understanding the Student," page 269.

within the special fields of natural science, nor only within science so interpreted as to cover the activities concerned with developing all organized knowledge, but in every case where intelligent postponed decision is called for. Whether one considers the military officer planning a campaign, the lawyer drawing up a brief, the merchant analyzing the sources of his overhead expenses, the child deciding which parent is most likely to grant a special privilege and just how and when the subject should be broached, there one meets a problem situation demanding analysis, a call for reflective thinking.

Although sharing with others the task of developing the disposition and ability to use reflective thinking as well as other characteristics of personality, nevertheless it is here that teachers of mathematics can make their major unique contribution. Part II of the Report discusses a number of major concepts which are involved in problem solving and shows how mathematics viewed from this angle often plays a fundamental rôle in the process.

Bibliography

I. *General*

It is of course impracticable as well as undesirable to attempt to list the major contributions to current thought in the broad field of basic educational philosophy. The following recent works have proved of special interest to some members of the Committee:

Bode, Boyd, *Democracy as a Way of Life* (New York, The Macmillan Co., 1937).

Brewer, John M., *Education as Guidance* (New York, The Macmillan Co., 1932).

Counts, George S., *Dare the School Build a New Social Order?* (New York, The John Day Co., 1932).

Day, Edmund E., "Basic Responsibilities of General Education in America," *The Educational Record,* Vol. 17, Supplement No. 10, October, 1936, pp. 8–22.

Demiashkevich, Michael, *An Introduction to the Philosophy of Education* (New York, American Book Co., 1935).

Dewey, John, *The Way Out of Educational Confusion,* The Inglis Lecture (Cambridge, Harvard University Press, 1931).

DOUGLASS, Harl R., *Secondary Education for Youth in Modern America,* a report to the American Youth Commission (Washington, D. C., American Council on Education, 1937).

DURANT, Will, *The Mansions of Philosophy* (New York, Simon and Schuster, 1929).

EVERETT, Samuel, editor, *A Challenge to Secondary Education* (New York, D. Appleton-Century Co., 1935).

———, *Democracy Faces the Future* (New York, Columbia University Press, 1935).

HANNA, Paul R., and others, *Youth Serves the Community,* a publication of the Progressive Education Association (New York, D. Appleton-Century Co., 1936).

HOPKINS, L. Thomas, and others, *Integration, Its Meaning and Application,* Committee on Integration of the Society for Curriculum Study (New York, D. Appleton-Century Co., 1937).

RAINEY, Homer P., and others, *How Fare American Youth?* A report to the American Youth Commission of the American Council on Education (New York, D. Appleton-Century Co., 1937).

Recent Social Trends in the United States, Report of the President's Research Committee on Social Trends (New York, McGraw-Hill Book Co., 1933). Especially Introduction.

RUGG, Harold, *Culture and Education in America* (New York, Harcourt, Brace and Co., 1931).

Social Change and Education, Thirteenth Yearbook of the Department of Superintendence of the National Education Association (Washington, D. C., 1935).

THAYER, V. T., ZACHRY, Caroline B., and KOTINSKY, Ruth, for the Commission on Secondary School Curriculum (Progressive Education Association), *Reorganizing Secondary Education* (New York, D. Appleton-Century Co., 1939). See this book in particular.

The Unique Function of Education in American Democracy, Educational Policies Commission and the Department of Superintendence of the National Education Association, 1937.

WHITEHEAD, Alfred North, *The Aims of Education and Other Essays* (New York, The Macmillan Co., 1929).

———, *Science and the Modern World* (New York, The Macmillan Co., 1925).

II. *References on the Rôle of Mathematics in Education*

The principal literature on the rôle of mathematics in education is to be found in the Yearbooks of the National Council of Teachers of Mathematics, and in numerous articles published in *The Mathematics Teacher, School Science and Mathematics,* and *The American Mathematical Monthly.* See in particular:

BETZ, William, "Mathematics as a Universal and Permanent Element

in Education," *Teachers College Record,* Vol. 39, November, 1937, pp. 132–152.

HEDRICK, E. R., "Crisis in Economics, Education, and Mathematics," *The Mathematics Teacher,* Vol. 29, March, 1936, pp. 109–114.

———, "The Contribution of Mathematics to General Education," *Secondary Education,* Vol. VIII, September, 1939, pp. 229–234.

MOULTON, E. J., "The Future of Mathematics," *School Science and Mathematics,* Vol. 36, February, 1936, pp. 124–137.

The Place of Mathematics in Modern Education, Eleventh Yearbook of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1936).

The Place of Mathematics in Secondary Education, Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, *Fifteenth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1940).

PRESSEY, S. L., *Psychology and the New Education* (New York, Harper and Bros., 1933), Ch. XIV.

REEVE, W. D., "Mathematics and the Integrated Program in Secondary Schools," *Teachers College Record,* Vol. 36, March, 1935, pp. 497–507.

Part II

MAJOR UNDERSTANDINGS GROWING OUT OF MATHEMATICAL EXPERIENCE

III

INTRODUCTION TO PART II: SOME CONCEPTS BASIC TO PROBLEM-SOLVING, CRUCIAL IN DEMOCRACY, PERVASIVE IN MATHEMATICS

Part II of this report is devoted to a discussion of the following major concepts, with special reference to their mathematical aspects:

Chapter IV. Formulation and Solution
Chapter V. Data
Chapter VI. Approximation
Chapter VII. Function
Chapter VIII. Operation
Chapter IX. Proof
Chapter X. Symbolism

In the judgment of the Committee, an understanding of these concepts is of inestimable value to the teacher in helping the student to learn the nature of both mathematics and the problem-solving process, and to appreciate the values of a democratic society. It is also the belief of the Committee that the student should gradually develop an always more mature understanding of the meanings of these concepts and that they should acquire richer and more discriminating content for him as his study of mathematics progresses.

THE RELATION OF THESE CONCEPTS TO THE PROBLEM-SOLVING PROCESS

As intimated at the conclusion of the previous chapter, the major rôle of mathematics in developing desirable characteristics of personality lies in the contribution it can make to growth in the abilities involved in reflective thinking, or

problem-solving. In the opinion of the Committee the study of mathematics is of educational value because mathematics can be made to throw the problem-solving process into sharp relief, and so offers opportunity to improve students' thinking in all fields. As one traces the biography of any problem situation from its inception in perplexity and conflict through to its mastery by means of human intelligence, those places in which mathematics has a special contribution (which for some problems means throughout) may be seen to come under one or more of the headings of the chapters in Part II.

Formulation and solution are discussed first among the concepts related to problem-solving because they are central to the process and in a way primary. To solve problems successfully, one must have a clear notion of what it means to formulate a problem, and also of what is meant by a solution. Before one can lay out an intelligent plan of campaign in the resolution of a difficulty, identify significant factors, make hypotheses, define terms, make assumptions, know where to look for information, or what to use in the way of technique, both the impelling problem and the aim sought must be reasonably clear. Formulation and solution are discussed together because the foreseen end affects the formulation of the problem, and the way the problem is formulated affects both the final solution and the process by which it is secured. In the course of solving a problem the more specific purpose originally laid down may be redefined or even rejected, but some end is always kept in view, and this end-in-view directs all steps in the process.

When a problem has been formulated, the next step is usually to collect the *data* necessary for arriving at a solution. But not all data are equally relevant, representative, accurate, and reliable. In order to solve many problems competently, it is necessary to be discriminating about the acceptability of data in these terms, and to collect, record, and organize them in ways appropriate to the problem.

Acceptable data ordinarily must not only be collected and

organized, but analyzed and interpreted before it is possible to draw conclusions from them. In order to analyze and interpret many types of data it is necessary to understand *approximation.* Approximation is always inherent in any problem that involves data collected by measurement. It is also usually inherent in the solution of problems that require the description of groups of data or the discovery of trends and relationships. In the latter case the analysis and interpretation call for the use of statistical concepts and methods. The attempt to analyze approximate quantitative data without a clear understanding of appropriate methods is likely to be futile. Moreover, many qualitative statements implicitly involve some degree of quantitative approximation, and without recognition and understanding of this fact conclusions drawn from such statements are likely to prove misleading. Some examples of statements of this kind are: "Americans enjoy a better standard of living than any other people in the world"; "Germans are sentimental and given to regimentation"; "Diabetes is inherited"; "Normal boys are not interested in marriage."

In the solution of many problems the notion of *function* —of some sort of determinate correspondence between two (or more) sets of data—underlies the entire process. The particular correspondence may in some cases rest upon definition (as in a table of squares) or on arbitrary assignment (as in the listing of telephone numbers to correspond to the names of subscribers). In other problems the correspondence is not arbitrary but is like the relationship between distance covered and elapsed time in the case of a falling body. In analyzing and interpreting data the investigator seeks to discover a determinate correspondence of this kind; he hopes to find such relationships among the variables that knowledge of the values of one or more of them serves to determine uniquely the values of some one other variable. If such relations can be shown to exist, more precise and often more far-reaching conclusions can be drawn; sometimes formulas

can be constructed that serve to facilitate and extend the analysis and to summarize relationships in compact symbolic form. Understanding of the concept of function as an ideal toward which the investigator strives in his work lends guidance in the attack on quantitative problems. Furthermore, some aspects of the concept of approximation can be completely understood and appreciated only in the light of comparison or contrast with the concept of function.

Problems cannot be solved without some degree and kind of "doing"—hypotheses must be tested by actual experiments, by "imaginary experiments," by computation, or some other *operation.* Each field of study has developed relatively unique kinds of operations appropriate to the analysis and interpretation of its own particular type of data. Mathematical operations are the active processes—techniques and methods—by which mathematical symbols, representative of data, are manipulated. The concepts of approximation and function serve primarily to guide the selection of procedure and methods, but the actual carrying out of the process requires the performance of operations upon the data.

In order to judge the validity of the solution of a problem it is necessary to understand the nature of *proof,* the basic principles of inductive and deductive reasoning. Without insight into the relation of conclusions to initial assumptions and to defined and undefined terms it is impossible either to work through to a solution upon which one can rely with assurance, or to accept or reject with confidence the solutions proposed by others.

Finally, the process of problem-solving involves *symbolism* of some sort. Without symbols—which may be words or may be other signs or marks representing concepts—many forms of reflective thinking are hardly possible. The use of symbols facilitates the manipulation of ideas and is essential for the communication of ideas to others.

The student's growth in understanding of these concepts as related to problem-solving means no mere superficial ex-

tension of his vocabulary, no mere contact with undigested information. Instead it should result in more effective and mature ability to resolve complex problems, and be manifested through more appropriate behavior in the face of problem situations in all the aspects of living. The development of such ability is not the task of teachers of mathematics exclusively. Teachers of many fields share this responsibility and can contribute toward students' well-rounded growth in the ability to solve problems.[1] Thus this constitutes a common ground for coöperative development of an essential characteristic of personalities equipped to function creatively in a democracy.

THE RELATION OF THESE CONCEPTS TO DEMOCRACY

But what, it may be asked, have the formulation and solution of problems, data, approximation, and the like, to do with "fostering the ideals of democracy"? In the judgment of the Committee the connections are immediate as well as fundamental. They become clear when the ways in which problems are formulated and solved in countries where there is less democracy (as in Germany, 1939) are contrasted with the formulation and solution of problems in a country where there is more democracy (as in the United States, 1939).

In the first place, the difference lies in the very possibility of formulating problems at all. In dictatorships like Germany (in 1939) the man on the street is not permitted even to formulate many problems, such as the following, which di-

[1] Teachers of English, science, and the social studies are also turning their attention to aspects of this objective. See the Report of the Language Section of the Committee on the Function of English in General Education, *Language in General Education;* the Report of the Committee on the Function of Science in General Education, *Science in General Education,* Ch. VII; The Report of the Committee on the Function of the Social Studies in General Education, the *Social Studies in General Education.* All are published or forthcoming reports of Committees of the Commission on the Secondary School Curriculum of the Progressive Education Association, published by D. Appleton-Century Company, New York.

rectly concern him: We Germans are told we have got rid of unemployment, but are we really better off? How does it go with the Russian worker and with the American worker? Is all this rearmament and military drill really for peace? Is it possible to believe that Jews are actually in control in all the countries that are against us, or that the Jews were trying to poison us? With one victory after another achieved by our Leader, why are we still so miserable? Secondly, in Germany the Leader, having permitted some problems to be formulated after a fashion, takes upon himself the right to provide the only acceptable answers. Finally, in dictatorships these problems are so formulated that they cannot be solved by the use of reflective thinking, and only mystical answers are possible to the questions raised. If the Leader is asked his opinion on an important issue, he replies, "No one can prevent me from fulfilling the mission entrusted me by Destiny." If asked what his mission is, he replies, "My mission is to save the country, the country must and will be saved. Nothing will prevent me from saving my country. No one can oppose Destiny. That, brothers, is my mission." Objective formulation, the need for checking apparent results, emphasis on method of formulation and solution stressed in Chapter IV, are all negated as completely as possible in anti-democratic countries.

The United States is called more democratic at least partly because in this country the situation with reference to the formulation and solution of problems is different. In far fewer instances is the possibility of formulating our pressing problems restricted. True, there are groups that do not wish certain questions to be asked, and these groups are sometimes powerful (advertisers exert a large measure of control over all the widely circulated media of expression, for example, and so-called "patriotic" groups try to forbid the objective study of Marxist theories of society). But there is still a considerable area of freedom for the asking of important questions and for the demand that answers be reliably and re-

sponsibly supplied. There is more respect for experts who are judged by their previous success, and less for those who solve problems by revelation. More questions that concern actual living are asked, and they are so formulated that objective answers are possible. And one of the most important privileges accorded the individual in the democratic countries is that of being allowed to check apparent facts, at least in some important areas. What Ford says is happening in his plants can be checked against what others say. There is no way of checking on the statements of industrialists in Germany (1939).

The United States and the other democracies will increase their measures of democracy when there is organized and widespread demand for it. The teacher of mathematics who helps his students realize the necessity of the types of freedom discussed above is building their allegiance to democratic ideals and at the same time fostering the fuller realization of these ideals in the future. If the child, accustomed to rely on the judgment and decisions of adults, is to achieve a status of mature independence, he must learn to identify his difficulties for what they are, to state his problems, and to describe clearly the nature of appropriate solutions. Questions such as "Exactly what is the problem that I am facing? What am I hoping to do? What sort of results should I seek?" must become habitual and fruitful, not only in relation to the more immediate aspects of the adolescent's life, but in his wider social relationships. It goes almost without saying that the teacher who does not permit his students to raise questions which are susceptible to mathematical treatment and which also seem important to them will not be able to make clear that mathematics has these direct connections with the ideals of a democratic society. Nor will he be able to develop in his students the allegiance to these ideals which alone assures their preservation.

A comparable analysis holds in regard to Chapter V on "Data." The very possibility of merely gathering and organ-

izing data in many important areas is precluded in anti-democratic societies. Neither authoritarian societies nor authoritarian groups within more democratic societies can permit the collecting of certain kinds of data about themselves or about subjects on which reliable data would impede the attainment of their ends. It is a striking fact that the availability of reliable statistical data on the standards of living of a country are a direct measure of the degree of democracy prevailing in that country as judged by other standards. The difficulty of securing data, however, applies not only in relation to such problems as the objective comparison of standards of living in Germany, Russia, France, and the United States, but also in relation to so-called business secrets, hidden accountancy, and advertising methods of certain groups within this country. Clearly, the path toward increased democracy lies in the direction of insisting on the right to collect and organize data for analysis and interpretation.

The crucial importance of widespread understanding of the concept of "Approximation" (Chapter VI) in a democracy is clarified when it is recognized that the word *statistics* originally meant "facts about the state." Without certain basic statistical notions, judgments concerning social phenomena are impossible, for social phenomena are group phenomena, and study of them necessarily involves approximations. The maintenance of democracy today is predicated upon the ability of large numbers of people to think clearly about problems that are essentially statistical in character. There is, in addition, a more specialized way in which understanding of basic statistical notions may serve to stem the inroads of anti-democratic forces. Such forces tend to create a spurious group divisiveness—mutual suspicion and mistrust among persons of various social, economic, or racial backgrounds. Valid reasoning about any characteristic of a group (a school class, an income group, a social class, or a race) depends upon knowl-

edge of how this characteristic is distributed both through the group under discussion and through the total population. The facts of wide variability among individuals and of the wide range of every so-called distinguishing character in every group that has been called a race establish the invalidity of prevailing concepts of race, and so point to the importance of developing widespread understanding of statistical concepts. Those who propound theories of racial superiority, of course, need more than a requisite knowledge of statistics to induce them to renounce their theories. It is likely, however, that the number of adherents they recruit will be diminished when more young people understand how to check their claims from a statistical point of view.

The concepts discussed in Chapters VII and VIII on "Function" and "Operation" are perhaps too specifically mathematical in reference to have an identifiable specialized bearing on democracy, aside from the contribution they may make to the common man's ability to think for himself in all areas of experience. But the ideas and concepts treated in Chapter IX on "Proof" are again typically related to democratic living. The notion of proof, as discussed there, is one that could not be encouraged in an anti-democratic country, and it would be well if students were to realize this fact at the time that they are learning something about the nature of proof.

The emphases of Chapter X on "Symbolism" are particularly important at a time when democracy is threatened by those who would consciously undermine it through the use of flamboyant symbols and specious phrases to delude and mislead the public, obfuscating its attempts to think clearly for itself. In order to offset the effects of propaganda, advertising, and special pleading, the man on the street must learn to test the soundness of what is proposed in words, numbers, and graphs by what these proposals mean—their antecedents and probable consequences in action. He must learn to be

wary, too, of the symbols he uses in his own thinking, lest they undermine his intellectual integrity and leave him a ready instrument for anti-democratic forces.

It may be noted that the bearing of each of these concepts on democracy and its maintenance under contemporary conditions is related more closely to "the free play of intelligence" than to either of the other two strands of the democratic tradition outlined in Chapter II—"optimum development of personality" and "reciprocal individual and group responsibility for promoting common concerns." Nor is this surprising, in view of the fact that the concepts themselves were chosen for emphasis partly on the basis of their relevance to reflective thinking and problem solving. Yet it must be clear from the above discussion that the maintenance of these other two strands of the tradition is dependent upon facilitating the free play of ideas among the people at large, and hence that the teaching of mathematics may be so conducted that it contributes to the enhancement of democracy in all of its aspects.

The contention of the argument above might be summarized in this way: The fruits of scientific (and mathematical) thinking in terms of application to the problems of human living require democratic societies for their full development. More important still, the kind of scientific and mathematical thinking that the Committee advocates developing in students—that is, thinking applied to the real problems in the basic relationships of living—provides a valuable tool for the preservation of a democratic way of life. Students who are in the habit of formulating real problems, and of insisting on genuine solutions, who know how to judge, collect, and interpret data, who are not misled by inaccurate or misleading statistics, and who know how to recognize valid proof, will not so easily be misled by propaganda, suppression of evidence, systematic calumny, demagoguery, or mystical symbols.

It may be remarked in conclusion that the fundamental operations of arithmetic, the formal solution of algebraic equations, the memorization of Euclidean propositions are taught in Germany (in 1939) possibly as well as they are taught anywhere. It is clear that mathematics taught as an abstract science contains little that is anti-authoritarian or pro-democratic; the same is not true for the kind of teaching of mathematics urged in this Report.

THE RELATION OF THESE CONCEPTS TO THE DEVELOPMENT OF AN UNDERSTANDING OF MATHEMATICS AS A UNIFIED FIELD

A set of major concepts worthy of special emphasis in the mathematics curriculum should serve to unify instruction in mathematics regarded not only as a tool for problem-solving, but as a science considered apart from possible applications as well. The number of concepts should be small in order that the list may be readily held in mind, and the concepts should have the property of systematic recurrence in diverse problems—or, stated negatively, they should not be the sort of concepts which, while necessary or at least useful in treating some problems, are not needed at all in many others.

If asked to state the major mathematical concepts involved in problem-solving many teachers might think first of labels appropriate only to some special aspect of mathematics, and might speak of equations, fractions, congruence, similarity, area, and the like. Important as such ideas are, they fail singly and as a group to meet the conditions stated above. They are not particularly helpful in the process of problem-solving in general; they are useful in some problems but not in others; they do not serve to unify instruction in mathematics; and finally, an inventory of concepts of this type results in an unwieldy list of items ranging in significance from the trivial to the profound. Thus the unquestionable usefulness of some of these concepts in many problem situations is

in itself not enough to make them major concepts in the sense intended here.

Most of the objections that hold against the use of ideas like "congruence" or "equation" as major concepts also hold against the use of the fields into which they are commonly classified—the subject-matter divisions of arithmetic, algebra, geometry, trigonometry, and so on. This classification reduces the number of concepts, but fails to assure the unification of instruction. The essential interdependence of arithmetic, algebra, geometry, and trigonometry that becomes obvious to advanced students easily escapes the perception of students pursuing these subjects separately and for the first time. Differences in special techniques are often allowed to hide the common aspects which alone justify applying the single name *mathematics* to all these seemingly diverse studies. Even within a single subject, such as algebra, the unity is easily forgotten amid a multiplicity of topics. When these special subjects are emphasized as such, not only does the student frequently fail to gain a sense of unity in mathematics as a science, but the unity in its methods of approach to problem-solving is also obscured.

Recognizing these difficulties, many teachers have sought eagerly for unifying principles. For several years the function concept has seemed to enjoy special popularity. Yet certainly this concept, viewed narrowly and treated rigorously, does not unify all of mathematics. Distinguished writers have been cited as authoritative sponsors for other unifying ideas such as that of logical implication, of group properties, of relations, of quantitative variation, and so forth. Undoubtedly some among these might be stretched to cover by devious extensions most of the important concepts of mathematics, but in the judgment of the Committee none of them is adequate for the purposes of the teacher. To the teacher one all-inclusive concept is less important than a few important ones permeating mathematics irrespective of special topic and

reasonably comprehensive of what is involved in problem-solving.

The list of unifying concepts to be emphasized in this Report might be altered in several respects without departing essentially from the conditions stated above, but in a general way these concepts seem to cover the subject and yet to deal with aspects so separate that a student with sound training in lines concerned with one may be deficient in his understanding of others. All are involved in dealing with mathematical questions that arise in analyzing many problem situations, and almost all (even all, by extension of meaning) are important in the solution of every problem whether or not it calls for characteristically mathematical treatment.

THE RELATION OF THESE CONCEPTS TO THE ORGANIZATION OF THE CURRICULUM

The Committee does not mean to suggest that these unifying concepts are to be considered separately or are to be made the basis of organization of separate groups of units devoted to each. The arrangement here adopted is intended to be, in the main, logical for anyone who surveys the field in retrospect. The order is not intended to represent psychological steps in learning or to parallel stages of adolescent growth.

In the ideal program these concepts would be employed repeatedly in the course of solving different problems. From time to time the emphasis would necessarily shift from one to another, since it would probably not be expedient for teachers to attempt to dwell simultaneously upon more than a few fundamental concepts. But it does seem possible to develop more than one at a time, and it is highly important that at least one be continually kept in mind. It is also to be remembered that the nature of these fundamental concepts is such that the power to give them formal expression and full appreciation comes only as the crowning step in a process

that may be lifelong. Some of the more abstruse ideas here mentioned in elaborating one or another of the concepts should perhaps never constitute explicit foci of attention for most students at the secondary-school level. They are included here because the teacher should understand all these ideas and should consciously direct his own teaching practice in accordance with them.

The mathematician seeks to obtain generalizations and rules of procedure that are applicable to a wide variety of specific situations. This characteristic of mathematics has influenced the Committee to organize the chapters of Part II in such a way as to show eventually how certain concepts apply in the various aspects of living rather than organizing about the latter and showing the concepts that apply to each. Although the Committee subscribes to the view that mathematics has something to contribute to the meeting of needs in every aspect of living, presentation of these contributions built around the basic aspects of living would have entailed considerable repetition in the Report. Thus, for example, the ability to interpret data expressed in graphical form is useful in relation to needs in each area; the specific abilities involved are essentially the same whether the data to be interpreted have to do with personal living, immediate personal-social relationships, social-civic relationships, or economic relationships.

It should perhaps be made doubly clear at this point that the Committee advocates planning curricular sequences primarily on the basis of concrete problems encountered in meeting educational needs in these four areas, rather than on the basis of logical sequences of the familiar sort, or separate subjects like algebra, plane geometry, solid geometry, and so on, or unifying concepts presented here or elsewhere. Secondary-school mathematics has been criticized as being "too general and too abstract" for secondary-school students. This criticism is justified less by the nature of mathematics itself

than by the tendency to impose certain general and abstract concepts upon the student—without his having had any responsible part in the gradual process of generalization and abstraction from concrete and specific instances arising in problems real to him.

The position of the Committee, briefly stated, is essentially this: A mathematics curriculum may be built by locating and studying concrete problem situations which arise in connection with meeting needs in the basic aspects of living. The major concepts here emphasized play a fundamental rôle in the analysis of these problems. They help to clarify the method of attack, and they tend to recur systematically in diverse problems. This recurrence in itself provides for the development of a sense of unity in mathematics as a method of dealing with problems. But in addition these major concepts serve to unify sub-concepts and related abilities customarily classified in separate subject fields—such as algebra and geometry. These sub-concepts, encountered first in concrete situations, should eventually be abstracted and generalized, and, in similar fashion, the major concepts should eventually serve to throw light on the analysis of problems arising in many different fields of thought.

In conclusion, it should be hardly necessary to remind the teacher that his contributions to the achievement of the major purposes of education are not confined to the development of the concepts and abilities particularly emphasized in the material of Part II. This Part is intended to supplement rather than to supplant Part I. The discussion is confined primarily to a detailed analysis of the specific contribution of mathematics to the development of ability in reflective thinking or problem-solving—indispensable both in meeting needs and in conserving democratic values. But the teacher of mathematics must build his total program toward an affirmative answer to such questions as the following:

Does it help the adolescent meet his needs in the major

aspects of living (personal living, immediate personal-social relationships, social-civic relationships, and economic relationships)?

Does it promote, enrich, and refine democratic ideals through the development of related desirable qualities of personality?

IV

FORMULATION AND SOLUTION

"If Mr. Horton's automobile averages 20 miles to the gallon of gasoline, and if gasoline costs 17 cents per gallon, how much will it cost him to drive 100 miles?" This kind of "problem" is fairly typical in mathematics classes. It is a comparatively simple task for the student to calculate that the expected answer is 85 cents, but he might solve many exercises of this kind without ever realizing that the relative ease of solution depends upon the way the problem as stated is already formulated for him.

In contrast, the problems confronted in the course of living from day to day are almost always complex, involving a number of evaluative and emotional elements as well as questions of fact. Take, for example, what might be any older student's perplexity as to whether or not he should go with a friend on an automobile trip during his holiday. There are some reasons why he would like to stay with his family, and perhaps he should—he may be under some obligation to do so. On the other hand, he may feel inclined to get away from home at all odds. But he cannot go if the trip will cost too much. Thus the solution of the problem is entangled in a number of issues each of which must be recognized and approached in an appropriate way.

It is not the specific function of the mathematics teacher to help the student resolve the "should" aspects of his problem, nor to decide how much cost is "too much." Neither is it his peculiar function to help the student come to grips with the non-rational elements in the situation—the student's own compelling desires and inclinations. But it is his responsibility to coöperate with other teachers in helping the

student to analyze the issues and to decide upon appropriate methods of approach to each, making clear which aspects of the total problem are amenable to mathematical formulation and best solved by mathematical means. And it is his opportunity so to guide the analysis of these strictly mathematical aspects of the total problem as to throw light upon the processes of problem formulation in all areas, and to increase the student's ability in this aspect of problem-solving.

The teacher of mathematics may recognize the values inherent in improving students' ability to formulate their problems, foreseeing the nature of appropriate solutions, and yet wonder how the teaching of a technical subject such as arithmetic, algebra, or geometry is to make any significant contribution. This is due in large measure to the fact that at present few mathematics courses provide adequate opportunities for students to practise the analysis of problem situations in other than a restricted sense. Most of the "problems" presented are so simplified and idealized that all that remains to be done is to recognize what operations will lead to the answer explicitly called for, and then to perform these operations—a routine task to be completed by a prescribed set of often relatively meaningless steps. Rarely does the student begin with a more comprehensive situation and go through the experience of simplifying and idealizing the problem for himself, so formulating it that he can work upon it and arrive at a solution which he himself conceives to be appropriate. Under these conditions he has no opportunity to realize the many assumptions and restrictions that have to be made in order to formulate and solve even the simplest problems capable of mathematical treatment.

Yet if the mathematics teacher recognizes an obligation to develop the student's ability in formulating and solving problem situations in many aspects of life, there is much that the study of mathematics may contribute. If this end is to be achieved, two steps are essential. First, students must be helped to formulate and solve their own problems—real

problems arising in connection with their needs. Second, their attention must be brought to focus on the processes of formulation and solution, so that they may recognize the characteristics of promising formulations and the nature of acceptable solutions.

FOCUSING STUDENTS' ATTENTION ON THE PROCESS AND NATURE OF FORMULATION

Experience in formulating problems is essential to the student, but this experience will bear its full fruit only if attention is brought to bear upon what has been involved in it. Only if the mathematics teacher guides his students in understanding the process and nature of formulation may he hope to assure an increase in their ability to formulate all problems in always more promising ways.

Restricting and Idealizing the Problem

It was assumed in Part I that problems arise in connection with basic human needs, and it was stated that personal health is one of these. As an illustration of problem formulation, then, students might consider some of the problems they encounter in maintaining physical and mental health. As they seek an answer to this question they may be helped to see that one naturally begins to think of factors which influence health—diet, exercise, rest, disease, accidents, and so on. This listing of factors (in some cases hypotheses) is a common procedure in connection with problem formulation and offers a convenient means of approach. But one can scarcely state the sort of situation or need one has in mind without involving some analysis or classification. This is illustrated by the phrase *physical and mental health* used above. Even at this early stage a breakdown into two types has been indicated. Formulation often depends upon the ability to recognize categories like these.

A second common step consists in selecting particular categories or factors for special study. Thus one may decide to restrict the study to problems primarily affecting physical health and concentrate upon accidents as one of the important factors that interfere with healthful living. In so doing it should be recognized that a painful accident may seriously affect mental health, but nevertheless the problem may be idealized by ignoring this phase at least temporarily. Although restrictions of this sort serve to reduce the scope of the inquiry, no problem has as yet been formulated. But progress has been made toward explicitness in the final formulation, for one now begins to ask questions like the following: "What sorts of accidents are most frequent in occurrence? Under what circumstances are these accidents likely to occur? How may these accidents be avoided?"

The first of the above questions calls for quantitative data, and hence is appropriate for study in a mathematics class. But the question is still vague in several ways. Are the "accidents" to be restricted to fatal ones, or are minor accidents, such as cutting one's finger, to be included? And what is the lower bound of frequencies included under the word "most"? Decision on these matters might lead to a revised question stated in some such words as the following: "What ten causes of fatal accidents in the United States rank highest in frequency?" The word *cause* in this formulation may harbor difficulty, but perhaps enough has been said in this brief sketch to illustrate in simple fashion the isolation of a reasonably precise sub-problem from an initially vague suggestion.

Similarly, in order to solve the mathematical aspect of the problem cited earlier, "How much will a holiday trip by automobile cost?" the student must recognize and take into account a considerable number of variables, among them being the distance, the gasoline and oil consumption of the car, the cost of gasoline and oil, the cost of meals on the road, and depreciation of the car. He must consciously decide to ignore

some factors while restricting his attention to a few of the others at a time. He must introduce certain concepts, such as "the average distance the car will travel on one gallon of gasoline," and so on. As a result, he eventually breaks up the problem into several parts that are capable of quantitative formulation—as, for example, "What will be the cost of gasoline per hundred miles?" Finally, by combining the results of a number of such parts, he obtains an approximate answer on the basis of which—in combination with "answers" to other aspects of his original problem—he may make his plans and decisions. If he is given experience in thus formulating a mathematical problem he may at the same time be helped to identify what is involved in problem formulation. This experience, together with the generalizations he draws from it, should improve his ability to cope with unformulated problem situations of other types.

The elimination of subjective, emotional, and evaluative factors in a problem situation for the purposes of scientific formulation. In the problem of whether or not to go on an automobile trip, cited above as an illustration, it was stated that a student might want to go at all odds, just to get away from his parents for a while and prove his independence, but that he cannot go if the trip costs too much. It was noted that mathematics as such has nothing to contribute in helping the adolescent to come to grips with his impelling inclinations. Furthermore, though mathematics helps to determine approximately how much a projected trip will cost, the question of whether or not it is "too expensive" depends on other factors.

Noticing such simple facts as these and generalizing upon them, the student may be led to recognize that those aspects of the problem situations he encounters which can be scientifically formulated are always non-evaluative and non-emotional. Stated differently, scientific and mathematical formulations are always precisely of such a nature as to minimize emotional, evaluative, and in fact all subjective elements.

They emphasize the common and the abstract, rather than the personal and the concrete.

This does not mean that the formulation and solution of those aspects of a total problem that are capable of mathematical treatment does not lead to solutions that are essential for proper evaluation. The adolescent cannot decide if the cost of his trip is "too much" until he knows "how much" it is. Moreover, answers on the scientific and mathematical aspects of a total problem may help to objectify for many individuals some situations that were at first approached oversubjectively and overemotionally. The adolescent can no longer feverishly declare that "he must get away at all costs" if he discovers that his projected trip will cost more than he can possibly spend.[1]

The main idea for students to gather from their study of this phase of what is involved in problem formulation is that neither mathematics nor science can tell any one what it would be desirable to do, or what it would be better to do. But they often can help to provide a *basis* for such decisions.

The effects of restricting the number of factors considered in the formulation of a problem. Attention has already been called to the importance of giving the student practice in ignoring certain pertinent factors in a total problem situation while he focuses his attention on others in order to formulate a solvable problem. Many problems cannot be solved at all unless idealized. The student should of course become conscious of this aspect of the process upon which he is engaged. But he should recognize also that in so restricting the factors to which he gives his attention, he is in many instances arbitrarily changing the initial problem into a new

[1] There is another way in which subjective, individual, emotional, and evaluative elements play a part in scientific problem formulation—a way perhaps too subtle and intricate to be called to the attention of any but more mature students. To the extent that there is some possibility of choice, the situations selected for study, the factors in these situations chosen for special emphasis, the way the final formulation is phrased, all reflect to some degree the individuality of the investigator, and so are influenced by emotional factors in his personality.

one. This changing and idealizing of the problem has several effects:

1. *The problem becomes thereby simplified.*
2. *The solution obtained to a problem so formulated may be inapplicable because of the unintentional suppression of crucial elements.*[2]
3. *The solution obtained to a problem so formulated may be applicable to an extensive class of problems.*[3]

The student should eventually come to realize that mathematics finds one of its chief justifications in its extreme idealization of problems and the resultant wide range of application of mathematical solutions.

Formulation on the basis of economical hypotheses. Different formulations are not necessarily equally acceptable, even if they lead to the same formal answer. Ordinarily that formulation and line of attack are to be preferred which observe economy of initial material most effectively. To drag in extraneous data, to invoke unneeded hypotheses, or use roundabout techniques, all these reveal lack of understanding, of esthetic taste, or of knowledge. The teacher should commend economy, discourage useless extravagance of detail, and make clear what is meant by "neatness" of proof—its clarity, directness, cleverness, and economy.

Of two possible formulations of a problem, that which leads to the more general result is usually to be preferred, and the more generally applicable result is usually that obtained on the more economical hypothesis. It often happens that a given theorem or relation turns out to be much more specialized than is really necessary. Thus the student may well form the habit of asking himself whether the type of end-result originally suggested in the formulation is not overrestricted.

[2] In a sense, the solution to an idealized problem applies only to such a problem and never, as it stands, to a problem encountered in the actual world.

[3] See below, subsection on "Wide Applicability," page 87.

The Statement of the Problem: The Choice of Helpful Concepts

To define a problem by reducing its expression to words or symbols requires some acquaintance with the set of concepts that have a bearing upon the problem. Moreover, among the concepts concerned a conscious choice must usually be made in terms of which to phrase the problem. The student should be trained to realize that such a choice should not be casual or accidental but should be governed by the relative simplicity, clarity, familiarity, or importance of the several concepts available.

The choice of concepts was illustrated in the examples cited above. In the first instance the phrase *fatal accidents* was selected in formulating the statement of a sub-problem, although *minor accidents, serious accidents,* etc., were available. Here clarity or ease of definition, as well as importance, played the dominant rôle. In discussing the second example the notion of "the average distance the car will travel on one gallon of gasoline" was mentioned. The use of this idea could be avoided, but expressed as "miles to the gallon" it is more familiar than, for example, "number of gallons per 100 miles."

Explicitness and Clarity of the Premises

Back of the formulation of any problem in terms of words or symbols there lies a framework of preconceptions, a mode of thinking, a fabric of assumptions. These assumptions, often largely or wholly tacit, provide the matrix in which any formulation takes its particular character. In discussing the financial loss entailed by accidents, for example, it might be assumed that no loss of income is involved if an injury does not keep one away from work. This assumption would probably be challenged by a salesman or any other person who has lost money through ineffectiveness as a result of nervousness following an accident. When intelligent and dis-

passionate disputants disagree about the solution of a problem, the source of their disagreement may frequently be traced to differences in the premises assumed by each—the statement of the problem under consideration either left assumptions tacit or obscured differences among them by the ambiguity of the terms used. Students should realize that if conclusions are to be clearly understood the premises require careful statement.[4]

The Relation of the Formulation to the Possibility of Solution

The student should be helped to recognize the intimate relationship between the formulation of a problem and the possibility of solution. In elementary arithmetic one should not ask what number increased by 9 is 5, since the number system understood at that level provides no answer. The problem: "What three consecutive integers have 100 as sum?" clearly has no solution, since 100 is not a multiple of 3, or since by trial neither 32, 33, 34, nor 33, 34, 35 will work, and any other triad of consecutive integers would have a sum deviating even more from 100. To say that a man bought apples at a nickel each and grapefruit at two for a quarter, and so spent a dollar all together, does not suffice to tell exactly how many of each kind of fruit were purchased, since more than one answer is possible.

Save when questions are employed as a rhetorical device for emphasis, the formulation of a problem is to be regarded as a call for a solution, whether or not any immediately practicable method of securing one is apparent. The type of final result to be obtained by solving a problem and the possibility of finding such a result are ordinarily foreshadowed in the way the problem is formulated. One of the thread-worn but perennially effective devices for confusing a witness under cross-examination is to demand a "yes or no" response to a

[4] Further discussion of the relation between assumptions and conclusions is included in Chapter IX, on "Proof."

question intended to becloud the issue. The oft-cited question, "Have you stopped beating your mother?" is more transparently unreasonable than many. Questions such as: "How can a given individual live to a ripe old age?" may well be prefaced by an inquiry as to whether solutions exist.

In examining the formulation of a problem, therefore, the careful student will scrutinize the nature of its presuppositions. There should be at least a plausible presumption that an answer of the type invited by the query actually exists. If on investigation the question is found erroneously to prejudge the nature of the solution, there should be no hesitation in rejecting that line of inquiry or at least in rephrasing the problem.

FOCUSING STUDENTS' ATTENTION ON THE NATURE OF ACCEPTABLE SOLUTIONS

It is no less important for the student to understand the nature of solutions and to distinguish between acceptable, less acceptable, and non-acceptable solutions than it is for him to understand the nature of promising formulations and to acquire facility in devising them.

Types of Solutions and the Possibility of Solution

In the sense here intended a "solution" is conceived as any objective, any end sought, any goal toward which the process of reflective thinking is directed. In this sense, the attainment of a solution means a disposal of the problem, its removal out of the field of active inquiry. Hence the *type* of solution sought varies with the problem being considered.

The convenient word *solution,* when used in connection with mathematical "problems," may suggest some sort of numerical answer to a formal exercise, but even when attention is confined to mathematical problems, what is meant by the *solution* depends upon the problem. The solution may indeed be a number, as the result of a process of elementary

computation, but mathematics holds much beyond the stage of arithmetic. The solution of a problem may be a sought-for formula, a function expressing the relationship between given variables, a table exhibiting associated values, a graph, or a statistical chart. But the problem need not be confined even to this level. The solution of a problem may be a figure, a method of construction, an algorithm for arranging computation, a statement of procedure. The solution may be a proof establishing a surmised theorem; it may be itself a theorem expressing the scope of generality of a relation long the object of explorative investigation. The problem may be the search for a postulational basis, or the reduction to orderly sequence of a collection of known facts. Euclid's problem in organizing a mathematical science was of this type.

Yet even at the secondary level the student may be led to understand the different types of solution involved in such tasks as the following: formulating a definition for a familiar concept; selecting among many data those which seem pertinent to a given study; cataloguing a complete census of cases of a given relation; displaying in orderly sequence all the essential steps in a proof; elaborating as many distinct elementary corollaries of a given theorem as one can discover, carrying through steps (described abstractly) in particular numerical cases, and so forth.

Among all the varieties of types of solutions elicited by mathematically formulated problems, the type least often understood by the uninitiated is doubtless the solution which consists in revising the formulation of the original problem to correspond to a more enlightened viewpoint. In such reformulation a frequent phase is the suppression of inappropriate restrictions, the widening of the definitions adopted, the extension of the inquiry to a broader field, a reopening of the inquiry as to the form of the solution desired. For example, the equation $x^2 + x + 1 = 0$ has no solution in the system of real numbers. But after extending the number system by admitting complex numbers a solution is readily obtainable.

Sometimes a problem may be "solved" by showing that no solution of the form originally asked for can exist, as is the case of the irrepressible problem of trisecting a general angle by Euclidean means, and the problem of "squaring the circle." The student should understand that within mathematics proper there are certain proofs of impossibility, proofs so carefully and so frequently checked as not to be reasonably open to doubt.

The student also often fails to realize that in many cases a "solution" consists of a restatement of the problem in more convenient form, a rendering explicit of what is implicit. For example, the equation $ax + b = 0$ (where a is not zero) is said to have as a solution $x = -b/a$, which is here given as an equation neither more nor less general than the original, and entirely equivalent to it.

In some cases a problem subsides as the result of showing that the inquiry was motivated by a misunderstanding, that rightly viewed there never was such a problem—only a lack of comprehension. A problem may also be disposed of by showing it to be but one interpretation or even but one aspect of a larger situation, itself perhaps substantially solved, or still constituting a well-recognized and worthy object of investigation.

In an appropriate sense it seems reasonable to expect that every mathematical problem has a solution of some sort although of course there remain classical problems as yet neither completely solved nor proved impossible of solution. Old problems once regarded as effectively embalmed revive in new aspects and with richer generalizations and invite afresh the ingenuity and intuition of mathematical discovery. As the horizon of inquiry broadens, the list of unsolved problems lengthens, but in large part the ancient strongholds of baffling mystery have yielded to assault and have proved effective vantage points for attacks upon the newly found and yet unconquered frontiers.

Criteria for the Acceptability of Solutions

Among the various criteria of acceptability which may be listed, several are sufficiently simple and important that students may be expected to understand them and to grow in their ability to use them.

Wide applicability. The generality of the foreseen solution—the possibility of applying it in a wide range of situations—is one criterion of desirable solutions that the members of a class should learn to recognize and use. By eliminating from consideration actual but not relevant differences—not relevant, that is, to the idealized problem as formulated—a number of distinct practical problems may find a common generalized formulation and solution, so that results obtained for one apply to the others as well. But the mere recognition of the possible wide applicability of a solution, once obtained, is not enough. Students must develop the ability to foresee the possibility of wide applicability, to formulate and solve problems accordingly, and to make appropriate applications. They must also recognize, however, that the moment a new situation with a somewhat different set of variables is accepted as an object upon which to apply previous generalizations, old questions recur. One cannot be sure, without analysis and perhaps without practical testing, that factors previously negligible and suppressed in the theoretical solution, continue to be negligible in this new situation. Rich experience in varied fields, however, often provides a degree of reasonable faith, amounting in some cases to deep conviction, in the applicability of a proposed solution.

Appropriateness as to form. The student should learn that the solution to a mathematical problem, as to any other problem, should be given in an appropriate form. This means, for example, that in theoretical work of an abstract nature, it is proper to use such symbols as π, $\sqrt{\ }$, etc., where otherwise only approximations would result. In work of a practical or

engineering turn, on the other hand, the demands of common usage are to be observed: the carpenter, mason, or farmer are not to be offered such answers as $\sqrt{17}$ acres, or $7\sqrt{\pi}$ inches, or log 37 seconds. The units involved in a result must be clearly indicated, whether feet, foot-pounds, dollars, or whatever. In connection with graphical work, similar standards demand that not only must graphs be clear and neat, but that all scales should be clearly marked, and all charts should be appropriately labeled.

Appropriate degree of precision. Solutions differ in degree of precision, and an approximate or incomplete result may be entirely adequate for the purpose of the inquiry. Much of advanced mathematical analysis contents itself with approximations, more or less refined. A given estimate is often later replaced by results based upon the original investigation but carrying the computation out to a finer degree of accuracy, or else by considering cases which by their special character permit the formulation of more exact estimates. Often even very crude estimates obtainable by use of little more than common-sense reasoning may be sufficient for the object in hand.

Students of mathematics, more perhaps than those of other fields, have come to expect ready-made complete data and are prone to accept the defeatist attitude of refusing to consider a problem for which additional data are desirable. But a probable conclusion, adequate for immediate action, may be needed in the face of incomplete information. To be conscious of the limitations encountered represents an important stage of intellectual grasp of a theoretical problem, but willingness to proceed as far as the data permit, and willingness to act, if need be, on incomplete information must characterize the shouldering of adult responsibilities.

BIBLIOGRAPHY

BURTON, William H., *The Nature and Direction of Learning* (New York, D. Appleton-Century Co., 1929), pp. 200–236.

Columbia Associates in Philosophy, *An Introduction to Reflective Thinking* (Boston, Houghton Mifflin Co., 1923).

GOOD, Carter V., BARR, A. S., and SCATES, Douglas E., *The Methodology of Educational Research* (New York, D. Appleton-Century Co., 1935). See particularly pp. 77–87, and Ch. IV, "The Formulation and Testing of Hypotheses."

SCHLUTER, W. C., *How to Do Research Work* (New York, Prentice-Hall, Inc., 1927).

V

DATA

The collection of data usually follows the formulation of a problem as a next step, and work toward the solution depends upon analysis of the data called for by the formulation. Hence an adequate understanding of the problem-solving process includes comprehension of the nature of data and of what is involved in the gathering and organization of them. This chapter is devoted to a discussion of some ways in which the teaching of mathematics can contribute to such understanding of data as will improve problem-solving ability in this and other areas.

If students are to understand the significance of data they must have experience in dealing with them. But just as students have customarily been presented with preformulated problems in secondary-school mathematics, so too the statement of these problems has almost always provided them with the data essential to securing a solution—no more and no less. Consequently their work in mathematics has afforded them little opportunity to deal with the difficulties associated with collecting, selecting, and organizing material basic to discussion or inference. Although there is a question as to whether or not these tasks belong in mathematics under a strict definition of the field (they are usually completed prior to formal mathematical analysis), a reorganization of secondary-school mathematics which emphasizes the nature of problem-solving would seem to call for attention to these aspects of the process.

In other areas of instruction during recent years there has been a marked increase in attention to activities of this sort,

and if current educational trends continue, still greater emphasis upon them is to be expected. This makes it possible for teachers of mathematics to coöperate advantageously with teachers in other fields in helping students build an understanding of the nature of data and of the principles governing their collection, organization, and interpretation in fields as diverse, for example, as the social studies and the physical sciences. In such cases the mathematics teacher also assumes primary responsibility for helping students in collecting, organizing, and analyzing mathematical data pertinent to problems originating in other courses. Without coöperative work of this kind the mathematical aspects of problems arising in other fields are often neglected. But even in those situations where coöperative work is not possible the teacher of mathematics should take responsibility for focusing attention on the nature of data and their rôle in the problem-solving process.

It is perhaps unnecessary to add that data collected and organized by students should be relevant to problems arising in connection with their needs, should be plentiful and reasonably accessible, and ordinarily should be socially significant on their own account. Problems that demand data satisfying these conditions are not difficult to find.

BUILDING STUDENTS' UNDERSTANDING OF THE NATURE OF DATA

According to the basic position of the committee a clear conception of the nature of data is helpful not only in connection with the processes of collecting and organizing material for the solution of problems but also in connection with reflective thinking in general. It is here proposed that the teacher of mathematics help students gain an understanding of some of the characteristics of acceptable data (relevance, representativeness, accuracy, and reliability) and of the variety of possible types of data.

Characteristics of Acceptable Data

The view is sometimes held that the investigator starts by gathering facts indiscriminately and, having accumulated a large number, inspects them to see to what conclusion they may point. When few clues concerning the nature of the situation are available, it is sometimes necessary to gather facts indiscriminately, but more as a means of facilitating the formulation of the problem than as part of the actual process of obtaining a solution. In most instances the gathering of data does not begin until after the problem has been carefully formulated and a detailed plan of work toward a solution has been made. Only in this way can strict standards of suitability be maintained when the collection of data begins.

Relevance. The intelligent investigator hopes to get all of the relevant data necessary to find an acceptable solution, and to omit all the irrelevant data. He collects his data in accordance with the principle that the manner in which each item contributes to the solution determines its pertinence as a factor in the study. If, for example, the problem is to determine the number of hours of sleep a person should have for healthful living, one probably needs not only information concerning the number of hours that a representative group of healthy people sleep but, in addition, other facts, such as the age of these persons, the type and hours of their employment, and the conditions under which they sleep. On the other hand, data on their financial condition or educational background would appear to be irrelevant.

If the given situation is of a familiar type, it may not be difficult to distinguish between the relevant and the irrelevant, or the more relevant and the less relevant. In a novel situation, however, one may be obliged to undertake an elaborate program of checking before its most significant features are confidently isolated and the criteria for relevance thereby determined.

Representativeness. The nature of many problems is such that the collection of a complete set of data would be extremely difficult if not quite impossible. The investigator must resort to a process of sampling. In such cases the extent to which the sample is a fair representation of the complete set of data becomes important. Thus suppose a class or committee in a large school wishes to take a poll of opinion on some question of school-wide interest. It would be satisfactory for them to question only a sample group of students, provided that they make sure that the opinion of this smaller group is representative of the opinion of the student body.

Various precautions may be taken to assure representativeness. For instance, effort may be made to choose a supposedly random sample. Such a sample of the total school population could probably be secured by taking every tenth name from an alphabetically arranged roster of the school. But a sample chosen in similar fashion from the alphabetical listing of the telephone directory would not be similarly representative of the total city population if the question involved economic factors, since this sample would include only persons financially able to subscribe to telephone service.

When the various factors that are likely to influence the result are known, the sample may be so selected that each of these factors is represented, and in approximately the same proportion as it would be in a complete set of data. In the poll of opinion mentioned above, for example, opinions might vary with the grade or school class of the students questioned. If there are about twice as many freshmen as seniors in the school, the sample should maintain about the same ratio of freshmen to seniors.

In some cases the representativeness of a sample is checked by increasing it through the collection of more cases. If the new data do not bring about appreciable changes in the conclusions, there is added reason to regard the original sample as representative. A more general procedure consists in drawing a number of different samples and comparing the con-

clusions to which they lead. If these are the same, the samples are probably representative. Furthermore it is possible by the use of advanced statistical methods to determine what fluctuations are to be expected when successive samples are taken.[1]

Accuracy. Acceptable data must be accurate in a degree that is appropriate to the solution of the problem. Judgment as to what constitutes a reasonable degree of quantitative accuracy is based upon experience in gauging the demands of the problem. Securing data of the required degree of accuracy depends upon knowledge of the techniques for reporting and instruments for determining the appropriate measurements, the precision of these instruments, and acuteness of observation in their use. The student should, of course, realize that complete accuracy in quantitative measurement is but an unattainable ideal, and he should understand how appropriate approximations serve to give the results demanded by scientific work.[2]

Reliability. To say that data are reliable means essentially that they are trustworthy and free from bias. Any doubt as to the reliability of the data leads to doubt concerning the conclusions based upon them. Therefore clearly unreliable data must be either rejected entirely or given very little weight. Allowance may be made for minor uncertainties, but either inadequate technique or systematic bias in gathering data vitiates the reliability of the conclusion.

It is unnecessary to explain here the way in which inadequate techniques leading to inaccuracy of some significant sort impair the reliability of the data. Bias may operate in two ways. The investigator who is more interested in presenting a certain conclusion presumably based upon facts than he is in the truth of his results often consciously rejects all data which tend to disprove the point he wishes to make.

[1] Cf. G. Udny Yule, *An Introduction to the Theory of Statistics* (London, Charles Griffin and Co., Ltd., 1924), Part III.

[2] For a further discussion of this point, see Chapter VI, "Approximation."

In more subtle fashion, the scientific investigator often unconsciously overlooks relevant data which point to a conclusion or solution incompatible with his existing pattern of ideas. Always acutely conscious of this latter possibility, the careful investigator not only admits into his calculations all relevant data significantly affecting his problem but also makes conscious and deliberate efforts to seek data that his recognized bias might cause him to overlook.

The Variety of Types of Data

Since data consist of any material used as a basis of discussion or inference, it is possible to distinguish among many types of data appropriate to different problems and different formulations. For example, an historical problem, such as "What caused the Civil War?" obviously requires data of a sort quite different from the sort required by a medical problem phrased in much the same way, such as "What caused a certain epidemic of typhoid fever?" In view of the fact that different methods of treatment are necessary in connection with different types of data, the student must learn to recognize the type he is using in the solution of any given problem.

Within mathematics itself there are not only the easily recognized examples of numerical data, but also a great variety of other types. Thus in the study of formulas, of mathematical relations, or of the laws of arithmetic, symbols for variables constitute part of the data. Again, the data for a given problem may be a set of relationships from which new ones are to be found, or they may be rules for construction, or a system of assumptions, or a collection of proved propositions. In theoretical geometry, for example, the data consist of certain things "given" with each theorem or construction, namely certain conditions on a figure, as well as an array of assumptions, definitions, and previously proved theorems. The solutions for one set of problems may also become the data for another. The advanced mathematician sometimes

starts with a number of mathematical sciences, each reasonably complete in itself, as data, and seeks a more inclusive formulation admitting each of the given sciences, or at least a large part of each, as special cases. Data of these and other types may be classified in other ways. One may, for example, distinguish between exact and approximate data, or quantitative and non-quantitative, or geometric and algebraic data, as well as make other less obvious distinctions.

In order to help students become conscious of the variety of types of data it is neither necessary nor desirable to set up formal illustrative exercises. It is sufficient to call attention in each instance to the fact that the material they are using as a basis of discussion or inference is a set of data, and to discuss with them the differences among various types as they arise in seeking the solution of different problems. The emphasis should be upon appropriateness of type to the formulation, rather than upon mere variety.

FOCUSING STUDENTS' ATTENTION ON THE PROCESSES OF COLLECTING AND RECORDING DATA

Developing a clear understanding of the rôle of data in problem-solving and increased ability to use them demands that some attention be devoted to the processes of collecting and recording data. Furthermore, it calls for some actual experience in collecting and recording of data on the part of students. These processes will lead to questions concerning the accessibility of data, the choice of units, the use of measuring and recording devices, and the choice and use of sources of data.

The Question of Accessibility

The investigator planning to gather data must take into consideration their accessibility. Practical considerations are not to be ignored. The securing of data that are very incon-

venient of access may not be justified. Mere multiplicity of data is not inherently valuable. The investigator must take into consideration questions of cost in dollars, in effort, in time, in demands on other people, when data are to be gathered. The use of a questionnaire, the museum expedition, the trip to a distant library, the modified repetition of former experiments, are questions of policy not to be answered on the mere basis of what further data are thereby securable. Biological investigations of far-reaching import have been carried out using but a single species such as the fruit fly.

On the other hand, some very inacccessible items of data may be considered so essential to the solution of the problem that they cannot be neglected. In such cases the investigator must often spend time, money, and effort without stint until the necessary data have been collected. If, however, essential data are extremely inaccessible or cannot be secured, the problem is sometimes reformulated so that a solution of the modified problem may be obtained.[3]

The Choice of Units

In many problems the quantitative data are expressed in terms of standard units, and attention must be given to the choice of appropriate units and to considerations related to their arbitrary nature.

A unit may be appropriate in several different senses. In the first and perhaps most important sense, appropriateness may mean that the unit is dimensionally suitable for expressing the given quantity—for example, inches, feet, miles, centimeters, meters, etc., for distance; square inches, square feet, square centimeters, etc., for area; cubic inches, cubic centimeters, etc., for volume; and pounds, grams, etc., for mass. Sometimes through carelessness but often through lack of understanding, the student writes "feet" when "square feet" is appropriate, "inches" instead of "cubic inches," and so forth. This frequently occurs when operations result in num-

[3] Cf. Chapter IV, "Formulation and Solution," page 83.

bers representing units of a dimensionality different from the dimensionality of units previously involved in the problem, as in the calculation of area or volume from linear measures. In addition, students have particular difficulty with complex units such as dollars per hour, feet per second, knots, price per hundred-weight, foot-tons, cost per passenger-mile, apples per boy.

In a second sense, to say that a unit is appropriately chosen may mean that the magnitude so given is easier to comprehend than it would be in some other unit which is, however, appropriate in the first sense. To express the distance between two cities in inches, or the width of the school-room in miles, would be extreme instances of inappropriateness in this second sense. In less obvious settings such inappropriateness is far from uncommon. This happens in most cases because the student has a very imperfect notion of the actual magnitude of the units he is using. For instance, some students when asked to draw a square inch freehand, present a square with an error far too great to be explained on the basis of the essentially approximate nature of the task. Success in the intelligent solution of many real problems demands not only knowledge of the units in common use but also a notion of their size that is a reasonably close approximation of their actual magnitude.

In a third sense appropriateness may refer to the choice between the English and the metric systems. The metric units are particularly appropriate for scientific work not only because they facilitate computation through adherence to the decimal system of notation, but also because of the way different units are related—one gram, for example, is defined as the mass of one cubic centimeter of pure water at its greatest density. Whereas it has been customary to have students spend some time in learning to transform from one of these systems to the other, it might be better if this time were to be spent in helping them gain a clearer conception of the actual magnitude of the metric units on the basis of direct experience with them.

One of the most cogent reasons for emphasizing appropriateness in the choice of units is that appropriate units facilitate the recognition of relationships among the data pertinent to any given problem. It is often necessary to change the representation of data from one unit to another, and such changes are a source of difficulty to many students. In the past this difficulty has probably been due at least in part to the fact that students have become acquainted with standard units through tables and formal exercises, rather than through the kind of concrete experience in gathering and dealing with data advocated here. Through this experience students should acquire such familiarity with the units as leads to better judgment in regard to their appropriateness in any given situation and to increased ability in changing from one unit to another.

In connection with the development of a genuine understanding of standard units, it is important to direct attention to their arbitrary nature. Whatever their historical origins, basic standard physical units are now fixed by arbitrary legal definitions stated in terms of the length, mass, etc., of certain objects maintained at prescribed conditions in some government agency. Other standard units, like the gallon, ton, etc., are legally defined in terms of these. Even the monetary unit depends upon basic standard units, since the dollar is defined by law as a certain number of ounces of gold of specified purity. The arbitrary nature of the dollar has been strikingly brought to general attention by recent change in the number of ounces of gold in terms of which it is defined. The purchasing power of the dollar, and consequently real wages, are immediately affected by such shifts in legally set standards, as well as by shifts in economic conditions. The Italian word *lira* referred originally to a pound of silver, as did the British *pound sterling,* but it is now equivalent to only a few cents in purchasing power. In gathering data on standards of living, therefore, the dollar should not be taken naïvely as a standard unit of constant value, for lack of understanding of the meaning of changes in the value of monetary units works against all

who receive fixed wages. This instance suggests how necessary it is for students to understand units and their arbitrary nature.

Whereas standard units are always arbitrary, the zero point of a uniform scale in some cases has inherent physical significance, and in other cases is essentially arbitrary. For example, the zeros for length, or weight, or pressure, or duration of periods of time are intrinsic. On the other hand the zero points for ordinary Fahrenheit and Centigrade temperature readings are based upon accepted convention alone. This is in contrast to the measure of "absolute temperature" used by the physicist. The zero of longitude and the zero of latitude are matters of convention only, and while zero frequency is an intrinsic notion, "Middle C" is "middle" by convention. Measuring historical time as B. C., or A. D., or otherwise, is again a matter of mere agreement. This distinction between an intrinsic and a conventional zero is important when one speaks of percentage variation from the arithmetic average. Such a percentage variation has no scientific value unless the average is itself measured from an intrinsic zero. There is no justification for speaking of one day as being twice as hot or twice as cold as another, since obviously one does not mean to suggest twice the temperature reading on the absolute scale. To say one refrigerator is twice as cold as another is presumably only justifiable in the sense that the temperature drop from normal room temperature is twice as great for one refrigerator as for the other.

The Use of Measuring and Recording Devices

The study of certain types of real problems requires the collecting of primary data through the use of relatively complex measuring and recording devices. The foot-rule, the yardstick, and the protractor are familiar instruments in mathematics classrooms; less common are weighing devices, hypsometers, transits, and other practical measuring instruments. Meters such as those for gas, water, and electricity are examples of still more elaborate measuring, counting, and recording

devices. The teacher of mathematics should assume at least part of the school's responsibility for helping students learn how such instruments are used, both because they may be needed in the collection of primary data essential to the solution of a given problem, and also because they illustrate how mathematics is used in the work-a-day world.

The importance of keeping a systematic record of the quantitative data collected requires emphasis. This record keeping calls for care in writing down the results of counting or measuring and for acquaintance with simple but highly serviceable devices like using tally marks in groups of five. The habit of recording data systematically is seldom developed spontaneously, but by discussing its value and persistent attention to its proper use the teacher of mathematics can do much to help the student acquire it.

The Choice and Use of Sources

Other types of problems require data that must be taken from primary sources such as the national census, or secondary sources such as a world almanac, atlas, or textbook. Since students are likely to have more occasion to depend upon sources of data than to collect primary data, they should learn how to locate and use sources and to judge the acceptability of the data they present. Those sources are to be preferred which have maintained high standards of relevance, representativeness, accuracy, and reliability in collecting and presenting their data—a consideration which makes it doubly important that students understand and appreciate the importance of these standards. The acceptability of the data presented in a source may be judged in part by the methods used by the original investigators, their training or status as authorities in the field, and the consistency of their findings with those of others.[4] At appropriate points the teacher should be prepared to assist students in judging the acceptability of the

[4] For a discussion of the evaluation of authorities, see *Science in General Education,* pp. 101–102.

data they are using through discussion and the use of auxiliary sources such as *Who's Who*. In this way he can contribute to the development of sound critical judgment of evidence.

FOCUSING STUDENTS' ATTENTION ON THE PROCESSES OF ORGANIZING DATA

Complex problems usually present an intermediate stage between the gathering and the using of data, namely, the organizing of data. Whenever it becomes important for the relationships among a large number of separate items to be comprehended, it is essential that some principle of organization be introduced. A dictionary consisting of definitions written on separate cards might be usable if the cards remained arranged alphabetically, but obviously a mere sackful of cards, with a word and its definition on each, would be of no service to a person desiring a particular definition. The naming of streets and the use of telephone numbers are necessities in a large city, although in an isolated hamlet they may be superfluous. The rooms in a large hotel are numbered but it would be absurd to assign numbers to the rooms in a family apartment. In the case of data—particularly scientific data—organization is especially important. If teachers of mathematics hope to have their subject recognized as a helpful tool in problem-solving by students and by teachers in other fields, they must devote more attention to the problem of so organizing data as to facilitate the discovery of relationships.

In most investigations, the data are first arranged in tabular form, and this method of representation may be regarded as basic.[5] From a study of the table, the student is sometimes able to make certain verbal statements concerning the nature of the relationship shown. Often, however, the relationship is far from obvious, and in such cases the transformation of the data

[5] In present mathematics courses the common sequence of steps in scientific procedure is often studied in almost the reverse order. Beginning with verbal statements, students are required to translate them into formulas, to use the formulas to compute tables, and finally to translate the tables into graphs.

to graphical form may yield valuable clues as to the nature of the variation involved.

The Construction and Use of Tables

In simple problems that involve only a few items of data tabulation may not be necessary. In many problems, however, this is almost an essential step. There are often several ways of arranging a given set of data in tabular form, and the advantages and disadvantages of the various alternative organizations should be considered. This may involve merely a question of horizontal or vertical display, or it may involve more serious questions.

It is important to note that the discovery of a convenient and helpful tabular form demands logical analysis. The following quotation emphasizes this point: [6]

> A necessary initial step in any statistical study is the orderly arrangement of data into classes having one or more common characteristics. Until this is done we are not in a position to make comparisons, to see trends, to draw inferences. This orderly arrangement generally takes the form of a tabulation.
>
> *Logical Function of Tables.* Tabulations are undertaken for two main purposes: (1) to enable the investigator to discover relationships which are not clearly discernible in the unclassified data; and (2) to facilitate the presentation of facts. Thus tabulation is both a step in the logical process of analyzing data and a summary of the results of such analysis.
>
> The planning of tables is an important aspect of the logical analysis of any set of data, whereas the execution of the plan is sometimes a routine task.

The following sections discussing one-parameter data, multiple-parameter data, and frequency tabulations deal with rather elementary notions, but many teachers of mathematics

[6] H. M. Walker and W. N. Durost, *Statistical Tables, Their Structure and Use,* (New York, Bureau of Publications, Teachers College, Columbia University, 1936). This book contains a chapter on "Characteristics of a Good Table" which includes a number of criteria adaptable for use in Secondary Schools. Section headings include the following: "When is a table needed?" "The title," "Ruling," etc.

have not had an opportunity to study them. The ideas treated are useful when questions relative to the organization of data in tables arise. The Committee does not propose that these topics be made the objects of formal study by the students. But as examples of each topic are found in the process of solving problems, the fundamental notion may be suggested to the students, and they may gradually be helped to recognize examples for themselves and to apply the appropriate principles of organization in the construction of tables.

Tables of one-parameter data. Data relative to a given problem often consist of a set or "series" of single items—for example, a list of numbers. Such a list may be called one-parameter data. Thus in determining a typical height for the boys and girls in a class at school, only the height of each student might be recorded, without the associated name or other identifying information. Collections of mere names, such as that of registered voters, or of students in a school class, or of the States in the Union, illustrate non-quantitative one-parameter aggregates.

The order of one-parameter data might remain wholly haphazard, but usually the nature of the data suggests a basis for organization. For example, a list of names may be arranged alphabetically. There are, of course, other arrangements. For instance, the States of the Union are sometimes given in an order suggested by geographical distribution, the list starting "Maine, New Hampshire, Vermont," and so on. A collection of dates is usually arranged chronologically. A collection of numerical measures is ordinarily listed according to increasing or decreasing magnitude. The appropriateness of a possible organization depends both upon the nature of the data and also upon the purpose of the investigation as it is involved in the formulation of the problem and the anticipated nature of the sought-for solution.

Tables of two-parameter and n-parameter data. In important problems one thing is usually studied in relation to another, and instead of one-parameter data there are lists of

paired items composing a two-parameter [7] series. Although a mere list of telephone subscribers might serve for a small central exchange where each is called for by name, a typical telephone directory is chiefly useful because each entry is a pair of associated items, the subscriber's name matched against his telephone number. For chronological series a mere set of dates is seldom enough—usually each entry matches a date against the name of an event occurring on this date. In short, instead of a mere collection of items each appropriately designated as a "value" of a general variable, x, one is ordinarily interested in a collection of pairs (x, y) with the x ranging in one list, the y in another.

In some problems sets of data of more than two parameters are needed. In studying the cost of certain articles, for example, one might use a mail-order catalogue which lists both price and shipping weight along with the name of each article. This is an example of three-parameter data; in general, there may be n-parameter sets of data. Not all such lists are quantitative. For example, a list of the States, their governors, and the cities in which their capitols are located would constitute a non-quantitative three-parameter set of data. For a single independent variable, one may have several dependent variables, as in the case of biographical lists where against each name is the date of birth, date of death, profession, etc. In the example of three-parameter data consisting of names, prices, and shipping weights of articles, one variable is independent and the remaining two are dependent. Tables may also have more than one independent variable. A table showing the population of the States by decades since 1860 would have two variables independent and one dependent.

Data of more than one parameter are arranged in tables of more than one column or row. In case there is only one independent variable, the methods of organization mentioned above in connection with one-parameter data (alphabetical,

[7] Here the term "two-parameter" is used in a wide sense, not restricted to the case in which the parameters are independent.

chronological, etc.) may be applied to it. Since the items in the other lists are often associated with those of the first by some determined correspondence,[8] they cannot be rearranged if they appear to lack systematic organization among themselves. In fact, often one of the chief tasks of the investigator is to discover whether such systematic organization does exist. In the case of three variables of which two are independent and one is dependent, the lists for both independent variables may often be arranged in some systematic order. Whether or not this is possible, one list is usually arranged vertically and the other horizontally. If rulings are drawn between the different items a rectangular network results, and each value of the dependent variable is placed in the "cell" corresponding to the proper row and column. Thus in one of the examples cited above the names of the States could be listed in alphabetical order vertically, and the decades 1860, 1870, 1880, etc., in chronological order horizontally. Then the population of Arkansas in 1870 would be found in the third row and second column. This device cannot be used in case there are more than two independent variables, but separate tables of this kind may be prepared for each value of other independent variables.

Frequency tabulations. In many statistical problems frequency tabulations are useful. In any particular one-parameter list, a given item may recur repeatedly. This fact need not be emphasized, each repeated occurrence retaining its separate listing. When, however, a long list consists of relatively few distinct items, most of which appear frequently, there is obvious economy in grouping like items together. For example, the following randomly arranged list (4, 7, 5, 6, 9, 4, 2, 5, 6, 5, 5, 8, 7, 6, 6, 4, 7, 8, 8, 5) may be rearranged in increasing order as follows (2, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 8, 8, 8, 9). This is still a one-parameter list. Indicating the frequencies with which the distinct numbers appear leads to a table such

[8] Cf. Chapter VII, "Function," page 142.

as the following: (2, 1), (4, 3), (5, 5), (6, 4), (7, 3), (8, 3), (9, 1), where the second number in each parenthetical pair designates the frequency of occurrence of the first number in the pair in the given list. This is a two-parameter list, more conveniently exhibited by a table arranged as follows: [9]

Number	2	3	4	5	6	7	8	9
Frequency	1	0	3	5	4	3	3	1

It is interesting to note that by means of a frequency tabulation one passes from originally isolated items of one-parameter data to paired items and a two-parameter set of data.

Sometimes after data have been collected, it is found that for some given purpose the graduations recorded are too fine for convenience. To preserve all these minute distinctions in the final table might entail so extensive a display that no simple direct story would be told at a glance. In such cases data are grouped into class intervals defined in terms of a larger unit. For example, if children are measured and heights are recorded to the nearest sixteenth of an inch, it might be found that no two children in a given group are reported as having identical heights. On grouping the results to show heights to the nearest inch only, the number of distinct items recorded is greatly reduced, but even if all the original items of data were distinct, each now appears with an associated frequency usually greater than unity. When a double classification is made according to both height and weight, the results may be exhibited as explained above by using a rectangular system of "cells." Every cell corresponds to a given height and weight, each measured to the nearest unit in a convenient scale. For example, heights may be measured in inches and weights in multiples of five pounds. There might then be a cell for the

[9] The frequencies themselves are necessarily positive integers although a zero frequency is often introduced to maintain unbroken the recorded regularity of the measured items.

pair (61 inches, 75 lbs.). The number of individuals in the group whose measurements fall in the same cell would be the frequency of this cell. Thus from a two-parameter array by heights and weights, one is led to a three-parameter table of heights, weights, frequencies, where heights and weights are independent variables, and associated frequencies are then determined.

Cumulative frequency tabulations. In connection with a table having one independent quantitative parameter, one may ask how many items fall below (or above) a certain measure. For example, one may wish to know how many students in a given group get less than nine hours sleep per day, how many drink less than one pint of milk a day, how many live within three blocks of the school, how many see less than two movies a week, and so on. It is not difficult to answer such questions when the collected information is of the more usual sort. It is done by forming cumulative tables, showing for each recorded measure how many items have this measure or less. The operation is one of mere addition. Thus suppose in a given school class the heights recorded to the nearest inch ran as follows:

Heights in inches	53	54	55	56	57	58	59	60	61	62	63
Frequencies	1	0	2	1	3	6	10	9	3	4	1

The table showing the number of students in the group whose heights do not exceed the listed values (cumulative frequencies) will be as follows:

Heights	53	54	55	56	57	58	59	60	61	62	63
Cum. frequencies	1	1	3	4	7	13	23	32	35	39	40

From such a table of cumulative frequencies, one can tell at a glance how many of the group do not exceed a given height.

The Construction and Use of Graphs

Graphical methods, despite essential limitations upon their refinement of accuracy, have many advantages over tables. With the help of a graph it is usually easy for an experienced interpreter to see at a glance any general trend or type of variation within the data, and thus to recognize relationships. Gross deviations, possibly due to mistakes in observing or recording, show up at once, and can be examined critically and sometimes disregarded.

The graphical method of presenting data has recently become so prevalent that the ability to interpret the common types of graphs occurring in newspapers, magazines, and books may be considered as a necessary extension of basic reading skills.[10] In present courses students learn how to read graphs, but do not always have sufficient experience in interpreting them. Students also learn how to construct graphs of various types. Often, however, the work on drawing graphs is carried on outside the context of a real problem on which they are working, and is essentially a drill type of experience. Little attention is given to the relative advantages of different types of graphs for representing different sorts of data, and the students have little opportunity to choose the type of graphical representation to be used—a kind of opportunity they must have if they are to grow in their ability to organize and present data bearing on the problems they are solving.

The following are illustrative of principles governing the choice of various graphical techniques. Circular and rectangular distribution graphs are used to show the relation of parts to a whole, and are appropriate only when the whole is clearly defined. Bar graphs are appropriate when the values of the independent variable are discrete and not continuous. If, for example, one wishes to show graphically the number of farm

[10] Graphs may be taken to include such diverse work-charts as organization charts, scale drawings, road maps, genealogical diagrams, weather maps, as well as figures using a coördinate system. These latter are more typical of the graphs traditionally studied in mathematics courses.

animals—hogs, cattle, sheep, horses, and so on—in a given State, a bar graph is appropriate, but a line graph is not. Line graphs are appropriate when the independent variable is continuous—representing, for example, time, weight, or height.

In using graphs a second consideration of some importance has to do with the principles of good construction. These include the choice of scale units, the labeling of the graph itself and of the axes, the avoidance of misrepresentation (such as sometimes occurs when the origin or base line does not appear), and so forth. Graphs are often consciously used to mislead the public, and students should learn to detect some of the devices used for this purpose. Pictograms in particular are often misleading. For public appeal, a brief convincing story is needed, showing perhaps startling contrast, attractive charm, or some other quality that arrests attention and sustains interest. But this aim should not be allowed to supersede the basic purposes of graphic technique, nor to make difficult or impossible the ready drawing of sound and appropriate conclusions.

Scale drawings and geometric figures are also invaluable aids in the organization of some kinds of data. A thorough understanding of scale drawings includes, of course, knowledge of relations among similar figures, especially those which are also similarly placed. For some students it may be extended to include the ability to read architect's blue-prints and mechanical drawings. Furthermore the student should become accustomed to think of geometric drawings as one way of organizing data—since their purpose of helping to keep in mind the relations being studied and of suggesting relations that might not otherwise occur to him is essentially that of the organization of data in general.

The organization of data is seldom an end in itself. Data are organized for some purpose. In solving problems that purpose is usually to facilitate the making of interpretations and the drawing of conclusions. Little has been said in this chapter

on these all-important topics, since the following four chapters are devoted to a detailed discussion of ways of operating upon data, interpreting them, and judging the validity of inferences and conclusions drawn from them.

Bibliography

Brinton, Willard C., *Graphic Presentation* (New York, Brinton Associates, 1939).

Garrett, Henry E., *Statistics in Psychology and Education* (New York, Longmans, Green and Co., 2nd edition, 1937).

Helmholtz, H. Von, *Counting and Measuring,* translated by Charlotte Lowe Bryan (New York, D. Van Nostrand Co., 1930).

Karsten, Karl G., *Charts and Graphs, an Introduction to Graphic Methods in the Control and Analysis of Statistics* (New York, Prentice-Hall, Inc., 1923).

Modley, Rudolf, *How to Use Pictorial Statistics* (New York, Harper and Bros., 1937).

Monroe, Walter S., and Englehart, Max D., *The Scientific Study of Educational Problems* (New York, The Macmillan Co., 1936), Ch. III.

Report of the Joint Committee on Standards for Graphic Presentation (New York, American Society of Mechanical Engineers, 1918).

Ruch, G. M., and Knight, F. B., *Standard Service Algebra* (Chicago, Scott, Foresman and Co., 1932), Ch. III.

Schaaf, William L., *Mathematics for Junior High School Teachers* (Richmond, Johnson Publishing Co., 1931), Ch. VI.

Shuster, Carl N., and Bedford, Fred L., *Field Work in Mathematics* (New York, American Book Co., 1935).

Sorenson, Herbert, *Statistics for Students of Psychology and Education* (New York, McGraw-Hill Book Co., 1936).

The Story of Weights and Measures, Committee on Materials of Instruction of the American Council on Education (Washington, D. C., American Council on Education, 1932).

Walker, H. M., and Durost, W. N., *Statistical Tables, Their Structure and Use* (New York, Bureau of Publications, Teachers College, Columbia University, 1936).

VII

APPROXIMATION

A popular American textbook on arithmetic published in the middle of the last century contains such exercises as the following: [1]

(1) Bought of Queen Victoria 9 acres of land for which I paid 157.753125£. Required the price per acre.
Ans. 17£. 10*s.* 6¾*d.*

(2) From $5.\dot{1}2345\dot{6}$ take $2.3\dot{5}23456\dot{}$.
Ans. $2.7\dot{7}1105582166692777798888859999\dot{4}$.

(3) Required the contents of the earth, supposing its circumference to be 25,000 miles.
Ans. 263858149120.06886875 miles.[2]

(4) Supposing the earth to contain 4,000,000,000,000,000,000,000 cubic feet, each foot weighing 100 pounds,[3] and that the earth was suspended at one end of a lever, its center being 6000 miles from the fulcrum or prop, and that a man at the other end of the lever was able to pull, or press with a force of 200 pounds; [and] . . . be able to move *his* end of the lever 100 feet per second, how long would it take him to raise the earth *one inch?*
Ans. 52,813,479,690 y. 17 d. 14 h. 57 m. 46⅔ sec.

(5) . . . At length she told her father that, after dinner, she would begin and place the [72] brilliants in all the situations they would admit of, and then she would be sure of finding the correct way of adjusting them, and that she would not take her tea until she had effected it. Now, supposing she were to place them in all the various ways they would admit, how long

[1] From Benjamin Greenleaf, *The National Arithmetic on the inductive system combining the analytic and synthetic methods together with the cancelling system; forming a complete mercantile arithmetic* (Boston, Robert S. Davis and Co. New edition, 1856, "revised, enlarged, and much improved"), pp. 147, 158, 326, 336, 354, 359 respectively.

[2] As recorded, not "cubic miles." The author seldom bothers to distinguish between linear, square, and cubic units.

[3] Of course modern estimates of which the author was ignorant would place this value at more nearly 350 pounds.

would she be obliged to wait for her tea, providing she could make one change each minute?
Ans: 6123445837688608686152407038527467274077809178469732898382301496397838498722168927420416000000000000000 minutes.

(6) A problem about the ages of a man and his three sons in which the student is asked to find, "how much older is the first than the second?" and for which the recorded answer is: 4 y. 5 mo. 3 w. 0 d. 0 h. 47 m. 45 sec.

(7) A problem on proportionate division of expenses for which the recorded answer is:
"A . . . pays $\$28.93^{12287}/_{28741}$, B . . . pays $\$25.83^{11997}/_{28741}$, C . . . pays $\$23.48^{16132}/_{28741}$, D . . . pays $\$21.74^{17066}/_{28741}$."

With an air of independent sovereignty the author fashions problems suggested by facts about the physical universe and then provides data that seldom adhere to these facts. The circumference of the earth is taken as 25,000 miles, and the diameter sometimes as 7964 miles, and sometimes as 7957¾ miles, and all of these figures are treated as though they were exact. The author does not hesitate to make such erroneous statements as that "a cubic foot of water weighs 1000 ounces" and that "the earth's density is 400." A problem on the height of a tide raised on the moon by the attraction of the earth is given "on the supposition that the moon has seas and oceans similar to those on the earth." There is no hint that physical measurements are necessarily approximate, and the author's whole contribution to the study of significant figures is covered by the single misleading sentence, "The first nine . . . figures [the digits] are called *significant,* as distinguished from the cipher, which is of itself insignificant."

The freedom to disregard facts exercised by this author contrasts strangely with the excessive computational demands he makes upon the student. Although the instances of answers computed to a ridiculous degree of "accuracy" quoted above are somewhat extreme, exercises of a similar sort are by no means infrequent either in the book from which the quotations are taken or in other texts of the period. They were

evidently designed to carry into practice one of the avowed intentions of the author, namely, to further "the discipline of the mental powers." The student was expected to accept data without inquiring about their accuracy, and to compute answers without regard for the dictates of either common sense or critical judgment.

As modern science developed prestige and began to influence daily thought, and as the disciplinary aim in education declined, the schools could no longer ignore the scientific investigator's recognition of the approximate nature of most quantitative data and of the necessity for treating them accordingly. Such treatment involves both eliminating unnecessary labor and avoiding the introduction of spurious accuracy in computation. Not infrequently it also involves taking repeated measurements of a quantity and thereby securing groups of items all representing this same quantity. Some statistical measure of central tendency among these items, such as the mean or arithmetic average, is then employed as a better approximation to the quantity than any single measurement. Statistical measures are also often used to arrive at approximate descriptions of large groups of data in which not all the items are measures of the same quantity (for example, the group of numbers each of which represents the height of a student in a class). These descriptions are approximate even when both the individual items of the group and the computation involved are unquestionably exact.

In the schools those aspects of approximation which deal with elimination of unnecessary labor and avoidance of spurious accuracy have usually been referred to under the heading, "Approximate Computation." Those aspects which deal with describing groups of numbers have been called "Statistics." Both topics have received increased attention recently in secondary-school mathematics,[4] and both are here discussed under the more inclusive heading, "Approximation."

[4] The immediate cause of this increasing emphasis on approximation is doubtless to be found in the fact that the Reports of both the *National Commit-*

In spite of the greatly increased use of statistics and approximate computation in other fields, and in spite of the authoritative backing of the Reports of national committees, many teachers have hesitated to do more than touch upon these topics. Extramural examinations have as yet given them little attention, and teachers have deemed them less important than other parts of mathematics which seem to them more obviously contributory to immediate success in college. Moreover, many teachers are themselves not familiar with the concepts and methods of approximate computation and statistics. Even today the list of courses leading to a major in mathematics in liberal-arts colleges often ignores statistics, although the study may be offered as an elective which falls somewhat outside the usual pattern of courses. Teachers trained under these conditions cannot be blamed for failure to put proper emphasis upon important but unfamiliar topics. Many of them, however, are gradually acquiring an understanding and appreciation of approximation which should result in further experimentation with materials and methods for developing this concept.

BUILDING STUDENTS' UNDERSTANDING OF APPROXIMATION IN MEASUREMENT AND COMPUTATION

Approximations Originating in Measurement

As the student attempts to gather data and compute results on problems of real importance to him, he should be helped to

---

tee of 1923 and the *Commission on Examinations in Mathematics* of the College Entrance Examination Board commented favorably upon the teaching of approximate computation and statistics, and that the yearbooks of the *National Council of Teachers of Mathematics* have devoted considerable attention to these topics. See: The *Reorganization of Mathematics in Secondary Education*, Ch. II, III, and IV; "Report of the Commission on Examination in Mathematics," *The Mathematics Teacher*, Vol. XXVIII, March, 1935, pp. 154–166; Aaron Bakst, *Approximate Computation, Twelfth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1937).

realize that approximations are forced upon the user of physical measurements by limitations of instruments and observation, and by the necessity for ignoring minor variations. For example, the measures from which the acceleration of gravity [5] is found can never be made with absolute exactness because of limitations of instruments and observation. Furthermore, this quantity must frequently be treated as a constant,[6] ignoring the fact that it varies (with latitude and height above sea level), and the same holds for quantities representing the diameter of the earth, the distance from the earth to the sun, the density of sea water, the length of a year in days, the speed of sound in air, and so on. If the student is to be able to work competently on problems involving measurement, then, he must acquire some of the fundamental notions of approximation, such as those indicated by the terms *precision, accuracy, rounding off,* and *significant digits.*

The basic notion for the student to acquire is that measurements are at best approximate. A measurement is expressed to within a given unit (such as to the nearest inch, or to the nearest tenth, hundredth, or thousandth of an inch). The magnitude of the unit chosen defines the degree of *precision* and depends upon the purpose for which it is to be used and the possible precision of the measuring instruments available. Thus the acceleration of gravity may properly be assigned the value 32 in one problem, 32.2 in another, and 32.16 in a third.

The concept of precision is not to be confused with those of *relative error* and *accuracy.* When a measure is taken "to the nearest unit," this means that the quantity recorded may err to the extent of ± 0.5 of the unit. Thus, in measuring the width of a sheet of paper to the nearest tenth of an inch and recording this width as, say, 8.4 inches, one must recognize

[5] Denoted by g in the formulas $S = \frac{1}{2} gt^2$ and $T = 2\pi \sqrt{\frac{l}{g}}$.

[6] In other cases, even when the facts warrant extraordinary precision, the facilities of the observer may be so restricted that noticeable discrepancies continue to persist despite reasonable precautions, and quantities substantially constant must be treated as variables. See page 114.

that the actual width lies between 8.35 and 8.45 inches, and that the possible error of the number recorded is therefore ± 0.05 of an inch (or ± 0.5 of the original unit, which was 0.1). The ratio of this possible error to the approximate number representing the measurement is known as the *relative error* and is usually expressed as a per cent. The *accuracy* of measurements is judged in terms of their relative error, those having the least relative error being the most accurate.

Unlike the precision of a measurement, its relative error and hence its accuracy are independent of the unit, but dependent upon the number of significant digits in the measurement. Thus 42.3 inches is much more precise than 423 inches, but their relative error and accuracy are the same, since $0.05 \div 42.3$ is the same as $0.5 \div 423$. On the other hand 423 inches and 4231 inches have the same precision, but the relative accuracy of the second is greater than that of the first, since $0.5 \div 4231$ is less than $0.5 \div 423$. It is for this reason that, in practice, accuracy is usually gauged at a glance by the number of significant digits in the measurement as recorded.

Clear understanding of the necessarily approximate nature of measurements not only helps the student to exercise due precaution in recording and reporting them; it also makes possible an effective economy of labor in computing with approximate numbers. Only with understanding of the nature of precision, relative error, and accuracy is it possible to be intelligent in the rounding off of numbers either for the purpose of reporting them or for that of computing with them.

As examples arise, students may be helped to understand that the rounding off of numbers to a given precision may result in loss of accuracy, or that rounding off to a given accuracy may involve loss of precision. In practice both economy and minimum error are effected if attention is focused on the precision of terms in addition or subtraction, and on the accuracy of numbers which are to be multiplied or divided. In the addition or subtraction of two numbers of different

precision, the more precise of the two should first be rounded off to the same precision as the less precise. In adding more than two numbers not all of the same precision, they should be first grouped in sets which have the same precision. The most precise group is to be added first, the result rounded off to the precision of the next group and added to it, and so on until the complete sum has been found. Loss of precision which results from the operations of addition or subtraction should be taken into account by rounding off at least one digit in reporting the final result.

In multiplication it is the accuracy of the factors as revealed by the number of significant digits in them that determines the accuracy of the product. The product may contain no more significant digits than the least accurate factor contains, and may be accurate to even one less significant digit. Consequently, prior to multiplication both factors may be rounded off to the number of significant digits in the least accurate factor. Similar rules hold in the case of division.

When the student sees how failure to observe rules like these results in spurious precision, or spurious accuracy, he may on occasion be interested in following out further rules of operation relating to genuine precision and accuracy in computation.[7] But even should he not be interested in further study of this topic, he at least rounds off with intelligent understanding of what is involved.

There are various ways in which teachers can help students develop an understanding of approximate numbers and the rules for computation with them. It is hardly necessary to state that the provision of opportunity for actual experience with measurement arising in connection with problems real to the students is one of the most effective ways of developing some of these concepts. Laboratory work in science courses, shop work, or "field work" can all contribute.

An excellent method of developing common sense in approximate computations is to help students learn how to use

[7] *Cf.* Bakst, *op. cit.*, for an extensive treatment.

a slide rule. The use of this eminently practical instrument crystallizes, as few other methods do, the concept of limited accuracy. The rounding off of numbers, the recognition of significant digits, and intelligent decision on the placement of the decimal point become integral aspects of the process. The methods for "contracted" multiplication and division are also helpful. They not only make possible economy of labor when slide rule accuracy is inadequate, but also serve to focus attention on significant figures and on the avoidance of spurious accuracy.

Approximations Originating in Mathematical Theory

The student should be fully aware of the fact that approximations also arise as a result of the nature of the number system. Many instances of mathematical computation illustrate operations which may be said to be "closed" or "in finite form." Addition, subtraction, multiplication, exact division, raising to positive integral powers, finding square roots of perfect squares, and the like, are all of this finite sort. But there are other types of operations that are essentially incomplete and nonterminating. Some of these are familiar, but the contrast between the types is not always clearly grasped. Division when not exact leads to nonterminating decimals, such as $\frac{1}{3} = 0.3333 \cdots$, $1\frac{2}{7} = 1.7142857142857 \cdots$, and so forth. In these cases the fractional forms are exact and the decimal notation is not (unless dots are placed above appropriate digits to indicate the beginning and end of a "period"). The student may fail to realize the successive refinements in using 0.3, 0.33, 0.333, 0.3333, $\cdots$ to approximate $\frac{1}{3}$. In an entirely similar manner the arithmetic process of finding square roots leads to successive approximations except for perfect squares. For example $\sqrt{2}$ is approximated to successive decimal places by 1, 1.4, 1.41, 1.414, 1.4142, and so forth. The student should realize that the completeness implicit in the use of operational symbols such as $\sqrt{\ }$, log, and the like, should often be consciously sacrificed in favor of the

concrete convenience of explicit decimal expression. When he should accept $3\frac{1}{7}$ or 3.1416 instead of π, for example, should be a matter of consciously justified decision on his part, and not of universal rule.

The representation of many rational and all irrational numbers by terminating decimals are not the only cases in which legitimate mathematical processes involve approximations. In the theory of equations, for example, Horner's or Newton's method provides means of finding approximate real roots. There are also certain undeservedly neglected formulas which are frankly approximate in most cases. Thus the formula $\sqrt{a^2+b} = a + \frac{b}{2a}$ may be used for finding the square roots of numbers approximately when b is small compared with a. Simpson's formula often provides a fair approximation to the volume of solids. These and similar formulas will yield approximate estimates for numbers obtainable exactly by other and usually more complicated methods.

There is one place of general popular interest in which at best one has only approximations to the true relations, and concerning which there is widespread misunderstanding, namely, in connection with maps of large portions of the earth's surface. Mathematical theory proves that it is impossible to make an accurate fixed-scale mapping of the curved surface of the earth upon a plane. Distortion of scale is inevitable. The more familiar types of mapping usually preserve angles and also preserve ratios of distances with reasonable accuracy within any small region. But at best a correct scale for one portion is inapplicable for some other portions. Hence on a map of a continent one ought not to expect to find a "scale of miles" printed to help measure distances. In somewhat similar manner, diagrams that use perspective to picture on a plane objects which are in space have different scales for different objects depending upon how far in the background the figures are supposed to be.

BUILDING STUDENTS' UNDERSTANDING OF STATISTICAL CONCEPTS [8]

Thus far in this chapter, the concept of approximation has been considered in relation to the individual items measured or computed during progress toward the solution of a problem. But one of the major fields of application of the concept of approximation lies in the general domain of statistics, and there are many reasons for believing that the scope of statistics as treated in the secondary schools may well be extended.

Although the science of statistics was perhaps once a backward commercialized step-child of the philosophically dubious subject of mathematical probability, it has now so matured as to attain a conspicuous position in modern life. Investigators began to study and develop statistical methods when they realized that it is valuable to be able to know central tendencies, measurable trends, partial associations in groups of items, even though this information does not ordinarily suffice to predict the behavior of any particular item in the group. Today workers on widely diverse problems collect data and compute averages, determine measures of dispersion and of skewness, find coefficients of correlation and lines of regression. Even the general public has by this time become thoroughly accustomed to seeing charts and graphs displaying statistical information.

If the student is to understand the importance of statistical approximations in varied types of problems—social, physical, biological, and many others—and use them himself to aid clear thinking about group phenomena, he must be thoroughly acquainted with what is meant by such terms as measures of *central tendency,* measures of *dispersion, associated variables, trend,* and *correlation.*

Some of the topics here discussed have been treated in recent secondary-school textbooks, but the emphasis has been

[8] Organization of data for a statistical inquiry is discussed in Chapter V, "Data."

largely upon their computational aspects. The interpretation of statistics and discussion of the relative advantages and disadvantages of various measures have been seriously neglected. It is, of course, true that by far the greater part of statistical theory cannot be developed at the secondary-school level. This Report therefore omits entirely or passes over with bare mention many topics which are important parts of the theory. For example, in connection with frequency tabulations [9] teachers may find it possible to help students develop an understanding of the notion of probability and of the major characteristics of the normal probability distribution, and to apply these understandings in the study of other statistical concepts.

Measures of Central Tendency

One hears on all sides the expression of such judgments as: "She is tall for her age"; "He is rich"; "He lives far away"; "It is a hot day." To say that a girl is tall means, of course, that she is tall with reference to a group of supposedly comparable children, and in every one of these statements there is an implicit awareness of a range or distribution of the particular characteristic mentioned—from the shortest to the tallest, from the poorest to the richest, and so on.

It is interesting to note that each range of characteristics is ordinarily subdivided into contrasted types, such as short versus tall, rich versus poor, near versus far, hot versus cold. Examined in greater detail, the contrary types may be regarded as separated by some central measure. A tall child's height exceeds this central measure, a short child's height falls below it. For a day in midsummer in a given locality there is approximate agreement as to what is regarded as ordinary temperature, even although no one may have expressed this "ordinary" temperature in terms of degrees Fahrenheit; a hot day is, then, one hotter than the ordinary, and

[9] Cf. Chapter V, "Data," page 106.

a cool day one cooler than the ordinary for that time of the year.

For scientific work it is obviously desirable to specify the critical level quantitatively, and to state it in terms of standard units. This quantitative description is achieved by the use of some average or mean value, which also serves to describe the group as a whole. When one says "the average height of this class is 63 inches," this number not only provides a critical level on the basis of which comparisons of tallness and shortness may be made, but it also serves as one way of describing the height of the class thought of as a group.

There are a number of different sorts of averages each of which has certain advantages for certain purposes. Among the averages most used are the arithmetic mean, the median, the mode, and the geometric mean.

The arithmetic mean. The arithmetic mean of a set of n measurements is defined as the nth part of their sum. This average is so widely used that the term *average* is often assumed to refer to this special type alone. The arithmetic mean is easily computed.[10] It is readily expressible as a formula of the usual sort, and one which is easily transformed by operational techniques. Thus if the formula is written $M = \frac{\Sigma x}{n}$ it is a simple matter to solve for n. These reasons alone suffice to make it popular with computers.

For scientific purposes, the arithmetic mean has also the signal advantage of being rigidly defined, and not left to the mere estimation of the observer. When a series of measurements of the same quantity is taken the mean can be computed and yields a determinate value that is more likely to

[10] The use and theory of the so-called "short-method" for computing the arithmetic mean are an illustration of useful applications of mathematics now usually neglected in secondary schools. In computing the arithmetic mean much labor may be saved at times by recourse to this method, and negative numbers are usually involved in a way which emphasizes their meaning and usefulness. The algebraic theory illustrates the concept of symbolism and the usefulness of algebra as a generalization of the arithmetical processes.

represent the true value than any one of the individual measurements.

But for certain purposes the arithmetic mean is not free from disadvantages. The magnitude of the arithmetic mean is influenced by all items of the group, and the extreme items sometimes exert an influence which makes the mean misleading.[11] For example, the arithmetic mean of the contributions to a school fund by members of the several classes may easily mislead as to the size of individual contributions where a single very generous donation overshadows many more modest gifts. The arithmetic mean of the heights of the students in a class will show conspicuously the influence of a few overgrown youths or some of retarded physical development. The distribution of income is another important example. The calculation of the arithmetic mean in this case presents no difficulties, but its interpretation is more complex. More than half the population receives *less* than the average. A few high incomes raise the average but have little effect upon the real standard of living of the population. In cases like these the arithmetic mean alone is not an adequate measure of central tendency. A student interested in the facts of income distribution would want to know what income level had as many families above it as below, and what income level was the commonest.

The median. The median measure may be defined as the middle or central item when the values are arranged in order of magnitude. If there is an odd number of items it is definitely determined. But if the group has an even number of items an additional convention is needed: if the two centrally placed items are distinguishable, a value half-way between them is usually taken as the median. This measure can often be found without resort to arithmetic. For example, the

[11] A type of average recently introduced and known as the *mid-mean* avoids this major disadvantage of the mean. The mid-mean is found by dropping out the bottom quarter and top quarter of the items arranged in order of magnitude, and then finding the arithmetic mean of the remaining central half of the original list.

median height of a class may be found by proceeding as follows. Arrange the students according to height, then march them off by twos, the tallest with the shortest, the next to the tallest with the next to the shortest, and so on. The last person or pair of persons will have the median height, which can then be marked on the blackboard.

Unlike the arithmetic mean, the median is clearly unaffected by extreme variations at the ends of the range. If the tallest student in the room were replaced by a giant, or the shortest by a midget, no change would result in the median. For problems pertaining to wages, gifts, taxes, etc., it is usually more informative to use medians than arithmetic means. If a class of one thousand alumni gave a total of $14,997 to their Alma Mater, the class contributed $15 each "on the average." But if one person gave $12,000 and the other 999 gave $3 each, the median donation ($3) would be more descriptive of individual contributions than the arithmetic mean.

The median suffers from the disadvantage of not being defined in algebraic terms, and from the necessity of using a convention supplementary to the definition when the number of items is even. It also has certain other disadvantages.[12]

The mode. The mode is that measure which occurs most frequently (if there be such a one). It is even more independent of unusual measurements than is the median, since even the number of extreme items is not likely to influence the mode. But in many cases it is hard to ascertain with reasonable assurance, and there is no simple algebraic method of treating it.

The geometric mean. The geometric mean is the nth root of the product of the n measures. It is thus always determinate and rigidly defined and is expressible by an algebraic formula. Another of its important properties is that the geometric mean of the ratios of corresponding observations in two series is equal to the ratio of their geometric means, and hence the

[12] Cf. G. Udny Yule, *An Introduction to the Theory of Statistics* (London, Charles Griffin and Co., Ltd., 1924), pp. 116–120.

geometric mean is regularly used in connection with comparison by ratios. The following example shows this property in a simplified concrete case:

Year	*Cash on Hand at Beginning*	*Cash on Hand at End*	*Gain*	*Ratio of Gain to Cash at Beginning*
1937	100	150	50	.50
1938	150	300	150	1.00
Arithmetic Mean	125	225	100	.75
Geometric Mean	$\sqrt{100 \cdot 150}$	$\sqrt{150 \cdot 300}$	$\sqrt{50 \cdot 150}$	$\sqrt{.50 \cdot 1.00}$

The ratio of the arithmetic mean gain to the arithmetic mean of the cash on hand at the beginning is $^{100}/_{125}$ or 0.80, and this does not equal the arithmetic mean of the ratios, which is 0.75. But the ratio of the geometric mean of the gains to the geometric mean of the cash on hand at the beginning (i. e., $\sqrt{50 \cdot 150} \,/\, \sqrt{100 \cdot 150}$) does equal the geometric mean of the ratios. Economists dealing with index numbers (which depend on ratios) properly favor the use of the geometric mean, and are deterred from even wider employment of it only by the relatively inconvenient details of computation which are entailed in its use.

The geometric mean is also of particular significance for studying data about a group of approximately similar geometric figures. If one is comparing a number of objects all of approximately the same shape and density, various types of measures of them might be considered: linear measures, areas, volumes, weights, etc. To take a simple case, consider three cubes of like material measuring 4, 6, 9, units respectively on an edge. The table on page 127 exhibits some of the properties of these cubes.

Suppose a cube is constructed so that its edge is the arithmetic mean ($6\frac{1}{3}$) of the edges of the members of the collection. The other properties (e. g., area, volume, and weight) of this object will not be the arithmetic means of the corresponding

Edge (linear units)	*Area of Each Face* (square units)	*Volume* (cubic units)
4	16	64
6	36	216
9	81	729
Arithmetic mean $6\frac{1}{3}$	$44\frac{1}{3}$	$336\frac{1}{3}$
Geometric mean 6	36	216

measures of the objects of the collection. The area of a face of this cube would be $40\frac{1}{9}$ square units and not $44\frac{1}{3}$; the volume would be $254\frac{1}{27}$ cubic units, and not $336\frac{1}{3}$. It is evident that by the use of geometric means throughout, complete harmony results. The geometric mean of the three edges is 6 units; of the areas of the faces, 36 square units; and of the volumes, 216 cubic units. In the sense of geometric mean one can properly speak, among similar objects, of the object of mean magnitude without distinguishing among length, area, volume, weight, etc.

The geometric mean is, however, not free from limitations. It is not useful if zero or negative measures occur, and it is difficult to compute without the aid of logarithms.

The teacher of mathematics has an important responsibility and opportunity in helping students to recognize the special uses and limitations of each of these measures of central tendency. Any one of them may be used either intentionally or inadvertently to give rise to false impressions. Without a clear understanding of each, students are unable to interpret intelligently the statistical information presented to them in books, newspapers, magazines, over the radio, and from the platform.

Measures of Dispersion

After having secured some measure of the central tendency of a distribution of quantitative items, one is ordinarily next

concerned with the extent to which the items deviate from the chosen mean or average. In some cases the values cluster closely about the mean, and in other cases they are widely scattered. Measures of dispersion provide a means of giving an approximate description of the characteristics of the group in this respect. They are helpful also in describing the position of a particular item in relation to the group. In speaking of the height of a girl it is often not enough to say merely "she is tall for her age." It is important to know whether the deviation of the height from the mean is exceptional, or whether it falls within the range of variations to be expected in a given proportion of cases. The most familiar measures of dispersion are the *range,* the *quartile deviation,* the *mean deviation,* and the *standard deviation.*

The range. The range is the interval between the largest and smallest items of the group. It is very easily found, but is not ordinarily a satisfactory measure of dispersion since a few items at the extremes may create a totally false impression of the way in which most items tend to cluster. Moreover, the addition of new items to the group or the elimination of a few which occur at the extremes may effect a marked change in the range.

The quartile deviation. The median, as explained above, separates the ordered list into two sublists of equal number of items. The "first quartile," Q_1, is the median of the lower half-list, and the "third quartile," Q_3, is the median of the upper half-list. The median and the two quartiles divide the observed values into four classes of equal frequency. The interval from Q_1 to Q_3, which is $Q_3 - Q_1$ units in length, is called the semi-interquartile range, and half of the items fall within this range. Half this interval is called the semiquartile range or *quartile deviation.*

The quartile deviation is easy to compute and is not affected by variations in the extreme items. It is, however, not adapted to convenient algebraic treatment, and in general

its advantages and disadvantages are comparable to those of the median.

The mean deviation. The mean or average deviation (more properly "arithmetic mean absolute deviation") is defined as the arithmetic mean of the absolute values [13] of the deviations of all items from a chosen average value. The mean deviation when found is always referred to the chosen average as origin. Although the mean deviation is easy to compute and is expressible as a formula (e. g., $A = \frac{\Sigma|d|}{n}$), it is not adapted to convenient algebraic treatment.

The standard deviation. The standard deviation, which is ordinarily preferred over other measures of dispersion, is usually defined in terms of deviations measured from the arithmetic mean. It is the square root of the arithmetic mean of the squares of the deviations from the arithmetic mean, and it is designated by σ (sigma). Because of the use of the squares of the deviations in determining σ, the effect of remote items is striking, and questions as to doubtful observations should be examined systematically prior to the computation. One reason for the popularity of σ is that it is readily expressible in algebraic symbols and the formula lends itself readily to algebraic treatment.[14]

The standard deviation has the disadvantage of being more difficult to compute than the other measures of dispersion mentioned, and its nature is less easily comprehended. The

[13] The arithmetic mean of the algebraic deviations (taking account of signs) is always zero.

[14] The formula may be written $\sigma = \sqrt{\frac{\Sigma x^2}{n}}$, when x denotes the deviation of an item x from the mean and n denotes the number of items. Algebraic treatment is illustrated by the case in which an "assumed mean" is used for short-cut computation. If s designates the "root-mean-square" deviation about an assumed mean d units from the arithmetic mean, then $s = \sqrt{\frac{\Sigma(x+d)^2}{n}}$, and it can be shown algebraically that $\sigma^2 = s^2 - d^2$, or $s^2 = \sigma^2 + d^2$. From this it follows that s as a function of the location of the assumed mean has its minimum value, σ, when $d = 0$. That is, the "root-mean-square" deviation is least when deviations are measured from the mean. Cf. Yule, *op. cit.*, p. 134.

meaning of the standard deviation is illustrated by a comparison of two otherwise similar distributions which have unequal standard deviations—in the one with smaller standard deviation the values cluster more closely about the mean. The student can be helped to understand why standard deviation is defined as it is by calling his attention to the fact that the algebraic sum of the deviations is zero, but that squaring them before adding yields a positive sum, and the eventual taking of the square root partially compensates for the initial squaring.

Approximate Relations Among Associated Variables

In attacking problems the question of association between variables arises frequently. Although an extensive theory exists for dealing with association between variables, the idea itself is simple, and secondary-school students can readily grasp the essential notion. Examples from daily life are plentiful. The toast turns darker the longer it is left exposed to heat. The older children are bigger and stronger than the little tots. As the baby passes out of infancy, it needs less sleep. The bigger oranges cost more than the smaller ones.

Thus in many cases there is an association between two given variables, such that an increase in one tends to be accompanied by a change in a definite direction (an increase or a decrease) in the other. But it is frequently very difficult or even impossible to foretell the exact amount of increase or decrease in any given instance. The toast grows darker, but two pieces of toast in the same toaster for the same period of time almost never turn out exactly the same shade. It is true as a general principle that as the individual grows older during childhood and adolescence, he also grows taller, heavier, and stronger, and hence that older children are taller, heavier, and stronger than younger children, but this does not mean that every thirteen-year-old is taller, heavier, and stronger than every twelve-year-old. It is certain that every man will die, but the exact date or cause of death of a given

healthy individual cannot be predicted. It is almost certain that a planted crop will yield some harvest, but it is difficult to estimate the size of the crop and impossible to predict the fate of any given grain of corn planted in the soil.

In the conduct of experimental research the scientist attempts to isolate the significant factors affecting the observed phenomenon, and to identify the relationships among these factors. Where the achievement of these intermediate ends is attainable, it makes possible the gradual discovery and successive extension of universal laws or principles, expressible perhaps in mathematical formulas, by which it is possible to predict what will happen in any given instance. But it often happens that a situation to be studied is admittedly so complex that it is practically impossible to isolate all of the factors influencing it, and the determination of an exact formula for the relationship among them would involve insuperable difficulties. Moreover, even were a formula to be discovered, its intricacy would make application hazardous or impossible in most daily situations. On the other hand, by means of statistical approximation one may measure the cumulative effect of many variable factors on a group of items without identifying every factor, or measuring the exact contribution of each factor to the observed result on the group, or attempting to predict what will happen in the case of each member of the group. Mortality tables, telling nothing certain about the decease of any given individual, provide the actuary with a reliable guide in forecasting the number of deaths to be expected among a large group of insurance policy holders, and the farmer can base estimates upon the general relation of quantity of seed corn to gathered crop.

Associations determined statistically are often phrased in much the same language as universal scientific "laws." In fact one sometimes mistakes what is intended to be merely a statistical generalization for an exact statement of universal validity. But statistical generalizations are not intended for application to any given selected individual or even to an in-

dividual chosen "at random." Yet when associations are observed and checked statistically, although perhaps not explained in the light of accepted fundamental principles, one may often proceed to forecast about groups tentatively with reasonable hope of approximate success. The laws of probability rather than invariable law or strict functional dependence [15] become the ground-work of faith in the reliability of conclusions based on groups of data.

In introducing students to these broad and important notions, it is necessary first to put them on guard against assuming an association exists where actually there is none. After the *existence of association* has been established, the student may proceed to study how one variable changes in relation to another, that is, the *trend,* and also the *correlation* between them.

Existence of association. The student should be helped to understand how he may proceed to investigate whether a suspected association between variables is likely to be an actual one. In simple cases the existence of a relationship may be shown by procedure like the following. Divide the values of each variable into two mutually exclusive groups—for example, separate boys from girls, natives from foreigners, those who are in the senior class from those who are not, those who have been inoculated against a disease from those who have not. Then the data may be arranged in a two-by-two frequency table similar to the one below.[16] The table deals with association between inoculation against cholera and exemption from attack.

	Not Attacked	*Attacked*	*Total*
Inoculated	276	3	279
Not Inoculated	473	66	539
Total	749	69	818

[15] Discussed in Chapter VII, "Function."

[16] Quoted with minor adaptations from Yule, *op. cit.,* pp. 31–32.

Here the important question is, How far does inoculation protect from attack? The most natural comparison is therefore—

Percentage of inoculated who were not attacked 98.9
Percentage of not inoculated who were not attacked . . . 87.8

or we might tabulate the complementary proportions—

Percentage of inoculated who were attacked 1.1
Percentage of not inoculated who were attacked 12.2

Either comparison brings out simply and clearly the fact that *inoculation* and *exemption from attack* are *positively associated* (*inoculation* and *attack* negatively associated).

The terms *positively associated* and *negatively associated* are used in a technical sense. If inoculation and exemption from attack were independent (i. e., not associated) both the first pair of per cents and the second pair of per cents would be approximately equal; but since $98.9 > 87.8$, positive association of inoculation and exemption is said to be indicated, and since $1.1 < 12.2$, negative association of inoculation and attack is likewise said to be indicated. It is important to observe that it is not enough to say that some (in fact most) of the inoculated cases were not attacked. One must also investigate whether the ratio of those inoculated but not attacked to the total number inoculated exceeds the ratio of those who were both not inoculated and not attacked to the total not inoculated. In this discussion comparison by rows in the table was used; association might also have been investigated by comparisons based on the columns of the table.

The methods of analysis discussed above may be applied to the investigation of many cases of supposed association. Students may want to find out whether those who do well in the study of languages do poorly in the study of mathematics as is sometimes said, whether income as an adult and number of years of schooling are really associated, and so on, for many similar questions real to them. But when students work on problems real to them, they can be encouraged to seek rea-

sons for the associations that they find. If there is an association between income as an adult and the number of years of schooling, does this mean that length of schooling is the crucial factor affecting income level as an adult? May it not be that both years of schooling and income level are closely related to the socio-economic status of the parental home? Perhaps higher income is not due to more schooling but rather to a higher initial socio-economic status that also makes longer schooling possible. Thus the teacher may help students recognize that the meaning of association must be examined carefully before broad conclusions are based upon it.[17]

The separation of the data on each of two presumably associated variables into only two groups serves to establish the presence or absence of an association between them, but it is often too crude to make possible a fuller study of the association when it exists. The data are therefore usually grouped into a larger number of classes by dividing the ranges of the variables into smaller parts called "class-intervals." The recognition of possible relationships is further facilitated by the construction of scatter-diagrams in which a tally representing each item is recorded in the proper cell of a rectangular frequency table. Often the tallies seem to cluster within a linear or curved strip, and so suggest study of the relationship in terms of trend or of correlation.

Trend. When one variable changes in approximately regular or systematic fashion in relation to the variable time, the data are said to exhibit a trend. The word *trend* is also often used to describe a systematic relation between any two variables. For example, if one records and classifies measurements of the heights of a group of children in terms of their respective ages, one can measure ages along the horizontal axis of a graph and heights along its vertical axis, and then mark a dot

[17] Many questions relating to the existence and strength of an association, as measured by means of "coefficients of association," or by Pearson's "contingency coefficient," although of great theoretical interest and practical importance, fall outside the scope of general education at the secondary-school level, and consequently are not discussed here.

recording the pair (age, height) for each individual measured. The marks on this graph, although scattered, would tend to cluster along a curve or straight line which would serve to indicate the general trend of the data. Though trends may also be discovered from tables of data or from scatter-diagrams, they are more readily recognized when the data are presented in graphical form as indicated.

The investigator seeks to find a definite line or curve of "best fit" which represents the trend approximately. In the case of linear trend it is possible with the aid of a transparent ruler to draw a straight line that seems to differ from the plotted points by as small an amount as possible. There are a number of methods by which a line of "best fit" may be computed, the simpler ones among them well within the comprehension of secondary-school students.[18] Non-linear trends —those represented by quadratic (or "parabolic") curves, exponential curves, and curves of even less simple analytic form —require more complex analysis for identification. Training in computing such trends is ordinarily left to advanced courses, since understanding of the methods used calls for considerable theoretical background.

Although students have little difficulty in recognizing many trends shown graphically, they must have special help on certain points. Thus they should learn to look for maximum and minimum values. They should be made sensitive to the effect of choice of scales on the appearance and position of the graph. For example, they should be put on guard against interpreting an approximately straight line on graph paper of non-uniform scales (as on semi-logarithmic graph paper) in the same way that a straight line on graph paper with uniform scales is interpreted.

In particular, they should be helped to understand the usefulness, justification, and dangers of interpolation and extrapolation. To determine action in the face of complicated

[18] See J. P. Guilford, *Psychometric Methods* (New York, McGraw-Hill Book Co., 1936), Ch. X.

and largely uncertain factors, human intelligence relies upon indications of trend. The best guess is usually that unknown instances of a given relationship will follow the trend indicated by known instances. Sometimes the trend is so extraordinarily uniform as to invite considerable confidence in this guess, although the guess must always be modified by any known or believed changes in the conditioning circumstances. In such cases the computation of intermediate values (interpolation) or the extension of the curve which best fits the known data (extrapolation) serves as an indication of the unknown data. In other cases the trend is at most suggestive, the variations being individually unpredictable even on the basis of the best methods of analysis. It would seem well within the scope of secondary-school work for students to learn to form tables of first or second differences, to relate the differences to changes in the graph, and to detect flagrant errors by the occasional irregularity of these differences. In the cause of this work, the assumptions underlying valid interpolation and extrapolation should become reasonably clear.

Linear correlation. When the data on two associated variables seem to exhibit a linear trend, it is frequently important to find some measure of the extent to which the smaller and the larger values of one variable tend to be associated with the smaller and the larger values of the other respectively (or the larger values are associated with the smaller and the smaller values with the larger). Such measures are called *coefficients of correlation.*

Understanding of the formulas for the computation of coefficients of correlation requires considerable mathematical maturity, and the effort required to develop this understanding with any but exceptional secondary-school students is usually not worth while. It is important, however, for students to be able to interpret the meaning of coefficients of correlation when these are encountered in the reading of statistical reports.

Without any detailed knowledge of how these measures are

computed, students can learn that the formulas are so constructed that coefficients of correlation are abstract numbers ranging from -1 to $+1$. If the smaller and larger values of one variable are associated with the smaller and larger values of the other respectively, the coefficient of correlation is a positive number. If the smaller values of one variable are associated with larger values of the other (and conversely), the coefficient is negative. If the variables are independent or uncorrelated, the coefficient is zero. When the numerical value of a coefficient of correlation is near unity (say larger than 0.90) one can often predict the value of the dependent variable corresponding to a given value of the independent variable with fair approximation. If the coefficient is $+1$ or -1, there is perfect linear correlation. In the general case of perfect correlation (not necessarily linear) a definite value of the dependent variable is associated with each value of the independent variable. In all such cases there is a *functional* relationship, and to this concept in its more general aspects the entire next chapter is devoted.

BIBLIOGRAPHY

BAKST, Aaron, *Approximate Computation, Twelfth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1937).

BARBER, Harry C., *Teaching Junior High School Mathematics* (Boston, Houghton Mifflin Co., 1924).

BOYER, Lee Emerson, "Elementary Approximate Computation," *The Mathematics Teacher,* Vol. 32, October, 1939, pp. 249–258.

GUILFORD, J. P., *Psychometric Methods* (New York, McGraw-Hill Book Co., 1936).

MITCHELL, V. G., and WALKER, Helen M., *Algebra: A Way of Thinking* (New York, Harcourt, Brace and Co., 1936), Chs. X and XI.

Selected Topics in the Teaching of Mathematics, Third Yearbook of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1928): Ch. X, "Measurement and Computation," George W. Finley; Ch. XI, "The Teaching of Direct Measurement in the Junior High School," William Betz; Ch. XII, "The Use of Measuring Instruments in Teaching Mathematics," C. N. Shuster.

The Teaching of Algebra, Seventh Yearbook of the National Council

of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1932). Ch. IV, "The Function Concept and Graphical Methods in Statistics and Economics," W. Lietzmann; Ch. VI, "Mathematics and Measuring of World Trends and Forces, A.D. 1932," W. S. Schlauch.

SHUSTER, Carl N., and BEDFORD, Fred L., *Field Work in Mathematics* (New York, American Book Co., 1936).

WALKER, Helen M., *Mathematics Essential for Elementary Statistics* (New York, Henry Holt and Co., 1934).

YULE, G. Udny, *An Introduction to the Theory of Statistics* (London, Charles Griffin and Co., Ltd., 1924).

VIII

FUNCTION

By use of the methods of statistics the student is enabled to describe certain characteristics of groups of quantitative data. But powerful as these methods are, they rarely reveal such a close relationship between the items of two sets of data that knowledge of a given item in one set carries with it knowledge of a uniquely determinate item of the other. Although a relation may be shown to exist between the ages of a group of children and their heights, this relation is not of such a nature that when the age of a particular child is known, his height is uniquely determined by the relationship. Yet the discovery of precise relationships of this latter sort—known as functional relationships—is one of the aims of investigators in many fields.

The topics treated in this chapter are more definitely mathematical than most of those in the preceding chapters. For many students who will eventually succeed in college, the study of functional relationships [1] expressible as formulas and reasonable proficiency in algebraic technique is easily justified. In a good many college courses, and certainly in science and engineering, such knowledge is pre-supposed. It is also true that students who do not go to college but eventually enter a

[1] Educators are likely to be misled by the use of the term *function* when teachers of mathematics are discussing their teaching problems and purposes, and teachers of mathematics are likely to be misled by the approval with which educators greet the proposal to "teach functional thinking." In most educational literature to say that learning (or subject-matter) is functional implies that the thing learned has or promises to have real significance for the student, in that it will make a difference to him in his behavior (including thinking) or in his command over his environment. Mathematics may or may not be functional for a given student in this sense. This Report aims throughout to discuss the teaching of mathematics so that it becomes functional in the educational sense. But the term *function* is used in its technical mathematical sense.

variety of technical or semi-technical occupations find such training useful. It must be granted, however, that the so-called average citizen very rarely has occasion to use formulas or algebraic techniques.

Yet the study of functional relationships is important in general education for at least two reasons. In the first place, it helps the student to see how the concept of function has enabled scientists and engineers to gain an effective control (or at least an explanation) of phenomena that would otherwise seem unmanageable or mysterious. The contrast between electricity as lightning and as a source of controlled power in an electric motor is due in no small measure to the fact that it has been found to behave in ways that can be described and predicted by means of functional relations. The student can make use of the fundamental notions involved as he seeks to understand the world around him and the remarkable achievements that science and mathematics have together made possible. He can understand some of the methods being used by scholars and technicians to solve problems which are beyond his own powers of attack, but in which he has a vital interest and on whose solution his welfare depends.

Secondly, the fact that in some cases functional relationships exist furnishes a standard in terms of which less exact relations are recognized as approximate. The problem-solver who deals with quantitative data often hopes to discover such relationships implicit in his data. But without knowledge of the existence and properties of linear or other formulas gained in other connections he would have little to guide him in his analysis. The irregularities of the usual statistical table or graph are noticeable in contrast to the regularity of the table or graph of linear or other simple formulas. Thus the study of functional relationships expressible by simple formulas may furnish the background that makes many of the procedures in problem analysis intelligible.

Finally, it must be noted that the idea of function permeates modern mathematics, and has been suggested by some author-

ities as the concept best suited to unify instruction in this field. Under a narrow quantitative definition of function, complete unification of mathematics is not possible. However, by employing the term *function* in a broad sense much of the system of elementary mathematical concepts may be embraced. Since the publication of the *Report of the National Committee* of 1923 there has been greatly increased emphasis upon the function concept in the teaching of secondary-school mathematics. Teachers and students have become familiar with some of the basic notions, but misconceptions are still prevalent concerning important points. The discussion which follows is not intended to be an introduction to the subject, but rather a commentary upon some aspects of the function concept which are most often overlooked.

DEVELOPING STUDENTS' UNDERSTANDING OF MATHEMATICAL FUNCTIONS OF ONE VARIABLE

Many students of mathematics who can repeat an acceptable definition of *function* may be unfamiliar with certain closely related ideas which, if not essential to an understanding of the concept, are at least of fundamental importance in clarifying it. In addition to acquiring the notions of *variable, domain* and *range, formula,* and *invariance,* the student should be able to recognize the simpler types of functional relationships and to describe their variation in general terms.

Fundamental Notions Related to the Function Concept

Function and variable. If a group of persons proposes to play tennis so that each player meets all of the others at singles, how many matches must be played? At first glance such a question does not suggest a functional relationship to most people. But after a little preliminary investigation as to the possible pairings, it becomes clear that two people can play one match, three people can play three matches; if there are four

people, six matches will be required; if there are five people, then ten matches will be required, and so on. These facts may be conveniently represented in tabular form as follows:

No. of players	2	3	4	5	6	7	8	etc.
No. of matches	1	3	6	10	15	21	28	etc.

This is a simple example of a mathematical function. In contrast to the situation which often arises in statistical studies, there is here a determinate pairing between the items in the two sets of data. By means of such examples, students may be prepared for a technical definition of function. As a rule, systematic investigators aim to arrange their materials in some sort of order, numerical or chronological or whatever, and they try to relate these ordered materials to some other set or sets of ordered materials. Any determinate correspondence from one set to another, such that each object in the first set corresponds to a determinate object in the second, is called a *one-valued function* or, more briefly, a function.

Although the concept of variable may not be strictly necessary to an understanding of the meaning of *function,* it is useful to introduce the concept, to distinguish between an "independent" and a "dependent" variable, and to speak of "a function of a variable." The number of players in the example above is a variable, and the number of matches which would be played under the conditions given is a function of this variable. The meaning of *variable* is of general logical and psychological interest and concerns not only all branches of science, but belongs no less to general philosophical study and daily conversation.[2] To try to find concepts at the same time more

[2] The notion of variable need not be restricted to classical mathematics. Not only time and distance, but every measurable character, property, or quality may serve as mathematical variable—for example, density, temperature, salinity, price, cost, energy output, intensity of illumination, and so on. One may even extend the notion to qualities not usually thought of as numerically graded, such as taste, moral value, degree of discomfort, persistency, punctuality, and other characteristics personal or impersonal. To define *variable* is thus a task beyond the confines of pure mathematical theory.

familiar and more fundamental in terms of which to fashion a definition for the concept "variable" seems to be an unprofitable task. It would appear to be sufficient here (and in the classroom) to make this elementary concept clear by examples and remarks, and to use the idea of variable as a basis for describing and defining other notions.

Domain and range. An independent variable is always to be thought of as in connection with a collection or "class" of constant objects called "arguments" over which it may be said to vary. This class is the "domain" of the variable and of the function. Although many writers leave the domain to be determined largely by inference from tacit assumptions suggested by the context, the identification of the domain is as much a part of the definition of a function as is the statement of a formula. Take for example a function expressed by use of the formula $y = 2x$. This formula does not of itself identify the function. If x is restricted to the domain of natural numbers (i. e., the class of numbers 1, 2, 3, 4 . . .) then the values of y will be the positive even numbers. But x might be restricted to whole numbers say from 1 to 10 inclusive. Again the domain might be the class of all rational numbers, or all real numbers, or any of numerous other familiar possible choices. A function is defined for each choice of argument in the domain and, properly speaking, for no other quantities. The formal expression of a formula may continue to have a plausible interpretation beyond the domain, but such an interpretation does not involve the given function.

The identification of a given function not only involves the identification of the domain, but also requires the recognition of the sort of quantity to which each argument is made to correspond. If $y = f(x)$ is an expression for the function as far as a formula is concerned, and the class (say A) over which x is to vary is given, the function is completely defined. But in giving f, one is informed of the nature of the variable, y, which is called the dependent variable. For each given argument (say a) in the class A, there is a unique "value" (say b) such that

$b = f(a)$, and these values, b, all lie in a specified class, (say B) so that f is said to be *"on A to B."* For the first of the domains mentioned above in connection with the formula $y = 2x$, the class B consists of the even integers. If the domain of x is the class 1, 2, 3 $\cdots$ 10, the class B consists of 2, 4, 6, $\cdots$ 20. If the domain of x consists of the rational numbers, class B also consists of the rational numbers.

The nature of the problem of correspondence is such, however, that while one is entitled to select an argument a from A at random and ask for the associated value b in B, the reverse process of choosing b and seeking the a to which it corresponds may present difficulties. It may turn out that not every object in B is actually assumed as value for the function. The same object B may be assumed as value for many distinct choices of a, and the question of what a's give rise to a given b may prove in practice almost insoluble, even when the original problem of finding the value $b = f(a)$ for given a is easy. The postage p for first class mail as a function of the weight w in ounces is an example. The domain of w is theoretically any real number less than 1120.[3] The formula for the postage is $p = 3[w]$, where $[w]$ denotes the integer $\geqq w$ but $< w + 1$. Given the weight of a letter it is easy to compute the postage; but to know that the postage is say, 9 cents, does not suffice to find the exact weight of the letter, although presumably $2 < w \leqq 3$.

The class of all values actually represented by the function, for all possible choices of arguments in the domain, is known as the "range" of the function. The range of the postage function defined above consists of the class of positive integers 3, 6, 9 $\cdots$ 3360. Thus a function establishes a correspondence on domain to range, such that for each argument in the domain there is made to correspond exactly one value in the range.[4]

[3] Under the postal regulations in effect in 1939.

[4] Works on mathematical analysis designed for the collegiate level regularly deal with the concept of one-valued function (in the mathematical sense) usually confined, however, to numerical variables, as alone of immediate interest. A reader with persistent mathematical curiosity will find the subject treated in thorough manner in G. H. Hardy, *Course of Pure Mathematics* (New York, The Macmillan Co., 7th edition, revised, 1938).

Unless otherwise implied explicitly or by the context, a function is usually assumed to be a function of one independent variable. A function, say f, then has a single variable argument, say x, and variable (or constant) value y, where x varies freely over a specified domain. While it is not feasible to enter here into any general discussion of possible types of domains, it is probably worth while to mention the two types conspicuous in most scientific applications: (1) domains of individuals, (where *individual* is used to denote any discrete object of attention, not necessarily a person), and (2) domains of variation of a single individual. In the former case counting is employed, tables of frequencies are constructed, classification into genera and species are made. Here questions of approximate measurement and relative precision play no rôle. The "object" may be non-mathematical, but mathematical "objects" are included since in classical number theory the natural numbers serve as abstract individuals. In the case of domains of variation of a single individual, there is usually continuity of variation. Though purely qualitative differences might be emphasized, or mere simple order without numerical gradation may be invoked, ordinarily there is a unit of measure as in the case of variations in weight, height, temperature, and so forth. Though the values are then at best approximate, they are treated as though they were exact: slight distinctions are ignored by systematic grouping into convenient class intervals.

Function and formula. To identify functions and formulas or to treat a formula as but an expression of a function betrays a familiar misconception concerning the nature of functions. A formula is not essential to the definition of a function. A table of corresponding values may serve as definition. Many functions of advanced analysis formerly describable only as solutions of given differential equations with assigned boundary values are not only named, but are expressed by abbreviations or symbols which now serve as formulas. For example, the exponential function satisfying $dy/dx = y$, with $y = 1$ when $x = 0$, is representable by an infinite series which may be denoted by

the symbol e^x; then the function is also representable by the formula $y = e^x$. Whether or not a formal analytical expression has been assigned to a given function depends in large part upon the importance of the function in current investigations of mathematical analysis or in the applications of mathematics. A function may also be defined by using more than one formula. The absolute value of x, for x a real number, is given by x for $x \geqq 0$, and by $-x$, for $x \leqq 0$. It is customary nowadays to introduce a special symbol, $| x |$, and by this device represent the absolute value of x for all real values.

Not only may a single formula be insufficient to cover the domain of a given function, it may also be too extensive. A given geometric locus may be so defined as to be made up of arcs of several familiar curves or straight lines, without including the whole of any one of these separate complete figures. The equation $| x | + | y | = 1$, when graphed, gives a square given also by selected segments of the four lines $x - y = 1$, $-x + y = 1$, $x + y = 1$, $-x - y = 1$. The postage for letters as function of weight is discontinuous, piece-wise constant, and is defined only for a limited domain. Even the locus of the vertex of a triangle for given acute vertical angle and fixed base consists of an arc of a circle and not of a complete circle. The function which is defined as 1 for values of x rational and restricted by $0 \leqq x \leqq 1$, and which is zero for all other real values of x, is familiar in analysis.[5] The formula even when appropriate seldom suggests the exact domain desired, although many mathematical writers fail to be completely explicit concerning the domain.

Despite the fact that the statement of a formula is not the same as the definition of a function, the value of formulas in the study of functions should be apparent. A formula is (1) compact, (2) easy to grasp, (3) suggestive, and (4) readily subject to mathematical operations, sharing thus in the common advantages of all symbolism.

Invariance. There are certain functions (sometimes called

[5] For some interesting discontinuous functions consult *ibid.*

stationary, constant, or *invariant*) which have a common fixed value no matter how the independent variable varies over the domain. Such functions are naturally of special significance. The distance from the center is such a function of a variable point on a circle. The product of the roots of the quadratic equation $x^2 + bx + c = 0$ is such a function of the parameter b, although the sum of the roots, being equal to $-b$, is not a stationary function of b. The distance between specified points of a figure is a stationary function of all rigid displacements of the figure. The sum of the angles is a stationary function of all variations of a triangle. The "nature" of the roots (as to reality) is a stationary function of the parameters of the quadratic equation $ax^2 + bx + c = 0$ so long as $b^2 - 4ac$ remains real and of one sign.

The concept of invariance is seldom mentioned in secondary mathematics although examples of invariant relations abound. The Newtonian law of gravitation asserts that amid various masses at varying distances, each instance of mutual attraction F is such that F times the square of the mutual distance, divided by the product of the masses, is a universal constant. Computation of the areas of circles is possible by virtue of the fact that for all of them area/radius^2 = constant. Trigonometric functions are of service because, for a fixed angle, each is independent of the size of the particular right triangle used in its initial definition. Such special points associated with a triangle as the centroid, orthocenter, circumcenter, incenter, etc., are remarkable because when determined by use of two vertices, or two sides, or altitudes, etc., the result is independent of the choice of two out of three possible elements. For example, the centroid of a triangle is the point of concurrence of the three medians of a triangle. Though any two medians serve to determine this centroid, the particular two medians selected are optional. It is invariant under permutations among the medians. The fact that the value of a fraction is unaltered by multiplying or by dividing numerator and denominator by the same non-zero quantity is one of the

important items concerning a fraction. That an equation remains valid when the same quantity is added to or subtracted from both members, lies at the base of algebraic manipulation. That the sequence of digits of the square root of a number is a function left unaltered if the decimal point be moved any even number of places in the number, makes possible the practical construction of effective tables of square roots. The theorem that if a pair of parallel lines be cut by a transversal, then alternate interior angles are equal is a theorem on invariance.[6]

Common Types of Quantitative Functions

In mathematics functions play a conspicuous rôle and may be said to constitute the major objects of study in many branches of the subject. Many formulas corresponding to functions exist which facilitate the determination of the value of the dependent variable. The area of the circle as function of the radius, the content of a cube as function of the length of an edge, the square of a number as function of the number, the profit on a sale as function of the total cost of articles sold, the interest on a loan of one dollar as function of the time, and so forth, are simple instances from several fields of elementary mathematics. The properties of a given function depend upon the type to which it belongs, and consequently the student

[6] Much of the study of elementary geometry is concerned with certain aspects of *invariance of form* under changes in material and dimension. The simpler plane and space figures such as segments, lines, broken lines, angles, triangles, parallelograms, polygons, squares, planes, angles, circles, circular arcs, circular sectors, circular segments, spheres, spherical segments, great circles, rectangular solids, cubes, cylinders, prisms, cones, pyramids, should become familiar independent of size, orientation, place or material of exemplification. It is probably appropriate that certain less traditional figures which the young person will frequently encounter outside of the classroom be discussed also in his mathematical studies. These include the ellipse (as an orthogonal projection of a circle), the logarithmic spiral, the helix, and the cycloid. Similar figures and proportionality furnish suitable material for impressing the significance of invariance. Illustrations of similar figures and of the value of studying relations among them are to be found among the biological sciences as well as in physics and technological applications.

should not only learn to recognize different types but also to associate special properties with the appropriate type.

Proportional variation; linear functions. The case in which the dependent variable changes in proportion to the independent variable is the simplest but surely most important case of linear function. For the purpose of illustrating a general method of approach, it is appropriate to consider in some completeness some pedagogical steps useful in introducing this simple functional relationship.

Although it has not been customary to introduce formulas from the standpoint of tabular representations of the data they summarize, it is suggested that this approach be adopted occasionally at least. This may be done in several ways. First, if the relationship is relatively simple, the students may build the table for themselves. For example, the students may be asked to prepare a table showing the wages received by a man who gets \$0.80 for each hour he works. By using no more than simple arithmetic, they can prepare the following table.

Numbers of Hours Worked	*Wages (in dollars)*
1	.80
2	1.60
3	2.40
4	3.20
5	4.00
6	4.80
.	.
.	.
.	.

The students may then be assisted in the formulation of a concise statement of the method of obtaining the wages corresponding to any number of hours worked. This statement constitutes the verbal statement of the relationship. At this point some discussion of the nature and advantages of symbols is appropriate, and the students may be shown how the formula $W = .80n$ summarizes the data.

Following the treatment of statistical graphs the preparation of a graph of this relationship is relatively easy. Teachers who, without giving the secret away, have led students to plot their first linear graph will recall the surprise and even delight which many experience on seeing the points take their places along a straight line. The fact that the graph is a straight line and passes through the origin, in contrast to the irregular nature of the usual statistical graph, now assumes special significance. Explanation of this phenomenon is in order, and the students may be led to see that the differences between the values of the dependent variable in the table are the same for constant differences in the independent variable. Graphically interpreted, this means that for uniform increase in the independent variable there is uniform increase in the dependent variable. The situation is therefore analogous to a staircase in which the steps have uniform tread and rise.

Summarizing the discussion thus far, the student may be led from simple arithmetic associated with some relationship to the formulation of both verbal and symbolic statements of a general rule of procedure. By translation of the data to graphical form his attention may be focused upon the linearity of the relationship, and this in turn related to constant differences in the table.

Suppose that this process has been repeated with other similar relationships until the various associated understandings have been thoroughly established. The student is then ready for what is, from the point of view of pure mathematics, perhaps the most significant step in the process—that is, abstraction and a higher stage of generalization. By study or comparison of various concrete formulas like the one above, he may be led to see that all of the examples so far studied in the group may be summarized in the abstract formula $y = ax$. He may well devote some thought to the value of abstraction. He may form the tentative generalizations that the graphs of such relationships are invariably straight lines passing through the origin, and that if the first differences of the independent

variable are constant, then the first differences of the dependent variable are also constant. Finally, he may discover that the steepness of the line $y = ax$ is apparently governed by the parameter a. From the point of view of method, it may now be wise to conduct a short study of this phenomenon in the abstract. This is readily done by drawing upon the same axes the graphs of the sequence of functions $y = a_i x$ where a_i takes the value $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, 1, 1.5, 2, 3, 4, and 5. Further minor generalizations may be drawn from this graph—namely, if the slope $a_i = 1$, the graph makes an angle of 45° with the x-axis. If $a_i < 1$, but positive, the slope is "gentle," if $a_i > 1$, the slope is "steep"—in general, as a_i increases, the slope increases.

The student may now be given experience in applying these generalizations to *new* situations. He may now be expected to predict or assert something about the table, the graph, or the formula of a new relationship. For example, given the formula for the total wages W earned after n hours of labor at \$1.50 per hour (i. e., $W = 1.5n$), he should be prepared to state that first differences in the table of this formula will be constant, that the graph will be a straight line through the origin, and that the steepness will be somewhat greater than that of the 45° line. He should be able to make similar statements about the cost of n baseballs at \$1.25 each from the formula $c = 1.25n$.

An introduction to certain related understandings may be made in this connection. First, if the student has already learned how to compute the value of the dependent variable when the independent variable is given this is an appropriate place for review. If, however, this has not been taught in previous work, this is an appropriate place to teach the technique of substitution. Similar remarks apply concerning the elementary skill now generally known as "changing the subject of the formula," that is, solving for x in terms of y. Most teachers will probably prefer to treat this by first starting with concrete formulas and assigning particular values to the dependent variable, leading eventually to the generalization sym-

bolically indicated by $x = \frac{y}{a}$ or $a = \frac{y}{x}$. In other words, this may be a good place to begin the study of the solution of equations.

Secondly, interpolation and extrapolation may be discussed in connection with linear graphs, the emphasis being upon arriving at the understanding that these processes have greatly increased accuracy and validity for functional data than for statistical data because of the known linearity of the data.

Thirdly, questions concerning the domain of the independent variables may be discussed. For example, in the formula $c = 1.25n$ mentioned above, the domain of n is restricted to non-negative integral values. That is, n can take only the values 0, 1, 2, 3, etc., since obviously one cannot buy half a baseball or 7.5 baseballs, or -3 baseballs, or any non-integral or negative number. In general, in the cost formula $c = pn$ for the total cost, c, of n things at $\$p$ each, n can have only integral values in some cases (e. g., n vases or n pictures) while in others (e. g., n yards of carpet), n could also take certain fractional values. Moreover, the domain for some functions is bounded above, or below, or both above and below. Thus in all of the examples so far cited the independent variable cannot be negative. The formula for the cost of sending first class mail, i. e., $p = .03\,[n]$, in which n is the number of ounces in the package and p is the postage in cents, is often taught (and graphed) as though p were a continuous variable and as though n could take an unbounded sequence of values, whereas legally n cannot exceed 1120. As noted above, the notions of domain and range, which are tremendously significant from both the purely mathematical and the practical points of view, have up to the present received practically no attention in mathematical work below the college level.

For some students and some schools an elementary analytical study of functional variation may contribute much to an understanding of functional relationships. The types of

relation considered in secondary mathematics are restricted to the very simple cases, and the types of variation of the independent variable usually considered have for the most part been restricted to those indicated by the words *increase, decrease, double, treble, halve.* Little consideration has been given to the quantitative effect of increase (or decrease) by an additive constant, or of transformations which involve multiplying the independent variable by constants other than 2, 3, ½, and ⅓. These variations may and should be generalized. In the discussion which follows an elementary method of treatment will be applied to the simple form $y = ax$.

Suppose the student has a concrete formula of the type (1) $y = ax$, and that x is to be increased by some constant k. At this point an understanding of major mathematical significance must be achieved, namely, that whatever the change in x, the *form* of the relationship remains the same. Denoting, therefore, the new value of x by X, and the new value of y by Y, one can write (2) $Y = aX$. Needless to say, this concept will require considerable discussion and numerous illustrations before it is fully understood by the students. Assuming that it has been made clear, the hypothesis above in symbolic form becomes (3) $X = x + k$. Substituting this in (2), one obtains (4): $Y = a\ (x + k)$. At this point the situation calls for the elementary operation of multiplying a binomial by a monomial—the application of this distributive postulate of algebra. Assume that this will now be taught, so that the students will follow the logic of writing (4) in the form (5) $Y = ax + ak$.

The student should next observe that the problem at hand calls for expressing the new value of the dependent variable Y in terms of its former value rather than in terms of the independent variable x. The use of formula (1), however, by mere substitution yields (6) $Y = y + ak$. This expression should now be interpreted by the students in words. Upon this verbal interpretation of a new result obtained by manipulation depends most of the value of the procedure. Without such interpreta-

tion one of the major values of algebraic treatment of the data of experience may be lost.

The above ideas have been developed in the abstract, and although simple they may be new to many teachers. Further detailed illustrations may be helpful. The application of the process to a particular formula such as a relationship between profit and selling price expressed by the formula $p = .08s$ will be discussed. Expressed in words, the formula states that "the profit on a certain article is 8% of the selling price." What happens to the profit if the selling price is increased by $12.00? The algebraic work follows:

Let p designate the initial profit and s the initial selling price, while P designates the changed profit and S the changed price. Then

$$(1)\quad p = 0.08s$$

and

$$(2)\quad P = 0.08S.$$

Owing to the announced increase, one has

$$(3)\quad S = s + 12.$$

Substituting (3) in (2), yields

$$(4)\quad P = 0.08\,(s + 12), \text{ or}$$

$$(5)\quad P = 0.08s + 0.96; \text{ hence using (1),}$$

$$(6)\quad P = p \quad + 0.96.$$

Interpreted, this means that if the selling price is increased by $12.00 the profit is increased by $0.96.

It may be argued that there is nothing striking about the result just found, and that it is readily calculated by simple arithmetic without resorting to algebraic manipulation. This must be granted in the case of this simple illustration. The point to be kept in mind, however, is that here it is the *method* which is important, and application of the same method in more complicated situations will yield results which would be difficult to find by the processes of ordinary arithmetic.

By way of further illustration consider the multiplicative type of transformation applied to the same formula. Suppose the selling price is doubled. What happens to the profit in this case? Following through the steps in the same order, one has,

(1) $p = 0.08s$ (4) $P = 0.08\ (2s)$
(2) $P = 0.08S$ (5) $P = 2\ (0.08s)$
(3) $S = 2s$ (6) $P = 2p.$

The fairly obvious result that the profit is also doubled has thus been shown in algebraical setting.

If the problem asks what happens when the selling price is increased 20%, a slightly more difficult situation results. In this case,

(1) $p = 0.08s$
(2) $P = 0.08S$
(3) $S = s + 0.20s.$

The addition of like terms, if it has not already been studied, must now be learned. Applying the skill in this case yields

(4) $S = 1.20s$

Then (5) $P = 0.08\ (1.20s)$, or
$= 1.20\ (0.08s).$

Thus (6) $P = 1.20\ p.$

Since the interpretation is more difficult here, an alternative method like the following may be preferred. Writing (2) and (3) as above, then substituting $S = s + 0.20s$ in (2), one has

(4) $P = 0.08s + 0.08\ (0.20)s$
(5) $= 0.08s + 0.20\ (0.08s)$
(6) $= p + 0.20\ p.$

This shows that p is also increased by 20% of itself. In general, if $y = ax$

Then $Y = aX$
Assume: $X = kx$
Then $Y = akx$
Or $= k(ax)$
$Y = ky.$

This generalization expressed in words should be the end product arrived at after numerous concrete examples have been studied. The student should come to realize that the language is technical and quantitatively exact in contrast to a qualitative sense of concurrent increase or decrease. Such a statement as the following "As tax rates are raised, popular

discontent increases," should be sharply differentiated from the precise functional relationship expressed by "the absolute temperature varies as the pressure."

In his study of problems based on actual data, the student may encounter tables that approximate a linear relationship. After study of the kind outlined in the preceding pages he should be able to recognize this fact. The graph of the data will help, and computation of first differences will strengthen the hypothesis. The student may even be ready for some very elementary "curve fitting," and seek to find a formula which will approximate the tabular or graphical data.[7] If so, he is making a start toward the acquisition of certain understandings which underlie much scientific work and in particular, mathematics. For the ability to find an algebraic formulation of a relationship provides a powerful tool for further investigation. It facilitates extrapolation and interpolation. It brings to light relationships formerly unsuspected. These ideas, which motivate the mature investigator, should eventually be clear to the students in the secondary school as they also seek to solve problems by less sophisticated methods.

The form $y = ax + b$. Detailed discussion quite similar to the above could be given concerning the form $y = ax + b$, which comes next in the scale of complexity. But since the simple transformation $y' = y - b$ reduces it to $y' = ax$, little more need be said here. The student should of course understand that this transformation moves each point of the line b units upward or downward (according to whether b is negative or positive). Although the study of transformations has not been customary in secondary-school mathematics, the importance and usefulness of the notion, coupled with the simplicity of the operations, suggest that it should receive more attention in the future.

The quadratic function; powers and polynomials. Most investigators are pleased if they can establish a reasonably strong linear correlation between two sets of data. The techniques

[7] Cf. Appendix I, page 396.

for treating non-linear relationships exist, but are not often used except in highly technical investigations. But for reasons similar to those which were used to justify the study of linear formulas, the student should have some experience with non-linear relationships. The properties of linear relationships are scarcely appreciated by those who are unfamiliar with non-linear variations according to some simple formula, such as $y = ax^2$. This formula may be studied in essentially the same way as was described in connection with the form $y = ax$, and the points of contrast between the two types of relationship may easily be made clear.

Starting with a simple "square law," $y = ax^2$, one may by shift of origin examine the representation of the general quadratic function $y = ax^2 + bx + c$, investigating the significance of the several coefficients, noting what happens to the form of the relation if either variable is multiplied by a constant, interpreting the operation of finding roots, illustrating the law of falling bodies (in a vacuum) and in general making the quadratic relation comprehensible to the student. The fact that in an equispaced table of arguments for a quadratic function, the first differences of the values of the function form an arithmetic progression and second differences are constant always interests the students. The corresponding fact that any equispaced table with constant (non-zero) second differences may provide values for a quadratic function proves no less interesting.

From the quadratic case one may for some students lead the discussion through cases of "power-laws" of the form $y = ax^n$, which upon the change of variables, $Y = \log y$, $X = \log x$, becomes $Y = c + nX$. The power laws (although for them the exponent n is not necessarily a natural number) suggest polynomial laws of the form $y = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ where n is by hypothesis a natural number. Even if but little is done with graphs of such relations the existence of maxima and minima, and their determination at least in the case of quadratic expressions, may be worth studying. For a few stu-

dents the idea of "synthetic division," with its obvious economy for the problem of locating zeros, may merit development.

Special graph paper, especially the simple and double logarithmic scale paper (the former called also arithmetic-log or semi-log) merit discussion and use as illustrating the simplicity of the straight-line graph. The purpose of such special paper and non-uniform scales is to exhibit a function not ordinarily thought of as linear in the form of a straight line for suitably changed variables. The power laws, and the exponential laws to be discussed next, assume linear shape when rightly plotted on such special paper.

Exponential functions. Functions of the type $y = ac^x$, $(c \neq 0,1)$ arise inevitably in connection with compound interest (to mention a financial application) and theoretical curves of growth or decay. That such a function reduces to a linear form under the transformation $Y = \log y$, $X = x$, adds much to the attractiveness and simplicity of the study. The cases of $c = 2, 10, \frac{1}{2}, \frac{1}{10}$, help to suggest what may be expected to happen in other cases. Problems on the time interval in which a variable will double itself at constant proportional rate of increase, or a radioactive substance reduce to half its original mass, may prove of interest. While it may be well to mention the Naperian constant $e = 2.7 \cdots$, as of wide scientific use, no details of the theory of natural logarithms need be given. Some properties of the Archimedean or logarithmic spiral may well be made the topic of brief discussion, or assigned as a special project to an individual.

Periodic phenomena. In secondary-school mathematics it is customary to ignore all reference to the vast body of periodic phenomena of such fundamental importance in science and engineering. Yet even at the level of high-school mathematics the student may become accustomed to recognizing periodic phenomena and their representation. Seasons, the time of day, vibrations, and wave shapes are familiar instances, and certain unsuspected vibratory phenomena may be demonstrated,

as in the field of sound. The modern boy is likely to be an enthusiast concerning radio transmission, and his enthusiasm may be capitalized and increased by some scientific understanding of certain common terms such as kilocycle, period, amplitude. The graph of the sinusoid is one of those very interesting and at the same time simple things which can well fall within the range of a broad program of secondary-school mathematical instruction.

DEVELOPING STUDENTS' UNDERSTANDING OF FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES

The study of several independent variables might seem at first formidable and abstruse. But the utility of algebra lies largely in its introduction of possible variables for the numerical constants of arithmetic. The formula for the volume of a right circular cone involves two independent variables, and finding the volume of so simple an object as a brick—a rectangular parallelopiped—by the formula $V = lwh$ involves three independent variables. In various applications, more variables, such as pressure, density, temperature, etc., may also be encountered. The quadratic expression $ax^2 + bx + c$ has x as independent variable and the three "parameters" a, b, c, are fixed in any given problem. For theoretical purposes, where problems themselves are subject to scrutiny and change, the coefficients a, b, c, although indeed constant for any single instance of application, are true variables. In so simple a figure as that of two triangles in a cartesian plane, perspective from a point, there are eleven independent variables.

It is customary for mathematicians, physicists, and engineers to use such terms as *number of independent variables, number of parameters, dimensionality, and degrees of freedom* essentially interchangeably. It is important that the teacher at least has a clear notion of what is meant. A point moving along a prescribed curve may be said to have one de-

gree of freedom—its position is one-dimensional. The curve may be in particular a straight line—for modern mathematics a straight line is a special case of a curve. A bead on a wire, a point on the rim of a wheel free to turn in a fixed plane about a fixed center, suggest one degree of freedom. The variable need not be spatial. It may, for example, be time, or density, or pressure, or velocity, or pitch. A "T-Square" used with the head against one side of the drawing board has one degree of freedom: it can move up and down on the board.

An object with two degrees of freedom is subject to motion of such a sort that when one point is always on a given curve, motion is still possible, but when any other point still free to move is then fixed, the entire object is fixed in space. A ruler merely lying on a drafting board has three degrees of freedom. It requires two independent specifications to fix one end at a given point on the board, and still another to fix the other end. A rigid object free to move in space has six degrees of freedom. The count may be made as follows: A point fixed in the object has three degrees of freedom of motion. Having placed this point in space, there are still two degrees of freedom for any other point fixed in the solid. Finally there is one degree of freedom of rotation about the line joining these two.

The notion of dimensionality or degrees of freedom carries over into formulas for area, volume, and many others. The area of a triangle has two degrees of freedom, as indicated by the variables b and h in the formula $A = \frac{1}{2}\, bh$. But the area of an equilateral triangle has only one degree of freedom, since in this case the variables b and h are not independent. It should be kept in mind that one cannot select certain independent variables and say that these are *the* inherent variables. Their number is fixed, but the particular variables chosen are not. Thus the usual formula for the area of an equilateral triangle, $A = \frac{\sqrt{3}}{4}\, e^2$, uses the edge as the independent variable. But the distance r from the center to a vertex may also be used, and in this case the formula becomes $A = \frac{3\sqrt{3}}{4} r^2$. That dimension-

ality remains fixed but choice of variables may be altered is also illustrated by the fact that one may use (x, y) coördinates to locate a point in the plane with respect to a particular frame of reference, or again (r, θ) coördinates giving distance and angle from an assigned initial direction.

It may also be noted that two-dimensional spaces, while alike as to dimensionability, may differ notably in respect to further characterization. Examples of surfaces (by definition with two degrees of freedom) include instances without boundaries, such as the entire plane, the surface of a sphere, the surface of a ring, the surface of a tetrahedron, the surface of a knotted and self-spliced rope. Surfaces may also be partly or completely bounded, as is the strip in a plane between two parallel lines including (or perhaps excluding) the points of these lines themselves, the interior of a circle, the surface of a sphere with the exception of a single point, the inner surface of a cup, and many other examples.

DEVELOPING STUDENTS' UNDERSTANDING OF THE GENERALITY OF THE FUNCTION CONCEPT

Despite the generality of such notions as variable, class, function, argument, domain, value, range, the student is likely to encounter these terms for the first time as objects of study in connection with mathematical theory. If a proper perspective and breadth of interest is to be maintained, the teacher of mathematics must be alert to prevent the misconceptions that are likely to ensue from the manner in which these terms are introduced to the student's consciousness. The function concept which has proved so fruitful in mathematics will suffer even for mathematical study if one confines the definition to numerical quantities. It is advantageous to retain a rather wide generality for the concept "function," allowing it to apply to logical situations rather than restricting it to purely quantitative correspondences familiar in classical mathematics. This broadness of scope of the term *function* makes it possible for teachers of mathematics to levy on many fields in addition to

the purely mathematical in order to illustrate the mathematical method of discovering and establishing correspondence between variables.

Generality as to Arguments

The arguments that compose the domain of a variable need not be numbers, nor need the values assumed by a function be numbers. The arguments may obviously be taken as expressions of various sorts—equations, formulas, and so forth, even geometrical figures—and the values may be numbers, proofs, constructions, and numerous other things. The teacher should find abundant examples from non-mathematical fields. The problem of "How do you spell the word ——?" is a function whose arguments are words, and whose values are spellings. When a word to fill the blank is given, the spelling is, as a rule, determined—there is a correspondence between words as sounds and their spellings. The dictionary is, among other things, a table of this function. The problem of naming the capital city of a State involves the function whose arguments are the States of the Union (or countries of Europe, etc.) and whose values are names of cities. To each state there corresponds a capital city, and a table of this function may be found in an atlas or *World Almanac*. The play being presented at a given theater is a function of the time (where the domain is restricted at times at which plays are being presented) and whose values are the corresponding plays. The homeroom teacher is a function of the student, as is the period of the student's lunch hour, and the recorded number of tardinesses.

Propositional Functions

Perhaps one of the most important types of function, not only for general purposes but for mathematics as well, is the "propositional function." A propositional function is illustrated by the statement "That x likes his homeroom teacher," where the domain of x may be all the children in school. The

substitution of particular names for x yields propositions such as "That John likes his homeroom teacher," or "That Mary likes her homeroom teacher." In general, a propositional function is one whose value is a proposition, namely a concept, idea, or notion, which definitely is or is not true.[8]

Now a declarative statement about specific individuals is an assertion of a proposition. Any such statement is true or false, and the proposition itself may be regarded as free of any essential assertive element. The assertion "John likes ice cream," in which "John" refers to a given individual boy, and "ice cream" is regarded as suitably restricted to familiar varieties, would of course be usually regarded as a true statement. If one fears that John's tastes may change, as for example when he has indigestion, one may make the more specific statement: "John liked the ice cream at the party on the occasion of his tenth birthday anniversary." This assertion corresponds to a proposition which can be expressed in some such form as "That John likes ice cream"—it can be followed by "is true" or "is false." In other words, the proposition is of a sort that may be asserted, denied, believed, or doubted. But the propositional *function* "that x likes his homeroom teacher" has no determinate truth-value, since it is neither true nor false until a specific child is selected as x. The truth or falsity is likely to depend upon which child is chosen. Thus two functions are here involved. One is the propositional function itself, the other is the related "truth-function," whose values are merely "true," or "false." A "true-false" question on an examination is for each person tested a function of the reply marked, and has for values "true" or "false," and is thus a function of the mark "true" or "false" recorded.

Relations

A numerical mathematical function is but a special case of a mathematical relation. Such a formula as $y = x^2$ expresses a

[8] The notion of truth is, however, extraneous to many ideas. Cf. Chapter IX, "Proof," page 204.

relation between x and y, which relation is of the special sort which constitutes a function. But the expression $x < 1$, while no less a statement of a mathematical relation, fails to represent a function. Indeed no value of x such that $x < 1$ is uniquely identified. If one were to adopt a narrowly numerical characterization of "function," then it would not be reasonable to attempt to relate the whole of mathematics to the function concept. But in the wider logical sense, a relation is but a special case of a function—namely of a propositional function. "That x is less than unity" is a propositional function of x, true for some values of x such as $0, -3, \frac{1}{2}$, and false for others, such as $2, \pi, 1$, and $\sqrt{-1}$.

Every propositional function of two variables, say "that X likes Y," expresses a relationship, and every relationship is expressible through a propositional function. For example, the filial relation is represented through "that X is the son of Y," and the paternal relation through "that X is the father of Y." The importance of perceiving and utilizing relations in attacking scientific problems is too well known to require emphasis. In the relatively tractable problems of geometry, relations of parallelism, of congruence, of similarity, and so forth, may be made basic to the whole study. It is through emphasizing relations as propositional functions that one can readily embrace geometry no less than algebra under the notion of function.

In the problem of acquiring and transmitting knowledge, relationships have an important rôle. Where possible, of course, one prefers to describe objects and events in terms of properties or characteristics, with no explicit reference to other entities. But a familiar property may not be at hand to characterize the object under study. Often it is convenient if not necessary to employ terms of *relation* and by establishing contacts with related entities the desired description may be made. Thus to describe an ellipse one may show its relation to the circle—children sometimes say "an ellipse is a flat circle." This idea may be made explicit by comparing the ordi-

nates of the ellipse to those of the circle whose diameter is the major axis of the ellipse. For example, for the ellipse $4x^2 + 9y^2 = 36$ the ordinates are $y = \pm \frac{2}{3}\sqrt{9 - x^2}$; in other words, each ordinate is two-thirds of the corresponding ordinate of the circle $y = \pm \sqrt{9 - x^2}$. The teacher may awake responsive understanding of the rôle of relationships by having the class analyze descriptions—pointing out the specific properties and also the relations employed, and noting that many terms are relative.

Among relations between two arguments, or otherwise expressed, among propositional functions of two arguments, certain ones are of special importance owing to exhibiting one or more of the following properties: reflexiveness, symmetry, and transitivity. Suppose that one has a relation R between the arguments x, y, z in a given common domain. This relation may be indicated by the symbol $x\ R\ y$, read "x has the relation R to y." A relation R such that x has this relation to itself (xRx) is called reflexive. A relation such that xRy implies that yRx is called *symmetric*. A relation such that xRy and yRz imply xRz is called *transitive*.

The relation of similarity in the domain of triangles has all three properties. For a triangle is similar to itself—hence, similarity is reflexive. If triangle x is similar to triangle y, then y is also similar to x, hence similarity is symmetric. Finally, if triangle x is similar to triangle y, and triangle y is similar to triangle z, then triangle x is also similar to triangle z; hence the relation is transitive.

On the other hand, the relation of perpendicularity [9] is symmetric but not reflexive or transitive. Line x may be perpendicular to line y, but not to itself. But if line x is perpendicular to line y, then y is perpendicular to x; the relation is symmetric. If line x is perpendicular to line y, and y is perpendicular to z, then x is not perpendicular to z (it is in fact parallel to z), so transitivity is not a property of the relation in this case.

[9] In the plane.

The relations of equality and congruence have all three properties. The relation "is brother of" (among men) is symmetric and transitive,[10] but not reflexive. The relations "is older than" and "is greater than" (for numbers) are transitive, but not reflexive or symmetric.

Many other relations could be mentioned. The filial relation ("is son of" or "is daughter of") and the paternal relation, parallelism, "to the east of," "can defeat—at war, or at tennis," are but a few which may be studied to discover whether they have the properties above.

These properties became important because fallacious thinking is sometimes based upon the assumption that they hold when as a matter of fact they may not. To say "China fears Japan, Japan fears Russia, and hence China fears Russia" is to assume the property of transitivity for the relation "fear." The advertisement "Chevrolet leads the sixes, and the sixes lead the world" was based upon a hoped-for assumption of transitivity. No boy or girl is likely to assume symmetry for the relation "love," or to believe that because John loves Mary that Mary also loves John. But in the less familiar fields of labor, or race questions, or international politics, the student may more easily go astray. Is the relation "get fair treatment from" symmetric between employers and laborers? Questions of this sort studied with the nature of relationships in mind may be more easily disposed of. To know that not all relationships are symmetric, reflexive, or transitive may help one to reject a plausible but fallacious line of reasoning. This does not mean that one needs to know the technical words for these relations, but only that one should be on guard against assuming they hold in particular cases.

BIBLIOGRAPHY

BRESLICH, E. R., *Problems in Teaching Secondary-School Mathematics* (Chicago, University of Chicago Press, 1931), Ch. IX. Includes an extensive list of references to magazine articles.

COOLEY, Hollis R., GANS, David, KLINE, Morris, and WAHLERT, Howard

[10] Among distinct persons.

E., *Introduction to Mathematics* (Boston, Houghton Mifflin Co., 1937). Part II, "The Function Concept."

HAMLEY, Herbert Russell, *Relational and Functional Thinking in Mathematics, Ninth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1934).

HASSLER, Jasper O., SMITH, Rolland R., *The Teaching of Secondary School Mathematics* (New York, The Macmillan Co., 1930). Ch. X, "Subject Matter and Objectives of Elementary Algebra," pp. 231–238.

Selected Topics in the Teaching of Mathematics, Third Yearbook of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1928), "Developing Functional Thinking in Secondary School Mathematics," E. R. Breslich, pp. 42–56.

The Reorganization of Mathematics in Secondary Education, Report of the National Committee on Mathematical Requirements (Oberlin, Ohio, The Mathematical Association of America, Inc., 1923). Ch. VII, "The Function Concepts in Secondary School Mathematics," E. R. Hedrick, pp. 64–73.

The Teaching of Algebra, Seventh Yearbook of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1932). "The Function Concepts and Graphical Methods in Statistics and Economics," W. Lietzmann, pp. 74–92; "The Function Concept in Elementary Algebra," N. J. Lennes, pp. 52–73; "Measuring the Development of Functional Thinking in Algebra," E. R. Breslich, pp. 93–118; "Mathematics and Measuring of World Trends and Forces," W. S. Schauch, pp. 119–136.

YOUNG, John Wesley, *Lectures on Fundamental Concepts of Algebra and Geometry* (New York, The Macmillan Co., 1911).

VIII

OPERATIONS [1]

Previous chapters have pointed out how such concepts as formulation and solution, data, approximation, and function permeate mathematical thinking. But the mathematical approach to problems involves more than formulating them in ways to facilitate the securing of acceptable solutions, collecting, recording, and organizing raw data, and recognizing relations among the data so treated. Problem-solving usually also demands the performance of various operations: numbers must be added or subtracted, equations must be solved, and many other operations must be performed. The techniques involved are so frequently demanded and the concepts basic to the operations they help to perform are so essential that they cannot be omitted in a consideration of mathematics as a method of problem-solving.

It is hardly necessary to stress the fact that in the past secondary-school mathematics has often emphasized facility in the performance of operations to the neglect of other important aspects of the field. Courses of study have tended to overemphasize techniques, and the most common feature of classroom method has been drill to develop skill in the performance of operations. There seems to be little doubt that such practices result in narrow views of the nature of mathematics, and in part account for the fact that many people who have devoted considerable time to mathematical study think that the subject consists largely of a body of operational techniques. The emphasis of this Report is designed to help correct

[1] This chapter differs from the others in Part II in that it concentrates less upon the concept of operation itself and more upon certain sub-concepts basic to intelligent performance of operations.

this situation by stressing concepts basic to operations no less than the techniques by which they may be performed, and by considering these concepts among a number of others fundamental in the study of mathematics.

The recurrent necessity of solving quantitative problems quickly and with a minimum of labor has led men to codify operational techniques into sets of formal rules. The fact that these rules can be learned and applied mechanically has made it possible for students to solve problems without insight concerning the nature of the operations they perform. But it is also possible to understand the nature of an operation and fail in its performance because of inability to recall some fact or for lack of some particular skill. In order to secure both understanding and skill there must be appropriate emphasis on each. There are, however, strong reasons for believing that attention to understanding will not only motivate acquisition of skill, but will also aid retention. Although the invention of computing machines tends to reduce the importance of skill in the performance of arithmetical operations, there is no escape from the necessity for understandings which throw light on the nature of the operations and serve as a guide to proper procedures governing their use. Except in routine work the intelligence of the operator must function in choosing the proper machine and the proper operations, and such choices depend upon a modicum of insight.

For these reasons this chapter begins with a discussion of the problem of developing among students understanding of concepts basic to operations, and proceeds to a brief consideration of a point of view in regard to the development of technical skill. The topics treated should not be considered in isolation from those discussed elsewhere in this Report. Above all they should not be regarded as suggesting units of a course of study. They suggest desirable concepts to emphasize as appropriate occasion arises in the course of solving problems.

BUILDING STUDENTS' UNDERSTANDING OF CONCEPTS BASIC TO OPERATIONS

Concepts Basic to Counting and the Fundamental Operations

Grade-school pupils are expected to learn to count, to add and subtract, to multiply and divide, and to use a host of related techniques. Yet they seldom really understand the nature of these processes, or see the relations between them and other aspects of mathematics. If the performance of the fundamental operations is to be more than a mere mechanical process, the student should eventually come into possession of certain important understandings about them. The full development of these understandings probably requires at least as much maturity as is found in the secondary school, and discussion of the following topics is therefore pertinent at this level.

Counting. Though seldom viewed as one of the fundamental operations, counting is the simplest operation in mathematics. The process is so common and familiar that to describe it as the operation of putting the positive integers into one-to-one correspondence with a set of objects is more likely to confuse than to clarify. Students can more readily grasp the nature of counting by studying the process as it is performed by mechanical means ranging from the primitive abacus to modern turnstiles, revolution counters, mileage recorders, and other similar devices. They may thus be brought to recognize that there is a distinction between the process of counting and the notation used to record the results; to appreciate the economy effected by a number system which employs a zero, place-value, and a convenient base; and to realize that the operation of counting is independent of the structure of the number system.[2]

[2] The number of individuals in a group is an invariant property of the group.

See Morris R. Cohen and Ernest Nagel, *An Introduction to Logic and Scientific Method* (New York, Harcourt, Brace and Co., 1934), pp. 291–292.

The fundamental operations in the system of natural numbers. In respect to the fundamental operations the student should come to understand that in the system of natural numbers (1, 2, 3, . . .):

Addition may be regarded as a short method of counting.

Multiplication is a short method of finding the sum of a group of numbers all of which are the same. Thus while the product of 273 by 67 may be found by adding a column of numbers consisting of 273 repeated 67 times, the multiplication process greatly reduces the labor.

Subtraction is the inverse process of addition.

Division is the inverse process of multiplication. It may be regarded as a short method of performing repeated subtraction.

The teacher can clarify the notion of inverseness by giving such simple illustrations of mutually inverse operations as: to increase, to decrease; to build up, to break down; to simplify, to complicate. Students who understand inverse operations will announce the result of examples like $632 - 548 + 548$ or $632 \times 548 \div 548$ at once without resorting to computation. Those who have not grasped the notion usually carry through the indicated operations, and the fact that they thereby obtain the number they started with may be used to clarify the concept of *inverse*. Eventually the student should not only sense the fact that subtraction and addition, division and multiplication, are mutually inverse operations, but should be able to express their relation in appropriate words.

The teacher should also be on the alert to help students overcome difficulties arising from the way in which limitation to the natural number system restricts the scope of possible operations. In the system of natural numbers subtraction is possible only when the minuend exceeds the subtrahend, but this fact seldom gives rise to confusion in students' understanding of the concept. On the other hand, the fact that "exact division" in the system of natural numbers is possible only when the dividend is a multiple of the divisor is seldom fully understood. For example, within the system of natural

numbers the "exact division" of 17 by 3 is not possible, since the concept calls for the integer which multiplied by 3 produces 17, and no such integer exists. The formal process for division, however, leads to the number 5, the integer which on multiplication by 3, gives the closest attainable approximation to 17 with a positive integral error. That is to say, when the dividend is not a multiple of the divisor, the process for division yields an integer as "quotient" and also a "remainder." This "quotient" is the integer which, multiplied by the proposed divisor, produces an integer approximately equal to but less than the dividend, and the error (the "remainder") is less than the divisor. The interpretation of the "remainder" in the division process often causes difficulty even when the number system is extended to include fractions. The clearing up of the difficulty may be effected by developing a clear concept of the nature of division, and making the distinction between the process for "division with remainder" and "exact division."

Opportunities for clarifying concepts of the fundamental operations occur in innumerable situations. For example, the problem of finding the number of city blocks in a district which is 17 blocks long and 14 blocks wide may be solved by counting the "blocks" on a map. To shorten the process one might note that each row contains 17 blocks, and the total number could be found by adding a column of numbers composed of 17 written 14 times. A still shorter process involves multiplying 17 by 14. It thus becomes clear that multiplication reduces the labor involved in repeated addition of the same quantity.

The nature of division may be clarified in connection with examples like the following. A farmer has a silo which he estimates contains 3000 cubic feet of ensilage (chopped corn). If he has 37 cows and plans to feed each of them 1 cubic foot of ensilage a day, how long will the feed last? This problem may be solved by a process of repeated subtraction: 37 from 3000, then 37 from 2963, etc., until a remainder of less than

37 is obtained. The number of times the subtraction must be performed corresponds to the number of days the feed should last, and it is important to note that the process used corresponds exactly to the way the problem could be solved in practice without the use of any arithmetic other than counting. The farmer may actually remove 37 cubic feet each day until the feed is exhausted. If he knows arithmetic, however, he can quickly solve the problem in advance by division. The "quotient" 81 corresponds to the number of days the feed will last, but in this instance the "remainder" of 3 cannot sensibly represent $\frac{3}{37}$ of a day, but rather indicates that after 81 days about 3 cubic feet would remain if measurements were rather accurately made. This illustrates how division may be regarded as a short method of doing repeated subtraction, and it emphasizes that the interpretation of the result is a matter of common sense quite independent of operational facility.

Fundamental operations in other number systems. New kinds of numbers have been added to the number system in order to make possible operations which would be impossible without them. New numbers adjoined to a number system are so defined that the laws of operation applicable to the original system apply to the new numbers with as few changes or exceptions as possible. The situation is analogous to a scheme of colonization where, as new colonies are added, the laws of the mother country are made to apply with a minimum of reinterpretation required by local conditions.[3] Thus when the number system is extended to include, say, the totality of positive fractions in addition to the system of natural numbers, and so becomes the system of positive rational numbers, some concepts gained in the study of operations on natural numbers need to be extended or reinterpreted. Division continues to be defined as the inverse of multiplication, but is now always

[3] "The principle of permanence of form," which expresses these ideas technically, is discussed in readable fashion by Tobias Dantzig, *Number, The Language of Science* (New York, The Macmillan Co., 1930).

possible (with the single exception of division by zero, which is never possible). The number system may also be extended by the inclusion of negative numbers, and this makes subtraction always possible. It also involves a new set of rules of operation. For example, the product of two negative numbers must be taken as a positive number in order to obtain consistent results in operating with both positive and negative numbers.

Fundamental operations in elementary algebra. The fundamental operations of elementary algebra do not differ in kind from those of arithmetic, but since algebra deals with generalized number, modifications of the rules are often necessary.

Operations in both arithmetic and algebra are performed under the commutative, associative, and distributive postulates or "laws." The commutative and associative postulates ordinarily give students little trouble in either arithmetic or algebra unless they are asked to learn them formally. To the students the commutative law is obvious, and it seems quite unnecessary to specify that the sum is independent of the order of the terms, or that the product is the same regardless of the order of the factors. Students do, however, often get into difficulty by assuming that the distributive law holds more generally than it does. Although $2(a - b) = 2a - 2b$, the expression $a^2 - b^2$ is not the same as $(a - b)^2$, nor is $\sqrt{a - b}$ equal to $\sqrt{a} - \sqrt{b}$. Although the mean of a collection of measurements in inches may be divided by 12 to find the mean in feet, the arithmetic mean of the members in one collection plus the mean in a second collection may not be the same as the mean of the pooled collections. Mistakes made in applying the distributive law are due largely to the inertia engendered by drill on techniques without sufficient attention to the development of concepts basic to them.

One of the most noticeable differences between arithmetic and algebra is the increased importance of *indicated* operations in the latter. In arithmetic not only can the sum of 3 and 5 be indicated by the symbols $3 + 5$, but the operation can also

be immediately carried out and the result expressed by the symbol 8. In the generalized arithmetic of algebra, however, the sum of two different literal numbers a and b can only be indicated, i. e., written $a + b$. Similarly the operation indicated by $6\frac{3}{9}$ or $63 \div 9$ can be immediately carried out, whereas in x/y or $x \div y$ the indicated operation of division must frequently be postponed until later stages of a problem, or even indefinitely. The immediate result of such differences in technique is that symbols of operation play a far more important rôle in algebra than in arithmetic. Such symbols in arithmetic usually express commands which are to be executed immediately, whereas in algebra some orders must be held in abeyance while certain preliminary manoeuvers are carried out. It not infrequently happens that at strategic points squads of symbols annihilate one another, so that certain orders expressed as indicated operations never need to be formally executed. The expression $\frac{x}{y} + \frac{y - x}{y}$ is a typical example. Instead of attempting to carry out the indicated divisions first (as might be done with known numbers in arithmetic), in algebra addition would take precedence, and the expression would eventually be reduced to 1.

In the case of explicit formulas, symbols of operation tell directly what is to be done to compute the dependent variable when particular values of the independent variables are given. In solving equations they play a dual rôle. In some cases, they state direct commands which must be carried out in order to reduce the equation to an equivalent one which more closely resembles a so-called standard form. This is illustrated by such an equation as

$$x^2 - 3x - 2 - (x - 2)(x - 3) = 0,$$

in which the usual first operation is actually to carry out the indicated multiplication of the two binomials. This makes it possible to carry out the indicated subtraction which reduces the equation to the form

$$2x - 8 = 0.$$

At this point the indicated operations assume their other rôle, namely, they guide one to a process in which the operations are the inverses of those indicated. No further direct reduction is possible, but the symbolism indicates that if the unknown number x is multiplied by 2, and 8 is then subtracted, the result is zero. To find x one therefore performs the inverse operations in reverse order. That is, 8 is added to 0 and the result is divided by 2. The rôle of the equation axioms relative to addition, subtraction, etc., is merely to formalize these processes, i. e., state general rules which in the long run economize on the thinking necessary to solve particular equations. The development of a clear understanding of the dual aspect of operational symbolism should be an important feature of instruction. Such understanding is rarely achieved by mere drill on technique. It is one of the component understandings of the concept of mathematical operation which should underlie the formal skills.

In arithmetic the student usually understands that the solution of a problem is some function of the known numbers given as data. But the corresponding fundamental notion of elementary algebra, namely, that the roots of equations are functions of the coefficients, is seldom understood. Emphasis on technique tends to submerge this notion, but increased attention to the concepts underlying the operations should be helpful in bringing it to the surface. If operations are actually carried out with numerical coefficients the functional relationship between root and coefficients is so disguised as to be practically unrecognizable. Thus an approved method of solution of the equation $3x - 5 = 7$ appears as follows: $3x - 5 = 7$; then $3x = 12$; and finally $x = 4$. If, however, the operations are for the moment merely indicated, one obtains $x = \frac{7+5}{3}$, a form which not only shows explicitly that the root is a function of the original coefficients but also makes the actual relationship apparent.

The solution of equations which have literal coefficients of

course involves this notion in more general form. Thus the process of solving $ax + b = c$ may be regarded as finding the particular combination of a, b, and c which may be used as a formula for finding x. This idea is usually first approached in connection with the quadratic formula, and even then students are not always conscious that the solution is a function of the coefficients. This important notion should be emphasized from the time when equations are first solved, and the processes used should be regarded as operations which yield the particular combination (or combinations) of the coefficients which satisfies the equation. The teacher of more advanced classes should help students apply this notion in connection with the solution of systems of linear equations. Solutions of such equations are often formally obtained by methods which use only the detached coefficients, as when they are expressed by means of determinants. The solution of rational integral equations in one unknown by synthetic division, though of doubtful appropriateness in general education, is another illustration. Understanding of these methods not only hinges upon the realization that the solution is a function of the coefficients; it depends also upon a clear recognition of the fact that such methods are merely convenient schemes for systematizing sequences of formal operations.

Concepts Basic to Comparisons: Difference, Ratio, Proportionality

Among the many types of applications of the fundamental operations in the solution of problems few are more frequently demanded or more important than those involved in making comparisons. Two basic methods are available for comparing numbers—the difference method and the ratio method. Though difference and ratio are quite elementary notions, secondary-school students often fail to understand them fully or to appreciate the relative usefulness of each in attacking problems of different sorts. When taught in connection with real problems, they take on meaning rarely at-

tained when they are taught in a special unit devoted to them alone.

Suppose, for example, that the class knew that in a given region in one month there were 204 cases of pneumonia and 112 cases of scarlet fever, and wished to compare these two numbers. It is possible to say that there were 92 more cases of pneumonia than of scarlet fever. It is also possible to say that there were nearly twice as many cases of pneumonia as of scarlet fever. Which of these means of comparison is to be preferred? The answer to this question depends in part upon the nature of the data and the purpose of making the comparisons, but at least two considerations are usually involved in making the decision.

First, if the difference between the numbers is large in comparison to one of them, the ratio is generally used. In the absence of additional information on the purpose of the inquiry, this suggests that to say, "There were about twice as many cases of pneumonia as of scarlet fever" is more revealing than to say, "There were 92 more cases of pneumonia." In comparing the weight of father and son who weigh 180 and 20 pounds respectively, the tendency is to say the father weighs 9 times as much as the son rather than to say he weighs 160 pounds more. In general, it seems to be true that scientists prefer to use the ratio of the two numbers, although there are certain types of problems in which they use differences.

Second, the difference method expresses the comparison in terms of the unit, but the ratio is independent of units. Thus in the example of comparative weights of father and son the comparison by the difference method is expressed as "160 pounds more," but the ratio method results in the phrase "9 times as much" with no reference to the unit. The ratio remains the same as long as father and son are both weighed in the same unit, whether this is ounces, pounds, grams, or kilograms. The ratio method is, therefore, frequently preferred when a number of comparisons are to be made and some of the pairs of numbers are expressed in different units.

Ratio, is of course, indicated in computing rates. Thus, to find the mortality rate in a given city it is necessary to find the ratio of deaths during a stated period to the population in the same period.

Starting with these basic methods of comparison of two numbers, the fundamental conceptions involved may be extended to cover the comparison of ratios and differences themselves. If two ratios are equal, one has an example of proportion. For instance, consider the following table showing the weights of two boys:

	Jan. 1938	*Jan.* 1939	*Gain*
John	140	147	7
Bill	120	126	6

To compare their gains one procedure would be to find the differences, 7 and 6, between their initial and final weights. According to this method, John gained one pound more than Bill; but since their initial weights were different, for some purposes this difference between differences does not provide a fair comparison. Using the ratio method, one may compare their gains to their initial weights as follows: $\frac{7}{140}$ and $\frac{6}{120}$. Since these ratios are equal, the growth of the boys was proportional. Or one may convert the ratios to per cent, and observe that since each gained 5 per cent, their gain was, from one point of view, equal. When two ratios are not equal they may be compared by finding either their difference or their ratio.

Differences and ratios are also used in the comparison of averages. Two groups may be compared by finding how much larger the mean of one is than the mean of the other. In some problems one must use statistical tests in order to say whether or not the difference between the means of two groups is "significant." These tests are based upon ratios—for example, the ratio of the difference of the means to the probable error of

the difference. Although the use of such tests is doubtless beyond the abilities of most secondary-school students, the other ideas about the use of differences and ratios discussed above are definitely within their powers of comprehension.

Concepts Basic to Other Operations

It is obviously impossible to discuss here the concepts basic to all of the operations and techniques used throughout mathematics, but one or two illustrations may indicate the scope of the topic. The construction problems of geometry, for example, make use of processes—the drawing of circles and straight lines—which may be considered operations. Many students fail to comprehend why constructions are made as they are in theoretical geometry because they do not understand the assumptions and conventions that dictate the methods and limit the scope of constructions restricted to the use of straight-edge and compasses. When they are asked to make a Euclidean construction of a tangent to a circle from a point outside, for example, they frequently attempt to adjust the straight-edge so that when the line is drawn through the point outside the circle it appears to the eye to touch the circle also; they cannot see why such visual adjustment is excluded from theoretical Euclidean construction, or why they are called upon to perform a number of elaborate—and to them seemingly unnecessary—preliminary steps. If the standard procedure in this construction is to be intelligible to them they must not only have the concept of tangent clearly in mind, but must also understand the classical tools (e. g., the straight-edge is ungraduated and no marks may be placed upon it) and the conventions governing their use. Otherwise the steps of the construction are relatively meaningless to them.

The distinction between techniques for performing operations and the concepts basic to them is important at all levels. Even the student who has advanced to the study of calculus may become expert in the techniques of differentiation without a clear notion of the limit concept which is basic to the

operation. Only by careful attention to the development of the underlying concepts can teachers make operations really meaningful to students.

HELPING STUDENTS DEVELOP THE NECESSARY SKILLS AND TECHNIQUES

Detailed discussion of methods and principles relating to drill on mathematical techniques lies beyond the scope of this Report. The Committee recognizes, however, an obligation to state its point of view concerning drill and correctness in computation.

A Point of View on Drill

In the opinion of the Committee, techniques of operation are subsidiary to more basic considerations.[4] The present Report emphasizes the study of problem situations in secondary-school mathematics, and consistency makes it necessary to hold that only those operations and techniques should be taught that are necessary or at least very convenient in the solution of real problems.

It must be recognized, on the one hand, that the great range and variety of real problems that may be encountered in advanced mathematics and its applications in science and engineering demand for their solution not only all of the operations now taught in secondary-school mathematics but many others as well. On the other hand, traditional secondary-school

[4] The Report of the National Committee, *op. cit.*, p. 11, gave an admirable statement concerning the emphasis upon techniques in algebra: "The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. . . . *Drill in . . . manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived as a means to an end, not an end in itself. Within these limits, skill in . . . manipulation is important, and drill . . . should be extended far enough to enable students to carry out the essential processes accurately and expeditiously." This statement seems to have been influential in bringing about a marked reduction in emphasis upon complicated techniques of operation in recent years.

training in mathematical techniques has been far more extensive than is justified by the requirements of problems students are likely to encounter as they meet their needs. For example, students have in the past learned to multiply polynomials of far greater complexity than they were likely to encounter in any problems real to them. This is not to say that certain problems real to others do not sometimes require the multiplication of polynomials of even greater complexity, but only that from the point of view of this Report practice in any given technique should be postponed until it is required in the study of some real problem. Though many students may thus never learn some of these techniques, their time may far more profitably be spent in gaining rich mathematical experiences of other kinds.

Even when the individual is affected by the solution of certain problems, he may yet have little need for knowledge of the technical operations which led to the solution. A new mechanical process, an improved technological application of known chemical principles, a harnessing of fresh sources of energy, these are but examples of solutions to problems that may intimately affect the comfort, health, or livelihood of part or all of the nation's population. The details in most cases are of more or less popular interest, but lie far beyond the scope of direct action for most people. Even within mathematics it may be well for students to learn that certain things have been done which are of interest and are worth understanding but these do not call for repetition by every student. The computation of π and the building of trigonometric or logarithmic tables are instances. That there exist arithmetical methods for extracting cube roots may be accepted without training in carrying out such computations. That there exist (in the application of continued fractions) direct methods for finding excellent rational approximations to irrational numbers, such as $2\frac{2}{7}$ for π, may be stated and admitted without asking the student to find such approximations as evidence of understanding the subject.

On the other hand, carrying out several cases of a simple operation is likely to insure a vivid and enlightened understanding seldom attainable by merely listening or looking on, and clearly some drill is needed as each new technique is encountered. But it does not follow that drill need be extended to the point where the skill is considered to be both adequate for all future needs and permanently established. When, as frequently happens, a real problem requires the repetition of the same technique with a number of different items, drill is inherent to the process of solution. If an operation is recurrently needed in the solution of a succession of real problems, spaced drill is almost automatically provided. On the other hand, if an operation is never needed again or is demanded only infrequently, it is clearly a questionable procedure to drill on it for the purpose of developing skill or for the sake of a highly problematical retention of the ability. Some operations recur so frequently that they become almost automatic, but this semi-automatic character should be a natural outcome of use, and is neither a virtue to be directly sought on its own account nor a vice to be avoided.

A Point of View on Correctness in Computation [5]

The student of mathematics must learn that adequate concepts with inadequate technique, correct methods with careless computation, do not lead to trustworthy results. The worse than useless character of results punctuated by careless mistakes should be made vivid to him: if computation is to have any value, the answer must be correct. In the judgment of the Committee the student should develop a conscious and dignified reliance upon the correctness of his computations; he should learn where to look for trouble and how to trace and correct mistakes that are likely to occur.

Many teachers have not approved of teaching methods for checking and have even restrained the student's desire to

[5] For an excellent discussion of some broader aspects of workmanship, see *Science in General Education*, pp. 128–135.

check his own results. One reason for this is the vain effort to develop proficiency in techniques of operation to the point at which they are so automatic as to preclude error. But though experience with numerous similar problems and sustained drill may engender in the student a feeling of confidence in his powers of computation and a false assurance that whatever result he obtains is unquestionably correct, further experience should make clear that all such automatic responses are fallible and that there is no substitute for care and good judgment in checking results.

A second factor that accounts for some teachers' negative attitude toward checking results is the demand for speed. A student who does not pause to review his work has more time for the next exercise. But this whole emphasis seems to have little justification. Speed comes naturally with practice, but clear, well-directed thinking takes precedence over rapid automatic response as an aim of mathematical instruction.

In the third place, students today are often allowed to develop an almost contemptuous attitude toward correctness of work as one ill consequence of a generally healthy reaction from the tradition of excessive drill. It is certainly well for students to become acquainted with mechanical computing devices and to realize that few persons can compete in speed and accuracy with these sturdy products of man's mechanical inventiveness. It is also true that work in the mathematics classroom should be largely for the teaching of concepts and methods, and that the particular situation being studied is to be accepted as primarily a tool for training in solving more significant or more complex problems, although supposedly not without value on its own account. Still, careless workmanship is never to be condoned, and all results obtained should be tested for apparent reasonableness.

In work involving relatively few steps the student should be able to glance over his computations and by carrying through mentally the steps already indicated catch most mistakes due to carelessness, whether in computing, in copying figures, or

in misreading his own writing. For example, few mistakes seem so gross to an experienced computer as missing even the approximate order of magnitude of the answer. The widespread notion that the placing of a decimal point, which takes so little technique, absorbs so little time, and makes so little difference in the general appearance of a written answer, is after all trivial should be thoroughly condemned. In longer operations systematic methods of checking should be known and intelligently applied. The less traditional the problem, the greater the need for checking.

For this reason the student should be familiar with routine checking devices. Short-cuts, visual checks, scale drawings, slide-rule estimates, substitution of purported answers, application to numerical data, examination of limiting cases, innumerable methods for testing and retesting each stage of his work should be part of his habitual practice. Growth of confidence in his work should not rest upon his feeling that a familiar piece of reckoning has been performed in a traditional way, but in his knowledge (insofar as the situation presents familiar aspects) that the results agree at each stage with what trained common sense, aided by systematic checking, declares to be reasonable.[6]

BIBLIOGRAPHY

A General Survey of Progress in the Last Twenty-five Years, First Yearbook of the National Council of Teachers of Mathematics (New York, Published by the Council, 1926). See particularly: "Suggestions for the Solution of an Important Problem that Has Arisen in the Last Quarter of a Century," Raleigh Schorling, pp. 84–99.

BRESLICH, Ernst R., *Problems in Teaching Secondary-School Mathematics* (Chicago, University of Chicago Press, 1931).

[6] The importance of checking successive stages of work has a bearing upon planning and constructing tests and examinations. Time should be allowed not only to finish but to check. Furthermore, it is unfair to expect a student under the pressure of an important examination to be led astray through unreasonable data assigned by an examiner. A student who rejected as implausible an answer of 120 miles per hour as the speed of a yacht seems hardly deserving of a poor mark from an examiner who set a problem that did "correctly" lead to such an answer.

DANTZIG, Tobias, *Number, The Language of Science* (New York, The Macmillan Co., 3rd edition, revised and enlarged, 1939).

EVERETT, John Phelps, *The Fundamental Skills of Algebra* (New York, Bureau of Publications, Teachers College, Columbia University, 1928).

FINE, Henry Burchard, *A College Algebra* (Boston, Ginn and Co., 1904), Chapter I, "The Natural Numbers—Counting, Addition, and Multiplication."

———, *The Number-System of Algebra* (Boston, D. C. Heath and Co., 1907).

HASSLER, J. O., and SMITH, Rolland R., *The Teaching of Secondary Mathematics* (New York, The Macmillan Co., 1930).

HELMHOLTZ, H. Von, *Counting and Measuring*, translated by Charlotte Lowe Bryan (New York, D. Van Nostrand Co., 1930).

IX

PROOF

Arriving at conclusions involves the characteristic process of rational thought, namely inference. Inference is a typical intellectual process, ranging from an almost immediate and impulsive realization, as when a child infers its mother's intentions, to highly analytic and subtle behavior, as when a detective solves a murder mystery or a philosopher develops a metaphysical formulation of the world order. Formal mathematics is largely concerned with drawing conclusions from premises within a specialized field, so that inference of one form or another underlies most of modern mathematical exposition, but inference describes also the work of the historian who reads original documents, of the jurist who interprets the law, of the investor who surveys market fluctuations, of the doctor who observes symptoms, of the physicist who pieces out the story of his pointer-readings.

The ultimate test of the validity of the solution of a problem lies in whether or not it works when put into practice. But frequently problems are of such a nature that the test of the solution in action must be deferred for some time, and people wish assurance that the solution will work when the time comes. Furthermore, very often people seek to use tested solutions of particular problems as rules or principles of further action; they desire assurance that a solution that holds in those cases in which they have actually tried it out holds also in all other similar circumstances.

The concept of proof in its broad sense refers to all the ways in which this assurance is gained or may be communicated to others. In its narrower sense it refers to methods of reasoning

such that any doubt as to the validity of a solution must refer to the assumptions upon which it is based, rather than to the logical necessity of the inferences drawn from them.

Few concepts have had greater influence on the thinking of man than that of proof, and few objectives of mathematical instruction on the secondary level are more frequently mentioned than those related to training in logical thinking. Since the time of the ancient Greeks there has been a persistent belief that the study of geometry affords better training in deductive reasoning and leads to better understanding of proof than does the study of any other field or subject now taught in secondary schools. Today millions of boys and girls study geometry according to a logical pattern which is essentially that of Euclid, and many teachers of mathematics still believe that demonstrative geometry (as distinguished from "informal" or "intuitive" geometry) is a good means of developing the ability to think logically. They claim that the rigor of the proof in this field sets a standard which careful demonstrations in other fields of thought may well attempt to emulate, and that students should therefore learn geometry in order to learn to reason with equal rigor in other fields.

Fundamentally the end sought is for the student to acquire both a thorough understanding of certain aspects of logical proof and such related attitudes and abilities as will encourage him to apply this understanding in a variety of life situations. There can be little disagreement concerning the value of this goal, but it must be freely admitted that the claims made for the contribution of geometry and of mathematics in general, as taught in the past, to the development of the ability to reason in any field have frequently been exaggerated. There has been meager objective evidence of material improvement in students' ability to think deductively as a result of exposure to traditional courses in formal geometry. This may be due in part to the fact that the geometry classroom has all too often failed to afford actual opportunity for independent thinking; teaching methods have permitted and even encouraged stu-

dents' reliance upon memory, rather than upon deductive reasoning in making demonstrations.

Perhaps the chief reason why so slight a measure of success has sometimes attended instruction in geometry is not to be found in the inherent difficulty of the subject but rather in vagueness of aim in teaching and lack of prior sensitization to what is meant by and involved in proof. In marked contrast to the circumstances under which geometry was studied in Alexandria and by scholastic philosophers throughout the Middle Ages, the boys and girls of today usually approach formal demonstrative geometry with no background of training in the theory of logic. The student is usually ignorant of the technique of sound argument, and to him the whole problem of deductive proof is frequently meaningless. Unfortunately most of arithmetic is taught, explained, and practised much as shopwork technique might be; the student is convinced of its correctness by analogy, illustration, unfailing success, approval of the teacher, or in other ways, but not by inference from prior principles. Algebra is usually taught as though it were a practical technique devoid of logical basis and of rational interconnections.[1] And a great deal of mathematical analysis is learned in this same way, without sufficient attention to the nature of the reasoning involved. Not only application of principles but also an introduction to these principles is required.

Furthermore, some of the assumptions concerning the nature of learning that are implicit in the argument for the study of geometry as a means of improving the ability to think deductively in other fields have been seriously questioned during the last generation. Students of the learning process are highly dubious as to whether the repetition of deductive geometric proofs of itself automatically builds insight into the nature of proof or a generalized ability to think deductively

[1] This failure to realize the inherent logic of algebra is apparently due in some measure to historical circumstance. The mathematicians who developed algebra a few centuries ago were content with far lower logical standards than those maintained by Euclid in ancient Alexandria.

which will transfer outside the confines of geometry itself.

It is here proposed that the teaching of mathematics can help secondary-school students achieve these ends, provided it is consciously and intelligently directed toward them, and provided certain essential conditions are observed. In the first place, if instead of merely studying proofs presented to them in geometry, students were more often led to engage in deductive reasoning of their own in geometry, they would at least gain first-hand experience in a process which one would hope to have become habitual with them. In the second place, opportunity for experience in deductive reasoning must be broadened beyond the confines of geometry to include the other aspects of mathematics in which it is appropriate. When the student has broad first-hand experience in deductive reasoning the likelihood of the transfer of this ability to other fields is increased.[2] But it is assured to far greater degree when a third condition is observed, namely, when the student is led to focus his attention on the nature of proof.

DEVELOPING STUDENTS' UNDERSTANDING OF THE NATURE OF PROOF IN THE BROAD SENSE

This chapter focuses in the main upon deductive proof in the relatively formal logical sense. Yet the student must not be encouraged to develop a hypercritical attitude which will cause him to resist all non-deductive processes.

Proof, in the broad sense, consists of any means of gaining assurance that a conclusion is valid or any orderly exposition of communicable bases for belief. We all have numerous beliefs of one sort or another, faith in cherished concepts that we regard as true. Many of these convictions we are not called upon to substantiate, but we are inclined to defend them with vigor. Proof is usually in order only when convictions require

[2] See Gilbert Ulmer, "Teaching Geometry to Cultivate Reflective Thinking: An Experimental Study With 1239 High School Pupils," *Journal of Experimental Education,* Vol. 8, September, 1939, pp. 18–25.

protection and support either because of our own distrust or for the sake of convincing others. It is of course obvious that grounds which may be adequate for conviction at one level of discussion may appear either superficial and fragmentary under more searching inquiry or labored and truistic among persons whose tastes and experiences largely coincide.

Cases in Which Proof of Any Kind Is Unnecessary, Impossible, or Premature

In order to appreciate the accepted scope for proof, it is desirable to glance at some areas in which conviction properly remains undefended. When it comes to communicating emotional states, for example, there is no place for proof. It is unnecessary to prove that we have such and such tastes and impossible to prove that we have such and such sensory experiences, although our statements in matters of this kind are not wholly undebatable and may show lurking inconsistencies. There are no communicable bases for belief, and consequently no proof is fitting or possible in regard to claims of immediate revelation in religious matters, experiences supposedly transcending all rational processes. The same obtains in respect to beliefs based on convention, authority, or precedent, for the acceptance of beliefs on these bases is the antithesis of the search for proof.

In some instances, the demand for proof may be relevant but premature. "Hunches"—shrewd, semi-inspired guesses, endowed with an element of conscious conviction of an intimately personal sort—are a case in point. While one might argue about a "hunch" and try to make it seem reasonable, the force of the conviction is not readily justifiable to an unsympathetic auditor. Then again there are tentative hypotheses [3] offered in a spirit of cautious inquiry—also a kind of

[3] By *hypothesis* is here meant a general principle, derived inductively, observed to hold approximately throughout the sample tested and proposed for consideration as perhaps holding true universally for cases idealized in specified respects. In this sense one may speak of the "nebular hypothesis," the "hypothesis of organic evolution," the "quantum hypothesis."

"hunch"—for which the demand for any kind of immediate proof would not be pertinent. These hypotheses furnish the structure upon which proofs may be molded, but may rest on no prior groundwork of evidence as to their validity.

Inductive Proof

In those cases where it is appropriate and possible to question bases of belief, there are several familiar and reasonable methods for establishing conviction. In stressing proof in its narrower sense, the teacher should guard against leaving the impression that pure deduction is the only justifiable form of arriving at beliefs. In natural science, for example, it is immediately apparent that, important as is the rôle of strict deduction, there yet remain other ways of establishing conviction, ways which the student cannot afford to despise or ignore. Induction plays a fundamental (although not all-inclusive) rôle in establishing convictions in general; the solution of most problems involves induction and deduction [4] alternately.

Inductive reasoning is the process by which generalizations or conclusions are derived from observation of a series of particular instances. Most inductive generalizations assume that the sample examined provides reasonably valid evidence as to the more extensive class from which it may have been selected at random. The probable truth of a conclusion reached by induction increases with the number of diverse instances for which the conclusion is verified—assuming that no case of failure has been encountered. The scientist makes generalizations based on data, but hesitates to accept these generalizations as valid until they have been subjected to elaborate testing. In this connection convergence of evidence, the method of agreement, the method of difference, and the method of concomitant variation are of crucial importance.[5]

[4] The rôle of deduction in scientific generalization is discussed on page 203 below.

[5] Cf. Morris R. Cohen and Ernest Nagel, *An Introduction to Logic and Scientific Method* (New York, Harcourt, Brace and Co., 1934), Ch. XIII, "The Methods of Experimental Inquiry."

Some scientific generalizations are proposed as universally valid "laws" that admit of no exception. Theoretically one recalcitrant experiment—one instance of failure—suffices to shatter the entire edifice of such "laws." No amount of *agreement* is decisive. In practice an instance apparently at variance to an announced universal principle is accepted as a challenge to further research and a hint as to the eventual limitations of the range of variation within which the law operates. The laws themselves are usually accepted as only first approximations, not to be rejected because of slight discrepancies in extreme circumstances. In fact as the frontiers of science are pushed outward peripheral inconsistencies continue to present themselves, but ordinarily any particular difficulty is eventually mastered.

Other scientific generalizations are proposed as only statistically valid, indicative of trends and tendencies but not applicable for prediction in individual cases. These are not to be confused with universal laws. Even for the student of mathematics, notions of probability and of statistical trends obtrude to call attention to the fact that principles of action may in many cases have no more secure a basis than data that suggest correlation. From observed and computed correlations, the step to the interpretation in terms of causes and of principles is one of the most difficult and hazardous in the use of the inductive method. But although understanding of the nature of statistical generalizations serves as a check upon extravagant claims for the absolute certainty to be achieved by the methods of natural science, it also serves to make clear that inductive methods play an important rôle in establishing conviction. The remark is often made that deductive logic gives certainty whereas inductive methods provide only probability. The suggestion that rational analysis discovers truth while modern science yields only guesses is probably at best misleading. There are those with strong pragmatic leanings who claim complete agnosticism with regard to any kind of truth save that to which one arrives through the method usually called

scientific, which employs hypotheses and experiments, and continually tests, adjusts, infers, predicts, generalizes, and applies.

The cultivation of the ability to generalize and the study of the values and dangers of induction have by no means received the attention that they deserve in connection with secondary-school mathematics. In an enthusiasm over the deductive aspects of mathematics, the teacher may easily be led to ignore its inductive features, and to regard the subject as "the science in which one draws necessary conclusions," treating mathematics as a branch of (or field of application for) logic, if not identifying the two lines of study. A glance at the history of mathematics, or an inquiry into the thinking processes of the mathematical discoverers, should suffice to bring out the fact that deduction is but one of the many aspects of mathematics.[6] Even if all deductive proof were suppressed, there would still remain in mathematics definitions, techniques, propositions, relations, figures, problems, postulates, and so forth; mathematics would then be on a par with much of natural science.

The whole of Part II up to this point has suggested ways in which mathematics can be used in the solution of important life problems by means primarily inductive in nature. Here it is appropriate to point out only that the experimental methods of the laboratory may be invoked to study even the more abstract (although elementary) aspects of mathematics itself. In particular, many of the elementary relations of geometry may be inferred from data obtained by measurement of a set of figures. Thus observation of two parallel lines cut by a transversal may lead the student to the hypothesis that the alternate-interior angles found are equal. If a number of figures are drawn in which the distance between the parallels and the inclination of the transversal are allowed to vary, a series of measurements of the alternate-interior angles will

[6] Even in the current century the remarkable Hindu mathematical genius, Ramanujan, failed to appreciate the value of rigorous demonstration. He was interested in discovering numerical, not logical, relations.

yield a set of data. In treating these data many of the concepts discussed in the preceding chapters of Part II are applicable, but in addition the principles of induction involved should be emphasized. Growth in understanding of induction and in the related abilities depends upon focusing attention upon the process, instead of regarding it as incidental to the learning of subject-matter. This understanding should result in an increase of such behaviors as the following: When the student is confronted with a problem new to him and requiring some generalization as a solution, he attempts to construct several examples of the phenomenon being studied and to observe and identify those characteristics that are common to them; he recognizes that solutions (generalizations) arrived at in this way are tentative and seeks for possible exceptions; in some cases, he attempts to establish his generalizations rigorously by deductive methods.

DEVELOPING STUDENTS' UNDERSTANDING OF THE NATURE OE PROOF IN THE NARROWER SENSE

As noted above, through much of the study of mathematics there is no felt need for deductive technique. One learns certain rules for procedure, acquires a mathematical vocabulary, solves problems, tests conjectured principles experimentally, discovers seemingly general relations, and by practice achieves a settled sense of conviction as to many mathematical factors in experience. This is the stage in which all mathematics was in some periods and in some civilizations, and in which much of elementary mathematical study is today. To understand the meaning of proof in the narrower sense the student must advance beyond this stage.

The If-Then Principle

Basic to proof in its narrower sense is the principle of formal implication, often known as the "if-then" principle because it

applies to situations in which *if* certain assumptions (premises, propositions) are granted, *then* the conclusions (inferences, deductions) that necessarily follow must also be granted.

Now if the assumptions change, the conclusions usually likewise change, and in many cases deductive conclusions depend upon assumptions which may not even be explicitly stated. Almost any group decision which is not unanimous is illustrative. Many of the Supreme Court decisions are socially significant examples of how clear thinkers arrive at opposing conclusions because of different assumptions or different interpretations of the situations. The different members of the Court may seem to start their arguments from assumptions explicitly the same, but because of different tacit assumptions, often social rather than legal, their conclusions disagree.

Hence the student should develop the ability to recognize both stated and unstated assumptions affecting reasoned conclusions, and should learn to state his own assumptions explicitly. This entails increasing his sensitivity in recognizing cases in which emotional bias hampers recognition of assumptions on his own part or the part of others. Moreover, it entails increasing his awareness of the fact that if the meaning of the words or symbols used in the statement of the assumptions changes, the assumptions themselves change, and consequently may lead to different conclusions.[7]

Again, students should learn that if the meaning of the words in which the assumptions are expressed is different for each of two persons, the assumptions that each holds will be different from those held by the other, and each will draw different conclusions from what seems to be the "same" set of assumptions. For example, if members of "high-school fraternities" are ineligible for athletic competition, the conclusion that no club member may represent the school in athletic competition follows when "high-school fraternity" is defined to embrace all high-school clubs. But not so if high-school

[7] See Chapter X, "Symbolism," and also Chapter III, "Formulation and Solution," pp. 82–83.

fraternities are regarded as special cases of school clubs nor again if the term "school club" does not apply to fraternities. Unless students are sensitized to the rôle of definitions in deduction, they are usually unable to select the key words which need to be clearly defined if different people are to draw the same necessary conclusions from them. Finally, students should be helped to understand that in any deductive system there are so-called "primitive terms" (e. g., *between, point, line, plane*) which it is fruitless to attempt to clarify by definition. This is not to say that such terms are intrinsically undefinable, but only that the effort to define introduces other terms which in turn cause difficulty, so that the situation of having some undefined terms must always remain.

As students grow in effective understanding of the "if-then" principle they may be expected to scrutinize assumptions and definitions as the source of inconsistency if two or more arguments lead logically to incompatible conclusions, to reject at least one of two arguments which are based upon identical assumptions and definitions but which lead to inconsistent conclusions, and to accept tentatively or for the sake of argument certain assumptions which are contrary either to fact or to their personal beliefs in order to trace their implications and to "see where they lead." This last involves recognition of the fact that an argument may be logical even though the conclusion is contrary either to evidence or to what one believes or would like to believe.

The operation of the "if-then" principle is of course illustrated by any instance of mathematical proof. What is needed, from the point of view of this Report, is decidedly more emphasis on the principle itself and relatively less on applying it blindly in specific formalized situations. It is the responsibility of the teacher to encourage students to focus attention on important logical principles, and eventually to assist them in making these principles explicit. These ends may be achieved by giving students some opportunity to help formulate their own assumptions and definitions in selected parts of

geometry and to draw their own conclusions from them, learning from this how the choice of assumptions and definitions affects the logical argument and the conclusions. This process should properly be extended to work in other fields, both mathematical and non-mathematical, and may even take its departure from materials definitely non-mathematical.[8]

Instead of presenting each original exercise or theorem in geometry as a unique entity, and with both the conditions on the figure and the conclusion given, teachers should offer students opportunity to discover for themselves implied relations in geometric figures. Suppose, for example, their attention is focused upon the figure formed by two intersecting pairs of parallel lines—a "parallelogram"—and they are asked to discover, by measurement or otherwise, all of the relations which appear to be true in the figure without having been told anything about it. When they are encouraged to draw as many additional lines, such as diagonals, as they find helpful, different students usually present different lists of propositions, but the composite list of the class includes most of the propositions usually found in an entire unit on parallelograms. The distinctive feature of such an approach is that instead of studying a logical pattern presented to them by the teacher or textbook, students are actively and creatively engaged in drawing their own deductions from their own assumptions.

Now the acceptance of certain of the listed relations implies the acceptance of others. If, for example, it is assumed that opposite segments are equal, it is possible to deduce that they are necessarily parallel. If, on the other hand, it is assumed that the opposite segments are parallel, it is possible to deduce

[8] Harold P. Fawcett, a member of the Committee, has carried on extensive experimentation in helping students to understand logical proof and to apply their understanding in a variety of fields. The manuscript of his book *A Description and Evaluation of Certain Procedures Used in a Senior High School to Develop an Understanding of the Nature of Proof,* was available when the first draft of this chapter was written, and so influenced profoundly the discussion presented here. The book later appeared as the *Thirteenth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1938).

that they are necessarily equal. In either case, of course, other properties of the figure can also be deduced. Any one set of properties may be used to define the figure. Traditional choice focuses upon the parallelism of the sides, following which the other properties appear as theorems. There is no fundamental reason, however, why some other set of properties of this same figure should not be used for definition. If freedom of choice is provided and individuality of approach is encouraged, the discussion of the different developments that ensue illustrates the fact that if certain assumptions and definitions are made, certain conclusions necessarily follow. It also illustrates the fact that the details of the logical structure change as one or another set of properties is chosen as a point of departure, and that the choice of assumptions and terms to be defined is to a certain extent arbitrary and a matter of taste or custom.

In the teaching situation outlined above some students may assume that adjacent segments (as well as the opposite segments) are equal, and others may assume that they are not equal. The former may deduce some properties not deduced by the latter—for example, that the diagonals are perpendicular. This property follows from the assumptions held by the first group, but it does not follow from the assumptions held by the second group. Such situations can be used to illustrate how different assumptions lead to different conclusions.

Situations of this kind may also be used to call students' attention to the distinction between necessary and sufficient conditions. On examining what has happened, they will discover that the first group has made such assumptions that their conclusions apply to the figure ordinarily known as a rhombus, and that the assumptions of the second group apply to all figures known as a parallelogram, of which the rhombus is a special case. All of the conclusions that apply to a parallelogram necessarily apply also to a rhombus, but they do not suffice to distinguish a rhombus from all other parallelograms. As ordinarily defined, to know that a figure is a parallelogram it is sufficient but not necessary to know that it is a rhombus.

On the other hand, to know that a figure is a rhombus, it is necessary but not sufficient to know that it is a parallelogram. In any valid "if-then" deduction the conclusions are drawn from sufficient conditions. Valid "if not—then not" deductions follow from necessary conditions. In the solution of mathematical problems it is considered desirable to have a minimum set of conditions, both necessary and sufficient, to prove the case.

One reason for introducing students to the notion of necessary and sufficient conditions is that it provides them with another point of view from which to examine the relation between assumptions and the conclusions drawn from them through the application of the "if-then" principle. If a proposition in simple "if-then" form and its converse both hold, the assumptions in the statement of the proposition form a set of necessary and sufficient conditions. This is one reason why students should learn to formulate converses of propositions and prove them if possible. The fact that a converse of a given demonstrable proposition may not hold is merely another way of stating that the conditions stated in the "if" clause of the given proposition are not both necessary and sufficient.

Another illustration of the way in which the teaching of geometry may be modified in order to place more emphasis than usual on logical principles is afforded by study of the restrictions involved in the precise use of specified principles of geometrical construction. The nature of these restrictions may be readily illustrated by discarding the compasses and using a straight-edge on which it is permissible to place marks for the transferring of segments.[9] The resulting changes in the elementary construction exercises are instructive and exemplify the "if-then" principle at work. Other changes in the conventions of construction may be used to similar advantage; for example, use of the parallel ruler may be permitted. The

[9] David Hilbert, *The Foundations of Geometry*, translated by E. J. Townsend (Chicago, The Open Court Publishing Co., 1902); George Bruce Halstead, *Rational Geometry;* a Textbook for the Science of Space (New York, J. Wiley and Sons, 2nd edition, 1907).

fact that "an arbitrary angle cannot be trisected by straightedge and compasses" depends upon the definition of these tools and certain conventions or assumptions concerning their use. If these assumptions or conventions are appropriately changed it becomes possible to trisect an angle and to solve other construction problems that are impossible under the traditional assumptions.

Many other modifications of usual practice in the teaching of geometry could be suggested, but these should be enough to indicate possible types of changes in method and content that emphasize the logical principles underlying the subject. The analysis of common figures by methods of the types suggested eventually leads to clarification of most of the concepts found in the usual modern introduction to demonstrative geometry, but this completeness is less important than the first-hand experience with logical principles it makes possible.

But study of the "if-then" principle need not be confined to geometry. Logical deduction also plays a rôle in algebraic theory, although, as indicated above, explicit attention to the deductive aspects is often neglected. Deductive principles come to the fore in the proofs of the laws of exponents and the choosing of definitions for expressions involving negative, zero, and fractional exponents, for example. Such topics offer opportunity to study the rôle of definitions and assumptions with a logical rigor usually reserved exclusively for geometry.

Quantitative problems also illustrate the "if-then" principle. It should be evident that the process of formulating problems in quantitative terms involves assumptions, definitions, and the drawing of conclusions. If certain assumptions are made, and if certain data are at hand, then a certain solution may be obtained. But different assumptions may lead to other solutions, and the different solutions may be quite inconsistent with one another. The prevalent tendency to provide explicitly all the data needed to solve problems, and no more, has obscured the fact that the selection of the data presented involved numerous assumptions that may markedly affect the

validity of the solution from a practical point of view. Discussion of these matters with the students and their recognition of unrevealed assumptions is fully as important, from an educational point of view, as obtaining the numerical results called for.

Nor should the mathematics teacher hesitate to draw upon non-mathematical material to clarify these principles. In fact, as previously suggested, he may on occasion start with the analysis of logical principles inherent in the study of non-mathematical problems, introducing mathematical situations later. Mathematical subject-matter is relatively free from the personal bias that in other domains often tends to make difficult the application of logical principles, and so may be used to clarify them and to exhibit their operation under ideal conditions. Although, as always, the approach should be made through relatively simple situations which are of immediate interest to the students, and although the analysis should focus upon all of the logical aspects involved, it may be desirable on occasion to single out some particular aspect, such as the definition of terms, for separate attention. An unsolved local problem may be suggested—perhaps one of school or civic interest about which there has been some controversy and where terms have been used ambiguously. Advertisments, editorials, political speeches, and similar articles are fruitful sources of material for the work of this kind on definitions, and may be used to set the stage for the specific study of mathematical concepts. Students may be asked to suggest some mathematical terms with which they are familiar—*triangle, denominator, decimal, square, exponent, circle,* and the like. At this point the emphasis may be upon making notions more precise, and upon recognizing that the choice of the particular name assigned to the concept is a relatively arbitrary matter. *Triangle* might as well be *trilateral,* for example, and *parallelogram* might as well be *equalogram.*

Questions as to whether logical principles should be made explicit first in connection with mathematical work, and then

later applied in the analysis of non-mathematical situations, or whether the opposite order should be followed, cannot be answered without further experimentation, and without reference to specific circumstances. Perhaps the practice of beginning with familiar non-mathematical situations, making a transition to mathematical examples as a means of clarification, and then applying to additional non-mathematical problems may in most instances be superior to either of the above alternatives. In any case it is desirable that whenever possible the students themselves decide the terms to be defined and the assumptions to be made. Their efforts to formulate definitions or assumptions should be a creative activity rather than a mere memorization of definitions and assumptions presented to them by the teacher or in the textbook. The teacher may ask questions and guide the discussion, but should avoid giving the impression that a unique set of definitions and assumptions is being sought, or that one form of statement is superior to all others. Finally, it is unnecessary to attempt to reduce the number of undefined terms and assumptions to a minimum. All terms which are selected by the students as clear and unambiguous may remain undefined, although in some cases a later need for refinement may initiate the formulation of a definition. Similarly, propositions which seem obvious to the student may be accepted without proof (as postulates), although later they may be fitted into the logical structure as theorems. On the other hand, as the work proceeds the list of postulates may be augmented as hitherto tacit assumptions are discovered by the students. In brief, the teacher must be careful not to force students too rapidly into logical considerations for which they are not ready.

The Interplay of Deduction and Induction

As indicated earlier in this chapter, induction and deduction play alternate rôles in the process of arriving at scientific generalizations. The rôle of deduction should now be clear. In solving many problems the investigator reasons that if a

given hypothesis is valid then a certain result or conclusion should be obtained. The investigator gathers data to check his various assumptions, and if the assumptions are thus verified, he not infrequently attempts to construct a rigorous deductive argument for the conclusions. Often with ingenuity he finds a proof of some kind. By careful examination of this first proof he sometimes discovers that it is unnecessarily circuitous or that certain of the initial assumptions are not really used. This calls for a restatement of the problem with a sloughing off of all unessential assumptions, and this in turn has the effect of generalizing the conclusion. Thus finally he achieves a far more direct and economical sequence of steps, and at the same time a more general or sweeping result.

This description applies not only to the process of arriving at generalizations of a statistical or approximate sort, but also to the establishment of universal "laws" and mathematical theorems. In the latter case deduction plays a further rôle: it establishes that if the stated assumptions are satisfied, no exceptions to the announced conclusion are possible.

Logical Deduction, Truth, and Fact

Formal logic is not primarily concerned with discovering truth, or with problems of fact and truth. It is primarily concerned with the formal validity of any given inference on the basis of prior assumptions, rather than with the internal truth of any particular statement. Although the important applications of logic are concerned with true statements, that is, with statements of facts, or at least with statements purportedly true, consideration of the way in which a true statement represents facts while a false statement does not, lies beyond the bounds of formal logic. It is therefore unnecessary here to attempt to specialize some one meaning of the word *truth* and to adopt it, rejecting all other interpretations.

According to widely accepted views, facts subsist independent of proof—they are not appropriately classified as either true or false. What is called true is but a proposition;

the true proposition supposedly corresponds in some sense to fact, and replaces the fact for purposes of thought.

What, then, are the aims of deductive proof? If proof in the narrower sense is not for the avowed purpose of establishing truth, what end does it serve? Formal logic contributes in substantiating belief in two ways. First, it sometimes helps in determining whether a given statement does in fact follow from an initial set of supposedly true statements. Second, it aids in bringing to light inconsistencies which may have lurked in a set of supposedly true statements and thus offers grounds for rejecting from a given system certain statements purporting to be true. In brief, consistent statements are possible concerning any recognizable fact, and any system of mutually inconsistent statements cannot properly represent a fact. It is possible to show that many proposed systems are not consistent and must be rejected, even if some long-accepted system cannot be proven to be free from undetected contradictions. It may not be possible to prove a given man honest beyond all doubt, but a single instance of dishonesty on his part undermines belief in his honesty as a person. No statement of a supposed general law of nature may be amenable to rigorous proof, but a single substantiated deviation may be sufficient to force the abandonment of a long-accepted principle. But formal logic cannot be used to determine internally whether a self-consistent body of statements does or does not represent a fact.

Four Stages of Abstraction with Regard to Definitions and Assumptions

Even one who is accustomed to detailed rigorous analysis and who is versed in the concepts and methods of traditional logic may easily fail to grasp the point of view from which modern mathematicians consider the logical bases of their subject. The story is a long one marked by bitter controversy at many stages and seldom presenting so many unsettled aspects as at the present time. The modern point of view in

mathematical logic, however, is far more sophisticated than that of Euclid's followers through many centuries. Even to suggest the most immediately pressing distinctions calls for a subdivision of the inquiry. Four stages of abstraction with respect to definitions and assumptions may be distinguished.

The first stage. At the first stage mathematics is treated as a rigorous logical study of facts of certain sorts. The aim of mathematical investigation is then considered to be the discovery of facts as to quantity and space, and proof is largely the technique for establishing conviction when experimental verification would be unwieldy.[10] Numbers are accepted without definition and their nature is unquestioned—as when one says: "As certain as that $2 + 2 = 4$." In daily conversation, when speaking of points and lines, one has no question in mind as to what he is talking about—he may even attempt to visualize them—although they remain undefined. The "space" one studies is supposedly unique, and has properties which are believed to be "true." At this stage one might be free to replace one set of assumptions by another, but only by a second set consistent with the first. Indeed the assumptions are then regarded as stating obvious familiar truths, and the questions that might arise on denying one of them would be considered merely absurd. For example, for those who are still at this stage non-Euclidean geometry can have no meaning. For them, although perhaps not for Euclid, there is clearly only one parallel to a given line through a point outside the line. This is the stage at which almost all secondary-school students remain during most of their mathematical study, and therefore governed the discussion of teaching methods stressing the "if-then" principle above.

[10] Theorems concerning the nature of space are not absolutely true. They are determined, not by some unseen force inherent in nature, but by assumptions arbitrarily made concerning the space in which we live and by the patterns of thought we use. All human thinking is carried on with reference to some set of assumptions and the conclusions reached in any area of thought are true only to the degree that these assumptions are true. Demonstrative geometry is peculiarly qualified to cultivate this point of view and to develop in students a sound approach to the scientific attitude.

The second stage: known terms, assumptions uncertain; indirect proof. An insistence upon the truth of all the assumptions from which inferences are to be drawn sounds so reasonable and seems so much in accord with ideals for daily conversation that the student, if not the teacher, finds difficulty in reconciling the method of indirect proof with his ordinary habits of thought. In this method one consciously takes up for examination assumptions eventually to be rejected. One investigates what conclusions follow from assumptions which do not merely fail to be obviously true, but which are found later to be false. This point of view toward assumptions is in contrast to that of the first stage, although the final objective —the establishment of facts—is the same. At the second stage all terms continue to be regarded as clearly understood, but some assumptions temporarily adopted need not be obviously true.

The indirect method is based upon several principles which need to be clearly understood if they are to be successfully applied in new situations. One of the more important of these principles is that the validity of an indirect proof depends upon whether or not all of the alternative assumptions have been recognized. Since the method consists in eliminating as logically untenable all of the alternative assumptions save one, if by chance one (or more) possible assumption has been overlooked, the proof is not conclusive. Discussion of these and many other principles may be found in the literature.

The use of indirect proof has often been attacked, and for a variety of reasons. It has been said that it is difficult for students to understand, "it isn't really a proof," and that it is never used in life situations. Such statements have led teachers of mathematics to search for non-mathematical illustrations of the use of the indirect method, and discussion of these illustrations in class has helped students to understand the method. Such illustrations have been included in a number of published textbooks, making it unnecessary to offer any here.

The third stage: terms abstract, generalized variables. It is

important that the teacher at least should have a more mature view of the nature of postulational thinking than is likely to be suggested by the discussion above. It may well be that geometry should be studied by the student in essentially the same spirit as it was being studied by the immediate disciples of Euclid. This represents such a momentous logical advance over the standpoint of most prior mathematical study and indeed of most of arithmetic, algebra, and trigonometry as taught even today, that some of the shortcomings of the classical point of view may be overlooked in elementary instruction. The teacher, however, should not find himself in the intellectual status of those who cannot accept the point of view that led to the development of non-Euclidean geometry. He at least should have reached the third stage.

At this stage the undefined terms and the assumptions now present themselves in a new light. The student has supposedly learned to confine himself rigorously to inferences validly derived from the data—undefined terms, definitions, assumptions, and prior theorems. But now he acknowledges the formal nature of the whole investigation and realizes that no shred of preconceptions clinging to the undefined terms can affect the validity of the conclusion save insofar as the meaning of the undefined terms is limited by the statement of the assumptions. From this viewpoint the undefined terms are treated as meaningless symbols, except that they belong to some general type, such as objects, classes, qualities, properties, etc. For example, a point, although regarded as an object, is any object for which the assumptions can be asserted. Similarly, a line is regarded as a class of points, the only usable properties of which are to be covered by explicit assumptions; parallelism is a relationship between two lines in a pair, a relationship concerning which the reader may remain completely ignorant save as the assumptions cast light upon the situation. For example, a point may be an ordered pair of numbers, a line may be an equation of the type $ax + by + c = 0$. What one

student calls a point might be what another justifiably thinks of as a circle.

From this viewpoint, undefined terms are not clear and unambiguous—rather, they are systematically ambiguous. Assumptions stated as asserted propositional functions of the undefined terms are no longer obviously true, but are on the other hand devoid of imagery and are accepted by agreement. The situation suggests a game with counters in which the postulates and definitions stipulate the conditions for winning. Whether one thinks of the counters as representing wealth, or persons in a campaign, or factors in biological evolution, does not affect the method of playing the game.

At this stage, logical validity is the primary requisite, and rigorous deduction characterizes each step in demonstration. Proof is now no longer aimed at truth, but only at correctness of inference, usually called *validity*. But although consistency must mark the developing system at every step of its progress, consistency alone is not usually acceptable as adequate justification for the effort of mastering the intricacies of a subject field. One plans always so that the logical structure admits of an application on the level of stage one, although the possibility of other applications is not ignored. The student working at this stage usually does not start out to construct mathematical systems at will. He has a familiar objective constantly before him and tries to produce an abstract system fitted for immediate application to a subject studied inductively in prior work. Modern treatments of plane Euclidean geometry and of surveying are of this type. Double-elliptic non-Euclidean geometry can also be studied from this viewpoint. A "line" may then be interpreted as a great circle on a fundamental sphere. The interpretation changes the usual meaning of some of the terms, but provides a concrete factual basis for scientific investigation. Postulates for natural sciences are formulated either from this viewpoint or from the viewpoint of stage one.

The fourth stage: abstract postulate system. At stage four the student cuts himself free from accepted results of inductive observation, and allows his imagination free play with abstract systems of postulates. This stage is distinguished from stage three in that the system of assumptions does not grow out of consideration of a given system of phenomena or relations to which they are to be applicable. One now starts with the assumptions and attempts to find instances to which they may be shown to be applicable. Independence proofs are pertinent at this stage. One examines the consequences of contradicting one of the postulates constructed for a study from the viewpoint of stage three. If the system of postulates thus modified is consistent—which would be established if one could find an example to which this modified system is applicable—then certainly the postulates whose denial resulted in the modified system of postulates could not have been implied by the others, so that the original system of postulates must have been independent.[11]

From this viewpoint, for example, various types of non-Euclidean geometry present themselves as on an equal footing with the abstract geometry of Euclid.[12] For students who attain to this maturer level of understanding of the nature of assumptions some discussion of these developments is possible and highly desirable. If one of the basic postulates of traditional geometry is altered, then changes appear in the theory. The classical example of this centers about the postulate of parallels and the non-Euclidean geometries associated with the alternative postulates. It is also possible to alter other postulates. For instance, instead of assuming that a line segment may be extended as desired, one might assume that segments may only be extended for some arbitrary distance. The tracing

[11] For an easily accessible discussion of independence of postulates see J. W. Young, *Fundamental Concepts in Algebra and Geometry* (New York, The Macmillan Co., 1911).

[12] For an excellent and exhaustive (but formidable) discussion of the postulates of Euclidean geometry the reader is referred to H. G. Forder, *The Foundations of Euclidean Geometry* (Cambridge, Eng., The University Press, 1927).

of changes in the usual theory which must be made if such an assumption is adopted is a valuable exercise. In particular, it calls for a change in the usual definition of parallelism, and if the traditional theorems are to be readily established, some definition based on the equality of angles cut by a transversal is suggested. Such changes in theory should do more to make clear the nature of a logical structure and of the principles of deduction than can the solving of numerous exercises based on a unique set of postulates and definitions. However, the teacher should not forget that such notions were so contrary to the spirit of the first three stages, that not long ago many professional mathematicians of extensive experience, deep study, originality, and integrity failed to see any justification for treating non-Euclidean geometry as on a logical parity with Euclidean.

The existence of several viewpoints of this general sort is not peculiar to mathematics. One encounters like stages, for example, in the novel. A simple account of actual happenings without character motivation would correspond to a pre-demonstrative stage. Stage one is reached in the historical novel, in which actual events are portrayed, where the effort is made to render all characters plausible, and where the action is adequately motivated throughout. "What happened" is explained, and some effort is made to show "why" it happened. A novel of current life in which the episodes are genuine but the names are fictionalized would correspond to stage three. The author has certain definite persons in mind but the reader is entitled to see himself and his acquaintances in the novel. At stage four all relation to history is abandoned. The novelist then no longer feels bound to possible documentation. His interest is wholly centered in characterization, description, motivation, and plausible action, without regard to the actual existence of the characters portrayed. Similar stages may be seen in portraiture. At viewpoint one, the painter makes a portrait study of a familiar person and the spectator seeks to

identify known characteristics of personality through facial form and position deftly captured by the artist. One has also portrait studies of interest on their own account, genuine but anonymous, and these correspond roughly to stage three. At other times, "portraits" seem to have been conceived from stage four. All interest in the physical existence of the subject is forgotten. One has mythical characters, actors seized upon to produce effects which give the competent critic a sense of consistent character analysis.

BIBLIOGRAPHY

BEATLEY, Ralph, "The Second Report of the Committee on Geometry," *The Mathematics Teacher,* Vol. 26, October, 1933, pp. 366–371.

———, "The Third Report of the Committee on Geometry," *The Mathematics Teacher,* Vol. 28, October, 1935, pp. 329–379, November, 1935, pp. 401–450.

BELL, E. T., *The Queen of the Sciences* (Baltimore, The Williams and Wilkins Co., 1931).

BLISS, G. A., "Mathematical Interpretations of Geometrical and Physical Phenomena," *The American Mathematical Monthly,* Vol. 40, October, 1933, pp. 472–480.

CHRISTOFFERSON, Halbert Carl, *Geometry Professionalized for Teachers* (Menasha, Wis., George Banta Publishing Co., 1933).

COHEN, Morris R., and NAGLE, Ernest, *An Introduction to Logic and Scientific Method* (New York, Harcourt, Brace and Co., 1934).

Columbia Associates in Philosophy, *An Introduction to Reflective Thinking* (Boston, Houghton Mifflin Co., 1923).

FAWCETT, Harold P., *The Nature of Proof, Thirteenth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1938).

HUDSON, Hilda P., *Ruler and Compasses* (New York, Longmans, Green and Co., 1916).

JEPSON, R. W., *Clear Thinking* (New York, Longmans, Green and Co., 1936).

KEYSER, C. J., *Thinking About Thinking* (New York, E. P. Dutton and Co., 1926).

LAZAR, Nathan, *Logic in Geometry* (New York, The Mathematics Teacher, 1938).

Mathematical Association, Report, *The Teaching of Geometry in Schools* (London, G. Bell and Sons, Ltd., 1929).

SHANHOLT, Henry H., "A New Deal in Geometry," *The Mathematics Teacher,* Vol. 29, February, 1936, pp. 67–74.

SHIBLI, J., *Recent Developments in the Teaching of Geometry* (State College, Pa., Published by the author, 1932).

STABLER, E. Russell, "Teaching an Appreciation of Mathematics: The Need of Reorganization in Geometry," *The Mathematics Teacher,* Vol. 27, January, 1934, pp. 30–40.

The Teaching of Geometry, Fifth Yearbook of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1930).

THOULESS, R. H., *Straight and Crooked Thinking* (New York, Simon and Schuster, 1930).

YOUNG, J. W. A., *The Teaching of Mathematics in the Elementary and Secondary Schools* (New York, Longmans, Green and Co., 1924). Ch. XI, "Definitions and Axioms," and Ch. XIII, "The Teaching of Geometry.

ULMER, Gilbert, "Teaching Geometry for the Purpose of Developing Ability to Do Logical Thinking," *The Mathematics Teacher,* Vol. 30, December, 1937, pp. 355–357.

———, "Teaching Geometry to Cultivate Reflective Thinking: An Experimental Study with 1239 High School Pupils," *Journal of Experimental Education,"* Vol. 8, September, 1939, pp. 18–25.

X

SYMBOLISM

A word, a nod of the head, a motion of the hand, a traffic light—all of these are symbols or signs by convention. People could not get along in the world without using symbols. The most important symbols, for men, are spoken and written words and signs. Men would be crippled if they did not have words—if they had to point to everything they thought about or talked about.

Men use words to solve most of their perplexities, if not all of them. But it is not easy to use words properly in solving problems. Since words are not identical with the things they stand for, and since one word may stand for many different things, clear thinking requires the most careful relating of words to things. In helping students to think reflectively, therefore, the teacher should help them to understand the use of words. At the same time he would be helping them to communicate clearly, for successful communication among people also requires understanding and mastery of symbols.

Teachers of mathematics hardly need to be told about the importance of words and signs in their own subject. Mathematicians use symbols with unusual skill. They are especially adept at using abbreviations and written signs. They use letters, such as *a, b, c, x, y, z,* in algebra, and signs such as $, ¢, @, in commercial arithmetic. They use many abbreviations and signs in geometry. There is a great saving of time in the use of such symbols. More important than the saving of time is the clear exhibition of formal relations that is effected by symbolism, and the flexibility gained in handling mathematical ideas, as, for example, through the Hindu-Arabic notation. Mathematics has become an effective mode of thinking as it has

acquired an effective form of expression. Mathematics illustrates the gaining of intellectual power through control of symbols, and as applied in physics and engineering it illustrates the gaining of material power through control of symbols.

Teachers of mathematics use symbols skilfully in their own subject, but few of them think about their use of symbols, and many of them would be as surprised to find that they are masters of symbolism as was Molière's gentleman when he discovered that he had been writing prose all his life. Teachers seldom realize that they can help students to develop their own powers of reflective thinking through a more conscious attention to symbols in the mathematics classroom. It is the purpose of this chapter to suggest ways in which this help may be given.

During the first few years of their lives children develop as much fluency with words as the human race developed during thousands of years of its early existence. They do this unconsciously, without thinking about symbols. By the time they reach the secondary school they often have an astonishing glibness. Partly to gain a feeling of self-assurance with their age mates and with their elders, they tend to use slogans and stock phrases to meet almost every situation. Insofar as they do this, they are ceasing to use language for the purposes of reflective thinking. There are, of course, situations in which the use of stock verbal patterns is highly effective. Language is not always used for the purposes of reflective thinking. But there are also many situations in which stock phrases are used consciously or unconsciously to avoid thinking. This is a difficulty which the teacher of mathematics must face, together with other teachers who hope to encourage skill and discrimination in the use of words.

The teacher of mathematics has an opportunity to help his students avoid some of the pitfalls that lurk in the language they speak with such ease and fluency. He can do this by helping them to study the nature and use of symbols in mathematics and to compare the uses of symbols in mathematics and in

ordinary language. This does not mean that the class should study special units on symbolism. Though such units may be useful in some schools and with relatively mature students, they would in no case serve the whole purpose. The teacher should be on the alert for suitable opportunities in all parts of his course, so as to make a conscious attention to symbols part of the everyday life of his students.[1]

IMPROVING STUDENTS' UNDERSTANDING OF SYMBOLS THROUGH STUDY OF MATHEMATICAL USAGE AND COMPARISON WITH NON-MATHEMATICAL USAGE

In general, the Committee's advice to teachers is that they show explicitly how symbols are used in mathematics, and make continual comparisons and contrasts with the uses of symbols in non-mathematical language. The following pages contain some illustrations of ways of putting this general recommendation into effect.

A symbol for a given idea may be either a sign or a word. For example, the word *curve* beside a highway means much the same thing as the sign "S." A given sign or a given word may be used at one time in mathematics and at another time in non-mathematical language. In algebra, x is usually a symbol for a number whose value is to be found. In a mystery story, "Dr. X" may be a symbol for a mysterious person. The word "multiply" is used mathematically in the sentence, "When the Pilgrims multiplied 5 by 6, the result was 30."

[1] In fostering the student's ability to use symbols correctly, the teacher should be sensitive to the range of individual differences among his students in their preference for one type of symbol as against others. Some students will find far more pleasure than others in manipulating words and in studying their uses. Some will express themselves most clearly and vigorously in non-verbal ways—through drawing, working with wood, metal, clay, or stone, through dancing, or music. Though a boy or girl who prefers a non-verbal means of expression may never do well at analyzing words, he may symbolize his ideas in other ways with unusual power and clarity. But they will be ideas of other kinds, with other uses in his life. Some aptitude in handling the general ideas which require verbal analysis is necessary to all people.

This word is used non-mathematically in the sentence, "When the Pilgrims multiplied, the result was the Society of Mayflower Descendants."

Inspection of a given symbol out of its context will not always tell one whether it is a mathematical or a non-mathematical symbol.[2] The distinctions are not in the symbols themselves, but in the ways they are used and in the purposes for which they are used. Perhaps the only useful way to define a mathematical symbol is to say that it is any symbol used by mathematicians in what they call mathematics.

One of the first lessons about symbols for secondary-school students might well be the lesson that symbols are instruments or tools of thought. Another is the lesson that a given symbol may often serve a variety of purposes. The way to find out what purpose a symbol is serving, and whether it is used in mathematical or in non-mathematical language, is to study the setting and context in which the symbol is used.

Some Uses of Symbols in Mathematics

In suggesting ways by which the teacher may help students to see how symbols are used, the Committee will discuss the following three uses of symbols in mathematics.[3]

1. For describing physical things precisely in terms of size, order, and number
2. As shorthand for other symbols
3. As symbols of other symbols

In the discussion of each of these uses of symbols in mathematics, there will be many references to "mathematical uses" and "non-mathematical uses." Therefore, these phrases will be abbreviated to "M uses" and "NM uses" respectively.

[2] This statement applies generally to elementary mathematics. But in the calculus and in many branches of higher mathematics symbols have been invented which are seldom, if ever, used in non-mathematical discourse.

[3] These three uses are not mutually exclusive. Furthermore, they do not cover all the uses of symbols in mathematics. They are merely convenient categories for an elementary discussion of the common uses of symbols in mathematics.

Symbols as instruments for describing physical things precisely in terms of size, order, and number. Through the arts and crafts, through commerce, and in the last few centuries through the natural sciences and their applications in engineering, men have tried to control the energy and the materials of the physical world. Symbols such as numbers, units of measure, and geometrical figures have served as intellectual tools for this conquest of nature. Hogben writes of mathematics as the language of *size,* and of ordinary language as the language of *sorts.*[4] This distinction brings out an important M use of symbols—namely, to describe the size, order, and number of concrete things in the world.

The teacher might encourage students to draw up a list of characteristics which symbols must possess in order to serve as useful tools for dealing with physical objects. One of the principal characteristics is a correspondence of symbols with real or imagined physical events, such as the correspondence of numbers with the operation of counting, the correspondence of units of weight with the operation of weighing, the correspondence of certain geometrical figures with related constructional operations in architecture, the correspondence of triangles drawn in a ship's chart room and the path of the ship, and the correspondence of the graph of a particular equation with the path of a cannon-ball.

There are relatively few symbols in NM-usage dealing with size, order, and number. A few vaguely quantitative words, such as *large, small, few, several, many,* are valuable because they express ideas that numbers alone cannot readily convey. But symbols used in mathematics are much more important in the quantitative description of physical things than are symbols in NM-usage.

Symbols as shorthand for other symbols. The following examples show how symbols taken from the language of everyday life and used for an algebraic equation were supplemented in

[4] Lancelot Hogben, *Mathematics for the Million* (New York, W. W. Norton and Co., 1937), Ch. III, "The Grammar of Size, Order, and Number."

the course of two centuries by symbols consisting of letters and operational signs.[5]

Regiomontanus, A. D. 1464:
3 Census et 6 demptis 5 rebus aequator zero.
Pacioli, A. D. 1494:
3 Census p 6 de 5 rebus ae 0.
Vieta, A. D. 1591:
3 in A quad – 5 in A plano + 6 aequatur 0.
Stevinus, A. D. 1585:
3 ② – 5 ① + 6 ⊙ = 0.
Decartes, A. D. 1637:
$3x^2 - 5x + 6 = 0.$

Mathematics uses many ideographic symbols, which are symbols that do not depend upon a phonetic language, but convey meanings directly by signs which are sometimes but not always pictorial. A sign "S" indicating a curve in a highway is an ideographic symbol. A money-bag with a dollar sign on it is an ideographic symbol of wealth. Arithmetic and algebraic symbols for the digits and for the elementary operations of addition, subtraction, collection of terms, and so forth, are ideographs, as are charts and graphs and symbols commonly used in geometry.[6]

Ideographic symbols serve to condense what would otherwise be hopelessly extended. For example, the quadratic equation briefly expressed by $ax^2 + bx + c = 0$ would require some such statement as the following to be expressed in words: "Take the product of the unknown by itself and by a first given constant coefficient; to this product add the product of the unknown and a second given constant coefficient; to this sum of

[5] From *ibid.,* p. 303. The distinction between symbols for numbers and symbols for mathematical operations is explained in readable fashion by Hogben, pp. 70–94, and is consequently not developed here.

[6] The extent to which ideographic symbols are employed in modern life is suggested by the long list, covering eleven large finely printed pages (3001 to 3011) in *Webster's New International Dictionary* (Springfield, Merriam Co., 2nd edition, 1937). See also R. Modley, *How to Use Pictorial Statistics* (New York, Harper and Bros., 1937), pp. 46, 47. Otto Neurath has recently written a history of mankind largely in ideographic symbols. See *Modern Man in the Making* (New York, Alfred A. Knopf, 1939).

products add a third given constant term; and equate the result to zero." The verbal statement clearly renders further mathematical operations on the basis of this formula extremely difficult if not impossible. Prolix and inappropriate symbols block reflective thinking. Formulation in an appropriate and very compact symbolic form, on the other hand, facilitates further operations.

Aware of the values of compact symbols for his purposes, the mathematician frequently devises briefer notations for long and cumbersome expressions which turn up again and again in his work. Thus instead of the lengthy expression

$$x_1 + x_2 + x_3 + \cdots x_n \text{ he often uses } \sum_{i=1}^{i=n} x_i.$$

This symbol by definition means "The sum of the n quantities x_1, x_2, etc." A bar graph, a circular distribution graph (or "pie chart"), and a curve plotted on coördinate paper, are shorthand for arrays of data and for relations among data.

Students might be encouraged to invent shorthand methods of representing symbols. Where they do this without encouragement, as in geometry, the teacher might point out the fact that they are creating symbols to serve as shorthand for other symbols. This is the process mathematicians have followed for centuries. Those abbreviations which proved most useful have been taken up by others. The class in mathematics might follow this same process, discussing abbreviations suggested by students and adopting as a class those that promise to be useful.

The process of translating one set of symbols into a more convenient shorthand set should be accompanied systematically by the reverse process, namely, that of translating shorthand symbols back into longhand. For example, the translation of $ax^2 + bx + c = 0$ into words is a useful corrective for students who tend to memorize formulas without thinking about their meanings.

The use of ideographic symbols, abbreviations, and other shorthand devices gives more than economy of space in writing. It gives new effectiveness to the mind. As a simple illustra-

tion of this, students might be asked to multiply XVII by VIII in Roman numerals, and to compare this process with multiplying 17 by 8 in Arabic numerals.[7] Whitehead [8] says the following about the importance of a good notation:

> By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race. Before the introduction of the Arabic notation, multiplication was difficult, and the division even of integers called into play the highest mathematical faculties. Probably nothing in the modern world would have more astonished a Greek mathematician than to learn that, under the influence of compulsory education, the whole population of Western Europe, from the highest to the lowest, could perform the operation of division for the largest numbers. This fact would have seemed to him a sheer impossibility. The consequential extension of the notation to decimal fractions was not accomplished till the seventeenth century. Our modern power of easy reckoning with decimal fractions is the almost miraculous result of the gradual discovery of a perfect notation.

Before the invention of the decimal notation with the sign "0," calculation had to be done laboriously and for the most part by hand on the abacus. To the Greeks and Romans the number scripts were merely labels to record the result of doing work with an abacus. But the invention of "0" made it possible to calculate with paper and pencil. In addition, this invention made it possible to express any number, however large, with nine numerals and zero, whereas hitherto new signs like the Roman X, C, and M had to be introduced for every power of ten.

The teacher of mathematics might regularly, at every level in the junior and senior high school, call attention to the intellectual effectiveness which the symbol "0" has given to man. With boys and girls in the seventh or eighth grades, this can

[7] The teacher may wish to illustrate this with Greek as well as the Roman notations. For Greek number notations see Vera Sanford, *A Short History of Mathematics* (Boston, Houghton Mifflin Co., 1930), pp. 81–83.

[8] A. N. Whitehead, *Introduction to Mathematics* (New York, Henry Holt and Co., 1911), p. 59. See also Hogben, *op. cit.*, pp. 283–285.

be illustrated by the study of various number notations. With high-school students of algebra, the teacher can point out that algebraic equations as now used could not exist without a symbol for zero.[9]

Reference has already been made to graphs and curves on coördinate paper as shorthand symbols.[10] These are not only convenient time savers. They enable a person to hold arrays of facts in his mind, to discover relations among facts, and to employ arrays of data in solving problems. For example, a curve of the population growth in America, or a curve of price levels over the past century, gives one a command of facts in significant relationships that he could hardly obtain from laboriously examining a table of figures.

The term *shorthand* has been used in the sense of *abbreviation*. There are many NM as well as M uses of symbols as shorthand in this sense. There are several systems of shorthand for note-taking and letter-writing. There are many ideographic symbols in NM usage, such as traffic lights, coats-of-arms, and trade-marks. In everyday language people are constantly using conventional formulas to serve as shorthand for long phrases. "New Deal," "WPA," "TVA," are examples from current journalism. The gossip column of a metropolitan newspaper coins several new ones every month. Thus symbols serve in both M usage and NM usage as shorthand for other symbols.

Symbols used as symbols of other symbols. Some of the examples given in preceding paragraphs showing symbols as shorthand have hinted at the use of symbols in another sense —namely, in the sense of one general symbol standing for many particular symbols. This is another of the uses of symbols in mathematics.

The symbol "△" stands for all particular triangles. An al-

[9] A. N. Whitehead, *Introduction to Mathematics*, pp. 63–68.

[10] These are cases of rigidly controlled metaphor. When a man says "the price went up and then went down," he refers to what he would do if he were pointing to the figures arranged on a vertical scale. The graph gives a picture of that pointing. The teacher of mathematics has an opportunity to help students understand the uses of metaphor through discussing examples like this one.

gebraic symbol, "*a*," stands for a class of numbers. A graph is a generalization of a table of particular symbols. These are all symbols of other symbols. The teacher may point out that algebraic symbols are used to state general arithmetic principles. In using symbols in arithmetic the mathematician is restricted to illustrative examples; algebraic symbols permit him to formulate general principles and rules of operation. Instead of being confined to a series of examples of the addition of fractions, for instance (such as $\frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15} = \frac{22}{15}$, $\frac{1}{7} + \frac{2}{9} = \frac{9}{63} + \frac{14}{63} = \frac{23}{63}$), he can say once and for all $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$ where *a, b, c, d,* are regarded as symbols for selected positive integers.

Symbols when used in mathematics for other more particular symbols help to free the mind from attention to unnecessary details. Thus they facilitate generalization and the discovery of relationships among quantities. Indeed by their use otherwise unsuspected facts about the nature of connections among quantities may be deduced—for example, the "quadratic formula" which expresses the values of x in the equation $ax^2 + bx + c = 0$.

In using symbols to refer to other symbols, mathematicians show their desire to *consider general things generally.* They are unwilling to restrict their use of symbols to descriptions of physical entities, even though they recognize the use of symbols to describe physical things precisely is important. They also think of symbols as referring to things that are not physical entities and cannot be pointed to. For example, most mathematicians think of the diagrams in geometry as referring to abstractions. The word *circle* refers to all physical circles and also to an ideal curve which is without breadth and whose central point is without area. Discussion of ideal circles might well be encouraged at least for the more capable students, and they might be led to see the philosophical issue involved. Study of the beliefs of Plato about the nature of mathematical symbols might help the abler students to grasp some of the

problems found in using symbols for contemplative purposes.

Mathematicians start with certain initial sets of symbols, such as numbers, and perform operations on those symbols so as to get new symbols, such as logarithms, or sixth powers of numbers, or fourth roots of numbers. These new symbols are more abstract, further removed from physical entities, than the symbols with which the mathematician starts. Thus it is possible to speak of different "levels of abstraction" on which mathematicians work. At any given level of abstraction, the symbols of a lower level are considered as if they were objects. For example, algebra operates at a level of abstraction which considers particular numbers as objects; but the particular number, 16, is itself an abstraction, referring at its level of abstraction to all classes of physical objects made up in a certain way.

Symbols of symbols are found in NM as well as in M usage. Symbolic logic contains symbols which have the same general and abstract character that algebraic symbols possess. This might be pointed out to the more mature students. It is also important for students to understand that many symbols in ordinary usage are general and abstract. For example, *descendants* stands for "sons, daughters, grandsons, granddaughters, etc." The teacher of mathematics can help students to analyze abstractions by showing them how abstract symbols are built up in mathematics, and then comparing this process with the building up of abstract symbols in NM usage,[11] such as *justice, democracy, wealth.*[12]

[11] Most teachers of mathematics will have to learn a great deal about symbols if they are to help their students in this way. The following books may be useful in this connection: *Language in General Education* (New York, D. Appleton-Century, 1940), Ch. 4; I. A. Richards, *Interpretation in Teaching* (New York, Harcourt, Brace and Co., 1938), Part 3; A. Korzybski, *Science and Sanity* (Lancaster, Pa., Science Press, 1933).

[12] The most able students can probably consider such abstractions as infinity and complex numbers. They can also study the psychological bases of symbols. For example, they might be given opportunity to examine critically Aristotle's dictum, "Spoken words are the symbols of mental experience and written words are the symbols of spoken words. Just as all men have not the same writing, so all men have not the same speech sounds, but the mental ex-

Distinctions Between M and NM Uses of Symbols

The teacher might well spend a good deal of time with senior high-school students in helping them to answer such questions as: "Is it easier to use symbols accurately in M than in NM language?" "What are the differences between the two uses of symbols?" "What are the advantages of the various uses of symbols?"

Mention has been made of several characteristic uses of symbols in M language but it has been pointed out that symbols are used in NM language for similar purposes. Thus no hard and fast distinctions have been made between M and NM uses. Indeed, one of the lessons that students should learn is that the differences between M and NM uses of symbols are differences of degree, rather than of kind. In the next pages this will appear more clearly.

Students might have before them several passages for study. One might consist of a description largely in M language of a naval battle, written by an engineer and tactical expert. A second might consist of a description largely in NM language of the same naval battle, written by a reporter for a newspaper in a neutral country. A third might consist of a description largely in NM language of the same battle, written by a propaganda expert for a newspaper in the capital of one of the belligerents. A fourth might be a poem written in NM language by the poet laureate of one of the belligerents.

On careful analysis, these four passages might be found to have the same literal sense—that is, to present the same facts. But students would find many differences, which could be summarized as differences of:

Feeling, mood, or attitude toward the battle
Tone, or attitude toward the readers
Intention, or effect that is aimed at

---

periences, which these directly symbolize, are the same for all, as also are those things of which our experiences are the images." . . . *De Interpretatione,* 16a.

The engineer would probably betray little or no feeling about the battle. With his diagrams representing positions of ships at various times, speeds, number of pounds of explosives fired per minute, etc., he would put the emphasis upon literal sense of what he had to say. His tone would indicate respect for his readers as well as detachment from their interests in the battle. His intention would evidently be to give a clear and adequate description of the battle.

A reporter for a newspaper in a neutral country might show a feeling of excitement at the tremendous spectacle of the battle and perhaps of horror if the carnage was great. His tone would be that of the observer of a thrilling incident who seeks to share his experience with his friends. His intention would be to convey the facts in an interesting way. He might use diagrams, but he would also use picturesque phrases—"a red glow shone through her portholes," "listing to port, she fired one last shot before she rolled on her side and sank."

A propagandist for a newspaper in the capital of one of the belligerent countries would show such feelings as he wanted his readers to feel. He would describe the "bravery" of his country's men and the skill of their officers. His tone or attitude toward his readers would be that of a participant in the battle writing home to his friends about it. His intention would be to build up in his readers' minds the kind of picture of the battle that would do the most to make them feel as his government, or his editor, wanted them to feel. If necessary, he might suppress some of the facts so as to get the effect he wanted to achieve.

The poet laureate of one of the belligerent countries would attempt to convey his own feeling about the battle in such a way as to arouse the same feeling in his readers. He might feel uplifted by a great victory, or downcast over a defeat. He would use no diagrams, and probably no technical terms. He would use picturesque phrases, but his phrases would differ from those of the reporters. The intention of the latter would

be to *convey a picture* of what took place, but the poet would intend to *evoke a feeling* about what took place.

Again, students might have before them several pieces written about infinity. One might be written by a mathematician. One might be written by a popular magazine writer, one by a poet, and one by a priest. Here, too, there would be the same kinds of difference as those observed in the descriptions of the naval battle, with the added difference that the four authors would not be found to convey the same literal sense. They would not convey the same literal sense because the term *infinity* has different meanings for different people, and does not refer to a physical event like a naval battle.

The result of a good deal of experience of this sort would lead to the conclusion by students that symbols are generally used mathematically in certain restricted moods and tones, and with certain limited intentions on the part of the author. The students would discover that symbols in NM language are used with a much wider range of mood, tone, and intention, although usages with moods, tones and intentions quite similar to those in M usage would also be found.[18]

Referential and emotive language. Students would early make a distinction between *referential* and *emotive* uses of symbols. In mathematics, symbols are used ordinarily to convey facts, or to talk about things neutrally. This is referential discourse. Symbols of NM usage are also used in referential discourse, in such things as news stories, legal briefs, and accounts of botanical field trips. Symbols in M usage are espe-

[13] Exercises such as the following might be developed: Students might compare several descriptions of the growth and development of adolescent boys and girls. One description might be written by a scientist reporting his measurements of a group of boys and girls during the age period from ten to eighteen. Another might be written by a novelist, such as Booth Tarkington, describing the adolescent development of an individual boy or girl. Another one might be written by a poet, portraying a girl's feelings during her first love affair.

Students might prepare several essays on the symbol "7," one as would be written by a pupil in the sixth grade, another as by a mathematician, a third as by a gambler, and one as by a Seventh Day Adventist.

cially well adapted to referential language because they enable one to describe size, order, and number, of physical things.

In studying the uses of symbols students would find mathematical words and signs used very seldom to express attitudes and emotions or to engender them in the reader, which are the purposes of emotive discourse. Nevertheless, they would find a few interesting cases of emotive effects from symbols in mathematics. They would find mathematicians referring to certain sets of symbols as "elegant" proofs. They would also find that M usage was proving surprisingly emotive for many lay readers, even though the authors had not intended it, because of the despair aroused in some readers by the sight of a mathematical equation.

Unambiguity of symbols in M usage. Students would probably come to the conclusion that M usage has the advantage for referential use of being less ambiguous than most NM usage. This would not be a new discovery. Bacon said that in order to save themselves from the "great sophism of all sophisms, equivocation or ambiguity of words," men must "imitate the wisdom of the mathematicians" and make clear the sense in which they use their terms.[14]

Words in NM usage have many meanings, usually only partially distinguishable from one another. A dictionary lists numerous cognate meanings, and accepted usage provides innumerable gradations within and beyond the confines laid down in any given dictionary. A powerful novel or play or speech may lead to fresh associations of terms and tinge long familiar words with new shades of meaning. Hints of approbation or condemnation, suggestions of poetic emotion, flavor of some special trade, sport, race, or class tend to cluster around them.

In contrast with this, symbols in mathematics may be defined once and for all by fiat. They may be used always in the same sense. Their meanings are relatively independent of context and free from emotive variations. This is especially true

[14] Francis Bacon, *Works*, Edited by Robertson, Book I, pp. 118, 120.

of ideographic symbols in M usage. They require no translation, and so may be truly international and timeless.[15]

If students were to rank types of symbols in order of unambiguity, they would probably arrive at the following order:

Ideographic symbols in M usage
Ideographic symbols in NM usage (excluding Chinese ideographs)
Verbal symbols in M usage (e. g., *function, power, root, exponent, value, argument*)
Verbal symbols in scientific usage (e. g., *density, mass, force, power, energy, law, species*)
Verbal symbols in non-scientific usage.

Relative independence of context for the meaning of symbols in M usage. Students should also learn that the context of symbols in NM usage very largely determines their meanings. The two sentences, "The maid dusted the table" and "The baker dusted the pie" both use the word *dusted,* but in the first case it probably means "took dust off" while in the second it doubtless means "put dust (sugar or flour) on." Thus the meaning of words is profoundly influenced by the context, and confusion may arise when a meaning appropriate to one context is transferred without modification to another. Furthermore, a wider context than the mere sentence or paragraph in which words occur, namely, the whole setting in which they are used, also has a profound influence upon their meaning. The word "Fire!" shouted in a theater means something quite different from the same word shouted in a rifle range.

In contrast to the meaning of symbols in NM usage, that of symbols in M usage is relatively independent of context. The extent to which this holds true, however, is easily exaggerated. The teacher who asks "What is 7 times 8?" may insist that there is only one possible reply, namely, "56." But unless the student has been trained to respond in a particular way

[15] E. g., the symbols, 1,000,000,000 (= 10^9) and 1,000,000,000,000 (= 10^{12}) are easily distinguished in arithmetic, but the word "billion" usually means the former in the United States and in France, and the latter in England and Germany, causing frequent confusion in the popular press.

he might venture any one of the following in an attempt to answer the question: "A number," "a product," "8 times 7," "4 less than 60," "more than 50," "an arithmetical exercise," "the number of objects in that picture," "an even number," "a multiple of 7," and so forth. In certain contexts any one of these might be the appropriate answer.

Even within the narrower context of mathematical exposition itself, some symbols on occasion change their meanings as the setting changes. For instance, the superscript prime may mean feet, minutes of arc, the first derivative, a second parameter, and so forth. The letter s may indicate the semiperimeter of a triangle, the arc length of a curve, displacement in space, the limit of a series, the sum of given terms, and so forth. Furthermore, the mere position of a symbol in relation to other symbols may change its meaning. In -5, the minus sign indicates not an operation upon 5 but a negative magnitude, while in 5– the minus sign indicates an approximate estimate, the 5 erring by excess. In 2.5 the point means something other than the use of dots in $1.428 \cdots$ or in a series $a + a^2 + a^3 + \cdots$. In algebra the expression $f(x + y) = f(x)\, f(y)$ might be misinterpreted as an incorrect statement of the product of a monomial and a binomial, but in a more advanced course it is likely to indicate a property of a special function $f(z) = a^z$. Nor does the juxtaposition of two symbols always mean the same thing. Though the expression ax symbolizes the product of a and x, in calculus the expression dx usually designates not a product but a differential. In $2\frac{1}{2}$, the juxtaposition indicates the sum of 2 and $\frac{1}{2}$. In 23 the two digits acquire place value and the symbol means neither 6 nor 5 but 20 plus 3. In geometry AB usually indicates not a product but a line segment of which points A and B are endpoints. In the symbol \$2, the dollar sign indicates the unit and is placed first, while in 2°, the degree sign follows the numeral.

But the numerous meanings attached by custom to the same symbol in M usage need cause no confusion unless their contexts overlap. There is one source of difficulty in the fact that

students have had long experience with words which tend to carry over part of their meaning from one context to another. Consequently, it is frequently difficult for them to recognize that those mathematical symbols which change their meaning with the context change it entirely and carry over no part of their meaning from one context to another.

Comparison of mathematical and other usages. In the preceding few pages some of the distinctions between M and NM uses of symbols have been pointed out. But care has been taken to show that hard and fast distinctions cannot be made, and that such distinctions as exist are differences of degree rather than of kind. When students understand these points, they may proceed to describe the differences more definitely.

In describing differences in the usage of symbols students will probably find it useful to distinguish between pure mathematical, scientific, and other usages. A good deal of what has been referred to in this chapter as M usage is more strictly called scientific usage. For example, *quart, mile,* and *minute of arc* are terms in scientific use, although students are often introduced to them in a mathematics course. Speaking quite strictly, one might say that only such symbols as are derived from numbers are purely mathematical.

There are other distinctions which the more mature students might make. For example, there are *postulational* as distinguished from *empirical* terms. The circumference of a circle of given radius is determinable to any accuracy one may require. This is a postulational term. But "the circumference of the earth" is an empirical term known only within approximate limits set by the operations of physical measurement.

An important characteristic of symbols in mathematical and scientific usage is that they can be substituted with ease for the things to which they refer. Thus the mathematician and the scientist are able to work effectively with symbols for long stretches of time, without checking up to make sure that their symbols still refer to the same things they stood for at the be-

ginning of the process. This is due to the fact that mathematicians and scientists use symbols in relations to each other that are similar to the relations among physical things. For example, multiplying numbers has its physical counterpart.[16] Students can grasp the significance of this by comparing the grammar of arithmetic and algebra with the grammar of ordinary language. Hogben has done this systematically, with many examples which high-school students can follow.[17]

One result of attempts to compare mathematical with other usages of symbols will probably be a focusing of students' attention on the *precision* and *abstraction* that is obtained in M usage. Students should be led to realize that these are not obtained only in M usage. The better students will get a working knowledge of the conditions under which precision is obtained, and the conditions under which abstraction is obtained, in both M and NM usage.

The goal of the teacher as he helps the student to study the uses of symbols in mathematics and to compare M and NM usage should be to bring him to the frame of mind where he habitually thinks of symbols as tools, and almost unconsciously asks himself in each new situation: What kind of use must I make of symbols in this situation?

THE SOCIAL ORIGINS OF SYMBOLS

Although the origins and historical evolution of symbols are of interest to the specialist, these topics are remote from the needs of most adolescents and may therefore remain incidental in the secondary school. Yet their study would be of some use in helping the student to understand the symbols in current use. Some few students may make special studies of these topics, and the teacher at least should have some knowledge of them.[18] Here it is possible only to indicate one or two

[16] Cf. Chapter VIII, on "Operation."

[17] Hogben, *op. cit.*

[18] The position of the Committee in regard to teaching the history of mathematics is outlined in Chapter XI, "The Development and Nature of Mathematics."

of the conceptions which this history might help to develop.

For example, to a degree surprising to most persons the significance of symbols is dependent upon the culture and its social traditions. This holds true even of expressive bodily movements and exclamatory sounds. What is a negative shake of the head in America and western Europe is an affirmative sign in the Balkans. The clapping of hands, the nodding of the head for approval, and numerous other gestures polite or vulgar are meaningless to persons brought up in cultures in which they are not used. Likewise, imitative marks are to a considerable extent dependent upon accepted conventions for their significance. Whether it be the pictograms and sign language of primitive people or the diagrams and maps of modern engineering and commerce, one must be taught if one is to understand.

Another idea which acquaintance with the origins of symbols might build is that some traditional symbols have lost all trace of their original connotation. The orange blossoms, veil, rice, and old shoes at a wedding, the vesture of the officiating representative of the church, pranks at Hallowe'en or at Mardi Gras, Easter Eggs, the dollar mark, are but instances of hosts of symbols the present-day use of which does not depend upon their original significance. The story of the alphabet is not without interest in this connection; proficient use of the written letter A does not depend upon acquaintance with the fact that it was once written  to stand for the conspicuous long horns of the bull whose roar was supposed to suggest the sound "aah."

In the case of mathematical notation the history of particular forms of symbols is not always clear. Among many forms used at various times some have survived, but it is difficult to say with assurance just what led to the choice of a given notation or why it survived when others were ignored. On the other hand simple and suggestive symbols tend to replace complicated and directly representative ones, and one can easily show how some particular forms may have been derived.

For example, in keeping a tally an item that occurred 27 times might first have been marked off by 27 vertical strokes, thus, 111111111111111111111111111. Then in counting every fifth stroke might have been used to "cancel" the previous four: 𝍸 𝍸 𝍸 𝍸 𝍸 11. In recording for the future, this might have been written as V V V V V 11. When the number of 1111's or V's became too numerous for reckoning at a glance, a diagonal stroke might have been made to join these in pairs, thus, 𝍸̸ 𝍸̸ 𝍸̸ 𝍸̸ 𝍸 11, or more briefly XXVII as in the Roman notation. To take another example, it is also easy to see how 〓 for "two," as still used in China and Japan, could be slurred into Z and thence develop into 2, and how the oriental 〓 could similarly become ⫧ and then ⫧ and finally 3.

It is not essential to build conceptions like these in order to increase such understanding of symbols on the part of students as will improve their ability in reflective thinking. Nonetheless such conceptions are sometimes of value in throwing light on the characteristics of symbols in current use, and they increase students' appreciation of the cultural origins of his symbolic instruments of thought and communication.

SOME BROADER ASPECTS OF SYMBOLISM

The discussion has skimmed over with scanty reference many significant aspects of symbolism. It permeates all aspects of living. Through mere repeated association the observer learns to infer the presence of some aspect of his environment from the perception of another aspect familiarly concomitant with it, and thus the one serves as a symbol of the presence of the other. Modern psychology has so familarized the public with the theory of associated responses, conditioned reflexes, and the like, that a brief hint should suffice here.

Many symbols other than words carry a simple informational message; sounds, colors, and tactile sensations frequently symbolize other associated sensory patterns. The ringing of the

door bell is the symbol of the presence of some one at the door. Tunes of familiar songs come to suggest the accompanying words and sentiments. The striped pole marks a barber shop; the red and green traffic lights indicate the authorized direction of traffic-flow. The sound of the siren warns of an official interruption. The color no less than the engraving on a postage stamp may mark its designated value. Trade marks, code symbols, standard abbreviations, and other semi-literal symbols bear evidence to the importance of symbolic devices in commerce. The shape and position of a letter box, the uniforms and insignia of public servants from the general in the army to the red-caps in the depot, all these call attention to the pervasiveness of symbols in modern life.

The symbols mentioned in the preceding paragraph are conventional signs, made by man. There are also natural signs. For example, the odor of food is a symbol of its presence and even of its taste. The position of the sun in the sky is a symbol of the time of day. The angles subtended at the eyes symbolize the distances at which the subtending objects are situated.

Symbols also play a striking part in the emotional life of individuals and groups. There is much evidence that in dreams of an individual hopes and fears that he is unable to express even to himself take on symbolic forms and reassert their claims in strange guises. Then, too, many symbols are imbued with highly charged emotional meanings, and call forth responses that are almost purely emotional in nature: names of deities are held to be sacred; national flags and military uniforms evoke feelings deeper than those called forth by other insignia. The uses of symbols to give rise to emotional response in some situations are well known. For example, martial music, columns of marching soldiers in uniform, the waving flag arouse the fervor called "patriotic" and encourage enlistments in the forces of national defense. Furthermore, most superstition may be thought of as an emotionally stimulated belief, without scientific or factual basis, in the portentous symbolic meaning of trivial events, and one need not go back to primitive cultures

for occasions on which intelligent persons reveal some type or degree of faith in the mystical power of symbols. It is not only that gamblers place credence in lucky or unlucky tokens, or that faith in fortune telling of one kind or another is still widely prevalent, or that many persons still associate mathematical science with an occult and mystic background which it once treasured but now abjures.[19] A more subtle and insidious form of the same kind of faith in the mystical power of symbols lies in the tendency of many persons to find in a name for a phenomenon a sufficient explanation of it, particularly if the name has a forbidding, technical sound.

Though the emotional side of symbolism is too extensive to permit of more than this brief mention here, it is not remote from the objectives of this chapter. For when symbols are clearly understood and fully appreciated as necessary means for the deliberate conduct of rational analysis and for the logical process of inference from data in accordance with hypotheses, it is less likely that they will be used unintelligently as portents of events to which they are unrelated or that names will be accepted in place of analyses. The areas in which reflective thinking is brought to bear are widened, and symbols are increasingly used to further thinking rather than to preclude it by mystic faith or obfuscate it through the misuse of names and signs. Thus although sounds and marks may fail to call forth thought, and may even be used to interfere with thought, reflective thought cannot operate save through symbols of some kind; and focusing attention upon the essential rôle and proper uses of symbols in reflective thinking minimizes the likelihood of their misuse as in superstition or propaganda.

BIBLIOGRAPHY

BRIDGMAN, P. W., *The Logic of Modern Physics* (New York, The Macmillan Co., 1927).

[19] See E. T. Bell, *Numerology* (Baltimore, The Williams and Wilkins Co., 1933); H. Berkeley, *Mysticism in Modern Mathematics* (London, H. Frawde, 1910).

CHASE, Stuart, *The Tyranny of Words* (New York, Harcourt, Brace and Co., 1938).

CARNAP, R., *The Logical Syntax of Language* (New York, Harcourt, Brace and Co., 1937).

KORZYBSKI, A., *Science and Sanity* (Lancaster, Pa., Science Press, 1933).

Language in General Education, The Report of the Committee on the Function of Language in General Education of the Commission on the Secondary School Curriculum, Progressive Education Association (New York, D. Appleton-Century Co., 1940).

OGDEN, C. K., and RICHARDS, I. A., *The Meaning of Meaning* (New York, Harcourt, Brace and Co., 1926).

RICHARDS, I. A., *Interpretation in Teaching* (New York, Harcourt, Brace and Co., 1938).

For further reading in connection with the topics in this chapter, many references are to be found in books and articles dealing with language instruction, psychology, child-growth, philosophy, and semantics.

Part III

THE DEVELOPMENT AND NATURE OF MATHEMATICS

XI

THE DEVELOPMENT AND NATURE OF MATHEMATICS

There are reasons beyond the pedantic why the teacher of mathematics may well seek to develop in his students some perspective on the relation of mathematics to the civilizations in which it has played a part, its rôle through the ages, and the character of its development as a science.

Not the least of the values to be achieved by such study lies in reënforcing the notions of the nature of mathematics built in the course of work based on Part II. But perhaps more important is study of the history of mathematics so conducted as to give students understanding of the fact that this science is one of the main instrumentalities by which man has constructed for himself the kind of world in which he now lives. The adolescent needs understanding of his world and of the forces, scientific as well as physical and social, at work within it. In addition, he needs understanding of the workaday world of adults, and of how the subjects he studies in school function outside the school. For all students, then, insight into the broadly historical aspects of mathematics serves to give the subject new light and life. For some of them—those who are particularly interested in this field—it offers an especially acceptable approach to the study of civilization and its development.

There are, as a matter of fact, many common misconceptions of the nature of mathematics and its place in the history of civilization. Many students suppose that mathematics has always had its present form and that it will always keep this with few additions. On the other hand, they may have the idea that mathematics is the product of a steady inexorable prog-

ress from primitive prehistoric forms of counting and space perception to present complex techniques. Moreover, even among well-educated persons engaged in scholarly but non-mathematical lines of study mathematics has often been spoken of as though it were a certain system of propositions and relations without relation to personalities and human affairs. When pure mathematics divorced from applications is considered, the discoverer and the occasion of the discovery alike are usually treated as incidental if not accidental to the heart of the science. Conversely, most investigators into wider historical problems of human adjustment overlook mathematics as a facet of civilization. In a study of the Dark Ages, for example, the arts, metaphysics, feudalism, religious philosophy, the Crusades, and many other central themes are explored with rarely a casual remark about the state of arithmetical knowledge.

In order to correct these misconceptions as well as to attain the positive values of seeing mathematics in its human and historical perspectives it is here proposed that the teacher help students gain the understandings of the sort briefly outlined in this chapter.

UNDERSTANDINGS RELATED TO THE DEVELOPMENT OF MATHEMATICS IN RESPONSE TO THE NEEDS OF THE TIMES

Mathematics as a Direct Response to the Spirit and Needs of the Times

Much of mathematical progress has been a direct response to the immediate practical needs of the everyday life of the community. One who would attempt to engage in historical generalization would hope to show that the culture of each of the major historical periods may be thought of as manifesting itself in any one of a number of directions—through philosophy, literature, or art, for example. Mathematics also in each period helps to reflect the spirit of the age—in fact it has been

called a "mirror of civilization." [1] Mathematics has its own history which is not separate from but an essential phase of the history of civilization. To understand the nature of mathematical study at any given time one must learn something of the manner in which the people of that time lived. The story of mathematics reaches back to a period more remote than the earliest for which the formal historian is able to document his conclusions. It is doubtless difficult to be sure of the facts of prehistory, but it is also foolish to close one's eyes to imperative evidence that leads to the conclusion that knowledge of mathematics antedates historical records.

Every savage who erected a tent and fashioned artifacts for the chase surely possessed some rudiments of what might be called informal geometry—some idea of the vertical line and horizontal plane, and numerous concepts of form. Early indeed must man have also learned to count—to group objects of similar nature and to assign to them as grouped a quantitative numeral.[2] Indeed there is every reason to believe that the notion of number was one of the earliest concepts in human thought. In many cultural systems (though not in all) primitive man came to recognize that the number *seven* in seven fish is the same as in seven days.[3] To count the objects in a large collection may have been useless or impracticable: taking census of one's own tribe may have been unnecessary and of an attacking tribe impossible. But the distribution of food and shelter within the family circle, the division of the spoils of the hunt or of battle, called for some notion of counting. The rudest types of human society relied upon barter as a method for distribution of necessities, and barter involves the naming of numbers. In any type of culture involving the possession and exchange of property the development of some

[1] In the phrase of Lancelot Hogben, *Mathematics for the Million* (New York, W. W. Norton and Co., 1937), Ch. I.

[2] Students read with interest: David Eugene Smith, *Number Stories of Long Ago* (Boston, Ginn and Co., 1919).

[3] Cf. Alfred North Whitehead, *Science and the Modern World* (New York, The Macmillan Co., 1937), pp. 29–30.

ability even in the elementary operations of addition, subtraction, multiplication, and division, at least in simple cases involving small numbers, would seem to have been inevitable.

Any race whose architecture shows high designing skill, whose calendar corresponds faithfully to the apparent course of the sun, whose trade shows understanding of the principles of interest computation, may properly be regarded as having an appreciable measure of mathematical knowledge. And, as a matter of fact, some of the most ancient historical records reveal a high level of mathematical achievement. The "Ahmes Papyrus," which was copied from or founded upon a much earlier work, itself dates back to about 1700 B. C.[4] The problems in arithmetic and geometry treated in this and other ancient documents make evident that the Egyptians and Babylonians were well equipped to deal with mathematical questions involved in commerce, taxation, surveying, and architecture.

In ancient Egypt taxation was based in part on the area of arable land, and consequently the annual flood of the Nile compelled the development of mensuration. Much mathematics of a practical nature was needed for the construction and use of Nilometers to study these floods, for the planning of irrigation ditches to utilize them, and for the making of a calendar to predict them. Herodotus called geometry "the gift of the Nile." In Babylonia, too, practical geometry was necessary for irrigation projects, and a tablet in the Museum of the University of Pennsylvania shows a map of a small area that may be cited as an early scale drawing.

The theology of both Egyptians and Babylonians required the accurate orientation of temples, and in the art of palace and temple the study of conventionalized forms was given a substantial impetus. The need for mechanical manoeuvering and shopwork technique was answered in part by a practical study of the wheel and the lever.

[4] See Florian Cajori, *A History of Mathematics* (New York, The Macmillan Co., 1931), p. 9.

With the development of business, commercial mathematics reached a high point, and there are still extant instances of Babylonian promissory notes, mortgages, and letters of credit. Deferred payments and temporary slavery for debt were common, and there was some recognition of even the principles of compound interest. Both the Egyptians and the Babylonians had methods of recording large numbers and computing with them, and showed reasonable expertness with fractions. The abacus developed as a practical mechanical computing device.

Mathematics of a more theoretical nature was not lacking in either of these two civilizations. The mathematics of both countries included materials which would now be classed as algebraic, as well as theoretical geometry. And all of this development took place so early in point of time that we today are chronologically nearer to Euclid than he was to the Babylonians and Egyptians who devised a system of numerals and learned to use them in mathematics.[5]

It is not necessary here to continue with an elaboration of the way mathematics has responded to the needs of the times and reflected the spirit of successive ages. For this background the teacher has a number of books, written from various points of view, at his disposal.[6]

The Insufficiency of the Needs and Spirit of the Times as an Explanation of All Mathematical Development

Although the history of mathematics is indeed a part of the history of civilization, reflecting at each period the spirit of the age and contributing directly to the felt needs of the people, the spirit of the successive ages does not fully account for the checkered development of this field. New ideas have sprung up in unexpected quarters, and others have been dis-

[5] See Raymond Clare Archibald, "Babylonian Mathematics with Special Reference to Recent Discoveries," *The Mathematics Teacher*, Vol. 29, May, 1936, pp. 209–219.

[6] See Bibliography at the close of this chapter.

carded and forgotten for no assignable reason. One must exercise caution lest *post hoc* arguments establish spurious connections between any particular development in mathematics and the social, economic, or cultural conditions surrounding it. Thus in commenting on the history of the metric system Sarton [7] makes the following statement:

> It is easy enough to explain some facts retrospectively, especially if one be free to select the convenient facts and to abandon the inconvenient ones. Why did the most industrial and mercantile nation of Europe reject the metric system, while its use would have caused great economies in time and money? Suppose the situation had been reversed, how tempting it would have been to explain the creation of the metric system as a necessary result of the superior mercantilism of England.

Actually, of course, mercantile countries were loath to accept the metric system even after it was developed and established in France. It was imposed upon the territory which Napoleon conquered but discarded in favor of the old measures on his fall, and its subsequent adoption in Europe outside of France was slow, and is still not complete.

The development of the metric system also reveals the effects of political events and of chance. The making of this system is attributed to two main factors: scientists desired a set of units that would be universally understood and that could be re-created if the original standard units were destroyed, and economists wanted a single system to replace the great number of local measures then in use in order to facilitate the movement of goods, especially food-stuffs, in France. Both groups had been discussing the problem for over a century before the adoption of the metric system. The scientists had been interested in the possible use of the seconds pendulum as a unit of length and had undertaken considerable work to study the variations in its length at different localities on the earth's surface. The economists had been told more than

[7] George Sarton, *The Study of the History of Mathematics* (Cambridge, Mass., Harvard University Press, 1936), p. 15.

once that their project was too expensive for the Kingdom of France to undertake. The scientists, through the Académie, succeeded in bringing the matter to the attention of Louis XVI and later (1789) got funds from the Estates General to begin the work. The progress of the task was hindered by the necessity for getting permissions from the various revolutionary governments as they came into power. Meanwhile, the idea of using the length of the seconds pendulum had been discarded in favor of making a great survey using an instrument recently invented by a member of the Académie, and basing the unit of length on the length of a degree of a meridian. Ironically enough, the system urged as a weapon of democracy was finally adopted as the "Napoleonic measures." [8]

But chance and sporadic incidents do not account for all inexplicable aspects of the history of the development of mathematics. What led the Greeks to give up a passable numeral notation for a bad one? Why were the place system and the zero of the Hindu-Arabic notation so late of invention and so slow of adoption? Why did not the Greeks with their brilliant originality ever develop an algebraic notation? Why did not great algebraists prior to Newton discover the binomial theorem?

Nor should one, in enthusiasm for cultural interpretation, discount the mere zest for the subject which led many a mathematician to give free rein to his imagination in the search for interesting problems. The mathematicians of European courts of the eighteenth and nineteenth centuries delighted in enhancing their own prestige by proposing difficult problems to the world. Since the Renaissance competitive examinations crowned by substantial awards have in many cases led to results of theoretical importance, although topics proposed have frequently had no relationship to the practical needs of the times. In many a fertile period of mathematical thought there

[8] Cf. G. Bigourdan, *Le Système Mètrique des Poids et Mésures* (Paris, Gauthier-Villars, 1901).

Adrien Favre, *Les Origines du Système Métrique* (Paris, Les Presses Universitaires de France, 1931).

seem to have been some who were out of step, who thought along their own individual lines, sometimes with startling success and sometimes with a fumbling that left to greater minds the honors of acknowledged mastery.

And no simple deterministic formula seems to account for the sporadic nature of genius. It is possible only to guess at what led to the discovery of the Pythagorean theorem. One cannot tell why Diophantus, steeped presumably in a geometric tradition, was able to make his brilliant contributions in numerical analysis. How did Gauss, at the age of nineteen, chance upon his theory of constructible polygons? How did the youthful Galois happen to conceive his great contributions to algebraic theory?

UNDERSTANDINGS RELATED TO THE DEVELOPMENT OF MATHEMATICS AS A SCIENCE

As a result of mathematical study the student should gradually acquire certain general understandings concerning the development of mathematics as a science. These generalizations are historical in nature, and are consequently subject to exceptions and qualifications. Nevertheless they serve to throw light on aspects of mathematics which all too often are never made clear to students.

The Cumulative Character of Mathematical Development

The development of mathematics has been cumulative in character. Mathematical science has experienced a remarkable cumulative development throughout history and is advancing with unparalleled rapidity today. Despite the fact that the Egyptians used mathematics to construct the temples and the early Chaldeans carried out astronomical computations, mathematics was not a perfected instrument at the dawn of civilization. The peoples of antiquity carried out even elementary computations by cumbersome and often in-

exact methods; their ideas were crude and their techniques cramped.

Research largely incorporates and extends previous results. Arithmetic, geometry, algebra, and analytic geometry, calculus, higher analysis, and projective geometry arose in approximately the order given, and the more advanced results of later periods have usually been based upon the reliable elementary results of earlier periods. Though shortcomings and errors in the work of investigators of the past are sometimes discovered and corrected, to a remarkable degree progress in mathematics has been in a steadily forward direction with but few far-reaching rejections of what once seemed valid.

In many instances mathematical ideas seem to arise independently and often almost simultaneously. The long drawn controversy as to the prior claims of Newton and Leibniz as discoverers of the calculus is but a famous instance of what is today almost a commonplace. Men's minds, at least at a given historical period, tend to follow startlingly similar patterns. Analogous backgrounds suggest similar philosophical views, furnish similar incentives to attack certain particular types of problems, and even lead to the use of similar methods. The way in which non-Euclidean geometry was developed by different authors, the discovery of logarithms in Switzerland by Bürgi and in Scotland by Napier, are but outstanding instances of what in less famous achievement has been almost the rule. It is less surprising that such a device as the planimeter has been "invented" many times. Today only the rapid distribution of published results prevents extensive duplication of essentially identical work, and even so the same result is often republished by writers not adequately conversant with results previously obtained.[9] Among amateur mathematicians who do not have access to mathematical literature or who cannot or will not read what may be available upon problems in

[9] For evidence on this point, see Leonard Eugene Dickson, *History of the Theory of Numbers* (Washington, Carnegie Institution of Washington, 1919–1923), 3 vols.

which they are interested, the almost daily rediscovery of what was known even to the Greeks is a continuous source of amazement to more critical and disillusioned professionals.

Cumulative progress in mathematics has been facilitated by a changing attitude toward the dissemination of knowledge. In the sixteenth century and later, some mathematicians felt a necessity for concealing the knowledge by which they earned their livelihood. A striking example is the Cardan-Tartaglia controversy over the cubic equation. This stands in contrast with Stevin's detailed explanation of decimal fractions and Robert Norton's offer to travel about England at his own expense to teach all who wished to learn the new device. In extenuation of the attitude of Cardan and Tartaglia, it should be noted that the prestige of each rested largely on ability to worst all comers at mathematical contests while Stevin was sharing a discovery which helped people to do better the thing they were doing already. Galileo's use of an anagram in announcing a discovery and also the later use of the mathematical contest (Pascal's challenge in regard to the cycloid, for instance, or the elder Johann Bernoulli's in regard to the curve of quickest descent),[10] illustrate the concealment of knowledge while establishing a basis for claims as to priority of discovery.

As a means of spreading information in regard to discoveries, correspondence between people of like interest was important, as for example the exchange of letters between Fermat and Pascal in 1654 which laid the foundations for the study of probability. The discussions at informal meetings of scholars, such as the conferences of Father Mersenne's at which Pascal presented his "Essai pour les Coniques," and the formation of more formal groups, such as the Royal Society in England, were the precursors of modern scientific meetings.

As in the physical sciences, many mathematical subjects

[10] Besides the general histories of mathematics, the student should consult David Eugene Smith, editor, *A Source Book in Mathematics* (New York, McGraw-Hill Book Co., 1929).

once thought of as unrelated have later been found to be closely connected: the conic sections and the laws of motion,[11] logarithms and exponents,[12] the logarithmic function and the area under the rectangular hyperbola,[13] the trigonometric functions and the exponential,[14] Fibonacci series, phyllotaxis and the golden section,[15] the regular polygons and the complex roots of unity,[16] the concept of rate and infinite series,[17] compound interest and the shapes of horns,[18] postulates and the rules of arithmetic,[19] Desargues' theorem on perspective triangles and the existence of projective three-dimensional space,[20] the solution of algebraic equations and the theory of groups,[21] algebra and geometric curves,[22] rational approximations to the values of a quadratic surd and Euclid's algorithm for the highest common factor,[23] Euclid's algorithm and the possibility of unique decomposition into prime factors,[24] rational fractions and the musical scale,[25] Euclidean constructions and the extraction of square roots,[26] space-time and non-Euclidean geometry.[27] These are but a few of the numerous unexpected relations which have enriched mathematical theory.

Mathematical solutions once considered purely abstract

[11] Kepler and Newton.

[12] Unsuspected by Napier. Due to Jacques Bernoulli and Leibniz.

[13] Lord Brouncker and Leibniz.

[14] Roger Cotes and Euler.

[15] Kepler, Tait. See for example: D'Arcy Wentworth Thompson, *On Growth and Form* (Cambridge, Eng., University Press, 1917), pp. 643 ff.

[16] Gauss.

[17] Leibniz.

[18] Jacques Bernoulli. Also see Thompson, *op. cit.*, Chapter XI, "The Logarithmic Spiral," and Theodore Andrea Cook, *The Curves of Life* (New York, Henry Holt and Co., Ch. XII, "The Spirals of Horns."

[19] Peano and Hilbert.

[20] David Hilbert, *The Foundations of Geometry* (LaSalle, Ill., Open Court Publishing Co., 3rd edition, 1938), pp. 95–99.

[21] Galois.

[22] Descartes.

[23] Euler (by his theory of continued fractions).

[24] Dirichlet, Kummer and Dedekind, in their theory of ideals.

[25] Pythagoras.

[26] Gauss.

[27] Minkowski and Einstein.

have later found practical application. Imaginary numbers were characterized by Cardan as being "as subtle as they are useless," yet no physicist questions their usefulness today. Kepler found in the motions of celestial bodies the conic sections discovered by the Greeks. An abstract algebra became the physicist's tool. The trigonometric ratios, originally developed for astronomy, came to be used to study sound, light, and electricity, as well as the flow of heat. The logarithm, introduced as a labor-saving device in connection with multiplication, reappeared in the study of the spirals in nature. The number π, studied by the ancients, proved of value in the normal law of probability used in the kinetic theory of gases and in studying biological variations. The odds in betting became the cornerstone of actuarial science. And a theorist like Einstein gives new impetus to a mathematical study the physical applications of which could not have been dreamed of at the time of original discovery.[28]

Certain mathematical problems which were probably originated as a sort of recreation later assumed great practical importance. One such problem concerned a man who started out with a certain sum of money, visiting three cities in turn; in each city he doubled his money in a gambling game and in each he spent ten ducats; the ten ducats spent in the last city were the last of his funds; how much did he have at the start? If the three cities are replaced by twenty years, the doubling is changed to increasing by 6 per cent, and the deduction to $1000 a year, the problem becomes that of finding the present value of an annuity of $1000 a year for twenty years, money earning 6 per cent.

An outstanding illustration of the way in which a special problem may lead to results of wide significance is found in the work of Pascal and Fermat on the proper division of a gambling stake if a game is interrupted before its close. From this came the subject of probability with its many applica-

[28] Cf. E. T. Bell, *The Queen of the Sciences* (Baltimore, The Williams and Wilkins Co., 1931), pp. 121–123.

tions. One may mention in particular Pascal's Triangle and the Binomial Theorem. The Pascal Triangle, a device used in different times with varying applications, is the arrangement of numbers given in the accompanying diagram. Each number is equal to the sum of the two numbers in the line above that lie directly above the number and in the place to the left of it. This arrangement was known long before Pascal, and it was rediscovered a number of times afterward. It was printed by Apianus in 1527 and used at that period in finding the coefficients of the terms of a binomial raised to a power indicated by a positive integer. This diagram was important in solving equations of degree higher than the third, and these equations were essential in making tables of trigonometric functions for use by astronomers in preparing tables for navigators.

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Newton was certainly familiar with the numbers as they appeared in the table, and his work in trying to find the area under the curve $y = (1 - x^2)^{\frac{1}{2}}$ by interpolation led to the important step of suspecting that the series might not terminate. By interpolation, he derived a general expression for the terms of the expansion of $(P + PQ)^{m/n}$. The proof of the binomial theorem for general exponents was later given by Abel about 1825. Its application to compound interest was accomplished by Newton's contemporaries and its extension to determining the value of e was accomplished by his immediate successors. The application of the theorem to the normal distribution curve followed on the work of DeMoivre in 1734. Thus the idea was used in the construction of trigonometric tables, in the study of probability, in finding the area under a curve, and in the countless applications of the normal distribution curve.[29]

[29] As an extension of this old-time diagram, it is interesting to extend the table upward as was done by Lucas. To do this, one must assume that the

Some Non-Cumulative Aspects of Mathematical Development

In the course of the development of mathematics many topics once highly important have gradually been discarded because of subsequent mathematical discoveries. Various rules based on proportion (the Rule of Three, the Double Rule of Three, the Reverse or Backer Rule of Three) were discarded after algebraic equations were introduced. Dating also from the time when algebraic symbols were either non-existent or little understood are the since discarded Rule of False and Double Rule of False. On the other hand the "three famous problems" of Greek mathematics—to trisect an angle, to double a cube, and to square a circle—although no longer "problems" since they have been proved impossible of solution,[80] are still studied because of their historical interest.

Other topics are no longer given serious consideration because the problems they were used to solve no longer exist. The problem of determining the length of time between successive conjunctions of two planets if their times of one com-

vacant spaces are filled by zeros. The first part of the resulting diagram is as follows:

```
1 -4 +10 -20 +35 -56 ....
1 -3 + 6 -10 +15 -21 .... The rows then give the coefficients of the terms
1 -2 + 3 - 4 + 5 - 6 .... of a binomial when the exponent is a negative
1 -1 + 1 - 1 + 1 - 1 .... integer, and interesting properties of the num-
1  0   0   0   0   0 .... bers in the different columns may be easily de-
1  1   0   0   0   0.... duced.
1  2   1   0   0   0....
```

[80] It is unfortunate that the details of the reasoning that convinces a mathematician that these problems are impossible of solution is beyond the comprehension of students in secondary schools. Should the question come up (as it perennially does), it might be well to refer the students to accounts of the work done on these problems as given in one of the histories of mathematics in order that they may see the importance of the problems in their influence on the study of higher plane curves. They may also learn that in the majority of cases what has been tried repeatedly in a given way and has always failed will fail again unless an entirely new approach can be found, and that this remark is strikingly true of mathematical problems. See in particular F. C. Boon, *A Companion to Elementary School Mathematics* (London, Longmans, Green and Co., 1924). For a thorough but more advanced discussion consult Felix Klein, *Famous Problems of Elementary Geometry,* translated by W. W. Beman and D. E. Smith (Boston, Ginn and Co., 1897).

plete revolution around the sun (or earth) were known—once important from an astrological point of view—has become a mere puzzle. Astrology is now without credit as a science, but the invention of clocks with two hands prompted the extension of this once real problem to the puzzle of finding the length of time elapsing between a given position of the hands of the clock and the moment at which they will be together.

Certain complicated business transactions have passed out of use, and the mathematical problems associated with these are no longer taught. For example, in sixteenth-century Italy it was customary to employ a shepherd on agreement that after the expiration of a given term of years the flock was to be divided equally between shepherd and owner. If such a contract were terminated before the allotted time, the proper division of the sheep had to be determined mathematically by a complicated but now obsolete method.

In reflecting the spirit of the times, mathematics follows fashions. While in a sense respectable mathematics of any period retains the respect of succeeding mathematicians, yet it does not necessarily awaken their interest. Certain fields lie long neglected because they subsequently seem to offer little promise of yielding new and important results. A newly found powerful concept not infrequently finds many young workers eager to test its efficacy, but after a field has once been largely conquered, its outlying problems are often neglected for some more promising line of research. This is perhaps attributable to the fact that students under a great master do not seek in their youth to compete with their teacher on a ground which he has thoroughly plowed. Illustrations may be found throughout the centuries. Formal manipulation with numerically explicit series, which proved so fertile in the hands of Euler and Jacobi, is no longer studied so intensively. Invariant theory, which in the middle of the last century largely absorbed the activities of numerous mathematicians, has lost much of its charm of freshness. The Golden Age of synthetic geometry may be past. Divergent series suffered a long eclipse only to return

to fresh conquests. Diophantine analysis has had a checkered career as far as popularity is concerned. The theory of equations, which once was practically the whole of algebra, is usually slighted in modern algebraic studies.

Light Thrown on the Nature of Mathematics by Certain Aspects of Its Development

Mathematical truth has validity independent of place, personality, or human authority. Mathematical relations are not established, nor can they be abrogated, by edict. The multiplication table is international and permanent, not a matter of convention nor of relying upon authority of state or church. The value of π is not amenable to human caprice. The finding of a mathematical theorem may have been a highly romantic episode in the personal life of the discoverer, but it cannot be expected of itself to reveal the race, sex, or temperament of this discoverer. With modern means of widespread communication even mathematical notation tends to be international despite all nationalistic tendencies in the use of words or of type.

The desire to generalize, so to state propositions as to avoid exceptions and to include many previous results in a single blanket formulation, to abstract the essential while dropping accidental limitations, characterizes much of mathematical discovery. Mathematical concepts are formalized generalizations or limiting forms often suggested by but usually eventually divorced from practical experience. This fact is particularly evident in geometry. The points, lines, planes of Euclidean geometry do not exist as material objects although they are suggested by elements of familiar experience. Thus the point of a thorn, a star, a mote in a sunbeam, all suggest a figure of no length, breadth, or thickness. The straight line is suggested by a stretched string, or the edge of a beam of light, or of an edge of some types of knife-blade. A folded leaf, the path of a dropped pebble, the horizon, the homeward coursing of the pollen-laden bee are other familiar suggestions of a straight

line. The surface of a pond, the surfaces of stones rubbed long together, even the palm of the hand remind one of plane surfaces. For circular arcs one may turn to the ripples in a pond, the periphery of sun or moon, the right section of a tree trunk, or the path of a swinging object. Raindrops, bubbles, berries, and some larger fruit, as well as many nuts, suggest spheres, and even the human cranium may be nearly spherical. Tree trunks, the central portions of the long bones, the fingers and even the earth worm remind one of cylinders. In geometry these figures have all been idealized, and it is hardly necessary to explain in detail that the "figures" of a geometry text and the graphs of algebraic curves are merely diagrams of more theoretical relations, and that the concepts of higher analysis based upon the notion of limit are mathematical abstractions, in many cases only fictions, insofar as material objects are concerned.

The pressure to generalize tends to modify definitions, assumptions, and symbolism, as well as the formulation of results. This trend may be seen in all branches of mathematics from the successive extensions of the number system to the modern tensor calculus. Projective geometry and the powerful methods of the calculus all bear witness to the same tendency. Numbers, for instance, are not objects but are logical concepts. Whole numbers arose from the practice of counting finite collections of individual objects—usually material, but not necessarily so. The number system was then successively extended to include new types of concepts on equal footing with those already sanctioned by tradition. Zero, negative numbers, fractions, irrational numbers, imaginary numbers, all became "numbers."

As a consequence of this tendency, not a few thinkers have found convenient the scheme of classifying sciences in a series with mathematics at one end, as the most general and least dependent upon the particulars of the case, and with physics, chemistry, astronomy, botany, zoölogy, geology, history, economics, psychology, political science and sociology following

in some such order as here listed, each being regarded as less completely "mathematicized" than its predecessor, and each less affected by conduct and emotions than its successor.[31]

APPROACHES TO THE DEVELOPMENT OF THESE UNDERSTANDINGS

Some Connections in Which Historical Materials May Be Introduced

It is probable that the materials for the study of the evolution of mathematics are best introduced incidentally to other work, although this has serious dangers. The presentation may become a series of scattered incidents. The teacher must assume the task of gathering the threads together and of showing relationships among the different parts.

Occasionally students ask such questions as the following: "Did any one ever measure the height of a tree the way the textbook says, and what instrument did he use?" "Did the Romans really *use* Roman numerals?" "Who invented logarithms and how did he come to do it?" "How does it happen that we have two ways of indicating square root?" Such curiosity, encouraged, stimulated, and satisfied, brings an appreciation of mathematics as the product of long development, a process that may reasonably be expected to be still going on. The following paragraphs are intended to indicate how his-

[31] See Auguste Comte's classification of the sciences in his *Positive Philosophy* 1828 to 1842. Similarly, "The Tree of Knowledge: Basic and Applied Sciences," poster copy of mural painted by John Norton for the Century of Progress Exposition, Chicago, obtainable from Museum of Science and Industry, Jackson Park, Chicago. As another instance one may note the 17-part classification adopted by the International Catalogue of Scientific Literature (1901–1914). A. Mathematics, B. Mechanics, C. Physics, D. Chemistry, E. Astronomy, F. Meteorology (including Terrestrial Magnetism), G. Mineralogy (including Petrology and Crystallography), H. Geology, J. Geography (mathematical and physical), K. Paleontology, L. General Biology, M. Botany, N. Zoology, O. Human Anatomy, P. Physical Anthropology, Q. Physiology (including Experimental Psychology, Pharmacology, and Experimental Pathology), R. Bacteriology. This list is not designed to cover social sciences.

torical material can be introduced in connection with various topics of interest to students.

Standardized weights and measures. Certain understandings related to standardized weights and measures may be developed in a junior high-school class. The students may try to imagine how a free exchange of goods can take place without a definite set of units of measure. They may suggest the use of natural units, such as the length of the thumb joint, the forearm, the arm, the total span of both arms, the foot, or the pace. From this point on they may be helped to see that these might serve an individual, but they require standardization if they are to be used by many people who may be widely separated. Standardized measures are generally characteristic of efficient governments whereas confusion and multiplicity of standards tend to accompany disorder and chaos in government. Standardized measures are precious and require safe keeping. Thus the Romans placed their standard measures in the Temple of Castor and Pollux. Standardized weights require constant checking. In imperial Rome this task was assigned to the aediles; in the United States today it is assigned to the Bureau of Standards. Study of the reduction in the number of units of weight and measure in the past century by comparing the measures in use today with those listed in an old arithmetic book—perhaps one used by a grandparent or great grandparent—may prove interesting to the students. For example, in retailing in urban communities the peck and the bushel have been largely replaced by the pound.

Surveying instruments. A fruitful topic for study in connection with junior high-school mathematics is the subject of old-time surveying instruments. The A-shaped level used by the Egyptians and others, the carpenter's square, the hypsometer, the cross-staff, the quadrant and octant, and the plane-table may well be studied. The astrolabe is of particular interest although much of its theory is beyond the students at this level. However, as an instrument for measuring heights by

indirect methods, it is well within their ability to use. The construction and use of such instruments is a worth-while manual activity and motivates the study of indirect measurement.

The symbols of algebra. The study of algebra may well be the occasion for a survey of the development of its symbolism. No more fruitful exercise can be devised for this than to compare the symbols for the same idea as they appear in the works of different authors, studying the gradual evolution of the notation which is now in use.[32] The fact that the + and − signs first appeared in print in Widmann's work of 1489, that the sign = is due to Robert Recorde in 1557, that the sign x is probably due to Oughtred, and that ÷ is due to Rahn and to the publicity given his work by Pell—all these lend credence to the statement that our symbols today are a "mosaic of individual signs of rejected systems." [33]

The study of exponents presents interesting possibilities, for the extension of the idea to negative integers, fractions, zero, and transcendental numbers gives meaning to the statement that certain symbols in themselves seem to stimulate mathematical discovery. To contrast the exponential notation with the clumsy ways of representing roots by the use of radical signs is again a valuable activity. It is probably safe to conclude that symbols which are easy to print, and which lend themselves to the extension of concepts, have survived in the long run. That the development of symbols is not complete even for elementary mathematics is witnessed by the fact that in some cases we still have several symbols for the same thing (as ÷, :, /, and — for division) and certain symbols have different meanings according to the context. Even so simple a thing as the notation for decimal fractions is not yet uniform among different countries.

The applications of arithmetic. The teaching of the modern

[32] As was done by Hogben in the quotation given earlier in Chapter X, on "Symbolism."

[33] Florian Cajori, *A History of Mathematical Notations* (Chicago, Open Court Publishing Co., 1929), Vol. II, p. 337.

applications of arithmetic often gains in clarity from study of the similar applications in former times. Study of the use of credit instruments, not only within a single country but in international exchange, is made more comprehensible by citing certain of the conditions that have fostered it. For example, the need of paying Peter's Pence and other church dues collected in England to the authorities in Rome and the need of Florentine manufacturers to pay English sheep raisers or English wool merchants for the wool they purchased led the Lombard bankers who handled both transactions to transfer the credit rather than the actual coin. It is important also to trace the changing attitude toward the charging of interest on loans and the fall of the rate actually charged when the process was made legal. This has an important analogy in the way in which the legalizing of a relatively high rate of interest on small loans made on little or no security tends to prevent the extortionate charges formerly made.

Adaptation to the Maturity of Students

The type of work conducted on any particular topic properly depends in large measure on the maturity of the class which is considering it. For example, the subject of Roman numerals is often treated in the elementary schools in connection with the study of "life in far away times," or in similar units. Pupils not only learn to read and write these numerals, but forms depending on place value, such as IV and VI may be compared with the Hindu-Arabic idea of place value. If pupils inquire how people used Roman numerals in computation, they may be led to investigate the use of loose-counter abacus and to compare this method with that in use today.[34]

If the subject is to be treated in the secondary school, it

[34] See David Eugene Smith, *Number Stories of Long Ago* (Boston, Ginn and Co., 1919); David Eugene Smith and Jekuthiel Ginsburg, *Numbers and Numerals* (New York, Bureau of Publications, Teachers College, Columbia University, 1937); Bertha M. Parker, *The Story of Numbers*, Achievements of Civilization, No. 2 (Washington, D.C., The American Council on Education, 1932).

would be well to point out, in addition, the long period in which Roman numerals were in active use, the variations in their forms during that time, and the reluctance with which they were abandoned in favor of Hindu-Arabic numerals. For instance, the notation IIII, which still appears on some watch faces, was the accepted form for four in Cicero's time and the form IV did not develop until much later. During the transition period from the introduction of Hindu-Arabic numerals to their widespread adoption, number notations assumed compromise forms. Senior high-school students may be led to inquire into and try to explain the length of time it has taken for subsequent extensions of the number system to be developed and widely adopted. Roman numerals were in use in the accounts of the government of France until the French Revolution. Over eight hundred years elapsed between the invention of the zero (and its accompanying place-value) and the extension of the numerals to include decimal fractions. Although the convenient method of writing very large or very small numbers in standard form makes it possible to show the degree of accuracy of any approximate number, this device has been relatively slow in gaining wide usage.

Adaptation to the Special Interests of Individual Students

Clearly the best approach to the history of mathematics lies in the special interests of individual students, and these interests themselves may be developed through the study of mathematics and its history. This is obvious in the case of the student interested in history and social studies. To the mechanically minded student the study of old-time measuring and computing instruments is alluring. It is interesting to him to note that the A-shaped level of the Egyptians is identical with the level of Roman surveyors. A similar instrument was used in running Mason and Dixon's Line. Good work can be done by junior high-school students with replicas of their own manufacture. The budding philologist enjoys tracing number names and their cognates in other languages, or in seeing the

many expressions in English that stem from some old practice such as counter reckoning.

Students of science may be interested in tracing the ways in which progress in the natural sciences has both fostered and been furthered by mathematical developments. The invention of the telescope, the thermometer, the barometer, the anchor escapement of a watch, and the study of the laws of the pendulum and of falling bodies, made possible the closer study of physical phenomena and the use of algebraic formulas in expressing the results of this study.

Other students may be interested in following out the relations of mathematics to philosophy. These relations have in some cases been intimate and in others remote. The Pythagoreans founded their views of the universe on the concept of number. Numerological speculations have been dominant in occult thought ever since. The magic square [35] known to many ancient people still arouses more vivid interest than many other more important topics of number theory. The philosophical views of Descartes, or Kant's ideas about space and time, remind one that mathematics and philosophy may go hand in hand. In the field of symbolic logic, the work of Leibniz, Boole, and Schröder converted much of the formal logical heritage from Aristotle into something similar to elementary algebra.[36]

Students interested in personalities find value in learning about mathematicians as individuals. Mathematics as a study is not to be justified by any appeal to a long list of distinguished investigators in the field, yet for those who may be blind to the deeper lessons of mathematical history, there is likely to be much of lively interest in the biographical aspects of mathematical development. In these days when one hears so often the accusation that mathematics is dull and without human interest, the teacher can ill afford to neglect the historical side-

[35] Appearing, for example, in an etching, "Melancholia," by Dürer.

[36] See C. I. Lewis, *A Survey of Symbolic Logic* (Berkeley, University of California Press, 1918), pp. 3–4.

lights that may serve to illumine the whole subject for some students. In the long story of mathematical progress through the ages one may encounter many traditional anecdotal accounts reminiscent of the nobility and frailty of mathematical enthusiasts. Pythagoras sacrificing a hecatomb to the Muses who inspired him, in his great joy over his discovery of a proof for the theorem named in his honor; the story of Archimedes and his cry of "Eureka"; the rope-stretchers of Egypt, and the horoscope casters of France; Cardan and his theft from Tartaglia, and the resulting fifteen-day, thirty-one problem challenge; Henry IV of France and the forty-fifth degree equation; Napoleon and his compass construction; Pascal, Fermat and the gambler; Galileo and the swinging lamp; Euler and his reply to Diderot; the quarrel between Newton and Leibniz; Laplace and his political ambitions; John Milton's abuse of Vlacq; Poncelet's studies in a Russian prison—traditional items such as these, even if sometimes of somewhat dubious authenticity and of course trivial in themselves, may contribute to making the story of mathematics live in the imaginations of some students. Certainly the teacher will find interest and possible profit in reading the accounts of the lives of mathematicians.

Whoever attempts to generalize about mathematicians as a separate group of individuals faces, of course, the usual difficulties of all biographical study. The great names of antiquity, except in military and political fields, have been preserved often with practically no reliable biographical information. In a true sense they live only in their works. Such information as is available from ancient as well as more recent and better documented times seems to show, however, that no circumstances of family means or training, no selective characteristic of emotional attitude or social status, distinguishes great mathematicians from persons in other lines of activity.

Bibliography

Archibald, Raymond Clare, *Outline of the History of Mathematics* (Oberlin, O., Mathematical Association of America, Inc., 4th edition, 1939).

Ball, W. W. Rouse, *A Short Account of the History of Mathematics* (New York, The Macmillan Co., 5th edition, 1912).

Bell, E. T., *Men of Mathematics* (New York, Simon and Schuster, 1937).

Bowditch, Charles P., editor, "Mexican and Central American Antiquities, Calendar Systems, and History," *Bulletin of the Bureau of American Ethnology,* United States Smithsonian Institution (Washington, Government Printing Office, 1904), No. 28.

Cajori, Florian, *A History of Mathematical Notations* (Chicago, Open Court Publishing Co., 1928–1929), 2 Vols.

———, *A History of Mathematics* (New York, The Macmillan Co., 2nd edition revised and enlarged, 1919).

Curtis, D. R., "Fashions in Mathematics," *The American Mathematical Monthly,* Vol. 44, November, 1937, pp. 559–566.

Damper, William, *A History of Science and Its Relations with Philosophy and Religion* (New York, The Macmillan Co., 1930).

Dantzig, Tobias, *Number, the Language of Science* (New York, The Macmillan Co., 1939).

Davis, Harold T., *Philosophy and Modern Science* (Bloomington, Ind., The Principia Press, 1931).

De Villiers, Melius, *The Numeral-Words, Their Origin, Meaning, History and Lesson* (London, H. F., and G. Witherby, 1923).

Enriques, Federigo, *Problems of Science* (Chicago, The Open Court Publishing Co., 1914). Authorized translation by Katherine Boyce.

Heath, Thomas L., *A Manual of Greek Mathematics* (New York, Oxford University Press, 1931).

Hogben, Lancelot, *Mathematics for the Million* (New York, W. W. Norton and Co., 1938).

———, *Science for the Citizen* (New York, Alfred A. Knopf, 1938).

Hotelling, Harold, "Some Little Known Applications of Mathematics," *The Mathematics Teacher,* Vol. 29, April, 1936, pp. 157–169.

Marvin, F. S., editor, *Science and Civilization* (New York, Oxford University Press, 1923).

Sanford, Vera, *A Short History of Mathematics* (Boston, Houghton Mifflin Co., 1930).

Sarton, George, *Introduction to the History of Science* (Baltimore, Williams and Wilkins Co., Published for the Carnegie Institution of Washington, Vol. I, 1927; Vol. II, 1931).

———, *The History of Science and the New Humanism* (Cambridge, Mass., Harvard University Press, 1937).

———, *The Study of the History of Mathematics* (Cambridge, Mass., Harvard University Press, 1936).

Science at the Cross Roads, Papers presented to the International Congress of the History of Science and Technology held in London from June 29th to July 3rd, 1931, by the delegates of the U.S.S.R. (London, Kniga, Ltd., 1931), "The Social and Economic Roots of Newton's 'Principia,' " B. Hessen.

SMITH, David Eugene, *History of Mathematics* (Boston, Ginn and Co., 1923–1925), 2 Vols.

———, editor, *A Source Book in Mathematics* (New York, McGraw-Hill Book Co., 1929).

SPINDEN, Herbert J., "Maya Dates and What They Reveal," *Science Bulletin* of the Museum of the Brooklyn Institute of Arts and Sciences, Vol. 4, September, 1930.

SULLIVAN, J. W. N., *The History of Mathematics in Europe* (New York, Oxford University Press, 1925).

THORNDIKE, Lynn, *A History of Magic and Experimental Science,* Vol. I–II: *During the First Thirteen Centuries of Our Era* (New York, The Macmillan Co., 1923); and Vol. III–IV: *Fourteenth and Fifteenth Centuries* (New York, Columbia University Press, 1934).

WELLS, H. G., *The New and Revised Outline of History* (Garden City, N. Y., Garden City Publishing Co., Inc., 1931).

WHITEHEAD, Alfred North, *Science and the Modern World* (New York, The Macmillan Co., 1925). See particularly Ch. II, "Mathematics as an Element in the History of Thought."

WOLF, A., and others, *A History of Science, Technology and Philosophy in the 16th and 17th Centuries* (New York, The Macmillan Co., 1935).

Part IV

UNDERSTANDING THE STUDENT AND EVALUATING HIS GROWTH

XII

UNDERSTANDING THE STUDENT[1]

In concerning ourselves here with the nature of adolescent personality and behavior we may seem to be entering the province of the guidance specialist and leading the subject teacher far afield from his primary concerns. But the basic educational philosophy behind this Report and the larger study of which it is a part justifies this course. The entire attempt to reëvaluate secondary-school practice rests upon the conviction that education must concern itself primarily with the highest possible development of individual personality—that subject-matter and skills are merely means to that end. For the good of the individual and the social group of which he is a part we must learn to think in terms of personality needs and the effects of our educational demands and practices upon them.

All experience affects personality and is in turn affected by it. What any child learns in a given classroom situation is an individual matter that can be understood only in terms of the experience and attitudes which he brings to it. No two children in the same classroom are having exactly the same experience and the total learning situation for the group is continuously affected by each one of them.

We cannot think of the individual and his environment as two separate entities. There is a continual interaction between them. The nature of the individual student and his past experiences will profoundly influence his reaction to the present learning situation and the reaction of teachers and students to him. This applies not only to personal relationships in the

[1] For the preparation of this enlightening chapter the Committee is indebted to Dr. Peter Blos, of the Study of Adolescents, Commission on Secondary School Curriculum, Progressive Education Association.

classroom and outside it but equally to the subject-matter itself—the meaning it will have for the individual, his ability to accept it, and the purpose it will serve in his total development.

If, then, a teacher is to have any adequate understanding of his teaching in terms of its effect upon the individual personality growth of his students, he will need to have a basic knowledge of the development and functioning of human personality in general, of the problems and peculiarities of the particular age group with which he is dealing, and of the nature of the academic subject he teaches in terms of its special meaning for various types of personalities.

Most of us are accustomed to think of the subject-matter we teach as an objective body of knowledge and skills and of any student's ability to learn it as largely determined by the level of his intellectual capacities. We tend to forget that his emotions are involved, too. He must be interested in order to learn. What this actually means is that he must find in the nature of the subject-matter—or in the personality of the teacher—some positive contribution to his individual emotional needs. Failing this he will be indifferent. Or he may even find the demands of a particular subject or teacher directly opposed to his own inner needs, in which case he may develop a negative attitude that will tend to preclude learning. The emotional significance for *him* is all important in determining his interest and motivating his work.

It may seem somewhat contradictory to speak of the emotional significance of an abstract subject like mathematics. But one has only to remember the peculiar importance that has been attributed to numbers and geometric forms throughout the history of the race. The magic or mystic quality of certain numbers is of primary importance in the spells and rituals of all primitive peoples and is still apparent in the superstitions of our own day. Children quite spontaneously exhibit a primitive superstitious attitude in this regard. Who has not seen

their compulsive counting of steps, of windows, or of cars in a passing train—or noted the importance they attach to avoiding stepping on the lines of the sidewalk? Examples might be multiplied—from art, from philosophy, from abnormal psychology—but these are undoubtedly sufficient to illustrate the possibilities of emotional charge in connection with subject-matter that is in itself so apparently non-emotional.

Mathematics is, of course, the de-emotionalized subject *par excellence.* The *content* itself is abstract and any emotional implication is vested in the *process*—or in the symbols of which it makes use. Here we have symbolization and abstraction carried to the extreme. It is precisely this abstract quality that gives the subject its special appeal for some types of personalities and makes it anathema to others. Manipulation of symbols may carry with it satisfaction of certain power drives, of certain needs for ordering that which is confused, of certain needs for building and creating, which in some people find outlet most comfortably in an abstract situation. But this very divorce from the concrete and personal and operation on a purely abstract level renders mathematics almost totally meaningless for another type of personality—one which seeks its major satisfactions in the manipulation of concrete materials or the feeling tone of human relations.

Perhaps this explains the general expectation that boys will show greater interest in mathematics and science than girls. Certainly, in our culture, absorption in the personal and concrete is usually considered typically feminine, concern with generalities and abstractions, typically masculine. Whatever the truth of this in regard to boys and girls in general, there is no doubt that there are many individual instances of highly developed scientific and mathematical ability in girls, as every teacher can testify. But in our society there is little opportunity for it to find professional outlet. A girl who manifests strong interest in these fields is at present generally looked upon as somewhat masculine and unnatural. This, in itself,

constitutes an emotional problem that must be taken into consideration in the teaching situation.

Quite aside from this question of emotional response to an abstract subject like mathematics, there is considerable variation—as any experienced teacher knows—in intellectual capacity to think on this level. The power to deal with abstractions is a highly developed one which many individuals never achieve—either because of low-grade intellectual endowment or because of bad teaching. There is too little emphasis, in teaching the beginnings of arithmetic and reading, on establishing a true understanding of the connection between symbols and the real facts and relationships which they represent. This results in a type of utter confusion for many children that seriously hampers their ability to progress. As Prescott [2] points out in *Emotion and the Educative Process,* "Symbolization should not precede or replace experience. To short-circuit the process by introducing symbols at too early an age, too rapidly at appropriate times, or so extensively as to crowd out concrete experience, is to confuse the child and to render much of education futile." In his subsequent discussion Prescott suggests that such confusion is very common, that it results in hostile emotional attitudes which further complicate the problem of learning and which may be behind much of the present agitation for elimination of the compulsory study of mathematical subjects from the secondary-school curriculum.

It is impossible, then, to think of academic learning as something distinct from emotional development. The two are inextricably interwoven. Effective teaching requires that the teacher shall be aware of the emotional meaning of his subject for the individual student and that he shall measure his results in the light of their total contribution to the student's personality development. It is the purpose of this chapter to help the teacher to evaluate his teaching in these terms.

[2] Daniel Alfred Prescott, for the Committee on the Relation of Emotion to the Educative Process, *Emotion and the Educative Process* (Washington, American Council on Education, 1938), pp. 218–219.

THE ADOLESCENT AS THE TEACHER OBSERVES HIM

The secondary-school teacher deals almost entirely with adolescent boys and girls. Often he finds them difficult, for adolescents on the whole do not fit comfortably into adult expectations and plans. It requires considerable insight to understand their attitudes and actions.

Just what is meant by *adolescence?* Roughly speaking, it is the period between childhood and adulthood, a period hard to define in terms of exact age or measurable growth but easy to describe in terms of characteristic behavior. Any experienced teacher could paint the picture, but it would not be a static one—a rapidly shifting pattern is typical of this period.

The early or pre-adolescent phase is discouraging to many teachers and parents because at this time the child actually seems to be losing ground. At about the age of nine or ten most children seem to be exceptionally well-adjusted, happy, compliant, reasonably responsible, obedient, and through with most of the unpleasant habits and problems of early childhood. And then, quite unexpectedly, at eleven or twelve this beautiful adjustment breaks down. The child becomes more restless and unstable, less responsible, less obedient, often openly hostile to the adults he loves best. His carefully trained habits of order and cleanliness are lost. He is careless in his personal appearance, his language, his work—even, if he is a boy, deliberately dirty and greedy. Girls don't seem to go so far in this direction and often skip the sloppy stage entirely. All pre-adolescents profess a strong preference for members of their own sex and either indifference or hostility to the opposite one—but this again is a more marked and prolonged reaction in boys than girls.

Nervous habits like nail-biting often reappear at this time. Strange superstitions and rites are common. Adult standards and appeals seem to have little claim for the pre-adolescent, but he is almost slavish in his dependence on the approval of his gang and highly secretive about its affairs. Mostly he is

withdrawn and self-protective against adult intrusion, though occasionally he breaks down into surprising bursts of confidence. He is amazingly babyish and irresponsible in many ways—yet he demands the freedom and privileges of an adult, aping adults as he understands them and seeming to prefer their vices to their virtues.

At the same time he becomes increasingly difficult to teach by the traditional methods. His attention is fluctuating and hard to enlist. His creative powers seem to wane. Much of his energy goes into day-dreaming, and anything serves to distract him. His interest in abstractions is at a low ebb although he enjoys a concrete type of manipulation and is intrigued by mechanical gadgets of all sorts. He is at war with time, never beginning anything soon enough, but utterly refusing adult help in planning as in almost everything else.

But presently—at about fourteen or fifteen, sometimes even later—the adolescent pattern shifts. The child is still an unstable, moody, and rather unpredictable person—at war with adult authority and with himself—overanxious, self-conscious, and overconfident by turns, apparently sophisticated but frequently only on the verbal level, desperately wanting help and guidance but often quite unable to ask for it or to accept it from those he loves best. Yet in many ways he is much more grown up than he was and much easier to teach. By this time his interest in the opposite sex is ordinarily open and frankly expressed—he is as meticulous about appearance and manners as he was careless a few years ago; he is clean and orderly again and considerably more responsible; his creative powers reach a high point (often, in fact, they reach their highest level at this age and the adolescent seems to give promise of a genius that never materializes); his intellectual interests increase both in scope and intensity, and he throws himself into work with new vigor and direction. He still has problems and difficulties in plenty—he still lapses into a life of introspection or pure fantasy at times, and uses the most childish patterns of problem-solving at others—bragging, swaggering, sulking, storming, ly-

ing, weeping, exaggerating his bodily ills, and the like. But much of the time he is making an attempt to face and meet his problems and those of the world on a more mature level. He is greedy for information about the ways of the world and about the intricacies of human relationships; he is consciously reëvaluating his standards and ideals.

This is a period of deep confidences and intense emotional friendships, often between two members of the same sex, of hero worship, idealism and devotion. Usually the adolescent at this stage seeks an adult to copy—some older man or woman on whom to model his character and behavior—usually some one outside his own family and frequently very unlike them in standards and ideas. Through all this he remains a very baffling person, swinging rapidly from independence to dependence and back, fearful one day and overconfident the next, moody, oversensitive, never quite sure what he wants and apparently wanting quite opposite and irreconcilable things.

Not every adolescent shows all of these traits, but the picture is typical enough to be generally recognized. Most teachers could add details from their own experience. But its usualness makes adolescent behavior none the less strange. How can it be explained?

THE ADOLESCENT AS THE TEACHER UNDERSTANDS HIM

We can best understand adolescence as one phase in a long period of growth. Change is typical of all growth. In adolescence this change is so rapid, and the adjustments which the organism is called upon to make are so profound, that stresses are inevitable.

The Strains and Conflicts of Adolescence

The physical development of puberty is often profoundly disturbing to the child. His own body is in many ways strange

and hard to manage, and he experiences new and threatening feelings and sensations. During the years of middle childhood the feeling of life seems to be largely dormant and the child's interest is directed away from his own body, where it centered in infancy, outward toward the exploration of the external world. With the physical changes of puberty his interest in his own body is renewed, the emotions reawaken, and he finds himself in the grip of feelings and drives which he only partially understands, which he cannot openly gratify, and about which he almost invariably feels guilty. Though the degree of this guilt will be profoundly influenced by the attitudes that surround him in his early training, it is certain to be present in some degree. In a culture such as ours—which gives no direct recognition and status to puberty and makes a pretense of strict taboo of all that is sexual while actually exploiting sex feeling quite openly on all sides—the resulting conflict is considerable.

The adolescent usually gets little help from his parents and teachers. In many cases their own insecure attitudes toward sex leave them quite unable to welcome his normal development and to supply reassurance and helpful guidance. They don't like to believe that he is reaching this phase. They think he is too young. They deny the whole problem or pretend that it can be solved by "plenty of exercise." Their unspoken fears and their denial of his needs merely add to his anxiety. Many an adolescent is tortured by the fear that his perfectly normal sexual feelings denote abnormality or degrading wickedness.

On the other hand, those whose physical or emotional development is slow may be equally disturbed. In any junior or senior high-school class we find a surprisingly wide range of physical, emotional, and intellectual maturity, and the unevenness of development within the same child is often startling. Within wide limits this is quite normal but to the child himself it may seem to constitute a serious threat. The girl whose figure develops slowly, the boy whose voice does not

change as soon as it might or whose genitals are somewhat small for his age, the child (of either sex) who lags behind his friends in developing an interest in members of the opposite sex, may feel inferior or even doubt his own normality. If most of his friends are interested in dancing and dates, he dare not admit that such activities have no appeal for him.

Whether he is attempting to accept and manage new urges of which he is somewhat ashamed and for which he has no comfortable outlet in our culture, or straining toward a form of behavior for which he is emotionally unready, the most normal adolescent is under serious stress. Moodiness and introspection are not hard to understand under the circumstances, nor touchiness, self-consciousness, and inconsistent extremes of behavior.

Any living creature subject to uncomfortable strain from within or without will seek, by whatever means possible, to relieve it. If the forces arrayed against it seem too strong to permit direct attack, retreat or flight is inevitable. Retreat in the inner emotional life takes the form of *regression,* or a return to methods of achieving relief and satisfaction that proved comforting in an earlier phase of development.

This, indeed, seems to be just what occurs in the life of the child when he first faces the conflicts and strains of adolescence. The breakdown of good habits, the return to dirt and disorder and greediness, the recurrence of neurotic fears and nervous habits, of restlessness and instability, the renewed interest in concrete manipulation as opposed to abstract intellectuality —these and other seemingly inexplicable manifestations of the pre-adolescent period are understandable enough if we see them as an attempt to relieve strain via the route of infantile satisfaction. Temper tantrums, day-dreams, bragging, swaggering, sulking and weeping, lying, cheating, exaggerating one's bodily ills—these are the defenses and escapes of the small child faced by powers too strong for him, and these are the defenses to which he returns when the strains of adolescence are more than he can face. As temporary reactions they are

usual and entirely to be expected at this time. Many types of behavior which would be considered dangerously neurotic in adults, and even in children of elementary-school age, must be viewed as normal during the adolescent period.

The shift in pattern which occurs as the child proceeds from the pre-adolescent stage to later adolescence is a shift from escape and regression to a more direct attack on problems and difficulties—a more conscious striving toward the goals of maturity. But as even the stoutest and most healthy minded weary of the fight at times and slip back into childish methods when the odds seem too great, it is not surprising to find this type of behavior recurring all through the adolescent period —and, in fact, present to some degree in the lives of all but the most mature adults.

The adolescent has three main goals before him: (1) the establishment of independence—emotional, and in most cases economic as well; (2) the achievement of a sound heterosexual adjustment; and (3) identification with the social groups and ideals to which he will give his adult allegiance and through which he will find much of the meaning of his life. Obviously these are not distinct but closely interrelated tasks, all of them closely bound up with the attainment of physical maturity and to a large degree dependent upon it—all of them inevitably involving conflict and strain.

It is obvious that the adolescent struggle for independence will be accompanied by tension or conflict no matter how favorable the past history or present setting of the child. The adolescent is torn between his desire for adult freedom and his desire for childish protection. He longs for the comfortable security of that very authority which he is so vehemently fighting. Yet he is equally impelled to grow up and experience the joys of adulthood—economic independence, the right to live by his own standards and ideals, and sexual freedom. These desires he is often ashamed to admit even to himself because of the implied disloyalty to his parents or because of a deep-seated fear of his own emotions.

In addition he senses the pain which he inevitably causes his parents, for they also are of two minds about having him grow up and cannot relinquish their authority and protectiveness without some struggle. If he loves them deeply, it is hard for him to fight them. If he has had an unhappy childhood and still feels resentful, he may actually enjoy hurting them, but he cannot admit this even to himself without deep distress. When his parents give in too easily to his demands for freedom, he is no happier, for he feels abandoned and unready to accept the full responsibility. He may even construe their leniency as lack of interest and lack of love.

Parents and teachers unwittingly increase the adolescent's uncertainty and strain. They are themselves quite inconsistent in their treatment of him—urging him at one moment to be grown up and to accept more responsibility—demanding of him, in the next breath, a type of submissive obedience that implies that he is still a child. As a matter of fact, he is neither a child nor an adult and he has no real status of his own. Our whole society treats adolescents with the same inconsistency—refusing them the protection of children but allowing them no real part in its adult concerns—except perhaps when wars or emergencies suddenly increase the need for all available man power. No wonder that they are confused.

The rapid swings from independence to dependence and from hostile to loving behavior are but a reflection of this struggle, of alternating daring sallies into independence and fearful retreats to the comforting status of childhood.

In the midst of this inner turmoil, the adolescent is facing the necessity of making choices and decisions of far-reaching importance for his future. What is to be his vocation and how can he best prepare for it? How shall he find his mate? Will he marry early or late, and what prospect has he of being able to afford marriage at all? How is he to manage his sexual needs in the meantime? What is a workable code in sex behavior, in business ethics, in marital pattern? What, even, is the accepted rôle for a man or a woman in our society today? Can he accept

the religious and political affiliations of his parents? If not, what groups are to have his allegiance?

These have always been difficult decisions. But today they are unusually difficult, for young people have no clearly defined social customs and expectations to guide them. Their own parents and teachers are confused in the face of rapidly shifting economic and social patterns. They don't know how to advise the children, for their own values and standards seem inadequate. Many of their most cherished beliefs and virtues have failed them. Often the older adolescent has not only his own problems to face but must accept partial responsibility for his parents' support as well, and this in a world that offers him no assurance that hard work will bring success and satisfaction—or even that there will be any opportunity to work at all. Yet in this same society self-support is generally considered the measure of a man's adequacy. Not until he can establish financial independence will the boy feel that he has proved himself, and this is equally true for many a modern girl. Needless to say, the struggle for emotional and sexual independence is bound to be more difficult in later adolescence if the child remains economically dependent on his family, for it is a rare parent who can grant complete independence of thought and action to a child of any age whom he still supports and who still shares his home. The most mature adult finds many of his decisions difficult in the face of our present social uncertainties. For the adolescent the strain is acute.

Typical Adolescent Behavior in Response to Strain

The adolescent's more bizarre behavior is highly revealing of his inner conflicts for those who learn to see its true significance. They will not be satisfied to accept all at its face value. The swaggering bravado so typical of this age will be recognized as a thin disguise for self-doubt, inadequacy, or fear. Those who are really sure of themselves have no need to swagger. Those who are afraid and cannot admit it must have recourse to some pretense, designed as much to convince them-

selves as others. Attempting to deal with this type of reaction, as so many parents and teachers do, by belittling the swagger and "taking the conceit out of him," will only enhance his difficulties. He needs all the help he can get in achieving self-confidence and self-respect.

Just so, a great show of sophistication in adolescence is no sign that the child knows everything. It is more apt to be a bluff to cover his ignorance and guilty curiosity. He is ashamed to admit his ignorance to his fellows—ashamed to admit his curiosity to himself and to the adults whose disapproval he fears. Those who feel content with the knowledge (particularly sexual knowledge) they have or free to seek more have little need to parade their sophistication. Unless we see beneath the surface we can do little to meet the problem. Our task is to supply the knowledge he craves and the assurance that curiosity is normal and healthy.

It is easy, also, to misinterpret the withdrawal and secrecy of the adolescent as implying that he has attained a comfortable self-sufficiency and feels no further need of adult help. Some adolescent secrecy is, of course, simply self-protection from adult interference as boys and girls make their first ventures into independence and try out standards and activities which they feel will not be approved or permitted. This sort of secrecy is merely the cloak for disobedience or rebellion when open defiance becomes too uncomfortable. Not infrequently it involves not only the clam-like uncommunicativeness which adults find so baffling, but also deliberate lying and deceit by youngsters who have previously been honest and who will be so again. One often hears adolescents discussing what they "have to do to keep their parents happy," which in their language means how the absurd adult demands can be sidestepped without too much friction.

But sometimes the secrecy is more significant, revealing a deep sense of guilt or inferiority and a profound need for help which the child is afraid to express. Feeling himself deeply unworthy—sometimes because of sex interest or sex fears,

sometimes because of unpopularity, or school failure, or the new and painful struggle with his parents—he hesitates to reveal himself, fearing to lose adult love and respect. This type of withdrawal, which expresses itself in moody introspection and touchiness, in unhappiness and lack of sociability with old and young alike, is a distress signal, denoting not self-sufficiency, but quite the reverse. We cannot force the child out of this phase, no matter how fully we recognize his need for help—but it is sometimes possible to reassure him indirectly. If somehow we can convince him of our basic tolerance and of our acceptance of the naturalness of much that he fears within himself—then and then only can he seek the help he needs.

Typical adolescent aggressiveness and hostility are, similarly, mere camouflage for fear in many cases. Proceeding on the assumption that there is safety in the first attack, many insecure people meet all new experience in a hostile manner —with their fists up, literally or figuratively. This type of reaction is frequent, in adolescents especially, as an approach to those in authority. It is characteristic of those who are impelled to rebel and who at the same time fear retaliation. To reply to this type of hostile child with counter hostilities—punishment or ridicule, for example—is useless. His hostility can subside only if his fear is replaced by trust. The teacher who needs to retaliate or to uphold his authority for its own sake can do little for the hostile youngster.

Actually, the teacher must learn to interpret much of the rebelliousness and hostility of adolescents as a mere reflection of their struggle to free themselves from their parents. The battle for independence is not limited to the home circle. Whatever the child's reaction to home authority at this time —be it open rebellion, cringing fear, reasonable independence, or exaggerated dependence—it will be largely reproduced in his attitude toward his teachers as well. In a very real sense they become parent substitutes—symbolizing the parent for the child. He invests them, in his imagination, with many

of the attributes of his parents. This leads him to expect his teachers to react as his parents do and impels him to behave accordingly. The teacher who is himself mature and secure can accept this type of behavior for what it is worth and turn his attention to helping the child fundamentally with the problem it expresses. But if the teacher remains immature in his own attitudes toward authority, the adolescent's rebellions are interpreted as a personal insult or a serious threat, and no constructive relationship is possible.

Not only rebellious attitudes toward parents are thus projected, of course. Submissive attitudes are reflected in the same way and call for equal insight in their handling by the teacher. But perhaps the greatest call upon the teacher's maturity and balance is made when he is invested by the child with the loving and protective qualities of the parent. If the adolescent is not quite ready to stand alone as he breaks away from his parents, needing both something of the protective love he has known as a child and a model for suitable adult behavior toward which he is striving, he will seek a parent substitute in some older man or woman whom he admires. Teachers, naturally enough, are often called upon to play this rôle, and all high-school teachers are familiar with "the crush problem." There is no point at which the teacher's maturity is more important for the child. His wise handling of the crush will contribute to the child's healthy growth—unwise handling may cause great pain and even permanent damage. The teacher whose own emotional life is healthy and well developed can accept the child's love with understanding of its transitory character but temporary importance, supplying the security that is needed but helping the child on his way toward maturity and the type of independent, heterosexual love that should eventually replace the childish need.

Unfortunately some teachers are themselves so in need of affection and emotional outlet that they cannot resist the temptation to prolong and exploit the child's need in order to meet their own—or so frightened of their own impulses,

conscious or unconscious, that they are forced to repel such advances with frigid coldness or cruel ridicule. Teachers who find themselves in difficulties on this score have a serious responsibility and might well confer with the guidance workers in their schools if such counsel is available.

The adolescent meets many of his difficulties by these devices: regression to early childish forms of satisfaction and self-protection, camouflage of real feeling behind a mask of bluff, and projection of problems from one area of his life into another—but they are not his only resources in the face of acute tensions. Just as he can run away physically from a threatening situation, he can run away emotionally, escaping into the world of fantasy. Here temporarily he is successful, beloved, dominant, all his inadequacies and doubts are lost, all the stories have a happy ending, and he is inevitably the hero. Of course, he cannot go on escaping from his problems in this way forever, and there would be reason for concern if this were his predominant reaction, even temporarily. But within reasonable limits and as temporary relief, fantasy is a healthy safety device and the day-dreaming of adolescence serves an important function in easing strains that would otherwise be too intense. Adults are ordinarily too easily alarmed by the unhealthy possibilities of this typical adolescent behavior, failing entirely to credit its positive values or to capitalize its educational possibilities.

There are rich constructive uses for the newly stimulated fantasy life. Guided into channels of artistic expression, in writing, music, dramatics, painting, and the visual arts—or for the more mechanical type of child, construction, or designing, or applied science—it can express itself symbolically, contributing to cultural growth as well as to emotional relief. As one might expect, in the early or pre-adolescent stage the child is more given to pure day-dreaming and less able to translate his fantasies into artistic expression—but with the later shift in pattern the creative aspects become uppermost and should be capitalized. Such outlets will never entirely re-

place pure day-dreaming or the direct expression of emotion and feelings, of course, nor should they, but at the senior high-school level schools might profitably make more provision for recognizing and using the cultural possibilities of this phase of the adolescent personality.

Educators have been more successful, often too successful, in capitalizing another device which adolescents—and, in fact, all people—unconsciously employ in solving their problems: the device of compensation. An individual who is conscious of real or fancied inadequacy in one field will ordinarily seek to restore his self-esteem by feverish efforts to excel at some other point. Such compensation is often the explanation of a passionate drive for intellectual achievement, or athletic prowess, or popularity, business success, or social eminence, or sexual conquest. Up to a certain point this type of compensation is a healthy adjustment to the inadequacies from which all human beings suffer, and it is the power behind much of the best work they produce. But carried too far it results in one-sided, unbalanced personalities. Teachers have been all too ready to acclaim and stimulate this kind of behavior, especially when it produced scholastic or athletic excellence. In mistaken enthusiasm for good scholarship or good teams, they often fail in their responsibility for helping students to a more well-rounded social adjustment.

On the one hand, schools need to go further in providing reasonable opportunities for compensation for those who are incapable of high intellectual or athletic achievement; on the other, they must give more thought to the direct solution of those personality problems which should not be met in this indirect manner. Compensation has its legitimate uses in relieving strain and facilitating adjustment to real deficiencies. It should not be accepted as a substitute for a direct attack upon the many personality problems of young people which they could be helped to work through more profitably.

These various indirect devices for adjusting to strain—regression, denial or camouflage, projection, fantasy, and com-

pensation—are not actually separate and totally unrelated forms of behavior. They have been separately described only for clarity and convenience. Actually, in meeting any life problem, several or all of them may be employed together.

The Influence of Early Experience in the Family

Individual adolescents differ considerably, of course, in the pattern of their behavior and in the degree of strain of which it is a fair index. Some of them seem to meet the inherent problems of this stage of growth with comparative ease—others with greatly exaggerated difficulties. In attempting to account for these differences we must return again to the concept of adolescence as one phase in a continuous growth process; it is not hard then to realize that in dealing with his immediate problems the adolescent is greatly helped or hindered by the character of his early experiences. For it is an inescapable law of growth that each early phase of development must be successfully fulfilled if the next stage that grows out of it is to be satisfactorily achieved. We all know this in regard to physical growth and intellectual development. It is equally true, but often overlooked, in the sphere of the emotional life. It is not surprising, then, that the child's earliest relations to his parents and to his brothers and sisters, his earliest experiences of love and gratification, of discipline and denial, exert profound influence on his emotional development and are reflected in his approach to the problems of adolescence. We cannot understand his attitudes today without some conception of the way in which they were built up.

The human infant is at first entirely selfish and self-centered —interested only in the gratification of his immediate bodily needs and the pleasure or distress associated with them. His love for others is a purely selfish love determined by their usefulness to him and replaced by anger as soon as they thwart him in any way. But soon he begins to be interested in pleasing those who care for him. By this time he is prepared to give up some of his infantile desires and to learn to control some of his

primitive responses as a price for his parents' love and approval. This is the beginning of socialization.

It is inevitable, of course, that this training will often involve anger and frustration for the child, but angry hostile feelings can be kept at a minimum by patient, loving care or greatly accentuated by neglect or severity. If the child feels himself truly secure in the love of those who care for him—and if their training of him is neither too severe nor too lax—he is able to accept the necessary denials and gradually progress to more social and mature satisfactions. His parents' rules become his rules; their standards are gradually absorbed and become his conscience. As he matures his love grows gradually less selfish and includes not only those who serve him but others whom he accepts for their own sake. His interest is no longer centered on his own body alone, but broadens to include the world outside. In middle childhood he explores and experiments and welcomes new experience as a challenge, secure in the love and protection he has known since birth, but no longer totally dependent upon it. Gradually he builds up faith in his own powers and judgment, loyalties broader than the family group, power to contrast objectively the standards of his home with others that differ, and to choose and modify without too great conflict and fear.

If, however, he has reason to doubt his parents' love for him or has been completely overwhelmed by its intensity, or if their early demands were too severely enforced or not enforced at all, there will be difficulties in his development. Instead of satisfying and gradually outgrowing his infantile desires and his infantile ways of loving and hating the child may cling to them. They will remain part of his personality, often hidden from view and totally unrecognized, but hampering his future growth and his progress toward more mature attitudes and satisfactions. With the new intensity of feeling and emotion that is characteristic of adolescence many of these old problems become troublesome again and add greatly to the normal conflicts and strains.

The best adjusted adolescents find achieving independence is not easy. They are afraid at times, doubtful of their own adequacy to meet adult standards, guilty about their hostility to their parents. If they have known the security of satisfying love and wise control as little children, they are ordinarily able to face these fears and doubts without too much distress. But the inherent difficulties may be vastly increased. Those who have been too much protected will find the sheltering comfort of childhood unusually hard to leave behind; while for those who have felt unloved and insecure as little children all new experience will be terrifying, all responsibility charged with danger. For either type, adulthood with its responsibilities looms only as a threat.

Early in life some of these insecure children established a pattern of buying the love they craved by meek compliance, a method employed to some degree by all children, but often greatly exaggerated if love has been too hard to get or discipline too severe. These children avoid the slightest rebellion as threatening a loss of love but beneath their apparent compliance unexpressed and often unconscious resentments continue. More aggressive children sometimes adopt quite opposite tactics. If they feel unloved and unfairly treated, they respond with violent rebellion, a rebellion that brings upon itself repeated punishment and a deep-seated fear of all authority or a guilty sense of personal wickedness. In both groups hostile feelings are intense and guilt inevitable.

These early ways of responding become deep-rooted in the child's personality. They are the only patterns he knows for meeting authority, for approaching new experience, for dealing with his own hostile feelings toward adults. These, then, are the patterns which he must follow in his adolescent struggle for freedom, and they will be primary in deciding how difficult he will find it.

So too he approaches the delicate task of heterosexual adjustment dependent on patterns of behavior he has learned in

his childhood. His earliest experiences of affection, his earliest feelings about men and women, are the groundwork on which the child must build a mature love for a suitable mate and the comfortable acceptance of his own rôle as an adult. Before a person of either sex is capable of a mature love relationship, he must have passed through a complex development that includes first self-love, then a gradually maturing love for the mother, then a more equal and less dependent love for friends of his own age and either sex, and not infrequently romantic love for a much older person. Girls must take one additional step—transferring the intense affection of early childhood from the mother, who is the first love of all children, to the father, who must give his daughters their first experience of masculine affection. And finally, before any individual can achieve a satisfying adult love life, he must fully accept the implications of his own sex, and the privileges, limitations, and responsibilities that go with it in the particular culture in which he lives.

But unfortunately this complex development does not always progress smoothly. It may go astray in a variety of ways. It should not surprise us, then, to find many adolescents whose difficulties in this area are acute, or to learn that they are deeply rooted in early childhood experience. But similar problems and maladjustments may arise from very different life histories.

Children may be hampered by too much love or by too little. The child who has felt unloved or unwanted will grow up needing an infantile protective sort of love relationship; so too will the child whose early love experience has been too intense. As a result of their own immaturity or of unsatisfactory marital relationships, parents may be incapable of loving their children whole-heartedly or may seek from them satisfactions that really exploit them. Such parents are not monsters—as they are often pictured in popular plays and books —but merely unfortunate individuals who react as they must.

Nevertheless they cannot help but warp the development of their children. The boy who is loved too much or not at all by his mother will have difficulty in his adjustment to other women; similarly the girl's relationship to her father will determine her approach to other men. The boy's acceptance of his masculine rôle and the girl's acceptance of her femininity will depend to a large extent upon their feelings about their parents and the sort of people those parents are. It may be hampered too by jealousy of brothers or sisters or by the parents' unsatisfied longing for a child of the opposite sex.

It is impossible in this brief discussion to suggest the many intricate and subtle variations of these patterns that are found in actual life. But these few examples should suffice to suggest the extent to which childhood experience may complicate the adolescent's task of sexual adjustment—a task which he finds troublesome enough at best in our society.

The early history is equally important in determining the individual's ability to achieve a mature level of social identification. This implies a concern for the general social good and satisfaction in work which furthers it. It requires ability to cooperate whole-heartedly with equals or those in authority and the power to evaluate changing standards and trends without either a slavish subservience to an established pattern or a passionate need to overthrow it.

As has been explained above, the child progresses from complete self-love to a gradually maturing love of his family. The next step is allegiance to a gang or group of friends of his own age and sex. Here for the first time he develops an intense loyalty outside the family, and often in conflict with some of its standards. For a time he may be arrayed with the gang against all adults and their demands, a typical state for the preadolescent. This is a valuable step forward in breaking his complete dependence on the family group and in finding social satisfaction on a broader basis. But gangs are factional, often hostile to each other, and utterly unmindful of any

social good beyond their own. In later adolescence there should be a swing away from this type of loyalty toward more mature social identification. But this does not always take place. Curiously enough a child's failure to mature socially may be the result of either too much or too little satisfaction at earlier levels—too much parental loving or too little—too long or too short a period of infantile self-enjoyment.

Jealousy of brothers and sisters may also play its part in delaying social development. Where it has been extreme the child may transfer his hostile attitude to all his playmates and be unable to find satisfaction in their companionship or to make himself pleasing to them. He may be too aggressive or too servile with his fellows but whichever way he expresses his own insecurity, he will have difficulty in establishing a free give-and-take relationship. He is apt to be jealous and touchy, unable to accept group pressures. And this inability to coöperate with equals and to find full pleasure in group enterprise will be carried on to hamper his adult relationships as well. Just so his attitude toward parental authority will be carried forward to influence his relationship to all subsequent authority, be it the gang leader, the teacher, the employer, the church, or the state itself. Extreme rebellion or extreme submission rather than comfortable acceptance of authority and reasonable independence reflect patterns developed in early childhood. They are certain to hamper the adolescent and the adult in finding full satisfaction in social relationships and in reaching a mature view of social responsibilities. So too they hamper in the evaluation of and choice among changing social standards and ideals. It was made clear above that the child's early sense of security or lack of it greatly influences his attitude toward all that is new and untried and that his early relationship to his parents may impel him to cling slavishly to their standards or to reject them passionately. Only if the early childhood relationships were sound will he be free to modify the parental standards gradually and thoughtfully in the face of new experience. This applies, of course, not only to personal

behavior and independence, but to social vision and loyalty as well.

THE STUDY OF PERSONALITY

The Developmental Approach

Adolescent behavior is, then, the product of many factors —past and present experience, stimuli from within and from without. To understand the reactions of any individual adolescent we must think in terms of his response to the external pressures of his immediate environment—home, school, and general social setting—and the inner pressures that result from the maturing process. We must recognize the extent to which his past experience will influence his reaction to both inner and outer demands—and learn to *see all his behavior as a consistent expression of a personality pattern that has been built up in response to individual experience.* We must understand that all behavior, no matter how seemingly trivial or accidental, has meaning in terms of the needs of the individual, though he may be quite unaware of the purpose it serves or the motive behind it. Finally we must keep in mind the devices, direct and indirect, which all personalities employ in meeting their problems—if we are to interpret the real meaning behind the adolescent's overt behavior and appreciate its full significance for his total development.

In the case that follows, the history of one adolescent boy—an outstanding student in mathematics—is given in some detail. An attempt is made to explain the gradual development of his personality in response to this individual history and the consistency of its expression in all his adolescent interests and behavior. No case is ever repeated exactly—nor does the same behavior have the same implications in all cases. But it is hoped that the detailed presentation of one case may illustrate the basic principles of personality development and behavior, and aid the teacher in his understanding of other adolescent personalities.

AN ILLUSTRATIVE CASE: PAUL[3]

Introduction

It is not difficult, when one is teaching mathematics, to single out the mathematically talented, gifted, or interested student. His ability is taken for granted. It is just as simple to single out the student of good general intelligence whose work in mathematics is inferior or mediocre. He is disturbing to the teacher and requires much attention to discover the reasons for his learning difficulty. Teachers are therefore constantly concerned with failures, but very rarely analyze the factors that may account for satisfactory or even superior progress in any given field. Often the subject teacher readily accepts the exceptionally brilliant student in his field without looking at the child's record in the light of his whole intellectual and social development. This may be due to the fact that the subject teacher in many high schools has a very specialized contact with the students that prevents him from seeing the distribution of their interest and effort in the light of their entire school experience.

Since we are interested here especially in the development of adolescent personality or, better, in the processes which adolescents employ in response to conflicting stimuli, it will be our main concern to understand the adolescent's behavior and interests (whatever they may be) in terms of their psychological meaningfulness for him as a particular personality in a specific social setting. As has been pointed out previously, behavior and attitudes are not a question of chance. They will be explained as consistent reactions serving a definite purpose for the individual in a total situation. This will ultimately lead to an understanding of the adolescent personality.

In the following case discussion, an attempt is made to reconstruct the social and intellectual development of a boy from

[3] For more detailed discussion of this case, see Peter Blos, *The Adolescent Personality: a Study of Individual Behavior and its Meaning*, a forthcoming publication of the Study of Adolescents to be published by D. Appleton-Century Co.

the material which was available for study, in order to arrive at some larger understandings of personality growth and adolescent behavior. It seemed wise to choose a case that ordinarily would not receive too much consideration from teachers. The student under discussion is not only considered to be a normal boy and an asset to the school but he is also a good student throughout, and particularly able in mathematics.

It will not be surprising to find in the school records little attempt to account for the outstanding talent of the student. School records will not contain much more in regard to his mathematical brilliance than data in general terms of grades and casual comments. Therefore, in order to understand his special interest in mathematics, we have to consider the student's whole personality with its diversified emphasis in different fields. Furthermore, we must trace back interest and behavior patterns until we arrive at an understanding of the continuity, the consistency in growth and development, which will enable us to see his adolescent interest and behavior in terms of the purposes they serve and the experiences that determined them.

Identifying Data: Family and School

Paul is an only child. Both his parents are American-born and live in a large city where they were brought up. The father, the oldest of six children, comes from a poor Austrian immigrant family. He did not attend college but went into business after completing high school. After many years of work, he built up his own business, which was, however, very severely affected by the depression. The circumstances of the family have always been very moderate. The father married at the age of twenty-eight. The mother, who also comes from an Austrian immigrant family, is the seventh of eight children; she married at the age of twenty-five. She takes care of the household and is greatly interested in Paul's education, without exerting any undue pressure on him. The mother likes music and used to play the piano in the family circle.

She says, "I love music. It expresses all emotions to me." Relatives of both parents are close to the family, and reciprocal visits are frequent. The parents have no religious affiliation. The mother sometimes wishes she were religious. She says, "We all have a feeling that there is something." The rest of her family is orthodox, but she and her husband agreed to educate Paul without any religious creed. The parents have no strong political convictions or ideals. Their philosophy has been definitely influenced by the depression. The marriage is happy, and the family life is very satisfactory to everybody. The father has a cheerful disposition and forgets his business while he is at home. The family likes to do things together.

The school which Paul attends is a coeducational high school in a metropolitan area. Most of the children who attend the school come from families in better economic circumstances than Paul's. But there is no indication in the student body of any discrimination against Paul on this score. Paul himself does not seem much concerned with this matter. A large proportion of his classmates at the tenth-grade level are sophisticated metropolitans. The girls are inclined to follow fashions, and some of them can afford it adequately. The boys take girls out to dances, movies, and parties. The social life assumes great importance for most boys and girls.

Besides the academic program the school offers a great variety of creative work in various fields. Almost all the students intend to go to college, and the school helps them in planning for the future and in their choice of a vocation.

As the School Sees Paul [4]

Elementary school. The first sentence in the admission report reads: "Paul is very methodical, very independent, a typical businesslike child." From the time of his entrance into the first grade he was quick in grasping and mastering reading or number work. Writing always proved difficult for him, partly due to his muscular coördination, which was very poor

[4] The records were available from grade one to grade ten inclusive.

at the time of his entering school. This nervous condition dated back to his preschool years when he developed an overactivity for which he was treated temporarily. His behavior in the classroom (first grade) is described as follows: "Exceedingly restless, hence very disturbing to children around him. Very active. He is usually a wild effect of arms and legs and flying smock." His speech is considered very poor. He talks too loudly and does not pronounce correctly. This speech difficulty persists with variations during Paul's whole elementary-school career. His attitude toward the group changes. First it is described as very good, but soon he is found to be petulant. In the third grade he starts to feel superior with the expectation that every one else is going to consider him so. He takes the floor and keeps it for a while, trying to be witty, then looking about for applause. He soon finds out that the group as a whole does not like his clowning, and he tries to give it up. Nevertheless, he goes on to strain himself to do things that are beyond his age and comprehension, in order to give the impression of being unusually clever. This attitude interferes for a time with his steady progress. But he catches up again and in the fourth grade he develops unusual skill in number work. His teacher calls him a "natural mathematician." A year later his mathematical abilities stand out even more prominently and the records now call him a "mathematical genius." The report continues: "He rarely makes an error and the standard tests reveal his arithmetic age is over fifteen years. He is equally precise in statements, spelling, and punctuation, but in composition he has no style. In writing verse he works out the meter mathematically but it is not poetry."

Paul is liked by his classmates but he has no close personal friends. His special abilities are admired by the group, which gives him due credit for his contributions. Teachers find him intellectually responsive but personally somewhat unapproachable. He never develops into a leader in the group in spite of his outstanding qualities, much to the disappointment of his teachers, all of whom repeatedly remark that Paul

is a tireless and thorough worker. He never wastes any time; he tackles every task with sweeping zest. Whether it is reading, arithmetic, writing, map-drawing, weaving, shop work, he is completely absorbed by each task until it is completed. Even in subjects for which he has no particular ability he exhibits eagerness and persistence alike. In shop it is reported that he talks in a loud voice nearly all the time he is working. In the sixth-grade report, the last one of the elementary school, Paul's work is rated as above average except for shop and art. In purely academic work or discussions Paul is always at the top of the class but, in contrast to this, he is very slow to understand simple directions or to see a joke. It is this kind of astonishing incongruity in Paul's mental reactions which made a classmate in the fifth grade say of him, "Paul has brains, but he hasn't common sense."

Secondary school. As soon as Paul enters the seventh grade (age 12.4) his teachers become aware of his unusual academic capacities. Paul is soon far ahead of the class, especially in mathematics and languages. The teacher of general science considers him a most unusual student, and the mathematics teacher expresses his admiration for Paul's ability in mathematical reasoning, in abstract thinking and theoretical speculation. In each grade the mathematics teacher feels that Paul could just as well do the work of the next grade and is afraid of his being bored by the comparatively slow progress of the class. The comment of the ninth-grade mathematics teacher will illustrate what has been said. "Paul is brilliant, the most brilliant person in the ninth grade. He is rarely satisfied with the depth to which his group investigates a subject; he always wants to go much deeper and is perfectly capable of doing so." As indicated previously, Paul has a tendency to ask abstract and sometimes seemingly unrelated questions in mathematics and physics class. Occasionally he seems to ask questions just for the fun of composing them. This is, at times, somewhat exasperating for the teacher. However, his physics teacher indicates a little uncertainty as to whether even these unrelated

and sometimes incomprehensible questions can be dismissed as attention-gaining devices, as some other teachers believe. In physics, as in mathematics, he is considered brilliant and far ahead of the rest of the class in his understanding. The practical applicability of physical or mathematical principles does not attract his attention as much as the theoretical extension of them into the controversial field of hypotheses and new constructions.

In the field of language, Paul's aptitude is remarkable. When he enters high school, his English teacher comments on his great interest in words and his liking for English grammar. His Latin teacher in the ninth grade remarks, "Paul has keen delight in words. He learns declensions and conjugations with the greatest ease." In spite of this linguistic flair, Paul's compositions are never of a quality commensurate with his intellect. In fact English composition remains his weakest subject, while English literature, where he can make use of his verbal and analytical abilities, is up to his general high level of achievement. His memory for words encountered in reading is unusual. In a discussion in English class he explains that he never sees a person or a place when reading a book. "I just see the words," he says. He proves this soon thereafter on an occasion when the class is reading *Silas Marner*. A student refers to Dunstan as tall. Paul immediately jumps to his feet, saying "That's not right. It says on page 29, 'a thickset, heavy-looking man.' " Descriptions of scenes or objects are prevalent in his creative writing. His style is bare of any emotional tinge, except in his writing on one topic: war and peace. These papers indicate that he is a convinced pacifist, and he expresses his point of view very emphatically. "I am sure that nine out of ten of the members of our class will answer 'no' to this question ('Do you think students can do anything to prevent war?'), but I disagree." And later: ". . . I think it would be noble and more for the benefit of mankind if I were killed because I did not go to war than if I went. Of course it is very easy to say that now, but I hear that such acts are rare cases when war

does come . . . anyway you'll read about me then." He is an active collaborator in editing a school pamphlet on peace. He displays the same intense interest again when, during the summer vacation (age 13.2), he writes a manifesto against war. He adds a fictitious name to his own as co-author of the article, hoping that if another person's name appears, the people who buy the pamphlet will not think it is a biased opinion. He sells the pamphlet from door to door, avoiding any of his mother's friends in order to prove his ability as a salesman. He sells all of them. His first sentence after he rings the bell is: "Do you want your boy to be compelled to go to war?"

Paul's reactions to literature are also revealing. When asked (grade ten) whether he would rather read novels dealing with home life or adventures he answers, "On the whole, I enjoy novels that describe home life yet I cannot specifically describe why. It might be because my life has been wholly a family one, so I understand the plot workings rather easily even though they often are extraordinary . . . family life is always fresh and intimate even today."

In book reviews and discussions in class Paul shows a very great concern with ideas pertaining to reincarnation, the origin of the universe, and immortality. He is highly impressed by the philosophy that an individual's problems are small and insignificant compared with those of mankind. In a class discussion he says, "The trouble with people now is that they believe there's no life after death. They feel that death is the end of consciousness. You spend some seventy odd years being, then all of a sudden you stop and the world goes on without this particular part of consciousness. Then you wonder why all these consciousnesses come in and out of the world." He continues, ". . . while we live we need faith in the divine power to hold us together. . . . A good many don't believe. That's the trouble." This preoccupation with religious ideas and philosophical speculations is astonishing in the light of Paul's background and the predominantly factual and logical approach which he shows generally.

In Latin Paul shows such rapid progress that the teacher offers to teach him and another girl separately merely for the enjoyment of their intelligent mastery of the language. Paul's contributions in art are very poor, and he himself confesses that he does not like to work in the studio. Mechanical drawing, which is of great importance to him on account of his chosen career as an engineer, is difficult for Paul, probably owing to a disturbance in his motor coördination. During the seventh grade Paul is troubled by a sudden stiffness of the neck which starts during the summer holidays. The stiffness, which his parents feared might be infantile paralysis, subsides but a general lack of motor coördination and nervousness persists. For the same reason he has to refrain from physical education activities during the eighth and ninth grades. Then his condition improves, and he is allowed to play football and soccer again. For almost two years he had been restricted in his physical exercise, and at times had to rest when others played. In the tenth grade he is allowed to take up sports again. This is of great importance to Paul, as will be pointed out later. He throws himself enthusiastically into the competitive games, exhibiting, in spite of his interrupted training, a remarkable skill and courage. At this time his physical education teacher describes him as "one of those do or die boys." The gym teacher of the seventh grade describes him as follows, "You never knew which way a leg or an arm was going to move. He didn't either. I was expecting to see an arm going across the field and Paul coming after it. You might call him a modernistic boy—all angles."

Paul has always participated in such club activities as the science club, the camera club, and the orchestra. He has been a member of the student council for three years, holding various offices. In meetings Paul shows a tendency to obscure the issue with trivialities and to take the side of the opposition. This usually puts him on the defensive and in the minority. A teacher remarks about Paul's behavior in council meetings,

"The boy says some awfully stupid things. He is usually opposed; I would say he is probably always in the minority. I think it is because he shoots off too soon, he does not wait and does not think it through. Then he feels he has to back it up. . . ."

Paul has no intimate friends at school. Teachers remark again and again that he does not get along with his fellow-students as well as he should, considering the fact that he is well meaning and seemingly without intention to hurt anybody. He has a sharp tongue which the other students do not like. He is somewhat quick tempered and tends to be contentious. He is always sincerely contrite when cool. He is lacking in consideration for others and claims an undue portion of the teachers' attention and time "due to his exceptional academic capacity and desire to learn." In a way he seems to be content without personal friends and is objectively interested in people much as in ideas and books. Paul usually sits in the back of the classroom. Often, after lunch, he can be found alone at his desk, humming to himself and working at his Latin or mathematics. "Paul always dresses the same," remarks one tenth-grade teacher. "He has worn the same blue sweater the whole winter, the same brown trousers girded tightly around his waist, the same battered brown oxfords. Nevertheless he is always neat and his shirts are always spotlessly clean under the eternal blue sweater." Paul's voice, which is very unpleasant, "extremely raucous," annoys the class a great deal. The teachers have to remind him incessantly to use it with consideration. "Paul used to answer his recitations in a voice whose volume would have carried to the deepest recesses of the large school auditorium." This loud voice, which he uses without apparent reason, is finally brought under control after repeated efforts of the teachers. He still speaks in a high pitch fairly frequently. Besides his cutting remarks, his silly actions and clowning interfere with his popularity, especially in the junior high school years. Some teachers

believe that he acts this way to attract attention. His physical condition doubtlessly contributes to his restlessness, his nervousness, and his driving and aggressive attitude. He is considered mentally and physically overactive.

Paul is very indifferent about girls. He prefers the company of boys in orchestra club and in theoretical discussions. In the tenth grade he shows a romantic tenderness for a while toward the girl with whom he studies Latin. This is hardly apparent in his behavior except when he asks the teacher not to tell the girl how many pages of Latin he has done because she probably has not accomplished as much and may feel badly about it.

Paul is always willing to assist where his help is needed. He is in great demand as tutor before school and at noon. He takes a very professional attitude, "Now be sure you know the difference between the ablative and the dative there." In the light of some of his outgoing interests and his coöperativeness, it is even more astonishing that he is personally alone. In fact, this peculiarity needs further illumination.

Paul's annoying speech, his cutting or silly remarks that are resented by his fellow-students, are compensated for by his keen friendliness, his good intentions and his total lack of resentment. One teacher tries to describe this picture by saying, "There is a quality of sweetness in Paul, but there is no consideration of others. . . ." Though Paul is not very well liked during the junior high school, this changes during the tenth grade with his physical improvement and the subsequent participation in sports. The records very clearly show a different attitude of the group toward Paul dating from his return to the playing field.

Students and teachers alike are puzzled by Paul's attitude, which is partly withdrawn, partly aggressive, partly outgoing; but the discrepancy between his unusual mental powers, his adequate physical maturation, and his social immaturity does not escape the attention of most of his teachers.

As the Parents See Paul [5]

Paul was a wanted child and his parents always found great satisfaction in him. He is an only child. His parents wanted it so because they both came from very large families where they never enjoyed the intimacy of a small family group. For this very reason they relive their own life with their child and try to compensate for the shortcomings of their own childhood. At the same time they are carefully on their guard against spoiling him—which may account for some of the disciplinary measures employed in his early childhood.

At the age of five months Paul developed a severe eczema which made sudden weaning necessary. The child was completely weaned in one day and responded to this privation with incessant crying. He refused all food for twenty-four hours. By this time, completely exhausted, he accepted the bottle. To bring the eczema to a stop, food restrictions were necessary, and measures were taken immediately after the abrupt weaning. Experimentation with formulas resulted in a diet which has been described by the mother as "almost a starvation diet." The child who had been so upset after being weaned became placid and quiet. The diet may have contributed also to a physical exhaustion and general weakness. In spite of all the effort the eczema did not clear up as expected, which made the routine care for the child painful and distressing. Thumb-sucking became prevalent for some time. The inactivity and sluggishness of the child grew to such a degree that it was of severe concern to mother and doctor. Sitting, walking, and talking were delayed. But slowly he became more active and outgrew his passivity completely. In fact, he developed, around the fifth year, an undesirable hyperactivity with lack of muscular coördination. This motor disturbance cleared up when he was in second grade and recurred in early adolescence, as has been mentioned. In spite of the difficulties in Paul's physical development the parents were always eager to regard him

[5] From interviews with the mother.

as a normal child without spoiling him, to make demands upon him just as if he had no physical handicaps. The eczema, which subsided somewhat after one year, finally cleared up at about the age of six. Before he entered school he had tonsilitis (2.5), measles (4), and whooping cough (5).

Toilet habits were established at one year. Before that time great attention had to be paid to changing his diapers immediately, on account of his eczema. From two to four he used to amuse the family and friends because he would say when he needed to use the toilet, "Attention please, mother."

Until Paul was five years old, he was occasionally spanked by his mother. His father never used corporal punishment. The mother felt that there were times "when he had to know," and without spanking she felt "he wouldn't know." At the age of five the mother used to reason with him. This measure proved to be very successful. If he was sent to his room to "think it over," his mother always joined him after a while explaining to him why she had done it. Paul was never sullen or resentful afterwards. His disposition is that of a compliant, sunny child. Therefore the parents feel that they have never had any difficulty in his upbringing in spite of his difficult physical problems.

Outstanding from childhood on was Paul's undemanding attitude which was never deliberately inculcated by the parents. He never asked for things he wanted or needed. His parents sometimes wished he would ask for things; he accepted everything as it was. Even as an older child he was pathetically grateful for a tie or a shirt. At the age of thirteen he asked, when offered a present, whether he could possibly get a violin. He had wanted this instrument for three years. His parents fulfilled his wish, and he has been taking violin and piano lessons since that time. His interest in music is keen and genuine; he practises both instruments regularly. At the age of seven he had taught himself the notes of the piano from an encyclopedia.

When his family talked about going to see *Green Pastures*,

Paul asked his mother not to get him a ticket. He said, "I want to keep the little religion I have." He thought that *Green Pastures* was a "satirical play." When this misconception was corrected, he went and enjoyed it very much.

The family has no set expectations regarding Paul's vocation. They leave it to him to decide what he wants to do. In spite of their modest circumstances they do not press Paul to be an economic success. His father's advice to him is briefly, "Depend on yourself, have self-confidence, be modest, but have a certain amount of aggressiveness." Paul's relationship with his father has always been excellent. The father took plenty of time to play with him when he was a little child and still likes to do things with him. The family group is very closely knit, and for this very reason Paul probably never needed the company of other children and never had any close friends. "He seemed to be perfectly happy at home with his mother and father." During the last few years he has become more concerned about his lack of friends and resorts to talking the problem over with his mother. This makes him feel better, and he overcomes his feeling of bewilderment. In his casual contacts with boys and girls the mother has recently found him irritable and impatient with boys but most polite and on his good behavior with the girls.

As Paul Sees Himself [6]

Paul's intellectual interests are suggested by the school records. His interest in languages is based on his admiration for a close relative, quite young, with whom he loves to discuss problems of language study. His reading background is not especially extensive. He has always liked to read but he thinks, on looking back, that he was, like his teachers and his mother,

[6] This material is obtained from interviews with Paul over a period of one year (the year when Paul was in grade ten). Some of the interviews are available verbatim, which makes authentic quotation possible. In presenting this material it proved necessary to give up the historical sequence of the interviews and to organize the material, with its widespread references, around certain large areas of Paul's interests, opinions, concerns, feelings, and attitudes.

against the cheap literature which the other boys liked to read. He remarks regretfully, "I think I missed something." A lot of his thinking is taken up with the search for arguments against war. He is preoccupied by the war-peace alternative, and if anybody does not agree with his passion for peace, he mobilizes his whole intellectual arsenal to defend his position. When he is alone he spends much time thinking over mathematical problems, and thinking, he says, often prevents him from sleeping for hours. "I have so many thoughts I can't go to sleep." His early memories are full of geometrical or spatial details; e. g., "I played in a triangular park," or "a street which was at right angles to a main thoroughfare." He is not very communicative about his thoughts but he says he often wonders, "what is the first year of life when people know that they are alive." This is tied up in his mind with the problem of whether people become doctors, lawyers, etc., because of native ability, or because of outside stimuli. Paul considers himself "sort of a theoretician." He says about himself, "I'm not good with my hands." In spite of this lack of manual dexterity he has never disliked shop or mechanical work. Speaking of the school, he says, "There is some learning you swallow, some you chew neatly, and some you spit out." Asked which subjects belong to his different categories he says, "I swallow science, I chew neatly English composition and I spit out art." He first became interested in science in the sixth grade, when he made an electric circuit for electric trains. He wishes to be an engineer and would like to make electric generators "to create power." Of the sciences he disliked biology extremely. Referring to this subject he explains, "I'm interested in the way life begins. I'd have liked to have started with the way life begins and worked up, but we started with mammals and worked down." This approach did not hold his interest. The concern which Paul expresses in connection with his disappointing experience in biology is of a very fundamental nature.

In contrast to his logical and orderly approach to life and

work he is constantly puzzled about problems of a religious or philosophical nature. He says, "It is necessary to have beliefs even though not necessarily true beliefs." Some of the problems over which he occasionally broods are immortality, reincarnation, and universal law versus moral law which man creates for himself. One of the moral laws which Paul mentions to illustrate his concept is the law which says that it is wrong to kill. "The law, 'Thou shalt not kill,' was developed so that men wouldn't wipe out mankind." Though he thinks that we need faith in a divine power which holds us together, Paul does not believe that the power who created us is interested in these "millions of little ants." What we become is in ourselves. "We can go to the dogs or make something of ourselves." "But what happens with us after death?" he asks himself. Concerning immortality he says, "Now I'm inclined to believe that there is no special reason for not being mortal, I mean for living after death. Sometimes it seems unendurable but we generally can't help mankind after death. Of course," he continued emotionally, "that's very hard to believe, that there is only mortality. The belief in immortality helps mankind." Just as Paul extends his thoughts into the infinitely large—of space (universe) and time (immortality, destruction of consciousness by time)—he also extends his interest into the infinitely small. Conceivably he may, when alone, concern himself with the following: "Try and see the molecules. Even an electron, the smallest known unit, has a certain amount of mass, why can't anything have less mass?"

When the book *Man the Unknown* by Carrel was suggested to him as enlightening reading on many of his questions he replied, "I'm sort of half afraid to look into this problem too deeply for fear I'd lose completely any faith that there is something like that——." "Like what?" he was asked. "Like some divine power——."

In his relation to other people Paul's philosophy demands that he "find something good in everybody." He never criticizes or condemns anybody. He always tries to see the other

side. In the light of this human approach to people he is surprised at not having more friends. He thinks a course in "How to get along with people" might be a very helpful thing at school. He considers some adults as poor in manners and conduct, but this could be remedied if only they were taught how to do it. Paul admires boys who can make friends with adults. "I'd like to do it but I can't." He shrugs his shoulders and repeats helplessly, "I just can't." Paul's companionship with boys is based mainly on common interests in mathematics and athletics. He admits that he has no particular friends; he likes companionship in general. He very frequently changes the group with which he walks home from school.

Asked about friendship, Paul says: "I think friendship is one of the most important things in our lives." He has often thought about friendship. "About two years ago I was worried because it seemed to me that I didn't have any friends in the school. I never seemed to do things with other people. I felt all right about it when I talked to Mother." As has been mentioned before, his exclusion from athletics was partly responsible for this state of affairs; after he was allowed to take athletics again in the tenth grade, he ceased to be the onlooker and outsider. Referring to his return to athletics he says, "My, was I glad! You see we spend every afternoon in gym and the fellows kind of get together after sports."

Paul divides people into three categories: "1) people who are lots of fun at a party, who are witty in general, but who have no interest in 'good things'; 2) people who are interested in 'good things' but who are very hard to know; 3) people who have live interests but with whom one can become acquainted." He defines "good things" as a live interest in politics, athletics, hobbies, studies. For instance, "the way Bob and I enjoy theoretical mathematics." One of the relationships which he "valued" was with a boy in the neighborhood in which he formerly lived. This boy loved to discuss ideas. Paul and he had many discussions, particularly about religion. The friendship dissolved when Paul moved away.

Talking about gangs and groups at school Paul expresses great resentment about the way "boys get into huddles about sex." He does not like the way boys talk about sex in the school. He thinks they have a "warped point of view" and "their facts are inaccurate." He continues, "Their conversation could make you lose respect for certain phases of sex." Paul makes a distinction between a clean and a dirty boy. A dirty boy is one "who tells a lot of dirty jokes." He thinks "how it happened" (intercourse) is what confuses boys most. He continues: "I always went to my father if I wanted to know about sex. My father has a sane view. He says that everything comes in its time. He has perspective. He found that for himself a certain course was wise." Asked about that course, he answers, "To keep away from girls until you are quite old." This scheduled postponement of any interest in girls reflects itself in his attitude at school. When asked whether he is interested in girls, he answers, "I think I have a latent admiration for girls, for their grace and beauty, the way they walk and carry themselves." He feels this latent admiration especially for two girls, who, in general type, resemble his mother. Paul never takes girls out. "I don't think it's just money," he says. "I think if I were really interested in taking them out I would, but I'm not interested." He complains about modern women that "they are more interested in a job than in home life." In return, the girls naturally do not care much for his company. One girl comments, "He has a really incredible mind, but to tell you the truth I don't care for him socially. When the class gave a party, and every one was buzzing around enjoying themselves, Paul was talking about the comparative merits of Latin and Greek."

In accordance with the philosophy "that everything comes in time" Paul does not make demands on life or people but is content with what he gets. Since his philosophy also implies success in life as merit for one's honest efforts, he tries to plan his life so that he will be prepared. In an interesting comment about "when youth ends" he says, "You are young

as long as you are dependent for support from your father. I suppose you are still young as long as you are getting an education and still learning." This compliancy shows itself also in the extreme modesty of his demands. In spite of his happy family life, he has developed the idea that the allowance he receives, and the suits which he gets are investments which his parents make and for which they will get returns when they are old.

As far as his future is concerned, Paul is worried about two things: "Will I get a job?" and "Which course shall I choose?" Paul is a child who has felt the effects of the depression, and this experience has had a definite influence upon his outlook on life. "We live in a new situation," he says. "When my father was young, he was very poor, but everybody believed in those days that he could get rich and there was more opportunity. It was the period of the economic revolution and there was still room for expansion. Those were the days when the little bootblacks became millionaires, though my father didn't start with as low a position as a bootblack. Today there isn't the same opportunity to get rich. Boys of twenty in college don't think any more that they can go out and lick the world. They're more sensible. And boys in high school worry about what they will do after college." This insecurity concerns Paul a great deal. He ponders about the value of experience versus education, of getting a job after high school or attending college. "Well, that's one thing I've thought about a lot. I've thought sometimes that getting practical experience is more important than college. Like when you look through the classified ads you find out experienced men only need apply, you will find very few taken that haven't got experience. And the time to get experience is when you're young. But then again, college gives you training in certain things." In addition to this problem, Paul is not certain whether he should put his emphasis in life on the job or on his family. He expects that there will be a conflict, which is to be avoided by thinking it over carefully. He continues, "Well, you see it all de-

pends on which way you look at the job. I mean, some people look at a job as just a way to make money so that they can live happily with their friends and their family, and some people think that the job is sort of their calling, their main idea in life. That is, there are those two things. There is a social circle and your job, and the idea is which one you think important. Of course, I really haven't decided yet as I haven't had a job." On another occasion, when Paul spontaneously takes up a related problem, it is possible to get closer to the central conflict which expresses itself in his fears related to his future. "This is what I guess is a conflict I have," Paul says. "I used to be a realist. At the beginning of the depression I was very cynical, but lately I've become an idealist. I'll tell you what I mean. I'm supposed to be brilliant and I guess I am a bit—not a lot, but just a bit. Now, is it my duty to give myself to humanity at the expense of my own happiness? Or is it my duty to give myself to myself?" It became clear from the continuation of the interview that humanity and his family were used interchangeably, which goes back to a formulation of the problem previously mentioned.

When the question of marriage is discussed, he is asked whether he will find it important to marry. "Oh, yes," he replies. "Well, I mean I think that to marry is sort of to get a family circle of your own rather than being part of some other family circle, that is, I mean part of my mother's and father's and my aunts' and uncles' family circles, sort of starting of a new one. And of course the thing is if America is going to keep on having people, we have to have new children, and that's another thing." In the education of his own child Paul would more or less repeat what his parents did with him. In discussing discipline he repeats almost the exact words used by his mother when she explained her educational policy with him as a small child. He says, "I'd like to teach him [his son] his obligations and privileges so that he could understand relations with people. For instance, there are some things that people have to learn to do when they are told to do them."

When asked to be specific, he says, "I'm not prepared to talk. I haven't thought about the subject."

Paul himself thinks that he is "not entirely unspoiled." Asked about it he continues, "Well, I don't know. I'm a little self-centered." He implies on further questioning that he thinks a little too much about himself. He sometimes seems to be obsessed by the idea that everything can be resolved by "thinking it over." The process of thinking plays a very important part in his personality. "I think I'm a deep thinker," he said once, not at all smugly, "but I think sometimes I say silly things which make people not respect me." He cannot mention any of the silly things but describes the behavior which he has in mind as "acting the clown."

To complete the picture two of Paul's outgoing interests have to be mentioned: his musical activity and his work in a settlement during the summer vacation. After the ninth grade Paul, who was not going away for the summer, decided to help in a settlement as assistant to a teacher in a play school. He had the feeling that he was wasting his summers away and "after all I am getting a little older and I ought to do something. I've thought that—oh, I don't know where I got the idea." The work in the settlement brought Paul into contact with children and adults. He enjoyed the work and got rather interesting insights into the lives of children from underprivileged homes.

Paul plays two instruments, piano and violin. His interest in music started very early. His mother started to give him piano lessons when he was seven. His musical gift comes from his mother's side. As previously mentioned, Paul knew the notes at the age of seven, having learned them by himself from an encyclopedia. When he was graduated from elementary school he asked for a violin instead of a suit and a watch, which his parents and a relative wanted to give to him. He practised regularly through the years. In going through Paul's interviews and looking at his few writings it is astonishing to note how little reference is made to his music. In his auto-

biography he forgets to mention it at all. There is only one paper in which he deals with music. It is called "Operas? Pooh!" written at the age of 14.9. For the purpose of illustrating his point of view on a musical issue, as well as giving a sample of his writing, this paper follows in its original form:

I am very interested in music generally. It is a hobby of mine. Orchestras, chamber music, and soloists attract me, but I would not go out of my way to see an opera. First of all, the little plot there is, is always absurd, everybody often dying in the end. Then, as most operas are in a foreign language, one must read a libretto afterward to get the meaning and thus cannot appreciate the dialogue or the words of arias. Thus people do not go for the story of operas.

The sets, indeed, are beautiful, but for beautiful sets, I'd sooner see a real play where I can at the same time hear a story with dialogue.

Now with all this aforementioned nonsense going on, one really cannot fully appreciate the music that the orchestra (which you cannot see) is playing. Many people say that the best part of an opera is its overture, in which the orchestra plays alone.

Thus I say, if you are a lover of dialogue and plot, see a legitimate play; if a lover of sets and scenes, go to an art gallery or a play; and if you are a true lover of music, hear a plain, musical concert.

For the past two years, Paul has liked to play chamber music with some other boys. When one of the boys dropped out, a girl wanted to join them but she was turned down by the group. Paul comments upon this incident saying, "We didn't care to have a girl in the group. Well, I think in this case I'm glad there are just the three boys."

In conversations during the interviews Paul frequently has to make sure first that he understands the question correctly. Then, after he has given the answer he asks again for confirmation that his answer has been to the point: "This is what you mean, isn't it?" He shows a tendency to conform to an expected pattern rather than to respond spontaneously to the questions of the interviewer in terms of his present situation.

Reconstructing Paul's Social and Intellectual Development

It must be borne in mind that the data presented are data selected from material that is in itself necessarily incomplete. But in spite of this difficulty we can make an effort to reconstruct Paul's development on the basis of the material which is at our disposal. With the help of interpretive techniques, it is possible to discover enough indications and evidences to build up a picture of Paul's social and intellectual development in terms of fundamental tendencies or personality patterns. This reconstruction is attempted in order to explain the nature of his adolescent personality as consistently expressed in various aspects of his behavior, interests, and relationships.

From what has been said about Paul's relationship to adults, it becomes evident that he accepts their standards. They govern his behavior, and he conforms compliantly without any overt resentment. He identifies himself easily with the demands and expectations of the adult world. This is illustrated in his remarks about the books he reads and in his rather anxious attempt to answer correctly and to the point in talking with the interviewer. It is evident again in Paul's readiness to pass on to his child the same education which his parents provided for him and in his almost literal repetition of his mother's words on this subject. The same trend of accepting adult standards, without any seeming conflict, is further expressed in his resentment about boys who "get into huddles about sex."

This compliancy during adolescence becomes more understandable if we consider Paul's comment about how long youth lasts. According to him, he is young as long as his parents pay for him, as long as he is getting an education, as long as he is learning. While this lasts he is a child and must not make any demands. The compliancy mentioned is the expression of

Paul's identification with his parents, a process that had its origin in infancy.

It has been pointed out that the parents have always been greatly concerned with the child and provided a very warm and secure home for him. Then we must remember that early illness made it necessary for his mother to be very particular about his care, and to attend to him more frequently than is usual with an ordinary child. At the same time the mother, guarding herself against overprotectiveness, used more severe disciplinary measures than she probably would have under more ordinary circumstances. On the one hand the child, through his physical condition, became the center of attention and care; on the other hand, severe demands were made upon him in an effort to avoid spoiling. He lived for years in this contradictory situation between overprotection in bodily care and severe demands in behavior and training. It must not be forgotten that his physical development was seriously hampered by illness and that attention-getting was closely connected with his excretory functions on account of his eczema. ("Attention, Mother, please.") Paul's physical condition made him more dependent upon his mother and less outgoing to other children. The home and the family guaranteed the necessary protection which he needed. A contributing factor may have been the parents' enjoyment, with the child, of the intimacy of a small family group. This must account, in part at least, for Paul's withdrawal from other children and his complete identification with his family and adults in general.

This identification also becomes apparent in Paul's relationship with girls. He accepts unchanged the rules which his father once had laid down for his own conduct with the other sex. Paul postpones his heterosexual experimentation until he is "quite old." What was good for his father must also be good for him. As will be shown later, Paul succeeds in sacrificing the search for personal relationships only by investing other areas of his life with the uncertainty and dissatisfaction which

are inevitable as a result. He does not express directly or even recognize these dissatisfactions. In his attitude toward girls, Paul's struggle over his heterosexual adjustment becomes apparent. He still clings to childhood relationships and does not succeed in weaning himself from the family. His positive attachment to the family is reaffirmed by his remarks about home life in novels and his general references to family life. The same conflict becomes evident again in Paul's criticism that girls are not interested in home life—as his mother is—and also in his choice of girls who resemble her in general type. This phase is common in the struggle for heterosexual adjustment. Paul's latent admiration of girls' grace and beauty, coupled with his remarks concerning marriage (as a means to keep up the American population), disclose so obviously his rejection of emotionality that they need no further comment. Paul is said to be extremely polite and well behaved toward girls, while sometimes irritable and impatient with boys. This justifies some doubt of his statement about girls, when he says, "I am just not interested in them." In disregarding girls as he does, he contradicts the group standards at school and puts himself into the opposition and minority for the sake of "good things." He seeks satisfaction on an academic level, which naturally cannot entirely assuage his general tension.

In order to understand that Paul's adolescence did not evolve new methods of response, but only intensified existing character trends that were established at an early age, one must remember that, even in the first grade, Paul impressed people as a methodical, conscientious child who attacked and completed tasks without external pressures and depended heavily on accomplishment and accuracy for his feeling of adequacy and security. Many references scattered throughout the records further substantiate what has been said, e.g., Paul's accuracy in spelling and punctuation, his objection to wasting time, his liking for grammar, his unusual memory for words.

It has been reported that Paul received corporal punishment from his mother up to the age of five. The child responded to these disciplinary measures very calmly. He never became sullen, never infuriated, never resentful. He always accepted punishment as just. This submissiveness was combined with a tendency never to wish for anything and to be grateful for whatever he received. This undemanding attitude must have been very pronounced, since his parents wished he would ask for things.

The early lack of initiative and the absence of wishes and desires are surprising, especially if we consider Paul's intelligence. The extreme submissiveness which he showed after sudden weaning, and later in childhood, suggests that, as a result of this experience, he repressed his feelings in order to protect himself against a repetition of rejection, dissatisfaction, or frustration. It will be remembered that he became completely passive after the twenty-four hours of crying, and when subdued, finally accepted the bottle. Dietary factors partly conditioned his subsequent behavior.

It is clear from the records cited that in elementary school he found ways and means of compensating after one initial exploration of various forms of attention-getting and various methods of finding his level and securing acceptance by the group. In early adolescence his motor restlessness recurs and his social adjustment undergoes a change. In spite of his compliancy and friendliness, which are repeatedly mentioned, he is actually rather aggressive at school although in a disguised manner.

This aggressiveness at school is, characteristically, not directed against any individual; it is of a general, indirect, diffuse nature. Paul does not show much consideration for the group. He uses his voice with undue loudness, as if he had to impress a fictitious opponent. He has a sharp tongue and can make cutting remarks. He puts himself easily into the opposition, and then defends his points with verbal weap-

ons beyond reason and out of keeping with his highly developed intellect. He claims an undue portion of the teachers' time and has a tendency to ask seemingly unrelated questions, which the teacher consequently has difficulty in answering; some teachers think that he asks questions for the fun of composing them. These behavior patterns are described by the teachers as inconsistent with Paul's constructive and conforming personality.

We must try to understand the apparent incongruity in the whole context of Paul's experience. He uses verbal aggressiveness as an outlet for impulses which he has suppressed and suppresses at home, and by doing so he counterbalances his apparently submissive and compliant attitude. The fact that certain phases of Paul's behavior are determined by inner tensions that seek outlet on a verbal level in a disguised form of aggression can be demonstrated very well by the material.

A common reaction to unrecognized and unadmitted aggressiveness, which the foregoing material suggests as characteristic of Paul's personality, is a particular kind of self-punishment. In contrast to his intelligent responses and his logical thinking, there is repeated reference to his clowning, to his silly behavior. He himself says, "I say silly things which make people not respect me." This behavior, which Paul seems unable to control despite the fact that he regrets it very much, probably represents a self-induced punishment. According to his code of values, not being respected is a severe degradation. It is noteworthy that the self-degradation occurs on the intellectual level where he plays out most of his conflicts, as will be shown later.

One more reference must be made to the problem of aggressiveness. Paul has been preoccupied with the problem of war for many years. It will be recalled that he wrote a pamphlet with a fictitious co-author (to avoid being biased) and sold it from door to door, introducing his sale with the words: "Do you want your boy to be compelled to go to war?" This, as well as his crusading at school against war, and his concern

with the commandment "Thou shalt not kill," so that mankind "will not eradicate itself," indicates his fear concerning his own aggressiveness. Such projection of a personal problem into the abstract realm of ethics and morals is a device frequently employed during adolescence.

In the light of what has been said, Paul's concern about immortality is also of great interest. He finds it "unendurable that there is only mortality." He wonders "why all these consciousnesses come in and out of the world." Here again, as in the war problem and the like, he is predominantly occupied with the question of death, destruction, extinction. "The trouble with people is that they believe there is no life after death," that they believe in violence and extermination. These aspects of life frighten Paul. He can discuss problems like the ones mentioned for hours, loves to do it, but can never arrive at a solution of any of them. They represent on an intellectual level his immediate conflicts (his unrecognized aggressive feelings) which are projected into the universe or the abstract.

This situation has been aggravated by the return of Paul's motor difficulty and his exclusion from athletics. (Competitive sports serve a useful purpose in draining off aggressive feelings for most normal boys.) At least the coincidence of his poorest social adjustment and his exclusion from athletics suggests a dynamic relationship between those factors and the establishment of a vicious circle in which unrelieved aggressive feelings made for unsatisfactory social behavior and social maladjustment in turn increased the aggression. It will be remembered how Paul's attitude toward his schoolmates and theirs toward him changed as soon as he was allowed to participate in sports again. Paul's reaction to his return to the athletic field shows a very healthy progress in his social development. It assures us of his emotional responsiveness and flexibility, to which further reference will be made later. The athletic activity provided an opportunity to get together with boys. Even if it did not lead to personal friendships, Paul enjoyed the companion-

ship of the group. As has been said, Paul was extremely courageous and aggressive in games, which were all of a competitive nature.

The question arises, "Why did Paul make no friends in such situations?" The only answer which we can give, in the light of the foregoing interpretation, is his tendency to attach himself not on an emotional level but according to preconceived intellectual standards. Paul has a definite valuation scale for people; he chooses his friends on the basis of their interest in "good things." This implies that he builds his friendships on mutual interest in mathematics, peace, and philosophical or religious problems. From this we can conclude that these problems which he discusses represent for him not only "good things"—namely, things in which it is permissible to be interested, but also that they are emotionally charged topics, acting as substitutes for interests and feelings which cannot be openly faced because of the conflicts involved. Discussing them is the only way for him to establish a relationship with others. In all of Paul's friendships the discussion of problems stands in the foreground; in fact, discussion and intellectual companionship constitute the essential satisfaction which these relationships offer.

At this point attention should be turned to Paul's intellectual development and specifically to his interest in mathematics. Paul's mathematical bent is mentioned as early as the fourth grade. It gradually expanded until, during high school, it was manifested as an interest in the theoretical aspects of mathematics and physics. The prevailing characteristic of his interest in these topics is the process of mastering problems by abstract thinking. Paul considers himself a thinker. If anything bothers him it has to be thought over. In this belief that an attack with intellectual weapons will resolve any problem, Paul's thinking undoubtedly approximates a magic act. Early in his life he had to retreat from action to thinking; in doing so he assured himself of a satisfactory reconciliation with his mother. This, in conjunction with other experiences,

probably established thinking as his pattern of response to emotional conflict very early.

During adolescence, intellectualization assumes an outstanding rôle in Paul's emotional life. In mathematics Paul operates with symbols and logic and proves things without too much concern about their application. This, as well as his interest in words (single words, conjugation, declension, etc.) indicates the emphasis which he places on the intellectual process as such, neglecting somewhat the human implications of applied symbols. To use Paul's language: he swallows science like pleasurable food. It serves as a satisfaction on a primitive level. It gives him a feeling of power and security. But in spite of his belief in the power of thinking, he expresses repeatedly his feeling that people have to believe in the supernatural, and to resort to authority or to unprovable assumptions. The reason for this contradiction will be taken up later. It is only mentioned here to illustrate the conflict which exists in Paul's mind between reason and belief. From this conflict it becomes evident that the preoccupation with abstract operations does not satisfy Paul as fully as we might expect from the depth of his interest in them. But the satisfaction which he experiences in abstract thinking (mathematics, etc.) does decrease his need for and his readiness in finding valuable experiences in other fields, especially personal relationships. Thus we understand the teacher's remark that Paul is just as much interested in books and ideas as in people.

Paul dislikes English composition, art, and biology. The first one he "chews neatly." He "spits out" art. Obviously art is a field in which one can create only in as far as one is able to let emotional responses advance into the foreground of consciousness. This emotional relaxation is impossible for Paul, who is generally distrustful of emotions and bars them from consciousness. His writings, similarly, have little emotional tone, but are rather factual descriptions. His early poetry is little more than a construction of words with the syllables counted in order to fit the meter.

In this connection Paul's remarks about biology should be recalled. He disliked this subject extremely because the teacher began with mammals. He would have liked to study the primitive forms of life first, dwelling on the hypothetical issue of "how life begins." The study of mammalians brought him in too direct contact with his own present conflict over sexuality from which he retreated by withdrawing his interest, rationalizing that the teacher's approach was faulty.

In contrast to Paul's scientific approach to problems there is—as has been mentioned already—a religious-philosophical concern to which he transfers all his inquietude and insecurity about himself. Despite his interest in logical thinking, he is deeply concerned about a deity, immortality, reincarnation. The persistence and intensity with which he pursues these problems prove beyond doubt that they are of essential importance for his emotional life.

In order to understand those philosophical concerns, one must remember that Paul, with progressing physical maturation, finds himself confronted with emotions of which he cannot readily dispose. This maturation begins and progresses without any noteworthy deviation. But during these years of bodily change his thinking (preoccupation with himself on an intellectual level) assumes a predominant importance. "I think too much about myself," he says, almost with an implication of guilt. With the growth of his intellectual powers and progressing maturation, he extends his thinking into the realm of the infinitely great and the infinitely small. This displacement of emphasis from personal to abstract problems is well known through the study of many adolescents: the nucleus of the conflict can be recognized in the various philosophical and religious disguises.

Most enlightening in this respect is Paul's question as to whether he should give himself to humanity at the expense of his own happiness or whether it is his duty to give himself to himself. Here is the conflict very clearly stated: "Shall I accept the standards of my family, reject my impulses and be

compliant, or shall I throw over all restrictions and accept my maturity with all its consequences?" The same conflict is shown in his uncertainty over the choice between college and experience, if we recall his conception of youth and how long it lasts. For he says that you are young as long as you are getting an education, that you are a child as long as you get food (learning) which is paid for by your father. Accepting a college education means therefore prolonged submissiveness and compliancy, experience on a job would imply immediate independence and self-assertion on a mature level. The same alternative is expressed in his thinking about the comparative importance of native ability and outside stimuli. Paul's emotional conflict over his own maturation is further indicated in his worries about the future. He wonders whether a job should be a means toward building a satisfactory private life (family, friends) or whether it should be a calling and a main idea in life. In each of those alternatives there is the same conflict revealed: "Shall I follow my conscience and remain a child or shall I follow my impulses and grow up?" This conflict, with its alternating attitudes toward maturation, is also acted out in the many philosophical-religious speculations. More could be said to substantiate this statement, but it must be left to the reader to gather further points of reference which are abundant in the material.

After all that has been said it must be emphasized that in spite of the difficulties in his emotional adjustment Paul makes repeated efforts to extend his personality into other areas of experience. So we hear that he offered his services to a socially significant enterprise when he went into the settlement summer play school as an assistant. "After all, I'm getting a little older and I ought to do something." Furthermore Paul is interested in music and gets together with other boys in order to play chamber music. As we know, the musical interest in the family comes from the mother's side; she herself is a great lover of music and introduced him to the piano at the age of seven. There is no doubt that music plays an important rôle

in Paul's life. It opens an outlet to his emotionality on an esthetic level and serves as a socializing activity. This, as well as his enthusiastic participation in sports, indicates that he makes some effort to enrich his emotional experiences and to escape a too great intellectualization of his personal and immediate problems.

The Rôle of Mathematics in Paul's Personality

From the foregoing it has probably become evident that Paul's mathematical interest and talent, which is an intricate part of a total situation, can be singled out for consideration only artificially. Nevertheless, it may prove valuable to summarize what has been said in reference to mathematics.

In the first grades of the elementary school, which Paul entered in a physically unfavorable condition, he impressed the teachers as having a methodical, mature mind. In contrast to this he was babyish in manner and rather unstable in his behavior toward others. He managed reading easily. In the third grade he strove for attention and was rebuked by the group. Number work soon assumed an emotional significance for Paul which was assisted by a highly developed intellectual capacity. His physical condition probably caused Paul to turn toward compensatory strivings in other fields at an early age. Other determining factors have been mentioned in the foregoing discussion.

Paul's interest in mathematics and physics has become outstanding during adolescence. The theoretical side of everything absorbs him greatly. It is on the basis of knowledge and intellectual interests that he builds his relationships with schoolmates. Whereas the achievement in mathematics and physics gives him a feeling of accomplishment, it does not completely satisfy him. The conflict between reason and belief, conscience and impulse, causes his preoccupation with religion and philosophy. The exact sciences represent security and protection from his emotionality which attaches itself to the thinking process rather than to any content. The meaning

of thinking in Paul's personality has been stressed at length as a preoccupation with himself on an intellectual level. This as well as his attitude toward science, as something to be swallowed and enjoyed, indicates the persistence of infantile patterns. In consequence Paul's human relationships are impoverished, and his expressions in writing are inhibited, argumentative, factual, and bare of any feeling tone. There has always been the tendency to de-emotionalize relations and behavior.

Mathematics has served as a protective device against conflicting feelings; in fact, it has absorbed a great deal of his emotional life, providing for the expression of certain powerful drives in an impersonal and intellectualized form. Furthermore, mathematics has provided Paul with a feeling of self-assertion and adequacy which he was denied in other fields. With the help of other constructive influences in his life, Paul in later adolescence has offset the escape into intellectualization by enthusiastic sharing of group interests (sports, music, social work). These socializing experiences prevent him from investing an undue portion of his emotional drive in his intellectual interests. As could be seen at the close of the records, the investment of energy in other areas of experience has not made him less successful in the field of mathematics, and it has made him a happier and a better adjusted person.

This summary is necessarily incomplete. Owing to the intricate interdependence of all parts in any personality, the significance of a specific characteristic (like interest in mathematics) can be fully understood only if we consider it as part of a total constellation of traits and patterns. The summary therefore should be read in the light of what has been said before, and it must be left to the reader to interrelate again what has here necessarily been taken out of its context.

The Significance of the Individual Case

It would be a great mistake to conclude from the foregoing case that all gifted mathematics students are seeking these same

satisfactions or that all young people whose problems parallel Paul's will find their outlets in this same kind of work. There are many possible causes for an intense interest of this sort. One child will find mathematics emotionally satisfying because of his need to identify himself with an admired parent or mathematics teacher. Another may develop his mathematical bent in rivalry against a brother or sister against whom he feels unable to compete in some other way. It is not unusual for an unattractive or emotionally immature girl to compensate for her lack of social success through some such intellectual striving, and the same may be true of boys. One child may express through the abstract processes of mathematics or science a desire for power which, for some reason connected with his past, he cannot exercise more openly in personal relationships; another may find in the ordering of abstract symbols some relief from emotional confusion arising in disturbed family relationships. There are innumerable possible explanations for the emotional appeal that a particular subject may have for a particular child. Only in the light of his personal history is it possible to say which the true one is.

Nor can it be too strongly emphasized that similar problems will not always be expressed through similar interests. The projection of personal problems into the realm of the abstract and intellectual—which we noted as typical of Paul's behavior —can occur in any field of interest, not merely in mathematics or science. One boy becomes a lawyer or a politician because of close identification with a father in that profession, another for exactly the opposite reason, using the father's own profession as the ground on which to fight him. The same is true in many other fields. Aversions to certain subjects have similar explanations. They may express a disguised hostility to some person closely identified with the subject or a positive relationship to some one who also dislikes it; they may represent withdrawal from competition with an intellectually superior parent or brother or sister—or a disguised revolt against the pres-

sures and standards of an intellectual family. And so on and so on.

Paul's case serves only to illustrate the larger principals involved in interpreting interests and behavior. It is not offered as a ready-made explanation to fit other cases.

What then are some of these larger principles which this material illustrates?

First, perhaps, that all behavior is meaningful—that it serves some definite purpose for the individual, though often one quite unknown to him—and that even the most trivial or apparently inconsistent aspects are worthy of attention in the investigation of personality. Thus we noted that Paul's clowning was important as a bid for group attention in his early years—that its persistence in adolescence, despite his honest desire to eliminate it, amounted to a form of self-induced punishment for buried aggressive feelings. Similarly we found that his loud voice, his need to be in the opposition even at the expense of logic and common sense, his cutting remarks followed by contrition, his passionate concern with war, with death, and with life thereafter, were all indirect expressions of aggressive feelings in a personality which could not express aggression directly. We noticed that the apparent inconsistencies in this personality were not inconsistencies at all, but meaningful devices for meeting explainable needs and relieving tensions.

Second, and closely related to the above, is the principle that all aspects of personality fit into a consistent pattern and that this pattern is built up gradually in response to individual experiences. In Paul's case the personality pattern involved an unusual degree of submissiveness and compliancy toward adult demands coupled with a need to suppress natural aggressive feelings and to project all conflict upon an intellectual plane. Submissiveness, denial of emotionality, and disguised aggression were disclosed as a consistent whole that became understandable only in the light of Paul's early history. It was clear

from the case material not only that his earliest experiences in connection with weaning, physical disability, toilet training, and discipline were involved in setting this pattern, but that the tendencies thus established had been reëinforced by subsequent experiences—the unusually close and satisfying family relationship, on the one hand, which reduced the chances for other social contacts and increased the tendency to identify with adults; the physical handicap on the other, which increased the need to compensate intellectually, and decreased the usual outlets for aggressive feeling.

This case material illustrates, too, the typical conflict in adolescence between the inner pressures of maturation and the external demands of the family and the larger social group. And again we note that the response of the individual to those strains must be in terms of his own gradually evolved personality pattern. Paul betrayed a deep-seated conflict between his desire for independence and his sense of obligation to his family; between the demands of his reawakened emotions and his acceptance of his father's standards; and these conflicts were manifested characteristically in a disguised intellectual form, in his philosophic questionings, in worries about his obligation to society and the choice between college and experience, in denial of his own interest in girls. This need to deny and to intellectualize the conflicts, as well as their very intensity, are a direct reflection of the individual's early experience, as the case makes clear. The effect of the general social setting is also apparent in the pressure exerted by the pattern of boy-girl relations typical of Paul's school, and the intensified worries about his economic future resulting from his experience in the depression.

We see also in this case an illustration of the fact that when powerful emotions and feelings are suppressed, they still persist and exert their influence in the personality. When denied conscious expression, they betray themselves indirectly through various devices, some healthy and others undesirable. All education for civilized living involves suppression of some emo-

tions and impulses. Part of our task as educators is to help supply healthy, useful forms of indirect expression. It is only when too great a proportion of the emotional drive is thus diverted, or when the substitute expression is useless or detrimental to the individual, that there is cause for concern. Enough has been said already to make amply clear the ways in which suppressed aggressions and denied emotional conflicts betrayed themselves throughout Paul's behavior and interests.

Finally, this case material is valuable in illustrating many of the psychological devices that were described earlier and that explain typical adolescent behavior, and in fact, human behavior in general. We see in Paul's case an excellent example of intellectual *compensation* for social and physical inadequacies, of *projection* of problems from the emotional to the intellectual level, of the use of intellectuality as a form of escape. We note the use of submissiveness to secure love in infancy and the continuance of this pattern in all subsequent relations with adults. We have clear illustrations of a child's *identification* with loved parents in building up his conscience and standards; of his dependence on the mother relationship in formulating his attitude toward girls, on the father relationship in accepting his masculine rôle. We note the persistence of certain infantile patterns established on the basis of deprivation during childhood.

Thus devices and processes that have been explained theoretically above become more concrete as illustrated in the life of a particular child, and many of the general principles of human development may be recognized in connection with the study of Paul's development.

THE IMPLICATIONS FOR TEACHING

Obviously the individual classroom teacher has neither time nor training to investigate the past experience of each of his students, nor is there any need for him to do so. Case histories, like the one summarized in this chapter, are useful

in increasing our fundamental understanding of personality development and are presented to the teacher for study purposes.

What then is the application of all this to the work of the ordinary classroom? Though the teacher must of necessity put his emphasis on the demands of the immediate situation, his teaching will be immeasurably enriched both in its value to his students and in its satisfactions to himself if he learns to view day-to-day behavior in the classroom in terms of the clues it yields to the fundamental trends of his students' personalities and to their long range developmental needs. If he is able to recognize the purposefulness of all behavior, no matter how trivial, and if he learns to deal with it in terms of its deeper meanings rather than its surface aspects, much that has seemed merely annoying or extraneous to the real business at hand will become for him a significant part of his basic task; much that he has been quite ready to accept and even foster may call for reconsideration and redirection.

He may be teaching a subject that is new to his students, first-year algebra or geometry, for example, or he may be teaching individuals with whom he has had no previous contact, yet he cannot expect his students to approach either the subject-matter or himself with neutral attitudes and without feelings. As has been repeatedly pointed out in these pages, the use that the student will make of any learning experience, his positive or negative reaction to the material, to the teacher, to the other members of the class, will be in terms of his own previous experiences and the personality pattern to which they have given rise.

And equally true, though perhaps less stressed in our discussion thus far, is the fact that the teacher himself inevitably reacts to student behavior and student interest in terms of his own human needs and his own basic personality. He quite inevitably finds himself curiously in sympathy with certain students and unreasonably aggravated by others, challenged by certain types of behavior, or disproportionately irritated

or distressed or even threatened. He reacts to an extent he seldom suspects in terms of his own inner needs and his own problems. He tends to feel close to those students whose interests or problems parallel his own, to be less interested in or even unduly annoyed by others on equally personal grounds. Of this he may be totally unaware, recognizing only his likes and dislikes, not the reasons behind them. He may find hostile adolescent behavior peculiarly hard to face; he may be aggravated beyond reason by overdependent students, or he may be tempted to let them lean upon him too heavily; he may need the affections of his students too much, or be unable to accept them at all. To a large degree such reactions will depend on the maturity of his own adjustment to authority, to independence, and to love. So in a hundred other ways he may unwittingly inject himself into what seems to be an objective situation. He, like his students, has an emotional response to the subject he teaches, to teaching itself. His choice of profession and his field of work are important to him because he has chosen them in answer to his own needs. A slight to them becomes a slight to himself. Lack of interest or success on the part of a student presents a practical problem which the teacher must meet, and it may also involve a deeper threat to his own security or pride by indirectly calling into question compromises and solutions he has made in adjusting to his own problems.

Personal reactions of this sort cannot be entirely avoided. They are part and parcel of our basic psychological make-up. But the more consciously such aspects in a situation can be faced, the less they tend to distort and hamper our relations to others. The extent to which a teacher recognizes the deeper meanings in his own reactions and those of his students will be a fair measure of the objectivity which he achieves in dealing with them. He needs to understand not only his students' problems, but his own as well.

This by no means implies that the classroom teacher can or should assume full responsibility for dealing with the more

difficult problems of maladjustment which one sees occasionally in every high school. Without question his sympathetic and understanding attitude may be helpful and his observations highly valuable. It is a task for the professional guidance worker to unravel the hidden factors in such situations and it is fortunate that the need for such counselors is becoming more widely recognized and that our schools increasingly provide such services. Close coöperation between subject teachers and these professional counselors can be of immeasurable value both in meeting the more baffling student problems and in enriching the classroom teachers' understanding of all their students and themselves.

How then can the teacher distinguish between normal adolescent behavior with which he can deal in ordinary commonsense fashion and those more serious personality problems that indicate the desirability of professional help? While the trained observer learns to recognize the difference by countless minor clues in behavior, it is no easy matter to set up rule-of-thumb standards. Perhaps the best rough measure of a child's general adjustment at this level is his relationship to other boys and girls—his acceptance by the group—his ability to participate in group concerns with satisfaction. One can judge too by his prevailing mood—a general air of happiness and contentment as contrasted with a chronic moodiness or irritability—though occasional passing interludes of depression or irritability are quite normal at this age. In the school situation, an additional clue may be found in comparing teacher reactions. The child who seems to present markedly different fronts to different people, whose teachers disagree widely in their judgment of his personality and his abilities, is apt to be a child in difficulty. Finally, one might watch with concern the child who is constantly in disciplinary difficulties (one who seems to need to get himself punished) and his exact opposite, the child who never gets into any trouble at all (one who accepts adult standards too completely and makes no move toward independence). Where professional guidance is available these are the boys

and girls who should have such help and about whom the teacher may well seek advice.

But every secondary-school teacher must deal with countless minor problems that represent the temporary difficulties in adjustment of perfectly normal boys and girls subject to the peculiar stresses and strains of the adolescent years. It is hoped that the foregoing discussion of the nature of these difficulties and the behavior to which they give rise may be helpful to teachers in lessening needless anxieties and irritations and in suggesting suitable responses. For teachers of adolescents must learn to accept calmly and deal tolerantly with behavior that would be a legitimate cause for serious concern in adults or younger children. Some bizarre and even neurotic manifestations are to be expected in the most normal adolescents. Primarily it is hoped that this chapter may help the individual teacher to visualize the part that he can play in developing the child's total personality and lead him to adjust his demands —insofar as the school situation permits—to the long range needs of his individual standards.

Bibliography

I. *Adolescent Psychology*

Brooks, Fowler D., *The Psychology of Adolescence* (Boston, Houghton Mifflin Co., 1929). Gives the results (in usable form) of many investigations of all phases of adolescence. Stresses the need for observation, accurate measurement, and critical interpretation. Addressed to teachers and parents. Not especially interesting reading.

Conklin, Edmund S., *Principles of Adolescent Psychology* (New York, Henry Holt and Co., 1935). Intended for all people dealing with adolescents. Excellent bibliography from which author draws his data. As thoroughgoing as any book yet published on the subject. Comparatively easy reading.

Dimock, Hedley S., *Rediscovering the Adolescent* (New York, Association Press, 1937). Reports a study of adolescents, presenting comparative data on physical development, social adjustment, play activities, emancipation from parents, etc., but makes no attempt to explore the more deep-seated causes of behavior.

Hollingworth, Leta S., *The Psychology of the Adolescent* (New York, D. Appleton-Century Co., 1928). Formulates the problems of the

adolescent in our culture. Easy reading. Addressed to parents and teachers.

RICHMOND, Winifred, *The Adolescent Girl* (New York, The Macmillan Co., 1925). Discusses typical problems of normal, abnormal, and delinquent girls in simple terms and with real understanding.

SCHWAB, Sidney I., and VEEDER, Borden S., *The Adolescent, His Conflicts and Escapes* (New York, D. Appleton-Century Co., 1929). Describes the physical and mental changes of adolescence and the conflicts inevitable at this period. The joint work of a neurologist and a pediatrician. Difficult reading, but interesting.

TAYLOR, Katherine Whiteside, for the Commission on Human Relations (Progressive Education Association) *Do Adolescents Need Parents?* (New York, D. Appleton-Century Co., 1938). Discusses the parents' rôle in the guidance of adolescents and the needs and problems of young people today. Addressed to parents but helpful for teachers also. Easy reading. Contains excellent topical bibliography.

THOM, Douglas A., *Normal Youth and Its Everyday Problems* (New York, D. Appleton-Century Co., 1932). Discusses typical adolescent problems and common-sense guidance procedures. Friendly and helpful in tone and easy to read, but not exhaustive in its analysis of the problems presented.

WILLIAMS, Frankwood E., *Adolescence* (New York, Farrar and Rinehart, 1930). A mental-hygiene study of adolescents, stressing social adjustment. Includes educational implications. An important book. Well written and interesting.

II. *Youth in the World Today*

BELL, Howard M., *Youth Tell Their Story* (Washington, D.C., American Council on Education, 1938). Presents in popular, readable form the results of an investigation conducted for the American Youth Commission. A valuable survey of the actual conditions faced by young people today and their attitudes toward their own problems.

BROMLEY, Dorothy Dunbar and BRITTEN, Florence Haxton, *Youth and Sex: A Study of 1,300 College Students* (New York, Harper and Bros., 1938). Summarizes the results of a survey of changing sex attitudes and mores among American college students of both sexes. Journalistic in style but objective; interesting to all concerned in the guidance of young people.

DELL, Floyd, *Love in the Machine Age* (New York, Farrar and Rinehart, 1930). Interprets our present age and the attitudes of today's young people in a forward-looking and challenging manner. Very interesting reading.

LINDLEY, Betty and Ernest K., *A New Deal For Youth* (New York, The Viking Press, 1938.) Describes the work of the National Youth Administration and the problems it is trying to solve. Recommended

for its factual data on present-day conditions despite its somewhat overenthusiastic evaluation of the N. Y. A.

III. *Mental Hygiene and Family Backgrounds*

AICHHORN, August, *Wayward Youth* (New York, The Viking Press, 1935). Reports the work of a distinguished psychoanalyst in his training school for delinquent children. Helpful for parents and teachers of normal adolescents because of the insight it gives into the fundamental causes of adolescent maladjustment. Interesting reading.

Child Study Association of America, *Sex Education: Facts and Attitudes* (New York, 1934). Problems and practices in sex education discussed by nine authorities from the fields of medicine, psychiatry and parent education. Covers parental attitudes, the guidance of children and the problems of young people. Easy to read.

FISHER, Dorothy Canfield and GRUENBERG, Sidonie Matsner, editors, *Our Children* (New York, The Viking Press, 1932). Twenty-nine papers by authorities in medicine, education, and child guidance. See especially Ch. XV, "The Meaning of Maturity," Adolf Meyer; Ch. XVI, "The Family Drama," Bernard Glueck; Ch. XXII, "Character in Adolescence," Thomas J. McCormack; Ch. XXVII, "In Quest of Life Values for the Machine Age," Lawrence K. Frank.

FLÜGEL, J. C., *The Psycho-analytic Study of the Family,* 4th ed. (London, Hogarth Press, 1931). A basic book on emotion in relation to the family, family influences, and origins of conflict in the family. Solid reading.

FOLSOM, Joseph Kirk, *The Family* (New York, John Wiley and Sons, Inc., 1934). A scholarly integration of the various scientific approaches to the study of family life, sociological, psychological, and anthropological. Difficult but rewarding reading.

———, editor, *Plan for Marriage* (New York, Harper and Bros., 1938). Frank and illuminating discussions of the problems surrounding sex adjustment and marriage for young people today. Prepared by staff members of Vassar College and addressed to college students but invaluable for all concerned in guiding adolescents. Well written and interesting.

FREUD, Anna, *Psycho-analysis for Teachers and Parents* (New York, Emerson Books, Inc., 1935). Presents the psycho-analytic concept of child development in clear and simple form. Helpful for an understanding of the continuity of the emotional life.

GROVES, Ernest R., and BLANCHARD, Phyllis, *Introduction to Mental Hygiene* (New York, Henry Holt and Co., 1930). Draws together from psychiatry, psychology, and sociology material of value in the study of human conduct from a mental-hygiene point of view. Addressed to the college student and general reader. Easy reading.

HARTWELL, Samuel W., *Fifty-Five "Bad" Boys* (New York, Alfred A. Knopf, 1931). Psychiatric study and treatment of problem boys.

Presents an especially good treatment of techniques. Interesting and readable.

HEALY, William, *Personality in Formation and Action* (New York, W. W. Norton and Co., 1938). A scholarly and temperate presentation of the formation of personality, addressed to the lay reader of good educational background.

———, and BRONNER, Augusta F., *New Light on Delinquency and Its Treatment* (New Haven, Published for the Institute of Human Relations by Yale University Press, 1936). A study of delinquent children which stresses the emotional problems arising in the family as causative factors. Useful to all those concerned with child guidance.

LEE, Porter R., KENWORTHY, Marion E., and others, *Mental Hygiene and Social Work* (New York, The Commonwealth Fund, 1929). Prepared primarily for students of social work, but the chapters on child guidance in Part I are especially helpful for teachers. Not difficult to read.

MORGAN, John J. B., *The Psychology of the Unadjusted School Child* (New York, The Macmillan Co., Revised Edition, 1936). An application of mental-hygiene principles to the ordinary problems of school life. Addressed to teachers. Interesting and practical.

PLANT, James S., *Personality and the Cultural Pattern* (New York, The Commonwealth Fund, 1937). Analyzes home, school and community institutions as they operate to modify the personalities of our children. Difficult to read but an important book.

RICHMOND, Winifred V., *Personality, Its Study and Hygiene* (New York, Farrar and Rinehart, 1937). Outlines the modern approaches to the study of personality, emphasizing the normal rather than the pathological. For the lay reader of good educational background.

RYAN, W. Carson, *Mental Health Through Education* (New York, The Commonwealth Fund, 1938). Criticizes our present educational practices as tested by mental-hygiene principles and offers constructive suggestions. Easy reading.

SAYLES, Mary Buell, *The Problem Child at Home* (New York, The Commonwealth Fund, 1928). Gives case records illustrating common problems and a general treatment of child development and psychology, with special attention to the rôle of parents and the home. Interesting reading.

———, *The Problem Child in School* (New York, The Commonwealth Fund, 1925). Gives case records illustrating some common types of maladjusted school children, with attention to the work of the visiting teacher. Interesting reading.

SHERMAN, Mandel, *Mental Hygiene and Education* (New York, Longmans, Green and Co., 1934). Designed to increase the effectiveness of teachers in respect to the general adjustment of school children. Simple and easy to read.

STERN, Bernhard J., *The Family: Past and Present* (New York, D. Apple-

ton-Century Co., 1938). (Prepared for the Commission on Human Relations of the Progressive Education Association.) Gathers source materials for the study of changing forms and contemporary trends in family life. Interesting and readable.

STRAIN, Frances B., *New Patterns in Sex Teaching* (New York, D. Appleton-Century Co., 1934). Approaches sex as an integral aspect of all living. Describes the sex life of children from infancy to adolescence and suggests helpful ways of answering children's questions. Addressed to parents. Easy reading.

SYMONDS, Percival M., *Mental Hygiene in the School Child* (New York, The Macmillan Co., 1934). An introduction to mental-hygiene principles addressed to teachers, offering practical suggestions for the handling of typical classroom situations. Not exhaustive, but readable and helpful.

WHITE, William A., *The Mental Hygiene of Childhood* (Boston, Little, Brown and Co., 1919). Examines child psychology and the relations between parents and children from the psychoanalytic point of view. An outstanding contribution. Easy reading.

WICKMAN, E. K., *Children's Behavior and Teachers' Attitudes* (New York, The Commonwealth Fund, 1928). A study of teachers' attitudes toward children's behavior. Contrasts the judgments of teachers with those of mental hygienists on behavior commonly observed in the classroom. A significant book.

ZACHRY, Caroline B., *Personality Adjustments of School Children* (New York, Charles Scribner's Sons, 1929). Presents the points of view and techniques in education, psychology, and mental hygiene which bear upon personality integration. Illustrated with case histories. A book for classroom teachers. Interesting and easily read.

IV. *Young People in Various Cultures*

BENEDICT, Ruth, *Patterns of Culture* (Boston, Houghton Mifflin Co., 1934). An approach to the relation between the psychology of the individual and the patterns of culture through a study of three distinct American Indian cultures, each permeated by one dominating idea. Good but solid reading.

MEAD, Margaret, *Coming of Age in Samoa* (New York, William Morrow and Co., 1928). An anthropological study of adolescence in Samoa. Propounds the thesis that the characteristic behavior of adolescence varies with the culture. Excellent reading.

———, *Growing Up in New Guinea* (New York, William Morrow and Co., 1930). A study of the development of the Manus child from infancy to adulthood, comparing the cultural and educational ideals which surround him with our own. Interesting and readable.

WILLIAMS, Frankwood E., *Russia, Youth, and the Present-Day World* (New York, Farrar and Rinehart, 1934). Attempts to assess the effects of the Russian way of life upon the mental hygiene of masses of young people. Addressed to the interested public. Easy reading.

XIII

THE EVALUATION OF STUDENT ACHIEVEMENT

In other parts of this Report the Committee has given a statement of the purposes of general education and has discussed some of the more important concepts and special objectives of mathematical instruction. The belief that instruction in mathematics can make significant contributions toward the development of a number of desirable abilities and qualities of personality has also been expressed. It is therefore necessary to face another question of great importance. How can teachers determine the extent to which students are making progress toward the achievement of these objectives? How can teachers evaluate their work?

A carefully planned program of evaluation is important for several reasons. Whoever approaches the problem from the standpoint of the individual is interested in studying his development. What are his personality traits? What does he know? Is he growing in the ability to do reflective thinking? What are his attitudes? Answers to these and to many other questions are necessary in any attempt to draw a comprehensive picture of an individual. Society (in particular as it is represented by parents, employers, and institutions of higher learning) has a right to expect such a picture. The student has been under the school's guidance for some years. What has been the direction of his growth? How much has he changed? Evaluation plans are essential if a sufficient body of unbiased data on these points is to be available.

Evaluation is also seen to be important if the problem is approached from the standpoint of the educational program as a whole. How is one to judge among the relative merits

of different educational programs? If educators are to do more than offer unsupported opinions on such questions some means of obtaining evidence is necessary. Thus nearly every one today believes in evaluation techniques of some sort. But in order to form valid judgments concerning many educational questions one must devise, and use extensively, means of evaluation which are far more comprehensive and probing than those in general use today. Some of the characteristics of such an evaluation program are discussed in the following section.

A third important purpose of evaluation is to provide a basis for the guidance of students. School administrators and teachers now take into account the fact that not all students have the same abilities, interests, and needs and they no longer expect them all to achieve the same educational goals. In order, however, to discover as precisely as possible the particular abilities, interests, and needs of a given individual, plans for making careful observations and measurements are demanded. In the light of these data the student can be helped to plan an educational and life program designed to bring about the best possible realization of his potentialities. Guidance even on the basis of all available data is a complicated and delicate problem. But guidance without data is folly.

For reasons such as these, evaluation is now recognized as an educational task of crucial importance. Clearly, it should not be considered as something apart from the curriculum, or as a problem to be attacked and solved only after curriculum and method have been decided upon. The threads of these problems are so interwoven in the educational fabric that separation of them destroys the fundamental design, although unfortunately this point has not always been acknowledged in the past. Moreover, in order to achieve the purposes mentioned above to the fullest extent an evaluation program should have certain characteristics which are often not clearly recognized.

GENERAL CHARACTERISTICS AND METHODS OF EVALUATION

The teacher of mathematics has as a rule little difficulty with the construction of classroom tests for a limited range of objectives. But relatively few teachers have given much thought to more general considerations relating to the characteristics of satisfactory evaluation and to methods of securing it. The discussion that follows aims to provide background and breadth of vision rather than to give suggestions for achieving technical proficiency in test construction.

Some Characteristics of a Satisfactory Evaluation Program

Balance and comprehensiveness. The most frequent criticism of the typical educational test is that it measures little more than the ability to recall information. Scores on mathematics tests are often said to depend more upon the ability to carry through formal operations than upon successful achievement of any other objective. One need not pass upon the justice of these criticisms in order to recognize that the ordinary testing program gives little direct evidence about achievement of many of the objectives discussed in earlier chapters. Consider, for example, traits of personality such as creativeness, coöperativeness, social sensitivity, and the ability to think reflectively. To what extent do the tests given by the average teacher yield evidence of these qualities? If these and similar characteristics are considered the real goals of education, the school should obtain evidence that they are being achieved—evidence of growth with respect to such a range of objectives as will provide proper balance or distribution of emphasis among the various goals of education. Thus an ideal for a satisfactory evaluation program is the development of balance and comprehensiveness.

Measurement of growth. A satisfactory evaluation program should be designed to reveal more than status at a given time. It should yield evidence of growth or progress toward

the important objectives, and it should also tell something about the extent to which abilities once developed are retained. This means that the evaluation must sample the same sorts of behavior at regular intervals over a considerable range of time—ideally it would be an almost continuous process. Efforts must be made to determine initial status, provision should be made to measure periodically the increments of growth which it is hoped are being made, and data on the permanency of the traits developed should be sought. Only under such conditions can schools measure their efficiency properly and modify their practices intelligently.

The teacher of mathematics may hope that students from the seventh grade onward through the high school will make more or less steady progress toward the achievement of certain objectives—for example, the understanding of and ability to use the concept of proof. In order to evaluate such progress, tests should be so constructed, administered, and the data so recorded, that growth or lack of growth becomes apparent. Though most teachers give tests rather frequently (monthly, weekly, or even daily), the objectives measured are usually quite specific, and the tests are focused primarily upon knowledge supposedly acquired since the previous test was given. The measurement of growth or change should mean more than this—not in the sense that tests should be given more frequently, but rather that the testing program as a whole should be so designed as to make possible the discovery of the cumulative effect of work toward general and overarching objectives.

Teacher and student participation. Evaluation programs often have been imposed upon schools from without. Programs have been planned and tests have been prepared by examining bodies not always in close touch with actual classroom situations. In such cases it not infrequently happens that the objectives that the tests are designed to cover are not the same as those which the teachers are trying to help the students attain. Differences of this sort are not al-

ways the fault of those who construct the tests. They also occur because teachers in service have failed to keep in touch with developments in their field. Such difficulties can be at least partially removed if arrangements are made for close coöperation between examining bodies and teachers in the field. By these means the persons who construct extramural examinations can keep in touch with the thinking and practice of progressive teachers, and the teachers may become better informed concerning objectives and content which have recently been receiving more attention. Thus active participation in the planning of evaluation methods on the part of the teachers who are to use them is a highly desirable characteristic of an ideal program.

A second point sometimes overlooked is that evaluation depends upon student coöperation. Thus under ideal conditions the objectives of the teacher and examining board are accepted by the students and become their objectives also. When this is the case, students may participate in planning the examinations. This does not necessarily mean that students prepare specific questions, but it does mean that they may profitably share in determining the sorts of things which are to be tested. When this is achieved, one may expect a more wholesome attitude toward examinations than is now commonly found.

Curricular freedom. If an evaluation program is too rigidly prescribed, it tends to stifle fruitful experimentation. If objectives are stated in terms of specific content, or if "ground to be covered" is outlined in great detail, teachers have little freedom to adapt their courses to their individual situations. Accepting the view that this is undesirable, one is led to seek ways of stating objectives that are sufficiently specific to serve as definite goals, yet do not put the curriculum into a straight-jacket. A promising method of doing this is discussed in a later section. At the moment it suffices to say that insofar as possible an ideal evaluation program provides for a wide

range of freedom in the choice of particular curricular materials and teaching methods.

Diagnostic features. The acceptance of the doctrine that an evaluation program should be designed to obtain evidence about achievement with respect to all important objectives leads to the formulation of a fifth characteristic. Evaluation should provide for diagnosis of individual strengths and weaknesses. For example, such diagnosis may provide evidence that a certain student excels in the ability to recall information, and yet is relatively weak in the ability to *apply* facts and principles for the purposes of explanation and prediction of phenomena arising in situations new to him. Again students may have an excellent command of the technical terminology of a given subject and at the same time make serious errors in interpreting data encountered in the study of that subject. The early discovery of students who exhibit such variations in behavior makes it possible to take remedial measures designed to maximize achievement.

Diagnostic evaluation is important not only because it shows which objectives need special emphasis, but also because of the way it contributes to the writing of reports that give a more complete picture of the individual. The notion that a single numerical or letter grade is adequate for the description of educational achievement is rapidly passing. In the future reports to parents and to higher institutions may be expected to present data of a more descriptive character. A comprehensive diagnostic evaluation program is essential in order to secure data for these more revealing reports.

Technical qualities. The two most important technical characteristics which the evaluation instruments used in the program should have are validity and reliability. There are also other characteristics that are sometimes desirable. Thus the measurement of the growth that a given unit of instruction produces may demand an instrument capable of detecting small increments of ability. More tests of general

abilities are needed with technical precision of a high order.

The validity of an evaluation instrument (test, questionnaire, record, etc.) refers to the extent to which it measures what it is supposed to measure. In determining validity some criterion external to the evaluation instrument itself is needed. The nature of the criterion used varies widely in the case of different tests, but in general the basic consideration consists of a statement of the objectives for which the instrument is designed to measure achievement. If the statement of objectives is sufficiently precise and descriptive, and if the instrument is properly constructed, it becomes possible to determine whether the behavior revealed by the evaluation instrument corresponds to that specified by the objectives.

The reliability of an evaluation instrument is a measure of its accuracy. If a test has high reliability it will yield the same interpretation of behavior when taken again by the same students under similar conditions. In order for a test to be reliable it is usually necessary to insure that it be scored in the same way, and the results interpreted in the same way, by different competent persons.

Emphasis upon intelligent interpretation. An evaluation program involves more than the collection of good data. If the program is to realize its purposes, very careful attention must be given to problems of interpretation. It is not enough to obtain and record raw scores on a variety of tests. It is especially important to recognize that adding such scores and averaging them may actually conceal significant aspects of individual behavior. When scores are averaged, two students may have the same average score, yet vary widely in the achievement of particular abilities that contributed to the average. The data must also be studied in relation to the interests and attitudes of the pupils, their home, school, and community background, and many other such factors. An increment of growth on the part of a given student may far exceed that of another, yet the smaller growth may actually

be very significant because of the circumstances under which it was achieved. This analysis and synthesis of data to form a complete picture of individual and school achievement is the most difficult and at present one of the weakest parts of evaluation programs.

Major Steps in the Process of Evaluation

Most teachers are so thoroughly experienced in test construction that it has become almost a routine task. This is possible when evaluation is narrowly conceived and limited in scope. But evaluation from the point of view under discussion is far from simple—it is an educational problem of considerable complexity and difficulty. In solving a problem of this magnitude the formulation of a general method is often helpful, and the outlines of such a formulation will now be discussed. A more complete discussion may be found in the references given.[1]

When evaluation is approached from a broad point of view and the technical details are suppressed and it becomes clear that at least four fundamental steps are involved; namely,

1. Deciding upon the objectives with respect to which achievement is to be measured.
2. Finding situations related to the objective which provide opportunities for the student to exhibit the desired behavior.
3. Getting a record of what the student does when faced with the situation.
4. Interpreting the record.

The formulation of objectives. The first step in any evaluation program is to decide upon the objectives with respect to which achievement is to be measured. There are several

[1] See R. W. Tyler, *Constructing Achievement Tests* (Columbus, Ohio, Bureau of Educational Research, Ohio State University, 1934); H. E. Hawkes, E. F. Lindquist, and C. R. Mann, *The Construction and Use of Achievement Examinations* (Boston, Houghton Mifflin Co., 1936), Ch. I, "Identification and Definition of Objectives to be Measured," R. W. Tyler; *Thirty-fourth Yearbook* of the National Society for the Study of Education. (Bloomington, Illinois, Public School Publishing Co., 1935), Ch. VII, "Elements of Diagnosis," R. W. Tyler, and Ch. VIII, "Techniques of Diagnosis," L. J. Brueckner.

methods of obtaining lists of objectives. One method consists of making an analysis of the purpose of the course—that is, of trying to formulate answers to such questions as "Why should algebra be taught in the secondary school?" A second method frequently used consists of analyzing textbooks on the subject and attempting to identify the purpose of each topic. In practice a combination of at least these two methods is highly desirable.

What are some of the difficulties one meets in attempting to formulate a satisfactory list of objectives? One difficulty is that objectives obtained from the first type of analysis are often stated so vaguely that other teachers are not sure what is meant. The interpretations that are given to such statements depend upon the mathematical and educational training of the interpreter, and are thus subject to rather wide variation. If teachers do not clearly understand the implications of such statements, it is not likely that they are actively working upon such objectives in the classroom. Moreover, methods and materials for fostering achievement of these values are, for the most part, either not well known or are lacking entirely. This situation has led some writers to classify objectives as "ultimate" and "immediate," and for practical reasons to concentrate upon the measurement of achievement of the specific skills and concepts commonly recognized as of the latter type. This leaves the important problem of showing that the "ultimate" objectives are being achieved still to be solved.

The second method of obtaining objectives involves difficulties of a different sort. Instead of being too vague, the statements now tend to be too specific and too numerous. They are usually stated in terms of specific content; for example, "the ability to multiply signed numbers," or "the ability to square binomials." When objectives are stated in this way, the list tends to become so long that it is difficult to keep all of the items in mind. Points of major emphasis are often lost sight of because of the extensiveness of the list.

By mentioning specific content, statements of this kind te
to restrict the curriculum rigorously to the items menti
in the prescribed list. Teachers feel there is little oppor
for experimental efforts to develop broader aims that
methods and materials other than of the drill type.

In order to overcome these difficulties it is help
a form of statement of objectives which achieves
between concreteness and generality and such t
is reasonably complete yet not too extensive. O
that helps realize these conditions is the followi
jectives should be stated in terms of t ha
expected of students who make pro
the objective. The term *behavior* is
sense that includes mental and
criterion means that the teac
the students are to do, think,
the objective. The effort to for
results in clarification of the me ing
points the way to improved methods of
tion.

In previous chapters the Committee has sta many ob-
jectives, but not always in such a way as to satisfy this cri-
terion. For purposes of evaluation it may often be necessary
to make more detailed analyses of the behavior involved. By
way of further illustration it may be well to consider an ob-
jective now usually classified as "ultimate" or "intangible"—
for example, "appreciation of beauty in the geometrical forms
of nature, art, and industry." An attempt must be made to
tell what a person does when he appreciates—in a sense,
to define the word in operational terms. Among the symp-
toms of the desired behavior which different writers might
list are the following:

1. The student recognizes familiar geometrical forms in natural objects (such as crystals), in artistic products (such as stained glass windows and architectural design), and in industry (such as the Texaco and Chevrolet trade-mark designs, machine parts, etc.).

2. The student experiences a feeling of satisfaction or pleasure recognizing geometric form in natural or in fabricated objects. If occasion arises, the student is able to describe preferences ms of certain aspects of design (symmetry, proportion, asym- listortion, contrast, similarity).
   student calls the attention of his friends to examples or occurrence of geometric form. Occasionally, he may nples to class for the benefit of classmates and teacher. rl on a summer European trip took time to sketch the certain cathedral window and brought the drawing to all.

co be extended, but perhaps enough has hat is meant by description of an ob- vior.

econd step in the general outline me typical situations in which tunity to exhibit one or more l. It is at this point that one's training becomes important. If sted in securing transfer, then some of t... ...ould be bona fide life-situations. Consider the obje... suggested by the phrase "the ability to reason logically." If by this one means only that the student is to be able to give acceptable proofs of geometrical exercises, then perhaps the teaching and testing need not go beyond the confines of pure mathematics. But if one means that the student should also be able to reason logically as he reads newspapers and magazines, then the available evidence tends to show teaching and testing materials must be modified accordingly. A student who apparently understands the "if-then" type of argument in geometry may fail to reach the logical conclusion in a situation involving social or political issues. If the teacher wants to find out whether the student can make the transfer, it is necessary to study the student's reactions in situations which require the desired ability. Such situations can be found, although the teacher who is unaccustomed to looking for them often has difficulty at first. This

difficulty can be overcome with practice—that is, as the teacher himself learns to make the necessary transfer.

At various points in this Report the Committee has emphasized the idea that the objectives are to be applied in the solution of problems arising in the basic aspects of living. This principle suggests that the problem situations used for evaluation purposes should sample adequately the field of personal problems, socio-civic problems, etc. In this way, one may obtain a more representative sample of real problems and thus obtain more satisfactory data concerning the abilities of students to apply their learning to solve problems in the basic aspects of living.

Obtaining a record. The third step in the general procedure consists in obtaining a record of the student's behavior when faced with an appropriate situation. In some cases, this may involve observation and recording of the observed behavior. Suppose, for example, the objective is that the student should learn to handle his compasses, or perhaps a sextant, skilfully. The teacher can then make it a point to watch him as he attempts to draw circles or measure angles with the sextant and make notes of his progress as he becomes more skilful in the use of the instrument. These remarks illustrate the fact that it is not necessary to assume that evaluation involves "paper and pencil" tests. There are several other useful methods of getting a record of behavior. Of course, paper and pencil tests are an economical method to use if the objective is such that they can be used. The student then makes his own record. But in the development of a comprehensive evaluation program which includes "intangible" objectives, the notion that the aim at this stage is to obtain some sort of record is extremely valuable. Many people have given up in despair at this point because they never considered the possibility that there are other methods of obtaining a record of achievement in addition to paper and pencil tests written by the student.

Achievement with respect to many objectives can be readily

measured by the use of written tests. But here again much progress has been blocked by assuming that some form of short-answer technique should be used. The marking of essay-type tests is notoriously subjective. The same paper evaluated by different teachers may receive a wide range of scores. Essay examinations also require so much time to be spent in the mechanics of writing that they ordinarily sample only a small range of student behavior. To overcome these difficulties various short-answer techniques have been devised. Thus there are true-false tests, multiple-choice tests, matching tests, and the like. Ordinarily the use of these techniques reduces subjectivity in the scoring, but it involves other difficulties. Many teachers of mathematics reject tests of this type because they feel that the tests do not really measure achievement of the objectives for which they are designed. The true-false tests in particular have often been subjected to severe criticism. Many teachers prefer examinations in which students write out complete proofs or show all of the work necessary to solve a problem. Advocates of one or the other of these two types of tests thus appear to be in opposition to each other. It is commonly assumed that this is "an either-or situation." Some teachers thus reason as follows: Either we must continue to depend upon tests calling for reasonably complete solutions or proofs, or we must resort to short-answer techniques; but since we are not convinced that techniques such as the use of multiple-choice items give us satisfactory evidence of achievement, we must continue to use the traditional form of examination.

Fortunately, there is a way out of this difficulty, but it involves considerable work. For purposes of illustration, consider the problem of measuring the ability to prove originals in geometry. Suppose it is agreed that the ability to write out a proof is a valid measure of achievement of the objective. Suppose further that a test is prepared calling for the proof of a number of originals. When the test has been given to a group of students, if several teachers mark the papers

considerable variation in the scores may be expected. This occurs because as a rule the several teachers are scoring on different bases. Analysis of the objectives which underlie the scoring will bring these differences to the surface. It is then usually possible to prepare a careful set of scoring directions satisfactory to the several teachers. If the papers are now rescored according to the directions, it is ordinarily possible to get rather close agreement among the scores assigned by the different teachers. This same method may be applied to any good essay-type test, even when the objectives covered are decidedly of the "intangible" sort. In other words, it is possible to eliminate most of the subjectivity of scoring provided pains are taken in preparing a key and marking the papers. The scores thus obtained may be regarded as the *basic evidence* provided the teachers agree that this type of examination calls for abilities they regard as important.

Suppose now that a short-answer test is prepared covering exactly the same situations—for example, exactly the same originals. If both the longer essay-type test and the test using the new techniques are given to the same group of students, the results obtained from the two types of tests can be compared. If the results agree very closely, then it is possible in the future to use the shorter form which requires less writing by the student and is much easier to score. In general, then, the aim is to devise some short-answer form of test such that the coefficient of correlation between the results on this test and the original basic evidence is high—say, about .90 or higher. It is not necessary to claim that the behavior response is the same in each case. But one set of scores may serve as an index of the other. Such a process as this requires more detailed work than the average teacher is prepared to do. But once a few teachers have learned how to make newer types of tests which are satisfactory, it is often possible for others to save themselves much of the work.

In order to obtain a sufficiently high coefficient of correlation the new-type test may appear to differ very much from

the types which have been familiar. In order to test some of the objectives hitherto regarded as intangible, test situations which are somewhat more elaborate than those now given must be devised—the tests will appear to be more complex than most tests thus far published. But it seems reasonable that in order to measure complex behavior a more complex instrument is needed. It should be possible, however, to devise means of obtaining evidence about achievement of objectives now considered intangible, and study of this evidence should lead to marked changes in teaching procedure. The point at the moment is that in the approach outlined above it is not necessary to assume that any of the well-known short-answer techniques is to be used. The aim at the outset is to get some direct evidence—by actual observation if necessary. One then tries to find an easier way of getting essentially the same evidence. If a short-answer technique can be found which is satisfactory, it can be used. If not, one is forced to use the original or basic evidence, even if it is more difficult to get repeatedly.

It seems likely that for some time to come other means besides paper and pencil tests must be used in order to obtain evidence about achievement with respect to many of the more intangible objectives. It may, therefore, prove helpful to list some of these methods at this point. Among them are the following:

Anecdotal records by teachers, parents, and students [2] (The anecdotal record consists of a contemporary description of a significant reaction of a student and of the setting in which the reaction took place. This procedure has promise of developing into a good, practicable means of evaluating progress toward the less tangible objectives of education.)

Records by trained observers [3] (This is a rather specialized method, probably most useful for research purposes.)

[2] John A. Randall, "The Anecdotal Behavior Journal," *Progressive Education*, Vol. 13, January, 1936, p. 21. Synonymous terms used by various workers are: observational record, behavior episode, behaviorgram, behavior specific, sentence description.

[3] Willard Olsen and Elizabeth Cunningham, "Time-sampling Techniques,"

Questionnaires
Interviews
Critical study of student products (writing, art products, models etc.)
Students' diaries of reading, and check-lists of other activities (A student record of free reading is a particularly valuable basis for coöperation between teacher and student in a program of student self-evaluation.)

Interpreting the record. There is little to be gained from the process of constructing and giving tests and collecting other records of student behavior unless the data so obtained are carefully interpreted. This task of interpretation involves several steps in itself. In the first place, the teacher may study a particular student, analyzing his achievement and relating it to the experiences he has had both in and out of school. This should lead to some plan of action designed to remove deficiencies revealed and to promote growth.

In the second place, the teacher may examine the achievement of groups of students who exhibit similar behavior, or the class as a whole may be studied. This study should also lead to tentative plans for improving the curriculum and methods of instruction. In some cases it may be desirable to compare one class with another, and in the case of important general objectives, to examine whether or not the students are growing from year to year in desirable directions.

Finally, if a comprehensive set of evaluation data has been collected, the behavior with respect to different objectives may be related and compared. Reading abilities, interests, attitudes, appreciations, thinking abilities, emotional adjustment, and other characteristics may be summarized and interrelated in an effort to evaluate the student as a whole personality, rather than as an organism whose various aspects may be viewed in isolation.

Child Development, Vol. 5, March, 1934, p. 41. J. W. Wrightstone, "An Instrument for Measuring Group Discussion and Planning," *Journal of Educational Research*, Vol. 27, May, 1934, p. 641.

A later section of this chapter will discuss the interpretation of test results in more detail. For this purpose it will be necessary to refer to particular tests, and consequently some sample exercises are presented in the next section.

SAMPLE EVALUATION TECHNIQUES

The modern teacher is well acquainted with the commonly used test techniques. In the field of mathematics there are many published tests which employ true-false questions, multiple-choice questions, matching exercises, and similar devices. Moreover, books on methods of teaching and on test construction discuss the principles to be observed in preparing test exercises of various types. In this section, therefore, the sample exercises presented are for the most part so chosen as to illustrate means of obtaining evidence about the achievement of objectives which are not commonly tested at present. For a detailed discussion of certain technical aspects, and illustrations of the usual type of test items, the reader should consult the references.[4] As a rule only one exercise will be used for purposes of illustration. In order to be reliable, a test must usually include a number of exercises of the same type.

The Ability to Recognize Quantitative Factors in Problem-Situations

In Chapter IV it was mentioned that the ability to recognize quantitative factors is often one of the prerequisites of problem solving. Closely related to this is the ability to recognize relationships among factors. To get evidence about these abilities exercises of the following type are suggested.

The graduating class of the Woodrow Wilson High School is planning to buy something for the school as a class memorial. In discussing the matter during a class meeting a number of impor-

[4] Hawkes, Lindquist, and Mann, *op. cit.*, particularly Ch. VII, "Examinations in Mathematics"; Ernst R. Breslich, *The Technique of Teaching Secondary-School Mathematics* (Chicago, University of Chicago Press, 1930), Ch. VII.

tant questions were asked. Below are some of the questions members of the class asked. Some of these questions clearly call for *quantitative* answers. Check all such questions on the lines at the left.

___	1. What should we plan to buy; for example, books or pictures for the library?	___ 1.
___	2. How much will the memorial cost?	___ 2.
___	3. How long will the memorial last?	___ 3.
___	4. Will the faculty approve of what we buy?	___ 4.
___	5. How much money do we have in the treasury now?	___ 5.
___	6. Where will the memorial be placed?	___ 6.
___	7. Who should make the presentation speech?	___ 7.
___	8. What sort of things have classes given in previous years?	___ 8.
___	9. How many members of the class will contribute?	___ 9.
___	10. Whom should we appoint to buy the memorial?	___ 10.
___	11. Is each person to pay the same amount?	___ 11.

Suppose you were interested in knowing the amount you personally would have to pay. How many of these questions would have to be definitely answered before you could find out? Check the number of all such questions on the lines at the right.

Understanding of Some of the Qualities of Good Data

When a problem has been clearly formulated and certain factors to be studied have been selected, the next step is to collect data. Prior to this, however, it is usually wise to make certain tentative decisions concerning what will constitute a good set of data.[5] Exercises of the following type may be used to discover whether students understand some of the qualities of good data.

Franklin School has no lunch room. Some students go home for lunch. Others bring their lunch, or go to near-by eating places. A few of the students who thought the school should provide a lunch room went to the principal and suggested it to him. He said, "Would a lunch room make expenses? Do you know how many students would buy their lunches in the school and how much they would spend for lunch?" Since they did not have this information, they decided to get it.

Irene said, "At the next assembly we can ask all students who would buy their lunch here to raise their hands, and we can

[5] Cf. Chapter V, "Data."

count them. Then we can ask how many can afford to spend thirty cents for lunch."

Other students offered the suggestions listed below. You are to decide whether each suggestion is a good one, or not, and tell why you think so. Follow these directions:

Part I 1. If you think the suggestion is a good one, put a plus sign (+) on the line at the right.
2. If you think it is *not* a good suggestion, put a zero (0) on the line.
3. If you cannot decide, leave the space blank.

Part II 1. If each suggestion is followed, certain things may be true about the data obtained. For example,
a. The data would be more complete.
b. The data would be more accurate.
c. There would be more data than are needed for the purpose, or the data would be irrelevant—that is, they would not help to answer the problem stated.

These are really reasons why the suggestion is a good or a bad one. On the line at the right of each suggestion below, write the number of any of these reasons which explains why the suggestion is good or bad. If you have other reasons, write them in, number them, and use them also.

LIST OF SUGGESTIONS

PART I	PART II
___*a*. We should send letters to all of the parents to make sure they will allow their children to buy lunches and spend what they say they will.	___*a*.
___*b*. Since some people may not be at the assembly, we should get the information from the homerooms.	___*b*.
___*c*. We should ask some schools which have lunch rooms whether they are able to meet expenses.	___*c*.
___*d*. We must also find out how many teachers would buy their lunches at the lunch room.	___*d*.
___*e*. Several students should help in counting the number of people who will buy lunches in the school.	___*e*.
___*f*. We should ask the home economics department how much the food in a good lunch would cost.	___*f*.
___*g*. We should find out how many students there are in our school.	___*g*.
___*h*. We should find out how many people would spend fifteen cents, how many twenty cents, how many twenty-five cents, and so on.	___*h*.
___*i*. We should ask the restaurants and drug stores nearby how many students buy their lunches there.	___*i*.

___ *j.* We should check to find out who is absent when we get the data and see these people later. ..... ___ *j.*

Arrangement of Data in Tabular Form

Measurement of the ability to arrange data in convenient tabular form does not readily lend itself to short-cut techniques. This ability is probably best evaluated by studying the tables prepared by students to exhibit the data they have collected. It is, of course, possible to select sets of numerical data given in paragraph form and ask students to arrange them in convenient tabular form. The following exercise is one of that type.

The following is taken from the *Question and Answer* department of a newspaper.

Q. How many ships were built in Japan in 1934 and what was their tonnage? How did this compare with ship-building in the United States?

A. In 1934 the Japanese built 12 ships with a total tonnage of 89,480. In the same year the United States built 6 ships of which the total tonnage was 144,000.

Directions: Arrange these data in a convenient tabular form.[6]

The Proper Use of Approximate Numbers

In connection with the tabulation of raw data, and later as they are manipulated for purposes of interpretation, careful attention should be given to the principles of handling approximate numbers. Exercises of the following type have been designed to secure evidence that students know and can apply some of these principles.

This is test of your ability to apply certain principles of handling approximate numbers.

Alden was sports reporter for the school paper. In an article to be published a few days before the big football game between East High and Central High, he wished to compare the two teams. His data included the following table about weights of the regular linemen:

[6] Note that this is a 3-parameter set of data. Cf. Chapter V, "Data."

Position	*Weight in Pounds*	
	East	*Central*
Right End	149	163
Right Tackle	165	169
Right Guard	157	158
Center	158	160
Left Guard	171	171
Left Tackle	164	175
Left End	155	161
TOTAL WEIGHT:	1119	1157

He wished to report the average or mean weight of each line. His younger brother did the division for him as follows:

$$7\overline{)1119.000}\,159.857 \quad ; \quad 7\overline{)1157.000}\,165.285$$

Which of the following figures should be reported? Check the proper ones.

Average Weight

East	*Central*
___*a*. 159.857	___*f*. 165.3
___*b*. 159.8	___*g*. 165
___*c*. 160	___*h*. 165.285
___*d*. 159.85	___*i*. 165.28
___*e*. 159.9	___*j*. 165.29

Check any of the following reasons which apply to your decision.

___1. People will want to know the average very accurately.
___2. The numerical value of a measurement is expressed only approximately.
___3. The boys will each lose several pounds during a game.
___4. The original data are apparently accurate to the nearest pound or three significant figures.
___5. The figures I marked are more exact than the others.
___6. It is always a good rule to round off numbers to the nearest tenth.
___7. The results of computation with approximate data are generally not accurate to more significant figures than the original data.
___8. It is always a good rule to round off numbers to the nearest hundredth.

Choosing a Form of Graphical Representation and Drawing Graphs

Since it is often desirable to represent tabular data in graphical form, students should learn to choose a type of graph which is suited to the data. This ability is rarely tested in published examinations, although the following method or an adaptation may be used. In particular, the test may be made wholly objective by providing a list of reasons as well as a list of graph types, and having the students also indicate their reasons by use of a code letter or number. The complete test should include a variety of different types of tabular data.

Directions: There are several tables in this part of the test. You are to indicate which one of the following types of graph you would use in representing these data.

1. Circular distribution graph
2. Bar graph
3. Single line graph
4. Rectangular distribution graph
5. Multiple-line graph

You will find space provided below the tables for doing this. Give your reasons for selecting that particular type of graph.

TABLE I

IMPORTS OF CERTAIN FARM PRODUCTS IN 1935–36 AND 1936–37

	1935–36	*1936–37*
Cattle	10,890,000	11,694,000
Cheese	11,352,000	13,583,000
Meats	14,502,000	23,318,000
Wool	22,747,000	54,393,000
Cotton	9,547,000	17,055,000
Corn	13,117,000	45,428,000
Wheat	37,487,000	47,912,000

Table I

Type of graph _____

Reasons: _____

TABLE II

WEATHER CONDITIONS PREVAILING IN FATAL ACCIDENTS IN 1936

	Accidents	*Per Cent*
Clear	28,750	85.7
Fog	800	2.4
Rain	3,260	9.7
Snow	740	2.2
TOTAL	33,500	100.0

Table II

Type of graph _____

Reasons: _____

The obvious way to measure the ability to draw graphs of these and other data is to provide graph paper or grids and have the students draw the graphs. The type of exercise just discussed yields evidence as to the ability to choose a proper type of graph. Hence, for these exercises students may be told to draw a graph of a particular type. The following exercises illustrate the method.

Directions: To the right of each table you will be told what type of graph is to be used for the data of the table. Make the type of graph indicated for each table.

TABLE V

DEATH RATES PER 10,000,000 GALLONS OF GASOLINE USED

	1933	*1934*	*1935*	*1936*
Connecticut ..	18.9	18.8	18 3	14.0
Delaware	21.4	20.9	16.5	13.3
Florida	26.4	26.3	24.6	21.4
Maine	18.6	18.0	17.5	13.6

Graph for Table V
Multiple-Line Graph

The Interpretation of Data

The preceding test exercises have all been constructed to measure primarily abilities required by those who collect data and present them in one form or another to others. Even more important are the abilities required in order to be an

intelligent consumer of data. The following exercises are constructed more from the latter point of view than the former. The test situation below is designed to obtain evidence concerning achievement of the following objectives:

The student should be able to obtain numerical answers to simple problems.

He should use the method or methods best adapted to the problem at hand. This implies that he should have an understanding of the advantages and disadvantages of various ways of presenting data.

Existing mathematics tests yield evidence about achievement of the first of these, but little or no evidence about the second. It is probable that future courses will devote more time to teaching for the second objective than has been customary. It appears likely, then, that in the beginning of his experience with mathematics the student will make relatively little use of the more refined methods (e. g., the formula) and that as a result of his study changes will take place *not only* in his ability to get the solution, but also in his choice of method and his reasons for the choice.

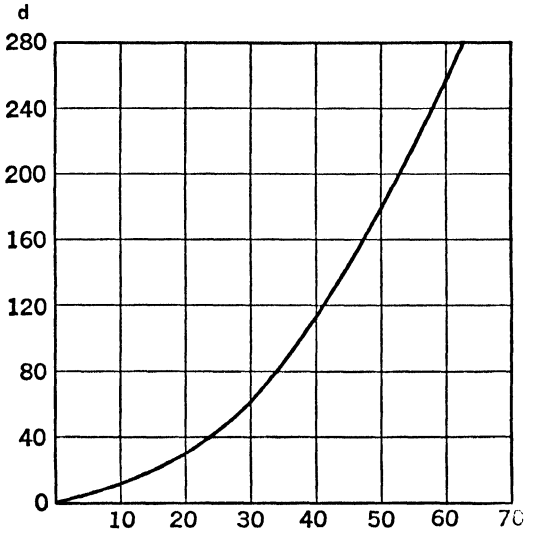
The distance required to stop a car after the brakes are applied is an important factor involved in automobile accidents. The relationship between "stopping distance" and speed for a certain car may be expressed in four ways as follows:

A. *Verbal statement:* The number of feet a certain car will travel after the brakes are applied is .07 times the square of the speed expressed in miles per hour.

B. *Table:*

Speed in miles per hour: v	*Distance in feet: d*
10	7
20	28
30	63
40	112
50	175
60	252

C. *Graph:*



D. *Formula:* $d = .07v^2$

Similar formulas hold for all cars with good brakes. You are to assume that these data apply in all of the problems which follow.

Directions: Below are a number of situations and problems all of which are related to the data given above. In most of them you are to do three things:

First: Answer the question or solve the problem.

Second: Tell which of the four ways of expressing the data you used to answer the question. In some cases you may not be able to answer the question, but perhaps you can tell which of the ways you think *should* be used. Sometimes more than one way may be used. You can indicate which way you used by writing one or more of the letters A, B, C, or D on the line marked *Method.*

Third: You are to tell why you chose the method you did.

A "list of reasons" is given on the next page. Read this list and decide whether any of them are ones you would give. Write the *number* of such reasons on the line marked *Reason* which follows each situation. If you have some reason not given on the list, write it out below the exercise.

LIST OF REASONS

1. The verbal statement is easiest to understand.
2. The table is easiest to use when the numbers in the problem are ones given in the table.
3. Study of the table shows that the stopping distance increases as the speed increases.
4. The graph shows vividly how the stopping distance changes as the speed changes.
5. The graph is convenient when only a rough estimate is required for values which lie between those given by the table or used in making the graph.
6. The formula is difficult to understand unless you have studied a lot of mathematics.
7. The formula gives more accurate results for values which lie beyond those given in the graph.
8. A formula usually holds for many values not given by the table or graph.
9. The formula gives more accurate results for values which lie between those given in the table than does "interpolation" in the table.
10. The formula usually gives more reliable results for values which lie beyond those given in the table.
11. Since the formula is more general than the table or graph, it is most useful for drawing precise conclusions about changes in d for any value of v.

Sample:

Problem: The speed limit in a town is 30 miles per hour. If you apply the brakes at this speed, how far will the car travel before coming to a stop? . . . . . 63 ft.

Method: Which of the four ways of expressing the relationship did you use? . . . . . B

Reasons: 2

Other reasons: _____

This shows that the distance traveled before stopping would be 63 feet; that the result was obtained from the *table,* and that the reason the table was used was No. 2 in the "List of Reasons."

Problem 1.
The speed limit in a town is 25 miles an hour. About how far will a car travel before stopping when the brakes are applied at a speed of about 24 miles an hour? . . . . . .____
Method: Which of the four ways of expressing the relationship did you use? . . . . . .____
Reasons: ____
Other reasons: ____

Problem 2.
A motorist was arrested for "speeding" at 54 miles per hour. The judge asked him if he knew how far his car would travel (after applying the brakes at that speed) before it would come to a stop. What answer (precise to the nearest foot) should he give? . . . . . .____
Method: Which of the four ways of expressing the relationship did you use? . . . . . .____
Reasons: ____
Other reasons: ____

Problem 3.
A highway crosses a railroad in a forest so that the approach of a train cannot be seen. The road is so straight and level that sometimes cars approach the crossing at almost 80 miles an hour. At what distance from the crossing should a sign be placed to warn drivers about the tracks ahead? . . .____
Method: Which of the four ways of expressing the relationship did you use? . . . . . .____
Reasons: ____
Other reasons: ____

A second type of interpretation of data test uses situations like the following:

Directions: This is a test to see how well you can draw inferences or make interpretations from data which are presented to you. In each problem you are given some data followed by a set of statements about the data or *interpretations of the data.* Check each statement in one of the columns on the right according to the following standards.

1. The evidence given is sufficient to prove the statement true.

2. The evidence given suggests that the statement is probably true.
3. The evidence given is insufficient to justify a decision concerning the statements.
4. The evidence given suggests that the statement is probably false.
5. The evidence given is sufficient to prove the statement false.

If you think the evidence given is sufficient to make the statement true, put a check (√) in the first column. If you think the evidence given is sufficient to make the statement probably true, put a check (√) in the second column, etc.

Problem.

The surface temperature, mean distance from the sun and the diameters of five planets are listed in the following table:

Planet	*Average Temperature (absolute)*	*Distance from the Sun (millions of miles)*	*Diameter*
Mercury	650	36	3000 miles
Earth	290	93	7900 "
Mars	250	141	4200 "
Jupiter	100	483	87000 "
Saturn	90	886	73000 "

Statements	(1)	(2)	(3)	(4)	(5)
1. The amount of heat falling on planets close to the sun is greater than that falling on planets farther away					
2. The temperature of the planets becomes less as their distance from the sun increases					
3. The temperature of a planet depends solely upon the radiant energy which it receives from the sun .					
4. The temperature of a planet becomes less with increasing diameters					
5. Planets farthest from the sun have lower temperatures than those near it					
6. Uranus, a planet at a distance of 1781 million miles from the sun, will have a lower surface temperature than Saturn					
7. The distance from Mars to the Earth is 48 million miles					
8. The temperature of Venus, at a distance from the sun of 67 million miles, is higher than that of Mercury					
9. Jupiter is the largest planet of the solar system .					

The Ability to Apply Several Rather General Principles of Variation

The exercise below illustrates one method of testing for the ability to apply principles of variation. In the complete test, situations which involve five different types of function are used. These functions are of the forms: (1) $y = ax$; (2) $y = ax^2$; (3) $xy = k$; (4) $y = ax + b$; (5) $y = ax^3$. The student is expected to be able to tell how a change in an independent variable will affect the dependent variable. The transformations applied to the independent variables include the following: (1) general "increase" of "decrease"; (2) additive, as "increased by 0.2 of an inch"; (3) multiplicative, e. g., "is doubled." About as many opportunities are given to check irrelevant or incorrect statements as are given to check correct ones.

If the students have learned the principles of such variation *in general,* they should be able to check the items rather quickly and without recourse to much computation. If, however, they have not this ability to *apply general principles* in these supposedly new situations, then they either must resort to computation on almost every statement, or else fail to respond at all. A convenient Tally Sheet permits analysis to determine (1) the types of *function* of those sampled for for which this ability is or is not well developed, and (2) the types of *variation* for which the ability is or is not well developed. One situation and items ten to twenty-five follow.

The following quotation is taken from an article printed in a daily newspaper:

". . . George L. Smith, an automotive engineer of the national capital, has made a study of the [automobile] brake situation in relation to speed fatalities and is far from being content.

"He points out that 10 years ago the usual stock passenger automobile sold to the general public rarely had a speed capacity in excess of 50 miles an hour. That, of course, is a high rate of speed, and the great weight of a motor car, striking at such a speed, creates an impact which is bound to cause severe results. But

Mr. Smith asserts that while today the speed of stock motor cars has been stepped up to 70 miles an hour, there has not been a comparable advance in braking power. The destructive power of a car going 70 miles an hour is almost exactly twice that of the same car going 50 miles."

—Columbus, Ohio, *Dispatch,* Jan. 7, 1936

The "destructive power" referred to in the article comes from the *kinetic energy* of the car, and this may be computed from the *formula:*

$$K = \tfrac{1}{2}\, mV^2$$

where K = kinetic energy
m = mass (sometimes called "weight") of the car
V = velocity or "speed" of the car

The article says "the destructive power of a car going 70 miles an hour is almost exactly twice that of the same car going 50 miles." Note that 70 is 1.4 times as great as 50.

Check (√) any of the following statements with which you agree.

___10. The article is incorrect. The destructive power of a car going 70 miles an hour cannot be almost twice that of the same car going 50 miles an hour.

___11. A certain car bought ten years ago had less destructive power then than it would have today.

___12. If one of two cars of the same weight is moving twice as fast as the other, its kinetic energy is four times as great.

___13. If one of two cars is 10% heavier than the other, its destructive power will be 10% greater provided both are moving at the same speed.

___14. Since 70 is not twice 50, the "destructive power" at 70 miles per hour cannot be twice as great.

___15. Since 70^2 is almost twice 50^2, the kinetic energy of a car going 70 miles an hour will be about twice that of one going 50 miles per hour.

___16. The kinetic energy of a car moving 50 miles an hour is 100 times as great as that of the same car moving 5 miles an hour.

___17. The article is correct. The destructive power of a car going 70 miles an hour is almost exactly twice that of the same car going only 50 miles an hour.

___18. Not enough information has been given to enable one to

decide whether the final statement in the newspaper article is accurate or not.

___19. The destructive power of two cars with the same "horsepower" will be the same.

___20. The kinetic energy of a car moving 40 miles an hour will be four times as great as that of the same car moving 20 miles an hour.

___21. The destructive power of a car weighing 4000 pounds will be twice as great as that of a car weighing 2000 pounds.

___22. If the weight of one car is twice that of a second car, the "destructive power" of the first is four times that of the second.

___23. The destructive power of a car does not depend upon its weight.

___24. The destructive power of one car which is 1.4 times as heavy as a second car will be 1.4 times as great.

___25. Car A is twice as heavy as car B, but is moving only half as fast when it strikes. How will the destructive power of the two cars compare?

___*a*. It will be the same.

___*b*. Car A will be twice as destructive as car B.

___*c*. Car A will be half as destructive as car B.

___*d*. No definite answer is possible.

The Nature of Proof

Limitations of space prevent the presentation of more than two types of test exercises designed to obtain evidence of pupils' understanding of the nature of proof. Both types use nonmathematical situations, and hence are adapted to discovering the extent to which notions acquired in connection with mathematical study transfer to other types of reasoning situations.

The first type of test exercise requires the conscious recognition of certain general principles of logic. One such principle often stressed in courses of geometry is sometimes called the "if-then" principle. The following exercise makes use of it.

In the course of a letter to a newspaper the president of a power and light company said, "The Federal Government is

building electric power lines which will compete with private power utilities in the Tennessee Valley. It is unfair for the Federal Government to compete with private power utilities."

Assuming these statements to be true, check (√) any of the following statements which in your opinion are consistent with them:

___1. It is unfair for the Federal Government to build competing power lines in the Tennessee Valley.
___2. It is quite fair for the Federal Government to build competing power lines in the Tennessee Valley.
___3. Further information is needed before any logical conclusion can be drawn.

Check (√) below any statements which you would use to explain or support your conclusion.

___*a*. The government will be doing a useful public service and should go right ahead.
___*b*. The power companies have a lot of money invested and the government will ruin their business.
___*c*. The word *utilities* needs to be more carefully defined.
___*d*. If a person accepts the original statements, then to be logical he should accept the conclusion which follows from them even if it is not necessarily true.
___*e*. We need to know whether the private power companies charge too much.
___*f*. The soundness of an indirect argument depends upon whether all the possibilities have been considered.
___*g*. The government does not have to pay taxes to itself so it is unfair for it to compete with private business.
___*h*. The people are benefited because competition helps keep the cost of power low.
___*i*. Changing a definition may change the conclusion although the argument from each definition is logical.

The principle is here stated as Reason *d*. Exercises of this kind thus require the student to decide which of the three conclusions is most logical, and to explain his choice. Students who do not understand the principle involved are often led astray by their attitudes, and tend to use statements such as *b* or *g* to explain their choice.

The test from which this situation is taken samples four important logical principles. Three of them are stated above

as items *d*, *f*, and *i*. The fourth deals with the fallacy *argumentum ad hominem*. It may be stated for secondary school students in some such form as the following: An argument cannot be disproved by a personal attack on the arguer, or by questioning his motives, etc.

The ability of students to recognize the rôle of assumptions in daily reasoning is one of the most difficult of behaviors to measure. One technique which has been devised for this purpose is illustrated by the following set of directions and one exercise from a complete test. In this test the student may choose either of two conclusions, but he should associate with his choice one or more major assumptions which are involved. He is then asked to put himself in the position of one who chooses an opposite conclusion, and to associate with it the underlying assumptions. Success on both of these types of responses would seem to indicate that a student has some understanding of the rôle of assumptions in reasoning.

TEST: THE RÔLE OF ASSUMPTIONS IN REASONING [7]

Do you understand the importance of underlying assumptions in your everyday thinking?

You are constantly meeting situations in which differences of opinion arise. Different people, equally intelligent and logical in their thinking, often reach opposite conclusions in the same problem and from the same data. Why does this happen?

These discrepancies are partly due to the varying points of view or basic assumptions with which different individuals begin their thinking. Often they are quite unconscious of these assumptions which underlie their thinking and lead to their decisions. Can you analyze your own thinking and discover the principles and assumptions upon which you base your conclusions? Can you analyze the conclusions of others and point out the basic assumptions from which they logically result?

This test gives you a chance to see how well you can do these two things. Each problem is a controversial situation in which either of two opposite conclusions may be reached. In order to form a conclusion logically one or more basic principles must be

[7] By Lewis Taylor, North Shore Country Day School, Winnetka, Ill.

assumed and applied to the problem. The underlying principles are usually not stated in the problem itself.

In each problem you are to do three things:

Part I. Read the situation carefully and, on the basis of the data given, try to form your own conclusion about the issue stated. Of the two opposing conclusions offered choose the one that you most nearly agree with and place a check mark (✓) on the answer sheet immediately under the letter of that conclusion.

Part II. Read the statements given in Part II of the problem and select those two statements upon which your conclusion could most logically be based. Place check marks on the answer sheet opposite the numbers of those statements in the column under the conclusion that you have checked.

Part III. Go over the list of statements and select those two upon which the opposite conclusion could most logically be based. Place check marks on the answer sheet opposite the numbers of these statements in the column under this other conclusion.

PROBLEM X

	Conclusion	
	A	B
	✓	
1.		
2.	✓	
3.		✓
4.		
5.		
6.	✓	
7.		
8.		
9.		
10.		✓

A sample answer is shown on this page. In Problem X, this student has checked conclusion A as the correct conclusion. In support of that conclusion, he has marked Statements 2 and 6 as the two which best support that conclusion. Then in the other column he has checked Statements 3 and 10 as the two which seem most logically to support the opposite conclusion.

Since this is a test on your ability to see logical relationships, try to be very accurate and precise in your thinking. Do not check statements merely because you think they are true, but because they form a logical basis for decision in the problem stated.

PROBLEM I

In a recent advertising campaign this argument was presented:

America's Largest-Selling Gasoline

1,000 Motorists a minute drive in for Mobilgas—the biggest army of motorists using a single brand of gasoline—stop and fill up with Mobilgas!

Part I. Assuming that the data given in this advertisement are true, does it show that you should use Mobilgas in preference to other gasolines?

Conclusion A. Yes.
Conclusion B. No.

Part II. Why do you think so? From the statements below select the two upon which your conclusion could most logically be based. Check those statements on the answer sheet in the column under the conclusion you have chosen.

1. Every automobile driver wants to use the best gasoline.
2. If you accept an assumption you must accept the conclusion that logically follows from that assumption.
3. The brand of a product which is most widely used is usually the best in the field.
4. Perhaps Mobilgas is not America's largest-selling gasoline.
5. The popularity of a product is very often due to the extent to which it has been advertised.
6. Since the same company has made one of the best motor oils for years, its gasoline should be superior also.
7. Mobilgas may be cheaper than other gasolines.
8. Researches have shown that widely advertised and widely used products may be of inferior quality.
9. If so many drivers find Mobilgas satisfactory for their cars it will probably be a good gasoline for mine.

Part III. Suppose that a friend of yours disagrees with your conclusion but takes the opposite one. Upon what might he base his conclusion? From the above list of statements select those two upon which he might most logically support his decision. Check them on the answer sheet in the column under that conclusion.

Evaluation of Qualities of Personality Other than Reflective Thinking

The evaluation of creativeness, appreciations, interests, social sensitivity, coöperativeness, and similar qualities offers many difficulties. One of the most effective methods at present is the use of anecdotal records. Data may also be obtained by careful analysis of the work of students with these qualities in mind. In some cases questionnaires yield useful data, and

in others it is possible to set up check-lists of typical behaviors that reveal these qualities. The following quotation [8] shows how such a device can be used for obtaining a record of cooperative behavior:

A number of schools have spent considerable effort in collecting evidence on coöperativeness. Several of them have used the "check sheet" . . . to distinct advantage. A list of all the opportunities for coöperative behavior—such as close attention to discussion, courtesy to person speaking, helping other students, cleaning up laboratory desks, and care of the room—is placed opposite a class roster and the items checked for each student.

In all evaluations of this type, care must be taken to see the total picture. It is so easy to be impressed by non-coöperativeness and to take coöperativeness for granted that anecdotal records or check sheets may not give an accurate picture. Obviously a teacher should not make anecdotal records or use a check sheet when feeling disgruntled or depressed. In using such techniques it is often desirable to write a summary of the data obtained in a short paragraph, lay it aside and review it later with the questions: "Does this give an accurate picture of John or Betty?" "Is this picture a sample of customary behavior, or did Betty and I have a bad day when this record was made?" Again, the record of a single teacher may be of low reliability, but a composite picture from a number of teachers will be more useful.

Although the use of questionnaires is open to certain objections, a carefully developed technique of this kind yields valuable information concerning appreciations, interests, and similar characteristics. Such questionnaires should be based upon careful analyses of the behaviors involved. Then questions may be so formulated that appreciation may be inferred from the responses on a number of such questions. It is practically useless to ask a pupil: "Do you appreciate geometric form as it occurs in nature, art, and the products of industry?" But he may be asked a number of questions such as the following:

"Do you notice the use of squares, hexagons, circles, etc., in the decorative stonework of buildings?"

[8] *Science in General Education,* pp. 425–426.

"Have you ever called the attention of a friend to an example of geometric form occurring in a natural object—such as a crystal?"

A suitable code for responses may aid the ultimate interpretation. Thus three columns headed respectively *often, occasionally,* and *never* could be provided and the student directed to put a check mark in one of the columns. An analysis of the responses on a properly validated questionnaire of this type should yield valuable information on the achievement of the underlying objective. This example gives an indication of a field for research in evaluation related to mathematical objectives.

Similar techniques are now in use for discovery of the interests of students. Only in rare cases could one be sure of the reliability of an affirmative response to the question "Are you interested in mathematics?" when asked by the teacher in the regular classroom situation. The student can hardly be blamed for trying to make a favorable impression by overstating his interest. But one may get evidence of interest indirectly by listing a large number of widely different activities and asking the student to indicate whether he *likes* them, is *indifferent* to them, or *dislikes* them. Some of the activities scattered through the list may be mathematical in nature. These may be sorted out in analyzing the responses, and if a considerable proportion are liked, one may infer that the student is interested in mathematics. One such questionnaire summarizes interests into fifteen different categories of which mathematics is one. Music, fine arts, writing, and sociable activities are among the other categories. The following are typical of the 300 activities listed:

92. To visit an observatory to look at stars
93. To read about famous people of the past
94. To direct a play
95. To read short stories
96. To practise speaking a foreign language

97. To solve problems in algebra
98. To attend a party where one is expected to mix with the whole crowd
99. To study how English words are derived from Latin
100. To study ideas about how poverty can be abolished

Of these, number 97 is obviously classified under "Mathematics," and the per cent of "Like" responses in the totality of items so classified is taken as one index of interest in this field.[9]

These examples of evaluation techniques not widely used at present should give an intimation of the possibilities of measurement of some of the so-called "intangible" objectives of education. Much research must be done before they are perfected and simplified, but clearly this is a field in which energetic teachers can work for the improvement of teaching.

SAMPLE INTERPRETATIONS OF TEST DATA

Interpretations Based on a Single Test

The fourth step in the general evaluation procedure involves the interpretation of the data which have been collected. In the case of many of the exercises discussed above the question first arises as to how the tests are to be scored. The second example of an interpretation of data exercise will be used as an illustration of methods of scoring not in common use. It will also be useful for showing how general behavior tendencies of a student may be discovered by means of test data. The first step in scoring this test is the preparation of a scoring key. Assume that a number of competent teachers have agreed as to how the exercise should be marked. The responses of a given student may then be tallied in the following table:

[9] A similar technique is being used in the evaluation of personal and social adjustment. See George V. Sheviakov and Jean Friedberg, "The Evaluation of Personal and Social Adjustment" (Chicago, Progressive Education Association, Evaluation in the Eight Year Study, 1939).

Statements in this test which SHOULD *have been marked as:*	*This student marked these statements as:*					
	True	*Prob. True*	*Ins. Data*	*Prob. False*	*False*	*Omitted*
True	*a.*	*b.*	*c.*	*d.*	*e.*	
Probably True	*f.*	*g.*	*h.*	*i.*	*j.*	
Insufficient Data	*k.*	*l.*	*m.*	*n.*	*o.*	
Probably False	*p.*	*q.*	*r.*	*s.*	*t.*	
False	*u.*	*v.*	*w.*	*x.*	*y.*	

Total Omitted _____

Complete agreement of the student with the key would result in tally marks in the cells on the main diagonal of this table—i. e., tally marks in cells *a, g, m, s,* and *y.* This would represent 100 per cent agreement of judgment with the jury which determined the key. Usually tally marks appear in other cells. The ratio of the number of tallies in the cells of the main diagonal to the total number of statements may be expressed as a per cent. This may be interpreted as a score on "general accuracy," and may range from 0 to 100 per cent. All departures from agreement with the jury reduce the score on general accuracy. Consider statement 6 in the test exercise above. This statement involves an extrapolation; suppose the jury keyed it as a (2) or "probably true" statement. A student who checked statement 6 in column (2) would receive a tally in cell *g* of the table above. A student who checked this statement in column (1) (i. e., as "true") would receive a tally in cell *f* of the table. This student has "gone beyond the data" to a greater extent than the jury was willing to go. If a student checks this as a (3) type of statement, a tally would be recorded in cell *h* of the table, and indicates that the pupil is unwilling or unable to go as far as the jury in making inferences from the data. For convenience such behavior is called "cautiousness." When a student marks a "true" statement as "false" or "probably false" tally marks appear in cells *e* or *d.* Such a re-

sponse is called a "crude error." In the same way it is possible to classify tally marks occurring in any cell as indicating accuracy, going beyond the data, caution, or crude errors.[10]

With these remarks in mind the responses of several twelfth-grade students who have taken a complete test consisting of ten samples of data accompanied by 119 statements may be presented.

Pupil	*General Accuracy*	*Accuracy With PT & PF*	*Beyond Data*	*Caution*	*Crude Errors*
A	51%	34%	36%	27%	14%
B	64	38	19	35	6
C	31	0	53	30	25
Class Range	11–65	0–58	12–69	10–53	6–29
Class Median	40	16	34.5	2.4	16

All Scores in Per Cents.

What can be said about the ability of these students to interpret data, using this phrase to describe the complex of abilities ranging from ability to read numbers from a table or chart to those involved in making judgments of trend, representativeness of sampling, cause and effect relationships and similar aspects involved in the statements made about the data? The following are sample descriptions of students:

Student A is a better than average interpreter. She is well above the median of her class in the ability to make the finer distinctions involved in judging "probably true" and "probably false" interpretations, and is about at the median in the tendency to go beyond the data, to be overcautious, and to make crude errors in judgment.

[10] If an electrical test scoring machine is available it is not necessary to record the tally marks in a table, count them, and convert to per cent. By use of the appropriate scoring keys the number of tally marks which *would* appear in such a table is automatically converted to per cent and read from a dial on the machine. Since the machine will handle hundreds of scores of this type per hour, it is possible to get the four different types of scores indicated for a whole class in a relatively short time.

Student B ranks very high in the ability to interpret data, being second in the group in general accuracy and near the top in accuracy with PT and PF statements. This student tends to be rather cautious in making judgments.

Student C is a rather poor interpreter, with a low score in general accuracy and extremely low ability to make fine distinctions. This student errs by "going beyond the data" and hence shows a tendency to be gullible. The record is further marred by a high score on crude errors, the pupil grossly misjudging 25 per cent of the statements for which truth and falsity may be confused. This may indicate a reading difficulty.

These three cases, selected almost at random from thousands available, illustrate three different general types of reaction to statements about data. When the group as a whole is considered, other sorts of things may be said. Thus the median of 40 on general accuracy made by this twelfth-grade class is rather low, since medians for this grade range from 29 to 60, the norm being 46. This class as a whole has a tendency to go beyond the data. This suggests that the teachers should lead the class to give some attention to those safeguards of thinking which must be applied in dealing with trends, interpolation, and extrapolation, data derived from sampling, cause and effect relationships, and other similar factors which the test incorporates. Discussion of this sort could be continued at some length. Perhaps enough has been said to at least indicate some of the sorts of information which may be obtained from reliable tests of this kind.

Interpretations Based on Several Tests

In the course of a comprehensive evaluation program the time periodically arrives at which the teacher must attempt to summarize the student's status or progress toward various objectives. It will be taken for granted that it is inadvisable to attempt to lump together the scores from a variety of tests for different objectives and to arrive at a single numerical or letter "grade." The alternative seems to be to report achievement of each objective separately. One of the present trends

is toward a "paragraph description" of the student. The extensiveness of such a description depends, of course, upon the extent of the data available. A reasonably complete description of a given student or class would involve treatment of data from a number of different types of tests. It would then be possible to comment upon the characteristics of the student with respect to many of the major objectives of instruction in mathematics, including the interests, attitudes, reasoning ability, informational background, technical facility, and other components of well-rounded educational development.

As an illustration of this method of reporting, a student is described below. The description to be given is of the sort that would be built up gradually in the school files. The sort of report which would be transmitted to students, parents, or to a college would differ in several respects, although the data and reports in the school files would be used as a basis in these cases. The description below is based upon actual data, but for the sake of brevity is confined to discussion of the following four aspects: (*a*) interests, (*b*) ability to interpret data, (*c*) certain abilities related to logical thinking, and (*d*) certain abilities related to the function concept.

Student Ann B. (12th Grade, class of 41 pupils)

Ann has the highest interest in science, mathematics, and engineering activities of any pupil in her class—she likes 85 per cent of the activities in the field sampled in a General Interest Questionnaire. (Class median 31, range 6–85.) This is unusual for a girl. She also has high interests in sports, industrial arts, and reading. Her interest in social studies and English is just average, but she seems to have strong likes for foreign languages and fine arts activities. Her interest in home arts and music is rather low, especially for a girl. She definitely dislikes activities which involve taking leadership, and is at the bottom of her class in liking sociable sorts of activities. These data, based on her responses to 300 widely different sorts of activities, suggest that Ann may not be very well adjusted socially. This point should be studied more carefully, however.[11]

[11] It should be borne in mind that this description is based upon only a part of the data available. The girl was chosen for illustrative purposes on the basis

This girl is a better than average interpreter of data (general accuracy 51 per cent, class median 40, range 11–65). She does especially well in making the finer distinctions involved in judging "probably true" and "probably false" interpretations.

Ann is considerably better than average in recognizing the assumptions underlying conclusions in arguments, and in distinguishing statements of facts from assumptions. She also does well in organizing logical arguments, with no tendency to use irrelevant facts and contradictory statements. But she is not the best student in her class in either the ability to interpret data or in the aspects of logic mentioned above.

In the ability to handle certain functional relations Ann is again well above the median but not superior. On a test of "functional thinking" abilities her score was 17, compared with a group median of 10 and range of –6 to 27. Her weakness here is with multiplicative transformations, particularly with the function $y = ax^2$.

A more abbreviated description of a student may be recorded on forms especially prepared for the purpose. The form which most closely corresponds to the objectives and philosophy of this Report is the one which has been prepared by the Committee on Reports and Records of the Commission on the Relation of School and College of the Progressive Education Association, of which Eugene Randolph Smith is Chairman.

Descriptions of these kinds in place of the usual numerical or letter grades are still in rather primitive form and are not as yet common in the schools. To many teachers they may appear to be confusing rather than helpful, or at least to be impractical. The problem of practicality can be solved, however, if the reports are thought to be sufficiently useful and an advance over present practice. Some schools are now using reports analogous to the example given. In certain cases they have reduced the number of reports per year in order to make

of her high interest in science, mathematics, and engineering before data from other tests were examined. Mention of other interests was included to give a hint as to the possibilities for more general descriptions on which different teachers coöperate.

those prepared more complete. Moreover, the belief is spreading that the job of the teacher is to plan work with the students, discover and overcome their difficulties, study their progress and plan new work in the light of data on their needs and interests. This sort of teaching stands in sharp contrast to that in which the teacher does little more than assign work in the textbook, hear recitations, and check homework and tests *ad infinitum*. But objectives, curricular materials, and ways of teaching are changing so that routine methods will no longer suffice. Perhaps the time has come when traditional methods of reporting are no longer really practical in progressive schools. Thus new forms of reporting in harmony with a modern conception of teaching are being devised and will gradually become "practical" as they are improved and as the concept of practicality also changes. The basic issue, therefore, is whether the newer forms of reporting are more useful than the old.

The major advantage of these newer forms of reporting is that they are focused upon objectives and attempt to show the student as a thinking, feeling individual, rather than as a specimen to which a mere label is attached. In the second place, no particular organization of material, such as that now commonly included under the phrase "the first semester of plane geometry" is presupposed or prescribed by the form of reporting. This leaves a particular school free to organize materials in harmony with its own situation. The reports tend to shift the emphasis from achievement on a limited range of specific objectives—recall of facts, mastery of skills, knowledge of technical vocabulary, etc.—to those more basic ultimate objectives which are supposed to be achieved. This shift in emphasis on the report ordinarily results in increased attention being given to the basic objectives during the instruction, and encourages more comprehensive views of the purposes of educational activity. For many teachers such considerations outweigh less fundamental points centering around the increase in time required to prepare the more comprehensive

reports. The job of the teacher is to promote the welfare of the students, and a progressive evolution of systems of reporting is a means to this end.

Bibliography

BRESLICH, Ernst R., *The Technique of Teaching Secondary-School Mathematics* (Chicago, University of Chicago Press, 1930), Ch. VII.

HAWKES, H. E., LINDQUIST, E. F., and MANN, C. R., *The Construction and Use of Achievement Examinations* (Boston, Houghton Mifflin Co., 1936). Contains a bibliography.

SEDER, Margaret, "An Experimental Study of a New Mathematics Test for Grades 7, 8, and 9," *The Mathematics Teacher,* Vol. 32, October, 1939, pp. 259–264.

TYLER, R. W., *Constructing Achievement Tests* (Columbus, Ohio, Bureau of Educational Research, Ohio State University, 1934).

UPTON, C. B., "Standardized Tests in Mathematics for Secondary Schools," *The Reorganization of Mathematics in Secondary Education* (Oberlin, Ohio, The Mathematical Association of America, 1923). A summary of the status of testing in 1923, including many illustrative exercises and an extensive bibliography.

Appendices

II

A SOURCE UNIT[1] ON NORMAL VARIABILITY

How can I get up to normal weight? Why can't I be normal in height? Is my mind normal? Why isn't my complexion normal? Questions such as these reveal the existence of personal problems in the lives of adolescents and of adults as well. Many people are preoccupied by questions which concern their real or fancied variation from norms or averages of one kind or another.[2]

One of the needs of adolescents in the area of personal living is for assurance that they are developing normally. In the words of the Committee on the Function of Science in General Education: [3]

There is a phase of the problem of increasing the self-assurance of adolescents that might be indicated by stating that everybody needs a *sense of proportion.* Many students are worried by aspects of their own development which seem to them abnormal or subnormal. For the sake of their self-assurance it is often desirable to develop understandings concerning physical development over and above those necessary for the maintenance of physical health. But many worry about their be-

[1] The term *source unit* as used here means a preliminary exploration of a problem showing its possibilities for study. A *teaching unit* would involve the selection and further development of certain of the suggestions included in the source unit.

[2] This Report does not present the sorts of data that have led many experts in the field of mental hygiene to the conclusion that worries about abnormality of one kind or another are not only widespread in occurrence but have fundamental psychological significance and play an important rôle in emotional adjustment. In this unit the findings of experts in mental hygiene will be accepted, and attention will be directed to the question: Is there anything which teachers of mathematics can do about these problems? For more extended discussion of the significance of the problem the reader is referred to literature in the field of mental hygiene, particularly the publications of the Study of Adolescents of the Commission on the Secondary School Curriculum, and of the Commission on Human Relations of the Progressive Education Association. See Alice V. Keliher, with the Commission on Human Relations, *Life and Growth* (New York, D. Appleton-Century Co., 1938); Peter Blos, *The Adolescent Personality: A Study of Individual Behavior and Its Meaning* (New York, D. Appleton-Century Co., forthcoming publication).

[3] *Science in General Education,* pp. 95, 96.

havior and emotions as well as their anatomy and feel that they must hide or bluff about their mental or emotional inferiorities. Some of these are reassured by being shown that development is not uniform through time or for any of the characteristics of a given person, and that the human population presents a wide range of normal variability with respect to most of its attributes.

It would seem that students *need,* then, a functioning understanding of variability. Any subject-matter that deals with human characteristics may be used as a vehicle for the satisfaction of this need, and no one special subject-matter can be singled out as best. Uniformity of trend, central tendency, and dispersion are the basic concepts of the science of statistics, and elementary applications of statistical methods are one of the best ways to put the growing and generally "abnormal" adolescent at ease with respect to the characteristics of his own peculiar self. The converse aspect of this point of view, namely the variability of other people, develops concomitantly. . . . Many physical and mental characteristics show a wide range of normal variation, both between individuals and in the same individual at different times.

The need often arises because the individual feels that he is "not like other people." Failure to recognize the wide range of individual differences, and misconceptions about the nature of published or intuitive "norms" or averages of weight, height, intelligence, and the like, aggravate worries about personal status for which there may be little justification. These worries often lead to behavior reactions which may or may not be desirable. Thus people who believe themselves to be "overweight" or "underweight" sometimes resort to diets, exercises, or nostrums that may actually be injurious. It seems probable that such worries and behavior reactions would at least partially disappear or would never arise if people had a clearer understanding of the nature of normal variability. Considerations of this kind and the data which support them indicate that to acquire this understanding and the related behavior modifications is a real need of a great many boys and girls. The purpose of this appendix is to show the contribution which instruction in mathematics can make toward the satisfaction of this need.

WHAT POSSIBILITIES FOR MATHEMATICAL WORK DOES THE UNIT HAVE?

Discovering and Formulating the Problem

Problems relating to normal variability are so close to the surface that little ingenuity is required on the part of the teacher to start

the students thinking about them. Questions may even come up spontaneously in connection with other work, and be scheduled for later, or perhaps immediate study. There are many possible introductions to the work—for example, the teacher might read aloud the very first chapter of *Life and Growth,* and then ask the questions which are answered in the third chapter: "How fat is fat? How tall is tall?" Some such method may be used to set the stage for active study of the unit.

A first step in attacking problems relating to normal variability might be a preliminary clarification of what is involved. Two questions which immediately arise are: "What does the word *normal* really mean?" and "Normal with respect to what?" The latter question may, of course, be phrased in other ways; for example, "What factors shall we study to see what normal means in particular cases?" Class discussions of questions like these should bring out the notion that "normal" means a sort of average, and when accurately defined may be found by arithmetical operations on data obtained from the measurement of large groups of people. It is, therefore, necessary to decide at the outset what characteristics are to be studied, and students may be encouraged to suggest possible characteristics either orally or in writing. The list thus obtained may contain some items which are not readily susceptible to quantitative treatment. For example, complexion and digestion seem to involve judgments more qualitative than quantitative. The inclusion of such characteristics provides an opportunity to help students develop understanding and powers of recognition of the quantitative aspects of the problem. They may, in fact, be asked to indicate which characteristics in the complete list suggest quantitative data, and certain items in the reduced list thus obtained may then be selected for class or individual study. This list of characteristics will usually include weight, height, and age; but it may also include a great many others such as strength (e. g. grip), reach, reaction time, and mental age ("intelligence").[4]

It should hardly be necessary to mention that study of the nonquantitative factors may also be very desirable. But such study is probably best carried on in coöperation with teachers of other departments. The teacher of mathematics is primarily responsible for the quantitative aspects of the problem. It is worth noting, however, that many of the general ideas which may be developed

[4] Ordinarily this preliminary work leading to the selection of particular variables should not require more total time than one class period, although in some cases it may require part or all of several days' instruction. General remarks on time requirements will be made in a later section.

are applicable to the qualitative characteristics, although with less precision.

Collecting Data

When the problem has been formulated, and particular factors, such as weight, have been selected for study, the next step is to collect data. But some preliminary discussion may be advisable.

1. Certain concepts about the nature of measurement may need to be reviewed or perhaps learned for the first time. Students should understand, for example, that measurements are expressed in terms of arbibitrarily chosen units, and that all measurement yields only approximate data.

2. Decisions must be made as to the desired precision, accuracy and completeness of the data. Precision will depend upon the apparatus available. Thus in the case of weight the school may have a scale, or perhaps one of the pupils may bring an ordinary bathroom scale from home. But it is not likely that weights may be measured more closely than to the nearest pound, nor is it desirable in view of the range of daily variation.

The question of how complete the data need to be is a difficult one. If the class is small it may be advisable to try to secure data from other classes in the school. This is usually necessary in order to secure sufficiently varied domains for the variables. Moreover, if the results for a given group or person are to be compared with published norms, data on the latter must also be secured. When this is done the appropriateness of the data must be taken into account; for example, it should be based upon measurements of children rather than of adult groups. Such points should be considered and some decisions, even if only tentative, should be made prior to the actual collection of data.

3. As what is to be done becomes clarified the size of the job may appear rather formidable. If so, different tasks may be assigned to committees. One committee may undertake to weigh the group to be studied. Another may measure heights, a third may secure data from books on age-height relationships, and so on. When this is done each group must then decide upon the particular responsibilities of individual members. It may be remarked in passing that such methods sometimes seem to lead to confusion and disorganization. If, however, students are to develop qualities of personality such as those discussed in Chapter II they must have opportunities for practising them. The aim, therefore, is to use such situations to help the students develop qualities of personality which tend to reduce the confusion and disorganization. This can be achieved only by careful planning and willingness to coöperate on the part of all concerned.

4. As the data are collected they should be recorded in some systematic fashion. This means that tables must be planned and the data entered

with proper attention to the number of significant figures and similar details. Study or review of the characteristics of good tables and of approximation thus may become a part of the learning activity. Specific instruction on these matters will be required in many cases.

Interpretation of the Data

Study of the data obtained by measurement and from published books or pamphlets will ordinarily involve most of the following types of work.

1. Study of tables should lead to the discovery of the more obvious relationships and variations. For example, as the height increases, the weight tends to increase. Statistical concepts and computations which may be found useful include the range, median, mode, and mean. All of these will be concretely illustrated by the type of data usually found in this study.
2. Some of the data found in the various sources are likely to be in graphical form. Thus the reading and interpretation of these graphs will provide an opportunity either to apply abilities previously acquired or to learn and practise unfamiliar skills and principles of interpretation.
3. In order to facilitate the recognition of relationships which exist the tabular data may also be represented in graphical form. Development of the ability to construct graphical representations will thus become one of the important objectives associated with this problem. Since a variety of different types of graphs may be demanded, discussion of the bases of choice will be in order. Particular emphasis is likely to fall upon the line graph and the histogram.
4. If the problem is being analyzed by relatively advanced students, study of measures of dispersion, standard error, and some elementary curve fitting may be found useful. There are a number of formulas by which certain physical indices may be computed. Some of the pupils may wish to compute the values of these indices using their personal data. The ability to evaluate a formula is needed for this purpose, and some understanding of symbolism is also involved.

Interpretation of the data will, of course, involve discussion of the relationships discovered, and comparison of individuals with the group and with national norms. But special emphasis should be placed upon the recognition of the wide range of individual differences disclosed. Students should be helped to understand that normality expressed in terms of a mean is a statistical concept, and that variation from the mean is to be expected—in fact that it is the "normal" thing. Students must understand that the likelihood that a given person should have the mean weight for his height or age is small, and that there is ordinarily no cause for alarm if one's weight (or some other factor) varies from the "norm."

Only in cases where the individual is at the extremes of the distribution is there much cause for concern. Even then one should examine the relationships among different factors before taking action. Body types, hereditary traits, and similar factors may be such that the variation is to be expected. On the basis of such study one can usually decide whether special treatment may be required, and in these cases the student may be referred to specialists. Thus if the student seems to be greatly overweight or underweight he may be advised to consult a physician who can study his case and make recommendations as to the proper treatment if any is in fact desirable.

It is also important that students come to realize that "norms" often change with time. When data have been collected over a period of years the averages are frequently found to change in ways that cannot be explained on the basis of chance alone. A typical instance is the well known decline in the rate of infant mortality, perhaps as a result of better pre-natal care and improvements in medical practice.[5] It is not wise to become too complacent if one is within the range of tolerance, above or below the norm. Perhaps the norm itself can be raised (or lowered) appreciably with profit for the social welfare.

WHAT ARE SOME ACTIVITIES WHICH THIS PROBLEM SUGGESTS?

The preceding section has given an overview of work which is related to this problem. The present section will supplement the previous discussion by listing various activities which students may undertake.

Activity 1. Listening to a reading of Chapter I of *Life and Growth*[6] or individual silent reading of the chapter.

Activity 2. Suggesting factors which might be studied, deciding which of these are readily susceptible to quantitative treatment, and choosing those which are to be studied in detail.

[5] In attempting to identify the causes of statistical changes great caution must be used. For example, the rate of infant mortality is defined by the formula $r = D/n$ where D represents the number of deaths of infants under one year of age, and n represents the number of thousands of births registered. But the value of a fraction decreases as the denominator increases. Hence, it is possible to reduce the death rate r by merely seeing to it that a larger fraction of the total number of births is registered.

[6] Keliher, *op. cit.*

Sample factors which may be studied: [7]

Age
Height
Weight
Strength of grip
Reach
Reaction time
Mental age

As an indication of the possibilities for extension of this list attention may be called to the fact that books related to the teaching of physical education frequently suggest a variety of measurements which an instructor of physical education may make. One such list follows: [8]

Weight
Height standing
Height sitting
Girth of chest in repose
Girth of chest after expiration
Girth of chest after inspiration
Girth of abdomen
Lung capacity
Breadth of chest
Depth of chest
Strength of right forearm
Strength of left forearm
Strength of upper arm and back as measured by the push-up and pull-up
Strength of back
Strength of legs.

Activity 3. Collecting data on one or more of the factors chosen for study. This may involve measurement of some sort (including mental tests), and it may require the use of sources to secure data obtained by others on a more extensive scale. The following are also involved:

(1) Understandings about measurement (e. g., choice of units).
(2) Decisions as to the desired accuracy and completeness of the data.

[7] Mental age may be found by the usual type of intelligence test. Strength of grip and reaction time require special apparatus. Descriptions of such apparatus may be found in books on measurements in physical education and psychology. Data may also be found in such sources.

[8] Jackson R. Sharman, *The Teaching of Physical Education* (New York, A. S. Barnes and Co., 1936), p. 190.

(3) Tabulation of the data, with attention to the proper handling of approximate numbers.
Tables may be arranged:
a. Alphabetically
b. According to rank
c. Frequency (See Activity 4.)

Activity 4. Analysis of tabular data. In order to simplify the discussion in this section only the following variables will be considered: age, height, weight, and body type. Assume that data on these factors are available for about one hundred pupils whose ages range from twelve to nineteen years. The data are here given rounded off to the nearest unit. For purposes of illustration only portions of the complete tables will be used. In class work, of course, the complete table would be studied. The original data might be tabulated as follows:

Boys

Name	*Age*	*Height*	*Weight*	*Type*
Allan	14	63	106	Average
Ben	13	65	119	Tall, slender
Charles	12	57	84	Average
Dan	15	60	98	Short, stocky
Edward	14	63	111	Average
Frank	12	63	107	Tall, slender

(1) The mean, mode, and median of the ages, heights, and weights may be found and interpreted.

(2) Because these data are assumed to be fairly extensive there are many possibilities beyond those just listed in (1). The alphabetical arrangement and the averages for the group may conceal certain important relationships.

Suppose that the relationship (if any) which exists between height and weight is the first one to be sought. The students may note that the table of heights contains many different numbers. In order to be able to refer to them readily as a group the term *variable* may be introduced. It is desirable to have the heights arranged in increasing (or decreasing) order. Since the first step in this process is to identify the least and the greatest value of the variable to be tabulated, the term *range* will be found convenient. When the heights have been arranged in order the frequency of occurrence of each may be tabulated. In the brief table given above the height 63 inches has a frequency of 3. If one now asks what weight corresponds to a height of 63 inches,

there are a number of possible answers. The usual procedure is to compute the mean of these numbers and to report the mean or "average" weight. For the data given here 108 pounds, which is the mean of 106, 111, and 107, might be reported. This procedure may be carried out in detail with the complete set of data. A brief excerpt from the resulting table would appear as follows:

Height	*Frequency*	*Average Weight*
62	18	102
63	29	110
64	21	114

The frequency column could, of course, be omitted, although it shows the number of cases from which the mean was computed.

(3) The observation that 29 students who were 63 inches tall (to the nearest inch) were of different ages suggests the tabulation of the frequency of these various ages as follows:

	Age							
Height	12	13	14	15	16	17	18	19
63	2	5	7	6	4	3	0	2

If one asks for the weight of the 7 students who are both 63 inches tall and 14 years old, there are again a number of possible answers. This suggests the computation of the mean weight of these 7 students, and by repeating the process for the other ages one line of a new table could be prepared. It would be similar to the following:

Height	*Average Weight*	*Age*							
		12	13	14	15	16	17	18	19
62	102								
63	110	106	107	108	110	113	118	123	127
64	114								

It may be noted in passing that the members of the 12 and 13 year age groups would probably be of the "tall, slender" type, the members of the 14 and 15 age groups would be of average build, and the group from 15 to 19

years old would be the "short-stocky" type. One should also note at this point that the average weight (110) of the whole group is not the average of the eight averages which correspond to different age groupings. The statistical fallacy involved in the contrary assumption is one which is frequently made. It is clear that in a problem such as this the opportunity to clear up this puzzling situation arises naturally. It is significant that this is a point frequently overlooked in formal units on statistics.

These computations and retabulations are not only valuable mathematical exercises. They should help students to understand tables of these types found in books. Perhaps enough has been said to show the learning possibilities involved in the search for relationships which exist in data like these. Further work may involve the recognition of quantitative relationships (e. g., as the height increases the weight increases), and the extension of a similar study to other variables.

Activity 5.[9] Graphs may be drawn to represent the data. Graphs drawn by the students or found by them in the sources may be interpreted.

Activity 6. Books on teaching and testing in the field of physical education describe certain formulas which yield various physical indices. While the computation of these indices may not contribute greatly to the basic purposes of this problem, it may in some cases be undertaken for reasons such as the following. First, the formulas involved may be found to deal with matters of considerable interest to the students, and may stimulate interest in the formula as a tool. Secondly, these formulas illustrate the fact that mathematics is now being applied in a wide variety of different situations. Leaders in physical education are making use of formulas in their work, although this is a field in which the uninitiated would hardly expect much use for mathematics other than arithmetic. Thus work on formulas may broaden the field of interest and increase students' appreciation of the applications of mathematics. The following are examples:

(1) "McCloy has developed formulas for the classification of boys by equalizing height, weight, and age. Each of his formulas gives a classification index which is a measure of the athletic group to which each individual be-

[9] This activity need not follow the analysis of tabular data. It may be carried on at the same time. For example, as soon as a frequency tabulation has been made it may be represented by a bar graph or histogram.

longs. He points out that these three factors (age, height, and weight) do not account for the whole of athletic proficiency. He lists at least seven other physiological, mental, and character factors which enter into expert performance in athletics. Age, height, and weight are three of the most fundamental elements in determining athletic ability and a plan for equalizing these is a definite contribution toward the complete measurement of athletic ability.

"Three classification indices have been proposed by McCloy:

(1) $20A + 6H + W$ (Classification Index I)
(2) $6A + W$ (Classification Index II)
(3) $10A + W$ (Classification Index III)

"The first formula is recommended for use with all age groups. In cases where the age is more than seventeen the age is counted as seventeen. The second formula is for use with groups, such as those in college, where all individuals are more than seventeen years of age. The third formula is for use in elementary schools where all pupils are below fifteen years of age. It was found that in boys under fifteen the height was not an important factor. Since in many elementary schools there are boys under fifteen and in colleges there are boys under seventeen, the first formula is preferred in all cases. The only advantage of using the other formulas is the small amount of time saved in making the computations." [10]

Students may evaluate these formulas for the boys in a class (or school), and so determine possible groups which would tend to equalize athletic abilities.

(2) Index of arm strength.

Let P = number of pull-ups.[11]
p = number of push-ups.
W = weight in pounds.
h = height in inches.
I_a = the index of arm strength.

$I_a = (P + p)\ (W + h - 60)$

Students may obtain the necessary data and compute their index of arm strength.

[10] Sharman, *op. cit.*, pp. 191–192.

Note that, strictly speaking, these are usually not considered formulas as written. A dependent variable should be specifically introduced; e. g., $C_2 = 6A + W$.

[11] Books on physical education explain methods of performing these exercises. Methods for boys and girls differ. $P + p$ is seldom greater than 40.

F. R. Rogers, *Fundamental Administrative Measures in Physical Education* (Newton, Mass., The Pleiades Co., 1932), pp. 181–182.

(3) Type of stature index.[12]

Let I_s = the type of stature index

S = sitting height

h = total height

Then $I_s = \frac{100S}{h}$

Students may compute their type of stature index.

Activity 7. Comparison of data on individuals with the means or medians of the group may be undertaken. This may involve finding the numerical differences, or the variation may be expressed in terms of the per cent by which a given individual differs from the mean of his group.

Activity 8. Comparisons of individual and group data with similar data published in books and other sources may be made. The question is: How do these students as a group compare with published norms?

Activity 9. Measures of dispersion, including average deviation, standard deviation, and quartile deviation may be computed and interpreted. Since one of the most important understandings to be derived from study of this problem centers around the dispersion of the data, this is a very desirable activity if the mathematical maturity of the pupils is sufficient to carry it out successfully.

Activity 10. In cases where there appears to be a relationship between two or more variables, the students may undertake some simple curve-fitting. The coefficient of linear correlation between height and weight is usually about .87. It is possible to obtain a linear formula which may be used to predict the weight corresponding to a given height. This work may be done without making any extensive use of correlation or more advanced curve-fitting methods, such as least squares. As an illustration a brief discussion of the development of a height-weight formula for tall slender girls ranging in age from 12 to 18 years will be given. Assume the data on the following page are at hand.

The graph is approximately linear and the first differences are approximately constant. Assume a formula of the type

$$w = ah + b$$

where a and b are to be determined from the data. If we arbitrarily choose the line which passes through the first

[12] John F. Bovard and Frederick W. Cozens, *Tests and Measurements in Physical Education* (Philadelphia, W. B. Saunders Co., 1930), p. 144. Other similar formulas are given.

HEIGHT-WEIGHT TABLE FOR TALL SLENDER GIRLS [18]

Height	*Av. Weight*	△	*w*	*d*
60	95	5	95	0
61	100	5	100	0
62	105	5	105	0
63	110	5	110	0
64	115	5	115	0
65	121	6	120	1
66	127	6	125	2
67	133	6	130	3
68	136	3	135	1
69	140	4	140	0

△ = tabular difference in weight; d = difference between tabular weight and w.

five points, the slope a is 5 by inspection. Then substituting the pair (60, 95) we have

$$95 = 5 \cdot 60 + b$$

and $b = -205$; hence

$$w = 5h - 205$$

would be one approximate formula. In the table w has been computed by this formula, and the discrepancies or residuals d between values of w as predicted by the formula and those given in the original data have been calculated. It happens in this case that these are all positive, although in general some will be positive and some negative. By choosing certain other pairs of values, formulas with different coefficients may be obtained. Various tests of goodness of fit may then be applied. For high school students the best-fitting curve may be defined as the one which yields the minimum value of A, where $A = \frac{\Sigma|d|}{n}$.

Or the root-mean-square test may be used, although it is more difficult to compute. Then $A = \sqrt{\frac{\Sigma d^2}{n}}$.

Other more refined methods of finding a suitable formula may also be used.[14]

The purpose of such work extends beyond the calculation of a formula which approximately fits the data. Students may acquire an understanding of the methods used by scholars in many fields to handle their data. Emphasis must be placed upon the various advantages which follow from

[18] For a more extensive table see J. F. Williams, *Healthful Living* (New York, The Macmillan Co., 1927), table inserted at p. 278.

[14] For a good elementary discussion of these methods consult J. P. Guilford, *Psychometric Methods* (New York, McGraw-Hill Book Co., 1936).

the possession of a suitable formula in many types of investigation.[15]

Activity 11. Such a study as has been described should be carefully summarized and reviewed at the close. Students may write small booklets or reports which contain the data, graphs, mathematical results, and summaries of the conclusions. Some students may make reports before the class or a school assembly explaining the study and its findings. Large charts and graphs may be prepared for this purpose. If the school or class arranges exhibits for the parents and general public, such materials add greatly to the effectiveness of the presentation.

REMARKS ON GRADE-PLACEMENT, TIME-ALLOTMENT, AND DRILL

Although only a few of the many possibilities for significant mathematical work which study of this problem provides have been indicated, the extensiveness of what has been outlined is certain to raise questions as to time allotments, grade-placement, and the like. The Committee has recognized the obligation to comment on these matters, and will attempt to state its views.

In the first place, it is clearly possible to teach much of this work in piecemeal fashion under the usual organization of courses in mathematics. Thus, most good modern courses in junior high school mathematics include some work in measurement. In connection with such work students can easily measure their heights and weights. Students at the seventh grade level can compute averages, and many students at the ninth grade level can formally compute standard deviations, draw graphs, and even find linear formulas to fit data. Thus, it is possible to analyze the problem further in terms of the difficulty of the concepts and skills involved, and then to distribute portions of the work at different levels of instruction, supplying data as needed. Formal instruction can be given as usual in the necessary skills, and they may then be applied to appropriate parts of this general problem. Such an organization of the materials involves little change from existing practices, and has many arguments in its favor. The principal argument against it is that the piecemeal attack on the broad problem lacks coherence.

In order to achieve the fundamental objectives outlined by this Report, and in particular to achieve the purpose set forth in the

[15] See Chapter VII, p. 140.

opening paragraphs of this appendix, some effective provision for integrating is necessary. The device recommended is the study of broad problems by the methods of reflective thinking. When concepts and skills are acquired in isolated contexts the pupil may be unable to put them together when faced with a complex problem. A student may know the definitions and some of the properties of the mean and the standard deviation, and be able to compute them, yet fail to see their meaning in a problem such as this. Moreover, he needs practice in the methods of reflective thinking. These are believed to be cogent reasons for studying this and similar problems in a continuous block of time.

If general problem-solving is held to be the dominant objective, and the problem under discussion is chosen for study, then it appears wise to postpone it until many of the necessary concepts and skills have been developed in connection with more simple problems. In many schools this should mean that the problem can be successfully dealt with in the latter part of the ninth year. In that case many of the basic concepts and skills should be available. Those which are not available can be taught in connection with this work, and certain other more advanced aspects can be omitted or postponed. The problem is not too difficult for any grade in the senior high school, and can be dealt with thoroughly at the twelfth-grade level. The concepts and skills are no more difficult than much of the traditional content of mathematics courses at these levels.

The question of time-allotment is a perplexing one, since it depends upon the mathematical maturity which the pupils bring to the problem, and the extent to which the analysis is carried. The time required might range from about one or two weeks in the case of students in the twelfth grade who have previously acquired most of the abilities involved and are merely *applying* them to a new problem, up to nine or ten weeks in case many abilities need to be developed as the work proceeds. If, for example, the students are drawing graphs for the first time in connection with this work, or are learning how to determine the coefficients a and b so that a formula like $w = ah + b$ will fit the data, then considerable time will be necessary. Since interest is likely to lag when the same problem is studied for too long a stretch, the problem should be attacked at a time when the students have enough skills and concepts to carry it to completion in at most four or five weeks.

As noted above, even under most traditional courses of study students should be ready for this problem by the latter part of the

ninth year. They should have already become familiar with the concepts of median, mean, and mode. They should be able to figure percentages, draw graphs, evaluate simple formulas, solve simple equations, and use negative numbers. The work on this problem can then introduce frequency tabulations, histograms or frequency curves, and measures of dispersion. At the same time review of the topics mentioned above would be naturally included. The nature of this review is indicated by the activities described above. Under these circumstances about four weeks of time should suffice for the study of this problem.

The question of drill must also be considered. If problems are studied in accordance with the methods here suggested, little formal drill should be necessary. The following two illustrations show how "drill" and review became a part of the actual problem solving process. In Activity 4 described above a large number of *arithmetic means* would be computed. This furnishes drill on addition, division, and the formal rule for computing the arithmetic mean. If other factors are studied, such as reach or girth of chest, additional work of this kind would be involved. As a second illustration consider Activity 10. If the straight line which is being determined is assumed to pass through two points not both on the line $w = 5h - 205$, then a different equation will of course be found. For example, if the points chosen to determine the line are (63, 110) and (66, 127), the system

$$110 = 63a + b$$
$$127 = 66a + b$$

must be solved. The approximate solution is $a = 5.7$, and $b = -249$, whence $w = 5.7h - 249$. Similarly, the choice of other pairs of points will lead to other sets of equations to be solved, providing "drill" on this operation. The particular table illustrated in the exposition of Activity 10 happens to lead to a number of equations for a and b which are not independent. In the general case, not more than two of the points determined by the data would fall on the same straight line. Then 10 points would lead to 45 different sets of equations for a and b, and 45 different empirical formulas for predicting w from h. This should be more than enough drill if all are found. Finally, the determination of the best-fitting curve involves much use of signed numbers and the formula in terms of which the criterion is expressed.

This opportunity for "drill" which arises naturally in the course of the study is not characteristic of ordinary mathematical "problems." This is because each "problem" of the traditional type con-

tains only enough data to obtain a unique solution. Broad problems of the type here under discussion usually involve more data, and this characteristic makes possible what might be called *inherent drill.* It is easily possible, of course, to use these data as a basis for constructing drill exercises of the usual type. For example:

A sixteen-year-old girl of the tall slender type is 68 inches tall and weighs about 130 pounds. By what per cent does she differ from the normal weight for that height?

This exercise has one missing item of data which could be found in a table similar to the one given in Activity 10. A still more formal exercise may be made by adding to the exercise the sentence: "The normal weight for this height is 136 pounds."

Since this entire problem tends to emphasize individual differences it is pertinent to remark that provision should be made for students of different mental ability. One method of doing this is to encourage students to investigate factors such as those listed in Activity 2 as special individual or small group projects. Some students may thus investigate more factors or more difficult factors than others. As a second way of caring for individual differences some students could study percentiles, ogive curves, the method of averages for curve-fitting, and similar statistical methods not taken up by the group as a whole.

HOW DOES THE STUDY OF THE PROBLEM CONTRIBUTE TO THE OBJECTIVES OF GENERAL EDUCATION?

In the course of this discussion only occasional references have been made to some of the qualities of personality discussed in Chapter II. There are, however, many opportunities in the study of this problem to develop some of these qualities. Creativeness, appreciation, and a wider range of interests can be fostered by encouraging students to discover and investigate certain factors not studied by the group.[16] Social sensitivity may be developed by making students conscious of individual differences and their implications, and by emphasizing the generality of this phenomenon.

[16] As an example: In one class a boy investigated the effect of exercise on the pulse rate. This happens to be the basis of the Schneider Test for Physical Fatigue and Efficiency, but there is reason to believe that this boy was unaware of that fact and that the formulation of the hypothesis and planning the experiment were creative experiences for him. Prior to this experience the boy had shown little interest or aptitude in mathematics, being solely interested in music. He has since become a graduate student in mathematics and engineering.

There is a wide range of differences not only in physical characteristics, but also in many broad social variables. Coöperativeness is clearly needed in order to carry on an investigation of this type successfully. Students must not only allow themselves to be measured—in many cases they can help others in parts of the work. Finally, reflective thinking applied to the analysis of a problem situation is the core of the study. Properly handled the experiences should constitute an excellent example of the use of this quality. Readiness to act on the basis of tentative judgments and self-direction are additional qualities which may be developed if the work is planned in ways which permit them to function.

CONCLUSION

This appendix illustrates the application of the general theory of Parts I and II of the Report to a specific problem. It is believed that study of this problem provides many opportunities for developing desirable qualities of personality. The analysis of the problem will call for the use of the concepts of formulation and solution, data, approximation, function, operation, proof, and symbolism which were discussed in Part II. The accepted principles of teaching methods which apply to questions of grade-placement, drill, and provision for individual differences can be applied in ways which should make them function effectively rather than formally. The hypothesis underlying the discussion is that a mathematical curriculum built by arranging problems of this kind in proper sequence should prove to be a rich and valuable experience for the great majority of adolescent boys and girls.

III

ILLUSTRATIVE SHORT ACTIVITIES

The Committee recognizes that some teachers and some schools will find it difficult to establish immediately the sort of program suggested in the Report. A period of transition is to be expected. At first a teacher may find it possible to introduce only one or two comprehensive problems for study in a given school year, gradually extending and organizing the program into a coherent whole. During such a transition period, or even if the organization of the mathematics curriculum were to remain much as it is, the major concepts discussed in Part II would still be worthy of special emphasis. Numerous activities may be developed for the purpose of throwing light upon aspects of these concepts. A few samples are given below. They should indicate the possibilities for enriched teaching in schools unable as yet to embark upon as fundamental a reorganization of the curriculum as is suggested by the Report.

FORMULATION AND SOLUTION

Students may be presented with the setting of a problem situation and be asked to list elements which they think should be studied. The elements thus listed may then be discussed, and finally various related sub-problems carefully formulated. During this work the various objectives discussed in Chapter IV should be touched upon, and eventually become explicit in the minds of the students. The following situations are illustrative:

Situation 1

"Jim Smith was in the hospital as a result of an auto accident. One of the students in his homeroom at Newton High School suggested that they should buy some flowers for him. As the group discussed this proposal, a number of questions were asked. What questions do you think should be asked in such a discussion?"

If students are allowed to write their responses, such questions as the following may be expected:

1. What kind of flowers shall we get?
2. What will the flowers cost?

3. Who will take the flowers to the hospital?
4. How many flowers shall we buy?
5. How many students will pay for the flowers?
6. Will we get to see Jim at the hospital?
7. Will each student pay the same amount?

One of the specific problems which should emerge might be phrased as follows: How much will each student have to pay if flowers are purchased? Among the assumptions or agreements might be the following: We shall assume that each student pays the same amount. It might be understood in advance that an assessment of $.25 would be reasonable, but proposed solutions of $25 or 2.5 cents would not satisfy the criterion of reasonableness. These remarks indicate the manner in which the objectives relative to Formulation and Solution may be applied.

Situation 2

The graduating class of the Woodrow Wilson High School is planning to buy something for the school as a class memorial. In discussion a number of important questions were asked. What questions do you think should be asked in such a discussion? What things should the class consider in making a decision as to what to do?

Situation 3

Springville is an inland town. There is no river, lake, or ocean near, so the people had no place to swim. A group of young people got the idea that the town should have an artificial swimming pool such as many other communities have. One evening they got together to see what could be done about it. Many suggestions were made and discussed, and many questions were asked. What do you think would be some sensible questions to ask at such a meeting? What would you suggest that the group should study before trying to make definite plans?

Situation 4

In Middletown there are two factories which employ boys and girls as soon as they have been graduated from high school. A group of young people were discussing which company they would rather work for, and several different factors were mentioned which would influence their choices. Suppose you knew very little about these companies and wished to know more in order to make

a choice as to the one in which you would prefer to work. What things would you take into consideration?

Situation 5

The Jefferson Township High School had a good football team but people seemed to think their record of games won was not as good as that of teams from other similar schools against whom they played. Players were injured or got sore muscles. Boys caught cold and were unable to play. Every year the team lost several games they might have won if all their best men had been able to play. Finally, a group of students in the school decided to study this problem. They wanted to find out whether the Jefferson team really had a poorer record and if so, whether anything could be done about it. What are some of the questions they might ask before they could really solve this problem? What are some of the things they should consider studying in order to solve it?

Situation 6

Marion Jones was chairman of a committee which was assigned the problem of arranging a debate tournament. The committee had to decide just when each team was to meet its various opponents, and make this clear both to the different teams and to the audience. How should they do this? What type of solution does this problem suggest?

Situation 7

The Board of Education of Lodi has just appropriated money for a new athletic field. The athletic department wants the field to provide a quarter-mile running track with 120 yard straight-away, football field, baseball diamond, pole vault pit, broad and high jump pit, shot put circle, and discus circle. The available piece of land is so small that some of these must overlap. Some of the boys were trying to decide how the field should be arranged. What *form of solution* do you think would be best for this problem? Why?

Situation 8

Some students in a Science class made a large number of measurements of the current flowing in an electrical circuit when the voltage and resistance were changed. The data were recorded in their note-books as follows:

Voltage	*Resistance*	*Current*
110	110	1.0
110	100	1.1
110	90	1.2
110	80	1.4
110	70	1.6
110	60	1.8
110	50	1.2
etc.	...	...
...	...	...
100	110	0.9
100	100	1.0
100	90	1.1
100	80	1.2
etc.	...	...

The students would like to find some method of presenting these data which would require much less space. What type of solution would it be wise for them to seek?

DATA

1. The sorts of activities suggested above should provide opportunities for practice in classification. For example, the students in one school undertook the study of the cost of a trip. They individually listed the factors which they thought needed to be taken into account. These lists were then mimeographed so that each student had the complete list of factors suggested by the class. They then faced the problems of classification, elimination of duplicates, clarification, etc. One list of factors was classified under *transportation,* another under *lodging,* a third under *food,* etc. As the study continued, the students discovered that their original classifications needed revision. This exercise proved to be a highly interesting, stimulating, and profitable experience for the class.

2. The following exercise illustrates situations in which a knowledge of the qualities of good data is useful.

Franklin School is located on Washington Avenue, a street which is very heavily travelled. The students and the teachers felt that a "stop and go" traffic light should be installed so that they could get across the street more safely. Miss Brent's mathematics class decided to get some data on the amount of traffic in front of the school in order to convince the city council that a signal was needed.

After some discussion, Miss Brent said, "The problem is: How much

traffic is there on Washington Avenue in front of our school? How shall we obtain this information?"

John said, "We can appoint some one in our class to count the cars going by in one day."

The class criticized John's plan. Some of the things they said are given below. You are to decide whether each suggestion is a good one or not, and tell why you think so. Follow these directions:

Part I. 1. If you think the suggestion is a good one, put a plus sign (+) on the line at the right.
2. If you think it is *not* a good suggestion, put a zero (0) on the line.
3. If you cannot decide, leave it blank.

Part II. If each suggestion is followed, certain things may be true about the data obtained. For example:
1. The data would be more complete.
2. The data would be more accurate.
3. There would be more data than is needed for the purpose, or the data would be irrelevant—that is, it would not help to answer the problem stated.

These are really reasons why the suggestion is a good or a bad one. On the line at the *right* of each suggestion below, write the number of any of these reasons which explains why the suggestion is good or bad. If you have other reasons, write them in, number them, and use them also.

List of Criticisms

_____*a.* More than one person should count the traffic	_____*a.*
_____*b.* The speed of the traffic should be taken into account	_____*b.*
_____*c.* The traffic should be counted on more than one day ..	_____*c.*
_____*d.* The traffic does not need to be counted while school is in session	_____*d.*
_____*e.* The traffic does not need to be counted on Sunday ..	_____*e.*
_____*f.* The number of people crossing the street should also be counted	_____*f.*
_____*g.* The number of people in each car should also be counted	_____*g.*
_____*h.* The number of people enrolled in school should be taken into account	_____*h.*
_____*i.* Since school doesn't begin until nine o'clock, the traffic does not need to be counted before eight-thirty in the morning	_____*i.*
_____*j.* Some one should go along as timekeeper and help "tally" or record the amount of traffic	_____*j.*
_____*k.* The traffic should be counted for a month	_____*k.*
_____*l.* The traffic should be counted for a whole week	_____*l.*
_____*m.* The traffic doesn't need to be counted after five o'clock	_____*m.*

Situations of this kind may be used in several ways.

a. The class may work the exercise as it stands, and the results may be used as a basis for discussion of the relevancy, accuracy, and completeness of the data.
b. The situation only may be given, and the class may criticize the plans orally making suggestions as to other modes of procedure which would yield better data.
c. The situation may suggest similar ones appropriate to a given discussion. If conditions are favorable, after discussions of the type indicated, the students may proceed to the actual collection of data.
d. They may be used for the evaluation of achievement with respect to several of the objectives under consideration.

APPROXIMATION

As noted in Appendix I, many of the activities there suggested could be used apart from a consideration of problems relating to normal variability as a whole. Data from various sources may be supplied to the students and activities analogous to those listed in Appendix I may be used to clarify the concepts related to approximation.[1]

FUNCTION

Although the literature on the teaching of mathematics is rich in illustrative materials related to the concept of function, the following may prove helpful to some teachers.

1. Relationships between units of measure yield an extensive class of linear functions. For example, twelve linear inches correspond to one linear foot, or expressed as a formula: $n_i = 12n_f$.
(*a*) Students may make a list of as many such relationships as they can discover.
(*b*) Part or all of the resulting list may be used as material for work on the objectives discussed in Chapter VII. For example, each of these functions may be exhibited in the form of a table, graph, formula, and verbal statement.
(*c*) The set of tables, graphs, and formulas may be studied to arrive at

[1] For an extensive list of activities see: Aaron Bakst, *Approximate Computation, Twelfth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1937). See also Lee Emerson Boyer, "Elementary Approximate Computation," *The Mathematics Teacher,* Vol. 32, October, 1939, pp. 249–259.

general statements about the properties of functions of the type $y = ax$.

(*d*) The general principles derived may be used deductively to describe the properties of other formulas of the same type.

2. The study of various *rates* leads to a class of linear functions. Thus, the statement that "the birth rate was 17 per thousand of the population" may be put in the form $B = 17n$ where B represents the number of births, and n the number of thousands of people in the population. In general, the birth rate is the parameter r in the formula $B = rn$, and r takes different values from year to year, or month to month, or for other time intervals.

(*a*) Students may seek other examples of rates; e. g., death rate (general and from specific causes such as tuberculosis or accidents), rate of interest, "average" or uniform rate of speed, etc.

(*b*) These examples may become the basis for study directed toward each of the objectives mentioned above.

(*c*) Students may collect data on birth rates, death rates, accident rates, etc., and attempt to interpret the changes observed. The social and economic implications of these changes may be considered to an extent depending upon the maturity of the pupils.

3. (*a*) Students may measure various physical characteristics of the members of the class and try to discover relations among the variables studied. The age-height, age-weight, and height-weight relationships are probably the ones most frequently observed. Many other relationships, such as age-chest circumference, are observable if the range of ages is sufficiently extended at the lower end.

(*b*) Since the limited range of age obtainable from a single class somewhat restricts the interpretations, it is advisable to obtain data from other sources covering the period from infancy to adulthood.

4. The extension of a spring corresponding to various weights may be measured. If the weights do not exceed the elastic limit, the data should be approximately linear.[2]

5. Secure five or more fairly new dry cells, or borrow two or three storage batteries. Borrow an ammeter and a buzzer or bell from the science laboratory. Connect the apparatus in series, and take readings of the current which flows when one cell (one cell of a storage battery or one dry cell), two cells, three cells, etc., are used. The unit for the independent variable may be taken as 1 cell, or as 1.5 volts for each dry cell, or as 2 volts for each storage battery cell. The data may be recorded in tabular form, the graph should be approximately a straight line. The relation between the current (usually denoted by I) and the voltage (E) is of the form $I = CE$. The constant C is sometimes called

[2] See Lancelot Hogben, *Mathematics for the Million* (New York, W. W. Norton and Co., 1937), p. 416.

the conductance, and is the reciprocal of the resistance of the circuit. The constant may be determined from the data to give an empirical formula which fits the data approximately. Simple criteria may be used to determine the formula of "best fit."

6. Draw circles on cross section paper such that the radii are one unit, two units, three units, etc. For each circle obtain an approximation to the area by counting the number of square units enclosed. Tabulate and graph the resulting data, and observe that the formula connecting the radii and areas which the data suggest is of the form $A = cr^2$. Determine the constant c by statistical methods.

7. Reproduction by cell division is a common biological phenomenon. The number (n) of individual descendants of a given cell after x divisions is given by the formula $n = 2^x$. Students may derive this formula by drawing simple diagrams and building up the first few terms of the geometric progression by counting. The functional relationship may then be studied from various points of view.

8. The compound amount on one dollar may be computed by simple arithmetic for a number of intervals, and the compound interest law may be derived and studied. This inductive work need not presuppose formal work on geometric progressions, but may in fact serve as an introduction to them.

9. Data on the law of cooling are readily obtainable.[3] An empirical formula connecting cooling time and temperature may then be found. Since the data should approximately follow an exponential law, the use of semi-logarithmic graph paper and logarithms will facilitate the work.

10. The students may check a swinging bob to verify that the time of vibration is approximately independent of the amplitude of the swing. They may examine the period of vibration as function of the length of the sustaining cord, and of the weight of the bob.

11. The students may attempt to discover a formula for the minimum height of the expression of $x^2 + bx + c$ as a function of b and c, by examining many cases.

12. The students may construct (with suggestions from the teacher) some simple nomograms.

13. Using a table of square roots, the students may learn how to construct a logarithmic scale, such as would be used on a slide-rule.

14. By counting small squares, the area of a right segment of a given parabola as function of the chord may be discovered, the parabola being taken in the form $y = ax^2$.

15. A common question related to the study of polygons deals with the number of diagonals that can be drawn in a polygon with a given number of sides. To secure data pertinent to the solution of this problem involves the actual construction of a number of polygons and counting all the diagonals that can be drawn. Two variables are in-

[3] Cf. *Ibid.*, p. 419.

volved and the data thus collected may be presented as in the accompanying table where n represents the number of sides of the polygon and d the number of diagonals. The question may then be raised as to whether or not it is possible to determine from these data a relation of the variables which will make it possible to determine d once n is known. The method of finite differences indicates that this relationship is of the second degree and by solving three equations in three unknowns the general formula can be obtained and the problem solved.

n	d
3	0
4	2
5	5
6	9
7	14
.	.
.	.
.	.

OPERATIONS

Since so large a part of secondary-school mathematics consists of work that aims to develop operational facility, it is hardly necessary to suggest activities related to this concept. What is needed is not new types of activities so much as new types of emphasis as the usual activities are taught. In order to develop fully the concept of operation it may, however, be desirable at appropriate points to focus almost exclusive attention upon it for brief intervals. The following types of activities may be suggestive to teachers who undertake such work.

1. In order to emphasize the mechanical aspect of operations, pupils may try to list as many mechanical counting and computing devices as they can discover. The principles on which some of these operate may be actively studied as individual projects. Manufacturers of computing machines publish booklets which may be helpful in developing such a project. References to encyclopedias should also prove helpful.

2. Students may find it interesting and instructive to learn how underlying generalizations about the operations are useful in practice. For example, a booklet entitled "Short Cuts That Save Valuable Time in Operating Adding Machines," published by the Burroughs Adding Machine Company, Detroit, Michigan, prints in large type: "Multiplication is merely repeated addition," and shows how the principle is used in operating the machine. On another page short-cuts commonly taught in mathematics courses are illustrated. The multiplication of 4087×198 is done by two methods. In one "20 operations" are required, while the other, based on the relation $4087 \times 198 = 4087\,(200 - 2)$ requires only "6 operations." Opportunities for study of such booklets may be provided in mathematics classes.

3. Some exercises may be solved by uneconomical methods (e. g., repeated subtraction in place of division) and also by more direct methods in order to emphasize the economy thus effected.

4. Some equations with numerical coefficients may be solved by indi-

cating the operations but not combining the numbers except perhaps as a final step.

PROOF

The recent literature on the teaching of mathematics contains many illustrations of non-mathematical activities designed to help develop the concept of proof. For this reason none need be given here.[4] For activities calling for mathematical proofs one need only turn to any textbook. What is most needed in this connection is not so much new activities as better teaching and new emphases.

SYMBOLISM

1. Since pantomime is a form of symbolism, students may attempt to communicate certain ideas by this means. Class discussion should emphasize the *symbolic* aspects of the activity.
2. Students may invent ideographic marks to symbolize many common activities, such as going up and down in elevators, running around race tracks, and the like.
3. Students may collect a set of the diagrams often published in newspapers to depict the events of a football game, naval or land battle, etc. An elaborate symbolism is usually involved, with different types of lines used to represent running plays, passes, kicks, etc.
4. Students may make maps depicting the path they usually take in going from home to school, interesting hikes, etc.
5. By inserting pins in a map, students may prepare a symbolic representation of the location of their homes with respect to the school.
6. Students may make lists of different mathematical notations which are essentially equivalent in meaning; e. g., $1 \div 5$; 1:5; 1/5; $\frac{1}{5}$; 5^{-1}.

[4] See in particular Harold P. Fawcett, *The Nature of Proof, Thirteenth Yearbook* of the National Council of Teachers of Mathematics (New York, Bureau of Publications, Teachers College, Columbia University, 1938); and Henry H. Shanholt, "A New Deal in Geometry," *The Mathematics Teacher,* Volume 29, February, 1936, pp. 67–74; and R. W. Jepson, *Clear Thinking* (New York, Longmans, Green and Co., 1936).

INDEX

Abbreviations, as symbols, 214, 219-222
Abstractions, application of, 232, 251-253; as characteristic of mathematical development, 256-258; understanding through proportional variation, 150-151; indirect proof in, 207; relation to adolescent development, 271-272; stages of, with regard to definitions and assumption, 205-211; understanding through study of symbols, 224
Abstract postulate system, 210
Accidents, illustrative problem on, 78 ff.
Accuracy, of data, 94; of measurement, 117
Act of the Commonwealth of Massachusetts, A.D., 1827, An, cited, 3
Activities, illustrative, on approximation, 408; on data, 406-408; on formulation and solution, 403-406; on function, 410-411; on operations, 411-412; on proof, 412; on symbolism, 412; in illustrative source unit, 390-398
Addition, as fundamental operation in natural number system, 171
Adolescence, characteristics of, early, 273-274; later, 274-275; meaning of, 273; strains and conflicts of, 275-280
Adolescent needs, determination of, 17-19; in economic relationships, 22-24; in immediate personal-social relationships, 21; in personal living, 20-21; in social-civic relationships, 22
Adolescents, affectation of sophistication by, 281; aggressiveness in, 282-283; and crushes, 283; and fantasy, 284; and lack of status, 279; and parents, 279; compensations for inadequacies by, 285-286; difficult decisions for, 279-280; difficulties in teaching, 274; effect of social and economic change on, 7; effect of early experience on, 286-291; goals of, 278; reduction of employment opportunities for, 6; variation in development of, 276-277; withdrawal of, 281-282
Adulthood, approach to, 288
Advanced mathematics, use of, 46-48
Aggressiveness, as characteristic of adolescent behavior, 282-283
Ahmes Papyrus, 244
Aims, of adolescents, 278; of general education, 42-43; of mathematics education, historical sketch of, 3 ff.
Algebra, 4, 5, 49; conventional teaching of, 189; fundamental operations in, 174-177; ideographic symbols in, 219-220; rôle or deduction in, 201; symbolism in, 175-176, 214, 260; symbols of, teaching development of, 260; understanding functions of independent variables through, 159; understanding functional variation through, 153
Algorithm, as type of solution, 85
Anecdotal records, as evaluation technique, 352
Appreciation. *See* Esthetic appreciation
Approximation, 112-137; building students' understanding of, in measurement and computation, 115-120; in statistics, 121 ff.; disregard of, in secondary-school mathematics, 115; illustrative short activities on, 408; originating in mathematical theory, 119-120; originating in measurement, 115-119
Architecture, as indication of mathematical development, 244; geometric form in, 49
Arguments, and variable, 143; generality as to, 162
Arithmetic, 49; conventional teaching of, 189; ideographic symbols in, 219; illustrative problems from, 112-113; study of applications of, 260-261; symbolism in, 214; versus algebra, 174-177
Arithmetic mean, as measure of central tendency, 123-124; formula for, 123

Art, geometric form in, 49
Associated variables, approximate relations among, 130-137; linear correlation of, 136-137; negative, 133; positive, 133; trend in, 134-136
Assumptions, analysis of, in effecting a reasoned conclusion, 196; as data, 95; clarity of, in formulating problem, 82-83; involved in selection of data, 201-202; stages of abstraction with regard to definitions and, 205-211
Astrology, 255
Automobile trip, illustrative problem on, 75 ff.
Average. *See* Arithmetic mean

Babylonia, evidences of mathematical development in civilization of, 244-245
Bacon, Francis, 228
Bar graphs, 109-110
Barter, numbers needed in, 243
"Basic Responsibilities of General Education in America," Edmund E. Day, 36
Behavior, adolescent, in response to strains, 280-285; illustrative case history, 293-329; as consistent expression of individual personality, 292; characteristics of, essential to democratic living, 31-43; classroom, interpretation of, in view of students' personality and needs, 329-333; statement of evaluation objectives in terms of, 347
Bias, operation of, in use of data, 94-95
Blos, Peter, 269 n
Bode, Boyd H., on coöperation for common concerns, 29
Body, adolescents' interest in, 276
Boy, acceptance of masculine rôle by, and parental relationship, 290; case history of, 293-329; relative interest of in mathematics, 271

Case study, of outstanding student in mathematics, 293-329; conclusions from, 325-329; implications of for teaching, 329-333
Centigrade temperature, as arbitrary unit of measurement, 100
Central tendency, measures of, 122-127; geometric mean, 125-127; median, 124-125; mode, 125
Chance, rôle of, in mathematical development, 246-247
Change, effect of, on secondary education, 6-8; in adolescence, 275
Chart, statistical, as type of solution, 84
Checking, in computation, 183-185
Circle graphs, 109
Civilization, neglected in mathematics, 243-245
Class discussion, and growth of tolerance, 50
Class-intervals, 134
Coefficient, of correlation, 136
Coefficients, literal, 177; understanding root as function of, 176
College, mathematics a requirement for, 3
College Entrance Examination Board, established, 4
Committee of Ten, recommendations of, for mathematics curriculum, 4
Comparisons, concepts basic to, 177-180
Compensation, as characteristic of adolescents, 285
Computation, building students' understandings of approximation in, 115-120; checking, 183-185; importance of correct, 183-185
Concepts, choice of, in statement of problem, 82; approximation, 59, 112 ff.; data, 59, 90 ff.; formulation and solution, 59, 75 ff.; function, 59, 139 ff.; operation, 59, 165 ff.; of operation, versus techniques, 180-181; proof, 59, 187 ff.; relation of, to curriculum organization, 71-74; to democracy, 63-69; to problem-solving process, 59-63; to understanding of mathematics, 69-71; symbolism, 59, 214 ff.; statistical, building students' understanding of, 121 ff.
Conditions, necessary and sufficient, 199-200
Congruence, 166
Construction, geometric, 180; experimental approach to, 200; method of, as type of solution, 85; rules for, as data, 95
Context, independence of, for meaning of symbols in mathematical usage, 227-231
Coöperation, of teachers, essential to meet student needs, 45
Coöperativeness, as personal characteristic essential to democratic living, 36; evaluating, 373; rôle of

mathematics in developing, 50-52
Correlation, coefficients of, 136; linear, of associated variables, 136-137
Counting and fundamental operations, illustrative examples in, 172-173; as early concept, 243; basic concepts of, 170
Creativeness, as personal characteristic of democratic living, 37-39; rôle of mathematics in developing, 50-52; types of, 38-39
Criticism of student products, as evaluation technique, 353
Crushes, as characteristic of adolescent behavior, 283
Culture, dependence of non-mathematical symbols on, 233; value of mathematics for, 4
Cumulative frequency tables, 108
Curriculum, application of democratic ideals in, 30-31; needed flexibility of, 25; overemphasis on mathematical operations in, 168-169; relation of evaluation program to, 342-343; relation of major concepts to organization of, 71-74

Data, 90-111; assumptions in selection of, 201-202; building students' understanding of, 91-96; characteristics of, 92-95; accuracy, 94; relevance, 92; reliability, 94-95; representativeness, 93-94; collecting and interpreting, illustrative source unit on, 388-390; collection of, accessibility in, 96-97; choice of units in, 97-100; primary and secondary sources in, 101-102; to follow formulation, 92; use of measuring and recording devices in, 100-101; determining existence of association among, 132-134; economy of, in formulating problem, 81; evaluation test on, 355-357; functional relationships among, 139-166; illustrative short activities on, 406-408; inaccuracy of, in nineteenth-century arithmetic, 113; interpretation of, test on, 360-365; multiple-parameter, 104-106; one-parameter, 104; operation of bias in use of, 94-95; organization of, focusing students' attention on, 103-111; organization of, in geometric figures, 110; in graphs, 109-110; in scale drawings, 110; in tables, 103; test, interpretation of, 375 ff.; test of arrangement of, 357; test on graphical representation of, 359-360; types of, 95-96; understanding of, derived from all fields of learning, 91
Dates, chronological arrangement of, in tables, 104
Day, Edmund E., on coöperativeness, 36
Day-dreaming, 274; translating into artistic or mechanical expression, 284-285
Degrees of freedom. *See* Independent variables
Decimals, in approximation, 119
Deduction, and induction, 203-204; "primitive terms" in, 197; truth and fact, 204-205
Definitions, rôle of, in deduction, 197-198; stages of abstraction with regard to assumptions and, 205-211
Democracy, essential to full development of scientific thinking, 68-69; ideals of, as basis of educational values, 24-31; free play of intelligence, 29-31; optimum development of personality, 27-28; reciprocal individual and group responsibility for promoting common concerns, 28-29; personal characteristics essential to, coöperativeness, 36; creativeness, 37-39; esthetic appreciation, 33-34; reflective thinking, 39-41; rôle of mathematics in developing, 48-53; self-direction, 36-37; social sensitivity, 32-33; tolerance, 35-36; relation of major concepts to, 63-69; versus individualism, 27-28
Democracy as a Way of Life, Boyd H. Bode, 29
Democracy in Transition, on individualism in democracy, 28
Deviation, mean, 129; quartile, 128-129; standard, 129-130
Dewey, John, on democracy, 25-26
Diagnosis, in evaluation program, 343
Diaries, of reading, as evaluation technique, 353
Difference, as basic concept to comparisons, 177-180
Dimensionality. *See* Independent variables
Discipline, value of mathematics for, 4
Dispersion, measures of, 127-130; mean deviation, 129; quartile deviation, 128-129; range, 128; standard deviation, 129-130

Division, as fundamental operation in natural number system, 171; "contracted," 119; of approximate numbers, 118; with remainder, 172
Domain, of independent variables, 152; of variable, as fundamental notion related to function, 143-145; types of, 145
Drill, value of, 183
Durost, W. N. *See* H. M. Walker

Economic relationships, adolescent needs in, 22-24
"Education and Social Change," John Dewey, 25-26
Education, democratic ideals as basis of, 24-31; for development of individual personality, 269; mathematics in relation to purposes of, 16 ff.; purpose of, 43
Egypt, evidences of mathematical development in civilization of, 244-245
Emotion, and regression in adolescence, 277; and symbolism, 235-236; eliminating factors of, in formulating problem, 79-80; of language, as distinction between mathematical and non-mathematical uses of symbols, 227-228; rôle of, in learning, 270-272
Emotion and the Educative Process, Daniel Alfred Prescott, quoted, 272
Employment, exclusion of adolescents from, 6
Engineering, applicability of functional relationships in, 139-140; periodic phenomena in, 158-159
English system of measurement, versus metric, choice of, 98
Enrolment, in mathematics courses, decrease in, 9-11; secondary-school, increase in, 7
Environment, and determination of needs, 17-19; interaction of individual and, 269-270
Equality, 166
Equations, beginning study of, 151-152; ideographic symbols in, 219-220; operational symbols in, 175-176; solution of, 176-177
Essay examinations, disadvantages of, 350
Esthetic appreciation, as personal characteristic, essential to democratic living, 33-34; rôle of mathematics in developing, 49-50
Evaluation, eliminating factors of, in formulating problem, 79-80; of qualities of personality, 372-375; of reflective thinking, 354 ff.; of student achievement, 338-382; process of, 345-354; *see also* Tests
Evaluation program, as basis for guidance, 339; characteristics of, balance and comprehension, 340; curricular freedom, 342-343; diagnostic features, 343; intelligent interpretation of data, 344-345; interpretation of test data, 375 ff.; measurement of growth, 340-341; teacher and student participation in, 341-342; valid techniques, 343-344; illustrative techniques for, 354 ff.; methods in, formulating objectives, 345-348, choosing life situations for, 348-349; interpreting the record, 353-354; obtaining records, 349-353; planning essential in, 338
Existence of association, in variables, 132-134
Experience, and personality, 269; family, effect of, on adolescent, 286-291
Experiments, restriction of data in, 81
Exponential functions, quantitative nature of, 158-159
Exponents, teaching development of, 260
Extrapolation, 152; in trend of associated variables, 135-136

Fahrenheit temperature, as arbitrary unit of measurement, 100
Family, effect of early experiences in, on adolescent, 286-291
Fantasy, as device of adolescents, 284; constructive uses of, 284-285
Figures, as type of solution, 85; geometric, use of, in organization of data, 110
Form, appropriate, as criterion for acceptable solution, 87-88
Formula, as expression of function, 145; as type of solution, 84; data in study of, 95; dimensionality in, 160-161; for arithmetic mean, 123; multiplicative transformation in, 153-155; operation of symbols in, 175-176; use of, to express proportional variation, 149; value of, 146
Formulation and solution, 75-89; illustrative short activities in, 403-

406; clarity of premises in, 82-83; collection of data to follow, 92; effect of restricting factors in, 80-81; effect of, on solution, 75; eliminating extraneous factors in, 79-80; focusing students' attention on, 77-84; of problem, in illustrative source unit, 386-388; on basis of economical hypotheses and data, 81; relation of, to solution, 83-84; revision of, as type of solution, 85; statement of problem for, 82; *see also* Solution
Fractions, operations in, 173
France, development of metric system in, 246-247
Frequency tabulations, 106-108
Function, 139-166; as type of solution, 84; common types of quantitative, 148-158; exponential, 158; periodic phenomena, 158-159; proportional variation, 149-156; powers and polynomials, 156-158; the form $y = ax + b$, 156; developing students' understanding of generality of, 161-166; fundamental notions related to, 141-148; fundamental notions related to, domain and range, 143-145; function and formula, 145-146; function and variable, 141-143; invariance, 146-148; illustrative short activities in, 410-411; meaning of, 142; of one variable, developing students' understanding of, 141-159; value of knowledge of, 139-140

Generalizations, application of, 151; as characteristic of mathematical development, 256-258; as laws, 193; mathematical, understanding of through proportional variation, 150-151
Genius, rôle of, in mathematical development, 248
Geometric mean, as measure of central tendency, 125-127
Geometry, 4, 51; construction operations, 180; contribution of, to logical thinking, 188; data in, 95; examples of, as contribution to logical thinking, 200; figures of, as guide to relations, 198; ideographic symbols in, 219; occurrence of, in nature, art, industry, and architecture, 49; relations in, 164; symbolism in, 214; traditional teaching of, 188-189; use of, in Babylonian civilization, 245; in Egyptian civilization, 245
Girls, acceptance of feminine rôle by, and parental relationship, 290; relative interest of, in mathematics, 271-272
Graphs, and discussion of interpolation and extrapolation, 152; as ideographic symbol, 220; as type of solution, 84; bar, 109-110; circle, 109; deriving formula for, 156; importance of good construction in, 110; in organization of data, 109-110; line, 110; pictograms, 110; reading versus interpretation of, 109; test on, 359-360; use of, to express proportional variation, 150; to show trend of associated variables, 135
Greenleaf, Benjamin, arithmetic problems by, 112-113
Growth, as continuous process, 286; measurement of, in evaluation, 340-341
Guidance, evaluation as basis for, 339; in self-direction, 37
Guidance worker, coöperation of, with teacher, 332

Havighurst, Robert J., cited, 16 n
Heterosexual adjustment, as adolescent goal, 278; effect of childhood behavior patterns on, 288-289
Hero worship, as characteristic of adolescent behavior, 283
History, 262
Hogben, Lancelot, cited, 218, 232, 243
Hypotheses, economy of, in formulating problem, 81

Ideographic symbols, in mathematical usage, 219-222; in non-mathematical usage, 222; unambiguity of, 229
Immediate personal-social relationships, needs in, 21
Independence, adolescents' struggle for, 278; effect of childhood behavior patterns on, 288
Independent variables, developing students' understanding of, 159-161
Indirect proof, known terms assumptions uncertain in, 207; principles of, 207

Individual, adapting historical aspects of mathematics to special interests of, 263-264; and group responsibility for common concerns, as democratic ideal, 28-29; as chief concern of education, 269; characteristics of, essential to democratic living, 31-43; rôle of mathematics in developing, 48-53; development of, as democratic ideal, 27-28; effect of early experience on, 286-291; interaction of, with environment, 269-270; *see also* Adolescents and Students
Individualism, versus development of personality, 27-28
Induction, and deduction, 203-204
Inductive proof, 192 ff.
Industry, geometric form in, 49
Inference, process of, 187 ff.
Instruments, for measurement and recording. use of, in collecting data, 100-101
Intelligence, free play of, as democratic ideal, 29-31
Interest, rôle of, in learning, 270-272
Interpolation, 152; in trend of associated variables, 135-136
Interpretations, of test data, based on one test, 375-378, based on several tests, 378-382; rôle of, in evaluation program, 344-345
Interviews, as evaluation technique, 353
Invariance, as fundamental notion related to function, 146-148; examples of, 147; of form, 148 n
Inverse, developing understanding of, 171

"John Dewey's Philosophy," Max C. Otto, 26
Junior high school, teaching development of weights and measures in, 259

Klein, Felix, cited, 5
Kotinsky, Ruth. *See* Thayer, V. T.

Language, referential and emotive, as distinction between mathematical and non-mathematical usage, 227-228; symbolism in, 215-216
Learning, rôle of emotions and interest in, 270-272
Life history, by outstanding student in mathematics, 293-329; conclusions from, 325-329; implications of, for teaching, 329-333
Linear correlation, of associated variables, 136-137
Linear functions, quantitative nature of, 149-156
Line graphs, 110
Logarithms, 252; discovery of, 249; use of, to teach exponential functions, 158
Logic, background of, essential to concept of proof, 189
Logical principles, use of non-mathematical material in illustration of, 202

Maps, approximation in making of, 120
Mathematicians, interest in, 263-264
Mathematics, advanced, use of, 46-47; and truth, validity of, 256; as response to spirit and needs of the times, 242-245; compulsory teaching of, 3-4; development and nature of, 241-264; adapting teaching of, to students, 261-262; development of as science, application of abstractions in, 251-253; cumulative character of, 248-254; effect of changing attitudes toward knowledge, 250; fashions in, 255-256; noncumulative aspects in, 254-256; related ideas in, 250-251; development of interest in biographical aspects of, 263-264; disregard of approximation in secondary, 115; effect of chance in development of, 246-247; emotional significance of, 270-272; emphasis on teaching of function concept in, 140-141; history of, contribution of, to tolerance and coöperativeness, 51; implications of case history for teaching of, 329-333; in prehistoric times, 243-244; in relation to general education, 16 ff.; loss of confidence in value of, 9-12; often a basis for decision in individual problem, 80; overemphasis on operations in, 168-169; popular misconceptions of, 242; relation of major concepts to understanding of, 69-71; rôle of genius in development of, 248; rôle of, in meeting student needs, 44-48; social bearing of, 46; substitution

of symbols for referents in, 231-232; theory of approximations originating in, 119-120; use of, in vocation, 46; *see also* Concepts
Mathematics education, criticism of, 5-6; reasons for reëxamination of, 3-12
Mathematics teacher, and students' ability to formulate problems, 75-76; and students' use of language, 215-216; coöperation of, with teachers of other fields, 75-76; in building understanding of data, 91; emphasis on mathematical and non-mathematical symbols by, 225-232; implications of life history for, 329-333; interpretation of test data by, 378-382; knowledge of personality development essential to, 52; needed attention of, to gifted students, 293; rôle of, in conveying social and economic bearing of mathematics, 46; in focusing students' attention on formulation, 77; in meeting students' needs, 45-46; use of historical materials by, 258-264
Mean, arithmetic, 123-124; geometric, 125-127
Mean deviation, as measure of dispersion, 129; formula for, 129
Meaning, of mathematical symbols, changes in, 230
Measurement, approximations originating in, 115-119; accuracy, 117; precision, 116; relative error, 116; rounding off numbers, 117-118; significant digits, 118; building students' understanding of approximation in, 115-120; devices for, used in collecting data, 100-101; English versus metric as unit, 98; nature of, 116; of growth, as characteristic of guidance program, 340-341; units of, as symbols, 218
Measures, teaching historical aspects of, 259
Median, as measure of central tendency, 124-125
Methods, effect of new psychology on, 8-9; of evaluation, 345-354; illustrative, 354 ff.
Metric system, development of, 246-247; of measurement, versus English, choice of, 98
Mode, as measure of central tendency, 125
Moore, E. H., cited, 5
Motions, as symbols, 214
Multiple-parameter data, tables of, 104-106
Multiplication, as fundamental operation in natural number system, 171; "contracted," 119; of approximate numbers, 118

Names, alphabetical arrangement of, in table, 104
National Arithmetic, The, Benjamin Greenleaf, illustrative problems from, 112-113
National Committee on Mathematical Requirements of the Mathematical Association of America, cited, 5-6
Nature, geometric form in, 49
Needs, and democratic ideals, 24-31; as basis of educational planning, 17-20; classification of, 20-24; meaning of, 17; personal and social aspects of, 19-20; rôle of mathematics in meeting, 44-48
Negative numbers, operations with, 174
New-type tests, 351-352
Numbers, approximate, test on, 357-358; as early concept, 243; as symbols, 218; imaginary, 252; mystic quality in, 270; negative, 174; rounding off, 117-118; significant digits, 118; system of natural, operations in, 171-173
Nunn, T. Percy, 5

Objectives, formulating for evaluation, 345-348
One-parameter data, table of, 104
One-valued function. *See* Function
Operations, 168-185; concepts basic to, comparisons, 177-180; concepts versus techniques, 180-181; construction in geometry, 180; counting, 170; developing students' understanding of concepts basic to, 170-181; illustrative short activities on, 411-412; overemphasis on, 168-169; techniques of, subsidiary to study of problem situations, 181-182; value of checking, 183-185; value of drill in, 183
Otto, Max C., on democracy, 26

Parameters. *See* Independent variables

Parents, and adolescents' acceptance of masculine or feminine rôle, 289-290; and treatment of sex problem in adolescence, 276; relationships of, with adolescents, 279; rôle of, in emotional growth of child, 286-291
Periodic phenomena, examples of, in science and engineering, 158-159
Perry, John, cited, 5
Personality, and experience, 269; as chief concern of education, 8-9, 269; development of, as democratic ideal, 27-28; evaluation of, 372-375; influence of early family experience on, 286-291; study of, developmental approach to, 292; illustrative life history, 293-329
Personal living, adolescent needs in, 20-21
Philosophy, 263
Pictograms, 110
Polynomials, as quantitative function, 156-158
Powers, laws of, as quantitative function, 156-158
Precision, in mathematical usage, 232; of measurement, 116
Preliminary Exploration of the Field of Immediate Social Relations of Boys and Girls in Secondary Schools, on interaction of individual and environment, 17-18
Premises, explicitness of, in formulation, 82-83
Prescott, Daniel Alfred, on dangers of teaching mathematical symbols, 272
Primary sources, in collection of data, 101-102
Probability, study of, 250
Problem, complexity of, in everyday life, 75; illustrative, automobile trip, 75 ff.; on physical and mental health, 78; restricting and idealizing, 77 ff.; restatement of, as type of solution, 86; statement of, 82
Problem-solving, reflective thinking in, 39-41; relation of major concepts to, 59-63; through reflective thinking, rôle of mathematics in, 52-53; *see also* Concepts
Proof, as type of solution, 85; cases impossible or unnecessary of, 191-192; concept of, 187 ff.; with relation to logical thinking, 188; defined in broad sense, 190-191; four stages of abstraction, 205-211; if-then principle, 195 ff.; illustrative short activities on, 412; indirect, 207; inductive and deductive reasoning in, 192; test on nature of, 368-372
Proportional variation, as quantitative function, 149-156; understanding of mathematical generalizations through, 150-151; use of graph to express, 150-151; use of table to express, 149
Proportions, as concept basic to comparisons, 177-180
Propositional functions, 162-163
Propositions, as data, 95
Psychology, effect of new theories of on secondary education, 8-9
Puberty, 275-276

Quadratic function, quantitative nature of, 156-158
Quartile deviation, as measure of dispersion, 128-129
Questionnaire, accessibility of data for, 97; as evaluation technique, 353; restriction of data in, 81; use of, in evaluating personality traits, 373-374

Range, as fundamental notion related to function, 143-145; as measure of dispersion, 128; of independent variables, 152; semiquartile, *see* Quartile deviation
Ratio, as concept basic to comparisons, 177-180
Recording devices, use of, in collection of data, 100-101
Records, anecdotal, 352; as evaluation technique, interpreting, 353-354; by trained observers, 352; obtaining, 349-353; interpretation of, 378-382
Reflective thinking, and symbolism, 234; as personal characteristic essential to democratic living, 39-41; evaluation of, 354 ff.; nature of, 40; rôle of mathematics in developing, 52-53
Reflexiveness, as properties of relations, 165
Regression, due to strains of adolescence, 277
Relations, discovery of, by students, 158; functional, generality of, 163-166; in reasoning, 166; properties of, 165

Relative error, of measurement, 116
Relevance, of data, 92
Reliability, of data, 94-95
Reorganization of Mathematics in Secondary Education, 5 n, 141
Reorganizing Secondary Education, V. T. Thayer, Caroline B. Zachry, and Ruth Kotinsky, 39-40
Report of the Committee of Ten on Secondary School Studies, cited, 4 n
Representativeness, of data, 93-94
Roman numerals, study of development of, 261-262; teaching of, 259

Sampling, in collection of data, 93
Sarton, George, on development of metric system, 246
Scale drawing, use of, in organization of data, 110
Scatter-diagrams, 134
School, application of democratic ideals in, 30-31; development of creativeness by, 39; development of self-direction by, 37; promotion of esthetic appreciation by, 34; promotion of social sensitivity in, 33; *see also* Mathematics teacher and Teacher
Science, applicability of functional relationships in, 139-140; classification of, on basis of generalization, 257-258; democracy essential to full development of, 68-67; often a basis for decision of individual problem, 80; periodic phenomena in, 158-159; relations in, 164; substitution of symbols for referents in, 231-232
Science in General Education, Commission on Secondary-School Curriculum, 385-386; on use of historical material in development of coöperativeness, 51
Secondary education, application of democratic ideals in, 30-31; effect of new psychological theories on, 8-9; effect of social and economic change on, 6-8; mathematics in relation to, 16 ff.; survey of, by Committee of Ten, 4
Secondary sources, in collection of data, 101-102
Self-direction, as personal characteristic essential to democratic living, 36-37; rôle of mathematics in developing, 50-52
Semiquartile range. *See* Quartile deviation
Sex, adolescents' conflict over, 276
Sexual adjustment, effect of childhood behavior patterns on, 290
Similarity, 165
Skills. *See* Techniques
Social-civic relationships, adolescent needs in, 22
Social adjustment, as adolescent goal, 278
Social development, effect of early childhood behavior patterns on, 291-292
Social sensitivity, as personal characteristic essential to democratic living, 32-33; rôle of mathematics in developing, 48-49
"Social Sensitivity," Hilda Taba, 32-33
Social studies, 49, 262
Solution, and formulation, 75-89; illustrative short activities in, 403-406; as data, 95-96; as revision of formulation, 85; criteria for acceptability of, 87-88; appropriate form, 87-88; precision, 88; wide applicability, 87; defined, 84-85; dependence of, on formulation, 75; focusing students' attention on, 84-88; impossibility of, 86; possible understanding of, without knowledge of operations, 182; relation of formulation to, 83-84; restatement of problem as, 86; types of, 85-86
Sophistication, affectation of, by adolescents, 281
Sources, primary and secondary, in collection of data, 101-102
Source unit, on normal variability, 385-402
Square root, finding, as approximation, 119-120
Standard deviation, as measure of dispersion, 129-130; formula and symbol for, 129
Statistical Tables, Their Structure and Use, H. M. Walker and W. N. Durost, quoted, 103
Statistics, 66; approximate relations among associated variables in, 130-137; developing students' understanding of, 121 ff.; importance of today, 121; measures of central tendency in, 122-127; measures of dispersion in, 127-130
Students, adapting topics to maturity of, 261-262; coöperation of, in evaluation program, 341-342; essential scrutiny of problem by, 84; inter-

Students (*Cont.*)
preting classroom behavior of, in view of personality and need, 329-333; life history of, 293-329; rôle of, in planning school program, 25; *see also* Adolescents
Study of the History of Mathematics, The, George Sarton, quoted, 246
Subject-matter, mathematics, changes, in, 5-6
Subtraction, as fundamental operation in natural number system, 171
Superstition, and symbolism, 236
Surveying instruments, historical aspects in study of, 259-260
Symbolism, 214 ff., 271-272; emotional aspects of, 235-236; relation of, to mathematics understanding, 69-71; *see also* Formula
Symbols, and reflective thinking, 234; as data, 95; broad aspects of, 234-236; dangers in traditional teaching of, 272; distinctions between mathematical and non-mathematical uses of, 225-232; referential and emotive language, 227-228; relative independence of context for mathematical usage, 229-231; unambiguity of mathematical usage, 228-229; ideographic, in mathematics, 219 ff.; illustrative short activities on, 412; mathematical, precision and abstraction in, 232; substitution of, for referents, 231-232; non-mathematical, dependence of, on culture, 233; of algebra, teaching development of, 260; operation of, in algebra, 175-176; postulational versus empirical, 231; social origins of, 232-234; types of, in order of ambiguity, 229; uses of, in mathematics, 217-225; as shorthand for other symbols, 218-222; as symbols for other symbols, 222-224; for describing physical things, 218; use of, to avoid approximation, 119; various meanings of, in non-mathematical usage, 228
Symmetry, as property of relations, 165

Taba, Hilda, on social sensitivity, 32-33
Tables, as type of solution, 84; construction and use of, 103-108; cumulative frequency, 108; deriving general formula from, 156; frequency, 106-108; of multiple-parameter data, 104-106; of one-parameter, data, 104; test of arrangement of data in, 357; use of, to express proportional variation, 149
Teacher, and crushes, 283; as parent substitute, 282-283; coöperation of, essential to meet student needs, 45; knowledge of personality essential to, 270; needed coöperation of mathematicians and, 16; observation of adolescents by, 273-274; participation of, in evaluation program, 341-342; recognition of abnormal behavior by, 332-333; responsibility of, for development of reflective thinking, 41; typical reactions of, to students' behavior, 330-331; *see also* Mathematics teacher
Techniques, of operations versus concepts, 180-181; teaching of, when required by real problem, 182; valid, for evaluation, 343-344; illustrative, 354 ff.; value of checking, 183-185; value of drill on, 183
Test data, interpretation of, based on one test, 375-378; based on several tests, 378-382
Tests, discussion of, 349-352; marking, 350; of ability to apply principles of variation, 366-368; of ability to recognize quantitative factors in problem situations, 354-355; of arrangement of data in tabular form, 357; of graphical representation of data, 359-360; of interpretation of data, 360-365; of nature of proof, 368-372; of proper use of approximate numbers, 357-358; of understanding of data, 355-357
Thayer, V. T., Zachry, Caroline, and Kotinsky, Ruth, on crucial need for intelligence today, 39-40
Theorem, as type of solution, 85
Time, arbitrary units of, 100
Tolerance, as personal characteristic essential to democratic living, 35-36; rôle of mathematics in developing, 50-52; versus *laissez-faire* attitude, 35
Transformation, of formula, 156
Transivity, as property of relations, 165
Trend, in associated variables, 134-136
Trigonometry, 5; examples of, in variance in, 147

Units, appropriateness of, 97-99; changes in, in recording of data, 99; choice of, in collection of data, 97-100; standard, arbitrary nature of, 99

Variable, domain of, as fundamental notion related to function, 143-145; functions of one, developing students' understanding of, 141-159; meaning of, 142-143
Variables, associated, 130-137; independent, developing students' understanding of, 159-161
Variability, normal, source unit on, 385-402
Variation, proportional, as type of quantitative function, 149-156; test on ability to apply principles of, 366-368

Walker, H. M., and Durost W. N., on tables, 103
Weights, teaching historical aspect of, 259
Whitehead, A. N., on advantage of Hindu-Arabic system, 221
Withdrawal, as characteristic of adolescent behavior, 281-282
Words, as symbols, 214

$y = ax + b$, as quantitative function, 156

Zachry, Caroline B. *See* Thayer, V. T.
Zero, as arbitrary unit, 100; symbolic use of, 221