MOLECULAR FLOW OF GASES

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To

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P R E F A C E

Recent developments in supersonic aerodynamics have emphasized the importance of a molecular approach to the subject of gas dynamics. When only the macroscopic part of the motion of a gas is largely significant, the contribution made by the unseen internal motions of the molecules can be adequately taken into account by a phenomenological (or continuum) analysis. However, when the characteristics of a flow depend more essentially on the random motion of the molecules, as in the slip flow of a highly rarefied gas, or on the internal structure of the molecules, as indicated by the relaxation effects associated with a strong shock wave, then a theory of gas flow must be developed in which more attention is given to the role played by the molecules.

In this book the characteristics of a gas flow are determined from an assumed molecular model and the distribution of the velocities of the molecules. The macroscopic properties of a frictionless, compressible (isentropic) flow are obtained from a simple spherical molecule and Maxwell's distribution law. A more complicated molecular model (point center of force) and a small order modification of Maxwell's distribution function are required in the corresponding calculation for a viscous, compressible (slightly nonisentropic) flow. The weak shock transition and the boundary layer are examples of this type of motion. The molecular concept permits the determination of both the equations of motion of a gas and the boundary conditions at the surface of a body. These results lead to the concepts of slip flow and temperature accommodation in a rarefied gas. The same basic ideas are used to develop the theory of free-molecule flow.

At present the molecular theory is limited by the lack of details regarding encounters between complex molecules. Since sufficient collision information for diatomic molecules is not available, the mathematical development in this book is complete only for a monatomic gas. However, the results apply equally well to a diatomic gas (air) if the appropriate changes are made in the values of the ratio of specific heats and the Prandtl number. Some discussion of strong shock waves is included in which effects arising from more complex

molecules are considered. In free-molecule flow no intermolecular collisions occur, and the diatomic gas can be included in the discussion. The molecular theory of turbulent flow, which involves encounters between clusters of molecules, does not appear to be sufficiently developed for inclusion in this book.

No attempt is made to give a complete treatment of either gas dynamics or the kinetic theory of gases. The essential purpose of this book is to assist the reader in making a transition in outlook from the continuum to the molecular viewpoint. Sufficient material from both the kinetic theory of gases and fluid mechanics is included for adequate development and illustration of the molecular approach.

It is hoped that this book will be of assistance to engineers and physicists. For this reason an effort has been made to maintain a level of mathematics which does not differ greatly from that normally required in the continuum theory of fluid mechanics. For example, the intuitive method of S. Chapman has been adopted for the derivation of the basic equations for viscous, compressible flow since it emphasizes the physical process more than the direct mathematical approach. The material in this book was collected over a period of seven years as a series of lectures to advanced students in engineering, physics, and applied mathematics at the University of Toronto.

Experience indicates that a molecular approach to gas dynamics is conducive to a high standard of research and design. In solving the many problems of high-speed flight in a rarefied atmosphere, the aerodynamicist has no other course but to use the molecular theory of gases.

I wish to express my thanks to Dr. I. I. Glass for many helpful discussions and suggestions during the preparation of the manuscript. I am also indebted to Mr. J. de Leeuw and Mr. K. Enkenhus for preparing the figures.

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C O N T E N T S

CHAPTER

APPENDIX

CHAPTER ONE

The Fundamental Equations

1.1 MOLECULES AND STATES OF MOLECULAR MOTION

The fundamental hypothesis of the molecular theory is that solids, liquids, and gases are composed of finite particles called molecules, which are in various states of motion. The term *molecule* is taken generally to mean a regular arrangement of atoms and electrons, or it may signify a free atom, an electron, or an ion.

No microscope has sufficient power to make the molecule visible. However, many experiments have been performed that give indirect evidence of the existence of molecules. X-ray techniques indicate that all the molecules in solids and many in liquids are atoms and electrons arranged in a regular pattern. Particles of various substances suspended on water or in air exhibit an agitated motion called Brownian movement, which may be ascribed to the invisible, random motion of the molecules. Observations of the oscillations of a sensitive torsion balance in a rarefied gas show that the motion is irregular because of impacts by individual molecules.

Each molecule is the center of a field of force. Since two or more solid bodies cannot occupy the same space, a force of repulsion between the molecules must exist at close range. On the other hand, resistance to tension indicates that a force of cohesion also acts when the distance of separation is larger. Between these two molecular forces lies a position of equilibrium about which the molecules of a solid oscillate without leaving their respective groups.

If the vibration of a molecule about the position of equilibrium becomes large (for example, when heat is applied), the molecule may escape from one group of molecules to another. Changes of shape are now possible, and the molecular state becomes that of a liquid. A comparison of X-ray measurements on liquids and solids shows less regularity in the molecular pattern of liquids.

As the oscillation increases, some molecules will be set free, and ultimately a gaseous state is attained, in which all molecules are

permanently free. The molecules of a gas are in continual motion, moving independently of each other except during occasional encounters. The mean dimension of the molecules is small compared with the average distance between them, and their velocities vary widely.

1.2 MATHEMATICAL MODELS OF GAS MOLECULES

A basic assumption of the kinetic theory of gases is that the physical properties of a gas depend essentially on the motion of the molecules and not on their internal structure. Thus no important effect due to internal structure is considered to occur during a molecular encounter. It is possible, therefore, to represent the molecules by simple mathematical models which greatly facilitate the task of determining the macroscopic characteristics of a gas from the motion of its molecules. The correspondence of the calculated properties of a gas with those observed experimentally is a test of the correctness of the assumed molecular model.

A mathematical treatment of the molecular motion will become very complex unless simple models are adopted. It has been found that, although the mathematical model may deviate appreciably from actual molecules, as indicated by physical tests, nevertheless it leads to results of practical value. Two models have been used extensively: (*a*) a spherical molecule having perfect elasticity, the force of collision being that due to impact; (*b*) a molecule represented as a point center of a spherically symmetrical force, which is usually one of repulsion. These models are further simplified by the assumption that no friction develops during encounter.

We also assume that the motion of the molecules is consistent with the laws of classical mechanics, at least to a first order of approximation. Thus no consideration will be given to the quantum mechanics of internal structure, and molecular encounters will be investigated on the basis of the Newtonian laws of conservation of momentum and energy.

1.3 DEFINITION OF DENSITY

Consider an element of volume ($d\tau$) in physical space, selected in such a way that it is very small (infinitesimal) compared with the range of variation of the physical parameters of the gas, yet large enough to contain a great number of molecules. We define the number of molecules in $d\tau$ at time t as the average of the molecules in $d\tau$ taken over an interval of time dt (containing t) which is very short compared with the time variation of the gas properties but much longer than the time required for a molecule to cross through $d\tau$ if unaffected by collisions.

If $d\tau$ is a cube with a side of 0·01 mm, then the number of molecules contained in the cube at normal temperature and pressure is about 2.7×10^{10}, and the average time required for a molecule to travel without collision over a distance of 0·01 mm is approximately 10^{-7} second.

The number of molecules in $d\tau$ is proportional to its volume and independent of the shape. If n is the *number density*, or number of molecules per unit volume, then the element $d\tau$ contains $n\,d\tau$ molecules and the density of the gas is

$$\rho = mn \tag{1}$$

where m is the mass of a molecule.

So long as $d\tau$ and dt satisfy the conditions given above, they are described as being macroscopically small, and the normal processes of the mathematics of continua may be applied in which $d\tau$ and dt are regarded as infinitesimals and n (or ρ) is a continuous function of position and time.

1.4 THE VELOCITY DISTRIBUTION FUNCTION

In order to determine the properties of a gas from the motion of its molecules, it is necessary to know in a statistical sense not only the number of molecules in the element of volume $d\tau$ but, more specifically, the number of these which have velocities within a given range. At a particular instant of time, each of the $n\,d\tau$ molecules has a definite velocity which may be plotted in a velocity space having for coordinates the velocity components u, v, and w. The resulting points, which represent the motion of the gas in $d\tau$ at time t, are widely scattered throughout the (u, v, w) space since the molecular velocities differ appreciably. If $n\,d\tau$ is large, an element $d\omega$ of the velocity space contains sufficient molecules for the determination of a number density corresponding to n in the physical space.

The number of points in unit volume of the velocity space is proportional to $n\,d\tau$ since this quantity is the number of points plotted throughout the whole (u, v, w) space. Furthermore, the number density of points in the velocity space depends in general on the position of $d\tau$ in the physical space (x, y, z), the position of $d\omega$ in the velocity space (u, v, w), and the time (t). It may be written in the form $nf\,d\tau$ where $f(x, y, z, u, v, w, t)$ is called the *velocity distribution function*. Then the number of molecules in the element of physical space $d\tau$ having velocity components corresponding to a point in the element $d\omega$ of the velocity space is $nf\,d\tau\,d\omega$. This may be stated otherwise as follows: at time t the number of molecules in the element of volume $dx\,dy\,dz$ having

velocity components between u and $u + du$, v and $v + dv$, w and $w + dw$ is

$$nf\,dx\,dy\,dz\,du\,dv\,dw \tag{1}$$

where $d\tau = dx\,dy\,dz$ and $d\omega = du\,dv\,dw$.

The velocity distribution function is governed by certain requirements. The total number of molecules in the element of physical volume $d\tau$ is obtained by integrating the above number over all possible values of u, v, w. Thus

$$n\,d\tau = \int (nf\,d\tau)\,d\omega \tag{2}$$

where the symbol

$$\int \cdots d\omega$$

is an abbreviated form of the triple integration

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cdots du\,dv\,dw$$

Since $n\,d\tau$ depends only on position and time, it will be seen from Eq. 2 that

$$\int f\,d\omega = 1 \tag{3}$$

A further condition is

$$\iint_{\text{vol}} nf\,d\tau\,d\omega = N \tag{4}$$

where N is the total number of molecules in a gas, and the integration extends throughout the volume occupied by the gas and over all possible values for u, v, w.

Just as n is described as a number density of mass, we may interpret $nf\,d\tau$ as a number density of motion. Since we shall attempt to calculate all the macroscopic properties of a gas from the motion of its molecules, it is clear that the function f is of fundamental importance. The role played by the law of distribution of molecular velocities becomes apparent when we consider the formation of the mean values which define the state of a gas.

Let us consider the mean value of a quantity Q which is a function of the velocity components only. The number of molecules in the element of volume $d\tau$ having velocity vectors which terminate in the element $d\omega$ of the velocity space is $nf\,d\tau\,d\omega$. The amount of Q carried by these molecules is $Qnf\,d\tau\,d\omega$. The total amount of Q transported by all the molecules in $d\tau$ is

$$\Sigma Q = \int (Qnf\,d\tau)\,d\omega = (n\,d\tau)\int Qf\,d\omega \tag{5}$$

where the integration extends over all possible values of u, v, w. The number of molecules carrying this sum total of Q is

$$\int (nf\, d\tau)\, d\omega = n\, d\tau \int f\, d\omega = n\, d\tau$$

Then the mean value of Q is defined as

$$\bar{Q} = \frac{\int (Qnf\, d\tau)\, d\omega}{\int (nf\, d\tau)\, d\omega} = \int Qf\, d\omega \tag{6}$$

where it is to be noted that n is a function of x, y, z, t only.

The mean value of the x-component of velocity in the element $d\tau$ is, therefore,

$$\bar{u}(x, y, z, t) = \int uf\, d\omega \tag{7}$$

The velocity components of a molecule can be referred to their average values by writing

$$u = \bar{u} + U, \quad v = \bar{v} + V, \quad w = \bar{w} + W \tag{8}$$

where

$$\bar{U} = \bar{V} = \bar{W} = 0 \tag{9}$$

The translational energy of a molecule may be written

$$E = \tfrac{1}{2}m[(\bar{u} + U)^2 + (\bar{v} + V)^2 + (\bar{w} + W)^2] \tag{10}$$

Then the average value of the translational energy possessed by the molecules in the element $d\tau$ is

$$\bar{E} = \int Ef\, d\omega = \tfrac{1}{2}m\bar{q}^2 + \tfrac{1}{2}m\overline{C^2} \tag{11}$$

where

$$\bar{q}^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2, \quad \overline{C^2} = \overline{U^2} + \overline{V^2} + \overline{W^2} \tag{12}$$

Therefore, the mean translational energy in the element of volume $d\tau$ can be regarded as made up of two components: (*a*) the kinetic energy of the visible mass motion ($\frac{1}{2}m\bar{q}^2$); (*b*) the kinetic energy of the invisible molecular agitation. It will be seen later (Section 2.5) that the ratio of these components ($\bar{q}^2/\overline{C^2}$) has an important bearing on the flow properties of a gas.

The kinetic energy of thermal agitation ($\frac{1}{2}m\overline{C^2}$) may not be the only hidden energy in a real gas. Thus polyatomic molecules also possess internal motions which contribute to the average internal energy such as rotation and vibration.

1.5 MOLECULAR ENCOUNTERS

In general, the encounters between molecules have a considerable influence on the distribution of molecular velocities. Before an equation can be developed from which to determine the function f, it is necessary to give some attention to the theory of molecular encounters.

Let us represent the molecules of a gas as smooth, spherical bodies of unlimited rigidity and elasticity, all having the same diameter σ. For such molecules neither an interchange between the internal and translational energies nor a loss of energy due to deformation can occur during encounter. All molecular interactions are assumed to be governed by the Newtonian laws of conservation of momentum and energy, and effects due to molecular structure are neglected.

The analysis will also be limited to binary encounters; other forms of collision are considered to be too infrequent to affect the state of the gas. Between collisions, the molecules move freely at constant though widely different speeds, since no external field of force exists to produce acceleration.

The assumption of spherical symmetry implied in the choice of the molecular model means that the analysis will apply strictly to a monatomic gas. Diatomic and polyatomic molecules cannot be regarded as spherically symmetrical, and a much more complex model must be considered (see Section 4.9). It seems likely, however, that in the process of averaging over all orientations to obtain a mean value the resultant effect of deviations from spherical symmetry would be substantially reduced, and the results obtained from the spherical model would apply qualitatively at least.

The average distance traveled by the molecules of a gas between collisions is assumed to be very large compared with their diameter. This condition enables us to adopt Boltzmann's assumption of molecular chaos. Thus, before encounter, two molecules will have traveled relatively long distances from widely separated starting points, and therefore no relation can exist between the initial motions of the molecules. In collisions between two classes of molecules (two groups whose velocity vectors terminate in either the element $d\omega_1$ or the element $d\omega_2$ of the velocity space), the assumption of molecular chaos implies that both sets of molecules are distributed at random throughout the element of volume $d\tau$ and no correlation exists between molecular velocity and position.

In order to emphasize the physical characteristics of a molecular encounter, we first assume that the line of impact is orientated along the Ox axis of a rectangular system of coordinates. This restriction will be removed later to obtain more general mathematical results. Let the

initial velocity vectors of two colliding molecules (Fig. 1) be

$$\mathbf{c}_1 = \mathbf{i}u_1 + \mathbf{j}v_1 + \mathbf{k}w_1 \tag{1}$$

and

$$\mathbf{c}_2 = \mathbf{i}u_2 + \mathbf{j}v_2 + \mathbf{k}w_2 \tag{2}$$

where (u_1, v_1, w_1) and (u_2, v_2, w_2) are the components of velocity in a rectangular system of coordinates and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors with directions parallel to the Ox, Oy, and Oz axes, respectively. If the line of impact is directed along Ox, then velocity components normal to Ox are unaffected by the collision

$$v_1 = v_1', \quad w_1 = w_1', \quad v_2 = v_2', \quad w_2 = w_2' \tag{3}$$

where the primed quantities are the velocity components after collision.

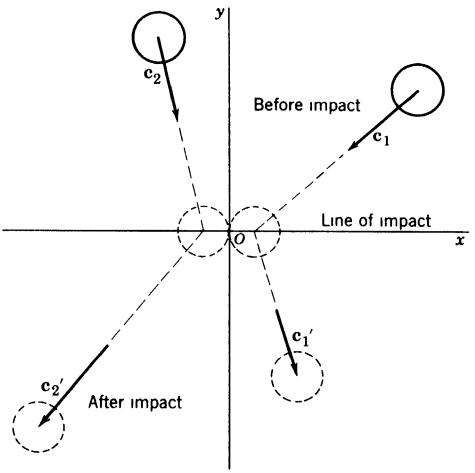

Fig. 1. A molecular encounter.

Then the velocity vectors after encounter may be written

$$\mathbf{c}_1' = \mathbf{i}u_1' + \mathbf{j}v_1 + \mathbf{k}w_1 \tag{4}$$

$$\mathbf{c}_2' = \mathbf{i}u_2' + \mathbf{j}v_2 + \mathbf{k}w_2 \tag{5}$$

Since the mass of each molecule is the same, the relations for the conservation of momentum and energy are, respectively,

$$\mathbf{c}_1 + \mathbf{c}_2 = \mathbf{c}_1' + \mathbf{c}_2' \tag{6}$$

and

$$\mathbf{c}_1^2 + \mathbf{c}_2^2 = \mathbf{c}_1'^2 + \mathbf{c}_2'^2 \tag{7}$$

If we square both sides of Eq. 6 and subtract Eq. 7,

$$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1' \cdot \mathbf{c}_2' \tag{8}$$

Substitution for $\mathbf{c}_2'$ from Eq. 8 in Eq. 6 yields

$$u_1'^2 - (u_1 + u_2)u_1' + u_1u_2 = 0 \tag{9}$$

which has the roots $u_1' = u_1$ or u_2. Similarly, $u_2' = u_2$ or u_1. The solution of interest is, therefore,

$$\begin{aligned} u_1' &= u_2, \qquad v_1' = v_1, \qquad w_1' = w_1 \\ u_2' &= u_1, \qquad v_2' = v_2, \qquad w_2' = w_2 \end{aligned} \tag{10}$$

It will be seen that the collision causes the molecules to exchange their x-components of velocity (along the line of impact), and the relative velocities before and after collision ($\boldsymbol{\Omega} = \mathbf{c}_2 - \mathbf{c}_1$ and $\boldsymbol{\Omega}' = \mathbf{c}_2' - \mathbf{c}_1'$, respectively) have the same magnitude but different directions.

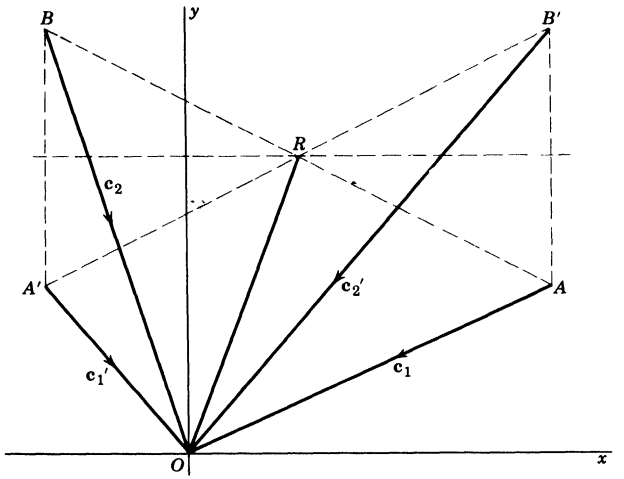

Fig. 2. Velocity vector diagram — (x, y) plane only.

To understand further the physical characteristics of a molecular encounter, let us investigate the velocity vector diagram. We first simplify the diagram by assuming that $w_1 = w_2 = w_1' = w_2' = 0$. Then, according to Eqs. 10, the vector diagram is that shown in Fig. 2. It will be seen that the final velocity vectors can be readily obtained geometrically from the initial velocities, since A' and B' are the reflections of B and A about a line drawn through the midpoint R of AB, parallel to the line of impact.

The general three-dimensional diagram (Fig. 3) in which w_1, w_2, w_1', w_2' are not zero can now be obtained from Fig. 2. Since $w_1 = w_1'$, $w_2 = w_2'$, then A and A' are displaced the same distance w_1 above the plane of the

paper (Fig. 2), and similarly B and B' are moved a distance w_2 above the (x, y) plane. The rectangle $AB'BA'$ still lies in a plane parallel to the line of impact but is now inclined to the (x, y) plane. Thus in general A' and B' are obtained from B and A by reflection about a plane perpendicular to the plane of $AB'BA'$ passing through the point R and parallel to the line of impact.

It is to be noted that all the velocity vectors are drawn through the origin O, and since the plane of the rectangle $AB'BA'$ does not necessarily

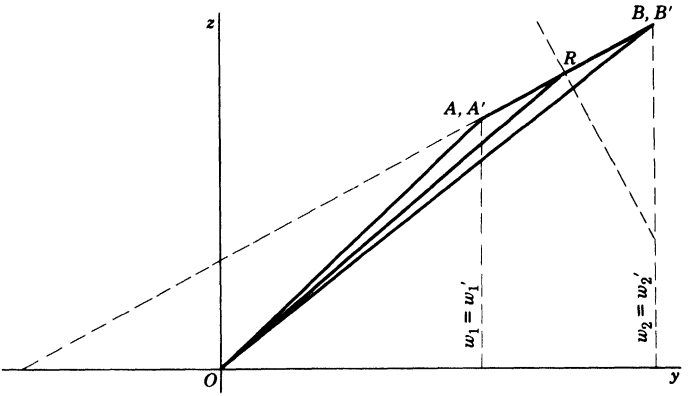

Fig. 3. Velocity vector diagram — effect of Oz components.

contain O, the encounter does not occur in general in one plane. The vector pairs $\mathbf{c}_1, \mathbf{c}_1'$ and $\mathbf{c}_2, \mathbf{c}_2'$ define two different planes each of which contains the line of impact (Fig. 3).

The vector OR has particular significance since it represents the velocity of the center of mass of the two molecules before and after the collision. The vector diagrams (Figs. 2 and 3) show that it is unaltered by the encounter. This can be verified analytically with the assistance of the law of conservation of momentum. The velocity of the center of mass before encounter is

$$2\mathbf{c}_0 = \mathbf{i}(u_1 + u_2) + \mathbf{j}(v_1 + v_2) + \mathbf{k}(w_1 + w_2) = \mathbf{c}_1 + \mathbf{c}_2 \qquad (11)$$

where $u_0 = \frac{1}{2}(u_1 + u_2), \cdots$. After collision, this vector becomes

$$2\mathbf{c}_0' = \mathbf{i}(u_1' + u_2') + \mathbf{j}(v_1' + v_2') + \mathbf{k}(w_1' + w_2') = \mathbf{c}_1' + \mathbf{c}_2' \qquad (12)$$

Then, according to Eqs. 10, $\mathbf{c}_0 = \mathbf{c}_0'$.

The energy equation can be expressed in terms of the relative velocity and the velocity of the center of mass as follows:

$$\mathbf{c}_1^2 + \mathbf{c}_2^2 = 2\mathbf{c}_0^2 + \tfrac{1}{2}\boldsymbol{\Omega}^2 = \mathbf{c}_1'^2 + \mathbf{c}_2'^2 = 2\mathbf{c}_0'^2 + \tfrac{1}{2}\boldsymbol{\Omega}'^2 \qquad (13)$$

Since $\Omega = \Omega'$, we conclude that the interaction between two molecules can be reduced to the motion of their center of mass and their relative motion. This result is based only on the laws of conservation of momentum and energy and is true for any collision irrespective of the direction of the line of impact.

Let us now remove the restriction that the line of impact is directed along Ox. We require the general equations by which the velocities after collision can be obtained from the initial velocities. If the line of

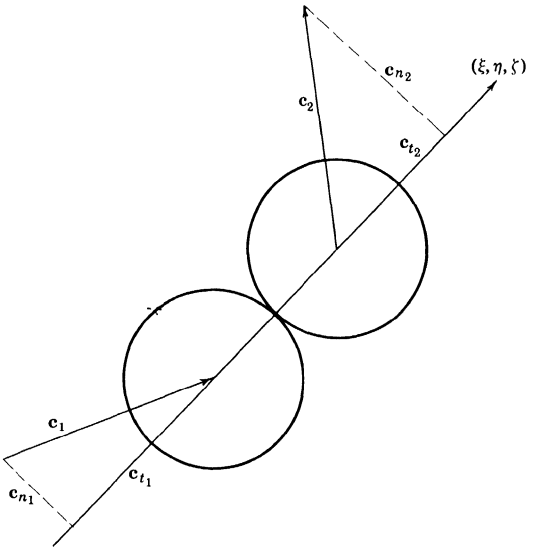

Fig. 4. Collision between two spherical molecules.

impact has a general orientation in space given by the direction cosines ξ, η, ζ (Fig. 4), then each velocity may be resolved into components perpendicular and parallel to the line of impact as follows:

$$\begin{aligned} \mathbf{c}_1 &= \mathbf{c}_{n_1} + \mathbf{c}_{t_1}, & \mathbf{c}_2 &= \mathbf{c}_{n_2} + \mathbf{c}_{t_2} \\ \mathbf{c}_1' &= \mathbf{c}_{n_1}' + \mathbf{c}_{t_1}', & \mathbf{c}_2' &= \mathbf{c}_{n_2}' + \mathbf{c}_{t_2}' \end{aligned} \tag{14}$$

Since the velocity components normal to the line of impact are unaltered by the collision, and those along the line of centers are interchanged, then

$$\begin{aligned} \mathbf{c}_{n_1}' &= \mathbf{c}_{n_1}, & \mathbf{c}_{n_2}' &= \mathbf{c}_{n_2} \\ \mathbf{c}_{t_1}' &= \mathbf{c}_{t_2}, & \mathbf{c}_{t_2}' &= \mathbf{c}_{t_1} \end{aligned} \tag{15}$$

and we may write

$$\mathbf{c}_1' = \mathbf{c}_{n_1} + \mathbf{c}_{t_2}, \qquad \mathbf{c}_2' = \mathbf{c}_{n_2} + \mathbf{c}_{t_1} \tag{16}$$

The first of these equations becomes

$$\mathbf{c}_1' = \mathbf{c}_1 + (\mathbf{c}_{t_2} - \mathbf{c}_{t_1}) = \mathbf{c}_1 + (\mathbf{c}_2 - \mathbf{c}_1)_t = \mathbf{c}_1 + \mathbf{\Omega}_t \tag{17}$$

where the relative velocity of the two molecules before collision is

$$\mathbf{\Omega} = \mathbf{c}_2 - \mathbf{c}_1 \tag{18}$$

and $\mathbf{\Omega}_t$ is the component of $\mathbf{\Omega}$ parallel to the line of impact. Similarly

$$\mathbf{c}_2' = \mathbf{c}_2 - \mathbf{\Omega}_t \tag{19}$$

Equations 17 and 19 have the component forms

$$\begin{aligned} u_1' &= u_1 + \xi\Omega_t, \qquad v_1' = v_1 + \eta\Omega_t, \qquad w_1' = w_1 + \zeta\Omega_t \\ u_2' &= u_2 - \xi\Omega_t, \qquad v_2' = v_2 - \eta\Omega_t, \qquad w_2' = w_2 - \zeta\Omega_t \end{aligned} \tag{20}$$

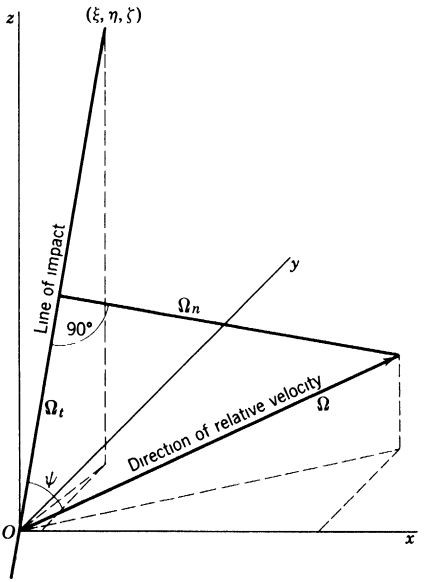

Fig. 5. Relative velocity diagram for two colliding molecules.

If ψ is the angle between $\mathbf{\Omega}$ and the line of impact (see Fig. 5), then

$$\Omega_t = \Omega \cos\psi \tag{21}$$

The direction cosines for the initial relative velocity vector are

$$\frac{u_2 - u_1}{\Omega}, \qquad \frac{v_2 - v_1}{\Omega}, \qquad \frac{w_2 - w_1}{\Omega}$$

Then
$$\cos\psi = \xi\left(\frac{u_2 - u_1}{\Omega}\right) + \eta\left(\frac{v_2 - v_1}{\Omega}\right) + \zeta\left(\frac{w_2 - w_1}{\Omega}\right) \tag{22}$$

and
$$\Omega_t = \xi(u_2 - u_1) + \eta(v_2 - v_1) + \zeta(w_2 - w_1) \tag{23}$$

Therefore, the final components of velocity are related to the initial components by the relations

$$\begin{aligned} u_1' &= u_1 + \xi^2(u_2 - u_1) + \xi\eta(v_2 - v_1) + \xi\zeta(w_2 - w_1) \\ v_1' &= v_1 + \xi\eta(u_2 - u_1) + \eta^2(v_2 - v_1) + \eta\zeta(w_2 - w_1) \\ w_1' &= w_1 + \xi\zeta(u_2 - u_1) + \eta\zeta(v_2 - v_1) + \zeta^2(w_2 - w_1) \end{aligned} \tag{24}$$

and

$$\begin{aligned} u_2' &= u_2 + \xi^2(u_1 - u_2) + \xi\eta(v_1 - v_2) + \xi\zeta(w_1 - w_2) \\ v_2' &= v_2 + \xi\eta(u_1 - u_2) + \eta^2(v_1 - v_2) + \eta\zeta(w_1 - w_2) \\ w_2' &= w_2 + \xi\zeta(u_1 - u_2) + \eta\zeta(v_1 - v_2) + \zeta^2(w_1 - w_2) \end{aligned} \tag{25}$$

In order to calculate the final velocities, we must know at least two of the quantities ξ, η, ζ ($\xi^2 + \eta^2 + \zeta^2 = 1$). Note that substitution from Eqs. 24 and 25 into the relations

$$\Omega = [(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2]^{1/2} \tag{26}$$

$$\Omega' = [(u_2' - u_1')^2 + (v_2' - v_1')^2 + (w_2' - w_1')^2]^{1/2} \tag{27}$$

verifies that $\Omega = \Omega'$. Since

$$\cos\psi' = \xi\left(\frac{u_2' - u_1'}{\Omega'}\right) + \eta\left(\frac{v_2' - v_1'}{\Omega'}\right) + \zeta\left(\frac{w_2' - w_1'}{\Omega'}\right) \tag{28}$$

substitution for the primed quantities on the right-hand side shows that the relative velocity vectors before and after collision have the same inclination to the line of impact (but different directions).

1.6 COLLISION FREQUENCY

The effect of encounters between class 1 and class 2 molecules (see Notation) on the velocity distribution function will depend on the frequency of the collisions. The number of such interactions during the interval dt in the element of volume $d\tau$ having a specific direction for the line of impact can be determined from the relative motion of the molecules.

Consider a collision between a class 1 molecule and a class 2 molecule, the center of the class 1 molecule being held fixed at the origin of coordinates (Fig. 6). Let the line of impact, with direction cosines ξ, η, ζ, lie within an elementary solid angle $d\chi$. Then the conditions for a collision are: (*a*) on contact the center of one molecule must be at a distance σ from the center of the other, where σ is the diameter of

each molecule; (*b*) a collision will occur in an interval of time dt if the center of the class 2 molecule is at or within a distance $\Omega\, dt$ of the surface of a sphere of radius σ, having the same center as the fixed molecule (see Fig. 6).

If an encounter occurs, then, at the moment of contact, the center of the class 2 molecule must lie on an elementary area $\sigma^2\, d\chi$ of the large sphere of radius σ (Fig. 6). At time dt before contact this molecule would be found on an area $\sigma^2\, d\chi$ moved a distance $\Omega\, dt$ in the opposite direction to that of $\boldsymbol{\Omega}$. Therefore, a collision will occur during the

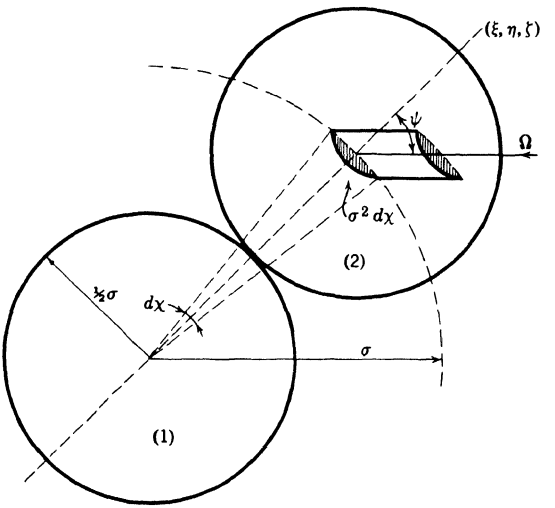

Fig. 6. Relative conditions for collision.

interval of time dt, if the center of the class 2 molecule is located on or inside a cylinder of base $\sigma^2\, d\chi$ and slant length $\Omega\, dt$. The volume of this elementary cylinder is $\sigma^2\Omega \cos\psi\, d\chi\, dt$.

We have seen that the number of class 1 molecules in the element of volume $d\tau$ of physical space is $nf_1\, d\omega_1\, d\tau$ where f_1 is the value of the distribution function in the element $d\omega_1$ of the velocity space (Section 1.4). An elementary cylinder of the type described above can be associated with each class 1 molecule in the element $d\tau$. Then, according to the assumption of molecular chaos, and since $d\chi$ and dt are infinitesimals, these cylinders will not overlap to any significant extent and they will therefore occupy a total volume

$$nf_1\sigma^2\Omega \cos\psi\, d\chi\, d\omega_1\, d\tau\, dt$$

The total number of class 2 molecules colliding with these class 1 molecules is, therefore, the number of class 2 molecules in the above volume. On the assumption of molecular chaos, this number may be written

$$n^2 f_1 f_2 \sigma^2 \Omega \cos \psi \, d\chi \, d\omega_1 \, d\omega_2 \, d\tau \, dt \tag{1}$$

This quantity is, therefore, the number of collisions between class 1 and class 2 molecules occurring in the element of volume $d\tau$ in the interval of time dt with lines of impact orientated within the elementary solid angle $d\chi$.

1.7 THE BASIC EQUATION FOR THE VELOCITY DISTRIBUTION FUNCTION

Having obtained the required collision information, we are now ready to consider Boltzmann's equation for the velocity distribution function. This relation can be obtained from a consideration of the variation in

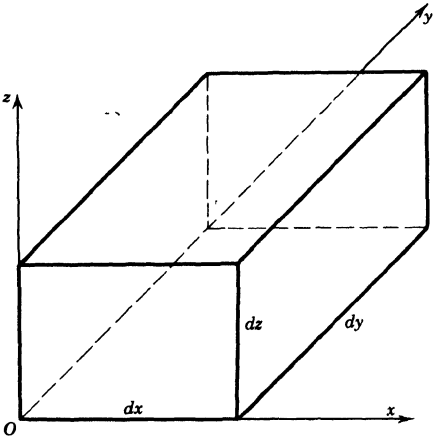

Fig. 7. Element of volume.

the number of molecules of a specific class in an element of volume of physical space over an infinitesimal interval of time. In a system of rectangular coordinates in physical space the element of volume $(d\tau)$ is $dx\,dy\,dz$. This macroscopically small volume is assumed to be such that, within its boundaries, the variation of f is infinitesimal, and the assumption of molecular chaos is valid.

The change in the number of class 1 molecules in the element of volume $d\tau$ during an interval of time dt arises from two sources: (*a*) the flux of class 1 molecules through the sides of the element $d\tau$ (Fig. 7); (*b*) the loss and gain of class 1 molecules due to encounters.

The number of class 1 molecules that enter $d\tau$ through the face perpendicular to x at the position x in time dt is

$$nf_1\, d\omega_1 \cdot u_1\, dt \cdot dy\, dz$$

and the corresponding number of molecules that leave $d\tau$ through the face perpendicular to x at the position $x + dx$ in time dt is

$$\left[nf_1 + \frac{\partial}{\partial x}(nf_1)\, dx\right] d\omega_1 \cdot u_1\, dt \cdot dy\, dz$$

where the bracketed quantity consists of the first and second terms of the Taylor expansion of the function $nf(u_1, v_1, w_1, x + dx, y, z, t)$ with respect to x, all terms involving differentials with powers greater than the first being neglected. Therefore the increment in the number of class 1 molecules due to the flux through these two faces in the interval dt is

$$-u_1 \frac{\partial}{\partial x}(nf_1)\, d\omega_1\, d\tau\, dt$$

and, therefore, the total change in the number of class 1 molecules due to flux through the boundaries of $d\tau$ is

$$-\left[u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + w_1 \frac{\partial}{\partial z}\right](nf_1)\, d\omega_1\, d\tau\, dt \tag{1}$$

We now determine the effect of collisions on the number of class 1 molecules. All class 1 molecules that enter into collision have their velocities changed and emerge in another class. On the other hand, some molecules in other classes have their velocities changed to the class 1 range by collision. We thus have a type a collision for which

$$\begin{aligned}(u_1, v_1, w_1) &\rightarrow (u_1', v_1', w_1')\\ (u_2, v_2, w_2) &\rightarrow (u_2', v_2', w_2')\end{aligned} \tag{2}$$

and a type b collision such that

$$\begin{aligned}(u_1', v_1', w_1') &\rightarrow (u_1, v_1, w_1)\\ (u_2', v_2', w_2') &\rightarrow (u_2, v_2, w_2)\end{aligned} \tag{3}$$

According to Section 1.6, the number of type a collisions in the element of volume $d\tau$ in an interval dt, having lines of impact orientated within an elementary solid angle $d\chi$, is

$$n^2 f_1 f_2 \sigma^2 \Omega \cos\psi\, d\chi\, d\omega_1\, d\omega_2\, d\tau\, dt \tag{4}$$

Similarly the corresponding number of type b collisions is

$$n^2 f_1' f_2' \sigma^2 \Omega \cos\psi\, d\chi\, d\omega_1'\, d\omega_2'\, d\tau\, dt \tag{5}$$

where $f_1' = f(u_1', v_1', w_1', x, y, z, t)$, $f_2' = f(u_2', v_2', w_2', x, y, z, t)$.

Note that the magnitude of the initial relative velocity (Ω) and its inclination to the line of impact (ψ) are common to both types of encounter, and n, σ are not affected by the interaction. Furthermore, the "spread" of velocity before collision ($d\omega_1\, d\omega_2$) can be shown to be the same as the spread after encounter ($d\omega_1'\, d\omega_2'$). Equations 1.5, 24, 25 (see Notation) may be regarded as the equations of transformation between two six-component velocity spaces in which the coordinates are u_1, v_1, w_1, u_2, v_2, w_2 and u_1', v_1', w_1', u_2', v_2', w_2', respectively. The elementary volumes of these two spaces are related by the expression

$$d\omega_1\, d\omega_2 = |J|\, d\omega_1'\, d\omega_2' \tag{6}$$

where J, the Jacobian, has the value

$$J = \frac{\partial(u_1', v_1', w_1', u_2', v_2', w_2')}{\partial(u_1, v_1, w_1, u_2, v_2, w_2)} \tag{7}$$

The equations for u_1, v_1, w_1, u_2, v_2, and w_2 in terms of the corresponding primed velocity components are the same as Eqs. 1.5, 24, 25, except that unprimed letters are primed and vice versa. Then, by carrying out the appropriate differentiation, it can be shown that $J' = J$, where

$$J' = \frac{\partial(u_1, v_1, w_1, u_2, v_2, w_2)}{\partial(u_1', v_1', w_1', u_2', v_2', w_2')} \tag{8}$$

But, according to the general theory of Jacobians, $JJ' = 1$. Therefore $J = J' = 1$. This result can be verified by the direct determination of J from Eqs. 1.5, 24, 25. Then

$$d\omega_1\, d\omega_2 = d\omega_1'\, d\omega_2' \tag{9}$$

which is a special case of Liouville's theorem (Ref. 1). We conclude that the increase in the number of class 1 molecules due to collisions of types a and b with molecules of class 2 in the element of volume $d\tau$ in the interval dt having lines of impact orientated within the elementary solid angle $d\chi$ can be expressed in the form

$$n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi\, d\chi\, d\omega_1\, d\omega_2\, d\tau\, dt \tag{10}$$

The total increase in the number of class 1 molecules due to collisions with the molecules of all classes in $d\tau$ in time dt for all possible directions of the line of impact is

$$\left[\iint n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi\, d\chi\, d\omega_2\right] d\omega_1\, d\tau\, dt \tag{11}$$

At time t, the number of class 1 molecules in the element of volume $d\tau$ is

$$nf(u_1, v_1, w_1, x, y, z, t)\, d\omega_1\, d\tau$$

and, at time $t + dt$, the corresponding number is

$$nf(u_1, v_1, w_1, x, y, z, t + dt)\, d\omega_1\, d\tau$$

or

$$\left[nf_1 + \frac{\partial}{\partial t}(nf_1)\, dt\right] d\omega_1\, d\tau$$

if we use the Taylor expansion and retain only the first-order differential. Then the change in the number of class 1 molecules in $d\tau$ during the interval dt is

$$\frac{\partial}{\partial t}(nf_1)\, d\omega_1\, d\tau\, dt \tag{12}$$

Equating this to the sum of expressions 1 and 11, we obtain the Boltzmann equation for the velocity distribution function

$$\frac{\partial}{\partial t}(nf_1) = -\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + w_1\frac{\partial}{\partial z}\right)(nf_1) + \iint n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega\cos\psi\, d\chi\, d\omega_2 \tag{13}$$

Since $nf_1 = nf(u_1, v_1, w_1, x, y, z, t)$, it has the differential

$$d(nf_1) = \frac{\partial}{\partial x}(nf_1)\, dx + \frac{\partial}{\partial y}(nf_1)\, dy + \frac{\partial}{\partial z}(nf_1)\, dz + \frac{\partial}{\partial t}(nf_1)\, dt \tag{14}$$

and the time derivative is

$$D_1(nf_1) = \left[\frac{d}{dt}(nf_1)\right]_{\text{class 1}} = \left(\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + w_1\frac{\partial}{\partial z}\right)(nf_1) \tag{15}$$

We shall regard D_1 (——) as a mobile operator signifying a time derivative "following the motion of the class 1 molecules." The Boltzmann equation may be written finally in the form

$$D_1(nf_1) = \iint n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega\cos\psi\, d\chi\, d\omega_2 \tag{16}$$

This is an integro-differential equation which indicates that the change in the number of class 1 molecules following their motion arises only from molecular encounters.

Equation 16 holds for a simple homogeneous gas. The above treatment can be generalized for a mixture of gases (Ref. 2).

1.8 THE GENERAL TRANSFER EQUATION

When the distribution function has been determined by means of the Boltzmann relation (Eq. 1.7, 16), the macroscopic properties of a flowing gas can then be found from Maxwell's transfer equation. Let $Q(u, v, w)$

be a quantity associated with each molecule, such as the momentum or energy (mu, $\frac{1}{2}mc^2$). Since each class 1 molecule possesses a quantity Q_1, the total amount of Q_1 carried by these molecules is $Q_1 \cdot nf_1\, d\omega_1\, d\tau$. The sum of these quantities at time t for molecules of all classes in the element of volume $d\tau$ is the aggregate amount of Q, that is,

$$\Sigma Q = \int nQ_1 f_1\, d\omega_1\, d\tau = n\bar{Q}\, d\tau \tag{1}$$

For example, if Q is the x-component of momentum (mu), then $\bar{Q} = m\bar{u}$ and $\Sigma Q = \rho\bar{u}d\tau$ or ΣQ is the total amount of the x-component of momentum in the element $d\tau$. The basic function of the general transfer equation is to describe the variation of any quantity ΣQ as the molecules transport Q in and out of $d\tau$ or exchange it during collisions.

Let us first investigate the variation of ΣQ due to the flux of molecules through the boundaries of $d\tau$. It has been shown in Section 1.7 that the total increase in the number of class 1 molecules during the interval dt arising from the gain and loss of molecules through the walls of $d\tau$ is

$$-\left[u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + w_1 \frac{\partial}{\partial z}\right](nf_1)\, d\omega_1\, d\tau\, dt$$

Then the total increment in the amount of ΣQ from this source is

$$-\left[\int Q_1\left(u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + w_1 \frac{\partial}{\partial z}\right)(nf_1)\, d\omega_1\right] d\tau\, dt \tag{2}$$

where $Q_1 = Q(u_1, v_1, w_1)$. The first term of the integral becomes

$$\int u_1 Q_1 \frac{\partial}{\partial x}(nf_1)\, d\omega_1 = \int \frac{\partial}{\partial x}(nu_1 Q_1 f_1)\, d\omega_1 = \frac{\partial}{\partial x}(n\overline{uQ})$$

Therefore, the increase in the amount of ΣQ due to the flux of molecules through the bounding surfaces of the element of volume $d\tau$ in time dt is

$$-\left[\frac{\partial}{\partial x}(n\overline{uQ}) + \frac{\partial}{\partial y}(n\overline{vQ}) + \frac{\partial}{\partial z}(n\overline{wQ})\right] d\tau\, dt \tag{3}$$

Molecular encounters also produce a change in ΣQ. In a type a collision the change in Q associated with the class 1 molecules is $Q(u_1', v_1', w_1') - Q(u_1, v_1, w_1)$, or $Q_1' - Q_1$. The number of type a collisions in the elementary volume $d\tau$ during the interval dt, having a specific direction for the lines of impact, is given by the expression 1.6, 1. Then the class 1 molecules that engage in these interactions account for a change in Q of

$$(Q_1' - Q_1)n^2 f_1 f_2 \sigma^2 \Omega \cos\psi\, d\chi\, d\omega_1\, d\omega_2\, d\tau\, dt \tag{4}$$

The total change in Q produced by all collisions in $d\tau$ with class 1 molecules during the interval dt is obtained by integrating the above expression over all directions of the line of impact and over all possible values of u_2, v_2, w_2. Finally the aggregate change in $\sum Q$ is determined by summing over all possible values of u_1, v_1, w_1. If ΔQ is the change in $\sum Q$ per unit volume per unit time, then

$$\Delta Q = \iiint n^2(Q_1' - Q_1) f_1 f_2 \sigma^2 \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{5}$$

The variation of $\sum Q$ in time dt can also be expressed in terms of the time derivative,

$$\frac{\partial}{\partial t}(n\bar{Q}) \, d\tau \, dt \tag{6}$$

following the method outlined in Section 1.7. Then Maxwell's general transfer equation is obtained by equating (6) to (3) and (5), and the resulting relation is

$$\frac{\partial}{\partial t}(n\bar{Q}) + \frac{\partial}{\partial x}(n\overline{uQ}) + \frac{\partial}{\partial y}(n\overline{vQ}) + \frac{\partial}{\partial z}(n\overline{wQ}) = \iiint n^2(Q_1' - Q_1) f_1 f_2 \sigma^2 \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{7}$$

1.9 MOLECULAR TRANSPORT OF MASS, MOMENTUM, AND ENERGY

Let us now investigate the form of the transfer equation when Q is placed equal to the mass of the molecule (m), each of the three components of momentum (mu, mv, mw), and the kinetic energy ($\frac{1}{2}mc^2$), respectively. Since the conservation laws apply to these quantities, ΔQ requires special consideration. When class 1 molecules were selected as the typical class, the expression for ΔQ was found to be

$$\Delta Q = \iiint n^2(Q_1' - Q_1) f_1 f_2 \sigma^2 \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{1}$$

If class 2 molecules are chosen as the typical class, then

$$\Delta Q = \iiint n^2(Q_2' - Q_2) f_1 f_2 \sigma^2 \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{2}$$

which is readily obtained by exchanging the subscripts 1 and 2. Adding these two results, we have

$$\Delta Q = \tfrac{1}{2} \iiint n^2[(Q_1' + Q_2') - (Q_1 + Q_2)] f_1 f_2 \sigma^2 \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{3}$$

When Q is the mass, momentum, or energy of a molecule, then the conservation laws require that $Q_1' + Q_2' = Q_1 + Q_2$ and, therefore, $\Delta Q = 0$. Then for these particular cases the transfer equation reduces to

$$\frac{\partial}{\partial t}(n\bar{Q}) + \frac{\partial}{\partial x}(n\overline{uQ}) + \frac{\partial}{\partial y}(n\overline{vQ}) + \frac{\partial}{\partial z}(n\overline{wQ}) = 0 \tag{4}$$

If $Q = m$, it will be readily seen that $\bar{Q} = m$, $\overline{uQ} = m\bar{u} \cdots$, and the mass-transfer equation is

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\bar{u}) + \frac{\partial}{\partial y}(n\bar{v}) + \frac{\partial}{\partial z}(n\bar{w}) = 0 \tag{5}$$

With the assistance of this equation and Eqs. 1.4, 8, the general transfer equation can be transformed to a simpler form. If we substitute from Eqs. 1.4, 8, then Eq. 1.8, 7 becomes

$$\bar{Q}\frac{\partial n}{\partial t} + n\frac{\partial \bar{Q}}{\partial t} = -\frac{\partial}{\partial x}[n(\overline{uQ} + \overline{UQ})] - \frac{\partial}{\partial y}[n(\overline{vQ} + \overline{VQ})]$$
$$-\frac{\partial}{\partial z}[n(\overline{wQ} + \overline{WQ})] + \Delta Q \tag{6}$$

If we expand this expression and note that the bracket multiplying $\bar{Q}$ is the left-hand side of Eq. 5, then

$$n\frac{d\bar{Q}}{dt} = -\left[\frac{\partial}{\partial x}(n\overline{UQ}) + \frac{\partial}{\partial y}(n\overline{VQ}) + \frac{\partial}{\partial z}(n\overline{WQ})\right] + \Delta Q \tag{7}$$

where d/dt is a mobile operator or time derivative "following the mass motion" having the expanded form

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z}$$

The momentum-transfer equations can be derived more conveniently from Eq. 7. If $Q = mu$, then $\bar{Q} = m\bar{u}$, $\overline{UQ} = m\overline{U^2}$ $(\bar{U} = 0), \cdots$. Since $\Delta Q = 0$, the transport of momentum by the molecules is expressed by the equations

$$n\frac{d\bar{u}}{dt} = -\left[\frac{\partial}{\partial x}(n\overline{U^2}) + \frac{\partial}{\partial y}(n\overline{UV}) + \frac{\partial}{\partial z}(n\overline{UW})\right]$$
$$n\frac{d\bar{v}}{dt} = -\left[\frac{\partial}{\partial x}(n\overline{UV}) + \frac{\partial}{\partial y}(n\overline{V^2}) + \frac{\partial}{\partial z}(n\overline{VW})\right] \tag{8}$$
$$n\frac{d\bar{w}}{dt} = -\left[\frac{\partial}{\partial x}(n\overline{UW}) + \frac{\partial}{\partial y}(n\overline{VW}) + \frac{\partial}{\partial z}(n\overline{W^2})\right]$$

When $Q = \frac{1}{2}m(u^2 + v^2 + w^2)$, then $\bar{Q} = \frac{1}{2}m(\bar{u}^2 + \bar{v}^2 + \bar{w}^2 + \overline{C^2})$ and

$$\overline{UQ} = m[\bar{u}\overline{U^2} + \bar{v}\overline{UV} + \bar{w}\overline{UW} + \tfrac{1}{2}\overline{UC^2}] \tag{9}$$

with similar expressions for $\overline{VQ}$ and $\overline{WQ}$. Again, $\Delta Q = 0$, and Eq. 7 becomes the energy-transfer relation,

$$n\frac{d}{dt}(\overline{C^2}) = -2n\left[\left(\overline{U^2}\frac{\partial\bar{u}}{\partial x} + \overline{V^2}\frac{\partial\bar{v}}{\partial y} + \overline{W^2}\frac{\partial\bar{w}}{\partial z}\right) + \overline{UV}\left(\frac{\partial\bar{v}}{\partial x} + \frac{\partial\bar{u}}{\partial y}\right) + \overline{VW}\left(\frac{\partial\bar{w}}{\partial y} + \frac{\partial\bar{v}}{\partial z}\right) + \overline{WU}\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right)\right] - \left[\frac{\partial}{\partial x}(n\overline{UC^2}) + \frac{\partial}{\partial y}(n\overline{VC^2}) + \frac{\partial}{\partial z}(n\overline{WC^2})\right] \tag{10}$$

where many terms are dropped through the application of the momentum-transfer equations (Eqs. 8).

Equations 5, 8, and 10 are therefore the fundamental relations governing the flow of a gas. Before they can be placed in a usable form, the velocity distribution function f must be found from the Boltzmann equation (Eq. 1.8, 7) so that appropriate functions can be substituted for the mean values $\overline{U^2}$, $\overline{UV}$, $\overline{UC^2}$, $\cdots$.

1.10 THE FUNDAMENTAL MACROSCOPIC QUANTITIES OF THERMODYNAMICS

The dynamical system of freely moving molecules, which constitutes a gas in the kinetic theory, must be identified with the thermodynamic concept of a gas. This can be done by expressing the above transfer equations in terms of the basic quantities of thermodynamics—density, pressure, and temperature.

The density of a gas has been defined in Section 1.3 above as the ratio of mass to volume for a macroscopically small volume (see Eq. 1.3, 1). Multiplying the mass-transfer equation (1.9, 5) by m, we have the well-known equation of continuity of the continuum theory,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\bar{u}) + \frac{\partial}{\partial y}(\rho\bar{v}) + \frac{\partial}{\partial z}(\rho\bar{w}) = 0 \tag{1}$$

If the gas is regarded as a continuum, the motion of the fluid element arises from pressures acting on its boundaries. In a rectangular system of coordinates, the fluid element has a volume $dx\,dy\,dz$, and the forces in the direction of x are due to a normal pressure ($-P_{xx}$) acting on the area $dy\,dz$ and two shearing stresses ($-P_{yx}$, $-P_{zx}$) acting on areas $dz\,dx$ and $dx\,dy$, respectively. (Note that the symbol P_{zx} indicates the force per unit area on the face perpendicular to the axis of z acting in the

direction of x.) The resultant normal force will be the difference between $+\left(P_{xx}+\frac{\partial P_{xx}}{\partial x}\,dx\right)dy\,dz$ at position $x+dx$ and $+P_{xx}\,dy\,dz$ at position x (see Fig. 8) or $\frac{\partial P_{xx}}{\partial x}\,dx\,dy\,dz$. Consistent with the theory of elasticity, the convention has been adopted that pressure is a negative (compressive)

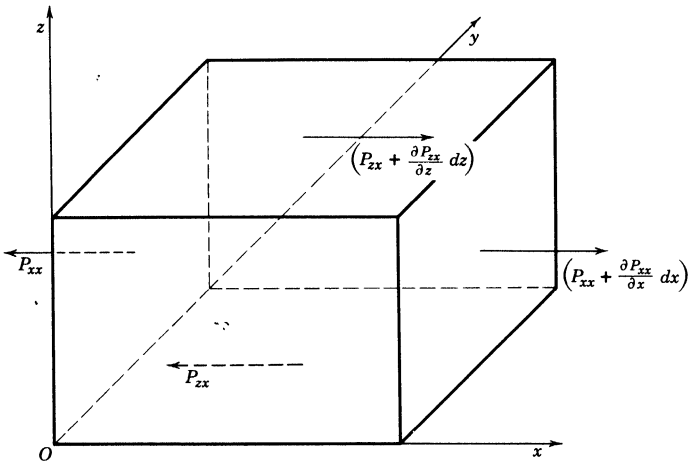

Fig. 8. Stresses on the fluid element.

force. In the same way the resultant shearing forces are found to be $\frac{\partial P_{yx}}{\partial y}\,dx\,dy\,dz$ and $\frac{\partial P_{zx}}{\partial z}\,dx\,dy\,dz$. Then the force acting in the direction of x on the fluid element is

$$F_x=\left(\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{yx}}{\partial y}+\frac{\partial P_{zx}}{\partial z}\right)dx\,dy\,dz \tag{2}$$

with similar expressions for F_y, F_z. According to Newton's second law, the equations of motion of the fluid element in the x, y, and z directions are, respectively,

$$\begin{aligned}
\rho\frac{d\bar{u}}{dt}&=\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{yx}}{\partial y}+\frac{\partial P_{zx}}{\partial z}\\
\rho\frac{d\bar{v}}{dt}&=\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_{yy}}{\partial y}+\frac{\partial P_{zy}}{\partial z}\\
\rho\frac{d\bar{w}}{dt}&=\frac{\partial P_{xz}}{\partial x}+\frac{\partial P_{yz}}{\partial y}+\frac{\partial P_{zz}}{\partial z}
\end{aligned} \tag{3}$$

A comparison with the momentum-transfer equations (1.9, 8) shows that

$$\begin{aligned} P_{xx} &= -\rho\overline{U^2}, \quad P_{yy} = -\rho\overline{V^2}, \quad P_{zz} = -\rho\overline{W^2} \\ P_{xy} &= P_{yx} = -\rho\overline{UV}, \quad P_{yz} = P_{zy} = -\rho\overline{VW}, \\ P_{zx} &= P_{xz} = -\rho\overline{WU} \end{aligned} \tag{4}$$

The static pressure (p) at a point in the gas is defined as the mean value of the normal pressures across three planes through the point parallel to the coordinate planes. Then,

$$p = \tfrac{1}{3}(P_{xx} + P_{yy} + P_{zz}) = \tfrac{1}{3}\rho\overline{C^2} \tag{5}$$

Equations 4 indicate that, from the point of view of kinetic theory, the pressure in a gas can be interpreted as the mean rate of flow of molecular momentum per unit area across a small element of area which is moving at the local velocity of the mass flow.

The concept of pressure is always associated with a surface. In the interior of a gas the surface is defined mathematically. The pressure on the surface of a solid body is not, in general, the same as the pressure within the body of the gas, since the theory at this point includes only the effect of collisions between molecules of the gas, and encounters between gas molecules and a wall have not been considered. The interaction between the molecules of a gas and a wall is quite complicated. For example, condensation or evaporation of molecules can occur. The problem of the pressure exerted by a gas on the surface of a body will be considered in detail at a later stage (see Sections 2.6 and 4.4).

When the molecules are represented as smooth, hard spheres with perfect elasticity, the conclusion is reached that pressure arises solely from momentum transfer. A more general model for the molecules would lead to a modified result. If intermolecular forces exist, part of the pressure must be due to the action of these forces across the surface under consideration. Furthermore, no allowance is made for the finite size of the molecules. If the field of force is strong only in regions adjacent to the molecule, and the diameter of the molecules is small compared with the average distance between them, then the effect of such forces may be neglected. In ordinary gases under normal conditions these effects are small, but, as the density of the gas increases, they must be taken into account in the momentum-transfer equations (see Section 3.10).

As the molecules of a gas move about, an organized component of their motion is discernible as mass flow, and the remaining random, molecular motion is invisible and beyond microscopic investigation by the ordinary methods of experimental physics. This hidden mechanical

energy is interpreted as heat energy in the kinetic theory. Thus the conversion of heat into mechanical work is essentially an increase in the visible mass flow at the expense of the hidden random motion.

The molecule represented as a hard, smooth sphere has only translational energy. More complicated molecular models may include other forms of hidden energy which can be exchanged during encounters, such as the energies of vibration and rotation. These effects will add further terms to the expression for the heat energy (see Section 4.9).

In the laboratory the heat energy is measured empirically relative to an arbitrary level by the expansion of mercury or some other device, and the reading is called a temperature. In thermodynamical considerations an absolute scale of temperature is defined, which is based on the second law of thermodynamics, and is therefore independent of the properties of substances. The definition of temperature in the kinetic theory of gases is based on the assumption that the molecules possess translatory energy only, that is,

$$RT = \tfrac{1}{3}\overline{C^2} \tag{6}$$

where T is the temperature and R is the gas constant referred to unit mass.

Combining Eqs. 5 and 6, we have the well-known equation of state of thermodynamics

$$p = \rho RT \tag{7}$$

Therefore, the state of a gas (apart from its mass motion) is specified thermodynamically by any two of ρ, p, and T, whereas in the molecular theory the state is given by n and $\overline{C^2}$.

NOTATION

$\mathbf{c}$	velocity of a molecule (vector)
$\mathbf{c}_n$	component of $\mathbf{c}$ normal to the line of impact
$\mathbf{c}_o$	velocity of the center of mass of two colliding molecules
$\mathbf{c}_t$	component of $\mathbf{c}$ parallel to the line of impact
C	speed of a molecule relative to the mass motion of the gas (in $d\tau$)
$D_1(\ )$	time derivative following the motion of the typical class (1) molecules
E	translational energy of a molecule
f	function specifying the distribution of molecular velocities
F_x, F_y, F_z	components of the resultant force acting on the fluid element
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of x, y, z, respectively
J, J'	Jacobians (see Eqs. 1.7, 7, 8)
m	mass of a molecule
n	number of molecules per unit volume
N	total number of molecules in a given volume of gas
p	static pressure at a point
P_{xx}, P_{yy}, P_{zz}	components of normal pressure acting on the fluid element
P_{xy}, P_{yz}, P_{zx}	components of shearing stress acting on the fluid element
$\bar{q}$	speed of the mass motion of a gas (in $d\tau$)

Q	a function of u, v, w (such as mu, muv)
ΔQ	the change in ΣQ per unit volume produced by molecular collisions in unit time
ΣQ	the aggregate amount of Q in the element of volume $d\tau$ at time t
R	gas constant per unit mass of gas
t	time
T	temperature
u, v, w	components of velocity of a molecule
$\bar{u}, \bar{v}, \bar{w}$	components of the mass velocity of the gas (in $d\tau$)
U, V, W	components of velocity of a molecule referred to the mass velocity (in $d\tau$)
x, y, z	space coordinates
ξ, η, ζ	direction cosines of the line of impact
ρ	density
σ	diameter of a molecule
$d\tau$	element of volume in physical space ($dx\, dy\, dz$)
$d\chi$	element of solid angle (Fig. 6)
ψ	angle between the relative velocity vector and the line of impact (Fig. 5)
$d\omega$	element of volume in the velocity space (u, v, w)
$\mathbf{\Omega}$	relative velocity of two molecules before collision
$\mathbf{\Omega}'$	relative velocity of two molecules after collision
Ω, Ω'	magnitudes of $\mathbf{\Omega}, \mathbf{\Omega}'$
$\mathbf{\Omega}_t$	component of $\mathbf{\Omega}$ parallel to the line of impact
Ω_t	magnitude of $\mathbf{\Omega}_t$

Note: A class 1 molecule has velocity components in the infinitesimal range $u_1, u_1 + du_1$; $v_1, v_1 + dv_1$; $w_1, w_1 + dw_1$. The subscripts 1 and 2 refer to class 1 and class 2 molecules, respectively. All primed quantities refer to conditions after collision. $\overline{Q}$ signifies the mean value of a function of u, v, w calculated according to Eq. 1.4, 6 (which denotes Eq. 6 in Section 1.4). If a reference in another chapter requires mention, this is indicated by the notation Ref. 1.2 (Reference 2 in Chapter 1).

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CHAPTER TWO

Isentropic Flow

2.1 MAXWELL'S VELOCITY DISTRIBUTION FUNCTION

The special case in which the collisions between the gas molecules no longer affect the distribution of molecular velocities is of great importance. The motion is such that, as we follow the class 1 molecules, the number lost by some collisions is regained by others, and a condition of equilibrium exists. Then Boltzmann's relation (1.7, 16) reduces to

$$D_1(nf_1) = \int\int n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi \, d\chi \, d\omega_2 = 0 \tag{1}$$

A solution of this equation will yield the required velocity distribution function which is unaffected by molecular interactions.

A sufficient condition for equilibrium is obtained by placing the integrand equal to zero (Ref. 1),

$$f_1'f_2' = f_1f_2 \tag{2}$$

However, it does not follow immediately that this is a necessary condition, since, to satisfy Eq. 1, we need only have an integrand with positive and negative values over the range of integration such that the value of the definite integral is zero. A discussion of this question forms part of a general investigation of equilibrium states in gases, and it is therefore reserved for detailed examination in Section 2.3 below.

We can write Eq. 2 in the form

$$\log f_1' + \log f_2' = \log f_1 + \log f_2 \tag{3}$$

Then $\log f$ is a summational invariant for molecular encounters. But, according to Section 1.5, the only summational invariants that occur in the theory of collisions are the quantities m, mu, mv, mw, and $\frac{1}{2}mc^2$ (see Eqs. 1.5, 6, 7). In general, therefore, $\log f$ must be a linear combination of these functions, or

$$\log f = k_1(u^2 + v^2 + w^2) + k_2u + k_3v + k_4w + k_5 \tag{4}$$

where $k_1, \cdots, k_5$ are functions of x, y, z, t since Boltzmann's equation refers to the molecules of a particular class in the element $d\tau$ (center x, y, z) of physical space at time t. No other independent functions of u, v, w will be involved, since, if the six velocity components before a binary encounter are known, the six resultant components can be calculated in terms of two parameters which establish the orientation of the line of impact. Further invariant functions of u, v, w would leave less than the required two parameters (see Section 1.5). Therefore, the velocity distribution function for molecular equilibrium, first found by Maxwell, may be written in the form

$$f = Ae^{-\beta[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]} \tag{5}$$

where A, β, u_0, v_0, w_0 replace the coefficients k_1, k_2, k_3, k_4, k_5 in Eq. 4.

This velocity distribution function must satisfy the condition set out in Eq. 1.4, 3,

$$A\int e^{-\beta[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]}\, d\omega = 1 \tag{6}$$

which leads to the result (see integrals in Appendix I)

$$A = \left(\frac{\beta}{\pi}\right)^{3/2} \tag{7}$$

Furthermore, the mean value of the x-component of velocity in $d\tau$ (Eq. 1.4, 7) is

$$\bar{u} = A\int u e^{-\beta[(u-u_0)^2+(v-v_0)^2+(w-w_0)^2]}\, d\omega = u_0 \tag{8}$$

Similarly $\bar{v} \quad v_0$, $\bar{w} \quad w_0$. Then, according to Eqs. 1.4, 8, the velocity distribution function may now be expressed in the form

$$f = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta C^2} \tag{9}$$

where $C^2 = U^2 + V^2 + W^2$.

The quantity β may be determined from a calculation of the mean values of U^2, V^2, and W^2. For example (see Appendix I),

$$\overline{U^2} = A\int U^2 e^{-\beta(U^2+V^2+W^2)}\, d\omega = \frac{1}{2\beta} \tag{10}$$

Similarly $\overline{V^2} = \overline{W^2} = \dfrac{1}{2\beta}$. Then

$$\beta = \frac{3}{2\overline{C^2}} \tag{11}$$

and Maxwell's velocity distribution function takes the final form

$$f = \left(\frac{3}{2\pi\overline{C^2}}\right)^{3/2} e^{-3C^2/2\overline{C^2}} \tag{12}$$

The function f refers only to the unordered portion of the molecular motion. It depends on the velocities of the unseen thermal motion (U, V, W) and on the internal energy per unit mass ($\frac{1}{2}\overline{C^2}$) which is in general a function of position and time (x, y, z, t).

We have seen that the number of molecules in the element of volume $d\tau$ having velocity vectors which terminate in the element $d\omega$ of the velocity space is $nf\,d\tau\,d\omega$. Now $d\omega$ may be written in the form $du\,dv\,dw$ or $dU\,dV\,dW$ (see expression 1.4, 1, and Eqs. 1.4, 8). If the rectangular coordinates U, V, W are replaced by the polar coordinates C, φ, θ, then

$$d\omega = C^2 \sin\varphi\, d\varphi\, d\theta\, dC \tag{13}$$

Therefore, the number of molecules in the element of volume $d\tau$, which have speeds in the range C, $C + dC$, and are orientated between the angles φ, $\varphi + d\varphi$ relative to the polar axis ($0z$), and θ, $\theta + d\theta$ relative to the reference plane for θ ($x0z$), may be written

$$nC^2 f \sin\varphi\, d\varphi\, d\theta\, dC\, d\tau \tag{14}$$

Finally, the number of molecules in the element $d\tau$ having speeds in the range C, $C + dC$ irrespective of direction can be obtained by integrating expression 14 with respect to φ and θ over all possible orientations ($0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$). The result is

$$4\pi nC^2 f\, dC\, d\tau \tag{15}$$

2.2 TRANSFER EQUATIONS CORRESPONDING TO MAXWELL'S DISTRIBUTION FUNCTION—ISENTROPIC FLOW

The transfer equations assume a special form when Maxwell's velocity distribution function is used to determine the various mean values. From Eq. 1.4, 6 and Eq. 2.1, 12,

$$\overline{U^2} = \overline{V^2} = \overline{W^2} = \tfrac{1}{3}\overline{C^2} \tag{1}$$

and

$$\overline{UV} = \overline{VW} = \overline{WU} = 0 \tag{2}$$

where it is to be noted that, when a mean value is formed with Maxwell's distribution function, the integration of all odd expressions in U, V, or W between the limits $-\infty$ and $+\infty$ yields a zero result. For example (see Appendix I),

$$\overline{UV} = A\int_{-\infty}^{+\infty} Ue^{-\beta U^2}\, dU \int_{-\infty}^{+\infty} Ve^{-\beta V^2}\, dV \int_{-\infty}^{+\infty} e^{-\beta W^2}\, dW = 0 \tag{3}$$

since the first two integrands are odd in U and V, respectively. Then the momentum-transfer equations (1.9, 8) become

$$n\frac{d\bar{u}}{dt} = -\frac{1}{3}\frac{\partial}{\partial x}(n\overline{C^2}), \quad n\frac{d\bar{v}}{dt} = -\frac{1}{3}\frac{\partial}{\partial y}(n\overline{C^2}),$$
$$n\frac{d\bar{w}}{dt} = -\frac{1}{3}\frac{\partial}{\partial z}(n\overline{C^2}) \tag{4}$$

Similarly, since

$$\overline{U^3} = \overline{UV^2} = \overline{UW^2} = \cdots = 0 \tag{5}$$

the energy-transfer equation (1.9, 10) in a Maxwellian flow is

$$\frac{d\overline{C^2}}{dt} = -\frac{2}{3}\overline{C^2}\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) \tag{6}$$

The energy-transfer equation can be reduced to a simple algebraic relation. If we write the mass-transfer equation (1.9, 5) in the form

$$\frac{dn}{dt} = -n\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) \tag{7}$$

and substitute for the bracketed quantity in Eq. 6, the result is

$$\frac{1}{\overline{C^2}}\frac{d\overline{C^2}}{dt} = \frac{2}{3}\frac{1}{n}\frac{dn}{dt} \tag{8}$$

or

$$\overline{C^2} = \kappa n^{2/3} \tag{9}$$

where κ is a constant of integration. This relation indicates that in a Maxwellian flow a decrease in the number of molecules per unit volume is accompanied by a reduction in internal energy.

The transfer equations (Eqs. 4, 9 and 1.9, 5) show that the mean-square speed ($\overline{C^2}$) and the components of the visible mass motion ($\bar{u}, \bar{v}, \bar{w}$) are the basic independent flow variables in a Maxwellian flow. All mean values of the type $\overline{U^2}$, $\overline{UV}$, $\overline{UC^2}$ are either zero or expressible in terms of the mean-square speed, and hence the contribution of the unseen random molecular motion to the macroscopic properties of the flow is made through $\overline{C^2}$ (or n, see Eq. 9).

If we use Eqs. 1.10, 5, 6, the special transfer equations above may be expressed in terms of the fundamental quantities of thermodynamics. The complete system of equations is

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\bar{u}) + \frac{\partial}{\partial y}(\rho\bar{v}) + \frac{\partial}{\partial z}(\rho\bar{w}) = 0$$
$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial p}{\partial x}, \quad \rho\frac{d\bar{v}}{dt} = -\frac{\partial p}{\partial y}, \quad \rho\frac{d\bar{w}}{dt} = -\frac{\partial p}{\partial z} \tag{10}$$
$$p = B\rho^{\gamma}$$

The analysis given above (Eq. 9) yields a value for γ of $1\frac{2}{3}$. This result will hold only for monatomic gases since the molecular model adopted in the preceding theory does not allow exchanges of rotational or vibrational energy during collisions.

In thermodynamics γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume,

$$\gamma = \frac{c_p}{c_v} \tag{11}$$

If the temperature of unit mass of gas is raised from T to $T + \delta T$ without change of volume, then the heat added is $c_v\,\delta T$. When the gas is flowing, $c_v\,\delta T$ is added without affecting the mass motion, and it appears as an increment of heat energy only. Now the internal energy per unit mass is $\bar{E}_i/m$, and, when the quantity of heat $c_v\,\delta T$ is added, this energy is increased by $\delta\bar{E}_i/m$. Therefore, $c_v\,\delta T = \delta\bar{E}_i/m$, or, in the limit,

$$c_v = \frac{1}{m}\left(\frac{d\bar{E}_i}{dT}\right)_{\text{vol}} \tag{12}$$

When the internal energy is translational only (see Eqs. 1.4, 11 and 1.10, 6),

$$\bar{E}_i/m = \tfrac{1}{2}\overline{C^2} = \tfrac{3}{2}RT \tag{13}$$

and

$$c_v = \tfrac{3}{2}R \tag{14}$$

If heat is added to unit mass of gas at constant pressure such that the temperature is increased from T to $T + \delta T$, mechanical work of amount $p\,\delta(1/\rho)$ is done while the internal energy is raised by $\delta\bar{E}_i/m$. In this event some of the heat added is converted to mass motion. Then, by definition,

$$c_p\,\delta T = p\,\delta\left(\frac{1}{\rho}\right) + \frac{\delta\bar{E}_i}{m} \tag{15}$$

According to Eq. 1.10, 7, $p\,\delta(1/\rho) = R\,\delta T$, and, in the limit,

$$c_p = R + \frac{1}{m}\left(\frac{d\bar{E}_i}{dT}\right)_{\text{vol}} = R + c_v \tag{16}$$

When the molecules have translational energy only, then from Eq. 14

$$c_p = \tfrac{5}{2}R \tag{17}$$

Experiment shows that the value $\gamma = 1\frac{2}{3}$ holds satisfactorily for monatomic gases such as helium and argon. If the appropriate value of γ is used, Eqs. 10 can be applied to polyatomic molecules for which

$$\bar{E}_i/m = \tfrac{1}{2}NRT \tag{18}$$

where N is the number of degrees of freedom possessed by the molecule.

This relation is one of the results of statistical mechanics which states that, when a gas is in equilibrium, an average energy $\frac{1}{2}kT$ may be associated with each degree of freedom of the molecule, where k is the gas constant for 1 molecule ($k = mR$). Then,

$$c_v = \frac{1}{2}NR, \quad c_p = R\left(1 + \frac{N}{2}\right) \tag{19}$$

and the ratio of specific heats is

$$\gamma = 1 + \frac{2}{N} \tag{20}$$

For a monatomic gas, $N = 3$, and $\gamma = 1\frac{2}{3}$. A diatomic molecule has 5 degrees of freedom, and the corresponding value of γ is $1\frac{2}{5}$. When Eqs. 10 are applied to a diatomic gas, this value of γ should be used.

The last equation of Eqs. 10 has a particular thermodynamic significance. The entropy per unit mass of gas is defined by the relation

$$dS = c_v \frac{dT}{T} - R\frac{d\rho}{\rho} \tag{21}$$

where $1/\rho$ is the volume of unit mass of gas. If Eq. 1.10, 7 is written in the differential form

$$\frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho} \tag{22}$$

then substitution for dT/T in Eq. 21 and integration leads to the last equation of Eqs. 10 when $dS = 0$. We conclude, therefore, that a Maxwellian flow is isentropic. The significance of entropy applied to a system of molecules is discussed in the next section.

2.3 EQUILIBRIUM OF A MOLECULAR SYSTEM

The condition for equilibrium will now be considered in more detail. When type a and type b collisions (see Section 1.7) occur with equal frequency, then a mechanism exists whereby equilibrium can be maintained. The perturbation produced by one type of encounter will be canceled by an equal and opposite effect due to the other type. This is a particular case of the principle of detailed balancing (Ref. 2, p. 660), according to which nature maintains a balance by providing an inverse process for each direct one. The implication of this principle is that balance is maintained in the interactions between the molecules of any two classes, and hence, for equilibrium, expression 1.7, 10 is zero. Under these circumstances Eq. 2.1, 2 is the necessary and sufficient condition for equilibrium.

A collision between two molecules is a reversible mechanical process. Although the effect of a single encounter may be reversed, the general trend of the whole molecular system, constituting the gas, is governed by the second law of thermodynamics, and is not reversible. As time progresses, the state of a gas (determined by all the molecular encounters) will tend in a definite direction, or remain "stationary." Before further progress can be made with a study of the equilibrium or stationary condition as it applies to the whole molecular system, we must first define a function the variation of which will indicate the trend of the state of a gas.

The number of class 1 molecules in the element of volume $d\tau$ is $nf_1\, d\omega_1\, d\tau$. If the state of the gas is changing, then at the end of an interval of time dt the corresponding number will be $[nf_1 + \delta(nf_1)]\, d\omega_1\, d\tau$, and the related fractional change is $\delta(nf_1)/nf_1$. As the state of a gas changes the fractional variation of the various classes will be $\delta(nf_1)/nf_1$, $\delta(nf_2)/nf_2$, $\cdots$, which may also be written $\delta(\log nf_1)$, $\delta(\log nf_2)$, $\cdots$. Therefore, we can associate with the molecules of each class a quantity $Q = \log nf$, the variation of which is a measure of the contribution made by the relevant class of molecules to the general change of state. The mean value of Q, averaged over all classes, will be a function of the type required, and we define the Boltzmann H function (Ref. 3) as follows,

$$H = n\bar{Q} = n\int f \log nf\, d\omega \tag{1}$$

The variation of H will indicate the trend of the state of a gas.

We shall now restrict the discussion of the condition for equilibrium to a gas for which no resultant flux occurs through the sides of the element of volume $d\tau$. Then Boltzmann's equation (1.7, 16) assumes the simpler form

$$\frac{\partial}{\partial t}(nf_1) = \int\int n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi\, d\chi\, d\omega_2 \tag{2}$$

The variation of H with respect to time is

$$\frac{\partial H}{\partial t} = \int \frac{\partial}{\partial t}[nf \log nf]\, d\omega = \int (1 + \log nf)\frac{\partial}{\partial t}(nf)\, d\omega \tag{3}$$

The derivative $\partial H/\partial t$ can be evaluated in terms of the collisions of class 1 molecules with all other classes,

$$\begin{aligned}\frac{\partial H}{\partial t} &= \int (1 + \log nf_1)\frac{\partial}{\partial t}(nf_1)\, d\omega_1 \\ &= \int\int\int n^2(1 + \log nf_1)(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi\, d\chi\, d\omega_2\, d\omega_1\end{aligned} \tag{4}$$

If the roles of class 1 and class 2 molecules are interchanged, then

$$\frac{\partial H}{\partial t} = \iiint n^2(1 + \log nf_2)(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \quad (5)$$

and the over-all value of the integral is unaltered. Adding these two results and using Eq. 1.7, 9, we have

$$\frac{\partial H}{\partial t} = \frac{1}{2}\iiint n^2(2 + \log n^2 f_1 f_2)(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \quad (6)$$

In the above calculations the derivative $\partial H/\partial t$ has been evaluated by taking first class 1 and then class 2 molecules as the typical group. In other words only a type *a* collision has been considered. But we may also take class 1′ and class 2′ molecules in turn as the typical groups corresponding to a type *b* encounter. Proceeding as above results in

$$\frac{\partial H}{\partial t} = \frac{1}{2}\iiint n^2(2 + \log n^2 f_1'f_2')(f_1f_2 - f_1'f_2')\sigma^2\Omega \cos\psi \, d\chi \, d\omega_1' \, d\omega_2' \quad (7)$$

If we add Eqs. 6 and 7, the final neutral value is

$$\frac{\partial H}{\partial t} = \frac{1}{4}\iiint n^2(\log f_1f_2 - \log f_1'f_2')(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \quad (8)$$

Since the logarithm is a monotonic increasing function, then $(\log f_1f_2 - \log f_1'f_2')$ is positive or negative according as f_1f_2 is greater than or less than $f_1'f_2'$. Therefore, the bracketed quantities in the integrand are always of opposite sign, and the product must be negative or zero. This means that, over the range of integration, the integral has no positive parts to balance the contribution from the negative portions. Hence the integral can vanish only if the integrand is zero for all values of the variables of integration. Therefore, Eq. 2.1, 2 is the necessary and sufficient condition for equilibrium in the element $d\tau$ when the gas has no mass motion.

This is Boltzmann's theorem. It shows that, for a gas at rest and in complete equilibrium, the Maxwellian distribution is the only steady one, and any other distribution will be adjusted by molecular collisions toward Maxwell's law.

It has been shown that the Maxwellian motion of kinetic theory corresponds to the isentropic change of state of thermodynamics. The condition for equilibrium can be deduced either from Boltzmann's H function for a molecular system or from the entropy function S of thermodynamics. It is evident, therefore, that some relationship exists between H and S.

An indication of the dependence of S on H can be obtained from their values for a gas in equilibrium having no mass motion. Substitution from Eq. 2.1, 12 in Eq. 1 yields

$$H = n\left[\log n + \frac{3}{2}\log\left(\frac{3}{2\pi\overline{C^2}}\right) - \frac{3}{2}\right] \tag{9}$$

This value of H refers to unit volume since H was defined in terms of the number of molecules per unit volume ($H = n\bar{Q}$). The value of H per unit mass of gas is

$$H_m = \frac{H}{mn} = \frac{1}{m}\log\left[n\left(\frac{3}{2\pi\overline{C^2}}\right)^{3/2}\right] + \text{constant} \tag{10}$$

The change in the entropy of a monatomic gas is (see Eq. 2.2, 21)

$$dS = R\left(\frac{3}{2}\frac{dT}{T} - \frac{d\rho}{\rho}\right) \tag{11}$$

The integral of this relation is

$$S = R\log\left(\frac{T^{3/2}}{\rho}\right) + \text{constant} \tag{12}$$

Since $\rho = mn$, and $RT = \frac{1}{3}\overline{C^2}$, then a comparison of Eqs. 10 and 12 gives the relation

$$S = -mRH_m + \text{constant} \tag{13}$$

The function H can be evaluated for any distribution function f, for example, an f corresponding to nonisentropic flow. Since S is not defined thermodynamically for gases not in equilibrium and having mass motion, Eq. 13 suggests a possible definition.

The above proof of the Boltzmann theorem cannot be extended to the gas having mass motion since an expression for the derivative of nf following the mass motion, $(d/dt)(nf)$, is required for the more general form of Eq. 3, but Boltzmann's equation (1.7, 16) provides the derivative $D_1(nf)$ following the motion of class 1 molecules. In many practical problems, however, the gas flow contains an undisturbed region in which the molecular motion may be regarded as Maxwellian. Thus the air essentially at rest in the inlet of a wind tunnel, or in the atmosphere far in front of an aircraft, contains molecules in Maxwellian (isentropic) motion. If the whole of a flow is isentropic, it seems reasonable to assume that, if the initial molecular motion of the gas at rest is Maxwellian, then Maxwell's law of distribution of molecular velocities also holds throughout that part of the gas which has mass motion.

2.4 FUNDAMENTAL PARAMETERS OF THE RANDOM MOLECULAR MOTION

The unseen, random motion of the molecules, as distinct from the ordered mass motion, is characterized by certain quantities which can be calculated from Maxwell's distribution function when the flow is isentropic. One of these is the mean speed. We have seen that the number of molecules in the element of volume $d\tau$ having speeds between C and $C + dC$ is $4\pi n C^2 f\, dC\, d\tau$ (see Eq. 2.1, 15). Then the mean random speed is

$$\bar{C} = 4\pi \int_0^\infty C^3 f\, dC = 2\sqrt{\frac{2\overline{C^2}}{3\pi}} \tag{1}$$

In terms of thermodynamic quantities,

$$\bar{C} = 2\sqrt{\frac{2RT}{\pi}} \tag{2}$$

Another quantity of interest is the most probable random speed. The value of C possessed by the greatest number of molecules corresponds to the maximum value of the function $C^2 f$, which is

$$C_m = \sqrt{\frac{2\overline{C^2}}{3}} = \sqrt{2RT} \tag{3}$$

It will be noted that Maxwell's distribution function contains C_m and may be expressed in the form

$$f = \left(\frac{1}{\pi C_m^2}\right)^{3/2} e^{-C^2/C_m^2} \tag{4}$$

The root-mean-square speed, which is fundamental to the transfer equations (2.2, 4, 6), is related to the two speeds above as follows:

$$\sqrt{\overline{C^2}} = 1.225 C_m = 1.086 \bar{C} \tag{5}$$

The function $C^2 f$ is plotted in Fig. 1 (see Eq. 2.1, 12, expression 2.1, 15) and the two above speeds are indicated.

When the molecules are assumed to be smooth, rigid spheres and no external field of force acts on them, each molecule travels freely in a straight line between impacts. The distance traversed between two successive contacts is called the free path. The average value of this quantity, or mean free path (L), is an important parameter in the kinetic theory of gases.

In order to calculate L, we first determine the total number of collisions which occur in the element of volume $d\tau$ during the interval of time dt. The number of encounters between class 1 and class 2 molecules in $d\tau$ over the interval dt having lines of impact orientated within the solid

angle $d\chi$ is given by expression 1.6, 1. If polar coordinates are used, then $d\chi = \sin\psi\, d\psi\, d\varphi$, and the corresponding frequency of collision for any orientation of the lines of impact is obtained by integrating over the ranges $0 \leq \psi \leq \pi/2$ and $0 \leq \varphi \leq 2\pi$ (see Fig. 1.6). The result is

$$\pi n^2 \sigma^2 f_1 f_2 \Omega\, d\omega_1\, d\omega_2\, d\tau\, dt \tag{6}$$

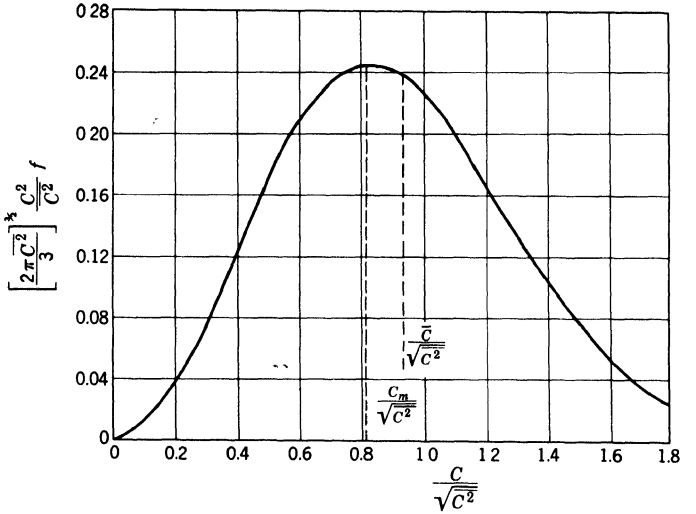


Fig. 1. Maxwell's distribution function.

Then the total number of collisions in $d\tau$ during the interval dt is

$$\tfrac{1}{2}\pi n^2 \sigma^2 \left[\int\int\int \Omega f_1 f_2\, d\omega_1\, d\omega_2\right] d\tau\, dt \tag{7}$$

where it should be noted that during the process of integration, each interaction is counted twice, once as the collision of a class 1 molecule with a class 2 molecule and again as an encounter between a class 2 and a class 1 molecule.

The integrals in the bracket are not immediately separable, owing to the presence of the relative velocity which has the form

$$\Omega = [(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2]^{1/2} \tag{8}$$

Separation can be achieved by the following change of variables,

$$\begin{aligned} X_1 &= U_2 - U_1, & Y_1 &= V_2 - V_1, & Z_1 &= W_2 - W_1 \\ X_2 &= U_2 + U_1, & Y_2 &= V_2 + V_1, & Z_2 &= W_2 + W_1 \end{aligned} \tag{9}$$

Then, $$\Omega^2 = X_1^2 + Y_1^2 + Z_1^2 \tag{10}$$

For convenience we shall also write

$$\Lambda^2 = X_2^2 + Y_2^2 + Z_2^2 \tag{11}$$

Then from Maxwell's distribution function (Eq. 2.1, 12)

$$f_1 f_2 = \left(\frac{\beta}{\pi}\right)^3 e^{-\frac{1}{2}\beta(\Omega^2+\Lambda^2)} \tag{12}$$

Following the convention adopted for the original variables (see Section 1.4), the symbol

$$\int(\cdots)dK_1$$

will denote the integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cdots)\,dX_1\,dY_1\,dZ_1$$

The elements of volume in the six-dimensional spaces specified by Eqs. 9 are subject to the transformation

$$d\omega_1\,d\omega_2 = |J|\,dK_1\,dK_2 \tag{13}$$

where the Jacobian is

$$J = \frac{\partial(U_1, V_1, W_1, U_2, V_2, W_2)}{\partial(X_1, Y_1, Z_1, X_2, Y_2, Z_2)} = \frac{1}{8} \tag{14}$$

Then $$\iint f_1 f_2 \Omega\,d\omega_1\,d\omega_2 = \frac{1}{8}\left(\frac{\beta}{\pi}\right)^3 \int \Omega e^{-\frac{1}{2}\beta\Omega^2}\,dK_1 \int e^{-\frac{1}{2}\beta\Lambda^2}\,dK_2 \tag{15}$$

If we introduce polar coordinates, the first integral on the right-hand side becomes

$$4\pi\int_0^{\infty}\Omega^3 e^{-\frac{1}{2}\beta\Omega^2}\,d\Omega = \frac{8\pi}{\beta^2} \tag{16}$$

and the second integral has the value $(2\pi/\beta)^{3/2}$ (see Appendix I). Therefore, the required total number of collisions is $\frac{\pi}{\sqrt{2}}n^2\sigma^2\bar{C}\,d\tau\,dt$ where $\bar{C} = 2/\sqrt{\pi\beta}$. Since each collision terminates two free paths, the corresponding number of free paths is $\sqrt{2}\pi n^2\sigma^2\bar{C}\,d\tau\,dt$.

The over-all distance traveled by the molecules in the element of volume $d\tau$ during the interval dt is $n\,d\tau\cdot\bar{C}\,dt$. Then the mean free path is obtained by dividing the total distance traversed by the number of free paths,

$$L = \frac{1}{\sqrt{2}\pi n\sigma^2} \tag{17}$$

Numerical values for the mean free path cannot be calculated immediately since the diameter of the molecule is unknown. A study of

viscosity in nonisentropic flows is necessary before estimates of L and σ can be made. Although a detailed discussion of these quantities must be deferred to Chapter 3, it can be mentioned here that under ordinary conditions $L = 6 \times 10^{-6}$ cm, $\sigma = 3.75 \times 10^{-8}$ cm for air, and for hydrogen $L = 1.125 \times 10^{-5}$ cm, $\sigma = 2.72 \times 10^{-8}$ cm.

The mean free path remains a definite quantity so long as σ can be defined. When the molecules are represented as point centers of force, the concept of the mean free path loses much of its significance.

Experiments on the flow of gases have shown that the ratio of the mean free path to some characteristic dimension of the apparatus (called Knudsen's number, *Kn*) has an important effect on the properties of the flow. When the diameter of a tube or the length of a body is comparable with the mean free path, the frequency of collision in $d\tau$ becomes insignificant, and the effect of a gas on a solid body arises essentially from encounters between free gas molecules and the surface.

2.5 A BASIC PARAMETER OF THE MASS MOTION—SPEED OF SOUND

We have seen that the random motion of the molecules can be assessed in terms of such speeds as $\sqrt{\overline{C^2}}$, $\bar{C}$, and C_m. In a similar way the visible mass motion of a gas is characterized by an important parameter called the speed of sound. To introduce this quantity, let us consider a gas initially at rest (no mass motion) in which a small disturbance is propagated in the direction of the x-axis. The equations for a one-dimensional, isentropic flow are

$$\begin{aligned} \rho_t + \bar{u}\rho_x + \rho\bar{u}_x &= 0 \\ \rho(\bar{u}_t + \bar{u}\bar{u}_x) &= -p_x \\ p &= B\rho^\gamma \end{aligned} \tag{1}$$

where the subscripts t and x signify the partial derivatives $\partial/\partial t$ and $\partial/\partial x$, respectively. If $p_x = a^2\rho_x$, where

$$a^2 = \frac{dp}{d\rho} \tag{2}$$

then the basic equations reduce to two partial differential equations from which the unknown variables ρ, $\bar{u}$ may be determined,

$$\begin{aligned} \rho_t + \bar{u}\rho_x + \rho\bar{u}_x &= 0 \\ \rho(\bar{u}_t + \bar{u}\bar{u}_x) &= -a^2\rho_x \end{aligned} \tag{3}$$

Since the disturbance is small, the density, mass velocity, and the quantity a vary only slightly from their initial values ρ_0, $\bar{u}_0(=0)$, and a_0. Let us write†

$$\rho = \rho_0 + \rho'', \quad \bar{u} = \bar{u}'', \quad a = a_0 + a'' \tag{4}$$

† The single prime is reserved for dimensionless quantities in this section.

where ρ'' and a'' are of low order compared with ρ_0 and a_0, respectively. Then, by applying the methods of the perturbation theory, the equations of motion reduce to

$$\begin{aligned} \rho_t'' + \rho_0 \bar{u}_x'' &= 0 \\ \rho_0 \bar{u}_t'' + a_0^2 \rho_x'' &= 0 \end{aligned} \tag{5}$$

If the first of Eqs. 5 is differentiated with respect to t and the second with respect to x, the difference between these two results yields a single differential equation of second order for ρ'',

$$\rho_{tt}'' = a_0^2 \rho_{xx}'' \tag{6}$$

It can be readily verified that the general solution of this equation is

$$\rho'' = F(x - a_0 t) + G(x + a_0 t) \tag{7}$$

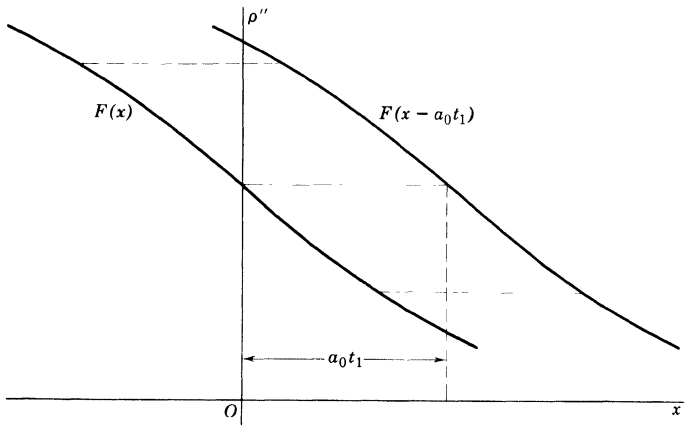


Fig. 2. Propagation of a small disturbance.

Consider first the case in which

$$\rho'' = F(x - a_0 t) \tag{8}$$

At $t = 0$, $\rho'' = F(x)$; that is, the perturbation density is initially any prescribed function of x. After a time t_1 elapses, $\rho'' = F(x - a_0 t_1)$, and it will be seen that the original distribution $F(x)$ is displaced unchanged a distance $a_0 t_1$ (Fig. 2). The displacement will be toward higher or lower values of x, depending on whether $\rho'' = F(x - a_0 t)$ or $\rho'' = G(x + a_0 t)$. It may be concluded, therefore, that small disturbances in the macroscopic properties of a gas are propagated without change of form at a speed a_0, called the speed of sound.

According to the transfer theory (see Section 1.8), the variation of any macroscopic property of a gas depends on both the flux of molecules and their collisions. In other words, the rate of transfer will depend on the average molecular speed and the frequency of encounters. The mean speed is a function of $\overline{C^2}$ only. Furthermore, the number of collisions per unit volume in unit time is proportional to $n^2\bar{C}$; hence the frequency depends only on $\overline{C^2}$ (see Eq. 2.2, 9). These physical facts indicate that the speed of propagation of a small disturbance through a gas in thermal equilibrium is a function of $\overline{C^2}$ only, and is independent of the form of the disturbance.

From Eqs. 2 and 1.10, 5

$$a^2 = \frac{1}{3}\frac{d(n\overline{C^2})}{dn} \tag{9}$$

According to Eq. 2.2, 8, $d\overline{C^2}/dn = \frac{2}{3}(\overline{C^2}/n)$ and the relation between a and $\overline{C^2}$ is

$$a = [\tfrac{5}{3}(\tfrac{1}{3}\overline{C^2})]^{1/2} \tag{10}$$

In terms of thermodynamic quantities

$$a = \sqrt{\gamma \overline{RT}} = \sqrt{\frac{\overline{\gamma p}}{\rho}} \tag{11}$$

If the various parameters are known at a point in an isentropic flow (a_1, T_1, ρ_1, p_1), then at any other point

$$\frac{a}{a_1} = \left(\frac{T}{T_1}\right)^{1/2} = \left(\frac{\rho}{\rho_1}\right)^{(\gamma-1)/2} = \left(\frac{p}{p_1}\right)^{(\gamma-1)/2\gamma} \tag{12}$$

The mean free path may be expressed in a similar form (see Eq. 2.4, 17),

$$\frac{L}{L_1} = \frac{\rho_1}{\rho} \tag{13}$$

or, in terms of the speed of sound,

$$\frac{L}{L_1} = \left(\frac{a_1}{a}\right)^{2/(\gamma-1)} \tag{14}$$

The basic quantity on which the isentropic flow of a gas depends can be obtained from the dimensionless forms of the equations of motion. Let us express all variables in terms of the known values at a point in the flow as

$$\begin{aligned} &\bar{u} = \bar{q}_1 u', \quad \bar{v} = \bar{q}_1 v', \quad \bar{w} = \bar{q}_1 w' \\ &x = L_1 x', \quad y = L_1 y', \quad z = L_1 z', \quad t = t_1 t' \\ &p = p_1 p', \quad \rho = \rho_1 \rho' \end{aligned} \tag{15}$$

where the known quantities are indicated by the subscript 1, and all primed quantities are dimensionless. Note that, by definition,

$$\bar{u} = \frac{\overline{dx}}{dt}, \quad \bar{v} = \frac{\overline{dy}}{dt}, \quad \bar{w} = \frac{\overline{dz}}{dt} \tag{16}$$

and we can write

$$u' = \frac{\overline{dx'}}{dt'}, \quad v' = \frac{\overline{dy'}}{dt'}, \quad w' = \frac{\overline{dz'}}{dt'} \tag{17}$$

only if $t_1 = L_1/\bar{q}_1$. The equations of motion now become

$$\begin{gathered} \frac{\partial \rho'}{\partial t'} + \frac{\partial}{\partial x'}(\rho' u') + \frac{\partial}{\partial y'}(\rho' v') + \frac{\partial}{\partial z'}(\rho' w') = 0 \\ \frac{du'}{dt'} = -\frac{1}{M_1^2}\rho'^{(\gamma-2)}\frac{\partial \rho'}{\partial x'}, \cdots, p' = \rho'^{(\gamma)} \end{gathered} \tag{18}$$

These dimensionless equations show that the isentropic motion of two gases will be dynamically similar if γ and the Mach number

$$M_1 = \frac{\bar{q}_1}{a_1} \tag{19}$$

are the same for each flow. The basic requirement for dynamic similarity between the motions of two monatomic gases is that the ratios of mass energy to mean thermal energy at two corresponding points must be the same (see Section 1.4).

2.6 SPECULAR REFLECTION FROM A SOLID BOUNDARY

Problems in fluid mechanics usually involve the motion of gases near the surface of a solid body. The influence of the surface on the gas flow is expressed in the boundary conditions which must be satisfied by the solution of the basic equations of motion. Let us consider a form of molecular interaction at a wall which is consistent with isentropic flow, called specular reflection, in which the molecules reflect in a perfectly elastic manner from a smooth surface (Ref. 1.4, p. 285).

A specular encounter produces no change in the components of velocity of a molecule which are tangential to the surface, and the normal velocity component is reversed in direction but unchanged in magnitude. Since there is no normal component of mass flow relative to the body at the surface, the components of velocity of an incident molecule are $\bar{u} + U$, $-|V|$, $\bar{w} + W$, and after reflection they become $\bar{u} + U$, $+|V|$, $\bar{w} + W$, where the frame of reference is taken to be fixed relative to the wall (see Fig. 3). Since Maxwell's distribution function is invariant with respect to changes of sign of U, V, and W (see Eq. 2.1, 12), then, if the

random motion of the incident molecules is Maxwellian, the U, V, W-components of the reflected molecules will also be distributed according to Maxwell's law.

The normal force on the wall can be determined by calculating the rate of transport by the molecules of the y-component of momentum to and from the surface. The number of molecules in unit volume carrying momentum mv is $nf\,d\omega$. The amount of momentum transferred by these molecules to unit area of the solid boundary is the number contained in a volume based on unit area of the wall having a height v.

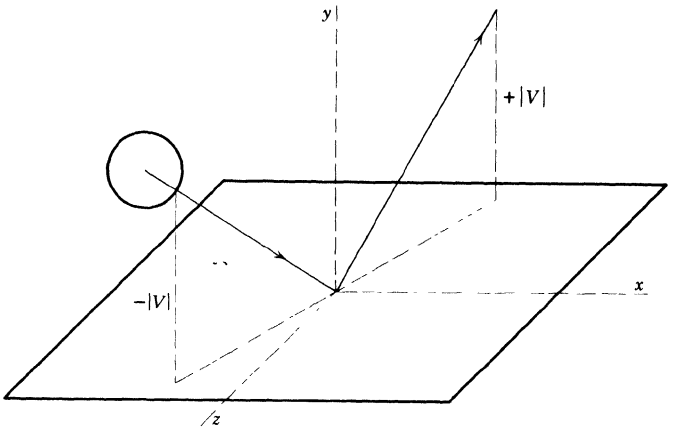


Fig 3. Specular reflection — a boundary condition for isentropic flow

The total amount of y-momentum transported by the molecules of all classes to unit area of surface in unit time is

$$mn\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v^2 f\,du\,dv\,dw \tag{1}$$

A similar expression can be obtained for the transfer of y-momentum away from the wall by reflection. Then the normal pressure acting on the surface is

$$P_{yy} = mn\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v^2 f\,du\,dv\,dw + mn\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} v^2 f\,du\,dv\,dw \tag{2}$$

Since $\bar{v} = 0$ at the surface, this equation reduces to

$$P_{yy} = mn\overline{V^2} = \tfrac{1}{3}\rho\overline{C^2} \tag{3}$$

(see Eq. 2.2, 1). Therefore, when the molecules reflect specularly from a smooth wall, the normal pressure on the surface is the same as the static pressure of the gas.

A consideration of the transport of mu and mw to and from the wall yields the results

$$P_{yx} = mn(\bar{u}\bar{V} + \overline{UV}) = 0 \tag{4}$$

$$P_{yz} = mn(\bar{w}\bar{V} + \overline{VW}) = 0 \tag{5}$$

Therefore, there are no shearing stresses, and the wall supports only normal pressure.

Since the incident and reflected random motion of the molecules is Maxwellian, then such mean quantities as $\overline{UC^2}$ are zero, and no heat conduction can occur (see Eq. 1.9, 10). We conclude that specular reflection does not permit any adjustment of the mass velocity and temperature of the gas toward those of the wall. The motion adjacent to the solid boundary is isentropic; it involves perfect slip flow and no accommodation between the temperatures of the gas and its boundary.

2.7 THE EXPANSION WAVE IN A ONE-DIMENSIONAL, UNSTEADY FLOW

When the disturbance propagated through a gas is not small, Eqs. 2.5, 3 must be retained in their exact form. From a study of these equations, we can deduce the properties of an important isentropic flow—the non-stationary expansion wave of finite amplitude in one dimension (Ref. 4, p. 99).

It is convenient to choose the macroscopic velocity $\bar{u}$ and the local speed of sound a as the two dependent variables. Since

$$\frac{1}{\rho}\frac{d\rho}{da} = \frac{2}{(\gamma - 1)a} \tag{1}$$

then Eqs. 2.5, 3 become

$$\begin{aligned} a_t + \left(\frac{\gamma - 1}{2}\right) a\bar{u}_x + \bar{u}a_x &= 0 \\ \bar{u}_t + \bar{u}\bar{u}_x + \left(\frac{2}{\gamma - 1}\right) aa_x &= 0 \end{aligned} \tag{2}$$

According to the method of characteristics, described in detail in Appendix II, the corresponding singularity equations are

$$\frac{dx}{dt} = \bar{u} + a, \qquad \frac{dx}{dt} = \bar{u} - a \tag{3}$$

and the associated regularity conditions are

$$\bar{u} + \frac{2a}{\gamma - 1} = \text{constant} \quad \text{when} \quad \frac{dx}{dt} = \bar{u} + a \tag{4}$$

and

$$\bar{u} - \frac{2a}{\gamma - 1} = \text{constant} \quad \text{when} \quad \frac{dx}{dt} = \bar{u} - a \tag{5}$$

The boundary conditions for the flow in an infinitely long tube produced by a piston which recedes from the gas under investigation are as follows:

(*a*) When $t = 0$, $\bar{u} = \bar{u}_1$, and $a = a_1$ for all $x \leq 0$.

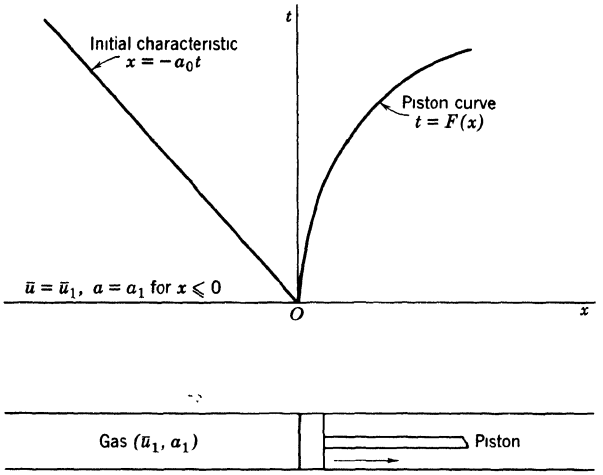


Fig. 4. Initial flow conditions in a tube.

(*b*) The motion of the piston is prescribed ($t = F(x)$) and is represented by a curve on a time-position diagram (see Fig. 4). Then the characteristic equations are

$$\bar{u} + \frac{2a}{\gamma - 1} = \bar{u}_1 + \frac{2a_1}{\gamma - 1} \quad \text{when} \quad \frac{dx}{dt} = \bar{u} + a \tag{6}$$

and

$$\bar{u} - \frac{2a}{\gamma - 1} = \bar{u}_1 - \frac{2a_1}{\gamma - 1} \quad \text{when} \quad \frac{dx}{dt} = \bar{u} - a \tag{7}$$

Instead of determining a solution of Eqs. 2 to satisfy the above boundary conditions in a direct manner, we shall employ the method of characteristics. Our object will be to investigate the characteristic lines associated with the given equations, and deduce the solution of the problem from their properties.

Some understanding of the nature of the flow can be gained from physical reasoning. If the gas is initially at rest in the tube ($a_1 \to a_0$), the first infinitesimal displacement of the piston must propagate a small disturbance through the gas at the speed of sound of the undisturbed

gas (a_0). This disturbance will therefore proceed along the characteristic line $x = -a_0 t$ which emanates from the origin of coordinates (see Fig. 4). Note also that the mass velocity $\bar{u}$ on the face of the piston must be the same as the piston speed.

The general properties of the characteristic lines now require investigation. The solution of the differential equation $dx/dt = \bar{u} - a$ is, in general, a family of curves in the (x, t) plane, each curve being specified by a particular value of the constant of integration. These curves will be designated the C_- characteristics. Similarly, the solution of the differential equation $dx/dt = \bar{u} + a$ leads to a family of C_+ characteristics.

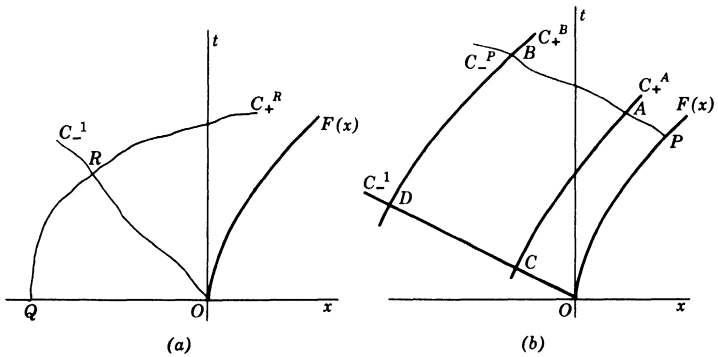


Fig. 5. Properties of characteristics in unsteady flow.

Let C_-^1 be the C_- characteristic through the origin in Fig. 5 (*a*). Let R be a point on C_-^1. Draw the C_+ curve through R, and let it cut the Ox axis at Q. Along C_-^1:

$$\bar{u}_R - \frac{2a_R}{\gamma - 1} = \bar{u}_1 - \frac{2a_1}{\gamma - 1} \tag{8}$$

where the right-hand side is evaluated at the origin. Along C_+^R:

$$\bar{u}_R + \frac{2a_R}{\gamma - 1} = \bar{u}_Q + \frac{2a_Q}{\gamma - 1} \tag{9}$$

But, from the boundary conditions, $\bar{u}_Q = \bar{u}_1$, $a_Q = a_1$ (Fig. 4). Therefore, $\bar{u}_R = \bar{u}_1$, and $a_R = a_1$. Since R is any point on C_-^1, then the characteristic C_-^1 is a straight line through the origin at all points of which $\bar{u} = \bar{u}_1$, and $a = a_1$.

We now turn our attention to the general C_- characteristic through any point P on the piston curve in Fig. 5 (*b*). Draw the C_+ characteristics through any two points A, B on C_-^P, and let these intersect C^1 at C, D. Then along C_+^A:

$$\bar{u}_A + \frac{2a_A}{\gamma - 1} = \bar{u}_1 + \frac{2a_1}{\gamma - 1} \tag{10}$$

Along C_+^B:

$$\bar{u}_B + \frac{2a_B}{\gamma - 1} = \bar{u}_1 + \frac{2a_1}{\gamma - 1} \tag{11}$$

where $\bar{u}_C = \bar{u}_D = \bar{u}_1$ and $a_C = a_D = a_1$. Hence

$$\bar{u}_A + \frac{2a_A}{\gamma - 1} = \bar{u}_B + \frac{2a_B}{\gamma - 1} \tag{12}$$

But along C_-^P,

$$\bar{u}_A - \frac{2a_A}{\gamma - 1} = \bar{u}_B - \frac{2a_B}{\gamma - 1} \tag{13}$$

From these two equations, $\bar{u}_A = \bar{u}_B$, and $a_A = a_B$. But A and B are any two points on C_-^P. Therefore, $\bar{u}$ and a are constant along C^P and are equal to their respective values on the piston curve ($\bar{u}_P$, a_P). Thus each C_- is a straight line having the equation

$$x - x_P = (\bar{u}_P - a_P)(t - t_P) \tag{14}$$

The corresponding characteristics in the ($\bar{u}$, a) plane are of particular interest. Each C_- characteristic in the (x, t) plane will transform to a point in the ($\bar{u}$, a) plane since $\bar{u}$ and a are constant along each C_-. Furthermore, each C_+ characteristic intersects C_-^1 and has the equation

$$\bar{u} + \frac{2a}{\gamma - 1} = \bar{u}_1 + \frac{2a_1}{\gamma - 1} \tag{15}$$

Then the C_+ lines all transform to the same straight line in the ($\bar{u}$, a) plane. But the C_- and C_+ characteristics are two families of intersecting lines in the physical plane, and the points representing the C_- lines must lie on the straight line given by Eq. 15. Therefore, the whole flow in the (x, t) plane transforms to a single straight line in the ($\bar{u}$, a) plane. For all isentropic, nonstationary flows in one dimension (see Eqs. 15 and 2.5, 12)

$$\frac{a}{a_1} = \left(\frac{T}{T_1}\right)^{1/2} = \left(\frac{\rho}{\rho_1}\right)^{(\gamma-1)/2} \left(\frac{p}{p_1}\right)^{(\gamma-1)/2\gamma} = \frac{1 + \dfrac{\gamma - 1}{2} M_1}{1 + \dfrac{\gamma - 1}{2} M} \tag{16}$$

where the ratios $M = \bar{u}/a$ and $M_1 = \bar{u}_1/a_1$ are the local and reference Mach numbers, respectively.

With the assistance of the C_- characteristics it is possible to obtain at least a graphical solution for any given piston curve. Completely analytical solutions can be determined in only a few cases, such as the centered expansion wave (Fig. 6). In this case all C_- lines meet at a single point (x_0, t_0), and for each characteristic

$$\frac{dx}{dt} = \frac{x - x_0}{t - t_0} = \bar{u} - a \tag{17}$$

If the gas is initially at rest, then $\bar{u}_1 = 0$, $a_1 = a_0$, and Eq. 15 becomes

$$\bar{u} + \frac{2a}{\gamma - 1} = \frac{2a_0}{\gamma - 1} \tag{18}$$

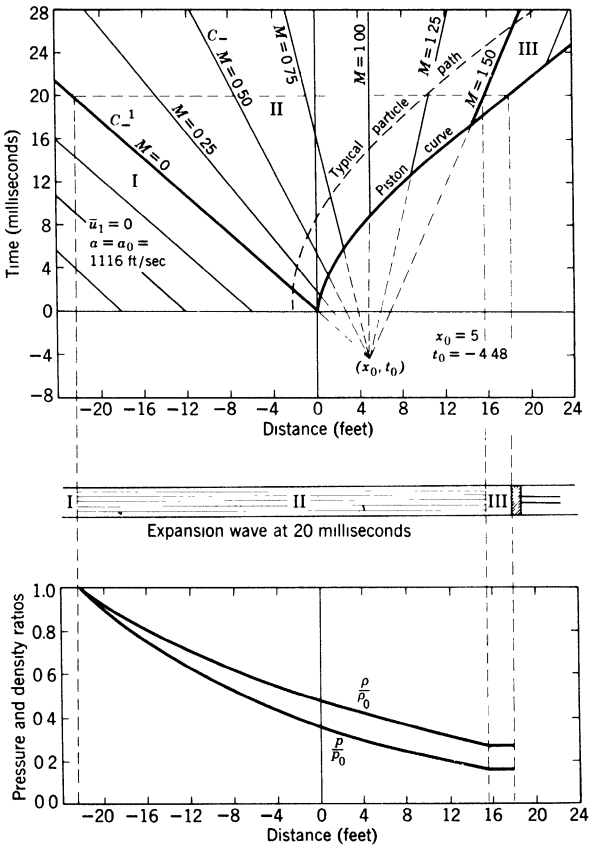


Fig. 6. Centered expansion wave in unsteady flow.

The mass velocity $\bar{u}$ and speed of sound a are readily obtained as functions of x and t,

$$\bar{u} = \frac{2}{\gamma + 1}\left(\frac{x - x_0}{t - t_0} + a_0\right) \tag{19}$$

$$a = \frac{2a_0}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1}\left(\frac{x - x_0}{t - t_0}\right) \tag{20}$$

The particle paths for the centered expansion wave may be obtained from the differential equation

$$\bar{u} = \frac{dx}{dt} = \frac{2}{\gamma + 1}\left(\frac{x - x_0}{t - t_0} + a_0\right) \tag{21}$$

This is an ordinary linear differential equation of the first order in the variables $(x - x_0)$ and $(t - t_0)$ and has the solution

$$x - x_0 = (t - t_0)\left[\kappa_1(t - t_0)^{-(\gamma-1)/(\gamma+1)} + \frac{2a_0}{\gamma - 1}\right] \tag{22}$$

where κ_1 is a constant of integration. The piston curve will be the particle path through the origin. In this case the constant of integration is

$$\kappa_1 = (-t_0)^{(\gamma-1)/(\gamma+1)}\left(\frac{x_0}{t_0} - \frac{2a_0}{\gamma - 1}\right) \tag{23}$$

where t_0 is zero or negative (see Fig. 6). The value of κ_1 for particle paths emerging from any other point on the C_-^1 characteristic can be obtained by substituting the coordinates of the point in Eq. 22.

The differential equation for the C_+ characteristics in the centered expansion wave is

$$\frac{dx}{dt} = \bar{u} + a = \frac{3 - \gamma}{\gamma + 1}\left(\frac{x - x_0}{t - t_0}\right) + \frac{4a_0}{\gamma + 1} \tag{24}$$

This is similar to Eq. 21 and has the solution

$$x - x_0 = (t - t_0)\left[\kappa_2(t - t_0)^{-2[(\gamma-1)/(\gamma+1)]} + \frac{2a_0}{\gamma - 1}\right] \tag{25}$$

where κ_2 is a constant of integration.

The general isentropic relations for the centered expansion wave are (see Eqs. 18 and 2.5, 12)

$$\frac{a}{a_0} = \left(\frac{T}{T_0}\right)^{1/2} = \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} = \left(\frac{p}{p_0}\right)^{(\gamma-1)/2\gamma} = \left[1 + \frac{\gamma - 1}{2}M\right]^{-1} \tag{26}$$

where

$$M = \frac{\bar{u}}{a} = \frac{a_0 + \dfrac{x - x_0}{t - t_0}}{a_0 - \left(\dfrac{\gamma - 1}{2}\right)\left[\dfrac{x - x_0}{t - t_0}\right]} \tag{27}$$

Figure 6 shows the C_- characteristics of a typical centered expansion wave and the corresponding variation of pressure and density. In the example shown, the piston accelerates from rest to a constant speed. Thus the expansion wave (region II) separates two regions of constant state (I, III). Note that, consistent with the properties of the characteristic lines, the derivatives of p and ρ with respect to x are discontinuous at the front and rear of the wave. In general, the expansion wave is a process by which the initial random energy of the gas molecules is converted to the ordered energy of mass motion.

An expansion wave can be readily generated in a shock tube (Ref. 5). In this tube the piston is usually replaced by a thin diaphragm which separates two compartments at different pressures. The sudden rupture of the diaphragm produces an expansion wave which spreads into the region of high pressure. The complete wave system is illustrated in Fig. 7, which shows a photograph of the (x, t) plane, obtained by means of a camera with the film mounted on a rotating drum.

The line extending to the left from the diaphragm position (see photograph, Fig. 7) is the C_-^1 characteristic, the slope of which is the speed of sound (a_0) in the stationary gas at the left. Measurements made on this line at room temperature and reduced to conditions at 0° C gave the following results (Ref. 6):

Air:	$a_0 = 1087 \pm 2$ ft/sec
Argon:	$a_0 = 1009 \pm 1$ ft/sec
Carbon dioxide:	$a_0 = 844 \pm 1$ ft/sec

These values were found to be in good agreement with the theoretical sound speed, $a_0 = \sqrt{\gamma R T_0}$.

According to Eqs. 26 and 2.5, 14, the mean free path at any point in the expansion wave is

$$\frac{L}{L_0} = \left[1 + \frac{\gamma - 1}{2} M\right]^{2/(\gamma-1)} \tag{28}$$

where L_0 is the known mean free path in region I (Fig. 6). Equations 27 and 28 show that M and L are largest in region III. As the final piston speed corresponding to this region is increased, the magnitude of L ultimately becomes comparable with the diameter of the tube. Under these conditions the frequency of the collisions between gas molecules is small compared with the number of direct encounters with the walls of the tube. In this kind of flow, momentum and energy are transported by freely moving molecules with exchanges occurring only on contact with the wall, and the transfer equations of Section 1.9 are no longer applicable. This limiting case, called free-molecule flow, is discussed in Chapter 5.

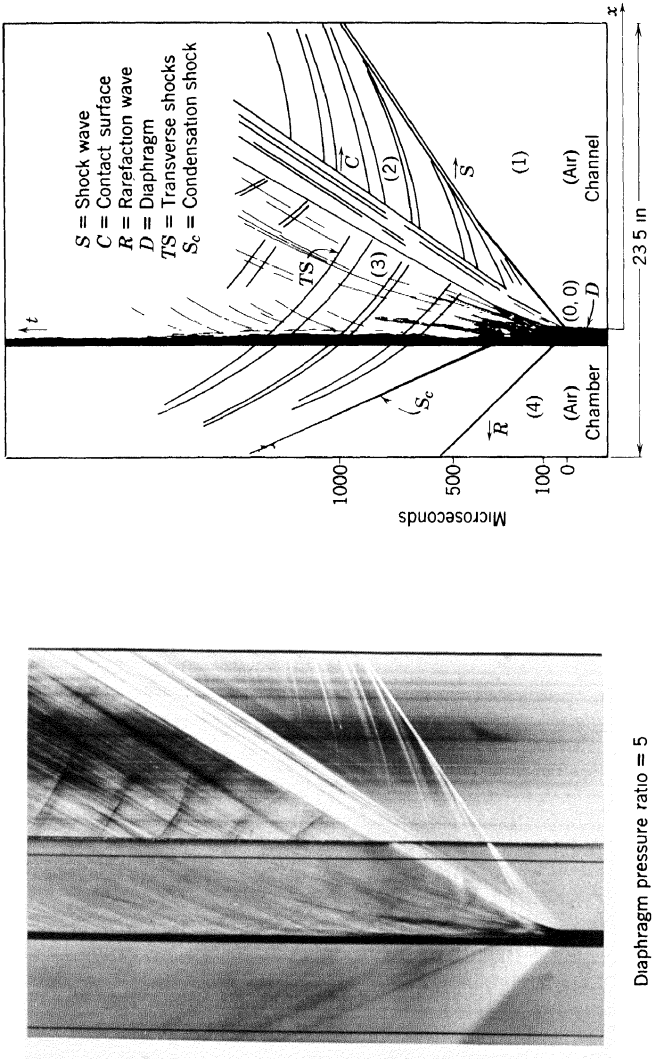


Fig. 7. Composite schlieren photograph and diagram of the waves in a shock tube, (x, t) plane.

The above treatment of the unsteady expansion wave has been idealized with respect to the influence of the walls of the tube and the face of the piston since the molecules are assumed to reflect specularly from these surfaces, resulting in an isentropic flow. As we shall see in Chapters 3 and 4, very little specular reflection actually occurs, and the flow is non-isentropic adjacent to the walls. However, the greater part of the motion in the interior of the gas is isentropic, and the solution given here represents a reasonably good first approximation.

2.8 THE EXPANSION WAVE IN A TWO-DIMENSIONAL, STEADY FLOW

The transfer equations for a steady, isentropic flow in two dimensions (see Eqs. 2.2, 10) are

$$\frac{\partial}{\partial x}(\rho\bar{u}) + \frac{\partial}{\partial y}(\rho\bar{v}) = 0 \tag{1}$$

$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial p}{\partial x}, \quad \rho\frac{d\bar{v}}{dt} = -\frac{\partial p}{\partial y} \tag{2}$$

$$p = B\rho^{\gamma} \tag{3}$$

where $d/dt \equiv \bar{u}(\partial/\partial x) + \bar{v}(\partial/\partial y)$. If we substitute for p from Eq. 3, the momentum-transfer equations become

$$\frac{d\bar{u}}{dt} = -F(\rho)\frac{\partial\rho}{\partial x}, \quad \frac{d\bar{v}}{dt} = -F(\rho)\frac{\partial\rho}{\partial y} \tag{4}$$

where

$$F(\rho) = \gamma B\rho^{\gamma-2} = \frac{a^2}{\rho} \tag{5}$$

Multiplying the left-hand sides of Eqs. 4 by $\bar{u}$ and $\bar{v}$ and the right-hand sides by the equivalent quantities dx/dt and dy/dt, respectively, and adding, we find that

$$\frac{1}{2}\frac{d}{dt}(\bar{u}^2 + \bar{v}^2) + F(\rho)\frac{d\rho}{dt} = 0 \tag{6}$$

or, in integrated form,

$$\frac{1}{2}\bar{q}^2 + \frac{a^2}{\gamma - 1} = \frac{1}{2}\bar{q}_1^{\,2} + \frac{a_1^{\,2}}{\gamma - 1} \tag{7}$$

where $\bar{q}_1$ and a_1 are the known values of the mass velocity and the speed of sound at a given point in the flow. Then, for all steady, isentropic flows (see Eqs. 7 and 2.5, 12),

$$\frac{a}{a_1} = \left(\frac{T}{T_1}\right)^{1/2} = \left(\frac{\rho}{\rho_1}\right)^{(\gamma-1)/2} = \left(\frac{p}{p_1}\right)^{(\gamma-1)/2\gamma} = \left[\frac{1 + \dfrac{\gamma-1}{2}M_1^{\,2}}{1 + \dfrac{\gamma-1}{2}M^2}\right]^{1/2} \tag{8}$$

where the local and reference Mach numbers are $M = \bar{q}/a$, $M_1 = \bar{q}_1/a_1$, respectively.

Equation 7 is sometimes called an energy equation. In the kinetic theory of isentropic flow, Eq. 2.2, 6 is the energy-transfer equation, and this relation, subject to the conservation of mass and the introduction of thermodynamic variables, becomes Eq. 3 above. Equations 7 and 2.7, 15 are solutions of the basic differential equations, and they express two different laws by which random molecular motion is converted to ordered mass flow during a steady and an unsteady expansion, respectively. From the point of view of the kinetic theory, Eq. 3 is a reduced form of the energy equation which is the same for both stationary and non-stationary, isentropic flow.

For convenience in solving steady-flow problems in two dimensions, the transfer equations may be expressed in terms of the velocities $\bar{u}$, $\bar{v}$, and a. From Eq. 1

$$\frac{d\rho}{dt} = -\rho(\bar{u}_x + \bar{v}_y) \tag{9}$$

If we substitute in Eq. 6, the result is

$$(\bar{u}^2 - a^2)\bar{u}_x + \bar{u}\bar{v}(\bar{u}_y + \bar{v}_x) + (\bar{v}^2 - a^2)\bar{v}_y = 0 \tag{10}$$

where a is expressed in terms of $\bar{u}$, $\bar{v}$ by Eq. 7.

To derive a second equation, we note that (Eqs. 4)

$$\frac{\partial}{\partial y}\left(\frac{d\bar{u}}{dt}\right) = \frac{\partial}{\partial x}\left(\frac{d\bar{v}}{dt}\right) \tag{11}$$

or

$$\frac{d}{dt}(\bar{u}_y - \bar{v}_x) = 0 \tag{12}$$

In many problems, the flow starts from rest or it is uniform in the undisturbed state. Under these circumstances we may take

$$\bar{u}_y - \bar{v}_x = 0 \tag{13}$$

throughout the flow.

Equations 10 and 13 are of the type treated by the method of characteristics in Appendix II. We shall therefore investigate the characteristics of these equations, and deduce from them the properties of an expansion wave in steady flow (Ref. 4, p. 273). The singularity equation is

$$\frac{dy}{dx} = \frac{\bar{u}\bar{v} \pm a^2\sqrt{M^2 - 1}}{\bar{u}^2 - a^2} \tag{14}$$

and the regularity conditions are

$$\frac{d\bar{v}}{d\bar{u}} = -\frac{\bar{u}^2 - a^2}{\bar{u}\bar{v} - a^2\sqrt{M^2 - 1}} \quad \text{when} \quad \frac{dy}{dx} = \frac{\bar{u}\bar{v} + a^2\sqrt{M^2 - 1}}{\bar{u}^2 - a^2} \tag{15}$$

and $$\frac{d\bar{v}}{d\bar{u}} = -\frac{\bar{u}^2 - a^2}{\bar{u}\bar{v} + a^2\sqrt{M^2 - 1}} \quad \text{when} \quad \frac{dy}{dx} = \frac{\bar{u}\bar{v} - a^2\sqrt{M^2 - 1}}{\bar{u}^2 - a^2} \tag{16}$$

These relations show that the characteristics can exist only if the flow is supersonic ($M > 1$).

Some properties of the characteristics can be deduced immediately from the above equations. Let us draw the C_+ characteristic and the velocity vector through any point P in the flow (Fig. 8). Then, if we refer to the tangent to C_+ through P,

$$\tan \alpha = \frac{\bar{u}\bar{v} + a^2\sqrt{M^2 - 1}}{\bar{u}^2 - a^2} \tag{17}$$

$$\tan \epsilon = \frac{\bar{v}}{\bar{u}} \tag{18}$$

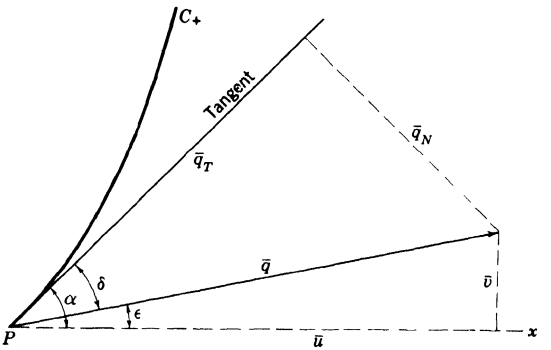


Fig. 8. Properties of the C_+ characteristic in steady flow.

The components of $\bar{q}$ normal and parallel to the direction of C_+ at P are

$$\bar{q}_N = \bar{q} \sin (\alpha - \epsilon) = \bar{q}[\sin \alpha \cos \epsilon - \cos \alpha \sin \epsilon] = a \tag{19}$$

and $$\bar{q}_T = a\sqrt{M^2 - 1} \tag{20}$$

A similar calculation for the C_- characteristic shows that the normal and tangential components of $\bar{q}$ are, respectively, $-a$ and $a\sqrt{M^2 - 1}$. We conclude that at any point in the flow the velocity vector bisects the angle between the tangents to the C_+ and C_- characteristics. Furthermore, the velocity of flow normal to the characteristic lines is the local speed of sound. Therefore, the characteristics represent infinitesimal disturbances of the mass flow moving with the speed of sound.

Useful alternative expressions for the C_+ and C_- characteristics may be determined from these results. Since $\tan \alpha = \tan (\delta + \epsilon)$ (see Fig. 8), then for C_+:

$$\frac{dy}{dx} = \frac{\bar{v}\sqrt{M^2 - 1} + \bar{u}}{\bar{u}\sqrt{M^2 - 1} - \bar{v}} \tag{21}$$

and for C_-:

$$\frac{dy}{dx} = \frac{\bar{v}\sqrt{M^2 - 1} - \bar{u}}{\bar{u}\sqrt{M^2 - 1} + \bar{v}} \tag{22}$$

The characteristic line (H_+) in the $(\bar{u}, \bar{v})$ or hodograph plane, corresponding to C_+, has the differential equation

$$\frac{d\bar{v}}{d\bar{u}} = -\frac{1}{(dy/dx)_{C_-}} = -\frac{\bar{u}\sqrt{M^2 - 1} + \bar{v}}{\bar{v}\sqrt{M^2 - 1} - \bar{u}} \tag{23}$$

This equation does not contain x, y, and can be integrated. To facilitate the integration, let us introduce the polar coordinates $(\bar{q}, \epsilon)$ in the hodograph plane. Then

$$\bar{q}^2 = \bar{u}^2 + \bar{v}^2, \quad \tan \epsilon = \frac{\bar{v}}{\bar{u}} \tag{24}$$

or, in differential form,

$$\bar{q}\, d\bar{q} = \bar{u}\, d\bar{u} + \bar{v}\, d\bar{v}, \quad \bar{q}^2\, d\epsilon = -\bar{u}\, d\bar{v} + \bar{v}\, d\bar{u} \tag{25}$$

But from Eq. 23

$$\sqrt{M^2 - 1}(\bar{u}\, d\bar{u} + \bar{v}\, d\bar{v}) = \bar{u}\, d\bar{v} - \bar{v}\, d\bar{u} \tag{26}$$

Then the differential equation for H_+ has the polar form

$$d\epsilon = \sqrt{M^2 - 1}\,\frac{d\bar{q}}{\bar{q}} \tag{27}$$

Similarly, for H_-,

$$d\epsilon = -\sqrt{M^2 - 1}\,\frac{d\bar{q}}{\bar{q}} \tag{28}$$

The velocity of sound can be eliminated with the assistance of Eq. 7. Let us write

$$\frac{1}{2}\bar{q}_1^2 + \frac{a_1^2}{\gamma - 1} = \frac{a_0^2}{\gamma - 1} \tag{29}$$

where a_0 is the velocity of sound corresponding to no mass flow. Then the differential equations for the H_+ and H_- characteristics are

$$d\epsilon = \pm \left[\frac{(\gamma + 1)\bar{q}^2 - 2a_0^2}{2a_0^2 - (\gamma - 1)\bar{q}^2}\right]^{1/2} \frac{d\bar{q}}{\bar{q}} \tag{30}$$

which have the integrated forms

$$\epsilon - \epsilon_1 = \left[\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left\{\frac{\gamma+1}{\gamma-1}\left(\frac{2a_0^2-(\gamma-1)\bar{q}^2}{(\gamma+1)\bar{q}^2-2a_0^2}\right)\right\}^{1/2} - \tan^{-1}\left\{\frac{2a_0^2-(\gamma-1)\bar{q}^2}{(\gamma+1)\bar{q}^2-2a_0^2}\right\}^{1/2}\right]_{\bar{q}_1}^{\bar{q}} \tag{31}$$

The H_+ and H_- characteristics are therefore reflections of each other about the $\bar{u}$-axis (Fig. 9). Since the flow must be supersonic, the minimum value of $\bar{q}$ will be the speed of sound.

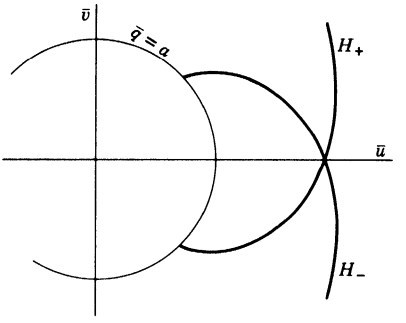


Fig. 9. H_+ and H_- characteristics — steady flow.

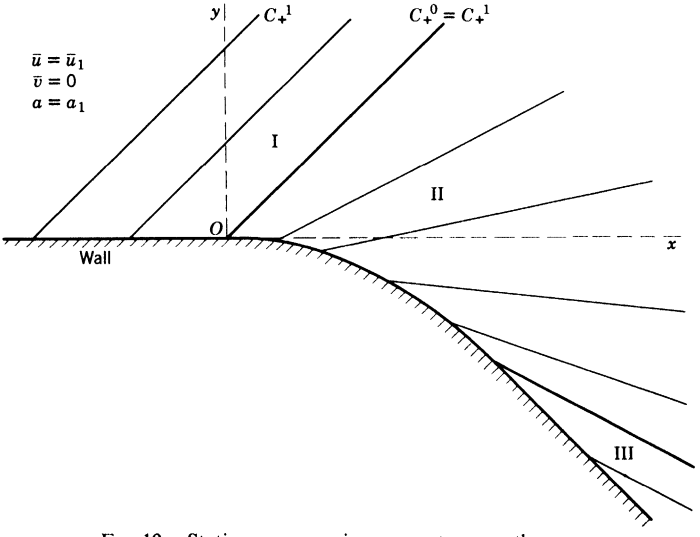
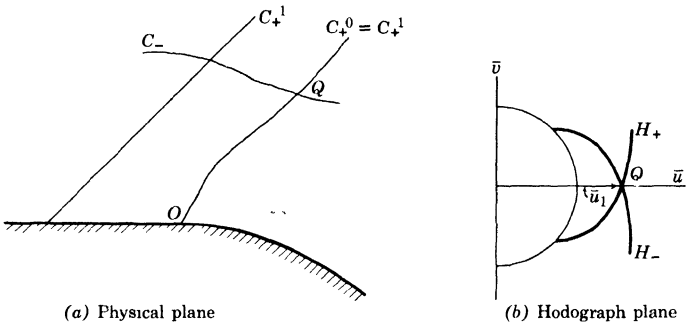


Fig. 10. Stationary expansion wave at a smooth corner.

Let us investigate the properties of the characteristics in a steady, two-dimensional flow around a bend. The wall is represented by the Ox axis for $x \leq 0$, and for $x \geq 0$ it curves away from the gas as shown in Fig. 10. At infinitely large negative values of x, the flow is uniform and parallel to Ox, and no disturbance of the flow occurs on the wall until it reaches the point O. In this region the C_+ characteristics are given by (see Eq. 21, $\bar{v}_1 = 0$)

$$\frac{dy}{dx} = \frac{1}{\sqrt{M_1^2 - 1}} \tag{32}$$

Thus the C_+ characteristics in region I are straight parallel lines which represent infinitesimal disturbances propagated at the speed of sound.



(a) Physical plane *(b)* Hodograph plane

Fig 11. Properties of the C_+^0 characteristic — steady flow.

It can be shown that region I extends as far to the right (Fig. 10) as the characteristic (C_+^1) through the origin O. Consider any C_- extending from region I and intersecting the C_+ through O at the point Q (Fig. 11). These two characteristics are represented in the hodograph plane by H_+ and H_- for which

$$\begin{aligned} H_+ &: F(\bar{u}_Q, \bar{v}_Q) - F(\bar{u}_0, \bar{v}_0) = F(\bar{u}_1, \bar{v}_1) \\ H_- &: G(\bar{u}_Q, \bar{v}_Q) = G(\bar{u}_1, \bar{v}_1) \end{aligned} \tag{33}$$

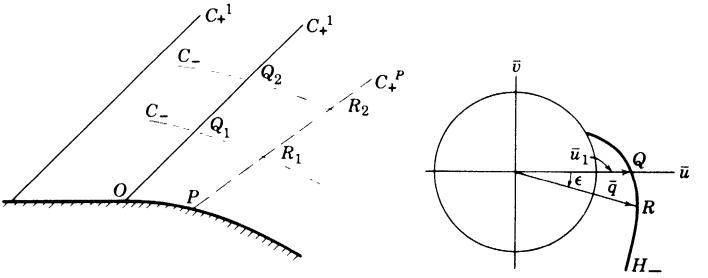
These relations show that the points $(\bar{u}_Q, \bar{v}_Q)$ and $(\bar{u}_1, \bar{v}_1)$ lie on both H_+ and H_-. Since there can be only one point of intersection of the H_+ and H_- curves, then

$$\bar{u}_Q = \bar{u}_1, \quad \bar{v}_Q = \bar{v}_1 \tag{34}$$

But Q is any point on the C_+ characteristic through O. Therefore, the characteristic through the origin (C_+^0) is the same as C_+^1, being a straight

line with a slope given by Eq. 32. It will be noted that the whole of region I in Fig. 10 is represented in the hodograph plane by the single point Q in Fig. 11 (b) at which $\bar{u} = \bar{u}_1$, $\bar{v} = 0$. Furthermore, all C characteristics emanating from region I are represented on the hodograph diagram by a single H_- curve which passes through Q.

In the case of the general characteristic through any point P on the curved portion of the wall (C_+^P, Fig. 12), two C characteristics, which intersect C_+^1 through the origin at points Q_1 and Q_2, are represented by a single H_- curve in the ($\bar{u}$, $\bar{v}$) plane, and the points Q_1, Q_2 transform



(a) Physical plane *(b)* Hodograph plane

Fig. 12. Properties of the general characteristic C_+^P — steady flow.

to the same point Q. A consideration of the equations for the H_- characteristics through the points Q_1, R_1 and Q_2, R_2 and the equation for H_+ through R_1, R_2 shows that the points R_1 and R_2 in the physical plane are represented by a single point R in the hodograph plane. Therefore,

$$\bar{u}_{R_1} = \bar{u}_{R_2}, \quad \bar{v}_{R_1} = \bar{v}_{R_2} \tag{35}$$

Since R_1 and R_2 are any two points on C_+^P, $\bar{u}$ and $\bar{v}$ have the same values along C_+^P, which must be a straight line (see Eq. 21) having a slope which can be determined from the values of $\bar{u}$, $\bar{v}$ on the wall ($\bar{u}_P$, $\bar{v}_P$). In the hodograph plane the C_+^P becomes the point R. We may conclude, therefore, that the whole physical flow in the (x, y) plane is represented by the H_- characteristic (Eq. 31) through the point $Q(\bar{u}_1, \bar{v}_1)$ in the ($\bar{u}$, $\bar{v}$) plane.

The details of the flow around a corner can be determined with the assistance of the characteristics. In many problems a graphical method must be used since analytical expressions for $\bar{u} = \bar{u}(x, y)$, $\bar{v} = \bar{v}(x, y)$ are obtainable in only a few special cases. The centered expansion wave in two-dimensional, steady flow, in which the C_+ characteristics radiate from a point, can be treated analytically.

The hodograph equation (31) may be placed in a more useful mathematical form by introducing the properties of the epicycloid. The differential equation for the hodograph characteristic H_- is

$$\frac{d\bar{v}}{d\bar{u}} - \frac{1}{(dy/dx)_{C_+}} = -\cot\alpha \tag{36}$$

On the other hand, if we refer to the notation on Fig. 13, the differential equation for an epicycloid is

$$\frac{d\bar{v}}{d\bar{u}} = -\cot\left[\frac{\pi}{2} - \frac{1}{2}(\eta - 2\zeta)\right] \tag{37}$$

Therefore

$$\alpha = \frac{\pi}{2} - \left[\left(\frac{r_l + 2r_s}{2r_l}\right)\eta - \Phi\right] \tag{38}$$

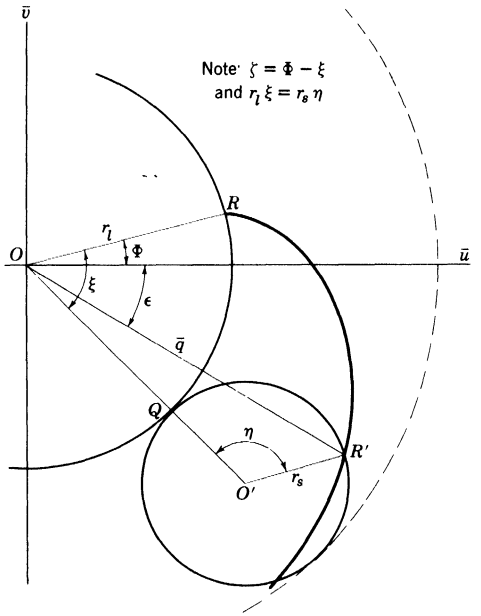


Fig. 13. The H_- characteristic — an epicycloid.

An application of the cosine law to the geometry of the epicycloid (Fig. 13) yields the following result

$$\frac{d\bar{q}}{\bar{q}\,d\epsilon} = -\frac{r_l}{r_l + 2r_s}\left[\frac{(r_l + 2r_s)^2 - \bar{q}^2}{\bar{q}^2 - r_l^2}\right]^{1/2} \tag{39}$$

A comparison of this equation with Eq. 30 shows that

$$r_l = a_0\sqrt{\frac{2}{\gamma + 1}}, \quad r_s = \frac{a_0(1 - b)}{\sqrt{2(\gamma + 1)}} \tag{40}$$

where

$$b^2 = \frac{\gamma - 1}{\gamma + 1} \tag{41}$$

The parametric equations for the hodograph curve (H_-) can now be found from the geometry of the epicycloid,

$$\begin{aligned} \bar{u} &= \frac{a_0}{\sqrt{2(\gamma - 1)}}[(1 + b)\cos\zeta - (1 - b)\cos(\eta - \zeta)] \\ \bar{v} &= \frac{a_0}{\sqrt{2(\gamma - 1)}}[(1 + b)\sin\zeta + (1 - b)\sin(\eta - \zeta)] \end{aligned} \tag{42}$$

where

$$\zeta = \Phi - (1 - b)\left(\frac{\pi}{2} + \Phi - \alpha\right) \tag{43}$$

and

$$\eta - \zeta = (1 + b)\left(\frac{\pi}{2} + \Phi - \alpha\right) - \Phi \tag{44}$$

The angle Φ is constant and depends only on the initial conditions. It can be determined from the relation

$$\Phi = \alpha_1 + \frac{\eta_1}{2b} - \frac{\pi}{2} \tag{45}$$

where

$$\tan\alpha_1 = \frac{\bar{u}_1\bar{v}_1 + a_1^2\sqrt{M_1^2 - 1}}{\bar{u}_1^2 - a_1^2} \tag{46}$$

and, from Fig. 13,

$$\cos\eta = \frac{(r_l + r_s)^2 + r_s^2 - \bar{q}_1^2}{2r_s(r_l + r_s)} = \gamma - \frac{\gamma^2 - 1}{2}\left(\frac{\bar{q}_1}{a_0}\right)^2 \tag{47}$$

An expansion wave in which the C_+ characteristics radiate from the origin of coordinates is shown in Fig. 14. According to Eqs. 19, 20, 29, and the relation (see Eq. 47)

$$\bar{q}^2 = \frac{2a_0^2}{\gamma^2 - 1}(\gamma - \cos\eta) \tag{48}$$

the polar components of velocity are

$$\bar{q}_N = a_0\sqrt{\frac{2}{\gamma + 1}}\cos\frac{1}{2}\eta, \quad \bar{q}_T = a_0\sqrt{\frac{2}{\gamma - 1}}\sin\frac{1}{2}\eta \tag{49}$$

where

$$\eta = 2b\left(\frac{\pi}{2} + \Phi - \alpha\right) \tag{50}$$

In accordance with the properties of the characteristics, the velocity components depend only on α.

The streamlines which specify the direction of the mass flow may be determined from the relation

$$\frac{dr}{r\,d\alpha} = \frac{\bar{q}_T}{\bar{q}_N} = -\frac{1}{b}\tan\frac{1}{2}\eta \tag{51}$$

which has the integrated form

$$r^{(\gamma-1)/(\gamma+1)}\cos\left[b\left(\frac{\pi}{2}+\Phi-\alpha\right)\right] = \text{constant} \tag{52}$$

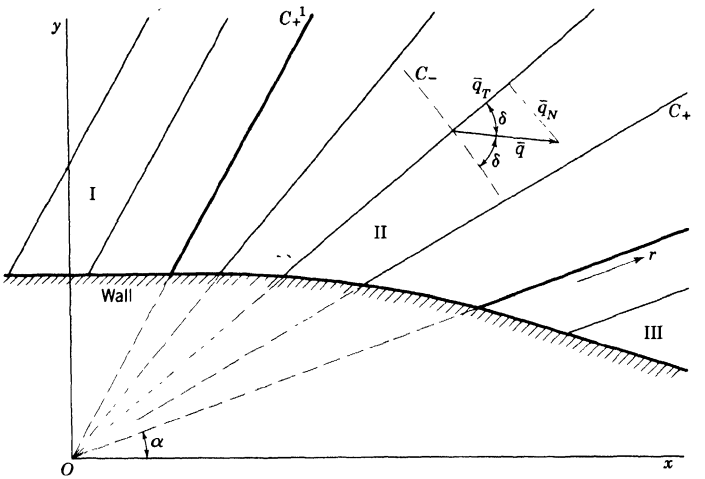


Fig. 14. Centered expansion wave in steady flow.

The constant can be evaluated in terms of the known values of (r, α) at a point on C_+^1 (Fig. 14). Any streamline may be selected as the wall around which the gas is flowing.

The C_- characteristics make an angle 2δ with the C_+ lines, and they are specified by the differential equation

$$\frac{dr}{r\,d\alpha} = \cot 2\delta = \frac{1}{2}\left[\frac{\bar{q}_N}{\bar{q}_T} - \frac{\bar{q}_T}{\bar{q}_N}\right] \tag{53}$$

If we substitute for $\bar{q}_N$, $\bar{q}_T$ from Eqs. 49 and integrate,

$$r^2 \sin\left[b\left(\frac{\pi}{2}+\Phi-\alpha\right)\right]\cos^{(\gamma+1)/(\gamma-1)}\left[b\left(\frac{\pi}{2}+\Phi-\alpha\right)\right] = \text{constant} \tag{54}$$

where the constant can be determined from the initial conditions.

The following expressions for the speed of sound and the Mach number may be deduced from Eqs. 29 and 48,

$$a = a_0 \left[\frac{1 + \cos \eta}{\gamma + 1}\right]^{1/2} \tag{55}$$

$$M = \left[\frac{2}{\gamma - 1}\left(\frac{\gamma + 1}{1 + \cos \eta} - 1\right)\right]^{1/2} \tag{56}$$

Equations 8 and 56 then give the distributions of pressure, density, and temperature throughout the flow. Typical variations of these quantities along the surface are presented in Fig. 15.

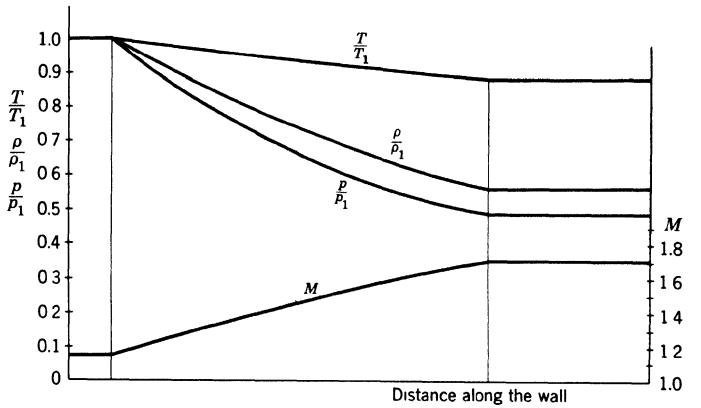


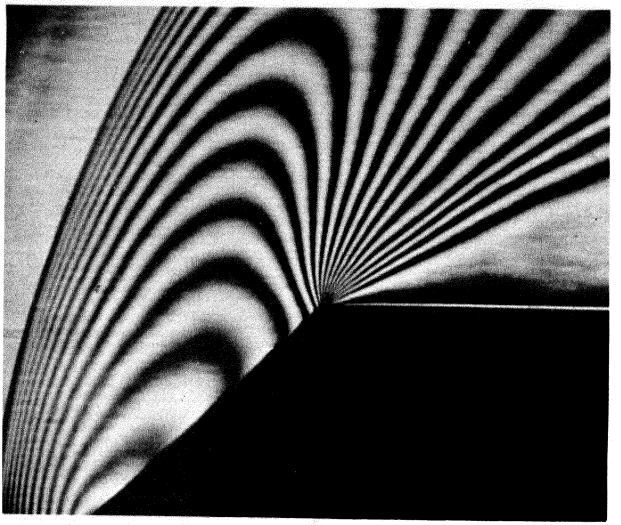
Fig. 15. Distribution curves for a centered expansion wave.

Photographs of the flow around expansion and compression bends obtained in a supersonic wind tunnel are shown in Fig. 16. As for the nonstationary wave, the theory assumes a nonaccommodating reflection of the molecules at the wall consistent with isentropic flow. The influence of a solid boundary will be reconsidered in Chapter 4.

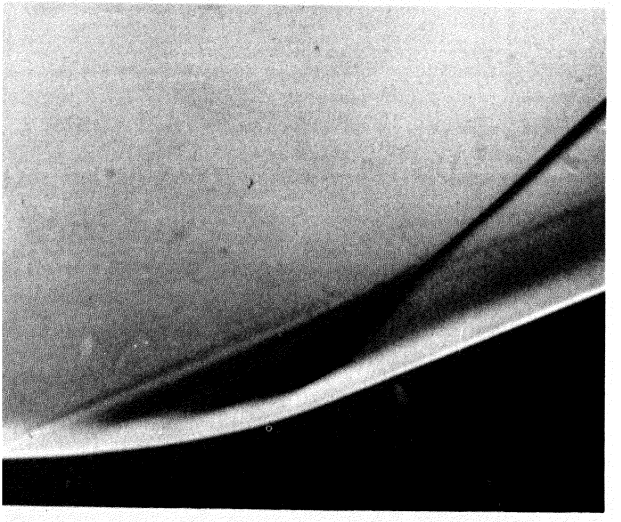
The mean free path at any point in the flow, expressed in terms of the known value in state I, Fig. 14, is given by the relation

$$\frac{L}{L_1} = \left[\frac{1 + \dfrac{\gamma - 1}{2} M^2}{1 + \dfrac{\gamma - 1}{2} M_1^2}\right]^{1/(\gamma - 1)} \tag{57}$$

As in the nonstationary wave, the mean free path increases as the expansion proceeds. Ultimately the collisions between gas molecules become unimportant compared with the direct encounters with the wall, and free-molecule flow is attained (see Chapter 5).



(*a*)



(*b*)

Fig. 16. (*a*) Centered expansion wave at a corner on a wedge (UTIA interferogram). (*b*) Centered compression wave at a bend (schlieren photograph, courtesy of the Aeronautical Research Institute of Sweden, Ref. 7).

NOTATION

a	speed of sound
A	coefficient in Maxwell's distribution function, a function of x, y, z, t
b	symbol for the quantity $\sqrt{(\gamma - 1)/(\gamma + 1)}$
B	constant in the energy-transfer equation (2.2, 10)
$\bar{C}$	mean value of the random molecular speeds in $d\tau$
C_m	most probable speed of a molecule in $d\tau$
c_p	specific heat at constant pressure for unit mass of gas
c_v	specific heat at constant volume for unit mass of gas
H	Boltzmann's function (Eq. 2.3, 1)
H_m	Boltzmann's function per unit mass of gas (H/mn)
k	gas constant per molecule ($k = mR$)
Kn	Knudsen's number, ratio of mean free path to characteristic body dimension
dK_1, dK_2	the elements $dX_1\,dY_1\,dZ_1$ and $dX_2\,dY_2\,dZ_2$, respectively
L	mean free path
M	Mach number, ratio of the mass velocity to the local speed of sound
N	number of degrees of freedom of a molecule
$\bar{q}_N$	magnitude of the component of mass velocity normal to a characteristic
$\bar{q}_T$	magnitude of the component of mass velocity parallel to a characteristic
r	distance to a point from the origin of coordinates
r_l	radius of the large circle in the epicycloid diagram (Fig. 13)
r_s	radius of the small circle in the epicycloid diagram (Fig. 13)
S	entropy per unit mass of gas
x_o, t_o	coordinates of the point of convergence of the characteristics in a centered expansion wave
X_1, Y_1, Z_1 X_2, Y_2, Z_2	variables of a transformation defined by Eqs. 2.4, 9
α	angle made by a characteristic with the x-axis
β	a function of x, y, z, t which occurs in Maxwell's distribution law, defined by Eq. 2.1, 11
γ	ratio of specific heats (c_p/c_v)
δ	angle between the mass velocity vector and a characteristic
ϵ	angle made by the mass velocity with the x-axis
ζ	the angle $\Phi - \xi$
η	angle subtended at the center of the small circle by the arc QR' in Fig. 13
θ	polar coordinate in the velocity space (C, φ, θ)
$\kappa, \kappa_1, \kappa_2$	constants of integration
Λ	symbol representing the sum $X_2^2 + Y_2^2 + Z_2^2$
ξ	angle subtended at the center of the large circle by the arc QR in Fig. 13
φ	polar coordinate in the velocity space (C, φ, θ)
Φ	angle defining the starting point of the epicycloid (Fig. 13)

Note: Symbols that do not appear above may be found in the Notation at the end of Chapter 1 (page 24). The subscript 1 applied to macroscopic functions of x, y, z, t (such as $L, M, \rho, \bar{u}, \cdots$) indicates a known reference value. The subscript 1 is changed to 0 when the reference mass velocity ($\bar{q}_1$) is zero. A dimensionless quantity is indicated by a prime (Eq. 2.5, 15). A perturbation quantity is denoted by a double prime (Eq. 2.5, 4).

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CHAPTER THREE

Basic Equations for Nonisentropic Flow

3.1 EVIDENCE OF NONISENTROPIC FLOW

All the isentropic flows treated in Chapter 2 have one property in common: They are expansion flows in which the pressure is reduced. Let us now consider briefly the properties of some compression flows (Ref. 2.4, pp. 116, 278).

The compression wave in a nonstationary, one-dimensional flow may be investigated by means of the method outlined in Section 2.7. In a flow of this type the piston accelerates into the gas which lies to the right of the origin (Fig. 1). The motion is described by the C_+ characteristics, and the basic characteristic in the $(\bar{u}, a)$ plane is

$$\bar{u} - \frac{2a}{\gamma - 1} = \frac{2a_0}{\gamma - 1} \tag{1}$$

Since the slope of each C_+ line is

$$\frac{dt}{dx} = \frac{1}{\bar{u} + a} = \left[\left(\frac{\gamma + 1}{2}\right)\bar{u} + a_0\right]^{-1} \tag{2}$$

then, as $\bar{u}$ increases with time on the face of the piston, the slopes of the succeeding C_+ characteristics emanating from the piston curve become smaller, and they must intersect. Now, from the properties of the characteristics, each point of intersection has more than one value of $\bar{u}$ and a. Therefore, the solution is no longer unique in the regions where these intersections occur, and it must be concluded that an isentropic motion can develop into a nonisentropic flow for which Maxwell's law of distribution of molecular velocities is not valid. Experimental evidence of this effect is clearly indicated in Fig. 2.7, which shows a compression wave and weak shock waves being propagated to the right in a shock tube. Ultimately this system forms the primary shock wave which involves nonisentropic compression (see Section 4.3).

A similar effect occurs in the case of a compression wave in steady, two-dimensional flow. If the wall "turns into" the flow (Fig. 2), the

relevant hodograph characteristic is H_- (see Fig. 2. 12*b*) from which we note that $\tilde{q}$ decreases as the flow proceeds (ϵ increases). Then,

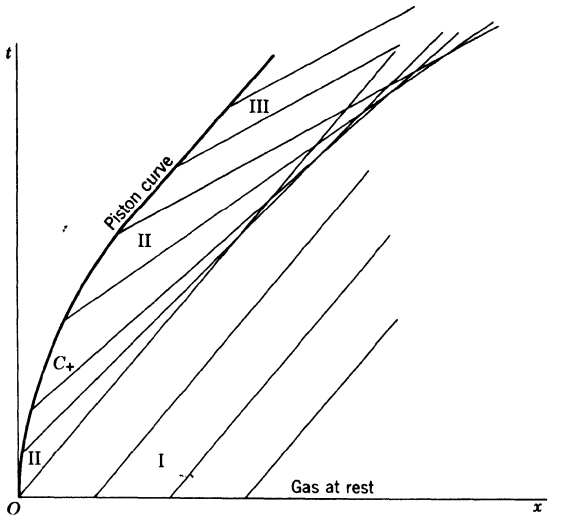

Fig. 1. Unsteady compression wave.

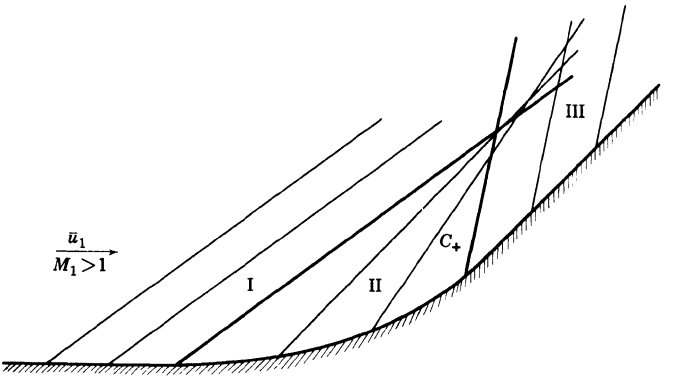

Fig. 2. Steady compression wave.

according to Eqs. 2.8, 48, 50, the angles of the C_+ characteristics (α) increase as $\tilde{q}$ becomes smaller, and again these lines must intersect. Photographs of this type of motion show that the isentropic compression wave develops into a nonisentropic shock wave (Fig. 2. 16*b*).

It has been pointed out that in its present form the theory assumes specular reflection of the molecules at a wall. When a streamline of the mass motion is considered to be a surface bounding the flow, the assumption is automatically made that the wall reflects the molecules in such a way that the normal components of velocity are reversed and the tangential components remain unchanged (Section 2.6). The solid boundary exercises no effect on the state of the gas flow, and the motion remains isentropic throughout. In practice, however, pure specular reflection does not occur, and the state of the gas adjacent to the wall is affected by the motion and thermal condition of the surface (Ref. 1.4, p. 285). Experimental investigation shows the existence of a thin boundary layer adjacent to the wall in which the motion of the gas is nonisentropic (see the flow adjacent to the curved surfaces in Fig. 2.16).

The kinetic theory of gas flow will now be extended on the following basis:

(*a*) In order to obtain a closer correspondence between theory and experiment for nonisentropic flow, a more accurate molecular model will be introduced. The information on collisions, restricted at present to the perfectly elastic sphere, will be extended to include molecules represented as point centers of force.

(*b*) The motion will be considered "slightly nonisentropic"; that is, the deviation of the distribution of molecular velocities from Maxwell's law is small. The equations for the transfer of mass, momentum, and energy now assume a more general form.

(*c*) The interaction of the molecules at a solid wall will be re-examined to provide boundary conditions for the equations of motion which are more consistent with experimental measurement. The reflection process will be assumed to be diffuse, with the result that at the surface of a body the mass motion and internal energy of the gas are in accord with the velocity and thermal condition of the wall.

3.2 ENCOUNTERS BETWEEN POINT CENTERS OF FORCE

The mathematical representation of a molecule as a perfectly elastic sphere is only a rough approximation, yet it is sufficient for the calculation of the essential properties of isentropic flow. Experimental investigations of the relation between pressure, density, and temperature (Eq. 1.10, 7) in dense gases indicate the existence of an additional term which arises from intermolecular forces. All the information available points to the conclusion that molecules exert a weak force of attraction on each other when they are separated by large distances, and a strong force of repulsion when they are short distances apart. As the next step in improving the mathematical representation of the molecule, it is

reasonable to adopt the point center of a spherically symmetrical field of force as the new molecular model (Ref. 1.1, p. 56). It will be found that the spherical molecule is a special case of the more general model.

An encounter between two molecules involves only their relative motion. The motion of a class 2 molecule (P) relative to a class 1 molecule (O) is illustrated by Fig. 3. Before the molecules begin to influence one another, the relative velocity is Ω. Let ROQ be a plane through O perpendicular to Ω, meeting the line of action of Ω at Q.

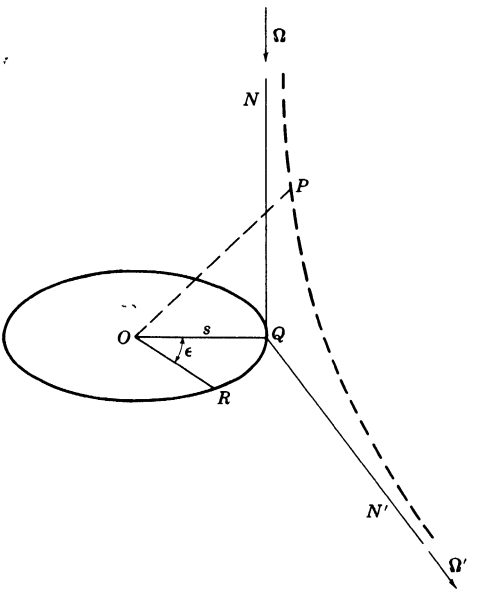

Fig. 3. Molecular encounter.

We select O as the origin of a system of polar coordinates (s, ϵ) in the plane ROQ, where s is the perpendicular distance between the line of action of Ω and the origin O, and ϵ is the angle ROQ. Since the force between the two molecules is directed along OP, the relative motion will be confined to the plane containing Ω and OP (NQO). It will be seen that the orientation of this plane is independent of the initial conditions, and the direction OR may be regarded as arbitrary.

The velocity components after a binary encounter must be calculated for the new molecular model. The construction of the velocity vector diagram is indicated in part in Fig. 4. The reader will find that a three-dimensional model of the vector diagram is of great assistance in

understanding the spherical trigonometry of subsequent calculations. Let OX, OY, OZ be the axes of a rectangular system of coordinates, and let ON be the line of action of $\boldsymbol{\Omega}$, drawn in accordance with Eqs. 1.5, 1, 2 ($\boldsymbol{\Omega} = \mathbf{c}_2 - \mathbf{c}_1$). The vertical plane containing OR (Fig. 3) is chosen as the plane through ON and OX in Fig. 4 so that the angle XNN' is the angle ϵ. The initial and final velocities (which are those at great distances of separation where the effect of the intermolecular forces is negligible) are governed by the conservation laws. It has been shown that the

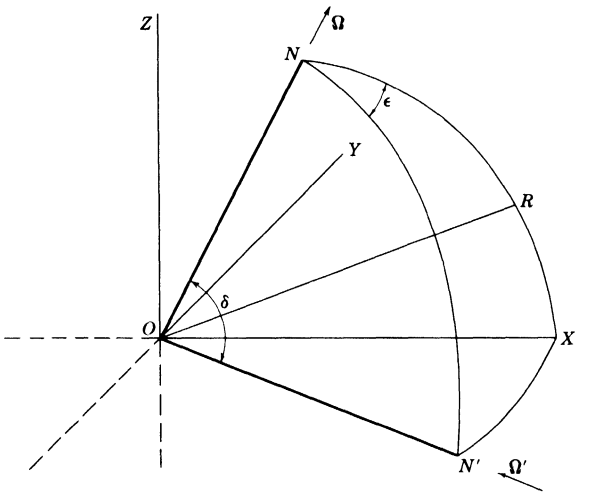

Fig. 4. Vector diagram for an encounter.

relative velocity after encounter ($\boldsymbol{\Omega}'$) has the same magnitude as $\boldsymbol{\Omega}$ but a different direction. The vector $\boldsymbol{\Omega}'$ lies in the plane of the motion (NOQ, Fig. 3), and is indicated in Fig. 4 by $N'O$. The angle between the initial and final velocity vectors (NON', Fig. 4) is called the angle of deflection (δ).

Now let X, Y, Z, R, N, N' lie on the surface of a sphere of unit radius, centered at O. If we apply spherical trigonometry to the triangle XNN', then

$$\cos N'X = \cos NX \cos \delta + \sin NX \sin \delta \cos \epsilon \tag{1}$$

where $N'X$ is the angle subtended at O by the arc $N'X$. This equation can be evaluated in terms of the components of the relative velocity,

$$\cos NX = \frac{u_2 - u_1}{\Omega}, \quad \cos N'X = -\frac{u_2' - u_1'}{\Omega} \tag{2}$$

where the direction of the relative velocity vector with respect to the origin must be taken into account (Fig. 4). Then Eq. 1 becomes

$$u_1' - u_2' = (u_2 - u_1)\cos\delta + [\Omega^2 - (u_2 - u_1)^2]^{1/2}\sin\delta\cos\epsilon \quad (3)$$

Similarly, for the spherical triangle YNN',

$$\cos N'Y = \cos NY\cos\delta + \sin NY\sin\delta\cos(\xi_2 - \epsilon) \quad (4)$$

where $\cos YNN' = \cos(\xi_2 - \epsilon)$, and ξ_2 is the angle XNY. From the spherical triangle NXY,

$$\cos XY = \cos NX\cos NY + \sin NX\sin NY\cos XNY \quad (5)$$

which may be written

$$(u_2 - u_1)(v_2 - v_1) + [\Omega^2 - (u_2 - u_1)^2]^{1/2}[\Omega^2 - (v_2 - v_1)^2]^{1/2}\cos\xi_2 = 0 \quad (6)$$

since $XY = \pi/2$. In terms of the components of the relative velocity, Eq. 4 has the form

$$v_1' - v_2' = (v_2 - v_1)\cos\delta + [\Omega^2 - (v_2 - v_1)^2]^{1/2}\sin\delta\cos(\xi_2 - \epsilon) \quad (7)$$

where $\cos\xi_2$ can be obtained from Eq. 6.

The same procedure, applied to the spherical triangles ZNN' and NXZ, yields the result

$$w_1' - w_2' = (w_2 - w_1)\cos\delta + [\Omega^2 - (w_2 - w_1)^2]^{1/2}\sin\delta\cos(\xi_3 - \epsilon) \quad (8)$$

where

$$(u_2 - u_1)(w_2 - w_1) + [\Omega^2 - (u_2 - u_1)^2]^{1/2}[\Omega^2 - (w_2 - w_1)^2]^{1/2}\cos\xi_3 = 0 \quad (9)$$

Expressions for the final components of velocity in terms of the given initial components can now be found. The equation for the conservation of momentum in the x-direction may be written in the following modified form:

$$2u_1' = 2u_1 + (u_2 - u_1) + (u_1' - u_2') \quad (10)$$

If we substitute for $u_1' - u_2'$ from Eq. 3, then

$$u_1' = u_1 + (u_2 - u_1)\cos^2\tfrac{1}{2}\delta + \tfrac{1}{2}[\Omega^2 - (u_2 - u_1)^2]^{1/2}\sin\delta\cos\epsilon \quad (11)$$

Similarly,

$$v_1' = v_1 + (v_2 - v_1)\cos^2 \tfrac{1}{2}\delta + \tfrac{1}{2}[\Omega^2 - (v_2 - v_1)^2]^{1/2} \sin \delta \cos (\xi_2 - \epsilon) \qquad (12)$$

and

$$w_1' = w_1 + (w_2 - w_1)\cos^2 \tfrac{1}{2}\delta + \tfrac{1}{2}[\Omega^2 - (w_2 - w_1)^2]^{1/2} \sin \delta \cos (\xi_3 - \epsilon) \qquad (13)$$

where $\cos \xi_2$ and $\cos \xi_3$ are given by Eqs. 6 and 9. The corresponding equations for u_2', v_2', w_2' can be obtained by a similar procedure.

In addition to the more general relations for molecular encounters given above, a new expression for the collision frequency is also required (1.6, 1). We have seen that the number of class 1 molecules in the element of volume $d\tau$ is $nf_1\, d\omega_1\, d\tau$. Associated with each point-center molecule in class 1 is an elementary volume given by $s\, d\epsilon \cdot ds \cdot \Omega\, dt$ (see Section 1.6), and the total of such elementary volumes in $d\tau$ is

$$nf_1 s\Omega\, d\epsilon\, ds\, d\omega_1\, d\tau\, dt \qquad (14)$$

The total number of class 2 molecules interacting with class 1 molecules in $d\tau$ during the interval dt, having paths relative to O (Fig. 3) lying in the range $s, s + ds$; $\epsilon, \epsilon + d\epsilon$ is the number of class 2 molecules in the above volume. Therefore, the required expression for the collision frequency is

$$n^2 f_1 f_2 s\Omega\, ds\, d\epsilon\, d\omega_1\, d\omega_2\, d\tau\, dt \qquad (15)$$

The total change in the quantity ΣQ per unit volume per unit time produced by collisions between molecules represented as point centers of force may now be expressed in the modified form (see Eq. 1.8, 5)

$$\Delta Q = \iiiint n^2 (Q_1' - Q_1) f_1 f_2 \Omega s\, d\epsilon\, ds\, d\omega_1\, d\omega_2 \qquad (16)$$

where the ranges of ϵ and s are, respectively, 0 to 2π and 0 to ∞. The more general form for Maxwell's transfer equation is

$$\frac{\partial}{\partial t}(n\bar{Q}) + \frac{\partial}{\partial x}(n\overline{uQ}) + \frac{\partial}{\partial y}(n\overline{vQ}) + \frac{\partial}{\partial z}(n\overline{wQ}) = \iiiint n^2 (Q_1' - Q_1) f_1 f_2 \Omega s\, d\epsilon\, ds\, d\omega_1\, d\omega_2 \qquad (17)$$

The introduction of the point-center molecule must be accompanied by a reconsideration of the definition of the thermodynamic quantities (Section 1.10). It will be seen that p retains its original definition, but the pressure equations (1.10, 4, 5) should now include small terms which arise from the action of the intermolecular forces across the surface on

which the pressure is required. In the analysis that follows, the gas will be assumed to be composed of "small-field molecules" such that the average effect due to the intermolecular forces is negligible compared with the contribution of the momentum to the pressure. The effect of intermolecular forces on pressure is discussed further in Section 3.10. Since the point-center molecules have only translational energy, the definition of temperature remains the same (Eq. 1.10, 6).

3.3 ANGLE OF DEFLECTION

The angle of deflection (δ), which occurs in the velocity relations (Eqs. 3.2, 11, 12, 13), may be found from a study of the relative motion of two point-center molecules, one in class 1 and the other in class 2.

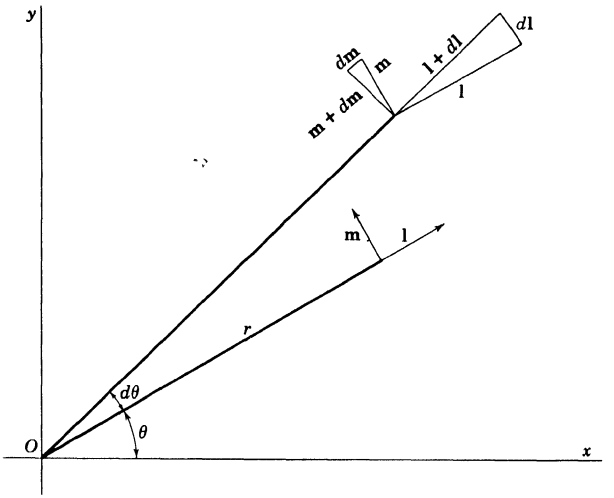

Fig. 5. Determination of acceleration vector.

It will be shown that δ is a function of the distance parameter and the magnitude of the relative velocity (s and Ω, see Fig. 3).

According to the results obtained in Section 1.5, the mutual interaction of two molecules can be deduced from their relative motion and the motion of their center of mass. The dynamics of a binary encounter between two point-center molecules of mass m is essentially that of a particle of mass $\frac{1}{2}m$ moving relative to a fixed center of force (Eq. 1.5, 13).

The components of acceleration of the particle can be conveniently expressed in terms of the polar coordinates (r, θ). Let $\mathbf{l}$ be a unit vector

in the direction of the radius vector $\mathbf{r}$ such that $\mathbf{r} = \mathbf{l}r$, and let $\mathbf{m}$ be the unit vector in the direction of increasing θ, $\boldsymbol{\theta} = \mathbf{m}\theta$. During an interval of time dt, the particle moves from position (r, θ) to position $(r + dr, \theta + d\theta)$, and $\mathbf{l}$, $\mathbf{m}$ become, respectively, $\mathbf{l} + d\mathbf{l}$, $\mathbf{m} + d\mathbf{m}$. From Fig. 5, $d\mathbf{l} = \mathbf{m}\, d\theta$, $d\mathbf{m} = -\mathbf{l}\, d\theta$. Then the velocity vector is

$$\frac{d\mathbf{r}}{dt} = \mathbf{l}\frac{dr}{dt} + \mathbf{m}r\frac{d\theta}{dt} \tag{1}$$

and the acceleration vector is

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{l}\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + \mathbf{m}\left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right] \tag{2}$$

The equations of motion for a particle of mass $\frac{1}{2}m$ moving relative to a fixed point under the influence of a force $g(r)$ directed along the radius vector are as follows:

$$\frac{1}{2}m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = g(r) \tag{3}$$

$$\frac{1}{2}m\left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right] = 0 \tag{4}$$

Equation 4 may be written in the form

$$\frac{1}{2}m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \tag{5}$$

which states that the angular momentum $\frac{1}{2}mr^2(d\theta/dt)$ is constant throughout the motion. Since the particle had the initial angular momentum $\frac{1}{2}ms\Omega$, then

$$r^2\frac{d\theta}{dt} = s\Omega \tag{6}$$

Equation 3 can now be expressed in an integrable form by multiplying throughout by dr/dt and using the condition that $r^2(d\theta/dt)$ is constant. The integrated result states that the difference between the kinetic energy at $t = 0$ and any subsequent time t is equal to the work done in bringing the particle from $r \to \infty$ at $t = 0$ to a distance r from the origin at time t,

$$\frac{1}{4}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - \frac{1}{4}m\Omega^2 = \int_\infty^r g(r)\, dr \tag{7}$$

The quantity

$$\Phi(r) = \frac{2}{m}\int_\infty^r g(r)\, dr \tag{8}$$

is called the force potential per unit mass. Rearranging the bracketed quantity on the left-hand side of Eq. 7 and substituting from Eq. 6,

we have

$$\frac{s^2\Omega^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] = \Phi(r) + \frac{1}{2}\Omega^2 \tag{9}$$

Then the angle of deflection is

$$\theta = \pm \int_0^{\eta} \frac{d\eta}{\left[1 - \eta^2 + \frac{2}{\Omega^2}\Phi\left(\frac{s}{\eta}\right)\right]^{1/2}} \tag{10}$$

where $\eta = s/r$, and initially $\theta = 0$ at $r \to \infty$.

The distance of nearest approach to the origin O may be obtained from the stationary condition $dr/d\theta = 0$, which requires that

$$1 - \eta^2 + \frac{2\Phi}{\Omega^2} = 0 \tag{11}$$

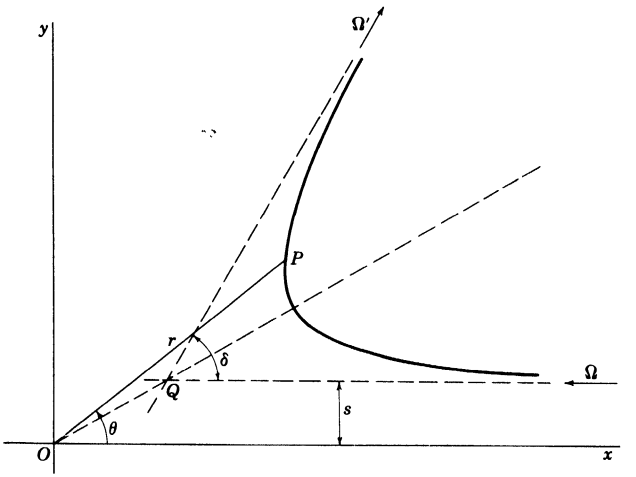

Fig. 6. Motion relative to fixed center of force.

The root of this equation (η_0) is the upper limit of the integral in Eq. 10 which specifies the angle of the line of symmetry (Fig. 6). The angle of deflection is twice this value,

$$\delta = 2\int_0^{\eta_0} \frac{d\eta}{\left[1 - \eta^2 + \frac{2\Phi}{\Omega^2}\right]^{1/2}} \tag{12}$$

Therefore, δ is a function of s and Ω only.

A repulsive force acting between two point-center molecules has the form α/r^ν, where α is a constant which includes the product of the masses of the colliding molecules (m^2). The corresponding force potential per unit mass is

$$\Phi(r) = -\frac{2\alpha}{m(\nu-1)r^{\nu-1}} \tag{13}$$

If we write

$$\zeta = s\left(\frac{m\Omega^2}{2\alpha}\right)^{1/(\nu-1)} \tag{14}$$

the angle of deflection is

$$\delta(\zeta) = 2\int_0^{\eta_0} \frac{d\eta}{\left[1-\eta^2-\left(\frac{2}{\nu-1}\right)\left(\frac{\eta}{\zeta}\right)^{\nu-1}\right]^{1/2}} \tag{15}$$

where η_0 is the root of the equation

$$1-\eta^2-\left(\frac{2}{\nu-1}\right)\left(\frac{\eta}{\zeta}\right)^{\nu-1} = 0 \tag{16}$$

It can be shown graphically that, for $\nu > 1$, this equation has only one real root. Thus δ is a function of ζ only, when a repulsive force exists.

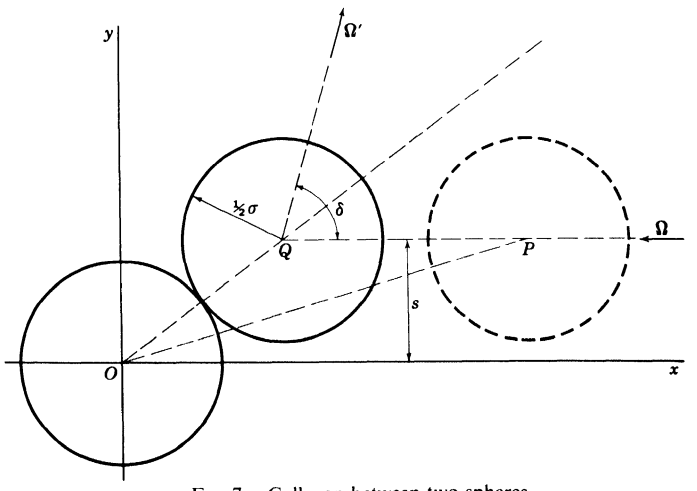

Fig. 7. Collision between two spheres.

In an interaction between two perfectly elastic spheres (Fig. 7) no collision occurs if $s > \sigma$, and no force acts until $OP = \sigma$. The velocity of the class 2 molecule relative to the class 1 molecule ($\boldsymbol{\Omega}$) remains

unchanged up to the point of contact at which it is instantaneously changed to Ω'. Thus the asymptotes of the more general encounter (Fig. 6) become the path of the relative motion in a collision between two spherical molecules, the distance of nearest approach (OQ) being σ. From the geometry of Fig. 7,

$$\sin \frac{1}{2}\delta = \frac{s}{\sigma} \tag{17}$$

3.4 VELOCITY DISTRIBUTION FUNCTION FOR NONISENTROPIC FLOW

In the preceding study of isentropic motion the macroscopic properties of a flowing gas were deduced from a mathematical model of the molecule and the appropriate distribution of molecular velocities. The following discussion of nonisentropic flow will proceed along the same lines. Collision information relevant to an improved molecular model is available, and consideration must now be given to a modified velocity distribution function.

The theory of nonisentropic flow must be sufficiently broad to include the flow of a gas near a solid boundary and the transition through a shock front. The molecular motion near a wall results from the interaction of incident and reflected molecules. The impinging molecules have motions consistent with the state of mass motion and internal energy of the gas, and the molecules emitted from the surface move in accordance with the velocity and thermal condition of the wall. The interactions of the incident and emergent streams of molecules produce a deviation in the distribution of molecular velocities from Maxwell's law in the neighborhood of the wall. According to this concept of the molecular motion, the effect of the wall is taken into account by introducing a new velocity distribution function, maintained essentially by collisions between gas molecules only.

If the molecules reflect specularly from the wall, the emergent molecules have motions consistent with the state of the gas, not the wall, and the intermingling of the two streams of molecules produces no departure from Maxwell's law. Nonisentropic theory must include a type of reflection which involves accommodation between the motions of the reflected molecules and the condition of the wall. This subject is considered in Section 4.4.

The nonisentropic flow in the shock wave may also be regarded as resulting from the intermingling and interaction of two streams of molecules emanating from two regions of the gas in two different states, one in front of and the other behind the shock. This process also results in a deviation of the distribution of molecular velocities from Maxwell's law in the transition region between the two states.

The direct method of determining the equations for nonisentropic flow is to solve the general Boltzmann equation for a modified velocity distribution function and apply it to Eqs. 1.9, 8, 10 to find the corresponding transfer equations. This method has already been employed in the theory of isentropic flow in Chapter 2. The more general problem has been solved by Enskog (Ref. 1).

An alternative method initiated by Maxwell (Ref. 2) and developed by Chapman (Refs. 3, 4, 5) is adopted here. Although tedious calculations are unavoidable, the mathematical treatment is simpler than that required for the direct method, and it is more descriptive of the physical facts.

Let us consider the motion of a gas in which the distribution of molecular velocities departs only slightly from Maxwell's law. We express the velocity distribution function in the form

$$f = f_0[1 + F(H_i)] \qquad (i = 1, 2, 3) \tag{1}$$

where

$$H_1 = \left(\frac{3}{\overline{C^2}}\right)^{1/2} U, \quad H_2 = \left(\frac{3}{\overline{C^2}}\right)^{1/2} V, \quad H_3 = \left(\frac{3}{\overline{C^2}}\right)^{1/2} W \tag{2}$$

f_0 is the Maxwell distribution function for which

$$f_0\, dU\, dV\, dW = Ke^{-\frac{1}{2}H_iH_i}\, dH_1\, dH_2\, dH_3 \qquad (K = (2\pi)^{-3/2}) \tag{3}$$

and $F(H_i)$ is a power series in H_1, H_2, H_3,

$$F(H_i) = a_iH_i + \tfrac{1}{2}a_{ij}H_iH_j + \tfrac{1}{6}a_{ijk}H_iH_jH_k \qquad (i, j, k = 1, 2, 3) \tag{4}$$

In Eq. 4, the order of the subscript numbers in a_{ij}, a_{ijk} is immaterial. Thus $a_{12} = a_{21}$, and $a_{112} = a_{121} = a_{211}$, and so on. Note that the repetition of a subscript implies the summation convention

$$(H_iH_i = H_1^2 + H_2^2 + H_3^2)$$

Sufficient terms have been retained in $F(H_i)$ so that it will make a contribution to the magnitude of each mean value that appears in the transfer equations (1.9, 8, 10).

The new distribution function (f) must satisfy certain conditions (see Section 1.4). The first requirement is expressed by Eq. 1.4, 3,

$$K\int F(H_i)e^{-\frac{1}{2}H_jH_j}\, dH = 0 \tag{5}$$

where, following the convention adopted in Section 1.4, we write the symbols

$$\int(\cdots)\, dH \tag{6}$$

for the triple integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cdots)\,dH_1\,dH_2\,dH_3 \tag{7}$$

If we substitute for $F(H_i)$ from Eq. 4, we find that all integrals are zero except those involving even functions of H_i (see Appendix I). Then Eq. 5 becomes

$$K\int(a_{ii}H_iH_i)e^{-\frac{1}{2}H_jH_j}\,dH = 0 \tag{8}$$

which reduces to

$$a_{ii} = 0 \tag{9}$$

Further relations governing the coefficients of $F(H_i)$ are obtained from the conditions indicated in Eqs. 1.4, 9. If $\overline{H_1} = 0$,

$$K\int H_1F(H_i)e^{-\frac{1}{2}H_jH_j}\,dH = 0 \tag{10}$$

which gives the relation

$$2a_1 + a_{1jj} = 0 \tag{11}$$

Similarly, if $\overline{H_2} = 0$, $\overline{H_3} = 0$, then $2a_2 + a_{2jj} = 0$, $2a_3 + a_{3jj} = 0$, respectively, and all three equations may be expressed in the form

$$2a_i + a_{ijj} = 0 \tag{12}$$

We may now calculate the average quantities required for the transfer equations (1.9, 8, 10). From the definition of the mean value (Eq. 1.4, 6),

$$\begin{aligned}\overline{H_1^2} &= 1 + K\int H_1^2F(H_i)e^{-\frac{1}{2}H_jH_j}\,dH = 1 + a_{11}\\ \overline{H_2^2} &= 1 + a_{22}, \qquad \overline{H_3^2} = 1 + a_{33}\end{aligned} \tag{13}$$

Condition 9 above can also be deduced from the relation $\overline{C^2} = \overline{U^2} + \overline{V^2} + \overline{W^2}$, or $\overline{H_1^2} + \overline{H_2^2} + \overline{H_3^2} = 3$. We also note that

$$\overline{C^2} = \int C^2f\,d\omega = \int C^2f_0\,d\omega \tag{14}$$

and hence the static pressure and the temperature have the same implication in both isentropic and nonisentropic flow. Furthermore, c_p and c_v continue to have the significance outlined in Section 2.2.

The remaining mean values which occur in the momentum-transfer equations (1.9, 8) are

$$\begin{aligned}\overline{H_1H_2} &= K\int H_1H_2F(H_i)e^{-\frac{1}{2}H_jH_j}\,dH = a_{12}\\ \overline{H_2H_3} &= a_{23}, \qquad \overline{H_3H_1} = a_{31}\end{aligned} \tag{15}$$

The following additional quantities are typical of those required for the energy-transfer equation (1.9, 10)

$$\overline{H_1^3} = a_{111}, \quad \overline{H_2^3} = a_{222}, \quad \overline{H_3^3} = a_{333}$$
$$\overline{H_1^2 H_2} = a_{112}, \quad \overline{H_1 H_2^2} = a_{122}, \quad \cdots \tag{16}$$
$$\overline{H_1(H_1^2 + H_2^2 + H_3^2)} = -2a_1, \quad \cdots$$

When the coefficients of $F(H_i)$ are known, the basic transfer equations for nonisentropic flow can be found. The assumed form of the modified velocity distribution function (Eqs. 1, 4) is valid only for a flow that departs slightly from Maxwellian motion. With the exception of $\overline{H_1^2}$, $\overline{H_2^2}$, $\overline{H_3^2}$, all mean values determined above are zero in an isentropic flow. Since the nonisentropic flow deviates from the isentropic by a small amount, all coefficients of $F(H_i)$ are of lower order than 1. Furthermore, the flow must remain slightly nonisentropic throughout all variations with respect to space and time, and all derivatives of the coefficients of $F(H_i)$ are assumed to have a small order of magnitude.

3.5 MEAN SPEED AND MEAN FREE PATH IN NONISENTROPIC FLOW

Although the coefficients of the modified distribution function (a_i, a_{ij}, a_{ijk}, Eq. 3.4, 4) have not yet been determined, it is possible to consider at this stage the expressions for the mean speed and the mean free path in a nonisentropic flow. According to the definition of the average value of a function of the velocity components (Section 1.4), the mean value of the random molecular speed is

$$\bar{C} = \int C f \, d\omega = 2\left(\frac{2\overline{C^2}}{3\pi}\right)^{1/2} + \int C f_0 F \, d\omega \tag{1}$$

In terms of the spherical coordinates in the velocity space (Section 2.1), the integral becomes

$$\int C f_0 F \, d\omega = \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} C^3 F(C, \varphi, \theta) e^{-\beta C^2} \sin \varphi \, d\varphi \, d\theta \, dC \tag{2}$$

where

$$U = C \sin \varphi \cos \theta$$
$$V = C \sin \varphi \sin \theta \tag{3}$$
$$W = C \cos \varphi$$

and $F(C, \varphi, \theta)$ can be obtained from $F(U, V, W)$ which is given in the expanded form in Appendix I (see also Eqs. 3.4, 2, 4). By considering only the integrals with respect to φ and θ, the reader can verify, with the

help of the definite integrals listed in Appendix I, that

$$\int Cf_0 F\, d\omega = 0 \tag{4}$$

Therefore the mean speed has the same value for both isentropic and slightly nonisentropic flows (Eqs. 2.4, 1, 2).

In an isentropic flow the most probable speed corresponds to the maximum value of the function $C^2 f_0$ where f_0 is Maxwell's distribution function. The same is true also for C_m in nonisentropic flows. In flows that are partly isentropic and partly nonisentropic, C_m can be a useful representation of molecular speed.

The calculation of the mean free path in a nonisentropic flow requires a reconsideration of the expression 2.4, 7 for the total number of collisions in the element of volume $d\tau$ during the interval of time dt. The integral

$$\iint \Omega f_1 f_2\, d\omega_1\, d\omega_2$$

must now be evaluated for the more general velocity distribution function. The substitutions introduced in Section 2.4 to obtain separation of the integrals are again necessary. Then,

$$\iint \Omega f_1 f_2\, d\omega_1\, d\omega_2 = \frac{1}{8}\left(\frac{\beta}{\pi}\right)^3 \iint \Omega(1 + G_1)(1 + G_2) e^{-\frac{1}{2}\beta(\Omega^2 + \Lambda^2)}\, dK_1\, dK_2 \tag{5}$$

where G_1 and G_2 are the expressions for $F(U_1, V_1, W_1)$ and $F(U_2, V_2, W_2)$, respectively, in the new variables X_1, Y_1, Z_1, X_2, Y_2, Z_2.

The product $(1 + G_1)(1 + G_2)$ may be reduced to $1 + G_1 + G_2$ since $G_1 G_2$ involves products of the coefficients a_i, a_{ij}, a_{ijk} which are of negligible order. The series $1 + G_1 + G_2$ is given in detail in Appendix I. Substituting this function in Eq. 5, we obtain a series of integrals. After integrating with respect to X_2, Y_2, and Z_2, we observe that only a few integrals remain since all odd functions in the integrand yield a zero result (see Appendix I), and other integrals disappear when Eq. 3.4, 9 is applied. The remaining integrals can be integrated with respect to X_1, Y_1, Z_1 by introducing spherical coordinates (as in the calculation of $\bar{C}$) and comparing the form of the integrals with respect to φ and θ with those listed in Appendix I. All the integrals reduce to zero except four, three of which disappear when Eq. 3.4, 9 is applied, leaving only the first integral corresponding to the first term in the series expression for $1 + G_1 + G_2$. The final result is the same as that previously determined for isentropic flow (see Eq. 2.4, 15). Therefore, the fundamental relation for the mean free path (Eq. 2.4, 17) has the same form for both isentropic and slightly nonisentropic flow.

The concept of a free path is based on the assumption that the molecules of a gas are smooth, perfectly elastic spheres which move freely except when actually in contact during a collision. If the molecules are represented by point centers of force, the mutual action between molecules does not vanish completely at any finite distance of separation, and the motion of these molecules is not free. However, when the density of the gas is low, so that the field of force surrounding each point center exerts a significant influence over only a small portion of the average distance of separation (small-field molecules), then it is possible to retain the concept of the free path by assigning an equivalent mean diameter to each molecule (see Section 4.1).

The definition of the speed of sound in a nonisentropic motion is based on the invariance of $\overline{C^2}$ with respect to the form of f. Since condition 1.4, 3 requires that $a_{ii} = 0$, then in both isentropic and nonisentropic flow, $\overline{C^2} = \overline{U^2} + \overline{V^2} + \overline{W^2}$, the mean values being formed with f_0 and f respectively, and $p = \frac{1}{3}\rho\overline{C^2}$, $RT = \frac{1}{3}\overline{C^2}$. Therefore, the speed of sound in a monatomic gas flowing nonisentropically is defined by the relations

$$a = [\tfrac{5}{3}(\tfrac{1}{3}\overline{C^2})]^{1/2} = \sqrt{\gamma R\overline{T}} \tag{6}$$

3.6 RATE OF FLOW OF MOLECULAR MOMENTUM

When the coefficients of the more general velocity distribution function (a_i, a_{ij}, a_{ijk}, Eq. 3.4, 4) are known, such quantities as $n\overline{U^2}$, $n\overline{UV}$, and $n\overline{UC^2}$ can be found from relations given in Section 3.4 and substituted in the transfer equations (1.9, 8, 10). The result will be the basic equations for slightly nonisentropic flow.

A method for determining the coefficients is suggested by Eqs. 3.4, 13, 15, 16. For example, it will be noted that a_{12}, a_{23}, and a_{31} are directly related to the pressure components $-\rho\overline{UV}$, $-\rho\overline{VW}$, $-\rho\overline{WU}$ (see Eq. 1.10, 4). According to the kinetic theory, the pressure in a gas is defined as the mean rate of flow of molecular momentum per unit area across the relevant surface. Thus the tangential pressure $-\rho\overline{UV}$ will arise from the transport of molecular momentum mu at rate v through the surface $dx\,dy$ (see Fig. 1.7). A detailed investigation of this transport process in slightly nonisentropic flow, resulting in a determination of a_{ij}, can be made by means of Maxwell's transfer equation.

Let us calculate the transfer equation for $Q = uv = (\bar{u} + U)(\bar{v} + V)$. In order to find the left-hand side of Eq. 1.8, 7, we require the quantities

$$\begin{aligned} &\overline{Q} = \bar{u}\bar{v} + \overline{UV}, \qquad \overline{UQ} = \bar{u}\,\overline{UV} + \bar{v}\,\overline{U^2} + \overline{U^2V} \\ &\overline{VQ} = \bar{u}\overline{V^2} + \bar{v}\overline{UV} + \overline{UV^2}, \qquad \overline{WQ} = \bar{u}\overline{VW} + \bar{v}\overline{UW} + \overline{UVW} \end{aligned} \tag{1}$$

If these mean values are expressed in terms of $\overline{C^2}$ and the coefficients a_i, a_{ij}, a_{ijk} and substituted in the left-hand side of Eq. 1.8, 7, and all second-order quantities involving these coefficients and their derivatives are neglected, then,

$$n\left(\frac{\overline{C^2}}{3}\right)(\bar{v}_x + \bar{u}_y) = \Delta uv \tag{2}$$

where some terms are eliminated by application of Eqs. 2.2, 4. Therefore, only the isentropic terms are retained on the left-hand side, and this procedure is justified only if the calculated value of Δuv is found to have a magnitude of the first order. According to Eq. 3.2, 16, if $Q = uv$, then

$$\Delta uv = \iiiint n^2(u_1'v_1' - u_1v_1)f_1f_2\Omega s\, ds\, d\epsilon\, d\omega_1\, d\omega_2 \tag{3}$$

where $0 \leq \epsilon \leq 2\pi$, $0 \leq s \leq \infty$. This multiple integral will now be evaluated for the more general distribution function.

Substitution for u_1', v_1' from Eqs. 3.2, 11, 12 and for f_1, f_2 from Eqs. 3.4, 1–4 results in an expansion of the right-hand side of Eq. 3 into a long series of multiple integrals. The integration can be reduced to a reasonable amount of work by observing the following rules:

(*a*) Integration should be performed with respect to ϵ, s, (X_2, Y_2, Z_2), and (X_1, Y_1, Z_1) in the order given.

(*b*) With the assistance of Appendix I, most of the integrals are zero by inspection.

(*c*) Further reduction of the results can be made on application of Eqs. 3.4, 9, 12.

Integrating first with respect to ϵ, we note that only the quantity $(u_1'v_1' - u_1v_1)$ contains functions of ϵ. Substitution from Eqs. 3.2, 11, 12 yields

$$\int_0^{2\pi}(u_1'v_1' - u_1v_1)\, d\epsilon = 2\pi(u_2v_2 - u_1v_1)\cos^2\frac{1}{2}\delta - \frac{3\pi}{4}(u_2 - u_1)(v_2 - v_1)\sin^2\delta \tag{4}$$

where the integrals are readily evaluated from Appendix I, and $\cos^4 \frac{1}{2}\delta$ is eliminated by means of the trigonometric relation

$$\cos^4\tfrac{1}{2}\delta = \cos^2\tfrac{1}{2}\delta - \tfrac{1}{4}\sin^2\delta \tag{5}$$

Only the angle of deflection δ contains s. In order that the final result may apply generally to any force between the point-center molecules, the integrals with respect to s will be evaluated in detail at a later stage.

For the present we write them in the form

$$I_1(\Omega) = \int_0^\infty s \cos^2 \tfrac{1}{2}\delta \, ds, \qquad I_2(\Omega) = \int_0^\infty s \sin^2 \delta \, ds \tag{6}$$

Then the right-hand side of the transfer equation becomes

$$\Delta uv = 2\pi n^2 \iint [(u_2 v_2 - u_1 v_1) I_1 - \tfrac{3}{8}(u_2 - u_1)(v_2 - v_1) I_2] \Omega f_1 f_2 \, d\omega_1 \, d\omega_2 \tag{7}$$

Since Ω appears in the integrand, it is necessary to transform from the variables $U_1, V_1, W_1, U_2, V_2, W_2$ to the new variables $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ in order to separate the integrals. The method is indicated in Section 2.4. Then

$$\Delta uv = \frac{1}{4}\left(\frac{n}{\pi}\right)^2 \beta^3 \iiint_{-\infty}^{+\infty} \iiint_{-\infty}^{+\infty} \left[\left(\bar{u} Y_1 + \bar{v} X_1 + \frac{1}{2} X_1 Y_2 + \frac{1}{2} Y_1 X_2\right) I_1 - \frac{3}{8} X_1 Y_1 I_2\right] G\Omega e^{-\frac{1}{4}\beta(\Omega^2 + \Lambda^2)} \, dX_1 \, dY_1 \, dZ_1 \, dX_2 \, dY_2 \, dZ_2 \tag{8}$$

where $G = 1 + G_1 + G_2$ is given in Appendix I.

The integration with respect to X_2, Y_2, Z_2 is performed first. Three types of integral are involved:

$$J_1(X_1, Y_1, Z_1) = \iiint_{-\infty}^{+\infty} G e^{-\frac{1}{4}\beta\Lambda^2} \, dX_2 \, dY_2 \, dZ_2 = \pi^{3/2}\left(\frac{2}{\beta}\right)^{1/2}\left(\frac{2}{\beta} + a_{11} X_1^2 + a_{22} Y_1^2 + a_{33} Z_1^2 + 2a_{12} X_1 Y_1 + 2a_{23} Y_1 Z_1 + 2a_{31} Z_1 X_1\right) \tag{9}$$

$$J_2(X_1, Y_1, Z_1) = \iiint_{-\infty}^{+\infty} X_2 G e^{-\frac{1}{4}\beta\Lambda^2} \, dX_2 \, dY_2 \, dZ_2 = \frac{\pi^{3/2}}{\beta}\left(\frac{2a_1}{\beta} + a_{111} X_1^2 + a_{122} Y_1^2 + a_{133} Z_1^2 + 2a_{112} X_1 Y_1 + 2a_{123} Y_1 Z_1 + 2a_{113} Z_1 X_1\right) \tag{10}$$

$$J_3(X_1, Y_1, Z_1) = \iiint_{-\infty}^{+\infty} Y_2 G e^{-\frac{1}{4}\beta\Lambda^2} \, dX_2 \, dY_2 \, dZ_2 = \frac{\pi^{3/2}}{\beta}\left(\frac{2a_2}{\beta} + a_{112} X_1^2 + a_{222} Y_1^2 + a_{233} Z_1^2 + 2a_{122} X_1 Y_1 + 2a_{223} Y_1 Z_1 + 2a_{123} Z_1 X_1\right) \tag{11}$$

As indicated in Appendix I, an integral is zero when an odd function multiplies the exponential term in the integrand. After integration the resulting expression is further reduced when Eqs. 3.4, 9, 12 are applied. Then Eq. 8 becomes

$$\Delta uv = \frac{1}{4}\left(\frac{n}{\pi}\right)^2 \beta^3 \iiint_{-\infty}^{+\infty} \left[(\bar{u}Y_1 + \bar{v}X_1)J_1 I_1 + \frac{1}{2}(X_1 J_3 + Y_1 J_2)I_1 - \frac{3}{8} X_1 Y_1 J_1 I_2 \right] \Omega e^{-\frac{1}{4}\beta\Omega^2} \, dX_1 \, dY_1 \, dZ_1 \tag{12}$$

The integrals with respect to X_1, Y_1, Z_1 can be evaluated in terms of spherical coordinates as specified by the equations

$$\begin{aligned} X_1 &= \Omega \sin\varphi \cos\theta \\ Y_1 &= \Omega \sin\varphi \sin\theta \\ Z_1 &= \Omega \cos\varphi \end{aligned} \tag{13}$$

($0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$, $0 \leq \Omega < \infty$). Then

$$dX_1 \, dY_1 \, dZ_1 = \Omega^2 \sin\varphi \, d\varphi \, d\theta \, d\Omega \tag{14}$$

A typical integral is

$$\iiint_{-\infty}^{+\infty} X_1 Y_1^2 \Omega e^{-\frac{1}{4}\beta\Omega^2} \, dX_1 \, dY_1 \, dZ_1 = \int_0^\infty \Omega^6 e^{-\frac{1}{4}\beta\Omega^2} \, d\Omega \int_0^\pi \sin^4\varphi \, d\varphi \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta \tag{15}$$

which is zero as an inspection of the integral with respect to θ will show. In nearly all cases the integral involving θ disappears, and many terms are immediately dropped by inspection with the assistance of Appendix I. In some integrations the integral with respect to φ is zero. It will be found that only one term (that containing $X_1^2 Y_1^2$) gives a non-zero integral, and hence

$$\Delta uv = -\frac{\pi^{1/2} n^2 a_{12}}{80} \left(\frac{\overline{C^2}}{3}\right)^{-5/2} \int_0^\infty \Omega^7 I_2(\Omega) e^{-\frac{1}{4}\beta\Omega^2} \, d\Omega \tag{16}$$

If we write

$$\frac{1}{\mu} = \frac{\pi^{1/2}}{80m} \left(\frac{\overline{C^2}}{3}\right)^{-5/2} \int_0^\infty \Omega^7 I_2(\Omega) e^{-\frac{1}{4}\beta\Omega^2} \, d\Omega \tag{17}$$

where μ is called the coefficient of viscosity, then the transfer equation for $Q = uv$, obtained by combining Eqs. 2 and 16, reduces to the following relation,

$$a_{12} = -\frac{\mu}{mn(\overline{C^2}/3)} \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) \tag{18}$$

This result will be valid for any nonisentropic flow provided the expression on the right-hand side is a quantity of second order similar to a_{12}.

The calculations of the transfer equations corresponding to $Q = vw$ and $Q = wu$ follow the same procedure as that outlined above, and similar equations for a_{23} and a_{31} are found. Finally, the three equations which describe the transfer of the momentum rates uv, vw, and wu may be written (see Eqs. 1.10, 4 and 3.4, 15)

$$\begin{aligned} -\rho\overline{UV} &= P_{xy} - P_{yx} = \mu\left(\frac{\partial\bar{v}}{\partial x} + \frac{\partial\bar{u}}{\partial y}\right) \\ -\rho\overline{VW} &= P_{yz} - P_{zy} = \mu\left(\frac{\partial\bar{w}}{\partial y} + \frac{\partial\bar{v}}{\partial z}\right) \\ -\rho\overline{WU} &= P_{zx} - P_{xz} = \mu\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right) \end{aligned} \tag{19}$$

The coefficients a_{11}, a_{22}, a_{33} can be determined by finding the transfer equations for $Q = u^2$, v^2, w^2, respectively. Consider first the transport of $Q = u^2$. The relevant quantities required for the left-hand side of Maxwell's general transfer equation (1.8, 7) are

$$\begin{aligned} \bar{Q} &= \bar{u}^2 + \overline{U^2}, \quad \overline{UQ} = 2\bar{u}\overline{U^2} + \overline{U^3} \\ \overline{VQ} &= 2\bar{u}\overline{UV} + \overline{U^2V}, \quad \overline{WQ} = 2\bar{u}\overline{UW} + \overline{U^2W} \end{aligned} \tag{20}$$

Furthermore,

$$n\frac{d}{dt}\left(\frac{\overline{C^2}}{3}\right) + 2n\left(\frac{\overline{C^2}}{3}\right)\frac{\partial\bar{u}}{\partial x} = \Delta u^2 \tag{21}$$

if only the first-order (isentropic) terms are retained. If we substitute from Eq. 2.2, 6, the transfer equation becomes

$$-\frac{2}{3}n\left(\frac{\overline{C^2}}{3}\right)\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) + 2n\left(\frac{\overline{C^2}}{3}\right)\frac{\partial\bar{u}}{\partial x} = \Delta u^2 \tag{22}$$

where

$$\Delta u^2 = \iiiint n^2(u_1'^2 - u_1^2) f_1 f_2 \Omega s \, ds \, d\epsilon \, d\omega_1 \, d\omega_2 \tag{23}$$

According to the rules for computation given in (a), (b), and (c) above, the integral with respect to ϵ is evaluated first,

$$\begin{aligned} \int_0^{2\pi}(u_1'^2 - u_1^2)\,d\epsilon &= 2\pi(u_2^2 - u_1^2)\cos^2\frac{1}{2}\delta \\ &\quad + \frac{\pi}{4}[\Omega^2 - 3(u_2 - u_1)^2]\sin^2\delta \end{aligned} \tag{24}$$

where u_1' is given by Eq. 3.2, 11. Then Eq. 23 reduces to

$$\Delta u^2 = \pi n^2 \iint\{2(u_2^2 - u_1^2)I_1 + \tfrac{1}{4}[\Omega^2 - 3(u_2 - u_1)^2]I_2\}\Omega f_1 f_2 \, d\omega_1 \, d\omega_2 \tag{25}$$

By substituting new variables according to Eqs. 2.4, 9 and introducing the integrals J_1 and J_2, we obtain the contribution of molecular collisions to the transfer equation,

$$\Delta u^2 = \frac{1}{2}\left(\frac{n}{\pi}\right)^2 \beta^3 \int\int\int_{-\infty}^{+\infty} \left[2X_1 I_1(2\bar{u}J_1 + J_2) + \frac{1}{4} I_2 J_1(\Omega^2 - 3X_1^2)\right] \Omega e^{-\frac{1}{4}\beta\Omega^2}\, dX_1\, dY_1\, dZ_1 \qquad (26)$$

If we transform to spherical coordinates (see Eqs. 13) and integrate, we find again that most integrals with respect to θ or φ are zero, and others disappear by virtue of Eq. 3.4, 9. Only the term involving X_1^4 makes a contribution, and we have finally

$$\Delta u^2 = -\frac{\pi^{1/2} n^2 a_{11}}{80}\left(\frac{\overline{C^2}}{3}\right)^{-5/2} \int_0^\infty \Omega^7 I_2(\Omega) e^{-\frac{1}{4}\beta\Omega^2}\, d\Omega \qquad (27)$$

Therefore, when $Q = u^2$, the transfer equation reduces to the following relation for the coefficient a_{11},

$$a_{11} = +\frac{2\mu}{mn(\overline{C^2}/3)}\left[\frac{1}{3}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) - \frac{\partial \bar{u}}{\partial x}\right] \qquad (28)$$

The corresponding expressions for a_{22} and a_{33} can be found in the same way by considering $Q = v^2$ and $Q = w^2$, respectively. If we combine these results with Eqs. 1.10, 4 and 3.4, 13, the transfer equations become

$$\begin{aligned} -\rho\overline{U^2} = P_{xx} &= -p - \frac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) + 2\mu\left(\frac{\partial \bar{u}}{\partial x}\right) \\ -\rho\overline{V^2} = P_{yy} &= -p - \frac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) + 2\mu\left(\frac{\partial \bar{v}}{\partial y}\right) \\ -\rho\overline{W^2} = P_{zz} &= -p - \frac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) + 2\mu\left(\frac{\partial \bar{w}}{\partial z}\right) \end{aligned} \qquad (29)$$

By substituting from Eqs. 19 and 29 into Eqs. 1.9, 8, we obtain the general momentum-transfer equations for nonisentropic flow (see Section 3.9).

3.7 RATE OF FLOW OF MOLECULAR ENERGY

In order to complete the information required for the determination of the basic transfer equations for nonisentropic flow, the expressions for the quantities $\overline{UC^2}$, $\overline{VC^2}$, and $\overline{WC^2}$ which appear in the energy-transfer equation (1.9, 10) must now be found. According to Eqs. 3.4, 16, we must now turn our attention to the a_i coefficients in the general velocity distribution function (Eq. 3.4, 4).

Consider the transfer equation for $Q = u(u^2 + v^2 + w^2)$. Following the procedure outlined in Section 3.6 above, we first calculate the left-hand side of Maxwell's transfer equation (1.8, 7). Neglecting all second-order quantities containing a_i, a_{ij}, a_{ijk} and their derivatives, the following first-order (isentropic) terms remain:

$$\left(\frac{n\overline{C^2}}{3}\right)\left[4\bar{u}\bar{u}_x - \frac{4}{3}\bar{u}(\bar{u}_x + \bar{v}_y + \bar{w}_z) + 2\bar{v}(\bar{v}_x + \bar{u}_y) + 2\bar{w}(\bar{w}_x + \bar{u}_z) + 5\frac{\partial}{\partial x}\left(\frac{\overline{C^2}}{3}\right)\right] = \Delta u(u^2 + v^2 + w^2) \tag{1}$$

where a number of terms disappear by virtue of the momentum and energy-transfer equations (2.2, 4, 6).

The next step is to evaluate the right-hand side of Maxwell's transfer equation,

$$\Delta u(u^2 + v^2 + w^2) = \iiiint n^2[u_1'(u_1'^2 + v_1'^2 + w_1'^2) - u_1(u_1^2 + v_1^2 + w_1^2)] \times f_1 f_2 \Omega s \, ds \, d\epsilon \, d\omega_1 \, d\omega_2 \tag{2}$$

It is convenient to separate Eq. 2 into three parts. Let us write

$$\Delta u(u^2 + v^2 + w^2) = \Delta u^3 + \Delta u v^2 + \Delta u w^2 \tag{3}$$

where the notation on the right-hand side is illustrated by the relation

$$\Delta u^3 = \iiiint n^2(u_1'^3 - u_1^3) f_1 f_2 \Omega s \, ds \, d\epsilon \, d\omega_1 \, d\omega_2 \tag{4}$$

The evaluations of Δu^3, $\Delta u v^2$, and $\Delta u w^2$ proceed according to the method outlined in detail for Δuv in Section 3.6. The following results are obtained:

$$\begin{aligned}
\Delta u^3 &= -\frac{\pi^{1/2} n^2}{80}\left(\frac{\overline{C^2}}{3}\right)^{-2} \times \left[a_1 + 3\left(\frac{\overline{C^2}}{3}\right)^{-1/2}\bar{u}a_{11} + \frac{3}{2}a_{111}\right] \times \int_0^\infty \Omega^7 I_2(\Omega) e^{-\frac{1}{4}\beta\Omega^2} \, d\Omega \\
\Delta u v^2 &= -\frac{\pi^{1/2} n^2}{80}\left(\frac{\overline{C^2}}{3}\right)^{-2} \times \left[\frac{1}{3}a_1 + \left(\frac{\overline{C^2}}{3}\right)^{-1/2}\bar{u}a_{22} + 2\left(\frac{\overline{C^2}}{3}\right)^{-1/2}\bar{v}a_{12} + \frac{3}{2}a_{122}\right] \\
&\qquad \times \int_0^\infty \Omega^7 I_2(\Omega) e^{-\frac{1}{4}\beta\Omega^2} \, d\Omega
\end{aligned} \tag{5}$$

$$\Delta uw^2 = -\frac{\pi^{1/2}n^2}{80}\left(\frac{\overline{C^2}}{3}\right)^{-2} \times\left[\frac{1}{3}a_1 + \left(\frac{\overline{C^2}}{3}\right)^{-1/2}\bar{u}a_{33} + 2\left(\frac{\overline{C^2}}{3}\right)^{-1/2}\bar{w}a_{13} + \frac{3}{2}a_{133}\right] \times\int_0^\infty \Omega^7 I_2(\Omega)e^{-\frac{1}{4}\beta\Omega^2}\,d\Omega$$

Then, according to Eq. 3,

$$\Delta u(u^2 + v^2 + w^2) = \frac{\pi^{1/2}n^2}{80}\left(\frac{\overline{C^2}}{3}\right)^{-2} \times\left[\frac{4}{3}a_1 - 2\left(\frac{\overline{C^2}}{3}\right)^{-1/2}(\bar{u}a_{11} + \bar{v}a_{12} + \bar{w}a_{13})\right] \times\int_0^\infty \Omega^7 I_2(\Omega)e^{-\frac{1}{4}\beta\Omega^2}\,d\Omega \tag{6}$$

where a number of terms disappear on application of Eqs. 3.4, 9, 12.

The a_{ij} coefficients may be introduced into Eq. 1 by means of such equations as 3.6, 18, 28,

$$\frac{n\overline{C^2}}{3}\left[5\frac{\partial}{\partial x}\left(\frac{\overline{C^2}}{3}\right) - \frac{2mn}{\mu}\left(\frac{\overline{C^2}}{3}\right)(\bar{u}a_{11} + \bar{v}a_{12} + \bar{w}a_{13})\right] = \Delta u(u^2 + v^2 + w^2) \tag{7}$$

If we introduce the quantity μ, defined by Eq. 3.6, 17, then Eq. 6 becomes

$$\Delta u(u^2 + v^2 + w^2) = \frac{n^2 m}{\mu}\left(\frac{\overline{C^2}}{3}\right)^{5/2} \times\left[\frac{4}{3}a_1 - 2\left(\frac{\overline{C^2}}{3}\right)^{-1/2}(\bar{u}a_{11} + \bar{v}a_{12} + \bar{w}a_{13})\right] \tag{8}$$

where the value of μ must be such that the expression on the right-hand side has a magnitude of first order. If the two expressions for $\Delta u(u^2 + v^2 + w^2)$ are equated, the following relation is obtained for a_1,

$$a_1 = \frac{15\mu}{4mn}\left(\frac{\overline{C^2}}{3}\right)^{-3/2}\frac{\partial}{\partial x}\left(\frac{\overline{C^2}}{3}\right) \tag{9}$$

From this result and similar expressions for a_2 and a_3 and Eqs. 3.4, 16, the following expressions are found for the three mean values that appear in the energy-transfer equation (1.9, 10),

$$\begin{aligned} \rho\overline{UC^2} &= -\frac{15}{2}\mu\frac{\partial}{\partial x}\left(\frac{\overline{C^2}}{3}\right) \\ \rho\overline{VC^2} &= -\frac{15}{2}\mu\frac{\partial}{\partial y}\left(\frac{\overline{C^2}}{3}\right) \\ \rho\overline{WC^2} &= -\frac{15}{2}\mu\frac{\partial}{\partial z}\left(\frac{\overline{C^2}}{3}\right) \end{aligned} \tag{10}$$

By substituting from Eqs. 3.6, 19, 29 and Eq. 10 into Eq. 1.9, 10, we obtain the general energy-transfer equation for slightly nonisentropic flow (see Section 3.9).

The above relations may also be expressed in terms of the temperature. Since, by definition, $RT = \frac{1}{3}\overline{C^2}$, and, for a monatomic gas, $c_v = \frac{3}{2}R$, we may write

$$\begin{aligned} \frac{1}{2}\rho\overline{UC^2} &= -\lambda\frac{\partial T}{\partial x} \\ \frac{1}{2}\rho\overline{VC^2} &= -\lambda\frac{\partial T}{\partial y} \\ \frac{1}{2}\rho\overline{WC^2} &= -\lambda\frac{\partial T}{\partial z} \end{aligned} \tag{11}$$

where λ is called the coefficient of heat conduction. According to our calculations

$$\lambda = 2.5c_v\mu \tag{12}$$

which is true for a monatomic gas in which the internal energy arises entirely from the translational motion of the molecules.

3.8 VISCOSITY AND HEAT CONDUCTION

The calculations in Section 3.6 show that the property of viscosity will be apparent in a flow only when the coefficients a_{ij} are not zero, that is, when the motion is nonisentropic (see Eq. 3.6, 18). Therefore, a gas exhibits viscous motion only when the distribution of molecular velocities departs from Maxwell's law.

A physical insight into the action of viscosity can be achieved from a consideration of the transfer of molecular momentum between two contiguous layers of the mass flow. Momentum is carried by the molecules from one layer to the other both by direct translation and by intermolecular collisions. If this transfer process is such that there is a tendency for the two layers to assume the same macroscopic velocity, then the molecular velocity distribution function is non-Maxwellian and viscous flow occurs.

Since an isentropic flow contains no viscous action, adjacent layers of mass motion slip freely by each other. As long as the molecular motion is Maxwellian, the gas has no inherent ability to adjust its mass motion toward that of a solid boundary. However, it is an experimental fact that, when a flat plate is introduced into an isentropic flow (aligned in the direction of the streamlines), the motion of the layers of mass flow near the plate is adjusted toward that of the wall. The distribution of molecular velocities adjacent to the surface differs from Maxwell's law, and the action of viscosity becomes evident by the formation of a boundary layer of nonisentropic flow.

From the macroscopic point of view the viscous action of a nonisentropic flow appears as a resistance to the distortion of an element of volume of the gas as it moves with the mass motion. Thus such quantities as $\frac{1}{2}(\bar{v}_x + \bar{u}_y)$ and $\frac{1}{3}(2\bar{u}_x - \bar{v}_y - \bar{w}_z)$, which occur in Eqs. 3.6, 19, 29, are typical components of the rate of strain tensor which specifies the deformation of the fluid element.

The coefficient of viscosity can be determined from Eq. 3.6, 17 when details of the molecules are specified. If the molecules are smooth, perfectly elastic spheres, then, from Eqs. 3.3, 17 and 3.6, 6,

$$I_2(\Omega) = 4\int_0^\sigma s\left[\frac{s^2}{\sigma^2}\left(1 - \frac{s^2}{\sigma^2}\right)\right]ds = \frac{\sigma^2}{3} \tag{1}$$

and

$$\frac{1}{\mu} = \frac{\pi^{1/2}\sigma^2}{240m}\left(\frac{\overline{C^2}}{3}\right)^{-9/2}\int_0^\infty \Omega^7 e^{-\frac{1}{4}\beta\Omega^2}\, d\Omega \tag{2}$$

The integral is a gamma function having the value $3(4\overline{C^2}/3)^4$ (see Appendix I). For spherical molecules

$$\mu = \frac{5m}{16\pi^{1/2}\sigma^2}\left(\frac{\overline{C^2}}{3}\right)^{1/2} \tag{3}$$

When the molecules are point centers of repulsive force,

$$I_2(\Omega) = \left(\frac{m\Omega^2}{2\alpha}\right)^{-2/(\nu-1)}\int_0^\infty \zeta \sin^2\delta\, d\zeta \tag{4}$$

where $\delta(\zeta)$ is given by Eq. 3.3, 15. The integral in Eq. 4 is a pure number dependent only on the value of ν in the law of repulsive force and may be denoted by $A(\nu)$. Then

$$\frac{1}{\mu} = \frac{\pi^{1/2}}{80m}\left(\frac{\overline{C^2}}{3}\right)^{-9/2}A(\nu)\left(\frac{m}{2\alpha}\right)^{-2/(\nu-1)}\int_0^\infty \Omega^{(7\nu-11)/(\nu-1)} e^{-\frac{1}{4}\beta\Omega^2}\, d\Omega \tag{5}$$

The integral is again a gamma function (see Appendix I) and has the value $\frac{1}{2}\left(\frac{4\overline{C^2}}{3}\right)^{(4\nu-6)/(\nu-1)}\Gamma\left(\frac{4\nu-6}{\nu-1}\right)$. Therefore, when the molecules are represented as point centers of repulsive force, the coefficient of viscosity is

$$\mu = \frac{5m}{8\pi^{1/2}A(\nu)\Gamma\left(4 - \frac{2}{\nu-1}\right)}\left(\frac{\overline{C^2}}{3}\right)^{1/2}\left(\frac{2m}{\alpha}\frac{\overline{C^2}}{3}\right)^{2/(\nu-1)} \tag{6}$$

It has been shown that the basic variables of isentropic flow are n, $(\bar{u}, \bar{v}, \bar{w})$, and $\overline{C^2}$ (Section 2.2). All the macroscopic properties of the flow were determined from these quantities. The above calculations of a_{ij} and μ indicate that, although a nonisentropic flow has the additional property of viscosity, the fundamental variables remain the same.

The above expressions for μ show that the coefficient of viscosity depends only on the mean kinetic energy of the unseen random molecular motion ($\overline{C^2}$). It is independent of the number of molecules per unit volume (n). In terms of thermodynamic quantities we may write

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^N \tag{7}$$

where μ_1 and T_1 are the known values of the coefficient of viscosity and the temperature at a point in the flow. For spherical molecules, $N = 1/2$, and, when the molecules are represented as point centers of force, $N = 1/2 + 2/(\nu - 1)$. The value of N can also be found experimentally. Investigations show that N is somewhat larger than $1/2$ for monatomic gases, and, therefore, the more complex point center of a weak field of force must be adopted as a molecular model in order to obtain a sufficiently accurate expression for μ in terms of T (see Section 4.1).

Our study of the rate of flow of molecular energy by means of Maxwell's transfer equation has shown that viscous action in a flow is accompanied by the conduction of heat (see Eqs. 3.7, 10). Consider the variation of $\overline{C^2}$ in an element of volume $d\tau$. If the molecular motion is Maxwellian, then, according to Eq. 2.2, 9, $\overline{C^2}$ depends only on the number density of molecules in $d\tau$. When the flow is nonisentropic, $\overline{C^2}$ is also affected by conduction arising from the departure of the distribution of molecular velocities from Maxwell's law. These effects are introduced by the terms $n\overline{UC^2}$, $n\overline{VC^2}$, and $n\overline{WC^2}$ in the energy-transfer equation (1.9, 10).

No adjustment between the internal energy of a gas and the thermal condition of a boundary is possible in an isentropic flow. Accommodation between $\overline{C^2}$ and the temperature of a wall can be brought about only by heat conduction, that is, by a departure of the molecular motion from the Maxwellian in the region adjacent to the wall.

According to Eqs. 7 and 3.7, 12, we may write

$$\frac{\lambda}{\lambda_1} = \left(\frac{T}{T_1}\right)^N \tag{8}$$

where λ_1 and T_1 are the known values of the coefficient of heat conduction and the temperature, respectively, at a point in the flow. It may be concluded that, similar to the coefficient of viscosity, the thermal conductivity depends on $\overline{C^2}$ and is independent of n.

Experimental measurement of the coefficient of viscosity (or heat conduction) provides an estimate of the mean free path. From Eq. 3

$$\sigma^2 = \frac{5m\bar{C}}{32\sqrt{2}\mu} \tag{9}$$

and Eq. 2.4, 17 becomes

$$L = \frac{32}{5\pi}\frac{\mu}{\rho\bar{C}} = \frac{16}{5}\frac{\mu}{\rho\sqrt{2\pi RT}} \tag{10}$$

3.9 NONISENTROPIC FLOW EQUATIONS IN VECTOR FORM

The general equations for a slightly nonisentropic flow can now be found by substituting for such quantities as $n\overline{UV}$, $n\overline{U^2}$, $n\overline{UC^2}$ (Eqs. 3.6, 19, 29 and 3.7, 11) in the transfer equations (1.9, 8, 10). The results may be expressed in the vector notation which, in addition to providing a convenient shorthand, facilitates the determination of the particular form of the equations of motion corresponding to a given system of orthogonal coordinates. Details of the vector notation applied to fluid mechanics may be found in Ref. 6.

In a generalized system of orthogonal coordinates, a point in space is located by the intersection of three planes: $\alpha_1 =$ constant, $\alpha_2 =$ constant, $\alpha_3 =$ constant. An infinitesimally small displacement is given by

$$d\kappa^2 = h_1^2\, d\alpha_1^2 + h_2^2\, d\alpha_2^2 + h_3^2\, d\alpha_3^2 \tag{1}$$

where

$$h_1^2 = x_{\alpha_1}^2 + y_{\alpha_1}^2 + z_{\alpha_1}^2 \tag{2}$$

with similar relations for h_2^2, h_3^2.

The vector operator ∇ (del) has the following form in generalized orthogonal coordinates,

$$\nabla \equiv \frac{\mathbf{l}_1}{h_1}\frac{\partial}{\partial\alpha_1} + \frac{\mathbf{l}_2}{h_2}\frac{\partial}{\partial\alpha_2} + \frac{\mathbf{l}_3}{h_3}\frac{\partial}{\partial\alpha_3} \tag{3}$$

where $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ are unit vectors aligned in the directions of the lines of intersection of the three planes. The operation of ∇ on scalar and vector quantities can be expressed in terms of a system of generalized orthogonal coordinates as follows:

(*a*) Operation on the scalar function b:

$$\nabla b = \frac{\mathbf{l}_1}{h_1}\frac{\partial b}{\partial\alpha_1} + \frac{\mathbf{l}_2}{h_2}\frac{\partial b}{\partial\alpha_2} + \frac{\mathbf{l}_3}{h_3}\frac{\partial b}{\partial\alpha_3} \tag{4}$$

(*b*) Operation on the unit vector $\mathbf{l}_1$:

$$\nabla\cdot\mathbf{l}_1 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\alpha_1}(h_2h_3) \tag{5}$$

$$\nabla\times\mathbf{l}_1 = \frac{1}{h_1}\left(\frac{\mathbf{l}_2}{h_3}\frac{\partial h_1}{\partial\alpha_3} - \frac{\mathbf{l}_3}{h_2}\frac{\partial h_1}{\partial\alpha_2}\right) \tag{6}$$

with similar expressions for $\nabla\cdot\mathbf{l}_2$, $\nabla\cdot\mathbf{l}_3$, $\nabla\times\mathbf{l}_2$, and $\nabla\times\mathbf{l}_3$.

(*c*) Operation on a vector, $\mathbf{b} = \mathbf{l}_1 b_1 + \mathbf{l}_2 b_2 + \mathbf{l}_3 b_3$:

$$\nabla \cdot \mathbf{b} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha_1}(b_1 h_2 h_3) + \frac{\partial}{\partial \alpha_2}(b_2 h_3 h_1) + \frac{\partial}{\partial \alpha_3}(b_3 h_1 h_2)\right] \tag{7}$$

$$\nabla \times \mathbf{b} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{l}_1 & h_2\mathbf{l}_2 & h_3\mathbf{l}_3 \\ \dfrac{\partial}{\partial \alpha_1} & \dfrac{\partial}{\partial \alpha_2} & \dfrac{\partial}{\partial \alpha_3} \\ b_1 h_1 & b_2 h_2 & b_3 h_3 \end{vmatrix} \tag{8}$$

The values of α_1, α_2, α_3 and h_1, h_2, h_3 for the three systems of orthogonal coordinates most frequently used are:

	α_1	α_2	α_3	h_1	h_2	h_3
Rectangular coordinates	x	y	z	1	1	1
Cylindrical coordinates	x	r	θ	1	1	r
Spherical coordinates	r	ϕ	θ	1	r	$r \sin \phi$

The vector forms of the equations for the transfer of mass, momentum, and energy in a nonisentropic flow are respectively:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{q}) = 0 \tag{9}$$

$$\rho \frac{d\mathbf{q}}{dt} = -\nabla p - \frac{2}{3}\nabla[\mu(\nabla \cdot \mathbf{q})] + 2(\nabla \cdot \mu\nabla)\mathbf{q} + \nabla \times [\mu(\nabla \times \mathbf{q})] \tag{10}$$

$$\rho c_v \frac{dT}{dt} + p(\nabla \cdot \mathbf{q}) = \mu[\nabla^2\mathbf{q}^2 - (\nabla \times \mathbf{q})^2 - \frac{2}{3}(\nabla \cdot \mathbf{q})^2 - 2\mathbf{q} \cdot (\nabla^2\mathbf{q})] + (\nabla \cdot \lambda\nabla)T \tag{11}$$

where the mobile operator d/dt is represented vectorially by $(\partial/\partial t) + \mathbf{q} \cdot \nabla$. The momentum-transfer equation (10) may be expressed in the following alternative form:

$$\rho \frac{d\mathbf{q}}{dt} = -\nabla p + \mu\nabla^2\mathbf{q} + \frac{1}{3}\mu\nabla(\nabla \cdot \mathbf{q}) + 2(\nabla\mu \cdot \nabla)\mathbf{q} + [(\nabla\mu) \times (\nabla \times \mathbf{q})] - \frac{2}{3}(\nabla \cdot \mathbf{q})\nabla\mu \tag{12}$$

The above equations combined with

$$p = \rho RT \tag{13}$$

and

$$\frac{\mu}{\mu_1} = \frac{\lambda}{\lambda_1} = \left(\frac{T}{T_1}\right)^N \tag{14}$$

constitute a set of six equations for the six unknowns $\mathbf{q}$, p, ρ, T, μ, λ.

3.10 MOLECULAR TRANSFER IN A DENSE MONATOMIC GAS

In the preceding theory of the flow of a monatomic gas, the diameter of the molecule was considered to be negligible compared with the mean free path. Therefore, the analysis holds for a somewhat rarefied gas. When a flow involves a region of highly compressed gas, σ may become comparable with L, and the transport of momentum and energy across a surface results not only from the motion of the molecules but also from collisions between molecules with centers on opposite sides of the surface. If the finite size of the molecule is taken into account, new expressions for the static pressure and the coefficients of viscosity and heat conduction are obtained.

When the molecules have a finite size, the physical effects are (*a*) to reduce the space in the element of volume $d\tau$ which is available to the centers of the molecules, (*b*) to produce shielding of one molecule by another, and (*c*) to introduce corrections to functions of position since the centers of two colliding molecules and the point of contact are no longer the same point in space.

When one molecule collides with another molecule, the center of the first lies on a sphere of radius σ concentric with the second molecule. No center can lie inside this sphere of contact. If the gas is not too highly compressed, the overlap in contact spheres can be neglected, and the total volume in $d\tau$ from which the centers of the molecules are excluded may be taken to be $\frac{4}{3}\pi n\sigma^3\, d\tau$. The ratio of the excluded volume to the total volume ($d\tau$) is

$$e = \tfrac{4}{3}\pi n\sigma^3 \tag{1}$$

A class 2 molecule will shield a class 1 molecule when their contact spheres overlap. The centers of all other molecules will be excluded from an area S on the sphere associated with 1 (Fig. 8). If the distance between the centers of the adjacent molecules is r, then

$$S(r) = 2\pi\sigma(\sigma - \tfrac{1}{2}r) \tag{2}$$

The shielded area is a maximum at $r = \sigma$ and zero at $r \geq 2\sigma$.

If the gas is only moderately dense, few molecules will be close enough to cause any shielding, and at most a molecule is shielded by only one other molecule. The total volume between r and $r + dr$ associated with all class 1 molecules in $d\tau$ is $4\pi r^2 dr \cdot nf_1\, d\omega_1\, d\tau$, and the total number of class 2 molecules having their centers in this volume is obtained by multiplying the above expression by $nf_2\, d\omega_2$. Multiplying again by $S(r)$, we obtain the total area of the contact spheres associated with the class 1 molecules in $d\tau$ which is shielded by class 2 molecules.

Integrating the final product over all classes of shielding and shielded molecules and over the range $\sigma \leq r \leq 2\sigma$, it is found that the average shielded area of the contact sphere associated with any molecule is $\frac{11}{3}\pi^2 n\sigma^5$. Therefore the average fraction of the area of a contact sphere which is protected from contact with the centers of colliding molecules is $\frac{11}{16}e$.

The effect of excluding the centers of the molecules from a fraction e of the volume $d\tau$ is to increase the frequency in the ratio $1/(1-e)$. On the other hand, the shielding of one molecule by another reduces the number of collisions in unit time in $d\tau$ by the factor $1 - \frac{11}{16}e$. If we apply these corrections to the expression for collision frequency and ultimately to the Boltzmann equation, and allow for the fact that the centers of colliding molecules and the point of contact are now three different points in space, a new velocity distribution function can be

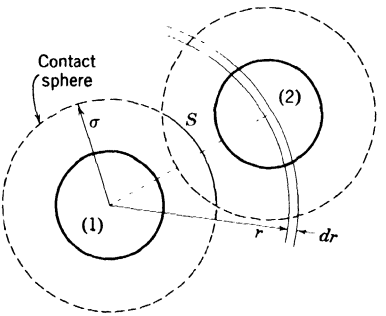

Fig. 8. Shielding of one molecule by another.

calculated, from which the transfer properties of the gas arising from the movement of the molecules from point to point may be determined.

In addition to the transport of momentum and energy produced by the motion of the molecules, we must consider the additional transfer of these quantities which occurs during collisions of molecules of finite size. Since σ can no longer be neglected compared with L, it is possible for two colliding molecules to have their centers on opposite sides of the reference surface, and a transfer of momentum and energy will take place along the line of impact. The frequency of such collisions can be calculated with corrections applied as outlined above. The resultant flux of momentum and energy across each face of $d\tau$ can be found from a typical collision between a class 1 and a class 2 molecule by integrating over all classes of molecules. This method of calculation is due to Enskog. Further details may be found in Ref. 1. 1 and Ref. 7.

By combining the results of the calculations of momentum transport by the point-to-point movement of the molecules and by transfer during collisions, a new pressure tensor is obtained having such components as

$$\begin{aligned} P_{xx} &= -p + \tfrac{2}{3}\mu'(2\bar{u}_x - \bar{v}_y - \bar{w}_z) \\ P_{xy} &= \mu'(\bar{v}_x + \bar{u}_y) \end{aligned} \tag{3}$$

where, if terms of order e^3, e^4, $\cdots$ are neglected, the static pressure is

$$p = \rho RT(1 + \tfrac{1}{2}e + \tfrac{5}{32}e^2) - \tfrac{3}{4}e^2\mu(\bar{u}_x + \bar{v}_y + \bar{w}_z) \tag{4}$$

and the new coefficient of viscosity is

$$\mu' = \mu(1 + 0.0875e + 0.217e^2) \tag{5}$$

In the above expressions μ is the coefficient of viscosity corresponding to rigid, spherical molecules in a somewhat rarefied gas (see Eq. 3.8, 3).

It is significant that in a dense gas the equation of state departs from the simple form $p = \rho RT$ which holds for both an isentropic and a nonisentropic flow in a somewhat rarefied gas. When the gas is at rest, the equation of state resembles van der Waals' relation if terms of order e^2 are neglected. When the gas is in motion, an important new effect called compression or bulk viscosity appears as a direct result of the finite size of the molecules. The coefficient of bulk viscosity has the form $\frac{3}{4}e^2\mu$. This viscosity effect will be important in flows involving high density and a large value of the dilitation $(\bar{u}_x + \bar{v}_y + \bar{w}_z)$.

A similar analysis of the transfer of energy by the translation and collision of rigid, spherical molecules shows that the coefficient of the derivatives T_x, T_y, and T_z in the expressions for the heat flux is

$$\lambda' = \lambda(1 + 0.287e + 0.216e^2) \tag{6}$$

where λ is the coefficient of heat conduction in a somewhat rarefied gas ($\lambda = 2.5c_v\mu$). Thus the relation between the coefficients of heat conduction and viscosity in a dense gas has the form

$$\lambda' = 2.5c_v\mu(1 + 0.200e - 0.018e^2) \tag{7}$$

The above discussion is based on the assumption that the molecules are rigid spheres which interact only on contact. If the molecules are regarded as point centers of spherically symmetrical force, then each of the momentum-transfer equations (Eqs. 1.9, 8) includes an additional term which indicates the over-all effect of the forces acting on the molecules in $d\tau$. Thus the first of the momentum equations has the form

$$n\frac{d\bar{u}}{dt} = \Sigma F_x - \left[\frac{\partial}{\partial x}(\rho\overline{U^2}) + \frac{\partial}{\partial y}(\rho\overline{UV}) + \frac{\partial}{\partial z}(\rho\overline{UW})\right] \tag{8}$$

where $\sum F_x$ is the total force acting on all the molecules in $d\tau$ in the x-direction. If no external field of force is present, then $\sum F_x$ arises solely from intermolecular forces. Owing to the principle of action and reaction, the forces acting between molecules within the walls of $d\tau$ will make no resultant contribution to $\sum F_x$. Only forces acting across the boundaries of $d\tau$ will provide an over-all effect which may be expressed in the form of a tensor with components f_{xx}, f_{xy}, $\cdots$ acting on unit area. Then the resultant force acting in the direction of x on all molecules in $d\tau$ is

$$\left(\frac{\partial f_{xx}}{\partial x} + \frac{\partial f_{yx}}{\partial y} + \frac{\partial f_{zx}}{\partial z}\right) dx\, dy\, dz \tag{9}$$

and the first of the momentum equations may be written

$$n\frac{d\bar{u}}{dt} = \frac{\partial}{\partial x}(f_{xx} - \rho\overline{U^2}) + \frac{\partial}{\partial y}(f_{yx} - \rho\overline{UV}) + \frac{\partial}{\partial z}(f_{zx} - \rho\overline{UW}) \tag{10}$$

A comparison with Eqs. 1.10, 3 shows that

$$P_{xx} = f_{xx} - \rho\overline{U^2}, \quad P_{yx} = f_{yx} - \rho\overline{UV}, \cdots \tag{11}$$

and, according to the definition of static pressure (Eq. 1.10, 5),

$$p = \tfrac{1}{3}\rho\overline{C^2} - \tfrac{1}{3}(f_{xx} + f_{yy} + f_{zz}) \tag{12}$$

A correction for the intermolecular forces acting across the faces of $d\tau$ therefore leads to a modified equation of state having a correcting term similar in form to the virial of Clausius (Ref. 1. 2, page 70). The detailed evaluation of $f_{xx}, f_{yx}, \cdots$ is a matter for further research.

A discussion of the physical significance of some of the results in this section is presented in Section 4.1.

NOTATION

a_i, a_{ij}, a_{ijk}	coefficients in the power series $F(H_i)$ which are functions of x, y, z, t $(i, j, k = 1, 2, 3)$
$A(v)$	the definite integral in Eq. 3.8, 4
b	any scalar function of x, y, z, t
$\mathbf{b}$	any vector function of x, y, z, t, magnitudes of the components being b_1, b_2, b_3
e	total volume occupied by the molecules in a unit volume
f	general velocity distribution function (Eq. 3.4, 1)
f_o	Maxwell's velocity distribution function (Eq. 2.1, 12)
f_{xx}, f_{yx}, f_{zx}	components of the tensor of total intermolecular force per unit area
$F(H_i)$	polynomial which indicates the deviation of f from f_o (Eq. 3.4, 4, see also $F(U, V, W)$, Appendix I)
$\sum F_x, \sum F_y, \sum F_z$	components of the total force acting on all the molecules in $d\tau$
$g(r)$	force acting between two colliding molecules

G	symbol denoting the sum $1 + G_1 + G_2$ (see Appendix I)
G_1, G_2	forms for $F(U_1, V_1, W_1)$ and $F(U_2, V_2, W_2)$, respectively, when $X_1, Y_1, \cdots, Z_2$ are introduced (Eqs. 2.4, 9)
h_1, h_2, h_3	coefficients of the displacement expression (3.9, 1)
H_i $(i = 1, 2, 3)$	dimensionless forms for the components of the random velocity of a molecule (Eqs. 3.4, 2)
I_1, I_2	definite integrals which depend on the force law in molecular collisions (Eqs. 3.6, 6)
J_1, J_2, J_3	definite integrals with respect to X_2, Y_2, Z_2 (Eqs. 3.6, 9, 10, 11)
K	the constant $(2\pi)^{-3/2}$
$\mathbf{l}$	unit vector in the direction of r (Fig. 5)
$\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$	orthogonal unit vectors in physical space
$\mathbf{m}$	unit vector in the direction of increasing θ (Fig. 5)
N	exponent in the law of variation of μ with T (Eq 3.8, 7)
r	polar coordinate, or distance between the centers of two molecules
s	perpendicular distance between the line of action of Ω and the origin of coordinates (Fig. 3)
S	area of contact sphere of one molecule shielded by another molecule (Fig. 8)
α	constant in the law of repulsive force (contains n.[2])
$\alpha_1, \alpha_2, \alpha_3$	parameters associated with three orthogonal surfaces
Γ	standard symbol for the gamma function
δ	angle between the initial and final relative velocities Ω, Ω' or the angle of deflection (Fig. 6)
ϵ	polar coordinate in a plane perpendicular to the plane containing the orbit (Fig. 3)
ζ	function of s, Ω defined by Eq. 3.3, 14
η	the ratio s/r
η_0	root of Eq. 3.3, 11
θ	polar coordinate in a plane (Figs. 5, 6) or space (Eqs. 3.5, 3)
$d\kappa$	displacement in physical space
λ	coefficient of heat conduction in a somewhat rarefied gas
λ'	coefficient of heat conduction in a dense gas
μ	coefficient of viscosity in a somewhat rarefied gas
μ'	coefficient of viscosity in a dense gas
ν	exponent in the law of repulsive force (α/r^ν)
ξ_2, ξ_3	angles in the collision diagram given by Eqs. 3.2, 6, 9
$\Phi(r)$	force potential per unit mass

Note: Symbols that do not appear above may be found in the Notation at the end of Chapters 1 and 2 (pages 24, 63). The subscript 1 applied to macroscopic functions of x, y, z, t denotes a known reference value. The notation Fig. 2.7 means Fig. 7 in Chapter 2.

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CHAPTER FOUR

Nonisentropic Flows

4.1 EXPERIMENTAL INVESTIGATIONS OF VISCOSITY AND HEAT CONDUCTION

The coefficient of shear viscosity can be expressed in terms of measurable quantities. A relation of this type may be obtained as a solution of the basic flow equations (3.9, 9, 10, 11) for given boundary conditions. The mathematical work can be greatly simplified and the experimental accuracy increased if the flow conditions and the boundary configurations are selected so that viscosity plays a dominant role, and other effects such as that due to compressibility are negligibly small. In order that the calculated relation for μ may apply, the tests must be made at densities that correspond to a mean free path which is much smaller than the dimensions of the apparatus ($Kn \ll 1$).

The coefficient of viscosity has been accurately measured in long cylindrical tubes in which viscous action produces a parabolic distribution of mass velocity rising from zero at the wall to a maximum value on the axis of the tube. The motion is slow, and the density and temperature of the gas are sensibly constant throughout. A calculation of the rate of flow of gas through the tube leads to the following expression:

$$\mu = \frac{\pi(p_1 - p_2)d^4}{128lQ} \tag{1}$$

where the tube has a diameter d and a length l, $p_1 - p_2$ is the difference in static pressure between the ends of the tube, and Q is the volume of gas which flows through the tube in unit time (Ref. 1. 2, p. 167). These quantities can be determined experimentally and values for μ obtained.

Instruments called viscometers have been designed to utilize the above relation (Ref. 1, p. 102). Corrections must be applied when the fully developed parabolic flow does not exist over a sufficient length of the tube.

Accurate measurements of μ have also been obtained from observations of the rotating shear flow between two cylinders (Ref. 1, p. 205). In this method the action of viscosity is found directly from the deflection

of a cylinder suspended coaxially inside a rotating cylinder which induces a viscous flow in the intervening annular space.

For most problems in fluid mechanics the coefficient of viscosity is considered to vary with temperature only (Ref. 1, p. 117). Equations 3.8, 3, 6 indicate that the relationship is of the type given by Eq. 3.8, 7, where N depends on the mathematical model chosen to represent the

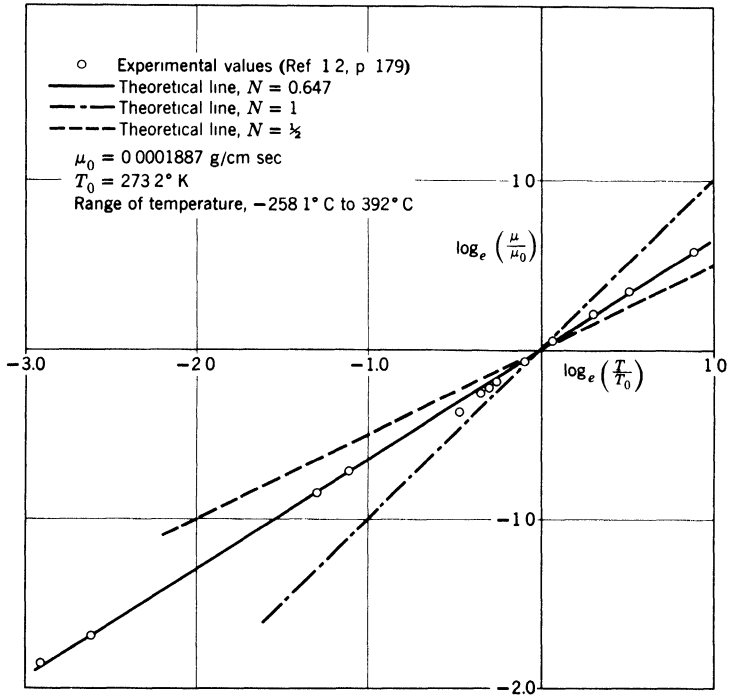


Fig. 1. Variation of viscosity of helium with temperature.

gas molecule. Since our derivation of the expression for viscosity applies strictly to a monatomic gas, let us investigate the variation of the viscosity of helium with temperature (Fig. 1). It will be seen that, over a wide range of temperatures, Eq. 3.8, 7 gives an accurate description of the dependence of μ on T if $N = 0.647$. Since N is appreciably larger than $\frac{1}{2}$, the relation between μ and T (Eq. 3.8, 3) based on the representation of the molecules as smooth, perfectly elastic spheres is not satisfactory (Fig. 1). The point-center molecule is more suitable for viscosity calculations ($\nu = 14.6$ for helium). Note that the values of N

determined experimentally for various monatomic gases are appreciably different (Ref. 1. 1).

It will be realized, however, that even a point center of repulsive force is probably an oversimplified model of the gas molecule. Although the dependence of μ on T for helium (and hydrogen) is given accurately by Eq. 3.8, 7, for many other gases the following relation, developed by Sutherland (Ref. 2), is more satisfactory:

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^{3/2} \left(\frac{T_1 + \kappa}{T + \kappa}\right) \tag{2}$$

where μ_1, T_1 are given. This result is based on the assumption that the molecules are smooth, elastic spheres surrounded by a weak, attractive field of force. The effect of adding the attractive force to the spherical molecule is to reduce the mean free path and increase the collision frequency. The analysis indicates that these effects can be accounted for if the apparent diameter of the molecules is taken to be the geometrical diameter multiplied by $\left(1 + \frac{\kappa}{T}\right)^{1/2}$. Thus the diameter depends on the temperature, becoming smaller as the temperature increases.

Sutherland's formula is used regularly to calculate the viscosity of air, which is important in aerodynamics (Ref. 3). Equation 2 provides an accurate value for μ over the range $0 \leq T \leq 300°$ C with $\kappa = 114$, $T_1 = 273.2°$ K, and $\mu_1 = 1.709 \times 10^{-4}$ g/cm sec.

Since Sutherland's work was based on a modification of the elastic sphere, it is to be expected that further progress might be made by a refinement of the field of force surrounding the point-center molecule. Lennard-Jones has considered a law of force such that two molecules attract or repel each other depending on whether the distance of separation is large or small (Ref. 4). He obtained a more general relation between μ and T, which may be expressed in the form

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^{3/2} \left(\frac{T_1^{(\nu-3)/(\nu-1)} + \kappa}{T^{(\nu-3)/(\nu-1)} + \kappa}\right) \tag{3}$$

where the forces of repulsion and attraction are assumed to be constant/r^ν and $-$ constant/r^3, respectively.

Although accurate formulae are required to calculate the viscosity for specific conditions, the introduction of even such relations as Eq. 3.8, 7 into the basic flow equations (3.9, 9, 10, 11) leads to considerable complication of the mathematical treatment. In many flow problems μ is assumed to be constant or to be given with sufficient accuracy by an expression of the type

$$\frac{\mu}{\mu_1} = \sigma\left(\frac{T}{T_1}\right) \tag{4}$$

where the constant σ is chosen to give the correct value of μ when T is selected somewhere in the range of interest in a particular problem. Frequently, in aerodynamic investigations, the relation for μ must be reasonably reliable over a large range of temperatures which includes the temperature of the undisturbed flow in the test section of a wind tunnel and the temperature of the atmosphere in free flight. Equation 4 has been found to involve an error of less than 5 or 6 per cent in boundary layer calculations over this range (Ref. 5).

As Eq. 3.7, 12 indicates, the coefficient of heat conduction follows closely the pattern set by viscosity, and λ is also a function of temperature only and independent of density unless the gas is highly compressed. The numerical coefficient 2.5 in Eq. 3.7, 12 holds for a monatomic gas since the theory has been developed for this case only. For other gases, Eucken (Ref. 6) has developed a semiempirical expression as follows:

$$\lambda = \tfrac{1}{4}(9\gamma - 5)\mu c_v \tag{5}$$

Experimental measurements of $\lambda/\mu c_v$ show that it is reasonably close to 2.5 for monatomic gases, and for many other gases the coefficient $\frac{1}{4}(9\gamma - 5)$ gives a satisfactory approximation (Ref. 1. 1, p. 241).

The macroscopic motion of a gas in a cylindrical tube is described as laminar when the radial distribution of mass velocity is parabolic. As the speed of the flow is increased, the motion ultimately becomes turbulent, and the distribution of mass velocity assumes a new form. In this type of flow, viscosity and heat conduction arise from a transfer process which is accompanied by interactions between large clusters of molecules. Since the equations of motion of Chapter 3 are based on the assumption that only binary molecular encounters are significant in the gas flow, they are not valid for the calculation of turbulent motion. A kinetic theory of liquids has been developed by Born and Green (Ref. 7) in which multiple molecular encounters are investigated.

The relations for the coefficient of viscosity derived in Section 3.8 show that in terms of the fundamental variables of the kinetic theory μ depends on $\overline{C^2}$ but not on n. An interesting verification of this result was obtained for a somewhat rarefied gas by Boyle as early as 1660, who observed that at ordinary pressures the degree of damping of an oscillating pendulum produced by the viscous action of the surrounding gas was independent of the density. However, it has been seen in Section 3.10 that, when the density of a monatomic gas is such that the mean free path is comparable with the diameter of the molecules, the shear viscosity varies with the density according to a relation of the type

$$\mu' = \mu[1 + k_1(T)\rho + k_2(T)\rho^2] \tag{6}$$

where

$$e = k(T)\rho, \quad k(T) = \frac{4\pi\sigma^3}{3m}, \quad k_1 = 0.0875k, \quad k_2 = 0.217k^2 \tag{7}$$

and we note that, in general, the effective diameter of the molecule (σ) is a function of the temperature (T). The ratio μ'/μ is plotted against pressure in Fig. 2 for carbon dioxide (Ref. 8) and nitrogen (Ref. 9).

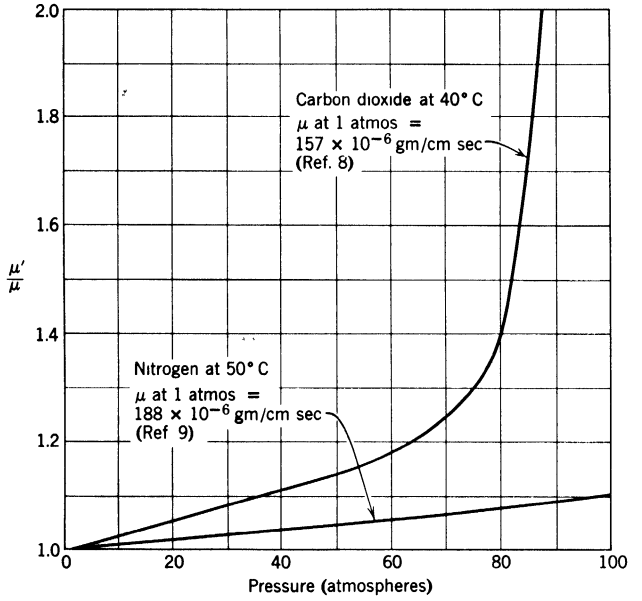


Fig. 2. Experimental determination of the variation of viscosity with pressure.

As the pressure of a gas is increased, a slow increment in the shear viscosity becomes apparent. At high pressures the increase in μ' over μ becomes significant. A rapid rise in the curve for CO_2 (Fig. 2) occurs when the gas begins to take on some of the characteristics of a liquid.

The static pressure in a dense, monatomic gas may be expressed in the form

$$p = \rho RT[1 + k_3(T)\rho + k_4(T)\rho^2] - \mu_b(\bar{u}_x + \bar{v}_y + \bar{w}_z) \tag{8}$$

where

$$k_3 = \tfrac{1}{2}k, \quad k_4 = \tfrac{5}{32}k^2 \tag{9}$$

When a dense gas has no mass motion, the second term on the right-hand side of Eq. 8 disappears and the equation of state is

$$p = \rho RT[1 + k_3(T)\rho + k_4(T)\rho^2] \tag{10}$$

If the term $k_4\rho^2$ is dropped, then Eq. 10 is a special form of van der Waals' equation of state which holds for spherical molecules.

It will be noted that e can be found experimentally from measurements of the variation of pressure with temperature since, if the variation of e with T can be neglected,

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \rho R\left(1 + \frac{1}{2}e + \frac{5}{32}e^2\right) \tag{11}$$

A summary outlined in Ref. 1. 1 (p. 289) indicates that, using measurements of e made in this way, the calculated and the experimental values of μ' for nitrogen agree reasonably well for pressures up to about 1000 atmospheres.

The monatomic theory may be sufficient to indicate the deviation of μ' from μ, but it provides only a qualitative idea of the effect of bulk viscosity in a gas flow. The form of Eq. 8 suggests that, where a very large dilitation occurs in a flow, we may expect the equation of state to differ noticeably from the idealized relation, $p = \rho RT$. No important effects due to bulk viscosity appear to have been found in monatomic gases, but ample evidence exists to show that strong deviations from the idealized equation of state can occur in diatomic and polyatomic gas flows.

Bulk viscosity effects have been encountered mainly in the propagation of ultrasonic sound waves in diatomic and polyatomic gases. Investigations have shown that the accompanying absorption of energy is due largely to viscous action, the associated heat conduction and radiation having little effect (Ref. 10). Stokes calculated an expression for the absorption coefficient in terms of the shear viscosity (Ref. 11). When it was found experimentally that the absorption was much greater than that calculated from the Stokes formula, a bulk viscosity was introduced empirically following the pattern of Eq. 8.

In the following discussion we shall treat μ_b in the more general sense in which it may be applied to all gases and represents essentially a measure of the deviation of the equation of state for a given gas flow from the idealized relation (Eq. 8). Although the monatomic theory suggests a bulk viscosity effect due to the finite size of the molecules, the available experimental evidence indicates that a much more important effect arises when the molecule has other degrees of freedom in addition to the three degrees of translation possessed by the simple point-center molecule. However, both theory and experiment show that a fundamental deviation from the idealized equation of state ($p = \rho RT$) is produced by any modification of the simple molecular model to a more complex form.

The determination of μ_b is still a matter for investigation, and, as yet, no direct method of measurement has been devised. An experimental method for measuring the ratio μ_b/μ has been developed (Ref. 12). It is based on the theoretical work of Eckart (Ref. 13), who considered the propagation of a sound beam without reflection along the axis of a tube. However, experimental results still lack consistency, and only orders of magnitude can be stated (Ref. 14). Tisza concludes from the available literature that μ_b/μ for carbon dioxide is roughly 2000, but for air the ratio is of order 1 (Ref. 15).

4.2 EQUATIONS FOR STEADY FLOW IN ONE DIMENSION

The properties of the transfer equations for a slightly nonisentropic flow can be conveniently investigated in one dimension. The problem is reduced to one dimension in order to minimize the mathematical complexity while retaining the general physical properties of the motion. If the gas flow contains a very large gradient of velocity, then, neglecting the van der Waals correction terms in Eq. 4.1, 8,

$$p = \rho RT - \mu_b \frac{d\bar{u}}{dx} \tag{1}$$

The relations required to evaluate the momentum-transfer equations (1.9, 8) are

$$\begin{aligned} -\rho\overline{U^2} + P_{xx} &= -p + \frac{4}{3}\mu\frac{d\bar{u}}{dx} \\ \rho\overline{V^2} &= \rho\overline{W^2} = p \\ \rho\overline{UV} &= \rho\overline{VW} = \rho\overline{WU} = 0 \end{aligned} \tag{2}$$

where the difference between μ' and μ is regarded as negligible. The terms necessary to complete the energy-transport equation (1.9, 10) have the values

$$\frac{1}{2}\rho\overline{UC^2} = -\lambda\frac{dT}{dx}, \quad \frac{1}{2}\rho\overline{VC^2} = \frac{1}{2}\rho\overline{WC^2} = 0 \tag{3}$$

Then the equations for the transfer of mass, momentum, and energy in the one-dimensional, steady flow of a monatomic gas are

$$\begin{aligned} \frac{d}{dx}(\rho\bar{u}) &= 0 \\ \rho\bar{u}\frac{d\bar{u}}{dx} &= -\frac{d}{dx}(\rho RT) + \frac{d}{dx}\left[\left(\frac{4}{3}\mu + \mu_b\right)\frac{d\bar{u}}{dx}\right] \\ \rho c_v\bar{u}\frac{dT}{dx} + \rho RT\frac{d\bar{u}}{dx} &= \left(\frac{4}{3}\mu + \mu_b\right)\left(\frac{d\bar{u}}{dx}\right)^2 + \frac{d}{dx}\left(\lambda\frac{dT}{dx}\right) \end{aligned} \tag{4}$$

These relations are referred to in the literature as the Navier-Stokes equations in one-dimensional form, although in many references μ_b is absent.

Equations 4 can be rearranged and the three following alternative relations obtained

$$\frac{d}{dx}(\rho\bar{u}) = 0$$

$$\frac{d}{dx}\left[\rho\bar{u}^2 + \rho RT - \left(\frac{4}{3}\mu + \mu_b\right)\frac{d\bar{u}}{dx}\right] = 0 \tag{5}$$

$$\frac{d}{dx}\left[\rho\bar{u}\left(c_v T + \frac{1}{2}\bar{u}^2\right) - \left(\frac{4}{3}\mu + \mu_b\right)\bar{u}\frac{d\bar{u}}{dx} - \lambda\frac{dT}{dx}\right] = 0$$

These equations are in integrable form. For all values of x

$$\rho\bar{u} = \rho_1\bar{u}_1$$

$$\rho RT + \rho\bar{u}^2 - \left(\frac{4}{3}\mu + \mu_b\right)\frac{d\bar{u}}{dx} = \rho_1 RT_1 + \rho_1\bar{u}_1^2 \tag{6}$$

$$\rho\bar{u}\left(c_v T + \frac{1}{2}\bar{u}^2\right) - \left(\frac{4}{3}\mu + \mu_b\right)\bar{u}\frac{d\bar{u}}{dx} - \lambda\frac{dT}{dx} = \rho_1\bar{u}_1\left(c_v T_1 + \frac{1}{2}\bar{u}_1^2\right)$$

where the constants of integration have been evaluated in terms of the known values $\bar{u}_1$, ρ_1, and T_1 at a point in the flow (state 1) at which

$$\left(\frac{d\bar{u}}{dx}\right)_1 - \left(\frac{dT}{dx}\right)_1 = 0 \tag{7}$$

Equations 6 combined with

$$\frac{\mu}{\mu_1} = \frac{\lambda}{\lambda_1} = \left(\frac{T}{T_1}\right)^N \tag{8}$$

and a relation for the bulk viscosity (μ_b) in terms of ρ, T constitute a set of six equations for six unknowns ($\bar{u}$, ρ, T, μ, λ, μ_b).

The complete set of equations can be expressed in dimensionless form according to the notation given at the end of this chapter and reduced to the following

$$\frac{K}{Re_1}\left(\frac{4}{3} + K_b\right)\frac{dU}{dX} = \frac{\Theta}{\gamma M_1^2 U} + U - \left(1 + \frac{1}{\gamma M_1^2}\right) \tag{9}$$

$$\frac{K}{(\gamma - 1)M_1^2 Pr Re_1}\frac{d\Theta}{dX} = \frac{1}{\gamma(\gamma - 1)}\frac{\Theta}{M_1^2} - \frac{1}{2}U^2 + \left(1 + \frac{1}{\gamma M_1^2}\right)U - \left(\frac{1}{2} + \frac{1}{(\gamma - 1)M_1^2}\right) \tag{10}$$

where
$$U = \frac{\bar{u}}{\bar{u}_1} = \frac{\rho_1}{\rho} \tag{11}$$

and
$$K = \frac{\mu}{\mu_1} = \frac{\lambda}{\lambda_1}, \quad K_b = \frac{\mu_b}{\mu} \tag{12}$$

The mean free path has been taken as the basic reference dimension. The dimensionless form of the equations shows that the nonisentropic flows of two gases having the same laws of viscosity and heat conduction will be dynamically similar if γ, M, Re, and Pr, calculated for state 1, are the same for both motions, where

$$\gamma = \frac{c_p}{c_v}, \quad M_1 = \frac{\bar{u}_1}{a_1}, \quad Re_1 = \frac{\rho_1 L_1 \bar{u}_1}{\mu_1}, \quad Pr = \frac{\mu_1 c_p}{\lambda_1} \tag{13}$$

The relative significance of these dimensionless parameters can be assessed for a monatomic gas in terms of the kinetic theory. When the energy of the molecule is wholly translational, the ratio of specific heats (γ) has the value $1\frac{2}{3}$ (see Section 2.2). According to Eqs. 2.4, 1 and 2.5, 10, the Mach number is given by the relation

$$M_1 = 2\left(\frac{6}{5\pi}\right)^{1/2}\left(\frac{\bar{u}}{\bar{C}}\right)_1 \tag{14}$$

and, on substituting for L from Eq. 3.8, 10, the Reynolds number is

$$Re_1 = \frac{32}{5\pi}\left(\frac{\bar{u}}{\bar{C}}\right)_1 \tag{15}$$

From Eq. 3.7, 12, we have $Pr = Pr(\gamma) = \frac{2}{3}$. Therefore, since γ and Pr are constant, the fundamental flow parameter is $\bar{u}/\bar{C}$ and we conclude that the one-dimensional motion of two somewhat rarefied, monatomic gases ($K_b = 0$) having the same value of N (Eq. 3.8, 7) will be dynamically similar if the ratio of the mass velocity ($\bar{u}$) to the mean random speed of the molecules ($\bar{C}$) is the same at two corresponding points in the flows. Eliminating $\bar{u}/\bar{C}$ from Eqs. 14, 15, we obtain the result

$$Re_1 = 1.648 M_1 \tag{16}$$

which is true at any point in the flow of a monatomic gas.

In many problems of fluid dynamics, diatomic gases are of more interest (air, in particular). The above equations are still satisfactory for determining the properties of the slightly nonisentropic flow of a diatomic gas provided the appropriate experimental values of γ, Pr, and N are used. Experiments on diatomic gases indicate that γ is about 7/5, and since, according to Eq. 4.1, 5,

$$Pr = \frac{4\gamma}{9\gamma - 5} \tag{17}$$

then Pr is approximately 3/4. More exact values for specific gases can be obtained from tables in Refs. 1.1 and 1.2. Since the energy of the diatomic molecule is not all translational, Eq. 16 must be revised. The Mach number and the Reynolds number usually vary appreciably throughout the flow, and they are the primary factors upon which the motion depends. However, if large ranges of temperature occur in the motion of a diatomic gas, the corresponding variations of γ and Pr may become significant and they must then be taken into account. This question is discussed further in Section 4.9.

4.3 THE SHOCK TRANSITION

In many problems of fluid mechanics the motion may be regarded as isentropic except in certain "layers" in which very rapid changes of state occur. The shock front and the boundary layer are two such regions. Let us investigate the properties of these nonisentropic layers from the point of view of the molecular theory of gases.

The plane shock wave is a sharp transition of density, pressure, temperature, and mass velocity between two states (ρ_1, p_1, T_1, $\bar{u}_1$) and (ρ_2, p_2, T_2, $\bar{u}_2$) in which all derivatives with respect to x are zero. This sudden change of state is accompanied by viscosity effects and heat conduction. When Eqs. 4.2, 6 are applied to states 1 and 2, they become the Rankine-Hugoniot conditions for the plane shock wave (Refs. 16, 17):

$$U_2 = \frac{1}{\gamma + 1}\left(\gamma - 1 + \frac{2}{M_1^2}\right) \tag{1}$$

$$P_2 = \frac{1}{\gamma + 1}(2\gamma M_1^2 - \gamma + 1) \tag{2}$$

$$\Theta_2 = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_1^2 - 1)\left(\gamma + \frac{1}{M_1^2}\right) \tag{3}$$

where U_2, P_2, and Θ_2 are, respectively, the mass velocity, pressure, and temperature in state 2 referred to their corresponding values in state 1.

The over-all change in entropy is

$$S_2 - S_1 = \log [P_2 U_2^{\gamma}] \tag{4}$$

According to the second law of thermodynamics, $S_2 \geq S_1$, and, on substituting for P_2 and U_2, it is found that this condition is fulfilled if $M_1 \geq 1$. Thus the transition involves a decrease in mass velocity and an increase in the density, pressure, and temperature. The velocity decrement through the shock front is $1 - U_2$ or $\frac{2}{\gamma + 1}\left(\frac{1}{1 + M_1^2}\right)$.

We shall now investigate the shock transition in detail from the point of view of the Navier-Stokes equations developed in Section 4.2. If the transition is a slightly nonisentropic process, the momentum and energy transport is specified by Eqs. 4.2, 9, 10. These relations can be expressed in a form which involves only U and the viscosity ratio K in terms of X,

$$U\frac{d^2U}{dX^2} + \left(\frac{dU}{dX}\right)^2 + \frac{1}{K}\left[A_1 - \left(A_2 - \frac{dK}{dX}\right)U\right]\frac{dU}{dX} + \frac{A_3}{K^2}(U-1)(U-U_2) = 0 \tag{5}$$

where A_1, A_2, and A_3 have the following (constant) values in terms of the quantities of state 1,

$$A_1 = \frac{3}{4}Re_1\left(1 + \frac{1}{\gamma M_1^2}\right), \quad A_2 = Re_1\left(\frac{3}{2} + \frac{Pr}{\gamma}\right), \quad A_3 = \frac{3(\gamma+1)Re_1^2 Pr}{8\gamma} \tag{6}$$

Equation 5 is valid for a gas in which the bulk viscosity may be neglected compared with the shear viscosity ($K_b \ll \frac{4}{3}$). The effect of bulk viscosity will be considered later in this section.

With a view to simplifying the mathematical problem to facilitate a preliminary discussion of the shock transition, let us investigate the circumstances under which the coefficients of viscosity and heat conduction may be regarded as sensibly constant through the transition ($N = 0$). If the increment in temperature through the shock front is small compared with T_1, then the order of the over-all change in K is $N(T_2 - T_1)/T_1$. It appears possible, therefore, to obtain some qualitative idea of the properties of the shock transition if we solve Eq. 5 for the case in which the coefficient of viscosity is constant.

If $K = 1$, Eq. 5 becomes

$$U\frac{d^2U}{dX^2} + \left(\frac{dU}{dX}\right)^2 + (A_1 - A_2U)\frac{dU}{dX} + A_3(U-1)(U-U_2) = 0 \tag{7}$$

This is a nonlinear differential equation, and a complete solution is still a matter of some difficulty. However, a particular solution which satisfies the boundary conditions

$$U > 1, \; X \to -\infty; \quad U \to U_2, \; X \to \infty \tag{8}$$

can be readily obtained for a Prandtl number of $\frac{3}{4}$. We note that Eq. 7 can be factored and written

$$\left[\frac{d}{dX} + \left(\frac{2A_1}{1+U_2} - A_2\right)\right]\left[U\frac{dU}{dX} - \frac{A_1}{1+U_2}(U-1)(U-U_2)\right] = 0 \tag{9}$$

provided $$\frac{A_1}{1+U_2}\left(\frac{2A_1}{1+U_2} - A_2\right) + A_3 = 0 \tag{10}$$

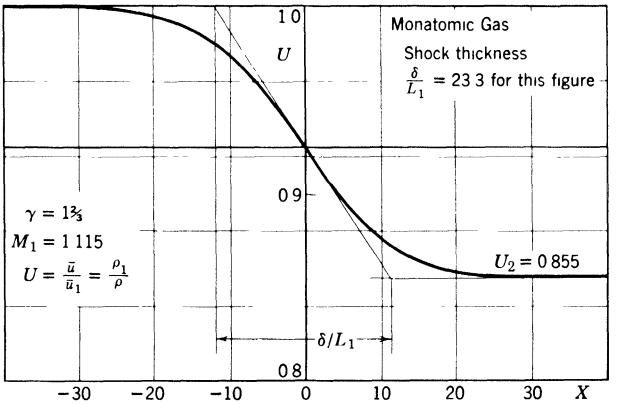
which reduces to $Pr = \frac{3}{4}$. At this particular value of the Prandtl number the differential equation for the velocity transition becomes

$$U\frac{dU}{dX} - \frac{3}{8}Re_1\left(\frac{\gamma+1}{\gamma}\right)(U-1)(U-U_2) = 0 \tag{11}$$

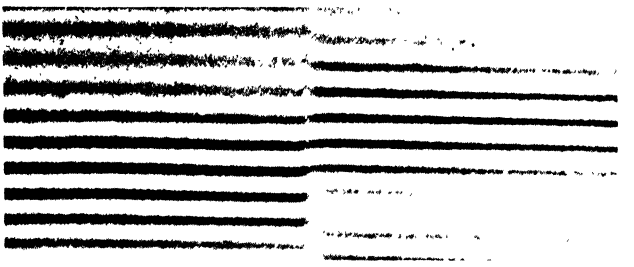
and the solution is

$$X = \frac{8}{3Re_1}\left(\frac{\gamma}{\gamma+1}\right)\left(\frac{1}{1-U_2}\right)\left[\log\left(\frac{1-U}{1-\sqrt{U_2}}\right) - U_2\log\left(\frac{U-U_2}{\sqrt{U_2}-U_2}\right)\right] \tag{12}$$

This solution coincides with Becker's result obtained for $Pr = \frac{3}{4}$ by different reasoning (Refs. 18, 19).



(*a*) U (or ρ_1/ρ) vs. X.



State 1 State 2

(*b*) UTIA interferogram of shock wave in air.

Fig. 3. The shock transition.

A typical velocity transition based on Eq. 12 is plotted in Fig. 3 (*a*) for $M_1 = 1.115$ ($Re_1 = 1.83$). From Eqs. 1, 3 the final velocity and temperature ratios are $U_2 = 0.855$, $\Theta_2 = 1.113$, which hold for a monatomic gas ($\gamma = 1\frac{2}{3}$, $Pr = \frac{2}{3}$). According to Mandl's general solution of Eq. 5 (for constant μ), the substitution of a Prandtl number of $\frac{2}{3}$ instead of $\frac{3}{4}$ in the calculation of the profile plotted in Fig. 3 has little effect on the profile when M_1 is in the neighborhood of 1.1 (Ref. 20). It will be noted that the transition proceeds from $-\infty$ to $+\infty$, the maximum gradient occurring at $X = 0$ ($U = \sqrt{U_2}$), and most of the change in U is concentrated in the region adjacent to this point.

The thickness of a shock wave may be defined in terms of the maximum gradient as follows (see Fig. 3 (*a*)):

$$\frac{\delta}{L_1} = \frac{1 - U_2}{\left|\frac{dU}{dX}\right|_{\max}} \tag{13}$$

The maximum absolute value of the velocity derivative can be obtained from Eq. 11, and it is found that

$$\frac{\delta}{L_1} = \frac{8}{3Re_1}\left(\frac{\gamma}{\gamma + 1}\right)\left(\frac{1 + \sqrt{U_2}}{1 - \sqrt{U_2}}\right) \tag{14}$$

where U_2 is given by Eq. 1. The dimensionless parameter δ/L_1 is significant from the point of view of the kinetic theory of gases since it relates the effective range of transition of the macroscopic properties of the gas to the initial mean free path.

Details of the internal process in the shock front are difficult to determine experimentally. However, Greene, Cowan, and Hornig have measured the thickness of weak shock waves in argon and nitrogen in a shock tube by a method based on the reflection of light produced by the sharp density gradients in the transition (Ref. 21). Sherman has measured the profiles and thicknesses of normal shock waves of moderate strength, using an equilibrium temperature probe (Ref. 22). The theoretical and experimental values of the reciprocal shock thickness ratio (L_1/δ) are compared for various Mach numbers in the initial state (M_1) up to 2 in Fig. 4. The accuracy of the experimental values given in Ref. 21 is not high (the error may be as much as $\pm$ 25 per cent), and the comparison indicates that the above theory predicts values of the shock thickness which are too small. On the other hand, Sherman concludes that the Navier-Stokes equations give an adequate description of the shock transition up to $M_1 = 2$.

For a more accurate consideration of the problem we must return to the original Eqs. 4.2, 9, 10. Eliminating X, a single equation in U and

Θ is obtained,

$$\frac{d\Theta}{dU} = (\gamma - 1)\left(\frac{4}{3} + K_b\right) M_1^2 Pr \left[\frac{\Lambda_1(U, \Theta)}{\Lambda_2(U, \Theta)}\right] \tag{15}$$

where

$$\Lambda_1(U, \Theta) = \frac{1}{\gamma(\gamma - 1) M_1^2} \frac{\Theta}{} - \frac{1}{2} U^2 + \left(1 + \frac{1}{\gamma M_1^2}\right) U - \left(\frac{1}{2} + \frac{1}{(\gamma - 1) M_1^2}\right) \tag{16}$$

and

$$\Lambda_2(U, \Theta) = \frac{\Theta}{\gamma M_1^2 U} + U - \left(1 + \frac{1}{\gamma M_1^2}\right) \tag{17}$$

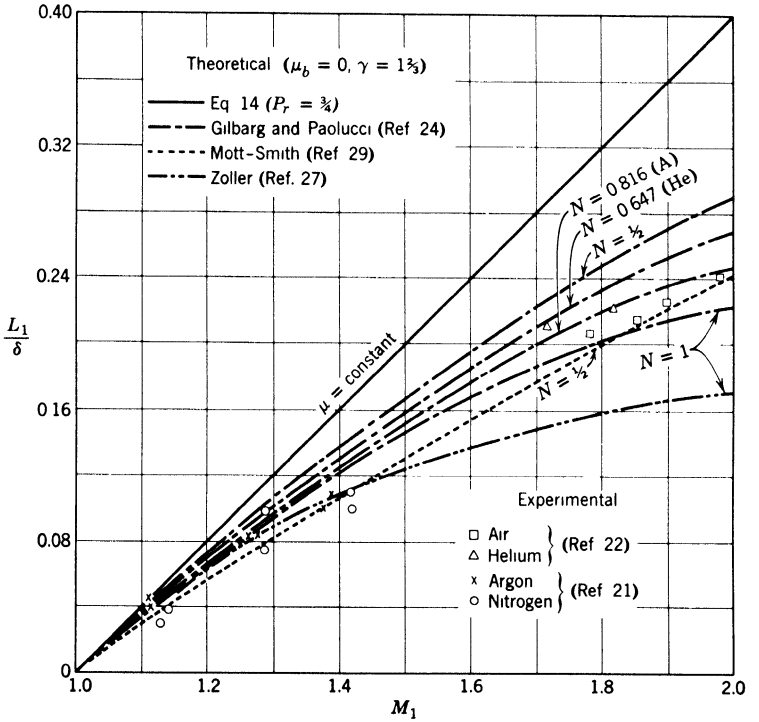


Fig. 4. Shock thickness for various Mach numbers.

A detailed study of Eq. 15 was made by Gilbarg for various values of the exponent N in the viscosity law (Eq. 3.8, 7), and a numerical method of solution was developed (Ref. 23).

The relations

$$\Lambda_1(U, \Theta) = 0, \quad \Lambda_2(U, \Theta) = 0 \tag{18}$$

which correspond to $d\Theta/dX = dU/dX = 0$, are parabolas in the (U, Θ) plane. They intersect at two singular points (1) and (2) (Fig. 5) which correspond to the initial and final states of the shock transition. The properties of the points (1) and (2) may be found from a consideration of the roots of the characteristic equation of the given system of relations (Eqs. 4.2, 9, 10). It is found that (1) is a nodal point and (2) is a saddle point in the field of integral curves of Eq. 15.

The problem is to find an integral curve which joins (1) and (2)—called the shock curve. When an integral curve crosses a parabola,

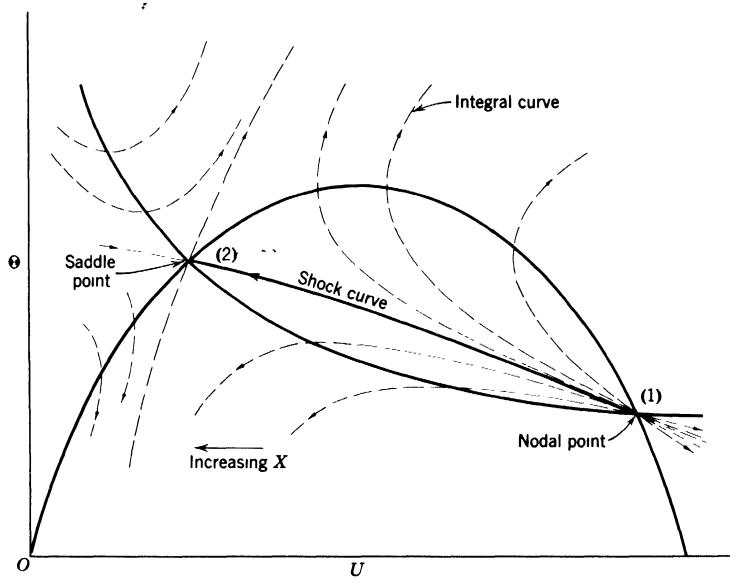


Fig. 5. Field of integral curves and the shock curve.

the sign of its slope is reversed. As the integral curves for X increasing are traced out, it is found that each curve passes through (1) but diverges out of the region enclosed by the parabolas (right-to-left in Fig. 5). On the other hand, proceeding in the direction of decreasing X, it is apparent that all integral curves are directed into this region and pass through (1). These facts suggest that a definite procedure for constructing the shock curve can be found by starting at point (2). The slope of the integral curve at (2) can be calculated from the characteristic equation of the system. By starting at a point adjacent to (2) lying on a straight line through (2) with this slope, we find that the basic equation can be

integrated by quadrature to obtain the shock curve and ultimately the transition functions $U(X)$ and $\Theta(X)$.

Using the above method, Gilbarg and Paolucci have determined the shock thickness ratios for a range of Mach numbers ($1.2 \leq M_1 \leq 4$), for $Pr = \frac{2}{3}$ and $N = \frac{1}{2}$ (spherical molecule), 0.647 (helium), 0.816 (argon), and 1 (Ref. 24). The results are given in Fig. 4. These calculations show that the shock thickness ratio depends quite significantly on the laws of variation of viscosity and heat conduction with temperature (Eq. 4.2, 8). As N increases, the thickness of the shock front becomes larger. For argon agreement with test results is improved if the more correct viscosity law ($\mu \sim T^{0.816}$) is used, but the theoretical value of δ/L_1 appears still to be too small.

It may be noted here that Grad (Ref. 25) has outlined an analytic method for approximating to the solutions of the basic equations (4.2, 9, 10). His procedure involves less labor than numerical integration if a systematic survey of the effect of various parameters is required.

In some gas flows the condition $K_b \ll \frac{4}{3}$ may not be fulfilled in either state 1 or state 2. In such flows the effect of bulk viscosity will not be negligible. Some idea of the influence of bulk viscosity on shock structure can be obtained from an estimate of the thickness of a shock wave in a gas for which $K_b \gg \frac{4}{3}$ (CO_2 or N_2O). The relation for the shock curve in the (U, Θ) plane now becomes

$$\frac{d\Theta}{dU} = (\gamma - 1)K_b M_1^2 Pr\left[\frac{\Lambda_1(U, \Theta)}{\Lambda_2(U, \Theta)}\right] \tag{19}$$

If K_b is a large quantity, the integral curve through the singular points (1) and (2) lies very close to the parabola $\Lambda_1 = 0$. In fact, if $K_b \to \infty$, the difference between the shock curve and the parabola approaches zero. Therefore, we may replace the shock curve by the parabola, that is, for $K_b \gg \frac{4}{3}$,

$$\frac{1}{\gamma(\gamma - 1)}\frac{\Theta}{M_1^2} = \frac{1}{2}U^2 - \left(1 + \frac{1}{\gamma M_1^2}\right)U + \left(\frac{1}{2} + \frac{1}{(\gamma - 1)M_1^2}\right) \tag{20}$$

By substituting for Θ in Eq. 4.2, 9, the differential equation for the variation of U with X when K_b is large may be written

$$U\frac{dU}{dX} = \frac{Re_1}{K_b K}\left(\frac{\gamma + 1}{2}\right)(U - 1)(U - U_2) \tag{21}$$

In order to estimate the probable effect of bulk viscosity when K_b is large, let us assume that both μ and μ_b are constant throughout the motion. Then $N = 0$, $K_b =$ constant, and Eq. 21 differs from Eq. 11

only in the coefficient on the right-hand side. Then for this case

$$\frac{\delta}{L_1} = \frac{K_b}{Re_1}\left(\frac{2}{\gamma+1}\right)\left(\frac{1+\sqrt{U_2}}{1-\sqrt{U_2}}\right) \tag{22}$$

If $\delta(0)$ is the thickness of the shock wave when no bulk viscosity effects are present ($\mu_b = 0$), and $\delta(\mu_b)$ is the corresponding value when the compression viscosity is significant, then

$$\frac{\delta(\mu_b)}{\delta(0)} = \frac{3}{4}\frac{K_b}{\gamma} \tag{23}$$

For carbon dioxide $K_b \approx 2 \times 10^3$, $\gamma = 1.3$, and the order of the ratio $\delta(\mu_b)/\delta(0)$ is about 10^3.

In other gases the orders of magnitude of μ and μ_b are about the same ($K_b \approx 1$). Under these circumstances the correct shock curve corresponding to Eq. 15 must be calculated, $U(X)$ found by integrating by quadrature, and δ/L_1 determined according to Eq. 13. Using $K_b = \frac{2}{3}$ and the basic information relevant to air ($c_r = 0.171$, $\gamma = 1.4$, $N = 0.768$), Gilbarg and Paolucci estimated that $\delta(\mu_b)$ was about one-third greater than $\delta(0)$ at $M_1 = 2$ (Ref. 24). Sherman finds good agreement between theoretical and experimental values of $\delta(\mu_b)$ for weak shock transitions in air and nitrogen when $K_b = \frac{2}{3}$ (Ref. 22). On the other hand, Sherman found no evidence of bulk viscosity in helium under similar test conditions.

Until more reliable information regarding bulk viscosity becomes available, it is difficult to assess the effect of this parameter quantitatively. However, the above calculations are sufficient to show that compression viscosity is an important factor in the flow of a gas in which large gradients of the macroscopic quantities occur.

When the macroscopic properties of a gas flow vary almost discontinuously as in the strong shock transition, the question arises whether the motion is still slightly nonisentropic. In other words, do the Navier-Stokes equations (4.2, 4) still apply? As M_1 (or P_2, the strength of the wave) increases, the shock thickness ratio (δ/L_1) decreases monotonically (Fig. 4), or, in other words, the maximum absolute slope of the transition curve [Fig. 3 (*a*)] becomes steadily larger. Then, according to Eqs. 3.6, 28 and 3.4, 13, $|a_{11}|_{\max}$ tends to become comparable with 1 for strong shock waves, and the velocity distribution function departs more and more from Maxwell's law. Ultimately third-order quantities in the expression for f (Eqs. 3.4, 1, 4) will not be negligible, and the transfer equations must be reconsidered since such quantities as $\rho\overline{U^2}$, $\rho\overline{UV}$, and $\rho\overline{UC^2}$ will have modified forms.

The more extended expression for f and the resultant transfer equations have been considered by Burnett (Ref. 26, see also Appendix III). The corresponding equations of motion now involve additional terms of a highly complex nature which raise the order of the system of partial differential equations. A numerical method for solving the Burnett equations subject to the boundary conditions for the shock transition has been developed by Zoller (Ref. 27) for the case $N = 1$ (Eq. 3.8, 7). Shock profiles were obtained which exhibited thicknesses as much as 35 per cent greater than the corresponding case based on the inclusion of second-order terms only (see Fig. 4, $N = 1$).

Further theoretical investigations of the transition through the shock front have been made by Wang Chang and Mott-Smith (Refs. 28, 29). Wang Chang considered the problem from the point of view of the sequence of Chapman-Enskog approximations (Ref. 1.1), but the resultant series was found to converge too slowly for practical purposes. Mott-Smith obtained an approximate solution by a superposition of two Maxwell distributions chosen to satisfy a transport equation. The method may be significant for strong shock waves but is somewhat arbitrary and cannot be readily incorporated into a sequence of better approximations. At the higher shock strengths, Mott-Smith obtained larger values for δ/L_1 than those calculated from the Navier-Stokes equations (Fig. 4).

The above discussion suggests that the greatest value of a study of the shock-transition problem may be in the check that it provides on the validity of the fundamental equations for a gas flow in which the molecular motion deviates substantially from the Maxwellian. A systematic treatment of the procedure for deriving the successive systems of transfer equations depending on the order of magnitude of the deviation of the distribution function from Maxwell's law has been given by Grad (Ref. 30). In Grad's method, Boltzmann's equation is replaced by a thirteen-moment approximation which forms a system of simultaneous equations. Successive iterations applied to this system yield in turn the equations of motion corresponding to (*a*) Maxwell's distribution function (isentropic or Euler flow), (*b*) the first modification of Maxwell's law (slightly nonisentropic, or Navier-Stokes flow), and (*c*) the second modification of the Maxwell distribution function (somewhat non-isentropic, or Burnett flow). Grad shows that, in general, the thirteen-moment approximation is more exact than either the Navier-Stokes or Burnett approximations.

Grad has considered the shock-transition problem from the point of view of the thirteen-moment approximation to Boltzmann's equation. Over the range of M_1 for which a solution was found to exist, the

calculated shock thicknesses were larger than those deduced for comparable conditions by all other methods.

An interesting result of the analysis of the shock front based on the Navier-Stokes equations is that a solution exists over the whole range of shock strengths ($0 \leq P_2 < \infty$). When the equations of motion are generalized to include other terms which become significant in a flow involving large gradients (Burnett equations), the calculated profiles are found to be subject to damped oscillations at $M_1 > 1.23$ (Zoller, Ref. 27), and no solution of the shock problem is obtainable when $M_1 > 2.36$. In Grad's calculations, the shock-transition solution broke down at about the value of M_1 at which the thirteen-moment approximation would be expected to become inapplicable ($M_1 = 1.65$). According to this method, the applicability of the Navier-Stokes solution is limited to $M_1 \leq 1.2$.

The question arises whether the shock thickness referred to the mean free path in the initial state is an adequate parameter for characterizing the shock structure. Both the definition of δ and the choice of L_1 are arbitrary. One difficulty with regard to δ as defined by Eq. 13 pertains to its sensitiveness to small changes in profile of the transition (Fig. 3). One can imagine a profile in which 95 percent of the transition in the macroscopic properties of the gas actually occurs over a distance far in excess of δ if the transition curve is such that $\left|\frac{dU}{dX}\right|_{\max}$ happens to be very large compared with adjacent values of this derivative. Grad has suggested a definition based on the integral properties of the profile. If $U(X)$ is a single-valued, monotonic function, and the area enclosed between it and the asymptotes $U = 1$, $U = U_2$ is finite, then the center of the transition can be located as the point at which the area between $U(X)$ and $U = 1$ on the left is equal to the area between $U(X)$ and $U = U_2$ on the right (A, Fig. 6). If a straight line is now drawn through the center point, intersecting the asymptotes so that each of the triangles so formed has an area A, then the thickness of the shock front is defined as

$$\delta = \frac{8A}{1 - U_2} \tag{24}$$

This definition of δ has many features to recommend it, but it is still essentially arbitrary.

Improvements in the shock-transition theory have been found to coincide with larger values of δ compared with L_1. Since L_1 was arbitrarily selected as the reference length, a further question for investigation is whether another choice of mean free path would sensibly affect the comparison of the various theories. Grad has considered this point and has suggested the use of a mean free path within the shock

transition. A calculation of the variation of L through the shock front gives results which depend appreciably on the molecular model. For spherical molecules L_2/L_1 is given by ρ_1/ρ_2 (see Eq. 2.5, 13), and it tends to a finite limit $(\gamma - 1)/(\gamma + 1)$ as the shock strength (P_2) becomes very large. If the molecules are point centers of force, L_2/L_1 is a function of both density and temperature, having the form $\frac{\sigma^2(T_1)\rho_1}{\sigma^2(T_2)\rho_2}$. Since the effective diameter, $\sigma(T)$, of the molecule is reduced as the temperature increases (see Section 4.1), the ratio $\sigma^2(T_1)/\sigma^2(T_2)$ can become very large

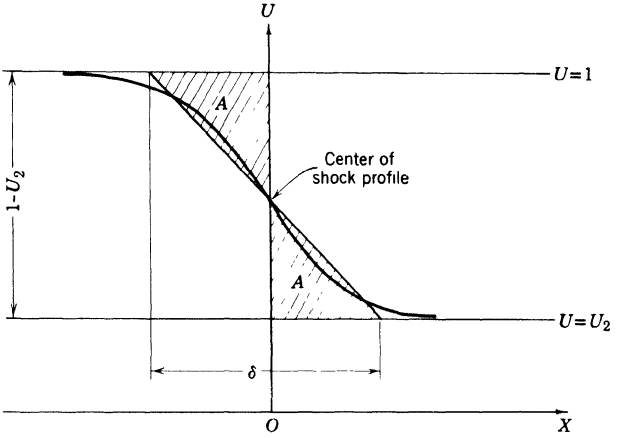


Fig. 6. Alternative definition of shock thickness.

for strong shock waves and may supersede ρ_1/ρ_2 as the dominant factor (as $P_2 \to \infty$, ρ_1/ρ_2 remains finite, but $T_2 > \infty$). Therefore, it is possible for the mean free path downstream from the shock front to be larger than the initial value in the undisturbed gas ($L_2 > L_1$).

Perhaps in the final analysis the best criterion of shock structure is the transition profile itself. The work of Sherman (Ref. 22) shows that it is possible to probe the shock front and plot the profile. Such techniques may also lead to the direct determination of the coefficient of bulk viscosity.

4.4 DIFFUSE REFLECTION FROM A SOLID BOUNDARY

The molecular motion in a layer of gas adjacent to the surface of a body is also non-Maxwellian. Before considering solutions of the basic equations for nonisentropic flow near a wall, we require the boundary

conditions which govern the flow on the surface. The motion of the gas on the surface depends on the transfer of mass, momentum, and energy by the gas molecules to and from the body.

Little is known about the complex mechanism of the interaction of a gas molecule with the surface of a solid body. It is difficult to coordinate existing experimental observations since the results depend on the arrangement of the apparatus. In particular, the physical process is not the same for different materials; for example, surface adsorption of the gas

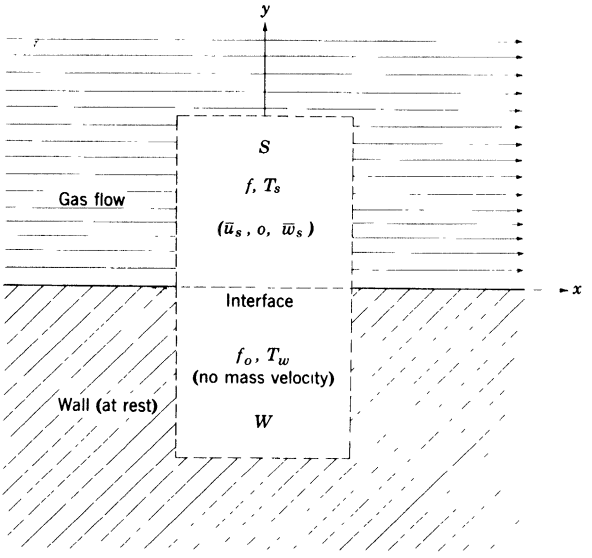


Fig. 7. Diffuse reflection.

can be a factor, or the molecules may condense on the surface and then evaporate.

When the molecules reflect specularly (Section 2.6), the surface is assumed to be perfectly smooth. In physical reality, however, the solid boundary is much more likely to be quite irregular, having interstices in which the molecules may be trapped temporarily. Reflection of the molecules from a surface of this type is diffuse. The molecules strike the boundary with complete loss of the tangential components of mass velocity relative to the wall, and they escape after attaining either partial or complete thermal equilibrium with the wall, the orientation of emission being independent of the direction of impingement. The speeds of the diffusely reflecting molecules are considered to be distributed according

to Maxwell's law corresponding to a temperature which is different, in general, from the temperature of the gas or the wall, and all directions of motion away from the surface are equally probable (Ref. 31).

The physical process involved in diffuse molecular reflection is illustrated in Fig. 7, in which the axes of reference are chosen fixed relative to the body. The element of volume S in the flowing gas is located with one face on the surface of the body. It contains gas molecules which have a mass velocity $(\bar{u}_s, o, \bar{w}_s)$ and a random motion governed by the more general law of distribution of molecular velocities (Eq. 3.4, 1). The reflecting molecules are assumed to act as though they emerged from an adjacent elementary volume W with a Maxwellian random motion and without mass motion (relative to the body). In the calculations that follow, the Maxwellian motion is assumed to be consistent with the temperature of the wall. The molecular motions in the elements of volume S and W are mutually compatible if the total mass, momentum, and energy transferred upward by the molecules from W to S are respectively equal to the total mass, momentum, and energy which would be transported upward if the molecules in W moved with the same mass velocity and random motion as the molecules in S.

Consider the flow of mass upward from the element of volume W. If W contains S-molecules, the number of molecules in unit volume having velocity components in the range $u, u + du$; $v, v + dv$; $w, w + dw$ is $n_s f\, du\, dv\, dw$. The number of these which cross unit area of the interface in unit time is the number in a volume based on unit area with a height v. Therefore, the total mass transported upward by S-molecules across unit area in unit time would be

$$mn_s \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} v f(T_s)\, du\, dv\, dw \tag{1}$$

On the other hand, the total mass transferred upward from W across the same area in unit time by the reflected molecules is

$$mn_w \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} v f_0(T_w)\, du\, dv\, dw \tag{2}$$

Since the component of mass flow normal to the interface is zero (Fig. 7), then for compatibility in the mass flow

$$\frac{n_s}{\sqrt{\beta_s}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} H_2(1 + F) e^{-\frac{1}{2}H_1 H_1}\, dH_1\, dH_2\, dH_3$$

$$= \frac{n_w}{\sqrt{\beta_w}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} H_2 e^{-\frac{1}{2}H_1 H_1}\, dH_1\, dH_2\, dH_3 \tag{3}$$

where $\beta = 3/(2\overline{C^2}) = 1/(2RT)$, $F = F(H_i)$ is given by Eq. 3.4, 4, and the notation used is that described in Section 3.4. Upon integration, Eq. 3 reduces to

$$\frac{n_w}{n_s} = \left(\frac{\beta_w}{\beta_s}\right)^{1/2}\left[1 + \frac{1}{2}a_{22}\right]_s \tag{4}$$

where, according to Eqs. 1.10, 6; 2.4, 2; 3.6, 28; 3.8, 10,

$$a_{22} = \frac{5}{6}\left[\frac{L}{\overline{C}}(\bar{u}_x + \bar{w}_z - 2\bar{v}_y)\right]_s \tag{5}$$

A similar treatment of the transfer of the normal component of momentum (mv) across the interface yields a relation between p_s and p_w. The total normal momentum transported upward from W by S-molecules through unit area in unit time would be

$$mn_s\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} v^2 f(T_s)\,du\,dv\,dw \tag{6}$$

and the corresponding transport of normal momentum in the upward direction from W by the reflected molecules is

$$mn_w\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} v^2 f_0(T_w)\,du\,dv\,dw \tag{7}$$

Then the transfer of mv at the boundary is compatible if

$$\frac{n_s}{\beta_s}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} H_2^2(1+F)e^{-\frac{1}{2}H_iH_i}\,dH_1\,dH_2\,dH_3$$

$$= \frac{n_w}{\beta_w}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} H_2^2 e^{-\frac{1}{2}H_iH_i}\,dH_1\,dH_2\,dH_3 \tag{8}$$

Since $p = mn/2\beta$, Eq. 8 reduces to a relation between the static pressures in S and W (Fig. 7),

$$\frac{p_s}{p_w} = \left[1 - a_{22} - \frac{2a_{222}}{3\sqrt{2\pi}}\right]_s \tag{9}$$

The individual values of the coefficients a_{ijk} have not been required in the calculations up to this point. They can be obtained by an extension of the theory given in Section 3.7. The particular form of Maxwell's transfer equation (1.8, 7) for $Q = u^3$ is

$$n\bar{u}\frac{d\overline{C^2}}{dt} + n\overline{C^2}\frac{\partial}{\partial x}\left(\bar{u}^2 + \frac{1}{3}\overline{C^2}\right) = \Delta u^3 \tag{10}$$

Substituting from Eq. 2.2, 6, this relation may be written

$$2n\bar{u}\overline{C^2}\left[\bar{u}_x - \frac{1}{3}(\bar{u}_x + \bar{v}_y + \bar{w}_z)\right] + n\overline{C^2}\frac{\partial}{\partial x}\left(\frac{\overline{C^2}}{3}\right) = \Delta u^3 \tag{11}$$

or

$$-\frac{n^2 m}{\mu}\left(\frac{\overline{C^2}}{3}\right)^2\left[3\bar{u}a_{11} - \frac{4}{5}\left(\frac{\overline{C^2}}{3}\right)^{1/2}a_1\right] = \Delta u^3 \tag{12}$$

But, from the first of Eqs. 3.7, 5,

$$\Delta u^3 = -\frac{n^2 m}{\mu}\left(\frac{\overline{C^2}}{3}\right)^2\left[\left(\frac{\overline{C^2}}{3}\right)^{1/2}a_1 + 3\bar{u}a_{11} + \frac{3}{2}\left(\frac{\overline{C^2}}{3}\right)^{1/2}a_{111}\right] \tag{13}$$

If we equate Eqs. 12 and 13, the result is

$$a_{111} = -\tfrac{6}{5}a_1 \tag{14}$$

In the same way, Maxwell's transfer equations for $Q = uv^2$ and $Q = uw^2$ are found to reduce to

$$a_{122} = a_{133} = -\tfrac{2}{5}a_1 \tag{15}$$

Similarly,

$$\tfrac{1}{3}a_{222} = a_{211} - a_{233} = -\tfrac{2}{5}a_2 \tag{16}$$

$$\tfrac{1}{3}a_{333} = a_{113} - a_{223} = -\tfrac{2}{5}a_3 \tag{17}$$

These relations confirm Eqs. 3.4, 9, 12.

Returning to Eq. 9 we have, from Eqs. 1.10, 6; 2.4, 2; 3.7, 9; 3.8, 10,

$$a_{222} = -\frac{45}{16}\left(\frac{\pi}{2}\right)^{1/2}\frac{L}{T}T_y \tag{18}$$

The application of the compatibility condition to the transfer of tangential momentum at the interface between the elements of volume S and W produces an equation for the slip velocity. For the transport of mu,

$$mn_s\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} uvf(T_s)\,du\,dv\,dw = mn_w\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} uvf_0(T_w)\,du\,dv\,dw \tag{19}$$

which becomes

$$\frac{n_s}{\beta_s}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}(H_1 + \sqrt{2\beta_s}\bar{u})H_2(1+F)e^{-\frac{1}{2}H_1H_1}\,dH_1\,dH_2\,dH_3 = \frac{n_w}{\beta_w}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}H_1H_2e^{-\frac{1}{2}H_1H_1}\,dH_1\,dH_2\,dH_3 \tag{20}$$

The reflected molecules make no contribution to the resultant tangential momentum since they emerge in directions all of which are equally

probable. Therefore, the right-hand side of Eq. 20 is zero. Then Eq. 20 reduces to

$$\left[\bar{u}(1 + \tfrac{1}{2}a_{22}) + \frac{1}{2\sqrt{2\beta}}(\sqrt{2\pi}a_{12} + a_{122})\right]_s = 0 \tag{21}$$

Neglecting all products of the a-coefficients, the mass velocity of the gas on the surface in the direction of the x-axis is

$$\bar{u}_s = -\tfrac{1}{2}\sqrt{RT_s}[\sqrt{2\pi}a_{12} + a_{122}]_s \tag{22}$$

where

$$a_{12} = -\frac{5}{4}\frac{L}{C}(\bar{u}_y + \bar{v}_x) \tag{23}$$

and

$$a_{122} = -\frac{15}{16}\left(\frac{\pi}{2}\right)^{1/2}\frac{L}{T}T_x \tag{24}$$

Finally, the temperature jump between the gas and the wall can be determined from the flow of translational energy of the molecules across the interface. The equation for compatibility of energy transport between W and S (neglecting radiation and energy input) is

$$\frac{1}{2}mn_s\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}vc^2f(T_s)\,du\,dv\,dw = \frac{1}{2}mn_w\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}vc^2f_0(T_w)\,du\,dv\,dw \tag{25}$$

which may be expanded into

$$\frac{n_s}{\beta_s^{1/2}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}H_2[(\bar{u}\sqrt{2\beta_s} + H_1)^2 + H_2^2 + (\bar{w}\sqrt{2\beta_s} + H_3)^2] \times (1 + F)e^{-\frac{1}{2}H_iH_i}\,dH_1\,dH_2\,dH_3 = \frac{n_w}{\beta_w^{3/2}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}H_2(H_iH_i)e^{-\frac{1}{2}H_jH_j}\,dH_1\,dH_2\,dH_3 \tag{26}$$

The integrals in this relation have already been determined with the exception of the two involving the quantity $H_2(H_iH_i)$. Upon integration Eq. 26 becomes

$$\frac{n_s}{2\beta_s^{3/2}}\left[\beta(\bar{u}^2 + \bar{w}^2)\left(1 + \frac{1}{2}a_{22}\right) + \bar{u}\left(\frac{\beta}{2}\right)^{1/2}(\sqrt{2\pi}a_{12} + a_{122}) + \bar{w}\left(\frac{\beta}{2}\right)^{1/2}(\sqrt{2\pi}a_{32} + a_{322}) + 2\left(1 + \frac{3}{4}a_{22} - \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}a_2\right)\right]_s = \frac{n_w}{\beta_w^{3/2}} \tag{27}$$

Substitution for $\bar{u}_s$ from Eq. 22 and for $\bar{w}_s$ from the corresponding expression for the slip velocity in the z-direction shows that all terms

on the left-hand side of Eq. 27 are of negligible order except the last term, and we have

$$\frac{T_s}{T_w} = \left[1 - \frac{1}{4}a_{22} + \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} a_2\right]_s \tag{28}$$

where the ratio n_w/n_s is given by Eq. 4, a_{22} is obtained from Eq. 5, and

$$a_2 = \frac{75}{32}\left(\frac{\pi}{2}\right)^{1/2} \frac{L}{T} T_y \tag{29}$$

The above results may be summarized as follows:

$$\frac{p_s}{p_w} = 1 - \frac{5}{6}\left[\frac{L}{\bar{C}}(\bar{u}_x + \bar{w}_z - 2\bar{v}_y)\right]_s + \frac{15}{16}\left[\frac{L}{T}T_y\right]_s \tag{30}$$

$$\bar{u}_s = L_s\left[\frac{5\pi}{16}(\bar{u}_y + \bar{v}_x) + \frac{15\pi}{128}\frac{\bar{C}}{T}T_x\right]_s \tag{31}$$

$$\bar{w}_s = L_s\left[\frac{5\pi}{16}(\bar{w}_y + \bar{v}_z) + \frac{15\pi}{128}\frac{\bar{C}}{T}T_z\right]_s \tag{32}$$

$$\frac{T_s}{T_w} = 1 - \frac{5}{24}\left[\frac{L}{\bar{C}}(\bar{u}_x + \bar{w}_z - 2\bar{v}_y)\right]_s + \frac{75\pi}{128}\left[\frac{L}{T}T_y\right]_s \tag{33}$$

We conclude from these equations that, when the molecules of a gas reflect diffusely from a boundary in a nonisentropic flow, small discontinuities of normal pressure, tangential mass velocity, and temperature occur between the gas and the wall. These discontinuities arise from the departure of the random molecular motion from the Maxwellian since, if Maxwell's law is valid for the flow adjacent to the wall, then, for diffuse reflection,

$$p_s = p_w, \quad \bar{u}_s = 0, \quad T_s = T_w \tag{34}$$

which are the relations for complete accommodation between the gas flow and arbitrary boundary conditions.

In the analysis of the boundary layer which follows, we shall consider first the flow of a gas at ordinary pressures for which the above discontinuities are negligibly small. The effect of slip flow and temperature jump in a rarefied gas is discussed later in Chapter 5.

4.5 THE BOUNDARY LAYER EQUATIONS

The theory of the nonisentropic flow of a gas around a flat plate aligned in the direction of the mass flow will now be developed. The non-Maxwellian motion of the molecules is confined to a region adjacent

to the surface of the plate, and everywhere outside this thin layer (beyond the influence of the wall) the molecular velocities are distributed according to Maxwell's law. The thickness of the boundary layer can be defined in terms of either the mass velocity or temperature. At any position along the plate, the origin being taken at the leading edge, the thicknesses of the nonisentropic momentum- and energy-transfer layers [$\delta_u(x)$ and $\delta_T(x)$, respectively] are different in general, but it is assumed in the following analysis that they have the same order of magnitude.

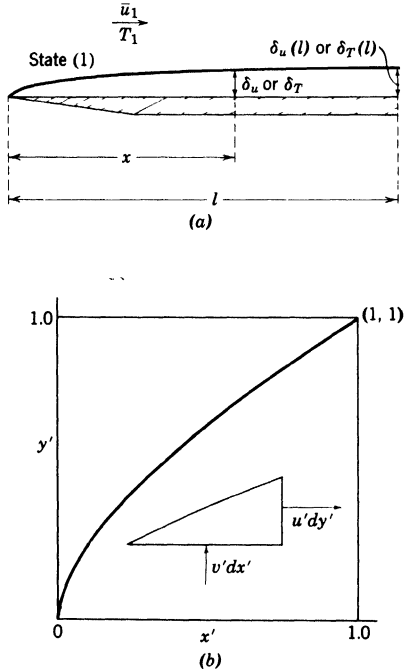


Fig. 8. Boundary layer on a flat plate.

Let us introduce the following dimensionless quantities,

$$\Delta = \frac{L_1}{\delta_u(l)}, \qquad D = \frac{\delta_u(l)}{l}, \qquad \delta' = \frac{\delta_u(l)}{\delta_T(l)} \tag{1}$$

Except in highly rarefied gases, Δ is a small number. Furthermore, the theory of the boundary layer is based on the assumption that D is also a small quantity, having about the same order of magnitude as Δ.

The ranges of variation of x along the plate ($0 \leq x \leq l$) and of y normal to the plate [$0 \leq y \leq \delta_u(l)$ or $\delta_T(l)$] are of different orders (Fig. 8*a*). If we write

$$x' = \frac{x}{l}, \quad y' = \frac{y}{\delta_u(l)}, \quad y'' = \frac{y}{\delta_T(l)} \tag{2}$$

then all the primed quantities have order 1 (Fig. 8*b*).

The mass velocity and the temperature have the ranges $\bar{u}_s \leq \bar{u} \leq \bar{u}_1$ and $T_1 \leq T \leq T_\delta$, respectively, where $\bar{u}_1$, T_1 are the mass velocity and temperature of the undisturbed flow, and T_δ is the maximum temperature in the boundary layer, having the same order of magnitude as the temperature of the gas on the wall (T_s). Then

$$u' = \frac{\bar{u}}{\bar{u}_1}, \qquad T' = \frac{T}{T_1} \tag{3}$$

and similarly

$$\rho' = \frac{\rho}{\rho_1}, \qquad p' = \frac{p}{p_1} \tag{4}$$

are also first-order quantities.

The flow across an element of a line within the boundary layer is $\rho(\bar{u}\,dy - \bar{v}\,dx)$, and, if this line is a streamline, then $\bar{u}\,dy = \bar{v}\,dx$. When this equation is written in dimensionless form, each term must have the same order of magnitude, and, therefore, the dimensionless component of velocity normal to the plate having order 1 is

$$v' = \frac{\bar{v}}{\bar{u}_1} \cdot \frac{1}{D} \tag{5}$$

With the assistance of the primed quantities above we may investigate the orders of magnitude of the terms in the boundary conditions and the basic equations of motion with a view to retaining only those terms which make a significant contribution to the physical properties of the flow. The boundary conditions (Eqs. 4.4, 30–33) may now be written in the following dimensionless form,

$$\frac{p_s'}{p_w'} = 1 - L_s'\Delta\left[\frac{\alpha_1 M_1 D}{\sqrt{T'}}(u'_{x'} - 2v'_{y'}) - \frac{\alpha_2 \delta'}{T'} T'_{y'}\right]_s \tag{6}$$

$$u_s' = L_s'\Delta\left[\alpha_3(u'_{y'} + v'_{x'}D^2) + \frac{\alpha_4 D}{M_1\sqrt{T'}} T'_{x'}\right]_s \tag{7}$$

$$\frac{T_s'}{T_w'} = 1 - L_s'\Delta\left[\frac{\alpha_5 M_1 D}{\sqrt{T'}}(u'_{x'} - 2v'_{y'}) - \frac{\alpha_6 \delta'}{T'} T'_{y'}\right]_s \tag{8}$$

where $L_s' = 1/\rho_s'$, and $\alpha_1, \alpha_2, \cdots, \alpha_6$ are constants. Since all primed quantities are of order one, then all the bracketed terms multiplied by Δ

are negligibly small (M_1 being assumed to be of order 1), and the boundary conditions reduce to

$$\bar{u}_s = 0, \qquad p_s = p_w, \qquad T_s = T_w \tag{9}$$

The equations of motion for a steady, two-dimensional flow expressed in rectangular coordinates can be derived either from the vector equations in Section 3.9 or directly from the transfer equations (1.9, 5, 8, 10) by substitution from Eqs. 3.6, 19, 29 and 3.7, 11. The required relations are

$$\frac{\partial}{\partial x}(\rho\bar{u}) + \frac{\partial}{\partial y}(\rho\bar{v}) = 0 \tag{10}$$

$$\rho\frac{d\bar{u}}{dt} = -\frac{\partial}{\partial x}\left[p - \mu\left(\frac{4}{3}\bar{u}_x - \frac{2}{3}\bar{v}_y\right)\right] + \frac{\partial}{\partial y}[\mu(\bar{u}_y + \bar{v}_x)] \tag{11}$$

$$\rho\frac{d\bar{v}}{dt} = \frac{\partial}{\partial x}[\mu(\bar{u}_y + \bar{v}_x)] - \frac{\partial}{\partial y}\left[p + \mu\left(\frac{2}{3}\bar{u}_x - \frac{4}{3}\bar{v}_y\right)\right] \tag{12}$$

$$\rho c_v\frac{dT}{dt} = -\left[p - \mu\left(\frac{4}{3}\bar{u}_x - \frac{2}{3}\bar{v}_y\right)\right]\bar{u}_x - \left[p + \mu\left(\frac{2}{3}\bar{u}_x - \frac{4}{3}\bar{v}_y\right)\right]\bar{v}_y + \mu(\bar{u}_y + \bar{v}_x)^2 + \frac{\partial}{\partial x}(\lambda T_x) + \frac{\partial}{\partial y}(\lambda T_y) \tag{13}$$

The following boundary layer theory for the flow of a gas near a flat plate is developed on the assumption that no pressure gradient exists in the direction of x ($p_x = 0$). The dimensionless form of the above equations of motion, subject to this condition, may now be determined.

When expressed in terms of the primed quantities defined above, both terms of the mass-transfer equation are found to be of order 1 and must be retained,

$$\frac{\partial}{\partial x'}(\rho'u') + \frac{\partial}{\partial y'}(\rho'v') = 0 \tag{14}$$

Turning our attention to the momentum-transfer equations, the three terms of Eq. 11 become

$$\frac{l}{\rho_1\bar{u}_1^2}\left[\rho\frac{d\bar{u}}{dt}\right] = \rho'(u'u'_{x'} + v'u'_{y'}) \tag{15}$$

$$\frac{l}{\rho_1\bar{u}_1^2}\frac{\partial}{\partial x}\left[p - \mu\left(\frac{4}{3}\bar{u}_{x'} - \frac{2}{3}\bar{v}_y\right)\right] = -\frac{\beta_1}{M_1}D\Delta\frac{\partial}{\partial x'}\left[\mu'\left(\frac{4}{3}u'_{x'} - \frac{2}{3}v'_{y'}\right)\right] \tag{16}$$

$$\frac{l}{\rho_1\bar{u}_1^2}\frac{\partial}{\partial y}[\mu(\bar{u}_y + \bar{v}_x)] = \frac{\beta_1}{M_1}D\Delta\frac{\partial}{\partial y'}\left[\mu'\left(v'_{x'} + \frac{1}{D^2}u'_{y'}\right)\right] \tag{17}$$

where $\mu' = \mu/\mu_1$, β_1 is a constant, and

$$\frac{\mu_1}{\rho_1 l\bar{u}_1} \qquad \frac{\beta_1}{M_1}D\Delta \tag{18}$$

Then, if Δ and D have the same order of magnitude, Eq. 11 reduces to the following first-order form,

$$\rho'(u'u'_{x'} + v'u'_{y'}) = -\frac{\beta_1}{M_1}\frac{\Delta}{D}\frac{\partial}{\partial y'}(\mu' u'_{y'}) \tag{19}$$

A similar treatment of Eq. 12 shows that this relation contains no first-order terms except that involving p_y, and hence this equation merely states that $p'_{y'} = 0$.

Application of the same procedure to the energy-transfer equation yields the result

$$\rho'(u'T'_{x'} + v'\delta'T'_{y'}) = -\beta_2 p'(u'_{x'} + v'_{y'}) + \beta_3 M_1 \frac{\Delta}{D}\mu'(u'_{y'})^2 + \frac{\beta_4 \delta'^2}{M_1}\frac{\Delta}{D}\frac{\partial}{\partial y''}(\lambda' T'_{y'}) \tag{20}$$

where

$$\frac{\lambda_1}{\rho_1 l \bar{u}_1} = \frac{\beta_5}{M_1} D\Delta \tag{21}$$

β_2, β_3, β_4, β_5 are constants, and $\lambda' = \lambda/\lambda_1$.

The above analysis indicates that the basic equations for steady mass flow in two dimensions with no gradient of pressure may be reduced to the following first-order forms,

$$\frac{\partial}{\partial x}(\rho\bar{u}) + \frac{\partial}{\partial y}(\rho\bar{v}) = 0 \tag{22}$$

$$\rho(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) = \frac{\partial}{\partial y}(\mu\bar{u}_y) \tag{23}$$

$$\rho c_v(\bar{u}T_x + \bar{v}T_y) = -p(\bar{u}_x + \bar{v}_y) + \mu\bar{u}_y^2 + \frac{\partial}{\partial y}(\lambda T_y) \tag{24}$$

$$p = \rho RT \tag{25}$$

$$\frac{\mu}{\mu_1} = \frac{\lambda}{\lambda_1} = f\left(\frac{T}{T_1}\right) \tag{26}$$

These are the boundary layer equations for the flow around a flat plate aligned in the direction of the mass motion.

4.6 MOMENTUM TRANSFER IN THE BOUNDARY LAYER

Let us now investigate the transfer of momentum by the molecules of a gas in the nonisentropic layer adjacent to a flat plate. A number of studies of the properties of the laminar boundary layer have been made (Refs. 3, 32). The method of Chapman and Rubesin (Ref. 5) is discussed here since it involves a minimum of restrictive assumptions.

However, for simplicity, the analysis will be limited to a plate with constant surface temperature. The mass- and momentum-transfer relations may be written in dimensionless form by referring all variables to their corresponding values in the undisturbed flow [state 1, Fig. 8 (*a*)]. In particular, the position variables (x, y) are expressed in terms of the mean free path (L_1) in state 1. The two transfer equations may be written

$$\frac{\partial}{\partial X}(\Gamma U) + \frac{\partial}{\partial Y}(\Gamma V) = 0 \tag{1}$$

$$\Gamma(UU_X + VU_Y) = \frac{1}{Re_1}\frac{\partial}{\partial Y}(KU_Y) \tag{2}$$

The dimensionless symbols in these relations are defined in the Notation at the end of this chapter.

Equation 1 can be satisfied identically by introducing the dimensionless stream function (Ψ) defined by the relations

$$\Gamma U = \Psi_Y, \qquad \Gamma V = -\Psi_X \tag{3}$$

Considerable simplification of Eq. 2 can be achieved by introducing two new independent variables (Φ, Ψ). The operators involved in this transformation (Ref. 33) are

$$\frac{\partial}{\partial X} = \Phi_X \frac{\partial}{\partial \Phi} + \Psi_X \frac{\partial}{\partial \Psi}, \qquad \frac{\partial}{\partial Y} = \Phi_Y \frac{\partial}{\partial \Phi} + \Psi_Y \frac{\partial}{\partial \Psi} \tag{4}$$

In the special case in which $\Phi = X$, these operators reduce to

$$\left(\frac{\partial}{\partial X}\right)_Y = \left(\frac{\partial}{\partial X}\right)_\Psi - \Gamma V\left(\frac{\partial}{\partial \Psi}\right)_X, \qquad \left(\frac{\partial}{\partial Y}\right)_X = \Gamma U\left(\frac{\partial}{\partial \Psi}\right)_X \tag{5}$$

Then the mobile operator in steady, two-dimensional flow is

$$U\left(\frac{\partial}{\partial X}\right)_Y + V\left(\frac{\partial}{\partial Y}\right)_X = U\left(\frac{\partial}{\partial X}\right)_\Psi \tag{6}$$

and the momentum-transfer equation may be written

$$U_X = \frac{1}{Re_1}\frac{\partial}{\partial \Psi}[K\Gamma U U_\Psi] \tag{7}$$

In general the product $K\Gamma$ is a function of temperature. However, since the pressure is sensibly constant throughout the flow, then, according to Eq. 1.10, 7,

$$\Gamma = \frac{1}{\Theta} \tag{8}$$

If we adopt Eq. 4.1, 4 as a sufficiently accurate representation of the variation of viscosity with temperature, then,

$$K\Gamma = \sigma \tag{9}$$

where σ is a constant, chosen to give the best values of μ over the relevant range of temperature. Following Chapman and Rubesin, we can calculate σ by matching Eq. 4.1, 4 with Eq. 4.1, 2 for the constant surface temperature Θ_w,

$$\sigma = \sqrt{\Theta_w}\left(\frac{1 + \alpha/T_1}{\Theta_w + \alpha/T_1}\right) \tag{10}$$

If $K\Gamma$ can be replaced by σ, then the momentum-transfer equation becomes

$$U_X = \frac{\sigma}{Re_1}\frac{\partial}{\partial\Psi}(UU_\Psi) \tag{11}$$

which is subject to the boundary conditions:

$$\begin{aligned} &\text{At } \Psi = 0\ (y = 0), \quad U = U_w = 0 \\ &\text{At } \Psi \to \infty\ (y \to \infty), \quad U = 1 \end{aligned} \tag{12}$$

where the flow in the boundary layer is such that Δ is negligibly small.

Equation 11 can be reduced to an ordinary differential equation. If the substitution

$$\xi = \sqrt{\frac{Re_1}{\sigma}}\,\Psi \tag{13}$$

is made first, the new form of the momentum-transfer relation (11) is

$$U_X = \frac{\partial}{\partial\xi}(UU_\xi) \tag{14}$$

which is now free of constant coefficients related to the initial flow conditions, and the boundary conditions are the same as Eqs. 12 with ξ replacing Ψ. Writing

$$U = \frac{1}{2}f'(\eta) \quad \left(f' = \frac{df}{d\eta}\right) \tag{15}$$

where η is defined by the relation

$$f(\eta) = \frac{\xi}{\sqrt{X}} \quad (0 \leq \eta < \infty) \tag{16}$$

then

$$U_X = -\frac{ff''}{4Xf'}, \quad U_\xi = \frac{f''}{2f'\sqrt{X}} \tag{17}$$

where

$$\eta_X = -\frac{f}{2Xf'}, \quad \eta_\xi = \frac{1}{f'\sqrt{X}}$$

and Eq. 14 reduces to

$$ff'' + f''' = 0 \tag{18}$$

This equation was first considered by Blasius in his early studies of the boundary layer (Ref. 34). It must be solved subject to the following boundary conditions:

$$f'(0) = f(0) = 0; \quad f'(\infty) = 2 \tag{19}$$

Since two boundary conditions are applied at $\eta = 0$, let us write $f(\eta)$ as a power series expansion about this point,

$$f(\eta) = B_0 + B_1\eta + \frac{B_2}{2!}\eta^2 + \frac{B_3}{3!}\eta^3 + \cdots \tag{20}$$

The power series forms for $f'(\eta)$, $f''(\eta)$, and $f'''(\eta)$ can be readily determined by differentiation. If $f'(0) = f(0) = 0$, then $B_0 = B_1 = 0$, and substitution in Eq. 18 gives

$$B_3 + B_4\eta + \tfrac{1}{2}(B_2^2 + B_5)\eta^2 + \tfrac{1}{6}(4B_2B_3 + B_6)\eta^3 + \cdots = 0 \tag{21}$$

This relation must be true for all values of η, which can be satisfied only if the coefficients are zero,

$$B_3 = B_4 = B_6 = 0, \quad B_5 = -B_2^2, \cdots \tag{22}$$

The series expansion for $f(\eta)$ becomes

$$f = \alpha g(\zeta) = \alpha\left(\frac{\zeta^2}{2!} - \frac{\zeta^5}{5!} + \frac{11\zeta^8}{8!} - \frac{375\zeta^{11}}{11!} + \cdots\right) \tag{23}$$

where

$$\zeta = \alpha\eta, \quad \alpha = B_2^{1/3} \tag{24}$$

According to the third boundary condition,

$$\lim_{\eta \to \infty} f'(\eta) = 2 \tag{25}$$

or, since $f'(\eta) = \alpha^2 g'(\zeta)$,

$$\lim_{\zeta \to \infty} g'(\zeta) = \frac{2}{\alpha^2} \tag{26}$$

By calculating $g'(\zeta)$ for ever-increasing, large values of ζ until the result repeats to a sufficient order of accuracy, the value of α can be found,

$$\alpha = \left[\frac{2}{\lim_{\zeta \to \infty} g'(\zeta)}\right]^{1/2} \tag{27}$$

This calculation has been considered by a number of authors, the accepted value being (Ref. 35)

$$\alpha = 1.09924, \quad \text{or } B_2 = 1.32824 \tag{28}$$

The functions $f(\eta)$, $f'(\eta)$, and $f''(\eta)$, based on tables given in Ref. 36, are plotted in Fig. 9.

In order to obtain $U(Y)$, the relation between Y and η is required. Integrating the first of Eqs. 3,

$$Y = \int_0^{\Psi} \frac{d\Psi}{\Gamma U} \tag{29}$$

At constant X, $$d\Psi = \sqrt{\frac{\sigma X}{Re_1}} f'(\eta)\, d\eta$$

(see Eqs. 13, 16), and, substituting from Eqs. 8 and 15,

$$Y = 2\sqrt{\frac{\sigma X}{Re_1}} \int_0^{\eta} \Theta(\eta)\, d\eta \tag{30}$$

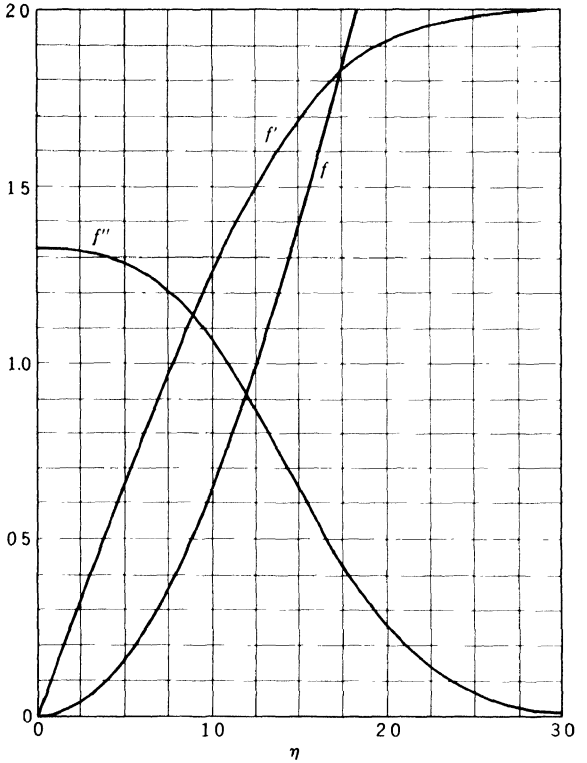


Fig. 9. Function f(η) and its derivatives (Ref. 36).

Therefore, $\Theta(\eta)$ must be known before $Y(\eta)$ can be found, and a calculation of the distribution of mass velocity through the boundary layer

must await a consideration of the transfer of molecular translational energy (see the next section).

If the motion is slow ($M_1 \ll 1$), and if $T_w \approx T_1$, the resulting small variations of temperature can be neglected ($\Theta \approx 1$). Then,

$$Y = 2\eta\sqrt{\frac{\sigma X}{Re_1}} \tag{31}$$

and

$$U = \frac{1}{2}f'\left(\frac{Y}{2}\sqrt{\frac{Re_1}{\sigma X}}\right) \tag{32}$$

This special solution was obtained by Blasius (Ref. 34).

According to Eqs. 3.6, 19, the shearing stress is $\mu\bar{u}_y$ since the remaining term ($\mu\bar{v}_x$) is negligible for boundary layer flow. Expressed in dimensionless form, the shearing stress is

$$\tau = \frac{2K}{Re_1}U_{1'} \tag{33}$$

According to Eq. 15,

$$U_{1'} = \tfrac{1}{2}f''(\eta)\eta_{1'} \tag{34}$$

and from Eqs. 3, 13, and 16

$$\eta_{1'} = \eta_\xi \xi_{1'}\Psi_{1'} = \frac{1}{2(\Theta)}\sqrt{\frac{Re_1}{\sigma X}} \tag{35}$$

Then

$$\tau = \frac{1}{2}f''(\eta)\sqrt{\frac{\sigma}{XRe_1}} \tag{36}$$

if the surface temperature is constant. On the surface of the plate $f''(0) = 1.328$, and the coefficient of local skin friction is

$$C_f = \tau_w = 0.664\sqrt{\frac{\sigma}{XRe_1}} \tag{37}$$

The over-all drag coefficient due to skin friction for unit width of plate of length l, based on the dynamic head $\frac{1}{2}\rho_1\bar{u}_1^2$, can be obtained by integrating τ_w over the relevant area,

$$C_F = \frac{L_1}{l}\int_0^{l/L_1}\tau_w\, dX = 1.328\sqrt{\frac{\sigma Kn}{Re_1}} \tag{38}$$

where

$$Kn = L_1/l \tag{39}$$

The quantity Kn is the Knudsen number for a flat plate. In the problem discussed here $Kn = D\Delta$, and the Knudsen number is necessarily very small if the boundary layer equations (4.5, 22–26) and the boundary conditions (4.5, 9) are valid. When Kn is of order 1, the mechanism of momentum transfer is quite different since collisions

between gas molecules have little part in the transport process which now involves a direct exchange between freely moving molecules and the surface of a body (see Section 5.3).

4.7 ENERGY TRANSFER IN THE BOUNDARY LAYER

The energy-transfer relation (4.5, 24) can be reduced to a simpler form with the assistance of the mass-transfer equation (4.5, 22) and Eqs. 4.5, 25; 4.6, 8. The first term on the right-hand side of Eq. 4.5, 24 can be transformed as follows,

$$p(\bar{u}_y + \bar{v}_x) = -\frac{p}{\rho}\frac{d\rho}{dt} = \rho R\frac{dT}{dt} \tag{1}$$

and the energy-transfer equation becomes

$$\rho c_p(\bar{u}T_x + \bar{v}T_y) = \mu\bar{u}_y^2 + \frac{\partial}{\partial y}(\lambda T_y) \tag{2}$$

or, in dimensionless form,

$$\Gamma(U\Theta_X + V\Theta_Y) = \frac{(\gamma-1)M_1^2}{Re_1}(KU_Y^2) + \frac{1}{PrRe_1}\frac{\partial}{\partial Y}(K\Theta_Y) \tag{3}$$

As in the momentum-transfer equation, further simplification can be obtained by introducing the stream function Ψ in place of the independent variable Y. From Eqs. 4.6, 5, 6,

$$\Theta_X = \frac{(\gamma-1)M_1^2}{Re_1}(K\Gamma UU_\Psi^2) + \frac{1}{PrRe_1}\frac{\partial}{\partial\Psi}(K\Gamma U\Theta_\Psi) \tag{4}$$

If the variation of viscosity with temperature can be expressed with sufficient accuracy by Eq. 4.1, 4, then Eq. 4 may be written

$$\Theta_X = \frac{(\gamma-1)M_1^2\sigma}{Re_1}UU_\Psi^2 + \frac{\sigma}{PrRe_1}\frac{\partial}{\partial\Psi}(U\Theta_\Psi) \tag{5}$$

A solution of this differential equation is required which will satisfy the following boundary conditions:

$$\Theta(X,0) = \Theta_w\ (\Theta_w \text{ constant}); \quad \Theta(X,\infty) = 1 \tag{6}$$

Introducing ξ in place of Ψ according to Eq. 4.6, 13, we have

$$\Theta_X = (\gamma-1)M_1^2UU_\xi^2 + \frac{1}{Pr}\frac{\partial}{\partial\xi}(U\Theta_\xi) \tag{7}$$

The velocity ratio and its derivative (U, U_ξ) are given in terms of η by Eqs. 4.6, 15, 17, so that

$$\Theta_X = \frac{\gamma-1}{8}\frac{M_1^2}{X}\frac{f''^2}{f'} + \frac{1}{2Pr}\frac{\partial}{\partial\xi}(f'\Theta_\xi) \tag{8}$$

The fact that f' and f'' depend only on η suggests an investigation of the form of Eq. 8 when η replaces ξ. We note that

$$\left(\frac{\partial}{\partial \xi}\right)_X = \frac{1}{f'\sqrt{X}}\left(\frac{\partial}{\partial \eta}\right)_X$$
$$\left(\frac{\partial}{\partial X}\right)_\xi = \left(\frac{\partial}{\partial X}\right)_\eta - \frac{f}{2f'X}\left(\frac{\partial}{\partial \eta}\right)_X \tag{9}$$

Introducing these operators into Eq. 8, the result is

$$\Theta_{\eta\eta} + Prf\Theta_\eta - 2Prf'X\Theta_X = -\frac{\gamma - 1}{4} PrM_1^2 f''^2 \tag{10}$$

This is a linear, partial differential equation of the second order.

Let us find a solution subject to the revised boundary conditions:

$$\text{At } \eta = 0, \quad \Theta = \Theta_w \ (\Theta_w \text{ constant}); \quad \text{at } \eta \to \infty, \quad \Theta = 1 \tag{11}$$

Applying a method of separation of the variables, we write

$$\Theta(X, \eta) = g_1(X) + g_2(\eta) \tag{12}$$

and Eq. 10 becomes

$$\frac{1}{2Prf'}\left[g_2'' + Prfg_2' + \frac{Pr}{4}(\gamma - 1)M_1^2 f''^2\right] = Xg_1' \tag{13}$$

Since the left-hand side of this equation depends on η only, and the right-hand side is a function of X only, both of which are mutually independent, then the two sides of Eq. 13 must be constant. Considering first the right-hand side,

$$g_1(X) = \text{constant} \cdot \log X \tag{14}$$

This relation contains a logarithmic singularity at the leading edge of the plate ($X = 0$) which is not consistent with the boundary conditions (11). Therefore, the constant must be zero, and from the left-hand side of Eq. 13,

$$\Theta'' + Prf\Theta' + \frac{Pr}{4}(\gamma - 1)M_1^2 f''^2 = 0 \tag{15}$$

for which the boundary conditions are now

$$\Theta(0) = \Theta_w, \quad \Theta(\infty) = 1 \tag{16}$$

This is an ordinary, first-order differential equation for Θ'. Using the standard method for solving an equation of this type, we obtain

$$\Theta' = -\frac{Pr}{4}(\gamma - 1)M_1^2 (f'')^{Pr} \int_0^\eta [f''(\zeta)]^{2-Pr} d\zeta + A(f'')^{Pr} \tag{17}$$

and, on further integration,

$$\Theta = \frac{Pr}{4}(\gamma - 1)M_1^2 \int_\eta^\infty (f'')^{Pr} \left\{ \int_0^\eta [f''(\zeta)]^{2-Pr} d\zeta \right\} d\eta$$
$$- \kappa_1 \int_\eta^\infty (f'')^{Pr} d\eta + \kappa_2 \tag{18}$$

where the limits of integration for Θ' and Θ are chosen to suit the boundary conditions at $\eta = 0$ and $\eta \to \infty$, respectively. The integrals in Eq. 18 will be denoted as follows:

$$r(\eta) = \frac{1}{2} Pr \int_\eta^\infty [f''(\eta)]^{Pr} \left\{ \int_0^\eta [f''(\zeta)]^{2-Pr} d\zeta \right\} d\eta \tag{19}$$

and

$$s(\eta) = \int_\eta^\infty [f''(\eta)]^{Pr} d\eta \tag{20}$$

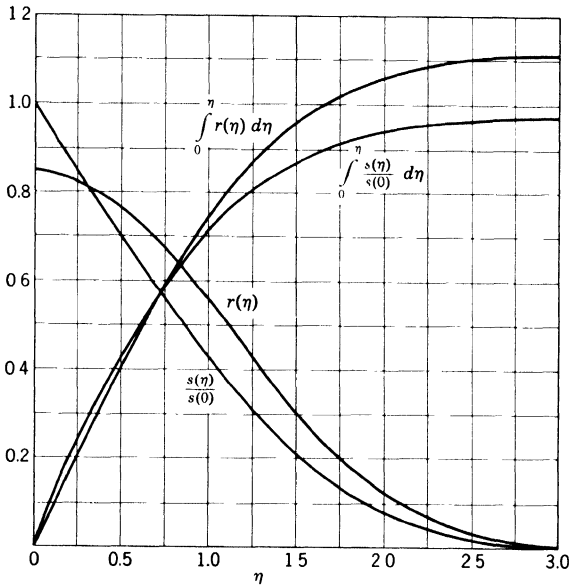


Fig. 10. Curves for $r(\eta)$, $s(\eta)$, and their integrals (Ref. 5).

Curves for these functions for air are given in Fig. 10. According to the boundary conditions,

$$\kappa_1 = -\frac{\Theta_w - \Theta_e}{s(0)}, \quad \kappa_2 = 1 \tag{21}$$

where
$$\Theta_e = 1 + r(0)\frac{\gamma - 1}{2} M_1^2 \tag{22}$$

Then
$$\Theta = 1 + r(\eta)\frac{\gamma - 1}{2} M_1^2 + (\Theta_w - \Theta_e)\frac{s(\eta)}{s(0)} \tag{23}$$

is a particular solution of the energy-transfer equation which is valid for constant surface temperature.

The quantity $r(0)$ is called the recovery factor. It has the values 1 and 0.85 for $Pr = 1$ and $Pr = 0.74$ (air), respectively. When $\Theta_w = \Theta_e$ ($\eta = 0$), then $\Theta' = 0$, and it will be seen that Θ_e is the equilibrium temperature corresponding to the case of no heat transfer at the surface of the plate. It is also called the adiabatic wall temperature.

We may now calculate the relation between Y and η. Substituting for Θ in Eq. 4.6, 30,

$$\frac{Y}{2}\sqrt{\frac{Re_1}{\sigma X}} = \eta + \frac{\gamma - 1}{2} M_1^2 \int_0^\eta r(\eta)\, d\eta + (\Theta_w - \Theta_e)\int_0^\eta \frac{s(\eta)}{s(0)}\, d\eta \tag{24}$$

Curves for the above integrals are also given in Fig. 10.

It will be seen that both η and Y tend to infinity as U approaches 1. Some idea of the effective thickness of the boundary layer can be obtained by determining the magnitude of Y at which U comes within 1 per cent of the free stream value. Thus, substituting the value of η correspondir.g to $U = 0.99$ into Eq. 24, the required value of δ_u can be found in terms of the known parameters in the undisturbed flow.

The distribution of Mach number in the boundary layer is of particular interest. The variation of M with η is given by

$$M = \frac{M_1 U}{\sqrt{\Theta}} = \frac{M_1}{2} f'(\eta)\left[1 + r(\eta)\frac{\gamma - 1}{2} M_1^2 + (\Theta_w - \Theta_e)\frac{s(\eta)}{s(0)}\right]^{-1/2} \tag{25}$$

Then $M(Y)$ can be found from Eqs. 24 and 25. This function is discussed further in Section 4.8.

The heat transferred by the molecules through unit area in unit time is $-\lambda T_y$ or, dividing by $\rho_1 \bar{u}_1 \cdot c_p T_1$, it can be placed in dimensionless form

$$q = -\frac{K\Theta_Y}{PrRe_1} \tag{26}$$

analogous to τ. Substitution from Eq. 4.6, 35 yields

$$q = -\frac{K\Theta'\eta_Y}{PrRe_1} = -\frac{1}{2}\frac{\Theta'}{Pr}\sqrt{\frac{\sigma}{Re_1 X}} \tag{27}$$

On the surface of the plate ($\eta = 0$)

$$C_h = q_w = 0.296\left(\frac{\Theta_w - \Theta_e}{Pr}\right)\sqrt{\frac{\sigma}{Re_1 X}} \tag{28}$$

where

$$[f''(0)]^{Pr}/s(0) = 0.5915 \tag{29}$$

The ratio of the local coefficients of heat transfer and friction on the surface is

$$\frac{C_h}{C_f} = \frac{0.445}{Pr}(\Theta_w - \Theta_e) \tag{30}$$

Reference to the expression for Θ_e (Eq. 22) shows that for large heat transfer at low skin friction loss, M_1 must be small (C_h/C_f positive and large). Aerodynamic heating occurs at high values of M_1, owing to the very considerable temperatures developed in the laminar boundary layer.

We can define a coefficient of total heat transfer at the surface along the lines of the definition of C_F,

$$C_H = \frac{L_1}{l}\int_0^{l/L_1} q_w\, dX = 0.5915\left(\frac{\Theta_w - \Theta_e}{Pr}\right)\sqrt{\frac{\sigma Kn}{Re_1}} \tag{31}$$

The Nusselt number is frequently mentioned in the literature on heat transfer. It is defined as follows:

$$Nu = \frac{hx}{\lambda_1} \tag{32}$$

where $h(x)$ is given by the relation

$$-[\lambda T_y]_w = h(T_w - T_e) \tag{33}$$

In terms of dimensionless parameters

$$Nu = -\frac{X[K\Theta_Y]_{\eta=0}}{\Theta_w - \Theta_e} = C_h\left[\frac{Re_1 X Pr}{\Theta_w - \Theta_e}\right] \tag{34}$$

4.8 EXPERIMENTAL INVESTIGATIONS OF THE BOUNDARY LAYER

Although the boundary layer has been the subject of experimental investigations at low Mach numbers for many years, the extension of the theory into the high subsonic and supersonic flow regions (Ref. 5) has led to further work with emphasis on the study of heat transfer. When the Mach number is small, it is possible to neglect the heat generated in the boundary layer and to concentrate on measurements of the skin friction, and there is little need for an approach to the subject from the point of view of the kinetic theory. In supersonic flow, however, this heating effect is quite important. The motion in the boundary

layer at large values of M is of special interest from the standpoint of molecular physics since it involves essentially a conversion of the energy of mass motion of the molecules to the energy of random motion at constant pressure.

The design of a flat plate for experiments on the boundary layer must necessarily depart to some extent from the infinitely thin plate assumed in the theory. The plate must have a small leading-edge radius and

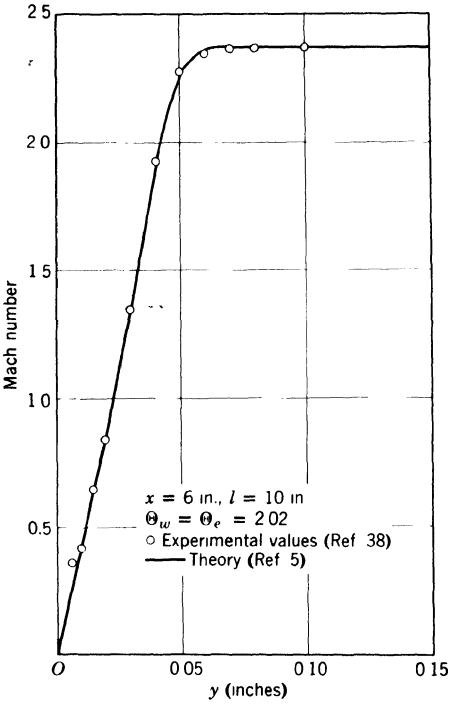


Fig. 11. Typical curve of M vs. y in the boundary layer.

sufficient thickness to house a cooling or heating system, heat flow meters, electric leads, and pressure tubes. Typical diagrams of such models are given in Refs. 37, 38. In addition to the deviation of the model from the idealized plate, problems of instrumentation also arise such as the interpretation of the readings of probing devices adjacent to the surface. Although much experimental investigation at high Mach numbers remains to be done, some results of interest can be mentioned briefly in support of the preceding theory.

A typical distribution of the Mach number through the boundary layer, measured by Maydew and Pappas (Ref. 38) by means of an impact-pressure traverse, is shown in Fig. 11. A comparison with the theory of Chapman and Rubesin (Ref. 5) is also included. A similar comparison of measured and calculated velocity distributions, given in Ref. 37, indicates somewhat less agreement.

A direct experimental approach to the determination of local skin friction has been developed by Liepmann and Dhawan, and Coles (Refs. 39, 40). Measurements of C_f were obtained by observing the force on

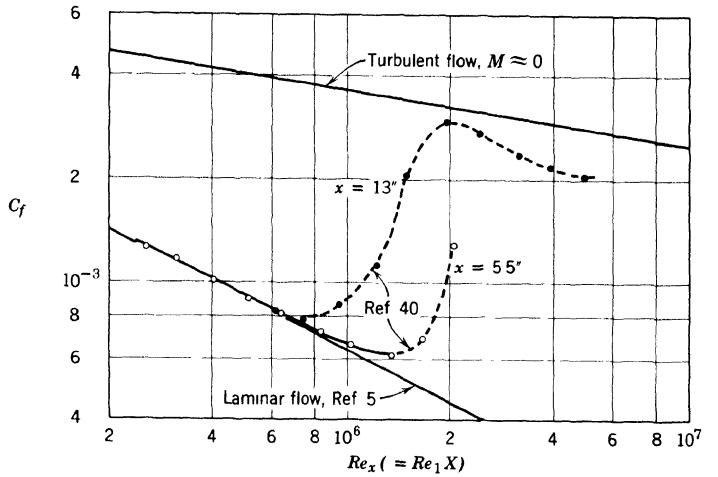


Fig. 12. Skin friction on a flat plate ($M = 2.6$).

a small movable insert in a flat plate by means of a reluctance measuring device. Tests were made at low Mach number to check the reliability of the floating element technique, and very good agreement with results obtained by other methods was found. Subsequent measurements at supersonic speeds by Coles at Mach numbers up to 4.5 show good agreement between theory and experiment (Fig. 12).

According to Fig. 12, a transition from laminar to turbulent flow occurs in the boundary layer at a value of $Re_1 X$ (or Re_x) of about 10^6. Since molecular theory does not yet provide an adequate description of turbulent motion, this subject is beyond the scope of this book. A theoretical study of turbulent flow from the point of view of kinetic theory presents an interesting challenge to the modern research worker.

Measurements of heat transfer in the laminar boundary layer on a flat plate in supersonic flow have also been made (Refs. 37, 41). The

temperature recovery factor was found to be 0.884, which is about 4 per cent higher than the calculated value (Section 4.7). Heat transfer data for a cooled plate obtained by Slack are given in Fig. 13. These results have been corrected for surface temperature gradients by means of the more general theory of Chapman and Rubesin. At positions 3, 4, and 5 in. back from the leading edge the theory is substantiated, but at $x = 2$ in. a noticeable discrepancy occurs. A complete assessment of the heat transfer theory of Section 4.7 must await further experimental data.

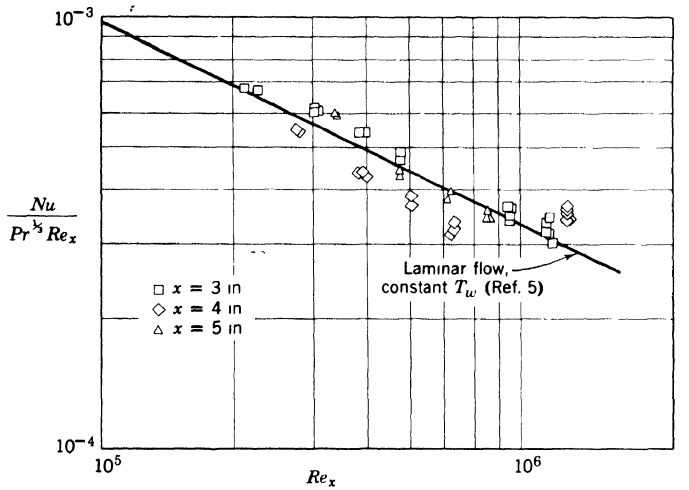


Fig. 13. Heat transfer in the laminar boundary layer on a cooled flat plate (Ref. 37).

4.9 EFFECTS ASSOCIATED WITH MORE COMPLEX MOLECULES

In the preceding discussion the properties of a flowing gas have been determined, assuming a simple monatomic form for the molecules. The internal energy of the gas is due entirely to the random translational motion of the molecules. More generally the internal energy will be larger if molecular rotation and vibration must be taken into account. When electronic excitation, dissociation, and ionization occur in a gas flow, these effects will contribute further to the content of internal energy. Let us express the total internal energy per unit mass in the form (Ref. 42)

$$\bar{e} = (\beta - 1)\frac{p}{\rho} \tag{1}$$

Then the energy equation for steady flow has the convenient form

$$\beta \frac{p}{\rho} + \frac{1}{2}\bar{q}^2 = \text{constant} \tag{2}$$

Let us first restrict our discussion to molecules having only translational, vibrational, and rotational motions. The flow is such that the equation of state remains the same as in the monatomic theory, and RT replaces p/ρ in Eqs. 1, 2. The contribution to β made by the translational motion of the molecules is

$$\beta_t = \frac{\bar{e}_t}{RT} = \frac{3}{2} \tag{3}$$

(see Section 2.2). According to the classical kinetic theory of gases, the contribution to β due to molecular rotation is $\beta_r = 0$ for atoms, $\beta_r = 1$ for diatomic molecules and polyatomic molecules whose atoms lie on a straight line, and $\beta_r = \frac{3}{2}$ for all other polyatomic molecules.

The vibrational energy of the molecules can be determined approximately by resolving the vibration into normal modes and considering each mode as an harmonic oscillator. A diatomic molecule has one normal mode; a polyatomic molecule with n_0 atoms on a straight line has $(3n_0 - 5)$ modes; all other polyatomic molecules have $(3n_0 - 6)$ normal modes. The energy per unit mass in one mode is $f(Z)RT$ where

$$f(Z) = \frac{Z}{e^Z - 1}, \quad Z = \frac{h\nu}{kT} \tag{4}$$

h is Planck's constant, k is Boltzmann's constant (gas constant per molecule), and ν is the frequency of molecular vibration. The frequencies of the various normal modes of vibration of the molecules can be determined from band spectra. It is found that the simple light molecules have relatively high frequencies of vibration. The frequencies for N_2 and O_2 are 2345 and 1570 waves per cm, respectively. On the other hand, CO_2 has four modes of vibration for which ν has the values 667, 667, 1336, and 2350 waves per cm.

The variation of β_r (per mode) with temperature is indicated in Fig. 14. When the temperature is low, the vibrational energy is negligibly small. However, the difference between the vibrational energies of different gases at the same temperature may be large. For example, at room temperature the vibrational energies of N_2 and O_2 are negligible but the vibrational energy of CO_2 is still appreciable.

As the temperature tends to become high, the vibrational energy per unit mass in one normal mode approaches the value RT. It should be noted, however, that at very high temperatures the vibrations cannot be

regarded as harmonic since an interaction occurs between the rotational and vibrational motions. At 5000° K the error in β_v resulting from the assumption of harmonic oscillation is 7 per cent for N_2.

The contribution of one normal mode of vibration to the specific heat at constant volume is

$$[c_v]_v \text{ (per mode)} = \frac{d}{dT}[f(Z)RT] \tag{5}$$

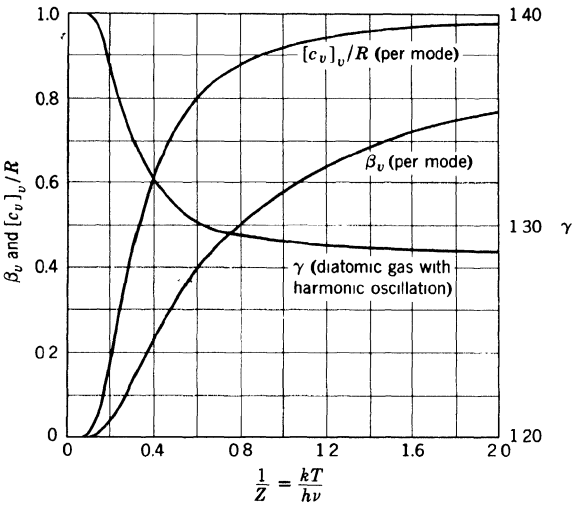


Fig. 14. Properties of the harmonic oscillator (Ref. 42).

For a diatomic gas (one mode of vibration) the specific heat at constant volume is

$$c_v = \tfrac{5}{2}R + [c_v]_v \tag{6}$$

and the specific heat at constant pressure is

$$c_p = \tfrac{7}{2}R + [c_v]_v \tag{7}$$

where $[c_v]_v$ is given in Fig. 14. The ratio c_p/c_v is the value of γ for a diatomic gas with harmonic oscillation. The variation of γ with T is shown in Fig. 14.

When the vibrational energy of the molecules makes a significant contribution to the specific heat, γ must be regarded as a function of temperature (Fig. 14). As the temperature increases, the vibration makes a larger contribution to the internal energy content, and c_p, c_v

increase (Eqs. 6, 7). From the differential equations for isentropic flow, we have

$$\frac{d\rho}{\rho} = \frac{c_v}{RT}dT, \quad \frac{dp}{p} = \frac{c_p}{RT}dT \tag{8}$$

Then, if c_v and c_p are functions of the temperature, the density and pressure depend on temperature in the following way:

$$\log \rho = \frac{1}{R}\int \frac{c_v\, dT}{T}, \quad \log p = \frac{1}{R}\int \frac{c_p\, dT}{T} \tag{9}$$

These functions are tabulated in Ref. 43. The effect of a variable specific heat on the nonstationary expansion wave is discussed in Ref. 44.

When a gas flow involves very rapid changes of state, the time required for a degree of freedom to adapt itself to the sudden transition in the energy content (called the relaxation time) becomes an important consideration. If the time in which large temperature changes occur in a flow become comparable with or even less than the relaxation time, then the partition of the internal energy content of the gas among the various degrees of freedom deviates initially from the distribution corresponding to the equilibrium state. The subsequent process of redistribution of the energy content among the degrees of freedom, as equilibrium is approached, is irreversible and an increase in entropy results.

The translational degree of freedom has the shortest relaxation time since even single collisions between molecules produce substantial changes in the translational motions of the molecules. Thus very rapid changes in the translational energy content are possible over distances of the order of the mean free path.

The exchange of rotational energy at an intermolecular collision is almost as effective as the transfer of translational energy, particularly in the case of elongated molecules like those of CO_2. When the molecules are more accurately represented as spherical (as in the case of N_2), the exchange is less efficient and some 10 to 100 collisions are required before rotational energy can be effectively transferred.

On the other hand, the process of exchange of vibrational energy between molecules is inefficient and equilibrium is attained very slowly by this degree of freedom. In CO_2 at 32.5° C and 1 atmosphere, about 33,000 collisions are needed to reduce the defect to $1/e$th of its initial value. This arises essentially from the fact that the amplitude of the vibrations is very small compared with the range of the intermolecular forces (Ref. 42).

We conclude, therefore, that the translational and rotational degrees of freedom are "active" but that molecular vibration is an "inert" degree of freedom. The internal energy content of the active degrees of freedom

changes almost discontinuously since only a few mean free paths are necessary for the transfer. We may say that at every point of the flow

$$\bar{e}_a + \frac{p}{\rho} = \beta_a RT \tag{10}$$

where $\bar{e}_a$ is the internal energy content of the active degrees of freedom per unit mass. This relation may be used as a basis for a definition of temperature at a point in a gas flow in which relaxation effects occur. If the gas is diatomic, $\beta_a = \frac{7}{2}$.

The energy content of the inert degrees of freedom per unit mass ($\bar{e}_i$) does not remain consistent with $\bar{e}_a$ and is not necessarily a function of the local temperature.

In a flow in which molecular vibration is the only inert degree of freedom, we write

$$\bar{e}_a = c_{v_a} T, \quad \bar{e}_i = c_{v_i} T_i \tag{11}$$

where for a diatomic gas $c_{v_a} = \frac{5}{2}R$, and c_{v_i} is the same as $[c_v]_t$ per mode (Fig. 14). The temperature (T_i) associated with the inert degree of freedom attains the value T when equilibrium is ultimately reached. We write further

$$\begin{aligned} c_v &= c_{v_a} + c_{v_i} \\ c_p &= c_{v_a} + c_{v_i} + R = c_v + R, \quad \gamma = c_p/c_v \end{aligned} \tag{12}$$

$$c_{p_a} = c_{v_a} + R, \quad \gamma_a = c_{p_a}/c_{v_a} \tag{13}$$

During a rapid change of T and before equilibrium is reached, T_i will follow T with a certain lag according to the relation

$$\frac{dT_i}{dt} = \chi(T - T_i) \tag{14}$$

where $\chi(\rho, T)$ is a retardation factor.

Let us consider the relaxation effect in a steady, subsonic compression flow (Refs. 45, 46). If s is directed along the axis of a stream tube at a point where the area of cross section is A_s, then the equations of motion may be written

$$\begin{aligned} \rho \bar{q} A_s &= \text{constant} \\ \bar{q} \frac{\partial \bar{q}}{\partial s} &= -\frac{1}{\rho} \frac{\partial p}{\partial s} \\ c_{p_a} T + c_{v_i} T_i + \tfrac{1}{2}\bar{q}^2 &= c_p T_0 \end{aligned} \tag{15}$$

where

$$p = \rho RT, \quad \bar{q} \frac{\partial T}{\partial s} = \chi(T - T_i) \tag{16}$$

The subsonic, compression flow in the stream tube from free stream to the stagnation point of a body is of particular interest (Fig. 15). If the compression time is very slow compared with the relaxation time, the pressure and temperature will rise isentropically from the free stream values p_1, T_1 to the stagnation values p_0, T_0. In the following analysis of subsonic flow, it will be assumed that the difference between T_1 and T_0 is not large and variations in the specific heats with temperature can be neglected. Then

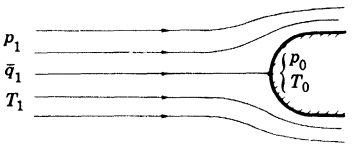
$$c_{p_a}T_1 + c_{v_i}T_1 + \tfrac{1}{2}\bar{q}^2 = c_{p_a}T_0 + c_{v_i}T_0$$

or

$$c_pT_1 + \tfrac{1}{2}\bar{q}^2 = c_pT_0 \tag{17}$$

and Eqs. 15, 16 reduce to the regular equations for isentropic flow for which

$$\frac{p_0}{p_1} = \left(\frac{T_0}{T_1}\right)^{\gamma/(\gamma-1)} \tag{18}$$



(a) Very slow compression – full adjustment in both T and T_i

(b) Very rapid compression – full adjustment in T, but no significant change in T_i ($T_i = T_1$)

Fig. 15. Subsonic compression flow.

If now the time required for the gas to attain the major portion of its temperature rise at the nose of the body is small compared with the relaxation time, then the vibrational part of the energy content of the gas does not follow the rise in temperature and comes to equilibrium after the compression is over. The associated increase in entropy results in a pressure p_2 lower than p_0. For a very rapid compression flow,

$$c_{p_a}T_1 + \tfrac{1}{2}\bar{q}^2 = c_{p_a}T_2 \tag{19}$$

The equations of motion now reduce to a form similar to the regular isentropic equations, except that c_{p_a} replaces c_p, and hence

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_0}\right)^{\gamma_a/(\gamma_a-1)} \tag{20}$$

Combining Eqs. 18, 20, we have

$$\frac{p_2}{p_0} = \left[\frac{c_p - c_{v_i}(T_1/T_0)}{c_p - c_{v_i}}\right]^{(c_p-c_{v_i})/R}\left(\frac{T_1}{T_0}\right)^{c_{v_i}/R} \tag{21}$$

The variation of this pressure ratio with the temperature is expressed by the derivative

$$\frac{d}{d(T_1/T_0)}\left(\frac{p_2}{p_0}\right) = \frac{c_{v_i}}{R}\left(\frac{p_2}{p_0}\right)\left[\frac{c_p(1 - T_1/T_0)}{c_p(T_0/T_1) - c_{v_i}}\right] \tag{22}$$

If $T_1/T_0 < 1$, the slope of the curve of p_2/p_0 vs. T_1/T_0 is positive, and, since $p_2/p_0 = 1$ when $T_1/T_0 = 1$, then, if $T_1/T_0 < 1$, we must have $p_2/p_0 < 1$.

During the attainment of equilibrium after the compression, a flow of energy takes place from the translational and rotational motions to the vibrational motion. The associated increase in entropy is

$$\Delta S = \int \frac{c_{v_a}\,dT}{T} + \int \frac{c_{v_i}\,dT_i}{T_i} \tag{23}$$

Throughout this process, $\bar{q} = 0$, and we have

$$c_{p_a}T + c_{v_i}T_i = c_pT_0 \tag{24}$$

from which

$$T = \frac{c_pT_0 - c_{v_i}T_i}{c_p - c_{v_i}} \tag{25}$$

Then

$$\Delta S = c_{v_i}\int_{T_1}^{T_0}\left[\frac{1}{T_i} - \frac{c_p - c_{v_i}}{c_pT_0 - c_{v_i}T_i}\right]dT_i \tag{26}$$

If we integrate and substitute from Eq. 21,

$$\Delta S = R \log\left(\frac{p_0}{p_2}\right) \tag{27}$$

The entropy loss can be considerable. The drag coefficient of a wing measured in a wind tunnel containing pure CO_2 may be as much as twice the drag coefficient determined in air at the same Mach and Reynolds numbers. Care must be taken in substituting one gas for another in a wind tunnel with a view to improving the range of *Re* and *M*.

The above analysis suggests that the relaxation effect in a subsonic compression flow may be investigated with an impact tube (Refs. 46, 47).

An experimental check on Eq. 21 can be made by the method illustrated in Fig. 16. Gas accelerates from rest in a large container to a subsonic speed in the free atmosphere and flows around an impact tube. The length of the nozzle is such that the change of state from p_0, T_0 to p_1, T_1 is very slow compared with the relaxation time; that is, it is an isentropic process. On the other hand, the flow is stopped suddenly at the inlet of the tube and the duration of this very rapid compression is comparable with the relaxation time.

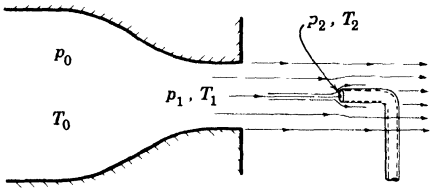


Fig. 16. Experimental investigation of the relaxation effect.

Kantrowitz (Ref. 46) has investigated CO_2 and found that Eq. 21 was valid, any difference between theory and experiment being due largely to the fact that the ideal instantaneous compression assumed in the above theory was not completely realized in the experiments. Some data on this investigation are as follows: gas velocities, 300 and 600 ft/sec; expansion times through the nozzle, 7×10^{-4} sec and 2.8×10^{-4} sec; compression times, 7×10^{-7} sec and 14×10^{-7} sec; the relaxation time of the CO_2 used, 10^{-5} sec (approx.). The difference between T_0 and T_1 was less than 30° F in this experiment, thus limiting the variation in c_{r_1} to less than 8 percent.

When the time for the temperature changes is of the same order as the relaxation time, a more general theory must be used. Kantrowitz has considered the more general problem (Refs. 46, 47) and developed a method for measuring the relaxation times of various gases (N_2, H_2O), using the impact tube method.

When the flow is supersonic, the compression occurs through a shock wave for which the following equations apply to states 1, 2, and 3 indicated in Fig. 17,

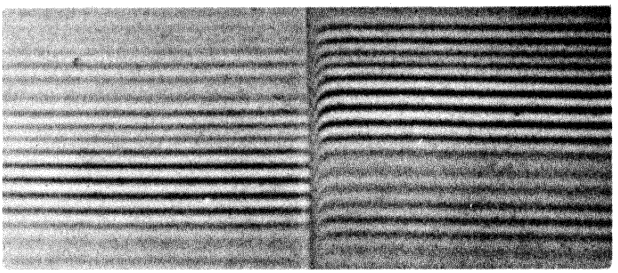
$$\rho_3\bar{u}_3 = \rho_2\bar{u}_2 = \rho_1\bar{u}_1$$

$$p_3 + \rho_3\bar{u}_3^2 = p_2 + \rho_2\bar{u}_2^2 = p_1 + \rho_1\bar{u}_1^2$$

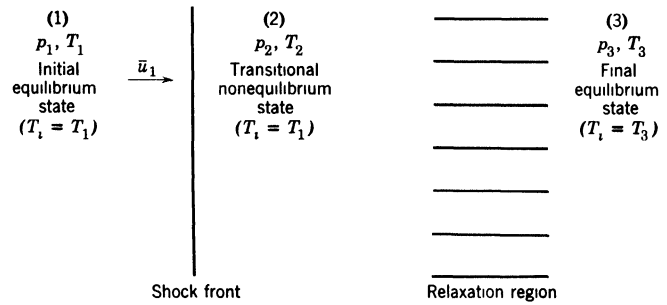
$$\beta_3\frac{p_3}{\rho_3} + \frac{1}{2}\bar{u}_3^2 = \beta_2\frac{p_2}{\rho_2} + \frac{1}{2}\bar{u}_2^2 = \beta_1\frac{p_1}{\rho_1} + \frac{1}{2}\bar{u}_1^2 \qquad (28)$$

$$\frac{p_3}{\rho_3 T_3} = \frac{p_2}{\rho_2 T_2} = \frac{p_1}{\rho_1 T_1}$$

where the functional form of β depends on whether complete equilibrium has been reached. If equilibrium exists, then $\beta = \beta(T)$ and β may be obtained from tables (Ref. 42). In the initial state 1 in front of the shock wave the gas is everywhere in equilibrium. Through the shock front the increase in internal energy content appears largely in the active degrees of freedom, and the gas is not in equilibrium in state 2. Between states 2 and 3 a relatively slow relaxation process takes place (Fig. 17*a*)



(*a*) UTIA interferogram of a shock wave in CO_2.



(*b*) States corresponding to (*a*).

Fig. 17. Relaxation effects in a shock wave.

in which energy is transferred from the active to the inert degrees of freedom. Equilibrium is regained asymptotically in state 3.

Application of the equations of motion between the two equilibrium states 1 and 3 shows that the macroscopic properties of the gas in state 3 are independent of all intervening processes associated with the establishment of equilibrium between the active and inert degrees of freedom. The relaxation effect in air can be illustrated by a calculation of the quantities in state 2 when T_1 is about 300° K or less (Ref. 42). At this value of T_1 the vibrational energy in state 1 is negligible; that is $\beta_1 = 3.483$

(Ref. 42, Table III). If now we assume that the time of transition through the shock front is very short compared with the relaxation time so that $T_i = T_1$ in state 2, then $\beta_2 = 3.483$. The results of these calculations are plotted in Fig. 18, which compares the macroscopic properties of the gas in states 2 and 3. The shock wave in air appears to consist of an almost instantaneous compression ($1 \rightarrow 2$) followed by a further gradual compression during the final attainment of equilibrium ($2 \rightarrow 3$).

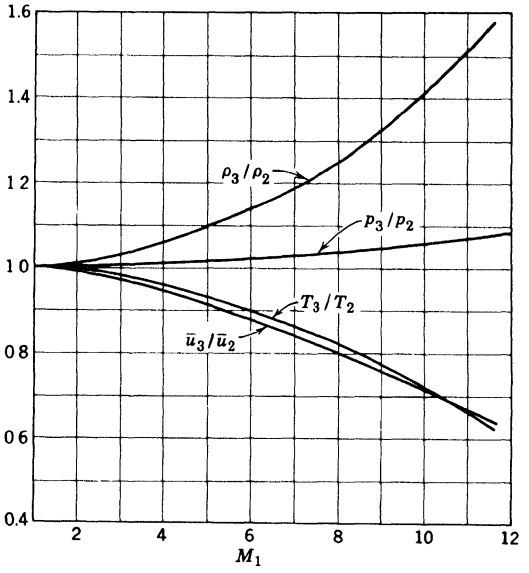


Fig. 18. Relaxation effect in a shock wave, calculated for air (Ref. 42).

The corresponding temperature changes consist of a "discontinuous" rise from T_1 to T_2 after which T_2 settles relatively slowly to a final lower value T_3. The final drop in temperature is associated with the transfer of internal energy content from the active to the inert degrees of freedom. Note that up to $M_1 = 3$ there is little difference between states 2 and 3. The transitions of the macroscopic properties of a gas through the shock wave are illustrated diagrammatically in Fig. 19.

When shock waves become very strong, exceedingly high temperatures occur in states 2 and 3. The high temperature produced behind a strong shock wave after reflecting from the end of a shock tube is illustrated in Fig. 20 (Ref. 48). In shock tubes temperatures well in excess of 5000° K are not difficult to obtain and luminescence such as that

shown in Fig. 20 is frequently observed. At such temperatures the electronic excitation, dissociation, and ionization of a gas become important. The contribution of electronic excitation to the internal energy content is usually small compared with contributions due to molecular vibration and dissociation.

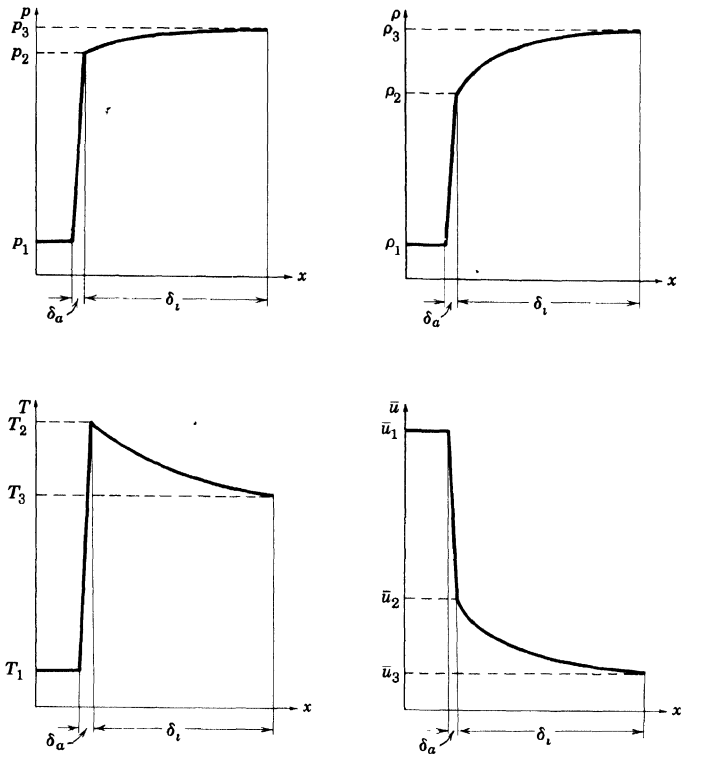


Fig. 19. Effect of relaxation in the shock transition (diagrammatic). δ_a, δ_i are shock thicknesses due to the active and inert degrees of freedom, respectively.

When a gas dissociates, the equation of state assumes the more general form

$$p = \rho RT(1 + \alpha_d) \tag{29}$$

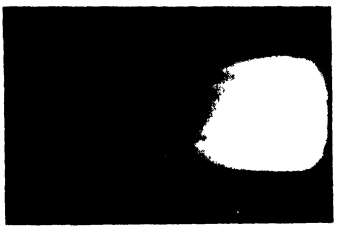
where α_d is the fraction of molecules dissociated or the degree of dissociation. According to the theory of dissociation, the partial

pressures due to atoms and molecules are, respectively,

$$p_A = \left(\frac{2\alpha_d}{1 + \alpha_d}\right) p, \quad p_M = \left(\frac{1 - \alpha_d}{1 + \alpha_d}\right) p \tag{30}$$

A characteristic quantity relevant to dissociation is defined as follows

$$\kappa_d = \frac{p_A^2}{p_M} = \left(\frac{4\alpha_d^2}{1 - \alpha_d^2}\right) p \tag{31}$$



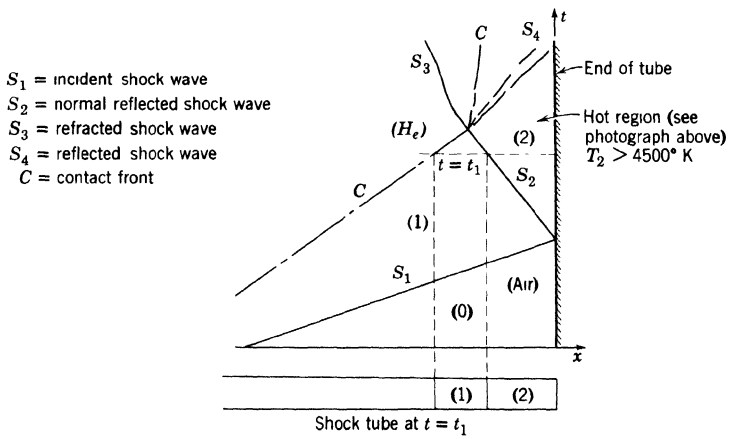


Fig. 20. Visible radiation in air produced by a reflecting shock front (Ref. 48).

This quantity can be found in tables or by calculation (Ref. 42). Then

$$\alpha_d = \left(\frac{\kappa_d}{\kappa_d + 4p}\right)^{1/2} \tag{32}$$

It is found that α_d is essentially a function of T, depending only slightly on p, ρ.

The coefficient of internal energy content in a dissociated gas is

$$\beta = \left(\frac{1 - \alpha_d}{1 + \alpha_d}\right)\beta_M + \left(\frac{\alpha_d}{1 + \alpha_d}\right)\left(\frac{E_d}{RT} + 2\beta_A\right) \tag{33}$$

where β_M and β_A are the coefficients of internal energy content for the molecular and atomic gas, respectively, and E_d is the energy of dissociation per unit mass of gas. The quantity β_M is obtained by adding the contributions to the energy content of the gas made by the translation, rotation, vibration, and electronic excitation of the molecules. On the other hand, β_A is due to translational energy only and has the value 5/2.

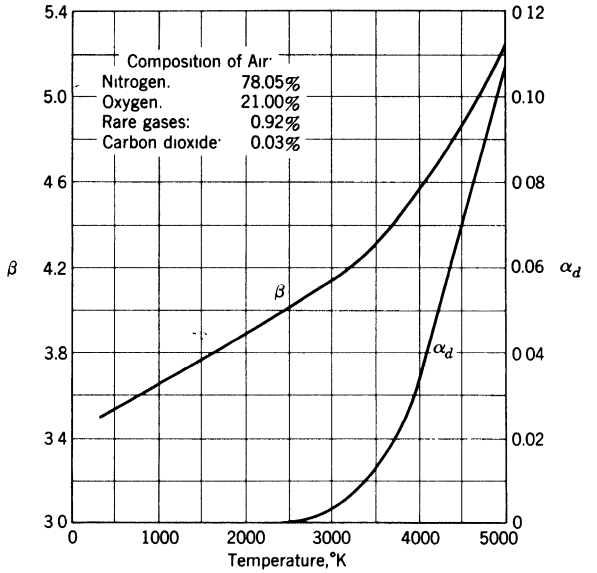


Fig. 21. Dissociation in air as a function of temperature.

Two of the terms in Eq. 33 are the values of β_M and β_A weighted according to the partial pressures. The remaining term takes account of the process of dissociation which involves the dissociation energy E_d.

The internal energy content and dissociation of air are plotted in Fig. 21 for the temperature range $300° \text{ K} < T \leq 5000° \text{ K}$. It appears that dissociation begins to make a significant contribution to the internal energy content of air when T is 2000–3000° K (Ref. 42).

NOTATION

A	area between the shock profile and one asymptote (Fig. 6)
A_s	area of cross section of a stream tube
A_1, A_2, A_3	coefficients in the shock transition equation which depend on the initial conditions, Eqs. 4.3, 6

$B_0, B_1, B_2, \cdots$ constant coefficients in the polynomial $f(\eta)$, Eq. 4.6, 20
$[c_v]_v$ contribution to c_v by one normal mode of vibration, Eq. 4.9, 5
C_f (or τ_w) coefficient of local skin friction, Eq. 4.6, 37
C_F coefficient of over-all skin friction drag, Eq. 4.6, 38
C_h (or q_w) coefficient of local heat transfer at the wall, Eq. 4.7, 28
C_H coefficient of total heat transfer, Eq. 4.7, 31
d diameter of a tube
D ratio of the thickness of the velocity layer at $x = l$ to the length of the flat plate $[\delta_u(l)/l]$
E_d energy of dissociation per unit mass of gas
$\bar{e}$ total internal energy per unit mass ($\bar{e} = \bar{E}_i/m$, Section 2.2)
h heat transfer per unit area and time and per degree temperature difference, Eq. 4.7, 33, or Planck's constant according to context
k the ratio e/ρ, or the gas constant per molecule according to context (see e, Notation, Chapter 3)
k_1, k_2, k_3, k_4 coefficients in the viscosity and static pressure relations for high pressure, Eqs. 4.1, 7, 9
K dimensionless coefficient of viscosity or heat conduction ($K = \mu/\mu_1 = \lambda/\lambda_1$)
K_b ratio of the coefficient of bulk viscosity to the coefficient of shear viscosity (μ_b/μ)
Kn Knudsen number (L_1/l or $D\Delta$ for a flat plate)
l length of tube or flat plate
n_o number of atoms in a molecule
Nu Nusselt number (hx/λ_1), Eq. 4.7, 34
P dimensionless pressure ratio (p/p_1)
Pr Prandtl number ($\mu_1 c_p/\lambda_1$) having the value $4\gamma/(9\gamma-5)$
q heat-transfer coefficient, Eq. 4.7, 26
Q volume of a gas flowing through a tube in unit time
$r(0)$ recovery factor, Eq. 4.7, 22
Re Reynolds number based on the mean free path ($\rho L\bar{u}/\mu$)
Re_1 Reynolds number at a reference point ($\rho_1 L_1 \bar{u}_1/\mu_1$)
Re_x Reynolds number based on the distance along the plate ($Re_1 X$)
s distance along the axis of a stream tube
T_e temperature of the gas at the surface of a body when no heat transfer occurs
T_i temperature associated with the inert degrees of freedom
T_δ maximum temperature in the boundary layer
U, V dimensionless velocity components ($\bar{u}/\bar{u}_1, \bar{v}/\bar{u}_1$)
X, Y dimensionless position coordinates ($x/L_1, y/L_1$)

α a constant (1·09924)
α_d degree of dissociation
$\alpha_1, \alpha_2, \cdots, \alpha_6$ constants in the general boundary conditions (4.5, 6–8)
β $3/(2\overline{C^2})$ or a dimensionless coefficient in the definition of $\bar{e}$, Eq. 4.9, 1
$\beta_1, \beta_2, \cdots, \beta_5$ constants in the dimensionless equations of motion (4.5, 18–21), or values of β for various states according to Eq. 4.9, 1
Γ dimensionless density ratio (ρ/ρ_1)
δ thickness of the shock wave, Eq. 4.3, 13
δ' ratio of the thicknesses of the dynamic and thermal layers, Eq. 4.5, 1
$\delta_T(l)$ thickness of the thermal layer at $x = l$

$\delta_u(l)$	thickness of the dynamic layer at $x = l$
Δ	ratio of the reference mean free path to the thickness of the velocity layer at $x = l$ $(L_1/\delta_u(l))$
ζ	symbol for the product $\alpha\eta$
η	parameter in the transformation 4.6, 16, related to the perpendicular distance from the plate at a given position x
Θ	dimensionless temperature ratio (T/T_1)
Θ_e	dimensionless equilibrium temperature (T_e/T_1)
κ	Sutherland's constant, Eq. 4.1, 2
κ_1, κ_2	constants of integration
μ_b	coefficient of bulk or compression viscosity
ν	frequency of molecular vibration
ξ	parameter in the transformation 4.6, 13
σ	constant in the approximate viscosity-temperature relation, Eq. 4.1, 4, or the diameter of a molecule (function of T)
τ	coefficient of shearing stress, Eq. 4.6, 33
χ	retardation factor, a function of ρ, T (Eq. 4.9, 14)
Ψ	dimensionless stream function, Eq. 4.6, 3

Note: Symbols that do not appear above may be found in the Notation at the end of preceding chapters (pages 24, 63, 97). The subscripts 1 and 2 applied to macroscopic functions of x, y, z, t indicate specific values corresponding to two different states of flow. Capital letters are used, in general, to indicate dimensionless quantities. However, in the discussion of the order of magnitude of the terms in the boundary conditions and the equations of motion in Section 4.5 a different system is necessary, and primes are employed to denote the required dimensionless ratios. Since the state of a gas and a body are not necessarily the same at the surface, the subscripts s and w are attached to the various symbols to indicate that they refer to either the state of the gas at the surface or the condition of the wall, respectively. In Section 4 9, the subscripts t, r, v refer to contributions made by translation, rotation, and vibration of the molecules, respectively, and the subscripts a, i indicate the active and inert degrees of freedom, respectively. Quantities with subscripts A and M refer, respectively, to the atoms and molecules of a gas in a dissociated state.

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CHAPTER FIVE

Mechanics of Rarefied Gases

5.1 FLOW AT LOW DENSITY

An understanding of the motion of rarefied gases at low speed is important in many industrial processes in which high vacuum is required, such as distillation, and a fair accumulation of information on this subject is now available to the designer. Interest in the mechanics of rarefied gases at high Mach numbers has been increased by the practical realization of supersonic flight at very high altitudes. In the "adopted atmosphere" discussed in Refs. 1 and 2, the mean free paths at altitudes of 60, 75, and 95 miles are approximately $\frac{1}{10}$, 1, and 10 feet, respectively. At altitudes above 100 miles the mean free path may be appreciably larger than the over-all dimensions of an aircraft. In both high-vacuum systems and high-altitude flight, the Knudsen number becomes large, and the transfer equations are different from those which hold at greater densities.

Investigations of the mechanics of rarefied gases have been made from two points of view. If the gas is highly rarefied, the frequency of collisions between gas molecules in the element of volume $d\tau$ (Section 1.6) becomes negligibly small. Nevertheless, even at very low density, there will still be sufficient molecules in $d\tau$ to permit the definition and determination of the macroscopic properties of the gas. Such motion is described as free-molecule flow. For example, when the mean free path in the upper atmosphere is 10 feet (that is, the chance of collision is small), the number of molecules in a cubic inch is about 10^{13}, thus permitting the calculation of pressure, temperature, and mass velocity by the methods of Section 1.4. Near the surface of a body the two streams of incident and emergent molecules (Section 4.4) experience little interaction. The boundary layer disappears, and the molecular motion adjacent to the surface remains Maxwellian. The properties of the motion of highly rarefied gases can be developed, therefore, from Maxwell's law of distribution of molecular velocities.

On the other hand, if the gas is only moderately rarefied, it is necessary to reconsider the general equations for viscous, heat-conducting,

compressible flow (3.9, 9, 10, 11) in conjunction with the more general boundary conditions (4.4, 30–33) in which discontinuities in velocity, temperature, and pressure occur at the surface of a body. The motion adjacent to a surface is non-Maxwellian, the boundary layer still exists, and the flow takes place with slip and a temperature jump at the wall.

The properties of highly rarefied gases were investigated extensively by Knudsen (Ref. 3) and Smoluchowski (Ref. 4), who explained the low-speed phenomena which they observed by theories based on the assumption of Maxwellian molecular motion. Their theories were extended by Sanger (Ref. 5), Tsien (Ref. 6), Ashley (Ref. 7), and others (Refs. 8, 9) so that a complete theory of free-molecule flow now exists. Knudsen has described his early experimental work at low speed in Ref. 10. Few experimental investigations of free-molecule flow at high speed exist at present, but the work of Schaaf, Stalder and their co-workers makes it possible to obtain some assessment of the validity of the theory of free-molecule flow.

Investigations of slip flow were first presented (1875) by Kundt and Warburg (Ref. 12), who found that the discharge of a gas from a tube at low density was larger than that calculated for the same pressure difference assuming laminar flow. These experiments led Maxwell (1879) to determine the boundary conditions at the wall from a consideration of the interaction of gas molecules with the surface of a solid body (Ref. 13). He found that the tangential velocity at the surface was small but finite and that slip flow could occur. Pioneering work on the aerodynamics of spheres in rarefied gases at low speeds was done by Epstein (Ref. 14) and Millikan (Ref. 15). More recently, efforts have been made to estimate the effect of slip flow on skin friction and heat transfer at high Mach numbers (Refs. 16, 17, 18, 19).

5.2 EFFUSIVE FLOW OF FREE MOLECULES

Let us consider the motion of free molecules through a small aperture in a diaphragm which separates two large compartments filled with gas (Fig. 1). The mean free path in either compartment is much greater than the diameter of the hole, but very small compared with the dimensions of the compartments. Then the molecules of each gas will pass through the aperture unhindered by collisions just as if the other gas were absent. On the other hand, there will be sufficient molecules in each compartment to permit the determination of the macroscopic properties of each gas (pressure and temperature, or n and $\overline{C^2}$).

We assume first that the pressures are different, and the temperatures are the same on each side of the aperture. The occasional loss of a molecule from either gas through the hole produces no appreciable

effect on the motion of the molecules in the body of the gas. Thus, neither gas develops a mass motion toward the opening. We may assume, therefore, that the molecular velocities are distributed throughout the motion according to Maxwell's law for a gas at rest.

Let us determine the number of molecules which stream through unit area of the aperture in unit time from one side only (e.g., side 1, Fig. 1).

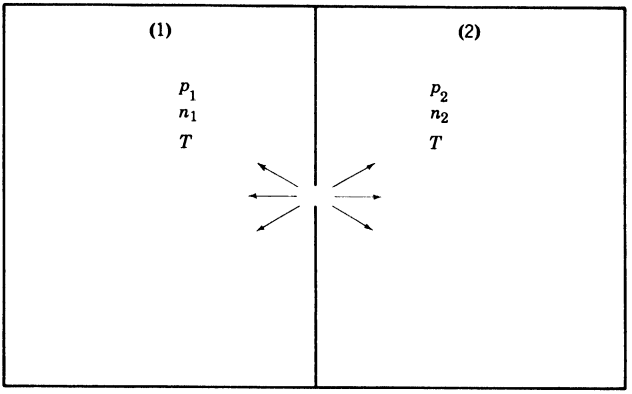

Fig. 1. Effusion of free molecules.

We have seen that the number of molecules per unit volume having velocity components in a particular velocity range is $n_1 f\, d\omega$ (see Section 1.4). The number of molecules of this class passing through unit area of the aperture in unit time is $n_1 u f\, d\omega$, and the total number is obtained by integrating this expression over all possible values of u, v, w possessed by molecules moving toward the opening (see Eqs. 3.4, 2),

$$N_1 = \frac{n_1}{(2\pi)^{3/2}} \left(\frac{\overline{C^2}}{3}\right)^{1/2} \times \int_0^\infty H_1 e^{-\frac{1}{2}H_1^2}\, dH_1 \int_{-\infty}^\infty e^{-\frac{1}{2}H_2^2}\, dH_2 \int_{-\infty}^\infty e^{-\frac{1}{2}H_3^2}\, dH_3 = \frac{1}{4} n_1 \bar{C} \qquad (1)$$

Then, the flow of mass through unit area leaving compartment 1 in unit time is

$$mN_1 = \frac{1}{4} \rho_1 \bar{C} = \frac{p_1}{\sqrt{2\pi RT}} \qquad (2)$$

(see Eqs. 1.10, 7 and 2.4, 2). Similarly, the corresponding flow of mass leaving compartment 2 is

$$mN_2 = \frac{p_2}{\sqrt{2\pi RT}} \qquad (3)$$

and the resultant flow of mass through area A in unit time is

$$Q_m = \frac{A(p_2 - p_1)}{\sqrt{2\pi RT}} \tag{4}$$

This type of free-molecule flow was investigated experimentally by Knudsen in 1908 (Ref. 10). His diaphragms were two pieces of platinum foil, 0.0025 mm and 0.0050 mm thick, in which irregular holes of area 5.21×10^{-6} cm^2 and 66.0×10^{-6} cm^2, respectively, were pierced.

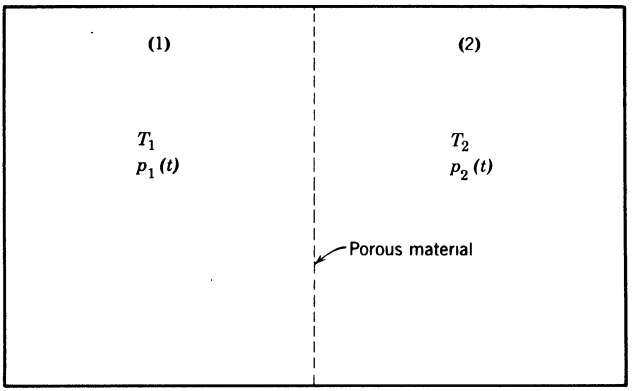

Fig. 2. Thermal transpiration

Knudsen measured the resultant flow of mass in terms of the volume and pressure of the effused gas collected in unit time. Since the mass mn_1 occupies unit volume, the volume flow corresponding to mN_1A is N_1A/n_1. The product of pressure times volume flow for compartment 1 is $\frac{1}{4}(Ap_1\bar{C})$ or $Ap_1\sqrt{RT/2\pi}$, and, therefore, the resultant value of this quantity for both compartments is

$$Q_p = A(p_2 - p_1)\sqrt{\frac{RT}{2\pi}} \tag{5}$$

Knudsen measured the value of $Q_p/(p_2 - p_1)$ at very low pressures for hydrogen, oxygen, and carbon dioxide at temperatures of 22° C and 100° C. When the pressure was sufficiently low to give a mean free path about ten times the mean diameter of the aperture or more, the magnitude of $Q_p/(p_2 - p_1)$ was found to be equal to $A\sqrt{RT/2\pi}$ within the limits of experimental accuracy.

The properties of free-molecule flow can be illustrated further by a second experiment in which initially the pressures in the two compartments are the same, but the temperatures are different (Fig. 2). If the

system is allowed to reach equilibrium after a period of time, then at $t = 0$ the flow of mass through area A in unit time is

$$Q_m(0) = \frac{pA}{\sqrt{2\pi R}}\left(\frac{1}{\sqrt{T_2}} - \frac{1}{\sqrt{T_1}}\right) \tag{6}$$

As the motion proceeds the pressures change to p_1 and p_2, and at any later time t the corresponding flow of mass is

$$Q_m(t) = \frac{A}{\sqrt{2\pi R}}\left(\frac{p_2}{\sqrt{T_2}} - \frac{p_1}{\sqrt{T_1}}\right) \tag{7}$$

The system will reach equilibrium when the flow of mass is zero, that is, when

$$\frac{p_2}{p_1} = \sqrt{\frac{T_2}{T_1}} \tag{8}$$

This effect is called thermal transpiration. It was investigated theoretically by Maxwell and experimentally by Reynolds in 1879. Reynolds observed that the phenomenon occurred only at low pressures. Equation 8 was checked directly by Knudsen (Ref. 10), the diaphragm being replaced by porous material. He found the difference between theory and experiment to be less than the experimental error.

These examples of the effusion of freely moving molecules provide proof of the existence of free-molecule flow and support the assumption that the molecular velocities are distributed according to Maxwell's law.

5.3 TRANSFER OF MASS, MOMENTUM, AND ENERGY BY FREE MOLECULES

We shall now investigate the transport of mass, momentum, and translational energy by free molecules to and from a surface of area A at constant temperature T_w. The surface moves through a macroscopically uniform gas at rest, having infinite extent and a mean free path much larger than the dimensions of the surface. As in molecular effusion it is assumed that the molecular motion of the gas is Maxwellian. Furthermore, the gas is not sensibly affected by the presence of the surface, and no local mass motion develops in the gas. The impinging molecules do not collide with the rebounding molecules sufficiently for an appreciable change in the law of distribution of molecular velocities to occur. We shall postulate further that the molecules reflect diffusely with velocities governed by Maxwell's law corresponding to a temperature (T_r) different, in general, from that of the gas (T_i) or the surface (T_w).

Since the motions of the incident and emergent molecules are assumed to be effectively independent, the transfer of mass, momentum, and energy by the incoming molecules may be considered separately from

the transport of these quantities by the reflected molecules. The velocity components u, v, w arise partly from the motion of the surface relative to the body of the gas ($\bar{u}$, $\bar{v}$, $\bar{w}$) and partly from the random motion of the molecules. For convenience let us choose a system of rectangular coordinates fixed relative to the surface with the y-axis normal to the surface (Fig. 3).

The number of incident molecules per unit volume having velocity components in the range u, $u + du$; v, $v + dv$; w, $w + dw$ is $n_i f\, du\, dv\, dw$ where f is Maxwell's distribution function, and the number of these which strike unit area of the surface in unit time is $n_i v f\, du\, dv\, dw$. Then

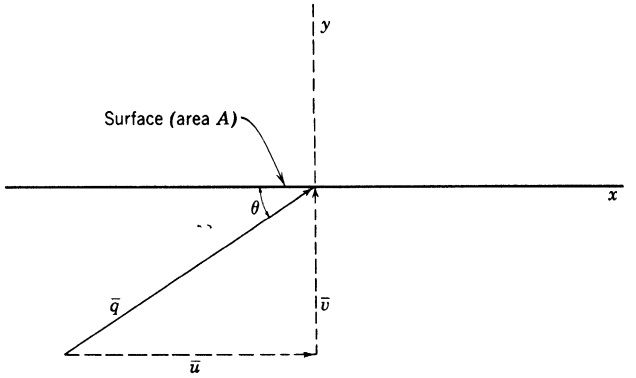

Fig. 3. Flow relative to a flat surface.

the total number of incident molecules which collide with the surface per unit area per unit time is obtained by integrating over all possible values of u, v, w which may be possessed by molecules approaching the surface:

$$N_i = n_i\left(\frac{\beta_i}{\pi}\right)^{3/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_1^2}\,du\int_0^{\infty} ve^{-\frac{1}{2}H_2^2}\,dv\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_3^2}\,dw \tag{1}$$

where $u = \bar{u} + H_1/\sqrt{2\beta_i}, \cdots$. The second integral may be written in the form

$$\frac{1}{\beta_i}\int_{-\bar{v}\sqrt{\beta_i}}^{\infty}(\bar{v}\sqrt{\beta_i} + x)e^{-x^2}\,dx$$
$$= \frac{1}{2\beta_i}\left[e^{-\beta_i\bar{v}^2} + \bar{v}\sqrt{\pi\beta_i}(1 + \operatorname{erf}\bar{v}\sqrt{\beta_i})\right] \tag{2}$$

where x is used here as a variable of integration ($x = H_2/\sqrt{2}$), and

$$\operatorname{erf}\bar{v}\sqrt{\beta_i} = \frac{2}{\sqrt{\pi}}\int_0^{\bar{v}\sqrt{\beta_i}} e^{-x^2}\,dx \tag{3}$$

is the error function. Since each of the two remaining definite integrals have the value $\sqrt{\pi}/\beta_i$, then

$$N_i = n_i\sqrt{\frac{RT_i}{2\pi}}\left[e^{-S_v^2} + S_v\sqrt{\pi}(1 + \operatorname{erf} S_v)\right] \tag{4}$$

where

$$S_v = \bar{v}\sqrt{\beta_i} = \frac{\bar{v}}{C_i} \tag{5}$$

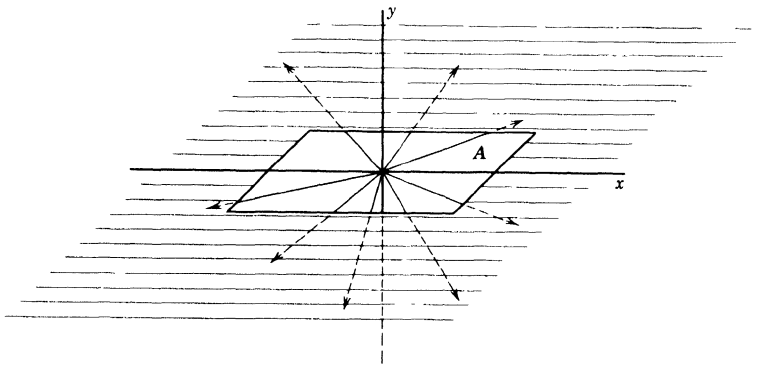

Fig. 4. Reflection of free molecules.

is the ratio of the component of mass velocity normal to the surface to the most probable speed of the incident molecules. Let us define a molecular speed ratio:

$$S = \bar{q}/C_i = \sqrt{\frac{\gamma}{2}}M_i \tag{6}$$

that is, it is a type of Mach number. Then

$$S_r = S\sin\theta \tag{7}$$

The reflected molecules may be regarded as issuing from a fictitious gas on the reverse side of the surface which now acts as a kind of window (Fig. 4). This gas is at rest relative to the surface; its molecules have only a Maxwellian random motion corresponding to a temperature T_r. Then the number of molecules reflected diffusely from the surface is the same as the number of molecules which escape through an equivalent aperture (Eq. 5.2, 1):

$$N_r = n_r\sqrt{\frac{RT_r}{2\pi}} \tag{8}$$

If the number of molecules is conserved during the process of reflection, then $N_i = N_r$ and

$$n_r = n_i\sqrt{\frac{T_i}{T_r}}\,[e^{-S_v^2} + S_r\sqrt{\pi}\,(1 + \operatorname{erf} S_r)] \tag{9}$$

The pressure on the surface produced by the impinging molecules of a particular velocity group is equal to the sum of their momenta normal to the surface just before collision. The number of molecules of a given class striking unit area in unit time is $n_i v f\,du\,dv\,dw$, and the normal momentum carried by each molecule is mv. Then the total pressure due to all the incident molecules is

$$p_i = \rho_i\left(\frac{\beta_i}{\pi}\right)^{3/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_1^2}\,du\int_0^{\infty} v^2 e^{-\frac{1}{2}H_2^2}\,dv\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_3^2}\,dw \tag{10}$$

(see Eqs. 3.4, 2) or

$$C_{p_i} = \sin^2\theta\left[\frac{1}{\sqrt{\pi}S_r}\,e^{-S_i^2} + \left(1 + \frac{1}{2S_r^2}\right)(1 + \operatorname{erf} S_r)\right] \tag{11}$$

The reflected molecules exert a pressure on the surface due to their normal momenta which is assumed to be the same as that possessed by molecules escaping from a fictitious Maxwellian gas at rest on the reverse side of the surface. Thus,

$$p_r = \frac{\rho_r}{2(2\pi)^{3/2}\beta_r}\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_1^2}\,dH_1\int_{-\infty}^{0} H_2^2 e^{-\frac{1}{2}H_2^2}\,dH_2\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_3^2}\,dH_3 = \tfrac{1}{2}\rho_r RT_r \tag{12}$$

or, substituting for n_r from Eq. 9,

$$C_{p_r} = \frac{\sin^2\theta}{2S_r^2}\sqrt{\frac{T_r}{T_i}}\,[e^{-S_r^2} + \sqrt{\pi}\,S_r(1 + \operatorname{erf} S_r)] \tag{13}$$

Therefore the coefficient of total pressure acting on the surface is

$$C_p = \sin^2\theta\left[\frac{1}{S_r}\left(\frac{1}{\sqrt{\pi}} + \frac{1}{2S_r}\sqrt{\frac{T_r}{T_i}}\right)e^{-S_r^2} + \left(1 + \frac{1}{2S_r^2} + \frac{\sqrt{\pi}}{2S_r}\sqrt{\frac{T_r}{T_i}}\right)(1 + \operatorname{erf} S_r)\right] \tag{14}$$

When the surface is at rest relative to the body of the surrounding gas, then $\bar{q} = 0$ and

$$p = p_i + p_r = \rho_i RT_i \tag{15}$$

if $T_r = T_i$. In general, the normal pressure coefficient depends on the molecular speed ratio, the temperature ratio T_r/T_i, and the inclination

of the surface relative to the direction of $\bar{q}$. It is independent of the Knudsen number.

Since the incident molecules are temporarily trapped by the surface (diffuse reflection), they lose all their tangential momentum on contact, and the shearing stress is

$$\tau_i = \rho_i\left(\frac{\beta_i}{\pi}\right)^{3/2}\int_{-\infty}^{\infty} ue^{-\beta H_1^2}\,du\int_0^{\infty} ve^{-\beta H_2^2}\,dv\int_{-\infty}^{\infty} e^{-\beta H_3^2}\,dw \tag{16}$$

which reduces to

$$C_f = \sin\theta\cos\theta\left[\frac{1}{\sqrt{\pi}S_r}e^{-S_r^2} + 1 + \operatorname{erf} S_r\right] \tag{17}$$

Since the reflected molecules have no preferred direction of emergence, they make no resultant contribution to the shearing stress,

$$\tau_r = \frac{\rho_r}{2(2\pi)^{3/2}\beta_r}\int_{-\infty}^{\infty} H_1e^{-\frac{1}{2}H_1^2}\,dH_1\int_{-\infty}^{0} H_2e^{-\frac{1}{2}H_2^2}\,dH_2\int_{-\infty}^{\infty} e^{-\frac{1}{2}H_3^2}\,dH_3 = 0 \tag{18}$$

when $\theta = 0$, the skin friction has the value $1/S\sqrt{\pi}$.

The molecules also convey translational energy to and from the surface. Each of the impinging molecules of a given class ($n_i vf\,du\,dv\,dw$ per unit area per unit time) carries an energy $\frac{1}{2}mc^2$ to the surface. Therefore, the total incident translational energy transported by the molecules of a gas to unit area of the surface in unit time is

$$\begin{aligned} E_i &= \frac{1}{2}\rho_i\left(\frac{\beta_i}{\pi}\right)^{3/2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} vc^2e^{-\beta H_jH_j}\,du\,dv\,dw \\ &= \frac{1}{2}mN_i\left[\bar{q}^2 + RT_i\left\{4 + \frac{1}{(\varphi+1)}\right\}\right] \end{aligned} \tag{19}$$

where

$$\varphi(S,\theta) = \frac{e^{-S_i^2}}{\sqrt{\pi}S_i(1+\operatorname{erf} S_i)} \tag{20}$$

and N_i is given by Eq. 4.

The emergent molecules transport the following energy away from unit area of the surface in unit time,

$$\begin{aligned} E_r &= \frac{1}{4}\frac{\rho_r}{(4\pi\beta_r)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} H_2(H_kH_k)e^{-\beta H_jH_j}\,dH_1\,dH_2\,dH_3 \\ &= \sqrt{\frac{2}{\pi}}\rho_r(RT_r)^{3/2} \end{aligned} \tag{21}$$

Substituting from Eq. 9 for n_r, we have

$$E_r = 2mN_iRT_r = \sqrt{\frac{2}{\pi}}\rho_iRT_r\sqrt{RT_i}[e^{-S_i^2} + \sqrt{\pi}S_r(1+\operatorname{erf} S_r)] \tag{22}$$

If the temperature of the fictitious gas is the same as the surface temperature ($T_r = T_w$), then the translational energy carried by the emergent stream of molecules is

$$E_w = 2mN_i RT_w = \sqrt{\frac{2}{\pi}}\,\rho_i RT_w \sqrt{RT_i}[e^{-S_v^2} + \sqrt{\pi}\,S_v(1 + \operatorname{erf} S_v)] \quad (23)$$

The resultant transfer of molecular translational energy is $E = E_i - E_r$.

5.4 MOMENTUM AND ENERGY EXCHANGE AT A SURFACE

Experimental evidence indicates that more general types of molecular reflection from solid bodies may occur in which only a fraction (g) of the incident tangential momentum is transmitted to the wall, and the degree of contact of the impinging molecules with the surface is not sufficient to give them a mean energy consistent with the wall temperature (T_w) when they emerge. Thus, in general, the ratio

$$\alpha = \frac{E_i - E_r}{E_i - E_w} \quad (1)$$

is different from 1. Maxwell (Ref. 13) regarded g as the fraction of the incident molecules which emerges diffusely from the surface, the remainder being reflected specularly. The ratio α was introduced by Knudsen (Ref. 10) and designated the coefficient of accommodation.

The value of g has been investigated for a number of gas-surface combinations. This quantity can be inserted into Eq. 4.4, 19 and a new expression for the slip velocity ($\bar{u}_s$) obtained. The application of this modified boundary condition to simple flow configurations provides a method for determining g in terms of measurable quantities. In an investigation of the viscosity of hydrogen, helium, and oxygen on burnished silver oxide, Blankenstein (Ref. 20) found g to be 0.99. Millikan (Ref. 21) obtained about the same value of g for air on machined brass. On the other hand, Van Dyke (Ref. 22) and Millikan found that g was somewhat less than 1 for some gas-surface combinations. A survey of the literature by more recent investigators (Refs. 8, 19) led to the general conclusion that, where the aerodynamics of high altitude flight is concerned, we may take $g = 1$.

During experiments on the flow of gases through tubes at low pressure, Kundt and Warburg (Ref. 12) found that the temperature of the gas at the surface was not the same as the wall temperature. Smoluchowski (Ref. 23) investigated this temperature discontinuity and came to the conclusion that, in general, $T_r \neq T_w$ and that this effect was due to insufficient contact between the impinging molecules and the wall to give the emergent molecules a mean energy corresponding to the wall

temperature. The same effect was observed later by Knudsen (Ref. 10) and Berry and Soddy (Ref. 24).

Knudsen developed a simple theory for the energy exchange by gas molecules between two plates of areas A_1 and A_2 placed parallel to and opposite each other. The gas has no mass flow, and its mean free path is much larger than the distance separating the two plates. According to the principles of free-molecule motion, the translational energy carried to unit area of plate 1 by the incident molecules is

$$(E_i)_1 = 2mN_iR(T_i)_1 \tag{2}$$

The energy transported away from unit area of plate 1 is

$$(E_r)_1 = 2mN_iR(T_r)_1 \tag{3}$$

Therefore, the total translational energy transferred to unit area of plate 1 is

$$E_1 = 2mN_iR[(T_i)_1 - (T_r)_1] \tag{4}$$

Similarly for plate 2 the corresponding resultant transfer of energy is

$$E_2 = 2mN_iR[(T_i)_2 - (T_r)_2] \tag{5}$$

Since, for a static gas,

$$N_i = n_i\sqrt{\frac{RT_i}{2\pi}} \tag{6}$$

the total exchange of translational energy per unit area between the two plates is $E_2 = E_1$, or

$$E = p_i\sqrt{\frac{2R}{\pi T_i}}[(T_r)_1 - (T_r)_2] \tag{7}$$

If we introduce the accommodation coefficient for a static gas,

$$\alpha = \frac{T_i - T_r}{T_i - T_w} \tag{8}$$

and assume that the plate surfaces are identical, then

$$E = \alpha p_i\sqrt{\frac{2R}{\pi T_i}}[(T_w)_1 - (T_w)_2] \tag{9}$$

Owing to the presence of residual gases and radiation, the absolute value of E is difficult to measure. Knudsen used a difference method based on measurements at two different pressures, the temperatures of the plates and the gas being kept constant. Then

$$\delta E = \alpha\sqrt{\frac{2R}{\pi T_i}}\,\delta p_i\,\delta T_w \tag{10}$$

In his determination of α, Knudsen used coaxial cylinders instead of parallel plates.

The accommodation coefficient has been measured by a number of investigators (Ref. 1.4) and found to range from 0.1 to 1.0 in static gases. No extensive measurements of α have been made in gases having a large mass flow ($\bar{q}$) compared with the mean molecular speed ($\bar{C}$). However, in tests at high speed reported in Ref. 25 the value of α was about 0.9.

The accommodation coefficient increases with molecular weight and wall temperature and depends on the material, finish, age, and history of the surface. It is probable that, when the gas possesses a large mass motion, α will also be found to depend on the molecular speed ratio and the incidence of the surface relative to the direction of the mass flow. In general, the experimental information regarding α is still incomplete, and only rough values are available for specific applications.

The value of α for metallic surfaces in air is important in high-altitude flight. Wiedmann and Trumpler (Ref. 26) have found that α varies from 0.87 to 0.97 for air on bronze, cast iron, and aluminium with various kinds of surface finish. For the manufactured surfaces of aircraft in service existing information indicates that α lies between 0.9 and 1.0.

When α and g can be compared under the same conditions, it is found that g is substantially larger than α. Thus, although a few collisions of each incident molecule while trapped within the interstices of the wall are sufficient to produce completely random emission, a much larger number of such encounters is necessary to adjust the energy of the incident molecules so that they emerge at a temperature consistent with the thermal condition of the wall. In subsequent calculations we shall assume that $g = 1$ and that the accommodation coefficient has the range $0.9 \leq \alpha \leq 1.0$.

5.5 DRAG AND HEAT TRANSFER TESTS IN FREE-MOLECULE FLOW

Experimental studies of free-molecule motion have been made at low mass velocities. Epstein and Millikan have considered the flow around a sphere. A comparison of the drag coefficient calculated by Epstein (Ref. 14), assuming diffuse reflection, with measurements made by Millikan (Ref. 15) shows reasonably good agreement. The difference was considered to be due to the presence of a small amount of specular reflection in the experiments.

Very little experimental work has been done on free-molecule flow at high values of the speed ratio (S). However, some systematic investigations of the drag and heat transfer of cylinders and spheres with diameters much smaller than the mean free path have been made (Refs. 11, 25, 27).

These tests provide further evidence of the validity of the concepts of free-molecule flow.

Let us calculate the drag and energy equations for a cylinder in a free-molecule flow. The momentum transfer to and from a plane surface of area A was determined in Section 5.3, and, in order to find the drag of a cylinder moving through the gas in a direction normal to its length, A is now replaced by dA or $rl\,d\theta$. We note also that when dA is located on the rear face of the cylinder, the transfer equations will be different from those already obtained for the front surface (Section 5.3), since the range of v in the relevant integrals is now $-\infty \leq v \leq 0$.

The rate of change of momentum per unit area in the direction of the mass motion ($\bar{q}$) due to molecules impinging on an element dA of the front face ($0 \leq v \leq \infty$) is

$$(F_i)_f = p_i \sin\theta + \tau_i \cos\theta$$

$$= \frac{1}{2}\rho_i \bar{q}^2 \sin\theta \left[\frac{1}{S_r\sqrt{\pi}} e^{-S_r^2} + \left(1 + \frac{\sin^2\theta}{2S_r^2}\right)(1 + \operatorname{erf} S_r)\right] \quad (1)$$

The contribution of the molecules impinging on the front face of the cylinder to the total drag coefficient is therefore

$$(C_{D_i})_f = \int_0^{\pi/2} \left[\frac{1}{S_r\sqrt{\pi}} e^{-S_r^2} + \left(1 + \frac{\sin^2\theta}{2S_r^2}\right)(1 + \operatorname{erf} S_r)\right] \sin\theta\, d\theta \quad (2)$$

Similarly, when dA is on the rear face ($-\infty \leq v \leq 0$), the corresponding relations are

$$(F_i)_b = -\frac{1}{2}\rho_i \bar{q}^2 \sin\theta \left[\frac{1}{S_r\sqrt{\pi}} e^{-S_r^2} - \left(1 + \frac{\sin^2\theta}{2S_r^2}\right)(1 - \operatorname{erf} S_r)\right] \quad (3)$$

and

$$(C_{D_i})_b = -\int_0^{\pi/2} \left[\frac{1}{S_r\sqrt{\pi}} e^{-S_r^2} - \left(1 + \frac{\sin^2\theta}{2S_r^2}\right)(1 - \operatorname{erf} S_r)\right] \sin\theta\, d\theta \quad (4)$$

The drag coefficient due to all incident molecules is

$$(C_{D_i})_f + (C_{D_i})_b = 2\int_0^{\pi/2} \left[\frac{1}{S\sqrt{\pi}} e^{-S_r^2} + \sin\theta \left(1 + \frac{1}{2S^2}\right) \operatorname{erf} S_r\right] d\theta \quad (5)$$

$$= \frac{\sqrt{\pi} e^{-\frac{1}{2}S^2}}{S}\left[I_0 + \left(S^2 + \frac{1}{2}\right)(I_0 + I_1)\right] \quad (6)$$

where

$$I_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{\frac{1}{2}S^2 x}}{\sqrt{1 - x^2}}\, dx \quad (7)$$

and

$$I_1 = \frac{S^2}{2\pi}\int_{-1}^{1} e^{\frac{1}{2}S^2 x}\sqrt{1 - x^2}\, dx \quad (8)$$

The integrals I_0 and I_1 are modified Bessel functions of order 0 and 1, respectively.

The rate of change of momentum per unit area in the direction of $\bar{q}$ due to molecules emerging from dA on the front face of the cylinder is

$$(F_r)_f = p_r \sin\theta = \frac{1}{2}\rho_i \bar{q}^2 \frac{\sin^3\theta}{2S_r^2}\sqrt{\frac{\overline{T_r}}{T_i}}\left[e^{-S_r^2} + \sqrt{\pi}S_r(1 + \operatorname{erf} S_r)\right] \quad (9)$$

The contribution of these emergent molecules to the drag coefficient is

$$(C_{D_r})_f = \frac{1}{2S_r^2}\sqrt{\frac{\overline{T_r}}{T_i}}\left[e^{-S_r^2} + \sqrt{\pi}S_r(1 + \operatorname{erf} S_r)\right]\int_0^{\pi/2}\sin^3\theta\, d\theta \quad (10)$$

Similarly, for the molecules emerging from the rear face,

$$(C_{D_r})_b = \frac{1}{2S_r^2}\sqrt{\frac{\overline{T_r}}{T_i}}\left[e^{-S_r^2} - \sqrt{\pi}S_r(1 - \operatorname{erf} S_r)\right]\int_0^{\pi/2}\sin^3\theta\, d\theta \quad (11)$$

The drag coefficient for all reflected molecules is

$$(C_{D_r})_f + (C_{D_r})_b = \frac{\sqrt{\pi}}{S}\sqrt{\frac{\overline{T_r}}{T_i}}\int_0^{\pi/2}\sin^2\theta\, d\theta = \frac{\pi^{3/2}}{4S}\sqrt{\frac{\overline{T_r}}{T_i}} \quad (12)$$

Finally, the coefficient of total drag is the sum of Eqs. 5 and 12,

$$C_D = \frac{\sqrt{\pi}}{S}\left\{e^{-\frac{1}{4}S^2}\left[I_0 + \left(S^2 + \frac{1}{2}\right)(I_0 + I_1)\right] + \frac{\pi}{4}\sqrt{\frac{\overline{T_r}}{T_i}}\right\} \quad (13)$$

Therefore, the drag coefficient of a cylinder in a free-molecule flow is a function of the molecular speed ratio and the temperature ratio T_r/T_i.

The effective temperature of the emergent molecules (T_r) can be found by determining the relation for the balance of energy between the cylinder and its surroundings. We shall consider this balance under the assumption that the circumferential distribution of temperature on the cylinder surface is constant. Furthermore, radiation to and from the cylinder is not affected by variation of the density of the gas, and it may become comparable with the convective heat transfer at low density. Therefore, radiation effects must be included in our calculations. Let us assume that the cylinder emits and receives radiant energy such that the emissivity is independent of wavelength; that is, it is a "gray body."

The translational energy per unit area conveyed by the incident molecules in unit time to the element dA on the front face of the cylinder is given by Eq. 5.3, 19. The corresponding energy transported to dA on the rear face of the cylinder is

$$(E_i)_b = \frac{1}{2}m(N_i)_b\left[\bar{q}^2 + RT_i\left(4 - \frac{1}{\varphi_b - 1}\right)\right] \quad (14)$$

where
$$\varphi_b = \frac{e^{-S_v^2}}{S_v\sqrt{\pi}(1 - \operatorname{erf} S_v)} \tag{15}$$

and
$$(N_i)_b = n_i\sqrt{\frac{RT_i}{2\pi}}\,[e^{-S_v^2} - S_v\sqrt{\pi}(1 - \operatorname{erf} S_v)] \tag{16}$$

These results may be obtained by the methods outlined in Section 5.3, where the limits of integration with respect to v for incident molecules are changed to $-\infty \leq v \leq 0$.

The corresponding emergent energy for front and rear faces can be found from the definition of the accommodation coefficient (Eq. 5.4, 1),

$$(E_r)_f = (1 - \alpha)(E_i)_f + 2\alpha mR(N_i)_f T_w \tag{17}$$

$$(E_r)_b = (1 - \alpha)(E_i)_b + 2\alpha mR(N_i)_b T_w \tag{18}$$

The equation for energy balance states that the total incident energy is equal to the total emergent energy,

$$\int(E_i)_f\,dA + \int(E_i)_b\,dA + \epsilon BT_c^4 = \int(E_r)_f\,dA + \int(E_r)_b\,dA + \epsilon BT_w^4 + J \tag{19}$$

where T_c is the temperature of the walls containing the gas and cylinder, and J is the output of internal energy from unit area of the cylinder in unit time which may be electrical or due to a coolant. If we substitute for E_r, the equation for energy balance becomes

$$\alpha\int_0^{\pi/2}[(E_i)_f + (E_i)_b]\,d\theta - 2\alpha mRT_w\int_0^{\pi/2}[(N_i)_f + (N_i)_b]\,d\theta + 2\pi[\epsilon B(T_c^4 - T_w^4) - J] = 0 \tag{20}$$

Reduction of the integrals is a lengthy but straightforward procedure, the result being

$$2\frac{T_w}{T_i}(Z_1 + Z_2) - \left[(S^2 + 2)Z_1 + \left(S^2 + \frac{5}{2}\right)Z_2\right] + \frac{\sqrt{2}}{\alpha\rho_i}\left(\frac{\pi}{RT_i}\right)^{3/2}[\epsilon B(T_w^4 - T_c^4) - J] = 0 \tag{21}$$

where
$$Z_1 = \pi e^{-\frac{1}{2}S^2} I_0(\tfrac{1}{2}S^2) \tag{22}$$

and
$$Z_2 = \pi S^2 e^{-\frac{1}{2}S^2}[I_0(\tfrac{1}{2}S^2) + I_1(\tfrac{1}{2}S^2)] \tag{23}$$

Equation 21 governs the heat transfer process between the cylinder and its surrounding gas and container. It specifies the cylinder temperature (T_w) in terms of the molecular speed ratio, the gas temperature and density ($\overline{C^2}$, n), and the temperature of the surrounding walls.

The transfer of energy under adiabatic conditions is of particular interest. If neither an exchange of radiant energy nor an output of internal energy from the cylinder occurs, then the energy balance involves only the transport of translational energy by the molecules to and from the cylinder. The reflected molecules now emerge from the fictitious gas with a temperature equal to the surface temperature ($T_r = T_w$), and the energy balance relation states that the resultant transfer of translational energy is zero. Then, for an insulated cylinder,

$$\frac{T_r}{T_i} = \frac{T_w}{T_i} = \frac{(S^2 + 2)Z_1 + (S^2 + \frac{5}{2})Z_2}{2(Z_1 + Z_2)} \tag{24}$$

and the drag coefficient becomes

$$C_D = \frac{\sqrt{\pi}}{S}\left\{e^{-\frac{1}{4}S^2}\left[I_0 + \left(S^2 + \frac{1}{2}\right)(I_0 + I_1)\right] + \frac{\pi}{4}\left[\frac{(S^2 + 2)Z_1 + (S^2 + \frac{5}{2})Z_2}{2(Z_1 + Z_2)}\right]^{1/2}\right\} \tag{25}$$

It will be seen that the drag coefficient of an insulated cylinder depends only on the molecular speed ratio.

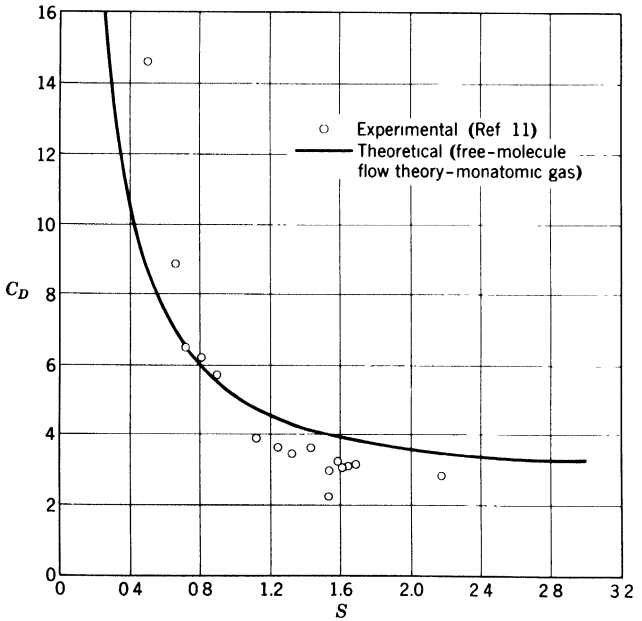

Fig. 5. Drag of an insulated cylinder in helium.

Tests have been made by Stalder, Goodwin, and Creager (Ref. 11) on a cylinder mounted transversely to the direction of mass motion of the gas in an open-jet, nonreturn wind tunnel at very low density. Test gases (He, N_2) were introduced into a settling chamber through a throttle valve and then passed through an axially symmetrical nozzle at high speed.

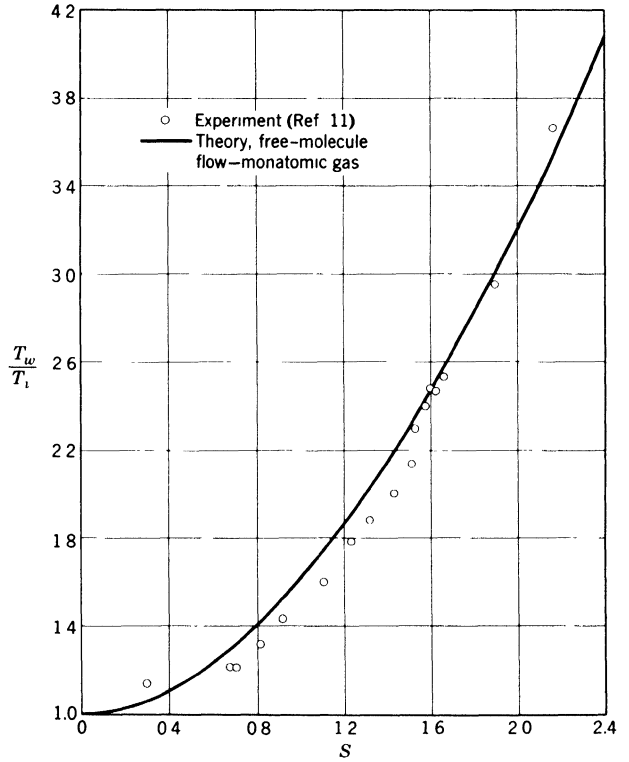

Fig. 6. Heat transfer tests on a cylinder in helium.

A cylindrical model was selected for test because it could be accurately constructed on a small scale of materials arranged to form a thermocouple for the measurement of temperature. The model was a right circular cylinder 0.0031 in. in diameter and $\frac{15}{32}$ in. in length, made of iron and constantin wires to form three thermocouple junctions along its exposed length. It was mounted on a microbalance in the test region of the wind tunnel at the nozzle outlet and covered with fine-grain soot to ensure an emissivity of known value (0.95).

A comparison between values of C_D measured in helium gas and a curve of C_D versus S based on Eq. 25 (Fig. 5) shows good agreement when allowance is made for difficulties in measuring such small forces, end effects ($S < 1$), and the possibility of a small amount of specular reflection ($S > 1$). The results show no dependence on the Knudsen number since, over the range $0.5 \leq S \leq 2.4$, Kn varied between 185 and 15. These results indicate that free-molecule flow existed over the whole range of S.

The results of heat transfer tests in helium are given in Fig. 6, which shows a comparison of measured values of T_w/T_i with those calculated from Eq. 24. The agreement is satisfactory when account is taken of heat losses due to conduction through the end supports and the presence of some radiation. The existence of such heat losses is indicated by the fact that the measured cylinder temperatures are lower than the corresponding theoretical values. Temperature gradients along the wire were detected, thus confirming that a small amount of heat was lost through the end supports.

Further work by Stalder, Goodwin, and Creager on six cylinders of different construction showed that fully developed free-molecule flow exists at $Kn \geq 2$ (Ref. 25). Under these conditions heat transfer data can be well correlated with theory if the accommodation coefficient is taken to be 0.9.

5.6 EFFECT OF THE KNUDSEN NUMBER ON HEAT TRANSFER

A comparison of the heat transfer properties of a gas at large and very small values of the Knudsen number can be obtained from the results given in Sections 5.3 and 4.7. The energy balance equation for the upstream side of an insulated flat plate in free-molecule flow has the form ($E_i - E_r = E_w$)

$$\frac{1}{2}\bar{q}^2 + 2RT_i\left[1 + \frac{1}{4(\varphi + 1)}\right] = 2RT_w \tag{1}$$

or

$$\frac{T_w}{T_i} = 1 + \frac{1}{4(\varphi + 1)} + \frac{1}{2}S^2 \tag{2}$$

When the plate is aligned in the direction of mass flow ($\bar{q} = \bar{u}$, $\theta = 0$, $\varphi \to \infty$), then

$$\frac{T_w}{T_i} = 1 + \frac{1}{2}S^2 \qquad \left(S = \sqrt{\frac{\gamma}{2}}\,M_i\right) \tag{3}$$

It has been shown in Section 4.5 that when the Knudsen number is very small, a laminar boundary layer exists, and according to Section 4.7

the equilibrium temperature for a monatomic gas is

$$\frac{T_w}{T_i} = 1 + \frac{\gamma - 1}{2} r(0) M_i^2 = 1 + 0.326 S^2 \tag{4}$$

For a monatomic gas, the stagnation temperature is given by (see Eq. 2.8, 8)

$$\frac{T_w}{T_i} = 1 + \frac{2}{5} S^2 \tag{5}$$

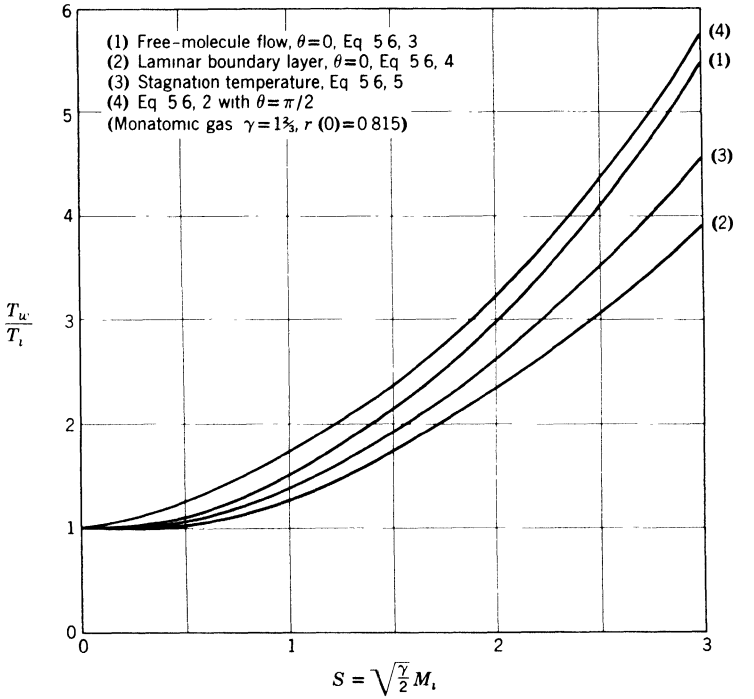

Fig. 7. Surface temperature of an insulated plate at large and small values of the Knudsen number.

Curves for Eqs. 3, 4, and 5 are shown in Fig. 7. It is evident that a higher adiabatic wall temperature will be attained when the Knudsen number is large (free-molecule flow) than when *Kn* is very small (flow with intermolecular collisions). Furthermore the equilibrium temperature of a flat plate in free-molecule flow is higher than the stagnation temperature of the undisturbed stream. The fundamental difference between the two forms of energy exchange near a surface is that the

incident and emergent streams of molecules in the flow of a highly rarefied gas do not interact sufficiently to produce an appreciable departure of the distribution of molecular velocities from Maxwell's law, whereas in the flow of a denser gas the interaction occurs to such an extent that the molecules have velocities distributed according to a non-Maxwellian law.

The equilibrium temperature of a surface in a free-molecule flow depends on the angle of incidence (Eq. 2). When the component of mass velocity normal to the surface is not zero ($\theta > 0$), more translational energy is carried by the incident stream of molecules perpendicular to the plate and T_w/T_i is still larger. This ratio is plotted against S in Fig. 7 for $\theta = \pi/2$. In the flow of a dense gas, however, the component of mass velocity normal to the plate is always zero on the surface.

5.7 MOMENTUM TRANSFER WITH SLIP FLOW

The mechanics of highly rarefied gases is discussed first in this chapter because the molecular motion is Maxwellian and the mathematical method is greatly simplified by the assumption that the transfer properties of the gas depend only on direct encounters between gas molecules and a surface. We shall now investigate the non-Maxwellian motion of a rarefied gas. It will be found that the present theory in this regime is much less satisfactory than in the free-molecule flow region.

It has been shown that when the motion is slightly nonisentropic, a more general velocity distribution function leads to both the equations for viscous, compressible flow (3.9, 9, 10, 11) and the general boundary conditions (4.4, 30–33). The theory of the laminar boundary layer indicates that Δ and D have the orders ($x = l$)

$$\Delta \approx \left(\frac{Re_1}{\sigma X}\right)^{1/2}, \qquad D \approx \left(\frac{1}{\sigma Re_1 X}\right)^{1/2}$$

where, for a monatomic gas, $Re_1 = 1.648 M_1$. At ordinary densities, X is a very large number, and the boundary conditions reduce to Eqs. 4.5, 9.

The orders of magnitude of both Δ and D become significant near the leading edge of the plate where the distance x may have the same order of magnitude as the mean free path. At a high Mach number Δ is more important than D. When the density is reduced so that the mean free path is comparable with the distance from the leading edge over most of the plate, then both Δ and D must be taken into account.

It will be noted that the orders of magnitude of the terms in the general boundary conditions are 1, Δ, $D\Delta$, and $D^2\Delta$, respectively. When terms

of order Δ (or D) are significant but those of order $D\Delta$ and $D^2\Delta$ are negligible, then, according to Eq. 4.5, 7,

$$u_s' = \alpha_3 \Delta L_s'(u_y')_s \tag{1}$$

or in the dimensionless form used in boundary layer theory ($\alpha_3 = 5\pi/16 \approx 1$),

$$U_s = \left(\mathscr{L}\frac{\partial U}{\partial Y}\right)_s \tag{2}$$

Thus the mass velocity of the gas at the surface of the plate is small but finite, and a "slip flow" is said to occur.

If terms of the same order of magnitude are retained in the boundary layer relations, Eqs. 4.5, 22–26 remain unaltered but the second momentum equation (4.5, 12) can no longer be omitted since it contains some terms of order Δ and D.

The momentum transfer in the boundary layer when slip flow occurs has been the subject of only preliminary investigations, and nothing more than an estimate of the probable effect of slip can be presented. Direct modification of the existing boundary layer theory has been attempted by Nonweiler (Ref. 19), and Schaaf and Mirels (Refs. 16, 18) have produced approximate results by extending Rayleigh's non-stationary problem (Ref. 28). The latter method is an interesting, indirect approach to the boundary layer problem and will be discussed first. We shall then consider an alternative approximate treatment based on Rayleigh's equation in stationary flow (Refs. 29, 30).

In Rayleigh's problem an infinite flat plate is instantaneously accelerated from rest to a constant speed in a direction parallel to its plane. The solution gives the change with time in the distribution of the mass velocity adjacent to the surface as the influence of the plate extends further into the flow. For convenience we consider here the equivalent problem in which a viscous, compressible gas is suddenly set in motion with a constant mass velocity $\bar{u}_1$ along the plate which remains fixed relative to the system of coordinates.

Since the plate is infinite, all x-derivatives are zero, and the transfer equations become

Mass:
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y}(\rho \bar{v}) = 0 \tag{3}$$

Momentum:
$$\rho \frac{d\bar{u}}{dt} = \frac{\partial}{\partial y}(\mu \bar{u}_y), \qquad \rho \frac{d\bar{v}}{dt} = \frac{4}{3}\frac{\partial}{\partial y}(\mu \bar{v}_y) \tag{4}$$

Energy:
$$\rho c_v \frac{dT}{dt} = -p\bar{v}_y + \frac{4}{3}\mu \bar{v}_y^2 + \mu \bar{u}_y^2 + \frac{\partial}{\partial y}(\lambda T_y) \tag{5}$$

where the pressure is assumed to be constant throughout the flow.

Equations 3 and 4 have the dimensionless forms

$$\frac{\partial \Gamma}{\partial \chi} + \frac{\partial}{\partial Y}(\Gamma V) = 0 \tag{6}$$

$$\Gamma\left(\frac{\partial U}{\partial \chi} + V\frac{\partial U}{\partial Y}\right) = \frac{1}{Re_1}\frac{\partial}{\partial Y}\left(K\frac{\partial U}{\partial Y}\right) \tag{7}$$

$$\Gamma\left(\frac{\partial V}{\partial \chi} + V\frac{\partial V}{\partial Y}\right) = \frac{4}{3}\frac{1}{Re_1}\frac{\partial}{\partial Y}\left(K\frac{\partial V}{\partial Y}\right) \tag{8}$$

Let us introduce a function $\psi(\chi, Y)$ analogous to the stream function in steady, two-dimensional flow (Section 4.6),

$$\Gamma = \frac{\partial \psi}{\partial Y}, \qquad \Gamma V = -\frac{\partial \psi}{\partial \chi} \tag{9}$$

Then Eq. 6 is satisfied identically. According to the von Mises transformation (Ref. 31), Eq. 7 can be placed in the simple form,

$$\frac{\partial U}{\partial \chi} = \frac{\sigma}{Re_1}\frac{\partial^2 U}{\partial \psi^2} \tag{10}$$

by means of the operational relations

$$\left(\frac{\partial}{\partial \chi}\right)_Y = \frac{\partial}{\partial \chi} - \Gamma V\frac{\partial}{\partial \psi}, \qquad \left(\frac{\partial}{\partial Y}\right)_\chi = \Gamma\frac{\partial}{\partial \psi} \tag{11}$$

and the assumption that $K\Gamma = \sigma$ (constant). Equation 10 provides the function $U(\chi, \psi)$ independently of the energy-transfer equation.

We investigate first the solution of Eq. 10 when no slip occurs ($u_s = 0$). The boundary conditions are

$$\begin{aligned} U &= 1 \quad \text{for } \chi = 0,\ \psi > 0 \\ U &= 0 \quad \text{for } \chi \geq 0,\ \psi = 0 \\ U &\to 1 \quad \text{for } \chi \geq 0,\ \psi \to \infty \end{aligned} \tag{12}$$

These boundary conditions indicate that initially ($t = 0$) the flow above the plate is uniform everywhere ($U = 1$, $\chi = 0$, $\psi > 0$), and the boundary layer thickness is zero. At time $t \geq 0$, no slip occurs at the surface of the plate and the velocity tends to the undisturbed value at large distances above the plate ($U \to 1$, $\chi \geq 0$, $\psi \to \infty$). Thus a thin layer of viscous, compressible flow develops near the surface having a thickness which increases with time.

Equations 10 and 12 are equivalent to those which occur in the theory of heat conduction in solids (Ref. 32), and a solution can be obtained

using the Laplace transformation. The subsidiary equation corresponding to Eq. 10 is

$$\frac{d^2U^*}{d\psi^2} - \frac{Re_1}{\sigma} U^* = -\frac{Re_1}{\sigma} [U^*(\psi)]_{\chi=0} \tag{13}$$

and the subsidiary boundary conditions are

$$\begin{aligned} U^* &= 0 \quad \text{at } \psi = 0 \\ U^* &\to \frac{1}{\nu} \quad \text{at } \psi \to \infty \end{aligned} \tag{14}$$

Equation 13 is an ordinary differential equation of second order with constant coefficients and inhomogeneous form, having the general solution

$$U^* = \frac{1}{\nu} + \kappa_1 e^{-\psi\sqrt{Re_1\nu/\sigma}} + \kappa_2 e^{\psi\sqrt{Re_1\nu/\sigma}} \tag{15}$$

When the boundary conditions are applied, this reduces to

$$U^* = \frac{1}{\nu}(1 - e^{-\psi\sqrt{Re_1\nu/\sigma}}) \tag{16}$$

According to Appendix III of Ref. 33 (Nos. 1, 83) this expression is the Laplace transform of

$$U(\chi, \psi) = \operatorname{erf}\left[\frac{\psi}{2}\left(\frac{Re_1}{\sigma\chi}\right)^{1/2}\right] \tag{17}$$

This result can be calculated by application of the inversion theorem for the Laplace transformation (Ref. 32, pp. 71, 93, 111).

According to Eqs. 11 and 4.6, 33, the shearing stress is

$$\tau = \frac{2K}{Re_1}\frac{\partial U}{\partial Y} = \frac{2\sigma}{Re_1}\frac{\partial U}{\partial \psi} \tag{18}$$

Now

$$\frac{dU^*}{d\psi} = \left(\frac{Re_1}{\sigma\nu}\right)^{1/2} e^{-\psi\sqrt{Re_1\nu/\sigma}} \tag{19}$$

which is the Laplace transform of

$$\frac{\partial U}{\partial \psi} = \left(\frac{Re_1}{\pi\sigma\chi}\right)^{1/2} \exp\left(\frac{Re_1\psi^2}{4\sigma\chi}\right) \tag{20}$$

(Appendix III of Ref. 33, No. 84). Therefore

$$\tau = 2\left(\frac{\sigma}{\pi Re_1\chi}\right)^{1/2} \exp\left(\frac{Re_1\psi^2}{4\sigma\chi}\right) \tag{21}$$

The shearing stress on the surface of the plate is

$$C_f = \tau_w = 2\left(\frac{\sigma}{\pi Re_1\chi}\right)^{1/2} \tag{22}$$

When slip flow occurs, the boundary conditions associated with Eq. 10 are

$$\begin{aligned} U &= 1 && \text{for} \quad \chi = 0, \quad \psi > 0 \\ U &= U_s = \left(\frac{\partial U}{\partial \psi}\right)_s && \text{for} \quad \chi > 0, \quad \psi = 0 \\ U &\to 1 && \text{for} \quad \chi \geq 0, \quad \psi \to \infty \end{aligned} \tag{23}$$

Thus the motion begins the same way as the flow with no slip except that, at $t = 0$, the mass velocity at the surface remains finite but small.

According to the method of the Laplace transformation, the subsidiary Eq. 13 is unchanged, but it must now be solved, subject to the boundary condition

$$\frac{dU^*}{d\psi} - U^* = 0, \qquad \psi = 0 \tag{24}$$

If we substitute from Eq. 15 (with $\kappa_2 = 0$) and place $\psi = 0$, then

$$\kappa_1 = -\frac{1}{\nu[(Re_1\nu/\sigma)^{1/2} + 1]} \tag{25}$$

and the Laplace transform of $U(\chi, \psi)$ is

$$U^* = \frac{1}{\nu}\left[1 - \frac{e^{-\psi\sqrt{Re_1\nu/\sigma}}}{(Re_1\nu/\sigma)^{1/2} + 1}\right] \tag{26}$$

Applying the inversion theorem of the Laplace transformation (Ref. 32, pp. 114, 115) or referring to a table of Laplace transforms (Appendix III of Ref. 33, No. 86),

$$\begin{aligned} U(\chi, \psi) = {} & \operatorname{erf}\left[\frac{\psi}{2}\left(\frac{Re_1}{\sigma\chi}\right)^{1/2}\right] + \exp\left(\psi + \frac{\sigma\chi}{Re_1}\right) \\ & \times \operatorname{erfc}\left[\left(\frac{\sigma\chi}{Re_1}\right)^{1/2} + \frac{\psi}{2}\left(\frac{Re_1}{\sigma\chi}\right)^{1/2}\right] \end{aligned} \tag{27}$$

The slip velocity at the wall is

$$U_s = \exp\left(\frac{\sigma\chi}{Re_1}\right) \operatorname{erfc}\left(\frac{\sigma\chi}{Re_1}\right)^{1/2} \tag{28}$$

To obtain the shearing stress when the flow slips at the wall, we first calculate the derivative in Eq. 18, that is,

$$\frac{dU^*}{d\psi} = \frac{e^{-\psi\sqrt{Re_1\nu/\sigma}}}{\nu[1 + (\sigma/Re_1\nu)^{1/2}]} \tag{29}$$

which is the Laplace transform of

$$\frac{\partial U}{\partial \psi} = \exp\left(\psi + \frac{\sigma\chi}{Re_1}\right) \operatorname{erfc}\left[\left(\frac{\sigma\chi}{Re_1}\right)^{1/2} + \frac{\psi}{2}\left(\frac{Re_1}{\sigma\chi}\right)^{1/2}\right] \tag{30}$$

(Ref. 33, Appendix III, No. 87). Then

$$\tau = \frac{2\sigma}{Re_1} \exp\left(\psi + \frac{\sigma\chi}{Re_1}\right) \operatorname{erfc}\left[\left(\frac{\sigma\chi}{Re_1}\right)^{1/2} + \frac{\psi}{2}\left(\frac{Re_1}{\sigma\chi}\right)^{1/2}\right] \tag{31}$$

On the surface of the plate ($\psi = 0$)

$$C_f = \frac{2\sigma}{Re_1} \exp\left(\frac{\sigma\chi}{Re_1}\right) \operatorname{erfc}\left(\frac{\sigma\chi}{Re_1}\right)^{1/2} \tag{32}$$

When no slip occurs, an interesting fact emerges from a comparison of Eqs. 22 and 4.6, 37. For stationary and nonstationary flow in the boundary layer, respectively, we have

$$C_f \sim \left(\frac{\sigma}{Re_1 X}\right)^{1/2} \quad \text{and} \quad C_f \sim \left(\frac{\sigma}{Re_1 \chi}\right)^{1/2} \tag{33}$$

Similarly, from Eqs. 4.7, 24 and 17, the corresponding expressions for the variations of the boundary layer thickness are

$$\delta_u \sim \left(\frac{\sigma X}{Re_1}\right)^{1/2} \quad \text{and} \quad \delta_u \sim \left(\frac{\sigma\chi}{Re_1}\right)^{1/2} \tag{34}$$

Thus the variations of skin friction and boundary layer thickness with time in Rayleigh's problem are similar to the variations of these quantities with position in the steady motion of the boundary layer on a semi-infinite flat plate (Ref. 34, p. 52). We may therefore match the solutions of these two problems by writing

$$\chi = k_1 X \tag{35}$$

where k_1 is determined by equating the relevant expressions. Since the skin friction is of major interest, let us match Eqs. 22 and 4.6, 37. Then $k_1 = 2.888$.

When the slip velocity is small ($U_s \ll 1$), Eq. 35 will provide an estimate of the skin friction on a flat plate in steady slip flow. Thus

$$U_s = \exp\left(\frac{\sigma k_1 X}{Re_1}\right) \operatorname{erfc}\left(\frac{\sigma k_1 X}{Re_1}\right)^{1/2} \tag{36}$$

and

$$C_f = \frac{2\sigma}{Re_1} \exp\left(\frac{\sigma k_1 X}{Re_1}\right) \operatorname{erfc}\left(\frac{\sigma k_1 X}{Re_1}\right)^{1/2} \tag{37}$$

These results can be further reduced since for small slip velocity Δ [or $(Re_1/\sigma X)^{1/2}$] is small but not negligible and hence the parameter $\sigma k_1 X/Re_1$ in Eqs. 36 and 37 is a large quantity. When any parameter ξ is large, the complementary error function has the asymptotic form (Ref. 18)

$$\operatorname{erfc} \xi = \frac{1}{\xi\sqrt{\pi}} e^{-\xi^2}\left(1 - \frac{1}{2\xi^2} + \frac{1 \cdot 3}{(2\xi^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2\xi^2)^3} + \cdots\right) \tag{38}$$

Then, for small slip (large values of X),

$$U_s = \left(\frac{Re_1}{\pi\sigma k_1 X}\right)^{\frac{1}{2}} \tag{39}$$

and

$$C_f = 0.664\left(\frac{\sigma}{Re_1 X}\right)^{1/2}\left(1 - \frac{Re_1}{2\sigma k_1 X} + \cdots\right) \tag{40}$$

Therefore, according to the above estimate, the effect of slip is to reduce the skin friction coefficient, but the amount will be small since no term containing $1/\sqrt{X}$ appears in the final bracket in Eq. 40.

When the gas is monatomic, $Re_1 = 1.648M_1$, and we may write

$$U_s = \exp\left(\frac{7\sigma X}{4M_1}\right) \operatorname{erfc}\left(\frac{7\sigma X}{4M_1}\right)^{1/2} \tag{41}$$

and

$$M_1 C_f = 1.214\,\sigma \exp\left(\frac{7\sigma X}{4M_1}\right) \operatorname{erfc}\left(\frac{7\sigma X}{4M_1}\right)^{1/2} \tag{42}$$

In free-molecule flow $U_s = 1$, and, according to Eq. 5.3, 17 ($\theta = 0$),

$$M_1 C_f = \left(\frac{2}{\pi\gamma}\right)^{1/2} \tag{43}$$

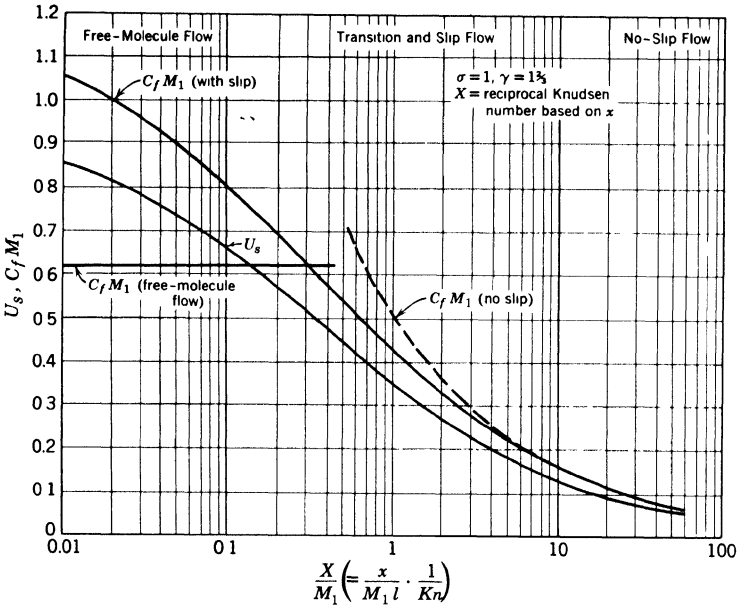

Fig. 8. Velocity ratio and local skin friction in slip flow.

The variations of U_s and $M_1 C_f$ with the basic parameter X/M_1 for $\sigma = 1$ are shown in Fig. 8. It will be seen that Eqs. 41, 42 span the entire range from no-slip flow to free-molecule flow in a closed form, though no accuracy can be claimed for them in the latter regime.

The smallest values of X occur near the leading edge of the plate, and it is to be expected that the influence of slip on the flow will be concentrated in this region. Two analyses of this problem, one by Lin and Schaaf (Ref. 17) and the other by Nonweiler (Ref. 19), indicate that the effect of slip is to reduce the thickness of the boundary layer near the leading edge, but there is no first-order change in the skin friction coefficient.

To obtain a comparison with measurements of the drag of flat plates in the slip flow regime (Ref. 30), we require the over-all drag coefficient due to skin friction. Let us define the drag coefficient (C_D) as follows:

$$C_D = \frac{D}{\frac{1}{2}\rho_1 \tilde{u}_1^2 \cdot bl} \tag{44}$$

where D is the total drag of the plate (both sides), $\frac{1}{2}\rho_1 \tilde{u}_1^2$ is the dynamic pressure, and bl is the area (one side). In dimensionless form the drag coefficient due to skin friction is

$$C_D = \frac{2L_1}{l} \int_0^{l/L_1} C_f \, dX \tag{45}$$

Substituting for C_f from Eq. 37 and integrating by parts results in

$$C_D = \frac{\Lambda}{z^2}\left[e^{z^2} \operatorname{erfc} z - 1 + \frac{2z}{\sqrt{\pi}}\right] \tag{46}$$

where

$$z^2 = \frac{\sigma k_1}{Re_1 Kn} = \frac{25\pi\sigma k_1 Re_l}{128\gamma M_1^2} \tag{47}$$

and

$$\Lambda = \frac{4\sigma}{Re_1} = \frac{5}{2}\left(\frac{\pi}{2\gamma}\right)^{1/2} \frac{\sigma}{M_1} \tag{48}$$

and Re_l is the Reynolds number based on the plate length ($Kn = Re_1/Re_l$). Note that, according to Eq. 3.8, 10,

$$Re_1 = \frac{16}{5}\left(\frac{\gamma}{2\pi}\right)^{1/2} M_1 \tag{49}$$

For a diatomic gas (air), $Re_1 \approx 1.5M_1$ and $Kn \approx 1.5M_1/Re_l$.

In the slip flow regime, z is large (or Kn is small) and Eq. 46 can be reduced to the approximate form

$$C_D = \frac{\Lambda}{z^2\sqrt{\pi}}\left[2z - \sqrt{\pi} + \frac{1}{z}\right] \tag{50}$$

In the limit, as z tends to a large value, only the first term in the bracket is significant, and C_D reduces to its value when no slip occurs (Eq. 4.6, 38, $C_D = 2C_F$).

Measured values of C_D, reported by Schaaf and Sherman in Ref. 30, are shown in Fig. 9. All the experimental points lie somewhat above the curve given by Eqs. 46, 47, 48, the agreement between the observed and calculated values of C_D tending to improve for the larger values of z (no slip).

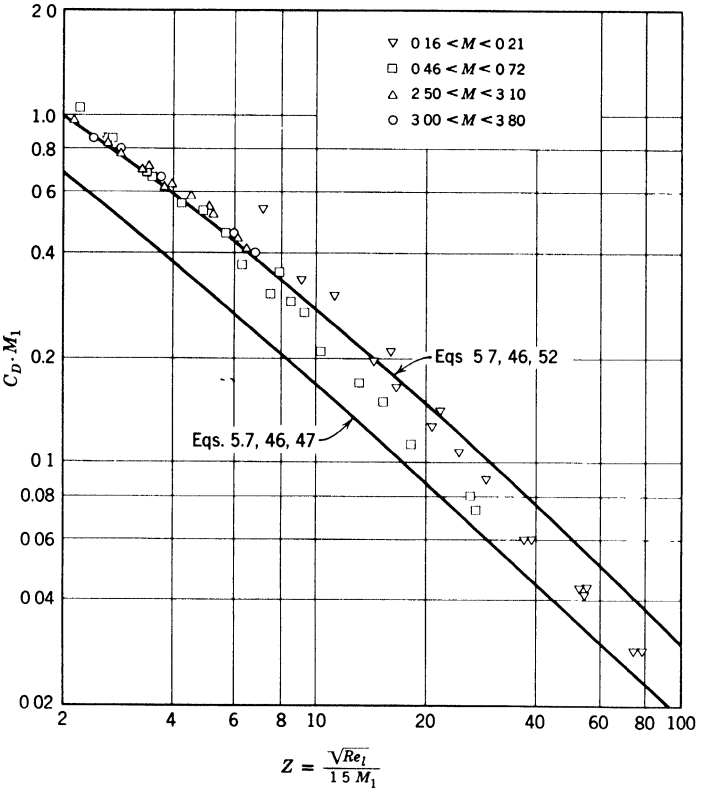

Fig 9. Skin friction drag of flat plates in slip flow.

Schaaf has outlined a direct approach to the approximate calculation of C_D in slip flow based on Rayleigh's equation for steady, two-dimensional flow (Refs. 28, 29, 30, 35). If the perturbation method described in Section 2.5 is applied to Eqs. 4.5, 10–13 and further approximation to the boundary layer of a flat plate is made, then the equations of motion reduce to

$$\rho_1 \bar{u}_1 \frac{\partial \bar{u}}{\partial x} = \mu_1 \frac{\partial^2 \bar{u}}{\partial y^2} \tag{51}$$

plus other equations not required for this analysis. This relation was used by Rayleigh in early studies of the boundary layer. Following a method of solution similar to that described above in this section, Eq. 51 can be solved subject to slip flow at the boundary and the result is an equation of the same form as Eq. 46 (or Eq. 50) where for air

$$z^2 = \frac{Re_l}{2.25 M_1^2}, \qquad \Lambda = 2.67 \tag{52}$$

A curve corresponding to Eqs. 46, 52 is also shown in Fig. 9. The agreement between the observed values of C_D and those calculated according to Eqs. 46, 52 is good at the lower end of the range of z (slip flow regime), but this solution does not reduce to the correct relation for C_D at large z (no-slip flow).

5.8 ENERGY TRANSFER WITH A TEMPERATURE JUMP

When terms of the order of Δ in the boundary conditions for non-isentropic flow are no longer negligible, the mass velocity and temperature of the gas on the surface are different from those of the wall, and a slip flow occurs accompanied by temperature and pressure jumps. We now investigate the effect of a temperature jump on the energy transfer.

The temperature jump may be obtained from Eq. 4.5, 8 for equilibrium conditions,

$$\frac{T_s'}{T_w'} = 1 + \left(\frac{\kappa \Delta \delta'}{T_w'}\right) L_s' \left(\frac{\partial T'}{\partial y''}\right)_s \tag{1}$$

in which terms of order lower than Δ (or D) are neglected. In the dimensionless variables of boundary layer theory,

$$\Theta_s - \Theta_w = \kappa \left(\mathscr{L} \frac{\partial \Theta}{\partial Y}\right)_s \tag{2}$$

In the preceding section it was possible to express the momentum-transfer equation in a form independent of the energy-transport relation. Since the momentum process contributes to the energy exchange, terms arising from the momentum will occur in the energy equation which has a more complex form than Eq. 5.7, 10. We can obtain some insight into the effect of a temperature discontinuity on the energy transfer at a wall by considering a very slow mass motion such that the derivatives of $\bar{u}$ and $\bar{v}$ in Eq. 5.7, 5 may be neglected. Then the dimensionless form of the energy equation becomes

$$\Gamma\left(\frac{\partial \Theta}{\partial \chi} + V \frac{\partial \Theta}{\partial Y}\right) = \frac{\gamma}{Pr Re_1} \frac{\partial}{\partial Y}\left(K \frac{\partial \Theta}{\partial Y}\right) \tag{3}$$

Transforming this relation according to Eqs. 5.7, 9, 11 reduces it to

$$\frac{\partial \Theta}{\partial \chi} = \frac{\gamma\sigma}{PrRe_1} \frac{\partial^2 \Theta}{\partial \psi^2} \tag{4}$$

Let us consider this equation first when no temperature jump exists. The boundary conditions are

$$\begin{aligned} \Theta &= 1 \quad && \text{for } \chi = 0, \psi > 0 \\ \Theta &= \Theta_w \quad && \text{for } \chi \geq 0, \psi = 0 \\ \Theta &\to 1 \quad && \text{for } \chi \geq 0, \psi \to \infty \end{aligned} \tag{5}$$

Following the method given in Section 5.7 above, the subsidiary equation is

$$\left(\frac{\gamma\sigma}{PrRe_1}\right) \frac{d^2\Theta^*}{d\psi^2} - \nu\Theta^* = -1 \tag{6}$$

for which conditions 5 become

$$\begin{aligned} \Theta^* &= \Theta_w/\nu \quad && \text{for } \psi = 0 \\ \Theta^* &\to 1/\nu \quad && \text{for } \psi \to 0 \end{aligned} \tag{7}$$

The solution is

$$\Theta^* = \frac{1}{\nu}\left[1 + (\Theta_w - 1) \exp\left(-\psi\sqrt{\frac{PrRe_1\nu}{\gamma\sigma}}\right)\right] \tag{8}$$

According to Ref. 33 (Appendix III, No. 83) this is the Laplace transform of

$$\Theta = 1 + (\Theta_w - 1) \operatorname{erfc}\left(\frac{\psi}{2}\sqrt{\frac{PrRe_1}{\gamma\sigma\chi}}\right) \tag{9}$$

The heat-transfer coefficient is (see Eq. 4.7, 26)

$$q = -\frac{\sigma}{PrRe_1} \frac{\partial \Theta}{\partial \psi} \tag{10}$$

Differentiating Eq. 8 with respect to ψ,

$$\frac{d\Theta^*}{d\psi} = -\sqrt{\frac{PrRe_1}{\gamma\sigma\nu}} (\Theta_w - 1) \exp\left(-\psi\sqrt{\frac{PrRe_1\nu}{\gamma\sigma}}\right) \tag{11}$$

According to Ref. 33 (Appendix III, No. 84),

$$\frac{\partial \Theta}{\partial \psi} = -\sqrt{\frac{PrRe_1}{\pi\gamma\sigma\chi}} (\Theta_w - 1) \exp\left(-\frac{PrRe_1\psi^2}{4\gamma\sigma\chi}\right) \tag{12}$$

At the surface ($\psi = 0$) the heat-transfer coefficient is

$$C_h = q_w = \frac{(\Theta_w - 1)}{\sqrt{\pi\gamma Pr}} \sqrt{\frac{\sigma}{Re_1\chi}} \tag{13}$$

When a temperature jump exists, the boundary conditions to be associated with Eq. 4 are

$$\begin{aligned} \Theta &= 1 && \text{for } \chi = 0,\ \psi > 0 \\ \Theta &= \Theta_w + \kappa \frac{\partial \Theta}{\partial \psi} && \text{for } \chi \geq 0,\ \psi = 0 \\ \Theta &\to 1 && \text{for } \chi \geq 0,\ \psi \to \infty \end{aligned} \tag{14}$$

The subsidiary Eq. 6 must now be solved subject to the condition

$$\kappa \frac{d\Theta^*}{d\psi} - \Theta^* = -\frac{\Theta_w}{\nu} \quad \text{at } \psi = 0 \tag{15}$$

Then

$$\Theta^* = \frac{1}{\nu}\left[1 + \frac{(\Theta_w - 1)\exp\left(-\psi\sqrt{\frac{PrRe_1\nu}{\gamma\sigma}}\right)}{\kappa\sqrt{\frac{PrRe_1\nu}{\gamma\sigma}} + 1}\right] \tag{16}$$

which is the Laplace transform of

$$\Theta = 1 + (\Theta_w - 1)\left[\operatorname{erfc}\left(\frac{\psi}{2}\sqrt{\frac{PrRe_1}{\gamma\sigma\chi}}\right) - \exp\left(\frac{\psi}{\kappa} + \frac{\gamma\sigma\chi}{\kappa^2 PrRe_1}\right) \times \operatorname{erfc}\left(\frac{1}{\kappa}\sqrt{\frac{\gamma\sigma\chi}{PrRe_1}} + \frac{\psi}{2}\sqrt{\frac{PrRe_1}{\gamma\sigma\chi}}\right)\right] \tag{17}$$

(Ref. 33, Appendix III, No. 86). The gas temperature at the surface of the plate is given by

$$\Theta_s = \Theta_w - (\Theta_w - 1)\exp\left(\frac{\gamma\sigma\chi}{\kappa^2 PrRe_1}\right)\operatorname{erfc}\left(\frac{1}{\kappa}\sqrt{\frac{\gamma\sigma\chi}{PrRe_1}}\right) \tag{18}$$

and the coefficient of heat transfer on the surface is found to be

$$q_w = \frac{\sigma}{PrRe_1}\left(\frac{\Theta_w - 1}{\kappa}\right)\exp\left(\frac{\gamma\sigma\chi}{\kappa^2 PrRe_1}\right)\operatorname{erfc}\left(\frac{1}{\kappa}\sqrt{\frac{\gamma\sigma\chi}{PrRe_1}}\right) \tag{19}$$

An estimate of Θ_s and q_w for stationary flow along the plate can be made by matching solutions for the case of no temperature jump, following the method outlined in Section 5.7. From the theory of stationary boundary layer flow (Section 4.7),

$$q_w = \frac{1}{2Pr}[f''(0)]^{Pr}\left\{\int_0^\infty [f''(\eta)]^{Pr}\,d\eta\right\}^{-1}(\Theta_w - \Theta_e)\sqrt{\frac{\sigma}{Re_1 X}} \tag{20}$$

where, as $U \to 1$, $\Theta_e \to 1$ (Eq. 4.7, 22). Substituting $\chi = k_2 X$ in Eq. 13, and matching the result with Eq. 20, we have

$$k_2 = \frac{4Pr}{\pi\gamma}[f''(0)]^{-2Pr}\left\{\int_0^\infty [f''(\eta)]^{Pr}\,d\eta\right\}^2 \tag{21}$$

Then, for small temperature discontinuities where the above value of k_2 may be expected to hold with reasonable accuracy,

$$\Theta_s = \Theta_w - (\Theta_w - 1) \exp\left(\frac{\gamma k_2 \sigma X}{\kappa^2 Pr Re_1}\right) \operatorname{erfc}\left(\frac{1}{\kappa}\sqrt{\frac{\gamma k_2 \sigma X}{Pr Re_1}}\right) \tag{22}$$

and

$$q_w = \frac{\sigma}{Pr Re_1} \frac{(\Theta_w - 1)}{\kappa} \exp\left(\frac{\gamma k_2 \sigma X}{\kappa^2 Pr Re_1}\right) \operatorname{erfc}\left(\frac{1}{\kappa}\sqrt{\frac{\gamma k_2 \sigma X}{Pr Re_1}}\right) \tag{23}$$

Since these solutions apply primarily at large values of X, the asymptotic form for the complementary error function given by Eq. 5.7, 38 may be used. Then approximately

$$\Theta_s = \Theta_w - \kappa(\Theta_w - 1)\sqrt{\frac{Pr Re_1}{\pi \gamma k_2 \sigma X}} \tag{24}$$

and

$$q_w = \frac{(\Theta_w - 1)}{\sqrt{\pi \gamma k_2 Pr}} \sqrt{\frac{\sigma}{Re_1 X}} \left[1 - \frac{1}{2}\frac{\kappa^2 Pr}{k_2 \gamma}\left(\frac{Re_1}{\sigma X}\right) + \cdots\right] \tag{25}$$

Therefore, the effect of a small temperature jump at the surface is to reduce the rate of heat transfer.

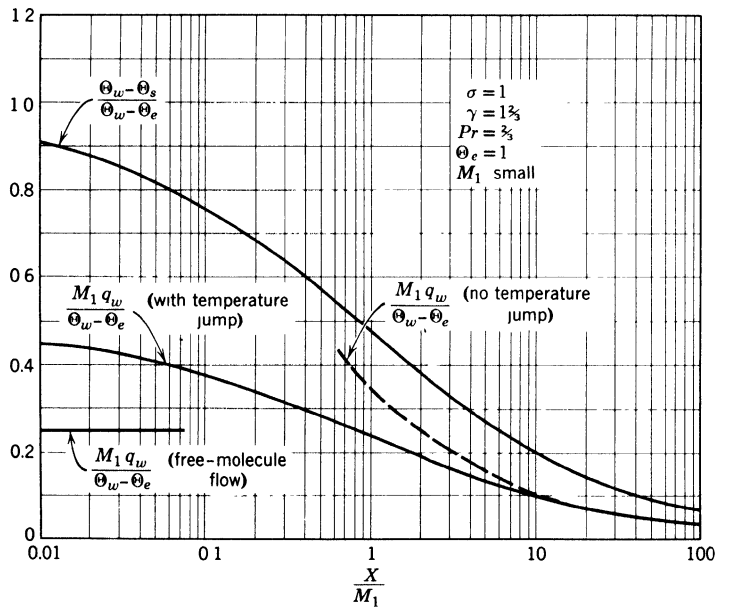

Fig. 10. Temperature jump and local heat transfer in slip flow.

For a monatomic gas, $Re_1 = 1.648M_1$, $Pr = \frac{2}{3}$, $\gamma = 1\frac{2}{3}$, and we may write

$$\frac{\Theta_w - \Theta_s}{\Theta_w - 1} = \exp\left(\zeta \frac{X}{M_1}\right) \operatorname{erfc}\left(\sqrt{\zeta \frac{X}{M_1}}\right) \tag{26}$$

and

$$\frac{M_1 q_w}{\Theta_w - 1} = \frac{\sigma}{1.648\,\kappa Pr} \exp\left(\zeta \frac{X}{M_1}\right) \operatorname{erfc}\left(\sqrt{\zeta \frac{X}{M_1}}\right) \tag{27}$$

where

$$\zeta = \frac{\gamma k_2 \sigma}{1.648\,\kappa^2 Pr} \tag{28}$$

According to the theory of free-molecule flow, $\Theta_s = 1$, and for low mass velocity (see Section 5.3)

$$\frac{M_1 q_w}{\Theta_w - 1} = \frac{\gamma - 1}{\gamma} \sqrt{\frac{2}{\pi\gamma}} \tag{29}$$

Relations 26 and 27 are plotted in Fig. 10. They cover the range from flow with no temperature jump to free-molecule flow in a closed form, but they can be regarded as indicative only for large Mach numbers or as the free-molecule flow regime is approached.

5.9 THE TRANSITION REGIME

In the preceding sections of this chapter the mechanics of rarefied gases has been investigated first for gases having a density so low that intermolecular collisions may be neglected and then for moderately rarefied gases in which slip flow occurs. Observation of free-molecule flow appears to indicate that this regime exists for $X/M_1 < 0.1$ (Refs. 11, 25). On the other hand, our calculations suggest that no significant slip effect can be expected if $X/M_1 > 10$ (roughly). These limits for X/M_1 (Fig. 8) can only be tentative and are subject to change with further analysis and observation of low-density flows.

Between the regimes of free-molecule motion and no-slip flow lies a transition region in which extremely complex transfer processes occur since intermolecular encounters and collisions between gas molecules and a wall are of equal importance. As yet no satisfactory theory exists for this regime.

As in strong shock transition (Section 4.3), one may ask whether the flow in the transition regime may be described as slightly nonisentropic. The second modification of the Maxwell velocity distribution function (Burnett flow, Appendix III) introduces additional terms in the equations of motion, which have been discussed by Tsien, Schaaf, Schamberg, and Truesdell (Refs. 6, 29, 36, 37). These terms are significant when the product of the Mach number and the Knudsen number is large (or X/M_1 is small). Therefore the flow in the transition region appears to involve more than a small deviation from Maxwellian motion. The

Burnett terms and their relation to slip flow are being studied. At present no conclusive experimental evidence exists to support the more general (Burnett) flow equations.

NOTATION

A	area of a plane surface
b	width of a flat plate
B	Stefan-Boltzmann constant, $3{\cdot}74 \times 10^{-10}$ foot-pound per square foot, °F^4
C_D	total drag coefficient of a body
C_i	most probable speed of the incident molecules
C_p	pressure coefficient $(p/\frac{1}{2}\rho_i\bar{q}^2)$
D	total skin friction drag of a plate (both sides)
F	rate of change of momentum per unit area
g	fraction of incident molecules which are emitted diffusely
I_0, I_1	modified Bessel functions of order 0 and 1 (Eqs. 5.5, 7, 8)
J	output of internal energy from unit area of a body in unit time
k_1, k_2	constants in the transformations from time to position (Eqs. 5.7, 35 and 5.8, 21)
K	dimensionless coefficient of viscosity or heat conduction $(\mu/\mu_1$ or $\lambda/\lambda_1)$
l	length of a cylinder
$\mathscr{L}$	dimensionless ratio of mean free paths (L/L_1)
N	number of molecules per unit area per unit time
Q_m	mass flow through unit area in unit time
Q_p	mass flow in terms of pressure and volume
r	radius of a cylinder
Re_l	Reynolds number based on the length (l) of a flat plate
S	speed ratio in free molecule flow $(\bar{q}/C_i$ or $M_i\sqrt{\gamma/2})$
S_v	component of S normal to a surface $(\bar{v}/C_i$ or $\bar{v}\sqrt{\beta_i})$
T_c	temperature of the walls containing a gas and body
U^*	Laplace transform of U
U_s	dimensionless slip velocity $(\bar{u}_s/\bar{u}_1)$
z	parameter in the equation for the flat plate drag coefficient defined by Eq. 5.7, 47
Z_1, Z_2	functions of S (Eqs. 5.5, 22, 23)
α	energy transfer or accommodation coefficient (Eq. 5.4, 1)
ϵ	emissivity, dimensionless
ζ	constant given by Eq. 5.8, 28
θ	inclination of a surface to the mass velocity
Θ^*	Laplace transform of Θ
κ	the constant α_6 or $75\pi/128$
κ_1, κ_2	constants of integration
Λ	coefficient in the equation for the flat plate drag coefficient defined by Eq. 5.7, 48
ν	parameter in the Laplace transformation
ξ	variable in the asymptotic expression 5.7, 38
φ	a quantity in the energy equation defined by Eq. 5.3, 20
χ	dimensionless time ratio $(\bar{u}t_1/L_1)$
ψ	dimensionless variable analogous to the stream function

Note: Symbols that do not appear above may be found in the Notation at the end of preceding chapters (pages 24, 63, 97, 154). The subscripts i and r refer to the incident and emergent (reflected) streams of free molecules, respectively.

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APPENDIX ONE

Mathematical Aids

1 DEFINITE INTEGRALS WITH ZERO VALUE

$$\int_{-\infty}^{\infty} x^n e^{-\beta x^2}\,dx, \qquad n = 1, 3, 5, 7, \cdots$$

$$\int_0^{2\pi} \sin x\,dx \quad \int_0^{2\pi} \cos x\,dx \quad \int_0^{2\pi} \sin x \cos x\,dx \quad \int_0^{2\pi} \sin^3 x\,dx$$

$$\int_0^{2\pi} \cos^3 x\,dx \quad \int_0^{2\pi} \sin x \cos^2 x\,dx \quad \int_0^{2\pi} \sin^2 x \cos x\,dx$$

$$\int_0^{2\pi} \sin x \cos^3 x\,dx \quad \int_0^{2\pi} \sin^3 x \cos x\,dx \quad \int_0^{2\pi} \cos^5 x\,dx$$

$$\int_0^{2\pi} \sin x \cos^4 x\,dx \quad \int_0^{2\pi} \sin^2 x \cos^3 x\,dx \quad \int_0^{2\pi} \cos (y - x)\,dx$$

$$\int_0^{2\pi} \cos x \cos^2 (y - x)\,dx \quad \int_0^{\pi} \sin x \cos^3 x\,dx \quad \int_0^{\pi} \sin^3 x \cos x\,dx$$

$$\int_0^{\pi} \sin^4 x \cos x\,dx \quad \int_0^{\pi} \sin^5 x \cos x\,dx$$

2 DEFINITE INTEGRALS WITH FINITE VALUE

$$\int_{-\infty}^{\infty} e^{-\beta x^2}\,dx = \sqrt{\frac{\pi}{\beta}} \quad \int_{-\infty}^{\infty} x^2 e^{-\beta x^2}\,dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}$$

$$\int_{-\infty}^{\infty} x^4 e^{-\beta x^2}\,dx = \frac{3}{4\beta^2}\sqrt{\frac{\pi}{\beta}} \quad \int_{-\infty}^{\infty} x^6 e^{-\beta x^2}\,dx = \frac{15}{8\beta^3}\sqrt{\frac{\pi}{\beta}}$$

Note:

$$\int_0^{\infty} x^n e^{-\beta x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x^n e^{-\beta x^2}\,dx, \qquad n = 0, 2, 4, 6, \cdots$$

$$\int_0^{\infty} x e^{-\beta x^2}\,dx = \frac{1}{2\beta} \quad \int_0^{\infty} x^3 e^{-\beta x^2}\,dx = \frac{1}{2\beta^2}$$

$$\int_0^{\infty} x^5 e^{-\beta x^2}\,dx = \frac{1}{\beta^3} \quad \int_0^{\infty} x^7 e^{-\beta x^2}\,dx = \frac{3}{\beta^4}$$

Gamma function:

$$\int_0^\infty x^n e^{-\beta x^2}\,dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\beta^{(n+1)/2}}$$

where $$\Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n-1}{2}\right)! \text{ for } n \text{ odd.}$$

$$\int_{-a}^\infty e^{-\beta x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,(1 + \operatorname{erf} a\sqrt{\beta})$$

$$\int_{-a}^\infty x e^{-\beta x^2}\,dx = \frac{1}{2\beta}\,e^{-\beta a^2}$$

$$\int_{-a}^\infty x^2 e^{-\beta x^2}\,dx = \frac{1}{2\beta}\left[\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,(1 + \operatorname{erf} a\sqrt{\beta}) - ae^{-\beta a^2}\right]$$

$$\int_{-a}^\infty x^3 e^{-\beta x^2}\,dx = \frac{1}{2\beta^2}\,(1 + \beta a^2)e^{-\beta a^2}$$

$$\int_{-a}^\infty x^4 e^{-\beta x^2}\,dx = \frac{1}{4\beta^2}\left[\frac{3}{2}\sqrt{\frac{\pi}{\beta}}\,(1 + \operatorname{erf} a\sqrt{\beta}) - a(2\beta a^2 + 3)e^{-\beta a^2}\right]$$

$$\int_0^{2\pi} \sin^2 x\,dx = \pi \qquad \int_0^{2\pi} \cos^2 x\,dx = \pi \qquad \int_0^{2\pi} \sin^4 x\,dx = 3\pi/4$$

$$\int_0^{2\pi} \cos^4 x\,dx = 3\pi/4 \qquad \int_0^{2\pi} \sin^2 x \cos^2 x = \pi/4$$

$$\int_0^{2\pi} \cos x \cos(y - x)\,dx = \pi \cos y \qquad \int_0^{2\pi} \cos^2(y - x)\,dx = \pi$$

$$\int_0^{\pi} \sin x\,dx = 2 \qquad \int_0^{\pi} \sin^3 x\,dx = \tfrac{4}{3} \qquad \int_0^{\pi} \sin x \cos^2 x\,dx = \tfrac{2}{3}$$

$$\int_0^{\pi} \sin^5 x\,dx = \tfrac{16}{15} \qquad \int_0^{\pi} \sin^3 x \cos^2 x\,dx = \tfrac{4}{15}$$

$$\int_0^{\pi/2} \sin x \cos x\,dx = \tfrac{1}{2}$$

Error function: $$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy$$

Complementary error function:

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\,dy$$

3 GENERAL VELOCITY DISTRIBUTION FUNCTION FOR SLIGHTLY NONISENTROPIC FLOW

$$f = f_0[1 + F(U, V, W)]$$

where

$$f_0 = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(U^2+V^2+W^2)}$$

$$\beta = \frac{3}{2\overline{C^2}} = \frac{1}{\overline{C_m}^2} = \frac{1}{2RT}$$

$$F(U, V, W) = (2\beta)^{1/2}(a_1U + a_2V + a_3W) + 2\beta[\tfrac{1}{2}(a_{11}U^2 + a_{22}V^2 + a_{33}W^2) + a_{12}UV + a_{23}VW + a_{13}UW] + (2\beta)^{3/2}[\tfrac{1}{6}(a_{111}U^3 + a_{222}V^3 + a_{333}W^3) + \tfrac{1}{2}(a_{122}UV^2 + a_{233}VW^2 + a_{113}U^2W + a_{112}U^2V + a_{223}V^2W + a_{133}UW^2) + a_{123}UVW]$$

$$G(X_1, Y_1, Z_1, X_2, Y_2, Z_2) = 1 + G_1 + G_2$$
$$= 1 + \sqrt{2\beta}(a_1X_2 + a_2Y_2 + a_3Z_2) + \beta/2\,[a_{11}(X_1^2 + X_2^2) + a_{22}(Y_1^2 + Y_2^2) + a_{33}(Z_1^2 + Z_2^2) + 2\{a_{12}(X_1Y_1 + X_2Y_2) + a_{23}(Y_1Z_1 + Y_2Z_2) + a_{13}(X_1Z_1 + X_2Z_2)\}]$$
$$+ \tfrac{1}{3}(\beta/2)^{3/2}[a_{111}(3X_1^2X_2 + X_2^3) + a_{222}(3Y_1^2Y_2 + Y_2^3) + a_{333}(3Z_1^2Z_2 + Z_2^3) + 3\{a_{122}(2X_1Y_1Y_2 + X_2Y_1^2 + X_2Y_2^2) + a_{233}(2Y_1Z_1Z_2 + Y_2Z_1^2 + Y_2Z_2^2) + a_{113}(2Z_1X_1X_2 + Z_2X_1^2 + Z_2X_2^2) + a_{112}(2Y_1X_1X_2 + Y_2X_1^2 + Y_2X_2^2) + a_{223}(2Z_1Y_1Y_2 + Z_2Y_1^2 + Z_2Y_2^2) + a_{133}(2X_1Z_1Z_2 + X_2Z_1^2 + X_2Z_2^2)\} + 6a_{123}(X_1Y_1Z_2 + X_1Y_2Z_1 + X_2Y_1Z_1 + X_2Y_2Z_2)]$$

APPENDIX TWO

Differential Equations and Their Characteristics

1 DEFINITIONS

Differential equation: an equation involving an unknown function and its derivatives from which the unknown function is to be determined.

Ordinary differential equation: a relation involving one independent variable, one dependent variable, and one or more derivatives of the dependent with respect to the independent variable.

Partial differential equation: an equation which involves one dependent, two or more independent variables, and partial derivatives of the dependent with respect to the independent variables.

Order of a differential equation: the order of the highest derivative the equation contains.

Degree of a differential equation: when a differential equation is polynomial in form with respect to its derivatives, then the power of the highest derivative is termed the degree of the equation.

A differential equation is linear if the dependent variable and its derivatives appear in the first degree only and are not combined as products in any term. Otherwise the differential equation is described as nonlinear.

2 SOME CLASSIFICATIONS OF DIFFERENTIAL EQUATIONS

The equation

$$a(x)\frac{du}{dx} + b(x)u = c(x) \tag{1}$$

is a first-order, first-degree, linear, ordinary differential equation. An ordinary linear differential equation of the nth order and first degree has the general form

$$\sum_{r=1}^{n} a_r(x)\frac{d^r u}{dx^r} = c(x) \tag{2}$$

A differential equation is homogeneous if each term taken as a product has the same over-all degree, for example,

$$a(x)\frac{du}{dx} = b(x)u \tag{3}$$

When the derivative of highest order occurs linearly in a nonlinear differential equation, the relation is described as quasilinear. Thus

$$a(x, u)\frac{du}{dx} = c(x, u) \tag{4}$$

is a quasilinear ordinary differential equation of the first order. A quasilinear ordinary differential equation of the nth order may be written

$$\sum_{r=1}^{n} a_r\left(x, u, \frac{du}{dx}, \cdots, \frac{d^{n-1}u}{dx^{n-1}}\right)\frac{d^r u}{dx^r} = c\left(x, u, \frac{du}{dx}, \cdots, \frac{d^{n-1}u}{dx^{n-1}}\right) \tag{5}$$

that is, the nth derivative occurs in the first degree only.

The nonlinear ordinary differential equation of the nth order may be expressed in the general form

$$f\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \cdots, \frac{d^n u}{dx^n}\right) = 0 \tag{6}$$

A differential equation of the nth order can always be reduced to a set of n simultaneous equations of the first order containing n unknowns. By introducing new dependent variables to replace the derivatives, the above general nonlinear equation can be replaced by the following:

$$\begin{gathered} u_1 = \frac{du}{dx}, \quad u_2 = \frac{du_1}{dx}, \quad u_3 = \frac{du_2}{dx}, \cdots, u_{n-1} = \frac{du_{n-2}}{dx} \\ f\left(x, u, u_1, u_2, \cdots, u_{n-1}, \frac{du_{n-1}}{dx}\right) = 0 \end{gathered} \tag{7}$$

which is a set of n first-order equations from which to determine the n unknowns $u, u_1, u_2, u_3, \cdots, u_{n-1}$.

3 INITIAL VALUE PROBLEMS

In general, a differential equation is not amenable to an elementary treatment. In many cases a method of numerical approximation must be used. Therefore, it is important to investigate whether a solution does exist. The determination of the necessary and sufficient conditions for the existence of a unique solution of a differential equation consistent with given initial values is called the initial value problem.

Consider the equation

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u + D(x) = 0 \tag{1}$$

where the values of u and du/dx at $x = x_0$ are given. Then we can determine a solution in the form of a Taylor series,

$$u(x) = u_0 + (x - x_0)\left(\frac{du}{dx}\right)_0 + \frac{(x - x_0)^2}{2!}\left(\frac{d^2u}{dx^2}\right)_0 + \frac{(x - x_0)^3}{3!}\left(\frac{d^3u}{dx^3}\right)_0 + \cdots \tag{2}$$

This is evident since u_0 and $(du/dx)_0$ are given and

$$\left(\frac{d^2u}{dx^2}\right)_0 = -\left[\left(\frac{B}{A}\right)_0\left(\frac{du}{dx}\right)_0 + \left(\frac{C}{A}\right)_0 u_0 + \left(\frac{D}{A}\right)_0\right]$$

$$\left(\frac{d^3u}{dx^3}\right)_0 = -\left\{\left[\frac{d}{dx}\left(\frac{B}{A}\right)\right]_0\left(\frac{du}{dx}\right)_0 + \left(\frac{B}{A}\right)_0\left(\frac{d^2u}{dx^2}\right)_0 + \left[\frac{d}{dx}\left(\frac{C}{A}\right)\right]_0 u_0 + \left(\frac{du}{dx}\right)_0\left(\frac{C}{A}\right)_0 + \left[\frac{d}{dx}\left(\frac{D}{A}\right)\right]_0\right\}$$

and so on.

In general, when the ordinary differential equation is of order n and the dependent variable and its derivatives up to the $(n - 1)$th order are given at $x = x_0$ as initial conditions, then the nth and all higher-order derivatives can be found from the differential equation and the solution can be given in the form of the Taylor series.

Under normal conditions the Taylor series is convergent. A function which can be expanded as a convergent power series in $(x - x_0)$ is termed a regular function at $x = x_0$.

However, there may be a value of the independent variable at which the nth and higher-order derivatives cannot be determined from the differential equation. In this case the differential equation has a singular point. We shall now investigate these singularities for various types of ordinary differential equations.

4 LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Let us consider the equation

$$a(x)\frac{du}{dx} + b(x)u = c(x) \tag{1}$$

with the initial condition that $u = u_0$ when $x = x_0$. This equation can be written in the form

$$\frac{du}{dx} = \frac{c(x) - b(x)u}{a(x)} \tag{2}$$

It will be seen that a singularity occurs if $a(x) = 0$ $(x = x_r; r = 1, 2, 3, \cdots$ unless $c(x) - b(x)u = 0$.

If $c(x_r) - b(x_r)u \neq 0$, then $du/dx \to \infty$ at all points $x = x_r$, and the Taylor series cannot be computed at $x = x_r$. Then $u(x_r)$ is not regular.

If $c(x_r) - b(x_r)u = 0$, or $u = \dfrac{c(x_r)}{b(x_r)}$, then $\left(\dfrac{du}{dx}\right)_r$ is finite, in general, but indeterminate—there may be more than one regular solution at the points $\left(x_r, \dfrac{c(x_r)}{b(x_r)}\right)$. These points are called the characteristics of the differential equation. The relation $u = \dfrac{c(x_r)}{b(x_r)}$ is the regularity condition.

As an example, let us investigate the equation

$$x(x - 1)\frac{du}{dx} + (1 - 2x)u + x^2 = 0 \tag{3}$$

This has the solution $$u = \alpha x(x - 1) + x \tag{4}$$

where the constant α is determined from the initial condition. The singularity equation is $x(x - 1) = 0$, or $x = 0$, 1, and the regularity condition gives $u = -\left(\dfrac{x^2}{1 - 2x}\right)_{x=0} = 0$, and $u = -\left(\dfrac{x^2}{1 - 2x}\right)_{x=1} = 1$. The characteristic points are, therefore, (0, 0) and (1, 1).

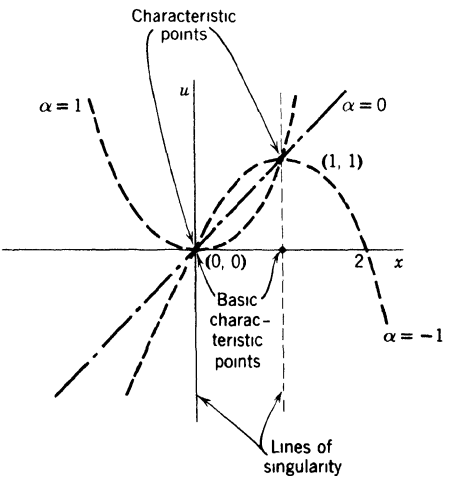

Fig. 1. Curves for Eq. 4, 4 (Appendix II).

The properties of the solution for various values of α are illustrated in Fig. 1. All the curves pass through the characteristic points since

Eq. 4 is identically satisfied by (0, 0), (1, 1) for all values of α. We see that for this particular example the characteristic points are common to all solutions. The curves cross the lines of singularity only at the characteristic points.

It is of interest to note that the characteristic points of the differential equation are also points on the curve of the solution. They are solutions of the differential equation since they were obtained by satisfying the differential equation identically.

It should also be noted that the equations used for the determination of the characteristic points are one order lower than the given differential

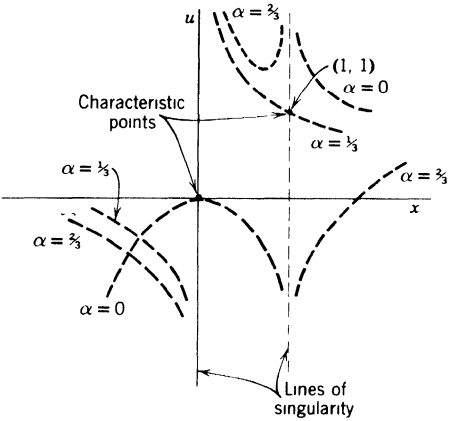

Fig. 2. Curves for Eq. 4, 6 (Appendix II).

equation. This fact will assume greater importance in subsequent discussions.

More generally the characteristic points lie on a curve corresponding to a specific value of the constant. The differential equation

$$x(x-1)\frac{du}{dx} - (1-2x)u - x^2 = 0 \tag{5}$$

has the solution

$$\frac{x^3}{3} - x(x-1)u = \text{constant} = \alpha \tag{6}$$

The singularity equation is $x(x-1) = 0$, and the regularity condition is $u = x^2/(2x-1)$. The characteristic points are (0, 0) and (1, 1). In this case only specific curves are able to cross the singularity lines (Fig. 2).

5 FIRST-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

The ordinary quasilinear differential equation of the first order has the general form

$$a(x, u)\frac{du}{dx} = c(x, u) \tag{1}$$

The equation for the singularities is

$$a(x, u) = 0, \quad \text{or } x = f(u) \tag{2}$$

In this case the line of singularities is a curve in the (x, u) plane.

To obtain the characteristic point we require in addition the regularity condition

$$c(x, u) = 0 \quad \text{when } x = f(u) \tag{3}$$

At this characteristic point the solution is regular but not unique.

As an example we consider the equation

$$\left(x - \frac{1}{u^2}\right)\frac{du}{dx} = x - u \tag{4}$$

This can be written in the form

$$(x - u)\,dx + \left(\frac{1}{u^2} - x\right)du = 0$$

which has the solution

$$f = \frac{x^2}{2} - ux - \frac{1}{u} = c \tag{5}$$

since $\partial f/\partial x = x - u$ and $\partial f/\partial u = (1/u^2) - x$. We have therefore: singularity equation, $x - (1/u^2) = 0$, or $x = 1/u^2$; regularity condition, $x - u = 0$ at $x = 1/u^2$, or $u^3 = 1$. The (real) characteristic point is therefore (1, 1).

Curves for the function $x = u \pm \left[u^2 + 2\left(c + \frac{1}{u}\right)\right]^{1/2}$ for various values of c are given in Fig. 3. We note that the two branch curves have a cusp at the characteristic point (1, 1) when $c = -\frac{3}{2}$. A continuous curve through the characteristic point can be obtained by a proper choice of branch solutions for $x < 1$ and $x > 1$ (see Fig. 3). All other regular solutions do not pass through the characteristic point or cross the two lines of singularity ($u = \pm 1/\sqrt{x}$). At $c = 0$, one branch does cross a line of singularities but does so at an infinitely large value for du/dx so that for one value of x there are two values of u. This example illustrates the importance of branch solutions which occur when the equation is nonlinear.

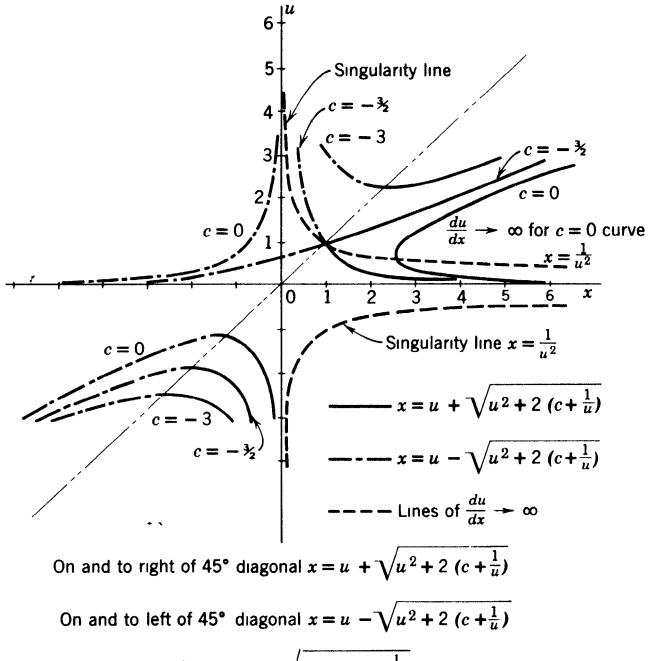

Fig. 3. Curves for $x = u \pm \sqrt{u^2 + 2\left(c + \frac{1}{u}\right)}$ (Appendix II).

6 PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Let us investigate the partial differential equation

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \tag{1}$$

where the distribution of the dependent variable u is given over a curve $\varphi(x, y) = \text{constant}$. Then, since the derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\cdots$ can be calculated in general from the differential equation, a solution in the form of a Taylor series, taken about a point on the initial curve (x_0, y_0), can be determined as follows:

$$\begin{aligned} u(x, y) = u(x_0, y_0) &+ \left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]_0 u \\ &+ \frac{1}{2!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]_0^2 u + \cdots \\ &+ \frac{1}{n!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]_0^n u + \cdots \end{aligned} \tag{2}$$

Let us write Eq. 1 in the form

$$\frac{\partial u}{\partial x} = \frac{c}{a} - \frac{b}{a}\frac{\partial u}{\partial y} \tag{3}$$

All derivatives of the type $\dfrac{\partial^{m+n} u}{\partial x^m \, \partial y^n}$ can therefore be calculated in terms of x, y, u, and $\partial u/\partial y$ except the partial derivatives $\partial^n u/\partial y^n$. We see that the Taylor series which formally satisfies Eq. 1 can be computed for arbitrary values of x_0, y_0, $\left(\dfrac{\partial u}{\partial y}\right)_0$, $\left(\dfrac{\partial^2 u}{\partial y^2}\right)_0$, $\cdots$, $\left(\dfrac{\partial^n u}{\partial y^n}\right)_0$, $\cdots$. The terms involving y in the series define an arbitrary function of y.

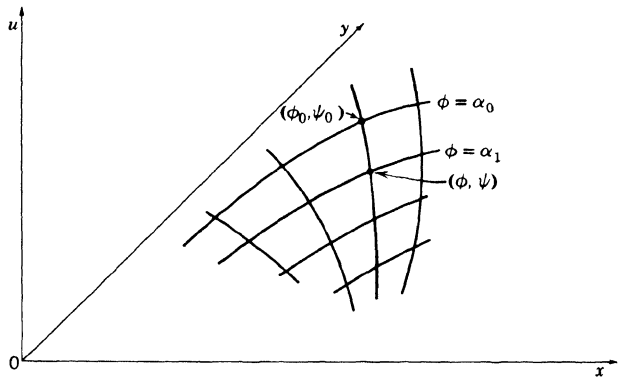

Fig. 4. Change of variables (Appendix II).

Under normal conditions the Taylor series is convergent and defines a unique, regular solution of the differential equation, consistent with the initial conditions. We can thus find u on a curve adjacent to the initial curve $\varphi(x, y) = \text{constant}$. This will be true except for certain singular initial curves for which the first and higher derivatives cannot be calculated from the given differential equation.

The initial curve $\varphi(x, y) = \text{constant}$ will, in general, be one of a family of curves corresponding to different values of the constant (Fig. 4). With this we may associate the orthogonal family of curves $\psi(x, y) = \text{constant}$.† Thus to each point (x, y) we can assign the corresponding values (φ, ψ). Then $u = u(\varphi, \psi)$.

† The slope of $\varphi(x, y) = \text{constant}$ is known at a point on its locus. We can therefore determine a new function $\psi(x, y)$ such that its slope is perpendicular to that of $\varphi(x, y) = \text{constant}$, e.g., $\varphi_x = \psi_y$, $\varphi_y = -\psi_x$.

On the initial curve $\varphi(x, y) = \alpha_0$ on which $u(x, y)$ is specified, only the parameter ψ will vary. Thus $\left(\frac{\partial u}{\partial \psi}\right)_0$, $\left(\frac{\partial^2 u}{\partial \psi^2}\right)_0$, $\cdots$ will be known for the Taylor series

$$u(\varphi, \psi) = u(\varphi_0, \psi_0) + (\varphi - \varphi_0)\left(\frac{\partial u}{\partial \varphi}\right)_0 + (\psi - \psi_0)\left(\frac{\partial u}{\partial \psi}\right)_0 + \frac{1}{2}(\varphi - \varphi_0)^2\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_0 + (\varphi - \varphi_0)(\psi - \psi_0)\left(\frac{\partial^2 u}{\partial \varphi\, \partial \psi}\right)_0 + \frac{1}{2}(\psi - \psi_0)^2\left(\frac{\partial^2 u}{\partial \psi^2}\right)_0 + \cdots \tag{4}$$

Thus the distribution of u on an adjacent curve $\varphi(x, y) = \alpha_1$ can be determined by this process from the assigned values of u on $\varphi(x, y) = \alpha_0$, provided $\frac{\partial u}{\partial \varphi}$, $\frac{\partial^2 u}{\partial \varphi^2}$, $\cdots$ can be calculated from the given differential equation.

Let us rewrite Eq. 1 in the form

$$a\left(\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial x}\right) + b\left(\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial y}\right) = c \tag{5}$$

which may be rearranged as follows

$$\left(a\frac{\partial \varphi}{\partial x} + b\frac{\partial \varphi}{\partial y}\right)\frac{\partial u}{\partial \varphi} + \left(a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y}\right)\frac{\partial u}{\partial \psi} = c \tag{6}$$

A singular curve exists if

$$a\frac{\partial \varphi}{\partial x} + b\frac{\partial \varphi}{\partial y} = 0 \tag{7}$$

Since $\varphi = \varphi(x, y) =$ constant, then

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0 \tag{8}$$

and the equation for a singular curve is

$$\frac{dx}{a} = \frac{dy}{b} \tag{9}$$

To obtain the characteristic lines in the (x, y, u) space, we require the regularity condition

$$\left(a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y}\right)\frac{\partial u}{\partial \psi} = c \quad \text{with } \frac{dx}{a} = \frac{dy}{b} \tag{10}$$

Multiplying the first term in the bracket on the left-hand side of Eq. 10 by dx/a and the second by dy/b, then

$$\left(\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy\right)\frac{\partial u}{\partial \psi} = \frac{c}{a}\,dx = \frac{c}{b}\,dy$$

or

$$\frac{\partial u}{\partial \psi}\,d\psi = \frac{c}{a}\,dx = \frac{c}{b}\,dy \tag{11}$$

Now since dx and dy define a displacement along the initial curve for which φ = constant, then $d\varphi = 0$ and

$$du = \frac{\partial u}{\partial \psi}\,d\psi$$

Therefore the equations for the characteristic lines are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \tag{12}$$

The above results may be obtained by the following alternative method. Let us consider

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c \tag{13}$$

and

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = du$$

as a pair of simultaneous equations. Then, solving for $\partial u/\partial x$, $\partial u/\partial y$ as required for the Taylor series (Eq. 2), we have

$$\begin{aligned}(a\,dy - b\,dx)\frac{\partial u}{\partial x} &= c\,dy - b\,du \\ (b\,dx - a\,dy)\frac{\partial u}{\partial y} &= c\,dx - a\,du\end{aligned} \tag{14}$$

Thus neither $\partial u/\partial x$ nor $\partial u/\partial y$ can be determined if

$$\frac{dx}{a} = \frac{dy}{b} \tag{15}$$

which is the singularity equation. The regularity condition is

$$\frac{dy}{b} = \frac{du}{c}, \quad \frac{dx}{a} = \frac{du}{c}, \quad \text{with } \frac{dx}{a} = \frac{dy}{b} \tag{16}$$

which is equivalent to Eq. 12.

7 DETERMINATION OF THE SOLUTION OF A FIRST-ORDER PARTIAL DIFFERENTIAL EQUATION BY MEANS OF THE CHARACTERISTICS

We now investigate the properties of the characteristics of a first-order partial differential equation. The solution of Eq. 6, 1 will be a surface in the (x, y, u) space which may be expressed in the form

$$F(x, y, u) = f(x, y) - u = 0 \tag{1}$$

Now the direction of the normal to this surface is specified by

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial u} \quad \text{or} \quad \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : -1 \quad \text{or} \quad \frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} : -1 \tag{2}$$

It will be seen that Eq. 6, 1 expresses the fact that the normal to the surface $F(x, y, u) = 0$ is orthogonal to the direction $a : b : c$. Therefore the values of a, b, and c at any point define a direction on the surface. As this direction is followed from point to point, a curve in the surface is obtained, and the equations for such curves are given by Eqs. 6, 12,† that is, these curves are the characteristic lines. We conclude, therefore, that any surface in the (x, y, u) space which is a locus of characteristics is a solution of the original differential equation (6, 1). Equations 6, 12 give a family of curves in (x, y, u) space which lie on a surface which is a solution of Eq. 6, 1.

This fact can be used to solve the differential equation. We first determine the characteristic curves and then group them into surfaces. Let us consider the two equations

$$\begin{aligned} a_1\,dx + b_1\,dy + c_1\,du &= 0 \\ a_2\,dx + b_2\,dy + c_2\,du &= 0 \end{aligned} \tag{3}$$

We can solve these equations for any two of dx, dy, du in terms of the third and so obtain Eqs. 6, 12 where $a = b_1c_2 - b_2c_1$, $b = c_1a_2 - c_2a_1$, and $c = a_1b_2 - a_2b_1$. It will be seen that the solution of the characteristic equations is the same as the solution of Eqs. 3 and hence has the form of two simultaneous equations

$$\begin{aligned} f(x, y, u) &= \alpha \\ g(x, y, u) &= \beta \end{aligned} \tag{4}$$

Each of these solutions taken alone represents a family of surfaces in the (x, y, u) space. The characteristic lines must therefore be the lines of intersection. A line of intersection will depend solely on the particular values of α and β. Once these are selected, we can trace out a definite curve in the (x, y, u) space. To obtain a whole surface containing

† Eqs. 12 in Section 6 of Appendix II.

characteristic lines, we treat α, β as variables, and the equation of the surface is

$$F(\alpha, \beta) = F(f, g) = 0 \tag{5}$$

where F is an arbitrary function.

Therefore, the solution of Eq. 6, 1 may be obtained as follows:

(1) Solve the equations for the characteristics—obtain two simultaneous equations.

(2) Place the solution in the form $f(x, y, u) = \alpha$, and $g(x, y, u) = \beta$.

(3) Then the required solution is $F(f, g) = 0$.

As an example, consider the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \tag{6}$$

The equations for the characteristics are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u} \tag{7}$$

The solutions are

$$\frac{x}{u} = A, \qquad \frac{y}{u} = B \tag{8}$$

Therefore, the solution of Eq. 6 is

$$F\left(\frac{x}{u}, \frac{y}{u}\right) = 0 \tag{9}$$

This result may be represented by a family of cones with vertex at the origin, the characteristic lines being straight lines through the origin.

The analysis of this section shows that the characteristics of a partial differential equation can be of considerable assistance in solving the equation. In the ordinary differential equation of the first order, the characteristics are specific points on the curve representing the solution. They yield much information regarding the character of the differential equation and its solution but they cannot be used to determine a complete solution. In partial differential equations of the first order, however, we have the important fact that the characteristics form the locus of the solution.

8 SIMULTANEOUS FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Let us now consider the following set of partial differential equations of the first order

$$\begin{aligned} a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} + c_1\frac{\partial v}{\partial x} + d_1\frac{\partial v}{\partial y} &= e_1 \\ a_2\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + c_2\frac{\partial v}{\partial x} + d_2\frac{\partial v}{\partial y} &= e_2 \end{aligned} \tag{1}$$

where $a_1, b_1, \cdots, c_2, d_2$ are, in general, functions of x, y, u, v, and where $u = u(x, y)$ and $v = v(x, y)$ are given on the initial curve $\varphi(x, y) =$ constant. The discussion given in Section 6 of this appendix now applies to both $u(\varphi, \psi)$ and $v(\varphi, \psi)$.

The coefficients of $\partial u/\partial\varphi$ and $\partial v/\partial\varphi$ are determined by rewriting Eqs. 1 as follows:

$$\begin{aligned} &u_\varphi(a_1\varphi_x + b_1\varphi_y) + v_\varphi(c_1\varphi_x + d_1\varphi_y) \\ &\qquad + u_\psi(a_1\psi_x + b_1\psi_y) + v_\psi(c_1\psi_x + d_1\psi_y) = e_1 \\ &u_\varphi(a_2\varphi_x + b_2\varphi_y) + v_\varphi)c_2\varphi_x + d_2\varphi_y) \\ &\qquad + u_\psi(a_2\psi_x + b_2\psi_y) + v_\psi(c_2\psi_x + d_2\psi_y) = e_2 \end{aligned} \tag{2}$$

Solving these for u_φ, v_φ, we have

$$Du_\varphi = M_1, \qquad Dv_\varphi = M_2 \tag{3}$$

where

$$D = \begin{vmatrix} a_1\varphi_x + b_1\varphi_y, & c_1\varphi_x + d_1\varphi_y \\ a_2\varphi_x + b_2\varphi_y, & c_2\varphi_x + d_2\varphi_y \end{vmatrix} \tag{4}$$

and

$$M_1 = \begin{vmatrix} e_1 - u_\psi(a_1\psi_x + b_1\psi_y) - v_\psi(c_1\psi_x + d_1\psi_y), & c_1\varphi_x + d_1\varphi_y \\ e_2 - u_\psi(a_2\psi_x + b_2\psi_y) - v_\psi(c_2\psi_x + d_2\psi_y), & c_2\varphi_x + d_2\varphi_y \end{vmatrix} \tag{5}$$

$$M_2 = \begin{vmatrix} a_1\varphi_x + b_1\varphi_y, & e_1 - u_\psi(a_1\psi_x + b_1\psi_y) - v_\psi(c_1\psi_x + d_1\psi_y) \\ a_2\varphi_x + b_2\varphi_y, & e_2 - u_\psi(a_2\psi_x + b_2\psi_y) - v_\psi(c_2\psi_x + d_2\psi_y) \end{vmatrix} \tag{6}$$

Then the singularity equation is $D = 0$, or

$$[ac]\varphi_x^2 + \{[ad] + [bc]\}\varphi_x\varphi_y + [bd]\varphi_y^2 = 0 \tag{7}$$

where $[ac] = a_1c_2 - a_2c_1$, and so on. But on $\varphi(x, y) =$ constant, $d\varphi = \varphi_x\,dx + \varphi_y\,dy$, so that

$$\frac{\varphi_x}{\varphi_y} = -\frac{dy}{dx} \tag{8}$$

and the singularity equation becomes

$$[ac]\left(\frac{dy}{dx}\right)^2 - \{[ad] + [bc]\}\frac{dy}{dx} + [bd] = 0 \tag{9}$$

or

$$\frac{dy}{dx} = \frac{1}{2[ac]}\{([ad] + [bc]) \pm \sqrt{([ad] + [bc])^2 - 4[ac][bd]}\} \tag{10}$$

The regularity condition is $M_1 = M_2 = 0$ with $D = 0$. If $M_2 = 0$, then

$$\begin{aligned} &(a_1\varphi_x + b_1\varphi_y)[e_2 - u_\psi(a_2\psi_x + b_2\psi_y) - v_\psi(c_2\psi_x + d_2\psi_y)] \\ &\qquad (a_2\varphi_x + b_2\varphi_y)[e_1 - u_\psi(a_1\psi_x + b_1\psi_y) - v_\psi(c_1\psi_x + d_1\psi_y)] \end{aligned} \tag{11}$$

Dividing out by φ_y, and writing

$$\frac{\varphi_x}{\varphi_y} = -\frac{dy}{dx} = \zeta \tag{12}$$

we have

$$[ab](\psi_x + \zeta\psi_y)u_\psi + \{([ac]\zeta - [bc])\psi_x + ([ad]\zeta + [db])\psi_y\}v_\psi - [ae]\zeta + [be] = 0 \tag{13}$$

Now it will be noted that

$$\psi_x + \zeta\psi_y = \frac{d\psi}{dx} \tag{14}$$

and, from the singularity equation (9),

$$[ad]\zeta - [bd] = [ac]\zeta^2 - [bc]\zeta \tag{15}$$

Therefore, the coefficient of v_ψ in Eq. 13 is

$$\{([ac]\zeta - [bc])\psi_x + ([ac]\zeta - [bc])\zeta\psi_y\}$$

or

$$([ac]\zeta - [bc])\frac{d\psi}{dx}$$

On $\varphi(x, y) =$ constant, $du = u_\psi\, d\psi$ and $dv = v_\psi\, d\psi$, and the regularity condition becomes

$$[ab]\frac{du}{dx} + ([ac]\zeta - [bc])\frac{dv}{dx} - ([ae]\zeta - [be]) = 0 \tag{16}$$

A consideration of Eqs. 4, 5, and 6 shows that with the help of $D = 0$, the relation $M_1 = 0$ can be converted to the same form as $M_2 = 0$. The equations for the characteristics are Eqs. 9 and 16.

It will be noted that there are two sets of characteristics corresponding to the two roots of Eq. 9 (ζ_+, ζ_-). In order for the quadratic equation (9) to have real roots,

$$([ad] + [bc])^2 - 4[ac][bd] \geq 0 \tag{17}$$

When this condition is valid, the given equations are described as being hyperbolic in character. When the expression in condition 17 is zero, the original equations are parabolic, and, if the condition does not hold, they are elliptic in character.

APPENDIX THREE

Summary of the Derivation of the Basic Equations of Motion of a Gas According to Molecular Theory — Burnett Equations

The procedure for obtaining the equations of motion of a gas from the molecular point of view may be summarized as follows. The function f which specifies the distribution of molecular velocities in the element of volume $d\tau$ is given by the Boltzmann equation

$$D_1(nf_1) = \iint n^2(f_1'f_2' - f_1f_2)\sigma^2\Omega \cos\psi \, d\chi \, d\omega_2 \tag{1}$$

where the subscripts refer to class 1 and class 2 molecules.

A quantity $Q(u_i)$ may be associated with each molecule and the mean value of this quantity in $d\tau$ is

$$\bar{Q} = \int Qf \, d\omega \tag{2}$$

For example, if $Q = u_i$, we obtain the velocity of mass flow in the direction of x,

$$\bar{u} = \int uf \, d\omega \tag{3}$$

The transport of Q by the molecules is given by Maxwell's transfer equation,

$$\frac{\partial}{\partial t}(n\bar{Q}) + \frac{\partial}{\partial x_i}(n\overline{u_iQ}) = \iiint n^2(Q_1' - Q_1)f_1f_2\sigma^2 \, \Omega \cos\psi \, d\chi \, d\omega_1 \, d\omega_2 \tag{4}$$

where the left-hand side of this relation accounts for the transfer due to the flux of molecules in and out of $d\tau$, and the right-hand side specifies the transport of Q arising from intermolecular encounters.

When Q is the mass, momentum, and energy of a molecule, these quantities are conserved during a collision and the integral in Eq. 4 is zero. We obtain the following equations for the molecular transfer of mass, momentum, and energy:

(*a*) $Q = m$ (mass) $\qquad \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial x_i}(n\bar{u}_i) = 0$ (5)

(*b*) $Q = mu_i$ (momentum)

$$n\frac{d\bar{u}_i}{dt} = -\frac{\partial}{\partial x_j}(n\overline{U_iU_j}) \tag{6}$$

(*c*) $Q = \frac{1}{2}mc^2$ (energy)

$$n\frac{d}{dt}(\overline{C^2}) = -2n\left(\overline{U_iU_j}\frac{\partial\bar{u}_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}(n\overline{U_iC^2}) \tag{7}$$

where $i, j = 1, 2, 3$; $\bar{u}_1 = \bar{u}$, $\bar{u}_2 = \bar{v}$, $\bar{u}_3 = \bar{w}$; $U_1 = U$, $U_2 = V$, $U_3 = W$.

When the molecular motion in the element of volume $d\tau$ attains an equilibrium state, the components of velocity of the molecules are distributed throughout the velocity space according to Maxwell's law

$$f_0 = \left(\frac{3}{2\pi\overline{C^2}}\right)^{3/2} e^{-(3C^2/2\overline{C^2})} \tag{8}$$

which is a solution of the Boltzmann equation (1). The mean values in the transfer equations (5, 6, 7) may be calculated according to Eq. 2, using this distribution function, the results being

$$\rho\overline{U_iU_j} = p\,\delta_{ij} \tag{9}$$

$$\tfrac{1}{2}\rho\overline{U_iC^2} = 0 \tag{10}$$

where $\delta_{ij} = 1$ if $i = j$ or $\delta_{ij} = 0$ if $i \neq j$. On substituting these results in Eqs. 5, 6, 7, we obtain the equations of motion for isentropic flow,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho\bar{u}_i) = 0$$

$$\rho\frac{d\bar{u}_i}{dt} = -\frac{\partial p}{\partial x_i} \tag{11}$$

$$p = B\rho^\gamma$$

The same procedure may be followed in deriving the equations for slightly nonisentropic flow. The new distribution function (f_1) can be determined directly by solving the Boltzmann equation (see Ref. 1.1) or indirectly from a consideration of Maxwell's transfer equation (4) for $Q = uv$, u^2, uc^2, etc. (see Chapter 3 and Appendix I). With this distribution function, which represents a small deviation from Maxwell's law (8), the mean values in Eqs. 5, 6, 7 are found to be

$$\rho\overline{U_iU_j} = p\,\delta_{ij} - 2\mu e_{ij} \tag{12}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial\bar{u}_k}{\partial x_k}\delta_{ij} \tag{13}$$

and

$$\rho\overline{U_iC^2} = -\lambda\frac{\partial T}{\partial x_i} \tag{14}$$

Substitution of these expressions into Eqs. 5, 6, 7 results in the equations for slightly nonisentropic flow in which effects due to viscosity and heat conduction now appear (see Eqs. 3.9, 9, 10, 11).

A further (second-order) correction of the Maxwell distribution function introduces additional terms which may be important in the strong shock transition and in rarefied-gas flow. The Hilbert-Enskog-Chapman-Burnett method derives these additional terms by determining a still more general distribution function (f_2) from the Boltzmann equation (1). Burnett has investigated the stresses and heat flux associated with f_2 and has deduced the following results:

$$\rho\overline{U_iU_j} = p\,\delta_{ij} - 2\mu e_{ij} + k_1\frac{\mu^2}{p}e_{ij}\frac{\partial\bar{u}_k}{\partial x_k} + k_2\frac{\mu^2}{p}\left[-\overline{\overline{\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right)}} - \frac{\partial\bar{u}_k}{\partial x_i}\frac{\partial\bar{u}_j}{\partial x_k} - 2\overline{\overline{e_{ik}\frac{\partial\bar{u}_j}{\partial x_k}}}\right] + k_3\frac{\mu}{\rho T}\overline{\overline{\frac{\partial^2 T}{\partial x_i\,\partial x_j}}} + k_4\frac{\mu^2}{p\rho T}\overline{\overline{\frac{\partial p}{\partial x_i}\frac{\partial T}{\partial x_j}}}$$
$$+ k_5\frac{\mu^2}{\rho T^2}\overline{\overline{\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}}} + k_6\frac{\mu^2}{p}\overline{\overline{e_{ik}e_{kj}}} \tag{15}$$

$$\rho\overline{U_iC^2} = -\lambda\frac{\partial T}{\partial x_i} + \theta_1\frac{\mu^2}{\rho T}\frac{\partial\bar{u}_j}{\partial x_j}\frac{\partial T}{\partial x_i} + 2\theta_2\frac{\mu^2}{\rho T}\left[\frac{1}{3}\frac{\partial}{\partial x_i}\left(T\frac{\partial\bar{u}_j}{\partial x_j}\right) + \frac{\partial\bar{u}_j}{\partial x_i}\frac{\partial T}{\partial x_j}\right]$$
$$+ \frac{\mu^2}{\rho}\left[\theta_3\frac{e_{ji}}{p}\frac{\partial p}{\partial x_j} + \theta_4\frac{\partial e_{ji}}{\partial x_j} + \theta_5\frac{e_{ji}}{T}\frac{\partial T}{\partial x_j}\right] \tag{16}$$

where to a first approximation (exact for Maxwell molecules)

$$k_1 = \frac{4}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right),$$
$$k_2 = 2, \quad k_3 = 3, \quad k_4 = 0, \quad k_5 = \frac{3T}{\mu}\frac{d\mu}{dT}, \quad k_6 = 8 \tag{17}$$

and
$$\theta_1 = \frac{15}{4}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \quad \theta_2 = \frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3$$
$$\theta_5 = 3\left(\frac{35}{4} + \frac{T}{\mu}\frac{d\mu}{dT}\right) \tag{18}$$

In Eq. 15 a tensor with a double bar over it has the following form

$$\overline{\overline{A_{ij}}} = \tfrac{1}{2}(A_{ij} + A_{ji}) - \tfrac{1}{3}A_{kk}\,\delta_{ij} \tag{19}$$

Burnett has found that Eqs. 17, 18 are only slightly different if the molecules are rigid elastic spheres.

I N D E X