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A TREATISE ON MAGNETISM AND ELECTRICITY

BY

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"All true and fruitful natural philosophy hath a double scale or ladder, an ascendent and descendent, ascending from experiments to the invention of causes and descending from causes to the invention of new experiments."

BACON, *Advancement of Learning*.

IN TWO VOLUMES

VOL. I

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PREFACE

DURING the preparation of the second volume of my *Treatise on Absolute Measurements*, I became strongly impressed with the desirability of a re-discussion of the whole subject of Electricity and Magnetism from the point of view of action in a medium. Since the publication of that work I have therefore tried to put together a statement which, from the beginning, should regard electric and magnetic forces as existing in a space-pervading medium in which the electric and magnetic energies are stored, and by which they are handed on from one place to another with a finite velocity of propagation.

Of course it is impossible to avoid abstractions. We cannot explain the electric and magnetic inductions in the sense of giving the *rationale* of their production by a mechanical system, but this may not be because we know less of things electric and magnetic than we do of the ordinary dynamical action of material systems, but because the explanation of electrical phenomena must be sought in the solution of those very difficulties which have to be faced when we try to account for the inertia of matter and ordinary forces between bodies, for example, gravitational attraction. The electromagnetic theory of light was established by Maxwell's dynamical discussion of the electromagnetic field, and verified by the experimental and theoretical researches of Hertz; but such dynamical theories seem to be possible only through the remarkable property of the Lagrangian method, by which the behaviour of the system may be qualitatively described and expressed in terms of generalised co-ordinates and their velocities, although we have no means of defining the connections between them and the co-ordinates and velocities of the particles of the system.

The conception of a system of conductors carrying currents as a dynamical system has been rightly regarded as one of the great steps in advance which Maxwell took, and has been of great service in

enabling mechanical explanations, or the construction of mechanical analogies of electrical action to be attempted with something of success, to the great profit of electrical science.

As to the mode of treatment I have adopted, a few words are necessary. The book is not a treatise on the mathematical theory of electricity merely, but is, I hope, to some extent successful in bringing the theory and practice together. Thus while in general assuming some elementary acquaintance on the part of the reader with electrical phenomena and their laws, I have endeavoured first to look at the phenomena as they present themselves, and then to show how they fall into their places in the general scheme of electrical action, and to point out the consequences to which they lead. As stated in the words of Bacon I have placed on the title-page, it is a double process by which natural philosophy advances; we ascend from experiments to causes, and descend from causes to experiments, and it is "most requisite that these two parts be severally considered and handled."

There are two chapters in the book the presence of which requires a word of explanation—the chapter on General Dynamical Theory, and the chapter on Fluid Motion. The former was written to provide in promptly accessible form, with references, all that the student could require for an intelligent apprehension of the special dynamical methods of the book, and so save him from having to disentangle what he wanted from a web of connected matter in a treatise on higher dynamics. I trust, however, that what I have written may lead to a wish to pursue the study of dynamics in the treatises and memoirs of the great masters of the subject. It is to be remembered that much of the chapter, especially the thermokinetic part, will find its application in Vol. II.

The same thing has to be said for the discussion of Fluid Motion. It is rather long, but it is, as far as seemed to me possible without disturbing the order and natural mode of demonstration, an account of those theorems which will be required in the discussion of electrical phenomena considered as manifestations of the motion of the ether. Besides, the general theorems of irrotational and vortex motion are those of the potential of electric and magnetic forces, and of the fields of distributions of currents, and are directly available in the electrical applications without further demonstration. A large range of applications will be found for them in Vol. II., and the chapter will there save much space.

In the preparation of these chapters I have consulted many original papers and authorities; but I must specially acknowledge the great help I have obtained from the papers of Dr. Larmor and the treatise of Professor Horace Lamb.

With regard to the mathematical treatment, I have, after consideration, decided not to use the vector analysis, but to endeavour to insist as far as possible on the physical meaning of the quantities symbolised. Some brevity of expression is no doubt lost by this process; but the discussion is, on the other hand, within the reach of a greater number of readers.

I am, of course, very deeply indebted to the writings of Lord Kelvin, Lord Rayleigh, and Clerk Maxwell; and, in connection with the Electromagnetic Theory of Light, I wish to express my great obligations to the papers of Mr. Oliver Heaviside. No other writer on Maxwell's Theory has done so much to elucidate and render consistent its various parts, or contributed so much to the practical solution of problems of wave propagation. Of the importance of Mr. Heaviside's views on general theory I am strongly convinced, and no adequate presentment of Maxwell's Theory can be obtained in which they are not largely adopted. Thus, Volume II., which will deal more with general discussions, will be affected to a still greater degree by the results of Mr. Heaviside's work.

Mr. Heaviside has strongly urged a recasting of our system of units which would get rid of the 4π factor, which appears in formulæ for the relations of quantities measured in the units now internationally adopted. There can be no doubt that such a reform would, on the whole, materially simplify mathematical expressions in the theory of electricity; but I have not felt justified in adopting it in the present volume. In Volume II. "rational units" will be employed in the more general theoretical discussions, and where the results of these come into comparison with those of quantitative experiments, the expressions will, if necessary, be modified to suit the C.G.S. units as at present defined.

As to notation, I have employed Clarendon type in general for directed quantities, and block type for some quantities such as energy, total magnetic induction through a circuit, and the like, which are merely scalar. In one point I have deviated from ordinary usage: k and μ here denote the electric and magnetic inductivities of a medium, while K and ϖ denote its specific inductive capacity and magnetic permeability, that is, the ratio of the inductivity, electric or magnetic as the case may be, to that of the standard medium.

This procedure is in accordance with the original signification of specific inductive capacity and magnetic permeability, as defined by Faraday and Lord Kelvin respectively; and whether or not the electric magnetic inductivity of the standard medium (ether *par excellence*) is taken as unity, it seems desirable, for the sake of clearness in considerations regarding units, to represent always the inductivity by its appropriate symbol where it properly occurs. Of course when in a particular range of applications of formulæ a certain assumed value of the inductivity is used throughout, as in Chapters III. and IV. of the present volume, where μ is taken as unity for air, the symbol may be conveniently omitted.

A good deal here and there of theoretical matter has been taken from what I have formerly written on electrical subjects, but it has all been thoroughly revised and corrected, as far as lay in my power, in the light of teaching experience and the advance of knowledge.

In the correction of proofs I have been aided in the most devoted manner throughout by Mr. Alfred Hay, B.Sc., of Liverpool College, by Dr. Maclean, of Glasgow University, and in the final revision of the latter part of the book I have had the great advantage of the help and criticism of my friend and former colleague, Mr. G. B. Mathews, F.R.S.

Volume II. will contain among other subjects an account of Electrolysis, of Magnetic Research, of General Theories of the Electromagnetic Field, of Distribution of Electricity on conductors at rest and in motion, and of recent work in the theory and observation of Electromagnetic Waves.

ANDREW GRAY.

UNIVERSITY COLLEGE OF NORTH WALES,
BANGOR, *Feb.* 23, 1898.

CONTENTS

CHAPTER I

PERMANENT MAGNETISM

SECTION I.—*Magnetic Phenomena and Elementary Theoretical Propositions*

Elementary Facts.—Magnetic Field and Lines of Force.—Magnetization by Induction.—Law of Magnetic Attraction and Repulsion.—Hypothesis of Magnetic Matter.—Magnetic Field of Uniformly Magnetized Magnet. Unit Quantity of Magnetism.—Definition of Magnetic Force or Field-Intensity.—Graphical Description of Lines of Force.—Lines of Force of a Very Small Magnet.—Magnetic Potential.—Equipotential Surface. Equipotential Curves.—Actual Magnets, Equivalent Surface Distribution.—Magnetic Moment. Axis of a Magnet.—Couple on Magnet in Magnetic Field.—Potential Energy of a Magnet.—Magnetic Poles.—Actions between Magnets. Simple Cases
Pages 1—28

SECTION II.—*Synopsis of Elementary Theory of Magnetism. Magnetic Potential of any given Distribution*

Potential of a Magnetic Shell.—Lamellar Distribution of Magnetism.—Complex Lamellar Distribution . . . . . *Pages* 29—34

CHAPTER II

MAGNETIC INTENSITY AND MAGNETIC INDUCTION

Influence of Medium occupying the Field.—Total Magnetic Induction over Closed Surfaces in Field.—Solenoidal Condition Fulfilled by Magnetic Induction.—Particular Cases of Magnetization.—Mutual Energy of Two Point-Charges. Apparent Force of Repulsion between them.—Magnetic Inductivity. Superposition of Magnetic Effects.—Field Containing Different Media.—Magnetic Intensity and Induction in Cavity cut in Magnetized Body.—Magnetic Inductivity, Permeability and Susceptibility.—Surface Conditions Fulfilled by Magnetic Intensity.—Magnetic Potential. Equations of Laplace and Poisson.—Method of Vector Potential.—Specification of Vector Potential.—Uniformly Magnetized Ellipsoid.—Uniformly Magnetized Ellipsoid of Revolution. Demagnetizing Forces.—Uniformly Magnetized Sphere.—Field External to Uniformly Magnetized Sphere . . . . . *Pages* 35—57

CHAPTER III

TERRESTRIAL MAGNETISM

Directive Force on Compass Needle. Magnetic Dip. Magnetic Equator.—Terrestrial Magnetic Poles.—True and False Magnetic Poles.—Magnetic Potential at Earth's Surface.—Horizontal Terrestrial Magnetic Force at Earth's Surface.—Determination of Surface Potential from Horizontal Force Relations between Horizontal Components.—Expression of Magnetic Potential at an External Point in Series of Spherical Harmonics.—The Magnetic Moment of the Earth.—Locality of Cause of Terrestrial Magnetic Phenomena.—Are the Terrestrial Magnetic Forces derivable from a Potential? Question of Current Perpendicular to Earth's Surface.—The Magnetic Elements.—Time Changes of the Magnetic Elements and their Causes.—Diurnal Changes of Intensity at Points on same Parallel of Latitude. Representation by Vector-Diagrams.—Diurnal Changes in Different Latitudes. Difference of Amount in Summer and Winter. Results of Observations on "Quiet Days."—Question of Derivation of Diurnal Changes of Intensity from a Potential.—Equipotential Lines of Diurnal Changes.—Magnetic Storms . . . . . *Pages* 58–84

CHAPTER IV

MAGNETISM OF AN IRON SHIP AND COMPENSATION OF THE COMPASS

Ship's Magnetism.—Soft Iron of Ship Represented by Iron Rods.—Expressions for Total Deviation of the Compass. Analysis of Deviations.—The Quadrantal Error.—The Semicircular Error.—Graphical Representation of Deviations. Determination of the Coefficients. The Dygogram.—The Heeling Error.—Compensation of the Compass . . . . . *Pages* 85—100

CHAPTER V

ELEMENTARY PHENOMEMA AND THEORY OF ELECTROSTATICS

SECTION I.—*Experimental Results and Action of Medium*

Elementary Notions.—Location and Transfer of Energy. Current in a Wire.—Electric Induction and Electric Intensity.—"Electrics" and "Non-Electrics." Conductors and Insulators. Electric Attraction and Repulsion.—Forces on Electrified Bodies regarded as due to Action of a Medium.—Faraday's Ice-Pail Experiment.—Division of Electric Field into Two Parts by Conducting Screen. Genesis of Field External to Closed Conductor.—Hypothesis of Incompressible Fluid.—Specification of Electric Induction and Electric Intensity. Energy of Field.—Surface Integral of Electric Induction.—Energy in Case of Æolotropic Medium.—Charged Spherical Conductor in Uniform Dielectric. Energy of System.—Spherical Condenser.—Tubes of Electric Induction. Unit Tubes.—Tension along Lines of Induction in an Electric Field. Traction on Surface of a Conductor.—Constancy of $\int Eds$ along any Path from one Conductor to another. Potential. Difference of Potential. Equipotential Surfaces.—General Problem of Electrostatics . . . . . *Pages* 101—119

SECTION II.—*Electrostatic Capacity and Electric Energy of Charged Conductors*

Electric Condensers or Leydens.—Energy in Terms of Electrostatic Capacity. Energy of a System of Charged Conductors.—Reciprocal Relation between two States of a System of Conductors. Applications.—Coefficients of Potential and Induction and Electrostatic Capacities of a System of Conductors. Reciprocal Theorem.—Properties of the Coefficients of Potential and Charge of a System of Conductors.—Energy of a System of Charged Conductors Expressed as a Quadratic Function of Potentials or Charges.—Reciprocal Relations. Exploration of an Electric Field. —Nature of Charges on Conductors on Different Cases. —Energy Change due to Relative Displacement of Conductors under Different Conditions.—Characteristic Equation of the Potential.—Electric Induction and Potential in Particular Cases.—Approximate Values of Coefficients of Potential and Induction.—Determination of Field within and without a Conductor by Potential Method.—Surface Distributions consistent with Surface Values of Potential. Green's Problem.—Green's Function.—Green's Function for a Spherical Conductor. Induced Distribution on a Spherical Conductor and on an Infinite Plane under the Influence of a Point-Charge. Electric Images.—Mutual Force between Two Charged Spheres. Explanation of an Apparent Anomaly.—Method of Inversion. Geometrical Inversion.—Electrical Inversion. Derivation of Induced Distribution from Equilibrium Distribution and *Vice-versa*.—Inversion of Uniform Spherical Distribution. Problem of Two Parallel Infinite Conducting Planes with Point-Charge between them.—Distribution on Two Spheres in Contact Obtained by Inverting Induced Distribution on Two Parallel Planes.—Case of Two Equal Spheres. Electric Kaleidoscope.

Pages 120—157

HAPTER VI

STEADY FLOW OF ELECTRICITY IN LINEAR CONDUCTORS

Process of Change from the State of Equilibrium to Another. Electric Current.—Steady Currents.—Analogue of an Electric Current.—Ohm's Law. Resistance. Direction of Current. Electromotive Force.—Flow in a Conductor containing an Electromotive Force. Heterogeneous Conductors and Circuits.—Meaning of Resistance. Rate of Production of Heat in Conductors. Joule's Laws.—Arrangements of Electric Generators in Series and in Parallel.—Arrangement of given Battery to produce Maximum Current through given External Resistance. —Arrangement for Maximum Current not that of Greatest Efficiency.—Theory of a Network of Conductors—Two Fundamental Principles : (1) Principle of Continuity.—(2) Sum of Electromotive Forces in any Circuit of Network equal to Sum of Products of Currents round Circuit into Resistances of Conductors. Examples. (1) Two Points connected by Conductors in Parallel. Conductance. Resistance and Conductance of Parallel Conductors. Specific Resistance and Conductivity.—(2) Bridge Arrangement.—Analytical Treatment of a General Network.—Solution of Equations. First Reciprocal Theorem. Conjugate Conductors. Second Reciprocal Theorem.—Cycle Method for a Network. Activity in a Network of Conductors.—Currents fulfilling Ohm's Law give Minimum Dissipation of Energy in Heat.—Elementary Discussion of Network of Conductors.—Reduction of Network to Bridge Arrangement. First Reciprocal Theorem. Conjugate Conductors. Second Reciprocal Theorem.—Effect of Joining a Wire between Two Points of a Network of Conductors.

Pages 158–179

CHAPTER VII

GENERAL DYNAMICAL THEORY

Generalised Co-ordinates and Velocities. Kinetic Energy.—Theory of Action.—Lagrange's Dynamical Equations. — Generalised Momenta. Reciprocal Theorems.—Ignoration of Co-ordinates. Modified Lagrangian Function.—Lagrange's Equations with Gyrostatic Terms.—Application of Lagrange's Equation to Theory of Gyrostatic Pendulum.—Generalised Forces. Applied Forces, and Forces of Constraint. Lagrange's Equations deduced from Cartesian Equations of Motion of Set of Particles.—Work done by Forces of Constraint in Variation of Kinematical Conditions.—Hamilton's Form of the Equations of Motion.—Impulsive Forces. Work done by System of Impulses.—Motional Forces. Dissipative Forces. Dissipation Function.—Controllable and Uncontrollable Co-ordinates. Thermokinetic Principle.—Dynamical Analogues of Thermodynamic Relations.—v. Helmholtz's Thermokinetic Theorems of Cyclic Systems . . . . . *Pages* 180—211

CHAPTER VIII

MOTION OF A FLUID

SECTION I.—*General Kinematical Theory*

Fundamental Assumptions. Lagrangian and Eulerian Methods.—Calculation of Acceleration of Fluid.—Velocity Potential.—Equation of Continuity. Theorem of Divergence.—Rotational and Irrotational Motion. Angular Velocity at Point in a Fluid.—Permanence of Velocity Potential.—Flow and Circulation. Circulation round Curve expressed as Surface Integral of Normal Spin.—Reducible and Irreducible Circuits.—Analysis of Motion at any Point in a Fluid.—Constancy of Flow along any Path moving with the Fluid.

Pages 212—225

SECTION II.—*Dynamical Theory*

Equations of Motion of a Fluid.—Kinetic Energy. Rate of Variation of Energy in Given Space.—Stream-lines.—Motion in Two Dimensions. Conjugate Functions.—General Theorems for Incompressible Fluid in Irrotational Motion. Integral Equation and Continuity. Tubes of Flow.—Periphractic Spaces.—Mean Potential over Spherical Surface . . . . . *Pages* 226—235

SECTION III.—*Green's Theorem*

Proof of Green's Theorem. Surfaces of Discontinuity.—Existence Theorem for Potential Function. Motion of Minimum Kinetic Energy. Uniqueness of Value of Potential Function for Given Space and Given Values at Surface.—Deductions from Green's Theorem. Sources and Sinks.—Kinetic Energy of Irrotational Motion. Expression as a Surface Integral. Cases in which Motion cannot exist.—Multiple Connection of Spaces. Reduction of Connectivity by Diaphragms. Number of Irreconcilable Circuits.—Cyclic Motion in Multiply Connected Space. Cyclic Constants.—Green's Theorem in a Multiply Connected Space.—Kinetic Energy of Cyclic Motion of Fluid.—Uniqueness of Motion in Multiply Connected Space.—Perforated Solids Moving in a Liquid. Ignoration of Co-ordinates . . . *Pages* 235—255

SECTION IV.—*Vortex Motion*

Vortex Lines and Vortex Tubes.—Permanence of a Vortex Filament.—Vortex Surface.—Determination of Velocities for Given Spin and Expansion of Fluid.—Auxiliary Function called Vector Potential. "Curling."—Velocity due to Long Straight Filament. Velocity due to Element of any Filament.—Velocity due to Closed Vortex Filament.—Velocity Potential of System of Vortices.—Electromagnetic Analogy.—Vortex Sheets. Cyclic Irrotational Motion regarded as due to Vortex Sheets on Bounding Surfaces. Electromagnetic Analogy.—Kinetic Energy of System of Vortices. Action of Rectilineal Vortices
Pages 255—275

CHAPTER IX

ELEMENTARY FACTS AND THEORY OF ELECTROMAGNETISM

Magnetic Fields of Currents. Electromagnetic Forces.—Örsted's Experiment.—Ampère's Theorem of Equivalence of a Current and a Magnetic Shell.—Definition of Unit Current. Proof of Ampère's Theorem.—Theorem of Work done in carrying a Unit Pole round Closed Path in Field of Current.—Effect of Medium occupying the Field.—Experiments Confirmatory of Ampère's Theory.—Theorem of Biot and Savart.—Magnetic Potential of a Current.—Theorem of Work done in Field of Current.—Equations of Currents.—Applications of Theorem of Work in Closed Path. Solenoidal Current. Cylindrical Distribution of Currents Parallel to Axis.—Examples of the Theorem of Equivalence of a Circuit and a Magnetic Shell.—Elementary Theory of Tangent Galvanometer.—Energy of Current-Carrying Circuit.—Mutual Energy of Two Current-Carrying Circuits.—Electrokinetic Energy.—Electromagnetic Force on Element of Circuit. Most General Specification of Current.—Mutual Energy of a Current and Magnetic Distribution.—Reaction of a Current-Element on a Magnetic System.—Magnetic Intensity of Straight Current Imbedded in Infinite Conducting Medium.—Magnetic Action of Radial Current in Infinite Medium.—Vector-Potential of Current.—Magnetic Intensity for (1) Case of Straight Vertical Conductor with One End on Surface of Earth, (2) for Plane Current Sheet with Lines of Flow Circles Round Point in it, and Current Inversely as Square of Radius of Circle.—Experimental Illustrations of Electromagnetic Action.—Expression of the Electrokinetic Energy of any System by means of the Vector-Potential.—Self and Mutual Inductance.—Rational Current Elements. Mutual Energy of Two Current Elements.—Forces between Two Current Elements.—Explanation of Actions on Conductors by Stresses in the Field.—Ampère's Experiment.—Weber's Experiments . . . . . *Pages* 276—323

CHAPTER X

INDUCTION OF CURRENTS

SECTION I.—*Experimental Basis of Theory of Current Induction*

Faraday's Experiments.—Felici's Experiments.—Induced Currents are due to Change of Magnetic Induction.—Faraday's Theory of Lines of Force (Induction).—Faraday's Experiments on "Unipolar Induction."—Numerical Estimation of Inductive Electromotive Force.—Electromagnetic Forces due to Induced Currents. Law of Lenz.—Self Induction.—Faraday's Experiments on Self Induction.—Faraday's Theory of Self Induction.—Henry's Experiments.—Application of Principle of Energy . . . . . *Pages* 324—336

SECTION II.—*Dynamical Theory of Current Induction*

Electrodynamics. Electrical Co-ordinates.—Electrokinetic Energy. Current or Electrokinetic Momentum.—Case of Two Mutually Influencing Circuits.—Unit Electromotive Force and Unit Resistance—Volt, Ohm, Ampère, &c.—Maxwell's Dynamical Illustration.—Lord Rayleigh's Dynamical Illustration.—Mutual Action of Two Invariable Circuits.—Single Circuit with Self-Inductance.—Theory of a Network of Conductors.—Primary with Secondary as Derived Circuit.—Oscillatory and Non-Oscillatory Discharge of a Condenser.—Condenser and Inductance-Coil in Series with Simple Harmonic Electromotive Force.—Inductance Counteracted by Capacity. Arrangement for Maximum Current. Impedance, Effective Impedance. Graphical Representation of Results.—Effect of Inductance on Signalling through a Cable or in Telephony.—Difference of Potential between Terminals of Condenser. Resonance.—Maxwell's Dynamical Analogies of Inductance and Capacity. Resonance.—Primary and Secondary. Inductance and Capacity in Primary with Harmonic Electromotive Force.—Conductors in Parallel, and Containing Resistance, Inductance, and Capacity. Equivalent Resistance and Inductance of Parallel Circuits.—Action of Induction Coil, or Inductorium, with Condenser across Break in Primary. Case I. Secondary Closed.—Action of Condenser in Induction Coil. Case II. Secondary Open.—Mechanical Illustration of Action of Induction Coil with Condenser.—Dynamical Theory of Vibrating System, having only Kinetic Energy and Acted on by Dissipative Forces. Effective "Resistance" and "Inductance." Illustration by Mutually Influencing Circuits *Pages* 337—384

CHAPTER XI

GENERAL ELECTROMAGNETIC THEORY

SECTION 1.—*Electromagnetic Theory of Light*

Electromotive Intensity at Element of Moving Conductor. Total Electromotive Force.—Flow of Energy in the Insulating Medium, and Dissipation in a Conductor. Displacement Currents.—Impressed Electric Force. Illustration by Ideal Magneto-electric Machine. Energy is received by System at Seat of Impressed Forces.—Complete Specification of Current in Imperfect Conductor or Imperfect Insulator. Fundamental Circuital Equations of Field.—Propagation of a Plane Electromagnetic Wave.—Plane Wave in Æolotropic Dielectric, Principal Wave-Velocities.—Question of Relation of Electrical Vibrations to Elastic Vibrations of Ether.—Determination of Velocity of Propagation.—Ratio of Units.—Electromagnetic Radiation. Hertz's Solution of Maxwell's Equations. Vibrating Electric Doublets.—Electric and Magnetic Intensities in Field of Doublet.—Direction of Vibration. "Longitudinal Light." Rate of Propagation. Different Directions and at Different Distances.—Electromagnetic Theory of the Blue Sky.—Propagation of Electrical Potential.—Graphical Representation of Field of Vibrator.—Experimental Verification of Theory of Vibrator.—Approximate Theory of Hertzian Receiver.—Experiments with Different Positions of the Receiver.—Exploration of Electric Field by Receiver.—Period of Vibrator. Determination of Wave Length and Velocity of Propagation in Rate of Radiation of Energy from Vibrator.—Reflection of Waves in Air from Metallic Surfaces. Standing Waves.—Multiple Resonance.—Effect of Size of Reflecting Surface.—Reflection of Electric Waves by Mirrors. Refraction by Prisms.—Polarization of Electromagnetic Beam. Relation of Plane of Polarization to Direction of Electric Vibration.—Transparency of Ordinarily Opaque Substances to Electromagnetic Waves . *Pages* 385—421

SECTION II.—*Flow of Energy in the Electromagnetic Field*

Motion of Energy across Bounding Surface of a Closed Space. Poynting's Theorem.—*Examples:* 1. Straight Wire with Steady Current. Energy Stream-Lines.—2. Discharge of a Condenser.—3. Radiation of Energy from a Hertzian Vibrator.—Distribution of Currents in Cross Section of Cylindrical Conductor. *Pages* 421—430

SECTION III.—*Moving Electric Charges*

Convection Currents.—Radiation in a Magnetic Field. The Zeemann Effect. Theory.—Experimental Verification . . . . . *Pages* 430—439

CHAPTER XII

THE VOLTAIC CELL

Volta's Experiments on Contact Electricity.—Interpretation of Volta's Results.—Objections to Volta's Method. His further Experiments.—Result for Chain containing Liquid.—Experiments of Kohlrausch.—Hankel's Experiments. Lord Kelvin's Experiments.—Lord Kelvin's Copper and Zinc Electric Machine.—Lord Kelvin's Induction Electric Machine, Founded on Contact Electrical Action.—Experiments of Ayrton and Perry. Clifton's Experiments.—Comparison of Results.—Pellat's Experiments.—Experiments on Metals Immersed in Different Gases.—Later Views on Contact Electricity. Differences of Potential Inferred from Flow of Energy.—Energy value of Electromotive Force of a Cell.—Explanation of Volta Effects as Air Metal—Metal-Air Differences of Potential. Thermoelectric Measure of Difference of Potential.—Energy Criterion of Existence of an Electromotive Force.—Summary of Results of Later Theory.—Appendix to Chapter XII . . . . . *Pages* 440—457

CHAPTER XIII

THERMOELECTRICITY

Elementary Phenomena.—Thermoelectric Inversion.—Thermoelectric Power.—Total Electromotive Force.—Peltier Effect.—Source of Energy in Thermoelectric Circuit.—Peltier Effect is Zero at Neutral Temperature.—Thomson Effect—Electric Convection of Heat.—Thermodynamic Theory.—Thermoelectric Diagram.—Appendix to Chapter XIII . . . . . *Pages* 458—473

INDEX . . . . . *Pages* 475—479

LIST OF PLATES

Pl. I.

THE EARTH'S MAGNETISM AS SHOWN BY THE DISTRIBUTION OF LINES OF EQUAL TOTAL FORCE IN ABSOLUTE MEASURE (British units, foot-grain-second-system); THE POSITION OF THE MAGNETIC POLES AND EQUATOR (Approximately for 1875).

Pl. II.

LINES OF EQUAL HORIZONTAL FORCE (Horizontal Force at Greenwich = 1).

From the *Admiralty Manual of Deviations of the Compass*, 1893 Edition.

Pl. III.

LINES OF EQUAL VERTICAL FORCE, 1895 (Horizontal Force at Greenwich = 1).

From the *Admiralty Manual of Deviations of the Compass*, 1893 Edition.

Pl. IV.

LINES OF EQUAL VARIATION (1895).

From the *Admiralty Manual of Deviations of the Compass*, 1893 Edition.

Pl. V.

SECULAR CURVES FOR DIFFERENT LATITUDES (BAUER).

From *Nature*, Dec. 23, 1897.

Pl. VI.

DIURNAL CURVES (BAUER) FOR EPOCHS 1780, 1885.

From *Nature*, Dec. 23, 1897.

Pl. VII.

VECTOR DIAGRAMS FOR DIFFERENT LATITUDES.

Pl. VIII.

CURVES OF EQUAL POTENTIAL FOR DIURNAL CHANGES.

CORRIGENDA

Page 60, last line, *for* Fig. 32 *read* Fig. 31

„ 61, line 10 from foot, *for* Fig. 33 *read* Fig. 32

„ 84, second last line, *for* annular *read* annual

„ 192, second footnote, *for* Art. *read* p. 60

„ 197, line 8 from foot, *for* $\frac{}{C'}\Big|$ *read* $\frac{\kappa}{C'}$

„ 328, line 4 from top, *for* **Force Induction** *read* **Force (Induction)**

„ 332, in Fig. 116 reverse the direction of motion of the slider

„ 359, equation (46), *for* $e^{-\frac{rr_1}{L(r+r_2)}t}$ *read* $e^{-\frac{\Sigma rr_1}{L(r+r_2)}t}$

„ „ „ (48), *for* $e^{-\frac{\Sigma rr_1}{L(r+r_2)}\tau}$ *read* $e^{-\frac{\Sigma rr_1}{L(r+r_2)}\tau}$

„ 360, line 2 from top, *for* T *read* L

„ 367, line 13 from top, *for* nL *read* γ

„ 374, line 14 from foot, *for* (75) *read* (74)

„ 380, line 18 from top, *for* one *read* the

„ 385, line 8 of Art. 498, *for* to be that *read* to be opposite to that

„ „ line 9 of Art. 498, *for* B *read* **B**

„ 404, lines 4, 5, and 9 from top, *for* where *read* when

„ 439, in line 3 of Art. 569, *after* moving charges are *insert the words* all of one sign, and have effective inertias proportional to their charges, or are

„ 454, line 20 from top, *for* conductor *read* element, per unit current,

„ 468, footnote, *for* *Ann. de Chin. read Ann. de Chim.*

MAGNETISM AND ELECTRICITY

CHAPTER I

PERMANENT MAGNETISM

SECTION I.—*Magnetic Phenomena and Elementary Theoretical Propositions*

Elementary Facts

1. The natural history of permanent magnetism began with the study of the properties of the lodestone, the so-called magnetic iron-ore. This is said to have been first found in Magnesia in Asia Minor, and hence to have given rise to the terms *magnet* and *magnetism*. It is occasionally found magnetic in the natural state, and probably, as we shall see presently, acquired its magnetism in consequence of terrestrial magnetic action.

The properties first observed were the power of the lodestone to attract pieces of iron, and to cause them to adhere to it when brought into contact with its surface, and the fact that when so adhering these pieces of iron had themselves the power of attracting other pieces. It was also noticed that under certain circumstances a piece of lodestone was repelled by another piece; and that a natural magnet left free, as when placed on a piece of wood or cork floating in water, or hung by a fibre without serious twist, took up always a certain definite position. This last property was made use of for the mariner's compass, which is said to have been known in Europe as early as the twelfth century, and to the Chinese at a date much more remote.

2. The observation that pieces of iron in contact with or near a lodestone had temporarily the power of attracting iron was of the greatest importance, as it led to the formation of artificial magnets. It was soon found out that pieces of steel brought into contact with or stroked by a

piece of lodestone retained the magnetic virtue conferred on them, and had the same power of acting on pieces of iron or other pieces of steel as was possessed by the lodestone itself. As pieces of steel could be forged conveniently, magnets of different shapes were made, and their behaviour studied.

In this way it was soon found to be possible to obtain much more powerful magnets than could be obtained by means of lodestones, which are therefore now regarded as of merely antiquarian interest.

3. It was early found that bar-shaped pieces of steel magnetized and used as compass needles, or otherwise placed so as to show freely their directional tendency, generally set themselves in a certain direction making a more or less acute angle with the astronomical north and south vertical plane, according to the place at which the experiment was made, and further that always the same end of the bar pointed north. It does not seem, however, to have been observed until about the time of Queen Elizabeth that such bars, suspended freely by their centre of gravity and *then* magnetized, not only set themselves in this north and south direction, but dipped their north pointing ends downwards at a certain angle to the horizontal as shown in Fig 1. This was noticed by Robert Norman, a maker of compasses, who found it necessary when a bar was thus suspended to weight the south-pointing end with a counterpoise, in order to maintain the bar in a horizontal position. This horizontality can easily be obtained without loading the bar by placing the suspension on one side of the centre of gravity as shown in Fig. 2. Thus any tendency of the bar to dip is counteracted by the couple brought into play by the weight of the bar. Sometimes also compass needles are, for the same reason, suspended by means of a cap resting on a point, so arranged that the centre of gravity of the bar is below the point of suspension (Fig. 3). This is in general the manner in which a compass card is kept horizontal.

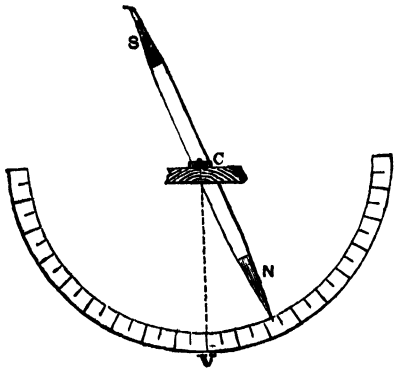


FIG. 1.

A magnet is also frequently suspended in a horizontal position by means of a double sling at the lower end of a suspension thread as shown in Fig. 4. This sling is very easily made by doubling the lower part of the thread on itself, then doubling again the doubled part, and securing the quadruple thread thus formed, by knotting it round itself.

4. Certain parts of both natural and artificial magnets are found

as a rule to exhibit the "magnetic virtue," as it used to be called, more intensely than others. These parts are sometimes called the "poles" of the magnet. For example if a bar of steel be stroked from end to end always in the same direction by a "pole" of some other magnet, it will be found to have been made into a magnet which possesses two "poles," or regions of relatively intense magnetic action, one near each end. Later we shall discuss processes of making magnets; at present we may suppose bars made and magnetized by the simple method just described.

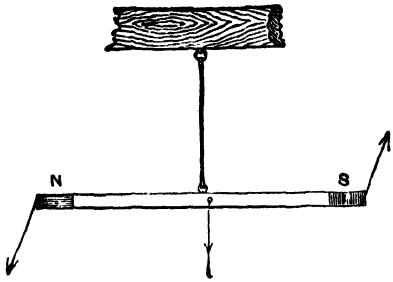


FIG 2.

5. Suppose then a bar magnet is suspended as shown in Fig. 4. It is found to rest in a nearly north and south direction, provided the thread be sufficiently free from torsion. The end which points north is marked say with an N, the other end with an S. (Or, according to another practice, the north-pointing end is painted red, the other end blue.) Another bar magnet is now made and suspended in the same way, and the ends marked as before. One of the magnets is dismounted and brought near to the suspended magnet in the different ways (A, B, C), shown in Fig. 5. Motion of the ends of the suspended magnet takes place in the direction shown by the arrows.

This result shows that a north-pointing end of one repels the north-pointing end of the other, and attracts the south-pointing end; and

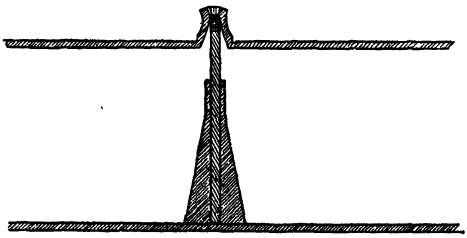


FIG. 3.

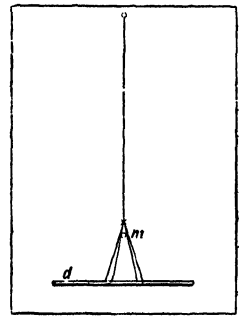


FIG. 4.

similarly the south-pointing end of one repels the south-pointing end of the other, and attracts the north-pointing end. It is usual to call the north-pointing end the north end or "pole" of the magnet, and

the south-pointing end the south end, or "S pole." This statement is expressed shortly by the rule "Like ends or poles of magnets repel one another, and unlike ends or poles attract one another."

6. The experiments also illustrate the fact that as a rule each magnet has at least two regions or poles at which magnetic action is most intense, which in a bar magnet are near its ends, and that one of these regions is characterised by a tendency to move towards the south, the other towards the north, when the magnet is placed in ordinary circumstances at a distance from other magnets.

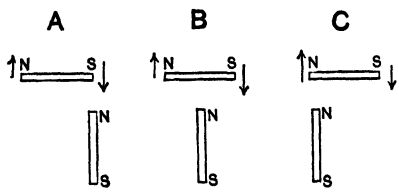


FIG. 5.

It is to be observed that the forces acting between the magnets are mutual. For example the two magnets, if suspended horizontally at not too great a distance apart, would be found to come finally to rest in stable equilibrium with their unlike ends turned the same way, and their suspension fibres deflected from the vertical towards one another.

The term "poles" has been used in this section in a somewhat vague sense to designate certain regions or parts of a magnet. An attempt is frequently made to use the term more definitely to designate certain points in the magnet. We shall deal with this usage of the term later. (See Art. 40.)

Magnetic Field and Lines of Force

7. Much of our knowledge of the elementary facts of magnetism is due to Dr. Gilbert, Physician in Ordinary to Queen Elizabeth. With him the modern science of magnetism may be said to begin. He first explained the directional tendency of a freely suspended magnet by supposing the earth to be a great magnet, and illustrated his theory by means of a *terrella* or "little earth" (Fig. 6) made from lodestone, which acted on a small needle, about the size of a grain of barley, placed in any desired position near it. Dr. Gilbert's researches are contained in a Latin treatise *De Magnete*,[1] published in 1600, which abounds in acute observations and valuable results of experiment.

8. One of Dr Gilbert's most important investigations was that which he made of the *orbis virtutis*, or, as we now call it, the *field of force* surrounding a magnet. His method was beautifully simple. Iron filings were dusted over a sheet of paper or cardboard placed above the magnet, then the paper was tapped so as to raise the filings from the

[1] An English translation has recently been made by Mr. Mottelay, and is published by Wiley and Son, New York. A translation with Notes is also in preparation by the Gilbert Club.

paper for an instant, and enable them to take up freely the positions in which the magnetic forces tended to place them. In Fig. 7 is given a copy of a photograph of curves made by Faraday.[1] The filings were first strewed on a sheet of dry gummed paper placed over the magnet, then fixed by softening the gum temporarily by means of a jet of steam. The nature of the magnet or magnets used in each case is described in the notes attached to the diagrams. It will be seen that the filings arrange themselves in distinct lines running round from one end of the magnet to the other. These curves indicate what Faraday called the lines of force of the magnetic field. We shall consider their exact meaning and their geometrical form in some simple cases presently.

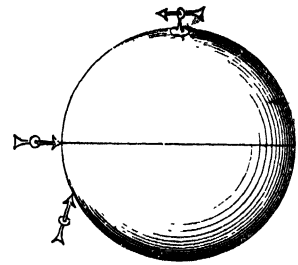


FIG. 6.

Magnetization by Induction

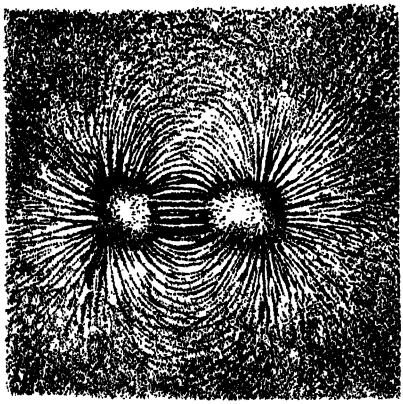
9. That the small pieces of iron attracted by a magnet become magnetized can be made clear in a number of ways. They have the power while in the field of the magnet of attracting other pieces, which also become magnets, and so on. Thus to one end of a bar magnet of moderate size it is possible to hang a succession of small nails, each clinging to its neighbour, and so on back to the bar. Also such pieces of iron have the power of deflecting suspended magnets, as may be proved by prolonging a steel magnet by a bar of iron, and presenting it to a test-magnet suspended in the earth's field of force.

If the bar is removed from a series of small pieces of iron thus clinging to one another, their magnetization disappears in great measure.

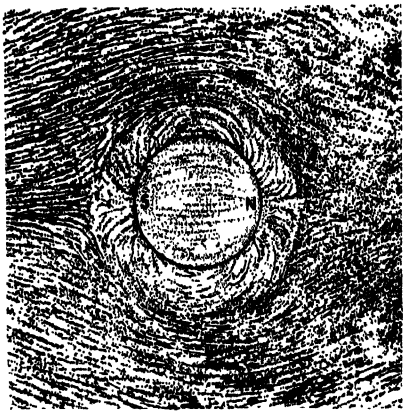
Again pieces of iron become magnetized while resting in various positions on the earth's surface. For example bars of iron standing in a corner of a smithy, and the iron of a ship's hull while the ship is on the stocks in a ship-building yard, are magnetized by the action of the earth.

10. Inductive magnetization thus produced in the earth's field, and indeed inductive magnetization in general, is facilitated by jarring the iron by striking it with a mallet, or otherwise. A common form of lecture-room experiment consists in taking an ordinary kitchen poker, holding it in the direction of the dipping needle, and striking it on the upper end with a wooden mallet. The iron becomes magnetized and generally retains to a considerable degree its magnetization; which, however, can be nearly all removed by placing the poker at right angles to the line of dip and jarring it in that position repeatedly with the

[1] These curves are the property of Lord Kelvin, P.R.S., who has kindly permitted their reproduction here.

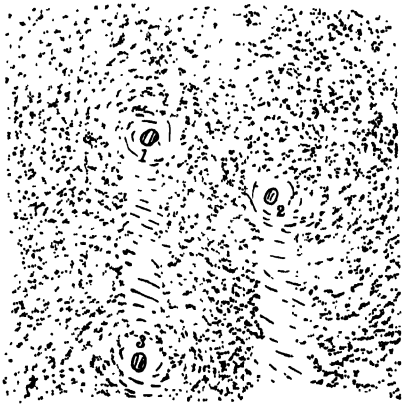


Lines of force of an ordinary bar magnet.



Lines of force of a spherical or disk-magnet (with "poles" at the extremities of a diameter), placed in a uniform field the lines of force of which are oppositely directed to those of the disk.

Lines of magnetic force for a system of three spherical or disk-magnets with like "poles" turned inwards towards the centre of the system.



Lines of magnetic force round three straight wires carrying currents, at right angles to the plane of the paper, and passing through it at the points 1, 2, 3. The currents at 1 and 2 are in the same direction, the current at 3 is in the opposite direction.

FIG. 7.

mallet. But the bar after having been magnetized in one direction can have its magnetization at once reversed by simply turning the bar end for end from its first position, and striking it again a few times with the mallet.

11. It is found in this experiment that the lower end of the poker repels the north-pointing end of a freely suspended magnet, and is therefore itself a north-pointing pole of the magnet which the poker has now become. Similarly the upper end is found to be south-pointing. The general rule of inductive magnetization is that the piece of iron, if placed in the field of force in the direction which a freely suspended magnet would assume at the place, is magnetized with the same polarity as the magnet.

Again in a common process of magnetization by stroking a steel bar with one end of a magnet, as shown in Fig. 8, always beginning and ending each stroke at the same ends of the magnet, the ends of the bar take the polarity shown, that is the end of the bar which is finally in contact with the stroking bar at the end of each stroke has the opposite polarity to that of the stroking end.

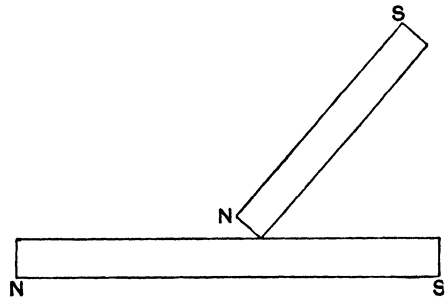


FIG. 8.

12. The small pieces of iron in the experiment with filings afford another example of inductive magnetization. Each little piece of iron becomes in the field a small magnet, and when raised from the paper momentarily by the tapping, takes the direction which a small permanent magnet would take there, except of course in so far as it may be disturbed by other magnetized particles. Thus the lines of force are given by little chains of small magnets, each successive pair in a chain having opposite poles adjoining.

13. The most powerful mode which has yet been devised of inductively magnetizing a bar of iron or steel is by surrounding the bar with a helix of wire, and causing a strong current of electricity to flow through the helix. Pieces of iron such as nails can be made to adhere to the extremities of the bar, and can be piled on one another, in positions in which they could not remain if the bar were unmagnetized. On the withdrawal of the current the demagnetization of the iron core of the helix is shown by the pieces of iron falling off.

In the older treatises on magnetism will be found elaborate accounts of various methods of magnetizing bars of steel by what was called "touch." Only one of these, that called "single touch," is described in Section 10. Since the invention of electro-magnets these methods have all become obsolete.

14. A greater or less amount of magnetism is always retained by a

piece of iron after it has been inductively magnetized. The amount depends, other things being the same, on the nature of the iron. Iron which is mechanically very soft retains in general only a small amount of magnetization, mechanically hard iron a considerably greater amount. Thus iron in which residual magnetism is small is commonly called *soft* iron. Mechanical softness and magnetic softness do not, however, uniformly coincide. Steel has great retentiveness for magnetism, and a specimen, after being magnetized by any process, say that here described, cannot be demagnetized or have its magnetization reversed except by being placed in a sufficiently intense field oppositely directed (relatively to the specimen) to that by which it was magnetized. In their power of taking and retaining magnetization different kinds of steel differ to a degree which can hardly be accounted for by differences in composition, but apparently depending on molecular constitution.

The property of retaining residual magnetism has been called coercive force. We shall find later in the chapter on Magnetic Measurements a numerical reckoning for coercive force, and see how to determine its amount.

Law of Magnetic Attraction and Repulsion

15. The natural philosophy stage of magnetic science may be said to have begun with the researches of Coulomb towards the end of last century. By means of his torsion balance he tested approximately the repulsion between two poles of the same name belonging to two different magnets. The balance as arranged for magnetic experiments is shown in Fig. 9. A wide glass case or box is prolonged upwards by a tube fixed over the centre of the cover, and carrying at its top a graduated torsion head for sustaining a fine silver wire. To the lower end of this wire is attached a stirrup in which a bar magnet can be placed horizontally. Another magnet can be inserted vertically through the cover so that one end shall be on a level with the suspended bar, and opposite one of its ends.

16. It was found by Coulomb, by careful experiments, that to turn the upper end of the silver wire through any angle relatively to the lower end required the application of a couple proportional to the angle. Thus it was possible to determine a couple turning the suspended magnet. This couple was produced by the action of the fixed magnet, which was introduced through the cover so that like poles of the two magnets should be opposed, and should have their poles on the circle in which that of the suspended magnet turned. First of all the suspended magnet only was placed in position, and the balance arranged so that the magnet rested in the earth's field of force without any disturbance from the torsion of the wire. The torsion head was then turned so as to deflect the magnet from that position through a small angle measured on the scale surrounding the case. It was found in one experiment that to turn the magnet 1° from its equilibrium position in the earth's field required a

twist of 35° between the two ends of the wire. The fixed magnet was then placed in position and gave a deflection of 24°, which, taking the couple required to produce by magnetic repulsion alone a deflection of 24° as twenty-four times the couple required for a deflection of 1°, corresponded to a twist of $24 \times 35°$. But the twisting was resisted by the 24° twist given to the fibre by the deflection, so that the total couple on the bar exerted by the magnet may be reckoned as that corresponding to 864° of torsion.

The torsion head was now turned so as to bring down the deflection to 12°, when it was found that the head had to be turned just eight

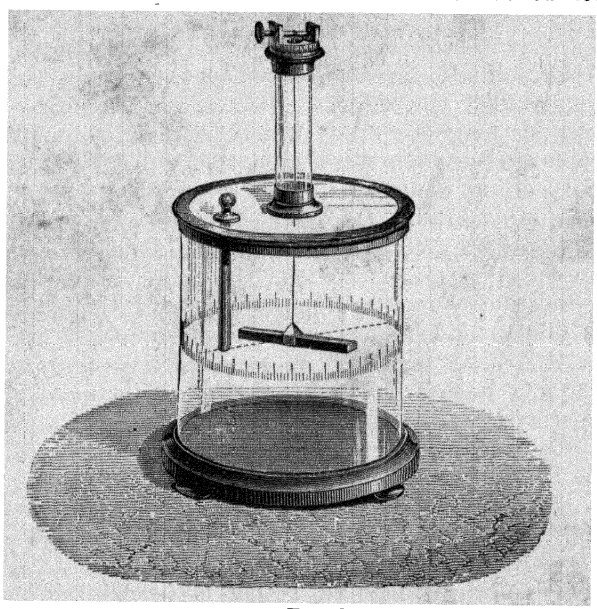


FIG. 9.

times round, that is through 2880°. Thus the couple on the wire was that due to $2880 + 12 \times 35 + 12$, or 3312, degrees of torsion. This is a little less than four times the former couple.

From such experiments Coulomb obtained the result that two poles of different magnets repel one another with a mutual force inversely proportional to the square of the distance between them. By finding the couples necessary to keep the magnet deflected through different angles when unlike poles were opposed, the magnetic attraction between the poles could also be measured.

17. But while such experiments gave a rough approximation to a law of inverse squares between two ordinary magnetic poles, it must be understood that for many reasons they could yield no exact result. The poles of the magnets could not be regarded as points, and hence no

inverse square law could possibly hold for them. Moreover the remote poles of the suspended and fixed magnets could not have been without effect; in fact the two bar magnets as wholes acted on one another, although the principal action was between the adjacent ends. Again, no account was taken of the earth's field in determining the forces acting on the magnet, or of the mutual inductive action of the two magnets in diminishing their magnetization.

A vibrational method used by Coulomb for determining the action of a magnet on a small needle placed at different distances from it will be described in the chapter on Magnetic Measurements.

Hypothesis of Magnetic Matter

18. A theory of magnetism was put forward at an early date in the later developments of the subject. It was supposed that there existed two imponderable magnetic fluids which pervaded the apparently active regions of a magnet, which were such that a portion of either had the property of repelling a portion of the same fluid, and of attracting a portion of the other kind. A hypothesis of imaginary magnetic matter has been applied with great advantage by Lord Kelvin to the discussion of the phenomena of magnetization. It gives a language for the expression of the theoretical results which have been arrived at, and enables an analogy to the probable constitution of a magnet to be pictured to the mind and usefully employed, without any danger of misunderstanding or pre-judgment of the actual facts of the case, if the investigator is careful not to be led by the mere words he uses to assign a reality to this imaginary matter which it does not possess.

19. It is of great importance also to remember that this hypothesis is only a short way of expressing certain facts of magnetism as they directly appear to us, and must not be permitted to lead to the conclusion that magnetic action is really action at a distance. There is nothing more certain than that magnetic action is propagated by means of a medium, and that that which appears to us the action of one magnet on another is really the action on each of the medium in contact with them. In describing magnetic phenomena it is convenient, however, to use language more or less descriptive of what is directly perceived. When we come to a discussion of theory we shall endeavour to keep action at a distance, and its methods and expressions, in their proper place as short cuts to results, or descriptions of the outcome of the more complete theory.

A magnetized body, we shall see, probably consists of some kind or arrangement of molecules in motion, producing also in the surrounding medium a motion which is the propagating cause of the apparent action at a distance of one magnet on another.

20. We shall make use at present of this hypothesis of imaginary magnetic matter, which we shall call for shortness simply magnetism, for the deduction of some results, useful in what follows. In the first

place let us suppose a long thin bar, whether straight or curved, to be made up of slices which have magnetism distributed on their ends as indicated in Fig. 10 by the shading. Then if we suppose each slice exceedingly thin, and to have equal quantities of the two kinds of magnetism at its ends as shown, the opposite magnetisms in contact at any junction of two adjoining slices will annul the action of one another on external magnetism, and there will be left only the unbalanced magnetism at their ends to produce external effect.

This distribution of magnetic matter corresponds to the molecular arrangement which has place in an actual magnet. Each molecule has a polarity directed in the same way in all the particles, or nearly so, corresponding to the two kinds of magnetism on the ends. If, as may possibly be the case, each molecule be in rotation round its axis, the polarity consists in the two aspects of the rotation, according as the particle is regarded from one end or the other. Whatever the nature of the molecule may be, the two kinds of magnetic matter must if used be taken as symbolising two aspects of one thing, neither of which can be regarded as existing alone, any more than either aspect of the rotation of a fly-wheel can be regarded as existing apart from the other.

That one kind of magnetism cannot exist without the other is shown by the experimental fact stated above, that if a bar magnet is broken each portion is a magnet having in general the same polarity as the original magnet. This can be easily verified by tempering glass-hard a straightened piece of clock-spring, magnetizing it, then breaking it into pieces, and testing them by means of a small horizontally suspended needle.

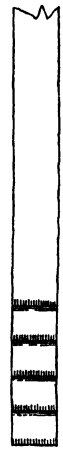


FIG. 10.

21. The magnetic distribution described above and illustrated in Fig. 10 is that of a uniformly magnetized magnet, and its poles are thus exactly at its ends. They may in this case, if the bar be thin, be taken as points coincident with the ends of the bar. Such a magnet may be approximated to very closely by carefully magnetizing a long thin knitting needle, by stroking it in the manner above described with another magnet.

Magnetic Field of Uniformly Magnetized Magnet. Unit Quantity of Magnetism

22. We propose now to investigate the lines of force of such a magnet on the supposition that the quantities of magnetism at the ends of the bar are equal and opposite, and are there concentrated at points. Also we shall suppose that two like quantities repel one another, and two unlike quantities attract one another, in each case with a force inversely as the square of the distance between the points at which the quantities are supposed concentrated, and directly as the product of the two quan-

tities. Any multiple of a quantity of magnetism may be imagined as produced by placing that number of equal thin bars together so that their like poles coincide. Thus we assume that the force, F, between two quantities m, m', of magnetism at points at a distance r apart in a medium of uniform constitution is given by the equation

$$F = \frac{mm'}{\mu r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ is a constant depending upon the medium. According to this specification F is taken positive if m and m' have the same sign, and negative if they have opposite signs. In the first case, the force is a repulsion, in the latter, an attraction, for we take one kind of magnetism, namely, that of the north-pointing end of a magnet, as positive, the other kind as negative. Later we shall obtain this result from a theory of displacement or motion in a medium filling the field.

23. Equation (1) defines unit quantity of magnetism as that which concentrated at a point in a medium for which μ is unity repels with unit force an equal quantity also concentrated at a point at unit distance from the former. If the force is a force of one dyne, that is to say the force which acting on a gramme of matter for one second gives it a velocity of one centimetre per second, and the distance be one centimetre, the quantity of magnetism is one unit in the centimetre-gramme-second (C.G.S.) system of units.

24. The value of μ is very generally taken as unity for air. If we wish to be quite definite we may take air at standard atmospheric pressure and temperature 0° C., as that substance for which μ is unity; but the alteration of the magnetic properties of air produced by any ordinary change of pressure or temperature is imperceptible. The value of μ however varies from medium to medium on account of some physical property which is different in different media; and when we know more of the inner mechanism of magnetic bodies and media generally we shall no doubt find some natural measure of μ depending on that property. At present all we can do is to find the ratio of the values of μ for any two different media.

It is usual to call μ the magnetic inductive capacity, or magnetic inductivity, of the medium, or its magnetic permeability. We shall however use the term permeability to designate the ratio of the value of μ for a given medium to the value for some standard medium. In the remainder of the present chapter we shall suppose $\mu = 1$.

Definition of Magnetic Force or Field-Intensity

25. By the force at a point in the field of a magnet is meant the force which a unit quantity of positive magnetism would experience if placed at that point. This is also called the intensity of the field at the point. We shall denote it in what follows by the symbol **H**. It is

clearly a directed quantity of such dimensions that when multiplied by a quantity m of magnetism it gives a product $m\mathbf{H}$ which is a force in the ordinary dynamical sense. (See the Chapter on the Units and Dimensions of Physical Quantities.)

If the unit of force here chosen is again the dyne, and the unit quantity of magnetism is 1 C.G.S. unit, $\mathbf{H}$ is measured in C.G.S. units of field intensity.

A uniform field is one for which $\mathbf{H}$ has the same value at every point. The reader will easily see after the discussion of tubes of magnetic force that if $\mathbf{H}$ has the same numerical value at every point of any region, it has the same direction at every point, and conversely.

26. A line of force is a curve so drawn in a magnetic field that the direction of the tangent at any point is the direction of the magnetic force, or field-intensity, at that point.

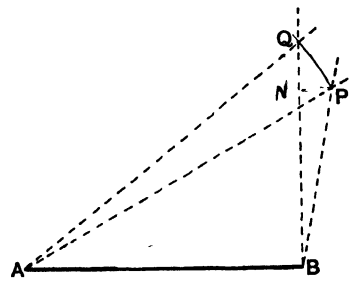


FIG 11

Consider the field of a uniformly magnetized filament. As we have seen the filament may be replaced by a quantity of magnetism $+ m$ at A, Fig 11, and another $- m$ at B. Let PQ be two points on a line of force at a distance ds apart, and let $AP = r$, $BP = r'$, $\angle PAB = \theta$, $\angle PBA = \theta'$. Then $\angle QAB = \theta + d\theta$, $\angle QBA = \theta' + d\theta'$. Also $\sin BQP = -r'd\theta'/ds$, so that if QN be a normal drawn to QP at Q, $\cos BQN = -r'd\theta'/ds$. Similarly, $\cos AQN = rd\theta/ds$. The forces on a positive unit of magnetism at Q are, neglecting small quantities, and taking μ as unity everywhere, a repulsion m/r^2 along AQ, and an attraction m/r'^2 along QB. Thus since there can be no component normal to a line of force, we get resolving along the normal QN

$$\frac{m}{r^2} r \frac{d\theta}{ds} + \frac{m}{r'^2} r' \frac{d\theta'}{ds} = 0$$

or

$$\frac{1}{r} \frac{d\theta}{ds} + \frac{1}{r'} \frac{d\theta'}{ds} = 0$$

Multiplied by the perpendicular from Q on AB this is

$$\sin\theta d\theta + \sin\theta' d\theta' = 0.$$

Hence integrating we obtain

$$\cos\theta + \cos\theta' = c \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where c is a parameter, constant for each line of force, but changing in value from line to line.

Graphical Description of Lines of Force

27. To describe these lines graphically we may proceed as follows. Draw a circle (Fig. 12) having unit diameter AC along AB, and draw any chord DA. Produce if necessary DA to E so that DE is made equal to the parameter c. From A with AE as radius describe an arc cutting the circle in F, then the angles DAC, CAF are angles θ, θ' for which

$$\cos\theta + \cos\theta' = c.$$

Thus it is only necessary to draw a line through B parallel to FA meeting AD produced if necessary in P. P is a point on the curve.

If the point E falls between A and D, the distance AF Fig. 13 is made equal to AE. A line through B drawn parallel to AF will meet

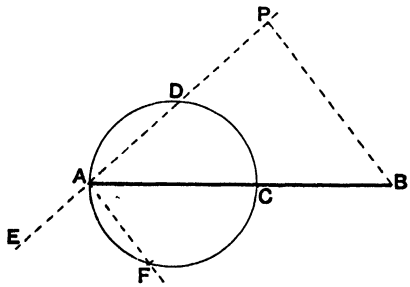


FIG. 12.

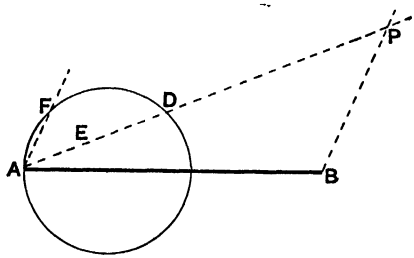


FIG 13.

AD in a point on the curve. The angle θ' is now obtuse, and is the supplement of FAB, so that $\cos\theta'$ is negative.

In this way different points on the curve can be found. In the neighbourhood of A or B the curve must be drawn from its inclination to AB. This is given by the equations $\cos\theta = c - 1$, $\cos\theta' = c - 1$. Fig. 14 shows curves for different values of c drawn for the magnetic filament and also illustrates the following other method of drawing the curves.[1] Describe a circle on AB as diameter, and lay off a distance AM such that $AM = c$. AB. Then from A draw any line to cut the circle in Q and lay off $Aq = AQ$ along AB. From B as centre with radius Mq describe a circle cutting the former circle in R. The point in which AQ and BR intersect is a point on the curve. The proof of the construction is left to the reader. A comparison of the form of these curves with the curves given by iron filings affords a rough confirmation of the law of magnetic force of which the curves of the Figure are a consequence.

[1] This cut and the construction here given are taken from *Constructive Geometry of Plane Curves*, by T. H. Eagles, M.A. [London, Macmillan and Co.].

Lines of Force of a Very Small Magnet

28. The equation of these curves when AB is taken very small, or, which is the same thing, the equation of a curve given by (2) when the parameter c is very small, is important. Let the origin of co-ordinates be taken at the middle point of AB, and the axis x along it. Let the length of AB be denoted by $2\delta a$ so that the co-ordinates of A are $-\delta a, 0$,

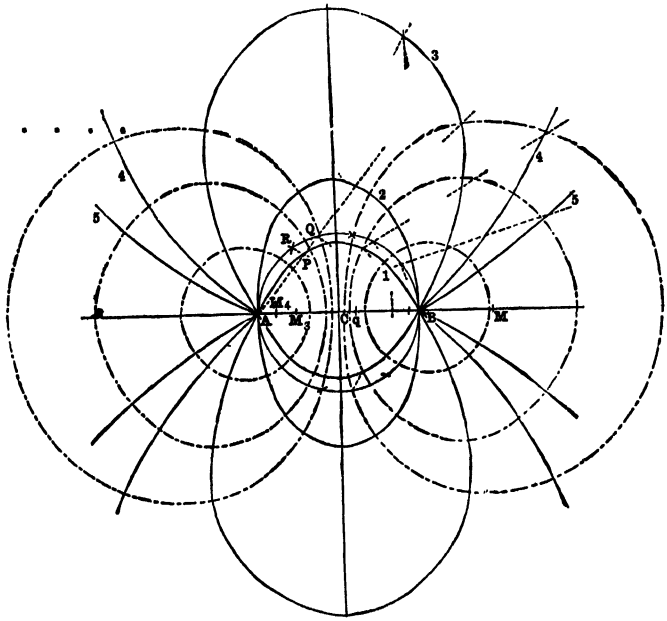


FIG. 14.

and of B, $+\delta a, 0$. Then if x, y, be the rectangular co-ordinates of P we have instead of (2)

$$\frac{x+\delta a}{\sqrt{(x+\delta a)^2+y^2}} - \frac{x-\delta a}{\sqrt{(x-\delta a)^2+y^2}} = c.$$

If δa be small this may be written

$$\frac{x+\delta a}{\sqrt{x^2+2x\delta a+y^2}} - \frac{x-\delta a}{\sqrt{x^2-2x\delta a+y^2}} = c$$

or

$$(x+\delta a)\left(1-\frac{x\delta a}{x^2+y^2}\right) - (x-\delta a)\left(1+\frac{x\delta a}{x^2+y^2}\right) = c\sqrt{x^2+y^2}$$

which reduced, and with $1/C$ put for $c/2\delta a$ becomes

$$\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{1}{C} \quad \ldots\ldots\ldots \quad (3)$$

where C is a parameter constant for any one curve. By varying C the whole family of curves is obtained.

This is the integral equation of the lines of force of a small magnet. It will be found of great importance in the theory of electrical radiation discussed later, as it is the equation also of the lines of electrical force in the neighbourhood of (but not too near) a dumb-bell electrical vibrator of the kind generally used in the study of the propagation of electrical waves.

29. If r be the distance of any point from the origin and θ its inclination to the axis of x, the equation can obviously be written in the form

$$r = C \sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From this expression the curve can be described with great facility. For draw a line OA (Fig. 15) making an angle θ with OX and meeting

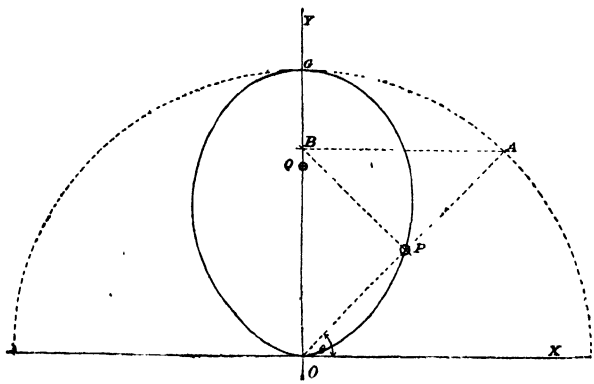


FIG. 15.

a circle described from O as centre with radius C in the point A. Then let fall a perpendicular AB on the axis OY, and a second perpendicular from B on OA meeting it in P. P is a point on the curve. For $OB = C \sin\theta$, and $OP = OB \sin\theta = C \sin^2\theta$.

This construction gives points on the curve with great ease except near the summit G. The part of the curve including G, can however be filled in sufficiently exactly by drawing a short circular arc with the proper radius of curvature for the point G. The expression for the radius of curvature at any point is $C\sin\theta\ (\sin^2\theta + 4\cos\theta^2)^{\frac{3}{2}}/3\ (\sin^2\theta + 2\cos^2\theta)$. For G, where $\theta = \pi/2$, this is $C/3$. Hence the arc is to be drawn from Q, where $GQ = C/3$.

With respect to the point on the axis, the equation of course does not apply as the magnet is there situated. Points fairly close to the origin are given quite well by the construction if carefully made.

The family of curves for different values of C is shown in Fig. 16.

Magnetic Potential

30. Let a unit of positive magnetism be supposed to be so far off from a given magnetic distribution that it may be regarded as being outside the field of force, and let it then be brought against magnetic repulsion to any point P of the field. Work must be spent on the unit in so doing, and the amount of work thus spent is the measure of the potential at P. If work on the whole is done by the magnetic system

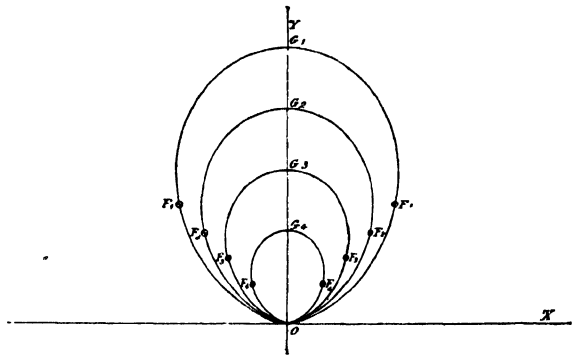


FIG. 16.

on the unit, while it is being brought to P, the work *spent* is negative that is the potential at P is negative.

For any distribution whatever, the work Ω done in bringing the unit from an infinite distance from the distribution to P is given by

$$\Omega = -\int \mathbf{H} \cos\theta \, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathbf{H}$ is the field-intensity at any element ds of the path, θ the angle between the direction of $\mathbf{H}$ and that of ds (taken positive in the direction *from* P) and the integral is taken along the path from an infinite distance from any point of the distribution to the point P.

If the distribution be a quantity m of magnetism situated at a point A, in a field of unit magnetic inductivity, and D be the distance of A from any element ds of the path along which the unit is brought, $\mathbf{H} = m/D^2$ and we have $\cos\theta = dD/ds$. Thus

$$\Omega = \int_r^\infty \frac{m}{D^2} \, dD = \frac{m}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The work done therefore in carrying a unit of magnetism from the point P to another point Q at distance r' from A is

$$\Omega = m\left(\frac{1}{r'} - \frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is called the *difference of potential* between P and Q. If we have any number of point distributions m_1, m_2, etc., at distances $r_1, r_2, \ldots, r_1', r_2', \ldots$ from P and Q respectively the difference of potential between P and Q is

$$\Omega = \Sigma m\left(\frac{1}{r'} - \frac{1}{r}\right) \quad \ldots \ldots \quad (8)$$

where Σ denotes summation of all the quantities $m_1(1/r'_1 - 1/r_1), m_2(1/r_2' - 1/r_2), \ldots$ which are given by the different distributions.

31. If the distribution is a continuous one on a surface or throughout a given volume the summation becomes an integration throughout the distribution, thus

$$\Omega = \int dm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad \ldots \ldots \quad (8')$$

where dm is any element of the distribution, and r_1, r_2, are the distances of P and Q from the element dm.

For a surface distribution of density σ per unit of area (that is amount of magnetism per unit of area) at an element dS of the surface, this is

$$\Omega = \int \sigma dS\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad \ldots \ldots \quad (9)$$

and for a volume distribution of amount ρ per unit of volume (volume-density) at an element dv it is

$$\Omega = \int \rho dv\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad \ldots \ldots \quad (10)$$

the integrals being taken over the surface and throughout the volume in the respective cases.

32. As a simple example consider the potential at P, Fig. 11, due to a uniform filament AB ($-m$ at A, $+m$ at B). Clearly we have, if $AP = r_1$, $BP = r_2$,

$$\Omega = m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad \ldots \ldots \quad (11)$$

that is the potential at P is equal to the work which would have to be spent if, there being a quantity m of magnetism at P, a unit of positive magnetism were carried from A to B.

As another example we shall calculate the potential at a point P of a magnetic doublet consisting of two equal and opposite point charges say $+m$ at B and $-m$ at A, Fig. 11. The potential of the doublet is

$$\Omega = m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = m\,\frac{r_1 - r_2}{r_1 r_2} \quad \ldots \ldots \quad (12)$$

But clearly if AB be small $r_1 - r_2 = AB\cos PAB = ds\cos\theta$, and

$r_1 = r_2 = r$, nearly. Hence if $\mathbf{m}$ denote the magnetic moment of the doublet we get

$$\Omega = \frac{\mathbf{m} \cos \theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If the centre of the doublet be at the point (x,y,z), the co-ordinates of P be ξ,η,ζ, and the direction cosines of the axis of the doublet be λ,μ,ν, the last equation becomes

$$\Omega = \mathbf{m} \frac{\lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)}{r^3} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

since $\cos \theta = \{\lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)\}/r$.

Clearly $\mathbf{m}$ may be taken as the moment of any combination of doublets at (x,y,z), if the resultant have its axis in the direction (λ,μ,ν).

Equipotential Surface. Equipotential Curves

33. The locus of an assemblage of points for every one of which Ω has the same value is called an equipotential surface. Any curve drawn on such a surface is called an equipotential line or curve.

Clearly there can be no component of field intensity tangential to an equipotential surface, and hence lines of force cut equipotential surfaces at right angles. Also the work done in carrying a unit of magnetism from any point of an equipotential surface to any other point of the surface along any path whatever is zero.

Also if δs be an element of a line of force between two equipotential surfaces, the potentials of which differ by $\delta\Omega$, the work done in carrying a unit of positive magnetism from the surface of less to the surface of greater potential is $\delta\Omega$. But this is numerically equal to $\mathbf{H}\delta s$, if $\mathbf{H}$ be the resultant magnetic force at δs. Hence if δs be taken positive in the direction from the surface of greater potential towards that of less

$$\mathbf{H}\,\delta s = -\delta\Omega$$

or in the limit

$$\mathbf{H} = -\frac{d\Omega}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

that is, $\mathbf{H}$ is equal at any point to the rate of diminution of the potential along an element ds of a line of force drawn through the point. From this we have for the components α,β,γ, of magnetic force parallel to rectangular axes x, y, z, the values

$$\alpha = -\frac{\partial\Omega}{\partial x}, \beta = -\frac{\partial\Omega}{\partial y}, \gamma = -\frac{\partial\Omega}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

34. As an example we may consider again the uniform magnetic filament AB. Since by (11) the potential is

$$\Omega = m\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

the equation of an equipotential surface is

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{c'} \quad \ldots\ldots\ldots \quad (17)$$

Taking the section of this surface by the plane of the paper in Fig. 14, it is easy to see that the curve of section cuts normally the family of curves of which the typical equation is

$$\cos\theta_1 + \cos\theta_2 = c.$$

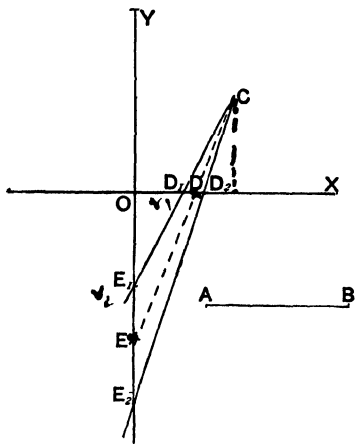


FIG. 17.

35. To describe an equipotential curve of parameter c'—see (17)—draw two rectangular axes OX, OY, Fig. 17, and lay down the point C the coordinates of which are c', c'. Then draw lines through C cutting off intercepts OD, OE, the first from the axis of x on the positive side of the origin, the other from the axis of y on the negative side of the origin. If r_1, r_2, be the numerical lengths of these intercepts we have by the Figure

$$r_1/r_2 = c'/(c' + r_2)$$

or

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{c'}$$

the equation of the equipotential surface, and also of its curve of intersection with the plane of the paper.

All values of r_1, r_2, thus obtained for a given value of c' will not belong to the equipotential curve which it is desired to draw. Only those are to be taken which lie between the two lines CD_1E_1 CD_2E_2 which give respectively $OE_1 + OD_1 = AB$, and $OE_2 - OD_2 = AB$, where AB is the distance between the extremities of the magnet. These lines can be drawn very exactly by calculating their inclinations to OX. If $m = \tan ODE$, we have in both of the limiting positions of the line $m = r_2/r_1 = (c' + r_2)/c'$. In the first case we have besides (putting $AB = a$) $r_1 + r_2 = a$, in the second $r_2 - r_1 = a$. Eliminating r_1, r_2, from the equations in the two cases we get for the first

$$(m^2 - 1)c' - ma = 0$$

and for the second

$$(m - 1)^2c' - ma = 0.$$

The greater roots of these equations are values of m which enable the limiting lines CD_1E_1, CD_2E_2, to be drawn. The lines ruled between them through C, as CDE, give intercepts which form the radii from A,B of points situated on an equipotential curve but on one side of the axis. To complete the curve round one pole of the magnet it is only necessary to lay down with the same pairs of radii the points on the other side of the axis. The same radii are of course available for the curve round the other pole.

Rotation of the whole diagram of equipotential curves round the axis of the magnet traces out the above family of equipotential surfaces be-

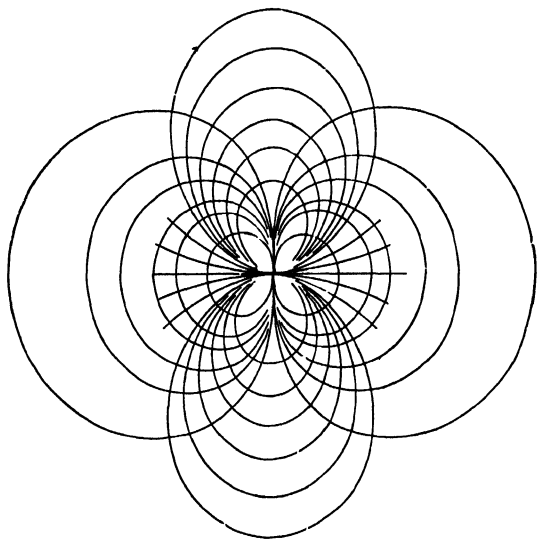


FIG. 18.

longing to the magnet. Fig. 14 shows lines of force and equipotential surfaces for the magnetic filament, Fig. 18 for an infinitely short magnet. Fig. 19 shows the nature of the lines of force in the field between two magnets with equal like poles turned towards one another. It will be noticed that, at points on a line drawn at right angles to the common axis of the magnets and through the centre of the space between them, the resultant force is along that line and from the magnet.

Actual Magnets. Equivalent Surface Distribution

36. Actual magnets are not in general uniformly magnetized, but have throughout their substance a volume density of magnetization representing unbalanced polarity. This is called *free* magnetism. Whatever the distribution of this may be, it is easy to prove experimentally,

what has been assumed above, that the quantity of free positive magnetism present in a magnet is exactly equal to the quantity of free negative magnetism.

It is only necessary to suspend the magnet in the field of, but at some considerable distance from, another magnet. It is then found that the magnet experiences no sensible translational force, but a very considerable resultant directional couple, unless it happens to be so placed that that couple vanishes.

The magnet above referred to as suspended in the field of the earth is a case in point. The most careful observation cannot detect any displacement of the magnet as a whole due to the earth's field, while the couple acting on it is very sensible, as much, perhaps for a square bar 60 cms. long and 1 cm. in diameter, hung at right angles to the earth's field-intensity, as nearly 20 grammes weight acting at an arm of 1 cm., or the weight of rather more than a quarter of an ounce acting at an arm of 1 inch.

For external points the action of a magnet, whatever its distribution, can be imitated by a certain distribution of positive and negative mag-

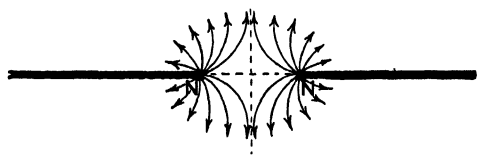


FIG. 19.

netism on the surface of the magnet. Grounds for this conclusion will be found in Chapter VI. with other general theorems regarding surface distributions. This surface distribution may be regarded as due to the poles of uniform magnetic filaments running through the magnet and having their ends on the surface of the magnet. It must however be clearly understood that we can obtain no knowledge of the actual distribution of magnetism *in* a magnet; all that can be obtained by the best of the so-called methods of determining magnetic distribution is a knowledge more or less accurate of the equivalent magnetization here referred to.

Magnetic Moment. Axis of a Magnet

37. Consider first a long thin bar uniformly magnetized in the direction of its length. The magnetic moment of such a bar is the couple which it would experience if it were placed with the line joining its poles at right angles to the lines of force of a uniform field of unit intensity. If C.G.S. units are used the couple gives the moment in C.G.S. units.

In the case of any other magnet there will be a position in which in a uniform field the couple acting on the magnet is a maximum.

The axis of the magnet is then at right angles to the lines of force of the field.

The axis may be found as follows. Let the magnet be freely suspended by its centre of gravity in a field uniform over the region occupied by the magnet. The magnet will set itself so that the couple acting on it is zero. Let then a line in the magnet have its position in space marked, or, which is the same, let the positions of two points in the magnet be marked. Now let the magnet be resuspended by its centre of gravity so that it rests in equilibrium in another position, and let the new positions of the same two points be marked. Draw then two planes bisecting at right angles the lines joining the two positions of each point. The line of intersection of these planes is the direction of the axis of the magnet. For, the axis of the magnet must have had the same direction in both cases, and hence it must have been possible to have turned the magnet from one position to a parallel one by simply turning it round an axis parallel to this direction, which is clearly given by the process described.

This construction is not convenient in practice, but instead the magnet may be hung in the field, and its position marked, and then be removed and a long thin needle hung in its place. If this be also suspended by its centre of gravity it will take the direction of the magnetic force, that is, the direction of the axis of the more complex magnet of considerable cross-section.

Determinations of magnetic axis have however very seldom to be made. We shall see later how uncertainty arising from want of exact knowledge of the position of the axis of the needle of a dip-circle is eliminated.

Couple on Magnet in Magnetic Field

38. If a magnet having taken up a position of equilibrium in a uniform field be turned through an angle of 180° round an axis at right angles to the magnetic axis it will again be in a position of equilibrium. It is clear from Fig. 20 that in one of these positions the equilibrium of the magnet is stable, in the other unstable. Any angular displacement of the magnet not compounded of a rotation through any angle round its own axis, and a rotation of 180° round a perpendicular axis, leaves it under the influence of a couple the moment of which depends on (1) the magnet itself, (2) the angle which the new direction of the magnetic axis makes with its direction of stable equilibrium, (3) the intensity of the magnetic field.

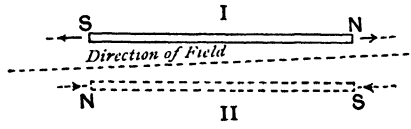


FIG. 20.

If the magnet be placed in a uniform field of intensity **H** so that its axis makes an angle θ with the position of stable equilibrium, that is with the direction of **H**, the moment of the couple is $\mathbf{MH} \sin \theta$ where

M is the magnetic moment of the magnet. Since the amount of free positive magnetism in the magnet is equal to the amount of free negative, the magnet may be regarded as made up of a very great number of uniformly magnetized filaments having their ends within or on the surface of the magnet. If δm be the free magnetism at either pole of one of these filaments the force exerted on that magnetism by the field is $\delta m\mathbf{H}$, and the forces on the poles are equal and in opposite directions. Hence if l be the distance between the poles, ϑ the angle between the filament and the direction of **H**, the moment of the couple on the filament is $\mathbf{H}\, l \sin \vartheta .\ \delta m$. Hence summing for all the filaments and putting **L** for the resultant couple we have

$$\mathbf{L} = \mathbf{H} \Sigma (l \sin \vartheta\ \delta m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

if the magnet be a thin plate with its breadth parallel to **H**, so that the axes of the elementary couples are all parallel.

But there is a position of the magnet and a corresponding value ϑ'' of ϑ for the particular filament considered for which

$$\Sigma (l \sin \vartheta' \delta m) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Now let θ be the angle through which the magnet must be turned from this position to attain its actual position in the field. Then $\vartheta = \theta + \vartheta'$ and

$$\begin{aligned} \Sigma (l \sin \vartheta\ \delta m) &= \Sigma \{l \sin (\vartheta' + \theta)\, \delta m\} \\ &= \cos \theta\, \Sigma (l\, \delta m \sin \vartheta') + \sin \theta\, \Sigma (l\, \delta m \cos \vartheta'). \end{aligned}$$

The first term on the right is zero by (19). The second term gives

$$\mathbf{L} = \mathbf{H} \sin \theta\, \Sigma (l\, \delta m \cos \vartheta') \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus the moment **M** of the magnet is given by the equation

$$\mathbf{M} = \Sigma (l\, \delta m \cos \vartheta') \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Similarly any other more complicated case may be treated.

Potential Energy of a Magnet

39. If a magnet is placed in a given position in a magnetic field, the field and magnet have mutual potential energy measured by the work which must be done against magnetic forces in bringing the magnet to the given position from another, defined as that for which the mutual potential energy is zero. We shall assume that this potential energy is zero when the axis of the magnet is at right angles to the lines of force of the field. If the magnet be so small that throughout it the field may be taken as one of the same intensity, no work will be done in bringing it from outside the field to any given position, if it be kept always at right angles to the lines of force; and therefore no work will be done in bringing it from outside the field in any manner whatever, and leaving it with its axis at right angles to the lines of force. It is to be understood in this connection that the magnetization of the

magnet remains unchanged when it is moved in the field, otherwise the statements here made will not be correct.

If now without change of position of the magnet as a whole it be turned round until the positive direction of its axis (from the south-pointing to the north-pointing end) makes an angle θ with the position of stable equilibrium of this axis, the couple due to magnetic forces acting on the magnet is $\mathbf{M}\,\mathbf{H}\sin\theta$ for the angle θ, and this tends to diminish θ. The work done *by* magnetic forces in bringing the magnet from the zero position to the final one is thus

$$-\int_{\pi/2}^{\theta}\mathbf{M}\,\mathbf{H}\sin\theta d\theta = \mathbf{M}\,\mathbf{H}\cos\theta$$

Hence the work done against magnetic forces is $-\mathbf{M}\,\mathbf{H}\cos\theta$ and if E denote the potential energy according to the specification

$$\mathsf{E} = -\mathbf{M}\,\mathbf{H}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If the components of the magnetic force $\mathbf{H}$ referred to three rectangular axes, x, y, z, Fig. 21, drawn in the true north, the east, and

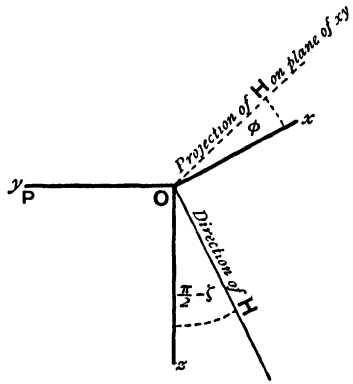


FIG. 21.

the vertically downward directions respectively, be α, β, γ, and the direction cosines of the magnetic axis referred to the same axes be l, m, n, we have instead of (22)

$$\mathsf{E} = -\mathbf{M}(l\alpha + m\beta + n\gamma) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Also substituting the value of $\sin\theta$ in terms of α, β, γ, l, m, n, we have

$$\mathbf{L} = \mathbf{M}\{(m\gamma - n\beta)^2 + (n\alpha - l\gamma)^2 + (l\beta - m\alpha)^2\}^{\frac{1}{2}} \quad . \quad . \quad (24)$$

This is plainly the resultant of three couples

$$\mathbf{M}(m\gamma - n\beta),\ \mathbf{M}(n\alpha - l\gamma),\ \mathbf{M}(l\beta - m\alpha)$$

round the axes of x, y, z, respectively.

Using polar co-ordinates, putting (see Fig. 21) ζ for the angle which **H** makes with a horizontal plane, ϕ for the angle between the plane of x, z, and a vertical plane containing **H**, and η and ψ for the corresponding angles for the magnetic axis we get

$$\alpha = \mathbf{H}\cos\zeta\cos\phi,\ \beta = \mathbf{H}\cos\zeta\sin\phi,\ \gamma = \mathbf{H}\sin\zeta$$
$$l = \cos\eta\cos\psi,\ m = \cos\eta\sin\psi,\ n = \sin\eta,$$

and instead of (23)

$$\mathsf{E} = -\mathbf{MH}\{\cos\zeta\cos\eta\cos(\phi - \psi) + \sin\zeta\sin\eta\} \quad . \quad . \quad . \quad . \quad (25)$$

Also for the component couples we obtain

$$\mathbf{M}(l\beta - m\alpha) = \mathbf{MH}\cos\zeta\cos\eta\sin(\phi - \psi) \quad . \quad . \quad . \quad . \quad (26)$$

with two similar equations for the other components.

Magnetic Poles

40. The determination of the positions of the "poles" of an ordinary bar magnet has been the object of much experimental research. Properly speaking, there are no definite poles in an ordinary magnet, if by poles are meant points at which the whole of the free magnetisms of the bar may be supposed concentrated, the negative at one, the positive at the other, so as to produce the actually existing external field. They exist only in the ideal case of an infinitely thin and uniformly magnetized filament, in which case they are the extremities of the bar.

As a matter of approximation, however, the positions of such points can be found; and one or two examples will be given in the next Chapter.

41. When a magnet is hung in a uniform field, it may be regarded as acted on by two sets of parallel forces, one set acting on the positive the other on the negative magnetism. The resultants of the two systems of parallel forces give the couple acting on the magnet; and the centres of these systems, or, what is the same thing, the "centres of mass" of the two distributions of magnetism, may be regarded as poles. But this idea of pole is not of any use, as all we are concerned with is the moment of the couple on the magnet, which, as we have seen, is the product $\mathbf{MH}\sin\theta$, where **M** is the magnetic moment, **H** the field intensity, and θ the angle between the direction of **H** and the magnetic axis.

The term "pole" in the sense of a quantity of magnetism concentrated at a point is also frequently used in specifying magnetic field intensity, or when discussing the mutual action between a magnet and a field, as, for example, when we speak of the force on a unit magnetic pole (that is, unit quantity of magnetism) placed at a point in the field.

Actions between Magnets. Simple Cases

42. We calculate first the field intensity produced at any point by a straight magnetic filament AB ($-m$ at A, $+m$ at B). Let the centre of the filament O be taken as origin of co-ordinates, and x, y, as indicated in Fig. 22, for the co-ordinates of the point P at which the intensity is to be found, and $2l$ the length of the filament. Then $AP^2 = r_1^2 = (x + l)^2 + y^2$, $BP^2 = r_2^2 = (x - l)^2 + y^2$. The forces at P due to the ends A and B respectively are numerically m/r_1^2, m/r_2^2, the former acting in the direction towards A, the latter in the direction from B. The direction cosines of the lines in which they act are thus $-(x+l)/r_1$, $-y/r_1$, $(x-l)/r_2$, y/r_2, respectively. The total component X along the magnet is thus given by

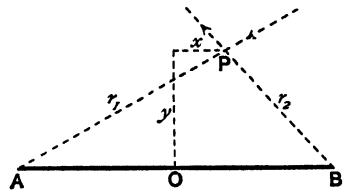


FIG. 22.

$$X = -mx\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) - ml\left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right) \quad . \quad . \quad . \quad . \quad (27)$$

and the component Y in the direction of y by

$$Y = -my\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence if ϕ be the angle which the resultant makes with the axis of x, we have

$$\tan\phi = \frac{y(r_2^3 - r_1^3)}{x(r_2^3 - r_1^3) + l(r_1^3 + r_2^3)} \quad . \quad . \quad . \quad (29)$$

If $r_1 = r_2$, that is if the point be on the axis of y, as in Fig. 23, we get

$$X = -\frac{2ml}{r_1^3}, \quad Y = 0.$$

$$\tan\phi = 0.$$

In this case X is the field intensity, and it can be written

$$X = \frac{-2ml}{(y^2 + l^2)^{\frac{3}{2}}} = \frac{-\mathbf{M}}{(y^2 + l^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad (30)$$

if $\mathbf{M}$ be the magnetic moment of the filament.

Again, if P be on the axis of x, as in Fig. 24, so that $r_1 = x + l$, $r_2 = x - l$, X becomes once more the field-intensity, and we have

$$X = -m\left\{\frac{1}{(x+l)^2} - \frac{1}{(x-l)^2}\right\} = \frac{2\mathbf{M}x}{(x^2 - l^2)^2} \quad . \quad . \quad . \quad (31)$$

Thus if the magnet be an ordinary bar magnet we can find the

intensity of its field, at distances from it great in comparison with its half-length, by the formulas

$$X = -\frac{\mathbf{M}}{y^3}, \; X = \frac{2\mathbf{M}}{x^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

according as the point is on the axis of y or on that of x, $\mathbf{M}$ denoting the magnetic moment of the magnet.

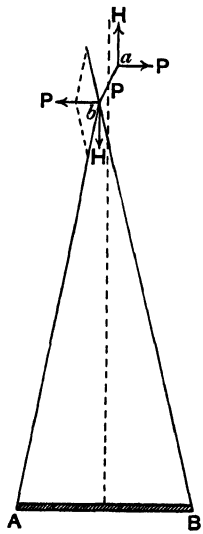


FIG. 23.

43. If a short needle of moment $\mathbf{M}'$ be horizontally suspended in the field with its axis making an angle θ with the axis of x, the couple exerted upon it by the magnet we have supposed is $\mathbf{M}\mathbf{M}' \sin\theta' (y^2 + l^2)^{\frac{3}{2}}$, in the first of the cases stated above (see Fig. 23), and $2\,\mathbf{M}\mathbf{M}' x \sin\theta' (x^2 - l^2)^2$ in the other (see Fig. 24).

If another field of intensity $\mathbf{H}$ say, due to some other distribution of magnetism, exist, having its direction parallel to y, another directive couple of magnitude $\mathbf{M}\,\mathbf{H}\cos\theta$ will act on the needle; and if this equilibrates the couple due to the magnet, in each of the cases supposed, we have

$$\frac{\mathbf{H}}{\mathbf{M}} = \frac{1}{(y^2 + l^2)^{\frac{3}{2}}} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

in the first case, and

$$\frac{\mathbf{H}}{\mathbf{M}} = \frac{2x}{(x^2 - l^2)^2} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

in the other.

These become in the approximative cases referred to above

$$\frac{\mathbf{H}}{\mathbf{M}} = \frac{1}{y^3} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\frac{\mathbf{H}}{\mathbf{M}} = \frac{2}{x^3} \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

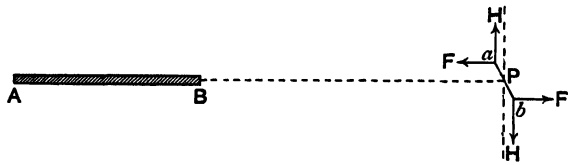


FIG. 24.

where $\mathbf{M}$ denotes the moment of the magnet used whatever the distribution of magnetism in it may be.

We shall see in the chapter on Magnetic Measurements, how, by using the horizontal component of the magnetic field intensity as $\mathbf{H}$ the values of $\mathbf{H}$ and $\mathbf{M}$ can be determined.

SECTION II.—*Synopsis of Elementary Theory of Magnetism. Magnetic Potential of any given Distribution*

44. It is an indirect but well established result of experiment, that a magnetized body may be considered as made up of a very large number of magnetized particles, the magnetic axes of which in any small element of the body have a common direction. Further, we suppose these particles so small that there is a very large number of them in a small element of the body. Whatever their nature may be, they must be regarded as *atomic* or indivisible magnets, in order to explain the fact that a portion of a broken magnet is itself a magnet, differing in no essential particular, other than moment and peculiarity of distribution, from the original magnet. In all such cases the algebraic sum of the magnetisms of each portion is zero; a fact which indicates that the positive and negative magnetisms of each magnetic particle are really two different aspects of one and the same physical phenomenon.

In what follows we shall consider only bodies of isotropic substance, that is which do not show differences of magnetic quality in different directions, and further only states and changes of states of a system of bodies which are maintained at a common constant temperature. All consideration of the magnetic phenomena of crystals and other æolotropic substances, and of thermodynamic consequences, are at present omitted. We shall consider the latter at least in a later Chapter.

45. Consider then a small rectangular prism of the body with edges parallel to the axes of x, y, z; let it contain n particles, of which the average magnetic moment is $\mathbf{m}$; then the magnetic moment of the element is $n\mathbf{m}$. If the lengths of the edges of the element are dx, dy, dz, the magnetic moment per unit of volume is $n\mathbf{m}/dxdydz$. Let this be denoted by $\mathbf{I}$, then $\mathbf{I}$ is what is called the intensity of magnetization of the substance.

The direction of $\mathbf{I}$ is the common direction of the axes of the particles. If λ, μ, ν, be the direction-cosines this direction we have $\lambda\mathbf{I}$, $\mu\mathbf{I}$, $\nu\mathbf{I}$, for the components of $\mathbf{I}$ along the axes of x, y, z respectively. We shall write $\lambda\mathbf{I}$, $\mu\mathbf{I}$, $\nu\mathbf{I} = A, B, C$. A, B, C are called the components of magnetization.

If the centre of the element $dxdydz$ be at (x, y, z) the potential $d\Omega$ which it produces at another point (ξ, η, ζ) is by the result (14) obtained at p. 19 above

$$d\Omega = n\mathbf{m}\frac{\lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)}{r^3}$$

or

$$d\Omega = \frac{A(\xi - x) + B(\eta - y) + C(\zeta - z)}{r^3}dxdydz \quad . \quad . \quad (37)$$

since $\mathbf{m}n = \mathbf{I}dxdydz$, and $\lambda\mathbf{I}$, $\mu\mathbf{I}$, $\nu\mathbf{I} = A, B, C$.

46. To find the potential Ω at (ξ, η, ζ) due to the whole magnetized mass we have only to integrate this throughout the whole space in which A, B, C, are different from zero. We thus obtain

$$\Omega = \iiint \frac{A(\xi - x) + B(\eta - y) + C(\zeta - z)}{r^3} dxdydz \quad . \quad . \quad . \quad (38)$$

Remembering that $-\partial r/\partial x = (\xi - x)/r$, &c., we find by integration by parts

$$\Omega = \iint \frac{A}{r} dydz + \iint \frac{B}{r} dzdx + \iint \frac{C}{r} dxdy - \iiint \frac{1}{r}\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz \quad . \quad (39)$$

47. The first three integrals relate to the surface of the body. Consider the first. It expresses that the whole body is to be supposed divided into thin rods of rectangular cross-section $dydz$, with their lengths parallel to the axis of x; and for each such rod the value of $Adydz/r$, for the negative end is to be subtracted from that for the positive end. Or, in the general case in which the body is of re-entrant boundary, or consists of two or more detached pieces, so that a line parallel to Ox, passing completely from one side to the other of the body, enters and emerges more than once, the sum of the values of $Adydz/r$ for the entrances is to be subtracted from the sum for the emergences; and this is to be repeated for every strip of section $dydz$ into which it is possible to divide the body.

If then dS_1, dS_2, be the areas intercepted at an entrance and an emergence respectively by a given strip, and l_1, m_1, n_1, l_2, m_2, n_2, be the direction cosines of the outward drawn normal at each, we have $l_2 dS = -l_1 dS_1 = dydz$, so that if we integrate $AldS/r$ over the whole surface of the body we get exactly the sum here specified. Equation (39) thus becomes when the other two terms are treated in the same way

$$\Omega = \iint \frac{1}{r}\left(Al + Bm + Cn\right) dS - \iiint \frac{1}{r}\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz \quad . \quad . \quad (40)$$

where l, m, n are the direction cosines of the outward drawn normal to the element dS of the surface.

48. It is obvious that we can interpret the first term of the expression in the right of (40) as the potential due to a surface distribution of density

$$\sigma = Al + Bm + Cn \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

at dS, and the second as the potential due to a volume density

$$\rho = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

at $dxdydz$.

If the angle between the outward normal to the surface at any point and the direction of $\mathbf{I}$ there be θ, the equation for the surface density becomes $\sigma = \mathbf{I} \cos \theta$.

49. It is to be noted that since the *whole* quantity of magnetism in the distribution is zero we ought to have

$$\iint (Al + Bm + Cu)\, dS - \iiint \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydx = 0.$$

That this is identically true is obvious.

50. The results here arrived at are of course consequences of the suppositions made as to the structure of the magnetized body. The surface and volume distributions are to be regarded as the unbalanced polarities of the magnetic molecules which abut on the surface, and are within the body respectively.

If throughout the body the equation

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0 \quad \ldots\ldots\ldots \quad (43)$$

hold, the distribution is said to be *solenoidal* or, sometimes, the body is said to be *uniformly* magnetized.

Potential of a Magnetic Shell

51. A magnetic shell is a thin surface magnetized at every point in a direction at right angles to its surface. If $\mathbf{I}$ be the intensity of magnetization of the shell at any point, $\delta\nu$ its thickness, the product $\mathbf{I}\,\delta\nu$ is called the strength of the shell, and is generally denoted by Φ. If this is the same at every point of the shell it is said to be a simple or uniform shell.

The potential of a simple shell at any point P is numerically equal to the product of the strength of the shell by the solid angle subtended at the point by the boundary of the shell. For let a small portion of the shell of area dS be considered, the distance of which from P is r, and the normal to which makes an angle θ with the direction of r. The moment of the element is $\mathbf{I}\,\delta\nu\, dS$, and hence the potential is

$$d\Omega = \mathbf{I}\delta\nu \frac{dS \cos\theta}{r^2} = \Phi \frac{dS \cos\theta}{r^2} \quad \ldots\ldots \quad (44)$$

But $dS \cos\theta$ is the projection of dS at right angles to r, and this divided by r^2 is the solid angle which the element subtends at P. Hence integrating over the shell, and putting ω for the total solid angle subtended by the bounding curve we have

$$\Omega = \Phi\omega \quad \ldots\ldots\ldots \quad (45)$$

The solid angle ω is taken positive when the point is on the positive side of the shell, that is, is so placed that a straight line drawn from the point, and intersecting the shell once or an *odd* number of times, meets first the positively magnetized surface.

It is to be noticed that the value of Ω remains unaltered however the shell may be deformed, provided its strength and the position of its boundary remain invariable, and the point remain on the same side of the shell.

The value of ω is zero for a closed shell; and hence the potential of a closed simple shell is zero at every point. On the other hand for a point within a closed shell ω is clearly $\pm 4\pi$ according as the positive side is turned inward or outward.

52. The difference of potential between two points very closely adjoining but on opposite sides of a shell is $4\pi\Phi$. This may be seen in the following manner. Let the shell be made a simple closed shell by a cap fitting the boundary. The potential at the point outside will be made zero, that of the point inside $\pm 4\pi\Phi$. Let $\Phi\omega_i$ be the original potential at the latter point. Then ω_i has been changed by the amount $\pm 4\pi - \omega_i$ which is the solid angle subtended at the internal point by the cap. But the solid angle subtended by the cap is the same at both points, and hence if ω_e be the solid angle subtended by the shell at the external point we have

$$\omega_e \pm 4\pi - \omega_i = 0$$

or

$$\omega_e - \omega_i = \pm 4\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Lamellar Distribution of Magnetism

53. When the magnetism of the body may be regarded as made up of simple magnetic shells, either closed or having their edges on the surface of the body, the magnetization is said to be lamellar. Let, in this case, ϕ denote the sum of the strengths of the shells passed through by a point carried within the magnet from any chosen zero position to any other position (x, y, z): then

$$\phi = \int \mathbf{I} \cos\theta ds.$$

where $\cos\theta\, ds$ denotes the thickness of any shell passed through, $\mathbf{I}$ the intensity of magnetization, and θ the angle between ds and the direction (λ, μ, ν) of $\mathbf{I}$. But since

$$\cos\theta = \lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds},$$

the equation for ϕ may be written

$$\phi = \int \mathbf{I}\left(\lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds}\right)ds \quad . \quad . \quad . \quad . \quad (47)$$

and therefore

$$\frac{d\phi}{ds} = A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}.$$

Thus we find

$$A = \frac{\partial\phi}{\partial x},\ B = \frac{\partial\phi}{\partial y},\ C = \frac{\partial\phi}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The function ϕ is called the potential of magnetization. When $A, B, C,$ are thus derivable by differentiation the magnetization is said to be *lamellar*.

If the magnetization is also solenoidal, that is, is lamellar-solenoidal, the condition

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = \nabla^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad (49)$$

holds, or ϕ satisfies Laplace's equation (p. 46 below).

Complex Lamellar Distribution

54. If a shell be not simple, that is, if its strength varies from point to point, it is said to be complex. If a magnet be made up of complex shells, the only condition to be expressed is that the direction of magnetization is normal to a family of surfaces. The equation of a line of magnetization is

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and if this be at right angles to a family of surfaces $\phi(x, y, z) = c$, where c is a variable parameter, the condition must hold

$$hA = \frac{\partial\phi}{\partial x},\ hB = \frac{\partial\phi}{\partial y},\ hC = \frac{\partial\phi}{\partial z},$$

where h is a function of the co-ordinates x, y, z. Hence we have the conditions

$$\frac{\partial(hC)}{\partial y} = \frac{\partial(hB)}{\partial z}, \text{ \&c.}$$

and these give

$$C\frac{\partial h}{\partial y} - B\frac{\partial h}{\partial z} + h\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) = 0$$

with two similar equations. Multiplying these equations by $A, B, C,$ respectively, and adding, we find the condition

$$A\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) + B\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + C\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) = 0 \quad . \quad (51)$$

which is that of complex lamellar magnetization.

55. If the magnetization is also solenoidal we have besides

$$A\frac{\partial h}{\partial x} + B\frac{\partial h}{\partial y} + C\frac{\partial h}{\partial z} = \nabla^2\phi,$$

since

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.$$

Multiplying this equation by h we find

$$hA\frac{\partial h}{\partial x} + hB\frac{\partial h}{\partial y} + hC\frac{\partial h}{\partial z} = h\nabla^2\phi,$$

or

$$\frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial h}{\partial z} = h\nabla^2\phi \quad . \quad . \quad . \quad (52)$$

This may be taken as the condition fulfilled by a lamellar solenoidal distribution.

A number of theorems equally applicable both to magnetism and to electricity will be proved in chapter VI., which relates more particularly to electrostatic phenomena. Further developments, however, of the mathematical theory of permanent magnetism sketched above have not in general any very direct bearing on electromagnetism, and they are therefore not given here. The preceding brief discussion will be supplemented in what follows as additional information is required, and in another chapter which will contain some applications of thermodynamics to the explanation of magnetic phenomena.

CHAPTER II

MAGNETIC INTENSITY AND MAGNETIC INDUCTION

Influence of Medium occupying the Field

56. We have defined field intensity above, in accordance with the usual practice, as the force which unit quantity of magnetism experiences when placed at the point considered. We now propose to look at the matter from another point of view so as to distinguish clearly between what it is now usual to call magnetic induction and magnetic intensity or magnetic force. To do so we must call more direct attention than hitherto to the fact that magnetic action must be propagated by changes which take place in the medium occupying the field. These consist in a certain effect produced at every point of the field by the presence of the magnetic system, and propagated outwards from the magnetic system by mutual actions between the different parts of the medium. What the precise nature of this effect is we cannot tell, but it is probably a species of motion of the medium which causes such parts of the medium or portions of matter in the medium as are magnetizable to exhibit magnetic phenomena. We shall take as the measure of this effect a quantity which we shall call the magnetic induction. The ordinary mode of reckoning this quantity will be specified presently.

As the result of the existence of this change in the medium, the field will be the seat of a certain distribution of energy, which we shall regard as kinetic, and we begin by specifying that if $\mathbf{B}$ denote the magnetic induction at any point, the energy per unit of volume there, or T_v, shall be equal to $\mathbf{BH}/8\pi$. This introduces a second quantity $\mathbf{H}$ the specification of which so far is simply that multiplied into $\mathbf{B}/8\pi$ it gives T_v. $\mathbf{H}$ will afterwards be identified with what we have called the magnetic force or magnetic intensity.

We shall see later when considering the magnetic field produced by a current of electricity, that there is a perfect mathematical analogy between the current and the magnetic force on the one hand, and a vortex filament or system of filaments in a perfect fluid, and the velocity at any point in the surrounding fluid on the other. The energy of a magnetic

field therefore corresponds exactly to that of the motion of the fluid in the field so to speak of the vortex. We shall use the analogy here to some extent by adopting a phraseology founded upon it for the specification of certain quantities with which we have to deal.

Comparison with ordinary kinetic energy suggests that we may regard $\mathbf{B}/4\pi$ as the magnetic momentum per unit volume of the medium at the point considered, and $\mathbf{H}$ as the corresponding velocity; and some such convention seems preferable to the alternative view which might be adopted, in which $\mathbf{B}/4\pi$ is regarded as a magnetic displacement analogous to an elastic displacement in the medium, and $\mathbf{H}$ as the corresponding "force" producing it. We shall generally use the term magnetic intensity for $\mathbf{H}$, in order to avoid the "begging-the-question" influence of the name magnetic force. Still if the latter term is used it ought to be remembered that magnetic force like electric force and electromotive force is not a force in the ordinary dynamical sense at all; and we shall see that in the generalised dynamical method of Lagrange, which we shall use in dealing with the subject of current-induction, there is an indefinite number of kinds of "forces" in an extended sense which are obtained by a simple process from the expressions for the kinetic energy.

57. We denote the component of the magnetic induction at any point by a, b, c. Further we call the product $\mathbf{B} \cos\theta dS$, where dS is an area of an interface drawn in the field through the point in question, and θ is the angle which the normal to the element makes with $\mathbf{B}$, the total magnetic induction across the area dS. By integration we can calculate the total induction across a closed surface drawn in the field. This will have the value

$$\int \mathbf{B} \cos\theta dS,$$

where the integral is taken over the surface.

We further assume that the magnetic induction fulfils the condition that its integral taken over any spherical surface drawn in the field of a quantity of magnetism placed at the centre, is numerically equal to this quantity multiplied by 4π: that is we assume that

$$\int \mathbf{B} \cos\theta dS = 4\pi m$$

if m be the quantity of magnetism. We suppose for the present m to be the only magnetism, so that the total induction is the same over every concentric spherical surface drawn from the position of m, and equal to $4\pi m$. We are further led by symmetry to take as the direction of the induction the radial direction drawn through the point considered and the position of the point-charge of magnetism.

Total Magnetic Induction over Closed Surfaces in Field

58. From this we can prove that the total induction across any closed surface drawn in the field and not including the charge within it is zero. For consider a surface drawn in the field and bounded by parts

of two concentric spherical surfaces having their common centre at the charge, and by a surface swept out by moving a radius so that a point in it passes round a closed curve. The total induction across the inner spherical part of the boundary is obviously equal to that across the outer portion, while there can be none at right angles to a radial portion.

Consider now a cone of small angle and let dS'_1 be an oblique section of the cone the outward normal to which makes an angle θ_1, with the axis of the cone. The total induction outward across this is $\mathbf{B}_1 \cos\theta_1 dS'_1 = \mathbf{B}_1 dS_1$ where dS_1, is the normal section at the same place. Consider another oblique section dS'_2 for which the corresponding inclination of the normal to the axis is θ_2, and normal section dS_2. The total induction there is $\mathbf{B}_2 \cos\theta_2 dS'_2 = B_2 dS_2$. But we have seen that $\mathbf{B}_2 dS_2 - \mathbf{B}_1 dS_1 = 0$, and hence

$$\mathbf{B}_2 \cos\theta_2 dS'_2 - \mathbf{B}_1 \cos\theta_1 dS'_1 = 0,$$

that is the total induction outwards from the surface contained between the two oblique sections is zero.

From this the conclusion at once follows that the total induction across a closed surface of any form drawn in the field so as not to include the point charge is zero.

59. Moreover it clearly follows from the equation

$$\mathbf{B}_1 \cos\theta_1 dS'_1 = \mathbf{B}_2 \cos\theta_2 dS'_2$$

that the total induction across a closed surface of any form drawn so as to include the magnetic charge is equal to that taken over any spherical surface having its centre at the charge, in other words is equal to $4\pi m$.

We now assume that the induction at each point produced by any given distribution is the resultant of the inductions that would be produced at the point by each elementary part of the distribution acting separately. Hence we can prove that the total induction over any closed surface drawn in the field is equal to 4π times the total quantity of magnetism within the surface. For the total induction due to any element outside the surface is zero, and so therefore is the induction across the surface for the whole distribution. For any element within the surface, on the other hand, the total induction across the surface is equal to 4π times the quantity of magnetism at the element. Hence we get

$$\int \mathbf{B} \cos\theta dS = 4\pi \Sigma m \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where Σm is the total quantity of magnetism within the surface.

Solenoidal Condition Fulfilled by Magnetic Induction

60. In every actual magnetic distribution the total quantity of magnetism within a closed surface drawn entirely in the medium filling the space between the magnetic molecules must be zero; for we suppose the magnet, as already stated made up of molecular magnets

every one of which possesses equal and opposite quantities of magnetism. Thus

$$\int \mathbf{B} \cos\theta dS = 0.$$

This may be written, if λ, μ, ν be the direction cosines of the normal,

$$\int (\lambda a + \mu b + \nu c) dS = 0,$$

or

$$\int (a dydz + b dzdx + c dxdy) = 0.$$

This clearly may be regarded as the result of partial integration throughout the interior of the surface, of the quantity

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z},$$

so that we have

$$\iiint \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \right) dxdydz = 0 \quad . \quad . \quad . \quad (54)$$

the triple integral being extended throughout the whole space within the closed surface.

Now this equation holds for every closed surface we may draw in the field, so long at least as we do not come down to dimensions comparable with those of the magnetic molecules. If, however, our elements descend to molecular dimensions, we have to deal with discontinuities of structure analogous to those which would modify the equations of fluid motion, if the fluid instead of being treated as a *continuum* were regarded as the body of grained structure which no doubt it is in reality.

With this limitation we have the equation

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0,$$

that is the induction fulfils what is called the solenoidal condition. This result is sometimes expressed in words by saying that the convergence of the induction is everywhere zero. It really asserts the fact that at no point of the field can there be an isolation of one kind of magnetism from the other, in other words a unipolar magnetic molecule cannot exist.

61. For any given case lines of magnetic induction can be drawn in the field. These are defined *mutatis mutandis* just as are lines of magnetic force (p. 13). A tubular surface formed by lines of induction is called a tube of induction. The total induction over a cross-section of a tube is the same everywhere: if it is unity the tube is called a unit tube.

The intensity of the magnetic induction can be expressed by the number of unit tubes or *lines of induction* per unit of area of a surface drawn at right angles to the lines of induction in the field, that is to say it is inversely as the cross section of a unit tube.

Particular Cases of Magnetization

62. It is to be understood that in what follows the medium taken as standard of reference is the luminiferous ether. Ordinary magnetizable or diamagnetizable matter will be regarded as existing in the ether, and as forming when polarized part of the magnetic system producing the momentum at each point of the ether, and the magnetization wherever it exists.

63. We now consider one or two important particular cases of magnetization. These, like the results of the preceding discussion (with the exception of the solenoidal condition fulfilled by the magnetic induction, which requires modification when applied to the corresponding quantity, the electric displacement), are at once *mutatis mutandis* applicable to dielectric polarization.

Considering as before an isotropic field we take first the case in which the state of the field is symmetrical about one point, at which we suppose a point-charge of magnetism, for example one pole of a very long uniform filamental magnet, to be situated. The induction in this case is outwards from the pole at every point. Thus by what has gone before we get

$$4\pi m = 4\pi r^2 \mathbf{B}$$

by taking the total induction across a spherical surface of radius r having the point-charge at its centre.

The magnetic intensity corresponding to the induction we shall suppose proportional to the latter according to the specification given on p. 36, and in an isotropic field it will have the same direction. Thus for the case of a single point-charge of magnetism just specified $\mathbf{B} = m/r^2$, and we take as the intensity the quantity $\mathbf{H} = m/\mu r^2$, where μ is a multiplier depending on the medium. If we regard $\mathbf{B}/4\pi$ as the momentum per unit volume of the medium at the point considered, and $\mathbf{H}$ as the corresponding velocity, the two will grow up together from zero, and we get $\mathbf{BH}/8\pi$ as the energy of the medium per unit volume at the place where the induction is $\mathbf{B}$ and the intensity $\mathbf{H}$. Thus the energy spent in the volume $4\pi r^2 dr$ constituting a shell of radius r and thickness dr having its centre at the point charge is

$$\frac{1}{8\pi}\mathbf{BH} \cdot 4\pi r^2 dr = \frac{1}{8\pi}\frac{m}{r^2}\frac{m}{\mu r^2} \cdot 4\pi r^2 dr.$$

The energy thus spent in unit volume in setting it into motion is

$$\frac{1}{8\pi}\mathbf{BH} = \frac{m^2}{8\pi\mu r^4},$$

and the whole energy spent in the field is

$$T = \int \frac{m^2}{8\pi\mu r^4} dv \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where dv is an element of volume, and the integral is taken throughout the whole medium. We may take the integral for example throughout all space except that of a small sphere of radius a, which we may suppose to surround m. Thus we get

$$T = \frac{m^2}{8\pi} \int_a^\infty \frac{4\pi r^2 dr}{\mu r^4} = \frac{m^2}{2\mu a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Mutual Energy of Two Point-Charges. Apparent Force of Repulsion between them

64. It is clear that this integral will become infinite if a be made infinitely small. This however is for our present purpose of no moment; and the formula just found will be of use afterwards, especially in the theory of electrical action. What we wish to calculate at present is the change of energy of the medium produced by bringing a second point-charge into the field of a first.

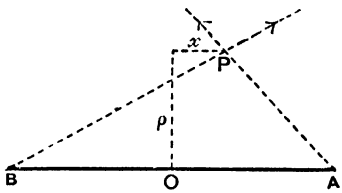


FIG. 25.

To find this let the distance apart of the two charges in their final positions be $2a$, and take the point half way between them as the origin, and the line joining them as axis of x. The direction of the induction or magnetic momentum at any point due to either charge is radially outwards from the charge. To find the resultant momentum due to both charges we have simply to compound the two momenta. Let m and m' be the charges at the points A and B, Fig. 25, and let x be the distance of the point considered from the origin along the line OA, ρ its distance from the axis. The component momentum along the axis of x due to the two point-charges is easily found to be

$$\frac{m}{4\pi} \frac{x-a}{\{(x-a)^2+\rho^2\}^{\frac{3}{2}}} + \frac{m'}{4\pi} \frac{x+a}{\{(x+a)^2+\rho^2\}^{\frac{3}{2}}}$$

The component at right angles to the axis is

$$\frac{m}{4\pi} \frac{\rho}{\{(x-a)^2+\rho^2\}^{\frac{3}{2}}} + \frac{m'}{4\pi} \frac{\rho}{\{(x-a)^2+\rho^2\}^{\frac{3}{2}}},$$

and by the symmetry round the axis there is no other component.

Thus the square of the resultant momentum is

$$\frac{m^2}{16\pi^2}\frac{1}{(x-a)^2+\rho^2}+\frac{m'^2}{16\pi^2}\frac{1}{(x+a)^2+\rho^2}$$
$$+\frac{2mm'}{16\pi^2}\frac{x^2-a^2+\rho^2}{\{(x^2+a^2+\rho^2)^2-4a^2x^2\}^{\frac{3}{2}}}.$$

The two first terms taken together and multiplied by $4\pi/2\mu$ make up the work done per unit volume in establishing the point charges separately. The last term also multiplied by $2\pi/\mu$ is the mutual energy of the two charges per unit of volume at the point considered. Thus for the whole mutual energy we have

$$T_m = \frac{mm'}{4\pi\mu}\int_0^\infty 2\pi\rho d\rho\int_{-\infty}^{+\infty}\frac{(x^2-a^2+\rho^2)dx}{\{(x^2+a^2+\rho^2)^2-4a^2x^2\}^{\frac{3}{2}}}.$$

This integral can be evaluated without much difficulty and the result is

$$T_m = \frac{1}{\mu}\frac{mm'}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Differentiating this we get for the energy spent on the system in separating the charges through a distance $2da$ the value

$$-\frac{mm'}{\mu(2a)^2}2da,$$

or the force between the charges is a repulsion of amount

$$\frac{mm'}{\mu(2a)^2}=\frac{m^2}{\mu(2a)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

if the charges are equal. If they have opposite signs the force is of course an attraction.

Magnetic Inductivity. Superposition of Magnetic Effects

65. The magnetic repulsion on unit quantity of magnetism, that is the *intensity of the magnetic field produced by a point-charge* m, is thus $m/\mu r^2$ where r is the distance of the point considered from that at which the pole or point charge is situated. The quantity μ is what we have called the *magnetic inductivity* of the medium.

The greater the magnetic inductivity is the smaller is the force $m/\mu r^2$ experienced by unit quantity of magnetism placed at distance r from a point-charge m.

Thus we identify the quantity $m/\mu r^2$, the magnetic intensity referred to above, as corresponding to the magnetic induction at the point, with the field intensity **H** at a distance r from the point-charge m as defined at p. 12 above.

Further the magnetic potential Ω at a distance r from a point-

charge, that is the function from which the magnetic intensity, or its components, can be found according to the specification $\alpha = -\partial\Omega/\partial x$, $\beta = -\partial\Omega/\partial y$, $\gamma = -\partial\Omega/\partial z$, is given by

$$\Omega = \frac{m}{\mu r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

66. Now by the principle of superposition the magnetic intensity for any distribution of point-charges in the field can be found by calculating the resultant of the radial intensities due to the assemblage of point-charges. Thus the magnetic intensity can be calculated on action at a distance principles for any distribution whatever, and the methods already discussed are at once applicable.

The induction $\mathbf{B}$ is connected with $\mathbf{H}$ by the equation

$$\mathbf{B} = \mu\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The same relation is expressed also by the equations connecting the components a, b, c, of $\mathbf{B}$ with those α, β, γ, of $\mathbf{H}$, namely

$$a = \mu\alpha, \; b = \mu\beta, \; c = \mu\gamma \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The quantity μ is a physical quantity, not a mere constant of comparison of media, and its dimensions depend on the physical property which defines it. At present we are unable to fix those dimensions, all we can say is that the dimensions of $\mathbf{BH}$ or $\mu\mathbf{H}^2$ must be those of energy per unit volume of the medium, that is $[ML^{-1}T^{-2}]$ in the ordinary dimensional notation (see the Chapter on Dimensions). If, however, $\mathbf{H}$ be regarded as the velocity of the medium at the point considered, $\mu/4\pi$ must be regarded as the density of the medium at the same point.

We shall denote the inductivity of the standard medium by μ_0.

Field Containing Different Media

67. So far we have dealt only with a single medium: we must now consider the case in which the field contains different media. We shall suppose each of these to be isotropic, and that the intensity at each point is in the direction of the momentum or induction there.

In the first place we can show that the normal component of magnetic induction is continuous on the two sides of an interface separating two media. For describe a closed surface forming a shell one face of which is just within, the other just outside the interface, the surface being supposed so drawn as not to cut through any magnetized molecule, if the media contain such. (This condition, it may be remarked, must always be fulfilled when any physical interface in a medium is constructed, say by cutting a narrow crevasse in it.) Then, since there is no magnetism within the shell, the total induction across the surface is zero. The edges of the shell contribute nothing to the integral, and hence the integrals over the two faces are equal and opposite; that is, since we

may take the shell of as limited area of face as we please, the normal components of induction have the same numerical value and the same direction on the two sides of the interface. Thus we get denoting the resultant inductions on the two sides of the surface by $\mathbf{B}_1, \mathbf{B}_2$, and the angles which the normals to the interface drawn towards the media on the two sides make with the directions of the induction by θ_1, θ_2,

$$\mathbf{B}_1 \cos\theta_1 = \mathbf{B}_2 \cos\theta_2, \qquad (62)$$

or if N_1, N_2 denote the normal components of induction taken from the interface towards the medium on each side

$$N_1 + N_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62')$$

If l, m, n be the direction cosines of the normal to the surface drawn towards either medium, $a_1, b_1, c_1, a_2, b_2, c_2$ the components of the magnetic induction on the two sides of the surface, we may express this surface condition in the form

$$a_1 l + b_1 m + c_1 n = a_2 l + b_2 m + c_2 n \quad . \quad . \quad . \quad (63)$$

It follows from the relation stated in these different ways that since

$$\mu_1 \mathbf{H}_1 \cos\theta_1 = \mu_2 \mathbf{H}_2 \cos\theta_2$$

the normal components $\mathbf{H}_1 \cos\theta_1$, $\mathbf{H}_2 \cos\theta_2$ of field-intensity are discontinuous.

Magnetic Intensity and Induction in Cavity cut in Magnetized Body

68. From the results arrived at we can draw some important consequences. Let a narrow crevasse be cut in a medium in the field, so that the walls of the crevasse are at right angles to the magnetic induction, and therefore also at right angles to the magnetic intensity. We may suppose to fix the ideas, the crevasse to be filled with the standard medium. The induction, as we have seen, has the same value in the crevasse at any point that it has at a near point in the medium itself. Thus if μ be the magnetic inductivity of the medium, $\mathbf{H}$ the magnetic intensity at a point near the wall of the crevasse, and $\mathbf{H}_0$, the field intensity at a near point in the crevasse, we have

$$\mu \mathbf{H} = \mu_0 \mathbf{H}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$\mathbf{H}_0$ may be regarded as the magnetic force due to the distribution of magnetism in the field. But this distribution consists of that on the walls of the crevasse set free by their formation and the magnetism elsewhere, and since the surface density of magnetism at either wall is $\mathbf{I}$, the intensity, which as we have seen above may be calculated on action at a distance principles, due to the first part, is $4\pi\mathbf{I}/\mu_0$. This may be established by the following investigation, which gives other useful results.

69. Let a cylindrical cavity of length $2l$, and radius r be cut within

a uniformly magnetized body and let its axis make an angle θ with the direction AB of magnetization as shown in Fig. 26. Let the intensity of magnetization be denoted by $\mathbf{I}$. The density of the distribution set free on the curved surface is $\mathbf{I}\sin\theta$ at points in a plane through the axis parallel to the direction of magnetization, if for simplicity the standard medium filling the cavity be supposed devoid of magnetization. At points in another axial plane making an angle ϕ with the former the density is $\mathbf{I}\sin\theta\cos\phi$. This distribution gives at the centre of the axis a field intensity at right angles to the axis, and in the plane through the axis and the direction of magnetization of amount

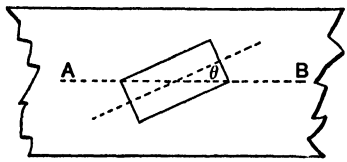


FIG. 26.

$$\frac{1}{\mu_0}2r^2\mathbf{I}\sin\theta\int_{-\pi/2}^{+\pi/2}\cos^2\phi d\phi\int_{-l}^{+l}\frac{dx}{(r^2+x^2)^{\frac{3}{2}}} = 2\pi\frac{\mathbf{I}}{\mu_0}\sin\theta\,\frac{l}{(l^2+r^2)^{\frac{1}{2}}}$$

Each end of the cylindrical cavity may be regarded as a disk of magnetism of density $\mathbf{I}\cos\theta$ and radius r. Now considering a ring of radius z and breadth dz and integrating, we get for the magnetic intensity due to the whole disk at a point on the axis distant l from its plane

$$2\pi\frac{\mathbf{I}}{\mu_0}l\cos\theta\int_0^r\frac{zdz}{(l^2+z^2)^{\frac{3}{2}}} = 2\pi\frac{\mathbf{I}}{\mu_0}\cos\theta\left\{1-\frac{l}{(l^2+r^2)^{\frac{1}{2}}}\right\}$$

70. If r is very small in comparison with l this becomes zero, and if l is small in comparison with r it becomes $2\pi\mathbf{I}\cos\theta/\mu_0$. Thus the ends of the cylinder give in the former case an intensity along the axis of zero value and in the latter of amount $4\pi\mathbf{I}\cos\theta/\mu_0$. If $\theta=0$, so that the axis of the cylinder is parallel to the direction of magnetization the intensity due to the ends in the latter case is $4\pi\mathbf{I}/\mu_0$.

The intensity due to the walls in any given case is the resultant of the components due to the curved surface and the ends, and hence if l be great in comparison with r, the intensity is at right angles to the axis in the plane through the axis and the direction of magnetization, and is $2\pi\mathbf{I}\sin\theta/\mu_0$ and when r is great compared with l it is $4\pi\mathbf{I}\cos\theta/\mu_0$.

We see therefore that in the case of a right cylindrical hollow of length small compared with r, and cut at right angles to the axis, that is in the crevasse supposed above, the intensity is $4\pi\mathbf{I}$. This holds for a narrow crevasse whether cylindrical or not, provided the point considered is at a distance from the edges great in comparison with the length.

It will be seen from Art. 87 below that within a spherical hollow the force is at every point in the direction of $\mathbf{I}$, and is

$$\frac{4}{3}\pi\mathbf{I}/\mu_0.$$

Magnetic Inductivity, Permeability and Susceptibility

71. Hence, returning to the discussion of the induction in the crevasse, we find, denoting by $\mathbf{H}'$ the field intensity due to the magnetism elsewhere than on the surface, that is the intensity in a long narrow cylinder with its axis in the direction of magnetization,

$$\mu\mathbf{H} = \mu_0\mathbf{H}_0 = \mu_0\left(\mathbf{H}' + 4\pi\frac{\mathbf{I}}{\mu_0}\right)$$

or if we write $\mathbf{I}/\mu_0 = \kappa\mathbf{H}'$

$$\mu\mathbf{H} = \mu_0(1 + 4\pi\kappa)\mathbf{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Now we have seen above that the intensity $\mathbf{H}$ is the magnetic field intensity calculable by action at a distance principles, from the magnetic distribution elsewhere than at the point in question, and is therefore to be identified with $\mathbf{H}'$. Thus we get the equation

$$\mu = \mu_0(1 + 4\pi\kappa) \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The factor κ is called the magnetic susceptibility of the substance relatively to the medium filling the crevasse (which we have supposed to be some medium taken as standard) and is essentially a numeric.

72. The multiplier $1 + 4\pi\kappa$ we call the magnetic permeability of the medium of magnetic inductivity μ. If we denote it by ϖ we have

$$\varpi = \frac{\mu}{\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

that is, it is the ratio of the inductive capacity of the medium to that with which the crevasse is filled. That it should depend upon the latter medium is obvious from the fact that in the investigation in Art. 71 $\mathbf{I}$ is the magnetization intensity with reference to that medium, in other words, $\mathbf{I}$ is the magnetic surface density set free by cutting the crevasse, and replacing the medium taken out by the standard medium.

73. The usual mode of stating the relations seems much less clear and hardly correct, and attention has recently been called to the desirability of avoiding confusion in this connection.[1] The relation (66) was first given by Lord Kelvin,[2] who called the force in a narrow crevasse cut in a magnetized body at right angles to the direction of magnetization the force in the magnetized body according to the electromagnetic definition, while that in a narrow cylinder cut with its length along the direction of magnetization he called the force according to the polar definition. The standard medium filling the crevasse was air. The ratio of the force according to the former definition to that according to the latter he called the magnetic *permeability*.

Afterwards the name permeability was extended to designate an

[1] O. Heaviside, *Electrician*, October 30, 1891, or *Electromagnetic Theory*, Vol. I. p. 126.
[2] Rep. of Papers on Electrostatics and Magnetism. § 517.

absolute quantity depending on some property of the medium itself and therefore only justly expressed by the proper absolute measure of that quantity, and not by arbitrarily assuming that μ_0 for the standard medium had the value unity. This assumption has led to great confusion in questions regarding units.

The mode of presenting the matter given above returns to the view taken by Lord Kelvin, and regards the intensity in the crevasse what it properly is, a magnetic field intensity and reserves the name *induction* for the totally distinct physical quantity obtained by multiplying the field intensity at any point by the magnetic inductive capacity.

Surface Conditions Fulfilled by Magnetic Intensity

74. We can now show that the tangential component of the field intensity is continuous on the two sides of an interface separating two media. For the field intensity on one side of the surface only differs from that on the other by the intensity due to the surface distribution of magnetism given by the change of medium. This, if the interface is of continuous curvature and the magnetization is continuous, is clearly an intensity normal to the surface. Hence the normal component of the magnetic intensity only is discontinuous, the tangential component is the same on the two sides of the surface.

Magnetic Potential. Equations of Laplace and Poisson

75. If we denote by Ω the potential of the whole magnetic distribution in the field, whether that at the surface of separation between two different media, or magnetism distributed throughout a medium in any given manner, the surface integral of magnetic induction taken over any surface in the field fulfils the equation

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0,$$

and therefore so also does the magnetic field intensity where the medium is uniform. But the field intensity, as we have seen, fulfils the equations

$$\alpha = -\frac{\partial \Omega}{\partial x}, \quad \beta = -\frac{\partial \Omega}{\partial y}, \quad \gamma = -\frac{\partial \Omega}{\partial z}.$$

Hence we find

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{\partial^2 \Omega}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (68)$$

which is called the characteristic equation of the magnetic potential. It was first given by Laplace for the case of gravitational attraction. It is to be carefully noticed that it holds only at points where there is none of the attracting or repelling matter.

For the case of symmetry of distribution of magnetism round a centre the equation takes the form

$$\frac{\partial^2\Omega}{\partial r^2} + \frac{2}{r}\frac{\partial\Omega}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and for symmetry round a straight line as axis (say the axis of z) the form

$$\frac{\partial^2\Omega}{\partial z^2} + \frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

as can easily be proved directly or by transformation from the standard form.

76. A more general equation given by Poisson holds in all cases in which we have a distribution of magnetism (or other attracting or repelling matter) of finite density at the point in question. Let the density of this distribution be denoted by ρ, and let it be supposed to vary continuously from point to point. Then we can take an element so small that the density may be regarded as uniform throughout the element. For a point in the element the potential may be divided into two parts Ω_1, Ω_2, the former due to the matter outside the element, the latter due to the matter of the element. Take the element a sphere of small radius r, and let the point considered be at a distance ζ from the centre. The force along the radius is $-\partial\Omega/\partial\zeta = -(\partial\Omega_1/\partial\zeta + \partial\Omega_2/\partial\zeta)$. This force in so far as it depends on Ω_2 is produced by the matter internal to the sphere of radius ζ, since that forming the shell between its surface and that of the concentric sphere of radius r produces no effect. Thus if the distribution be situated in a medium of inductivity μ_0

$$-\frac{\partial\Omega_2}{\partial\zeta} = \frac{4}{3}\pi\rho\frac{\zeta^3}{\mu_0\zeta^2} = \frac{4}{3}\pi\rho\frac{\zeta}{\mu_0},$$

and therefore

$$\mu_0\left(\frac{\partial^2\Omega_2}{\partial\zeta^2} + \frac{2}{\zeta}\frac{\partial\Omega_2}{\partial\zeta}\right) + 4\pi\rho = 0,$$

or returning to the standard form of the equation

$$\mu_0\left(\frac{\partial^2\Omega_2}{\partial x^2} + \frac{\partial^2\Omega_2}{\partial y^2} + \frac{\partial^2\Omega_2}{\partial z^2}\right) + 4\pi\rho = 0.$$

But since there is none of the matter producing Ω_1 at the point considered, Laplace's equation holds for Ω_1 at that point. Hence we get finally

$$\mu_0\left(\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2}\right) + 4\pi\rho = 0 \quad . \quad . \quad . \quad (71)$$

which is Poisson's equation.

77. The characteristic equation of the potential at the surface of a

magnetized body is easily found from the considerations above as to the discontinuity of the normal component of magnetic force at the surface. If we take lengths along normals to the surface drawn from the surface to the media on the two sides, we can calculate the surface density σ which would be consistent with the *actual* normal intensities $-\partial\Omega/\partial\nu$, $-\partial\Omega'/\partial\nu'$, if the medium on both sides of the surface were the standard medium of inductivity μ_0. This would be given by the equation

$$\mu_0\left(\frac{\partial\Omega}{\partial\nu}+\frac{\partial\Omega'}{\partial\nu'}\right)+4\pi\sigma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Thus, μ, μ' being the inductivities of the media in which the intensities are $-\partial\Omega/\partial\nu$, $-\partial\Omega'/\partial\nu'$, respectively, we have for the continuity of the normal component of the induction and the discontinuity of the magnetic force the equations

$$\mu\frac{\partial\Omega}{\partial\nu}+\mu'\frac{\partial\Omega'}{\partial\nu'}=0,$$

$$\mu_0\left(\frac{\partial\Omega}{\partial\nu}+\frac{\partial\Omega'}{\partial\nu'}\right)+4\pi\sigma=0.$$

These give

$$\sigma=\mu_0\frac{\mu'-\mu}{4\pi\mu}\frac{\partial\Omega'}{\partial\nu'}=\mu_0\frac{\mu-\mu'}{4\pi\mu'}\frac{\partial\Omega}{\partial\nu} \quad . \quad . \quad . \quad . \quad (73)$$

The density thus found is from its definition of a fictitious nature, and the expressions for it are not of very much practical use. But since they are usually given they are repeated here with the addition of the factor μ_0 necessary to give σ its proper dimensions.

Method of Vector Potential

78. We have seen that the surface integral of magnetic induction over a closed surface in a magnetic field is zero. It follows from this result that the surface integral over an unclosed surface depends only on the boundary. For it is equal and opposite to the surface integral over every cap which can be fitted to the boundary so as to close the surface. The integral over the cap must depend only on its boundary and so therefore does that over the surface to which it is applied.

Hence it follows that the integral of magnetic induction over the surface can be expressed as an integral in terms of the boundary, that is by a line integral, of some quantity A round the boundary. Since this quantity must change sign with the induction, it is a directed quantity. Let it make an angle θ with the element ds of the boundary, its components along the axis be F, G, H, the projections of an element ds of the boundary on the axes dx, dy, dz, and

the direction cosines of the normal to an element dS of the surface l, m, n, then we have

$$\int(la + mb + nc)dS = \int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds$$

$$= \int \mathbf{A}\cos\theta\, ds \quad . \quad . \quad . \quad (74)$$

79. To find the relation between F, G, H and a, b, c, consider a triangle $A\,B\,C$ (Fig. 27) formed by the three mutually rectangular faces of a tetrahedron, the three edges OA, OB, OC of which are the axes of x, y, z. Then according to the theorem the surface integral of $\mathbf{B}$ over the triangular area is equal to the line integral round ABC. We shall take the latter integral round the boundary in the direction in which it would be necessary for a person to go round in order to have the area on his left hand. Now we see at once that the integration round ABC can be converted into integration round the three closed paths $OABO$, $OBCO$, $OCAO$, since the integrations along the lines OA, OB, OC are cancelled by integrations along AO, BO, CO.

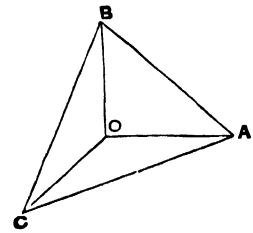


FIG. 27.

Thus we get

$$\int_{ABC}\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds$$

$$= \int_{OABO}\left(F\frac{dx}{ds} + G\frac{dy}{ds}\right)ds + \int_{OBCO}\left(G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds + \int_{OCAO}\left(H\frac{dz}{ds} + F\frac{dx}{ds}\right)ds.$$

Now let the tetrahedron be small and the lengths of its edges dx, dy, dz, then we can use the values of F, G, H for the middle points of OA, AB, BO, &c., in calculating the quantities on the right. Thus the first of the three integrals is

$$\left(F + \frac{1}{2}\frac{\partial F}{\partial x}dx\right)dx - \left(F + \frac{1}{2}\frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial F}{\partial y}dy\right)dx$$

$$+ \left(G + \frac{1}{2}\frac{\partial G}{\partial x}dx + \frac{1}{2}\frac{\partial G}{\partial y}dy\right)dy - \left(G + \frac{1}{2}\frac{\partial G}{\partial y}dy\right)dy,$$

which reduces to

$$\tfrac{1}{2}\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)dxdy = \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) \times \text{area } AOB.$$

The same method gives corresponding results which we can at once

write down for *OBCO, OCAO.* Now take *ABC* as dS, then since area $AOB = ndS$, &c.

$$(la + mb + nc) = l\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right) + m\left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right) + n\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right).$$

Thus we find the equations

$$\left.\begin{aligned} a &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} \\ b &= \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} \\ c &= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (75)$$

Specification of Vector Potential

80. These give the components of magnetic induction in terms of those of *vector-potential*, which is the name given to the quantity **A**. To specify the vector-potential consider that due to an element of the magnetic distribution of volume dv, and therefore of magnetic moment $\mathbf{I}dv$. Then the vector-potential due to this element is $\mathbf{I}dv \sin\phi/r^2$, at a point P distant r from the element on a line making an angle ϕ with the positive direction of magnetization. The direction of the vector-potential is at right angles to the plane of the angle ϕ, and, in accordance with the direction chosen as that of integration, appears to an eye regarding the path of integration of **A** in the direction opposed to that of magnetization of the element to be directed round the curve in the counter-clockwise direction.

We shall see that the specification of the vector-potential thus given corresponds precisely to that of the direction of the magnetic force at P, due to an element of a circuit replacing $\mathbf{I}dv$ so that the direction of flow of current is the same as that of magnetization.

81. To verify the specification let p, q, r be the direction cosines of $\mathbf{I}$, x, y, z the coordinates of $\mathbf{I}dv$, ξ, η, ζ those of the point considered, then

$$\mathbf{I}dv\frac{\sin\phi}{r^2} = \frac{\mathbf{I}dv}{r^3}[\{q(\zeta - z) - r(\eta - y)\}^2 + \&c.]^{\frac{1}{2}},$$

the r inside the brackets only being a direction cosine. This gives

$$dF = \frac{\mathbf{I}dv}{r^3}\{q(\zeta - z) - r(\eta - y)\},$$

$$dG = \frac{\mathbf{I}dv}{r^3}\{r(\xi - x) - p(\zeta - z)\},$$

$$dH = \frac{\mathbf{I}dv}{r^3}\{p(\eta - y) - q(\xi - x)\}.$$

Integrating throughout the whole distribution of magnetism we get putting u for $1/r$, and for $\mathbf{I}p$, $\mathbf{I}q$, $\mathbf{I}r$ their values A, B, C

$$\left.\begin{aligned} F &= \int\left(B\frac{\partial u}{\partial z} - C\frac{\partial u}{\partial y}\right)dv \\ G &= \int\left(C\frac{\partial u}{\partial x} - A\frac{\partial u}{\partial z}\right)dv \\ H &= \int\left(A\frac{\partial u}{\partial y} - B\frac{\partial u}{\partial x}\right)dv \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (76)$$

Hence we have since $\partial u/\partial\xi = -\partial u/\partial x$, &c.

$$a = \frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} = -\frac{\partial}{\partial \xi}\int\left(A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + C\frac{\partial u}{\partial z}\right)dv - \int A\nabla^2 u dv \quad . \quad (77)$$

with similar expressions for b and c.

The first term of the expression on the right is the value of $\mu_0 a$, where a is the magnetic force at the point considered, due to the magnetic distribution elsewhere, and μ_0 is the inductive capacity of the medium with respect to which the values A, B, C of the components of magnetization are taken.

The remaining term is zero unless the point considered falls within the limits of integration. If the latter is the case the value is $4\pi A'$, if A' is the value of A at the point considered. For it is clear that $Audv_0$ is the potential at the point considered of a volume element of attracting or repelling matter of density A, and therefore that the potential at the same point, U say, of the whole distribution is given by taking the volume integral, so that

$$U = \int A u dv.$$

Hence

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} + 4\pi A' = 0,$$

where A' is the density at the point considered.
But

$$\nabla^2 U = \int A\nabla^2 u dv,$$

and therefore

$$\int A\nabla^2 u dv = -4\pi A'.$$

Thus we obtain finally

$$a = \frac{\partial H}{\partial \eta} - \frac{\partial G}{\partial \zeta} = \mu_0 a + 4\pi A'$$

by equation (77). Similarly we should find

$$b = \frac{\partial F}{\partial \zeta} - \frac{\partial H}{\partial \xi} = \mu_0 \beta + 4\pi B',$$

$$c = \frac{\partial G}{\partial \xi} - \frac{\partial F}{\partial \eta} = \mu_0 \gamma + 4\pi C',$$

and the specification is verified.

The use of the vector-potential is sometimes convenient as an analytical expedient. But it is not a physical quantity which can be observed experimentally, and its use is sometimes attended with difficulty owing to the introduction of certain arbitrary functions which there is some trouble in interpreting.

Uniformly Magnetized Ellipsoid

82. Many special cases of magnetization might be described as examples of the principles set forth above, but we shall consider only one or two of great practical importance. The first we choose is that of an ellipsoid of uniform quality magnetized entirely inductively in a uniform field of inductivity μ_0; but before dealing with this particular example we consider some results which hold generally for inductively and uniformly magnetized bodies.

It is clear that any case of uniform magnetization may be imagined as produced by creating two volume distributions of magnetism, of uniform density numerically the same in the two cases but opposite in sign, coincident with the body, then displacing them relatively to one another through a small distance parallel to the direction of magnetization, and so that the motion of the positive distribution relatively to the negative is in the direction of magnetization. The density ρ must be imagined great so that when it is multiplied by the displacement, δs say, the product $\rho \delta s$ may be finite. For since the surface density of the magnetic distribution is $\mathbf{I} \cos \theta$, where θ is the angle between the direction of magnetization and the normal to the surface at any point, and the surface density of the magnetic distribution produced is $\rho \delta s . \cos\theta$, it is clear that $\mathbf{I} = \rho . \delta s$.

If now we call $\rho U/\mu_0$ the magnetic potential at any point P produced by the positive volume distribution, the potential at the same point due to the negative distribution will be opposite in sign to, but the same numerically as, the value which $\rho U/\mu_0$ would take if the point P were displaced a distance δs in the positive direction, that is it is $-\rho(U + \partial U/\partial s . \delta s)/\mu_0$. Hence if Ω be the potential at P due to the magnetized body we have

$$\mu_0 \Omega = -\rho \delta s . \frac{\partial U}{\partial s} = -\mathbf{I} \frac{\partial U}{\partial s}.$$

Putting p, q, r for the direction cosines of $\mathbf{I}$ we have $A, B, C = \mathbf{I}p, \mathbf{I}q, r$, and since $p, q, r = dx/ds, dy/ds, dz/ds$,

$$\mu_0\Omega = -\left(A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + C\frac{\partial U}{\partial z}\right) = AX + BY + CZ \quad . \quad . \quad (78)$$

if X, Y, Z be the components of magnetic force at P due to the positive distribution when $\rho = 1$. This equation enables the components of magnetic intensity for the actual case to be found by differentiation.

83. We can now apply these results to the case of a uniformly magnetized ellipsoid. We have only to find first the components X, Y, Z of force exerted at the external point by a homogeneous ellipsoid of repelling matter.

It can be proved[1] that for an external point the values of X, Y, Z are given by

$$X = 2\pi abc\xi\int_{\phi_1^2}^{\infty} \frac{d(\phi^2)}{\{(a^2+\phi^2)^3(b^2+\phi^2)(c^2+\phi^2)\}^{\frac{1}{2}}} \quad . \quad . \quad . \quad (79)$$

and two similar equations for Y, Z, that may be written down by symmetry. Here a, b, c denote the semiaxes of the given ellipsoid, and ϕ_1^2 the positive root of the cubic

$$\frac{\xi^2}{a^2+\phi^2} + \frac{\eta^2}{b^2+\phi^2} + \frac{\zeta^2}{c^2+\phi^2} = 1.$$

If now we choose multipliers L, M, N so that $X = L\xi$, $Y = M\eta$, $Z = N\zeta$, we have

$$\mu_0\Omega = AL\xi + BM\eta + CN\zeta \quad . \quad . \quad . \quad . \quad . \quad (80)$$

Thus the components α, β, γ of magnetic force are given by

$$\mu_0\alpha = -A\left(L + \xi\frac{\partial L}{\partial \xi}\right), \quad \mu_0\beta = -B\left(M + \eta\frac{\partial M}{\partial \eta}\right),$$

$$\mu_0\gamma = -C\left(N + \zeta\frac{\partial N}{\partial \zeta}\right),$$

where $L\xi$, $M\eta$, $N\zeta$ are given by (79) and similar equations.

On the other hand for an internal point the components of magnetic force are calculable from a potential Ω given also by (78), but with values of X, Y, Z given by

$$X = 2\pi abc\xi\int_{0}^{\infty} \frac{d(\chi^2)}{\{(a^2+\chi^2)^3(b^2+\chi^2)(c^2+\chi^2)\}^{\frac{1}{2}}} \quad . \quad . \quad (81)$$

[1] See *Absolute Measurements in Electricity and Magnetism*, Vol. II. Part I. p. 52, *et seq.*

and similar equations for Y, Z.[1] Hence

$$\mu_0\alpha = -AL, \quad \mu_0\beta = -BM, \quad \mu_0\gamma = -CN \quad . \quad . \quad (82)$$

since L, M, N are independent of ξ, η, ζ.

The magnetic force is thus uniform within the ellipsoid, and is the same for the same intensity of magnetization within similar ellipsoids.

The direction cosines of the magnetic force are proportional to AL, BM, CN, and those of $\mathbf{I}$ to A, B, C. Thus the force does not coincide in direction with the magnetization except when the latter is along one of the axes of the ellipsoid. In the latter case the force is in the opposite direction to the magnetization, and tends to demagnetize the body.

Uniformly Magnetized Ellipsoid of Revolution. Demagnetizing Forces

84. The integral can be easily obtained in finite terms if the ellipsoid have two axes equal, that is, is an ellipsoid of revolution. Thus to find L we make the substitution $(a^2+\chi^2)^{\frac{1}{2}} = 1/v$, and integrate. Similarly M and N can be found. We find (1) for a prolate ellipsoid of eccentricity ϵ [$b = c = a\sqrt{(1-\epsilon^2)}$]

$$\left.\begin{aligned} L &= 4\pi\frac{1-\epsilon^2}{\epsilon^2}\left(\frac{1}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon} - 1\right) \\ M = N &= 2\pi\frac{1}{\epsilon^2}\left(1 - \frac{1-\epsilon^2}{2\epsilon}\log\frac{1+\epsilon}{1-\epsilon}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (83)$$

(2) for an oblate ellipsoid of eccentricity ϵ ($b = c = a/\sqrt{1-\epsilon^2}$)

$$\left.\begin{aligned} L &= \frac{4\pi}{\epsilon^2}\left(1 - \frac{\sqrt{1-\epsilon^2}}{\epsilon}\sin^{-1}\epsilon\right) \\ M = N &= 2\pi\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}\left(\frac{1}{\epsilon}\sin^{-1}\epsilon - \sqrt{1-\epsilon^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (84)$$

If the ellipsoid be infinitely long $\epsilon = 1$, and therefore $\mathrm{L} = 0$. Hence if the magnetization is parallel to the axis the force within an infinitely long ellipsoid (or which is the same within a uniformly magnetized cylinder of great length, at a point at a distance from either end great in comparison with the diameter) is zero, and if the magnetization is transverse it is $-2\pi\mathbf{I}/\mu_0$ in the same direction.

If the ellipsoid is so oblate as to be capable of being regarded as a flat disk, $M = N = \pi^2 a/c$, and the force at right angles to the disk is

$$-\frac{\mathbf{I}L}{\mu_0} = -4\pi\frac{\mathbf{I}}{\mu_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

[1] See *Absolute Measurements in Electricity and Magnetism*, Vol. II. Part I. p. 52 *et seq.*

Lastly, if the ellipsoid is spherical $\epsilon = 0$, and if the magnetization is parallel to the axis of x, the force within it is

$$-\frac{\mathbf{I}L}{\mu_0} = -\frac{4}{3}\pi\frac{\mathbf{I}}{\mu_0} \quad \ldots\ldots\ldots \quad (86)$$

All these forces are in the opposite direction to that of **I**. They therefore tend to diminish the magnetization by producing magnetization of the opposite sign to that existing. Hence they are called demagnetizing forces. We shall have to take them into account when we deal with experimental determinations of magnetic induction in the magnetizable metals.

Uniformly Magnetized Sphere

85. Consider now a uniformly magnetized sphere placed in a field of uniform intensity **F** and inductivity μ_0, and let the directions of **I** and **F** coincide. Then for the force within the sphere we have

$$\mathbf{H} = \mathbf{F} - \frac{4}{3}\frac{\pi\mathbf{I}}{\mu_0}$$

and if κ be the magnetic susceptibility

$$\mu_0\kappa\mathbf{H} = \mathbf{I} = \mu_0\kappa\mathbf{F} - \frac{4}{3}\pi\kappa\mathbf{I}.$$

Therefore

$$\mathbf{I} = \frac{\mu_0\kappa}{1 + \frac{4}{3}\pi\kappa}\mathbf{F} \quad \ldots\ldots\ldots \quad (87)$$

Or if μ denote the inductivity of the sphere so that $\mu = \mu_0(1 + 4\pi\kappa)$

$$\frac{\mathbf{I}}{\mathbf{F}} = \frac{3\mu_0}{4\pi}\frac{\mu - \mu_0}{2\mu_0 + \mu} = \frac{3\mu_0}{4\pi}\frac{\varpi - 1}{\varpi + 2} \quad \ldots\ldots \quad (88)$$

if ϖ denote the permeability μ/μ_0 of the sphere.

If ϖ be great this approximates to $3\mu_0/4\pi$, so that if the sphere is very susceptible the intensity of magnetization is always very approximately $3\mu_0\mathbf{F}/4\pi$. Hence it is useless to attempt to measure the susceptibility or permeability of a highly magnetizable substance by experiments on a spherical portion of it. The results for different specimens would be much more affected by slight deviations from the spherical figure than by actual differences in susceptibility.

In an important class of cases ϖ is less than unity, and then the value of **I** is given by the equation

$$-\mathbf{I} = \frac{3\mu_0}{4\pi}\frac{1 - \varpi}{2 + \varpi}\mathbf{F} \quad \ldots\ldots\ldots \quad (89)$$

so that **I** has the opposite sign to that which it had in the former case. Substances belonging to this class are called diamagnetics. Their nature will be more fully considered in a later chapter.

Field External to Uniformly Magnetized Sphere

86. It is to be noticed that the field external to the sphere is disturbed by the magnetization. Figs. 28 and 29 show the field both

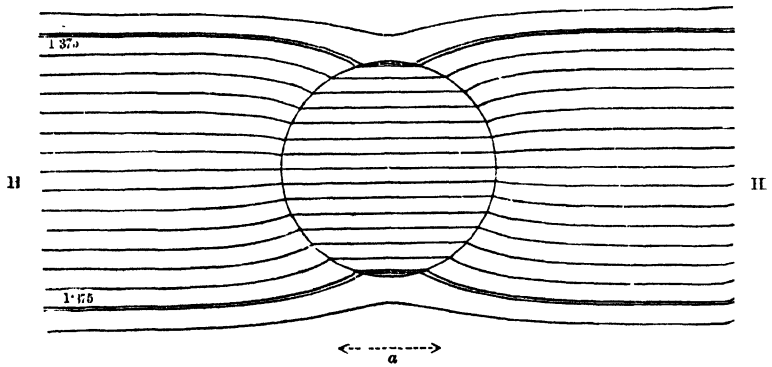


FIG. 28.

outside and inside a paramagnetic and a diamagnetic sphere respectively. The value of ϖ for the former is 2·8, for the latter ·48.

The equation of the lines of force is easily found. The magnetic force due to the magnetization of the sphere is, from the synthesis given above, clearly the same at all external points as that due to a doublet

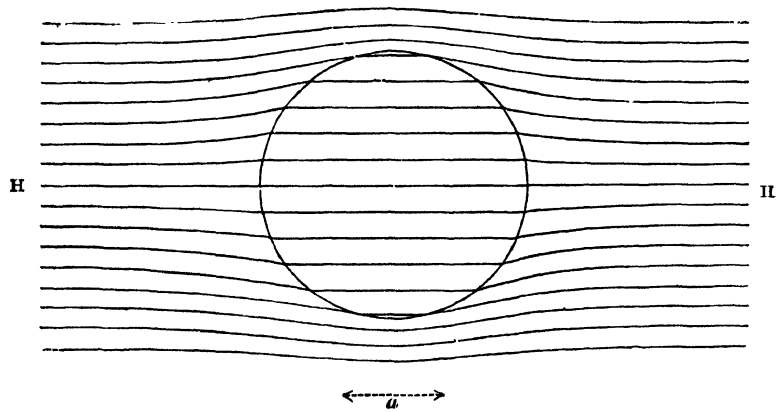


FIG. 29.

of moment = $\mathbf{I} \times$ volume of sphere, placed at the centre with its axis in the direction of magnetization. Denoting the moment of the doublet by $\mathbf{m}$, then we know by the theorem given at p. 19 that the magnetic

potential of the doublet, supposed situated at the origin with its axis in the direction of x, has at the point (x,y,z) the value

$$\Omega = \frac{\mathbf{m}}{\mu_0} \frac{x}{(x^2 + y^2)^{\frac{3}{2}}}.$$

The total component force in the direction of x is thus

$$X = -\frac{\partial \Omega}{\partial x} + \mathbf{F} = \frac{\mathbf{m}}{\mu_0} \frac{2x^2 - y^2}{(x^2 + y^2)^{\frac{5}{2}}} + \mathbf{F},$$

and that in the direction of y is

$$Y = -\frac{\partial \Omega}{\partial y} = \frac{\mathbf{m}}{\mu_0} \frac{3xy}{(x^2 + y^2)^{\frac{5}{2}}}.$$

But the equation of a line of force is $dx/X = dy/Y$, which gives

$$\frac{\mathbf{m}}{\mu_0} \frac{3xy}{(x^2 + y^2)^{\frac{5}{2}}} dx + \left(\frac{\mathbf{m}}{\mu_0} \frac{y^2 - 2x^2}{(x^2 + y^2)^{\frac{5}{2}}} - \mathbf{F} \right) dy = 0,$$

an equation at once integrable when multiplied by y. The integral is

$$\frac{\mathbf{m}y^2}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{\mu_0}{2} \mathbf{F} y^2 = C \quad . \quad . \quad . \quad . \quad . \quad (90)$$

where C is a parameter variable only from line to line.

Either the positive or the negative sign may be given to the radical on the left. With the positive sign the equation suits the case of a paramagnetic sphere magnetized by the field, with the negative sign it expresses the form of an external line of force when the sphere is diamagnetized by the action of the field.

The equation may be written in the form

$$y^2 = b^2 - \frac{a^3 y^2}{(x^2 + y^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

by putting $2\mathbf{m}/\mathbf{F}\mu_0 = a^3$, and $2C/\mathbf{F}\mu_0 = b^2$. From this form of the equation the curves given in Figs. 28 and 29 are plotted. These curves are taken from Lord Kelvin's paper on Lines of Force (*Reprint of Papers on Electrostatics and Magnetism*, § 632), in which the theory stated above was first given and illustrated.

CHAPTER III

TERRESTRIAL MAGNETISM

Directive Force on Compass Needle. Magnetic Dip. Magnetic Equator

87. THE tendency of a suspended lodestone or magnet, or compass needle to set itself at a given place in a particular direction, was at a very early date attributed to some action of the earth. It was recognized after the discovery of the magnetic dip by Hartmann about 1544 and Norman in 1576, that at every point of the earth's surface there is a magnetic force in a definite direction generally inclined to the horizontal, and further observation showed that the direction and magnitude of this force varied from point to point on the surface of the earth.

The whole matter was discussed in the clearest manner by the famous Dr. Gilbert, of Colchester, Physician in Ordinary to Queen Elizabeth, in his Latin treatise, *De Magnete magneticisque corporibus*. There he put forward the extremely important idea that the earth is a great magnet, and that therefore, in the language of Faraday, there exists a terrestrial magnetic field (or *orbis virtutis*, as Gilbert called it), in which a small needle, except in so far as it is disturbed by gravitational or other forces, sets itself with its axis in the direction of the resultant force at the place where it is situated.

The different positions of a small needle freely suspended at different parts of the earth's surface are illustrated by the drawing of a terrella, Fig. 6 above, which is taken from the second edition of Gilbert's book. It is instructive to compare this with the lines of force due to a uniformly magnetized sphere, which we have seen are at external points coincident with those due to a small magnet at the centre. Here it is clear that at the points N. and S. the needle would stand vertical, and at points on the circle midway between these horizontal.

This corresponds roughly to what is found by magnetic surveys to take place at the earth's surface. At different points on a sinuous line surrounding the earth in the region of the equator the dipping needle (that is a bar magnet suspended by its centre of gravity) remains horizontal. This line is called the magnetic equator.

The theory that terrestrial magnetic phenomena are due to a small

magnet at the earth's centre seems to have been held by Tobias Mayer, who flourished between 1723 and 1762. He worked out according to this theory the dip and force on a small needle at different places on the earth's surface. The results do not agree with observation; but it was a noteworthy attempt to explain on a simple hypothesis the facts of terrestrial magnetism.

Terrestrial Magnetic Poles

88. The conclusion which has been arrived at from a study of observations of terrestrial magnetism made at different parts of the earth's surface is that there are two distinct points, one in each hemisphere, at which the horizontal component of magnetic force vanishes and the dipping needle rests vertical. These are called magnetic poles. They are marked on the chart, Plate I (at the end of this volume), showing lines of equal total magnetic force for the date 1875.

It will be seen that the magnetic poles are not diametrically opposite and do not coincide with the extremities of the earth's axis. According to the calculations of Gauss, who investigated this subject, they should lie at latitude 73° 35′ N., longitude 264° 21′ E., and latitude 72° 35′ S., longitude 152° 30′ E., respectively. The north magnetic pole was, however, reached in 1831 by Sir James Ross, and found to be at latitude 70° 5′ N. and longitude 263° 17′ E. Again, on the *Erebus* and *Terror* expedition of 1839–43, a dip of 88° 56′ was attained, and it was inferred from the observations that the south magnetic pole was situated in latitude 73° 5′ S. and longitude 147° 5′ E. These positions agree very closely with those given by Gauss. As will be seen later, there is reason to believe that the poles are gradually changing their positions.

It is to be noted that the line joining these points is not parallel to the magnetic axis of the earth, that is the axis of greatest magnetic moment. This will appear later in the sketch we propose now to give of Gauss's great memoir on Terrestrial Magnetism.[1]

True and False Magnetic Poles

89. It has been held by many people that there are two north poles and two south poles of terrestrial magnetism. It is easy to show that if there are more north poles or more south poles than one, there must be an odd number of each. For consider surfaces of equal magnetic potential. A north or south pole must be a place where such a surface touches the surface of the earth. If then there are two north poles, these must be due either to the fact that two equipotential surfaces touch the earth, or that one touches in two points. If two surfaces touch, then, since equipotential surfaces cannot cut one another, they must each have two protuberances, as shown by the dotted lines in the diagram (Fig. 30). The intersection of the horizontal surface of the

[1] *Allgemeine Theorie des Erdmagnetismus. Result. d. Magnetischen Vereins*, Leipzig, 1839. Werke, 5ter Bd., S. 121.

earth with the equipotential surfaces must therefore give the other curves represented in the diagram, namely, two series of closed curves passing into a figure of 8 curve, and then into larger curves inclosing the

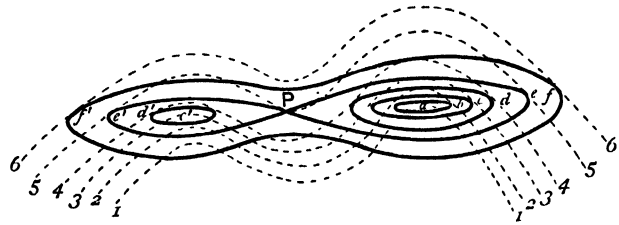


FIG. 30.

others. Thus the points a and c' are true poles in the sense of the definition, that is, the horizontal component of the magnetic intensity vanishes at these points. But the point P is also a point at which an equipotential surface touches the surface of the earth at a point of depression, and is therefore also a magnetic pole. There is, however, a distinct difference between the pole P and either of the other two. Supposing the latter to be true north poles, that is points towards which the north pointing end of a magnet turns, it will be seen that a north pointing pole near P, but within the looped figure, will point away from P, while if it be outside it will point towards P. Hence such a pole has been called, though not quite appropriately, a *false* north pole. It is easy thus to see that counting both true and false poles,

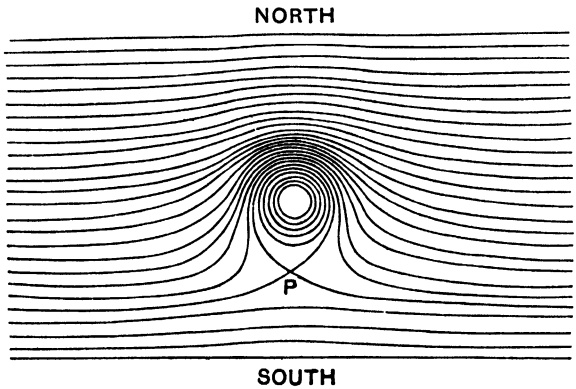


FIG. 31.

there must be an odd number if there are more than one pole of either kind. This result is also due to Gauss.

90. A false pole may also be produced by the presence in the earth's strata, near the surface, of a quantity of magnetic iron ore. The illustration of such a case given in Fig. 32 is due to Gauss. The

diagram shows the effect, on the east and west equipotential lines, of a magnet buried under the surface with its north pointing pole uppermost. It will be seen at once that this is a particular case of Fig. 31, in which one of the two poles there shown is at an infinite distance from the other. Since the horizontal forces of the earth and magnet cancel one another at the double point P, it is here again a so-called false pole of the same kind as the earth's south pole for a needle placed outside the looped figure, that is north of the equipotential line through P, of the opposite kind for a needle placed elsewhere.

Magnetic Potential at Earth's Surface

91. The question as to whether the magnetic intensities in the case of terrestrial magnetism are derivable from a potential is one not yet quite settled, and we shall return to it in Art. 105 below. Assuming that they are so derivable, let Ω be the magnetic potential, and consider the force at a point on the earth's surface. If H be the horizontal component and $\mathbf{H}$ the total force we have $H = \mathbf{H} \cos \psi$ where ψ is the dip. If now ds be any element of a line drawn on the earth's surface making an angle θ with the direction of H, that is with the magnetic meridian, the force along ds is $\mathbf{H} \cos \psi \cos \theta$.

Hence

$$-\frac{d\Omega}{ds} = \mathbf{H} \cos \psi \cos \theta = H \cos \theta \ldots\ldots (1)$$

Thus if Ω, Ω' be the values of Ω at the beginning and end of any line drawn on the surface of the earth

$$\Omega - \Omega' = \int H \cos \theta \, ds \ldots\ldots (2)$$

where the integral is taken from end to end of the line.

This integral has the same value along whatever path the integral is taken from a given initial to a given final point, and therefore, on the assumption made above, the integral taken round a closed curve is zero.

Horizontal Terrestrial Magnetic Force at Earth's Surface

92. Let now $P_0 P_1$, P_2, ... Fig. 33, be the angular points of a polygon drawn on the surface of the earth, the sides being arcs of great circles of the surface regarded as spherical, each small in length in comparison with the circumference of a great circle. We may take as the integral along each side the length of the side multiplied into the mean of the values of $H \cos \theta$ for the beginning and end. Thus if θ_1 be the inclination of the first side $P_0 P_1$, at its initial end, to the magnetic meridian, and θ'_1, the angle for the other end, we have for the integral along the side the approximate value.

$$\tfrac{1}{2}(H_0 \cos \theta_1 + H_1 \cos \theta'_1) \, P_0 P_1.$$

Denote by ζ the variation at any place, that is the angle in azimuth between the astronomical north and the magnetic north, reckoning it positive when toward the west and negative when toward the east. Then if $H_0, H_1, H_2, \ldots$ be the horizontal forces at $P_0, P_1, P_2, \ldots, \zeta_0, \phi_{01}$, the variation and the azimuth of the line $P_0 P_1$ at P_0, ζ_1, ϕ'_{01}, the cor-

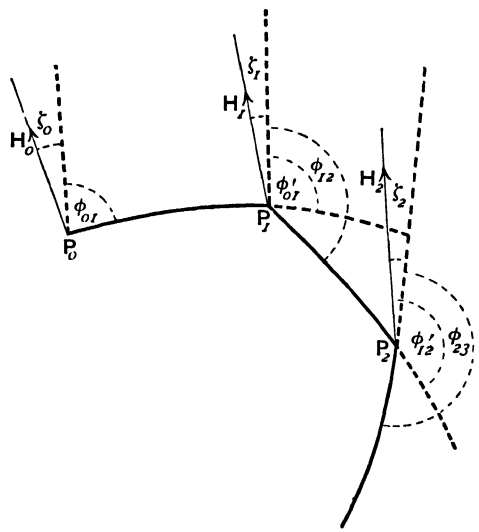


FIG. 32.

responding angles at P_1, and so on, the integral along $P_0 P_1$ is nearly

$$\frac{P_0P_1}{2}\left\{ H_0 \cos(\phi_{01} + \zeta_0) + H_1 \cos(\phi'_{01} + \zeta_1) \right\}.$$

Again the integral along $P_1 P_2$ is

$$\frac{P_1P_2}{2}\left\{ H_1 \cos(\zeta_1 + \phi_{12}) + H_2 \cos(\zeta_2 + \phi'_{12}) \right\},$$

and so on.

These added round a closed polygon ought to give a zero result provided, as we shall see, that there is on the whole no electric current flowing across the enclosed area; and therefore if the polygon be a triangle for which the lengths of the sides, the angles specified, and the horizontal forces at two of the angular points, are known, we are able to calculate the horizontal force at the third point. Gauss performed this calculation for a geodesic triangle having its vertices at Göttingen, Milan, and Paris. From the latitudes and longitudes of the places it was easy to calculate the angular distances of the places from one another along great circles on the earth's surface, and hence from the known radius of the earth to find their actual distances. The value

of H for Paris was thus found from those for Göttingen and Milan to within $\frac{1}{5}$ per cent. of the observed value.

93. We may express the horizontal magnetic force at the earth's surface in terms of two components, one along the astronomical meridian of the place, the other at right angles to the meridian. Let λ denote the longitude of the place, u the complement of the geographical latitude, and let the meridian be an ellipse of semi-major axis R, and semi-minor axis $R(1-\epsilon)$. Then the length ds of an arc of the meridian is $Rdu\,(1-\epsilon)\,[1+\{(1-\epsilon)^4-1\}\sin^2 u]^{\frac{1}{2}}/\{1-(2\epsilon-\epsilon^2)\sin^2 u\}$ and an element of the parallel of latitude has the length $d\sigma = R\,(1-\epsilon)\sin u\,d\lambda/\{1-(2\epsilon-\epsilon^2)\sin^2 u\}^{\frac{1}{2}}$. Taking now the component force X in the direction of the astronomical meridian and Y that along the parallel of latitude in the westward, here the positive, direction of λ, we have

$$\left.\begin{aligned} X = -\frac{d\Omega}{ds} &= \frac{\{1-(2\epsilon-\epsilon^2)\sin^2 u\}^{\frac{1}{2}}}{R(1-\epsilon)\,[1+\{(1-\epsilon)^4-1\}\sin^2 u]^{\frac{1}{2}}}\,\frac{\partial\Omega}{\partial u} \\ Y = -\frac{d\Omega}{d\sigma} &= -\frac{\{1-(2\epsilon-\epsilon^2)\sin^2 u\}^{\frac{1}{2}}}{R(1-\epsilon)\sin u}\,\frac{\partial\Omega}{\partial\lambda} \end{aligned}\right\}. \quad (3)$$

The horizontal force is therefore $\sqrt{X^2+Y^2}$, and if ζ be the variation $\tan\zeta = Y/X$.

If ϵ be put $=0$, that is if the surface be regarded as spherical,

$$X = \frac{1}{R}\frac{\partial\Omega}{\partial u},\quad Y = -\frac{1}{R\sin u}\frac{\partial\Omega}{\partial\lambda}. \quad . \quad . \quad . \quad . \quad (4)$$

are force components at the surface. If the point (supposed external) be at distance r from the centre, R in these expressions must be replaced by r.

Determination of Surface Potential from Horizontal Force Relations between Horizontal Components

94. We can now prove a number of theorems of great interest. First of all, if the northward component of force is known all over the surface of the earth, and the magnetic potential at one point, the potential at any other point can be found. For if Ω be the potential at the given point, and Ω_0 that at any other point of a latitude u_0, then by integrating along the meridian through the latter point to the north pole, and thence to the given point along its meridian, we find the required potential, thus

$$\Omega = \Omega_0 - \int_{u_0}^{0} RXdu + \int_0^u RXdu.$$

The first two terms give the potential at the north pole, Ω_n say. Thus

$$\Omega = \Omega_n + \int_0^u RXdu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From this we can find the westward component of the magnetic force at the given place. For

$$Y = -\frac{1}{R\sin u}\frac{\partial\Omega}{\partial\lambda} = -\frac{1}{\sin u}\int_0^u \frac{\partial X}{\partial\lambda}du \quad . \quad . \quad . \quad (6)$$

95. Again, if Y be given for all points the northward component at each point can also be found, provided the value of X along any line running from the north pole to the south is known, say along a meridian. For, integrating along the meridian first from the north pole, then along a parallel of latitude to the given point, we find

$$\Omega = R\int_0^u X du - R\int_{\lambda_0}^{\lambda} Y\sin u d\lambda + \Omega_n \quad . \quad . \quad . \quad (7)$$

Thus Ω is known along the earth's surface and therefore so also is X. The value of the constant Ω_n so far as X is concerned is of no importance.

96. Consider now a closed path on the earth's surface, joining along parallels of latitude and meridians, four points defined by their longitude and colatitude as follows, λ,u, λ_1,u, λ_1,u', λ,u'. The line-integral along this path must vanish, and, therefore, putting X,X_1 for the forces along the meridians of longitude λ, λ_1, and Y, Y' for the forces along the parallels of colatitude u, u', respectively, we get

$$\sin u\int_\lambda^{\lambda_1} Y d\lambda - \int_u^{u'} X_1 du - \sin u'\int_\lambda^{\lambda_1} Y' d\lambda + \int_u^{u'} X du = 0.$$

If we take the parallels of latitude very close together, so that $u'=u-du$, we have $Y'=Y-\partial Y/\partial u.du$, and the equation just written becomes

$$X_1 - X + \int_\lambda^{\lambda_1}\frac{\partial}{\partial u}(Y\sin u)d\lambda = 0 \quad . \quad . \quad . \quad . \quad (7')$$

which is of course derivable from (7). From this it follows that if the northerly component of the horizontal magnetic force is known for any meridian and colatitude u, and the values of Y and $\partial Y/\partial u$ are known along the parallel of latitude from that meridian to another, the value of the northerly component can be found for that other meridian.

If the meridians differ in longitude by the amount $d\lambda(=\lambda_1-\lambda)$ only, (7′) becomes

$$\frac{\partial X}{\partial\lambda} + \frac{\partial}{\partial u}(Y\sin u) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7'')$$

which is only (6) in another form.

If the longitude be expressed in terms of local time t, measured, say, from the meridian of Greenwich where the time is t_0, we have

$\lambda = 2\pi(t_0 - t)/T$ where T is the length of the day. Thus we may write (7′), (7″) in the alternative forms

$$X_1 - X + \frac{2\pi}{T}\int_t^{t_1} \frac{\partial}{\partial u}(Y \sin u)dt = 0, \quad -\frac{\partial X}{\partial t} + \frac{2\pi}{T}\frac{\partial}{\partial u}(Y \sin u) = 0.$$

Expression of Magnetic Potential at an External Point in Series of Spherical Harmonics

97. Whatever the distribution of magnetism may be to which terrestrial magnetic phenomena are due, if it is wholly within the earth we can express the magnetic potential in a series of spherical harmonics. Thus if S_0, S_1, S_2, ... be spherical surface harmonics the potential at a point P distant r from the centre is given by

$$\Omega = S_0\frac{R}{r} + S_1\frac{R^2}{r^2} + S_2\frac{R^3}{r^3} + \ldots\ldots \quad (8)$$

which is convergent for $R \leqq r$, that is, if the point for which Ω is expressed is external to or on the surface of the sphere. If now dm be a portion of the magnetic distribution at a distance r_0 from the centre of the earth, and at colatitude and longitude u_0, λ_0, and u, λ, be the colatitude and longitude of P. the distance ρ between the points is

$$\rho = [r^2 - 2rr_0\{\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0)\} + r_0^2]^{\frac{1}{2}}.$$

Expanding $1/\rho$ in a series of harmonics we get

$$\frac{1}{\rho} = \frac{1}{r}\left(T_0 + T_1\frac{r_0}{r} + T_2\frac{r_0^2}{r^2} + \ldots\right) \quad (r > r_0).$$

Integrating throughout the whole distribution we have

$$\Omega = \int\frac{dm}{\rho} = \frac{1}{r}\left(T_0\int dm + \frac{1}{r}\int T_1 r_0 dm + \ldots\right) \quad \ldots \quad (9)$$

Comparing this with the series for Ω we see that

$$S_0 = \frac{T_0}{R}\int dm(=0), \quad S_1 = \frac{1}{R^2}\int T_1 r_0 dm, \quad S_2 = \frac{1}{R^3}\int T_2 r_0^2 dm, \quad (10)$$

98. At any external point at distance r from the centre we can find the forces X, Y, Z, the first two being taken as formerly, the last outwards along the radius vector r. Thus we have

$$\left.\begin{aligned} X &= \frac{R^2}{r^3}\left\{\frac{dS}{\partial u} + \frac{\partial S_2}{\partial u}\frac{R}{r} + \frac{\partial S_3}{\partial u}\frac{R_2}{r^2} + \cdots\right\} \\ Y &= -\frac{R^2}{r^3 \sin u}\left\{\frac{\partial S_1}{\partial \lambda} + \frac{\partial S_2}{\partial \lambda}\frac{R}{r} + \frac{\partial S_3}{\partial \lambda}\frac{R^2}{r^2}\cdots\right\} \\ Z &= \frac{R^2}{r^3}\left\{2S_1 + 3S_2\frac{R}{r} + 4S_3\frac{R^2}{r^2} + \cdots\right\} \end{aligned}\right\} \quad (11$$

At the surface of the earth where $R=r$, these take the form

$$\left.\begin{aligned} X &= \frac{1}{R}\left\{\frac{\partial S_1}{\partial u} + \frac{\partial S_2}{\partial u} + \frac{\partial S_3}{\partial u} + \ldots\right\} \\ Y &= -\frac{1}{R\sin u}\left\{\frac{\partial S_1}{\partial \lambda} + \frac{\partial S_2}{\partial \lambda} + \frac{\partial S_3}{\partial \lambda} \ldots\right\} \\ Z &= \frac{1}{R}\left\{2S_1 + 3S_2 + 4S_3 + \ldots\right\} \end{aligned}\right| \quad (12)$$

and the value of Ω at the surface is given by

$$\Omega = S_1 + S_2 + S_3 + \ldots + S_m + \ldots \quad . \quad . \quad (13)$$

99. The following conclusions follow from these results:

1. That if Z is known and can be expressed in a series of spherical harmonics $Z_1, Z_2, \ldots, \Omega$ can be found. Thus

$$Z = \frac{1}{R}(Z_1 + Z_2 + \ldots + Z_m + \ldots) \quad . \quad . \quad . \quad (14)$$

Comparing this with (12) we get

$$Z_m = (m+1)S_m \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which proves the proposition.

2. That if X is known for every point on the earth's surface the value of Ω can be calculated. For

$$\Omega = R\int_0^u X du + \Omega_n$$

and developing the integral in a series we find

$$-R\int_0^u X du = \Omega_n - S_1 - S_2 - \ldots - S_m - \ldots \quad . \quad . \quad (16)$$

By taking a sufficient number of different values of u and λ the different terms of this series can be calculated. It is to be observed that $S_1, S_2, \ldots$ are functions of u and λ.

3. If Y be known over the whole surface and X for all points on a line drawn from one pole to the other, a meridian for example, the value of Ω can also be found. This follows from (7).

100. Using results for twelve different places successively equidistant along each of seven parallels of latitude Gauss calculated the values of $S_1, S_2, \ldots$, and therefore the values of Ω for these places, as far as terms involving the fourth power of $\sin u$. The coefficients of the different terms in the values of S_1, S_2, &c., thus obtained, enabled the values of these quantities at other places to be found to the degree of accuracy involved in the number of terms of the series taken. The values of ψ and $\mathbf{H}$ were then calculated for a large number of places at which observations had been made, so that the calculated dip and total magnetic force for

those places could be at once tabulated. The results thus obtained are given in Gauss's memoir, side by side with those of observation, and show a very remarkable agreement. The results used were those given for total magnetic force at different places by Sir Edward Sabine,[1] for variation by Mr. Barlow,[2] and for dip by Horner.[3]

The graphic representation of Gauss's calculated results shows that there are only two magnetic poles, or places where the dip becomes 90° and the horizontal force vanishes. The positions of these have already been specified. Gauss gives as the value of Ω at these places numbers which reduced to C.G.S. units give

$$\text{at N. pole, } \Omega = -\cdot 313\, R,$$
$$\text{at S. pole, } \Omega = \cdot 360\, R,$$

R is of course to be taken in centimetres. It is approximately $6{\cdot}4 \times 10^8$ centimetres.

The Magnetic Moment of the Earth

101. We have seen from the expansions of $1/\rho$ and Ω that

$$R^2 S_1 = \int T_1 r_0 dm,$$

and therefore from the value of T_1 it can be shown that

$$R^2 S_1 = \alpha \cos u + \beta \sin u \cos \lambda + \gamma \sin u \sin \lambda,$$

where

$$\alpha = \int r_0 \cos u_0 dm \quad \beta = \int r_0 \sin u_0 \cos \lambda_0 dm \quad \gamma = \int r_0 \sin u_0 \sin \lambda_0 dm \quad (17)$$

the integrals being taken throughout the whole magnetization.

The first of these is the component of the magnetic moment of the earth in the direction of the earth's axis, the second the component in the direction of the equatorial radius of zero longitude, the third of the equatorial radius of longitude 90°. According to Gauss's calculated results reduced to C.G.S. units,

$$\alpha = \cdot 323476 R^3, \beta = \cdot 031106 R^3, \gamma = -\cdot 062456 R^3 \ . \quad (18)$$

102. To compare the magnetization with that of a bar of steel we calculate the maximum magnetic moment of the terrestrial magnet, that is the moment with reference to the magnetic axis. This axis is parallel to the diameter of the earth, intersecting its surface at the points of latitude and longitude 77° 50′ N., 296° 29′ E., and 77° 50′ S., 116° 29′ E. respectively. The moment relatively to this axis is in C.G.S. units $\cdot 33092 R^3$.

The moment per cubic centimetre, that is the mean intensity of magnetization, is thus $\cdot 33092/\frac{4}{3}\pi = \cdot 99276/12{\cdot}5614 = \cdot 07903$. Hence the moment of a cubic metre is 79000, nearly, in C.G.S. units.

Now a certain steel bar, a pound in weight, used by Gauss in a deter-

[1] *B.A. Report*, 1833. [2] *Phil. Trans. R.S.*, 1833. . [3] *Physik. Wörterbuch*, Band 6.

mination of the horizontal component of the earth's field intensity had a moment of 10087·7 C.G.S. Hence such bars uniformly distributed throughout the earth's substance at the rate of 7·83 per cubic metre, and having their like poles all turned in the same direction, would just give the above magnetization intensity.[1]

Certain long bars of steel of comparatively high magnetizability have been found by the author to take a magnetic moment of about 780 per cubic centimetre (that is, an induction in the steel of over 10,000, about four and a half times that taken by Gauss's bar!) Consequently the magnetic moment of a cubic centimetre of such steel is about ten times as great as that of a cubic decimetre of the earth; that is, the mean magnetization intensity of the earth's substance is about $\frac{1}{10000}$ of that of very highly magnetized hard steel.

Locality of Cause of Terrestrial Magnetic Phenomena

103. A question of great interest and importance discussed by Gauss is whether the cause of terrestrial magnetism is wholly within or partly without the earth's surface. If part of the magnetic distribution is external to the earth, the potential at any external point nearer the centre than any part of the external distribution, or at any point of the surface, can be expressed in a double series of spherical harmonics, one giving the effect of the internal, the other that of the external distribution. Thus

$$\Omega = S_1\frac{R^2}{r^2} + S_2\frac{R^3}{r^3} + \ldots + S_i\frac{R^{m+1}}{r^{m+1}} + \ldots$$
$$+ T_1\frac{r}{R} + T_2\frac{r^2}{R^2} + \ldots\ldots + T_i\frac{r^m}{R^m} + \ldots \quad . \quad . \quad (20)$$

where T_1, T_2, . . . are the harmonics expressing the surface value of the potential due to the external distribution.

Then we have at points on the earth's surface

$$-\frac{\partial\Omega}{\partial r} = Z = \frac{1}{R}\{2S_1 + 3S_2 + \ldots + (m+1)S_m + \cdots$$
$$- T_1 - 2T_2 - \ldots - mT_m - \ldots\} \quad (21)$$

Hence if Z_m denote the general or mth term of Z

$$RZ_m = (m+1)S_m - mT_m \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Again if Ω_m be the general term of Ω

$$\Omega_m = S_m + T_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From observations of vertical force the value of Z_m can be found and from observations of horizontal force that of Ω_m. It was found by Gauss that T_m is sensibly zero, so that so far as his researches went there was no sign of any external agency producing terrestrial magnetic force.

[1] *Intensitas vis Magneticae, &c.*, Art. 21; and *Allgem. Th. d. Erdmag.*, Art. 31.

104. We can now find a distribution of magnetism on the surface of the earth, which at all external points will produce the same potential and force as the actual distribution produces.

It is proved below (see Arts. 192—201, 315—319), that if we have a distribution of electricity (or magnetism) on one side of a closed surface, it is possible so to distribute electricity (or magnetism) over the surface that the action of this distribution on the other side of the surface shall be the same as that of the actual distribution. If the potential of the actual distribution at each point of the surface be Ω, and Ω' be that of the substituted surface distribution, the condition to be fulfilled is

$$\Omega - \Omega' = 0$$

if the distribution is within the closed surface, and

$$\Omega - \Omega' = \text{constant}$$

if the distribution be outside the closed surface.

Both these conditions give space rate of variation of Ω' in any direction in the equivalent field equal to the space rate of variation of Ω in the same direction, that is the fields due to the two distributions are undistinguishable; the constant is zero in the former case simply because the potential is zero at an infinite distance from the surface.

It will be proved that if σ be the surface density at any point of the distribution, and $d\Omega/d\nu_1$, $d\Omega/d\nu_2$, the rates of variation of the potential in the direction *from* the surface on the two sides at the point

$$\frac{d\Omega}{d\nu_1} + \frac{d\Omega}{d\nu_2} + 4\pi\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

whether the surface be open or closed. Or we may state the theorem thus

$$N_1 - N_2 = 4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where N_1 is the normal force from the surface on one side and N_2 the normal force toward the surface on the other side, the positive direction of force being taken as that in which an element of positive electricity (or magnetism) tends to move.

Now by (8) above (since $S_0 = 0$) we have for the potential at a point at distance r from the centre of the earth produced by the equivalent surface distribution,

$$\Omega' = S_1 \frac{R^2}{r^2} + S_2 \frac{R^3}{r^3} + S_3 \frac{R^4}{r^4} + \ldots$$

This gives at a point just outside the surface

$$Z = \frac{1}{R}(2S_1 + 3S_2 + 4S_3 + \ldots)$$

as in (12).

Considering now the potential at an internal point at distance r

from the centre, by the surface distribution we have since the internal and external potentials must agree at the surface

$$\Omega' = S_1 \frac{r}{R} + S_2 \frac{r^2}{R^2} + S_3 \frac{r^3}{R^3} + \ldots$$

This gives for the radial force, Z' say, at that point

$$Z' = -\frac{d\Omega'}{dr} = -\frac{1}{R}\left(S_1 + 2S_2 \frac{r}{R} + 3S_3 \frac{r^2}{R^2} + \ldots\right) \quad . \quad . \quad (26)$$

Taking now two points infinitely near to one another, and on opposite sides of the surface, we have by the values of Z and Z' just found, and the theorem stated in (25)

$$4\pi\sigma = Z - Z' = \frac{1}{R}(3S_1 + 5S_2 + 7S_3 + \ldots),$$

that is from (12) and (13)

$$\sigma = -\frac{1}{4\pi}\left(\frac{\Omega}{R} - 2Z\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

This result was also given by Gauss.

Are the Terrestrial Magnetic Forces derivable from a Potential? Question of Current Perpendicular to Earth's Surface.

105. The question whether the magnetic intensity external to the earth's surface is derivable from a potential, as the theory of Gauss supposes, has engaged the attention of several investigators, among others Schuster, Schmidt, von Bezold, Rücker, and V. Carlheim-Gyllenskiöld. The matter was dealt with by Schmidt as follows.[1] It has been shown by Gauss[2] that Y can be developed in a series of the form $l + l' \cos\lambda + L' \sin\lambda + \ldots$. Hence if the line integral of Y round the earth along a parallel of latitude vanishes we must have $2\pi l \sin u = 0$. But this of course is $(\Omega_{\lambda+2\pi} - \Omega_\lambda)/R$.

As far as available data would allow this line integral has been computed by Schmidt for the epoch 1885, and for every fifth parallel of latitude from 60° N. to 60° S. As a result it has been found that $2\pi l \sin u$ is at latitude 45° N. or S., apparently about 8 per cent. of the difference between the maximum and minimum values of Ω/R for the parallel of latitude in question. At the equator it is less than 1 per cent. of this difference. Nearer the poles than 45° N. or S. it seems again to diminish.

The question has also been tested by Rücker, and by V. Carlheim-Gyllenskiöld, who have determined the value of the line integral of magnetic intensity round closed circuits in regions for which exact

[1] *Abh. d. K. B. Akad.* II. Cl. xix. Bd. i. 1895.
[2] Gauss u. Weber, *Resultate u.s.w. im Jahre* 1838.

magnetic surveys have been made, but without obtaining for any circuit a value sensibly different from zero. The same thing has been done also by von Bezold[1] for a trapezium bounded by the meridians 4° W. and 34° E., and the parallels of latitude 35° N. and 65° N., using numbers given by Neumayer in the collection of tables published by Landolt and Börnstein; but again the result has been a nearly zero value of the integral.

If these line integrals were of sensible amount we should, as we shall see later, be forced to the conclusion that electric currents flow between external space and the surface enclosed by the path of integration.

That the theory of Gauss is approximately correct seems certain, but its exactness can only be settled when accurate observations made at a greater number of stations are available.

Also it would be of great interest to have for all the magnetic observations in Europe, observations taken simultaneously, especially at some instant during a magnetic storm as suggested by Eschenhagen.[2] It has also been suggested by von Bezold that electric currents may exist over inland seas and lakes, and that it might be worth while to obtain accurate values of the line integral round the Black Sea and the Caspian.

The Magnetic Elements

106. The horizontal force, the dip (or inclination), and the variation (or declination) are called the magnetic elements, and are regularly observed and recorded at magnetic observatories. The methods of determining them will be fully discussed in the chapter on Magnetic Measurements. We shall here only deal with the distribution of their values over the surface of the earth, and their secular and other changes.

Lines of equal horizontal force drawn on a Mercator's chart of the earth's surface are shown in Pl. II., at the end of this volume. Lines of equal vertical force in Pl. III., and lines of equal variation in Pl. IV. These charts are taken from the *Admiralty Manual of Deviations of the Compass*, 1893 edition. The numbers at the right-hand side of the diagram opposite the corresponding lines are the values of the horizontal forces in terms of that at London taken as unity. To reduce to C.G.S. units they must be multiplied by ·1832.

By an inaccurate analogy the lines of equal magnetic dip have been called lines of magnetic latitude. The term however more properly belongs to lines of equal magnetic potential, since these are cut at right angles by the magnetic meridians, or lines of horizontal force.

The chart of lines of equal magnetic variation is of great use to the navigator, as it enables him in the absence of sights of sun or stars, when he knows his approximate locality, to tell the direction in which

[1] *Zur Th. d. Erdmag. Math. u. Naturw. Mitth. K. Phy. Akad.* 2., Berlin. April, 1897.
[2] L. A. Bauer, *Terr. Magnetism.* April, 1896.

his compass needle (supposed freed from the effect of the magnetism of the ship) actually points.

The chart of lines of equal total magnetic force, shown in Pl. I. at the end, has some interesting features. It will be seen that in both the Northern and Southern hemispheres, the lines bend in towards one another in the middle on the two sides, until the two sides meet to form a two-looped or figure of 8 curve. Then within each loop the lines become small closed curves surrounding points of maximum or minimum total force. These points, it is to be observed, do not coincide with the magnetic poles, but fulfil an entirely different definition, namely, that of being points of maximum or minimum total force; while a magnetic pole is a place at which the horizontal component of the magnetic force vanishes.

There are of course irregularities in these curves due to local disturbing forces which are hardly taken account of in the charts. For example in parts of the British Islands where there are large masses of basalt in the underground strata, there are deflections of the north pointing pole towards such places from distances of as much as fifty miles. There seems little doubt that local deviations of the compass needle on board ship have been thus produced. It is possible, as pointed out by Rücker (*Rede Lecture, Nature*, Dec. 23, 1897), that the surface strata containing magnetic matter are indications of much greater masses of similar rock beneath, from which the strata have been extruded.

Again there is an attraction of the north pointing pole towards places where rocks of the carboniferous or precarboniferous period have been thrust out through the newer strata, as at various coalfields. Thus a ridge line of magnetic matter runs along the Pennine Range, the so-called "Backbone of England," and another runs from Nuneaton to Dudley, and then south to Reading. Thence lines run in different directions; one of these passes southwards to the Channel at Chichester, and appears again in France near Dieppe, whence it has been traced by M. Moureaux to a point fifty miles south of Paris (See Rücker, *loc. cit.*).

Some remarkable irregularities have lately been observed by M. Moureaux in South Russia. At a village called Kotchetovka, in lat. 57° N, and long. 6° 8′ east of Poulkowa, the extreme values of the elements at fifteen stations scattered over an area of about a square kilometre were Declination + 58° to − 43°, Dip 79° to 48°, and Horizontal Intensity ·166 to ·589 C.G.S. Thus the horizontal intensity is actually greater at some stations than at the equator, and since the dip does not fall below 48° there must be at such places an extremely high value of the total intensity.

Time Changes of the Magnetic Elements and their Causes

107. Perhaps the most mysterious and perplexing of the phenomena of terrestrial magnetism are the secular and other changes which they undergo. Chief of these is the secular periodic change in the variation. An unbroken series of observations of the variation at Paris made nearly

every year from 1680 to the present time, and at longer intervals for some time previously, shows that in 1666 the variation was zero there, and acquiring a westerly value. This westerly value continued to grow up until in 1814 it attained a maximum of 22° 34′, when the needle began again to turn toward the east. In 1892 it was 15° 26′·9 W.

The same changes have been going on throughout Europe. In 1659 the compass pointed north at London, and was gradually moving round towards the west. The westerly variation continued to increase until in 1823 it had attained a maximum of 24° 30′, and had begun to diminish. In 1894 it was 17° 23′ W. at Kew.

While this change in the variation has been going on, the magnetic dip at London has no doubt been gradually diminishing. At the present time the decrease is about 1′·5 annually. This gradual diminution of the dip has been carefully observed at Paris. In 1671 the dip was about 75°, in 1776 it was 72° 25′, in 1876 65° 37′, and in 1892 it was 65° 9′·2.

Along with this diminution of the dip has taken place, as was to be expected, an increase in the horizontal magnetic force. According to Thorpe and Rücker,[1] there is a mean yearly increase of about ·00022 C.G.S. for England, ·00018 for Scotland, and ·00020 for Ireland.

It appears at first sight as if the whole terrestrial magnetic system were slowly turning round the earth's axis in a period of nearly 1000 years,[2] and in the direction opposed to the earth's rotation. In a model made by Mr. Henry Wilde one of two sets of currents contained within a globe, which represents the earth, rotates round the polar axis and produces changes in fair agreement with those observed at several places.[3] There are however great difficulties in the way of accepting any such theory as that here indicated.

108. The secular changes in variation and dip have been graphically represented by L. A. Bauer[4] by drawing the curve which the north pointing end of a magnet freely suspended at its centre of gravity would to an observer situated at the centre of the magnet seem to describe. Thus the curve shows the change of dip as well as that of declination. The curve for London, from 1540 to 1890, is shown in Fig. 33, and for comparison, the curves for London, Rome, Ascension Island, and Cape Town, places within a range of longitude of about 30°, but of latitudes varying from 52° N. to 35° S., are shown at the end of this volume in Plate V. It will be seen that the direction of motion of

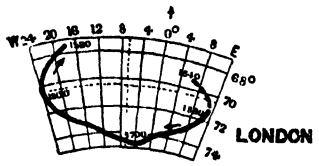


FIG. 33.

[1] *Magnetic Survey of the British Isles. Phil. Trans.*, 1890.
[2] See Lord Kelvin's *Popular Lectures and Addresses*, Vol. III. p. 25.
[3] *Proc. R.S.* June 19, 1890. See also Rücker, B.A. Address, 1894.
[4] *Beitr. z. Kenntn. d. Wesens d. Säcular Variation d. Erdmagn.* (Inaug. Dissert, Univ. of Berlin.)

the pole is clockwise, and this is found to be the case at nearly all the places for which observations have been recorded.

From the dates marked on the curve it will be seen that the speed of the pole in the secular orbit is not constant. Let us suppose, as was done by Mr. Wilde, that the changes are due to the superposition on a perfectly constant magnetic system, rotating with the earth, of a second magnetic system, also turning with the earth but at the same time describing a secular orbit round the earth's axis. If a needle could be suspended so that the earth turned beneath it, successive nearly identical sets of daily changes of the position of the needle would be observed, and each of these might be represented by a curve like that just described. The curve thus obtained for a single day ought, if the second system were invariable and its successive positions were symmetrical about the earth's axis, to agree with the secular variation curve so far as known for past time for the latitude of observation, and could then be used to predict the remainder of the secular curve.

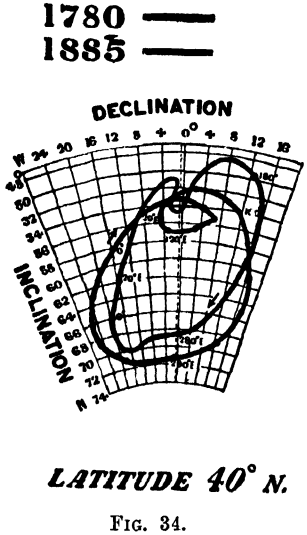


FIG. 34.

The daily curves would be exactly repeated, but the longitudes associated with various positions of the pole of the needle would continuously change. Without symmetry of position and constancy of the disturbing system the curves would slowly change in form as well, which Fig. 34 and Pl. VI. show is to some extent the case. Fig. 34 shows such curves drawn by Bauer for the epochs 1780, 1885, and a place in latitude 40° N., from recorded observations. A comparison of curves for latitudes 40° N., 0°, and 40° S. is given in Pl. VI. The longitudes for different positions of the needle are marked on the curves.

It will be seen from Pls. V. and VI. that these diurnal curves resemble the secular curves in form and in respect of their difference of size for equal north and south latitudes. The hypothesis must however still be regarded as very uncertain.

109. Besides these secular changes there are changes of comparatively short periods both in the variation and in the other magnetic elements.

The annual change in variation in the northern hemisphere is easterly from May to October, and westerly during the remaining six months, with a range at Greenwich of 2′ 25″. The maximum easterly deviation from the mean takes place in August, the maximum westerly in February. In the southern hemisphere the changes are simultaneous but in the opposite direction.

The daily march of the variation at Greenwich is shown in Fig. 35.

The straight line running along the diagram and marked 0 denotes the mean position of the needle; the curve shows the deviation from the mean positions at the hours marked along the top of the figure beginning at midnight. It will be seen that at midnight the needle is $1'\frac{1}{2}$ to the

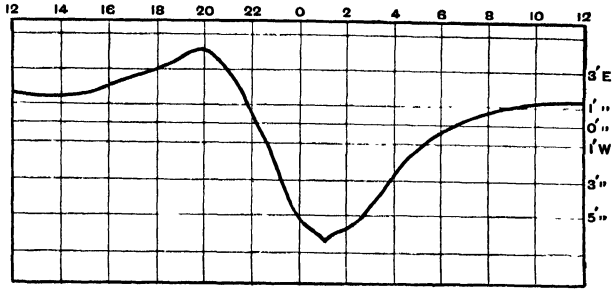


FIG. 35.

east of its mean position, and remains there until about 2 A.M. when it begins to move still further towards the east, reaching a maximum deviation of 4′ about 8 A.M. Then it moves rapidly towards the west and attains a maximum westerly deviation at 1 P.M., after which it returns through zero towards the initial deviation of $1'\frac{1}{2}$ east, reaching it about midnight.

The cause of this daily change of variation is of course the changes which take place in the forces towards the geographical north and west. The changes in the latter during twenty-four hours are shown for Greenwich and Lisbon in the curves of Figs. 36 and 37, which are taken from Professor Schuster's recent paper on the Diurnal Varia-

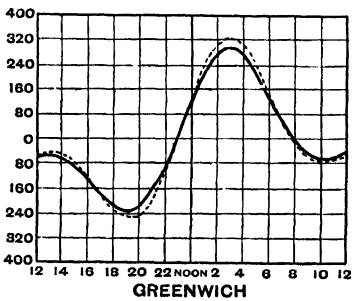


FIG. 36.

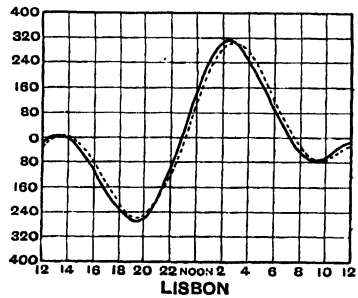


FIG. 37.

tion of Terrestrial Magnetism.[1] It will be seen that the change is a combination mainly of a diurnal and a semidiurnal constituent.

The dotted curves in the figures were drawn by Professor Schuster from results tabulated from the variable part of the magnetic potential,

[1] *Phil. Trans. R.S.*, Part A., 1889.

as found by him after the manner of Gauss, from observations of magnetic force at different parts of the earth's surface. It will be seen that the two sets of curves in each case very nearly coincide.

110. Professor Schuster also investigated the changes in the vertical force. His results for Greenwich and Lisbon are given in Figs. 38 and 39, and illustrate fairly his results. The object of the research was to decide whether the cause of the diurnal variation of terrestrial magnetism was external or internal to the earth's surface. The answer obtained was that the exciting cause is external to the earth, but is accompanied

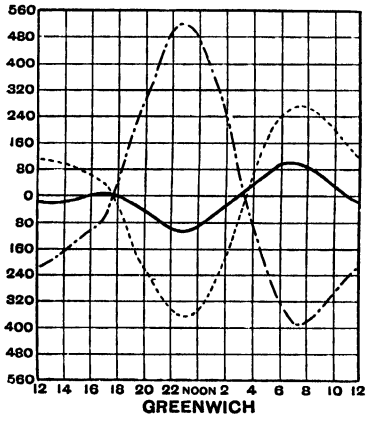


FIG. 38.

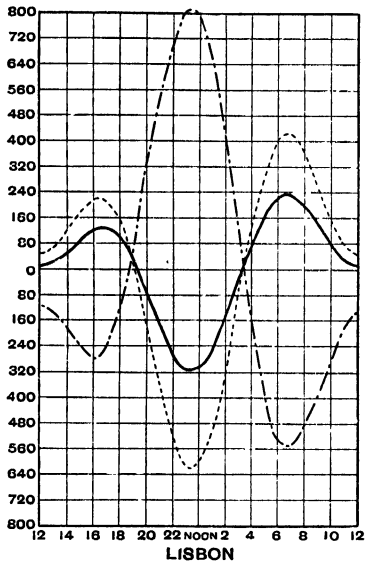


FIG. 39.

Comparison between calculated and observed curve of vertical force. Abscissæ denote astronomical time, ordinates denote vertical forces, to unit 10^{-6} C.G.S. The full line is the observed curve; the dotted line is the calculated curve on the hypothesis of outside force; the chain dotted line is the curve calculated on the hypothesis of internal force.

by an internal source of change to which it stands in fixed relationship. This is shown very well in Figs. 38 and 39. The full curves show the observed variation of vertical force, the lightly dotted curves the changes of vertical force on the supposition of an external cause, the chain dotted curves the same thing on the supposition that the cause is internal. It will be recognized at once that in each case (and the same holds for curves given in the paper for Bombay and St. Petersburg) the curve calculated on the former supposition agrees in character with the observed while the other calculated curves do not. The difference in the amounts of the observed and calculated effects is due to the internal agency already alluded to as connected with the external cause

of change. Such an internal source of periodic change of magnetic force was of course to be expected: for such changes must necessarily give rise to induced currents within the substance of the earth which will react on the magnetic field intensities.

111. The currents induced in a conducting sphere by periodic changes in magnetic potential are investigated by Professor Horace Lamb in an appendix to Professor Schuster's paper. The theory shows that if the whole earth were of good conducting material, the effect of the induced currents would tend to equality with the primary effect, and the phase of the resultant would approximate to a difference of 45° from that of the primary disturbance.

The results given in the curves show that there is good agreement of phase between the actual effect and the computed primary effect: so that the secondary action of the induced currents is only effective in reducing the amplitude. It has been suggested by Professor Lamb that this result would be produced if the currents circulated mainly in the internal parts of the earth, and were only slight in comparison in the outer strata. This is very likely to be the case, as the high internal temperature will undoubtedly have the effect of reducing the resistance of many of the materials of which the earth is composed.

Besides reducing the amplitude of the variable part of the vertical force, the currents reduced in the earth's substance must be effective in increasing the amplitude of the periodic changes of horizontal force.

112. The following is Professor Schuster's own summary of the results of his investigation.

"1. The principal part of the diurnal variation is due to causes outside the earth's surface, and probably to electric currents in our atmosphere.

"2. Currents are induced in the earth by the diurnal variation, which produce a sensible effect chiefly in reducing the amplitude of the vertical component and increasing the amplitude of the horizontal components.

"3. As regards the currents produced by the diurnal variation, the earth does not behave as a uniformly conducting sphere, but the upper layers must conduct less than the inner layers.

"4. The horizontal movements in the atmosphere which must accompany a tidal action of the sun or moon, or any periodic variation of the barometer such as is actually observed, would produce electric currents in the atmosphere having magnetic effects similar in character to the observed daily variation.

"5. If the variation is actually produced by the suggested cause the atmosphere must be in that sensitive state in which, according to the author's experiments, there is no lower limit to the electromotive force producing a current."

Diurnal Changes of Intensity at Points on same Parallel of Latitude. Representation by Vector-Diagrams

113. The daily variations in terrestrial magnetism have been discussed to some extent recently in connection with Schuster's theory by

v. Bezold. This writer first deals with the assumption that the mean normal daily changes, as they are obtained from days free from irregular disturbances, are equivalent to a system of magnetic forces which may be regarded as revolving round the earth, without alteration of itself in a period of twenty-four hours; then he offers some observations on the desirability and method of testing whether this system of forces is derivable from a potential function.

According to the assumption referred to, if X_d, Y_d, Z_d be the component magnetic intensities which express the daily variations, each of them must be a periodic function of λ, that is of the local time of the place at which observations are made. Thus if t be that local time, t_0 the corresponding time at Greenwich, and T the period of revolution (twenty-four hours expressed in terms of the chosen unit of time) we have

$$X_d = f_x\{u, \frac{2\pi}{T}(t - t_0)\}.$$

with similar equations for Y_d, Z_d.

To compare this result with observation, the diagram first used for the representation of X_d, Y_d, by Airy is of great importance. This v. Bezold calls a vector-diagram. In it the change of the horizontal force in direction and magnitude is laid down for each hour of the day, by drawing straight lines from an origin O (see Fig. 40), so that each line represents by its length the magnitude of $\sqrt{X_d^2 + Y_d^2}$, and makes an angle $\tan^{-1} Y_d/X_d$ with the axis of the Y forces. The extremities of these lines or vectors lie on a closed curve from which the amount of change for any interval during the day can be at once obtained. The successive hours are marked round the curve, and of course occur at different intervals depending on the varying rate of change during the day.

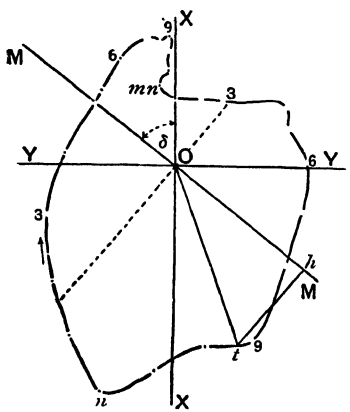


FIG. 40.

Clearly the radii-vectores in such a diagram are the directions in which a horizontally suspended needle freed from the mean action of the earth would successively place itself in consequence of the diurnal change of horizontal intensity.

Airy and Lloyd, who both used this diagram, took as components of the change those along and perpendicular to the magnetic meridian. For the present purpose this is not so convenient, and the axes are taken as indicated by the component X_d along the astronomical meridian at any place, and Y_d along the parallel of latitude there. When this is done it is found that the supposition as to its being the same succession of changes that occur at

all places along a parallel of latitude is confirmed. The vector-diagram at all such places is the same for the same local time at each.

Fig. 40 is the vector-diagram for lat. 60° N., and the diagrams for latitudes differing successively by 20°, from 80° N. to 80° S., are shown in Pl. VII. as obtained for the summer months of the northern hemisphere. These diagrams are taken from v. Bezold's paper, as deduced by him from Schuster's values of the potential. It will be seen that for a place in lat. 60° N., where the magnetic meridian is astronomically north and south, the daily change in H, the magnetic intensity along the magnetic meridian, has its maxima at noon and 9 p.m., while the other component has its maxima at about 6 a.m. and 3 p.m. The component along OX is zero at the times corresponding to the intersections of the axis YOY with the edge of the curve, that is at 6 p.m. and a little before 4 p.m.

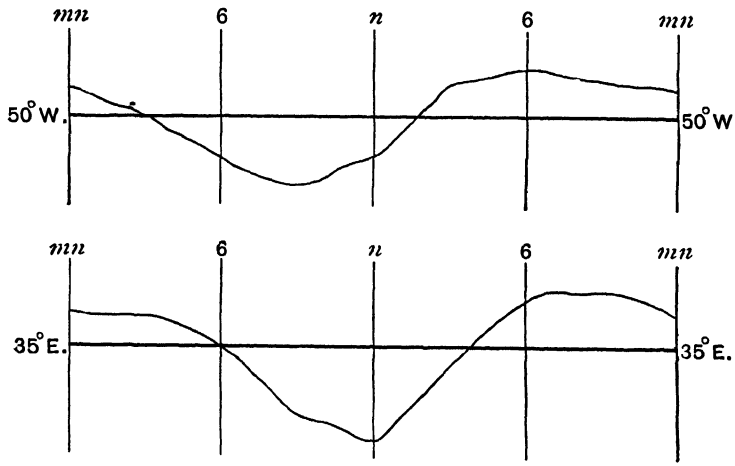


FIG. 41.

To find the components along and perpendicular to the magnetic meridian at any other place, it is only necessary to draw the meridian on the diagram, making the proper angle with the axis of X. Thus let MM be the magnetic meridian. The maximum daily variations along MM, that is the variations in H, are obtained by drawing tangents to the vector-diagram at right angles to MM. Thus on the diagram they occur about 9 a.m. and 6 p.m.

The components along MM are zero for the times corresponding to the intersection of a line perpendicular to MM with the curve, that is for Fig. 40 about 3.30 a.m. and 2 p.m.

The maximum magnetic east and west variations of intensity are obtained by drawing tangents to the curve parallel to MM, and accordingly have place about midday and 9 p.m.

It is clear that from this curve the march of the change in H, that is of the component along MM, can be obtained and compared with that observed. Thus Fig. 41 gives such derived curves for the latitude 60° N. and the magnetic meridians for which the declinations are 50° W. and 0°. The vector-diagram, however, shows the whole phenomenon of which the periodic curves in this figure show only part. As that has different forms for places of different values of the mean declination its graphical representation does not clearly display the progressive march of the changes along the parallel of latitude.

Diurnal Changes in Different Latitudes. Difference of Amount in Summer and Winter. Results of Observations on "Quiet Days"

114. An examination of Pl. VII. brings some curious facts to view. First there is the great difference between the summer and the winter results as regards the daily changes, and this difference as it alternates with summer and winter from hemisphere to hemisphere, suggests a cause connected with meteorological change. The difference would have been still more marked if the results had not been a mean for the half year, but had been taken for the time of the summer solstice, or for the months of June and July. It would be well also to obtain results for the winter solstice, or the month of December, and likewise for the equinoctial months.

Another point of considerable interest is the comparative smallness of the north and south components of the daily change of magnetic intensity at 40° N. and S. Above and below that latitude in both hemispheres the two components give an irregular oval curve. The direction of rotation of the radius-vector round the diagram is negative (or clockwise) in higher northern latitudes than 40°, positive in latitudes between 40° N., and some parallel south of the equator, then again negative to lat. 40° S., and finally once more apparently positive in higher southern latitudes.

115. It is clear that for a more complete discussion of the daily variations of terrestrial magnetism a set of careful observations at places on parallels of latitude every 5° or 10° apart would be of great value; and though the daily progression of the phenomena along each parallel has been established, the degree of exactness of this repetition of the phenomena as λ varies would be tested by observations made for the same interval of time at different places on each of a sufficient number of parallels. The results of such observations, so far as regards horizontal force, should be for ease of interpretation represented by means of vector-diagrams, and to exhibit the variations in dip as well as in declination, Bauer's mode of representation (Art. 109) should be employed.

Observations are now made of the daily changes at English Observatories on five days of each month selected by the Astronomer-Royal. These days are chosen for their freedom from irregular disturbances, and

are therefore called "quiet days." It has been found however by Chree (*B. A. Reps.* 1895, 1896) that when the changes of direction of the needle during a solar day are represented in the manner suggested by Bauer (Art. 108 above) the needle does not come back at the end of the day to its original direction, that is the path is not closed. It is to be remembered that the path of the pole of the needle while showing the diurnal change must be affected by the secular change, which would carry the pole in the opposite direction, and so prevent the path from being closed. But this is not sufficient to account for the whole effect, and thus it appears that on the quiet days the effect of magnetic storms is to check the secular motion of the needle.

Question of Derivation of Diurnal Changes of Intensity from a Potential

116. With regard to the derivations of X_d, Y_d, Z_d, from a potential Ω_d, it is obvious of course that the daily revolution of the changes round the parallels of latitude makes Ω_d a periodic function of period T, that is:—

$$\Omega_d = F(\lambda, u, t_0 + uT),$$

if change of the revolving value of Ω_d with longitude, is permitted, and

$$\Omega_d = F\left(u, \lambda + \frac{2\pi t_0}{T}\right)$$

if the revolving system is invariable.

The vanishing of $\int Y_d\, dt$, that is of the line integral of Y_d round a parallel of latitude is necessary, but is not sufficient to establish that Y_d is derivable from a potential function. To settle the question of the existence of a potential, what is needed is an evaluation of the line integral round a sufficient number of closed paths of other forms, drawn on the surface of the earth. We have already given above the expression for the line integral round a path consisting of two arcs of meridians joined by parallels of latitude, and found (equations (7′), (7″) above) what the expression becomes when (1) the distance along the meridians is made infinitesimal, (2) when the distance along the parallels is also made infinitesimal.

These formulas could easily be tested at places at which the daily changes are observed, and the question settled. For this purpose simultaneous observations at a number of places properly placed would as in the case of the ordinary more slowly changing intensity be of great value.

Equipotential Lines of Diurnal Changes

117. Von Bezold has also thrown Schuster's recalculated values of the potential Ω_d into a system of lines of equal potential drawn on a Mercator's projection of the earth's surface. This chart is reproduced as

Pl. VIII. at the end of the present volume. It will be seen that these lines show four well marked poles of the system of daily magnetic intensities for noon at Greenwich. The chain-dotted line shows the boundary between the illuminated and non-illuminated parts of the earth's surface, and it is to be observed that one of the poles lies in lat. 40° N. on the meridian of the place for which the non-circle is 11 h. 20 m., or thereabouts, while an opposite pole, in the same latitude nearly, is west of Greenwich almost on the chain-dotted line. The two other poles lie on the south of the equator, one of opposite sign to that first mentioned nearly on the same longitude and in about 30° south latitude; the fourth lies about 10° further south, 7 h. 30 m. west of Greenwich, or nearly in the same longitude as the second mentioned pole, and is of the opposite sign to the latter. This system of lines of course travels round the earth in a solar day.

Von Bezold points out that these poles cannot be produced by a system of currents inside the earth, though of course the magnetic intensities may be modified by such a system. This he shows by considering the vertical and horizontal forces, and pointing out that an internal system, which accounted for the observed vertical component, would give horizontal forces opposite to those observed, and *vice versâ*.

The difference between the phenomena in the summer and winter hemispheres is here again brought out very markedly, and suggests as before that the daily changes are greatly modified by meteorological influences.

It is suggested by v. Bezold that currents depending on the great atmospheric currents, the cyclones, and the anticyclones, would produce these poles in or near 40° of north and south latitude.

Magnetic Storms

118. Besides these regularly periodic variations in the magnetic elements there are irregular and violent changes which take place from time to time, and are well described as magnetic storms. Such disturbances have been found to be simultaneous, often with auroral displays, a fact which seems to point to electric discharge in the upper regions of the atmosphere as a cause of magnetic storms, as well as of the regular changes. It has also been observed, though the correspondence has not been unmistakably made out, that magnetic storms seem to be more frequent at times of great solar activity as shown by the outburst of sun-spots. Hence some physicists have been led to credit magnetic action exerted by the sun with such magnetic disturbances and also with the annual and diurnal variations of the magnetic elements.

In this theory there is at the outset a very serious difficulty. It may be true that the sun is a powerful permanent or electro-magnet, exerting a steady effect, but to produce suddenly changes of the magnetic forces experienced on the earth, comparable with the total amount

of these forces, this magnet must be subject to enormous changes of intensity rapidly produced and rapidly dying away. The subject has been discussed recently by Lord Kelvin in his Presidential Address (St. Andrew's Day, 1892) to the Royal Society, and the conclusion arrived at is adverse to the hypothesis of a direct solar origin of these magnetic changes. The following is a short summary of his treatment of the subject. (See also Lloyd, *Magnetism*, p. 233, and G. J. Stoney, *Phil. Mag.*, Oct., 1861.)

119. Supposing the sun magnetized to the same mean intensity as is the earth, the magnetic force produced by it at an external point can easily be calculated. Let the direction of magnetization be at right angles to the plane of the ecliptic, which it roughly is no doubt, if the sun's magnetization is connected with its rotation, as Lord Kelvin thinks is the case with the earth. The force then at a distance D from the sun at a point in the ecliptic is $\frac{4}{3}\pi R^3\mathbf{I}/D^3$, if $\mathbf{I}$ be the intensity of magnetization, that is the magnetic moment per unit of volume, and R be the radius of the sun. If D be equal to the earth's distance from the sun so that $D^3 = 228^3 R^3$, we get for the force the value $\frac{4}{3}\pi\mathbf{I}/228^3$. The force at the earth's surface along a meridian at the equator is $\frac{4}{3}\pi\mathbf{I}$. Hence the force produced by the sun would be about $1/228^3$, or 1/12,000,000, of the earth's force.

It is clear therefore that the sun must be something like 12,000 times as intensely magnetized as the earth in order to produce a force perceptible with ordinary instruments used in a magnetic laboratory. This according to the estimate given above (Art. 102) would be about the intensity of magnetization of well magnetized hard steel.

Since the moon's apparent diameter is approximately the same as the sun's, the ratio of the moon's radius to the distance of the earth from the moon is nearly equal to the corresponding ratio for the sun, and so the estimate just made for the sun holds also for the moon.

120. Now considering the earth at the vernal equinox, the magnetic force due to the sun may be resolved into two components, one, ·92 of the whole amount, parallel to the earth's axis, the other, ·4 of the whole, at right angles to the axis. The former contains a constant part, and a part varying with the sun's distance and having therefore a period of a year. The latter component would affect the variation, supposing the magnetisms in the northern and southern hemispheres of the sun to be of the same sign as those on the corresponding hemispheres of the earth, as if the south-pointing ends of needles on the earth were attracted towards a star in the plane of the equator and at 270° Right Ascension. This would give rise to a change in the declination of period one sidereal day.

This diurnal constituent, it is to be observed, is distinct from that discussed by Schuster, which had for period a solar day, and is smaller in amount than the latter. So far it has not been sought for by the harmonic analysis; but even if a change in the declination were found, it would not follow that it was due to the action of the sun as a magnet.

The whole effect might be due to effects of currents circulating in the earth's atmosphere, and therefore depending on the period of rotation, and moreover containing solar diurnal terms depending on difference of temperature produced by the sun's radiation.

121. But the chief objection urged by Lord Kelvin to solar agency as the cause of magnetic storms is the enormous expenditure of electrical energy necessary to produce by the action of the sun the oscillations of magnetic force observed in a magnetic storm. Thus in a magnetic storm which took place on June 25, 1885, the horizontal force at the following eleven places: St. Petersburg, Stonyhurst, Wilhelmshaven, Utrecht, Kew, Vienna, Lisbon, San Fernando, Colaba, Batavia, and Melbourne, increased considerably from 2 to 2.10 P.M., and fell from 2.10 to 3 P.M. with irregular changes in the interval.

The mean value at all these places was ·0005 above par at 2.10 and ·005 below par at 3 P.M. The changes as shown by the photographic records were simultaneous at the different places. Assuming these electrical oscillations of the sun, Lord Kelvin estimates that the electrical activity of the sun during the storm, which lasted about eight hours, must have been about 160×10^{24} horse power, or about 12×10^{35} ergs per second; that is about 364 times the activity of the total solar radiation, which is estimated at about 3×10^{33} ergs per second. The electrical energy thus given out by the sun in such a storm would supply, if transformed to the electrical vibrations of shorter period concerned in its ordinary radiation, the whole light and heat radiated during a period of four months. This, as Lord Kelvin remarks, is conclusive against the hypothesis that these violent magnetic disturbances are due to direct action of the sun.

It is of course conceivable that the earth might be immersed in a ray of abnormally great solar energy; but it is extremely unlikely that unless the radiation were abnormally great all round, the earth would remain in such a ray for a period of several hours.

The cause of magnetic storms therefore remains undiscovered. But it is only one of the mysteries of terrestrial magnetism, for the great fact of the earth's magnetism itself, not to speak of all its wonderful periodic changes, secular, annular, and diurnal, remains unaccounted for by any satisfactory theory.

CHAPTER IV

MAGNETISM OF AN IRON SHIP AND COMPENSATION OF THE COMPASS

Ship's Magnetism

122. It is not possible to discuss fully here the magnetism of an iron ship, and the compensation of a mariner's compass for use on board such a vessel; but these subjects are so important, both practically and from the point of view of pure science, that it seems desirable to give a short account of them.

The magnetization of an iron ship is (1) permanent or sub-permanent, and (2) transient. The permanent magnetization is independent of the change of place of the ship or the lapse of time, except in so far as the iron is subject to strains or shocks due to unusual stress of weather or other cause. It is set up mainly by terrestrial magnetizing force during the building of the ship, and its distribution depends on the position of the ship while being built.

The sub-permanent magnetism slowly changes with time, and its variations show effects only observable after the lapse of intervals of days or even weeks.

The transient magnetization is induced by the varying magnetic force to which the iron is subject, and follows the changes of these forces caused by the varying position of the ship.

The deviation of the compass needle in a ship is the angle which its direction makes with that which it would have if the needle were under the earth's force alone. It may be regarded as made up mainly of two parts, called the *semicircular* deviation or error, and the *quadrantal* deviation or error. How these arise and the manner of their compensation will form the first part of the present discussion.

123. The ship forms a large magnet or rather combination of magnets. It has (*a*) longitudinal magnetization, (*b*) transverse magnetization, and (*c*) vertical magnetization. The first must obviously exist, as the ship's hull is a great iron or steel girder, with its length in the fore and aft direction; the second is principally the longitudinal magnetization of the transverse beams and bulkheads; the third component is mainly due to the ribs and vertical beams and bulkheads. These produce magnetic

forces within the ship, to which must be added also those produced by the magnetization of iron or steel masts or spars, or of iron or steel carried as cargo.

Soft Iron of Ship Represented by Iron Rods

124. The magnetic forces at any point in the ship may be referred to as fixed in the ship, and the action on the compass needle for any position of the ship's head ascertained. Let the axes be drawn from the centre of the compass needle as origin, that of x from the stern to the head of the vessel, that of y from port to starboard, and that of z from the compass needle downwards, as shown in Fig. 42. Also let the components of magnetic force due to the earth be denoted by X, Y, Z, the forces due to the permanent magnetism by P, Q, R, and the total components by X', Y', Z'. The earth's force will magnetize the iron of the vessel tem-

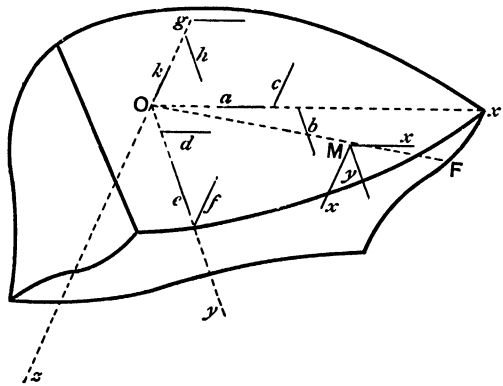


FIG. 42.

porarily, and every one of the three components X, Y, Z, will produce its own part of each of the three total components X', Y', Z'. For consider the first component X. It will produce at the compass needle, in consequence of the induced magnetization, a force which experience shows may be taken as proportional to X, and which may therefore be denoted by aX, where a is a constant. If a be positive this force will produce the same effect upon the needle as would a long soft iron rod placed in the ship in a fore and aft direction, beginning at a point in front of the needle and running towards the bow (as shown at a in Fig. 42), or beginning behind the needle and running towards the stern, and magnetized by the force X. If a be negative, the force is equivalent to that which would be produced by a short rod beginning in front of the binnacle, passing under the binnacle, and ending behind.

The same force X will produce, in consequence of unsymmetrical distribution of the iron of the ship about the longitudinal plane of

symmetry of figure (that is, the plane of the keel), an effect similar to that of a long bar beginning at a point, say, to port of the needle and running towards the stern, or to that of two bars, one situated as just specified, and the other beginning at a point to starboard of the needle and running towards the bow (d in Fig. 42). This will give a force dX in the direction of y, where d is positive. If the force dX due to this want of symmetry be negative, the rod or rods producing an equivalent effect must be reversed in direction.

Also there will similarly be produced a force gX in the direction of z, which will be equivalent to that produced by two rods, one beginning above (g of Fig. 42), the other below the needle, and running, the first towards the bow, the other towards the stern of the vessel, or *vice versâ*, according as g is positive or negative. Of course, instead of the pair of rods here specified, either rod singly may be substituted.

In like manner the forces produced by Y and Z can be accounted for by three iron rods thwart-ship, and three vertical, according to the schemes shown in Fig. 42.

Thus we have the equations

$$\left.\begin{aligned} X' &= X + aX + bY + cZ + P \\ Y' &= Y + dX + eY + fZ + Q \\ Z' &= Z + gX + hY + kZ + R \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

where a, b, c ... are constants. These equations were first given by Poisson. We now proceed to consider the effect of the forces thus specified in producing deviations of the compass.

Expressions for Total Deviation of the Compass. Analysis of Deviations

125. Let the ship be on even keel, and the direction of the head be towards a point at angular distance ζ eastward from the magnetic north, and ζ' eastward from the north point as indicated by the compass. Then ζ shows the "magnetic course," ζ' the "compass course." Clearly $\zeta - \zeta'$ is the deviation, δ say, of the compass needle from the true magnetic north, that is, the compass error.

If H be the total horizontal force of the earth, H' that in the ship, and ψ the dip, we have

$$X = H\cos\zeta, \quad Y = -H\sin\zeta, \quad Z = H\tan\psi$$

and

$$X' = H'\cos\zeta', \quad Y' = -H'\sin\zeta',$$

since the direction of the needle in the ship is that of H'. Thus from equations (1) we have

$$H'\cos\zeta' = (1 + a)H\cos\zeta - bH\sin\zeta + cH\tan\psi + P \quad . \quad . \quad . \quad (2)$$

$$-H'\sin\zeta' = (-\sin\zeta + d\cos\zeta)H - eH\sin\zeta + fH\tan\psi + Q \quad (3)$$

Multiplying the first of these by $\sin\zeta$, the second by $\cos\zeta$, adding and reducing we get

$$\frac{H'}{H}\sin\delta = \frac{d-b}{2} + \left(c\tan\psi + \frac{P}{H}\right)\sin\zeta + \left(f\tan\psi + \frac{Q}{H}\right)\cos\zeta + \frac{a-e}{2}\sin 2\zeta + \frac{b+d}{2}\cos 2\zeta \quad . \quad (4)$$

Again, multiplying the first equation by $\cos\zeta$, the second by $\sin\zeta$, and subtracting we find

$$\frac{H'}{H}\cos\delta = 1 + \frac{a+e}{2} + \left(c\tan\psi + \frac{P}{H}\right)\cos\zeta - \left(f\tan\psi + \frac{Q}{H}\right)\sin\zeta + \frac{a-e}{2}\cos 2\zeta - \frac{b+d}{2}\sin 2\zeta \quad . \quad . \quad . \quad (5)$$

Denoting $1 + (a + e)/2$ by λ, and writing

$$\lambda A = \frac{d-b}{2}, \quad \lambda B = c\tan\psi + \frac{P}{H}, \quad \lambda C = f\tan\psi + \frac{Q}{H},$$

$$\lambda D = \frac{a-e}{2}, \quad \lambda E = \frac{b+d}{2},$$

we get

$$\tan\delta = \frac{A + B\sin\zeta + C\cos\zeta + D\sin 2\zeta + E\cos 2\zeta}{1 + B\cos\zeta - C\sin\zeta + D\cos 2\zeta - E\sin 2\zeta} \quad . \quad . \quad . \quad (6)$$

which enables the deviation to be calculated in terms of the magnetic course.

Substituting $\zeta' + \delta$ for ζ, and $\sin\delta/\cos\delta$ for $\tan\delta$ in the last equation, and simplifying, we obtain

$$\sin\delta = A\cos\delta + B\sin\zeta' + C\cos\zeta' + D\sin(2\zeta' + \delta) + E\cos(2\zeta' + \delta) \quad (7)$$

or

$$\sin\delta = \frac{B\sin\zeta' + C\cos\zeta' + (A + D\sin 2\zeta' + E\cos 2\zeta')\cos\delta}{1 - D\cos 2\zeta' + E\sin 2\zeta'} \quad . \quad . \quad (8)$$

The coefficient E, it will be seen presently, is comparatively small, and we may neglect the term $E\sin 2\zeta'$ in the denominator. If the denominator be then expanded, and δ be not too great, so that we may put $\delta = \sin\delta$, $\cos\delta = 1$, we may write the above equation in the form

$$\delta = A_1 + B_1\sin\zeta' + C_1\cos\zeta' + D_1\sin 2\zeta' + E_1\cos 2\zeta' + F_1\sin 3\zeta' + G_1\cos 3\zeta' + \&c. \quad . \quad . \quad . \quad (9)$$

The higher terms in the second line may generally be neglected.

126. This result could of course have been reached by noticing that if

the ship's head were turned round to successive points of the compass, the same deviation, δ, of the needle would recur every time the ship's head took a given direction, except in so far as the changes in the ship's magnetism lagged behind the change in the magnetic force. Hence δ is a periodic function of ζ', and is expressible in the manner indicated. The coefficients, A_1, B_1, C_1, &c., can be found from the preceding A, B, C, &c., by expanding the denominator $(1-D\cos 2\zeta)^{-1}$ and substituting for the powers of $\cos 2\zeta'$ cosines of multiples of $2\zeta'$.

Provided the compass error is not more than about 20°, which it hardly ever is, as the deviation is either less in itself, or is reduced to a smaller value by partial compensation, we may take only the first five terms of the series above, and put

$$A = \sin A_1,\ B = \sin B_1(1+\tfrac{1}{2}\sin D_1),\ C = \sin C_1(1-\tfrac{1}{2}\sin D_1),\ D = \sin D_1,\ E = \sin E_1$$

where A, B, C, &c., are the values given above. (See *Admiralty Manual of Deviations of the Compass*, 1893 edition, p. 132, *et seq.*)

Considering the different terms which make up the deviation we see first that the constant term, $A = (d-b)/2\lambda$, is the effect of the unsymmetrical distribution of the soft iron of the ship with respect to the plane of the keel. This is the deviation when $\zeta' = 0$, that is, when the ship's head is on the north point as shown by the compass.

The Quadrantal Error

127. The constants b and d enter also into the coefficient E, the term which gives the variable part of the deviation depending on this want of symmetry. It is to be noticed that since the distribution of the iron of the ship is very nearly symmetrical with respect to the plane referred to, the constants b and d are very small numerically, and therefore the terms $A_1 + E_1 \cos 2\zeta'$ are relatively small parts of the total deviation.

The term $D_1 \sin 2\zeta'$ depends on a and e, and represents the effect of the fore and aft, and thwart-ship components of induced magnetization on the corresponding components of disturbing force at the compass needle. Since the longitudinal effect is greater than the thwart-ship effect, $(a-c)/2\lambda$, or D, is always positive. The pair of terms

$$D_1 \sin 2\zeta' + E_1 \cos 2\zeta'$$

make up what is called the *quadrantal deviation*. If we put in these

$$D_1 / \sqrt{D_1^2 + E_1^2} = \cos 2\phi,\quad E_1 / \sqrt{D_1^2 + E_1^2} = \sin 2\phi,$$

so that

$$D_1 \sin 2\zeta' + E_1 \cos 2\zeta' = \sqrt{D_1^2 + E_1^2}\sin 2(\zeta' + \phi),$$

we see that this part of the deviation has a maximum when $2(\zeta'+\phi) = 90°$, and again when $2(\zeta'+\phi) = 360°+90°$, that is, when

$\zeta'+\phi=45°$, and $\zeta'+\phi=225°$. Hence if $\zeta'+\phi$ is changed from 0 to 360° (that is, since ϕ is a small angle, when ζ' is changed through 360°) this part of the deviation has two equal maxima. It has also two minima of numerical value equal to the maxima, namely when $2(\zeta'+\phi)=270°$, and when $2(\zeta'+\phi)=630°$, that is for $\zeta'+\phi=135°$, and $\zeta'+\phi=315°$. Thus a numerical maximum of amount, $\sqrt{D_1^2+E_1^2}$, or D_1 approximately, is obtained on four compass courses, successively a quadrant distant from one another. Hence the name "quadrantal error."

This part of the compass error does not in general amount to more than from 5° to 10° in iron vessels, unless the compass has been badly placed, or soft iron carried as cargo is stowed too near the binnacle.

The Semicircular Error

128. Considering now the remaining pair of terms, namely,

$$B_1 \sin \zeta' + C_1 \cos \zeta'$$

we see that, since approximately

$$\sin B_1 = B = \frac{c \tan \psi + P/H}{\lambda}, \quad \sin C_1 = C = \frac{f \tan \psi + Q/H}{\lambda}$$

(ψ being the dip), the coefficients of these terms depend partly on the effect produced by vertically induced magnetism in consequence of unsymmetrical arrangement of the soft iron of the ship, (1) with reference to the cross section of the hull at the compass needle, which gives the constant c, (2) with reference to the plane of the keel, which gives the constant f. The latter constant is small in comparison with c.

The remaining parts of the coefficients B_1, C_1, depend upon the permanent magnetism of the ship, since

$$\frac{P}{\lambda} + \frac{c}{\lambda} H \tan \psi = BH, \quad \frac{Q}{\lambda} + \frac{f}{\lambda} H \tan \psi = CH$$

where $B=\sin B_1$ $C=\sin C_1$ approximately. We shall see presently how these parts can be determined.

The sum

$$B_1 \sin \zeta' + C_1 \cos \zeta'$$

is what is called the *semicircular deviation.* For writing it in the form

$$\sqrt{B_1^2 + C_1^2} \sin (\zeta' + \phi')$$

where

$$\cos \phi' = B_1 / \sqrt{B_1^2 + C_1^2}, \quad \sin \phi' = C_1 / \sqrt{B_1^2 + C_1^2}$$

we see that its numerical value is a maximum for $\zeta'+\phi'=90°$, and $\zeta+\phi'=270°$. Thus when the compass course is changed through 360°

this part of the deviation attains a maximum numerical value on two courses 180° apart.

The maximum semicircular error of the compass may amount to 30° or 40° in an armour-clad vessel, and to over 20° in an ordinary iron vessel. But as pointed out above, the total error can be kept down to less than 20° by partial compensation, in cases in which no attempt is made to completely annul it.

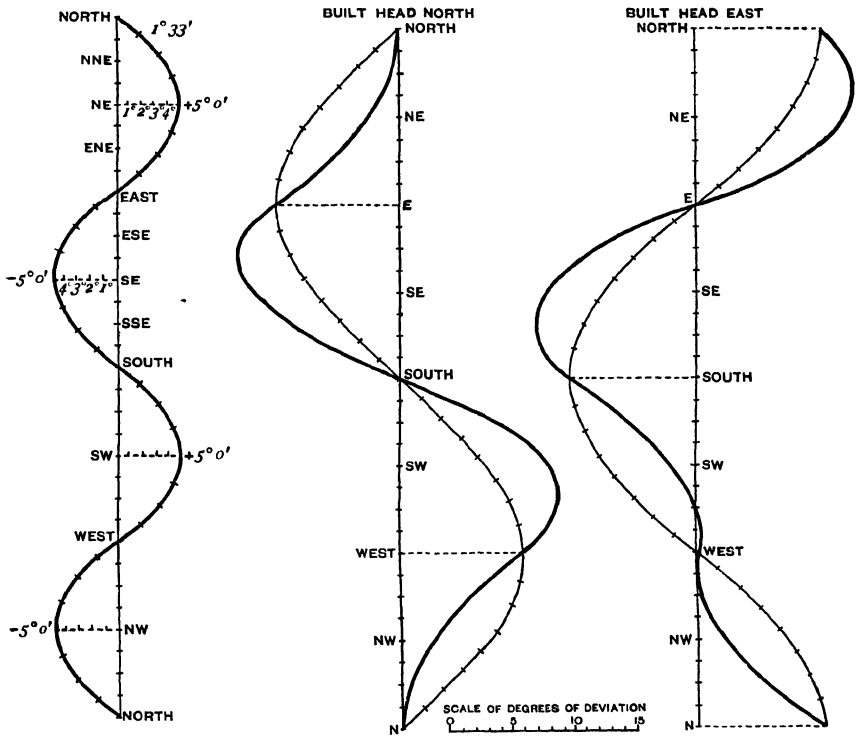


FIG. 43.

Graphical Representation of Deviations. Determination of the Coefficients. The Dygogram [1]

129. The semicircular and quadrantal errors for two ships, one built head north, the other built head east, are shown in Fig. 43. The successive points of the compass are laid down at equal distances apart along a straight line, and the errors on the various courses, as taken directly from the compass which is being observed, are shown by

[1] This somewhat unintelligible-looking word is a contraction of *dynamogoniogram*, that is, the diagram of force and of angle. The curve is well known as the Limaçon of Pascal.

ordinates drawn at right angles to this line. The quadrantal errors are the same in both cases, and have a maximum of 5° on the N.E., S.E., S.W., and N.W. courses. This is shown by the diagram on the left. The other two diagrams give the semicircular errors, which are shown by the lighter full lines for the cases specified. It will be seen that the maximum semicircular error is here about 10°. The dark full lines in these diagrams show the resultant error obtained by compounding the quadrantal and semicircular errors.

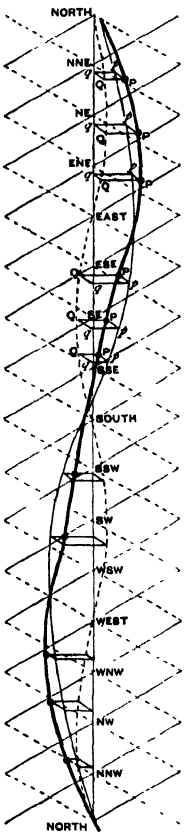


FIG. 44.

130. Fig. 44 illustrates a mode of representing the errors graphically devised by the late Mr. J. R. Napier, F.R.S. Instead of laying off the deviations observed on given courses, at right angles to the line of courses, they are laid off along lines drawn at angles of 60° to this line. These lines are shown dotted in Fig. 44. Westerly deviations are laid off towards the left, easterly deviations towards the right of the line of courses. It is to be noticed that in drawing this diagram the courses taken are courses as shown by the compass on board, and are not correct magnetic courses.

The convenience of Napier's diagram lies in this, that the curve of deviations having been thus drawn, the correct magnetic course to be laid down on the chart as corresponding to the compass course which has been steered or *vice versâ* can be at once found. A series of full lines intersecting the dotted lines at angles of 60°, and cutting the line of courses at the same points are drawn on the diagram. The rule (reversed for the other problem) is then as follows:—

Take on the line of courses the compass course actually steered, then pass from that point parallel to the diagonal dotted lines to the curve, then from that point parallel to the diagonal full lines, back to the line of courses. The point arrived at is the true magnetic course to be used on the chart.

This rule is obviously true. The distance from the starting point of the final point arrived at on the line of courses is equal to the deviation, since the three points are the vertices of an equilateral triangle.

The curves shown in Fig. 44 show the quadrantal and semicircular deviations, and the resultant error for H.M.S. *Achilles*,[1] a now obsolete ironclad. The maximum quadrantal error is about 6°·9′, the maximum semicircular error 21° 15′.

131. By determining the semicircular error on different compass courses the values of B_1, C_1, can obviously be found to any necessary degree of accuracy. If the observations are made at positions of

[1] *Elementary Manual of the Deviations of the Compass*, Edition 1870.

the ship sufficiently far apart in magnetic latitude, where the dip and the horizontal component of the earth's field intensity are known, the different parts of the exact coefficients B, C, can be calculated. For let B_1, B_2 be values of the former coefficient at two such places where H_1, H_2, ψ_1, ψ_2 are the values of H and ψ. Then we have

$$\frac{P}{\lambda} + \frac{c}{\lambda} H_1 \tan \psi_1 = B_1 H_1$$

$$\frac{P}{\lambda} + \frac{c}{\lambda} H_2 \tan \psi_2 = B_2 H_2,$$

which give

$$\frac{P}{\lambda} = \frac{H_1 H_2 (B_1 \tan \psi_2 - B_2 \tan \psi_1)}{H_2 \tan \psi_2 - H_1 \tan \psi_1} \quad . \quad . \quad . \quad (10)$$

$$\frac{c}{\lambda} = \frac{B_1 H_1 - B_2 H_2}{H_1 \tan \psi_1 - H_2 \tan \psi_2} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Similarly

$$\frac{Q}{\lambda} = \frac{H_1 H_2 (C_1 \tan \psi_2 - C_2 \tan \psi_1)}{H_2 \tan \psi_2 - H_1 \tan \psi_1} \quad . \quad . \quad . \quad (12)$$

$$\frac{f}{\lambda} = \frac{C_1 H_1 - C_2 H_2}{H_1 \tan \psi_1 - H_2 \tan \psi_2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The quantity λ $(=1+(a+e)/2)$ is easily interpreted as the average value of $H' \cos \delta / H$ for all possible courses. To prove this we recall the equation

$$\frac{1}{\lambda} \frac{H'}{H} \cos \delta = 1 + B \cos \zeta - C \sin \zeta + D \cos 2\zeta - C \sin 2\zeta.$$

Multiplying both sides by $d\zeta$, and integrating from $\zeta = 0$, to $\zeta = 2\pi$, we obtain

$$\frac{H'}{\lambda H} \int_0^{2\pi} \cos \delta \, d\zeta = 2\pi,$$

that is

$$\lambda = \frac{1}{2\pi} \frac{H'}{H} \int_0^{2\pi} \cos \delta \, d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which proves the proposition. Hence

$$H\lambda = \overline{H' \cos \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where the bar denotes the mean value of the product. But H' is the component of the local horizontal force, and this is in the direction of the needle. Hence $H' \cos \delta$ is the component horizontal magnetic force

in the direction of the head of the vessel. Thus $H\lambda$ denotes the mean value of this component for all possible directions of the vessel's head.

We are unable here to go into particulars regarding the mode of observing compass deviations in actual cases. Full information will be found in the *Admiralty Manual of Deviations of the Compass*, and in treatises on Compass Adjustment.[1] It must suffice to state that the different coefficients can be found by harmonic analysis if a sufficient number of values of δ are obtained for different compass courses.

132. When the coefficients, *A*, *B*, *C*, &c., have been found, the deviations on different courses are capable of a very elegant graphical representation invented by Archibald Smith,[2] to whom the complete working out of the theory of compass deviation is in great measure due.

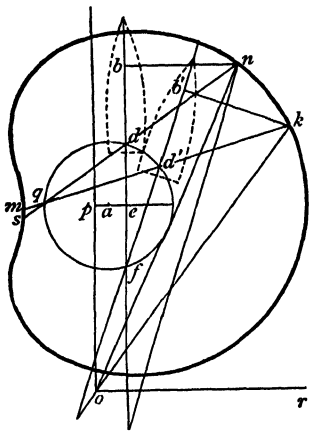


FIG. 45.

Draw first (Fig. 45) two lines *op*, *or*, to represent the magnetic north and the magnetic east directions. Then take a length *op* as great as is convenient to represent the value of λH; and from *p* take a length *pa* eastward or westward to represent a positive or negative value of $A \cdot \lambda H$ on the same scale, and a second length *ae* to represent *E*. Next draw from the point *e* a line *ed* northward or southward to represent $D \cdot \lambda H$, according as *D* is positive or negative, and a second length *db* from *d* to represent $B \cdot \lambda H$. Lastly, from *b* draw to the east or west a line *bn* to represent $C \cdot \lambda H$ according as it is positive or negative.

The angle *nop* is the deviation for $\zeta = 0$, that is, when the ship's head is due north. For we have

$$op + db + ed = \lambda H(1 + B + D),$$

$$pa + bn + ae = \lambda H(A + C + E).$$

Thus if we take *op* as unity, the first length represents $1 + B + D$, the second $A + C + E$. Now by equations (4) and (5)

$$\left.\begin{aligned} \frac{H'}{\lambda H}\sin\delta &= A + C + E \\ \frac{H'}{\lambda H}\cos\delta &= 1 + B + D \end{aligned}\right\} \quad \ldots\ldots \quad (16)$$

[1] See *Traité de la Régulation et de la Compensation des Compas*, par A. Collet. Paris; Challamel Aîné.

[2] Other forms of dygogram are also employed in practice. For their construction see the *Admiralty Manual*.

when $\zeta = 0$. Therefore for this value of ζ

$$\tan nop = \tan \delta_0$$

and

$$on = \frac{H'}{\lambda H}.$$

To find the value of δ for any other value of ζ, from a as centre and with ad as radius, describe a circle. Join nd and produce the line to cut the circle again in q. If now a slip of paper with straight edge be made of a length $2nd$, and be placed in different positions so that the middle point of its straight edge always lies on the circle, and the edge passes through q, the extremities for each position will give two points on a curve, called by Archibald Smith the dygogram. Thus n, s are two positions, k, m other two, and so on.

133. Taking the angle kqn as representing ζ, we can easily show that the value of $H'/\lambda H$ is represented by ok, and the deviation for the corresponding magnetic course ζ by the angle pok. For each of the points n, k we have diametrically opposite points s, m, corresponding to course $\zeta + 180°$, for which the dygogram gives the deviation and magnetic force.

To prove these statements it is only necessary to observe that if $\angle kqn = \zeta$, the projection of ok on the line op is op + projection of ad' + projection of $d'k$. The second of these is the radius of the circle ad turned through 2ζ; hence its projection on op is the sum of the projections of ac and ed on op, after each has been turned through an angle 2ζ in the same direction. Hence the projection is $ed \cos 2\zeta - ae \sin 2\zeta$. Thus the projection is $op + ed \cos 2\zeta - ae \sin 2\zeta$ + projection of $d'k$. But $d'k$ is dn turned through the angle ζ, and its projection on op is equal to the sum of the projection of the components db and bn when turned through the same angle, that is, is equal to $db \cos \zeta - bn \sin \zeta$. Thus the total projection sought is

$$op + db \cos \zeta - bn \sin \zeta + ed \cos 2\zeta - ae \sin 2\zeta,$$

which since $op = \lambda H$, $db = B.\lambda H$, $bn = C.\lambda H$, $ed = D.\lambda H$, $ae = E.\lambda H$, may be written

$$\lambda H (1 + B \cos \zeta - C \sin \zeta + D \cos 2\zeta - E \sin 2\zeta).$$

which we have seen (p. 88 above) is the value of $H' \cos \delta$.

Similarly it can be shown that the projection of ok upon or is

$$\lambda H (A + B \sin \zeta + C \cos \zeta + D \sin 2\zeta + E \cos 2\zeta),$$

which we have seen is $H' \sin \delta$. Hence

$$\angle pok = \delta, \; ok = H'.$$

Since angles at the circumference of a circle standing upon the same arc are equal, the angle nqk is equal to the angle dfd'. Hence since the angle nqk represents ζ, and the ship's head, when $\zeta = 0$, is in the direction fd, when the compass course is ζ the direction is fd', and all the lines showing the direction of the ship's head pass through f. Thus the diagrammatic plan of the hull in Fig. 45 shows the position of the ship's head, d, d', the positions of the compass.

It is important to remark that a knowledge of the five exact coefficients A, B, C, D, E permits the dygogram to be traced. Then a single determination of the ratio H'/H of the horizontal force within the ship, at the place of the compass, to the value of the earth's horizontal force gives the value of λH, the mean value of the component of horizontal force in the direction of the ship's head.

134. There are also what are called sextantal and octantal errors which are sensible in compasses of which the needles are long, and are represented by the group of terms—

$$F \sin 3\zeta' + G \cos 3\zeta' + K \sin 4\zeta' + L \cos 4\zeta'.$$

The two first terms are the sextantal error, and arise from too near approach of the needle to permanent magnets. It attains six equal numerical maxima, positive and negative alternately when the compass course is changed through 360°. These are at successive distances of 60° apart. The other two terms constitute the octantal error, so called because it passes through eight alternately positive and negative maxima, which occur at successive angular distances of 45°. These terms arise from the too near approach to the needle of pieces of soft iron, which are magnetized by and re-act on the needle.

With modern compasses, such as that of Lord Kelvin, which have short needles, these terms are not of importance and may be neglected.

The Heeling Error

135. Hitherto the ship has been supposed to be on even keel. When however she is heeled over to one side or the other the quantities a, b, c, &c., require modification. For the effect of the inclination is to raise one side of the ship and depress the other, hence altering generally the positions of the equivalent soft iron rods relatively to the compass.

Supposing the ship heeled over through an angle i, the fore and aft soft iron bar which gives a does not have its effect on the compass altered, but the pair of fore and aft bars on the port and starboard sides do not act in the same way as before. The bar formerly producing the effect dX along y has now the effect $d \cos i . X$. On the other hand, the fore and aft soft iron bars above and below the compass which produced the effect gX are displaced by the heel, one to port the other to starboard of the compass, so that they produce a force in the

direction of y of amount $-g \sin i \,.\, X$, supposing the heel is to starboard Thus we have to replace d by a quantity δ_i given by

$$\delta_i = d \cos i - g \sin i$$

If the heel is in the opposite direction i is reckoned negative.

In a similar manner it can be shown that the other quantities, b, c, &c., are replaced by those given in the following table :—

$$\left.\begin{aligned} b_i &= b \cos i - c \sin i, \quad d_i = d \cos i - g \sin i \\ c_i &= c \cos i + b \sin i, \quad g_i = g \cos i + d \sin i \end{aligned}\right\} \quad . \; . \; (17)$$

$$\left.\begin{aligned} e_i &= e - (f + h) \cos i \sin i - (e - k) \sin^2 i \\ f_i &= f + (c - k) \cos i \sin i - (f + h) \sin^2 i \\ h_i &= h + (e - k) \cos i \sin i - (f + h) \sin^2 i \\ k_i &= k + (f + h) \cos i \sin i + (e - k) \sin^2 i \end{aligned}\right\} \quad . \; . \; (18)$$

If i be so small that we can put $\sin i = i$, $\cos i = 1$ and neglect $\sin^2 i$, and the iron be symmetrical about the fore and aft midship line, these give for the coefficients when the angle of keel is i,

$$\lambda_i = \lambda, \; A_i = \frac{c - g}{2\lambda}, \; C_i = -\frac{c + g}{2\lambda}, \; D_i = D,$$

$$E_i = E + \frac{1}{\lambda}\left(e - k - \frac{R}{Z}\right) i \tan \psi = E + iJ.$$

Then the deviation is

$$\delta_i = \delta + \frac{c - g}{2\lambda} + iJ \cos \zeta - \frac{c + g}{2\lambda} i \cos 2\zeta \quad . \; . \; . \quad (19)$$

136. The terms here added to δ constitute the *heeling* error. The most important part is the term $Ji \cos \zeta'$, where

$$J = \frac{1}{\lambda}\left(e - k - \frac{R}{Z}\right) \tan \psi = -\left(D + \frac{\mu}{\lambda} - 1\right) \tan \psi \quad . \quad (20)$$

where $\mu = 1 + k + R/Z$, and D has the value given above, namely $1 - (e + 1)/\lambda$.

The total heeling error can be written in the form in which it is convenient to consider it when discussing its correction :—

$$\delta_i - \delta = Ji \cos \zeta + \frac{c}{\lambda} i \sin^2 \zeta - \frac{g}{\lambda} i \cos^2 \zeta \quad . \; . \; . \quad (21)$$

A smaller error due to pitching is not of sufficient importance to be taken into account.

Compensation of the Compass

137. With regard to compensation of the compass a great deal might be said. We shall not here enter into any discussion of how the various coefficients of the expression for the deviation are determined but merely describe shortly the process followed in the adjustment of

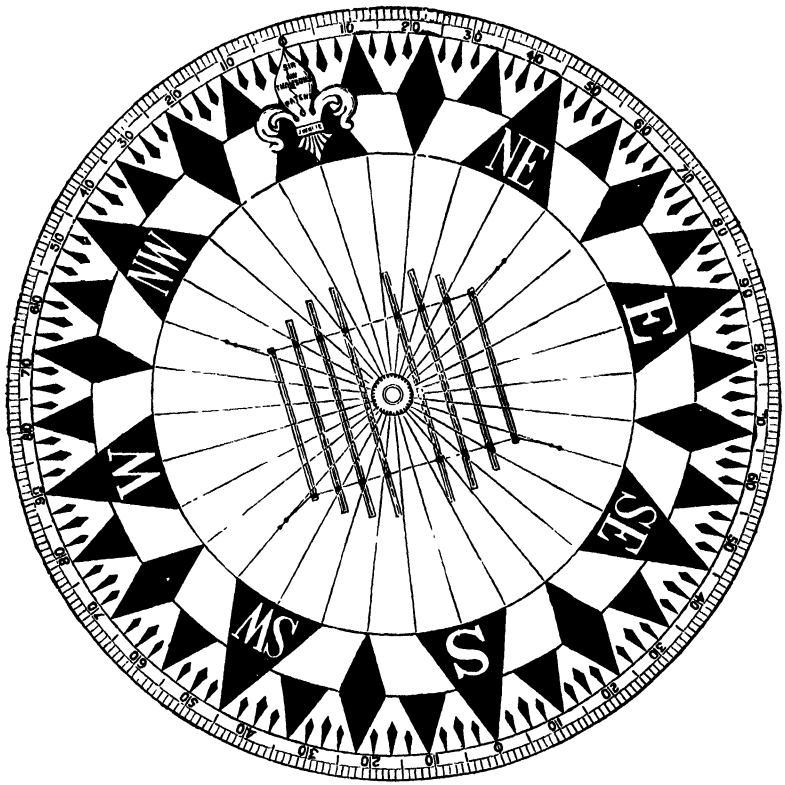


FIG. 46.

Lord Kelvin's compass, which is now very generally adopted on board large iron ships in this country.

The card of this compass is shown in Fig. 46. It consists of a paper ring, on which are marked the points and degrees in the ordinary manner, attached to a light rim of aluminium which keeps it in shape. Radial threads connect the ring to a central boss containing a sapphire cap, by which the compass is supported on an iridium point fixed below to the compass bowl. Below the card, strung like the steps of a rope ladder on two silk threads attached to the radial threads, are eight small magnets of glass-hard steel, which form the compass needle. These vary

in length from $3\frac{1}{4}$ inches to 2 inches, and are symmetrically arranged in the manner shown in the diagram.

The entire weight of the card, including needles, is $170\frac{1}{2}$ grains. This extreme lightness, combined with the relatively great moment of inertia obtained by the distribution of mass, ensures a long period of free vibration and therefore great steadiness. It also enables the frictional error on the supporting point to be made very slight.

138. The semicircular error is corrected by placing steel magnets under the needle in the binnacle, so as to annul the error on the north and south and east and west courses, due to the two horizontal components of disturbing magnetic force, arising principally from the permanent magnetism of the ship. In the correction of these errors two sets of permanent magnets are used in the binnacle, with their centres vertically under that of the needle, one set placed with their lengths in the fore and aft, the other with their lengths in the thwart-ship direction.

In the process of correction, when marks on the shore are available the true bearings of which from the ship are known, the ship is first placed with her head in the magnetic north direction, and the thwart-ship magnets are moved so as to bring the compass needle to the north point on the card. For it is clear that the fore and aft magnetic force cannot have any effect on the needle when the ship's head and needle are both in the same direction. When this is the case we have, since $\zeta = 0$,

$$A_1 + C_2 + E_1 = 0,$$

where C_2 denotes the value of C_1 as modified by the presence of the correcting magnets.

The ship's head is now placed on the magnetic east (or west) point, and after an interval of about five minutes has been allowed to elapse, the fore and aft magnets are placed so that the compass needle points also due magnetic east. This gives

$$A_1 + B_2 - E_1 = 0,$$

where B_2 is the value of B_1 as modified by the correcting magnets.

If A and E are negligible the semicircular error has been corrected.

139. The ship's head is now changed to one of the quadrantal points, say the S.E. point (magnetic). If on this course, or the N.W. course, there is a deviation of the compass needle to the west (see Fig. 44), the coefficient D is positive, and a pair of equal spheres of soft iron are placed one on each side of the compass, at equal distances from the centre, and so that the line joining their centres is at right angles to the plane of the keel and passes through the centre of the needle. The distance of the spheres apart is adjusted until the deviation has disappeared.

If, on the other hand, the deviation before the spheres are placed in position is found to be easterly on N.W. or S.E. courses, or westerly on N.E. or S.W. courses, then the coefficient D is negative, and the spheres must be placed in the plane of the keel, that is, afore and abaft the compass. This is, however, an altogether exceptional case.

When the ship is new the permanent magnetism of the iron will slowly change, and it will be necessary to alter from time to time the position of the compensating magnets correcting the semicircular error. When, however, the quadrantal error has been annulled the adjustment remains correct to whatever part of the world the ship may go.

The induced magnetism of the ship, after long sailing in one direction, say east or west, generally lags behind a change of course. Thus when the ship's head is changed to north or south, from east say, a temporary error arises which the reader can easily trace the nature of. This (which is known as Gaussin's error) cannot conveniently be corrected, and must be allowed for.

140. The vertical force of the earth induces magnetism in the ship's iron, which gives a horizontal component of force on the compass needle, namely, the part depending on c in the term $B_1 \sin \zeta'$. This varies as the ship goes from one latitude to another, and no provision for a corresponding soft iron compensator is made in the arrangement described above. Lord Kelvin has adopted in his compass the method, proposed long ago for the correction of this error by Captain Flinders, of placing an upright bar of soft iron exactly (in the case of a distribution symmetrical relatively to the plane of the keel) in front of or behind the binnacle, with its top about 2 inches above the needles. The bar used is round and about 3 inches in diameter, and of length varying from 6 to 24 inches, according to the requirements of the case.

The Flinders bar corrects the term of the heeling error $ci \sin^2 \zeta'/\lambda$, which is due to alteration of the positions of the equivalent soft iron rods representing cZ caused by the list given to the ship, and has its maximum when the ship's head is east or west. It also corrects partially the error $Ji \cos \zeta'$ by correcting the part—$ki \tan \psi \cos \zeta'/\lambda$ of this term.

The part $-Ri \tan \psi \cos \zeta'/Z\lambda$ is corrected by a vertical magnet placed in a tube immediately below the compass needle. The strength and distance of magnet required is determined by a comparison of the vertical force within the ship to the vertical force on shore.

141. Lord Kelvin has also shown how by means of an ingenious instrument called a deflector, which he has perfected, a comparison can be made of the directive forces on different courses. The adjustment then consists essentially in equalizing the directive force of the ship on a sufficient number of courses to make sure that it has as nearly as possible the same value on all courses, when it is certain that the compass is correct. This mode of adjustment is useful when sights of sun or stars or bearings of terrestrial objects are not available, and it can be carried out with great accuracy. For details the reader is referred to Captain Collet's treatise referred to on p. 94 above, or to Lord Kelvin's *Instructions for Adjusting the Compass*, to be obtained of James White, optician, Glasgow. On the whole subject of compass error and adjustment, the reader should consult, in addition to the treatise mentioned, the *Admiralty Manual of Deviations of the Compass.*

CHAPTER V

ELEMENTARY PHENOMENA AND THEORY OF ELECTROSTATICS

SECTION I.—*Experimental Results and Action of Medium*

Elementary Notions

142. Before proceeding to a discussion of electrical and electromagnetic theory it will be convenient to give here a short account of the elementary phenomena of electrostatics and the steady flow of electricity. These phenomena are capable of being regarded from two different points of view, one in which the charge or the current of electricity is the property only of the conductor, and another in which the charged conductor is simply part of the boundary of a state of strain in the dielectric surrounding it, and the wire along which a current flows is merely a guide for the transference of energy through the dielectric, from which also it receives in general a portion of energy to be dissipated in heat.

The first electrical phenomenon observed was the attraction which substances, such as glass and sealing-wax, exerted on light bodies, as small feathers, the dried pith of elder, and the like. This, on the face of it, is action at a distance; but it will be seen on consideration that such apparent attractions may be due to a medium in which the bodies are immersed, and which acts in such a manner that the two bodies are brought closer together, unless they are prevented from approaching by forces applied to the bodies by some other system, which, it may be noticed, must be some material system extending from one body to another. In fact the bodies are pushed towards one another in consequence of a state of stress existing in the medium surrounding the two bodies, and this can only be prevented from having any effect in displacing the bodies by another stress, set up in a material system by which the bodies are, so to speak, connected.

143. According to the old idea a piece of smooth glass rubbed with silk had developed upon it a certain fluid, which it was agreed to call electricity. This fluid had the properties of repelling other portions of the same fluid, and attracting portions of another fluid which was at the same time developed on the silk rubber. That idea is now replaced by

the conception, much more in accordance with the phenomena which have been observed, that by the rubbing a state of strain accompanied by internal stresses is set up in the non-conducting medium, or dielectric, as we shall call it, between the glass and the rubber; that what were regarded as the electric charges on the glass and silk, and to which the apparent attractions or repulsions of the electrified bodies were attributed, are simply the surface manifestations of this state of the medium at the surfaces of the bodies immersed in it; and that the attractions and repulsions observed are really the result of a system of internal stresses in the medium, which are the natural accompaniment of the state of strain.

144. Of this system of strains and stresses we shall endeavour later to give some more detailed account, though it is unfortunately the case that a completely satisfactory explanation, or even specification, of it is as yet impossible. Nevertheless it will be found to conduce to clearness of statement, and the prevention of false ideas, for example that of the real existence of a material something developed on the surface of an electrified body, or flowing through the substance of a wire carrying a current of electricity. The latter phenomenon, according to this more modern view of the subject, is an accompaniment of a progressive change in the state of the non-conducting medium, a change which in certain very important cases is continually made good in virtue of certain other changes going on in the state of a material system, so that the medium remains to our observation in a steady state, and what we call a steady flow of electricity, or a steady current, goes on along the conductor.

Location and Transfer of Energy. Current in a Wire

145. The setting up of this state of the dielectric involves the expenditure of work, part of which has its equivalent in energy, which has its seat in the medium while the state lasts, is conveyed to the medium as the state grows up, and leaves it as the state evanesces, in a manner which we shall seek to investigate. As a very important part of the new view to which reference has been made, a battery which furnishes a current to actuate a telegraph instrument, or an electric motor, or a voltameter or secondary battery in which electro-chemical change is effected, or other arrangement in which useful work is done, does not transmit energy along the wire, but sends it out into the medium, from which it flows again upon the arrangement in which the energy is utilised, and at the same time upon the wire to supply the energy which in all cases is dissipated in the conductor. The medium has its state changed, and the energy required for that is thrown out into the medium from the battery. In consequence of the presence of the conductor completing the circuit the state of the medium is continually breaking down, through the passage of energy from the medium to the conductor and included instrument; so that for the maintenance of the steady state energy is continually being thrown out into the medium by

the battery. Thus the wire and connecting conductors merely act as a guide, and the instrument or machine included in the circuit receives the energy which is given out by the battery, diminished by the energy which flows back upon the connecting conductors and is converted into heat. It is thus the medium, not the conductor, which acts as the carrier of energy; for example, in the case of a submarine cable, the vehicle of energy is the layer of gutta-percha which separates the conducting wire from the sea water in which the cable is laid.

146. What according to the newer theory goes on in the passage of a current along a wire will be more clearly understood by a short consideration of the discharge of a Leyden jar. According to the old and still common statement, the jar (which consists of a glass bottle coated inside and outside, about three-fourths of the way up, with tinfoil) is charged by allowing positive electricity (see below, p. 105) to flow into its inner coating from an electric machine; this induces negative electricity on the inner side of the outer coating supposed connected with the earth, the negative electricity thus induced enables more positive to flow into the inner coating, this induces further negative electricity on the outer coating, and so on.

Electric Induction and Electric Intensity

147. What really happens is that a state of strain is set up within the glass separator of the two coatings; the opposite surface manifestations of this are the two opposite charges of electricity. The state of strain is in a certain sense measured by a quantity called *electric induction*, directed from the inner coating towards the outer in the case supposed. This quantity we shall specify completely later. At each point in the dielectric the induction has a definite direction, and a line drawn so that its direction at each point is the direction of the induction is called a *line of induction*. A line of induction thus starts from the coating which we say is positively charged and ends in the negatively charged inner surface of the outer coating.

A part of the dielectric bounded laterally by lines of induction is called a *tube of induction*. Thus in the Leyden jar tubes of induction start from the inner electrified surface and end on the outer, and the course of induction is such that each tube, whatever its scope on the surfaces may be, has quantities of electricity at its ends which are complementary not only in amount but in actual fact, that is the portion of the surface aspect of the strain which is enclosed by the tube at one end physically corresponds in the state of strain to that at the other end. The positive charge from which the tube starts is thus equal in amount and opposite in sign to that in which the tube ends.

148. The energy stored up in the jar thus exists within the dielectric while the system is in equilibrium. Now let the coatings be connected by a wire. The tubes of induction move outwards sideways from the condenser with their ends on the wire, so that the positive and

negative electricities move along facing one another (thus moving in opposite directions round the circuit), while each tube gradually shortens as it advances, being swallowed up at its ends in the wire, thereby yielding up its energy to the conductor, until it becomes infinitely short and disappears. If a telegraph instrument or other arrangement in which energy is utilised is actuated, the tubes are only partially absorbed by the joining conductors, and there is a finite remainder of each tube which is absorbed by the arrangement.

If the condenser have its terminals connected to earth, the earth forms in no sense a reservoir into which the electricity of the condenser is discharged, but only part of the guiding conductor.

In the case of a battery the tubes are thrown out from the battery laterally into the medium, and then they move along as in the case just described with their ends on the conductor, being absorbed wholly or partially in it as they proceed. (See Chapter XI. below.)

149. When an electrified conductor is in electrical equilibrium the tubes of induction start normally from its surface, and exert on every element of it an outward pull, which is balanced, not by electrical strain of the conductor, for no such strain can be set up in perfectly conducting material, but by force due to ordinary mechanical strain of its material. For example, an electrified soap bubble is pulled everywhere normally outwards by the medium outside it, and the outward force on each element of the surface is balanced by part of the inward force on the same element due to the contractile force in the curved film.

A perfect conductor is in this theory a body which cannot endure strain of the kind which exists in the dielectric, and at which therefore the state of strain to which we have referred suddenly terminates. Conductors, however, may be more or less imperfect, and the state of strain capable of temporarily existing within them to some extent.

150. Dielectrics as well as ordinary conductors are, we have reason to believe, merely material systems imbedded in the ether which permeates their structure as it does all space, and the ether is itself the standard or ultimate dielectric medium to the action of which all electric and magnetic phenomena are to be referred. This view of the matter will become clearer as we proceed; the preceding discussion will serve to introduce the ideas, and at present we go on to a short sketch of elementary electrical phenomena, bringing them as far as possible into relation with the ideas which have just been explained.

It is to be clearly understood that at present we consider only the electrification of bodies which are at rest, relatively to the insulating medium in which they are immersed. Also it is to be supposed, unless the contrary is stated, that when conductors are referred to as brought into contact with one another, they are of the same material, so that there is no question of contact difference of potential. Further it is assumed that no change of the internal physical state of any of the bodies such as temperature, volume, or the like, accompanies the electrical actions or changes considered.

"Electrics" and "Non-Electrics." Conductors and Insulators. Electric Attraction and Repulsion

151. When a rod of glass has been rubbed with silk it is found by suspending the silk and glass side by side that they apparently attract one another. Again, if two small pith-balls be hung by silk threads so as to rest as nearly as may be at the same level a short distance apart, and one of them be touched with the rubbed glass, the other with the silk rubber, they will appear to attract one another.

Again, when a piece of sealing-wax is rubbed with a dry woollen cloth and then made to touch one of the balls, while the other is touched by the glass rod, attraction between the balls is observed, just as when the silk rubber and glass were used to touch the balls. Also when the balls are touched by the rubbed sealing-wax they seem to repel one another, and moreover one touched by the silk rubber repels one touched by the sealing-wax.

The result obtained by rubbing other substances is always in a similar way either a repulsion or an attraction, which would have been obtained by rubbing smooth glass, or sealing-wax, or both, the glass with silk, the sealing-wax with wool.

152. From these facts, which were long ago observed, arose the idea of two kinds of electricity, that of smooth glass rubbed with silk, and that of sealing-wax rubbed with a woollen cloth. The former of these was called positive the latter negative electricity. A portion of one of these electricities was regarded as having the property of repelling another portion of the same kind, and of attracting a portion of the other kind.

This qualitative result, which is still given in many elementary treatises on electricity as "the law of electrical attraction and repulsion," was practically all that was known before the forces between electrified bodies were quantitatively investigated by Coulomb. In the meantime it had been discovered by Stephen Gray that certain bodies such as silk, glass, sealing-wax, &c., acted as insulators, that is, did not allow electricity to pass off from themselves, or from bodies held or supported by them and excited by rubbing, and that certain other bodies, for example all metals, acted as conductors, that is when used as supports for electrified bodies allowed the electricity to escape to other bodies with which the supports were connected. This broke down the old distinction between electrics and non-electrics, or bodies which could be electrified by rubbing, and those which apparently could not; for it was immediately found that all bodies could be electrified, provided the body were held by a proper support to prevent the excitation from being dissipated as fast as it was produced.

153. The quantitative result obtained by Coulomb is in part expressed by the statement that the repulsion between two small similarly electrified conducting spheres is inversely proportional to the square of

the distance between their centres.[1] Here spheres are considered small if the ratio of the distance of the centres apart to the radius is in each case large compared with unity; for example, Coulomb's result would apply without sensible error to a pair of pith-balls, each 5 millimetres in radius with their centres 20 centimetres apart.

154. When a conducting sphere is charged with electricity and placed at a distance from other conductors great in comparison with its radius and the linear dimensions of those other conductors if they are charged, the distribution of electricity on the sphere is found to be symmetrical round the centre. This is proved experimentally by applying to the surface of the sphere a small disk of thin metal held by a glass insulating stem so as to coincide with the surface of the sphere, and then removing it and observing the repulsion between the charge on the disk when held at a fixed distance from a small insulated charged sphere, say a charged pith-ball, hung by a single silk fibre. It is found that wherever the disk is applied to the sphere, the charge removed is the same, inasmuch as the force, as measured by the deflection of the pith-ball pendulum against the action of gravity, is the same. For if the sphere were more intensely electrified at one part of its surface than another, the disk when applied at such a place would be more intensely electrified, and a greater repulsion would be produced.

Further, if such a proof-plane (as this insulated disk is called) is applied to the interior of a hollow conducting sphere, within which no electrified bodies are insulated, no charge is taken by it, showing that there is no electrification on the inner surface of such a sphere. Also, on the inner surface of a closed hollow of any shape within a conductor no electrification is found if there be no electrified bodies within the hollow.

Forces on Electrified Bodies regarded as due to Action of a Medium

155. These results are consistent with the theory of the action of a medium referred to above. When the spherical conductor is at a great distance from other conductors, the state of the medium near the sphere is quite symmetrical all round it. The sphere, being a conductor, supports no electrical strain within its substance, and none is transmitted through its substance to the medium existing within it, that is, no surface aspect of the state of strain in the dielectric is found to exist anywhere except at the external surface of the sphere, unless there are electrified bodies insulated in the hollow space within it. If however the sphere experience force from the field, this arises from dissymmetry of induction round its surface, due to the presence of other conductors.

156. Let us now suppose that we have a conductor, A, charged with electricity, and that it is connected by means of a wire with another conductor, B. B will also become charged with electricity. The state of the field is, in fact, not one of equilibrium with A charged and B not,

[1] For an account of the torsion balance experiments by which this result was established see the author's *Absolute Measurements in Electricity and Magnetism*, Vol. I., p. 254, or any good elementary treatise on electricity.

and the two in contact through a conductor, inasmuch as the region of strain in the field which abuts against the charged conductor, A, tends to spread itself out laterally, in a manner to be considered later, with the wire and the surface of the other conductor as guide, until this tendency is balanced by a distribution of the strain all round; so that the lateral action on the sides of each element of a tube is balanced.

According to Maxwell the dielectric is affected by stresses consisting of a tension along the tubes and an equal pressure in all perpendicular directions. The error, however, is to be avoided of identifying the strains above referred to with those in an elastic solid. We shall deal with this subject later.

The electrification is thus extended over the surfaces of both A and B, and if, as we suppose, the wire be very thin, it may be neglected or removed without disturbing the distribution on either conductor. We shall show that in a certain sense there is the same total quantity of electricity on the two conductors that there was originally on A.

Faraday's Ice-Pail Experiment

157. To prove this we make use of a celebrated experiment of Faraday, called his ice-pail experiment. A nearly closed vessel, such as a deep metal vessel, P, like that represented in the diagram, (Fig. 47), is hung by silk threads, or more conveniently supported, as shown in the diagram, by a block of solid paraffin or other non-conducting material, and has, in conducting connection with it as shown, two pith-ball pendulums, which are supported from one point by thin wires of metal. When the outside of the vessel is charged with electricity the balls become charged also and separate in consequence of their apparent repulsion. (This repulsion will find its explanation in a pull exerted on the surface of a charged conductor by the surrounding medium, as more fully stated below.) Any change in the electrification of the vessel naturally leads to a corresponding change in that of the balls and an alteration of their equilibrium distance apart.

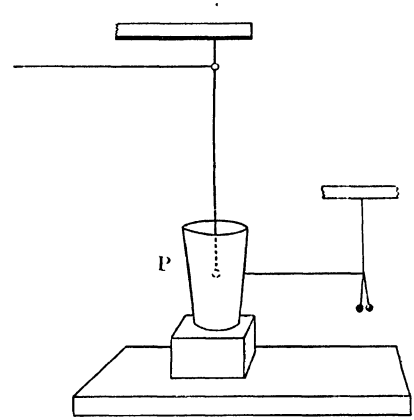


FIG. 47. Ice-pail on paraffin block, connected by thin wire with two pith-balls hung from a point by wires. A charged ball can be lowered by a silk thread to touch the bottom of the pail. A lid, with a hole in it through which the thread passes, may be supposed to close the pail.

Now if the vessel be initially uncharged, and a conducting ball be attached to a silk thread and charged, and then lowered within the vessel P, the pith-balls, which originally hung with the threads vertical,

move further apart, showing the acquisition of a charge by the exterior of the vessel. It is found, as the charged body descends lower and lower in P, that the deflection of the balls goes on increasing more and more slowly until, if P be fairly deep, it becomes practically constant. If now the conductor be made to touch the bottom of P and be then removed, no change will take place in the deflection of the pendulums, and the body will be found to be completely discharged.

158. Now let the experiment be varied by first lowering the ball near the bottom of the cylinder, then lowering by a silk thread another ball uncharged beside it. No change of the deflected position of the pith-balls will take place. Next bring the two balls into contact; it will be found that again there is no alteration of the deflection of the pendulums. Now withdraw the second ball, and test it for electrification by bringing it near a pith-ball pendulum, and it will be found to be electrified as was the first ball. It will also be found that the pendulums attached to P do not diverge so far as before, and that the former deflection is restored by reinserting the second ball.

Further, if the two balls be equal, and when brought into contact be similarly placed with respect to the interior surface of P, it will be found that the pendulum deflection is the same for either ball left charged alone within the vessel, showing that the charge on the first has, so far as the effect on P is concerned, been equally divided between the two. The same result will be obtained more conveniently by bringing the two balls into contact outside the vessel and at a distance from other conductors, after which each will be found to have the same effect on the pith-ball pendulum outside.

159. Again, if the first ball with the original charge be kept at a certain distance from a similarly charged pith-ball forming the bob of a pendulum, and the force of repulsion be measured by the deflection of the ball from the vertical, then the same experiment be repeated with each of the balls after division of the charge, it will be found that the force in each of the latter experiments is half that observed in the first.

Similarly, if the original charge be shared between three equal conducting spheres, the force shown by the pith-ball pendulum for the same distance of its bob from the charged ball will be one-third of that observed in the case of the ball with the original charge. The ice-pail experiment will assure us, however, that the total charge is unaltered.

160. We thus see how and in what sense we can subdivide charges into equal amounts while the total charge remains unaltered. We can also give by means of the same apparatus any multiple of a given charge to a conductor. Take an ice-pail, P', so small as to be capable of being placed entirely within P, and mount it on insulating supports so as to be easily moved about. Charge two balls with electricity, and introduce them together into P, and note the deflection of the pith-ball pendulum. Then lower them in succession into P', and bring them into contact with it at the bottom, and note that each, when withdrawn, is completely discharged. Lower now P' into P and note that the effect on P is the same as when the two charged balls were placed in it. The

charges on the balls have thus been transferred to P' without change of their aggregate amount.

161. Now perform the following experiment. Insert a charged ball within P and note the deflection of the pith-ball pendulums. Then withdraw the ball, taking care not to discharge it. Insulate P' well within P, and provide a piece of wire held by a handle of vulcanite or glass to make contact between the outer surface of P' and the inner surface of P. Now insert the charged ball within P', without bringing it into contact, and note that the deflection of the pith-ball pendulums is the same as before. Then while the ball is within P' make the connection indicated between P and P'. The deflection of the pendulums will not be affected. Withdraw both the charged ball and P'; still the deflection of the pith-balls is unchanged. Thus the deflection is the same as if the charged ball had been at once brought into P, and discharged by being brought into contact with its interior surface.

The same series of operations can be repeated as often as may be desired, and each time the same quantity of electricity is given to the vessel, as may be verified by comparing the result of a number of these operations, made with a single charged ball, with that of placing the same number of equally charged balls together within the vessel.

It is also to be noticed that when uncharged conductors are insulated within P their presence does not affect the distribution on P or its external field, nor is there found any electrification or any induction whatever at any point within P.

Division of Electric Field into Two Parts by Conducting Screen. Genesis of Field External to Closed Conductor

162. These experiments, besides illustrating the idea of quantity of electricity, show that the field of electric force, when a closed conductor contains electrified bodies, is divided into two parts, the region internal and the region external to the conductor, and that no matter how the connections among the electrified bodies in the interior may be varied, the external field undergoes no change; that is to say, the external field is quite independent of the arrangement of the tubes of induction in the interior, so long as the same number starts from the internal conductors.

163. The mode in which the external field arises will appear from the consideration of a single charge first insulated on a sphere at a distance from other conductors, and then introduced within the closed conductor. At first the lines of induction are directed outwards along the radii of the sphere produced, curving round, however, at a distance from it so as to terminate on the other conductors, whatever these may be. Now let an insulated and uncharged hollow sphere be brought into the field, into the position shown in Fig. 48. Hardly any of the tubes of induction will enter the shell by the opening o; but their arrangement will be altered, and a number will be intercept by the external surface of the shell, as shown. Where these terminate

are places of negative electrification on the sphere. By the bringing in of the hollow sphere, the radial direction of the tubes of induction has been disturbed, and a considerable number of them severed each into two parts, of which one extends from the ball to the outside or inside of the hollow conductor, the other from the hollow conductor outwards to other conductors. For, consider a tube passing very close to the hollow conductor. At the surface of the conductor the resistance to lateral motion of the tubes does not exist, and hence a tube close to the surface is moved nearer to the surface. As soon as it comes into contact it breaks into two parts, having their ends, one positive, the other negative, on the surface of the hollow conductor. These shorten at once, one runs down to its shortest length, for example, c, between the charged ball and the conductor; the other contracts into the part d,

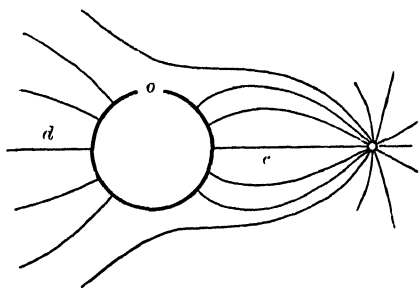


FIG. 48.

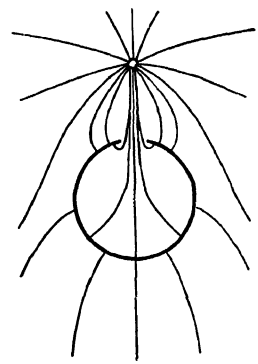


FIG. 49.

running from the hollow conductor to the original termination of the tube; and so with other tubes until a distribution on both ball and conductor is arrived at in which there is equilibrium of lateral action of the tubes. This is possible with tubes ending on part of the conductor C, and tubes leaving the rest of the surface, since the lateral action does not depend on the direction in which the tubes run.

If now the ball be brought nearer to and finally through the opening in C (see Figs. 49, 50), the tubes of induction will be drawn in with it, those which terminate on the conductor will shorten by the motion of their negative extremities round the edge of the opening until they terminate on the inside. Other tubes which pass by the conductor will be pushed up to it, will part as already described; one portion will run down to its shortest length within the conductor, the other will take up an equilibrium position outside, conditioned only by the arrangement of the tubes there. Finally, if the opening be closed up with the charged

body inside, all the tubes will have been divided into two parts, and, clearly, just as many will leave the outside and terminate on remote conductors as start from the enclosed body, inasmuch as these are the parts of the original tubes which, with altered length, retain the terminations the system of tubes starting from the charged ball had before the closed conductor was brought into the field.

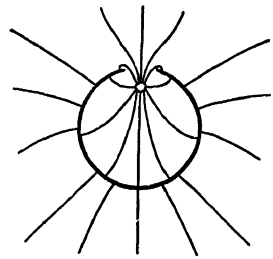


FIG. 50.

164. The modes of distribution outside and inside are independent, and we see how the contact of the ball with the interior of the closed conductor completely discharges it. At the place of contact or of spark between the conductors, the resistance to the lateral motion of the internal tubes is removed; the ends of these run along the conductors towards the place of contact, and the tubes shorten and finally disappear into the connection between the conductors, giving up their energy there in producing a spark and in heating the substance of the conductors.

Hypothesis of Incompressible Fluid

165. Another hypothesis, at first illustrative rather than a real way of accounting for the phenomena, may be mentioned here. It is that the dielectric-space outside a conductor is filled with an incompressible fluid, which also pervades the interior of the conductor. If then any additional quantity of incompressible fluid be forced into the conductor, an equal quantity must be forced out across the outer surface of the conductor, and across any closed surface which may be drawn in the dielectric so as to enclose the conductor. Thus if the space have an outer conducting boundary, an equal quantity of fluid must be forced inwards towards the conductor across that boundary.

The fluid displaced outward across any element of the inner conductor may be regarded as the charge of electricity on that element, the equal and opposite negative charge which is its complement is the fluid displaced inward across the corresponding element of the outer conductor. Corresponding elements are those connected by a tube of displacement, and the displacement across any cross-section of a tube is proportional to the electric induction at that cross-section.

This hypothesis is helpful in the description of electrical facts, but by itself is inferior to some others as regards the explanation which it affords of phenomena. A theory, however, has been recently put forward by Larmor, which aims at accounting for electrical phenomena by supposing that a fluid ether pervading space is endowed with an elasticity which resists rotational displacement, and explains a linear current as a vortex ring in the ether, and magnetic force as velocity of flow of the medium.

Some account of this and other ether-theories will be given in Vol. II., and with this in view a fairly complete sketch of the theory of both irrotational and vortex motion of an incompressible fluid is given in a later Chapter. We shall therefore not enter further into fluid theories at present.

Specification of Electric Induction and Electric Intensity. Energy of Field

166. We now assume, according to the statement made above, that what we, by an analogy with the ordinary elasticity of matter, regard as the electric strain in the dielectric medium at any point is measured by a directed quantity which we shall call the *electric induction* and denote by **D**, and which we shall also assume to be connected in general by a linear relation with another quantity called the *electric force* or *electric intensity* and denoted by **E**. These two quantities are precisely analogous to the magnetic induction and magnetic intensity defined and discussed above in section 56, Chapter II.; and the electric induction is 4π times the quantity called by Clerk Maxwell in his treatise the *electric displacement*. In the case of an isotropic dielectric the electric induction is taken as in the same direction as, and in simple proportion to, the electric intensity.

We can now express the energy of the system of electrification as the potential energy of the state of electric strain in the dielectric, which we suppose to have its seat where that strain exists. We take the amount of the energy per unit volume as measured by the work done by the electric intensity in producing the electric displacement, or for the case of an isotropic medium, as given by the equation

$$\mathsf{E} = \frac{1}{4\pi}\int \mathbf{E}\,d\mathbf{D} = \frac{1}{8\pi}\mathbf{E}\mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the more general case, in which **D** is proportional to **E**, and inclined to it at the constant angle θ, as they grow up together, we have

$$\mathsf{E} = \frac{1}{4\pi}\int \mathbf{E}\cos\theta\,.\,d\mathbf{D} = \frac{1}{8\pi}\mathbf{E}\mathbf{D}\cos\theta \quad . \quad . \quad . \quad . \quad (2)$$

Surface Integral of Electric Induction

167. In the case of magnetism the surface integral of magnetic induction is equal to 4π times the quantity of magnetism within the surface, and in a similar way we have here the surface integral of the outward normal component of electric induction taken over a closed surface drawn in the electric field equal to 4π times the quantity of electricity within the surface. The theorems that have been proved above in pp. 35 to 43 with respect to magnetic intensity and magnetic induction all hold for the electric quantities, when **E** is put for **H**, **D** for **B**, with

the corresponding substitutions of components, and k, which we call the electric inductivity of the medium, is put for μ, so that

$$\mathbf{D} = k\mathbf{E}.$$

Thus we identify in the same way as at p. 40 the electric intensity **E**, produced at any point in an unlimited uniform dielectric by a point-charge of amount q of electricity distant r from the point, as the quantity q/kr^2, which measures the repulsive force exerted on unit quantity of electricity placed at the point in question. Also, as before, the field, in any given case, is the resultant of the fields thus produced by the individual point-charges of which the given distribution may be regarded as built up. This synthesis leads, it will be found, to correct results.

Energy in Case of Æolotropic Medium

168. Putting f, g, h for the components of electric induction (that is 4π times the corresponding components of electric displacement denoted by the same letters by Maxwell), and P, Q, R, for those of electric intensity, we have for the most general linear relation between induction and intensity

$$\left.\begin{aligned} f &= k_{11}P + k_{12}Q + k_{13}R \\ g &= k_{21}P + k_{22}Q + k_{23}R \\ h &= k_{31}P + k_{32}Q + k_{33}R \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The energy per unit volume has now the value

$$d\mathsf{E} = \frac{1}{4\pi}\int(Pdf + Qdg + Rdh) \quad . \quad . \quad . \quad . \quad (4)$$

that is

$$d\mathsf{E} = \frac{1}{4\pi}\{(k_{11}P + k_{21}Q + k_{31}R)dP + (k_{12}P + k_{22}Q + k_{32}R)dQ + (k_{13}P + k_{23}Q + k_{33}R)dR\}.$$

But we suppose the energy to depend only on the state of the medium: hence $d\mathsf{E}$ must be a perfect differential in the variables P, Q, R. Thus we have

$$\frac{\partial \mathsf{E}}{\partial P} = \frac{1}{4\pi}(k_{11}P + k_{21}Q + k_{31}R),$$

with two other equations for $\partial\mathsf{E}/\partial Q$, $\partial\mathsf{E}/\partial R$. Hence since $\partial^2\mathsf{E}/\partial P\partial Q = \partial^2\mathsf{E}/\partial Q\partial P$, &c., we get the relations

$$k_{12} = k_{21},\ k_{23} = k_{32},\ k_{31} = k_{13}.$$

The energy per unit volume can thus, from (4), be written

$$\mathsf{E} = \frac{1}{8\pi}(k_{11}P^2 + k_{22}Q^2 + k_{33}R^2 + 2k_{23}QR + 2k_{31}RP + 2k_{12}PQ) \quad (5)$$

which by a suitable choice of axes can be reduced to the form

$$\mathsf{E} = \frac{1}{8\pi}(k_1P^2 + k_2Q^2 + k_3R^2) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where P, Q, R are now the components of electric intensity with reference to the new axes, and k_1, k_2, k_3, the electric inductivities in these directions, are called the principal electric inductivities of the substance.

Charged Spherical Conductor in Uniform Dielectric. Energy of System

169. Now let us consider an infinitely extended uniform dielectric surrounding a conducting spherical surface, of radius a, charged with electricity. As we have seen above, the tubes of induction issue normally from the surface of the sphere and extend radially outwards, having their farther extremities on conductors, which for our present purpose may be regarded as infinitely distant from the sphere, so that the field round the latter may be taken as symmetrical about the centre.

To find the energy of the field we have, taking the surface integral of $\mathbf{D}$ over a spherical surface of radius r concentric with the given one, and putting $4\pi Q$ for the value of this integral, so that Q is the charge of electricity, $4\pi r^2\mathbf{D} = 4\pi Q$, or $\mathbf{D} = Q/r^2$. The value of $\mathbf{E}$ is $\mathbf{D}/k$, so that $\mathbf{E} = Q/kr^2$, and the directions of $\mathbf{D}$ and $\mathbf{E}$ are the same. The energy of electric strain in the field is thus (if $d\varpi$ be put for an element of volume)

$$\int \mathsf{E}\,d\varpi = \frac{1}{8\pi}\int_a^\infty \mathbf{ED} \cdot 4\pi r^2 dr = \frac{Q^2}{2ka} \quad . \quad . \quad . \quad . \quad (7)$$

Spherical Condenser

170. If now a concentric spherical surface of radius b round the former have uniformly distributed over it a charge $-Q$, the tubes of induction will terminate there, and there will be no external electric field. This surface of course in practice would be the inner surface of a hollow shell of conducting material. The negative charge upon it merely represents the external ends of the tubes of induction.

The energy of this arrangement is given by

$$\mathsf{E} = \frac{Q^2}{2k}\left(\frac{1}{a} - \frac{1}{b}\right),$$

as may be seen either by substituting b for ∞ as the superior limit of the integral in (7), or by observing that a uniform distribution $-Q$ on a spherical surface of radius b, would produce external to itself an induction everywhere equal and opposite to that produced by $+Q$ on a concentric surface of radius a, so that the resultant induction is everywhere zero. Thus we have simply to subtract from $Q^2/2ka$, the energy external to the outer surface due to $-Q$, that is $Q^2/2kb$. This latter

way of regarding the matter is, however, a mathematical fiction: what we have physically is a conductor as internal and external boundary, and across that the electric strain is not propagated in either direction, inasmuch as the conductor will not sustain any such strain.

Tubes of Electric Induction. Unit Tubes

171. Let us consider now a tube of electric induction, and take first a closed surface made up of a portion of the tube bounded by end surfaces everywhere at right angles to the line of induction. Since there is no component of induction at right angles to the sides, the sum of the fluxes of induction across the ends must be equal to 4π times the quantity of electricity within the surface. Thus if $\mathbf{D}_1$, $\mathbf{D}_2$ be the inductions at any point of the ends at which the lines of induction enter and leave respectively the closed surface, and q be the quantity of electricity within the tube, we have, integrating over the ends,

$$\int \mathbf{D}_2 dS_2 - \int \mathbf{D}_1 dS_1 = 4\pi q \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence if the tube be thin, and there be no electric charge within the portion considered, we get, putting S_2, S_1 for the areas of the ends, now very approximately plane,

$$\mathbf{D}_2 S_2 = \mathbf{D}_1 S_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again, if the tube pass through a continuously electrified surface, we see by taking one end on one side of the surface, and the other end on the other side, but infinitely near the surface on the two sides, that

$$\mathbf{D}_2 - \mathbf{D}_1 = 4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where σ is the electric density (or quantity of electricity per unit of area) at the part of the surface at which the tube is taken.

If $\mathbf{D}_1 = 0$, that is if the induction be zero behind the surface, as in the case of a conductor in the substance of which one end of the tube is taken, this gives

$$\mathbf{D}_2 = 4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\mathbf{E} = \frac{4\pi\sigma}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The direction of $\mathbf{E}$ is normal to the surface of the conductor.

If the cross-section of the tube be so chosen that $\mathbf{D}_1 S_1 = 1$, this relation will by (9) hold for every part of the tube which does not contain any electric charge. Such a tube is called a *unit tube*, and the surface integral of electric induction taken over any closed surface in the electric field is equal to the number of unit tubes which cross the surface in the outward direction *minus* the number of those which cross the surface in the opposite direction. The quantity of electricity within the surface is

thus equal to 4π times this excess of the number of outward directed unit tubes over the number of inward directed unit tubes, or, as it is sometimes put, is 4π times the excess of the outward over the inward drawn *lines* of induction. When the word *line* is thus used it is to be understood as signifying a unit tube.

Tension along Lines of Induction in an Electric Field. Traction on Surface of a Conductor

172. We can now see that the energy contained in a portion of a narrow tube, (if there be no electricity within the portion) is

$$\frac{1}{8\pi}\int \mathbf{ED}S ds = \frac{1}{8\pi}\int \frac{\mathbf{D}^2}{k} S ds,$$

where S is the area of cross-section of the tube at any place, ds an element there of the length of the tube, and the integral is taken along the part of the tube considered. But $\mathbf{D}^2/8\pi k \,.\, Sds$ is the work done by a force $\mathbf{D}^2/8\pi k \,.\, S$ in a displacement ds, so that the force per unit area is $\mathbf{D}^2/8\pi k$.

In fact we see that we may regard a tube as formed by drawing out the negative charge on one end against an inward pull of amount $\mathbf{D}^2/8\pi k$ per unit of area. This is equivalent to supposing that a tension of amount $\mathbf{D}^2/8\pi k$ exists at every point in the medium along the lines of induction.

173. As an example let the radius b of the outer spherical surface in the case considered above be increased by an amount db. The change of energy is

$$\frac{Q^2}{2k}\left(\frac{1}{b} - \frac{1}{b+db}\right) = \frac{Q^2}{2k}\frac{db}{b^2}.$$

The tension in this case overcome through the distance db at each element of the outer surface in carrying out the outer ends of the tubes is therefore $Q^2/8\pi k b^4$. But $-\sigma$ being the density on the outer surface $\sigma^2 = Q^2/16\pi^2 b^4$, so that the inward tension along the lines of induction just inside the outer surface is $2\pi\sigma^2/k$. The outward tension on the inner surface is obviously also $2\pi\sigma^2/k$ where σ is the electric density there.

The traction on the outer surface of a conductor being taken as $\mathbf{D}^2/8\pi k$ we have its equivalent $2\pi\sigma^2/k$ as the value of the outward pull exerted by the external medium per unit area on the surface of the conductor, and likewise of the equal and opposite pull exerted per unit area by the conductor on the medium. We have a good example in an electrified soap-bubble, which if the electric surface density be σ, is pulled on by the external medium with a force per unit area $2\pi\sigma^2$,

causing an apparent diminution of amount $\frac{1}{2}\pi\sigma^2 r/k$ in the tension of each surface of the film where r is the radius of the bubble.

It may be noticed here that, in the particular case considered above, by increasing the radius b of the outer spherical surface from a to infinity, we get for the work done and therefore for the energy of the system the equation

$$\mathsf{E} = \int_a^\infty \frac{Q^2}{2k}\frac{db}{b^2} = \frac{Q^2}{2ka} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Constancy of $\int \mathbf{E}ds$ along any Path from one Conductor to another. Potential. Difference of Potential. Equipotential Surfaces

174. Next consider a single conductor of any form on which tubes of induction originate, but none terminate, and which is surrounded by a single closed conducting surface. The tubes of induction for this conductor leave its surface normally and pass outwards until they terminate on the surrounding conductor. The energy contained in a narrow tube, of cross-section dS at any point, is

$$\frac{1}{8\pi}\int \frac{\mathbf{D}^2}{k}\, dS\,.\,ds,$$

in which the integral is taken along the tube from one end to the other. Since $\mathbf{D}dS$ is constant along the tube and is equal to $4\pi\sigma dS'$, where dS' is the element of area intercepted on the surface of the conductor by the tube, and σ is the density there, we obtain

$$\frac{1}{8\pi}\int \frac{\mathbf{D}^2}{k}\, dS ds = \frac{\sigma dS}{2}\int \frac{\mathbf{D}}{k}\, ds = \frac{\sigma dS}{2}\int \mathbf{E}ds.$$

The value of $\int \mathbf{E}ds$ taken along a tube of induction must be the same for every induction tube passing from the inner to the outer bounding surface. Remembering that $\mathbf{E}$ is the electric intensity at a point of the field, that is the mechanical force on a very small conductor charged with unit quantity of electricity and placed at the point, we can prove this proposition in the following manner. Let such a point-charge be carried round a closed quadrilateral path consisting of two curves along lines of induction, and two connecting pieces at right angles everywhere to lines of induction. The work done in carrying the charged body round this path is zero; for this would clearly be the case in a field due to a single point-charge, and therefore also in one obtained by superimposing the fields of a system of individual point-charges (Art. 167). The work done in carrying it along each element of the parts of the path, which are at right angles to the lines of induction, is

also zero. Hence the work spent in carrying the charge along one of the two sides of the curved quadrilateral which lie along the lines of induction is equal to that gained in carrying it along the other. Since the lines of induction leave or enter the conductors at right angles to the surface, the value of $\int \mathbf{E} ds$ taken along any line of induction from one surface to the other has the same value, and therefore has the same value along any path whatever, whether a line of induction or not, leading from one surface to the other.

175. $\mathbf{E}$ must therefore in cases of electrostatic equilibrium be a function only of the co-ordinates, and we may denote it by $-dV/ds$, where V is a single-valued function of the co-ordinates only. $\partial V/\partial x.dx + \partial V/\partial y.dy + \partial V/\partial z.dz$ or dV is thus a perfect differential.

The energy within a tube of induction has the value

$$\int \frac{\sigma dS}{2} (V_1 - V_2),$$

where V_1, V_2, are the values of the function V at the inner and outer surfaces. The quantity $V_1 - V_2$ is called the *difference of potential* between the two surfaces.

Integrating now over the surface of the inner conductor, and putting Q for the charge of electricity upon it, as measured by 4π times the flux of induction across a closed surface in the dielectric surrounding it, we get for the energy

$$\mathbf{E} = \frac{Q}{2} (V_1 - V_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the outer surface be everywhere at a very great distance from the inner conductor the value of $\mathbf{E}$ will vanish, that is V will cease to vary with displacement of the point considered along a line of intensity; in fact, all lines of intensity will at a great distance have become untraceable in consequence of the smallness of magnitude of the electric intensity, and the impossibility of determining its direction.

176. We may if we please define Vdq for any point as the work done in carrying a small charge dq of electricity (so small that it does not appreciably affect the distribution on the conductors) from an infinite distance from the electrified conductor to the point in question, along any path against the electric intensity. The work thus done depends only on the initial and terminal points of the path, inasmuch as no work on the whole would be done in carrying the small charge dq round a closed path on which those two points are situated. This makes V zero for all points infinitely distant from every part of the electrification.

The value of V at any conductor is in this reckoning called the *potential* of the conductor, and its value for any point is called the potential at that point. The meaning and use of this function will be further illustrated in what follows, and especially in the chapter on

Fluid Motion which is given below as a preliminary to the discussion of general theories of electrical action.

A surface at every point of which the potential has the same value is called an *equipotential surface.* Such a surface can evidently be drawn through every point of the electric field. Any equipotential surface may be taken as the surface of zero potential, and the potential at any point is then the difference between the potential at that point and the potential at the surface.

Lines of electric intensity obviously meet equipotential surfaces at right angles, that is, the resultant electric intensity at any point of such a surface is normal to the surface.

It is clearly a property of the function V that it cannot have a maximum or minimum value in space void of electric charge. For if there could be a point or region of maximum or minimum of potential it would be possible to describe round it a surface at every point of which $\mathbf{E}$ would be directed outwards in the case of a maximum, and inwards in the contrary case. Thus the surface integral of electric induction would in the former case be positive and in the latter negative, that is, the surface would contain a charge of electricity, which is contrary to the supposition.

General Problem of Electrostatics

177. The general direct problem of electrostatics is ordinarily the determination for a given system of conductors insulated with given charges in presence of certain other conductors maintained at zero potential, or at given potentials, of the value of V for every point of the field, and the surface density of the electric distribution at every point of the conductors. In this problem there may be given for some or all of the insulated conductors, not the charges, but, what is equivalent, the outward normal component of the electric induction for every point external to and infinitely near the surface.

A still more general problem than this is obtained by replacing some or all of the conductors by surfaces, not generally equipotential, over which an arbitrary surface distribution of V or of the electric induction, or of V over some and of the electric induction over others, is made. The possibility of always finding a solution of this more general problem has been the subject of discussion; but in any case in which we may have to deal with it the question of the existence of a solution will be answered by finding one. It can easily be shown that for this problem, as well as for the other, if there exist one solution for a given set of conditions, there exists no other.

The problem first stated is sufficiently general for most electrical purposes, and will be treated as fully as is necessary in a later chapter. At present we shall consider only some general propositions regarding systems of conductors, and the properties of the potential function.

SECTION II.—*Electrostatic Capacity and Electric Energy of Charged Conductors.*

Electric Condensers or Leydens

178. The simple arrangement of two parallel conducting plates of the same material (copper or brass for example), in which the induction tubes extend across a uniform dielectric filling the space between them, is of considerable importance in practice. If, as is here supposed to be the case, the plates be of considerable dimensions in every direction in their own plane, and be opposite to one another, the lines of induction anywhere at a distance from the edge great in comparison with the distance between the plates may be assumed to be straight and at right angles to the plates; and being straight and parallel at every such place the tubes of induction will be uniformly distributed. Thus the values of **D** and **E** are constant at every place well under shelter of the plates.

Taking an area A in the dielectric parallel to the plates and crossed by such tubes of induction, we see from what has been said, that if d be the thickness of the dielectric the energy corresponding to the induction across A is

$$\frac{A}{8\pi}\frac{\mathbf{D}^2 d}{k} = \frac{A\mathbf{D}}{8\pi}\mathbf{E}d = \frac{A\sigma}{2}(V_1 - V_2) = \frac{Q}{2}(V_1 - V_2)$$

where $V_1 - V_2$ is the difference of potential of the two plates and Q the charge on the area A of the plate from which the tubes start.

We have $V_1 - V_2 = \mathbf{E}d = \mathbf{D}d/k$, and $Q = A\mathbf{D}/4\pi$. Hence we obtain

$$\frac{Q}{V_1 - V_2} = \frac{Ak}{4\pi d} \quad \ldots\ldots\ldots \quad (15)$$

This ratio is called the electrostatic capacity of the area A of the plate from which the lines of induction emanate, for the case of electrification here considered.

179. In general the electrostatic capacity of a conductor is defined as the ratio of the charge on the conductor to the potential of the conductor when all other conductors in the field are maintained at potential zero. Thus if $V_2 = 0$, the capacity of the area A in the case just considered would still be $Ak/4\pi d$. In all cases the potential of a conductor, and therefore its capacity, is affected by the presence of neighbouring conductors, unless the conductor in question is surrounded by a closed conducting screen maintained at zero potential. We shall denote the capacity of a conductor by C.

The electrostatic capacity of a conductor can easily be calculated in a number of simple cases. For example by (14) above the energy of the spherical distribution alone in its own field is $Q^2/2ka$, so that taking the potential at an infinite distance from the centre of the sphere as zero we have for the potential V of the sphere Q/ka. The capacity of the sphere is thus ka.

Similarly we could show that the capacity of a sphere radius a enclosed in a concentric sphere of radius b, and at zero potential, is $kab/(b-a)$, which for b infinite reduces to ka, as it ought. We shall return to the calculation of capacities in particular cases later.

Energy in Terms of Electrostatic Capacity. Energy of a System of Charged Conductors

180. The energy of the electrification thus considered can be expressed by either of the equations

$$\mathsf{E} = \tfrac{1}{2}C(V_1 - V_2)^2 = \tfrac{1}{2}\frac{Q^2}{C} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From this and (15) it is clear that for a given induction between the plates, the energy varies directly as d, that is the energy of the medium varies directly as d, for a given charge of electricity on the plate from which the tubes start. The physical reason for the slight amount of energy in this case is obvious: the extent of medium strained to a given intensity varies directly as d. The ordinary explanation by the proximity of the negative charge at the final extremities of the lines of induction refers to the same fact, but does so somewhat obscurely.

On the other hand for a given value of $\int \mathbf{E}ds$ along a line of intensity from one plate to another, that is, for a given value, U say, of $\mathbf{D}d$ or, which is the same thing, a given difference of potential between the two plates, the energy is given by

$$\mathsf{E} = \frac{A}{8\pi}\frac{U^2}{kd} = \frac{Ak}{8\pi}\frac{(V_1 - V_2)^2}{d} = \tfrac{1}{2}Q(V_1 - V_2) \quad . \quad . \quad (17)$$

and is *inversely* proportional to d.

The charge in this latter case is $Ak(V_1 - V_2)/4\pi d$ or, as before, $C(V_1 - V_2)$. This illustrates the so-called condensing action of the arrangement. The smaller d, the greater is the induction required to produce a given value $\mathbf{D}d$, or of $\mathbf{D}^2d/8\pi$, that is of the energy contained in a unit tube.

An arrangement of this kind is generally called a *condenser*, but there is, properly speaking, no condensation of any sort. Lord Kelvin and Lord Rayleigh have recommended the substitution of the name *leyden*.

181. We now consider the more complicated case of a number of charged conductors of any form insulated from one another. Let the system for definiteness be supposed enclosed within a single conductor S_0, and let S_1, S_2, S_3,.....denote the surfaces of the conductors, and let tubes of induction only *proceed from* S_1, and terminate on S_2, S_3,..... and on the surrounding conductor S_0. Further let tubes of induction proceed also from S_2, and terminate on S_3, S_4,.....and on S_0, and so for the other

conductors. The tubes of *resultant induction* will fall into groups, one for which the tubes originate in S_1, another for which they originate in S_2, and so on.

The energy of the system will be obtained by calculating first the energy E_1 for the resultant tubes which start from S_1, next the energy E_2 for those which start from S_2, and so on until all the tubes in the field have been taken into the account. We get by what has gone before

$$\mathsf{E}_1 = \tfrac{1}{2}\int\sigma dS_1(V_1 - V_0) + \tfrac{1}{2}\int\sigma dS'_1(V_1 - V_2) + \tfrac{1}{2}\int\sigma dS''_1(V_1 - V_3) + \ldots\ldots \quad (18)$$

where dS_1 is an element of the part of the surface S_1 from which tubes pass to S_0, dS'_1 an element of that part of the same surface from which tubes pass to S_2, &c., and the surface integrals are taken over these parts only; while $V_1 - V_0$, $V_1 - V_2$,.....are the differences of potential between S_1 and S_0, S_1 and S_2,.....But this may be written

$$\mathsf{E}_1 = \tfrac{1}{2}\int\sigma dS_1(V_1 - V_0) + \tfrac{1}{2}\int\sigma dS'_1(V_1 - V_0) - \tfrac{1}{2}\int\sigma dS'_1(V_2 - V_0) + \tfrac{1}{2}\int\sigma dS''_1(V_1 - V_0) - \tfrac{1}{2}\int\sigma dS''_1(V_3 - V_0) + \ldots\ldots \quad (19)$$

Similarly we have

$$\mathsf{E}_2 = \tfrac{1}{2}\int\sigma dS_2(V_2 - V_0) + \tfrac{1}{2}\int\sigma dS'_2(V_2 - V_0) - \tfrac{1}{2}\int\sigma dS'_2(V_3 - V_0) + \ldots\ldots$$

and so on. Thus adding, we obtain

$$\begin{aligned}\mathsf{E} &= \mathsf{E}_1 + \mathsf{E}_2 + \mathsf{E}_3 + \ldots\ldots \\ &= \tfrac{1}{2}(V_1 - V_0)\left\{\int\sigma dS_1 + \int\sigma dS'_1 + \int\sigma dS''_1 + \ldots\ldots\right\} \\ &\quad + \tfrac{1}{2}(V_2 - V_0)\left\{\int\sigma dS_2 + \int\sigma dS''_2 + \ldots\ldots - \int\sigma dS'_1\right\} \\ &\quad + \tfrac{1}{2}(V_3 - V_0)\left\{\int\sigma dS_3 + \int\sigma dS'_3 + \ldots\ldots - \int\sigma dS''_1 - \int\sigma dS'_2\right\} \\ &\quad + \ldots\ldots \end{aligned} \quad (20)$$

The sums of integrals in brackets are the charges on S_1, S_2,.....respectively, inasmuch as $-\int\sigma dS'_1$, $-\int\sigma dS_1''$, for example are the amounts of negative electricity, which correspond to the termination of the tubes from S_1 on S_2, S_3 respectively, $-\int\sigma dS'_2$ is the negative charge on S_3 due to tubes starting from S_2, which terminate on S_3, and so on. Thu

denoting the total charges on S_1, S_2,.....by Q_1, Q_2,.....we obtain the result,

$$\mathbf{E} = \tfrac{1}{2}Q_1(V_1 - V_0) + \tfrac{1}{2}Q_2(V_2 - V_0) + \ldots . \quad . \quad . \quad (21)$$

which, if $V_0 = 0$, becomes

$$\mathbf{E} = \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2 + \tfrac{1}{2}Q_3V_3 + \ldots .$$

or as we may write it for shortness

$$\mathbf{E} = \tfrac{1}{2}\Sigma QV \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Reciprocal Relation between two States of a System of Conductors. Applications

182. Now consider two equilibrium states of one and the same system of conductors, and let Q_1, Q_2,..... V_1, V_2,.....be the charges and potentials for one state, Q'_1, Q'_2,..... V'_1 V'_2,.....the charges and potentials for the other state. In both cases tubes of induction pass out from part of the surface of each conductor, and others pass inward toward the rest of the surface. Take the group of tubes of resultant induction which are connected with any conductor, that is the whole group of which part leaves and the other part arrives at that conductor. Describe round the conductor a closed surface everywhere normal to these tubes of induction. This can clearly be drawn anywhere in the field.

Now take one of the products, QV' say, of the charge of a conductor in one of the distributions by the potential in the other. This product may be written

$$\frac{1}{4\pi}\int \mathbf{D}dS \int \mathbf{E}'ds'$$

where the first integral is taken over the closed surface, and the second (in which $\mathbf{E}'$ is the *resultant* electric intensity) is taken along any line of intensity extending between the conductor and infinity or between the conductor and any other. Putting together the sum of products $\Sigma QV'$ for all the conductors we have,

$$4\pi\Sigma QV' = \Sigma\int \mathbf{D}dS \int \mathbf{E}'ds',$$

and the integrals exhaust the whole field. We may write,

$$\Sigma\int \mathbf{D}dS \int \mathbf{E}'ds' = \int k\mathbf{E}\mathbf{E}'d\varpi \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $\mathbf{E}$ is the resultant electric intensity due to the distribution Q_1, Q_2,..... and $\mathbf{E}'$ is that for the distribution Q'_1,Q'_2....., $d\varpi$ is an element of volume, and the integral is taken throughout the whole field, wherever $\mathbf{E}\mathbf{E}'$ is not zero.

But the volume-integral on the right of (23) could have been equally well deduced from the sum of products $\Sigma Q'V$. Hence we have the important reciprocal theorem for the two distributions.

$$\Sigma QV' = \Sigma Q'V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The theorem here proved is a particular case of a very general dynamical theorem, which is demonstrated in the chapter which follows on General Dynamical Theory.

183. From the results which have been obtained, we can now draw some important conclusions. Let all the charges Q_1, Q_3, Q_4,....., Q_2', Q'_3,.....be zero so that the two systems consist of Q_2, Q'_1 alone. Then the reciprocal theorem (24) gives

$$Q_2V'_2 = Q'_1V_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

or, in words, if all the conductors be without charge except one, A, the potential produced at any one, B, of the uncharged conductors by a given charge on A is equal to the potential which would be produced at A by the same charge on B, when all except B are uncharged.

Again, if the state of a system as expressed by charges Q_1, Q_2,..... and potentials V_1, V_2,.....be changed to one determined by charges Q'_1, Q'_2,.....and corresponding potentials V'_1 V'_2,.....the change in energy is $\frac{1}{2}\Sigma V'Q' - \frac{1}{2}\Sigma VQ$. But by (24) we have

$$\begin{aligned} \tfrac{1}{2}(\Sigma V'Q' - \Sigma VQ) &= \tfrac{1}{2}(\Sigma V'Q' - \Sigma VQ' + \Sigma V'Q - \Sigma VQ) \\ &= \tfrac{1}{2}\Sigma(V' - V)(Q' + Q) \quad . \quad . \quad . \quad . \quad . \quad (26) \\ &= \tfrac{1}{2}\Sigma(V' + V)(Q' - Q) \end{aligned}$$

This result is graphically illustrated in Fig. 51, and shows that the work done in changing the state of the system is numerically equal either to the area of the trapezium $A\ B\ C\ D$, or to the area of a trapezium $E\ C\ D\ F$. This illustrates the fact that the potential in-

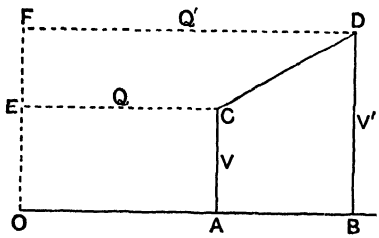


FIG. 51.

creases *pari passu* with the charge of the conductor, or, in the case of a system of conductors, that equal proportionate changes in the charges of all the conductors are associated with equal proportionate changes in the potentials.

Coefficients of Potential and Induction and Electrostatic Capacities of a System of Conductors. Reciprocal Theorem

184. The potential of any conductor is thus a linear function of the charges of all. Hence we obtain the series of equations

$$\left.\begin{aligned} V_1 &= p_{11}Q_1 + p_{21}Q_2 + \ldots + p_{n1}Q_n \\ V_2 &= p_{12}Q_1 + p_{22}Q_2 + \ldots + p_{n2}Q_n \\ &\cdots\cdots\cdots\cdots \\ V_n &= p_{1n}Q_1 + p_{2n}Q_2 + \ldots + p_{nn}Q_n \end{aligned}\right\} \quad \ldots \quad (27)$$

with the condition expressed by the theorem stated in (25), that $p_{21}=p_{12}, \ldots . p_{kh}=p_{hk}, \ldots .$ The coefficients $p_{11}, p_{12}, \ldots .$ are called coefficients of potential. Their physical meaning is clear: a coefficient of the form p_{kk} is the potential produced at the conductor distinguished by the suffix k, A_k say, by unit charge on the conductor itself when all other conductors are without charge on the whole: a coefficient of the form p_{hk} is the potential produced at the conductor A_k by unit charge on A_h when all the other conductors are without charge; and this, as we have seen, is equal to the potential produced at A_h by unit charge on A_k, when all the other conductors are without charge.

The number of independent coefficients of the form p_{kk} is of course n, and the number of the form p_{hk} is $\frac{1}{2}n\ (n-1)$, so that there are $\frac{1}{2}n(n+1)$ in all.

By solving equations (27) we obtain of course a set of equations

$$\left.\begin{aligned} Q_1 &= c_{11}V_1 + c_{12}V_2 + \ldots + c_{1n}V_n \\ Q_2 &= c_{21}V_1 + c_{22}V_2 + \ldots + c_{2n}V_n \\ &\cdots\cdots\cdots\cdots \\ Q_n &= c_{n1}V_1 + c_{n2}V_2 + \ldots + c_{nn}V_n \end{aligned}\right\} \quad \ldots \quad (28)$$

for which also there hold the conditions $c_{kh}=c_{hk}$, &c.

The coefficients of the form c_{kk} are the electrostatic capacities of the conductors. Each denotes the charge on the conductor indicated by the suffix when that conductor is at potential unity, while all other conductors are at zero potential. The coefficient c_{hk} denotes the charge on the conductor A_h when the conductor A_k is at unit potential and all others are at zero potential, and, being equal to c_{kh} is also the charge on A_k when A_h is at unit potential and all the others are at potential zero. Such coefficients are called coefficients of induction.

Properties of the Co-Efficients of Potential and Charge of a System of Conductors

185. The following additional general results are easily proved for the two kinds of coefficients. Every coefficient of potential is positive, and any one of the form p_{hk} is intermediate in value between zero and p_{hh} or p_{kk}. For if A_h be charged with a unit of positive electricity, and all the rest of the conductors be insulated without charge, tubes corresponding to total induction of amount 4π pass outwards from A_h. At any other conductor, A_k say, just as many unit tubes originate as end upon it, since the charge is zero. The potential, therefore, in certain directions must increase from A_k, in all other directions diminish. There must therefore be a conductor, the potential of which is the highest potential in the field, and this clearly must be A_h, since, as we have seen, there cannot be a place of maximum potential in space void of electricity, and from any other conductor the potential increases outwards in certain directions. The potential of A_k is therefore less than that of A_h and greater than zero, that is

$$p_{hh} > p_{kh} (= p_{hk}) > 0.$$

Similarly we can show that

$$p_{kk} > p_{hk} (= p_{kh}) > 0.$$

If an uncharged conductor be enclosed within another whether uncharged or not, for example A_l within A_k, no lines of induction pass from one of these conductors to the other, for as explained (Art. 141), the distribution on the outer conductor is unaffected by the presence of an internal but uncharged conductor. The potential of A_l is then the same as that of A_k, and we have

$$p_{hk} = p_{hl}.$$

As regards the capacities it is clear from (28) that they are all positive. For let the potential of A_h say be unity, and all the other conductors be at potential zero, we then have

$$Q_h = c_{hh},$$

The potential diminishes in every direction outwards from A_h, and therefore Q_h must be positive.

The coefficients of induction, c_{hk}, are however all negative. For let A_h be charged as just specified. Then since the potential of A_k is zero, and the only other conductor not at zero potential is A_h, there can be at no point near the surface of A_k a diminution of potential in the direction outwards from the conductor. Hence tubes of induction can only terminate on A_k and the charge on it is negative, that is c_{hk} is negative.

Again, since all the tubes in the case supposed originate on A_h, the sum of the negative charges on the other conductors must be numeri-

cally less than the charge on A_h; unless A_h be completely enclosed by the other conductors, when the positive charge on A_h is exactly equal and opposite to the sum of the charges on the other conductors. Hence

$$c_{hh} \geq -\Sigma c_{kh}$$

where the expression on the right denotes the sum of all the coefficients of induction for the system.

Energy of a System of Charged Conductors Expressed as a Quadratic Function of Potentials or Charges

186. By equations (27) and (28), the energy (Art. 161) of the electrified system can be expressed as a homogeneous quadratic function of the potentials or of the charges of the conductors. For physical reasons the energy of the system cannot be negative, and hence both quadratic functions must be positive whatever may be the values of the variables. Clearly if we form the functions we have by (22), (27), (28)

$$\mathsf{E} = \tfrac{1}{2}\{p_{11}Q_1^2 + 2p_{12}Q_1Q_2 + \ldots + p_{22}Q_2^2 + 2p_{23}Q_2Q_3 + \ldots\} \quad (29)$$

and

$$\mathsf{E} = \tfrac{1}{2}\{c_{11}V_1^2 + 2c_{12}V_1V_2 + \ldots + c_{22}V_2^2 + 2c_{23}V_2V_3 + \ldots\} \quad (30)$$

The condition which must hold in order that E may be positive whatever be the values of $Q_1, Q_2, \ldots V_1, V_2, \ldots$ is simply that if each of the homogeneous quadratic functions be converted into a sum of squares each of these squares must be positive. We therefore write for (29)

$$2\mathsf{E} = \frac{1}{p_{11}}\{(p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + \ldots)^2 + (a_{22}Q_2 + a_{23}Q_3 + \ldots)^2 + (a_{33}Q_3 + a_{34}Q_4 + \ldots)^2 + \ldots\} \quad (31)$$

so that the first square contains all the Q's, the second all except Q_1, the third all except Q_1 and Q_2, and so on. Equating coefficients from (29) with those on the right of the expression just written for E, we get exactly $\frac{1}{2}n(n-1)$ equations for the determination of $\frac{1}{2}n(n-1)$ unknown coefficients, so that the resolution (which in the general case can be effected in an infinite number of ways) is unique.

It is clear that since E is positive whatever values of $Q_1, Q_2, \ldots$ may be taken, we have

$$p_{11} > 0, \; a_{22}^2 > 0, \; a_{33}^2 > 0, \; \ldots.$$

Determining the coefficients we find easily that these conditions will be satisfied if

$$p_{11} > 0, \quad \begin{vmatrix} p_{11}, p_{12} \\ p_{12}, p_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} p_{11}, p_{12}, p_{13} \\ p_{12}, p_{22}, p_{23} \\ p_{13}, p_{23}, p_{33} \end{vmatrix} > 0, \; \ldots.$$

In the same way precisely similar conditions are found to be satisfied by the c coefficients.

It is to be observed that by partial differentiation of (29) and (30) we obtain by (27) and (28)

$$\frac{\partial \mathsf{E}}{\partial Q_h} = V_h, \quad \frac{\partial \mathsf{E}}{\partial V_h} = Q_h.$$

Thus $\partial \mathsf{E}/\partial Q_h$, $\partial \mathsf{E}/\partial V_h$, are what are called reciprocal functions in dynamics. The properties of such functions will be discussed in Chapter VII., which is devoted to general dynamical considerations.

Reciprocal Relations. Exploration of an Electric Field

187. The reciprocal relation $p_{hk} = p_{kh}$ asserts, as we have seen above, that if a charge exist on A_h and all the other conductors be insulated without charge, the potential at the conductor A_k is the same as that which would be produced at A_h if the same charge were on A_k, and all the conductors except A_k were insulated without charge. We can show that the same result holds if any or all of the conductors other than A_h, A_k are maintained at zero potential. For let all the conductors which are at zero potential be regarded as constituting a single conductor A_s, of which the charge is Q_s; and let Q be the charge on A_h. Then

$$V_k = p_{hk}Q + p_{ks}Q_s$$
$$V_s = p_{hs}Q + p_{ss}Q_s = 0,$$

and therefore

$$V_k = \left(p_{hk} - \frac{p_{hs}}{p_{ss}}p_{ks}\right)Q.$$

If the charge were transferred to A_k, and A_h insulated without charge, we should get in the same way

$$V_h = \left(p_{kh} - \frac{p_{ks}}{p_{ss}}p_{hs}\right)Q,$$

and since $p_{kh} = p_{hk}$, we get

$$V_h = V_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

188. The reciprocal relation just discussed enables the electric field due to a charged conductor to be conveniently explored. The conductor is insulated but is kept uncharged, and one electrode of a delicate electrometer is connected with the conductor, while the other electrode is connected with the earth, or with some conductor the potential of which is taken as zero. Then a small charged sphere carried by an insulating handle is placed with its centre at any point of the field. While this is in position the electrodes of the electrometer are connected for an instant, so that the conductor is reduced to zero potential. The sphere is then moved from point to point in the field, and the positions

noted for which the electrometer shows no deflection. These lie on a surface which is an equipotential surface for the conductor when the latter is charged. For the potential at the conductor due to the electrification of the sphere is equal to the potential which would be produced at the sphere by a charge on the conductor equal to that on the sphere; and this part of the potential is the same for all positions of the sphere for which there is zero deflection. By the principle of superposition this must be an equipotential surface for all charges of the conductor.

The convenience of the method consists in the zero potential of the conductor, which therefore does not lose or gain charge, while the exploring sphere, which can be insulated so that its charge remains practically constant, is changed in position in the field.

Nature of Charges in Conductors in Different Cases

189. The full discussion of equilibrium distributions of electricity we defer until the subject of potential has been more fully treated, as it will be in the chapter on Fluid Motion, which is given below. But from what has gone before there flow easily a number of useful propositions.

It will be readily seen that of a system of conductors there is always at least one, the electrification of which has at all points the same sign. If the conductors in the field have all positive potentials, or all negative potentials, that one is the conductor whose potential has the greatest numerical value apart from sign. If, however, the potentials of the conductors are some of them positive, and some negative, the two conductors which are respectively at the algebraically greatest and the algebraically least potential have each electrification of the same sign at every point.

The consideration of tubes of induction will show readily whether the electrification is all of the same sign or of opposite signs on one conductor. For example, if there be only two conductors in the field, and these have equal and opposite charges, all the tubes which originate on one terminate on the other, and, as may be seen by considering a surface distant from and enclosing both conductors, there can be other tubes in the field, hence the electrification is wholly positive over one and negative over the other.

If an insulated conductor have zero charge and be placed in presence of another charged conductor, the electrification of the latter will have the same sign at every point, that of the latter will have the positive sign at some points, the negative at others.

Energy Change due to Relative Displacement of Conductors under Different Conditions

190. Another theorem to which the reciprocal relation (24) at once leads is that if an electric system be displaced, subject to the condition that the charges are kept constant, the *loss* of energy entailed on

the system *minus* the gain of energy when the same displacement takes place, subject to the condition that the potentials remain constant has the value $\frac{1}{2}\Sigma\{(Q'-Q)(V'-V)\}$ where Q is the constant value of the charge of a conductor in the first case, and $V'-V$ is the change of potential, while V is the potential in the second case, and $Q'-Q$ the change of charge.

The *loss* of energy in the first case is $\frac{1}{2}\Sigma Q(V-V')$ and the *gain* in the second is $\frac{1}{2}\Sigma(Q'-Q)V$. Hence the difference specified above is

$$\tfrac{1}{2}\{\Sigma Q(V-V') - \Sigma(Q'-Q)V\}.$$

But $Q_1, Q_2, \ldots, V'_1, V'_2, \ldots$ are charges and corresponding potentials after the displacements in the first case, and $Q'_1, Q'_2, \ldots V_1, V_2, \ldots$ the charges and potentials after the displacement in the second case. The configurations being the same this represents two states of the same system, and we have by the reciprocal relation $\Sigma QV = \Sigma Q'V'$. Hence $\Sigma(Q-Q')V$ becomes $\Sigma Q'(V-V')$. Thus

$$\begin{aligned}\tfrac{1}{2}\{\Sigma Q(V-V') - \Sigma(Q'-Q)V\} &= \tfrac{1}{2}\{\Sigma Q(V-V') - \Sigma Q'(V-V')\} \\ &= \tfrac{1}{2}\Sigma(Q'-Q)(V'-V) \quad . \quad . \quad (33)\end{aligned}$$

which was to be proved.

If the displacement be very small $Q'-Q$ in the one case and $V'-V$ in the other will be very small quantities for each conductor, and their product is a quantity of higher order of smallness. Hence to the degree of approximation involved in neglecting the quantity on the right of (33) the loss in the first case is equal to the gain in the second.

191. If now the conductors be allowed to alter their configuration mechanical work will be done, and the potential energy will be diminished to the same extent. Then if the potentials be restored to their former values, and the charges be correspondingly altered to enable this result to be effected, the energy supplied to the system must be sufficient to make up the energy consumed in doing the mechanical work and to bring up the energy to its former value, and the further amount needed to supply the gain which the energy would have received if the potentials had been kept constant during the displacement. Thus if W be the work which must be spent on the system in the latter case, there must be supplied from without to the system in the latter case a quantity of energy $2W+\frac{1}{2}\Sigma(Q'-Q)(V'-V)$ or approximately $2W$, which also must be the energy supplied when the condition of constancy of potential is imposed throughout the change.

Characteristic Equation of the Potential

192. The characteristic equation of the potential has been discussed above (p. 46), and will be otherwise demonstrated later. It expresses simply the surface-integral of electric induction for a small rectangular parallelepiped of the medium. For the sake of clearness we give a proof in this connection also, taking in the case of an æolotropic medium.

Let the element be taken with its centre at O, the origin of co-ordinates, and its edges parallel to the axis of co-ordinates x, y, z, and of lengths dx, dy, dz. Let f be the normal component of electric induction (D) at the centre of the face (of area $dydz$) to the left of O, and f_2 that at the centre of the opposite face, and similarly let g_1, g_2, h_1, h_2, be the normal components at the centres of the other pairs of faces. If the values of $f_1, f_2, \ldots$ vary continuously over the faces of the element, we have, neglecting infinitesimals of a higher order, for the part of the surface integrals due to the three pairs of faces,

$$(f_2 - f_1)dydz + (g_2 - g_1)dzdx + (h_2 - h_1)dxdy.$$

If ρ be the average amount of electricity per unit volume within the element, equating the sum of these expressions to $4\pi\rho\, dx\, dy\, dz$ and dividing by $dx\, dy\, dz$ gives the result

$$\frac{f_2 - f_1}{dx} + \frac{g_2 - g_1}{dy} + \frac{h_2 - h_1}{dz} = 4\pi\rho \quad . \quad . \quad . \quad (34)$$

By the definition of the electric induction it is plain that if ρ be everywhere finite within the element the values of f, g, h will be continuous, and we shall have

$$f_2 = f_1 + \frac{\partial f}{\partial x}dx, \quad g_2 = g_1 + \frac{\partial g}{\partial y}dy, \quad h_2 = h_1 + \frac{\partial h}{\partial z}dz,$$

and therefore instead of (34)

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} - 4\pi\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

If the medium be æolotropic, that is if the relation between electric induction and electric intensity be that given by (3), we have

$$\frac{\partial}{\partial x}(k_{11}P + k_{12}Q + k_{13}R) + \frac{\partial}{\partial y}(k_{21}P + k_{22}Q + k_{23}R) + \frac{\partial}{\partial z}(k_{31}P + k_{32}Q + k_{33}R) - 4\pi\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

with the conditions $k_{12} = k_{21}, \ldots\ldots$

If the medium be isotropic, we have $k_{11} = k_{22} = k_{33}$, $k_{12} = k_{21} = 0$, $\ldots\ldots$ and therefore

$$\frac{\partial}{\partial x}(kP) + \frac{\partial}{\partial y}(kQ) + \frac{\partial}{\partial z}(kR) - 4\pi\rho = 0,$$

or since (Art. 155) $P = -\partial V/\partial x, \ldots\ldots$

$$\frac{\partial}{\partial x}\left(k\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial V}{\partial z}\right) + 4\pi\rho = 0 \quad . \quad (37)$$

At every point of space at which $\rho = 0$ the equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial V}{\partial z}\right) = 0 \quad . \quad . \quad (38)$$

holds. For the case of k uniform throughout the field this is what is known as Laplace's equation, from its having been first discussed by Laplace in connection with the subject of gravitational attraction.

It will be proved later (p. 139) that if the potential have an assigned system of values over the bounding surface or surfaces of a simply connected space, then to a function V found to fulfil this equation as well as the surface conditions there will correspond a value of the electric energy less than that for any other value of V which fulfils the surface conditions but does not satisfy (37).

It will be shown, moreover, that (as has already been stated) if there can be found one function V which satisfies definite surface conditions and (37) throughout the field, it is the only function that can be found to fulfil them, and is therefore the solution of the problem;—given the surface distribution of potential (or its equivalent) find the corresponding value of V for all points of the field.

From (38) is to be found of course equations (10), (11), (12) which have already been established for an electrified surface. It is only necessary to take the direction of the axis of x as the normal to the surface at the point considered, and to put σ for ρdx, supposed finite when dx, the thickness, so to speak, of the surface layer, is taken infinitely small.

If we draw normals n_1, n_2 from the surface towards the spaces on the left and right of the surface respectively, and suppose k_1, k_2 to be the values of k on the two sides of the surface, we get instead of (10) the equation

$$k_1 \frac{dV}{dn_1} + k_2 \frac{dV}{dn_2} + 4\pi\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or calling N_1, N_2 the components of electric intensity at right angles to the surface on the two sides

$$k_1 N_1 - k_2 N_2 - 4\pi\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (39')$$

Equation (39) or (39′) is usually called the surface characteristic equation. Different cases of it have already been exhibited above (p. 115).

Electric Induction and Potential in Particular Cases

193. We shall now consider some simple particular cases of electric charges and the resulting electric induction and potential. Let first the electric system consist of a point-charge q_1 of electricity situated at a point the co-ordinates of which are a_1, b_1, c_1, a quantity q_2 at (a_2, b_2, c_2), ... , in a medium of uniform inductive capacity k, and consider the electric induction at (x, y, z). The induction at (x, y, z) due to q_1 at (a_1, b_1, c_1), is radially directed from (a_1, b_1, c_1), and is q_1/r_1^2 where $r_1^2 = (x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2$. Hence its components are

$$(f_1, g_1, h_1) = \frac{q_1}{r_1^3}(x - a_1, y - b_1, z - c_1).$$

The components of induction due to the other charges are obtained from this by simply substituting the suffixes corresponding to the co-ordinates of their positions. Calling f, g, h the components of the resultant induction, we obtain

$$(f, g, h) = \Sigma\left\{\frac{q}{r^3}(x - a, y - b, z - c)\right\}$$
$$= \Sigma\left\{\frac{q}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where Σ denotes summation for all values of a_1, b_1, c_1, a_2, b_2, c_2, ... of a, b, c.

If the point-charges form a continuous volume distribution we may denote the volume density at any element of the space by ρ, and replace the summations in the formulæ above by integrations taken throughout the space occupied by the charges. Thus we get putting $d\varpi$ for the element of the space at which ρ is the density

$$(f, g, h) = \int\frac{\rho}{r^3}(x - a, y - b, z - c)d\varpi$$
$$= \int\frac{\rho}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)d\varpi \quad . \quad . \quad . \quad . \quad . \quad (41)$$

If σ denote the electric surface density at a point a, b, c of an electrified surface, and dS the area of an element including the point, these equations become

$$(f, g, h) = \int\frac{\sigma}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)dS \quad . \quad . \quad . \quad . \quad (42)$$

The resultant induction is of course given in all cases by the equation

$$D^2 = f^2 + g^2 + h^2.$$

In the case of a uniform isotropic medium with induction and electric intensity both in the same direction, we have for the intensity the relation

$$\mathbf{E} = \frac{\mathbf{D}}{k}, \quad \text{or} \quad (P, Q, R) = \frac{f, g, h}{k} \quad . \quad . \quad . \quad . \quad (43)$$

by means of which the value of $\mathbf{E}$ and its components are to be obtained from the formulae above.

But since if V be the electric potential

$$(P, Q, R) = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right),$$

we obtain

$$(f, g, h) = -k\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Having regard to the definition of potential given above (Art. 156) we see that the potential at P due to a point-charge q of electricity situated at any point O, the distance of which from P is r, is q/kr. For we have for the electric intensity at any point Q, the distance of which from O is x, the value q/kx^2. The work spent by external forces in moving a unit charge nearer to q a small distance dx is qdx/kx^2. Hence if V be the potential at P

$$V = \int_r^\infty \frac{q}{kx^2}\,dx = \frac{q}{kr} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The difference between the potential V at P, and the potential V' at the point P' at a distance r' from O, or the work spent in carrying a unit of positive electricity from P' to P, is therefore

$$V - V' = \frac{q}{r} - \frac{q}{r'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

It is important to remark that this value is independent of the path pursued between P' and P. It depends only on the distances of the points at which the charges are situated from O.

Further, the potential due to a series of point-charges q_1, q_2, q_3 at distances r_1, r_2, r_3, . . . from P is given by the equation

$$V = \Sigma\left\{\int_r^\infty \frac{q}{kx^2}\,dx\right\} = \Sigma\frac{q}{kr} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

For a continuous volume distribution we have

$$V = \int\frac{\rho}{kr}\,d\varpi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and for a distribution partly volume and partly surface

$$V = \int\frac{\sigma}{kr}\,dS + \int\frac{\rho}{kr}\,d\varpi \quad . \quad . \quad . \quad . \quad . \quad (49)$$

In all these expressions the respective integrals are taken throughout the distribution, surface or volume as the case may be.

In the particular case (already considered in Art. 158) of a sphere uniformly charged with a quantity q of electricity, and alone in the field, the potential is given by

$$V = \frac{q}{kr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where r is the distance of the point from the centre of the sphere, and is thus the same as if the charge were collected at the centre of the sphere. The potential at the surface of the sphere is q/ka, if the radius is a, and the capacity is ka.

Approximate Values of Coefficients of Potential and Induction

194. We can now find an approximation to the coefficients of potential and induction in one or two particular cases of a system of conductors. For example, let the system consist of a sphere A of radius a, and a sphere B of radius b at a distance r from the centre of the former. Let first the distance apart of the centres of the spheres be very great in comparison with the radius of either. Let the charge of A be q_1 and of B q_2. Then if we regard the actual field as due to the superposition of the field due to A upon that due to B, it is clear that on account of the great distance of any part of the surface of A from B, the field around A is to a correspondingly small extent influenced by the charge on B; and similarly the field round B is influenced to a like small extent by the charge on A. The potentials of A, B are therefore, approximately,

$$\left.\begin{aligned} V_1 &= \frac{q_1}{ka} + \frac{q_2}{kr} \\ V_2 &= \frac{q_1}{kr} + \frac{q_2}{kb} \end{aligned}\right\} \quad \ldots\ldots\ (51)$$

These equations give the charges if the potentials are known. Thus—

$$\left.\begin{aligned} q_1 &= \frac{kar}{r^2 - ab}(rV_1 - bV_2) \\ q_2 &= \frac{kbr}{r^2 - ab}(-aV_1 + rV_2) \end{aligned}\right\} \quad \ldots\ldots\ (52)$$

If $V_2 = 0$, (52) give

$$q_1 = \frac{ka_1r^2}{r^2 - ab}V_1, \quad q_2 = -\frac{kabr}{r^2 - ab}V_1 \quad \ldots\ (53)$$

or to a rougher approximation

$$q_1 = kaV_1, \quad q_2 = -\frac{b}{r}q_1 \quad \ldots\ldots\ (54)$$

which can be seen at once to be values which will nearly satisfy the conditions.

The effect of the alteration of the distribution on each sphere due to the presence of the other is of course what is here neglected.

195. Regarding a condenser as an arrangement of two conductors, such and so close together that the coefficient of induction of either on the other is very large, we might find in a precisely similar manner the action of one condenser, A, on another, A', at a distance very great in comparison with that anywhere between the constituent conductors of either condenser. Denoting the capacities of the conductors of the condenser A by a, c, the mutual coefficient by b, and the corresponding

quantities for the condenser B by e, f, g, we get for the coefficients of potential of the system of two conductors A

$$p_{11} = \frac{c}{ac - b^2}, \quad p_{22} = \frac{a}{ac - b^2}, \quad p_{12} = p_{21} = -\frac{b}{ac - b^2},$$

and for those of the system B

$$p'_{11} = \frac{g}{eg - f^2}, \quad p'_{22} = \frac{e}{eg - f^2}, \quad p'_{12} = p'_{21} = -\frac{f}{eg - f^2},$$

values which will not be appreciably altered by bringing one system into presence of the other at a great distance r.

Supposing the charges q_1, q_2, q'_1, q'_2 of the two pairs of conductors A, B given, the coefficients just written down enable the corresponding potentials V_1, V_2, V'_1, V'_2 to be calculated by the introduction of $1/kr$ as the mutual coefficient of potential of either conductor of A on either conductor of B. Thus four linear equations are obtained for the potentials, which can then, if the potentials be supposed known, be solved for the charges. It is easy to verify that this process gives the following coefficients of capacity and induction in which D is written for $k^2r^2 - (a + 2b + c)(e + 2f + g)$,

$$\left.\begin{aligned} c_{11} &= a + (a + b)^2 (e + 2f + g)\frac{1}{D} \\ c_{22} &= c + (b + c)^2 (e + 2f + g)\frac{1}{D} \\ c_{12} = c_{21} &= b + (a + b)(b + c)(e + 2f + g)\frac{1}{D} \end{aligned}\right\} \quad . \quad (55)$$

with similar coefficients for B obtained by interchanging a, b, c with e, f, g respectively.

The coefficients of induction between a conductor of one condenser and a conductor of the other are

$$\left.\begin{aligned} c_{A_1B_1} &= -kr(a + b)(e + f)\frac{1}{D}, \quad c_{A_1B_2} = -kr(a + b)(f + g)\frac{1}{D}, \\ c_{A_2B_1} &= -kr(b + c)(e + f)\frac{1}{D}, \quad c_{A_2B_2} = -kr(b + c)(f + g)\frac{1}{D}. \end{aligned}\right\} \quad (56)$$

If B consists of only one conductor $f = g = 0$, so that if D be now $k^2r^2 - (a + 2b + c)c$

$$\left.\begin{aligned} c_{11} &= a + e(a + b)^2\frac{1}{D}, \quad c_{22} = c + e(b + c)^2\frac{1}{D} \\ c_{12} &= b + e(a + b)(b + c)\frac{1}{D} \\ c_{A_1B_1} &= -ker(a + b)\frac{1}{D}, \quad c_{A_2B_1} = -ker(b + c)\frac{1}{D} \end{aligned}\right\} \quad . \quad (57)$$

and the other coefficients vanish.

If A also consist of only one conductor, then $b = c = f = g = 0$, and we have $D = k^2r^2 - ae$,

$$c_{11} = \frac{k^2ar^2}{k^2r^2 - ae}, \quad c'_{11} = \frac{k^2er^2}{k^2r^2 - ae}, \quad c_{A_1B_1} = -\frac{kaer}{k^2r^2 - ae}.$$

From these last equations the coefficients for the particular case, already discussed, of two spherical conductors placed at a great distance apart may be at once obtained.

196. It is interesting to note the effect on the potential of a conductor A produced by bringing an uncharged conductor B into the field. Let all conductors in the field, except that the effect on which is to be estimated, be without charge. The inductive action of the field on B is such as to assist the bringing on of B, and work is therefore done by the electric system. Thus the energy of A is diminished, that is, $\frac{1}{2}p_{11}q_1^2$ is diminished. But q_1 remains unchanged, so that p_{11} is diminished.

It is clear in the same way that if an uncharged conductor B be connected to A, p_{11} will be diminished, and it is obvious that B produces a greater effect than does any conductor which can be inscribed within it, and a less effect than does any conductor which can be described about it.

It follows that the introduction of a body B without charge increases the capacity of A. For in this case the capacity c_{11} of A, that is the ratio of its charge to its potential, is $1/p_{11}$. But it has been shown that p_{11} has been diminished, consequently c_{11} has been increased.

The addition of a conductor B without charge to A also increases the capacity, and the effect of B in this respect is greater than that of any conductor which can be inscribed in B, and less than that of any conductor which can be described about B.

It is clear thus that the capacity of a conductor, or system of conductors, all parts of which are at the same potential, is less than that of the circumscribing spherical conductor. The capacities of a system of spherical conductors which circumscribe a given system of conductors form also a superior limit to the capacities of the conductors.

As an example consider either of the condensers discussed above. If both of its conductors be at the same potential, unity say, the charges of the two conductors will amount to $a + 2b + c$, or $e + 2f + g$ as the case may be. By what has just been stated this cannot exceed half the greatest linear dimension of the condenser.

Determination of Field within and without a Conductor by Potential Method

197. We have seen that since a conductor cannot sustain or transmit dielectric strain, its substance forms an effective barrier against the penetration to the space within it of any effect due to external electrification. There is thus no electric field within a closed conductor

which contains no insulated electrified bodies. Now if we assume that we may apply the theory of the potential (which is, as we have pointed out, an action at a distance method of procedure) to the space within a closed conductor, the distribution on the surface must be such as to produce constancy of potential throughout the internal space. In so doing we consider the conductor as non-existent, so that the whole of the internal space is regarded as having the same property of transmitting electric strain as the external dielectric, and consider only the electric distribution and the external and internal fields so regarded as due to it. The distribution on the external surface of a closed conductor is thus always such as to produce constancy of the electric field within the whole space contained by it, inasmuch as it is independent of any internal charges. This will not in general be true for the distribution on the internal surface of a closed conductor, as this must be such as, with the distribution insulated within the internal space, to maintain the conductor at a constant zero potential. For it is clear that if the conductor be brought to zero potential, that is, produces no external field whatever, no tubes of induction terminate on its external surface, that is, there is upon its external surface no charge whatever, and the internal charge is unaffected by this circumstance.

Now it is found by experiment in all cases that have been examined that the equilibrium distributions on the external or internal surfaces are such as to fulfil these conditions; and we further find by this method, and the conditions as to constancy of potential, solutions of problems which are consistent with the results of experience. Electrical distribution in general will be treated in a subsequent chapter, but we may here illustrate the potential method by a few problems, although by doing so we anticipate to a slight extent the subsequent discussion.

198. The theorem of the surface integral of electric induction which we have seen holds for any system of point-charges, applied to a concentric spherical surface of radius z described within the space, internal to a spherical conductor, shows that since there is supposed to be no electricity within the surface, the electric induction and intensity are there zero. For by symmetry of circumstances the induction $\mathbf{D}$, if not zero, must be at every point at right angles to the concentric spherical surface referred to. Hence we have, if z be the radius of this surface,

$$4\pi z^2 \mathbf{D} = 0,$$

that is

$$\mathbf{D} = 0,$$

or since $\mathbf{D} = -k\partial V/\partial z$

$$\frac{\partial V}{\partial z} = 0.$$

The potential is therefore constant. The value of the potential must,

therefore, throughout the interior of the spherical surface, coincide with the value at the surface.

It may be noticed that, if proof were needed, the same method might be applied to show that the electric intensity at any external point, due to the spherical distribution, is the same as if the whole charge were collected at the centre; but this is quite sufficiently evident from the considerations adduced above.

Surface Distributions consistent with Surface Values of Potential Green's Problem

199. The problem which will occupy us to a great extent in the discussion of distribution is the determination in certain soluble cases of the distribution over a closed surface produced by the presence of a given external or internal distribution. An example of such a problem is the calculation of the induced distribution on a spherical surface connected with the earth produced by a single point-charge at an internal or external point.

We have seen that if we have an electrified surface the normal components of induction on the two sides of it are connected (see (39) above) by a certain relation with the density of the distribution. Thus the equation

$$V = \frac{1}{k}\int\frac{\sigma}{r}\,dS$$

may be replaced for a medium everywhere of uniform inductivity k, by

$$V = -\frac{1}{4\pi}\int\frac{1}{r}\left(\frac{dV}{dn_1} + \frac{dV}{dn_2}\right)dS.$$

The distribution over a surface, or system of surfaces, given in an electric field (of uniform inductivity k), which is consistent with an arbitrarily chosen potential V_S at each point of the surface, and with the fulfilment of Laplace's equation wherever there is no electrification by the potential V elsewhere than on the surface, is that of which the density is given for each point of the surface by the equation

$$\sigma = -\frac{k}{4\pi}\left(\frac{dV_S}{dn_1} + \frac{dV_S}{dn_2}\right) \quad . \quad . \quad . \quad . \quad . \quad (58)$$

For assuming that the arrangement of potential proposed can physically exist, let us suppose it made. There will be called into existence *some* distribution on the given surfaces. (A particular case which would arise under certain circumstances would of course be a zero surface distribution.) The condition in (58) holds for every case of a possible surface distribution: it must accordingly hold for this. It remains therefore only to show that there cannot be more

than one distribution fulfilling this condition, and the other conditions of the problem.

If possible, let another potential V, at other points of the field than those at which the value of the potential has been assigned, be consistent with the given surface values of the potential, and the given distribution of electricity elsewhere in the field. Then V, at the given surfaces, must coincide with V_s. It is clear that a potential $-V_1$, having the value $-V_s$ at the surfaces, is a solution consistent with change of sign of the electrification everywhere from that which produces $+V_1$. Hence the potential $V - V_1$ is a solution consistent with zero electrification everywhere in the field, with zero potential at the given surfaces, and zero potential at an infinite distance. Hence the potential must be zero everywhere else, otherwise there would be a region of maximum or minimum of potential in space void of electrification, which we have seen above to be impossible. Thus if V_1 coincides with V at the surfaces it coincides with V everywhere else, and the solution obtained is the only one.

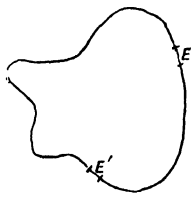


FIG. 52.

200. We come now to an important method given by Green for the calculation of distributions. Let σ be the density at any element E of a distribution over the surface which produces by its own action a potential at E equal to that produced at the same point by a unit of electricity at a point P in the field. [See Fig. 52.] Then the potential V at P due to a surface distribution which creates an arbitrary potential V_E at E is given by the equation

$$V = \int_S \sigma V_E dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

in which, as usual, the suffix S indicates that the integral is taken for every element of the surface.

For if E' be any other element, dS' its area, σ' the density there corresponding to σ at E, k times the potential at E due to the distribution of density σ is $\int_s \sigma' / E'E \, . \, dS'$, and this by hypothesis has the value $1/EP$. If σ_1 be the density at E of the distribution required to create the potential V_E at E, the corresponding potential V at P is given by

$$V = \int_S \frac{\sigma_1}{k \, . \, EP} dS = \int_S \sigma_1 \Big(\int \frac{\sigma' dS'}{k \, . \, E'E} \Big) dS.$$

But clearly from this we have

$$V = \int_S \sigma' dS' \Big(\int \frac{\sigma_1 dS}{k \, . \, E'E} \Big) = \int_S \sigma' V_{E'} dS'$$

$$= \int_S \sigma V_E dS.$$

If the surface under the influence of the unit charge at the point P be maintained throughout at zero potential, the potential will be zero at every point of it, and the induced distribution must be such as, with the unit charge at P, to maintain the surface at zero potential. The surface density at E is therefore $-\sigma$. Hence if σ be used to denote the density at E of the electrification induced by the point-charge at P, we must write instead of the formula just obtained,

$$V = -\int \sigma V_E dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

when the given value of the potential at E is not zero, but the arbitrarily specified value V_E.

Green's Function

201. The problem here solved is known as Green's problem, and is frequently stated somewhat as follows :—It is required to find a function U, which shall (1) fulfil Laplace's equation (or to adopt a common designation shall be *harmonic*) throughout a certain definite space ϖ bounded by a single surface or surfaces, except at a certain point P in that space where it becomes infinite; (2) be such that $U - 1/kr$ is harmonic throughout the whole space ϖ, the point P included; (3) give the value V_P at P of any function which is harmonic throughout the assigned space and has an assigned value V_E at each element E of the surface, by the equation

$$V_P = \frac{k}{4\pi}\int_S V_E \frac{dU}{dn} dS \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where dn denotes an element drawn outwards from E towards the space ϖ.

From what has been stated above it is plain that the value of U is, to a constant, the potential due to a unit charge situated at P together with that of the charge (called induced charge) which must exist on the bounding surface if that is everywhere maintained at zero potential in presence of the unit charge at P. For clearly σ in (60) is the density that exists at the element E in these circumstances, so that we have

$$-\frac{k}{4\pi}\frac{dU}{dn} = \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The value of U is, except as to the constant referred to, unique.

The function $U - 1/kr$ is generally denoted by G, and the name Green's function has been applied to it by Maxwell and others. Thus G is the potential at P due to the induced distribution on the bounding surface.

It follows from the reciprocal relation established above that the

potential $G_{PP'}$ produced at any point P' by the induced distribution on the bounding surface when there is unit charge at P, is equal to the potential $G_{P'P}$ at P due to the induced surface distribution when there is unit charge at P'. This may be proved independently as follows:— Let σ, σ' be the densities of the distribution at E in the first case and the second case respectively. We have

$$G_{PP'} = \int_S \frac{\sigma}{k \cdot EP'} dS = \int_S \sigma \left(\int_S \frac{\sigma' dS}{k \cdot EE'} \right) dS$$

$$= \int_S \sigma' \left(\int_S \frac{\sigma dS}{k \cdot EE'} \right) = G_{P'P} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Obviously likewise

$$U_{PP'} = U_{P'P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Green's Function for a Spherical Conductor. Induced Distribution on a Spherical Conductor and on an Infinite Plane under the Influence of a Point-Charge. Electric Images

202. As an example we shall find Green's function for a sphere maintained at zero potential in presence of a positive unit charge at an external or internal point P. First let P be external to the sphere; the function will relate to the space external to the sphere. We note that since the potential at the surface is zero, the surface distribution must be such as to exactly annul $1/kr$ for each surface element. Let a line (Fig. 53) be drawn joining the centre O of the sphere with P;

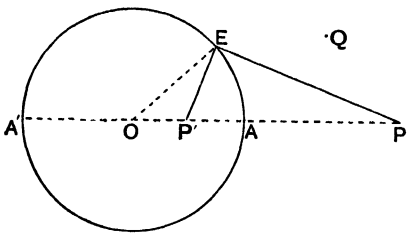


FIG. 53.

let f denote the length of this line, and a that of the radius of the sphere. On OP' take a point P' such that $OP'/a = a/f$. Then if P' be joined to any element E of the spherical surface, the two triangles $OP'E$, OEP are similar, and we have, writing r' for $P'E$, r for PE, $r' = ra/f$. Thus if a charge $-a/f$ be placed at P', the potential produced by it at E will be $-a/kfr'$ or $-1/kr$, which exactly annuls the potential $1/kr$ due to the positive unit at P.

The charge $-a/f$ at P' thus produces the required distribution of potential at the surface, and the potential due to it at external points is of course a harmonic function. Hence the field due to it cannot at any external point differ from that due to the surface distribution, since we have seen that the distribution of potential is in the circumstances unique throughout the external space.

Thus if Q be at any point of this space

$$G = -\frac{a}{kf}\frac{1}{P'Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

and the potential at Q is given by

$$V_{P'} = \frac{1}{k.PQ} - \frac{a}{kf}\frac{1}{P'Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

For any element E of the surface (since $P'E = r'$) we have

$$\frac{dG}{dn} = -\frac{a}{kf}\frac{d}{dn}\left(\frac{1}{r'}\right)$$

and

$$\sigma = -\frac{k}{4\pi}\frac{dU}{dn} = -\frac{1}{4\pi f}\frac{a}{f}\frac{d}{dn}\left(\frac{1}{r'} - \frac{f}{a}\frac{1}{r}\right) \quad . \quad . \quad . \quad (67)$$

To calculate this we notice that the intensity due to the unit charge at P is $1/kr^2$ and is in the direction of PE, and that the intensity due to the charge $-a/f$ at P' is a/kfr'^2 acting in the direction EP'. Resolving each of these into components along EO and PO we see that those of the former are in the directions PO, OE, and those of the latter in the directions OP, EO. Moreover the two components along the line OP exactly cancel one another, while the sum of the others is $(a^2/fr'^3 - a/r^3)1/k$ in the direction EO. This may be written $(f^2-a^2)/kar^3$. Hence

$$\sigma = -\frac{1}{4\pi}\frac{f^2-a^2}{a}\frac{1}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

If A, A' be the points in which the line PO meets the circle $PA = f - a$, $PA' = f + a$, so that the result may be written

$$\sigma = -\frac{1}{4\pi}\frac{PA.PA'}{OE}\frac{1}{PE^3} \quad . \quad . \quad . \quad . \quad . \quad (68')$$

It is very easy to verify by direct integration that this result gives, as it should, for the whole induced charge on the sphere, the value $-a/f$.

Of course if instead of a unit charge at P, q units were there placed, the value of σ would have to be increased in the same ratio.

Let now the unit charge be at an internal point P, so that OP, or f

is now less than a. We see at once that if P' be an external point such that $OQ = a^2/f$, and if a charge $-a/f$ be there situated, the sphere will be at zero potential. The corresponding diagram may be obtained from Fig. 53 by interchanging the letters P and P', and placing Q inside. The potential at any internal point P' will then be

$$V_Q = \frac{1}{k \cdot PQ} - \frac{a}{kf}\frac{1}{P'Q} \quad \ldots\ldots \quad (69)$$

Proceeding in the same manner as before we easily find for this case

$$\sigma = -\frac{1}{4\pi}\frac{a^2 - f^2}{a}\frac{1}{r^3} \quad \ldots\ldots \quad (70)$$

and the total induced charge of electricity on the sphere is now -1.

The point P' in these solutions is called the *electric image* of the point P, with respect to the spherical conductor, and from the two cases we see that P and P' are conjugate to one another, that is while P' in the first case is the electric image of P for the sphere influenced by a charge at P, P is the image of P' for the same sphere influenced by a charge at P'.

If we suppose a to be very great in comparison with the distance, h, say, of P from A, we have $f = a + h$, and (68) becomes

$$\sigma = -\frac{1}{2\pi}\frac{h}{r^3} \quad \ldots\ldots \quad (71)$$

approximately. This may be regarded as holding exactly when a (and therefore also f) is infinite. But when this is the case we fall on the distribution on an infinite plane maintained at zero potential under the influence of a positive unit point-charge situated at P. The image-charge $-a/f$ is here -1, and is situated at a point P', on the normal PA produced behind the plane, such that $PA = AP'$, that is at the optical image of P in the plane regarded as a reflecting surface. Thus the density, at any element E of the plane, of the negative induced distribution varies inversely as the cube of the distance PE of the element from P. A figure is unnecessary.

That this arrangement of inducing charge and its image produces zero potential at every element of the plane is obvious since the distances PE, $P'E$ are equal. The electric intensity outward from E, into the region in which P is situated, given by the arrangement is clearly $-2h/r^3$, and hence since this is $4\pi\sigma$ we get the result expressed in (71) which is thus verified.

Nothing is more easy than to verify by direct integration that the total charge on the plane is -1.

203. If the sphere be now supposed uninsulated and charged to a uniform potential V, the distribution upon it will be the distribution just determined together with a uniform distribution of density $\sigma = kV/4\pi a$. For the two distributions being separately possible may

be superimposed to give a possible distribution which will make the potential at the sphere V, and afford an induced distribution due to the point-charge (q say) at P. It will therefore be the only possible solution.

The potential at a point Q at distance R from the centre of the sphere will therefore be (1) when P is external ($f > a$),

$$\left.\begin{aligned} V_Q &= V\frac{a}{R} + \frac{q}{k \,.\, PQ} - \frac{a}{kf}\frac{q}{P'Q} \\ \text{for all external points, and} \quad V_Q &= V \end{aligned}\right\} \quad . \quad . \quad (72)$$

for all internal points;

(2) when P is internal ($f < a$)

$$\left.\begin{aligned} V_Q &= V\frac{a}{R} \\ \text{for all external points, and} \quad V_Q &= V + \frac{q}{k \,.\, PQ} - \frac{a}{kf}\frac{q}{P'Q} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (73)$$

for all internal points.

The surface density at any element E will in case (1) be

$$\sigma = \frac{1}{4\pi a}\left\{kV - (f^2 - a^2)\frac{q}{r^3}\right\} \quad . \quad . \quad . \quad . \quad (74)$$

and in case (2) (if the two distributions be taken together, as they may be if they are supposed to be on an ideal single surface)

$$\sigma = \frac{1}{4\pi a}\left\{kV - (a^2 - f^2)\frac{q}{r^3}\right\} \quad . \quad . \quad . \quad . \quad (75)$$

Of course in the actual physical case of a hollow spherical conductor the uniform distribution, represented by the first term on the right of the last equation, is on the external surface, and that represented by the second term is on the inner surface, the two being physically independent in the manner already explained.

The reader may prove for P external that if the total charge $Q(=kaV - qa/f)$ of the sphere fulfil the inequality

$$qa^2\frac{3f - a}{f(f - a)^2} > Q > -qa^2\frac{3f + a}{f(f + a)^2},$$

there is a circle of points on the sphere at which the density of the distribution is zero; and may find its position and may verify that the densities at A and A' are given by the equations

for A $$\sigma = \frac{1}{4\pi a}\left(kV - q\frac{f + a}{(f - a)^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (76)$$

for A' $$\sigma = \frac{1}{4\pi a}\left(kV - q\frac{f - a}{(f + a)^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (77)$$

It will be noticed that when the inducing charge is q at P, we have for Green's function, or the potential produced at a point Q by the induced charge,

$$G = -q\frac{a}{f}\frac{1}{k \cdot P'Q} \quad \ldots\ldots \quad (78)$$

whether P be external or internal to the sphere. It is to be remembered that in the former case $f > a$, in the latter $f < a$.

In the case of a plane, $a = f$, and the value for G for either side of the plane is given by the last equation.

Mutual Force between Two Charged Spheres. Explanation of an Apparent Anomaly

204. We can now find the actual dynamical stress between a sphere of radius, b, very small compared with a and f, and the sphere of radius a, supposed charged to potential V. Let the small sphere have its centre at P, and a total charge $+q$. If the radii of the spheres were comparable with one another, and with the distance between the centres, the influence of either sphere on the other could only be expressed by an infinite series of electric images, in the manner investigated below in the chapter on electric distribution on conductors. But since b is very small compared with a, the effect of q on the distribution on the larger sphere may be taken as nearly represented by the single image at the point P'. Also, since the induced density on the small sphere due to the distribution on the larger is approximately that caused by the charge kV/a at the centre of the larger, and $-qa/f$ at P', and these vary inversely as the cube of the distance of an element of the small sphere from O, or P as the case may be, the distribution on the small sphere may be taken as uniform.

The charge and potential of the small sphere are thus q, $q\{1/b - a/(f^2 - a^2)\}k + Va/f$, and those of the larger sphere $Q(= kVa - qa/f)$, V. The electric energy of the system is therefore given by the equation

$$\begin{aligned}\mathsf{E} &= \frac{1}{2k}q^2\left(\frac{1}{b} - \frac{a}{f^2 - a^2}\right) + \tfrac{1}{2}V^2ka \\ &= \frac{1}{2k}\left(\frac{Q^2}{a} + \frac{2Qq}{f}\right) + \frac{1}{2k}q^2\left\{\frac{1}{b} - \frac{a^3}{f^2(f^2 - a^2)}\right\} \quad . \quad (79)\end{aligned}$$

The work done against electric forces in separating, by externally applied force, the spheres through a further small distance df, will alter E by an amount $d\mathsf{E}/df \,.\, df$; that is, $-d\mathsf{E}/df$ is the mutual repulsive force between the spheres in the line of centres. But

$$-k\frac{d\mathsf{E}}{df} = \frac{Qq}{f^2} - q^2a^3\frac{2f^2 - a^2}{f^3(f^2 - a^2)^2} = kVq\frac{a}{f^2} - q^2\frac{af}{(f^2 - a^2)^2} \quad (80)$$

The force is therefore an attraction, zero, or a repulsion, according as either of the equivalent expressions on the right of (80) is negative, zero, or positive. Hence it is an attraction if V, or Q, is zero, that is, if the sphere is uninsulated, or if it has no charge on the whole. Also, if f be but little greater than a, that is, if the small sphere be very near the surface of the larger, $f^2 - a^2$ is very small, and the force is an attraction. If Q and q have the same sign, and the numerical value of Q be greater than that of $qa^3(2f^2 - a^2)/f(f^2 - a^2)^2$, or if V and q have the same sign, and the numerical value of kV be greater than that of $qf^3/(f^2 - a^2)^2$, the force is a repulsion.

The force is zero when $f^3/(f^2-a^2)^2 = kV/q$; and when the small sphere is in the position given by this relation, it is obviously in unstable equilibrium.

These results explain the attraction which is found to exist between a small charged sphere and a similarly charged conductor, when the distance between them is small, and their mutual repulsion when they are placed at a sufficiently great distance apart.

Method of Inversion. Geometrical Inversion

205. In the examples discussed above, we have instances of the discovery of a distribution of electricity over a closed surface which produces at each point of the surface, and at each point of space beyond the surface on one side, the same potential as is produced by a distribution within the space on the other side of the closed surface, and it has been seen that the surface distribution found is unique. This is always possible for any surfaces open or closed, or infinite in the electric field, that is, one distribution and only one over these surfaces can be found which shall produce at each point of them, and at each point of space entirely separated from a given distribution of electricity by those of the surfaces which are closed or infinite, the same potential as results from the given distribution. We shall deal fully, however, with this subject in a later chapter; but the applications of the method of images afford some excellent examples of equivalent distributions; and the solutions can be greatly multiplied by the method of inversion, first applied to electrical problems by Lord Kelvin.[1]

206. It may be convenient to recall here in the briefest possible manner the meaning, and some of the results of geometrical inversion. In Fig. 54 the distances OP, OP', OQ, OQ' fulfil the relation

$$OP \cdot OP' = OQ \cdot OQ' = a^2 . \quad . \quad . \quad . \quad . \quad (81)$$

The point P' is called the *inverse* of the point P with respect to O, which is called the centre of inversion; and similarly Q' is the inverse of Q with respect to O. If any system of points $P, Q, \ldots$ be given, a corresponding system of inverse points $P', Q', \ldots$ can be found, and if the

[1] See *Electrostatics and Magnetism*, 2nd Edition, p. 144 *et seq.*

first form a definite locus, the latter will form a corresponding derived locus. We shall call the first the direct system, the latter the inverse system of points. Of course, if P', Q', ... be regarded as the direct system of points, the corresponding inverse system is P, Q, ... with regard to the same centre. Each point P' is the *image* of the point P in the sphere of radius a and centre O. This is called the *sphere of inversion* and its radius the *radius of inversion*.

The triangles OQP, $OP'Q'$ in Fig. 54 are similar, and therefore the angle OQP is equal to the angle $OP'Q'$. Thus if P, Q be very near points, so that OP, OQ are nearly parallel to one another, the angle OQP is nearly equal to the angle $P'Q'Q$, that is, the line QP is inclined to QQ' at the same angle as that at which $Q'P'$ is inclined to $Q'Q$. Hence the inverses of any two lines or surfaces intersect at the same angle as do the original lines or surfaces.

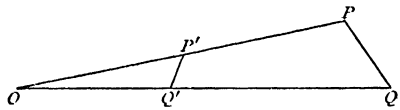


FIG. 54.

The inverse of a circle is another circle, and therefore that of a sphere is another sphere. For let PQ (Fig. 54) be the extremities of a diameter of the circle, and R any other point on the circle, then PRQ is a right angle. The inverse points are P', Q', R' and the angle $P'R'Q'$ is equal to a right angle $\pm$ the angle POQ, according as OR does or does not intersect PQ. Hence, as R moves round the circle, R' moves round another circle which is the inverse of the former.

If b be the radius of the circle (or sphere) and C its centre, the circle (or sphere) inverts into itself when $OC^2 - b^2 = a^2$.

The inverse of a straight line is a circle passing through the centre of inversion. For let P be a point on the straight line such that OP is at right angles to PQ, where Q is any other point on the line. Then if P', Q' be the corresponding inverse points, P' is fixed, and $OQ'P'$ is a right angle for every position of Q', and Q' is the inverse of all points on the line which are infinitely distant from P. Hence the locus of Q' is a circle of which OP' is the diameter.

It follows from this that the inverse of an infinite plane is a spherical surface passing through the centre of inversion.

According as the centre of inversion is without or within the surface inverted, the space within the inverse is the inverse of the space within or without the original surface, and the space without the inverse is the inverse of the space without or within the original surface.

Electrical Inversion. Derivation of Induced Distribution from Equilibrium Distribution and *Vice-versa*

207. The point P' which we have called the electric image of P with regard to the sphere AEA' is, it will be observed, the inverse of the

point P with regard to the same sphere, taken as sphere of inversion. It has been shown that a charge qa/f at P' will produce a potential at every point of the spherical surface equal to that produced by q at the point P, while, according as P is external or internal to the sphere, the potential due to the charge at P' will be a harmonic function for all external or all internal points. We shall call this the inverse of the charge q at P, reserving the term image-charge or image-distribution for the inverse distribution with sign reversed.

If instead of a single point-charge at P there be a system of point-charges $q_1, q_2, \ldots$ at points $P_1, P_2, \ldots$ without or within the sphere of inversion, a system of point-charges $q_1a/f_1, q_2a/f_2, \ldots$ situated at points $P'_1, P'_2, \ldots$ will be the inverse distributions, and will produce the same potential at the sphere as does the former system of charges. Thus to any distribution without or within the sphere of inversion corresponds another distribution which is within or without the sphere, and is the inverse of the former.

The space or surface, as the case may be, occupied by the inverse distribution is the inverse of that occupied by the given distribution. We easily see that if $d\varpi$, $d\varpi'$, dS, dS' be elements of volume and elements of surface in the direct and inverse distributions, and ρ, ρ', σ, σ' denote the volume densities and the surface densities in the two cases,

$$\left.\begin{aligned} &\frac{d\varpi'}{d\varpi} = \frac{a^6}{f^6} = \frac{f'^6}{a^6} = \frac{f'^3}{f^3}, \quad \frac{dS'}{dS} = \frac{a^4}{f^4} = \frac{f'^4}{a^4} = \frac{f'^2}{f^2} \\ \text{and} \qquad &\frac{\rho'}{\rho} = \frac{f^5}{a^5} = \frac{a^5}{f'^5}, \quad \frac{\sigma'}{\sigma} = \frac{f^3}{a^3} = \frac{a^3}{f'^3} \end{aligned}\right\} \qquad (82)$$

208. If V be the potential at any point Q due to the point-charge q at P, we have $V = q/(k.PQ)$. The potential produced at Q', the inverse of Q, by the corresponding image-charge at P', is $qa/(fk.P'Q')$. Hence if r, r' be put for OQ, OQ' we easily find

$$\frac{V'}{V} = \frac{r}{a} = \frac{a}{r'}.$$

Accordingly, since this is a ratio independent of the position of P, if V, V' denote respectively the potentials at the points Q, Q' due to the whole direct and inverse distributions respectively, we obtain

$$V' = \frac{r}{a}V = \frac{a}{r'}V. \quad \ldots \ldots \quad (83)$$

Thus if V is a constant over any surface S, V' is not a constant over the inverse surface S' unless r is a constant, that is, unless the equipotential surface is a sphere concentric with the sphere of inversion, when its inverse is concentric with it and is an equipotential surface of the inverse or image-distribution.

In the case in which V is a constant and r variable, we may reduce the potential of the inverse surface S' to zero by placing at the centre of inversion a charge $-kaV$. This will produce at any point at distance r' from O the potential $-aV/r'$ which is equal and opposite to V' or aV/r'. Thus if S be the surface of a conductor on which the direct charge is in equilibrium, the addition of this point-charge at O to the inverse distribution shows that the latter is the induced distribution on the inverse surface produced by a charge $-aV$ at O. The image-charge on S' corresponding to that on S is thus the induced distribution on S' due to a point-charge aV at O.

Thus by the process of inversion we obtain an induced distribution on the inverse surface from a given equilibrium distribution all at one potential, or conversely obtain from a given induced distribution on a surface, a natural equilibrium distribution on the inverse surface.

If S be not the surface of a conductor, but one of the equi-potential surfaces of the given distribution, its inverse S' is maintained at zero potential by the inverse distribution and the charge $-aV$ at O. Generally the given distribution lies part within part without any chosen surface S. Let these parts be denoted by q_1, q_2 respectively, and their inverses by q'_1, q'_2. Then if O be without S, q'_1 is within and q'_2 without S', and *vice-versa* if O is within S.

Now consider the distribution made up of the two parts of the inverse distribution q'_1, q'_2, and the charge $-aV$ at O, and suppose all points of S' to be at zero potential. The whole quantity of electricity within S' properly distributed over that surface will in each case produce the same potential at all points outside the surface, as is produced by the internal distribution. If the internal distribution be annulled and this surface distribution be substituted, the total potential at all external points will be unaltered, while that of each point on the surface and within it will be zero. The amount of the charge thus distributed will be q'_1 if O be without S, and $q'_2 - aV$ in the other case; while in each the density will be $(-kdV/dn)/4\pi$, where dV/dn is the rate of change of the total potential outwards from the surface S'.

Inversion of Uniform Spherical Distribution. Problem of Two Parallel Infinite Conducting Planes with Point-Charge between them

209. We may illustrate the method of inversion by applying it to one or two simple examples, reserving more recondite cases for later discussion.

First of all we shall invert a uniformly charged sphere. Let the potential of the sphere be denoted by V, its radius by β, the radius of inversion OA (Fig. 55) by a, the distance of the centre of inversion O from any point P' of the image by r', the distance of the same point O

from the centre of the inverse sphere by f, and the radius of the inverse sphere by a. We have

$$\beta = \pm \frac{a^2 a}{f^2 - a^2}$$

according as O is external or internal to the given sphere. But

$$\sigma' = \frac{a^3}{r'^3}\sigma = \frac{a^3}{r'^3}\frac{kV}{4\pi\beta}.$$

Hence, substituting the value of β just found we obtain

$$\sigma' = \pm \frac{f^2 - a^2}{4\pi a}\frac{kVa}{r'^3}$$

according as O is external or internal.

According as O is external or internal to the given sphere, and is therefore external or internal to the inverse, the spaces external and

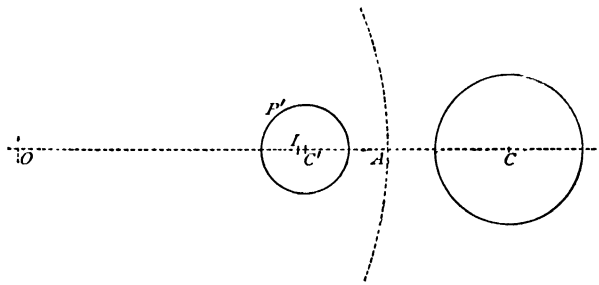


FIG. 55.

internal to the former are respectively the spaces external and internal or internal and external to the latter. Hence, according as O is external or internal, the potential of the inverse distribution at every internal point or at every external point of the inverse surface is the same as that of a charge kVa at O.

Also, since the potential of the given sphere is the same for all external points as if its charge $V\beta$ were concentrated at its centre C, the potential of the inverse distribution is the same at every point, external to the inverse sphere when O is external, and internal when O is internal, as that of a charge $q' = kV\beta.a/OC$ concentrated at the image I (Fig. 55) of the centre of the given sphere. But $OC = \pm a^2 f/(f^2 - a^2)$ and $\beta = \pm a^2 a/(f^2 - a^2)$ according as C is external or internal. Hence

$$q' = \frac{a}{f}kVa,$$

that is, the charge is the image in the inverse sphere of the charge $-kVa$ at O.

For OI we have $OI \cdot OC = a^2$, or

$$OI = \pm \frac{f^2 - a^2}{f}$$

that is, I is the image of O in the inverse sphere.

These results are those which have already been obtained above (pp. 143 *et seq.*).

210. Next we shall find the induced distribution for the case of two infinite parallel planes with a point-charge between them. This will afford an example of the method of successive influences introduced first by Murphy [1] for the solution of the problem of the mutual influence of conductors. We shall then invert this system and from it obtain the distribution on two mutually influencing spheres.

Let AB, Fig. 56, be the traces of two parallel planes on a perpendicular plane through P, a point between them at which a charge of

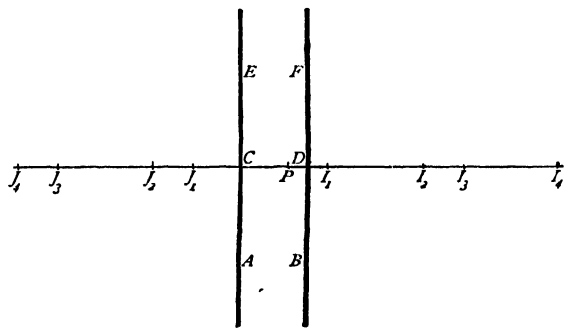


FIG. 56.

amount q is situated. Let a, β be the respective distances of P from C, D, the points in which a perpendicular through P meets the planes, and E a point on the plane A at a distance γ from C. The planes are to be supposed maintained at zero potential, and it is required to find the induced distribution upon them and the field at any point between them.

If the plane B were removed the density at E due to q at P would by (71) be $-qa/2\pi(a^2 + \gamma^2)^{\frac{3}{2}}$. But the induced electrification which q at P would induce if A were removed, produces the same field to the left of B as that due to a charge $-q$ at a point I, distant β from D on PD produced, and the corresponding electric density at E is therefore

$$\frac{q}{2\pi} \frac{a + 2\beta}{\{(a + 2\beta)^2 + \gamma^2\}^{\frac{3}{2}}}.$$

These two electrifications of A produce respectively the effects on the electrification of B of charges $-q$, $+q$ to the left of A on PC pro-

[1] Murphy's *Electricity*, Cambridge, 1833.

duced, the former at a point J_1, distant a, the latter at a point J_2 distant $a + 2\beta$ from C. The electrifications of B thus produced have on A the effects of charges $+q$, $-q$ at points I_2, I_3 distant $3a + 2\beta$, $3a + 4\beta$ to the right of A on CD. The corresponding densities at E are therefore

$$-\frac{q}{2\pi}\frac{3a + 2\beta}{\{(3a + 2\beta)^2 + \gamma^2\}^{\frac{3}{2}}}, \quad \frac{q}{2\pi}\frac{3a + 4\beta}{\{(3a + 4\beta)^2 + \gamma^2\}^{\frac{3}{2}}}.$$

In the same way another pair of densities at E could be found corresponding to point-charges $+q$, $-q$ at the respective distances $5a + 4\beta$, $5a + 6\beta$ to the right of A, and so on.

The electrification of A is that which would, if B were removed, be produced by $+q$ at P and an infinite trail of images I_1, I_2, ... of charges $-q$, $+q$, $-q$, ... at points to the right of P on CD produced. The potential at every point of A or to the left of it is plainly the potential due to $+q$ at P, and the image-charges to the right of P; and this is equal and opposite to the potential at the same point produced by the electrification of A. Similar results hold for B and the images to the left.

The potential at any point between the planes produced by the electrification of either is that due to the trail of images behind that plane, and the total actual potential at any such point is the sum of the potentials due to q at P and the two trails of images.

To verify that the potential of each of the planes is zero let V be the potential at any point E of the plane A. Then

$$V = \frac{q}{k}\left(\frac{1}{PE} - \frac{1}{J_1E}\right) + \frac{q}{k}\left\{\Sigma\left(\frac{1}{I_{2n}E} - \frac{1}{I_{2n+1}E}\right) - \Sigma\left(\frac{1}{J_{2n}E} - \frac{1}{J_{2n+1}E}\right)\right\}. \quad (84)$$

where n has every integral value from 0 to ∞.

Since $J_1E = PE$, the first term is zero; further, each series is convergent, and the terms of the two series (which are arranged in the same order in both) are identically equal. Hence V is zero, and the above process gives the required result.

The charges and distances of the images I_1, I_2, ... are given by the table

Images	I_{2n-1}	I_{2n}
Charges	$-q$	$+q$
Distances from P	$2(n-1)a + 2n\beta$	$2n(a + \beta)$

where n has every positive integral value from 1 to ∞. The charges and distances of the images J_1, J_2, ... are given by the same table when a and β are interchanged.

Clearly the density σ at E is given by the equation

$$\sigma = \frac{q}{2\pi}\sum_{n=0}^{n=\infty}\left[\frac{(2n+1)\alpha + 2(n+1)\beta}{\{[(2n+1)\alpha + 2(n+1)\beta]^2 + \gamma^2\}^{\frac{3}{2}}} - \frac{(2n+1)\alpha + 2n\beta}{\{[(2n+1)\alpha + 2n\beta]^2 + \gamma^2\}^{\frac{3}{2}}}\right] . \quad (85)$$

where n is any positive integer. The density at any point F on B is given by this equation with α and β interchanged, and γ taken as the distance from D to F.

Distribution on Two Spheres in Contact Obtained by Inverting Induced Distribution on Two Parallel Planes

211. We now invert the solution just discussed. Let the centre of inversion be P, the radius of inversion a, and let the planes and successive images be inverted, omitting the charge at P. The inverses of the planes are spheres touching at P, as shown in Fig. 55, and the

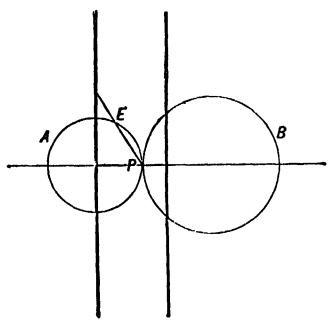


FIG. 57.

distribution on either sphere is the inverse of the distribution on the corresponding plane. For the inverse charges corresponding to the trail of images $I_1, I_2, \ldots$, and their distances we have

Images	I_{2n-1}	I_{2n}
Charges	$\dfrac{-q\alpha}{2(n-1)\alpha + 2n\beta}$,	$\dfrac{+q\alpha}{2n(\alpha+\beta)}$
Distances from P	$\dfrac{a^2}{2(n-1)\alpha + 2n\beta}$,	$\dfrac{a^2}{2n(\alpha+\beta)}$

where n has every positive integral value from 1 to ∞.

The table for the images J_1, J_2, is formed from this by interchanging α and β.

The diameters of the spheres A, B are respectively a^2/a, a^2/β, and therefore the inverse charges corresponding to $I_2, I_2, \ldots$ are within the sphere B, and the other series within the sphere A.

The potential at any point on the planes or behind them is zero, and therefore the potential at any such point due to the distribution on the planes is equal to that of a charge $-q$ situated at P, that is $-q/r$, where r is the distance of the point from P. The potential at any point on or within the spheres is therefore $-q/a$, a constant quantity. Again, since the potential produced by the electrification of either plane at any point on the plane or in front of it is the potential due to the trail of images behind the plane, the potential at any point on or external to either sphere produced by the distribution on the sphere is the potential at that point of the trail of images within the sphere. The charge on each sphere is therefore equal to the sum of the image-charges whose positions fall within it; and the distribution thus found is the equilibrium distribution when the spheres are freely electrified in contact.

In carrying out the solution we shall in inverting reverse the signs of all the charges, so that V will be $+q/a$.

Denoting the charge on the sphere B by Q_B, and the radii of the spheres A, B by r_1, r_2 respectively, summing the image-charges, and substituting in the result V for $+q/a$, $a^2/2r_1$ for a, $a^2/2r_2$ for β, we get

$$Q_B = kV\frac{r_1r_2^2}{r_1 + r_2}\sum_{n=0}^{n=\infty}\frac{1}{(n+1)\{r_1 + n(r_1 + r_2)\}} \quad . \quad (86)$$

where, as indicated, n has every integral value from 0 to ∞.

A similar expression with r_1, r_2 interchanged holds for Q_A.

The capacities of the spheres are of course equal to these expressions divided by V.

Multiplying the expression for the density at any point of the plane A by a^3/r'^3 [see (82) above] where r' is the distance from P of the corresponding point E on the sphere A, making the substitutions already specified above, and besides putting $a^4(1/r'^2 - 1/4r_1^2)$ for γ^2, we get for the density at E

$$\sigma = \frac{2kVr_1^2r_2^2}{\pi}\sum_{n=0}^{n=\infty}\left[\frac{(2n+1)r_2 + 2nr_1}{[r'^2\{(2n+1)r_2 + 2nr_1\}^2 + 4r_1^2r_2^2 - r'^2r_2^2]^{\frac{3}{2}}}\right.$$
$$\left. - \frac{(2n+1)r_2 + 2(n+1)r_1}{[r'^2\{(2n+1)r_2 + 2(n+1)r_1\}^2 + 4r_1^2r_2^2 - r'^2r_2^2]^{\frac{3}{2}}}\right] \quad (87)$$

This expression is convergent (unless $r' = 0$), and from it the density at any point can be approximately calculated in terms of the potential V, the radii r_1, r_2, and the distance, r', of the point considered from P. But if r' be very small the value of the density is known from other considerations to be very small also, hence the calculation need not be

carried out except for moderately large values of r'. That the density is zero when $r' = 0$, is evident from the fact that the spheres are there in contact, and since in the immediate neighbourhood of that point the surfaces are very close and very nearly parallel, a surface element there is practically within a closed conductor, and there can be no sensible charge upon it.

The charge on each sphere may be regarded as the difference of the sums of two harmonic series. Each series is divergent, but the two sets of terms taken together as in (86) constitute a convergent series, and hence the charge can be approximately calculated for given values of V, r_1, r_2. Equation (86) may be written

$$Q_B = kV\frac{r_1r_2}{r_1 + r_2}\sum_{n=0}^{n=\infty} \frac{r_2}{(n+1)\{(n+1)(r_1+r_2) - r_2\}} \quad (88)$$

and therefore may be further transformed to

$$Q_B = kV\frac{r_1r_2}{r_1 + r_2}\int_0^1 \frac{\theta^{\frac{r_2}{r_1+r_2}-1}}{1-\theta}d\theta \quad . \quad . \quad . \quad . \quad (89)$$

as may be verified by taking as successive terms under the sign of integration the product of the numerator into successive terms of the series $1 + \theta + \theta^2 + \ldots$, and then integrating term by term.

This is the form in which the result was given by Poisson.[1] Of course the corresponding value of Q_A is obtained by interchanging r_1 and r_2 in the exponential.

Case of Two Equal Spheres. Electric Kaleidoscope

212. When $r_1 = r_2$, we have

$$\begin{aligned} Q_A = Q_B &= kVr_1(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots) \\ &= kVr_1 \log_e 2 = \cdot 693147\, kVr_1 \quad . \quad . \quad . \quad . \quad (90) \end{aligned}$$

or the charge on each sphere is to the charge on the sphere when alone in the field and at potential V, as $log_e 2$ is to 1.

If r_1 be small in comparison with r_2

$$Q_B = kV\frac{r_2^2}{r_1 + r_2} + kV\frac{r_1r_2}{r_1 + r_2}\sum_{n=0}^{n=\infty} \frac{1}{n(n+1)} = kVr_2 \quad (91)$$

since $\Sigma\{1/n(n+1)\} = 1$. The charge is therefore nearly the free charge

[1] *Mémoires de l'Institut*, 1er partie, p. 1. See also Plana, Mém. de l'Acad. des Sci. de Turin, Sér. II., t. vii, p. 71, for a fuller development of Poisson's method and results.

which the sphere would have if alone and at potential V. The mean density is $kV_2/4\pi r_2$. On the same supposition we have

$$Q_A = kV\frac{r_1^2}{r_2}\sum_{n=0}^{n=\infty}\frac{1}{(n+1)^2} = kV\frac{r_1^2}{r_2}\frac{\pi^2}{6} \quad . \quad . \quad . \quad (92)$$

and the mean density is $\frac{1}{6}kV\pi^2/4\pi r_2$. Thus the mean density of the small sphere is to that of the large sphere as $\pi^2/6$, or 1·645, is to 1.

This result is important in its application to the interpretation of the result of the method, sometimes employed, of determining electric distribution by bringing a small conducting ball into contact with the charged conductor at different points, so as to receive charges the amounts of which are compared. The electric density at the point before contact is to the mean density on the ball very approximately as 1 is to 1·645. This result holds whether the charged conductor be spherical or not, provided its curvature be continuous round the point of contact over a distance great in comparison with the radius of the ball, and be small compared with that of the ball.

In the case of two infinite planes intersecting at an angle $2\pi/n$, (n a whole number) and influenced by a point-charge at a point P between them, the number of image-charges is $n-1$. They are placed like the images in a kaleidoscope, of which the planes are the mirrors, and P is the object, and consist of charges $-q$, $+q$, alternately, when taken in order round the circle on which they lie. The inversion of this gives the distribution on two spheres cutting at an angle $2\pi/n$.

We leave the theory of electrostatics for the present at this point, and proceed to a short discussion of steady flow of electricity as a preliminary to a chapter on electromagnetism. Other cases of distribution and questions regarding a field occupied in different regions by different dielectrics will be considered in a later chapter.

CHAPTER VI

STEADY FLOW OF ELECTRICITY IN LINEAR CONDUCTORS

Process of Change from the State of Equilibrium to Another. Electric Current

213. Consider a condenser so charged that while one plate is at zero potential the other has a charge Q and is at potential V. According to the theory given above, the energy E of the condenser is given by the equation

$$\mathsf{E} = \frac{1}{8\pi}\int\int\frac{\mathbf{D}^2}{k}\,dSds = \tfrac{1}{2}QV \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which one integral is taken over the surface of the charged conductor, and the other along a line of induction from the charged conductor to the other.

Now consider two condensers perfectly similar in size and all other respects, and so situated relatively to one another that they are without mutual influence when independently charged. Let one of these be charged as just described, and let its charged plate be connected by a fine wire to the corresponding plate of the other, while the second plate of the latter is kept at zero potential. It is found by experiment that when this is done a new state of the condensers is reached, in which the field between the plates of the condenser becomes the same in both cases, and that the intensity, at any given point, of the field of the originally charged condenser is, as nearly as can be observed, half what it formerly was.

Thus we have half the former field-intensity and therefore also half the induction ; but, as the area of the charged plate has been doubled, the value of the enregy given by the equation written above has only been halved. In other words the difference of potential between the plates has been halved, the charge has remained constant.

If the condensers are not equal in capacity the potential resulting from the operation stated has the ratio to the original potential of the capacity of the originally charged condenser to that of the condenser

made up of the two when placed in contact. In all sharing of charges between conductors the resulting distributions are consistent with constancy of the total charge, that is with constancy of the surface-integral of induction across a closed surface surrounding all the charged conductors, and described in their field.

This is the result of the equalisation of potential between the charged and uncharged plates connected together: we have to inquire what has become of the difference of energy. That there is expenditure of energy is due to the fact that a spark may be produced when the contact is made, and in any case heat is produced in the connecting wires.

The mode in which the transfer of energy is imagined to take place is described in general terms in Arts. 127, 128 above. The tubes of induction move outwards laterally (from a disturbance of the balancing lateral action set up by making the connection) with their ends on the wire, and the transference goes on until they have taken up the new equilibrium arrangement, with the same number of tubes in each of the two equal condensers. As the tubes move their energy is in part absorbed by the connecting wires and conductors, along which the tubes are guided, and passes from the medium across the bounding surface of the conductor. The energy thus absorbed is dissipated in heat in the conductor, and its total amount is independent of the nature of the connecting wire, provided the latter is not such as to make any addition to the united capacity of the two condensers. The rate, however, at which the energy is thus transformed and at which the redistribution is effected depends very much upon the wire and its arrangement, as we shall see later. The whole question of flow of energy in the dielectric medium will be fully considered in the chapter which follows on general electromagnetic theory.

The amount of energy thus lost from the system may be calculated from the considerations put forward in Section II. of the preceding chapter (Art. 180). Let the two condensers, instead of being equal, have capacities C_1, C_2, and have initially charges Q_1, Q_2 before they are put in communication. Before contact the energy of the system was $\frac{1}{2}Q_1^2/C_1 + \frac{1}{2}Q_2^2/C_2$, after contact it is $\frac{1}{2}(Q_1 + Q_2)^2/(C_1 + C_2)$. The loss of energy is therefore $\frac{1}{2}(Q_1C_2 - Q_2C_1)^2/\{C_1C_2(C_1 + C_2)\}$, which is essentially positive. If t be the time of transition the average rate of passage of energy from the conductors to whatever form or forms it may take, is $\frac{1}{2}(Q_1C_2 - Q_2C_1)^2/\{tC_1C_2(C_1 + C_2)\}$.

The second conductor has now charge $C_2(Q_1 + Q_2)/(C_1 + C_2)$, while the first has $C_1(Q_1 + Q_2)/(C_1 + C_2)$. The quantity of electricity which has been gained by the second conductor is thus $C_2(Q_1+Q_2)/(C_1+C_2) - Q_2$ or $(C_2Q_1 - C_1Q_2)/(C_1 + C_2)$, and this is equal to the loss of charge from the first. The average rate of transference of charge has thus been $(C_2Q_1 - C_1Q_2)/t(C_1 + C_2)$, and this we call the average *current* of electricity which has flowed along the wire during the transition.

Steady Currents

214. When the rate of flow varies with the time the current at any cross section of the conducting wire is measured of course at each instant of time by $-dQ/dt$, the time-rate of loss of charge by the conductor at higher potential. The comparison of currents experimentally will be dealt with later, at present it is sufficient to obtain a specification of the amount of a current.

In the greater number of cases with which we are concerned in practice, the current, whether constant or varying with the time, has at each instant the same value at every cross-section of the connecting wire; but when the capacity of this conductor is not negligible in comparison with those of the conductors it connects, and the state of the latter varies with the time, the current may vary very seriously from one cross-section to another at any one instant. For example, the current at different cross-sections of a submarine cable for some time after a terminal of a battery has been applied (and kept applied) to one of its extremities, generally has very different values at different cross-sections, owing to the charge required to raise the potential of each part of the internal conducting wire to the equilibrium value. Such a cable, with its internal conductor separated from the conducting sea-water by a coating of gutta-percha, forms a condenser which has a very sensible capacity for every unit of its length, and this must be taken into account in discussing the flow of electricity along it. We shall return to this later; but at present we confine our attention to *steady* or unvarying currents in linear conductors, that is to say conductors so thin that the flow at any cross-section may be regarded as being at every part of that cross-section in the same direction. The distribution of the current over the cross-section of a wire in certain cases is a very important problem; but this does not arise either when the flow is steady, as the current is then found by experiment to be equally distributed over the cross-section, as will presently appear.

Analogue of an Electric Current

215. When the flow is steady an electric current is in a sense the analogue of a current of an incompressible fluid from one vessel to another along a canal or pipe which opens into the vessel, and is kept full by the current. Here the difference of potential between the conductors connected by the wire is the analogue of the difference of pressure between the two vessels. Since the fluid is incompressible and the channel is kept full and unaltered in dimensions, the time-rate of flow, however it may vary with the time, will have at any one instant the same value at every cross-section. This analogy has led to the adoption in the language of practical electricity of the term pressure to denote the difference of potential applied to the mains of an electric lighting system.

Other analogies are found in the conduction of heat and the diffusion of liquids and gases; and these phenomena moreover have a mathematical theory which can be employed to give results in the problem of flow of electricity. As we shall see presently, the rate of flow depends upon the nature of the conducting material, just as the flow of heat depends on the thermal conductivity of the substance. If a difference of temperature be taken as the analogue of a difference of potential, rate of flow of heat as the analogue of a current of electricity, and thermal conductivity of a medium (taken as independent of temperature) as that of a quantity which we shall call the *specific electric conductivity* of a substance, we may transfer the equations of heat conduction bodily to the theory of flow of electricity.

This apparently complete parallelism of theory is not, however, to be regarded as evidence of any relation between the phenomena in the different cases, though no doubt hitherto undiscovered connections between them may exist.

Ohm's Law. Resistance. Direction of Current. Electromotive Force

216. It is possible by proper appliances to obtain a steady current of electricity along a conducting wire; for example, by means of a voltaic battery, a thermo-electric pile, or a dynamo-electric machine, the function of which may be taken to be that of maintaining two conductors of the same material connected by the wire, or two cross-sections of the wire itself near its extremities at a constant difference of potential. We must here anticipate what follows, so far as to suppose that we have an experimental means of comparing such differences of potential, and also of measuring currents. These methods, as well as the action of voltaic cells and other electrical generators, will be fully discussed in later chapters.

If, then, between any two cross-sections of a homogeneous wire, which is not in motion in a magnetic field, and is all at one temperature, a difference of potential be maintained of constant amount, it is found that in the wire a constant current ultimately flows, and that this current varies in simple proportion to the difference of potential, provided there is no sensible heating of the wire. This result expresses the so-called law of Ohm, which thus expresses a physical fact capable of being verified by experiment.

Again, if a wire of homogeneous material and uniform cross-section, at rest in a magnetic field, and at the same temperature throughout, have a constant current maintained in it, the difference of potential between any two of its cross-sections A B, is proportional to the length of wire between them. Further, if the difference of potential between A and B be maintained constant, while experiments are made with different lengths of wire included between them, the current is found to vary inversely as the length of wire to which the difference of potential is thus applied. Also if the length of wire

and the difference of potential between A and B be kept constant, while experiments are made with wires of the same material of different areas of cross-section, the current is found to vary directly as the area of cross-section. Finally, if wires of given length and cross-sectional area, but of different materials, be used with a given difference of potential, the current is found to vary with the material.

Hence the wire is said to oppose a *resistance* to the passage of a current, which is directly proportional to the length of wire between A and B, inversely proportional to the cross-sectional area of the wire, and depends on the material of which the wire is composed.

The positive direction of the current is taken as that from higher to lower potential, and is therefore from the copper plate to the zinc plate of an ordinary voltaic cell, such as Daniell's, along the external connecting wire (Arts. 221, 366, and Chap. XII below).

Putting γ for the measure of the current, that is, of course, the number of units of charge transferred along the wire per unit of time, V_A, V_B for the potentials at A, B ($V_A > V_B$), and R_{AB} for the resistance of the wire between A, B, we express all these results, including Ohm's law, by the equation—

$$\gamma = \frac{V_A - V_B}{R_{AB}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If a, b be other two points in the same conductor, and R_{ab} be the resistance between them, we have

$$\gamma = \frac{V_A - V_B}{R_{AB}} = \frac{V_a - V_b}{R_{ab}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, the slope of potential per unit of resistance along the wire is the same at every point. This gives also

$$V_a - V_b = (V_A - V_B)\frac{R_{ab}}{R_{AB}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is to be observed that equation (2) or (3) defines resistance of a conductor between the cross-sections at which $V_A - V_B$ is applied, and also unit of resistance. The former is that coefficient which multiplied into the measure of the current gives the measure of the difference of potential between A and B; the latter is the resistance existing between A and B when unit difference of potential exists between them, and unit current flows in the wire. It also expresses the physical fact just stated above as Ohm's law, inasmuch as it asserts the proportionality of γ to $V_1 - V_2$ when the wire is constant, or the proportionality of γ to the *slope of potential* along the wire.

It is not unusual to apply the term *electromotive force* to the difference of potential between two points or two equipotential surfaces in a homogeneous conductor, when thus considered with reference to flow of electricity from one to the other. We shall, however, generally use the

phrase in a somewhat different but not inconsistent sense. It is to be carefully observed that neither electromotive force nor resistance is a *force* in the ordinary dynamical sense.

It is not generally necessary, though it may be often convenient, to regard electromotive force as the *cause* of a current. The two things really exist together, and either, if it serves any purpose, may be regarded as producing the other.

Flow in a Conductor containing an Electromotive Force. Heterogeneous Conductors and Circuits

217. Equation (2) cannot be taken as fulfilled by a conductor made up of different homogeneous portions put end to end, or by a conductor moving across the lines of force in a magnetic field. For such cases we have

$$\gamma = \frac{V_A - V_B}{R} + \frac{E_{AB}}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V_A, V_B denote as before the potentials between cross-sections A, B, and R is the sum of the resistances of the homogeneous portions of the conductor contained between these cross-sections in the former case, or the actual resistance of the conductor in the latter. In such cases the conductor is said to *contain*, or to be the *seat of*, an electromotive force E_{AB}, or, as we say frequently, an electromotive force E is said to be in the conductor. The total electromotive force producing a current in the conductor is now $V_A - V_B + E_{AB}$, of which the part $V_A - V_B$ is frequently called the applied electromotive force.

218. Since in a heterogeneous conductor supposed at rest in a non-varying magnetic field (2) applies in the first case to every part, except any, however small, which includes a surface of discontinuity, the electromotive force is said to have its seat at the surface or surfaces of discontinuity. Its presence is manifested by the existence of a finite step of potential across the surface of contact. In the other case the electromotive force has its seat in every part of the conductor moving in the field, according to a law which we shall discuss in connection with electro-magnetic induction.

Consider a closed circuit made up of different homogeneous linear conductors placed end to end, and let E be the sum of the electromotive forces which have their seat in the circuit. Let adjacent points be taken on opposite sides of each surface of discontinuity, so that two points in each homogeneous part close to its extremities are thus obtained. Let the difference of potential between each latter pair of points be measured, taking them in order round the circuit in the direction in which the current flows. The sum of these differences taken in order is equal to the sum of the parts of E contributed by the discontinuities. For going round in the direction of the current from

a point in one of the homogeneous parts to the same point again we have $V_A = V_B$, and (5) gives

$$\gamma = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But denoting the successive homogeneous parts in their order round the circuit by the suffixes 1, 2, . . . , n and the differences of potential between the pairs of points in each near their extremities by $V_1 - V_1'$, $V_2 - V_2'$, . . . , $V_n - V_n'$, and the corresponding resistances by R_1, R_2, . . . , R_n, we have

$$\gamma = \frac{V_1 - V_1'}{R_1} = \frac{V_2 - V_2'}{R_2} = \ldots . = \frac{V_n - V_n'}{R_n} = \frac{\Sigma(V - V')}{R} \quad (7)$$

where $R = R_1 + R_2 + \ldots + R_n$. Hence

$$\Sigma(V - V') = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

E is called the electromotive force in the circuit.

For $V_A - V_B$ the difference of potential between two points A, B in a homogeneous part of the circuit we get evidently

$$V_A - V_B = E\frac{R_{AB}}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The example of *contact* electromotive forces, as the electromotive forces at the surfaces of contact of the heterogeneous parts of the circuit are called, most usually given is that of a voltaic cell; but as the question of the existence of the electromotive forces observed in this case is not without difficulty, we shall not discuss it at present, but pass on to some general statements and some results regarding certain arrangements of homogeneous conductors which are of great importance in practice.

Meaning of Resistance. Rate of Production of Heat in Conductors. Joule's Laws

219. First, the meaning of the resistance of a conducting wire may be put in a somewhat different light. In a homogeneous conductor, to which (2) applies, a quantity of electricity measured by γ is transferred from potential V_A to potential V_B per unit of time. The loss of energy is thus $\gamma\,(V_A - V_B)$ per unit of time, and this in the case supposed is found to take wholly the form of heat in the wire. Thus if $\mathbf{A}$ be the activity in the wire expended in heat we have

$$\mathbf{A} = \gamma(V_A - V_B) = \frac{(V_A - V_B)^2}{R} = \gamma^2 R \quad . \quad . \quad . \quad (10)$$

Thus R may be regarded as the amount of energy transformed into heat per unit of time in the portion of the conductor of which R is the

resistance when unit current flows, or $1/R$ is the activity spent in heat in the same portion of the conductor when unit difference of potential is maintained between its extremities.

The fact that the heat developed per unit of time in different conductors is proportional to the resistances of the conductors and to the squares of the currents flowing in them was established by the experiments of Joule,[1] and the law of development of heat is therefore generally referred to as Joule's law. In no circumstances is a portion of a homogeneous conductor, in which there is no gradient of temperature, cooled by the passage of a current through it, though heat may be absorbed by the passage of a current across a junction of two dissimilar metals or along an unequally heated conductor. Thus R is always a positive quantity.

In the more general case to which (5) applies we have for the rate of transformation of electrostatic energy as before $\gamma(V_A - V_B)$. But

$$\gamma^2 R = \gamma(V_A - V_B) + \gamma E_{AB} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We interpret the second term on the right as the rate at which energy is evolved in consequence of the existence of the electromotive force E_{AB} in the conductor. The sum of this and the rate at which electrostatic energy is yielded by the system is the rate of evolution of heat. Either of the terms on the right may be negative but not both; that is, the electromotive force E may enable work to be done against a difference of potential, thus increasing the electrostatic energy, or work may be done by the electrostatic difference of potential $V_A - V_B$ against the internal electromotive force if that opposes the current. But in all cases a positive value of $\gamma^2 R$ results, that is to say, work of this amount is always spent in the conductor in producing heat.

220. We may consider also a part of the circuit, between the terminals A, B, of which there exists an applied difference of potential $V_A - V_B$, and in which electromotive forces, for example, those due to a cell or cells of a voltaic battery, aiding the current, as well as other electromotive forces, for example, those due to voltameters or storage cells in which energy is spent in producing electrolytic decomposition, have their seat. This part of the circuit may or may not be heterogeneous. If the sum of the former or positive electromotive forces be ΣE_{AB}, and that of the latter, or negative electromotive forces, be $\Sigma E'_{AB}$, and R be the total resistance of the part of the circuit, the rate at which energy is spent in heat in it is

$$\gamma^2 R = \gamma(V_A - V_B) + \gamma\Sigma E_{AB} - \gamma\Sigma E'_{AB}$$

that is

$$\gamma^2 R + \gamma\Sigma E'_{AB} = \gamma(V_A - V_B) + \gamma\Sigma E_{AB} \quad . \quad . \quad . \quad (11')$$

The right-hand side of (11') shows the rate at which electric energy is

[1] *Phil. Mag.*, (S. 3), vol. xix, 1841, p. 260, and vol. xxiii, 1843, pp. 263, 347, 435, or *Collected Papers*, vol. i, pp. 60, 123.

evolved in the circuit, the left-hand side shows the rate at which energy is spent in heat, and in working against the negative electromotive forces respectively.

Equation (11′) applies of course also to a complete circuit, though in that case, since A and B are coincident, $V_A - V_B$ is zero.

Arrangements of Electric Generators in Series and in Parallel

221. Let us suppose that we have a number n of equal generators of electric currents, such as a number of equal voltaic cells, each of electromotive force E. If these be joined in series, that is, so that the electromotive forces of all act in the same direction along a single linear arrangement, the total electromotive force of the system is found to be n times that of one cell. Let the circuit be completed by an external conducting wire of resistance R. In general heat is generated within the cell as well as in the external joining conductor, that is to say, if the rate of generation of heat within the cell be $\gamma^2 r$, each cell has an internal resistance r. From what has been stated above the current γ is given by

$$\gamma = \frac{nE}{R + nr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If nr be great in comparison with R, which will always be the case if R is fixed, and n is taken great enough, but little advantage is gained by increasing n further. For the current is then approximately E/r, or that produced by a single cell when it is short-circuited, that is, has its terminals joined by a short piece of thick wire.

To join single cells in series is only advantageous when R is so large that the condition stated does not hold. But r may be virtually diminished by joining the cells in what is commonly called parallel. To fix the ideas let a Daniell's battery be supposed employed. Each cell consists of a plate of copper immersed in a solution of copper sulphate and a plate of zinc in a solution of zinc sulphate, in compartments of the containing vessel separated by a partition of porous earthenware which permits conducting contact between the liquids and retards their mixing together. A number m of equal cells of such a battery are placed abreast, and all the copper plates are joined together to form one terminal, and all the zinc plates to form the other terminal plate. The electromotive force of such a compound cell is E simply, but its internal resistance is r/m. If then n of these compound cells be joined in series, and the circuit completed by a resistance R the current obtained will be

$$\gamma = \frac{nE}{R + n\dfrac{r}{m}} = \frac{mnE}{mR + nr} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In practice it is sufficient, if the cells are similar in all respects, to join the zinc plates which form the terminal plates of each series of

n cells, and the copper plates which are the other terminal plates of these series, and to leave the intermediate zinc and copper plates of the different series unjoined.

Arrangement of given Battery to produce Maximum Current through given External Resistance

222. If R be not too great and the total number of cells mn be suitable, we can arrange the battery so that for the given value of R that of γ may be a maximum. The numerator of the above expression for γ is constant, since the available number of cells is mn, and it can be shown that $mR + nr$ is least when m and n are so chosen that $mR = nr$. For we have identically

$$mR + nr = (\sqrt{mR} - \sqrt{nr})^2 + 2\sqrt{mnRr}.$$

The second term on the right does not depend on the choice of m and n. Hence the right-hand side is least when the other term, which is essentially positive, is zero; that is, when $mR = nr$, or the external resistance R, is equal to the internal resistance nr/m of the battery as arranged.

If it is not possible to fulfil this condition exactly with the given number of cells, the arrangement which most nearly fulfils it should be chosen.

This theorem can only be applied when the resistance R and the battery which is to work through it are given. It is an entire fallacy to suppose, as is sometimes done, that of two batteries having equal electromotive forces, that which has the greater resistance is better adapted for working through a high resistance than the other, owing to its more nearly fulfilling the condition of external equal to internal resistance.

The arrangement just arrived at gives not only the maximum current in the external part of the circuit, but produces there also the greatest activity which can possibly be obtained, when the current is used only for the production of heat in the external part of the circuit. For the activity in R is given by

$$\mathbf{A} = mnE\gamma \frac{R}{mR + nr} = m^2n^2 \frac{E^2R}{(mR + nr)^2} \quad . \quad . \quad . \quad (14)$$

which is a maximum under the same conditions as γ.

Arrangement for Maximum Current not that of Greatest Efficiency

223. This, however, is not to be confused with the arrangement of maximum economy or efficiency. In fact, in the arrangement under discussion, as much energy is spent per unit of time in heat in the battery itself as in the external resistance; and therefore the ratio of the rate at which energy is usefully spent (in the external resistance) to the

whole rate of expenditure is $\frac{1}{2}$, that is, half the energy expended is wasted.

But the most economical arrangement is that in which this ratio most nearly approaches to 1, that is in which as little as possible of the energy given out by the battery is spent in the battery itself, and consequently as much as possible in the external part of the circuit. For economy of working, the internal resistance of the battery, and the resistance of the wires connecting the battery with the usefully working part of the circuit, must be made as small as possible. This subject will however be further dealt with in the discussion of electric motors.

Theory of a Network of Conductors—Two Fundamental Principles: (1) Principle of Continuity

224. We shall now consider shortly a network of linear conductors in which steady currents are flowing, and in which are situated any internal electromotive forces that may be possible. Besides the considerations advanced above, two principles, first stated explicitly and applied to this subject by Kirchhoff,[1] are available for the discussion of problems regarding such a system. The first is the principle of continuity already expressed, for a single wire, by the statement that the current has the same value at all cross-sections if the flow is steady. This expresses the fact that the rate of flow into any portion of the wire at any instant is precisely equal to the rate of flow out of the same portion. The same principle gives the result that, when steady currents are maintained in the various parts of a network of conductors, the total rate of flow of electricity towards the point at which several wires meet is equal to the total rate of flow from that point at the same instant. Thus the current arriving at A, Fig. 58, by the main con-

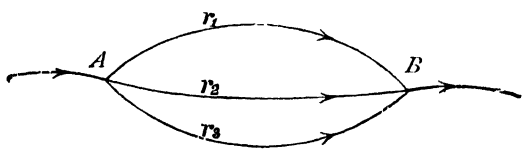


FIG. 58.

ductor is equal to the sum of the currents flowing away from A by the three parallel conductors which connect A with B.

(2) Sum of Electromotive Forces in any Circuit of Network equal to Sum of Products of Currents round Circuit into Resistances of Conductors

225. The other principle is contained in the following, which can be at once obtained by applying Ohm's law to any complete circuit which can be obtained in the network. In any closed circuit of conductors forming part of any linear system, the sum of the products

[1] *Pogg. Ann.*, bd. 72, 1847, and *Ges. Abh.*, p. 22.

obtained by multiplying the current in each part, taken in order round the circuit by its resistance, is equal to the sum of the electromotive forces in the circuit.

Thus let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the currents in m wires forming a complete circuit, and having resistances $r_1, r_2, \ldots, r_m$, and ΣE be the sum of the electromotive forces which have their seat in these conductors, we have

$$\gamma_1 r_1 + \gamma_2 r_2 + \ldots + \gamma_m r_m = \Sigma E \quad . \quad . \quad . \quad (15)$$

To prove this, consider equation (5) above applied to each conductor. Let V_1 be the potential at the first point of the circuit, which we shall suppose to be the initial point of the conductor of resistance r_1, V_2 the potential of the final point of this conductor, and the initial point of the conductor of resistance r_2, and so on. We suppose here, and throughout what follows on this subject, for simplicity (though without losing generality by so doing), that there are no electromotive forces just at the junctions so that the potential at each has a perfectly definite value. The first conductor gives

$$\gamma_1 r_1 = V_1 - V_2 + E_{12},$$

the second,

$$\gamma_2 r_2 = V_2 - V_3 + E_{23},$$

and so on. Adding these equations for the m conductors of the circuit we obtain (15), since the Vs disappear.

Examples. (1) Two Points connected by Conductors in Parallel. Conductance. Resistance and Conductance of Parallel Conductors. Specific Resistance and Conductivity

226. Taking first one or two simple examples, we shall now apply these principles to obtain some useful results. Consider the arrangement shown in Fig. 58. Let the point A be at potential V_A, the point B at potential V_B, and suppose that there is no electromotive force in the conductors to be taken into account. The currents from A to B by the wires of resistance, r_1, r_2, r_3, respectively, are $(V_A - V_B)/(r_1, r_2, r_3)$. Thus the total current is

$$\gamma = (V_A - V_B)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right).$$

If R be the resistance of a wire which, with the same difference of potential between A and B, might be substituted for the triple arc between A and B without altering the total current, we have $\gamma = (V_A - V_B)/R$, and therefore

$$\left.\begin{aligned} \frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\ \text{or} \qquad R &= \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The reciprocal of the resistance R of a wire, that is $1/R$, is called its *conductance*. The last equation, therefore, affirms that the conductance of a wire which, in the sense above indicated, is equivalent to the three conductors of conductances $1/r_1$, $1/r_2$, $1/r_3$, is equal to the sum of their conductances. The resistance of this wire is equal to the product of the three resistances divided by the sum of the different products which can be formed from the three resistances by taking them two at a time. These theorems are capable of obvious generalisation, with the following result: the conductance of a conductor, which is equivalent to any number n of distinct conductors joining two points, A, B, of a linear circuit, is the sum of the conductances of the separate conductors; the resistance of the equivalent conductor is equal to the product of the n resistances of the arcs divided by the sum of all the different products which can be formed by taking these resistances $n-1$ at a time.

If R be the resistance of a wire l centimetres long and of cross-section a square centimetres, the quantity Ra/l is called its *specific resistance*. The reciprocal of this is called the *conductivity*.

(2) Bridge Arrangement

227. As another example consider the arrangement shown in Fig. 59, and suppose that a single electromotive force, E, exists in the

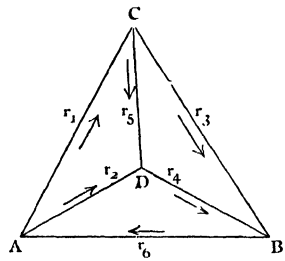


FIG. 59.

conductor of resistance, r_6. By the principle of continuity we get from the three points, A, C, D, the equations

$$\gamma_6 = \gamma_1 + \gamma_2,\ \gamma_3 = \gamma_1 - \gamma_5,\ \gamma_4 = \gamma_2 + \gamma_5 \quad . \quad . \quad . \quad (17)$$

where $\gamma_1\ \gamma_2, \ldots$ denote the currents in the wires of resistances $r_1, r_2, \ldots$ respectively, when the directions are as indicated in the diagram.

Applying the second principle to the circuits $BACB$, $ACDA$, $CBDC$, and noting that there is no electromotive force in any of these circuits except the first, we obtain by (15) and (17) the relations

$$\left.\begin{aligned} \gamma_1(r_1 + r_3 + r_6) + \gamma_2 r_6 - \gamma_5 r_3 &= E \\ \gamma_1 r_1 - \gamma_2 r_2 + \gamma_5 r_5 &= 0 \\ \gamma_1 r_3 - \gamma_2 r_4 - \gamma_5(r_3 + r_4 + r_5) &= 0 \end{aligned}\right\} \quad . \quad (18)$$

These give for the current in CD or γ_5,

$$\gamma_5 = \frac{E(r_2r_3 - r_1r_4)}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where
$$D = r_5r_6(r_1 + r_2 + r_3 + r_4) + r_5(r_1 + r_3)(r_2 + r_4) + r_6(r_1 + r_2)(r_3 + r_4) + r_1r_3(r_2 + r_4) + r_2r_4(r_1 + r_3) \quad (20)$$

The second two of (18) give

$$\gamma_5 = \frac{\gamma_6(r_2r_3 - r_1r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad . \quad (21)$$

It is interesting to notice that if R be the single resistance equivalent to the five resistances r_1, r_2, r_3, r_4, r_5 connecting A, B as shown in the diagram $\gamma_6 = E/(r_6 + R)$. Hence by (19) and (21) we obtain

$$R = \frac{r_5(r_1 + r_3)(r_2 + r_4) + r_1r_3(r_2 + r_4) + r_2r_4(r_1 + r_3)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad (22)$$

The arrangement here discussed is the well-known "bridge" arrangement of conductors used in the comparison of resistances. We have many examples of its use in what follows.

Analytical Treatment of a General Network

228. It is not difficult to deal with the problem of a network of linear conductors by an analytical method, but the main results are more instructively obtained by simple physical considerations. The chief steps of the analytical process are as follows, and may be fully worked out by the reader. Consider a set of n points, every one of which is directly connected with every other by a single conductor, the resistance and electromotive force in which are known. This will include all cases, as if one point is in reality connected with another by more than one conductor, these can be reduced to a single conductor of equivalent resistance carrying a current equal to the sum of the currents in the separate conductors; and if there be two points which are not directly connected a wire joining them can be imagined in which the current is zero. It is supposed, as before, that no electromotive forces exist *at* the points of meeting of the conductors.

Since there are n points of meeting, and $n-1$ conductors radiating from each there are $\frac{1}{2}n(n-1)$ distinct conductors. The n points give, by the condition of continuity, $n-1$ independent equations connecting the currents in the n conductors, which suffice to determine the $n-1$ differences of potential between them. For example, if we indicate the points by suffixes $1 \ldots n$ applied to the symbols for the various quantities, and denote by γ_{hk} the current along the wire connecting the point indicated by the suffix h with that indicated by the suffix k, we have as a type of these $n-1$ equations,

$$\gamma_{h1} + \gamma_{h2} + \ldots + \gamma_{hk} + \ldots + \gamma_{hn} = 0 \quad . \quad . \quad (23)$$

But by (5) $\gamma_{hk} = K_{hk}(V_h - V_k + E_{hk})$ where K_{hk} is the conductance of the wire joining the two points, and therefore the last equation becomes

$$\sum_{k=1}^{k=n} K_{hk}(V_h - V_k + E_{hk}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where the summation is taken for all integral values of k from 1 to n except h. If we write

$$-K_{hh} = K_{h1} + K_{h2} + \ldots + K_{h(h-1)} + K_{h(h+1)} + \ldots + K_{hn}$$

we shall have for the last equation

$$K_{h1}V_1 + K_{h2}V_2 + \ldots + K_{hh}V_h + \ldots + K_{hn}V_n = K_{h1}E_{h1} + \ldots + K_{hn}E_{hn} = \Phi_h \quad . \quad . \quad . \quad . \quad (25)$$

and there are $n-1$ such equations to be obtained by putting $1, \ldots n$ in succession for h.

So far we have considered a system complete in itself; but it is convenient sometimes to consider a system of conductors which receives current from without at definite points. We may therefore suppose that the system under consideration receives at the points $1, \ldots n$, electricity at the respective rates $Q_1, Q_2, \ldots, Q_n$. These rates must fulfil the relation

$$Q_1 + Q_2 + \ldots + Q_n = 0,$$

since the state of the system is supposed to be steady. The introduction of electricity at the rate Q_h at the point h will modify equation (25) so that we shall have instead of (25)

$$K_{h1}V_1 + K_{h2}V_2 + \ldots + K_{hh}V_h + \ldots + K_{hn}V_n = K_{h1}E_{h1} + \ldots + K_{hn}E_{hn} - Q_h = \Phi'_h \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If in (25) or (26) we put $V_n = 0$, the other quantities, $V_1, V_2, \ldots, V_{n-1}$, will become the excesses of the potentials at the different points above that at the point n, and the solutions of the $n-1$ equations of the form (25) or (26) will give these differences of potential in terms of the electromotive forces and the conductances. The values thus found for $V_1, V_2, \ldots$ being then substituted in the $\frac{1}{2}n(n-1)$ equations of the form (5) will give the currents in the conductors.

Solution of Equations. First Reciprocal Theorem. Conjugate Conductors. Second Reciprocal Theorem

229. One or two important results are easily obtained. First it is to be observed that, since $K_{hk} = K_{kh}$, the determinant of the system of equations (25) or (26) is symmetrical, so that the relations $\Delta_{hk} = \Delta_{kh}$ hold for its minors. Also the potentials V_h, V_k of any two points are given by the equations

$$V_h = \frac{\Sigma(\Delta_{gh}\Phi'_g)}{\Delta}, \quad V_k = \frac{\Sigma(\Delta_{gk}\Phi'_g)}{\Delta} \quad . \quad . \quad . \quad . \quad (27)$$

obtained from (26). The summations are taken for all integral values of g from 1 to $n-1$. The quantities, Q, may be, of course, all or any of them zero, so that the equations just written down include also the case of a self-contained system.

We obtain by (15)

$$\gamma_{hk} = K_{hk}\left\{\frac{\Sigma(\Delta_{gh} - \Delta_{gk})\Phi'_g}{\Delta} + E_{hk}\right\} \quad . \quad . \quad . \quad (28)$$

and similarly

$$\gamma_{lm} = K_{lm}\left\{\frac{\Sigma(\Delta_{gl} - \Delta_{gm})\Phi'_g}{\Delta} + E_{lm}\right\} \quad . \quad . \quad . \quad (29)$$

Now E_{lm} does not appear except in Φ'_l and Φ'_m, so that

$$\frac{\partial\gamma_{hk}}{\partial E_{lm}} = \frac{K_{hk}}{\Delta}\left\{(\Delta_{lh} - \Delta_{lk})\frac{\partial\Phi'_l}{\partial E_{lm}} + (\Delta_{mh} - \Delta_{mk})\frac{\partial\Phi'_m}{\partial E_{lm}}\right\}$$

$$= \frac{1}{\Delta}(\Delta_{lh} + \Delta_{mh} - \Delta_{lk} - \Delta_{mk})K_{hk}K_{lm} = \frac{\partial\gamma_{lm}}{\partial E_{hk}} \quad . \quad (30)$$

since $\Delta_{lh} = \Delta_{hl}$, &c.

This result expressed in words is the theorem, that if a given increase of the electromotive force E_{hk}, existing in the conductor hk, produce a certain increase of current from l to m in the conductor lm, an equal increase in the electromotive force E_{lm} in the conductor lm, will produce the same increase in the current from h to k in the conductor hk.

If the existence of an electromotive force in one of two of the conductors of the network does not affect the current in the other, this relation is also reciprocal, and the conductors are said to be conjugate. The analytical condition for conjugacy of the conductors is

$$\Delta_{lh} + \Delta_{mh} = \Delta_{lk} + \Delta_{mk} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Let us suppose that a quantity of electricity Q_h flows into the system per unit of time at the point h. Then the parts of V_l and V_m which depend on Q_h are $-Q_h\Delta_{hl}/\Delta$ and $-Q_h\Delta_{hm}/\Delta$ respectively. These show by their form that the effect on the potential of the point l, for example, produced by the flow at rate Q_h into the system at the point h is equal to the effect on the potential at h produced by an equal flow into the system at l.

The part of $V_l - V_m$ which depends on the entrance of electricity at rate Q_h at h is therefore $Q_h(\Delta_{hm} - \Delta_{hl})/\Delta$. Similarly the part of $V_l - V_m$ which depends on an outward flow Q_h at k is $-Q_h(\Delta_{km} - \Delta_{kl})/\Delta$. The total effect on $V_l - V_m$ is therefore $Q_h(\Delta_{hm} - \Delta_{km} - \Delta_{hl} + \Delta_{kl})/\Delta$. Obviously, since $\Delta_{hm} = \Delta_{mh}$, &c., this is equal to the part of $V_h - V_k$ which would arise from an equal inward rate of flow at l and outward flow at m.

Cycle Method for a Network

230. The following method, due to Clerk Maxwell, may in some cases be conveniently adopted for a network of conductors. The network is considered as made up of a series of meshes or cells in which each individual conductor, except those forming the outer edge of the network, is common to two meshes. A current is supposed to circulate round each mesh in the same direction, so that the actual current in each conductor is the difference of the current round two adjoining meshes. Thus each mesh is a closed circuit with its own current in it. Let γ be the current in any mesh, R the resistance, E the electromotive force, in its circuit, γ', γ'', . . . currents in adjoining meshes which have conductors in common with the mesh-circuit under consideration, r', r'', . . . the resistances of these conductors. Then we obtain at once for the mesh the equation

$$\gamma R - \gamma' r' - \gamma'' r'' - \ldots = E \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The reader may, as an example, apply this method to the bridge arrangement.

Activity in a Network of Conductors

231. Consider the sum $\Sigma V_h(\gamma_{h_1}+\gamma_{h_2}+ \ldots +\gamma_{h_n})$ taken for the n points of meeting. By the principle of continuity each term of this form is zero. But since $\gamma_{hk}=-\gamma_{kh}$ it is clear that the same quantity may be written $\Sigma\gamma_{hk}(V_h-V_k)$ where now the summation is taken for the $\frac{1}{2}n(n-1)$ distinct conductors. This is the electrostatic energy spent in the network, and we have just seen that its value is zero. It follows by (24) that

$$\Sigma R_{hk}\gamma_{hk}^2 = \Sigma E_{hk}\gamma_{hk} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

that is, the rate at which the energy spent in heat in the conductors is equal to the rate at which energy is furnished by the internal electromotive forces.

By equation (28) the currents are expressed as linear functions of the electromotive forces. Hence by substituting for γ_{hk} on the right of (33) the corresponding linear expression, the rate at which work is spent in heat may be expressed as a homogeneous quadratic function of the electromotive forces. On the left it is already expressed as a homogeneous quadratic function of the currents, in fact, as the sum of the products of the squares of the currents in the different conductors by the corresponding resistances, so that every term is essentially positive.

It must be observed that while the quantity on the left of (33) always gives the rate of expenditure of energy in producing heat in any conductors to which the summation is applied, whether these form the whole system or not, the summation on the right embraces all the electromotive forces concerned, and that if the equation is applied every conductor in which heat is produced must be included on the left.

Currents fulfilling Ohm's Law give Minimum Dissipation of Energy in Heat

232. We can show that the rate at which energy is spent in heat in a network of conductors is a minimum, if the currents while fulfilling the law of continuity, and producing heat according to Joule's law, each satisfy the equation $\gamma_{hk} = (V_h - V_k + E_{hk})/R_{hk}$. Let γ_{hk} have this value, and let the actual current be $\gamma'_{hk} = \gamma_{hk} + \xi_{hk}$, so that

$$\gamma'_{hk} = \frac{V_h - V_k}{R_{hk}} + \frac{E_{hk}}{R_{hk}} + \xi_{hk}.$$

Then we have

$$\Sigma\gamma'^2_{hk}R_{hk} = \Sigma\gamma'_{hk}(V_h - V_k) + \Sigma\gamma'_{hk}E_{hk} + \Sigma\gamma'_{hk}\xi_{hk}R_{hk},$$

in which the sum is taken for all the conductors of the system. The first term on the right vanishes, since it may be written $\Sigma V_h(\gamma'_{h_1} + \gamma'_{h_2} + \ldots + \gamma'_{h_n})$, in which the summation is taken with reference to all the points of meeting of the network, that is for all values of h from 1 to n, and the currents γ'_{hk} fulfil the law of continuity. Again, the last term may be written $\Sigma(\gamma_{hk} + \xi_{hk})\xi_{hk}R_{hk}$. But $\Sigma\gamma_{hk}\xi_{hk}R_{hk}$ may be written $\Sigma\xi_{hk}(V_h - V_k)$, which vanishes like the first term, since the currents ξ_{hk} fulfil the law of continuity. Thus for the activity spent in heat we have

$$\Sigma\gamma'^2_{hk}R_{hk} = \Sigma\gamma'_{hk}E_{hk} + \Sigma\xi^2_{hk}R_{hk}. \quad . \quad . \quad . \quad . \quad (34)$$

that is, the activity thus spent exceeds the rate at which work is done by the electromotive forces by the positive quantity $\Sigma\xi^2_{hk}R_{hk}$, which is the activity that would be spent in heat by the system of difference-currents ξ_{hk}. If ξ_{hk} be zero we fall back on the result already demonstrated.

Elementary Discussion of Network of Conductors

233. Most of the results obtained above with respect to a network of conductors can be obtained by elementary physical considerations. It will be instructive to treat the subject shortly in this way.

It is an easy inference from fundamental principles, and it can easily be verified by experiment, that the currents in the different parts of a system of conductors are not altered by connecting any two points, which are at different potentials, by a wire which contains an electromotive force equal and opposite to the difference of potential. The wire, before being put in position, will have the same difference of potential between its extremities as there is between the two points of the network under consideration. If then the end of the wire, which is at the lower potential, be joined to the point of lower potential the other extremity of the wire will have the potential of the other point, and may be made coincident with that point without changing the state

of the system. The new system obtained satisfies everywhere the principle of continuity, and equation (5) or, what comes to the same, the second relation formulated by Kirchhoff (Art. 224).

Again, it is easy to see that if an electromotive force in one conductor A in a linear system produces no current in another conductor B of the system, either conductor may be removed without affecting the current in the other. For if for example A were removed, the potentials at the points of the system at which it was attached would be altered. Let then an electromotive force equal and opposite to that difference of potential be placed in A; no current will flow in A, and the presence or removal of the conductor, after this has been done, will not affect the system. But it has been shown above (and it will be proved again presently in an elementary manner) that if an electromotive force in A produce no current in B, an electromotive force in B can produce no current in A. Hence, B can be removed also without affecting the current in A.

Reduction of Network to Bridge Arrangement. First Reciprocal Theorem. Conjugate Conductors. Second Reciprocal Theorem

234. Let A, B, C, D be four points of meeting in a network of conductors, such that besides any other connection there may be between the points A, B, a distinct wire between these points exists, and similarly for C, D, and let AB be the only conductor in which there exists an electromotive force. The network can be reduced to a system of six conductors arranged as in Fig. 59 and such that the wires, AB, CD and the currents in them remain unchanged. For currents will enter any one mesh of the network at certain points and leave it at certain other points. One of the former will be the point of maximum potential, one of the latter the point of minimum potential. The circuit formed by the mesh consists of two parts joining these points, and to any point in one of these parts will correspond a point of the same potential in the other part. Every point in one may then be supposed in coincidence with points of the same potential in the other; that is, the mesh may be replaced by a single wire joining the two points, and such that the currents entering or leaving it by wires joining it to the rest of the system are not affected by the change.

Clearly the current will enter the system at one extremity A of the wire AB and leave it at the other extremity B. Thus A and B are the points of meeting of the network which are at the highest and lowest potential respectively. Thus the meshes of the system can be reduced one after the other to two single wires, while CD is kept unaltered, until the network has been reduced to two meshes, one on each side of CD, connected by single wires to A, B respectively. Each mesh and connecting wire can be replaced by two wires joining A, or B as the case may be, with CD, and the whole system is reduced to an equivalent system of the form shown in Fig. 59.

Let now the electromotive force hitherto supposed acting in AB be transferred to CD while the resistances r_5, r_6 are maintained unchanged. The value of γ_6 will be got from (21) by simply interchanging r_5 and r_6, r_1+r_2 and r_1+r_3, r_3+r_4 and r_2+r_4, in D. But these interchanges leave D unaltered, and thus the new value of γ_6 is the same as the old value of γ_5. Hence an electromotive force which, placed in the conductor AB, produces a current in the conductor CD, will, if placed in AB, produce an equal current in AB.

The distribution of currents and electromotive forces in the system which has been supposed above, in order that the reduction described might be effected, may be superimposed on any other distribution which is possible, and the conclusion which has been reached will not be affected by the latter. Thus we have the general proposition, stated in Art. 229 above, from which the definition of conjugacy of two conductors is obtained as before.

235. Again, the five conductors AC, AD, BC, BD, CD in Fig. 59 may be regarded as the reduced equivalent of a network of conductors which a current γ_6 enters at A and leaves at B. The difference of potential between C and D $V_C - V_D$ is $\gamma_5 r_5$. But by (21)

$$\gamma_5 r_5 = \frac{\gamma_6 r_5 (r_2 r_3 - r_1 r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)}.$$

The resistance between the points C, D of the system of five conductors, is

$$\frac{r_5(r_1 + r_2)(r_3 + r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)},$$

and if a current of amount γ_6 were to enter at C and leave at D, the difference of potential between C and D would be the product of this expression by γ_6. The product multiplied by $r_1/(r_1+r_2)$ gives the difference of potential between C and A, and multiplied by $r_3/(r_3+r_4)$ gives the difference of potential between C and B. Hence the difference of potential between A and B is the difference of these products, or

$$V_A - V_B = \frac{\gamma_6 r_5 (r_2 r_3 - r_1 r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)},$$

the value given in (35) for the difference of potential $V_C - V_D$.

Hence, generalising as before, we have the following theorem, already proved in Art. 229 above. If a difference of potential $V_C - V_D$ between two points C, D of a linear system arise from a current entering and leaving at two other points A, B respectively, a difference of potential $V_A - V_B = V_C - V_D$ between the points A, B will arise from an equal current entering at C and leaving at D.

Effect of Joining a Wire between Two Points of a Network of Conductors

236. The following result is easily proved, and is frequently useful. If the potentials at two points A, B, of a linear system of conductors containing any electromotive forces, be V, V' respectively, and R be the equivalent resistance of the system between these two points, then if a wire of resistance, r, be added, joining AB, the current in the wire will be $(V-V')/(R+r)$. In other words the linear system, so far as the production of a current in the added wire is concerned, may be regarded as a single conductor of resistance R connecting the points AB and containing an electromotive force of amount $V-V'$. For let the points A and B be connected by a wire of resistance r, containing an electromotive force of amount $V-V'$ opposed to the difference of potential between A and B, no current will be produced in the wire, and no change will take place in the system of conductors. Now imagine another state of this latter system of conductors in which an equal and opposite electromotive force acts in the wire between A and B, and there is no electromotive force in any other part of the system. A current of amount $(V-V')/(R+r)$ will flow in the wire. Now let this state be superimposed on the former state, the two electromotive forces in the wire will annul one another, and the current will be unchanged. The potentials at different points, and the currents at different parts, of the system, will be the sum of the corresponding potentials and currents in the two states, and will therefore, in general, differ from those which existed before the addition of the wire.

As an example consider a circuit between two points of which there is a difference of potential V, and let r, r' be the resistances of the two parts of the circuit between the two points. (These two parts may of course be any two networks of conductors joining the points.) Then the equivalent resistance is $rr'/(r+r')$; and if another conductor of resistance R, and not containing any electromotive force, be connected between the two points, the new difference of potential V' will be given by

$$V' = V \frac{R}{R + \dfrac{rr'}{r+r'}} \quad \ldots\ldots\ldots \quad (35)$$

since it has been shown above that V'/R is the current in the conductor.

Hence in order that V' may be approximately equal to V, R must be great in comparison with $rr'/(r+r')$. But $rr'/(r+r')$ can be written in either of the forms $r/(1+r/r')$, $r'/(1+r'/r)$, which shows that r and r' are each greater than $rr'/(r+r')$. Hence if R be great in comparison with *either* of the two resistances r, r', V' will be approximately equal to

V, no matter how the electromotive force may be situated in the circuit. This result is useful in connection with the measurement of the difference of potential between two points of a circuit by a galvanometer, as it is only necessary to make the resistance in the circuit of the instrument great in comparison with that of either part of the circuit lying between the two points, to be sure that the difference of potential is practically unaltered by the application of the instrument.

CHAPTER VII

GENERAL DYNAMICAL THEORY

Generalised Co-ordinates and Velocities. Kinetic Energy

237. The application of general dynamical principles to the discussion of the properties of a system of currents or magnets, or of both combined, is due mainly to Clerk Maxwell, and is one of the chief peculiarities of his treatise on Electricity and Magnetism. Further progress has been made in this direction, and much light has been thrown on electromagnetic action by means of mechanical analogues which the dynamical method has suggested, and by the attempts which have been made to obtain some working dynamical hypothesis to explain the ether phenomena lately forced upon the closer attention of natural philosophers by the electromagnetic theory of light.

As it would be impossible without this dynamical treatment to present the theories which it is one of the principal objects of this book to give some account of, and as the discussions of general dynamical theories in treatises specially devoted to abstract dynamics have not in general such applications as the present in view, and are therefore for the ordinary student a little difficult to translate into electrical language, we devote a chapter here to such general methods as we shall require in what follows.

238. It is to be noticed in the first place that the law of conservation of energy is not sufficient by itself to enable us to find the mutual actions or relations of the different parts of a system—that is, to find a dynamical explanation of the phenomena. We require in addition a general dynamical process, from which, under certain assumptions to be stated, and as far as possible justified, these relations can be deduced. Such processes are furnished by the principle of Least and Stationary Action due to Maupertuis and Hamilton, and Lagrange's dynamical method. The generality of the former method is so great as to render its application to every type of dynamical problem a matter of some difficulty; but, as we shall see, it leads at once to the dynamical equations of Lagrange, which in many cases, not easily attacked by ordinary dynamics, give a ready means of solution.

239. First let us consider a system of bodies which are subject to certain kinematical conditions expressed by equations connecting the co-ordinates of the different particles of the system. These equations limit the freedom of the particles of the system and enable just as many of the co-ordinates to be determined in terms of the remaining co-ordinates as there are independent equations of condition. Thus if there are $3n$ co-ordinates, and m independent equations connecting them, there are in all $3n-m$ independent co-ordinates, or parameters, from which the position at any instant of any part of the system can be deduced. The system is then said to have $3n-m$ degrees of freedom.

As an example consider the motion of a rigid body. The kinematical conditions imposed on it render impossible any alteration of the relative configuration of the particles composing it. But any point of it is free to move in any one of three rectangular directions, or the body may turn round any one of three rectangular axes through any point. Thus the body has left by the condition six degrees of freedom; or, in other words, the position of the body is completely determined when there are given in position a point in the body, a line through the point, and a plane containing the line. First fix the point, this requires three co-ordinates; next fix the line, this requires two co-ordinates; last fix the plane, which requires one more co-ordinate. Thus there are six independent co-ordinates in all. If the point is constrained to remain in a fixed plane one freedom is removed, as two of the co-ordinates of the point, with the other three co-ordinates determine the position of the body; and so on.

240. To the term "co-ordinate" then we give, following Lagrange, an enlarged or generalised meaning. The co-ordinates are the $3n-m$ independent parameters, the variations of which give the motion of the system. They may be either $3n-m$ of the ordinary position co-ordinates in terms of which the remainder may be found by the m equations of condition already referred to; or they may be $3n-m$ other quantities $\psi, \phi, \chi, \ldots$, connected with the ordinary position co-ordinates by a set of $3n-m$ relations, making $3n$ known relations of condition in all.

In the former case we may write the m kinematical equations in the form—

$$\left.\begin{aligned} F_1(x_1, y_1, z_1, \ldots\ldots) &= 0 \\ F_2(x_1, y_1, z_1, \ldots\ldots) &= 0 \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

in the latter case the $3n$ equations are

$$\left.\begin{aligned} x_1 &= f_1(\psi, \phi, \chi, \ldots\ldots) \\ y_1 &= f_2(\psi, \phi, \chi, \ldots\ldots) \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

Thus we have either as in (2) $6n-m$ unknown quantities $x_1, y_1, \ldots \psi, \phi, \ldots$ with $3n$ kinematical equations, or as in (1) $3n$ unknown quantities $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, connected by m kinematical equations: in either

case the remaining $3n-m$ relations required for the determination of the unknown quantities are furnished by the equations of motion. These it is our main object now to establish.

It is to be observed that if (1) or (2) involve the time t explicitly the kinematical relations vary with the time, and the results obtained in the following analysis will not in general hold. Where the contrary is the case will be stated. If the time does not enter explicitly in these equations the kinematical conditions are said to be invariable.

241. Denoting time-rates of change of co-ordinates, or *velocities* in the ordinary and in the generalised sense, by the fluxional notation, $\dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{\psi}, \dot{\phi}, \dot{\chi}, \ldots$ we get from (2) the following—

$$\left.\begin{aligned} \dot{x}_1 &= \frac{\partial x_1}{\partial \psi}\dot{\psi} + \frac{\partial x_1}{\partial \phi}\dot{\phi} + \ldots \\ \dot{y}_1 &= \frac{\partial y_1}{\partial \psi}\dot{\psi} + \frac{\partial y_1}{\partial \phi}\dot{\phi} + \ldots \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

It is to be clearly understood that equations (2) connect $x_1, y_1, z_1, \ldots$ $\psi, \phi, \ldots$ without involving $\dot{\psi}, \dot{\phi}, \ldots$ It follows from (3) therefore that the kinetic energy $T\{=\frac{1}{2}\Sigma m(\dot{x}^2+\dot{y}^2+\dot{z}^2)\}$ is a homogeneous quadratic function of the generalised velocities $\dot{\psi}, \dot{\phi}, \ldots$ Thus we may write it

$$T = \tfrac{1}{2}\{(\psi, \psi)\dot{\psi}^2 + (\phi, \phi)\dot{\phi}^2 + \ldots + 2(\psi, \phi)\dot{\psi}\dot{\phi} + \ldots\} \quad (4)$$

in which $(\psi, \psi), \ldots(\psi, \phi)\ldots$ denote co-efficients which are functions of the co-ordinates and masses only, since $\partial x_1/\partial\psi, \ldots$ are such functions.

Theory of Action

242. The *action* of a system for the part of the motion which takes place in any interval of time from t_0 to t_1 is double the time-integral of the kinetic energy, or

$$A = 2\int_{t_0}^{t_1} T dt \quad \ldots\ldots\ldots \quad (5)$$

Since $2Tdt = \Sigma m\dot{s}ds$, this becomes

$$A = \Sigma\int_{s_0}^{s_1} m\dot{s}\, ds \quad \ldots\ldots\ldots \quad (6)$$

where s_0, s_1, are limits of s corresponding to t_0, t_1. Hence the action may be defined as the sum of the space-integrals of the momenta of the different particles of the system, or, which is the same thing, the sum of the products of the space average of the momentum of each particle by the length of the path described in the interval t_1-t_0.

According to the condition imposed on the system, there are two

theorems of stationary action: one, which is generally referred to as the Principle of Least Action,[1] for which the condition imposed is that the system should move from one specified configuration to another with constant sum $T + V$ of kinetic and potential energies; the other an analogous theorem, subject to the condition that the time of motion, not the total energy, is given. These theorems, expressed analytically, are that

(*a*)
$$\delta A = 2\delta\int_{t_0}^{t_1} T dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

with the condition

$$\delta T + \delta V = 0$$

and (*b*)

$$\delta\int_{t_0}^{t_1} (T - V) dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

with the condition

$$\delta t_1 - \delta t_0 = 0$$

provided the motion take place in accordance with the ordinary equation of motion for a system in which the forces are derivable by differentiation from a potential V. This equation is

$$\delta V + \Sigma m\dot{s}\delta s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It is to be noticed that in both these theorems, if the time do not appear explicitly in the kinematical equations, the sum of the kinetic and potential energies is a constant throughout any one mode of transition from one configuration to the other (see Art. 260); but that (when this is the case), while in (*a*) the energy is the same for all modes of transition, in (*b*) it may vary from one mode to another.

It is usual to denote $T - V$ by L and $\int_{t_0}^{t_1} L\, dt$ by S. S was called by Hamilton the *principal function* and A the *characteristic function* of the motion.

243. To prove theorem (*a*), let δA be the difference between the action for one set of paths of transition and that of another set infinitely little different. We have

$$\delta A = \Sigma\int_{s_0}^{s_1} m\delta\dot{s} ds + \Sigma\int_{s_0}^{s_1} m\dot{s} d\delta s$$

$$= \int_{t_0}^{t_1} \delta T \,.\, dt + \left[\Sigma m\dot{s}\delta s\right]_{s_0}^{s_1} - \Sigma\int_{t_0}^{t_1} m\dot{s}\delta s \,.\, dt$$

since

$$\Sigma\int_{s_0}^{s_1} m\delta\dot{s} ds = \Sigma\int_{t_0}^{t_1} m\dot{s}\delta\dot{s} \,.\, dt = \int_{t_0}^{t} \delta T \,.\, dt.$$

[1] For examples of the direct application of the Principle of Least Action to Dynamical Problems see a paper by Larmor, *Proc. L.M.S.*, xv., 1884, p. 158.

The integrated terms vanish, since the limiting configurations are the same for all sets of paths. Hence

$$\delta A = \int_{t_0}^{t_1} \{\delta T' - \Sigma m s \delta s\} dt \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But by the condition of constancy of energy we have

$$\delta T' + \delta V = 0$$

and by this the last result can be written

$$\delta A = -\int_{t_0}^{t_1} \{\delta V + \Sigma m s \delta s\} dt \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and δA vanishes if

$$\delta V + \Sigma m s \delta s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as stated above.

It is to be observed that this result shows that if the motion takes place according to the general variational equation of motion—that is, if the motion is unguided by any constraint applied to the system from without—the variation of the action from one mode of transition to another mode very near the first is of the second order of small quantities —that is, the action is stationary. The action can, however, be made as great as we please by rendering the path of transition for each particle sufficiently circuitous, so as to increase the time of motion. If then the variational equation of motion lead to only one possible motion, that must be a motion for which A is a minimum. When there are more than one possible motion none of these can give an absolute maximum of action, though some of them may involve neither maximum nor minimum action.[1]

244. The second theorem (*b*) may be proved thus—

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (\delta T' - \delta V) dt$$

since, the time being constant, there is no variation of dt. But

$$\int_{t_0}^{t_1} \delta T' . dt = \Sigma \int_{t_0}^{t_1} m \dot{s} \delta \dot{s} . dt = \Sigma \int_{t_0}^{t_1} m \dot{s} \frac{d\delta s}{dt} dt,$$

and therefore

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (\Sigma m \dot{s} \frac{d\delta s}{dt} - \delta V) dt \quad . \quad . \quad . \quad . \quad (12)$$

Hence, finally, integrating the first of the terms in brackets by parts, and remembering that the integrated terms vanish since the initial and final configurations are fixed, we get

$$\delta \int_{t_0}^{t_1} L dt = -\int_{t_0}^{t_1} (\Sigma m s \delta s + \delta V) dt \quad . \quad . \quad . \quad . \quad (13)$$

[1] For the discrimination of cases of minimum and stationary action see Jacobi, *Crelle*, Bd. 17, 1837, or Werke, Bd. 4, s. 46; or Routh, *Adv. Rigid Dynamics*, p. 250.

The right-hand term vanishes, and $\delta S = 0$, when

$$\Sigma m\dot{s}\delta s + \delta V = 0,$$

the same condition as before.

When the time of transition is not given we have

$$\delta\int L dt = \int L d\delta t + \int \delta L dt$$

$$= \left[L\delta t\right]_{t_0}^{t_1} - \int \frac{dL}{dt}\delta t\, dt + \int (\delta T - \delta V)dt,$$

the integrals being taken for any chosen time, $t_1 - t_0$, of transition. Now

$$\delta T dt = \Sigma m\dot{s}(d\delta s - \dot{s}d\delta t),$$

so that

$$\int \delta T dt = \Sigma m\dot{s}(\delta s - \dot{s}\delta t) - \Sigma\int m\ddot{s}(\delta s - 2\dot{s}\delta t)dt.$$

Also

$$\int \frac{dL}{dt}\delta t dt = \Sigma\int\left(m\ddot{s} - \frac{\partial V}{\partial s}\right)\dot{s}\delta t dt$$

and

$$\delta V = \Sigma \frac{\delta V}{\delta s}\delta s.$$

Hence, collecting terms, we find

$$\delta\int L dt = \left[L\delta t + \Sigma m\dot{s}(\delta s - \dot{s}\delta t)\right]_{t_0}^{t_1} - \Sigma\int_{t_0}^{t_1}\left(m\ddot{s} + \frac{\partial V}{\partial s}\right)\left(\delta s - \dot{s}\delta t\right)dt \quad (14)$$

At the limiting positions δs is zero, so that the integrated terms $\Sigma m\dot{s}\delta s$ vanish. Also if, as we suppose, the variational equation of motion hold, $m\ddot{s} + \delta V/\delta s = 0$, and we get

$$\delta S = \left[L\delta t - \Sigma m\dot{s}^2 . \delta t\right]_{t_0}^{t_1} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But, as will be shown below (Art. 260), if the kinematical equations do not contain the time explicitly the fulfilment of the variational equation of motion involves the constancy of the total energy $T + V$ during the motion. Hence in this case putting E for $T + V$ we have

$$\delta S = \delta\int L dt = - E\delta t \quad . \quad . \quad . \quad . \quad . \quad . \quad (15')$$

From this also the former result can be deduced. For let the energy be constant throughout any one mode of transition. Then, if t be any chosen value of the time of transition, we have

$$\int_0^t E dt = Et.$$

Hence

$$\delta\int(T+V)dt = E\delta t + t\delta E,$$

which by (15′) gives

$$\delta A = t\delta E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Thus δA vanishes if the energy is not subject to variation.

It is to be observed that $\delta s - \dot{s}\delta t$ is the variation which s undergoes when we pass from any point of the path considered to that point of an adjoining path which corresponds to the same instant of time. For in the passage from any chosen path to an adjoining one the change of s for an interval of time δt is δs, and therefore $s + \delta s - \dot{s}\delta t$ is the value of s in the new path for a point of time δt earlier, that is at time t. Hence the value of δS might have been obtained at once from (13) by inserting the integrated terms, writing $\delta s - \dot{s}\delta t$ for δs and adding the term $[L\delta t]$ taken between the limits t_1 and t_0. Of course it will be remembered that δt in (15′) is the variation of the whole time of transition.

Lagrange's Dynamical Equations

245. We shall now as a preliminary to a discussion of the ignoration of co-ordinates derive Lagrange's equations from the principle of least action. An independent proof of these equations by derivation from the Cartesian equations of motion of a set of free particles will be given later.

We have seen above that

$$\delta A = \Sigma\int m\delta\dot{s}ds + \Sigma\int m\dot{s}d\delta s$$

$$= \int\delta Tdt + \Sigma\int m\dot{s}d\delta s.$$

But by (4)

$$\Sigma m\dot{s}^2 = (\psi,\psi)\dot{\psi}^2 + 2(\psi,\phi)\dot{\psi}\dot{\phi} + \ldots. + (\phi,\phi)\dot{\phi}^2 + \ldots.,$$

so that

$$\Sigma m\dot{s}d\delta s = \frac{\partial T}{\partial\dot{\psi}}d\delta\psi + \frac{\partial T}{\partial\dot{\phi}}d\delta\phi + \ldots.$$

$$+ \left(\frac{\partial T}{\partial\psi}\delta\psi + \frac{\partial T}{\partial\phi}\delta\phi + \ldots.\right)dt \quad . \quad . \quad . \quad . \quad (17)$$

Also, if E be the total energy, $T = E - V$, so that for a conservative system

$$\delta T = -\left(\frac{\partial V}{\partial\psi}\delta\psi + \frac{\partial V}{\partial\phi}\delta\phi + \ldots.\right) \quad . \quad . \quad . \quad . \quad (18)$$

since V is a function of the co-ordinates only.

Hence, substituting from (17) and (18) in the value of δA and integrating by parts we obtain

$$\delta A = \frac{\partial T}{\partial \dot{\psi}}\delta\psi + \frac{\partial T}{\partial \dot{\phi}}\delta\phi + \ldots . \\ + \int \left\{ -\frac{\partial V}{\partial \psi} + \frac{\partial T}{\partial \psi} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} \right) \delta\psi + \ldots . \Big\} dt. \quad . \quad . \quad (19)$$

The integrated terms vanish, and if δA also vanishes we must have, in place of the variational equation of motion (9) above, since $\delta\psi$, $\delta\phi$, ... are independent,

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} + \frac{\partial V}{\partial \psi} = 0, \\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which are therefore the equations of motion of a conservative system in terms of generalised co-ordinates.

These equations were first given by Lagrange (*Mécanique Analytique*, Seconde Partie, Section IV.). We shall see later that if the system is not conservative they are subject to certain modifications.

If we put L for $T - V$ the equations may be written in the compact form

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} = 0 \\ &\quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20')$$

In this form they may very easily be deduced from the theorem (*b*) by a process similar to that used above.

Generalised Momenta. Reciprocal Theorems

246. The quantities $\partial T/\partial\dot{\psi}$, $\partial T/\partial\dot{\phi}$, . . . obtained by partial differentiation of the homogeneous function T of the velocities, are called the generalised components of momentum of the system, and we shall here denote them by the symbols $\xi, \eta, \zeta, \ldots$ They are evidently connected by the relation

$$\dot{\psi}\frac{\partial T}{\partial \dot{\psi}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \ldots . = 2T \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which we shall usually write in the less cumbrous form

$$\dot{\psi}\xi + \dot{\phi}\eta + \ldots . = 2T. \quad . \quad . \quad . \quad . \quad (22)$$

It is obvious that $\xi, \eta, \ldots$ are linear functions of the velocities as given in the equations

$$\left.\begin{aligned} (\psi, \psi)\dot{\psi} + (\psi, \phi)\dot{\phi} + \ldots &= \xi \\ (\psi, \phi)\dot{\psi} + (\phi, \phi)\dot{\phi} + \ldots &= \eta \\ \ldots\ldots\ldots\ldots & \end{aligned}\right\} \quad \ldots\ldots \quad (23)$$

which are independent and just as many as there are co-ordinates. By means of these $\dot{\psi}, \dot{\phi}, \ldots$ can be expressed as linear functions of the momenta $\xi, \eta, \ldots$ and hence T as a quadratic function of $\xi, \eta, \ldots$, the coefficients of which are functions of the coefficients of the system of equations (23), and are therefore functions only of the co-ordinates. Distinguishing these coefficients from those in (4) by square brackets we get

$$T = \tfrac{1}{2}\{[\psi, \psi]\xi^2 + 2[\psi, \phi]\xi\eta + \ldots + [\phi, \phi]\eta^2 + \ldots\} \quad (24)$$

We shall denote the kinetic energy by T_m or by T_v according as it is to be understood that it is expressed as in (24) or as in (4). We may therefore write (22) in the form

$$T_m = -T_v + \psi\xi + \dot{\phi}\eta + \ldots\ldots\ldots \quad (25)$$

Here T_m is the function obtained by supposing the velocities $\dot{\psi}, \dot{\phi}, \ldots$ on the right expressed in terms of $\xi, \eta, \ldots$ Hence we have

$$\begin{aligned} \frac{\partial T_m}{\partial \xi} &= -\left(\frac{\partial T_v}{\partial \dot{\psi}}\frac{\partial \dot{\psi}}{\partial \xi} + \frac{\partial T_v}{\partial \dot{\phi}}\frac{\partial \dot{\phi}}{\partial \xi} + \ldots\right) + \dot{\psi} \\ &\quad + \xi\frac{\partial \dot{\psi}}{\partial \xi} + \eta\frac{\partial \dot{\phi}}{\partial \xi} + \ldots \\ &= \dot{\psi}. \end{aligned}$$

We thus have the reciprocal equations

$$\left.\begin{aligned} \frac{\partial T_v}{\partial \dot{\psi}} &= \xi, \quad \frac{\partial T_v}{\partial \dot{\phi}} = \eta, \ldots \\ \frac{\partial T_m}{\partial \xi} &= \dot{\psi}, \quad \frac{\partial T_m}{\partial \eta} = \dot{\phi}, \ldots \end{aligned}\right\} \quad \ldots\ldots \quad (26)$$

Also we have

$$\frac{\partial T_m}{\partial \psi} + \frac{\partial T_m}{\partial \xi}\frac{\partial \xi}{\partial \psi} + \ldots = -\frac{\partial T_v}{\partial \psi} + \dot{\psi}\frac{\partial \xi}{\partial \psi} + \ldots$$

or

$$\frac{\partial T_m}{\partial \psi} = -\frac{\partial T_v}{\partial \psi}, \ldots\ldots\ldots \quad (27)$$

Ignoration of Co-ordinates. Modified Lagrangian Function

247. If the kinetic and potential energies be independent of certain co-ordinates $\chi, \chi', \ldots$ it is now obvious that the components of momen-

tum corresponding to these co-ordinates are constant. For $\partial T/\partial\chi$, $\partial T/\partial\chi'$, . . . , $\partial V/\partial\chi$, . . . are zero, and hence, by Lagrange's equations we have

$$\frac{d}{dt}\frac{\partial T'}{\partial\dot{\chi}} = 0, \quad \frac{d}{dt}\frac{\partial T'}{d\dot{\chi}'} = 0, \ldots.$$

$$\frac{\partial T}{\partial\dot{\chi}} = \kappa, \quad \frac{\partial T'}{\delta\dot{\chi}'} = \kappa', \ldots.$$

where $\kappa, \kappa', \ldots.$ are constants.

The last equations may be written in the form

$$\left.\begin{aligned}(\chi, \chi)\dot{\chi} + (\chi, \chi')\dot{\chi}' + \ldots. &= \kappa - \{(\psi, \chi)\dot{\psi} + (\phi, \chi)\dot{\phi} + \ldots.\} \\ (\chi', \chi)\dot{\chi} + (\chi', \chi')\dot{\chi}' + \ldots. &= \kappa' - \{(\psi, \chi')\dot{\psi} + (\phi, \chi')\dot{\phi} + \ldots.\} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \quad (28)$$

Our object is now to find what the theorem

$$\delta S = \delta\int L dt = 0,$$

with time of transition given, becomes when $\dot{\chi}, \dot{\chi}', \ldots.$ are eliminated by means of equations (28), in order to deduce therefrom the corresponding modified equations of motion. We shall not at first suppose that the momenta $\kappa, \kappa', \ldots.$ are necessarily constant.

Take the expression

$$\delta\int L dt = \int\left(\frac{\partial L}{\partial\psi}\delta\psi + \frac{\delta L}{\partial\dot{\psi}}\delta\dot{\psi} + \ldots. + \frac{\partial L}{\partial\chi}\delta\chi + \frac{\partial L}{\partial\dot{\chi}}\delta\dot{\chi} + \ldots.\right)dt;$$

write in it κ for $\partial L/\partial\dot{\chi}$, κ' for $\partial L/\partial\dot{\chi}', \ldots.$, and transpose, and the equation becomes

$$\delta\int\left(L - \kappa\dot{\chi} - \kappa'\dot{\chi}' - \ldots.\right)dt + \int\left(\dot{\chi}\delta\kappa + \dot{\chi}'\delta\kappa' + \ldots. - \frac{\partial L}{\partial\chi}\delta\chi - \frac{\partial L}{\partial\chi'}\delta\chi' - \ldots.\right)dt$$

$$= \int\left(\frac{\partial L}{\partial\psi}\delta\psi + \frac{\partial L}{\partial\dot{\psi}}\delta\dot{\psi} + \frac{\partial L}{\partial\phi}\delta\phi + \ldots.\right)dt \ldots \quad (29)$$

The expression on the right is the variation of S when $\psi, \dot{\psi}, \ldots.$ are varied, while $\chi, \dot{\chi}, \ldots.$ are left unchanged. Since the co-ordinates are independent, their variations are arbitrary, and this variation of S must vanish.

Hence (29) becomes

$$\delta\int(L - \kappa\dot{\chi} - \kappa'\dot{\chi}' - \ldots.)dt + \int(\dot{\chi}\delta\kappa - \frac{\partial L}{\partial\chi}\delta\chi + \dot{\chi}'\delta\kappa' - \ldots.)dt = 0 \quad (30)$$

For brevity we put

$$L' \equiv L - \kappa\dot{\chi} - \kappa'\dot{\chi}' - \ldots. \quad \ldots. \quad (31)$$

so that

$$L' = \tfrac{1}{2}(\xi\psi + \eta\phi + \ldots . - \kappa\dot{\chi} - \kappa'\dot{\chi}' - \ldots .) - V . \quad (31')$$

The first expression on the right is the difference between the kinetic energy depending on the momenta corresponding to the velocities $\dot{\psi}$, $\dot{\phi}$, ... and that due to the momenta κ, κ', ...

If now the co-ordinates χ, χ', ... do not appear explicitly in T or V, κ, κ', ... are constants, $\partial L/\partial\chi$, ... are zero. We have then

$$\delta\int L'dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

for the theorem expressed by (8) in which the time of transition is fixed. The equations of motion are therefore now obtained in the manner indicated above, but with L' substituted for L. Hence they are

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L'}{\partial\dot{\psi}} - \frac{\partial L'}{\partial\psi} &= 0 \\ \frac{d}{dt}\frac{\partial L'}{\partial\dot{\phi}} - \frac{\partial L'}{\partial\phi} &= 0 \\ \ldots\ldots & \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Lagrange's Equations with Gyrostatic Terms

248. It is to be observed in performing the operations indicated in equations (33) that κ, κ', ... are to be treated as constants, while $\dot{\chi}$, $\dot{\chi}'$, ... are explicit functions of $\dot{\psi}$, $\dot{\phi}$, ...

Now

$$T = \tfrac{1}{2}(\xi\dot{\psi} + \eta\dot{\phi} + \ldots . + \kappa\dot{\chi} + \kappa'\dot{\chi}' + \ldots .);$$

and if (κ, κ), (κ, κ') . . . . denote the ratios of the consecutive first minors of the determinant of the system of equations (28) to that determinant and

$$K \equiv \tfrac{1}{2}\{(\kappa, \kappa)\kappa^2 + 2(\kappa, \kappa')\kappa\kappa' + \ldots .\} \quad . \quad . \quad . \quad (34)$$

the values for $\dot{\chi}$, $\dot{\chi}'$, ... derived from (28) are

$$\left.\begin{aligned} \dot{\chi} &= \frac{\partial K}{\partial\kappa} - (\dot{\psi}M + \dot{\phi}N + \dot{\theta}O + \ldots .) \\ \dot{\chi}' &= \frac{\partial K}{\partial\kappa'} - (\dot{\psi}M' + \dot{\phi}N' + \dot{\theta}O' + \ldots .) \\ & \ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad . \quad . \quad (35)$$

where

$$\left.\begin{aligned} M &= (\kappa, \kappa)(\psi, \chi) + (\kappa, \kappa')(\psi, \chi') + \ldots . \\ M' &= (\kappa', \kappa)(\psi, \chi) + (\kappa', \kappa')(\psi, \chi') + \ldots . \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ N &= (\kappa, \kappa)(\phi, \chi) + (\kappa, \kappa')(\phi, \chi') + \ldots . \\ & \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad . \quad . \quad (36)$$

Hence, substituting in the value of T above, we get

$$T = \tfrac{1}{2}(\xi\dot{\psi} + \eta\dot{\phi} + \ldots - \dot{\psi}\Sigma_\kappa M - \dot{\phi}\Sigma_\kappa N - \ldots) + K$$
$$= T_1 + K, \text{ say} \quad \ldots \quad (37)$$

Hence

$$L' = T_1 + K - (2K - \dot{\psi}\Sigma_\kappa M - \dot{\phi}\Sigma_\kappa N - \ldots) - V$$
$$= T_1 + \dot{\psi}\Sigma_\kappa M + \dot{\phi}\Sigma_\kappa N + \ldots - K - V \quad \ldots \quad (37')$$

so that L' contains terms of the first degree in $\dot{\psi}, \dot{\phi}, \ldots$. We have therefore

$$\frac{\partial L'}{\partial \dot{\psi}} = \frac{\partial T_1}{\partial \dot{\psi}} + \Sigma_\kappa M$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{\psi}} = \frac{d}{dt}\frac{\partial T_1}{\partial \dot{\psi}} + \dot{\psi}\Sigma_\kappa \frac{\partial M}{\partial \psi} + \dot{\phi}\Sigma_\kappa \frac{\partial M}{\partial \phi} + \ldots$$

Again

$$\frac{\partial L'}{\partial \psi} = \frac{\partial T_1}{\partial \psi} - \frac{\partial K}{\partial \psi} + \dot{\psi}\Sigma_\kappa \frac{\partial M}{\partial \psi} + \dot{\phi}\Sigma_\kappa \frac{\partial N}{\partial \psi} + \ldots - \frac{\partial V}{\partial \psi};$$

and similar expressions are obtained in the same way for the variables $\phi, \dot{\phi}, \theta, \dot{\theta}, \ldots$. Hence equations (33) may be written

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\partial T_1}{\partial \dot{\psi}} - \frac{\partial T_1}{\partial \psi} + \dot{\phi}\Sigma_\kappa\left(\frac{\partial M}{\partial \phi} - \frac{\partial N}{\partial \psi}\right) + \dot{\theta}\Sigma_\kappa\left(\frac{\partial M}{\partial \theta} - \frac{\partial O}{\partial \psi}\right) + \ldots + \frac{\partial K}{\partial \psi} + \frac{\partial V}{\partial \psi} = 0 \\
&\frac{d}{dt}\frac{\partial T_1}{\partial \dot{\phi}} - \frac{\partial T_1}{\partial \phi} + \dot{\psi}\Sigma_\kappa\left(\frac{\partial N}{\partial \psi} - \frac{\partial M}{\partial \phi}\right) + \dot{\theta}\Sigma_\kappa\left(\frac{\partial N}{\partial \theta} - \frac{\partial O}{\partial \phi}\right) + \ldots + \frac{\partial K}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (38)$$

If in addition to $-\partial V/\partial\psi, -\partial V/\partial\phi, \ldots$ forces $\Psi', \Phi', \ldots$ act on the system, these quantities must be used instead of the zeros on the right of (38). $\Psi', \Phi', \ldots$ are here applied forces which tend to alter the total energy $T + V$ of the system.

249. The terms in these equations which have $\dot{\psi}, \dot{\phi}, \dot{\theta}, \ldots$ as factors are called gyrostatic terms for a reason which will appear below from an example or two which we shall give. It will be seen that in each equation no gyrostatic term with the velocity corresponding to that equation as a factor appears, and that in the ψ-equation the multiplier of $\dot{\phi}$ is the multiplier of $\dot{\psi}$ in the ϕ-equation with its sign changed, and so on.

Equations (38) were given by Lord Kelvin in 1873, and applied by him to important problems in fluid motion.[1] They are reproduced here for the sake of an important case of fluid motion to be considered later in connection with a vortex theory of the ether, and because the idea of a gyrostatically dominated medium will have to be discussed in its bearing on magneto-optic rotation.

[1] "On the Motion of Rigid Solids in a Liquid circulating irrotationally through Perforations in them or in a Fixed Solid." *Phil. Mag.*, May, 1873.

The equations may be established by transforming T into $T'' + K$ where T'' is a homogeneous quadratic function of the velocities $\dot{\psi}, \dot{\phi}, \ldots$ and K is the quadratic function of the momenta $\kappa, \kappa', \ldots$ obtained above, and thence calculating the terms in the Lagrangian equations expressed as in (20) above.[1] The modified Lagrangian function L' was given by Routh, and the method of obtaining it followed above is due to Larmor.[2]

250. When $\kappa, \kappa', \ldots$ are not constant the equations of motion (38) still hold; but besides (38) there exist now equations of motion for the co-ordinates $\chi, \chi', \ldots$ These may be put in the form

$$\left.\begin{aligned} \frac{\partial L'}{\partial \kappa} &= -\dot{\chi}, \quad \frac{\partial L'}{\partial \chi} = \frac{\partial L}{\partial \chi}, \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots \quad (39)$$

and there are as many pairs of such equations as there are quantities $\chi, \chi', \ldots$

To prove (39) we have first from (31)

$$\frac{\partial L'}{\partial \kappa} = \left(\frac{\partial L}{\partial \dot{\chi}} - \kappa\right)\frac{\partial \dot{\chi}}{\partial \kappa} - \dot{\chi} + \ldots = -\dot{\chi},$$

$$\ldots\ldots\ldots\ldots\ldots$$

and again

$$\frac{\partial L'}{\partial \chi} + \frac{\partial L'}{\partial \kappa}\frac{\partial \kappa}{\partial \chi} + \ldots = \frac{\partial L}{\partial \chi} - \dot{\chi}\frac{\partial \kappa}{\partial \chi} - \dot{\chi}'\frac{\partial \kappa'}{\partial \chi} - \ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

or

$$\frac{\partial L'}{\partial \chi} = \frac{\partial L}{\partial \chi}, \ldots$$

251. If the velocities $\dot{\chi}, \dot{\chi}', \ldots$ enter into the expression of the kinetic energy without being associated with the other velocities by terms of the form $(\psi, \chi)\dot{\psi}\dot{\chi}, \ldots$, then, by (36), $M, N, \ldots$ are all identically zero, that is we have

$$L' = T_1 - (K + V) \quad \ldots\ldots \quad (40)$$

where K is a homogeneous quadratic function of $\kappa, \kappa', \ldots$ as before, into which, however, the velocities $\dot{\psi}, \dot{\phi}, \ldots$ do not now enter, and T_1 does not contain the terms of the first degree $-\dot{\psi}\Sigma\kappa.M - \ldots$ We therefore have instead of (38)

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T_1}{\partial \dot{\psi}} - \frac{\partial T_1}{\partial \psi} + \frac{\partial K}{\partial \psi} + \frac{\partial V}{\partial \psi} &= 0 \\ \frac{d}{dt}\frac{\delta T_1}{\partial \dot{\phi}} - \frac{\partial T_1}{\partial \phi} + \frac{\partial K}{\partial \phi} + \frac{\partial V}{\partial \phi} &= 0 \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots \quad (41)$$

[1] *Thomson and Tait*, Vol. I. Part I. § 319, Example G.
[2] "Stability of Motion" Art., *Proc. Lond. Math. Soc.*, XV. (March 13, 1884).

Equations (40) and (41) are expressions of the theorem that if a dynamical system be specified by a group of position co-ordinates ψ, ϕ, ..., and partly by velocities of other co-ordinates of type χ, χ', ..., so that the kinetic energy is the sum of two parts, T_1, K, the first of which is expressed as a quadratic function of the velocities of type $\dot{\phi}$, $\dot{\psi}$, ..., and the other, K, as a quadratic function of the momenta corresponding to the velocities of type $\dot{\chi}$, $\dot{\chi}'$, ..., the motion of the system will be precisely the same as if the system had a quantity of kinetic energy T_1, and an additional quantity of potential energy K. In fact, we see that in any given case of motion the potential energy which exists may be regarded as the kinetic energy corresponding to velocities which are thus ignored.

As an example we shall see later that if we suppose a liquid to circulate round infinitely thin cores immersed in it, K is a quadratic function of the cyclic constants of the motion, and represents the kinetic energy of the fluid motion, while the kinetic energy of the cores themselves is T_1, and does not involve any terms of the first degree in the velocities of the cores. Hence in this case the modified Lagrangian function is

$$L' = T_1 - K - V,$$

so that L' is the same as if the circulation were zero and the potential energy of the motion were increased by K.

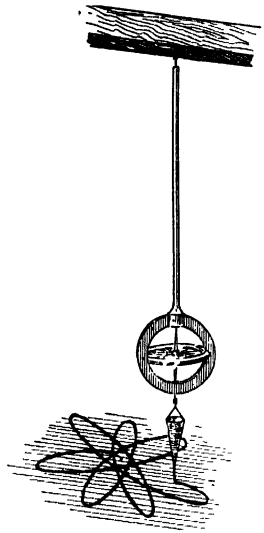
Application of Lagrange's Equation to Theory of Gyrostatic Pendulum

252. As an example of the process of forming the equations of motion of a dynamical system by Lagrange's method we consider here a case of some importance in the theory of magneto-optic rotation. Imagine a pendulum (Fig. 60) the bob of which is carried by a rod terminating at its upper end in one of the forks of a Hooke's universal joint and contains a gyrostat (Fig. 60) rotating about an axis coincident with the line joining the centre of inertia of the whole mass with the point of support—that is, the centre of the cross-link of the joint. We suppose the pendulum to be kinetically symmetrical about this line, and that the rod carrying the other fork of the joint is fixed in a vertical position.

In order that Lagrange's equations of motion may be used the kinetic energy must be expressed in terms of quantities which completely specify the position of every part of the system at any instant. Thus the expression of the kinetic energy in terms of the angular velocities with reference to axes fixed in the moving system—for example, the Eulerian velocities ω_1, ω_2, ω_3—is unsuitable. This is a point which is sometimes overlooked in the application of the Lagrangian method, and errors arise in consequence.

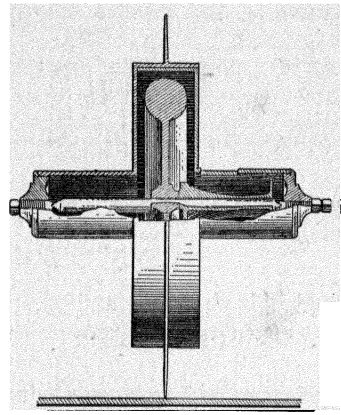
253. Let ϕ be the angle which the vertical plane ZOB, Fig. 62, through the point of support, O, and the centre of inertia, makes with a fixed vertical plane through the former point, and denote by $\dot{\phi}$ the

angular velocity with which this angle is increasing. Let θ be the inclination of the axis, OB, of kinetic symmetry to the vertical; $\dot{\theta}$ its rate of increase; C the moment of inertia of the pendulum, without the gyrostat, round OB; C' that of the gyrostat round the same axis; and A the moment of inertia of the pendulum, including the gyrostat, round



Instead of a kinetically symmetrical case which ought to be used in practice for such a pendulum, a ring-shaped bob is substituted in Fig. 60 to display the gyrostat. A short piece of steel wire having the upper end fixed and vertical, the other end fixed to the rod of the pendulum and directed along its axis, may be substituted with almost perfect equivalence for the Hooke's joint.

FIG. 60.



This Figure shows a gyrostat resting on a thin edge on a glass plate. The case is represented as cut open to show the fly-wheel, which is pivoted on a spindle turning in bearings attached to the case. As the section indicates, the fly-wheel is a thin disk with a massive rim. [This cut is reduced from Thomson and Tait's *Natural Philosophy* (Vol. I. Part 1, p. 397), to which the reader may refer for further information regarding gyrostatic action.]

FIG. 61.

any other principal axis through the point of support. Finally, to specify the position of the gyrostat at any instant, denote by ψ the angle which a plane fixed in the gyrostat and containing its axis makes with the plane ZOB.

The motion of the pendulum can be found from the kinematical

theory of the Hooku's joint;[1] but the following is perhaps simpler. Consider the equivalent suspension: a perfectly flexible untwistable wire, of which one end is soldered or screwed to the upper end of the pendulum rod, the other fixed so that the wire cannot turn. First let the inclination θ of the rod to the vertical remain constant, and the circle in which the centre of the bob moves be ABC (Fig. 63), and let $A, B,$ be the two positions of the bob at the beginning and end of an interval of time δt. Since the wire is untwistable for any position of the pendulum, it is simply bent. For the position A the bending is about an axis in the direction aa', for B about an axis in the direction bb', and these two axes include an angle $\dot{\phi}\delta t$.

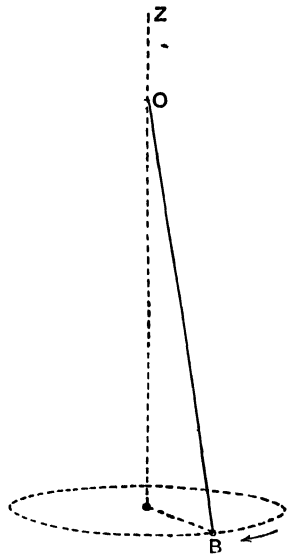


FIG. 62.

Now that line through the centre of the bob, and fixed in it, which was horizontal and a tangent to the circle at A makes with that which at B is horizontal and a tangent to the circle an angle $\dot{\phi}\delta t$, and both these lines are fixed in the bob and lie in a plane at right angles to the rod. For if the wire be unbent when the bob is at A, and the rod brought to the vertical, the line through the centre of the bob which was horizontal remains so. Then if the wire be bent about bb' from the vertical position a line parallel to bb' is brought up to be tangent to the circle at B.

These two lines, therefore, include an angle $\dot{\phi}\delta t$. But the direction of the axis has changed from OA to OB, so that the line which is now tangential to the circle makes an angle $\dot{\phi}\delta t$ with the former direction of the tangent. Now, if the pendulum had been simply carried round so as to make the line which was tangent at A also tangent at B, the pendulum as a whole would have rotated about the vertical at O through an angle $\dot{\phi}\delta t$, and therefore through $\dot{\phi}\cos\theta\,.\,\delta t$ about OB. Consequently the pendulum has rotated in the actual displacement through an angle $\dot{\phi}\,(1-\cos\theta)\delta t$ about OB apart from the angle turned through in consequence of the displacement of that axis. The angular velocity about the axis OB is thus $\dot{\phi}\,(1-\cos\theta)$ or $2\dot{\phi}\sin^2\frac{1}{2}\theta$.

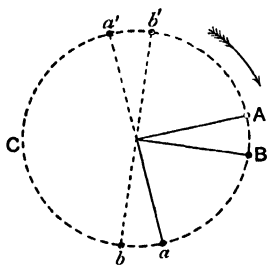


FIG. 63.

Combined with this motion round the axis OB is a rotation about a perpendicular axis in the plane ZOB. This is clearly due to the

[1] *Thomson and Tait*, Vol. I. Part I. p. 86.

motion of the pendulum which is carrying the bob in the direction of the tangent at B, and since the velocity of the bob in this direction is $\dot{\phi} . OB \sin\theta$ the angular velocity about the axis now referred to is $\dot{\phi}\sin\theta$. This angular velocity may be resolved into two components round axes fixed in the body and situated in a plane perpendicular to OB: one chosen so as to make an angle ϕ with the plane ZOB, and thus to coincide, when the pendulum is brought without other change of position to the vertical, with the initial position from which the azimuthal displacement of ZOB is measured; the other perpendicular to this direction.[1]

Besides having the motion just considered the pendulum may turn so as to alter θ without changing ϕ. This will give components round the axes just considered $+\dot{\theta}\sin\phi$, $-\dot{\theta}\cos\phi$ respectively. We have therefore the following results:

Angular velocities

$2\dot{\phi}\sin^2\frac{1}{2}\theta$, round the axis of symmetry OB.

$\dot{\phi}\sin\theta\cos\phi + \dot{\theta}\sin\phi$, round an axis perpendicular to OB in an axial plane fixed in the body.

$\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\cos\phi$, round the third rectangular axis.

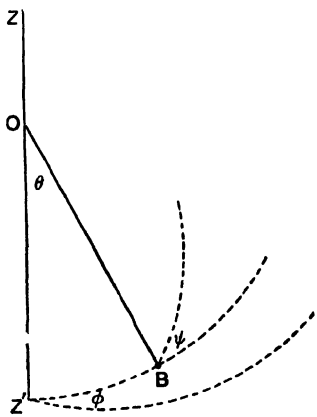


FIG. 64.

It remains to specify the motion of the gyrostat. Clearly it will have the same motion as the rest of the pendulum round the two rectangular axes last mentioned, together with a rotation round OB. The latter, by Fig. 64, in which ϕ is the angle ZOB makes with the fixed vertical plane, and ψ the angle which the plane fixed in the gyrostat makes with ZOB, is plainly $\dot{\phi}\cos\theta + \dot{\psi}$.

The kinetic energy is therefore given by the equation

$$2T = C(1 - \cos\theta)^2\dot{\phi}^2 + A(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + C'(\dot{\psi} + \dot{\phi}\cos\theta)^2 \quad (42)$$

in which the first term and the last are respectively twice the kinetic energy of rotation round OB of the pendulum without the gyrostat, and of the gyrostat alone, and the middle term is twice the kinetic energy of the whole system round the other two rectangular axes specified. If $C = 0$ the equation gives the kinetic energy of a kinetically

[1] This direction, in the more general case when the pendulum is not kinetically symmetrical about OB, must be so chosen that it is one of the principal axes of the body, while OB and the axis at right angles to ZOB are the other principal axes. The component angular velocity about the former is $\dot{\phi}\sin\theta\cos\phi$, about the latter $\dot{\phi}\sin\theta\sin\phi$.

symmetrical rigid body turning in any manner about a fixed point —that is, of a top.

254. If m be the mass of the pendulum and h the distance of the centre of inertia from O, the excess of the potential energy of the pendulum over that which it has when vertical is $mgh\ (1 - \cos\theta)$. Call this V; then the equations of motion are

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} &= 0 \\ \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} &= 0 \\ \frac{d}{dt}\frac{\partial T}{\partial \dot\psi} - \frac{\partial T}{\partial \psi} + \frac{\partial V}{\partial \psi} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

since no other applied forces than those due to gravity act.

We have here $\partial T/\partial\phi$, $\partial V/\partial\phi$, $\partial T/\partial\psi$, $\partial V/\partial\psi$ all zero. It will be noticed, therefore, that the last equation gives

$$\frac{\partial T}{\partial \dot\psi} = \text{constant},$$

so that the problem fulfils the condition stated above for ignoration of co-ordinates.

The three equations (43) are, as the reader may verify,

$$\begin{aligned} A\ddot\theta - \{C\sin\theta + (A - C - C')\sin\theta\cos\theta\}\dot\phi^2 + C'\sin\theta\,.\,\dot\phi\dot\psi \\ + mgh\sin\theta = 0, \\ \{C(1 - \cos\theta)^2 + C'\cos^2\theta + A\sin^2\theta\}\ddot\phi + C'\cos\theta\,.\,\ddot\psi \\ + 2\{C\sin\theta + (A - C - C')\sin\theta\cos\theta\}\dot\theta\dot\phi - C'\sin\theta\,.\,\dot\psi\dot\theta = 0, \\ \frac{d}{dt}(\dot\psi + \dot\phi\cos\theta) = 0. \end{aligned}$$

255. The last equation shows that the angular velocity, $\dot\psi + \dot\phi\cos\theta$, of the gyrostat round its axis remains constant during the motion, a well-known but remarkable result.

Putting $\partial T/\partial\dot\psi$ or $C'(\dot\psi + \dot\phi\cos\theta) = \kappa$, we have

$$\dot\psi = \frac{\kappa}{C'} - \dot\phi\cos\theta,$$

and therefore (p. 190 above), since $\dot\psi$ is the velocity for the ignored co-ordinate, $N = \cos\theta$, $M = 0$, &c., and

$$\begin{aligned} 2T &= \dot\psi\frac{\partial T}{\partial \dot\psi} + \dot\phi\frac{\partial T}{\partial \dot\phi} + \dot\theta\frac{\partial T}{\partial \dot\theta} \\ &= \frac{\kappa}{C'}(\kappa - C'\dot\phi\cos\theta) + \{C(1 - \cos\theta)^2 + A\sin^2\theta\}\dot\phi^2 \\ &\quad + \dot\phi(\kappa - C'\dot\phi\cos\theta)\cos\theta + C'\dot\phi^2\cos^2\theta + A\dot\theta^2 \\ &= \{C(1 - \cos\theta)^2 + A\sin^2\theta\}\dot\phi^2 + A\dot\theta^2 + \frac{\kappa^2}{C'} \quad . \quad . \quad . \quad (44) \end{aligned}$$

as might, of course, have been obtained from (42).

The kinetic energy is thus reduced to the sum of two quadratic functions, one, T_1, of the velocities $\dot{\phi}, \dot{\theta}$, the other, K, of the momentum κ corresponding to $\dot{\psi}$. Thus

$$2T_1 = \{C(1 - \cos\theta)^2 + A\sin^2\theta\}\dot{\phi}^2 + A\dot{\theta}^2, \quad 2K = \frac{\kappa^2}{C'}.$$

The equations of motion by (38) are reduced to two, which have the form

$$\left.\begin{aligned} \{C(1-\cos\theta)^2 + A\sin^2\theta\}\ddot{\phi} + 2\{C(1-\cos\theta) + A\cos\theta\}\sin\theta.\dot{\theta}\dot{\phi} - \kappa\sin\theta.\dot{\theta} = 0 \\ A\ddot{\theta} - \{C + (A - C)\cos\theta\}\sin\theta\dot{\phi}^2 + mgh\sin\theta = 0 \end{aligned}\right\} \quad (45)$$

The last term on the left of the first of these equations, is the gyrostatic term $\dot{\theta}\kappa\partial N/\partial\theta$, and there is no other.

These equations can be reduced, when θ is small throughout the motion, to x, y co-ordinates and take then a symmetrical form which exhibits better the gyrostatic terms. As they will be of use later, we give them here also. They are best obtained by transforming the kinetic energy to the new co-ordinates, viz. x, y, taken in a horizontal plane through the centre of inertia of the pendulum, with the projection of O upon this plane as origin.

We have approximately $1 - \cos\theta = r^2/2h^2$, where r $(= \sqrt{x^2+y^2})$ is the distance of the centre of inertia from the origin, $\dot{\theta} = (x\dot{x} + y\dot{y})/hr$ $\dot{\phi} = (x\dot{y} - y\dot{x})/r^2$, so that, neglecting terms of higher order than r^2/h^2, we get

$$2T = \frac{A}{h^2}(\dot{x}^2 + \dot{y}^2) + \frac{\kappa^2}{C'} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Hence

$$2T_1 = \frac{A}{h^2}(\dot{x}^2 + \dot{y}^2), \quad 2K = \frac{\kappa^2}{C'}.$$

For the calculation of the gyrostatic terms we have

$$\dot{\psi} = \frac{\kappa}{C'} - \dot{\phi}\cos\theta = \frac{\kappa}{C'} - \frac{x\dot{y} - y\dot{x}}{r^2}\left(1 - \frac{r^2}{2h^2}\right)$$

$$= \frac{\kappa}{C'} + M\dot{x} + N\dot{y},$$

where

$$M = y\left(\frac{1}{r^2} - \frac{1}{2h^2}\right), \quad N = -x\left(\frac{1}{r^2} - \frac{1}{2h^2}\right).$$

The equations of motion are, since $\partial T_1/\partial x$, $\partial T_1/\partial y$, $\partial K/\partial x$, $\partial K/\partial y$, are all zero,

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T_1}{\partial \dot{x}} + \kappa\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\dot{y} + \frac{\partial V}{\partial x} = 0 \\ \frac{d}{dt}\frac{\partial T_1}{\partial \dot{y}} + \kappa\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\dot{x} + \frac{\partial V}{\partial y} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (47)$$

Now

$$\frac{d}{dt}\frac{\partial T_1'}{\partial \dot{x}} = A\dot{x}, \quad \frac{d}{dt}\frac{\partial T_1'}{\partial \dot{y}} = A\dot{y},$$

$$\frac{\partial V}{\partial x} = mg\frac{x}{h}, \quad \frac{\partial V}{\partial y} = mg\frac{y}{h},$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{1}{h^2},$$

and therefore equations (47) become.

$$\left.\begin{aligned} A\dot{x} - \kappa\dot{y} + mghx = 0 \\ A\dot{y} + \kappa\dot{x} + mghy = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (48)$$

256. These are the equations of motion, referred to horizontal x, y co-ordinates, of the centre of inertia of a gyrostatic pendulum, the inclination of which to the vertical is always small, or (with change of sign of h) of a gyrostat supported, with its axis nearly vertical, by a piece of flexible wire or on gimbals placed in the prolongation of the axis *below* the centre of inertia. If the gimbal support is used both axes about which the gyrostat turns are supposed to be on the same level; if they are not, equations (48) require modification. Taking in this case the axes parallel to those about which the gimbal rings turn, and putting h_1 for the distance of the centre of inertia from the axis parallel to y, A_1 for the moment of inertia round the axis parallel to y and h_2, A_2 for the distance and moment of inertia with regard to the axis of x, we obtain the equations for this case by substituting for h and A in the first equation $-h_1$ and A_1, and in the second $-h_2$ and A_2.

It will be found later that equations (48) are similar in form to those which enter into the dynamical explanation of the effect, discovered by Zeemann, of a magnetic field on the spectrum of a gas.

Generalised Forces. Applied Forces, and Forces of Constraint. Lagrange's Equations deduced from Cartesian Equations of Motion of Set of Particles

257. We do not enter here on the solution of these equations of motion: many examples will present themselves later in electrical applications. For the sake of considering Lagrange's equations from different points of view we shall deduce them from the ordinary Cartesian equations of motion of a set of particles.

Let $F_1, F_2, \ldots$ denote forces in the ordinary sense which act on the particles $m_1, m_2, \ldots$; then in any possible displacements $\delta s_1, \delta s_2, \ldots$ of these particles the work done is

$$\delta W = F_1\delta s_1 + F_2\delta s_2 + \ldots = \Sigma F\delta s.$$

If $X_1, Y_1, Z_1, \ldots \delta x_1, \delta y_1, \delta z_1, \ldots$ be the components of $F_1, \ldots, \delta s_1, \ldots$ parallel to the axes we have

$$\delta W = \Sigma(X\delta x + Y\delta y + Z\delta z) \quad \ldots\ldots \quad (49)$$

We have supposed X, Y, Z, here to be the components of the force actually accelerating a given particle. This force is the resultant of the forces applied from without the system to the particle, and the forces of constraint due to the fulfilment of the internal conditions of the system. Thus for the components of applied force we may write X', Y', Z',...., and for the components of constraint X'', Y'', Z'',.... Hence, since $X=X'+X''$,....

$$\delta W = \Sigma(X'\delta x + Y'\delta y + Z'\delta z) + \Sigma(X''\delta x + Y''\delta y + Z''\delta z).$$

But, the forces of constraint referred to are due to the mutual actions of the particles of the system, and the resultant of the forces of constraint on any one particle is accompanied according to the Third Law of Motion, by an equal and opposite force on the rest of the system. Hence for the whole system

$$\Sigma(X''\delta x + Y'\delta y + Z''\delta z) = 0 \quad . \quad . \quad . \quad . \quad (50)$$

and we have

$$\delta W = \Sigma(X'\delta x + Y'\delta y + Z'\delta z) \quad . \quad . \quad . \quad . \quad (51)$$

There are also forces of constraint applied from outside the system for which the work in any possible arbitrary displacement vanishes. Such for example are the forces applied to certain particles of the system by frictionless unyielding guide-pieces, and they act at right angles to the guiding surfaces along which the particles to which they are applied travel. Since there is no displacement at right angles to these guiding surfaces these forces are taken as included with forces X'',.... which give (50).

In general we shall omit the accents in using (51), and it will be understood, unless the contrary is stated, that X, Y, Z,.... denote the externally applied forces and not the actual forces on the particles.

258. Calculating δx, δy, δz from (1), we have by (51)

$$\begin{aligned}\delta W &= \Sigma\left(X\frac{\partial x}{\partial \psi} + Y\frac{\partial y}{\partial \psi} + Z\frac{\partial z}{\partial \psi}\right)\delta\psi + \ldots . \\ &= \Psi\delta\psi + \Phi\delta\phi + \ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)\end{aligned}$$

where

$$\left.\begin{aligned}\Psi &= \Sigma\left(X\frac{\partial x}{\partial \psi} + Y\frac{\partial y}{\partial \psi} + Z\frac{\partial z}{\partial \psi}\right) \\ \Phi &= \Sigma\left(X\frac{\partial x}{\partial \phi} + Y\frac{\partial y}{\partial \phi} + Z\frac{\partial z}{\partial \phi}\right) \\ &\ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

The quantities Ψ, Φ,.... given by (50) are called the generalised components of external applied force corresponding to the co-ordinates ψ, ϕ,....

259. The Cartesian equations of motion of a set of particles $m_1, m_2, \ldots$ may be written briefly in the form

$$\left.\begin{aligned} m_1(\ddot{x}_1, \ddot{y}_1, \ddot{z}_1) &= (X_1, Y_1, Z_1) \\ &\ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

Here X_1, Y_1, Z_1, ... are the *actual* component forces on the particles indicated by the suffixes. If the particles are, as we suppose, subject to the conditions of constraint (2), these forces are due to the applied forces and the forces of constraint together.

Equations (54) and (53) give

$$\left.\begin{aligned} \Sigma m\left(x\frac{\partial x}{\partial \psi} + \dot{y}\frac{\partial y}{\partial \psi} + \dot{z}\frac{\partial z}{\partial \psi}\right) &= \Psi \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{aligned}\right\} \quad \ldots\ (55)$$

But

$$\dot{x}\frac{\partial x}{\partial \psi} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial \psi}\right) - \dot{x}\frac{d}{dt}\frac{\partial x}{\partial \psi} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{\psi}}\right) - \dot{x}\frac{\partial \dot{x}}{\partial \psi}$$

by (3), and similar equations hold for the y, z, co-ordinates. Hence we have

$$\Sigma m\left(x\frac{\partial x}{\partial \psi} + \dot{y}\frac{\partial y}{\partial \psi} + \dot{z}\frac{\partial z}{\partial \psi}\right) = \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi}$$

with similar equations in ϕ, θ, ... Thus we again obtain Lagrange's equations

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} &= \Psi \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} &= \Phi \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{aligned}\right\} \quad \ldots\ (56)$$

If the forces are conservative, so that $\Psi = -\partial V/\partial\psi$, ... these equations coincide with (20) or (20′) above.

One general remark we may make here to guard the reader against possible error in the use of Lagrange's equations. The co-ordinates ψ, ϕ, ... used must not only be independent, but they must be such as can express the position and, with the kinematical equations, the configuration of the system at any instant during the motion; that is, x_1, y_1, z_1, ... must be expressible as in (1).

Work done by Forces of Constraint in Variation of Kinematical Conditions

260. It is also to be observed that there is nothing in the process which equations (56) have been established which would be affected by the explicit appearance of t in equations (1) and (2), and that therefore the equations hold in this case also. If t, however, does explicitly appear in (2) we have

$$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \psi}\dot{\psi} + \frac{\partial x}{\partial \phi}\dot{\phi} + \ldots.$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

instead of (3). Thus the kinetic energy is no longer a homogeneous quadratic function of the velocities $\dot{\psi}$, $\dot{\phi}$, ..., but a quadratic expression involving terms of the second, first, and zero degrees in these velocities.

This can easily be shown to be inconsistent with constancy of the energy of the motion.

For from equations (1), if t explicitly appears therein, we have, if $\delta x_1\ \delta y_1\ \delta z_1, \ldots$. be any variations of the co-ordinates possible *under the particular kinematical conditions of the system which hold at time t*,

$$\left.\begin{array}{l} \dfrac{\partial F_1}{\partial x_1}\delta x_1 + \dfrac{\partial F_1}{\partial y_1}\delta y_1 + \ldots . = 0 \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots . \quad (57)$$

and we have besides the general variational equation of motion (derivable from (54) and (50)):

$$\Sigma\{(mx - X)\delta x + (m\ddot{y} - Y)\delta y + (mz - Z)\delta z\} = 0 \quad . \quad (58)$$

where $X, Y, Z, \ldots$. denote the *applied* forces and exclude constraints. Multiplying equations (57) by $\lambda_1, \lambda_2, \ldots$. respectively, and adding to (58), we obtain, by equating the coefficients of $\delta x_1, \ldots$. individually to zero,

$$\left.\begin{array}{l} \lambda_1\dfrac{\partial F_1}{\partial x_1} + \lambda_2\dfrac{\partial F_2}{\partial x_1} + \ldots . + m_1\ddot{x}_1 - X_1 = 0 \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (59)$$

Again, if variation of the time be permitted, the changes in the kinematical conditions are connected by the relations

$$\left.\begin{array}{l} \dfrac{\partial F_1}{\partial x_1}\dot{x}_1 + \dfrac{\partial F_1}{\partial y_1}\dot{y}_1 + \ldots . + \dfrac{\partial F_1}{\partial t} = 0 \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (60)$$

Multiplying (59) by $\dot{x}_1, \ldots$. respectively, and adding, we find by (60), since $dT/dt = \Sigma m\ (\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})$.

$$\frac{dT}{dt} = \Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) + \lambda_1\frac{\partial F_1}{\partial t} + \lambda_2\frac{\partial F_2}{\partial t} - \ldots . \quad (61)$$

The interpretation of this result is that the time-rate of increase of the kinetic energy is equal to the rate at which work is done by the forces $X_1, Y_1, Z_1, \ldots$ plus the activity

$$\lambda_1\frac{\partial F_1}{\partial t} + \lambda_2\frac{\partial F_2}{\partial t} + \ldots .$$

due to the forces brought into play by the varying of the kinematical relations. Hence, when $\partial F_1/\partial t, \partial F_2/\partial t, \ldots$ are all zero, the fulfilment of the kinematical relations has no effect on the energy of the system; and if the forces $X, Y, Z \ldots$. are conservative the sum of the kinetic and potential energies remains constant during the motion, provided t does not appear explicitly in the kinematical relations.

The quantities $\lambda_1, \lambda_2, \ldots$ are, it is to be noticed, all determinate, so that the forces $\lambda_1\partial F_1/\partial x_1, \ldots .$, which are the forces of constraint involved in the fulfilment of the equations of condition, can be evaluated.

Hamilton's Form of the Equations of Motion

261. Equations (56) express the fact that the time-rate of increase of each component momentum, $\partial T/\partial\dot{\psi}$ say, less the ψ-rate of increase of the kinetic energy, is equal to the applied force Ψ. In the case of a conservative system of forces the equations express the fact that the time-rate of increase of each momentum is equal to the value which the ψ-rate of increase of $T - V$ has when the velocities $\dot{\psi}, \dot{\phi}, \ldots$ are kept constant.

Thus we may write the equations in the form

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \frac{\partial(T_v - V)}{\partial\psi} \\ &\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (62)$$

We may contrast these with another form of the equations of motion given by Hamilton. Let us suppose that the kinetic energy is expressed as a homogeneous quadratic function of the momenta, then by (27) the last equations become

$$\left.\begin{aligned} \frac{d\xi}{dt} &= -\frac{\partial(T_m + V)}{\partial\psi} \\ &\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (63)$$

Also, by (26), $\dot{\psi} = \partial T_m/\partial\xi = \partial(T_m + V)/\partial\xi \ldots\ldots$ Thus we may write, as companions to (63),

$$\left.\begin{aligned} \frac{d\psi}{dt} &= \frac{\partial(T_m + V)}{\partial\xi} \\ &\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (64)$$

Equations (63) and (64) are the famous "canonical equations" of dynamics given by Hamilton and discussed and applied by Jacobi,[1] Bertrand,[2] Donkin,[3] Poincaré,[4] and others. It was pointed out by Jacobi that (63) and (64) hold also when V is an explicit function not only of the co-ordinates but of the time t, that is when the conservation of energy does not hold for the system.

The terms $-\partial T_v/\partial\psi, \ldots\ldots, \partial T_m/\partial\psi, \ldots\ldots$ are the external forces required to preserve the momenta constant, and depend on the appearance of the co-ordinates in the coefficients of the quadratic expressions for T. Their values depend therefore on the kinematical relations to which the system is subject.

The equations assert that the time-rate of increase of momentum is equal to the corresponding ψ-rate of decrease of the total energy, taken subject to the condition that the momenta are constant, and that any velocity $\dot{\psi}$ is equal to the ξ-rate of increase of the total energy taken subject to the condition that the velocities are constant.

[1] See Jacobi's *Werke* for several memoirs on this subject, but especially *Vorlesungen uber Dynamik*, Bd. VII., and *Ueber diejenigen Probleme der Mechanik in welchen eine Kraftefunction existirt*, Bd. V.

[2] Liouville, t. XVII. (1851, 1852), pp. 121, 393.

[3] *Phil. Trans.*, 1854, 1855.

[4] *Les Méthodes Nouvelles de la Mécanique Céleste*, t. I., 1892.

Impulsive Forces. Work done by System of Impulses

261. If, instead of a motion gradually changing, the system have its velocities suddenly altered by the application of forces during a very short interval of time τ, we can now easily calculate the work done. By (56), if we suppose each space-rate of variation $\partial T/\partial\dot{\psi}, \ldots$. of the kinetic energy to be finite, we have, denoting the time-integrals of the forces for the interval τ by $\dot{\Psi}, \dot{\Phi}, \ldots$,

$$\left[\frac{\partial T}{\partial\dot{\psi}}\right]_0^\tau = \dot{\Psi}, \quad \left[\frac{\partial T}{\partial\dot{\phi}}\right]_0^\tau = \dot{\Phi}, \ldots, \quad . \quad . \quad . \quad . \quad (65)$$

for the equations of motion. The quantities on the left denote, of course, the result of subtracting the initial value of $\partial T/\partial\dot{\psi}, \ldots$. from the value for the end of the interval τ.

The work done by the impulses can now be easily found. Writing (65) in the form

$$\xi_\tau - \xi_0 = \dot{\Psi}, \quad \eta_\tau - \eta_0 = \dot{\Phi}, \ldots . \quad . \quad . \quad . \quad (65')$$

multiplying the first equation by $\frac{1}{2}(\dot{\psi}_\tau + \dot{\psi}_0)$, the second by $\frac{1}{2}(\dot{\phi}_\tau + \dot{\phi}_0)$, ..., where $\dot{\psi}_0, \dot{\psi}_\tau, \ldots$ denote the velocities at beginning and end of τ, and adding, we obtain

$$\tfrac{1}{2}(\xi_\tau\dot{\psi}_\tau + \eta_\tau\dot{\phi}_\tau + \ldots.) - \tfrac{1}{2}(\xi_0\dot{\psi}_0 + \eta_0\dot{\phi}_0 + \ldots.) = \tfrac{1}{2}\dot{\Psi}(\dot{\psi}_\tau + \dot{\psi}_0) + \tfrac{1}{2}\dot{\Phi}(\dot{\phi}_\tau + \dot{\phi}_0) + \ldots. \quad (66)$$

since, identically,

$$\xi_\tau\dot{\psi}_0 + \eta_\tau\dot{\phi}_0 + \ldots. = \xi_0\dot{\psi}_\tau + \eta_0\dot{\phi}_\tau + \ldots. \quad . \quad . \quad (67)$$

Thus we have the theorem that the work done by the system of impulses is equal to the sum of the products of each time-integral of impulsive force, into the arithmetic mean of the velocities at the beginning and end of the interval τ during which the change of motion is effected. If $\psi_0, \phi_0, \ldots$ be all zero, that is if the motion has taken place from rest,

$$\tfrac{1}{2}(\dot{\xi}\dot{\psi} + \dot{\eta}\dot{\phi} + \ldots.) = \tfrac{1}{2}(\dot{\Psi}\dot{\psi} + \dot{\Phi}\dot{\phi} + \ldots.) \quad . \quad . \quad (68)$$

or the kinetic energy is equal to the sum of the products of the time-integrals of the forces into half the corresponding velocities acquired.

It is to be observed that, provided the interval of change τ is very small, the total work of the given system of impulses is independent of the order or manner of their application. The work done by any particular impulse, however, depends on this order.

The reciprocal relation expressed by (68) is very important. A similar theorem holds for forces and corresponding displacements of a system from stable equilibrium, provided the potential energy is a homogeneous quadratic function of these displacements.

Motional Forces. Dissipative Forces. Dissipation Function

262. In equations (38) are included what have been called gyrostatic terms, which may be regarded as forces depending on the first powers of the velocities $\dot{\psi}, \dot{\phi}, \ldots$, that is, as such forces are called, *motional forces.* It will be observed that if we multiply (38) (with forces, $\Psi', \Phi', \ldots$. included on the right, as there indicated) by $\dot{\psi}, \dot{\phi}, \ldots$, and add, they give

$$\frac{d}{dt}(T + V) = \Psi'\dot{\psi} + \Phi'\dot{\phi} + \ldots . \quad . \quad . \quad . \quad (69)$$

that is, the time-rate of increase of the total energy of the system is equal to the rate of working of the forces $\Psi', \Phi', \ldots$ It is to be observed that the presence of the gyrostatic terms causes no alteration of energy.

If the products $\Psi'\dot{\psi}, \ldots$ or any of them are negative these forces tend to diminish the total energy of the system. Such additional forces are called "frictional" or "dissipative" forces. A very important case of dissipative motional forces exists when $\Psi', \Phi', \ldots$ are linear functions of the velocities derivable by differentiation with respect to $\dot{\psi}, \dot{\phi}, \ldots$. from a positive homogeneous quadratic function given by the equation

$$F = \tfrac{1}{2}(R_{11}\dot{\psi}^2 + 2R_{12}\dot{\psi}\dot{\phi} + \ldots . + R_{22}\dot{\phi}^2 + \ldots .) \quad . \quad (70)$$

so that $\Psi' = -\partial F/\partial\dot{\psi}, -\Phi' = -\partial F/\partial\dot{\phi}, \ldots$. Adding these forces to (33) or (38), including on the right any further forces $\Psi'', \Phi'', \ldots$ we obtain

$$\left.\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial T}{\partial\dot{\psi}} - \dfrac{\partial T}{\partial\psi} + \dfrac{\partial F}{\partial\dot{\psi}} + \dfrac{\partial V}{\partial\psi} = \Psi'' \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad . \quad . \quad . \quad . \quad (71)$$

These give by the same process as before,

$$\frac{d(T + V)}{dt} + 2F = \Psi''\psi + \Phi''\dot{\phi} + \ldots . \quad . \quad . \quad . \quad (72)$$

$2F$ is therefore twice the rate at which energy is dissipated by the forces derived from F. F has been called by Lord Rayleigh the Dissipation Function. It is of great importance in the theory of mutually influencing currents, as we shall see below.

Controllable and Uncontrollable Co-ordinates. Thermokinetic Principle

263. In certain applications of dynamics to electrical and magnetic problems, and to the discussion of the laws of thermodynamics, it is convenient to consider the co-ordinates of the system as divided into two sets, those which from their nature, we may call *uncontrollable co-ordinates,* that is, co-ordinates which cannot be directly and individually affected from without in any specified manner, and *controllable co-ordi-*

nates, which can be so changed by the action of an external system. We have examples of uncontrollable co-ordinates in those which determine the positions and states of the individual molecules of a body; and of controllable co-ordinates, on the other hand, in the volume of a body, the position of its centre of inertia, the orientation of a plane fixed in the body, the state of the body as to electrical charge and the like, which are all under the influence of external systems.

264. Separating the independent co-ordinates of the system into two sets $\psi, \phi, \ldots\ldots, \chi, \chi', \ldots\ldots$ and supposing, for the present, both sets to enter into the expressions for the kinetic and potential energies, and to be acted on by corresponding external forces $\Psi, \Phi, \ldots\ldots \mathrm{X}, \mathrm{X}', \ldots$ we have for the equations of motion

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} + \frac{\partial V}{\partial \psi} = \Psi \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{\chi}} - \frac{\partial T}{\partial \chi} + \frac{\partial V}{\partial \chi} = \mathrm{X} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (73)$$

Multiplying these equations by $\dot{\psi}, \dot{\phi}, \ldots, \dot{\chi}, \dot{\chi}', \ldots\ldots$ in order, and adding we get, as in (69)

$$\frac{d}{dt}(T + V) = \Psi\dot{\psi} + \Phi\dot{\phi} + \ldots\ldots + \mathrm{X}\dot{\chi} + \mathrm{X}'\dot{\chi}' + \ldots\ldots \quad (74)$$

or

$$d(T + V) = \Psi d\psi + \Phi d\phi + \ldots\ldots + \mathrm{X}d\chi + \mathrm{X}'d\chi' + \ldots\ldots \quad (75)$$

where $d\psi, d\phi, \ldots\ldots, d\chi, d\chi', \ldots$ are any variations of the co-ordinates compatible with the kinematic conditions of the system.

The work done on the system by the forces acting on the controllable coordinates is $\Psi d\psi + \Phi d\phi + \ldots\ldots$ If this be denoted by $-dW$, $+dW$ will represent the work done by the system on external bodies. If with this we put dQ for $\mathrm{X}d\chi + \mathrm{X}'d\chi' + \ldots\ldots$ (75) may be written

$$dQ = dT + dV + dW \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (76)$$

which simply expresses the conservation of energy.

265. The equations of motion of a system which has a constant sum of energy are obtainable from the action-theorem (8) above, namely,

$$\int_{t_0}^{t_1} (\delta T - \delta V)dt = 0$$

where the time of transition $t_1 - t_0$ from one configuration to another is supposed constant. It may here be remarked that those of the more

general system, in which there is action between the system and external bodies, resulting in the passage of energy from one to the other with or without dissipation, can be obtained in like manner from the equation

$$\int_{t_0}^{t_1} (\delta T - \delta V - \Sigma\Theta\delta\theta)dt = 0 \quad . \quad . \quad . \quad . \quad (77)$$

in which θ denotes any coordinate, so that $\Sigma\Theta\delta\theta$ denotes $\Sigma\Psi\delta\psi + \Sigma X\delta\chi$ or $-\delta W + \delta Q$, that is, the whole work done on the system from without.

This equation written in the form

$$\int_{t_0}^{t_1} (\delta T - \delta V - \Sigma\Psi\delta\psi - \delta Q)dt = 0 \quad . \quad . \quad . \quad (78)$$

with δQ regarded as a quantity of heat furnished to the system from external bodies, has been called the *Thermokinetic Principle.*[1]

It may also be noticed that while (73) lead as already shown to (75) and (76), it is impossible to obtain from them (77), from which, on the other hand, they are derivable. In other words it is vain to attempt to deduce the principle of action from that of energy; in fact, as already stated, the principle of energy by itself is powerless to yield a dynamical theory of phenomena of any kind.

Dynamical Analogues of Thermodynamic Relations

266. Let us now suppose that χ, χ', ... are the uncontrollable co-ordinates, so that while X, X', ... are unknown dQ remains the quantity of energy introduced into the system from without through these co-ordinates. Further, let no product of the form $\dot{\psi}\dot{\chi}$, that is of the velocity corresponding to a controllable co-ordinate by a velocity of the other set, enter into the expression of the kinetic energy. If there were such terms the reversal of all the uncontrollable velocities would alter the energy of the system, a result which cannot hold in any of the applications we shall make of the method now under discussion.

Calling the kinetic energy corresponding to the velocities of the controllable co-ordinates T_c and that corresponding to the velocities of the uncontrollable co-ordinates T_u, the equations of motion for ψ, ϕ, ... will be, since $\partial T_u/\partial\dot{\psi} = 0$, &c.

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial T_c}{\partial\dot{\psi}} - \frac{\partial T_c}{\partial\psi} - \frac{\partial T_u}{\partial\psi} + \frac{\partial V}{\partial\psi} = \Psi \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (79)$$

Let us suppose also that if the coefficients of the terms in T_u

[1] See a paper *On the Laws of Irreversible Phenomena* by Dr. Ladislas Natanson, *Phil. Mag.*, May, 1896. See also v. Helmholtz, *Das Princip der Kleinsten Wirkung, Crelle*, Bd. 100, pp. 137, 213, and *Wied. Annalen*, Bd. 47 (1892), p. 1.

involve the controllable co-ordinates at all, it is only through a common factor $f(\psi, \phi, \ldots)$ of each coefficient, so that

$$T_u = f(\psi, \phi, \ldots)\{\tfrac{1}{2}[\chi, \chi]\dot{\chi}^2 + [\chi, \chi']\dot{\chi}\dot{\chi}' + \ldots\} \quad . \quad (80)$$

Thus (χ, χ), (χ, χ'), ... denoting the whole coefficients as above $(\chi, \chi) = f(\psi, \phi, \ldots)\,[\chi, \chi]$, and so for the others. When this is the case

$$\frac{1}{T_u}\frac{\partial T_u}{\partial \psi} = \frac{1}{f}\frac{\partial f}{\partial \psi}, \ldots \quad . \quad (81)$$

Thus the equations of motion become

$$\left.\begin{array}{l} \ldots \quad \dfrac{d}{dt}\dfrac{\partial T_c}{\partial \dot{\psi}} - \dfrac{\partial T_c}{\partial \psi} - \dfrac{1}{f}\dfrac{\partial f}{\partial \psi}T_u + \dfrac{\partial V}{\partial \psi} = \Psi \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad . \quad (82)$$

From the first of these, if the only variable quantity be supposed to be T_u, we obtain

$$-\frac{1}{f}\frac{\partial f}{\partial \psi} = \frac{\partial \Psi}{\partial T_u}, \ldots \quad . \quad (83)$$

Again, if we multiply

$$\frac{d}{dt}\frac{\partial T_c}{\partial \dot{\psi}} - \frac{\partial T_c}{\partial \psi}, \quad \frac{d}{dt}\frac{\partial T_c}{\partial \dot{\phi}} - \frac{\partial T_c}{\partial \phi}, \ldots$$

by $\dot{\psi}$, $\dot{\phi}$, ... respectively and add, we obtain (since $d\psi = \dot{\psi}dt$)

$$dT_c = \Sigma d\psi\left(\frac{d}{dt}\frac{\partial T_c}{\partial \dot{\psi}} - \frac{\partial T_c}{\partial \psi}\right).$$

Hence we have

$$dQ = \Sigma d\psi\left(\frac{d}{dt}\frac{\partial T_c}{\partial \dot{\psi}} - \frac{dT_c}{\partial \psi}\right) + dT_u + dV + dW. \quad . \quad (84)$$

This by (79) becomes

$$dQ = \Sigma d\psi\left(\Psi + \frac{\partial T_u}{\partial \psi} - \frac{\partial V}{\partial \psi}\right) + dT_u + dV + dW \quad . \quad (84')$$

But if now we suppose the potential energy to be independent of the uncontrollable co-ordinates we obtain

$$\Sigma \Psi d\psi = -dW, \quad \Sigma \frac{\partial V}{\partial \psi}d\psi = dV$$

and (84′) becomes

$$dQ = \Sigma \frac{\partial T_u}{\partial \psi}d\psi + dT_u \quad . \quad (85)$$

By (81) this may be written

$$\frac{dQ}{T_u} = \frac{1}{f}\Sigma\frac{\partial f}{\partial \psi}d\psi + \frac{dT_u}{T_u} \quad . \quad . \quad . \quad . \quad . \quad (86)$$

or dQ/T_u is a perfect differential. In this we have a dynamical analogue of the Second Law of Thermodynamics.

267. Using (83) in the result just obtained we find

$$dQ = -\Sigma T_u \frac{\partial \Psi}{\partial T_u}d\psi + dT_u \quad . \quad . \quad . \quad . \quad . \quad (87)$$

which becomes, if circumstances be such as to maintain T_u constant,

$$\frac{\partial Q}{\partial \psi}_{T_u \text{const.}} = -T_u \frac{\partial \Psi}{\partial T_u}_{\psi \text{const.}} \quad . \quad . \quad . \quad . \quad (88)$$

This is perfectly analogous to what is known as the third thermodynamic relation, if T_u be taken to represent the absolute temperature on the thermodynamic scale. Taking the relation as usually stated Ψ is the analogue of pressure, and the corresponding co-ordinate ψ is the analogue of the volume, which is the co-ordinate corresponding to pressure.

Further if dQ be zero, that is if all the energy passing from external bodies to the system or *vice versa* be communicated through the constrainable co-ordinates we have by (85)

$$T_u \frac{\partial \Psi}{\partial T_u}_{\psi \text{const.}} = \frac{\partial T_u}{\partial \psi}_{Q \text{ const.}} \quad . \quad . \quad . \quad . \quad (89)$$

These results are due to Prof. J. J. Thomson,[1] and his method of proof has been here followed.

V. Helmholtz's Thermokinetic Theorems of Cyclic Systems

268. Somewhat similar theorems have been obtained by von Helmholtz, though by a different method involving rather different assumptions. We give a sketch of his process also, preserving, however, the notation here adopted. Putting L as before for $T - V$ ($-H$ in v. Helmholtz's notation) we have for a typical co-ordinate θ, whether constrainable or not

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \Theta$$

[1] *Applications of Dynamics to Physics and Chemistry*, p. 98.

Writing s for $\partial L/\partial\dot{\theta}$ we have

$$2T = \Sigma\dot{\theta}\frac{\partial L}{\partial\dot{\theta}} = \Sigma\dot{\theta}s_\theta,$$

and so, if E be the total energy,

$$E = 2T - L = \Sigma\dot{\theta}s_\theta - L \quad . \quad . \quad . \quad . \quad . \quad (90)$$

269. The special theorems which v. Helmholtz obtains apply to what he calls cyclic systems, that is systems in which there are periodic or circulating motions. When the motion is definable by a single co-ordinate it is said to be monocyclic; when, however, it is only definable by a larger number of co-ordinates it is said to be polycyclic.

As an example of a monocyclic system we may take a circular disk rotating about its axis. Here the position of the disk depends on the angle which a plane containing the axis makes with a plane fixed in the space. But the energy depends only on the angular velocity, and not at all on this angle. As the disk turns its potential energy remains unaltered. Another example is found in the steady motion of fluid round a circular canal. A third example is to be found in the motion of the gyrostatic pendulum discussed in Arts. 252—256 above. The cyclic velocities in that case are, $\dot{\psi}$, the angular velocity of the flywheel of the gyrostat about its axes, and $\dot{\phi}$, the angular velocity of the precessional motion round the vertical.

Von Helmholtz assumes for these cyclic systems.

1. That neither the kinetic nor the potential energy depends on the uncontrollable (the cyclic) co-ordinates themselves, but only the velocities corresponding to them, so that these coordinates are what have been called *gyrostatic* or *speed coordinates.*

2. That the velocities corresponding to the controllable co-ordinates are always very small in comparison with the others, and further that the accelerations of the uncontrollable or cyclic velocities are also small.

Thus for the controllable co-ordinates the equations hold

$$\dot{\psi} = 0, \quad \dot{\phi} = 0, \ldots, \quad \frac{\partial L}{\partial\dot{\psi}} = 0, \quad \frac{\partial L}{\partial\dot{\phi}} = 0, \ldots \quad . \quad (91)$$

and the equations of motion are

$$-\frac{\partial L}{\partial\psi} = \Psi, \quad -\frac{\partial L}{\partial\phi} = \Phi, \ldots \quad . \quad . \quad . \quad . \quad (92)$$

For the uncontrollable co-ordinates we have in like manner

$$\frac{\partial L}{\partial\chi} = 0, \quad \frac{\partial L}{\partial\chi'} = 0, \ldots \quad . \quad . \quad . \quad . \quad . \quad (93)$$

$$\frac{ds_\chi}{dt} = \mathrm{X}, \quad \frac{ds_\chi}{dt} = -\mathrm{X}', \ldots \quad . \quad . \quad . \quad . \quad (94)$$

Thus according to the assumptions there are no forces depending on the cyclic co-ordinates. Thus changes of energy of the system cannot be introduced by change of the cyclic coordinates of the system, but change of the cyclic velocities may do so. A system fulfilling this condition is sometimes said to have an adiabatic motion.

The work communicated by the system to without or dW is $-\Sigma\Psi d\psi$, so that if dQ be the energy obtained through the unconstrainable co-ordinates

$$dQ = dE + dW = dE - \Sigma\Psi d\psi \quad . \quad . \quad . \quad . \quad (95)$$

From the equations of motion in terms of $s_\chi, \ldots$ just obtained for the unconstrainable co-ordinates

$$dQ = \Sigma \dot{s}_\chi \dot{\chi} dt = \Sigma \dot{\chi} ds_\chi \quad . \quad . \quad . \quad . \quad . \quad (96)$$

If the system be monocyclic and there is only one cyclic co-ordinate χ

$$dQ = \dot{\chi} ds_\chi.$$

Dividing by $2T = \dot{\chi} s_\chi$ we get

$$\frac{dQ}{T} = 2d(\log s_\chi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

so that $1/T$ is an integrating factor of dQ, and dQ/T is a perfect differential.

The above brief discussion of cyclic systems contains its general principles. Further information on the subject will be found in the examples which occur below.

We here close this dynamical chapter. Special forms of the Lagrangian function and their applications will be discussed in their proper connection, and we shall find in the dynamical analogues of electrical theorems further examples of the use of Lagrange's equations. The reader may, however, be referred to v. Helmholtz's papers in *Crelle's Journal*, Bd. 100, pp. 137, 213, to Professor J. J. Thomson's *Applications of Dynamics to Physics and Chemistry*, and to Professor G. H. Bryan's *Report on the Present State of Thermo-dynamics*, Part I., B. A., Report, Cardiff, 1891. Thomson and Tait's *Natural Philosophy* (Vol. I., Part I.) contains, besides a full account of general dynamical theory, a very valuable discussion of the cycloidal motions of gyrostatically dominated systems.

CHAPTER VIII

MOTION OF A FLUID

SECTION I.—*General Kinematical Theory*

Fundamental Assumptions. Lagrangian and Eulerian Methods

270. The analogy between the theory of electromagnetism and the theory of the motion of an incompressible frictionless fluid is so close, and later researches having for their object the dynamical explanation of electromagnetic action have emphasised so much more forcibly the fact that some vortex theory of electric currents is probably the true one, that it seems desirable to preface the discussion of electromagnetic theory by a chapter on fluid motion. Much space will thus be saved, since the proofs of general theorems here given need not be repeated; and the analogies and theories put forward will be rendered intelligible, without its being necessary to have recourse to treatises on hydrodynamics for a sufficiently full discussion of what are as much theorems of electrokinetics, as of vortex-motion of a fluid, if, in point of fact, the two things are not in some sense identical.

271. A fluid is a substance isotropic, and, if it is incompressible, perfectly uniform in quality, so far as it is here considered. The smallest part which in the course of mathematical analysis we have to deal with will be supposed to possess the properties of the fluid in bulk, so that no reference is necessary to the statistical treatment which becomes inevitable when the grained structure and relative motions of the molecules of the substance are taken into account. We suppose also that the mutual action between two portions of the fluid is at right angles to the interface separating them, so that tangential forces are regarded as non-existent, whether the fluid is at rest or in motion.

This action taken per unit of area at any point of an interface is called the "pressure in the fluid at the point." The forces acting on a portion of a fluid thus consist of forces calculable from the pressures at the different points of the bounding surface, and the *applied forces*, which, from whatever cause, act on the matter contained in the portion considered. Such forces are the force of gravity, electric and magnetic

forces, and forces arising from molecular attraction and repulsion, and are distinguished from the pressure through being applied from without the fluid.

272. The first problem, then, which arises in fluid motion is the calculation of the acceleration of the fluid at a particular instant. There are two ways in which this problem may be dealt with. We may consider either the acceleration, at the given instant, of the fluid at a specified point, or the acceleration, at the given instant, of a particular small part or "particle" of the fluid. In the first case the analysis does not directly follow the motion of a particle, but only does so by taking account of the variation of the motion at all the different points in the fluid; in the latter case the analysis follows each particle throughout the whole course of its displacement. Thus in the first case the co-ordinates, x, y, z, of a given point in the fluid are not regarded as variable, and hence x, y, z, as specifying such a point are not functions of the time t. On the other hand, if $x + dx, y + dy, z + dz$, be the co-ordinates of another point in the fluid infinitely near to x, y, z, on the path pursued by a particle of the fluid, and if dt be the interval of time taken by the particle to pass from the first point to the second, the ratios $dx/dt, dy/dt, dz/dt$, are the component velocities of the fluid parallel to the axes. To avoid any possibility of misunderstanding we shall, as is usual, denote these quantities by the symbols u, v, w. When other systems of co-ordinates are used we shall adopt the fluxional notation for velocities, and sometimes for accelerations; for example, in the polar system r, θ, ϕ, of co-ordinates, $\dot{r}, \dot{\theta}, \dot{\phi}$, will denote the time-rates of variation of r, θ, ϕ, for a particle of the fluid.

273. In the other method of dealing with the kinematics of a fluid the velocities u, v, w, along the axes at the point x, y, z, are functions merely of the time and the co-ordinates of the particle at the instant of time taken as the zero of reckoning. Thus, if a, b, c, be these co-ordinates

$$u = f_1(a, b, c, t), \quad v = f_2(a, b, c, t), \quad w = f_3(a, b, c, t),$$

and the acceleration is given by the equations

$$\dot{u} = \frac{\partial f_1}{\partial t}, \quad \dot{v} = \frac{\partial f_2}{\partial t}, \quad \dot{w} = \frac{\partial f_3}{\partial t}, \quad \ldots\ldots \quad (1)$$

where $\frac{\partial}{\partial t}\dot{}$ is used to denote partial differentiation with respect to t.

274. In general, in accordance with the notation elsewhere used in this book, we shall denote by $\frac{\partial}{\partial x}$, &c., differentiation with respect to variables x, &c., supposed explicitly appearing in the expressions for the quantities differentiated. This agrees with the notation now almost universally adopted for partial differentiation by Continental mathematicians. We shall denote complete time-rate of variation by $\frac{d}{dt}$ or

by the fluxional notation, as $\dot{u}$, $\dot{v}$, $\dot{w}$, for example. For convenience of printing we shall, as usual, write both kinds of differential co-efficients with the *solidus* notation where they occur in the text.

The two modes of dealing with the kinematics of a fluid explained above have been called the Eulerian and Lagrangian methods respectively. They were, however, both used by Euler; the latter was adopted by Lagrange in his great classical treatise on dynamics, the *Mécanique Analytique*.

Calculation of Acceleration of Fluid

275. To find the accceleration, in the so-called Eulerian mode of considering the motion, we have to notice that the velocity at a particular point may be perfectly invariable with the time, while the velocity at any instant is variable from point to point. We are thus led to consider the velocity of the fluid at a point x, y, z as a function, f say, of the time and the co-ordinates; so that, if x, y, z are regarded as constant, the time-rate of variation of the velocity at x, y, z is $\partial f/\partial t$. If the function does not involve t, $\partial f/\partial t$ is zero, and the motion is said to be *steady*.

By this latter mode of reckoning, if $u + \delta u, v + \delta v, w + \delta w$ be the component velocities of a portion of the fluid at time $t + \delta t$, u, v, w being their values at time t, we have approximately

$$u + \delta u = u + \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}\right)\delta t,$$

where dx/dt, &c., have the meanings explained above. Hence

$$\frac{\delta u}{\delta t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Thus, writing $\dot{u}, \dot{v}, \dot{w}$ for the component accelerations of the particle which is at x, y, z, we have the equations

$$\left.\begin{aligned} \dot{u} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \dot{v} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \dot{w} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

276. It is important to emphasize here the fact, already noticed above, that, while x, y, z, *considered as the co-ordinates of a particle of the fluid*, vary with the time, they are not in these equations considered as expressed as functions of t, but are the co-ordinates of a point in the fluid at which the velocity (u, v, w) and its rate of variation are reckoned.

It is to be observed that, if any quantity, F, whatever, characteristic

of the fluid or its motion, be expressible, like u, v, w, as a function of x, y, z, t, we have, precisely as in (2),

$$\dot{F} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \quad . \quad . \quad . \quad . \quad (3)$$

Velocity Potential

277. In certain cases of fluid motion the velocity is derivable from a function of x, y, z in the same way as the electric forces are in electrostatics derived from the electric potential. Thus if ϕ denote this function we have

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} \quad . \quad . \quad . \quad . \quad (4)$$

The function ϕ is called the *velocity-potential*. It may be single-valued or multiple-valued. In the latter case the motion is said to be *cyclic*, in the former *acyclic*.

In all other cases of fluid motion the velocity is not thus derivable from a potential function. All such motion is said to be rotational, in contradistinction to the motion for which a function ϕ exists, which is termed "irrotational motion." Rotational motion is also frequently called *vortex-motion*. The justification of these terms will appear in the discussions which follow.

Equation of Continuity. Theorem of Divergence

278. To complete this short kinematical discussion we have to establish an equation which is fulfilled by the motion, whether rotational or not, expressing the fact that, whatever rate of increase or diminution there may be of the quantity of fluid in any portion of space within which the motion of the fluid is considered, it must be equal to the rate of flow in or of flow out, as the case may be, of fluid across the boundary of the space.

Consider a rectangular parallelepiped of the fluid of edges dx, dy, dz, having its centre at x, y, z, where the velocity is u, v, w. If ρ be the mean density of the fluid within it at any time the mass of fluid contained by it is $\rho\, dx\, dy\, dz$. Consider the flow in the direction of x at the two ends of an infinitely thin filament of the volume, having its length parallel to x, and its left-hand and right-hand ends at the points $x - \frac{1}{2}dx, y + \frac{1}{2}\theta_1 dy, z + \frac{1}{2}\theta_2 dz$, $x + \frac{1}{2}dx, y + \frac{1}{2}\theta_1 dy, z + \frac{1}{2}\theta_2 dz$, where θ_1, θ_2, are quantities numerically less than unity. If σ be the cross-section of the filament, the rate of flow in at the left-hand end is to quantities of the first order of smallness

$$\left\{\rho u - \tfrac{1}{2}dx\frac{\partial(\rho u)}{\partial x} + \tfrac{1}{2}\theta_1 dy\frac{\partial(\rho u)}{\partial y} + \tfrac{1}{2}\theta_2 dz\frac{\partial(\rho u)}{\partial z}\right\}\sigma,$$

and the rate of flow out at the other end is

$$\left\{\rho u + \tfrac{1}{2}dx\frac{\partial(\rho u)}{\partial x} + \tfrac{1}{2}\theta_1 dy\frac{\partial(\rho u)}{\partial y} + \tfrac{1}{2}\theta_2 dz\frac{\partial(\rho u)}{\partial z}\right\}\sigma,$$

where $\partial(\rho u)/\partial x$, &c., are the rates of variation of ρu at the centre in the directions of x, y, z.

The excess of the rate of flow in above the rate of flow out is therefore

$$-dx\frac{\partial(\rho u)}{\partial x}\sigma,$$

and this is the same for each filament in the direction of x. Hence the total excess, for the parallelepiped, of rate of flow in, in the direction of x, above rate of flow out in the same direction is got by replacing σ by $dy\,dz$, and is

$$-\frac{\partial(\rho u)}{\partial x}dx\,dy\,dz.$$

Similarly for the directions of y, z, we get the excesses

$$-\frac{\partial(\rho v)}{\partial y}dxdydz, \quad -\frac{\partial(\rho w)}{\partial z}dxdydz;$$

and the total excess is

$$-\left\{\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right\}dxdydz.$$

This clearly must be equal to the time-rate of increase of the amount of fluid within the enclosure, that is to $\partial\rho/\partial t.dxdydz$, since ρ is a function of x, y, z, t and x, y, z is fixed for the element of volume considered. Hence we have the equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \quad . \quad . \quad . \quad . \quad (5)$$

If now we consider the time-rate of variation of the density of a portion of the fluid, or $\dot\rho$, we have by (3)

$$\dot\rho=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z},$$

so that (5) may be written

$$\dot\rho+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

279. If ρ is invariable, that is if the fluid is incompressible, $\dot\rho=0$, and we obtain the equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Equations (5), (6), and (7) are forms of what is commonly called the equation of continuity. The form (7) will be of most use in what follows, as expressing the kinematical condition fulfilled by the motion of an incompressible fluid, with which we shall have mainly to deal.

280. If $d\varpi$ denote the volume of an element of the fluid, equations

(5) and (6) express the invariability of $\rho d\varpi$ as the element changes its position in space. For $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$ is the rate at which the element increases per unit of its volume, or the *expansion* as it is called, and therefore $\rho\,(\partial u/\partial x + \ldots.)$ is the rate per unit volume at which the element is gaining mass in consequence of the expansion of volume alone, while $\dot\rho$ is the rate per unit volume at which the density of the fluid increases. The sum of these two rates of increase must be zero, since the element, preserving its identity, does not alter in mass. The quantity $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$ has also been called the *divergence* of the velocity at the point x, y, z, and the term is often used for other quantities the components of which satisfy the same relation. Thus Laplace's equation, Art. 192 above, states that the divergence of electric force in a dielectric in equilibrium is zero.

281. If the Lagrangian method is followed, the equation of continuity takes a somewhat different form. Let da, db, dc be the edges of a rectangular parallelepiped of the fluid when $t = 0$; at time t it will be an oblique parallelepiped. Let its centre be at x, y, z. The co-ordinates, relatively to the centre, of the eight corners of the original element were

$$\pm \tfrac{1}{2}da, \quad \pm \tfrac{1}{2}db, \quad \pm \tfrac{1}{2}dc.$$

The position of any particle of the element is a function of its original co-ordinates and the time t. Thus the particle, which had for original co-ordinates $a + \frac{1}{2}da, b + \frac{1}{2}db, c + \frac{1}{2}dc$, has at time t the co-ordinates

$$x + \tfrac{1}{2}\Sigma\frac{\partial x}{\partial a}da, \quad y + \tfrac{1}{2}\Sigma\frac{\partial y}{\partial b}db, \quad z + \tfrac{1}{2}\Sigma\frac{\partial z}{\partial c}dc,$$

where for brevity we write

$$\Sigma\frac{\partial x}{\partial a}da \equiv \frac{\partial x}{\partial a}da + \frac{\partial x}{\partial b}db + \frac{\partial x}{\partial c}dc.$$

and similarly for the others.
The relative co-ordinates of the particle are thus—

$$\tfrac{1}{2}\Sigma\frac{\partial x}{\partial a}da, \quad \tfrac{1}{2}\Sigma\frac{\partial y}{\partial b}db, \quad \tfrac{1}{2}\Sigma\frac{\partial z}{\partial c}dc,$$

and therefore the projections on the axis of x, of the sides which were originally da, db, dc are now

$$\frac{\partial x}{\partial a}da, \quad \frac{\partial x}{\partial b}db, \quad \frac{\partial x}{\partial c}dc.$$

Similarly those on the axes of y, z, are

$$\frac{\partial y}{\partial a}da, \quad \frac{\partial y}{\partial b}db, \quad \frac{\partial y}{\partial c}dc$$

$$\frac{\partial z}{\partial a}da, \quad \frac{\partial z}{\partial b}db, \quad \frac{\partial z}{\partial c}dc.$$

The volume of the parallelepiped is therefore now

$$\begin{vmatrix} \frac{\partial x}{\partial a}, & \frac{\partial x}{\partial b}, & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a}, & \frac{\partial y}{\partial b}, & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a}, & \frac{\partial z}{\partial b}, & \frac{\partial z}{\partial c} \end{vmatrix} da\, db\, dc,$$

or as we write it,

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} da\, db\, dc$$

and its density is ρ. Since the mass has not changed we have for the equation of continuity, denoting the original density by ρ_0,

$$\rho_0 = \rho \frac{\partial(x, y, z)}{\partial(a, b, c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If a, b, c be not the initial co-ordinates of the central particle, but parameters on the values of which these co-ordinates depend, then, denoting the initial co-ordinates by x_0, y_0, z_0, we get for the original volume of the element, by the same process as that used above, the expression $\partial(x_0, y_0, z_0)/\partial(a, b, c) \,.\, da\, db\, dc$, and for the equation of continuity

$$\rho_0 \frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)} = \rho \frac{\partial(x, y, z)}{\partial(a, b, c)} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Rotational and Irrotational Motion. Angular Velocity at Point in a Fluid

282. We now consider some of the characteristics of the different kinds of motion, reserving the establishment of the dynamical equations and their integration for a later Section. We take first the distinction between rotational and irrotational motion.

Let us suppose, first, that a velocity-potential $\phi(x, y, z)$ exists, so that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} . \quad . \quad . \quad . \quad (10)$$

Then we have

$$u dx + v dy + w dz = -d\phi \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad . \quad (12)$$

If the quantities on the left in the last three equations be denoted by 2ξ, 2η, 2ζ the equations may be written

$$\xi = 0, \quad \eta = 0, \quad \zeta = 0 \quad . \quad . \quad . \quad . \quad . \quad (12')$$

283. In the case in which the motion does not possess a velocity-potential the quantities ξ, η, ζ are at least not all zero. It is easy to show that they are the angular velocities about axes parallel to those of x, y, z, of an element of the fluid. For consider a small sphere of rigid material having its centre at the point x, y, z. Let its motion be parallel to the plane of y, z, and $\dot{\theta}$ be its angular velocity round the diameter parallel to x; let v, w denote the component velocities of its centre in the directions y, z respectively, $v+dv$, $w+dw$ those of a point on its surface the co-ordinates of which relatively to the centre are dy, dz. Then we have $v+dv=v-\dot{\theta}dz$, $w+dw=w+\dot{\theta}dy$, and therefore the relative velocities are $dv=-\dot{\theta}dz$, $dw=\dot{\theta}dy$.

Now the velocities parallel to y, z, of the fluid at the point x, $y+dy$, $z+dz$, taken relatively to (x, y, z) are $\partial v/\partial y \,.\, dy+\partial v/\partial z \,.\, dz$, $\partial w/\partial y \,.\, dy+\partial w/\partial z \,.\, dz$. If s denote $\frac{1}{2}(\partial w/\partial y+\partial v/\partial z)$ these relative velocities may be written $\partial v/\partial y \,.\, dy+sdz-\xi dz, sdy+\partial w/\partial z \,.\, dx+\xi dy$. The first two terms in each of these expressions are the velocities arising from a pure strain in the fluid; the last term in each is, by what we have just seen for the small sphere, the velocity arising from a rigid body rotation of the element of fluid, of angular velocity ξ, round an axis parallel to the axis of x. Similarly η, ζ, are the angular velocities round the other two axes.

It is to be observed that ξ, η, ζ are the angular velocities of rigid body rotation of an infinitesimal element of the fluid at x, y, z, but that, unlike a rigid body, the fluid has generally values of ξ, η, ζ which vary from point to point. It is also to be carefully noticed that an element of a fluid may move round an axis without having any rotational motion. As stated below, the quantities ξ, η, ζ have been termed the components of the *curl* of the velocity. The curl and the divergence of a quantity are intimately connected, being in fact together the result of a single vector operation. The curl of electric and magnetic quantities, like their divergence, plays a great part in electrical theory, and especially in the general theories which are discussed later in this treatise.

Permanence of Velocity Potential

284. We can show that under certain conditions, if a velocity-potential exist for any portion of fluid at any instant, the same portion of the fluid possesses a velocity-potential at any instant before or after.

It is obvious that if we integrate between 0 and t, and put u_0 for the x-component of the velocity when $t = 0$, the equation holds

$$u\frac{\partial x}{\partial a} - u_0 = \int_0^t \dot{u}\frac{\partial x}{\partial a}\,dt + \tfrac{1}{2}\frac{\partial}{\partial a}\int_0^t u^2 dt.$$

Here u is the same thing as $\dot{x}$, since the motion of a particle of the fluid is considered. Also $\partial x/\partial a = 1$, when $t = 0$.

We have likewise

$$v\frac{\partial y}{\partial a} = \int_0^t \dot{v}\frac{\partial y}{\partial a}\,dt + \tfrac{1}{2}\frac{\partial}{\partial a}\int_0^t v^2 dt$$

$$w\frac{\partial z}{\partial a} = \int_0^t \dot{w}\frac{\partial z}{\partial a}\,dt + \tfrac{1}{2}\frac{\partial}{\partial a}\int_0^t w^2 dt,$$

since $\partial y/\partial a = \partial z/\partial a = 0$, when $t = 0$.

It is to be observed that in these equations t may have any value positive or negative.

Adding these results, we obtain

$$u\frac{\partial x}{\partial a} + v\frac{\partial y}{\partial a} + w\frac{\partial z}{\partial a} - u_0$$

$$= \int_0^t \left(\dot{u}\frac{\partial x}{\partial a} + \dot{v}\frac{\partial y}{\partial a} + \dot{w}\frac{\partial z}{\partial a}\right)dt + \tfrac{1}{2}\frac{\partial}{\partial a}\int_0^t (u^2 + v^2 + w^2)\,dt. \quad (13)$$

Similarly two other equations can be written down by substituting in succession b and c for a, and at the same time v_0 and w_0 for u_0.

Multiplying these three equations respectively by da, db, dc, adding, and remembering that

$$dx = \frac{\partial x}{\partial a}da + \frac{\partial x}{\partial b}db + \frac{\partial x}{\partial c}dc, \text{ \&c.},$$

we find

$$udx + vdy + wdz - (u_0 da + v_0 db + w_0 dc)$$

$$= \int_0^t (\dot{u}dx + \dot{v}dy + \dot{w}dz)dt + \tfrac{1}{2}d\int_0^t (u^2 + v^2 + w^2)dt \quad . \quad (14)$$

Hence, if, when $t = 0$, $u_0 da + v_0 db + w_0 dc$ be a complete differential of a function of the variables a, b, c, the quantity $udx + vdy + wdz$ will also be a complete differential at time t (positive or negative) provided that $\dot{u}dx + \dot{v}dy + \dot{w}dz$ is one; for, u, v, w being functions of the coordinates and the time, the second part of the right-hand expression is a complete differential. We shall see later that this condition will be fulfilled if the applied forces acting on the fluid are conservative, that is are derivable from a potential function, and the density is a function only of the pressure. This proposition will be proved by another method later (see Art. 290 below).

285. It may be noticed that if the expression on the right of (14) be transposed to the left the expression obtained is (Art. 243) the change, dA, in the "action" involved in subjecting the terminal positions of the fluid particle to the variations dx, dy, dz, da, db, dc, in the given circum-

stances of the motion. By the principle of Least Action this should vanish; hence the equation may be deduced from this principle.

If the variations in the terminal positions be zero, this view of the equation shows that, if the fluid move from a given configuration, for which the velocities are specified, to another given configuration, the motion for which the action is least is that for which the work done upon the fluid in the passage is equal to the change which is produced in the kinetic energy.

Flow and Circulation. Circulation round Curve expressed as Surface Integral of Normal Spin

286. If ds be a line of elementary length drawn in the fluid, and q be the velocity of the fluid at its centre, we define the flow along the line as $q \cos\theta\, ds$, where θ is the angle between q and ds. Since

$$\cos\theta = \frac{u}{q}\frac{dx}{ds} + \frac{v}{q}\frac{dy}{ds} + \frac{w}{q}\frac{dz}{ds},$$

the flow becomes

$$\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right)ds,$$

or, as it is frequently written,

$$udx + vdy + wdz.$$

The value of $q\cos\theta$ is the component of velocity along ds, that is, $-\partial\phi/\partial s$ if the motion have a velocity-potential ϕ. Hence the flow along ds is $-d\phi$. Thus if we integrate along any curve s we get for the total flow along the curve

$$\int_s q\cos\theta\, ds = -(\phi_1 - \phi_0) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where ϕ_1, ϕ_0 are the values of ϕ at the final and initial ends of the curve. Thus, where there is a velocity-potential,

$$\int_s (udx + vdy + wdz) = \phi_0 - \phi_1 \quad . \quad . \quad . \quad . \quad (16)$$

in which the suffix attached to the sign of integration indicates that the integral is taken along the curve s.

If the curve is closed, then, as we shall prove presently, whether the potential is single- or multiple-valued, if it exist at every point of the path of integration, and a surface can be drawn having the path for its bounding edge and situated wholly in fluid possessing the velocity-potential, the flow is zero. The flow round a closed curve has been called by Lord Kelvin the *circulation* in the curve.

287. In considering circulation we shall take the ordinary left-handed system of axes represented in Fig. 65; that is, the axes will be supposed

so drawn that Ox can be turned into coincidence with Oy by a turn through 90° round Oz, in the counter-clockwise direction to an observer looking towards the origin from a point in Oz. Integration round the curve is taken, as explained at p. 49, in the direction in which an

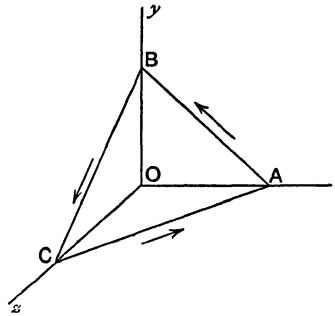


FIG. 65.

observer standing on that side of the area enclosed by the curve towards which a normal is considered drawn would have to pass round it so that he should have the area on his left hand. Thus in Fig. 65 the integration is taken in the direction of the arrows, and a normal, if drawn at any point of the triangular area would be drawn towards the side remote from O.

By precisely the same process as that adopted at p. 49 above we can show that

$$\int_{ABCA}(udx+vdy+wdz)=\left\{l\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+m\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+n\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}dS$$
$$=2(l\xi+m\eta+n\zeta)dS \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where dS is the area of the triangle ABC, and l, m, n are the direction cosines of the normal drawn to it in the direction indicated.

Now we can divide in this way any area S, whether plane or not, enclosed by a curve s, into elementary triangles, take the flow round each triangle in the manner indicated, and add the results. The flow along each side common to two triangles, being taken in opposite directions for the two, has no influence on the result, and there remains only the flow round the given bounding curve s. Thus we have the theorem

$$\int_s(udx+vdy+wdz)=2\int_S(l\xi+m\eta+n\zeta)dS \quad . \quad (18)$$

where the second integral is taken over the area S, and the first round its boundary s.

The second integral is thus zero over any area within which ξ, η, ζ are each zero, that is if the motion be there irrotational. It is to be noticed also that the integral taken over the surface S depends

only on the bounding edge, and that different surfaces having the same bounding edge and wholly situated within the moving fluid will give the same value of the integral, each of them being equal to the circulation in the edge. It follows, therefore, that if a surface S over which the integral is zero can be drawn with the given curve as bounding edge, all other surfaces whatsoever having the same bounding edge will give also zero integrals.

It is to be observed that the proof given above of the vanishing of the circulation only holds if the elementary triangles fill up the whole space within the circuit: in fact, the theorem only holds for a single closed circuit if the circuit is *reducible*, that is can be diminished to a point without passing out of the space within which the motion is irrotational.

Reducible and Irreducible Circuits

288. If the circulation be taken round a circuit which is irreducible, that is cannot be contracted to a point without passing somewhere out of the region in which the motion is irrotational, we can deal with the problem as follows. Let a surface be drawn having the given curve as

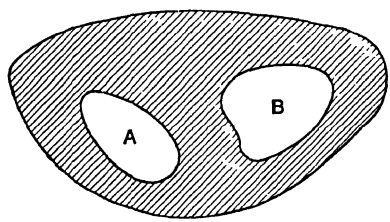


FIG. 66.

its outer bounding edge, and let the motion be irrotational only over the shaded portion of the enclosed surface as shown in Fig. 66. The theorem that the circulation is zero in the bounding curve will hold if the integral be taken round the outer curve and round the two regions

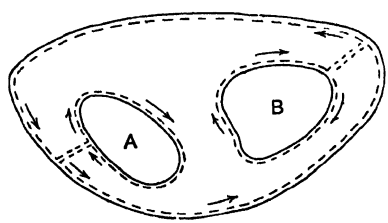


FIG. 67.

A and B in the opposite, or negative, direction with reference to their areas. This can be seen at once by considering the path indicated in Fig. 67, which is the former diagram repeated for clearness without the

shading. This path is clearly reducible and is all described in the positive direction with reference to the shaded area which it bounds. The circulation in it is therefore zero. The straight parts leading from the outer boundary to the inner curve are described each in opposite directions and therefore have no effect on the result. The circulation round the outer circuit is thus equal to the sum of the circulations in the opposite directions round the two smaller circuits in the interior.

The theorem of zero circulation in a fluid, in which there are any number of regions in which the motion is rotational, can thus be applied if the boundaries of these regions are taken into account as shown above.

Analysis of Motion at any Point in a Fluid

289. Having distinguished, as above, between rotational and irrotational motion, we consider now the nature of the motion at any point. If u, v, w be the velocities at time t at the point x, y, z, those at the same instant at $x + \mathsf{x}, y + \mathsf{y}, z + \mathsf{z}$, infinitely near the former, are

$$u + \mathsf{x}\frac{\partial u}{\partial x} + \mathsf{y}\frac{\partial u}{\partial y} + \mathsf{z}\frac{\partial u}{\partial z}, \text{ \&c., \&c.}$$

Thus the component velocities $\mathsf{u}, \mathsf{v}, \mathsf{w}$, at the second point relatively to the first are

$$\left.\begin{aligned} \mathsf{u} &= \mathsf{x}\frac{\partial u}{\partial x} + \mathsf{y}\frac{\partial u}{\partial y} + \mathsf{z}\frac{\partial u}{\partial z} \\ \mathsf{v} &= \mathsf{x}\frac{\partial v}{\partial x} + \mathsf{y}\frac{\partial v}{\partial y} + \mathsf{z}\frac{\partial v}{\partial z} \\ \mathsf{w} &= \mathsf{x}\frac{\partial w}{\partial x} + \mathsf{y}\frac{\partial w}{\partial y} + \mathsf{z}\frac{\partial w}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The first of these can be written

$$\mathsf{u} = \mathsf{x}\frac{\partial u}{\partial x} + \mathsf{y}\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mathsf{z}\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \mathsf{z}\eta - \mathsf{y}\zeta;$$

and similar expressions can be written down by symmetry for v, w, ξ, η, ζ, having the values stated above, p. 219. Hence, if for brevity we use the notation

$$\left.\begin{aligned} a = \frac{\partial u}{\partial x}, \quad b = \frac{\partial v}{\partial y}, \quad c = \frac{\partial w}{\partial z}, \\ f = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad g = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad h = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad (20)$$

we have for the relative velocities the equations

$$\left.\begin{aligned} \mathsf{u} &= a\mathsf{x} + h\mathsf{y} + g\mathsf{z} + \eta\mathsf{z} - \zeta\mathsf{y} \\ \mathsf{v} &= h\mathsf{x} + b\mathsf{y} + f\mathsf{z} + \zeta\mathsf{x} - \xi\mathsf{z} \\ \mathsf{w} &= g\mathsf{x} + f\mathsf{y} + c\mathsf{z} + \xi\mathsf{y} - \eta\mathsf{x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21)$$

Thus the motion in the most general case consists of two parts: a motion in the direction of the normal to the quadric surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = \text{const.} \quad (21')$$

and a rotation (of which the component angular velocities are ξ, η, ζ) round an axis having direction cosines $(\xi, \eta, \zeta)/(\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}$. The motion corresponds to the general strain of an elastic body. The former part may be called a motion of pure strain, and its axes are those of the quadric (21'). If we take the axes coincident with the principal axes of this quadric, we have for the velocities u', v', w' along them due to the pure strain the expressions

$$u' = a'x', \quad v' = b'y', \quad w' = c'z' \quad . \quad . \quad . \quad . \quad (22)$$

so that a', b', c', are the time-rates of elongation, per unit length, of lines in the direction of x', y', z'.

If we were to suppose that the motion consists of a pure strain and a rotation, both of them arbitrarily assumed, we should in endeavouring to suit this to the state of relative motion get precisely the same analysis of the motion as has been given above. Thus there is only one possible analysis of the motion into a motion of pure strain and a rotation.

Constancy of Flow along any Path moving with the Fluid

290. We shall now prove a theorem due to Lord Kelvin, which is of very great importance, as it embraces almost the whole theory of fluid motion. The statement of the theorem in dynamical language will, however, be given later, when we have dealt with the dynamical equations of motion.

Let us consider an element ds of a line moving with the fluid, and calculate the rate at which $udx + vdy + wdz$ is altering for the element. We get at once

$$\frac{d}{dt}(udx + vdy + wdz) = \dot{u}dx + \dot{v}dy + \dot{w}dz + udu + vdv + wdw \quad (23)$$

The time-rate of variation of the flow along a finite curve is obtained by integrating the expression on the right along the curve. Thus we obtain the theorem

$$\frac{d}{dt}\int_s (udx + vdy + wdz) = \int_s (\dot{u}dx + \dot{v}dy + \dot{w}dz) + \tfrac{1}{2}(q_1^2 - q_0^2) \quad (24)$$

where q_1, q_0 are the velocities at the final and initial ends of the curve.

We shall see that under the same conditions as stated in Art. 284 the first part of the expression on the right of (23) is a perfect differential, and hence that the integral taken round a closed curve vanishes, that is the circulation remains constant. Hence if the circulation round the curve is once zero it is always zero. Hence (18) if ξ, η, ζ be zero in a portion of a fluid they remain zero in that portion.

SECTION II.—*Dynamical Theory*

Equations of Motion of a Fluid

291. We now consider the dynamical equations, that is, the equations obtained by equating the value of the acceleration calculated above to its value obtained from a consideration of the density, pressure, and applied forces. Consider again the parallelepiped having its centre at x, y, z and its edges of lengths dx, dy, dz parallel to the axes. The mass of the element is $\rho dx\, dy\, dz$, if ρ be the density of the fluid at the element, and therefore, if X be the applied force per unit of mass in the direction of x, the total applied force in this direction on the element is $\rho X\, dx\, dy\, dz$. If p be the pressure at the centre, the difference of pressures on the two ends, reckoned positively when towards the right, is, by the process used in Art. 277, $-\partial p/\partial x\,.\,dx$. Hence the force to the right due to the pressure is $-\partial p/\partial x\,.\,dxdydz$. Thus we get the equation

$$\left(\rho X - \frac{\partial p}{\partial x}\right) dx\, dy\, dz = \rho \dot{u}\, dx\, dy\, dz,$$

or substituting the value of $\dot{u}$ from (2), and proceeding similarly for the other two pairs of forces, we obtain the three equations

$$\left.\begin{aligned} X - \frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ Y - \frac{1}{\rho}\frac{\partial p}{\partial y} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ Z - \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \quad . \quad . \quad (25)$$

If the forces are derivable from a potential function, Ω, the potential energy per unit of mass of the fluid, we can write these equations, using $\dot{u}$ $\dot{v}$, $\dot{w}$ for brevity,

$$-\left(\frac{\partial \Omega}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right) = \dot{u}, \quad -\left(\frac{\partial \Omega}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right) = \dot{v}, \quad -\left(\frac{\partial \Omega}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right) = \dot{w} \quad (25')$$

Hence we see at once that, if ρ be a function of p, $\dot{u}dx + \dot{v}dy + \dot{w}dz$ is a perfect differential. For, multiplying the equations in order by dx, dy, dz, and adding, we get, putting $dp/\rho = f'(p)dp$,

$$-d\{\Omega + f(p)\} = \dot{u}dx + \dot{v}dy + \dot{w}dz,$$

which proves the statements made at the end of Art. 284 above.

292. The statement of Lord Kelvin's theorem given in (24) now becomes

$$\frac{d}{dt}\int_s (udx + vdy + wdz) = -\int_s \frac{dp}{\rho} - \left[\Omega - \tfrac{1}{2}q^2\right]_s \quad . \quad (26)$$

where the suffix s after the bracket on the right denotes that the value of the enclosed quantity at the initial extremity is to be subtracted from that at the final extremity of s.

Let us suppose that a velocity-potential exists; then

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z},$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$$

and the equations of motion become

$$\left.\begin{aligned} X - \frac{1}{\rho}\frac{\partial p}{\partial x} &= -\frac{\partial^2\phi}{\partial t\partial x} + \tfrac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) \\ &\text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}\right\} \quad . \quad . \quad (26')$$

Multiplying the first of these by dx, the second by dy, and the third by dz, adding and integrating along any curve s drawn in the fluid, we get

$$\int_s (X dx + Y dy + Z dz) - \int_s \frac{dp}{\rho} = \left[-\frac{\partial\phi}{\partial t} + \tfrac{1}{2}q^2 \right]_s \quad . \quad . \quad (27)$$

If X, Y, Z have the potential Ω, this equation becomes

$$-\left[\Omega + \int\frac{dp}{\rho}\right]_s = \left[-\frac{\partial\phi}{\partial t} + \tfrac{1}{2}q^2 \right]_s \quad . \quad . \quad . \quad . \quad (28)$$

This result is generally put in the form of an indefinite integral, thus

$$\int\frac{dp}{\rho} = \frac{\partial\phi}{\partial t} - \Omega - \tfrac{1}{2}q^2 + F(t) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where $F(t)$ is an arbitrary function of t, which is added to complete the indefinite integral, and which if not expressed may be regarded as included in $\partial\phi/\partial t$.

The pressure is thus indeterminate; all that we can obtain by (27) or (29) is the difference of pressures between two points at the same time. If, however, the pressure be given throughout the fluid for any value of t, it is completely determinate for any other time.

The Lagrangian equations of motion of the fluid are comparatively rarely employed. They may be constructed by putting $\ddot{x}, \ddot{y}, \ddot{z}$ for the quantities on the right of the three equations of (25) respectively, multiplying then the three resulting equations by $\partial x/\partial a, \partial y/\partial a, \partial z/\partial a$ and adding for the first equation, then multiplying by the same quantities, but with the variable a, replaced first by b, then by c, for the remaining equations. The equations are thus

$$\left.\begin{aligned} (\ddot{x} - X)\frac{\partial x}{\partial a} + (\ddot{y} - Y)\frac{\partial y}{\partial a} + (\ddot{z} - Z)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{dp}{da} &= 0 \\ . \quad . \end{aligned}\right\} \quad (30)$$

Kinetic Energy.—Rate of Variation of Energy in Given Space

293. If T denote the kinetic energy of the fluid motion, and $d\varpi$ an element of volume, we have, integrating through the fluid,

$$T = \tfrac{1}{2}\int \rho(u^2 + v^2 + w^2)d\varpi \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Multiplying the equations of motion (25′), in the case in which the forces have a potential, by u, v, w, and adding, we obtain

$$\rho(u\dot{u} + v\dot{v} + w\dot{w})d\varpi + \rho\left(\frac{\partial\Omega}{\partial x}\dot{x} + \frac{\partial\Omega}{\partial y}\dot{y} + \frac{\partial\Omega}{\partial z}\dot{z}\right)d\varpi = -\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)d\varpi.$$

But it has been shown above (Art. 280) that

$$\frac{d}{dt}(\rho d\varpi) = 0,$$

and hence the last equation may be written, if $\partial\Omega/\partial t = 0$, in the form

$$\frac{d}{dt}(T + V) = -\int\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)d\varpi \quad . \quad . \quad (32)$$

where V is the potential energy of the fluid within the space and at the instant under consideration.

By integrating the expression on the right by parts, taking l, m, n as the direction cosines of the normal drawn *inwards* to the bounding surface at any element dS, we obtain

$$\frac{d}{dt}(T + V) = \int p(lu + mv + nw)dS + \int p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)d\varpi \quad (33)$$

If the fluid is incompressible the last term is zero and we get the theorem

$$\frac{d}{dt}(T + V) = \int p(lu + mv + nw)dS \quad . \quad . \quad . \quad (34)$$

which expresses the fact that then the time-rate of increase of the whole energy of the fluid within the space S is equal to the rate at which work is done by the pressure of the fluid at the bounding surface on fluid crossing it, the inward direction being taken as positive.

Stream-lines

294. A curve drawn in the fluid so that the tangent at every point is the direction of the flow there is called a stream-line. The equations of a stream-line are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{ds}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

or, if the velocities are derivable from a potential,

$$\frac{dx}{\frac{\partial\phi}{\partial x}} = \frac{dy}{\frac{\partial\phi}{\partial y}} = \frac{dz}{\frac{\partial\phi}{\partial z}} = \frac{ds}{\frac{\partial\phi}{\partial s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35')$$

which shows that, if the component velocities are so derivable, the stream-line at any point of a surface, for which ϕ has a constant value, is in the direction of the normal to the surface at that point.

In the case of steady motion (29) becomes

$$\int\frac{dp}{\rho} + \Omega + \tfrac{1}{2}q^2 = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where C is a constant, and this may be applied of course to a stream-line.

We may, however, integrate along a stream-line without assuming the existence of a velocity-potential. Since, the motion being steady, $\partial u/\partial t, \ldots$. are each zero, we have by the equations of motion

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial\Omega}{\partial x}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,$$

which by the equations of a stream-line may be written

$$u\frac{du}{ds} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{dx}{ds} - \frac{\partial\Omega}{\partial x}\frac{dx}{ds}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Adding these equations, we find for an element ds of a stream-line

$$\tfrac{1}{2}\frac{d(q^2)}{ds} = -\frac{1}{\rho}\frac{dp}{ds} - \frac{d\Omega}{ds},$$

of which the integral along the stream-line is

$$\tfrac{1}{2}q^2 + \int\frac{dp}{\rho} + \Omega = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where C is a constant which has the same value along each particular stream-line, but in the general case has different values for different stream-lines.

Motion in Two Dimensions.—Conjugate Functions

295. In the case of irrotational motion of an incompressible fluid which is independent of the values of one of the co-ordinates z, say, or, as it is called, motion in two dimensions, the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The equation of a stream-line may be written in this case

$$udy - vdx = 0,$$

and the equation of continuity is the condition that

$$u = -\partial\psi/\partial y, \quad v = +\partial\psi/\partial x, \quad \ldots\ldots \quad (38')$$

so that the last equation should be capable of being written in the form

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \quad \ldots\ldots \quad (39)$$

where ψ is a function of x, y, t in its most general form. It is called the stream-function.

If the motion be steady the integral equation of a stream-line is

$$\psi - C \quad \ldots\ldots \quad (40)$$

The whole system of stream-lines is given by taking C in the last equation as a parameter which varies from stream-line to stream-line.

If the motion be irrotational (38′) gives

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}, \quad \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} \quad \ldots\ldots \quad (41)$$

and the equation of continuity is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

Equations (41) give also

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad \ldots\ldots \quad (42)$$

which expresses the fact that $\zeta\,[=\frac{1}{2}(\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2)]$ is zero. Of course by (41) ξ, η, are identically zero.

The differential equation of an equipotential line is

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \quad \ldots\ldots \quad (43)$$

and by equations (41) that of a stream-line is

$$\frac{\partial\phi}{\partial y}dx - \frac{\partial\phi}{\partial x}dy = 0 \quad \ldots\ldots \quad (44)$$

so that the equipotential lines and the stream-lines are two systems of lines in the plane of x, y at right angles to one another.

Along an element δs of an equipotential line the variation $\delta\psi$ of the stream function is

$$\frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y = v\delta x - u\delta y \quad . \quad . \quad . \quad . \quad . \quad (45)$$

that is, $\delta\psi$ measures the rate of flow of fluid across δs in the direction from right to left to an observer looking along δs in the positive direction.

296. Apart from the possibility of realising the motions, it may be noticed that it follows from what has just been shown that the curves $\phi = \text{const.}$, $\psi = \text{const.}$, form a conjugate system, in the sense that when either set is taken as the equipotential curves the other set is the corresponding stream-lines. [Fig. 79 below shows such a conjugate system of lines.] Hence ϕ and ψ have been called conjugate functions. Their theory is very fully developed in modern treatises on the Theory of Functions of a Complex Variable,[1] and in works on Hydrodynamics and the Mathematical Theory of Electricity. Some account of their applications to electricity will be given later, but as excellent works on the Theory of Functions are now available no space will be devoted to proofs of their purely mathematical properties.

General Theorems for Incompressible Fluid in Irrotational Motion.—Integral Equation of Continuity.—Tubes of Flow

297. Several theorems of great importance, which are all analogues of well-known theorems in electricity and magnetism, can now be proved for the irrotational motion of an incompressible fluid. The theorems will hold also for the electric and magnetic applications when the corresponding quantities are substituted in the equations.

In the first place, for such a fluid the existence of the velocity potential ϕ causes the equation of continuity (7) to take the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

which is identical with Laplace's equation, already considered at p. 46 above. Denoting the expression on the left, as usual, by $\nabla^2\phi$, we see that throughout any mass of incompressible fluid moving irrotationally we have $\nabla^2\phi = 0$.

298. It follows at once that if $\partial\phi/\partial n$ denote the rate of variation of ϕ per unit of distance inwards along a normal to a closed surface S drawn in the fluid the equation

$$\int_S \frac{d\phi}{dn}\,dS = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

[1] See Forsyth, *Theory of Functions*, Camb. Univ. Press, 1893; Harkness and Morley, *Theory of Functions*, Macmillan, 1893; Klein, *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale;* Maxwell, *Electricity and Magnetism*, Vol. I.; Lamb, *Hydrodynamics*, 1895.

holds. For by direct integration of the three terms of $\nabla^2\phi$ with respect to x, y, z respectively

$$\iiint \nabla^2\phi \, . \, dxdydz = -\int_S \frac{d\phi}{dn} dS \, . \quad . \quad . \quad . \quad . \quad (47')$$

where the first integral is taken throughout the whole space enclosed by the surface S. If at every point of that space $\nabla^2\phi = 0$, we get (47). The second integral represents the total rate of flow across the bounding surface into the enclosed space. Hence (47) is sometimes called the integral equation of continuity for an incompressible fluid.

299. Consider any small closed curve drawn in the fluid, and let a stream-line be drawn through every point of the curve. These stream-lines will mark out a tubular surface, which we call a tube of flow. If the closed curve referred to lie on an equipotential surface each line will be at right angles to the plane of the curve at the point through which it is drawn. Consider then the portion of a tube of flow intercepted between two equipotential surfaces. Equation (47) holds for the portion considered if, as we here suppose, it does not include any discontinuity or fluid moving otherwise than irrotationally. But, except at the ends where the tube intersects the equipotential surfaces, $\partial\phi/\partial n$ is zero. Hence, denoting by S_1 and S_2 the ends of the portion of tube, the equation can be written

$$\int_{S_1} \frac{d\phi}{dn} dS_1 + \int_{S_2} \frac{d\phi}{dn} dS_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (48)$$

that is, the surface integral of normal flow outwards over one end face is equal to the surface integral of normal flow inwards over the other.

300. For any surface whatever the integral of normal flow across it may be regarded as the sum of integrals taken over portions σ_1, σ_2, σ_3, ... of the surface, so chosen that the value of each integral is unity. The tube of flow marked out by stream-lines drawn through the points of the boundary of any portion σ is called a unit tube. The flow across any surface is then equal to the number of unit tubes of flow which cross the surface. Equation (47) expresses the fact that the number of unit tubes which cross a closed surface within which $\nabla^2\phi = 0$ at every point is algebraically zero, that is, as many cross in one direction as in the other.

It follows from this result that a tube of flow cannot begin or end within the fluid, and must therefore be either endless or have its ends on the surface of the fluid. The endlessness of lines or tubes of flow is excluded in the case of single-valued velocity-potential, that is in irrotational motion in a simply connected region. For the potential increases or decreases continuously along a stream-line.

301. The condition that $\nabla^2\phi = 0$ at every part of the enclosed space renders it necessary that, when the closed surface includes any space or

spaces at any part of which this condition does not hold, the value of the surface integral of (47) taken over surfaces separating these regions from the rest of the fluid should be taken into account. These surfaces form, in fact, part of the bounding surface of the irrotationally moving fluid, and if we consider the values of the surface-integrals obtained by integrating, respectively, the terms on the left of (47′) with respect to x, y, z, we see that the statement made is correct. For example, when we integrate with respect to x the integral is taken along a filament at y, z, parallel to the axis of x, and a contribution to the surface integral is obtained wherever the filament meets the bounding surface.

Periphractic Spaces

302. It is supposed here, of course, that the space considered is simply connected, that is, that any circuit drawn in it can be contracted to zero without passing out of the space,[1] or that any two closed circuits drawn in the space are reconcilable by continuous change of one or both without passing out of the space considered. A simply connected space of this kind bounded by two or more unconnected surfaces is called *periphractic*. Such, for example, is the space included between two concentric spherical surfaces, or between an outer bounding surface of a fluid and any simply connected solids which may be immersed in the fluid. We shall consider multiply connected spaces later.

303. The truth of (47) for such cases as we have here supposed may be seen intuitively as follows, if it is admitted for a single bounding surface. Take any space, such as the space of which Fig. 66 (with A contracted to zero) may be regarded as a section, surrounding a closed space B throughout which $\nabla^2\phi$ is not zero. It is only necessary to connect the space B by a narrow tubular surface with the outer boundary. This tube can be made so fine as not to interfere sensibly with the motion of the fluid. The two surfaces are thus converted into a single surface enclosing the space between them, and the surface integral, taken over the whole bounding surface, is simply the sum of the integrals taken over the two surfaces, since that over the tube is vanishingly small.

When more surfaces are included it is only necessary to imagine a connecting tube for each with the outer surface (or tubes connecting all the enclosed spaces with one another, and one connecting one of these spaces with the outer surface) to enable the theorem given for a single surface to be applied.

From these considerations we see that (47) asserts in such cases that the integral over the outer enclosing surface is equal and opposite to the sum of the integrals over the enclosed surfaces; that is, that the total rate of flow into the space considered over the internal surfaces is equal to that outwards across the enclosing surface, or *vice versa*.

[1] A circuit which can be contracted to zero as here described is said to be *reducible*.

Mean Potential over Spherical Surface

304. Let $4\pi M$ denote the total rate of flow into the space considered, across the inner bounding surfaces, and let the outer surface be spherical. Integrating over the latter surface, we have

$$\int \frac{d\phi}{dr}\, dS = -\, 4\pi M,$$

or

$$\frac{1}{4\pi}\frac{d}{dr}\int \phi\, d\varpi = -\frac{M}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

where $d\varpi$ is the solid angle subtended at the centre of the sphere by the element of surface dS. Integrating with respect to r, we have

$$\frac{1}{4\pi}\int \phi d\varpi = \frac{M}{r} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where C is a constant which does not depend upon the radius, but yet so far as we have seen may depend upon the position of the spherical surface. This gives the mean value of ϕ over the sphere.

Now let the sphere, without alteration of the radius, be displaced through a small distance dx in the direction of x. The change of the mean potential is

$$\frac{1}{4\pi}\left\{\int \frac{\partial \phi}{\partial x}\, d\varpi\right\} dx = \frac{\partial C}{\partial x}\, dx \quad . \quad . \quad . \quad . \quad . \quad (51)$$

that is, $\partial C/\partial x$ is the mean value of $\partial\phi/\partial x$ taken over the sphere. This vanishes when r is infinite since we suppose the fluid at rest at infinity. Hence $\partial C/\partial x = 0$ also when r is infinite. Similarly $\partial C/\partial y = 0$, $\partial C/\partial z = 0$. But C has the same value for all concentric spheres enclosing the inner bounding surfaces. Consider then two such spheres, one of finite radius, the other so large that $\partial\phi/\partial x$ is zero at every point of it, and let them be displaced together. Since $\partial C/\partial x$ is zero for the large sphere it is also zero for the smaller. Thus C does not depend on the position of the centre of the sphere, and is the same for every spherical surface enclosing all the inner bounding surfaces.

If the sphere considered be wholly situated in the region of irrotational motion the value of M is zero, and we have

$$\frac{d}{dr}\left\{\frac{1}{4\pi}\int \phi d\varpi\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

that is, the average potential over the surface is independent of the radius of the sphere and is the same for all spheres having the same centre. The average potential over any sphere is therefore equal to that over an infinitesimal sphere surrounding the centre; that is, it is equal to the potential at the centre. This theorem is due to Gauss and,

with the results which follow, is of great importance in the electrostatic analogue.

305. It follows that the potential cannot be constant over any finite portion S of the non-rotating fluid without being constant over the remainder. For, taking this not to be the case, imagine a sphere described having its centre and part of its surface in the space S, and the remainder of the surface where the potential is either greater or less than the constant value. The constant potential is the potential at the centre, and the average over the surface must be greater or less than this value, which is impossible by the theorem. Hence no such region of greater or less potential can exist.

306. It follows also from the theorem that there can be no place of maximum potential in irrotationally moving fluid; for if there could be a point at which the potential is a maximum the mean potential over a sphere the centre of which is at the point would be less than the potential at the centre. For the same reason the value of the potential cannot be a minimum.

The same argument holds for $\partial\phi/\partial x$, since $\partial\phi/\partial x$ satisfies the same equations; that is (since the axis of x can be taken in the direction of the resultant velocity), the velocity cannot be a maximum or a minimum within the irrotationally moving fluid. But the numerical value of $\partial\phi/\partial x$, without regard to sign, while it cannot have a maximum, may have a minimum value. The maximum numerical value is obviously precluded by the theorem that there is neither a maximum nor a minimum in the algebraic sense; there is nothing, however, to prevent it from having a minimum numerical value. For example, this value may be zero at some point or points in the fluid, as we shall see later.

SECTION III.—*Green's Theorem*

Proof of Green's Theorem. Surfaces of Discontinuity

307. George Green of Nottingham gave in his famous *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* a theorem of pure analysis which is of the very greatest importance in all branches of physical mathematics. We give a proof here, with some examples of its application to particular problems. The extension of the theorem to multiply connected spaces will be given later, when fluid motion in such spaces is considered.

Let U, V, denote two finite, continuous, and single-valued functions of the co-ordinates x, y, z of a point within a closed surface S_0 (Fig. 68), and k any other arbitrary finite, continuous, and single-valued function of x, y, z (or a constant), and let the derivatives of these functions be finite and continuous also. Denote by E the integral

$$\iiint k^2 \left\{ \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right\} dxdydz$$

taken throughout the space enclosed by the surface S_0. Integrating by parts, we obtain

$$E = \iint Uk^2 \left\{ \frac{\partial V}{\partial x} dydz + \frac{\partial V}{\partial y} dzdx + \frac{\partial V}{\partial z} dxdy \right\}$$
$$- \iiint U \left\{ \frac{\partial}{\partial x}\left(k^2 \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(k^2 \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(k^2 \frac{\partial V}{\partial z}\right) \right\} dxdydz \quad (53)$$

and of course an exactly similar expression for E, which may be written down from this by interchanging U and V.

The triple integrals in this equation and its companion are taken throughout the space within the surface; and the elements of the

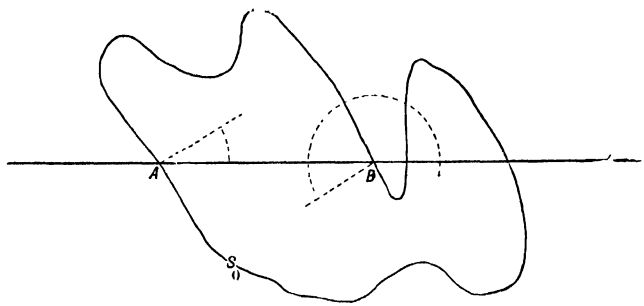


FIG. 68.

double integral, which are furnished only by the enclosing surface, are taken as negative where a point moving in the positive direction along x, y, or z, as the case may be, enters the surface, and as positive where the point emerges. Take first the motion of a point parallel to the axis of x. Draw a normal inwards from the surface at each of the points of entrance or emergence, and let l_1, l_2 denote the cosines of the angles which the normal makes at A, B, with the positive direction of the axis of x at an entrance and an emergence respectively. Let a straight rectangular filament of the space, of cross section $dydz$, intercept elements dS_1, dS_2, of the surface at the feet of these normals. Consider the positive sides of these elements to be those turned inwards to the enclosed space, then we have $dydz = l_1 dS_1$ at an entrance, and $dydz = -l_2 dS_2$ at an emergence. Each pair of elements therefore contributes to the integral the portion

$$-(Uk^2 l \frac{\partial V}{\partial x} dS)_1 - (Uk^2 l \frac{\partial V}{\partial x} dS)_2,$$

and, since we can exhaust the whole surface by pairs of elements, we obtain for the first term of the double integral the expression

$$-\int_{S_0} Uk^2 l \frac{\partial V}{\partial x} dS$$

taken over the surface.

Proceeding in the same manner for the directions y, z, and putting m, n, for the direction cosines of the inward-drawn normals at the corresponding elements of the surface, we find for the whole surface integral in (53) the value

$$-\int_{S_0} Uk^2\Big(l\frac{\partial V}{\partial x}+m\frac{\partial V}{\partial y}+n\frac{\partial V}{\partial z}\Big)\,dS.$$

In the companion equation to (53) we get, of course, a similar surface integral, except that U and V are interchanged.

Denoting the expression between the brackets in the surface integral just found by dV/dn, since it is the rate of variation of V inwards along the normal at dS, and using dU/dn in the similar sense in the companion equation, we obtain finally

$$E=-\int_S Uk^2\frac{dV}{dn}dS-\iiint U\Big\{\frac{\partial}{\partial x}\Big(k^2\frac{\partial V}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(k^2\frac{\partial V}{\partial y}\Big)+\frac{\partial}{\partial z}\Big(k^2\frac{\partial V}{\partial z}\Big)\Big\}\,dxdydz$$

$$=-\int_S Vk^2\frac{dU}{dn}dS-\iiint V\Big\{\frac{\partial}{\partial x}\Big(k^2\frac{\partial U}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(k^2\frac{\partial U}{\partial y}\Big)+\frac{\partial}{\partial z}\Big(k^2\frac{\partial U}{\partial z}\Big)\Big\}\,dxdydz \qquad (54)$$

which is Green's theorem.

308. The necessity for continuity and finiteness of the functions as specified above will be evident from the statement of the theorem in (54). If, for example, the value of $\partial U/\partial x$ is discontinuous within the limits of integration, that of $\partial(k^2\partial U/\partial x)/\partial x$ in the second expression for E becomes infinite, and the triple integral involving this term is indeterminate. Any region of such discontinuity must in the application of the theorem be excluded from the space considered. This may be done as follows:—

Let P (Fig. 69) be a point within the space at or near which $\partial U/\partial x$, &c., one or more, are discontinuous. Describe a small closed surface S round P so as to include the region of discontinuity. We can apply the theorem to the space included between S and S_0, if that be simply connected, provided we add to the surface integral the value of $-\int Vk^2(dU/dn)dS$ taken over S. The normal is, of course, supposed to be drawn from dS towards the space throughout which the volume integral is taken.

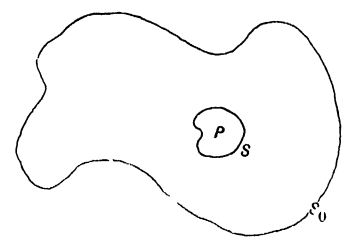


FIG. 69.

If the region of discontinuity is a mere point, P, then by supposing S shrunk down to infinitely small dimensions round P we can obtain as nearly as we please the proper finite value of E for the given case.

If there be a number of such points of discontinuity $P, Q, \ldots$ within the space considered, each must be dealt with in the manner

just described, that is, the triple integrals must be taken throughout the space included between the outer bounding surface and the infinitely small closed surfaces described round P, Q, &c., and the proper values of the surface integral over these latter surfaces added to that taken over the outer surface.

In the case of a cluster of such points or a finite region of discontinuity, a closed surface described round the cluster or region in question, and included in the surface integration, will enable the theorem to be applied to the remainder of the space.

309. It follows that if any discontinuity such as is here considered occur at points of a closed or unclosed surface, we may apply the theorem, provided we exclude the surface of discontinuity by a proper surface integration. Let, first, the surface of discontinuity be unclosed. Imagine a closed surface surrounding it described, and then shrunk down until it forms an infinitely thin shell, S (represented by the dotted line, Fig. 70), having the surface of discontinuity between its

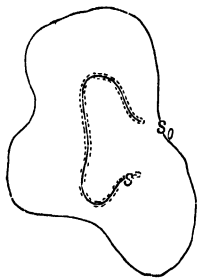


FIG 70.

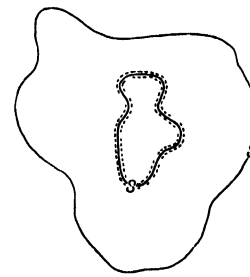


FIG. 71.

faces. We find E then for the rest of the space within the external containing surface S_0 by adding to the surface integral over S_0 the value of $-\int Vk^2 dU/dn \,.\, dS$ taken over the surface S, as already described.

If the surface of discontinuity be closed, we have only to suppose a surface S (Fig. 71) described round it, infinitely near it, and add to the integral over S_0 the value of $-\int Vk^2 \,.\, dU/dn \,.\, dS$ taken over S, to obtain the value of E for the space included between the outer closed surface S_0 and the inner S. The space within the surface of discontinuity may be treated separately by describing a closed surface within it, and infinitely near to it at every point, and using this as the outer bounding surface of the space now considered. Any other discontinuities within either space must of course be dealt with in addition by the method stated above.

Existence Theorem for Potential Function.—Motion of Minimum Kinetic Energy.—Uniqueness of Value of Potential Function for Given Space and Given Values at Surface

310. A question of considerable importance, though rather from the mathematical than from the physical point of view, may be shortly discussed here. Can a function ϕ be found which has a specified arbitrary, but continuous, system of values over the bounding surface or surfaces of a simply connected space, and satisfies the condition

$$\frac{\partial}{\partial x}\left(k^2\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k^2\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k^2\frac{\partial\phi}{\partial z}\right)=0 \quad . \quad . \quad (55)$$

throughout the space? Practically the same mathematical proof of this proposition has been given by Lord Kelvin and by Lejeune Dirichlet, though it has been objected to on certain grounds by some writers. Mathematically the question resolves itself into whether or not it is possible to determine ϕ so that it shall have an assigned value at every point of the bounding surface and satisfy the condition (55) in the interior. We shall give Lord Kelvin's proof here, as it will at the same time establish for us another very important theorem of fluid motion.

311. Let ϕ denote any function whatever of x, y, z which has the assigned system of values at the bounding surface and v a quantity which is zero at every point of the surface, and has the same sign at every internal point as the expression on the left of (55) has there, and therefore vanishes wherever this expression is zero. If $\phi'=\phi+\theta v$, where θ is any constant multiplier, then, putting Q for the integral

$$\iiint k^2\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}dxdydz,$$

taken throughout the portion of fluid considered, and Q' for the same integral with ϕ' used instead of ϕ, we have, by integration by parts, since v is zero at every point of the surface,

$$Q'-Q=-2\theta\iiint v\left\{\frac{\partial}{\partial x}\left(k^2\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k^2\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k^2\frac{\partial\phi}{\partial z}\right)\right\}dxdydz$$
$$+\theta^2\iiint k^2\left\{\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right\}dxdydz \quad . \quad . \quad (56)$$

Every term of the first integral on the right is positive by the condition imposed on v, and every term in the second integral is positive from its form. Hence we may write the result thus:

$$Q'=Q-m\theta(n-\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (56')$$

where m and n are both positive. If therefore θ be positive and less than n, Q' is less than Q; that is to say, unless (55) be satisfied a

function can be found which shall make $Q' < Q$. If, however, (55) be satisfied, $Q' > Q$; that is, ϕ then gives the smallest possible value to the integral Q.

But, since the integral is essentially positive (for every term is positive), there must exist for it a lower limit; that is to say, there exists a value of ϕ which makes it equal to this lower limit. This value satisfies (55).

312. One ground on which this *existence theorem*, as it is called, has been objected to is the assumption that a function v can be found to fulfil the condition stated. Whatever opinion may be held as to the validity of this and other objections, there is no doubt that if ϕ fulfil (55) Q' is greater than Q by the second integral on the right. This, stated in physical language, is the theorem that the kinetic energy of the motion given by the velocity-potential ϕ, which has the assigned values at the surface and satisfies (46), is smaller than that of any other motion fulfilling the surface condition by the kinetic energy corresponding to the difference between the velocity-potentials of the two motions. We here still call ϕ the velocity-potential, though the component velocities are

$$-k\partial\phi/\partial x, \ldots .$$

313. Further, if there exist one motion fulfilling the stated conditions, that motion is the only one that fulfils them. For, if possible, let ϕ_1 be the velocity-potential of another motion fulfilling the conditions, then $\phi - \phi_1$ also fulfils (55) and is zero at every point of the boundary. By a theorem which we shall prove immediately, the fluid under this potential is at rest; that is, the motion corresponding to the potential $-\phi_1$ is equal and opposite to that corresponding to ϕ; that is, $\phi = \phi_1$ throughout.

In proving the theorem on which this result depends, we suppose $k = 1$, which is the only case relevant to the fluid motion we are considering. But as the reader will see at once the theorem is true also in the more general case. Integrate the expression

$$q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$$

throughout any region the bounding surface of which is S. Integrating by parts, we get

$$\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}dxdydz = \int_S \phi\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}\right)dS$$

$$- \iiint \phi\nabla^2\phi \, . \, dxdydz \, . \qquad . \quad (57)$$

If the motion is irrotational throughout the space, $\nabla^2\phi = 0$, and the second integral on the right is zero. If then (1) $\phi = 0$ over

the surface, or (2) $d\phi/dn = 0$ over the surface, or (3) $\phi = 0$ over one part of the surface and $d\phi/dn = 0$ over the remainder, the other integral on the right will vanish. Hence so also will the integral on the left. But, as this is composed of elements which are all essentially positive, each element must vanish separately, that is $q = 0$ throughout the space. Hence, if condition (1), (2), or (3) is fulfilled, the fluid is at rest. The same conclusion follows also from the considerations set forth above as to lines and tubes of flow.

314. Again, if instead of an arbitrary distribution of velocity-potential we have given over the surface a continuous distribution of normal velocity, then, if there exist a solution of (46) fulfilling the former condition, there must also exist one fulfilling the latter. For conceive a fluid contained within a flexible envelope, and impress upon the envelope from without the distribution of normal velocity specified. This must satisfy the equation $\int_S q_n dS = 0$, where q_n is the normal velocity. If the normal velocity can be expressed at every point as the rate of variation in that direction of some function ϕ of the position of the point, the surface condition corresponds to a certain arbitrary distribution of ϕ, and by the last proposition there is one, and only one, solution fulfilling the prescribed condition.

Similarly there is one, and only one, solution if ϕ is given over one part of the surface and $d\phi/dn$ over the remainder.

With the proper changes, which will appear later, in the specification of the symbols, these theorems as well as those which follow, are of great importance in electricity.

Deductions from Green's Theorem.—Sources and Sinks

315. One or two important consequences of Green's theorem it is convenient to deduce here. Supposing k a constant, we get from (54)

$$\iiint \left(U\nabla^2 V - V\nabla^2 U \right) dxdydz = \int_S \left(V\frac{dU}{dn} - U\frac{dV}{dn} \right) dS . \quad (58)$$

where the first integral is taken throughout the space considered, and the second is taken over its surface, including those parts of the surface which exclude regions of discontinuity from the first integral. This equation is of great service in many parts of physics. It is sometimes, though wrongly, asserted to express Green's theorem.

If both U and V fulfil Laplace's equation the left-hand side of (58) is zero, and we have

$$\int_S U\frac{dV}{dn} dS = \int_S V\frac{dU}{dn} dS \quad . \quad . \quad . \quad . \quad . \quad (58')$$

316. As an example of the use of this result we shall put $U = 1/r$, where r is the distance of any point P from the element $dxdydz$, and for V a function ϕ of x, y, z which fulfils Laplace's equation and is every-

where finite and continuous within the region of integration. We have therefore $\nabla^2 V = 0$, throughout the space considered, and also $\nabla^2 U = 0$ for the same space except just at P. For U becomes infinite when $r = 0$, so that if P is within the space considered we must exclude it by describing an infinitely small sphere round it and extending the surface integral round that sphere. The value of dU/dn for this sphere is $dU/d\rho = -1/\rho^2$, where ρ is the radius; and the surface is $4\pi\rho^2$, so that the contribution to the surface integral is

$$-\int \phi \frac{dU}{dn} dS = 4\pi\phi_P,$$

where ϕ_P denotes the mean value of ϕ taken over the infinitely small sphere, centre P, that is, the value of ϕ at P.

For the rest of the space considered the left-hand side of (58) vanishes, and we have

$$4\pi\phi_P + \int_S \left\{\frac{1}{r}\frac{d\phi}{dn} - \phi\frac{d}{dn}\left(\frac{1}{r}\right)\right\} dS = 0,$$

or

$$\phi_P = \frac{1}{4\pi}\int_S \phi \frac{d}{dn}\left(\frac{1}{r}\right) dS - \frac{1}{4\pi}\int_S \frac{1}{r}\frac{d\phi}{dn} dS \quad . \quad . \quad . \quad (59)$$

317. To find a physical interpretation of this equation, imagine fluid to enter an indefinitely large space at a point, and to flow uniformly in all directions radially from the point. Let the amount of fluid introduced into the space per unit of time be m, then the same amount m crosses every concentric spherical surface in each unit of time. The rate of flow per unit of area at any place at distance r from the point is therefore $-m/4\pi r^2$. The function $m/4\pi r$ thus fulfils the analytical conditions for being the velocity-potential of the motion, and we shall adopt it as the value of ϕ for the motion.

We call the point from which the fluid diverges a *point-source* of intensity m, and we see that the potential of such a source, at distance r from it, may be taken as $m/4\pi r$.

Had we considered a flow uniformly converging radially to a point we should have obtained the same numerical value of the velocity, but with opposite sign. The potential for the same amount of fluid m carried inward per unit of time across each concentric spherical surface would in this case be $-m/4\pi r$. We call the point of convergence a *sink* of intensity m.

Instead of fluid we may have sources and sinks of heat and electricity, and in the latter connection we shall have to consider the subject fully later. We shall then see that the electrodes by which a current enters and leaves a conducting body play the parts of electric source and sink for the flow of electricity, the theory of which is precisely that of the irrotational flow of an incompressible fluid.

When ϕ is taken as electric potential, and sources and sinks are distributions of positive and negative electricity, we have corresponding theorems on the distribution of potential and lines of force in the electric field.

318. Now consider two equal and opposite point-sources A, B (Fig. 72), at a distance dn apart, the positive direction of dn being taken from the negative source at A to the positive at B, and let

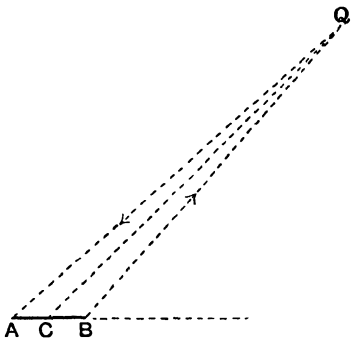


FIG. 72.

r, $r + dr$ be the (nearly equal) distances from these sources of any point Q in the moving fluid. Let also θ be the angle which the bisector CQ of the angle AQB makes with the positive direction of dn, that is with AB. Since the flow at any point Q is compounded of a flow of amount $-m/4\pi r^2$, towards A, and another of amount $m/4\pi(r + dr)^2$, from B, and these have potentials $-m/4\pi r$, $m/4\pi(r + dr)$ at Q, the velocity-potential there is $-m/4\pi \cdot \{1/r - 1/(r + dr)\}$, that is $-mdr/4\pi r^2$. But $dr = -dn \cos\theta$, and if we write m for $mdn/4\pi$ the potential is $\mathsf{m}\cos\theta/r^2$. We call this a double point source, or *doublet* source, of intensity m.

The potential of such a source at a point at distance r from the centre of the doublet can clearly be written also in the form

$$\mathsf{m}\frac{d}{dn}\left(\frac{1}{r}\right).$$

Going back now to (59), we see that the physical interpretation of the result there stated is that the potential at P is the potential that would be produced by a distribution over the surface S of simple point sources, so that the intensity at dS is $-(d\phi/dn)/4\pi$ per unit of area, and of doublet sources so that at dS the intensity is $\phi/4\pi$, where ϕ is the potential at dS.

319. The result here obtained is very important in many respects. The motion is shown to be producible by a surface distribution of sources which can be deduced as specified from the geometrical configuration

of the bounding surface S_1 and the values at every element of this surface of the potential, and of its rate of variation along the normal there. The actual sources may be very different; in fact, their nature may be quite unknown. The result is analogous to the replacement, in the theory of light, of the actual sources by what are called secondary sources, a notion due originally to Huyghens and developed very fully by Fresnel and his successors.

Let us suppose, however, that the potential at any point in the space included by the inner bounding surfaces or situated without the outer bounding surface, that is, throughout the rest of space, is given by a single-valued function ϕ'. For this space, since P is external to it, we have by (59)

$$\int_S \phi' \frac{d}{dn'}\left(\frac{1}{r}\right) dS - \int_S \frac{1}{r}\frac{d\phi'}{dn'} dS = 0$$

in which the normal n' is supposed drawn inwards from S towards the space to which ϕ' applies. Adding this equation to (59) we obtain, since $d/dn = -d/dn'$,

$$\phi_P = \frac{1}{4\pi}\int_S (\phi - \phi')\frac{d}{dn}\left(\frac{1}{r}\right) dS - \frac{1}{4\pi}\int_S \frac{1}{r}\left(\frac{d\phi}{dn} + \frac{d\phi'}{dn'}\right) dS \quad (59')$$

and if $\phi = \phi'$ at the surface S

$$\phi_P = -\frac{1}{4\pi}\int_S \frac{1}{r}\left(\frac{d\phi}{dn} + \frac{d\phi'}{dn'}\right) dS \quad . \quad . \quad . \quad . \quad (59'')$$

Thus the distributions, whatever they may be in the space separated from that in which P is situated by the surfaces S, may, so far as the motion of the fluid in the region in which P is situated be concerned, be annulled, and, if S be a surface of discontinuity of ϕ, the surface distribution of simple and doublet sources specified by (59′) substituted, or in the other case, that of continuity of potential, replaced by the arrangement of simple sources specified by (59″). If $d\phi/dn = -d\phi'/dn'$, that is, in the case of continuity of normal flow across S, the second integral in (59′) vanishes, and the distribution of sources is reduced to the doublets alone.

If the surface S consist of two parts, (1) one or more surfaces at finite distances from P everywhere, and (2) an outer spherical enclosing surface every part of which is at an infinite distance, the surface integrals may be taken to vanish for the latter part, since for this outer surface the second integral vanishes and the first becomes the constant, C, of (50). Since we have in any case only to deal with differences of potential, this C may without affecting any result be taken as zero.

For example take Green's problem (Art. 201 above). Let U be such a function that it is sensibly equal to $1/r$ in the immediate vicinity of P, is equal to zero at the internal bounding surface or surfaces S, and is

harmonic throughout the space for which the volume-integrals are taken. The existence of such a function is clear from the electric analogue. For let a unit charge be situated at P, and S be maintained at zero potential. The induced charge on S will be such as to give a potential exactly counteracting the value, $1/r$, of the potential at each element of S produced by the unit charge at P (k is here taken as unity). The value of U at any point is, then, the potential there due to the induced distribution, together with that due to the induced charge at P. This is harmonic throughout the space external to S, except just at P, where it is $1/r$. Thus let ϕ be the arbitrarily chosen potential at any point of the surface of S; since the expression on the left of (58′) (with the point P allowed for as in Art. 316) vanishes, the corresponding potential at P is

$$\phi_{\bar{P}} = \frac{1}{4\pi}\int_S \phi \frac{dU}{dn}\, dS \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Kinetic Energy of Irrotational Motion.—Expression as a Surface Integral.—Cases in which Motion cannot exist

320. We now pass to a consideration of the energy of the motion. By Green's theorem, or indeed by direct integration, we can express the kinetic energy by a surface integral. For if the motion is irrotational we have, by integration by parts, since the non-integrated term vanishes,

$$\tfrac{1}{2}\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dxdydz = -\tfrac{1}{2}\int \phi \frac{d\phi}{dn}\, dS. \quad (61)$$

the normal to the element of surface dS being supposed drawn inwards to the space occupied by the fluid, and the surface integral being taken over the whole bounding surface of the space considered in the volume integral.

Thus, if T denote the kinetic energy of the fluid,

$$T = -\tfrac{1}{2}\rho\int_S \phi \frac{d\phi}{dn}\, dS \quad . \quad . \quad . \quad . \quad . \quad . \quad (61')$$

Let the fluid extend to infinity, and the velocity tend to zero at every point very far from the inner bounding surface S_1, and let S_2 be a surface taken in the fluid so as to enclose S_1 and be everywhere very distant from it. We have

$$\int_{S_1} \frac{d\phi}{dn}\, dS_1 + \int_{S_2} \frac{d\phi}{dn}\, dS_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and

$$T = -\tfrac{1}{2}\rho\left\{\int_{S_1} \phi \frac{d\phi}{dn}\, dS_1 + \int_{S_2} \phi \frac{d\phi}{dn}\, dS_2\right\}.$$

But if the velocity may be taken as zero at every point of S_2 the second term on the right is

$$\int_{S_2} C \frac{d\phi}{dn} dS_2,$$

where C is the constant finite value of the potential ϕ at infinity, and by (62) this can be written

$$-\int_{S_1} C \frac{d\phi}{dn} dS_1.$$

Thus

$$T = -\tfrac{1}{2}\rho \int_{S_1} (\phi - C) \frac{d\phi}{dn} dS_1 \quad . \quad . \quad . \quad . \quad . \quad (63)$$

If there is no flux on the whole across the outer boundary, this reduces to

$$T = -\tfrac{1}{2}\rho \int_{S_1} \phi \frac{d\phi}{dn} dS_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

which is the result that would be inferred, were it legitimate to do so by the preceding equation (61), from the fact that S_1 is now the total boundary.

321. If the liquid fill the whole space so that S_1 does not exist, $T = 0$, and u, v, w are by Green's theorem zero at every point. Irrotational motion is thus impossible in a liquid filling infinite space and at rest at infinity.

It also follows from (61)—what has already been proved above, Art. 313,—that irrotational motion cannot exist within a finite, simply connected space with a boundary S_1 fixed at every point; for the velocity $-d\phi/dn$ at right angles to the bounding surface must be zero at every point, and hence u, v, w must be everywhere zero.

Multiple Connection of Spaces.—Reduction of Connectivity by Diaphragms.—Number of Irreconcilable Circuits

322. So far we have considered motion in simply connected spaces only; we have now to consider spaces of multiple connectivity. Such a space is represented in two dimensions in Fig. 73. A space of multiple connectivity admits of the drawing of at least one section or diaphragm so as to give a section having a closed curve as boundary, without dividing the space into disconnected parts. A surface for which one such diaphragm can be drawn is said to be 2ply connected, or to be of connectivity 2. If $n-1$ such diaphragms can be drawn it is said to be $n$ply connected, or to be of connectivity n.

The space enclosed by an anchor ring, or the space external to an anchor ring, is an example of a space of connectivity 2. If we suppose

an anchor ring to have a region such as A, Fig. 73, attached to it, it is changed into a space of connectivity 3; if two such regions are attached, as in Fig. 74, it becomes a space of connectivity 4; and so on. The diaphragms which in the different cases reduce the connectivity to 1 are shown by dotted lines in the figures.

323. We shall now prove that in a space of connectivity n it is possible to draw $n-1$ independent and irreconcilable irreducible

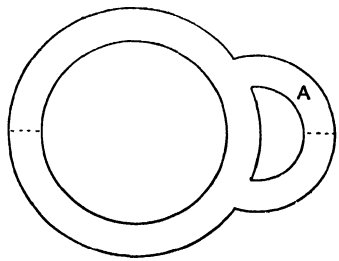


FIG. 73.

circuits. This can be proved in various ways. First it is to be observed that if a complete circuit cannot be drawn so as to cross one particular diaphragm without at the same time meeting another diaphragm an odd number of times, these two diaphragms divide the space into unconnected parts. This is clear from the fact that the circuit, having been carried across one diaphragm, cannot be completed without passing at least once through the other; that is, beyond the

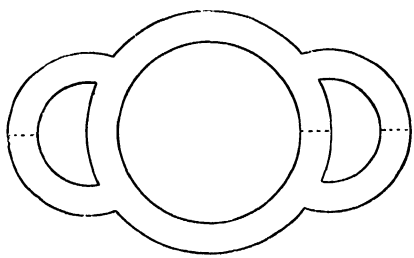


FIG. 74.

first diaphragm is a portion of the space from which the path cannot emerge into the remainder without piercing the second diaphragm.

It follows that, if the diaphragms are so drawn as to preserve connectivity of at least 1 for all the space, it is possible to draw a circuit so as to cross any particular diaphragm once, and any other an even number of times, of which as many are crossings in one direction as in the other. But this latter condition comes to the same thing

as prohibiting the passage of any other diaphragm at all. It is therefore possible, since there are $n-1$ diaphragms, to draw $n-1$ such circuits.

Also these circuits are irreconcilable. For if two of them were reconcilable—that is, if one could be changed into the other without somewhere passing out of the space considered—then in the process either each would have to be altered so as to pass through both diaphragms, or one would be withdrawn from one diaphragm and made to pass through the other. But, since each passes through its diaphragm only once, either alteration is impossible without taking the circuit across the closed curve which, as stated above, forms the boundary of the section made by a diaphragm.

324. The same thing may be proved more shortly thus. For every region added to a multiply connected space necessitating an additional diaphragm it is clear that a new circuit not reconcilable with any existing circuit can be drawn so as to pass through that diaphragm. But when there can be drawn only one diaphragm—that is, when the connectivity is 2, as in the anchor ring—only one independent circuit can be drawn; hence the number of independent irreconcilable circuits is equal to the number of diaphragms—that is, one less than the connectivity.

Cyclic Motion in Multiply Connected Space.—Cyclic Constants

325. The circulation in a circuit crossing only one of the diaphragms and crossing it once only, is the same for all such circuits. This is easily seen from Fig. 75, which represents part of a multiply

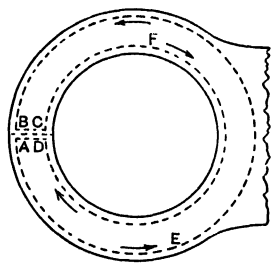


FIG. 75.

connected space. The circulation in the path $AEBCFD$ indicated by the arrows is zero, since the circuit is reducible; and, since the parts BC, DA are infinitely nearly equal and opposite, the circulations in the remaining parts taken in the same direction are equal. Let the circulation in either be κ; then κ is what is called the cyclic constant of the circuit and by (16) is equal to the change of the velocity-potential along the part of the closed path from B to A, or along the other part of the closed path from C to D.

326. Let now $\kappa_1, \kappa_2, \ldots . \kappa_{n-1}$ be the cyclic constants of the $n-1$ independent circuits, and consider any compound circuit in the space drawn so as to pass through the j^{th}, k^{th}, &c., diaphragms, and let the excess of positive crossings of the j^{th} diaphragm above negative crossings be p_j, the corresponding excess for the k^{th} diaphragm p_k and so on. Then the circulation in the circuit is

$$p_j\kappa_j + p_k\kappa_k + \ldots .$$

To see this we have only to notice that the compound circuit is reconcilable into the independent circuits of which one passes p_j times through the j^{th} diaphragm, another crosses p_k times the k^{th} diaphragm, and so on.

Green's Theorem in a Multiply Connected Space

327. In a multiply connected space Green's theorem must be modified by the reduction of the space to simple connectivity by barriers. Then the surface-integrals over the barriers must be introduced, and the theorem applied to all the compartments each simply connected, of which the space now consists. If the value of U on the positive side of a barrier exceeds that on the negative side by κ we have for the surface integral due to the two sides of the diaphragm $\kappa\int dV/dn \,.\, d\sigma$. Similarly if κ' be the difference of the values of V on the two sides of the same barrier we get $\kappa'\int dU/dn \,.\, d\sigma$ for the corresponding part of the surface integral in the companion expression of (58). The theorem is then (k being taken as a constant)

$$E = -\left\{\int_S U\frac{dV}{dn}\,dS + \Sigma\kappa\int\frac{dV}{dn}\,d\sigma\right\} - \iiint U\nabla^2 V\,dxdydz$$

$$= -\left\{\int V\frac{dU}{dn}\,dS + \Sigma\kappa'\int\frac{dU}{dn}\,d\sigma\right\} - \iiint V\nabla^2 U\,dxdydz\,. \quad (58')$$

where the first surface integral in each expression on the right is extended over the whole original bounding surface S, and the others are taken over the barriers. This extension of Green's theorem is due to Lord Kelvin.

Kinetic Energy of Cyclic Motion of Fluid

328. The kinetic energy of the fluid motion in a multiply connected space must now be expressed. It is clear that the energy will not be affected if we suppose each of the $n-1$ diaphragms we have imagined drawn in the fluid to move at each point with the motion which the fluid has at that point. The diaphragms will convert the space in which the motion takes place into a simply connected space; and to get the whole energy it is only necessary, since the motion is supposed irrotational,

to include the two sides of each diaphragm in the surface integral. Thus, drawing normals from the two surfaces of one of the diaphragms into the moving fluid, we have for that diaphragm the contribution to the surface integral

$$-\int_S \phi_0 \frac{d\phi}{dn_0}\,dS - \int_S \phi_1 \frac{d\phi}{dn_1}\,dS,$$

where the suffixes 0, 1 refer to the two sides of the surface. But, since the motion is the same on both sides of the surface,

$$\frac{d\phi}{dn_0} = -\frac{d\phi}{dn_1},$$

and the integral is

$$-\int_S (\phi_0 - \phi_1)\frac{d\phi}{dn_0}\,dS.$$

Now, as has been seen above, $\phi_0 - \phi_1$ is the same for every element dS of the surface, and is equal to what we have called the cyclic constant of the diaphragm for the state of motion considered. The kinetic energy of the whole motion is therefore given by the equation

$$T = -\frac{\rho}{2}\left\{\int_S \phi\frac{d\phi}{dn}\,dS + \kappa_1\int_{S_1}\frac{d\phi}{dn}\,d\sigma_1 + \kappa_2\int_{S_2}\frac{d\phi}{dn}\,d\sigma_2 + \ldots . + \kappa^n\int_{S_n}\frac{d\phi}{dn}\,d\sigma_n\right\} \quad . \quad . \quad (65)$$

where the first integral is taken over the bounding surface, properly so called, and the remaining integrals over the diaphragms.

Uniqueness of Motion in Multiply Connected Space

329. If the cyclic constants $\kappa_1, \kappa_2, \ldots .$ are given we can show that the irrotational motion in a multiply connected space is quite determinate. For let the space be rendered simply connected by means of diaphragms, and let there be two values of the velocity potential ϕ', ϕ'', which give for the difference of potential on the two sides of the successive diaphragms the values

$$\phi'_0 - \phi'_1 = \kappa_1, \quad \phi''_0 - \phi''_1 = \kappa_1$$

. . . . . .

Then we have

$$\phi'_0 - \phi'_1 = \phi''_0 - \phi''_1;$$

that is, the function $\phi' - \phi''$ has the same value on both sides of each diaphragm. Hence, if the velocity-potential be made $\phi' - \phi''$ at each point, the value of

$$\int\left\{\frac{\partial(\phi' - \phi'')}{\partial x}\,dx + \frac{\partial(\phi' - \phi'')}{\partial y}\,dy + \frac{\partial(\phi' - \phi'')}{\partial z}\,dz\right\}$$

will be zero for any closed curve, whether passing through a diaphragm or not. We can therefore apply Green's theorem and so get for each point of the space

$$\frac{\partial \phi'}{\partial x} - \frac{\partial \phi''}{\partial x} = 0 \ldots ;$$

that is, the two solutions are identical.

Perforated Solids Moving in a Liquid. Ignoration of Coordinates

330. When there exist moving solids in the liquid and some or all of these have perforations through which cyclical motion takes place, we have an excellent example of the principle of ignoration of co-ordinates discussed in Arts. 247, 248, above. Equations of motion are applicable to this case which are precisely analogous to the equations there established for a gyrostatic system. Let ϕ_s, ϕ_c denote the parts of the velocity-potential depending on the motion of the solids and the cyclical motion respectively, then the kinetic energy of the fluid becomes, since in (65) we must write $\phi_s + \phi_c$ for ϕ,

$$T = -\frac{\rho}{2}\left\{\int\int(\phi_s + \phi_c)\frac{d}{dn}(\phi_s + \phi_c)dS + \kappa_1\int\frac{d(\phi_s + \phi_c)}{dn}d\sigma_1 + \ldots .\right\} \quad (65')$$

But since ϕ_s, ϕ_c are both velocity-potentials, we have by (58)

$$\int\phi_s\frac{d\phi_c}{dn}dS + \int\phi_s\frac{d\phi_c}{dn}d\sigma + \ldots = \int\phi_c\frac{d\phi_s}{dn}dS + \int\phi_c\frac{d\phi_s}{dn}d\sigma + \ldots .$$

the first integral on each side being extended over the surfaces of all the solids, and the remaining integrals over both faces of each diaphragm. It is clear that all the integrals on the left are identically zero, since the cyclic constants of ϕ_s are all zero, and $d\phi_c/dn$ is zero at the surfaces of the solids. Hence we have, putting now $\kappa, \kappa', \ldots .$ for the successive cyclic constants, and $\sigma, \sigma', \ldots$ for the surfaces of the corresponding diaphragms, since suffixes are otherwise required.

$$\int\phi_c\frac{d\phi_s}{dn}dS + \kappa\int\frac{d\phi_s}{dn}d\sigma + \kappa'\int\frac{d\phi_s}{dn}d\sigma' + \ldots . = 0.$$

Substituting from this last result in (65') we get, since $d\phi_c/dn$ is zero at the surfaces of the solids,

$$T = -\frac{\rho}{2}\int\phi_s\frac{d\phi_s}{dn}dS - \frac{\rho}{2}\Sigma\kappa\int\frac{d\phi_c}{dn}d\sigma \quad . \quad . \quad . \quad (65'')$$

where the Σ denotes summation of the integrals taken over the diaphragms so as to convert the channels into simply connected spaces. The two terms on the right can, as we shall now show, be converted respectively into a homogeneous quadratic function of velocities sufficient to specify completely the motion of the solids, and a homogeneous quadratic function K of the cyclic constants $\kappa, \kappa', \ldots .$

331. The velocity-potential ϕ_s can be expressed in the form

$$\phi_s = \Sigma \dot{q}_a \phi_a + \Sigma \dot{q}_b \phi_b + \ldots .$$

where the suffixes do not relate to the summations but mark the different solids, and each contains as many terms as there are velocities $\dot{q}_a$, or $\dot{q}_b$, &c., necessary to fix the motion of the solid referred to (in ordinary coordinates six, namely, the linear velocities of the centre of inertia of the solid, and the angular velocities of the solid round three rectangular axes through that point), and the quantities ϕ_a, ϕ_b, ... are functions of the ordinary coordinates, from which the components of velocity normal to the surface at any point are to be calculated.

Again, we can write

$$\phi_c = \kappa\omega + \kappa'\omega' + \ldots .$$

where ω, ω', ... are functions of the coordinates to be determined from the conditions:—that ω is a cyclic function which diminishes in value by unity for each time its variation is taken from point to point in the positive direction round a closed curve passing once through an ideal barrier across the first channel, and returns to its former value when a circuit is completed not cutting this barrier, that $\nabla^2\omega = 0$ at all points, $d\omega/dx = 0, \ldots$, at infinity, and $d\omega/dn = 0$ at all points of the surfaces of the solids. The functions ω', ω'', ... fulfil similar conditions for their respective channels.

The justification of this specification of the potential lies in the fact that each constituent of the motion is in itself possible, and corresponds to an independent part of the motion which exists at the boundary of the system, that is the surfaces of the solids and the barriers of the channels, and that the combination of these partial motions forms a possible motion. If there were two possible motions of the fluid corresponding to generalised velocities $\dot{q}_1$, $\dot{q}_2$, ... of the solids, and cyclic constants κ, κ', ... it could be shown by reversal of one of them that the fluid would be at rest, since the kinetic energy would be zero (*see* Art. 328).

332. We thus have for the kinetic energy of the fluid

$$T = \tfrac{1}{2}a_{11}\dot{q}_1^2 + a_{12}\dot{q}_1\dot{q}_2 + \ldots . + \tfrac{1}{2}a_{22}\dot{q}_2^2 + \ldots . + \tfrac{1}{2}(\kappa, \kappa)\kappa^2 + (\kappa, \kappa')\kappa\kappa' + \ldots .$$

$$\text{where } a_{11} = -\rho\int\phi_1\frac{d\phi_1}{dn}\,dS, \quad a_{12} = -\rho\int\phi_2\frac{d\phi_1}{dn}\,dS = -\rho\int\phi_1\frac{d\phi_2}{dn}\,dS, \ldots .$$

(by (58) since $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$), and so on, the quantities ϕ_1, ϕ_2, ... being the functions of the coordinates associated with velocities $\dot{q}_1$, $\dot{q}_2$, ... respectively.

To include the kinetic energy of the solids it is only necessary to add another homogeneous function of the velocities $\dot{q}_1$, $\dot{q}_2$, ... This gives for the total kinetic energy depending on the solids a single homogeneous quadratic function but with different coefficients from

those specified above. Denoting the quadratic function of the velocities by T_1, that of the κs by K we have

$$T = T_1 + K$$

as at p. 191 above.

333. Now regarding $\rho\kappa$, $\rho\kappa'$, ... as components of momentum, let $\dot{\chi}$, $\dot{\chi}'$, ... be the corresponding generalised velocities. These will be given by equations of the same form as (35) p. 190 above, $\dot{q}_1$, $\dot{q}_2$, ... taking the places of the velocities $\dot{\psi}$, $\dot{\phi}$, ... and (for uniformity of notation) M_1, M_2, ... those of M, N, ... Thus we get

$$\kappa\dot{\chi} + \kappa'\dot{\chi}' + \ldots = \frac{2}{\rho}K - \dot{q}_1\Sigma\kappa M_1 - \dot{q}_2\Sigma\kappa M_2 - \ldots .$$

and therefore for the modified Lagrangian function

$$L' = T_1 + \rho\dot{q}_1\Sigma\kappa M_1 + \rho\dot{q}_2\Sigma\kappa M_2 + \ldots - K - V \quad . \quad (66)$$

from which the differential equations of motion are to be derived as already explained at p. 190 above. Applications will be found in the discussion of General Electromagnetic Theory given below.

The part of the kinetic energy which consists of terms involving κs is equal to the quadratic function K together with terms involving products of κs and velocities of coordinates. Now all the terms involving κs and no others, arise from integration over the barriers. Thus we have by (37) Chap. VII.

$$-\frac{\rho}{2}\Sigma\kappa\int\frac{d(\phi_s + \phi_c)}{dn}d\sigma = K - \frac{\rho}{2}\Sigma(\dot{q}\Sigma\kappa M).$$

Hence by the condition by which (65″) is derived from (65′) above, and the value of K, namely, $-\frac{1}{2}\rho\Sigma\kappa\int d\phi_c/dn.d\sigma$,

$$\Sigma(\dot{q}\Sigma\kappa M) = \Sigma\kappa\int\frac{d\phi_s}{dn}d\sigma = -\int\phi_c\frac{d\phi_s}{dn}dS. \quad . \quad . \quad . \quad (67)$$

Thus

$$L' = -\frac{\rho}{2}\left\{\int\phi_s\frac{d\phi_s}{dn}dS + 2\int\phi_c\frac{d\phi_s}{dn}dS\right\} - K - V \quad . \quad (66')$$

334. We shall show that the velocity $\dot{\chi}$ associated with any momentum $\rho\kappa$ is the rate of flow of liquid across the barrier. For taking the expression for the kinetic energy

$$T = \tfrac{1}{2}\rho\int\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}dx\,dy\,dz$$

and remembering that

$$\phi = \Sigma\dot{q}_r\phi_r + \Sigma\kappa\omega$$

and that $\nabla^2\omega = 0$, we have for the value of χ associated with a particular κ

$$\frac{1}{\rho}\frac{\partial T}{\partial \kappa} = \iiint\left(\frac{\partial\phi}{\partial x}\frac{\partial\omega}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\omega}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\omega}{\partial z}\right)dx\,dy\,dz$$

$$= -\int\phi\frac{d\omega}{dn}dS - \kappa\int\frac{d\omega}{dn}d\sigma - \kappa'\int\frac{d\omega}{dn}d\sigma' - \ldots.$$

$$= -\int\phi\frac{d\omega}{dn}dS - \kappa\int\frac{d\omega}{dn}d\sigma - \kappa'\int\frac{d\omega'}{dn}d\sigma - \ldots.;$$

for by (58) since $\kappa\omega$, $\kappa'\omega'$, ... , are possible potentials, and ω changes by unity for passage of a path of integration through the diaphragm σ, and ω', ω'', ... are similarly affected at the diaphragms σ', σ'', ... , and each is acyclic for every path of integration which does not pass through its special diaphragm, we have

$$\int\frac{d\omega}{dn}d\sigma' = \int\frac{d\omega'}{dn}d\sigma, \ldots.$$

But, also by the conditions fulfilled by the ωs, $d\omega/dn$ is zero at every point of the surfaces of the solids. Thus we have

$$\frac{1}{\rho}\frac{\partial T}{\partial \kappa} = -\int\frac{d}{dn}(\kappa\omega + \kappa'\omega' + \ldots.)\,d\sigma = -\int\frac{d\phi_c}{dn}d\sigma$$

the rate of flow of liquid across the barrier. It is easy to show that $\kappa\rho$ is the impulsive pressure that would have to be applied to generate from rest this part of the cyclical motion.

It is clear from (67) that if the solids be merely infinitely thin cores round which the fluid circulates, the terms of $\Sigma(\dot{q}\Sigma\kappa M)$ vanish on account of the smallness of the surface of the solids, and L' is the same as if there were no cyclic motion of the fluid, and the potential energy V were increased by K. This result leads to an important theorem as to the mutual forces between the cores which will be proved below (see Art. 359).

335. The process just explained of specifying the energy of the motion by one homogeneous quadratic function of the velocities specifying the motions of the solids, and another of the cyclic constants of the circulatory motion and thence forming the equations of motion of the solids, thus ignoring the coordinates of the fluid particles themselves, precisely corresponds to the case of motion of a rigid system explained in Arts. 247, 248, above. The only possible motion of the fluid corresponding to the motion of the solids thus defined and the cyclic motion specified by the constants κ, κ', ... is the actual motion. For consider two motions consistent with the motions of the solids and the motion of the fluid at the diaphragms. One of these motions reversed would reduce the solids and the fluid at their surfaces to rest, and the

flow across the diaphragms to zero. But by (65) above this would reduce the kinetic energy of the motion to zero, which is inconsistent with the motion of any part of the fluid. The two motions must thus be identical, that is there are not two possible motions fulfilling the conditions stated.

SECTION IV.—*Vortex Motion*

Vortex Lines and Vortex Tubes

336. We have seen above (Art. 283) that the quantities ξ, η, ζ are the components at time t of the angular velocity of an element of fluid round axes parallel to the axes of co-ordinates, and it has been proved that if any closed circuit s be drawn in the fluid the circulation round it is equal to twice the surface integral of (ξ, η, ζ) taken over any surface S bounded by the circuit. This theorem is expressed by (18), namely,

$$\int_s (udx + vdy + wdz) = 2\int_S (l\xi + m\eta + n\zeta)dS.$$

337. A vortex-line in the fluid is a line the direction of which at every point is that of the axis of resultant rotation at the point. If dx, dy, dz be the projections of an element ds of any vortex-line on the axes, the differential equations of the line are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

The region bounded by the vortex-lines passing through any closed curve drawn in the fluid is called a vortex-tube. If l, m, n be the direction cosines of the normal to the surface of a vortex-tube at any point they evidently fulfil the relation

$$l\xi + m\eta + n\zeta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (68')$$

The position and direction of the vortex-lines vary with the time in consequence of the motion of the fluid, and we shall see presently that we are entitled to regard the vortex-tubes as moving in the fluid from one place to another, preserving their identity, inasmuch as they are always made up of the same portions of the fluid.

Permanence of a Vortex Filament

338. If the closed circuit referred to above be drawn so as to embrace any system of vortex-tubes, and, while the circuit is kept in a fixed position, the surface be drawn so as to cut across the system at different places, the value of the surface integral on the right of(18) is the same for every such position of the surface. Consider now an infinitely thin vortex-tube, or vortex-filament, and let the circuit fit

close round it, while the surface is so drawn as to cut across the tube at right angles at any desired place, while the portion of the surface connecting this cross-section with the circuit lies everywhere close to the sides of the tube. The latter portion of the surface, by (57), contributes nothing to the surface integral, and we see that the integral over the cross-section is the same wherever the section is taken.

But if ω be the resultant angular velocity at any cross-section of area dS the surface integral is ωdS. If ω' and dS' be the angular velocity and area for any other normal section of the tube, we have

$$\omega dS = \omega' dS' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

that is, the angular velocities at different cross-sections are inversely as the cross-sectional areas. This theorem holds even when the values of ξ, η, ζ change abruptly so that there is a sudden bend in the vortex-filament, provided u, v, w are continuous.

It follows evidently from the theorem just proved that a vortex-filament cannot terminate within the fluid and must therefore either form a closed ring or have its ends on the surface of the fluid.

The constant product ωdS of the angular velocity into the cross section of the filament is called the *strength* of the vortex.

Vortex Surface

339. Any surface drawn in the vortically moving fluid, so that no vortex-lines cross it is called a vortex-surface. At every point of such a surface the condition (68′) is fulfilled. Now, for any circuit drawn in the fluid and moving with it, the circulation, by Lord Kelvin's theorem, remains constant, and therefore if the circuit is drawn on a vortex-surface the circulation remains zero, as the circuit moves with the fluid. Hence, as the fluid moves, vortex-surfaces remain vortex-surfaces and contain the same particles of the fluid. Further, as the intersection of two such surfaces is a vortex-line, vortex-lines move with the fluid and contain the same fluid particles.

340. Consider now any case of steady motion of a system of vortices. Draw vortex-lines through any stream-line whose equations are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} :$$

these lines define a vortex-surface. All stream-lines drawn through points on such a vortex also lie on the surface. For the particles on any vortex-line at any instant are there travelling along the stream-lines, and as vortex-lines move with the fluid they occupy in succession the positions of the series of vortex-lines which at any instant lie on the surface.

By (29), since the motion is steady, $\partial\phi/\partial t = 0$, and $F(t)$ reduces to a constant of integration invariable along each individual stream-line, but

in the general case variable from one stream-line to another. Hence on such a surface

$$\int\frac{dp}{\rho} + \Omega + \tfrac{1}{2}q^2 = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

along any chosen stream-line. We shall show that this equation holds also along a vortex-line, and therefore over the whole surface.

Since the motion is steady, we have three equations of the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial W}{\partial x}$$

in which W is put for $-\left(\int dp/\rho + \Omega\right)$. Multiplying these by ξ, η, ζ, respectively, and adding, we obtain

$$\xi\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) + \eta\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right) + \zeta\left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right)$$
$$= \xi\frac{\partial W}{\partial x} + \eta\frac{\partial W}{\partial y} + \zeta\frac{\partial W}{\partial z}.$$

But, since ξ, η, ζ are proportional to dx, dy, dz, the components of the element ds of a vortex-line, the expressions on the left and right of this equation are proportional respectively to the space-rates of variation of $\frac{1}{2}q^2$ and W along such a line. Hence $\frac{1}{2}q^2 - W$ is constant along a vortex-line, and the proposition is proved.

Determination of Velocities for Given Spin and Expansion of Fluid

341. We shall now consider how from the equations

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad 2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \theta,$$

where ξ, η, ζ, θ are known quantities, the velocities u, v, w are to be determined.

First we shall show that if the problem has a solution it has only one. For let there be two solutions u', v', w'; u'', v'', w'', which fulfil the given equations; then clearly $u'-u'', v'-v'', w'-w''$ will be velocities of another possible state of motion of the fluid. Let these be denoted by u_1, v_1, w_1. But for this motion

$$\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z} = 0, \ldots .$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0,$$

so that the velocities u_1, v_1, w_1, are derivable from a potential, and the motion is without divergence. Let the space in which the fluid is contained be simply connected, then the velocity at the bounding surface and at right angles to it must be zero; that is, if λ, μ, ν be the direction cosines of the normal to the surface, we must have for each possible motion

$$\lambda u' + \mu v' + \nu w' = 0$$
$$\lambda u'' + \mu v'' + \nu w'' = 0;$$

that is,

$$\lambda u_1 + \mu v_1 + \nu w_1 = -\frac{d\phi}{dn} = 0$$

if ϕ be the potential function from which the velocities u_1, v_1, w_1 are derivable. Thus the fluid has by (61) zero kinetic energy, that is, it is everywhere at rest. The same thing will hold by (64) if the fluid have only certain internal bounding surfaces, and extend thence to infinity, where it is at rest.

342. The solution of the problem of finding the velocities u, v, w may be split into two parts: the determination (1) of velocities u_1, v_1, w_1, which fulfil the equations

$$\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z} = 0, \quad \frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial x} = 0, \quad \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = 0 \quad . \quad . \quad (a)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

(2) of velocities u_2, v_2, w_2 which fulfil

$$2\xi = \frac{\partial w_2}{\partial y} - \frac{\partial v_2}{\partial z}, \quad 2\eta = \frac{\partial u_2}{\partial z} - \frac{\partial w_2}{\partial x}, \quad 2\zeta = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} \quad . \quad (c)$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

The velocities u_1, v_1, w_1 therefore correspond to an irrotational motion of expansion θ; u_2, v_2, w_2 are the velocities due to the vortices.

343. The first set are derivable from a potential ϕ, and this, from (b), has the value given by

$$\phi = \frac{1}{4\pi}\int\frac{\theta'}{r}\,d\varpi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where $d\varpi$ is an element of volume taken at a place at which the expansion is θ', r is the distance from that element to the point at which u_1, v_1, w_1 are taken, and the integral is taken throughout all space in which θ' is not zero. This may be verified by direct differentiation, or the reader may refer to the analogous theory developed above, Arts. 46, 47, 48, 76, 193.

Auxiliary Function called Vector Potential.—"Curling"

To find u_2, v_2, w_2, write

$$u_2 = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad v_2 = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad w_2 = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \quad . \quad (72)$$

so that the solenoidal condition is satisfied.

Again, by (c), if J denote the divergence $(\partial F/\partial x + \partial G/\partial y + \partial H/\partial z)$ of the quantity (F, G, H),

$$2\xi = \frac{\partial J}{\partial x} - \nabla^2 F, \quad 2\eta = \frac{\partial J}{\partial y} - \nabla^2 G, \quad 2\zeta = \frac{\partial J}{\partial z} - \nabla^2 H. \quad (73)$$

so that, if F, G, H can be found to satisfy the condition

$$J = 0$$

at every point, we shall have

$$\nabla^2 F + 2\xi = 0, \quad \nabla^2 G + 2\eta = 0, \quad \nabla^2 H + 2\zeta = 0 \quad . \quad (74)$$

As we have just seen, the last equations are satisfied if we take

$$F = \frac{1}{2\pi}\int \frac{\xi'}{r} d\varpi, \quad G = \frac{1}{2\pi}\int \frac{\eta'}{r} d\varpi, \quad H = \frac{1}{2\pi}\int \frac{\zeta'}{r} d\varpi \quad . \quad (75)$$

where ξ', η', ζ' are the values of ξ, η, ζ at any space-element $d\varpi$ of which the co-ordinates are x', y', z', and r is the distance of (x', y', z') from the point (x, y, z) for which u, v, w are to be found. These on differentiation give

$$J = -\frac{1}{2\pi}\int\left(\xi' \frac{\partial}{\partial x'} + \eta' \frac{\partial}{\partial y'} + \zeta' \frac{\partial}{\partial z'}\right)\frac{1}{r} d\varpi \quad . \quad . \quad . \quad (76)$$

since $\partial/\partial x\,(1/r) = -\partial/\partial x'\,(1/r), \ldots$. Integrating by parts we find from this

$$J = -\frac{1}{2\pi}\int \frac{1}{r}(l\xi' + m\eta' + n\zeta') dS + \frac{1}{2\pi}\int \frac{1}{r}\left(\frac{\partial \xi'}{\partial x'} + \frac{\partial \eta'}{\partial y'} + \frac{\partial \zeta'}{\partial z'}\right) d\varpi \quad (77)$$

The surface integral vanishes, since it is taken over the bounding surface of the vortex-tubes, and there $l\xi' + m\eta' + n\zeta' = 0$; the volume integral vanishes, since, from the definitions of ξ', η', ζ', $\partial\xi'/\partial x' + \partial\eta'/\partial y' + \partial\zeta'/\partial z'$ is identically zero.

The required conditions are thus fulfilled, and we have by (71)

$$\left.\begin{aligned} u = u_1 + u_2 &= -\frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} \\ v = v_1 + v_2 &= -\frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} \\ w = w_1 + w_2 &= -\frac{\partial \phi}{\partial z} + \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad (78)$$

344. The quantities F, G, H are the components of a directed quantity **A** (*cf.* p. 49 above), and, since they are derived from equations (74), which are identical with Poisson's characteristic equation for electric or magnetic potential, by the same process of integration as that used in the latter case, they are frequently called "components of vector-potential." The electromagnetic vector-potential will be further used and discussed in the chapters on electromagnetism which follow.

345. Clerk Maxwell suggested the term *curling* for the operation by which u_2, v_2, w_2 are derived from F, G, H, and that for shortness the connection (72) between (u_2, v_2, w_2) resultant q_2, and (F, G, H) resultant **A**, might be indicated thus:

$$q_2 = \text{curl } A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

According to this notation we have also (ξ, η, ζ), resultant ω, and (u, v, w), resultant q, connected by the equation

$$2\omega = \text{curl } q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

The condition for the existence of a velocity potential, that is, the vanishing of ξ, η, ζ may thus be expressed by the equation (in which q denotes the resultant of u, v, w)

$$\text{curl } q = 0.$$

We shall frequently use such expressions in what follows, in order to abbreviate the equations.

Velocity due to Long Straight Filament.—Velocity due to Element of any Filament

346. We may now work out two examples. First, let the vortex be a straight filament of strength m, lying along the axis of x, and extending from $x = -\infty$ to $x = +\infty$ in an unlimited fluid. Let us find for any point distant a from the filament the velocity due to the vortex.

From the conditions of the problem $G = H = 0$, and $2\pi F = m\int dx'/r$, where dx' is an element of the length of the filament. Hence, by (72),

$$v_2 = \frac{\partial F}{\partial z}, \quad w_2 = -\frac{\partial F}{\partial y}.$$

Again, let $z = 0$, so that the point lies in the plane of x,y, and we have $r = \sqrt{x^2 + a^2}$. Hence $v_2 = 0$, and

$$w_2 = -\frac{m}{2\pi}\frac{\partial}{\partial a}\int_{-\infty}^{+\infty}\frac{dx'}{\sqrt{x'^2 + a^2}} = \frac{m}{2\pi}\frac{1}{a} \quad . \quad . \quad . \quad . \quad (81)$$

The velocity is thus inversely as the distance a of the point from the axis, and at right angles to the plane determined by the axis and the point.

347. This result could have been obtained at once from the simplest

considerations. In whatever direction the velocity at the point in question may be, it is clear from the symmetry of the case that the direction of the velocity at any point d in the circle, radius a, drawn round the filament as axis, will be got from that at any other point c in the same circle by simply turning the whole system of fluid and vortex round the axis through the angle doc (Fig. 76). It is easy to show from this that there can be no component at any point in the plane through that point and the axis. For, take a closed path $abcd$ (Fig. 76), consisting of two equal circular axes, having the filament as axis, and two equal straight lines bc, da, both parallel to the axis. The circulation

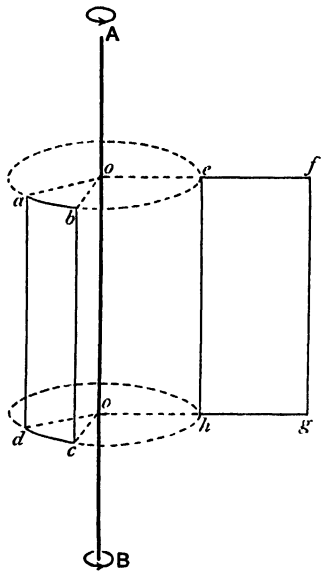


FIG. 76.

round this circuit is zero. The circulation along ab is clearly equal and opposite to that along cd. The circulation, if any, along bc must be equal and opposite to that along da. But this is impossible, since bc can be turned into the position ad by turning the whole system round the axis through the angle doc. This circulation is therefore zero.

That there is no circulation at right angles to the axis in the plane containing it and the point in question is clear also from the fact that the circulation in the circuit $efgh$ is zero. The velocity is therefore perpendicular to this plane.

Taking the velocity, then, as q, we have for the circulation round the circle of radius a the value $2\pi aq$. Hence

$$2\pi aq = m, \quad \text{or} \quad q = \frac{m}{2\pi}\frac{1}{a},$$

the same result as before.

348. From this example, also, we get with great ease an expression for the velocity at any point due to an element of a vortex-filament, namely:

$$dq = \frac{ma}{2\pi} \frac{dx'}{(x^2 + a^2)^{\frac{3}{2}}};$$

that is, if P (Fig. 77) be the point at which the velocity is required,

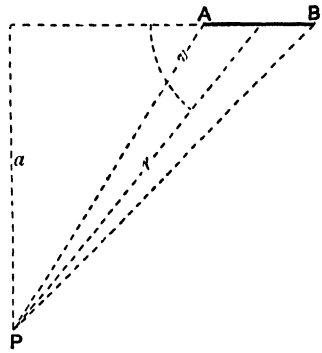


FIG. 77.

ds the length of the element AB, m its strength, and ϑ the angle between the line drawn from P to the centre of the element and the element itself,

$$dq = \frac{m}{2\pi} \frac{\sin \vartheta}{r^2} ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

The direction is at right angles to the plane ABP, and is that in which the point would be carried if it were situated on a rigid body rotating with the vortex element.

This is precisely the expression given by Ampère for the magnetic force due to an element of a conductor carrying an electric current of amount $m/2\pi$.

It must be observed that other expressions might be obtained for the velocity due to an element, which would, when integrated for a complete vortex-filament, give the same result as that of (82). Any term of proper dimensions which, integrated round a closed curve, would give a zero result might be added to the expression in (82) without affecting the velocity distribution due to the complete vortex.

Velocity due to Closed Vortex Filament

349. As another example, consider the velocity due to an endless vortex-filament of any form and of strength m. If ω be the resultant angular velocity, and σ the section at any point, so that $\omega\sigma = m$, and ds' be an element of length of the filament there, an element $d\varpi$ of volume of the filament will be $\sigma ds'$. Thus $\xi d\varpi = d\varpi \cdot \omega dx'/ds' =$

$\sigma\omega dx' = mdx'$. Similarly $\eta d\varpi = mdy', \zeta d\varpi = mdz'$. Thus, we get by (72) and (75), dropping the suffixes,

$$\left.\begin{aligned} u &= \frac{m}{2\pi}\int\frac{1}{r^3}\{(z-z')dy' - (y-y')dz'\} \\ v &= \frac{m}{2\pi}\int\frac{1}{r^3}\{(x-x')dz' - (z-z')dx'\} \\ w &= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (83)$$

where the integrals are taken round the filament.

In the case of the most general system of vortex-tubes, putting $dx'dy'dz'$ for $d\varpi$,

$$u = \iiint\left(\eta'\frac{\partial}{\partial z'}\frac{1}{r} - \zeta'\frac{\partial}{\partial y'}\frac{1}{r}\right)dx'dy'dz' \quad . \quad . \quad . \quad (83')$$

with the symmetrical equations for v and w. The volume integrals are taken throughout all space where ξ', η', ζ' are not zero.

Velocity Potential of System of Vortices

350. The velocity-potential due to a system of vortices can be found as follows. Take, first, a single endless vortex-filament. We have

$$u = \frac{m}{2\pi}\int\left(dy'\frac{\partial}{\partial z'}\frac{1}{r} - dz'\frac{\partial}{\partial y'}\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad (84)$$

a line integral taken round the filament. Writing the equation in the form

$$u = \frac{m}{2\pi}\int(Xdx' + Ydy' + Zdz'),$$

we see that it is equivalent to

$$u = \frac{m}{2\pi}\int\left\{l\left(\frac{\partial Z}{\partial y'} - \frac{\partial Y}{\partial z'}\right) + m\left(\frac{\partial X}{\partial z'} - \frac{\partial Z}{\partial x'}\right) + n\left(\frac{\partial Y}{\partial x'} - \frac{\partial X}{\partial y'}\right)\right\}dS',$$

in which the integral is taken over any surface S' having the filament as bounding edge, and l, m, n are the direction cosines of the normal to any element dS' of that surface.

Now we have

$$X = 0, \quad Y = \frac{\partial}{\partial z'}\frac{1}{r}, \quad Z = -\frac{\partial}{\partial y'}\frac{1}{r}.$$

Hence

$$\frac{\partial Z}{\partial y'} - \frac{\partial Y}{\partial z'} = -\left(\frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)\frac{1}{r} = -\frac{\partial^2}{\partial x\partial x'}\frac{1}{r},$$

$$\frac{\partial X}{\partial z'} - \frac{\partial Z}{\partial x'} = -\frac{\partial^2}{\partial x\partial y'}\frac{1}{r}, \quad \frac{\partial Y}{\partial x'} - \frac{\partial X}{\partial y'} = -\frac{\partial^2}{\partial x\partial z'}\frac{1}{r},$$

so that

$$u = -\frac{m}{2\pi}\int\frac{\partial}{\partial x}\left(l\frac{\partial}{\partial x'} + m\frac{\partial}{\partial y'} + n\frac{\partial}{\partial z'}\right)\frac{1}{r}dS',$$

and similar formulas hold for v, w, with the respective substitutions of $\partial/\partial y$, $\partial/\partial z$ for $\partial/\partial x$ after the integral sign. Hence the potential from which u, v, w are derived is given by the equation

$$\phi = \frac{m}{2\pi}\int\left(l\frac{\partial}{\partial x'} + m\frac{\partial}{\partial y'} + n\frac{\partial}{\partial z'}\right)\frac{1}{r}\,dS'$$

$$= \frac{m}{2\pi}\int\frac{\cos\vartheta}{r^2}\,dS'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where ϑ is there the angle between the line r and the normal (l, m, n). The integral is the solid angle subtended by the filament at the point at which u, v, w are to be found. Clearly the potential is cyclic, since the solid angle changes by 4π as the point considered is carried round from being close to the surface on one side to an infinitely near position on the other side—that is, passes round a closed circuit threaded through the filament.

The potential in the most general case is the sum of the values given by (84) for the potentials of the individual vortex-filaments into which the system can be regarded as divided.

The velocities and velocity-potential above found will, of course, be the actual velocities and potential in the case in which θ, the expansion, is zero. *Cf.* (71).

351. By the result obtained in Art. 317 above, the value obtained for ϕ in (85) may be physically interpreted as the potential due to a system of doublet sources of intensity, each $m/2\pi$ per unit of area, distributed over the surface with their positive ends all turned towards the same side of the surface, and their axes at right angles to the surface at every point. This is obviously consistent with the physical facts of the case. The stream-lines of the motion due to the vortex-filament are stream-lines threading through any surface bounded by the filament, and hence, wherever the surface may be drawn with the filament as edge, it seems physically possible that the motion may be imitated by supposing that fluid is given out at a certain rate normally at one side of the surface by each element, and swallowed up at the same rate by the other side of the element. The motion parallel to the surface at each point of it, compounded with this normal flow, gives the actual direction of motion along the stream-line crossing the element.

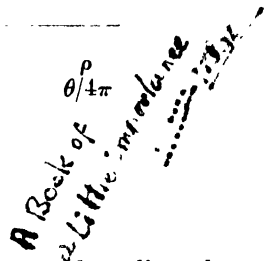
Electromagnetic Analogy

352. All the results above obtained with respect to vortex motion have perfect analogues in electromagnetic theory, so that the mathematical discussion here given will be available when we come to consider the magnetic effects of currents. The analogy is shown by the following table, in which corresponding quantities are bracketed.

The analyses and results, with these substitutions, hold in both theories.

	Components.	
Magnetic force	α, β, γ	
Fluid velocity	u, v, w	
Electric current	p, q, r	
Angular velocity of vortex $\div 2\pi$	$\xi/2\pi, \eta/2\pi, \zeta/2\pi$	
Electromagnetic momentum Auxiliary functions used for calculating u, v, w	F, G, H	in both theories.

Volume density of magnetism	ρ
Expansion of fluid	$\theta/4\pi$
Magnetic poles Sources and sinks	
Linear currents. Vortex-filaments.	
Magnetic potential. Velocity potential.	
Magnetic shell having linear current as bounding edge.	
Surface distribution of doublet sources equivalent to vortex-filament round edge of surface.	



Vortex Sheets.—Cyclic Irrotational Motion regarded as due to Vortex Sheets on Bounding Surfaces.—Electromagnetic Analogy

353. An idea of very great importance in the electro-magnetic theories which follow is that of *vortex-sheets*. Over a given surface let the velocity tangential to the surface be discontinuous from one side to the other, while the normal component is continuous. Then, if u, v, w, u', v', w', be the velocity-components on the two sides, the continuity of the normal component is expressed by

$$l(u - u') + m(v - v') + n(w - w') = 0,$$

where l, m, n are the direction-cosines of the normal. Consider a circuit consisting of two lines of length ds drawn on opposite sides of the surface parallel to the direction of the relative velocity

$$\{(u - u')^2 + (v - v')^2 + (w - w')^2\}^{\frac{1}{2}},$$

and two lines of length dn, infinitely short in comparison with ds, and perpendicular to the surface joining their extremities. The circulation in this path is

$$ds\{(u - u')^2 + (v - v')^2 + (w - w')^2\}^{\frac{1}{2}}.$$

This may be regarded as due to spin of amount $\omega ds dn$, the

direction of which is along the surface in the direction at right angles to ds. Hence we have

$$2\omega dn = \{(u - u')^2 + (v - v')^2 + (w - w')^2\}^{\frac{1}{2}}, \qquad (86)$$

which splits into three parts $\xi dn, \eta dn, \zeta dn$, so that

$$\xi dn = \tfrac{1}{2}(u - u'), \quad \eta dn = \tfrac{1}{2}(v - v'), \quad \zeta dn = \tfrac{1}{2}(w - w') \quad (86')$$

Here ξ, η, ζ (as well as ω) are quantities so great that their products by dn are finite when dn is infinitely small. They are components of strength of the vortex-sheet taken per unit distance in the direction of the relative velocity.

The surface of discontinuity may thus be regarded as covered with vortex-filaments, the vorticity of which is given by the above expressions, and which are everywhere perpendicular to the direction of relative flow. The relative motion may therefore be regarded as due to these vortices.

354. Whenever we have a boundary separating a region of irrotational motion from one of zero motion—as, for example, in the case of a vessel with fixed boundary in which cyclic motion is going on—in which the condition of continuity of the motion normal to the surface is fulfilled, that is, relatively to the surface there is no normal motion either inside or outside, we may regard the motion inside as due to a vortex-sheet covering the whole of the surface separating the moving fluid from the rest of space.

355. This is perfectly analogous to the case of a thin sheet of metal in which flow currents of electricity. The tangential components of magnetic force on the two sides of the sheet are discontinuous, and the line integral taken round a circuit composed of two lines lying close to the two sides of the sheet, with short pieces perpendicular to the sheet joining them, is equal to the current which flows through the circuit just as the circulation in the hydrodynamic case is equal to the vorticity enclosed by the circuit.

An important particular example is a uniform solenoidal distribution of current in a thin cylindrical sheet of metal with lines of flow everywhere at right angles to the generating lines of the cylinder. Here, at a great distance from the ends, and comparatively close to the outside of the cylinder, the magnetic force is zero, while in the interior it is of finite and nearly uniform value and is parallel to the generators of the cylinder. This solenoidal distribution of currents corresponds to a tube which separates a uniform flow of fluid through it from the space outside.

Fluid circulating through an endless tube has its analogue in an endless solenoid.

In all such cases the tube may be considered as a vortex-sheet in which the vortex-filaments surround the tube.

Kinetic Energy of System of Vortices.—Action of Rectilineal Vortices

356. We now consider the kinetic energy due to a system of vortices in an incompressible fluid. We have

$$T = \frac{\rho}{2}\iiint(u^2 + v^2 + w^2)dxdydz \quad . \quad . \quad . \quad . \quad (87)$$

where the integration is taken throughout the whole space in which the motion is not zero. By (78) this becomes

$$T = \frac{\rho}{2}\iiint\Big\{u\Big(-\frac{\partial\phi}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\Big) + v\Big(-\frac{\partial\phi}{\partial y} + \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\Big) + w\Big(-\frac{\partial\phi}{\partial z} + \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\Big)\Big\}dxdydz. \quad . \quad . \quad . \quad (87')$$

which, integrated by parts, gives

$$T = \frac{\rho}{2}\int\Big\{-\phi\frac{\partial\phi}{\partial n} + F(mw - nv) + G(nu - lw) + H(lv - mu)\Big\}dS + \rho\iiint(F\xi + G\eta + H\zeta)dxdydz \quad . \quad . \quad . \quad . \quad (88)$$

in which the first integral is taken over the surface of the enclosing vessel.

The first part of the surface-integral is zero if the fluid is enclosed by fixed walls; the whole surface-integral is zero if the vortices are all within a finite distance of the origin of co-ordinates and the fluid fills all space and is at rest at infinity. For, when quantities in the surface-integral which remain finite when the surface is taken at an infinite distance from the vortices are omitted, F varies as $1/(x^2 + y^2 + z^2)$ and u, v, w as $1/(x^2 + y^2 + z^2)^{\frac{3}{2}}$. Hence the surface integral vanishes. Inserting the values of F, G, H in the volume integral, we get for an infinite liquid of density ρ

$$T = \frac{\rho}{2\pi}\iiint\iiint\frac{\xi\xi' + \eta\eta' + \zeta\zeta'}{r}dxdydz\,dx'dy'dz' \quad . \quad (89)$$

It is worthy of notice that the volume integral in (88) is equal to the sum of the products obtained by multiplying the total rate of flow of matter through the circuit of each vortex-filament into the strength of the filament. This theorem the reader may easily prove for himself.

If the fluid is of finite extent the surface integral in (88) must be retained and taken over the whole bounding surface. It is to be noticed that the volume integral contributes nothing to the value of T except for those parts of the fluid where there is vorticity. Thus in the case of an infinite fluid the value of T in (87), which is an integral taken throughout the whole mass, is reduced by (88) to an integral which extends only to those parts of the fluid where vortex motion exists.

This mode of calculating the energy is that given by v. Helmholtz (*Crelle*, Bd. lv., 1858, s. 45) in his famous memoir on vortex motion, from

which, and Lord Kelvin's paper on the same subject (*Trans. R.S.E.*, vol. xxv., 1869), most of the results given in this section have been taken. As we shall see later, the expression given in (89) is, to a constant factor, precisely that for the energy of a system of electric currents replacing the vortex-system, and having components of current at any point x, y, z proportional to the values of ξ, η, ζ for the same point.

357. If, for example, the vortex-system reduce to two distinct endless filaments of strengths m $(=\omega\sigma)$, m' $(=\omega'\sigma')$ respectively, the corresponding distribution of electric current is that of distinct currents proportional to m, m' flowing in two linear circuits coincident with the filaments. In this case (89) takes a very simple form. We may replace, for any point x, y, z on either filament, dx, dy, dz by σds where ds is an element of length of the filament, and similarly $dx'dy'dz'$ by $\sigma' ds$. By Art. 349 $\xi dxdydz = mdx, \ldots$ and (89) becomes

$$T = \frac{\rho}{4\pi}\left\{m^2\iint\frac{\cos\theta_{12}}{r_{12}}ds_1ds_2 + 2mm'\iint\frac{\cos\theta}{r}dsds' + m'^2\iint\frac{\cos\theta_{12}}{r'_{12}}ds'_1ds'_2\right\} \quad (89')$$

where $\theta_{12}, \theta, \theta'_{12}$ are the angles between the positive directions of two elements, both on the first filament, one on the first filament and the other on the second, and both on the second filament respectively, and r_{12}, r, r_{12} the corresponding distances between the elements of each pair. The two integrations in the first term are both taken round the first filament, the two in the last term are both taken round the second filament, and those of the second term are taken one round the first filament, and the other round the second.

In the general case to which (89) applies, the whole system of vortices might be divided up into filaments, and dealt with by a process similar to that used to establish (89'). Thus we should obtain

$$T = \frac{\rho}{4\pi}\left\{\Sigma(m^2\iint\frac{\cos\theta_{12}}{r_{12}}ds_1ds_2) + 2\Sigma(mm'\iint\frac{\cos\theta}{r}dsds')\right\} . \quad (89'')$$

where the first term denotes the sum of the values of the expression in the brackets taken for each filament, and that of the values of the second expression in brackets taken for every distinct pair of filaments. The value of T is thus a homogeneous quadratic function of the strengths of the vortex filaments. We shall see that the electrokinetic energy of a system of linear currents exactly coinciding with the vortex filaments is a homogeneous quadratic function of the magnitudes of the currents, with precisely the double integrals as coefficients which are displayed in (89'').

358. Again, integrating (87) by parts, we get for a fluid filling all space

$$T = -\rho\iiint\left(xu\frac{\partial u}{\partial x} + yv\frac{\partial v}{\partial y} + zw\frac{\partial w}{\partial z}\right)dxdydz . \quad . \quad (90)$$

since the surface integral vanishes. By the equation of continuity for an incompressible fluid this may be written

$$T=\rho\iiint\left\{u\left(y\frac{\partial v}{\partial x}+z\frac{\partial w}{\partial x}\right)+v\left(z\frac{\partial w}{\partial y}+x\frac{\partial u}{\partial y}\right)+w\left(x\frac{\partial u}{\partial z}+y\frac{\partial v}{\partial z}\right)\right\}dxdydz \quad (91)$$

But, by integration by parts, also

$$\iiint u\left(y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}\right)dxdydz=-\iiint(u^2-u^2)dxdydz=0,$$

with two similar results for v, w. These give by subtraction from the expression on the right of (91)

$$T=2\rho\iiint\{u(y\zeta-z\eta)+v(z\xi-x\zeta)+w(x\eta-y\xi)\}dxdydz \quad (92)$$

an integral, again, confined to the vortex region of the fluid.

359. The fact that the elements of the volume integrals in the above expressions for the kinetic energy are zero except where there is spin of the fluid involves some important consequences. For example, take the case of a fluid in irrotational motion in a multiply connected space—that is, circulating round cores or through apertures in fixed solids. The motion normal to the bounding surfaces of the solids or vessel is everywhere zero, and we may suppose the fluid to extend throughout the rest of space if the velocity there is everywhere zero. Then we have simply a case of motion tangentially discontinuous at certain surfaces, and it has been shown that such a motion can be regarded as produced by a vortex-sheet extended over those surfaces. Hence the kinetic energy in all such cases can be calculated by (89) or (92) properly modified to suit the very great spin which must be regarded as existing within a very thin sheet of the fluid at the surface.

It is clear from the expressions for the relative velocity in Art. 352, that the case in which the solids immersed in the fluid are infinitely thin cores, round which the fluid circulates in irrotational motion, the strength of the vortex sheet directed along the surface of any core in the direction at right angles to that of the relative motion of the fluid, is equal to $\rho\kappa$ where κ is the cyclic constant for the core. For the line integral of the relative velocity round the core is the strength of the vortex, and this is also κ.

The electric analogue is a corresponding distribution of currents along the cores, and the force-systems between the cores are the same in amount as those between the conductors replacing the cores in that distribution. The forces are however opposite in sign in the two cases, as will be explained in the dynamical theory of currents given below.

360. We shall now consider briefly the action of vortices on one another taking only the case of parallel rectilineal vortices in an infinite incompressible fluid. The motion of the fluid due to such a system will be the same for all values of z, and we may therefore treat the motion as two-dimensional. Consider then any such system of vortices

having their axes all parallel to the axis of z. Let at any point (x, y) the angular velocity in the vortex-motion be ζ, and the components of velocity there be u, v. These must be component velocities with which the vortex-filament is changing in position there, inasmuch as vortex-filaments, as we have seen, move with the fluid.

It is clear that as any vortex-tube moves its cross-section remains unchanged. For the tube remains always composed of the same particles of fluid, the height parallel to z of any portion of it remains constant, and, the fluid being incompressible, the volume must remain constant. Hence the cross-section remains constant.

But, by Lord Kelvin's theorem, Art. 290 above, $\int \zeta \, dx \, dy$ must remain constant for any tube whatever. We may write this as $A\zeta_m$, where ζ_m is the mean value of ζ over the cross-section of area A. Since A remains constant, ζ_m must also remain constant. This holds for a tube, however small in dimensions of cross-section, taken in the system; hence ζ, the angular velocity of any point in the system, remains constant.

We can now prove that for the whole system the equations

$$\int u\zeta dS = 0, \quad \int v\zeta dS = 0 \quad . \quad . \quad . \quad . \quad . \quad (93)$$

hold, when the integrals are extended over all parts of the plane of x, y where there are vortices. The first integral may be written in the form

$$\tfrac{1}{2}\int u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy.$$

Integrating by parts we obtain

$$\int u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy = \int (uvdy - u^2dx) - \int\left(v\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y}\right)dxdy.$$

The line integral is to be taken round a curve encircling the whole system of vortices. If the vortices be all within a finite distance from the origin, and the curve of integration is taken everywhere at an infinite distance, the integral, as will be seen by considerations similar to those stated in Art. 355 above, must vanish. But by the equation of continuity the remaining part becomes

$$\int\left(v\frac{\partial v}{\partial y} + u\frac{\partial u}{\partial y}\right)dxdy = \tfrac{1}{2}\int (u^2 + v^2)dx,$$

a line integral which also clearly vanishes when taken round a curve at an infinite distance from the vortex system. Hence we have (93).

Since any element ζdS does not vary with the time, we have, integrating with respect to the time,

$$\int x\zeta dS = C, \quad \int y\zeta dS = C \quad . \quad . \quad . \quad . \quad . \quad (94)$$

where C, C' are constants. Take now mean values of x_0, y_0 for the system, such that

$$x_0\int\zeta dS = \int x\zeta dS, \quad y_0\int\zeta dS = \int y\zeta dS \quad . \quad . \quad . \quad (95)$$

the values of x_0, y_0 remain unchanged as the system of vortices moves with the fluid. Hence the point x_0, y_0, which may be called the centre of the system of vortices for the plane x, y, remains unaltered in position. The straight line parallel to z, through the point x_0, y_0, which we call the axis of the system, remains fixed in space.

It is to be remarked that x_0, y_0 are the co-ordinates of the centre of inertia of a thin stratum of matter imagined spread over the plane of x, y so that the density at each point is proportional to the value of ζ there.

361. We can now calculate the motion of the axis of any vortex-tube. To do this, it is only necessary to consider the motion due to the other vortices, for we have seen that the distribution of velocity due to any particular tube cannot affect the motion of that tube. The velocity at any point x, y due to any filament, the co-ordinates of which are x', y' and cross-section σ' at distance $r = \sqrt{(x-x')^2 + (y-y')^2}$ from x, y is, as we have seen above, $\zeta'\sigma'/\pi r$. This, if θ be the angle r makes with the axis of x, has components

$$du = -\frac{\zeta'\sigma'}{\pi}\frac{\sin\theta}{r}, \quad dv = \frac{\zeta'\sigma'}{\pi}\frac{\cos\theta}{r},$$

that is

$$du = -\frac{\zeta'\sigma'}{\pi}\frac{y-y'}{r^2}, \quad dv = \frac{\zeta'\sigma'}{\pi}\frac{x-x'}{r^2}.$$

Taking σ' as an element dS' of cross-sectional area of a vortex-tube, we get, integrating over the cross-section of the whole system,

$$u = -\frac{1}{\pi}\int\zeta'\frac{y-y'}{r^2}dS', \quad v = \frac{1}{\pi}\int\zeta'\frac{x-x'}{r^2}dS'.$$

Thus, if we write

$$\psi = \frac{1}{\pi}\int\zeta'\log r\, dS' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

we have

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Of course it is needless to introduce into ψ the value of ζ belonging to any part of a vortex-tube for the calculation of the motion of the axis of which ψ is to be used.

The function ψ is the stream-function and fulfils the differential equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 2\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

from which, according to the theory given above, the solution might have been derived. Of this equation (94) is the solution appropriate to this case.

When the fluid does not extend to infinity in all directions, but is bounded by a surface parallel to the axis of z we must add to this solution a complementary function ψ_0, so that

$$\psi = \frac{1}{\pi}\int \zeta' \log r \, dS' + \psi_0 \quad . \quad . \quad . \quad . \quad . \quad (98')$$

This function must be so chosen as to enable the boundary conditions to be satisfied, and to satisfy the equation

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

It represents the stream-function due to the vortex-sheet which, according to Art. 352 above, may be supposed to exist on the surface, and is such as to reduce the velocity to zero for every point external to the surface. Equation (98) is thus, by (98'), satisfied at every point inside and outside the boundary.

362. We shall now consider one or two particular cases. Suppose that there are two infinitely thin rectilineal vortices A, B, of strengths m_1, m_2, at a distance r apart in an unlimited fluid. Taking the axis of x along, r, and the origin at A, we have, for the velocity of B, $v_1 = m_1/\pi r$, and, for the velocity of A, $v_2 = -m_2/\pi r$. The velocities are thus inversely as the

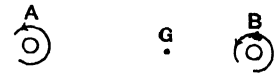


FIG. 78.

strengths of the vortex-filaments, and the pair of filaments (not the fluid around them) move as if they were rigidly connected with an axis A parallel to z, and in the plane of the vortices, at a distance from the origin $m_2 r/(m_1 + m_2)$. If the vortices are of the same sign this axis lies between the vortices at G; if the vortices are of opposite signs it lies on the line AB produced beyond the stronger vortex, and at a distance given by the same formula, account being taken of the signs of the values of m_1, m_2. If $m_1 = -m_2$, G is at infinity, and the pair of vortex-filaments move with constant velocity $m_1/(\pi . AB)$ at right angles to the line AB joining them.

The stream-lines due to a pair of vortex-filaments are given by the equation

$$m_1 \log r_1 + m_2 \log r_2 = C,$$

where r_1, r_2 are the distances of the filaments from any point the motion at which is under consideration. Different stream-lines are obtained by giving different values to the constant C.

363. The stream-lines of a pair of equal and opposite vortex-filaments are given by the equation $\log r_1 - \log r_2 = \text{constant}$, or

$$\frac{r_1}{r_2} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

and are two sets of circles surrounding A, B respectively, as shown in Fig. 79.

The straight line running up the centre of the diagram may be regarded as the limiting circle common to the two sets. Since there is

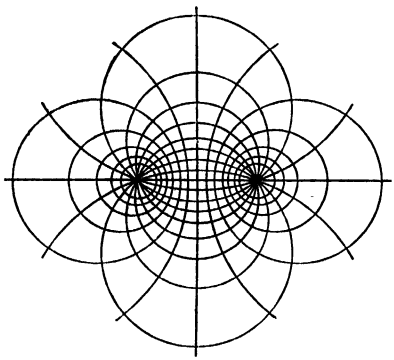


FIG 79.

no motion across the plane of which this straight line is the trace on the plane of x, y, no change in the motion of the fluid would be produced on either side by replacing it, after the motion has been set up, by an infinitely thin fixed sheet of matter cutting off communication between the portions of the fluid on the two sides. Thus the motion of a single rectilineal vortex which is parallel to and at a distance d from a fixed infinite plane wall is $m/2\pi d$.

Again, since there is no flow across the cylinder bounded by any circle of either set, we may suppose the surface of that cylinder replaced by a material wall cutting off all communication between the fluid on one side and the fluid on the other side of the surface. Then clearly, if we have a filament of strength m at A, or one of strength $-m$ at B, that filament must move always at right angles to the line drawn from it normal to the cylinder, and each point of it will therefore describe a circle round the axis of the cylinder.

It will be observed, however, that this solution, in the case in which the external filament is, say, A, presupposes a certain circulation, that due to the filament B, in a circuit round the outside of the cylinder but not embracing A. The amount of this is $-2m$. If a vortex-filament of strength $+m$ be placed along the axis of the cylinder, and we add its circulation everywhere to that of $-m$ at B, the circulation outside the

cylinder will be zero, and the section of the cylinder by the plane of x, y will remain a stream-line. The velocity of the filament at A will be $-m/\pi \,.\, (1/AB - 1/CA)$ at right angles to AB. But by the equation of the circular section of the cylinder

$$CB \,.\, CA = c^2,$$

where c is the radius, and therefore we have

$$AB = CA - CB = \frac{c^2 - CB^2}{CB} = \frac{CA^2 - c^2}{CA}.$$

The velocity of the filament A becomes therefore

$$-\frac{m}{\pi}\left(\frac{CA}{CA^2 - c^2} - \frac{1}{CA}\right).$$

If the solution is to provide for a specified circulation κ in the circuit referred to, the term $\kappa \,.\, CA/2\pi$ must be added to the velocity of A just found.

Since the radius c is the geometric mean of the distances CA, CB of the filaments from the axis, the filament B is called the image of the filament A in the cylinder.

364. It is an easy deduction from the preceding theory (for example from the theorem stated in the third paragraph of Art. 356), and it is of great importance in the electromagnetic analogue, that the kinetic energy T (taken for unit length parallel to z) of a system of rectilineal vortices is given by

$$T = \frac{\rho}{\pi}\iint \zeta_1 \zeta_2 \log r_{12}\, d\sigma_1\, d\sigma_2 \quad . \quad . \quad . \quad . \quad (101)$$

where ζ_1, ζ_2 are the angular velocities at filaments of infinitely small cross-section $d\sigma_1$, $d\sigma_2$ at a distance r_{12} apart, and the integration is so taken as to include *once* every distinct pair of elements in the system. It is to be observed, however, that in order that T may be finite the integral $\int \zeta d\sigma$ taken over all the filaments must be zero, and the value of ζ must be everywhere finite.

If for a set of isolated parallel rectilineal vortices of strength $m_1, m_2, \ldots$ situated at the points $x_1, y_1, x_2, y_2, \ldots$, for which the condition $\Sigma m = 0$ is not necessarily fulfilled, we write

$$P = \frac{1}{\pi}\Sigma m_1 m_2 \log r_{12},$$

where r_{12} is the distance of x_1, y_1 from x_2, y_2 and the summation is taken so as to include every distinct pair just once, we obtain by (96) and (97)

$$\left.\begin{aligned} m_1 u_1 &= -\frac{\partial P}{\partial y_1}, \quad m_1 v_1 = \frac{\partial P}{\partial x_1} \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (102)$$

which are clearly equivalent to

$$\Sigma mr\dot{r} = -\Sigma\frac{\partial P}{\partial\theta},$$

where r, θ are the polar co-ordinates of any filament, and m its strength. But, since P is independent of the choice of axes, turning them round through any angle does not alter P; that is, $\Sigma\,\partial P/\partial\theta = 0$, and the last result gives

$$\Sigma mr^2 = C,$$

where C is a constant.

Finally, from (102) we obtain

$$\Sigma m(xv - yu) = \Sigma\left(x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y}\right),$$

or, which is the same thing,

$$\Sigma mr^2\dot{\theta} = \Sigma r\frac{\partial P}{\partial r}.$$

The expression on the right is proportional to the change in P which would be produced by altering all the rs by amounts proportional to their actual values, that is producing a new configuration of the vortex-filaments geometrically similar to the former one. This change is equivalent to altering all the distances $r_{12}, \ldots$ by amounts proportional to their values. Hence $\Sigma r\,\partial P/\partial r = \Sigma m_1m_2/\pi$, and we have

$$\Sigma mr^2\dot{\theta} = \frac{1}{\pi}\Sigma m_1m_2.$$

This theory of a set of isolated rectilineal vortices is due to Kirchhoff (*Mechanik*, 20ste Vorles., § 3).

CHAPTER IX

ELEMENTARY FACTS AND THEORY OF ELECTROMAGNETISM

Magnetic Fields of Currents. Electromagnetic Forces

365. We have in Chapter VI. considered the flow of electricity in conductors, and stated the laws of distribution of a steady current in a network of wires. As a preliminary to the discussion of general electromagnetic theory, it is convenient now to deal with the magnetic effects of currents, that is with the ordinary dynamical actions between currents and magnets, and of currents on one another. We shall proceed in the order of discovery, and first describe these effects from the point of view in which they present themselves, that is as apparent actions at a distance, and endeavour to show in later chapters how they may be regarded as actions taking place in a medium filling the field.

Örsted's Experiment

366. The fundamental experiment of this part of electromagnetism is that made in 1820 by Örsted in Copenhagen. Fig. 80 shows the arrangement of apparatus commonly employed for its repetition in illustrated courses of lectures. A wire is stretched horizontally in the magnetic meridian, and above it or below it is placed a magnetic needle which rests parallel to the wire, provided no current flows in the latter.

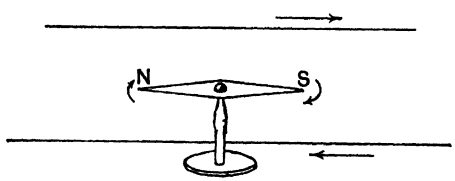


FIG. 80.

When, however, a current is made to flow the needle is deflected through an angle round its axis of suspension, and takes up a position of equilibrium intermediate between its original position and that at right angles to the wire. It is in fact acted on by a deflecting couple due to the current, and finally rests in stable equilibrium when

the restoring couple due to terrestrial magnetism balances the disturbing couple.

If without alteration of the current the conductor be turned "end for end," the needle is turned round in the opposite direction. This clearly shows that the so-called current has directional quality, whatever the real nature of the phenomenon may be.

Again, if the wire without reversal of direction relatively to the current is transferred from above the needle to below it, the direction of turning is also reversed. Thus if the wire run, say, from south to north above the needle and back from north to south below it, and a current be made to flow in the wire, the two parts produce effects on the needle which conspire to deflect it in the same direction. Hence by winding the wire a large number of times in the plane of the magnetic meridian, so as to make a coil surrounding the needle, it is possible to obtain a greatly enhanced effect, and a feeble current flowing in the circuit may be made to produce a large deflection of a magnetic needle properly suspended.

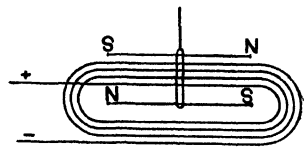


FIG. 81.

Each turn of wire exerts a couple on the needle, and the resultant couple is the sum of all the couples exerted by the simple turns of wire. Fig. 81 shows such an arrangement of wire, and is in fact a picture of a now obsolete form of "galvanic multiplier" or galvanometer.

To specify the direction in which the needle is turned we have first to specify that in which the current is considered to flow in the wire. Imagine the wire stretched north and south above the surface of a table, and let a magnet resting on the table, with its length east and west, and its north-pointing end turned towards the vertical plane through the wire, be brought up towards it from the west side of that plane. In consequence of the alteration of the magnetic field in the vicinity of the wire a current will be produced which is taken as flowing from *south to north* in the wire. If the wire be below the table the current will flow in the opposite direction, and this may be verified by noting that the deflections of a needle produced by the currents in the two cases are in opposite directions.

The direction here *assumed* for the current agrees with the convention based on the use of a voltaic cell, and, for the reason indicated in Art. 216, generally adopted. A simple form of voltaic cell consists of a plate of zinc amalgamated with mercury and a plate of copper placed side by side, but not directly in contact, in a vessel containing dilute sulphuric acid. When the plates are connected externally by a wire, a current flows in the circuit thus made up, the direction of which is assumed as being from the copper plate to the zinc plate along the wire. If then the current in the wire stretched in the south and north direction above the needle were produced by connecting the copper plate of such a cell to the south end of the wire, and the zinc

plate to the north end the deflection would be in the same direction as when the current is produced by moving a magnet along the table towards the wire (supposed above the table) in the manner already described.

The direction in which the current is supposed to flow being thus fixed, we can specify that in which the magnet turns round under the action of the current. Supposing the wire to run north and south above the needle and the current to flow from south to north, the north-pointing end of the needle will move towards the west, the other end towards the east. This rule may very easily be remembered by the following mnemonic device: Hold the right hand with fingers pointing along the wire in the direction in which the current flows, and with the palm turned towards the needle. The north seeking end of the needle will turn towards the outstretched thumb. Or, remembering that the northern regions of the earth have magnetism of the opposite kind to that of the north-pointing end of a needle, we may keep the rule in mind by remembering that the terrestrial magnet may be regarded as turned into position across the plane of the equator by currents circulating round the equator in the direction of the sun's apparent motion.

Ampère's Theorem of Equivalence of a Current and a Magnetic Shell

367 The explanation of Örsted's experiment is an important application of the general electrodynamic theory of Ampère, contained in the famous memoir[1] which is justly regarded as the *Principia* of electromagnetism. The fundamental theorem of Ampère's memoir, so far as this part of the subject is concerned, is contained in the following statement: *Every linear conductor carrying a current is equivalent to a simple magnetic shell, the bounding edge of which coincides with the conductor, and the moment of which per unit of area, that is the strength of the shell, is proportional to the strength of the current.* Thus in Ampère's view a current, whatever the form of the circuit in which it flows, is equivalent to a certain distribution of magnetism, that is to say, it produces a magnetic field affecting magnets placed in it just as the field of a certain actual system of magnets would. Indeed, as we shall see, he put forward the theory that an actual magnet is nothing else than a congeries of electric circuits of molecular dimensions carrying currents of electricity, and that the difference between a magnetized and a nonmagnetized body consisted in a similar orientation of these molecular circuits conferred on them in the case of the former body by the act of magnetization.

The direction of magnetization of Ampère's equivalent shell may be specified as follows. Let an observer stand on the shell near its edge, and face so that the surface of the shell is on his left hand, the edge on his right. Then, if he is looking in the direction in which the current is flowing the face on which he stands will be that covered with

[1] *Théorie des phénomènes électrodynamiques*, Mémoires de l'Institut, IV., 1823.

northern magnetism, that is magnetism of the opposite kind to that of the earth's northern regions. If he is looking in the opposite direction to that in which the current is flowing, the face of the shell on which he stands is that covered with southern magnetism. This result may also be remembered by the rule already given by means of the magnetization of the earth regarded as produced by currents flowing from east to west round the equator.

368. The theorem of Ampère stated above is founded on experiments proving the truth of the following more elementary theorem, which we shall now consider: *The magnetic field produced by the current in a plane closed circuit is the same at all points, the distances of which from every part of the conductor are great in comparison with every dimension of the circuit, as that produced by a small magnet placed anywhere within the circuit, with its centre in, and its axis at right angles to the plane of the circuit, and having a magnetic moment proportional to the current flowing, and to the area of the circuit.*

The following is an experiment by which this theorem may be verified. A circular circuit is mounted in a vertical plane on a sliding piece which can be moved along a horizontal slide. The circuit is arranged so as to be in the magnetic meridian, and the slide is therefore in the east and west magnetic direction. A needle is now set up with its centre on the east and west (magnetic) line through the centre of the circuit, and is provided with a pointer moving round a circular scale so as to show angles of deflection directly, or has rigidly attached to it a mirror by which a ray of light from a lamp is reflected to a scale, and which thereby measures the deflection of the needle. When a constant current is sent round the circuit (by means of wires from a battery at some distance and twisted together to prevent this part of the circuit from producing any direct effect on the needle) and the position of the circuit is changed along the slide, deflections of the needle are produced which show that the magnetic forces at the centre of the needle are very nearly in the inverse ratio of the cubes of the distances of the centre of the needle from the centre of the coil, when these distances are great in comparison with the dimensions of the circular conductor. Now we have seen above, p. 28, that this is exactly the result that would have been produced by placing a small magnet with its centre at the centre of the circuit, and its length at right angles to its plane.

If the circuit mounted is not circular the same result will be found to hold whatever be the form, provided the east and west line through the centre of the needle passes through the plane of the conductor within or near the circuit, and the distances are measured from the plane of the circuit to the centre of the needle. A small magnet placed along this line with its centre in the plane of the circuit would produce deflections of the needle following the same law of variation with distance. By properly choosing the moment of the magnet the deflections of the magnet and circuit may be made identical. By reversing

the magnet and using it along with the circuit, the two being displaced together, it can be shown that if there is an exact balance of effects at any distance, there is balance at all distances, provided of course the current is not altered from one experiment to another.

To complete the demonstration it is only necessary to notice that if the area of the circuit is altered in any given ratio the force at the centre of the needle is changed in the same ratio, and, further, to test whether a magnet and a current which produce the same magnetic force at distant points upon an east and west line passing through the circuit, as described above, also produce the same magnetic effect at all other distant points. Experiments to prove these facts can obviously be arranged with great ease, and it is not necessary here to enter into details regarding them.

Definition of Unit Current. Proof of Ampère's Theorem

369. It is now possible to define the measure of a current by means of its magnetic action. We take the numerical measure of a current as proportional to the intensity of the magnetic field produced by it at a given point. This mode of numerically reckoning current, as we shall find later, gives results consistent with those obtained from other methods which are sometimes used.

We define unit current as that current which flowing in a circuit of unit area can be replaced by a magnet of unit magnetic moment without altering the magnetic field produced at a distance from the circuit. Unit magnetic moment has already been defined, and in the C.G.S. system is the moment of a doublet composed of two opposite point-charges of magnetism, each 1 C.G.S. unit, placed at a distance of one centimetre apart. Thus, when the area of the circuit is one square centimetre, and it is replaceable as regards magnetic action by such a doublet, the current flowing is 1 C.G.S. unit.

It is important to observe that the magnet equivalent at distant points to the plane circuit may be supposed replaced by a very large number of equal small magnets uniformly distributed over the area enclosed by the circuit, with their centres in and their lengths at right angles to its plane. The same field as before will be produced if the total magnetic moment is the same as before: for, as has been seen, the force which a magnet produces at distant points is not affected by the position of the magnet within the circuit provided its direction is always the same. But the process of distribution of a large number of small magnets converts the equivalent magnet into an approximation to a uniform magnetic shell, the strength of which (that is its magnetic moment per unit area) is simply the measure of the current, and the larger the number of small magnets the closer is this approximation.

370. We can now prove Ampère's proposition stated above, that *any* linear circuit carrying a current is equivalent in external action to a magnetic shell, the edge of which coincides with the circuit. Let

a current of γ units flow in the circuit, represented by BAC in Fig. 82. As shown in Fig. 82, we may suppose the circuit converted by cross conductors into a network without any displacement of the boundary, and we may suppose that round each mesh a current γ also flows in the same direction as that flowing in the original conductor. These mesh currents give equal and opposite currents, that is zero current, in each conductor common to two meshes, and the system reduces at once to the current supposed to flow in the boundary and that alone. Thus we may suppose the action of the latter current the same as that of the system of mesh currents imagined. But each of the meshes may be taken so small that it may be regarded, with as little error as we please, as a small plane circuit: and each of these small circuits is by the preliminary theorem replaceable by a small magnet, or by a magnetic shell of strength equal to the current. But this replacing of each of the meshes by a shell would give a shell of strength γ bounded by the circuit. Hence the theorem.

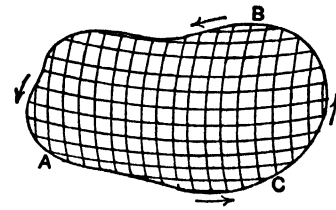


FIG. 82.

It is to be observed that any point at which the action of the finite shell is considered need only be at a distance from any part of the equivalent shell great in comparison with the dimensions of a mesh, hence the limitation as to distance imposed in the preliminary theorem does not here apply. It is only necessary to take into account the finite thickness of the wire, and therefore consider magnetic action at points at a distance of several diameters of the wire from the boundary.

It is also to be carefully observed that provided the boundary of the shell, that is the circuit, be undisturbed, the meshes may be supposed to have any position we please, in other words the shell is defined only by its boundary.

Theorem of Work done in carrying a Unit Pole round Closed Path in Field of Current

371. From the theorem of the magnetic shell we can at once derive a theorem of the greatest interest and importance in electrodynamics. Let a unit magnetic pole be carried in a closed path from any point P in the field of a circuit back again to the same point. The work done against or by magnetic forces is zero if the path does not pass through the circuit, and $4\pi\gamma$ if it does.

To prove these statements let the equivalent magnetic shell be constructed in such a position as not to intersect the closed path. Then it is clear that if work is done in carrying the pole from the point P to another Q on the path, just as much work will be gained in carrying the pole from Q to P along the remainder of the path. Thus the work is zero.

Again let the shell be imagined constructed so as to be infinitely near to the starting point P, and let the pole be carried round from P to another point Q infinitely near to P, but on the opposite side of the shell. Now we have seen at p. 32 that the work gained or spent in so doing is 4π times the strength of the shell, or $4\pi\gamma$. But although the shell was supposed fixed when the pole was carried round from P to Q, and the forces at each point were the same as those given by the current, it is obviously not necessary to suppose the shell in the same position when the pole is carried through the remaining short distance QP required to complete the closed path. We therefore now suppose the shell in a position clear of the element OP of the path: and it is plain that since the forces are finite and the element QP is infinitely short, the work done over QP is zero. Hence the work done in carrying the pole round the circuit is $4\pi\gamma$.

372. If the path be laced round the circuit any number of times, n, the work done in carrying the pole round the circuit is $4\pi n\gamma$. For let the full line in Fig. 83 represent the path, and let the different spires of the path Q, S, U, W, be connected by the dotted path P, R, T, V. Let the pole be supposed first carried round the closed path $PQRP$, next round the path $RSTR$, then round $TUVT$, and so on, the final path traversed being $PRT \ldots P$. By this process $4\pi\gamma$ work is done (or gained) in every closed path $PQRP, RSTR$, &c., and the paths PR, RT, &c., added are each traversed twice in opposite directions, so that the work done in them is zero. Hence the work done on the whole is simply that done in the path $PQRS \ldots P$ which was given.

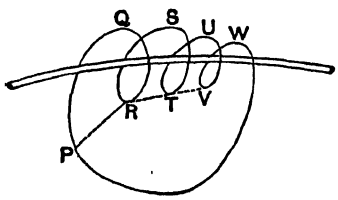


FIG. 83.

In the same way it can be shown that if the circuit pass any number of times n through the closed path the work done in carrying a unit pole round the path is again $4\pi n\gamma$. For, take the simple case of Fig. 84, in which the circuit passes twice through the path. Let the path be converted into two separate paths, each enclosing the circuit once, by drawing the line SQ. The work done in carrying the pole round SRQ is $4\pi\gamma$, and that done in carrying it round $QPSQ$ has the same value and sign. But in these two displacements of the pole the path SQ is traversed twice, in opposite directions, and hence the work done in it is zero. The work really done is therefore that in the path $RQPSR$, and is $2 \times 4\pi\gamma$.

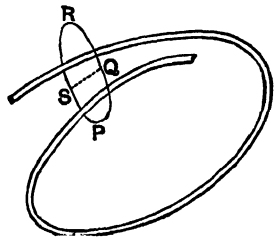


FIG. 84.

In a precisely similar manner it can be proved that for any number n of passages of the circuit through the path the work done is $4\pi n\gamma$.

Any complex case made up of interlacing of the path round the

circuit and the circuit through the path can be dealt with by combining the results of the two kinds of interlacing taken separately.

Effect of Medium occupying the Field

373. So far we have said nothing of the effect of the medium surrounding the circuit, and the theorems stated have been given without any limitation imposed by the medium or media occupying the field. The theorem of work done in any path in the field just discussed holds in all cases without modification, inasmuch as the magnetic intensity (not the magnetic induction) does not vary with the nature of the medium occupying the field, provided that medium is not different in different parts. This follows from the fact, verified by experiment, that the mutual action between a current flowing in a conductor and a distribution of magnetism is independent of the nature of the medium surrounding them. This theorem, however, does not hold for two distributions of magnetism.

In fact, the magnetic force is simply to be calculated from the magnetic shell of strength γ already defined, and that multiplied by the inductivity μ of the medium, gives the induction at the point considered. If the medium is not the same throughout the field, as when it consists partly of iron partly of air, the magnetization of the different parts must be taken into account in assigning the value of the magnetic force at any point.

The theorem, however, that the work done in carrying a unit pole round a circuit in which a current is flowing is $4\pi\gamma$, holds in all cases, for the reason that the line integral of the magnetic force depending on the magnetization of the medium is zero for any closed curve, since the potential of that magnetization is single valued so long at least as molecular circuits, if these exist, are not threaded by the line of integration. In the displacement of magnets such lines of integration have to be considered.

It will be noticed that in either case the quantity of magnetism of the shell replacing a current is not definite, but depends, even if the thickness of the shell is assigned, on the position chosen for the shell. For a given chosen thickness and position the quantity of magnetism varies with the medium, since the magnetic induction is different in the different cases. The advantage and consistency of thus regarding the action of a circuit will become more apparent when we consider the energetics of current-carrying circuits.

It is well here, however, to remark that if a given actual magnetic shell and a circuit are equivalent in one medium, they are not equivalent in another of different magnetic inductivity. The medium within the circuit is magnetized by the current and produces an effect at every point, and this varies when the medium is changed. The magnetic system on the other hand being composed of steel or some other solid magnetized substance does not admit of change of the medium within it as in the other case, and the magnetization in the space occupied by it is the same whatever the medium surrounding it may be.

Experiments Confirmatory of Ampère's Theory

374. Experiments made by Biot and Savart show that the magnetic force produced by the current in a long straight conductor at points at distances from the conductor small in comparison with its length and not near the ends, varies inversely as the distance of the point considered from the conductor. Its direction is at right angles to the plane through the conductor and the point considered. In fact the lines of force round the conductor, except near the ends, where they are affected by the connecting wires, are circles having the conductor for their common axis.

One set of Biot and Savart's experiments was made with a vertical conductor A B, near which was suspended by a cocoon fibre a small needle as shown in Fig. 85. The earth's force was neutralised at the needle as nearly as possible by a properly placed compensating magnet.

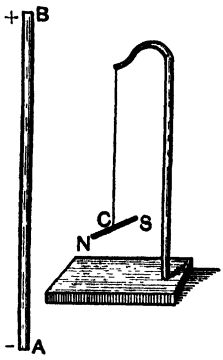


FIG. 85.

The needle was found to rest in the position shown in Fig. 85, so that lines drawn from the conductor to its extremities made equal angles with its length. This proved that the forces on its poles were, as shown in the figure, also equally inclined to the length of the magnet, so that the resultant force on the needle was at right angles to its length.

Experiments were made at different distances, and the forces at the different places compared by oscillating the needle. The results gave for the forces at different distances values inversely proportional to the distances.

Experiments were also made by stretching a wire horizontally at right angles to the magnetic meridian, and placing at different distances above and below it a horizontally suspended needle. They then observed the period of oscillation (1) when the needle was under the earth's force alone, (2) when a current was made to flow in the conductor. If T, T', denote the periods observed in the respective cases, K the moment of inertia of the magnet, and M its magnetic moment, it is easy to show that the intensity of the field due to the current is given by the expression $4\pi^2K/M.(1/T'^2 - 1/T^2)$. For since the magnetic force is at every point at right angles to the plane through the conductor and the point, the force due to the current and the horizontal force of the earth are in the same direction. Let H denote the earth's horizontal force, H_1 that due to the wire, we have

$$(H + H_1)/H = T^2/T'^2, \quad \text{or} \quad H_1 = HT^2(1/T'^2 - 1/T^2).$$

But by the theory of simple harmonic motion $4\pi^2/T^2 = MH/K$, or

$HT^2 = 4\pi^2 K/M$, and substituting this value of HT^2 in the formula for H we obtain the expression stated.

The results of the experiments showed clearly that the force varied in the manner stated, namely inversely as the distance of the point from the straight conductor.

375. The same law is shown very elegantly by an experiment due to Clerk Maxwell. The conductor is placed in a vertical position, and a light carriage of non-magnetic material is suspended so as to be free to turn round the conductor as an axis. A magnet is then placed on the carriage in any position. It is found that there is no moment tending to turn the carriage round the conductor, however great the current flowing in the wire. This proves that the turning moments due to the two different kinds of magnetism are equal and opposite.

For suppose that the magnet is a uniformly magnetized thin bar, and the conductor a thin wire, so that there may be no doubt as to the distance of the points considered from the conductor. Let the forces exerted on the extremities of the bar at right angles to the planes through them and the wire be F_1, F_2, and the distances of the poles from the conductor be r_1, r_2. Then since there is no turning moment

$$F_1 r_1 + F_2 r_2 = 0,$$

and therefore

$$-\frac{F_1}{F_2} = \frac{r_2}{r_1},$$

that is the forces have opposite signs, and are inversely as the distance from the axis. The actual magnet may be supposed made up of such thin uniformly magnetized bars, and the current of filamentary conductors, so that the theorem will by superposition of effects hold in this case also.

Theorem of Biot and Savart

376. Fig. 7 shows the lines of force as displayed by iron filings sprinkled on a card through which a straight wire is passed normally. Fig. 86 shows the relation between the direction of the current and the magnetic force. The current has the direction $A\,B$, while the magnetic force F is tangential to the circle passing through P the point considered, and having the wire as axis.

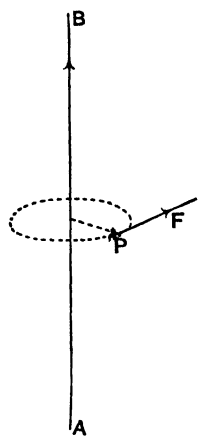


FIG. 86.

From the result obtained above as to the work done in carrying a pole in a closed path round a conductor in which a current is flowing, we can easily find the numerical value of the force at any distance. Calling γ the current, F the force at distance r, then we

have for the work done in carrying a unit pole round the circle of radius r

$$2\pi r F = 4\pi\gamma,$$

or

$$F = \frac{2\gamma}{r}.$$

Magnetic Potential of a Current

377. But the result may be obtained otherwise as follows:—Consider the magnetic shell replacing an infinitely long straight conductor. It is geometrically defined only by its edge, that is by the conductor, and therefore we may take it in any position we please and as a plane surface. Let the conductor, Fig. 87, be at right angles to the plane of the paper, so that A is its projection, AB that of the shell, P the position of the magnetic pole, $CP(=a)$ the distance of the pole from the plane of the shell, AC $(=b)$ the distance of C from A. We shall calculate first the potential of the shell at P, supposing the positive side of the shell turned towards that point.

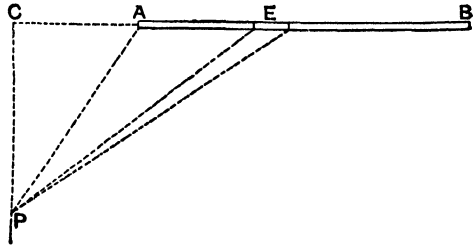


FIG. 87.

The solid angle subtended at P by the shell is the area of the lune cut out of a sphere of unit radius drawn from P as centre by planes drawn through P and the edges of the shell. The first edge is at A, the second is at an infinite distance on the right in the diagram. Hence the angle between these planes is $\pi/2 - APC$, or $\pi/2 - tan^{-1}b/a$. But the area of a lune cut out by planes inclined at an angle θ is 2θ, since that between planes inclined at 2π is 4π. Hence the solid angle subtended by the shell is $\pi - 2\,tan^{-1}b/a$, and the potential Ω of the shell is $\gamma(\pi - 2\,tan^{-1}b/a)$.

The components of the magnetic force at P are $-\partial\Omega/\partial a$, $-\partial\Omega/\partial b$ in the directions CP, AC. These respectively give a resultant the numerical value of which is

$$\{(\partial\Omega/\partial a)^2 + (\partial\Omega/\partial b)^2\}^{\frac{1}{2}} = \frac{2\gamma}{(a^2 + b^2)^{\frac{1}{2}}}$$

or putting r for $(a^2 + b^2)^{\frac{1}{2}}$ the force is $2\gamma/r$.

The direction of the force is in the plane of the paper and at right angles to PA, and towards that side of the plane through P and the conductor on which C lies. For

$$-\partial\Omega/\partial a = -2\gamma b/(a^2 + b^2), \quad -\partial\Omega/\partial b = 2\gamma a/(a^2 + b^2),$$

and the equation of the plane the projection of which is PA, is $bx-ay=0$, if x is taken from P in the direction PC. The x and y direction cosines of a normal to this plane are respectively proportional to $-b$ and a, as are also the component forces parallel to x and y. It is found by experiment that the direction which the current must have in order that a positive or north-seeking pole should move as here specified is from above downwards through the paper, which agrees with the rule already given.

It is to be observed that the magnetic potential of the current in the conductor is, if we put x for CP and y for AC,

$$\Omega = 2\gamma\left(C - \tan^{-1}\frac{y}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C is a constant. The potential is thus multiple valued, changing in fact by $4\pi\gamma$, along a closed path passing round the conductor. As, however, we have to deal only in practice with the difference of potential between two points, the ambiguity of value of the potential is of no consequence.

Theorem of Work done in Field of Current

378. From the fact, whether experimentally proved or deduced theoretically from other facts, that the force at a distance r from a straight conductor carrying a current is $2\gamma/r$ and has the direction specified, we can easily prove the theorem of work done in carrying a unit pole round a closed path of any form. Let $TUVW$, Fig. 88, be a circle surrounding the conductor C. Then clearly the work done in the circle is $4\pi\gamma$. Let now the given closed path $PQRSP$, which may or may not lie in a plane, be connected with the circle by VS, and QT. Thus we obtain two closed paths, neither of which passes round the conductor, namely $SRQTUVS$, $SVWTQPS$.

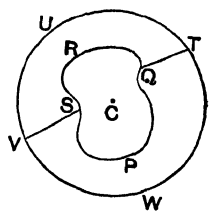


FIG. 88.

Let these be described by a unit pole in the order here stated. The work done on the whole is zero. But clearly the work done is that expended in the paths $TUVW$, $SRQPS$, which are traversed in opposite directions. The work done in the first is numerically $4\pi\gamma$, hence so also is that in the second.

So far we have considered a straight conductor, but if it is curved the theorem can still be proved very easily. Let the conductor be infinitely thin, and have finite curvature. Then taking a closed circular path infinitely near it, we see that the pole will be acted on only by the portion of the conductor which is near it as compared with the rest of the circuit, and this may be considered as a long straight conductor. For the circular path the work is $4\pi\gamma$. As before by connecting the circular path to one of another form the work done in carrying a unit pole round the latter may be shown to be $4\pi\gamma$.

If the conductor is not a thin filament, let it be imagined divided up into a large number of thin conductors coinciding with the lines of flow along the conductor. Then for each of these the work done in carrying a unit pole round it is $4\pi\delta\gamma$, if $\delta\gamma$ be the current in the filament, since the proof given above applies. Hence for a closed path embracing the whole current the work done is $4\pi\gamma$.

Equations of Currents

379. The theorem just discussed can be put into a form which is of great service in the higher parts of electromagnetic theory, where frequently we have to consider currents flowing in extended masses of matter, and it is convenient to resolve the current flowing across unit area at any point into components parallel to the axes of x, y, z. Let the current per unit area be denoted by q, and the cosines of the direction of flow by l, m, n; then we call lq, mq, nq, the components of current parallel to the axes and denote them by u, v, w. Hence the current in any direction of which the cosines are λ, μ, ν is $\lambda u + \mu v + \nu w$.

With this notation the theorem may be written

$$4\pi\int(\lambda u + \mu v + \nu w)dS = \int\left(\alpha\frac{\partial x}{\partial s} + \beta\frac{\partial y}{\partial s} + \nu\frac{\partial z}{\partial s}\right)ds \quad . \quad (3)$$

where α, β, γ are the components of magnetic force, ds an element the path traversed by the pole, dS an element of surface enclosed by the path.

By precisely the same process as that followed at p. 49 above, we can prove that

$$4\pi(\lambda u + \mu v + \nu w) = \lambda\left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right) + \mu\left(\frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}\right) + \nu\left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right)$$

and therefore

$$\left.\begin{aligned} u &= \frac{1}{4\pi}\left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right) \\ v &= \frac{1}{4\pi}\left(\frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}\right) \\ w &= \frac{1}{4\pi}\left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

equations which we shall find of great importance in what follows.

According to the notation of Art. 345, this theorem is expressed by saying that, to the factor $1/4\pi$, the current is the *curl* of the magnetic force. The x, y, z components of the curl of *any* directed quantity **H** of which the components are α, β, γ, are the quantities in the brackets of (4). These equations will, in what follows, be fre-

quently replaced by the single equation, in which q represents the resultant current per unit area.

$$4\pi q = \text{curl } \mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

For a full discussion of curling and related processes, see Heaviside's *Electrical Papers* and his Treatise on *Electromagnetism*, from which several examples which follow are taken.

Equations (4) and (4') are forms of what has been called by Heaviside the first circuital equation. It will be discussed fully in the next and later chapters in connection with another, and complementary circuital equation; but the theorem of work, in the simple form in which it is stated in Arts. 378, 379 above, can be used to give some interesting results as to arrangements of currents and their related magnetic fields.

Applications of Theorem of Work in Closed Path. Solenoidal Current. Cylindrical Distribution of Currents Parallel to Axis

380. For example, let it be required to find the current system which shall produce a uniform magnetic field of intensity $\mathbf{H}$ throughout the space within a cylindrical surface, with zero force outside that space. Let a unit pole be carried round the closed path $A\,B\,C\,D$, Fig. 89, the sides of which are parallel to the generating lines of the cylindrical space, $A\,B$ just outside, $C\,D$ just inside the space, while the ends $B\,C, D\,A$, are at right angles to the sides. The work done in carrying a pole round the path is simply $\mathbf{H}.C\,D$. But this must by the symmetry of the arrangement be 4π times the current passing through the area $A\,C$, in a direction at right angles to the area. Since the current is proportional to $C\,D$, the length of each side of the path, the current per unit length of the path must be constant. Let it be γ', then we have

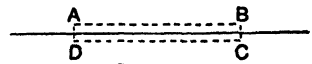


FIG. 89.

$$\gamma' = \frac{\mathbf{H}}{4\pi},$$

that is a current of amount $\mathbf{H}/4\pi$ per unit of length of the cylinder must flow round its surface at every point in a direction normal to the generating lines.

381. As another example consider a tubular space between two coaxial right cylindrical surfaces, in which the magnetic force is everywhere in the direction of the circle coaxial with the cylinders and passing through the point in question, while the force is zero outside and inside this space. Let the intensity of the force be inversely as the distance of the point considered from the axis, and be without variation from point to point in the direction parallel to the axis. Draw, as shown in Fig. 90, a closed path $A\,B\,C\,D$, the sides of which

$ABCD$ are, the former outside, the latter inside, the tubular space, and along circles coaxial with the tube, while the ends of the path are radial. Thus if $\mathbf{H}$ be the intensity along CD, we have for the work done round the closed path $ABCD$ the value $\mathbf{H}.CD$, and a current must flow along the outside of the tube parallel to its axis of amount $\mathbf{H}/4\pi$ per unit of length of its circumference.

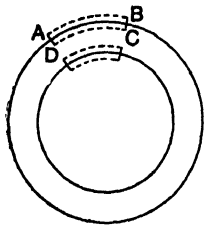


FIG. 90.

By considering a similar closed path at the inside of the tube it is easy to show that there must be a current also along the inside of the tube, but *in the opposite direction*, and of amount $\mathbf{H}r_2/4\pi r_1$ where r_1, r_2, are the radii of the inner and outer tubes respectively.

Thus the total current flowing along the outside of the tubular space is $\frac{1}{2}\mathbf{H}r_2$, and that along the inside $(\mathbf{H}r_2/4\pi r_1)2\pi r_1$, that is also $\frac{1}{2}\mathbf{H}r_2$. Hence the total currents are equal but in opposite directions. Equal and opposite currents, therefore, flowing along coaxial thin tubes produce no force external to the outer tube or internal to the inner tube, but only in the space between the tubes.

To verify that the solution is correct, we have only to take the closed path round the outside of the outer tube in a coaxial circle. By symmetry if there is magnetic force at one point of the path there must be a force at every other point directed relatively to the element of the path in the same way at each point. But the line integral round the path is zero, since there is no current on the whole through the path. Hence the tangential component at every point of the path must be zero, and as each current separately produces a tangential force, the forces due to the two currents annul one another.

This result also affords a proof of the theorem that the magnetic field produced by any distribution of currents symmetrical about a straight line parallel to which also the currents flow, is identical for all points external to the system of currents, with the field produced by an equal current flowing along the axis.

It should be noticed here that, no matter how thick the tubes are, there is no force at any point external to the space between them, or within the inner tube, provided the total currents are equal and opposite and symmetrically distributed about the axis. If the distribution be not thus symmetrical, the line integral for any external closed path is zero, but not so the force at every external point.

In a similar way it can be proved that there is no magnetic force within the hollow space in the interior of a conductor, formed by a solid tube bounded by two coaxial right circular cylindrical surfaces, and carrying a distribution of currents symmetrical about the axis of the cylinders, and flowing parallel to that direction.

Examples of the Theorem of Equivalence of a Circuit and a Magnetic Shell

382. We shall use this theorem later in many calculations regarding the magnetic fields of coils of different kinds, but before leaving it for the present we shall give one very simple application of it, namely:

To find the potential and force at a point P on the axis of a current flowing in a thin circular conductor of radius r.

To solve this problem we have only to calculate the solid angle subtended at the latter point by the circle and then find the rate of variation of this angle along the axis, since obviously by symmetry there is no force at right angles to the axis.

The area of a plane ring of breadth dx, and radius x concentric with the circle is $2\pi x dx$; and the area of each element of this projected at right angles to the line drawn from P to the element is the product of the area by $a/\sqrt{a^2+x^2}$, where a is the distance of P from the centre. Hence the solid angle subtended by the ring is $2\pi a x dx/(a^2+x^2)^{\frac{3}{2}}$. The total solid angle is therefore

$$2\pi a\int_0^r \frac{x dx}{(a^2+x^2)^{\frac{3}{2}}} = 2\pi\left(1 - \frac{a}{\sqrt{a^2+r^2}}\right) = 2\pi(1-\cos\theta)$$

where

$$\theta = \cos^{-1}(a/\sqrt{a^2+r^2}):$$

and, if γ be the current, the potential at P is

$$\Omega = 2\pi\gamma(1-\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence for the force we have

$$-\gamma\frac{\partial\Omega}{\partial a} = -\gamma\frac{\partial\Omega}{\partial\theta}\frac{d\theta}{da} = 2\pi\gamma\frac{r^2}{(a^2+r^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad (6)$$

Elementary Theory of Tangent Galvanometer

383. Suppose now the plane of the circle to be that of the magnetic meridian, and a needle very short in comparison with the radius r to be placed with its centre at P, any point on the axis of the circle.

The magnetic intensity just calculated will give equal and opposite forces parallel to the axis on the extremities of the needle, which under these forces and the directive forces due to the earth's magnetic field will take up a position of deflection from the magnetic meridian. Let ϕ be the angle of deflection, and M the magnetic moment of the needle, then the deflecting couple is $2\pi\gamma M r^2\cos\phi/(a^2+r^2)^{\frac{3}{2}}$ and the

directive couple due to the earth is $HM\sin\phi$. Hence for equilibrium we must have equality of these couples, which gives

$$\gamma = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2\pi r^2} H \tan\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The arrangement just described is that of an ordinary tangent galvanometer with eccentric needle. If there are n turns of wire arranged in a thin coil so that the dimensions of cross-section may be neglected, the equation becomes

$$\gamma = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2\pi n r^2} H \tan\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $a = 0$, so that the centre of the needle is at the centre of the coil, the last equation becomes

$$\gamma = \frac{r}{2\pi n} H \tan\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The multiplier of $H\tan\phi$ in these equations for γ is called the constant of the galvanometer. The value of this multiplier requires correction in general for the cross-section of the coil, and sometimes also for the length of the needle. These corrections, as well as the mode of using such instruments, will be fully discussed when we deal with the subject of galvanometry and the measurement of currents generally.

Energy of Current-Carrying Circuit

384. We shall now calculate the energy of a circuit carrying a current in a magnetic field. This will consist of two parts, one independent of the previously existing field, the other represented by the work which has been done in establishing the current in presence of the field.

Let the components of magnetic force of the previously existing field at any point be α, β, γ, and its inductivity μ (supposed at present independent of the magnetic state of the field), and let the circuit produce a field intensity α', β', γ', where the intensity previously existing was α, β, γ. By the specification of magnetic energy given in Art. 56 above, the energy in the medium is given by the equation

$$T = \frac{1}{8\pi}\int \mu\{(\alpha + \alpha')^2 + (\beta + \beta')^2 + (\gamma + \gamma')^2\} d\varpi \quad . \quad (10)$$

where $d\varpi$ is an element of volume, and the integral is taken

throughout the whole field. This breaks up into three integrals, so that

$$T' = \frac{1}{8\pi}\int\mu(\alpha^2 + \beta^2 + \gamma^2)d\varpi$$

$$+ \frac{1}{8\pi}\int\mu(\alpha'^2 + \beta'^2 + \gamma'^2)d\varpi$$

$$+ \frac{1}{4\pi}\int\mu(\alpha\alpha' + \beta\beta' + \gamma\gamma')d\varpi \quad . \quad . \quad . \quad . \quad (11)$$

Mutual Energy of Two Current-Carrying Circuits

385. The first of these is the magnetic energy of the field previously existing, the second that of the circuit introduced supposed existing alone, and the third that due to the introduction of the circuit into the previously existing field. It is this last expression we wish here to consider, and to change into another form which will be useful in discussions regarding the actions of magnetic fields on conductors.

The integral can be written in the form

$$\frac{1}{4\pi}\int\mu\mathbf{H}\mathbf{H}'\cos\theta\, d\varpi$$

where $\mathbf{H}$, $\mathbf{H}'$ are the previously existing and the induced force respectively, and θ is the angle between their directions.

Now we have

$$4\pi\gamma = \int\mathbf{H}'ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where γ is the current in the circuit, ds an element of a line of force where the intensity is $\mathbf{H}'$, and the integral is taken round a line of force due to the current, which, as we have seen, threads through the circuit. Further, the total induction through the circuit due to the previously existing field is the integral

$$\int\mu(l\alpha + m\beta + n\gamma)dS,$$

in which l, m, n denote direction-cosines of the normal to the surface element dS, and which has a constant value for all surfaces having the circuit for bounding edge, since the magnetic induction fulfils the solenoidal condition. Hence, multiplying both sides of (12) by this integral, and dividing by 4π, we obtain

$$\gamma\int\mu(l\alpha + m\beta + n\gamma)dS = \frac{1}{4\pi}\int\mu(l\alpha + m\beta + n\gamma)dS \times \int\mathbf{H}'ds.$$

Now let surfaces be so drawn as to be successive equipotential surfaces for the intensity due to the circuit, that is, for the intensity $\mathbf{H}'$.

Each of these has the circuit for bounding edge, and the first integral on the right of the last equation has the same value for each, that is, the value which it has for *any* surface having the circuit for bounding edge.

Hence we can evaluate the right-hand side of the last equation by multiplying the (constant) value of the first integral for each equipotential surface by the value of $\mathbf{H}'ds$ for the step from one equipotential surface to the next (since for a given pair of successive equipotential surfaces $\mathbf{H}'ds$ has the same value), and so on throughout the whole field. But plainly, since $l\alpha+m\beta+n\gamma=\mathbf{H}\cos\theta$, this gives simply the integral

$$\int\mu\mathbf{H}\mathbf{H}'\cos\theta\,d\varpi$$

where $d\varpi$, as here considered, is an element of volume, bounded by a tube of force between two successive equipotential surfaces; but may, of course, be any element, since a volume integral cannot depend on the manner in which the elements are taken. Thus we get for the mutual energy T_{cf} of the circuit and field

$$T_{cf}=\frac{1}{4\pi}\int\mu\mathbf{H}\mathbf{H}'\cos\theta\,d\varpi=\gamma\int\mu(l\alpha+m\beta+n\gamma)dS \quad . \quad (13)$$

that is, the mutual energy is equal to the product of the current into the surface integral of magnetic induction through any surface which has the circuit for bounding edge. This theorem is of enormous importance in electrodynamics. The proof here given is not that usually adopted, which treats the circuit as a distribution of magnetism acted on by the previously existing distribution: but it seems preferable to arrive at the theorem from the expression for the energy of the whole medium occupying the field.

Electrokinetic Energy

386. The magnetic energy we are here considering is energy depending on the state of the field at any instant; and it is not necessarily equal to the energy which has, up to that time, been thrown into the field from a battery or other source. That there is energy depending on the state of the field as distinguished from the total amount which has been furnished by the source is clear from the dissipation of energy in hysteresis; but we shall return to this subject later. According to the conclusion adopted above as to the nature of magnetic energy we regard it as kinetic or, as we call it, electrokinetic energy, and hence in a system not subject to dissipative forces we must so choose the sign of the energy that the mutual forces of the system will tend to cause the amount of energy to increase. Thus a circuit being brought into a field must tend in virtue of the forces exerted upon it by the field to move

so as to increase its electrokinetic energy, that is, we must so choose the sign of the surface integral of magnetic induction that in any actual case the mutual forces may tend to its increase. Thus if T_{cf} denote the mutual energy of the circuit and field, and $d\psi$ any small change of position or configuration of the circuit, and Ψ the force producing it, the work done by this force is $\Psi d\psi$. Thus

$$\Psi d\psi = dT_{cf},$$

or

$$\Psi = \frac{\partial T_{cf}}{\partial \psi} = \gamma \frac{\partial N}{\partial \psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

if N denote the magnetic induction through the circuit and γ be maintained constant in the change.

Electromagnetic Force on Element of Circuit. Most General Specification of Current

387. The force Ψ which thus tends to increase T_{cf} by increasing the magnetic induction through the circuit is called the electromagnetic force on the circuit. The circuit, if free to move as a rigid whole, will change its position in obedience to this force so as to increase N, and whether fixed in position as a whole or not, will tend to increase its area so as to include a larger total induction.

The resultant of electromagnetic force on each element of a circuit can only be in the direction at right angles at once to the magnetic force and to the element, because the element, if free to move in that direction, would increase the magnetic induction through the circuit at the greatest rate. Thus there is no electromagnetic force in the direction of the magnetic force on an element, since a displacement in that direction would not alter the electrokinetic energy of the circuit.

The direction of the force, Ψ, on an element of the circuit and the corresponding directions of the current and the magnetic induction at the element are shown in Fig. 91, in which ds is supposed perpendicular to the plane of B and Ψ. The value of N is to be taken positive or negative according as the direction of the magnetic induction through the circuit agrees with (as here), or is opposite to that in which a right-handed screw moves when the handle is turned round in the direction in which the current flows.

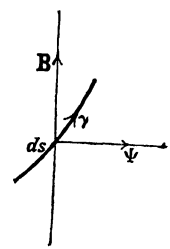


FIG. 91.

In general, however, the elements of the circuit are inclined to the direction of the magnetic induction. Let the angle between the direction of the current in an element of the circuit and the positive direction of the induction be θ, and let the element be displaced through a distance $d\psi$ in a

direction at once normal to itself and to the direction of the magnetic induction. The element may be supposed moved out along guiding wires placed in this direction at its extremities. Let ds be its length.

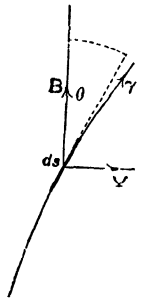


FIG. 92.

The change in N is the product of the induction $\mathbf{B}$ at the element into the component of the length of the element in a plane at right angles to $\mathbf{B}$ into the displacement; that is, $dN = \mathbf{B}sin\theta ds \,.\, d\psi$. Hence

$$dT_{cf} = \gamma \mathbf{B} \sin\theta ds \,.\, d\psi \quad . \quad . \quad . \quad (15)$$

and

$$\Psi = \gamma \mathbf{B} \sin\theta ds \quad . \quad . \quad . \quad . \quad (16)$$

This formula is applicable also in certain cases, when apparently no progressive change takes place in the induction through the circuit, as, for example, in the case of the Barlow wheel described below (Art. 402). The action on the element of the circuit in every case is due to the magnetic induction there existing, and the element experiences a force causing it to cut across the lines of induction, and of the amount given by equation (16).

If the direction cosines of ds be l, m, n, we have, using the components a, b, c of $\mathbf{B}$, the equation

$$\sin\theta = \frac{\{(mc - nb)^2 + (na - lc)^2 + (lb - ma)^2\}^{\frac{1}{2}}}{\mathbf{B}},$$

and therefore for (16) the alternative form

$$\Psi = \gamma\{(mc - nb)^2 + (na - lc)^2 + (lb - ma)^2\}^{\frac{1}{2}} ds \quad . \quad (17)$$

Substituting in this for the values $l\gamma/\sigma, m\gamma/\sigma, n\gamma/\sigma$ u, v, w, the components of the current in the directions of the axes taken per unit of the area σ of the cross-section of the conductor we find

$$\Psi = \{(vc - wb)^2 + (wa - uc)^2 + (ub - va)^2\}^{\frac{1}{2}} ds \,.\, \sigma.$$

From this, supposing ds in the direction of y, so that $u = w = 0$, and $\mathbf{B}$ in the plane of yz so that $a = 0$, we get

$$\Psi = (vc - wb)\sigma \,.\, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

that is $vc - wb$ is the electromagnetic force per unit of volume on the conductor in the direction of x. Denoting the components per unit volume in the directions of the axes by X, Y, Z, we find

$$\left.\begin{aligned} X &= vc - wb \\ Y &= wa - uc \\ Z &= ub - va \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which are called the equations of electromagnetic force.

It is to be particularly observed that u, v, w are here the components of the total current flowing from whatever cause, and include the

currents due to variation of electric displacement, and also the so-called convection currents produced by the motion of charged bodies. Currents due to the two last-mentioned causes have not yet been considered, but they are of great importance in the general theory of the electromagnetic field, and will be fully discussed in that connection.

Mutual Energy of a Current and Magnetic Distribution

388. An apparent difficulty in connection with the subject of the mutual energy of a circuit and a magnetic distribution arises here. It is not the case that when a circuit carrying a current in a magnetic field is interrupted so that the current is annulled, there is any work done in the circuit or in any other way which it is possible for us to detect, corresponding to an annulment of the mutual electrokinetic energy T_{cf}. The mechanical value of the current itself as thus tested is found to be quite independent of the existence of permanent magnets in its vicinity. The expression for the electrokinetic energy, however, enables us to calculate the forces on the circuit: and it must not be concluded that the energy, though not rendered available when the circuit is broken, does not exist. This subject will be further discussed in a later chapter.

389. The expression for the force on an element of a current-carrying conductor in a magnetic field may be applied to find the turning moment of a thin uniformly magnetized bar magnet on a conductor. Such a magnet, we have seen, may be regarded as made up of two equal and opposite point-charges of magnetism at its extremities. Choose an element ds of the conductor and draw lines to it from the extremities of the magnet. Let these lines make with the positive direction of the axis of the magnet angles θ_1, θ_2, and with the element angles ϕ_1, ϕ_2. First we shall suppose that all these angles are in one plane, and that the positive pole corresponds to the angle θ_1. The force exerted on the element by the field of the positive pole is by the expression found above $\mu\gamma m ds \sin\phi_1/r_1^2$ (where m is the pole-strength and r the distance of the pole from the element), and is in the direction of the normal to the plane through the element and the magnet. The product of this into the perpendicular distance of the element from the axis of the magnet, that is, into $r_1 \sin\theta_1$, is the moment turning the element round the magnet. Hence the total turning moment due to the two poles is

$$\mu\gamma m ds \left(\frac{\sin\theta_1 \sin\phi_1}{r_1} - \frac{\sin\theta_2 \sin\phi_2}{r_2}\right).$$

But we know by geometry that

$$r_1 \frac{d\theta_1}{ds} = \sin\phi_1, \quad r_2 \frac{d\theta_2}{ds} = \sin\phi_2,$$

and the moment on the element is therefore

$$\mu\gamma m(\sin\theta_1 d\theta_1 - \sin\theta_2 d\theta_2).$$

Integrating along the conductor we find for the turning moment round the magnet exerted on the whole conductor the value

$$\mu\gamma m(\cos\theta'_2 - \cos\theta'_1 - \cos\theta_2 + \cos\theta_1),$$

where θ_1, θ'_1 denote the angles which lines drawn from the positive pole to the beginning and end of the conductor make with the axis of the magnet, θ_2, θ'_2, the corresponding angles for lines drawn from the negative pole. The turning moment, therefore, depends only on the positions of the extremities of the conductor, and vanishes if the conductor forms a closed circuit.

If any element ds is not in a plane passing through the axis of the magnet it can be resolved into two components, one parallel to the axis, the other at right angles to the axis. (The justification for this statement will be given later.) The electromagnetic force in the latter component passes through the axis, and therefore has no moment round it. The force on the other component has the value already given.

The force on each element having been thus calculated, all the components of the conductor which are in planes through the axis may be transferred to one such plane by a rotation round the axis, and will give a continuous curve in that plane, the values of $\theta_1, \theta_2, \theta'_1, \theta'_2$, for the extremities of which will be the same as for the actual conductor.

It is clear from the above investigation that the turning moment round a given axis exerted on a given conductor in the field of a single point-charge depends only on the positions of the ends of the conductor, provided the axis passes through the pole. Hence the moment round any straight linear distribution of magnetism whatever, the field due to which produces the forces on the elements of the conductor, depends only on the positions of the ends of the conductor, and is zero if the conductor forms a closed circuit. This will also hold whether the linear distribution be straight or not, provided the magnetisation be uniform. If the magnetic filament is not straight, the axis to be considered will be the line joining the extremities of the filament.

390. The above analysis, combined with the form of the lines of force of a uniformly magnetized thin bar magnet, as given by (2), p. 13 above, leads to the following interesting result for the action of the field of a uniformly magnetized thin bar magnet on a conductor. Let the lines of force of such a magnet (for which see Fig. 14, p. 15) be supposed rotated round the axis of the magnet. They will sweep out coaxial surfaces. The turning moment on a conductor carrying a current of given amount is the same whatever be the arrangement of the conductor, provided it terminate on the same two surfaces, and the moment per unit current is measured by the difference of the parameters of the surfaces, and is therefore zero when the ends of the conductor lie on the same surface.

Reaction of a Current-Element on a Magnetic System.

391. Having thus calculated the force on an element of a circuit exerted by a magnetic field, and given some illustrations of the expression arrived at, we shall conclude this part of the subject by considering the reaction of the element on a magnetic system.

The force exerted on an element of a circuit in which there is a current γ, at a place where the induction is $\mathbf{B}$, is by the expression adopted above $\mathbf{B}\gamma \sin\theta ds$. Now we may suppose the induction produced by a single unit pole properly placed. The reaction on the field exerted *at the element*, is $\mathbf{B}\gamma \sin\theta ds$ in the opposite direction, and therefore the action on the pole is a parallel force $\mathbf{B}\gamma \sin\theta ds$, together with a couple of moment $\mathbf{B}\gamma r \sin\theta ds$, where r is the distance of the element from the pole. The total couple due to the whole circuit if closed is zero, as we have seen, and so the action on the pole may be calculated by finding the resultant of the forces $\mathbf{B}\gamma \sin\theta ds$ supposed acting at the pole.

Thus we obtain as the intensity of the field produced at the pole by the current in the element the value $\mathbf{B}\gamma \sin\theta ds$. But if $\mathbf{H}$ be the force produced at the element by the pole, we know that $\mathbf{B} = \mu\mathbf{H}$; and if r be the distance of the pole from the element, and the field be isotropic, we have $4\pi r^2\mathbf{B} = 4\pi$, or $\mathbf{B} = 1/r^2$, $\mathbf{H} = 1/\mu r^2$, so that

$$\mathbf{B}\gamma \sin\theta ds = \gamma \frac{\sin\theta ds}{r^2}.$$

The expression on the right-hand side is the intensity of field produced by the element ds at the point where the pole is supposed situated, and the resultant intensity is to be obtained by proper summation of the forces due to the several elements. The direction of this elementary force is at right angles to the plane through the element and the point considered.

The direction may be thus specified as to sign. Let the right hand be held open with the palm down, and the thumb pointing downwards. Let then the element be represented by the thumb of the right hand, and the current be supposed to flow in the direction in which it points, and the point considered be at the extremity of the forefinger, then the field intensity would tend to move a north pole there towards the next finger.

Magnetic Intensity of Straight Current Imbedded in Infinite Conducting Medium.

392. From the formula given above we may calculate the magnetic intensity due to a current in a straight conductor enclosed within a uniform medium extending indefinitely far in all directions. The current in the wire is the same at every point, and the return circuit is provided by the surrounding medium. We shall suppose that the current diverges radially and with equal intensity in all directions from

the end at which it enters the medium, and converges radially from the medium at the end at which it enters the wire. The resultant lines of flow in this case correspond exactly in form and distribution with the lines of force of a thin uniformly magnetized bar magnet, and may therefore be regarded as produced by two equal and opposite point-charges of magnetism at the extremities of the bar. (Art. 26 above.)

393. It will be convenient to consider separately the two radial distributions of current rather than the resultant current at any point. It is clear that the intensity of radial flow at any point, that is, the flow across unit area at any point of a spherical surface described from the extremity of the wire, from which the flow takes place as centre, is inversely proportional to the square of the radius of that surface. Thus if γ be the current flowing from the wire to the medium at B (Fig. 93), the outward radial current at distance r_1 from the end of the wire will be $\gamma/4\pi r_1^2$, and similarly at distance r_2 from the other end the inward radial flow will be $\gamma/4\pi r_2^2$. The rectangular components of outward flow at a point (x, y, z) on the surface of the sphere of radius r_1, if the centre of this sphere, that is B, be taken as origin, will be therefore given by

$$(u, v, w) = \frac{\gamma}{4\pi r_1^3}(x, y, z) \quad . \quad . \quad . \quad . \quad (19)$$

and similar expressions will hold for the inward flow.

We can calculate the magnetic intensity at any point P by direct application of the formula given in Art. 391. First of all we consider the straight wire. Let ds be an element of its length, r the distance of P from ds, and θ the angle the line joining ds with P makes with AB. The intensity due to the straight conductor will be the integral

$$\gamma \int \frac{\sin\theta ds}{r^2}$$

taken from A to B. But clearly, by Fig. 93, $\sin\theta = h/r$, where h is the dis-

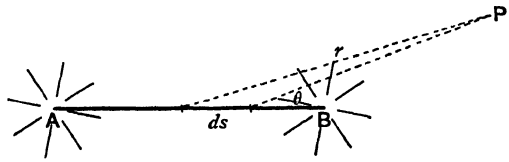


FIG. 93.

tance of P from the line $A\,B$, and therefore $\cos\theta = (-h/r^2)dr/d\theta = -dr/ds$ by the figure. Hence $ds = r^2 d\theta/h$, and the integral becomes

$$\gamma \int_{\theta_1}^{\theta_2} \frac{\sin\theta d\theta}{h} = \frac{\gamma}{h}(\cos\theta_1 - \cos\theta_2) \quad . \quad . \quad . \quad . \quad (20)$$

Magnetic Action of Radial Current in Infinite Medium

394. Now imagine the radial current from the wire to flow in equal narrow conical conductors (Fig. 94), having their apices at the end of the wire from which the current flows, and filling the whole space. Clearly by symmetry the force produced at P by the current in one of these will be neutralised by the force produced by the current in

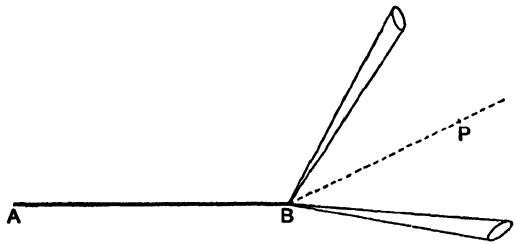


FIG. 94.

another in the same plane through, but at an equal distance on the opposite side, as shown in the diagram. Hence the intensity at P due to the radial current from B is zero.

This result and method for straight conductors will be useful later when we consider so-called unclosed circuits, as for example the wire connecting the spheres of a Hertzian vibrator.

Vector-Potential of Current

395. The quantity $\gamma ds \sin\theta/r^2$ is, by the specification given in Art. 83 above, the vector-potential of a magnetic element of moment γds, at a point P distant r from the element in a direction which makes an angle θ with the positive direction of the magnetic axis of the element (Fig. 95). The vector-potential of the current, however, is that which gives by equations (75) p. 50 above, the components of magnetic induction. Consider an element ds of the circuit in which the current flowing is γ. Let the cross-section (supposed small) of the conductor be σ, and u, v, the x and y components of the current in the plane of r and ds, so that $w=0$. Then putting

$$F = \mu\sigma ds\frac{u}{r}, \quad G = \mu\sigma ds\frac{v}{r},$$

we shall show that these are the x and y components of vector-potential at the point P. In order to prove this we have to show that $a=b=0$, and that

$$c = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}.$$

The first statement is obviously true, since $H=0$ and r does not contain z.

To prove the second, we perform the differentiations, and find

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = \frac{\mu\sigma ds}{r^3}(uy - vx).$$

But if ϕ be the angle which the line r makes with x we have $y/r = \sin\phi$, $x/r = \cos\phi$, also $u = \gamma\cos(\theta+\phi)/\sigma$, $v = \gamma\sin(\theta+\phi)/\sigma$. Hence

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = \frac{\mu\sigma ds}{r^2}\frac{\gamma}{\sigma}\{\cos(\theta+\phi)\sin\phi - \sin(\theta+\phi)\cos\phi\}$$

$$= -\frac{\mu\gamma ds}{r^2}\sin\theta.$$

But this by the conventions as to the axes with respect to which F, G, H are taken, states that $\mu\gamma ds \sin\theta/r^2$ is the magnetic induction in the negative direction of the axis of z. This agrees with the result as to the direction of magnetic force due to a current stated in the mnemonic rule in Art 391.

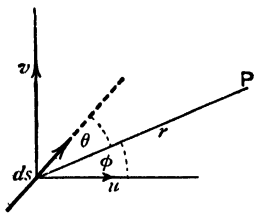


FIG. 95.

Of course any quantities χ, ψ, say, may be added to F and G respectively if they fulfil the condition

$$\frac{\partial\psi}{\partial x} - \frac{\partial\chi}{\partial y} = 0,$$

and are of proper dimensions. For example, if U be a function of the co-ordinates y, x, the values $\psi = \partial U/\partial y$, $\chi = \partial U/\partial x$ would fulfil the condition.

396. More generally, when the current is distributed through any space ϖ, the three components of vector-potential of the current are given by integrals taken throughout the whole space ϖ, namely

$$F = \mu\int\frac{u}{r}d\varpi, \quad G = \mu\int\frac{v}{r}d\varpi, \quad H = \mu\int\frac{w}{r}d\varpi \quad . \quad . \quad (21)$$

if μ has the same value throughout the medium. Should, however, the medium be different in different parts, the values of F, G, H will require modification by the addition of certain integrals taken over the separating surfaces of the different parts of the field.

It will be noticed that the expressions here given for the vector-potential are identical in form with those which we should write down for the ordinary scalar-potential of matter of density u, v, or w. The vector-potential is, however, a directed quantity (hence its name), which has x, y, z components, produced respectively by the x, y, z components u, v, w of the current. The expressions for these components will be modified in Chapter XI. for the case of propagation in the electromagnetic field.

397. In order to calculate the magnetic intensity we have to properly differentiate the integrals in (21). This may be done as follows: The integrals, it is to be noticed, are here regarded as taken throughout all space, since elements of space where there is zero current of course contribute nothing to the sum. Hence to find, for example, the value of F at a point $(x, y+dy, z)$ we may suppose the whole distribution of currents displaced through a distance dy in the direction of y negative. That will bring the current system into the same position relatively to (x, y, z) that it really occupies with respect to $(x, y+dy, z)$. But by this process F becomes $F+\partial F/\partial y \,.\, dy$, and u changes *at all points* to $u+\partial u/\partial y \,.\, dy$. Thus we get

$$\frac{\partial F}{\partial y} = \mu \int \frac{1}{r} \frac{\partial u}{\partial y} d\varpi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the integral being taken throughout all space as before.

Thus we obtain

$$\left.\begin{aligned} a &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} = \mu \int \frac{1}{r} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) d\varpi \\ b &= \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} = \mu \int \frac{1}{r} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) d\varpi \\ c &= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = \mu \int \frac{1}{r} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) d\varpi \end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

As an example we apply these equations to find the magnetic induction produced by the system of radial currents imagined in Art. 392. If r be the distance of the point P, at which the magnetic induction is to be found, from the element $d\varpi$ of the current system, we have

$$\begin{aligned} c &= \frac{\mu\gamma}{4\pi} \int \frac{1}{r} \left(\frac{\partial}{\partial x} \frac{y}{r_1^3} - \frac{\partial}{\partial y} \frac{x}{r_1^3} \right) d\varpi \\ &= \frac{\mu\gamma}{4\pi} \int \frac{1}{r} \left(- \frac{3xy}{r_1^5} + \frac{3xy}{r_1^5} \right) d\varpi = 0, \end{aligned}$$

and similarly a and b are zero. Thus the result already obtained is verified.

It is well worth noticing that equations (23) give a, b, c as the components of vector-potential of the quantity of which

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y}$$

are the x, y, z components respectively. We shall sometimes denote these for convenience by ξ, η, ζ, and write (23) in the form

$$(a, b, c) = \mu \int \frac{1}{r} (\xi, \eta, \zeta) d\varpi \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Thus, as expressed shortly by the use of the term *curl*, the magnetic induction at any point is the vector-potential of the curl of the current.

Magnetic Intensity for (1) Case of Straight Vertical Conductor with One End on Surface of Earth, (2) for Plane Current Sheet with Lines of Flow Circles Round Point in it, and Current Inversely as Square of Radius of Circle

398. The fact noticed at the end of the last article enables us to solve the problem of finding the magnetic intensity at any point when a straight conductor, carrying a current γ, has one end placed in contact with the indefinitely extended plane boundary of an otherwise infinite mass of conducting material.

Here the current is again radial and at any point at distance r_1 from the point of contact is $\gamma/2\pi r_1^2$ across unit of area of the hemispherical surface of radius r_1 drawn in the conducting material with the point of contact as centre. The components are therefore given by

$$(u, v, w) = \frac{\gamma}{2\pi r_1^3}(x, y, z) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and are each double of the components in the former case, and the currents within the mass give zero vector-potential as before. The discontinuity at the plane surface, where the components of current change from the value here given to zero, must however be taken into account. Thus if we take the axis of x along any radius on the surface and that of y, also in the surface, at right angles to that radius, we have

$$\eta d\varpi = \frac{\gamma}{2\pi a^2} dS$$

if a be the distance of the point for which η is calculated from the point of contact, and dS is the area of an element of the surface at that point.

We should obtain the same value for any radius, and therefore η is directed at every point tangentially to a circle having its centre at the point of contact, and has the same value at every point of that circle. It only remains to find by integration over the plane or otherwise, the potential of the quantity thus calculated at the point where the magnetic force is to be found.

399. Without direct integration the magnetic intensity can be found by the following ingenious method due to Oliver Heaviside.[1] Suppose two radial currents to flow, one in *all* directions from the point of contact B, and of density $\gamma/4\pi r_1^2$, at all points distant r_1 from the radiant point, and another of the same density everywhere as the first, but flowing towards the radiant point outside, and from the radiant point within the conducting body. The current outside is thus zero, that inside the conducting body is radial and of density $1/2\pi r_1^2$, which is the case supposed. The first current gives no magnetic intensity anywhere. The

[1] *Electrical Papers*, vol. i. p. 225.

magnetic force due to the second (which may be regarded as circuital) can be calculated by the ordinary theorem, thus: Let the point P at which the force is to be found be anywhere within the conducting body at a distance h_1 from the normal drawn to the plane surface through the point of contact. Describe a circle round this normal as axis, and calculate the current flowing through it. The area within this circle on the sphere with centre at the radiant point, on which the circle lies, is $2\pi r_1^2(1-\cos\phi)$, where ϕ is the angle between the axis of this circle and the line BP (see Fig. 96). The current through the circle is therefore $\gamma(1-\cos\phi)/2$. If β be the magnetic intensity its line integral, since it is tangential to the circle here considered, is $2\pi h\beta$, and this must be equal to $4\pi\gamma(1-\cos\phi)/2$. Hence

$$\beta = \frac{\gamma}{h_1}(1 - \cos\phi) \quad . \quad . \quad . \quad (26)$$

If the point be outside the conducting mass and ϕ be measured from the former direction of the normal, the equation is

$$\beta = \frac{\gamma}{h_1}(1 + \cos\phi) \quad . \quad . \quad . \quad (26')$$

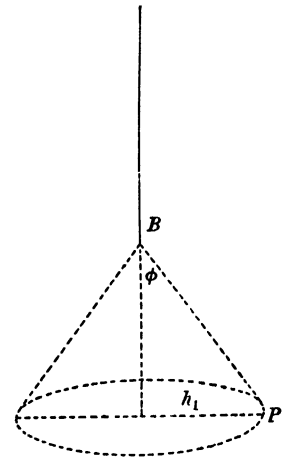


FIG. 96.

To this must be properly added, of course, the magnetic intensity due to the straight wire as given in (20), and that due to the currents radial or otherwise flowing to the further extremity of the straight wire. If the wire is not normal to the plane surface the resultant force must be found in the ordinary way.

The problem just discussed is that of a wire communicating with the surface of the earth, and the solution gives the magnetic intensity at any point in this case, on the supposition of uniform conductivity. It is of importance to notice that if we regard the distribution of η above specified as a current-sheet, we see that the magnetic intensity produced by it is the curl of the expression in (26) or (26'). This, a very little consideration will show, is radial from B (Fig. 96) below the surface, and towards B above, and of amount γ/r^2 at distance r from B.

Experimental Illustrations of Electromagnetic Action

400. The actions of currents on magnets and of magnets on currents can be illustrated by a variety of simple apparatus. A common form is that shown in Fig. 97. On a stand is mounted a horizontal circular coil. Along the vertical axis of the coil passes a metal stem on the upper end of which is a mercury cup. In this is placed a point attached to

the middle of a horizontal wire uniting two vertical pieces which dip into a horizontal circular trough just above the coil, and containing mercury.

A current is passed up the central stem to the mercury cup where it divides and passes by the two side rectangles to the circular trough.

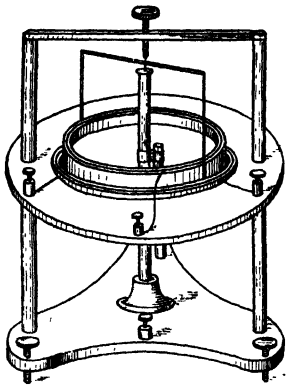


FIG. 97.

A current sent through the horizontal coil produces a magnetic field equivalent to that which would exist if a complex magnetic shell replaced the coil. The lines of force curve round from one face of the coil to the other, and the sides of the pivoted rectangle tend to cut across them. Since the currents are both outwards from the centre in the two horizontal upper parts of the frame, and both downwards in the side pieces, the wires to an observer looking to either side must appear to move towards the same hand, the right or the left. Thus the frame is set into rotation.

With a sufficiently strong current in the wire rectangle the current in the coil may be dispensed with, and the wire will rotate under the influence of the earth's force alone. The current in the two horizontal upper parts of the wire frame is in opposite directions, and these parts therefore cut across the earth's vertical lines of force also in opposite directions. The vertical sides of the wire rectangle tend to move both in the same direction.

The earth's force may be assisted by placing a bar magnet along the stem with its north pole upwards. If the magnet have its south pole up and be powerful enough to more than counteract the earth's field the rotation will be in the opposite direction.

If a shorter stem with mercury cup be substituted for that used with the wire frame, and the pair of magnets, united by a cross bar and provided with a vertical axle by which they can be pivoted in the mercury cup, is put in the place of the wire frame, as shown in Fig. 98, and a current is then sent along the central stem to the cup and thence to the circular trough by a connecting wire, the pair of magnets will rotate on the vertical axle. The vertical current in the central stem gives a circular line of force which moves the similarly directed poles, S, S', of the magnets both round in the same direction. The opposite poles at the upper ends tend to go round the other way; but as the field of the current at the lower ends is relatively stronger,

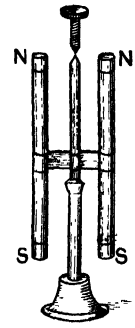


FIG. 98.

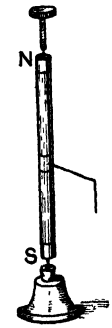


FIG. 99.

the pair of magnets rotates in the direction in which the S poles are moved.

A single magnet pivoted as indicated in Fig. 99 will behave in the same way. Obviously it may be regarded as a bundle of thin magnets, or rather as a combination of many sets of pairs like that last considered.

401. Fig. 100 shows the rotation of a magnet round a current and of a current round a magnet. The magnet NS on the left is attached at the bottom of a jar of mercury to a piece of copper by which a current is carried to the mercury. The current passes to the vertical wire shown dipping into the mercury and thence round to the other side, where by a wire attached at its upper end by a flexible joint it passes to a wide-mouthed vessel containing mercury, in the middle of which stands a vertical magnet. The action requires no explanation. The flexible joint at the lower end of the magnet swimming in the mercury on the left allows it to play round the current, while in a similar way the wire on the right plays round the magnet, both, to an observer looking from above, in the clockwise direction. This apparatus is due to Faraday.

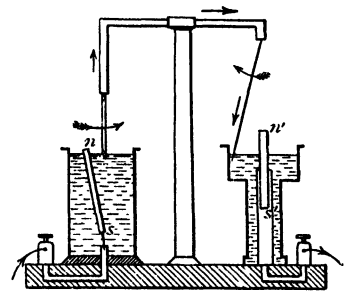


FIG. 100.

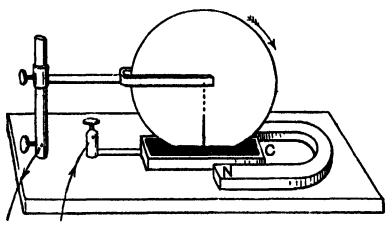


FIG. 101.

402. The apparatus shown in Fig. 101 is what is known as Barlow's wheel. A wheel of sheet copper is pivoted on a horizontal axis so that its plane is vertical, and its lower edge is made to dip in the mercury contained in a trough below as shown in the diagram. A horseshoe magnet is placed so that the lower part of the wheel is between its poles, and the lines of force pass across the wheel just above the part dipping into the mercury. A current is sent as shown from the rim of the wheel up the lower part to its centre, and if the magnet have the poles placed as marked the rotation is in the direction of the arrow.

Expression of the Electrokinetic Energy of any System by means of the Vector-Potential

403. The expression for the electrokinetic energy can be put into the form of a line integral by means of the vector-potential. Let at any point F, G, H be the components of this quantity due to the

currents in other circuits, and any magnetic distribution which may exist in the field. Then substituting the values of a, b, c in terms of these components, we obtain instead of (13)

$$T_{cf} = \gamma \int \left\{ l\left(\frac{\partial H}{\partial z} - \frac{\partial G}{\partial y}\right) + m\left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right) + n\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) \right\} dS \quad (27)$$

But by the process given in Art. 79 above, the surface integral on the right can, in the case of a linear circuit, be shown to be equivalent to the line integral.

$$\int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds$$

taken round the circuit. Hence

$$T_{cf} = \gamma \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds \quad . \quad . \quad . \quad . \quad (28)$$

If the circuit is non-linear the equation is

$$T_{cf} = \int (Fu + Gv + Hw) d\varpi \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where $d\varpi$ is an element of volume, and the integral is taken throughout all the space in which the current (components u, v, w) flows.

If F, G, H are due to the currents in other circuits only, their values are given by (21). In particular, if the magnetic intensities are due to the currents in one other circuit, this equation can be put into a simple and easily remembered form. Let γ' be the current in the second circuit, ds' an element of the circuit, then obviously by the definitions of the current-components, equations (21) may be written

$$F = \mu\gamma' \int \frac{1}{r}\frac{dx'}{ds'} ds', \quad G = \mu\gamma' \int \frac{1}{r}\frac{dy'}{ds'} ds', \quad H = \mu\gamma' \int \frac{1}{r}\frac{dz}{ds'} ds',$$

which give instead of (29) for the two circuits

$$T_{cf} = \mu\gamma\gamma' \iint \frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right) ds\, ds'$$

$$= \mu\gamma\gamma' \iint \frac{\cos\epsilon}{r} ds\, ds' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $\cos\epsilon$ is the angle between the two elements ds, ds' of the circuits.

If there be any number n of circuits, the total mutual energy of the system will obviously be given by summing the values of T_{cf} for all the different pairs of circuits in the system, or putting T_{cf} now for this total mutual energy

$$T_{cf} = \mu \Sigma\gamma\gamma' \iint \frac{\cos\epsilon}{r} ds\, ds' \quad . \quad . \quad . \quad . \quad . \quad (31)$$

By regarding the current in any single circuit as built up of small increments, each brought into the field of the previously existing current in the same circuit, we see that the self-energy of the circuit is given by the equation

$$T = \tfrac{1}{2}\mu\gamma^2 \iint \frac{\cos\epsilon}{r}\, ds\, ds' \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where ds, ds' are any two elements of the same circuit, r their distance apart, and ϵ the angle between them.

The total energy of any system of circuits can now be written down. Let $\gamma_1, \gamma_2, \ldots . \gamma_n$ be the currents, then clearly for the electrokinetic energy we have an equation of the form

$$T = \tfrac{1}{2}L_1\gamma_1^2 + M_{12}\gamma_1\gamma_2 + M_{13}\gamma_1\gamma_3 + \ldots . + \tfrac{1}{2}L_2\gamma_2^2 + M_{23}\gamma_2\gamma_3 + \ldots . + \tfrac{1}{2}L_n\gamma_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where the quantities $L_1, L_2, \ldots . M_{12}, M_{13}, \ldots .$ have the values indicated by

$$L_j = \mu \iint \frac{\cos\epsilon}{r} ds_j\, ds'_j, \quad M_{ij} = \mu \iint \frac{\cos\epsilon}{r} ds_i\, ds_j \quad . \quad . \quad (34)$$

where ds_j, ds'_j denote two different elements of the circuit, which is distinguished by the suffix j, ds_i, ds_j, elements of the different circuits marked by i and j, ϵ the angle, and r the distance between the elements in each case. The double integral on the right is F. E. Neumann's expression for what he called "the potential of a circuit on itself."

Self and Mutual Inductance

404. The quantity L_j is the total magnetic induction through the circuit distinguished by the suffix j produced by each unit of its own current; M_{ij} is the induction through the same circuit produced by each unit of the current γ_i in the current distinguished by the suffix i; or, which is the same, the total magnetic induction through the circuit of suffix i produced by each unit of the current γ_j. They have been called coefficients of induction from this fact, the Ls being coefficients of self-induction, the Ms coefficients of mutual induction. But the name *inductance* suggested by Heaviside is preferable; hence we shall call the Ls self-inductances, the Ms mutual inductances. We shall use the single word inductance when there is no doubt as to what is meant.

Rational Current Elements. Mutual Energy of Two Current Elements

405. The question of the action between two current elements pure and simple has given rise to an enormous amount of discussion, quite disproportionate to the importance of the question. For it is impossible to derive from theorems which are certainly true for complete

circuits expressions which apply without any ambiguity to elements which are supposed to be unclosed.

The addition of terms which integrated round the circuits give a zero result is always allowable and leads to no confusion; but when parts of circuits are considered, the existence of such terms renders the problem indeterminate.

Further, it is not the case that discrete current elements acting by themselves exist; there can be no doubt, on the other hand, that all circuits are closed. As already indicated in the theory here given, a current flowing in an apparently unclosed conductor is really flowing in a circuit completed through the surrounding insulating medium or dielectric, in which a flux takes place corresponding to, but different in nature from, the current in the conductor. This is the displacement current of Maxwell, and the hypothesis of its existence, and the consequences which flow from that hypothesis, are the main peculiarities of Maxwell's electromagnetic theory. Many of these consequences, as we shall see, have been verified by experiment.

The displacement current is assumed to produce magnetic effects according to the same laws as apply in the case of an ordinary conduction current; or, to speak more accurately, the measure of the displacement current is so assigned that the reckoning of current is the same both for conduction and displacement. Thus, taking the magnetic force at any point in a dielectric medium, the line integral of the magnetic force round a closed path drawn in the field is equal to 4π times the current flowing through the closed path. For example, taking the case discussed above, (p. 300), let the radial currents to and from the wire there described be displacement currents. We saw that neither system of radial currents produced any magnetic intensity, while the straight wire did. The line integral of the latter magnetic intensity round any closed path is then the measure of the radial current flowing across any surface enclosed by the path as bounding edge.

In fact, in this theory a current in a wire $A\,B$ is to be regarded as part of a complete circuital current consisting of the current in the wire, an outward radial displacement-current from B, and an inward radial displacement-current to A. This radiality of the displacement-currents is a consequence of the supposed isotropy and (practically) infinite extent of the surrounding dielectric.

It has been proved above that ordinary radial conduction currents produce no magnetic induction at any point, and by the assumption made above as to displacement-currents the same proposition holds also for them.

406. Suppose, then, that we are given two current elements, carrying currents γ_1, γ_2, which are rendered circuital by radial displacement-currents in the manner indicated in Fig. 102. Let ds_1 be the length of the element at A carrying γ_1, ds_2 the length of that at B carrying γ_2, (the positive directions of the elements being taken as those in which the currents flow) θ_1, θ_2 the angles which ds_1, ds_2 make with the line

AB, and r the distance AB of the centre of one element from the centre of the other. The total electrokinetic energy of the system can be calculated without ambiguity by finding the magnetic force at each point of the medium, squaring and multiplying by $\mu/8\pi$, and taking the result as the electrokinetic energy per unit of volume at the point considered. This quantity then integrated throughout the whole field would give the total energy required. From the form of the expression thus obtained the mutual energy of the elements could be at once inferred.

But this process, though perfectly straightforward and possible, would lead to certain integrals which it would be tedious to evaluate. In preference we make use of the theorem established above, that the mutual electrokinetic energy of two circuits is equal to the line integral

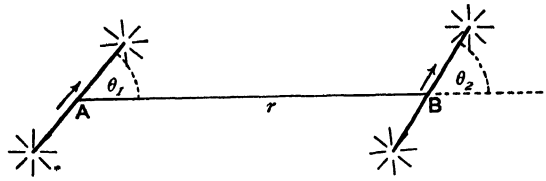


FIG. 102.

round either circuit of the vector-potential due to the other, multiplied by the current in the first; or, in the case of non-linear currents, to the sum of such line integrals for every narrow tube of flow into which the current system can be divided, which is the theorem expressed in equation (29).

Now the vector-potential at any point of the circuit of B may be divided into two parts: (1) the part due to the element A itself; (2) the vector-potential due to the radial system of currents of A. Part (1) gives two portions of the required integral, (a) that obtained by integrating over the element B, (b) that found by integrating over the radial currents of B. The latter vanishes by the symmetry of the radial currents. Again, part (2) of the vector-potential gives a portion (c) of the integral depending on the element B, and another (d) depending on the radial currents. The latter also vanishes for the same reason as before. Hence the problem is reduced to the determination of (a) and (c).

The value of (a) is easily obtained. The vector-potential of $\gamma_1 ds_1$ at B is $\gamma_1 ds_1/r$, and is in the direction of ds_1. The component along ds_2 is $\gamma_1 ds_1 \cos\epsilon/r$ if ϵ denote the angle between the elements. Hence this part of the energy is

$$\frac{\mu\gamma_1\gamma_2 ds_1 ds_2 \cos\epsilon}{r}.$$

To find the remaining part (c) of the energy we have to find the vector-potential at B of the radial system of currents completing the

circuit of A. Taking the radial currents flowing *from* A first, we see easily that we may regard only the component of the current along r, since the other components, which are at right angles to this direction, can by symmetry produce no vector-potential at B.

407. Consider, then, an outward current through a sphere of radius a drawn from the centre A of the element as centre, and supposed flowing radially from that point. Taking A as the origin of co-ordinates and the axis of x along AB, we have for the x-component of current at any point on that sphere $\gamma_1 x/4\pi a^3$; or, if θ be the angle which the radius drawn to the point P on the sphere considered makes with the axis of x, the current is $\gamma_1 \cos\theta/4\pi a^2$. The component of current has this value at all points on a small circle of radius $a \sin\theta$ described round AB as axis. Hence the flow through a narrow zone of breadth $ad\theta$, of which this circle is the middle line, is $2\pi a^2 \sin\theta d\theta . \gamma_1 \cos\theta/4\pi a^2$, or $\frac{1}{2}\gamma_1 \sin\theta \cos\theta d\theta$.

Taking, then, the current elements standing on this zone, and contained between the two spherical surfaces of radii a and $a+da$, we have for the vector-potential which they produce at B the value $\frac{1}{2}\gamma . \sin\theta \cos\theta \, d\theta \, da/\sqrt{a^2+r^2-2ar\cos\theta}$, and its direction is that of AB. To calculate the total vector-potential we have only to integrate this expression with respect to θ from $\theta=0$, to $\theta=\pi$, and from $a=0$, to $a=\infty$.

Integrating by parts we easily find the equation

$$\int \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{a^2+r^2-2ar\cos\theta}} = \frac{\cos\theta}{ar}\sqrt{a^2+r^2-2ar\cos\theta} + \frac{1}{3a^2r^2}(a^2+r^2-2ar\cos\theta)^{\frac{3}{2}}.$$

In taking this integral between the limits 0 and π, we must give it in two forms, one on the supposition that $r>a$, the other on the supposition that $r<a$. Thus, if $a<r$

$$\int_0^\pi \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{a^2+r^2-2ar\cos\theta}} = \frac{(a+r)^3}{3a^2r^2} - \frac{a+r}{ar} - \frac{(r-a)^3}{3a^2r^2} - \frac{r-a}{ar} = \frac{2}{3}\frac{a}{r^2},$$

and if $a>r$

$$\int_0^\pi \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{a^2+r^2-2ar\cos\theta}} = \frac{(a+r)^3}{3a^2r^2} - \frac{a+r}{ar} - \frac{(a-r)^3}{3a^2r^2} - \frac{a-r}{ar} = \frac{2}{3}\frac{r}{a^2}.$$

In integrating with respect to a we must use the first expression for the space within the sphere of radius r, and the second for all space external to that sphere. But

$$\frac{2}{3}\int_0^r \frac{a}{r^2}\, da + \frac{2}{3}\int_r^\infty \frac{r}{a^2}\, da = \frac{1}{3} + \frac{2}{3} = 1.$$

The vector-potential at B in the case supposed is therefore simply $\frac{1}{2}\gamma_1$, in the direction AB. The component in the direction of ds_2 is thus $\frac{1}{2}\gamma_1 dr/ds_2$, since $\cos\theta_2 = dr/ds_2$. This is due to a radial current starting from the centre of the element at A. But what we really have is an outward radial system at the positive end of ds_1, and an inward radial system at the other end. The vector-potential due to the former is therefore

$$\tfrac{1}{2}\mu\gamma_1\left(\frac{dr}{ds_2} + \frac{d^2r}{ds_1\,ds_2}\cdot\tfrac{1}{2}ds_1\right),$$

and that due to the latter is

$$-\tfrac{1}{2}\mu\gamma_1\left(\frac{dr}{ds_2} - \frac{d^2r}{ds_1\,ds_2}\tfrac{1}{2}ds_1\right),$$

so that the total vector-potential is

$$\tfrac{1}{2}\mu\gamma_1\frac{d^2r}{ds_1\,ds_2}ds_1\,ds_2.$$

The part (c) of the energy is therefore

$$\tfrac{1}{2}\mu\gamma_1\gamma_2\frac{d^2r}{ds_1\,ds_2}ds_1\,ds_2,$$

and the total mutual electrokinetic energy of the current elements supposed made circuital by radial displacement currents is given by

$$T_{cf} = \mu\gamma_1\gamma_2\left(\frac{\cos\epsilon}{r} + \tfrac{1}{2}\frac{d^2r}{ds_1\,ds_2}\right)ds_1\,ds_2 \quad . \quad . \quad . \quad (35)$$

This result was first stated by Oliver Heaviside in the *Electrician* for December 28, 1888, where it is given without proof.[1]

408. It can be shown by geometry that

$$r\frac{d^2r}{ds_1\,ds_2} = -\sin\theta_1\sin\theta_2\cos\eta,$$

when the differentiations are taken as here at opposite ends of the line AB. Hence (35) becomes

$$T_{cf} = \frac{\mu\gamma_1\gamma_2}{r}(\cos\epsilon - \tfrac{1}{2}\sin\theta_1\sin\theta_2\cos\eta)ds_1\,ds_2 \quad . \quad . \quad (36)$$

where η is the angle between the plane of r and ds_1, and the plane of r and ds_2. If $l_1, m_1, n_1, l_2, m_2, n_2$, denote the direction cosines of ds_1, ds_2, and we put $\cos\theta_1 = l_1, \cos\theta_2 = l_2$, the equation takes the form

$$T_{cf} = \frac{\mu\gamma_1\gamma_2}{2r}(2l_1l_2 + m_1m_2 + n_1n_2) \quad . \quad . \quad . \quad . \quad (37)$$

since

$$\cos\epsilon = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\eta = l_1l_2 + m_1m_2 + n_1n_2.$$

[1] See also Heaviside's *Electrical Papers*, vol. ii. articles xlviii. and xlix. pp. 501, 502, also article l., p. 506.

It is to be noticed that the formula of (35) for the mutual energy of two elements reduces to the expression given by the equation

$$T_{cf} = \mu\gamma_1\gamma_2 \iint \frac{\cos\epsilon}{r} ds_1\, ds_2 \quad . \quad . \quad . \quad . \quad . \quad (38)$$

if the circuits are closed, as in this case the second term for the action of the elements in the summation from element to element obviously gives a zero result.

In the case of two conductors of finite length AA', BB' (Fig. 103), closed by radial currents, we can easily calculate the contribution of

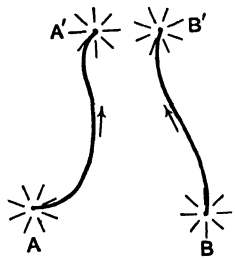


FIG. 103.

the second term in the expression of (35) for T_{cf}. For integrating $\frac{1}{2}d^2r/ds_1 ds_2$, $ds_1 ds_2$ along the two conductors in the directions of the arrows we easily find

$$\tfrac{1}{2}\iint \frac{d^2r}{ds_1\, ds_2} ds_1\, ds_2 = \tfrac{1}{2}(A'B' - A'B + AB - AB') \quad . \quad (39)$$

Hence the energy in this case is

$$T_{cf} = \mu\gamma_1\gamma_2 \iint \frac{\cos\epsilon}{r} ds_1\, ds_2 + \frac{\mu\gamma_1\gamma_2}{2}(A'B' - A'B + AB - AB') \quad (40)$$

The expression

$$T_{cf} = \mu\gamma_1\gamma_2\, ds_1\, ds_2 \left(\frac{\cos\epsilon}{r} + \frac{1-k}{2}\frac{d^2r}{ds_1\, ds_2}\right) \quad . \quad . \quad (41)$$

was given by von Helmholtz[1] for the mutual energy of two current elements. His expression, therefore, passes into that given by the theory of Maxwell, in which all currents are supposed closed by displacement currents, if k be put equal to 0. This has been noticed by Helmholtz himself, and by other writers, but in quite another connection.

[1] "Ueber die Theorie der Electrodynamik," *Crelle's Journ.* lxxviii. p. 309; *Ges. Werke* ii. p. 745.

It will be observed that the proper expression for the mutual inductance of two current elements is given by the equation

$$M = \mu ds_1 ds_2 \left(\frac{\cos\epsilon}{r} + \tfrac{1}{2}\frac{d^2r}{ds_1 ds_2}\right), \quad . \quad . \quad . \quad . \quad (42)$$

which leads to the equation (34) already given for two closed circuits

$$M = \mu \iint \frac{\cos\epsilon}{r} ds_1 ds_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Elements of the kind here considered of course can be combined to give a conductor of finite length, since the radial currents at two extremities in contact cancel one another.

Forces between Two Current Elements

409. The formulas found above enable us to calculate the force between two current elements which have been rendered circuital by radial displacement currents, or rational current elements, as they have been termed by Heaviside. For example, to find the force in the direction of the line joining the elements it is only necessary to differentiate (35), (36), or (37) with respect to r. The differential coefficient with minus sign prefixed, or $-\partial T_{cf}/\partial r$, is the applied force necessary to increase r, and therefore $\partial T_{cf}/\partial r$ is the internal force between the elements. Thus if we call this force X, we have

$$X = -\frac{\mu\gamma_1\gamma_2}{2r^2}(2l_1l_2 + m_1m_2 + n_1n_2). \quad . \quad . \quad . \quad (44)$$

This shows that if X be regarded as a force on, say, the element at B, and the elements be arranged as shown in Fig. 102, the force is in the direction to diminish r, that is, it is an attraction. Similarly, if we suppose the value of r to be subjected to increase in the direction from B to A, the applied force will be positive, and the internal force is an attraction of the element at A towards B.

A different law of force which was given by Ampère in his famous paper is expressed in the symbols here used by the equation

$$X = -2\gamma_1\gamma_2\frac{1}{r^2}(\cos\epsilon - \frac{3}{2}\cos\theta_1\cos\theta_2)ds_1 ds_2 \quad . \quad . \quad (44')$$

and was deduced by him on certain suppositions, and without taking into account the current in the dielectric. This law is easily deducible from Neumann's formula for the mutual inductance of two closed circuits, by an application of the calculus of variations.[1]

The result stated in (44) is in accordance with the observed fact that if currents be made to flow in two conductors, crossing one another at an acute angle so that the currents flow both towards the acute angle on one side and away from it on the other, the acute angle in consequence of the electromagnetic forces between the conductors tends to

[1] For further information see *Absolute Measurements*, vol. ii. part i. p. 125.

diminish. Forces act as shown in Fig. 104 so as to draw the conductors nearer to parallelism.

On the contrary, if one of the currents in Fig. 104 were reversed, as shown in Fig. 105, the forces would be changed to repulsions and the conductors would tend to set at right angles to one another.

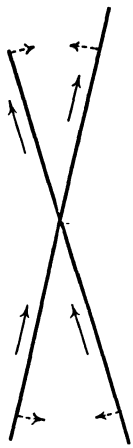


FIG. 104.

FIG. 105.

The particular case of two straight parallel conductors is illustrated in Fig. 106. The lines of force round the conductor AB are circles (of which one is indicated by the dotted line) of which AB is the common axis. The other conductor CD, carrying a current in the field of AB, tends to move so as to cut across the lines of magnetic induction most rapidly, that is, it tends to move along the radius, and in the direction shown by the arrow. The force on AB is equal and in the opposite direction. If the conductors be long and their circuits be each completed by a long wire at a great distance, the magnetic induction through either circuit will be increased by the approach of one wire towards the other if the currents are in the same direction, and diminished if they are in opposite directions. Thus there is attraction in the former case, repulsion in the other.

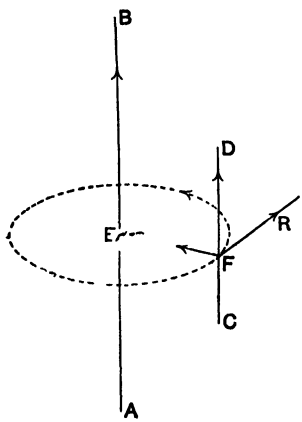


FIG. 106.

The same way of viewing the matter may obviously be applied to the case, which we have already referred to, of two conductors inclined to one another at an acute angle.

Explanation of Actions on Conductors by Stresses in the Field.

410. Later we shall see that, according to a theory of stress in the magnetic field, where magnetic force **H** exists in an isotropic non-magnetic medium, there probably exists also a stress of the nature of hydrostatic pressure combined with a tension along the lines of force. No completely satisfactory theory of stress has yet been elaborated, but according to that given by Maxwell a stress of the nature of hydrostatic pressure of amount $\mu\mathbf{H}^2/8\pi$ exists at every point, combined with a tension of amount $\mu\mathbf{H}^2/4\pi$ along the lines of force. Thus, according to Maxwell, there exists at every point a pressure across the lines of force and an equal tension along them. Such a theory as this gives at once an explanation of the apparent attraction of one conductor by the other. Consider for simplicity two long parallel straight conductors, the circuits of which are completed by wires at a great distance. The two sets of lines of force, which are in opposite directions round the circuits, conspire between the conductors, and are opposed in the rest of space: but if the currents are in the same directions the lines of force are opposed between the conductors, and conspire elsewhere. Thus in the former case the hydrostatic pressure is greater between the conductors than elsewhere, and the conductors are driven apart, in the latter case the pressure is less between the conductors than elsewhere and the conductors are pushed nearer to one another.

Ampère's Experiment.

411. But besides these qualitative results those of many careful quantitative experiments confirm the theory given above, and of a few of these we shall here give a short account. Some of the apparatus with which these experiments were made can be readily used to illustrate in a number of ways the attractions and repulsions referred to above. The general result demonstrated by the experiments is the truth, on which we have already insisted at some length, that a current system is equivalent to a distribution of magnetism.

The first experiments made of any importance were the famous four electrodynamic experiments of Ampère. These were made with the apparatus shown in Figs. 107—111.

The principal piece of apparatus is a stand arranged so as to allow a part of a circuit, generally itself very nearly a closed circuit, to turn freely round a vertical axis. The arrangement is clearly shown in Fig. 107. Two conductors, one a tube, the other a stout wire within it, are attached to binding screws in a heavy sole plate and carried vertically to some distance, then bent at right angles as shown in the drawing. The end of the wire projects beyond the mouth of the tube, which carries a projecting lip, so that two mercury cups at their extremities are in the same vertical. These cups bring the turned down

ends of a wire frame such as that shown in the figure into contact with the tube and wire, and therefore with the binding screws, and leave it free to turn round a vertical axis passing through the cups. A current can therefore be sent (by attaching the terminals of a battery to the screws or otherwise) round the circuit.

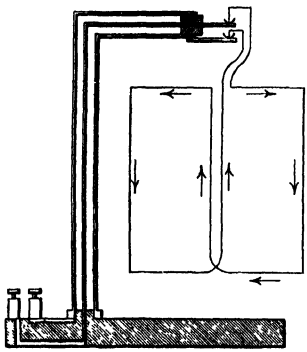


FIG. 107.

The frame shown is a double rectangle of wire, the parts of which are insulated from one another at the points of crossing. The current flows as shown by the arrows, and two nearly complete circuits are obtained, the areas of which are approximately equal.

The arrangement of tube and enclosed coaxial conductor prevents the communicating wires from having any appreciable effect, and limits any electromagnetic action experienced to the frame.

The modification of Ampère's stand shown in Fig. 108 is due to M. Nodot. A vertical platinum wire is hung by a silk thread as shown, passes down through a mercury cup, the bottom of which is a plate of mica perforated by a hole just large enough to give the wire enough of clearance. The frame is attached below, and a point at its lower end dips into a mercury cup vertically under the platinum wire above. The current is let in and out at the cups. The frame turns with great freedom, under even small forces.

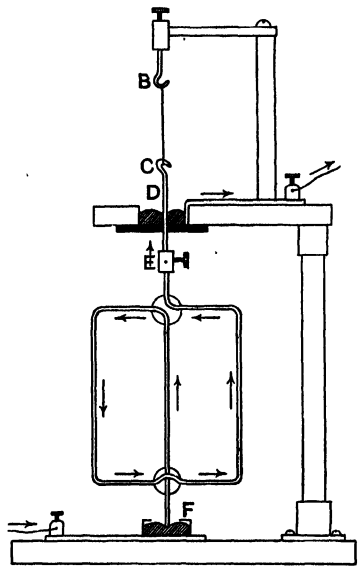


FIG. 108.

412. When the circuit as here arranged carries a current it shows no tendency to set itself in any particular position in the earth's magnetic field. If, however, the frame suspended consists of a simple turn of wire, it sets itself so that its plane is at right angles to the horizontal magnetic force of the earth, and so that the direction of the current in the circuit, if looked at from the north side, is in the opposite direction to the hands of a watch. This illustrates the equivalence of a current and a magnetic shell.

Ampère's first experiment.—A wire was doubled on itself as shown in Fig. 109, and attached to binding screws at its extremities. The two

parts of the wire, although close to one another along their whole length, did not touch.

When the wire carried a current and was brought near but not too close to one side of the suspended frame, there was no deflection of the latter. Thus the effect of the current in one part of the doubled wire neutralized the effect of the current in the other part.

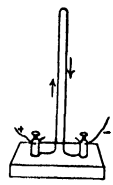


FIG. 109.

The neutralization is not in this case complete, but can be rendered quite exact by replacing one wire by a tube enclosing the other.

Ampère's second experiment.—In this one of the two parts of the doubled wire of the last experiment was made a zigzag close to the other, as shown in Fig. 110. There was still neutralization of the effect of one portion of the doubled wire by the other.

It follows that the effect of an element of a straight conductor can be replaced by that of a small crooked or sharply bent conductor, having the same beginning and end as the straight element, and the same current flow in both. This is in other words the important theorem that the electromagnetic force on a straight element may be considered as the resultant in the ordinary dynamical sense of those on any number of component elements at the same place. (See Art. 387.)

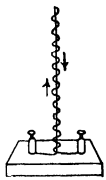


FIG. 110.

Ampère's third experiment.—A conductor was arranged so as to be free to move in the direction of its length. It consisted of an arc of wire supported on the convex surface of mercury in two troughs and attached to a light arm of wood pivoted at the other end, so that the wire was only free to move as specified. A current was sent along the wire from one trough to the other.

No arrangement of magnets or of circuits carrying currents brought near to the arc of wire was found to produce any displacement of the conductor. Ampère therefore inferred that the electromagnetic force on the arc of wire had no component in the direction of the length of the conductor. This agrees, it will be noticed, with the law of force given in Art. 391 for an element of a conductor in a magnetic field.

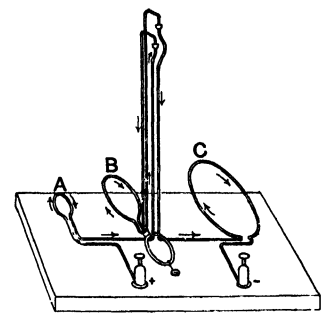


FIG. 111.

Ampère's fourth experiment.—A nearly closed conductor, *B*, Fig. 111, was hung on the stand, so as to be free to move round a vertical axis, and two others, *A* and *C* similar to the first, were arranged on the two sides of *B*. The dimensions were so chosen that each linear dimension of *B* was n times the corresponding dimension of *A*, and

$1/n$th of the corresponding dimension of C. Further, the positions of the conductors were made similar, that is the position of A with respect to B was similar to that of B with respect to C, and hence the distance of any element of C from any element of B was n times the distance apart of corresponding elements in B and A.

Currents of equal strength were sent in the same direction through A and C, and a current of any convenient amount through B in either direction. The actions of A and C on B were found to be opposed and exactly to balance one another.

413. From the result of the fourth experiment Ampère inferred that if the action on the movable conductor be made up of the actions, on each of its elements, of forces due to each element of the other conductors, the action of each pair of elements varies inversely as the square of the distance between them. To prove this in his manner we call r_1 the distance between an element b_1 in B and an element a_2 in A, and r_2 the distance between two similarly situated elements c_1 and b_2 in C and B, and let $f(r_1), f(r_2)$ denote the forces between the elements of the respective pairs per unit of length and per unit of current in each case. If ds be the length of each of the two elements of B chosen, those of the elements a_2 and c_1 of A and C are respectively ds/n, nds.

Since B is in equilibrium the forces for corresponding pairs of elements are to be taken as equal, and therefore if γ be the current in A and C, and γ' the current in B,

$$\frac{ds^2}{n}\gamma\gamma' f(r_1) = nds^2\gamma\gamma' f(r_2),$$

or

$$\frac{f(r_2)}{f(r_1)} = \frac{1}{n^2},$$

that is the law of the assumed force between a pair of elements is that it varies as the inverse square of the distance.

This is entirely an action at a distance way of looking at the matter. The truth no doubt is that there is no direct action of one element on the other. Each circuit produces a magnetic field, and the conclusion to be arrived at from the experiment, if B remains at rest and shows no tendency to deformation, is that the magnetic fields produced by A and C neutralize one another at every point of B.

From these four experiments Ampère deduced his law of force given above. We do not give the discussion here, as it depends on assumptions which it is impossible to justify, and further involves the fundamentally erroneous notion of an unclosed current element. Of course the law of force found gives the total action experienced by a circuit as a whole (in fact it can be found by supposing a circuit to be slightly distorted so that its elements are placed in slightly different relative positions). From it can be deduced various results which are known to be true, such as that a solenoid is equivalent to a uniformly magnetized magnet; but it is also the case that the same results are

obtained by using any one of several other laws of force between current elements which have been put forward by various mathematicians. In fact, an infinite number of laws of force between a pair of current elements are possible, and if their circuits are not supposed closed in some proper physical manner the search for the true law of force must be fruitless.

Weber's Experiments

414. An exceedingly valuable series of experiments was made by W. Weber[1] by means of his electrodynamometer. This is shown in the chapter below on Measurement of Currents. The instrument consisted of two circular coils suspended by bifilar wires so as to be free to turn round a vertical axis, and a fixed coil, which could by levelling have the planes of its windings made vertical. The current was conveyed to the suspended coil by the bifilar suspension.

Two forms of the apparatus were used: (1) One which had the movable coil system suspended within the fixed coil, so that their centres were as nearly as possible coincident. (2) One in which the fixed and movable coils were distinct, so that they could be placed at any required distance from one another, and in any relative positions.

The deflections of the movable coil were measured by the ordinary mirror and telescope method. (See the chapter on *Measurements of Currents*, Vol. II.)

In his first experiment Weber proved that the electromagnetic action on the suspended coil varied as the square of the strength of the current. The fixed coil was set up with its axis at right angles to the magnetic meridian, while the suspended coil had its plane in the meridian. The centres were coincident according to arrangement (1). Currents were sent through the coils, and to prevent too great a deflection, the current through the suspended coil was kept down to 1/246·26 of the current in the fixed coil, by carrying the rest of the current through a thick wire joining the terminals of the movable coil. A magnetic needle, consisting of a magnetized steel mirror hung within a vessel of sheet copper, was placed 58·3 centimetres due magnetic north of the centre of the fixed coil, and the tangents of the deflections of this needle gave a comparative measure of the different currents used. The results shown in the following table were obtained:—

No. of cells used.	Comparative values of force between coils $=A$.	Force on magnetometer needle in arbitrary units $=B$.	Force on needle found by formula $5{\cdot}15534\sqrt{A}$.	Difference $B - 5{\cdot}15534\sqrt{A}$
3	440·038	108·426	108·144	·282
2	198·255	72·398	72·589	− ·191
1	50·915	36·332	36·786	− ·454

[1] *Elektrodyn. Maasbest.* I. (1846).

It will be seen that the mutual action between the systems was proportional to the square of the current, that is to the product of the strengths of the two magnetic shells.

In another series of experiments Weber used the second form of apparatus; the movable coil was hung with its axis horizontal and parallel to the magnetic meridian, while the fixed coil was placed with its axis at right angles to the magnetic meridian and its centre, (*a*) in the (magnetic) north and south horizontal line, (*b*) in the (magnetic) east and west line through that of the suspended coil. Experiments were made in each case with distances between the centres of respectively 0, 30, 40, 50, 60 centimetres. A current was sent through both coils, and also through a coil set up about 8 metres from the fixed coil (so as to form a galvanometer with the magnetic needle mentioned in the other set of experiments), and through a reversing key, so arranged that the current in the suspended coil could be made to flow first in one and then in the opposite direction, without changing it in the rest of the circuit. The object of thus reversing the current was to determine and allow for the turning moment of the earth's magnetic field, when the axis of the suspended coil was deflected from the magnetic meridian. The corrected results of the experiments are shown in the table below, in which the second column for each series of positions gives the corresponding numerical values calculated by Ampère's formula for the force between two current elements.

	Positions of centres of coils.			
	In magnetic east and west line.		In magnetic north and south line.	
Distance of centres of coils apart	Couple observed.	Couple calculated.	Couple observed.	Couple calculated.
0	22960	22680	22960	22680
30	189·93	189·03	−77·11	−77·17
40	77·45	77·79	−34·77	−34·74
50	39·27	39·37	−18·24	−18·31
60	22·46	22·64	——	——

The results for the greater distances agree very fairly with calculation from Ampère's formula, and it has been shown that for a closed circuit (which each coil was very nearly) Ampère's formula and the magnetic shell theory give identical results.

It is to be remarked that in these experiments the two coils are not independent circuits; but that they may be so regarded is plain from the fact that the remaining portion of the circuit, if the wires are closed or twisted together, is of no effect, since it can be altered at pleasure

without affecting the action between the coils, provided the current be maintained constant.

The deflections θ, θ', in the two cases agree closely for the greater distances with the approximate equations

$$\tan\theta = \frac{2MM'}{D^3}\left(1 + \frac{d}{D^2}\right), \quad \tan\theta' = \frac{MM'}{D^3}\left(1 + \frac{b}{D^2}\right),$$

which give the deflections of a magnet of moment M', by another of moment M in the so-called "end-on" and "side-on" positions and at distances D apart, great in comparison with the dimensions of the magnets. This verified the proposition that the coils are replaceable by magnets.

Investigations of the mutual actions of circuits have been made with a great variety of experimental arrangements by Cazin, Boltzmann, and others. Full accounts of these are given in Wiedemann's *Elektricität*, Vol. III. Those of Weber, which we have described, are of special interest, as his electrodynamometer is in a modified form still a standard instrument for the absolute measurement of currents. But the surest experimental basis for the theory of electromagnetism is to be found in the accurate consistency of the results of electrical measurements made by means of instruments graduated by methods derived directly from it.

CHAPTER X

INDUCTION OF CURRENTS

SECTION I.—*Experimental Basis of Theory of Current Induction*

Faraday's Experiments

415. The chief experimental results on which the theory given below is based, were first obtained by Faraday, though it is also now certain that some of the most important of them were afterwards independently arrived at by Joseph Henry of Washington. The general nature of the phenomena observed may be understood by reference to a few simple experiments with easily obtainable apparatus. Two flat spirals or coils of wire are arranged parallel to one another as shown in Fig. 112. One, *A*, is in circuit with a galvanometer, the other, *B*, can be connected at pleasure to a battery, and a simple commutator enables the terminals of the battery to be reversed relatively to the coil *B* at pleasure, so that a current can be sent in either direction round the circuit of the latter.

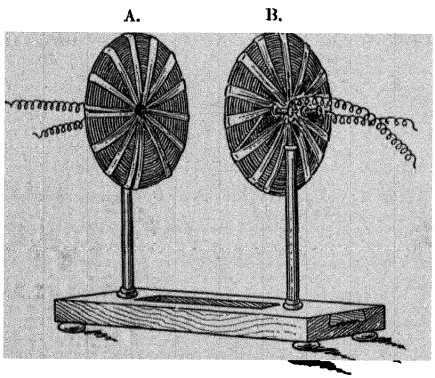


FIG. 112.

When the key is put down so as to send a current round the coil connected to the battery, a deflection of the galvanometer needle takes place showing that a current has been produced in the other coil as well. This current is transient; for the needle returns to zero, and so long as the battery current is not varied there is no disturbance of the needle. When however the key is released so as to stop the battery current a transient deflection of the needle again takes place, but in the opposite direction to the former. It will be found that, if the current stopped is the same in amount as the current started, the

transient deflection at the completion of the circuit, or "make", as we shall call it, is equal to that at the stoppage of the current, or "break."

This equality has been elaborately verified by v. Helmholtz and others. It may be roughly shown by releasing the key as soon as it is tapped down; when the transient current at make will be followed by that in the opposite direction at break, and the needle will scarcely move from the position of equilibrium, as the impulse received from one current is compensated by that received from the other.

416. Again let the circuit B after a current has been started in it be quickly placed nearer to and parallel to A. A transient current will again be produced in A. If the coil B after having been brought up be maintained in the same position relatively to A, no deflection of the needle will take place unless the current in B is quickly varied or stopped. If however B be quickly drawn farther away from A, a transient current will again be produced, but in the opposite direction to that shown by the needle when the coil was brought up. Further the transient currents produced by the approach of the coil and its withdrawal are in the same directions as those caused by the starting and stopping of the current respectively.

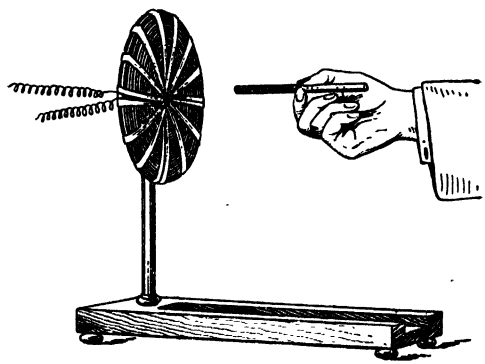


FIG. 113.

If instead of the coil B a magnet be used, as shown in Fig. 113, similar results are obtained. The effect of bringing a magnet from a distance up to or within a coil is to produce a transient current in one direction, and of its withdrawal from the coil to produce a current in the opposite direction.

Felici's Experiments

417. The laws of current induction are well illustrated by a method of experimenting used by Felici. Two coils A, B are arranged as shown in Fig. 114 in circuit with a battery, and so as to act inductively upon two coils X, Y, which are in series in a secondary circuit containing also a galvanometer. The coil A acts upon X and B upon Y, and it is possible so to adjust that these two actions exactly balance, and the galvanometer indicates no current when one is started or stopped in the primary coils A, B.

By means of such an arrangement, consisting however of a single turn of wire, serving as secondary to two primary turns, one on each side of the secondary, Felici proved a number of propositions which we

shall see follow immediately from the magnetic induction theory of Faraday. Instead of following in detail Felici's actual investigations, we shall shortly describe the experiments which can be performed with the apparatus represented in Fig. 114.

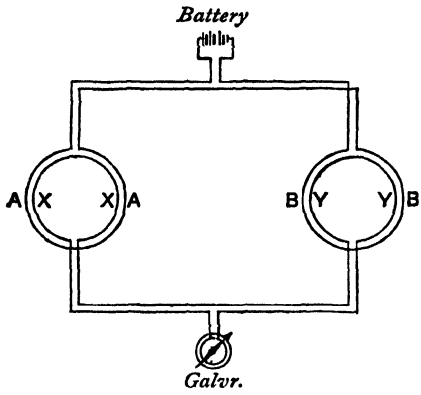


FIG. 114.

Balance is first obtained; then for one of the coils is substituted another differing only in area of cross-section of the conductor or in the material of the conductor. No disturbance of the balance is produced. The contrary result however follows if the form or dimensions of the coil are changed, unless the change is made up of different parts which compensate one another. Thus:

a. The induction of one circuit on another depends on the form and dimensions of the conductors, and not on their area of section or on their material.

By interchanging A and X, say, after balance has been attained it is found that the balance is not disturbed. Hence:

b. The inductive effect of A on X is equal to that of X on A.

The inductive effect of A on X is balanced by that of B upon Y; and again by that of a third primary coil C on a third secondary Z. Then the inducing current after passing through A is divided in any chosen ratio between B and C, while the secondary coils Y and Z are joined in series so that the currents induced in the latter are in the same direction, but are opposed to the induced current in X. They are then found to balance this last induced current. It follows that however the inducing current is divided between B and C each part produces an effect proportional to its strength, and such that the two effects added together precisely equal that which is produced by the whole current when flowing in one of the coils. Thus:

c. The inductive effect is proportional to the inducing current.

Further, the arrangement just described is so constructed that the sum of corresponding dimensions in B and Y, and C and Z, is equal to the corresponding dimension in A and X; when X is opposed to Y and C, thus arranged, there is no current through the galvanometer when the primary circuit is made or broken. This proves that:

d. The inductions between geometrically similar pairs of circuits are proportional to the linear dimensions of the different pairs.

418. In another series of experiments made with only one pair of coils, a primary and a secondary, Felici obtained further important

results. If the relative positions of a primary and secondary coil be changed, a series of positions will be found in which the starting or stopping of a current in the primary produces no effect on the secondary. The two coils are said to be then in conjugate positions. Felici found that if the secondary was quickly moved from any conjugate position with respect to the primary to another such position, without changing that of the primary, no induced current was shown by the galvanometer in the secondary, and this no matter how the transition was effected, or whether or not the current in the primary was altered during the change.

Again if the secondary was moved from a conjugate to a non-conjugate position, an induced current was produced exactly equal and opposite to that set up in the secondary by interrupting the primary circuit while in the latter position.

This was shown by moving the secondary quickly from the first to the second position and breaking the circuit immediately after, so that both induced currents had given their impulses to the needle before it had sensibly moved from its equilibrium position.

The effect of changing the position of the coils so as to allow inductive action to ensue, was equal and opposite to that due to stopping the current at the second position, and therefore precisely the same as that which would have been produced by starting the current in the coils after they had been placed in the final position.

Precisely similar results were found to hold when the primary was moved instead of the secondary.

Induced Currents are due to Change of Magnetic Induction

419. All these results are confirmatory of the view, put forward by Faraday, that the induced currents are the effect of the alteration of the total magnetic induction through the secondary circuit. For, (1) the mutual action of two circuits in producing currents in one another by induction depends only on the geometrical form and the dimensions of the circuits (unless the material be magnetic), (2) the magnetic induction at any point produced by the primary is proportional to the current strength, for different currents in the same circuit are compared by the magnetic intensities they produce at a given point, (3) the total magnetic induction through a circuit A produced by a given current in another circuit B, is equal to the total magnetic induction through B produced by the same current in A, or in other words the inductance of A through B is equal to the inductance of B through A (Arts. 403, 404), (4) the mutual inductance of two similar pairs of circuits is, as we shall show later, proportional to their corresponding linear dimensions.

Again when a primary and secondary circuit are in conjugate positions the magnetic induction through either due to a current in the other is zero, that is their mutual inductance is zero. Hence, changing the coils to a non-conjugate position when a current is flowing in

one, produces the same induction through the other as would be produced by first placing the coils in the non-conjugate positions and then causing the currents to flow in the primary.

Faraday's Theory of Lines of Force Induction

420. Faraday's idea that change of total magnetic induction through a circuit is the cause of induced currents is at the bottom of the whole mathematical theory of current-induction; but there can be no doubt that it relates to a real physical change which takes place in the medium when, owing to changes in currents or displacements of circuits or magnets, what we call the magnetic induction is altered, and it is the main object of this book to discuss consequences of such changes. Faraday began by attributing to a circuit in a magnetic field an "electrotonic state," and afterwards interpreted this view by means of his notion of "lines of force" belonging to or moving with magnets or current-carrying conductors. According to this idea lines of induction fill the space surrounding a magnet or a circuit in which a current is flowing, come into existence when the current is started in the conductor and disappear when the current is annulled, expand out from or contract down upon the circuit or magnet when the current is increased or diminished or the magnet is strengthened or the contrary, move with the magnet or circuit to which they belong when that is displaced, and by their passage across a conductor produce inductive electromotive force in it and set electricity in motion.

421. The whole modern theory of lines or tubes of induction, though not in its mathematical details, seems to have been clearly present in the mind of Faraday. By their relative concentration or sparseness of distribution he represented the intensity of magnetic induction from point to point of the field; and by the physical existence he rightly attributed to them, he obtained a much more vivid idea of the conditions and relations of magnetic and electric phenomena than was possible to men of far greater technical mathematical power, but inclined, perhaps for that very reason, to regard physical phenomena in too abstract a manner. The following extracts from his *Experimental Researches* will show how fully he stated the theory of induction which has only since the rise of practical electricity become part of the general knowledge of ordinary electrical students.

"The moving wire can be made to sum up or give the resultant at once of the magnetic action at many different places, *i.e.*, the action due to an area or section of the lines of force, and so supply experimental comparisons which the needle could not give except with very great labour, and then imperfectly. Whether the wire moves directly or obliquely across the lines of force in one direction or another, it sums up with the same accuracy in principle the amount of the forces represented by the lines it has crossed." (*Exp. Res.*, Series XXVIII, 3082.)

(In this statement is contained the method of exploration of a magnetic field by means of the induced current, which is so important theoretically, and which has been used with such good effect in practice for the determinations of the intensities of powerful fields such as those of dynamos, and for the investigation of fictive magnetic distributions.)

"If a continuous circuit of conducting matter be traced out or conceived of either in a solid or fluid mass of metal or conducting matter, or in bars or wires of metal arranged in non-conducting matter or space, which being moved crosses lines of magnetic force, or being still, is by the translation of a magnet crossed by such lines of force, and further, if, by inequality of angular motion, or by contrary motion at different parts of the circuit, or by inequality of the motion in the same direction one part crosses either more or fewer lines than the other; then a current will exist round it due to the differential relation of the two or more intersecting parts during the time of the motion: the direction of which current will be determined (with lines having a given direction of polarity) by the direction of the intersection combined with the relative amount of the intersection in the two or more efficient and determining or intersecting parts of the circuit." (Series XXVIII, 3088.)

Faraday's Experiments on "Unipolar Induction"

422. In further elucidation of this subject Faraday made a number of experiments with a bar-magnet and a loop of wire variously disposed relatively to one another. He found that when the magnet was rotated about its axis while the wire was held in a closed circuit (as in Fig. 115, on the left) no current was produced in the wire. When the wire was made to touch one end of the magnet with one extremity, and the centre of the magnet with the other (as in the other diagram of Fig. 115) so that the circuit of the wire was completed through the magnet, and the magnet was turned round its axis while the wire remained at rest, a current was produced, while none was observed when the circuit was rigidly connected with the magnet and turned with it. The direction of turning and the poles of the magnet being as represented in Fig. 115, on the right, the current is in the direction of the arrowhead on the wire.

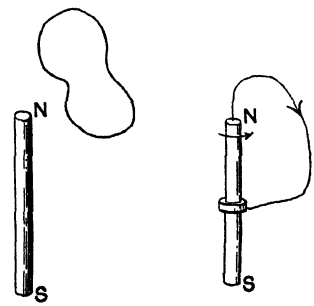


FIG. 115.

423. This is the so-called "unipolar induction," though there is nothing unipolar about it. A vast amount of discussion has been devoted to it, yet the phenomena are simple consequences of Faraday's principle. When the magnet moves its field of force moves with it. Hence when the loop turned with the magnet, there was no cutting of

tubes of induction and therefore no current. On the other hand when the loop was a detached circuit, there was cutting of lines of induction; but every line of induction which cut the wire once, cut it again at another point, so that the sum total of lines cutting the circuit was zero at every instant. Hence again there was no current.

When the loop made rubbing contact at its ends with the magnet, the lines moving with the magnet cut the wire as the magnet rotated, and the effect of this was not fully balanced by cutting elsewhere, since the lines passed from one pole of the magnet to the other in external space, and completed their circuit within the magnet. For the conductor through which the circuit was completed was the substance of the magnet, and this moved with the field. When a wire was led along close to the magnet from the end to the middle to complete the circuit, and this portion of wire turned with the magnet the result was the same, as clearly it ought to be.

Among his conclusions from these experiments Faraday says: "It is evident by the results of the rotation of the wire and magnet, that when a wire is moving among equal lines (or in a field of equal magnetic force) and with an uniform motion, then the current of electricity produced is proportionate to the *time*, and also to the *velocity* of motion."

"They also prove, generally, that the quantity of electricity thrown into a circuit is directly as the amount of curves intersected." (*Exr. Res.*, Ser. XXVIII, 3114, 3115.)

424. The last quotation but one is equivalent to a conclusion which may be drawn from the dynamical theory above, and which we shall make much use of: the electromotive force in a moving conductor or portion of a circuit is proportional to the rate at which it is cutting across lines of induction.

From this by proper summation we get the proposition practically stated in a former quotation, and used now in calculating total electromotive forces, viz.: each portion of the system of tubes cut across or moving across the conductor produces its own effect, which is added to that of the others, so that the effects of the whole system of tubes which have passed across the conductor are integrated by the circuit.

The last quotation states the rule followed for the estimation of the total quantity of electricity which flows in a transient current produced by the motion of a conductor in a magnetic field. This is simply the time integral of the current at any instant during the induction, a quantity which we can measure in an analogous way to that in which we measure an ordinary dynamical impulse, although we cannot estimate the impulsive force of which such an impulse is the time integral.

We shall see later how the time integrals of transient currents can be measured, and discuss the construction of instruments for that purpose. At present it is sufficient to assume that such measurements can be made.

Numerical Estimation of Inductive Electromotive Force

425. Faraday showed that by rotating the Barlow's disk placed in a magnetic field, as described in Art. 402 above, an electromotive force was produced in the circuit completed by the wire touching the disk at the centre and the mercury in the cup at the lower edge. We may now estimate, on Faraday's principle of the cutting of lines of magnetic induction, the electromotive force in the circuit. This will show how the electromagnetic unit of electromotive force is defined. (See also Art. 450.)

Let the induction be supposed of uniform amount $\mathbf{B}$, and to be directed at right angles to the disk, ω be the angular velocity, and r the radius of the disk. The impressed electric force $\mathbf{e}$ at a point distant x from the centre is $\omega x\mathbf{B}$, since the rate at which an element dx of the radius there is cutting across tubes is $\omega x\mathbf{B}dx$. Hence the whole impressed electromotive force in the circuit is

$$\omega\mathbf{B}\int_0^r x dx = \tfrac{1}{2}\omega\mathbf{B}r^2 = \tfrac{1}{2}vr\mathbf{B}$$

if v be the speed of the edge of the disk.

If the circuit be not closed there will be a difference of potential of this amount between the centre and the edge, and no work beyond that consumed in the friction of the bearings and the air will be required to drive it. If, however, $\mathbf{B}$ have not the same value at every point of the disk, internal currents will be set up in the metal, the mutual action between which and the magnetic field will oppose the motion (Art. 427). Thus energy will have to be expended in driving the disk across the lines of magnetic induction in the field, and will be dissipated in heat in the metal. Very marked heating effects can be produced in a copper disk by rotating it rapidly, so arranged that a portion of it only, between the centre and one part of the rim, is in an intense magnetic field.

If, however, an electromotive force equal and opposite to $\frac{1}{2}vr\mathbf{B}$ be placed in the wire the current will be reduced to zero, and the latter electromotive force will be obtainable from the rotation and other circumstances for the disk. This process has been applied to the determination of resistances in absolute units. (See *Abs. Meas.* Vol. II.)

The apparent constancy of the number of tubes of induction passing through the circuit may be explained by supposing the outer end of each radial portion as it leaves the mercury cup to remain connected with it round the edge. On this view $\pi r^2\mathbf{B}$ unit tubes are added in each turn to the total magnetic induction through the circuit.

Another arrangement, equivalent to that just described, is shown in Fig. 116. Two parallel rails in a uniform magnetic field directed at right angles to the plane of the rails, are connected by a slide which is moving with velocity v. The electromotive force is here $vl\mathbf{B}$ where l is the length of the slider between the contacts, and this is the difference of potential between the rails when the circuit is not closed by a second

connection AC. The directions of the motion, the induction, and the current are as shown by the arrows.

The general specification of the impressed electrical intensity at an element of a conductor, moving with velocity $\mathfrak{s}$ in a direction inclined at angle θ to the magnetic induction $\mathbf{B}$, is that it is the vector product of $\mathbf{B}$ and $\mathfrak{s}$, that is, $\mathbf{B}\mathfrak{s}\sin\theta$. Its direction may be inferred from Fig. 116. (See also Art. 498.

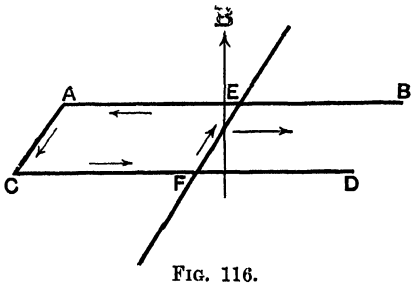


FIG. 116.

426. Faraday also investigated induced currents of higher orders than the first. The notion of magnetic induction shows at once that an induced current in a secondary will produce a magnetic field, transient like itself, but setting up induced currents in a conductor (which we may call tertiary) properly placed to receive it. In the case of a transient current in the secondary, the induced current in the tertiary *is first in one direction, then in the other*, as the secondary induction rises to a maximum, and again falls off to zero. From tertiary induced currents we can pass, of course, to induced currents of higher order.

Electromagnetic Forces due to Induced Currents. Law of Lenz

427. Induced currents produce electromagnetic forces between the inducing and the secondary circuits. These forces are always such as to oppose the motion of the circuit or magnet setting up the induced current. For example, the current excited by bringing a magnet or circuit nearer to a coil, is in the direction to produce a magnetic field which tends to oppose the magnet or circuit's motion of approach. If the magnet or circuit is being withdrawn, the magnetic field of the induced current tends to cause approach of the primary system. Or if the inducing field is set up by starting a current in the primary circuit, the magnetic field of the secondary tends to produce withdrawal of the circuit to a greater distance. This is the well-known law of Lenz.[1]

It should be noticed that if this were not the case there would not be stability of equilibrium of any arrangement of circuits. The motion would start a current which would set up a force on the circuits aiding the motion, and so the disturbance would go on increasing of itself.

On this law F. E. Neumann founded a theory of induction which may be regarded as a sequel to Ampère's theory of electromagnetic action. One of the most important results was the formula [Art. 403 (34)] for the mutual inductance of two circuits, or, as it used to be called, the electromagnetic potential of one circuit on the other. Any discussion of Neumann's theory would be foreign to the aim of the present work.

[1] *Pogg. Ann.* 31, p. 483 (1834).

Self Induction

428. The subject of self induction was also investigated by Faraday. It had been observed by Mr. Jenkin, who communicated the result to Faraday, that if a powerful electromagnet is included in the circuit of a battery, shocks can be obtained by making and breaking the circuit when the human body forms a shunt on the electromagnet coils. Thus let the body be included in the cross contact piece $A\,B$ of Fig. 117, and let the key K_2 be kept down while K_1 is depressed and raised. If the battery be powerful enough and the coil C have many convolutions the person making contact in $A\,B$ will experience smart shocks. If an iron core be included in the coils, the shock will be very decidedly more severe than if the coils are used alone.

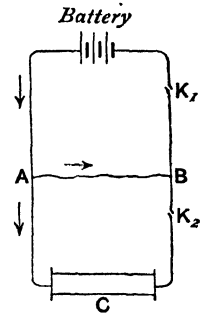


FIG. 117.

A phenomenon due to the same cause is perceived every time the circuit of an electromagnet is broken. When the circuit is broken a spark is seen, which, by the use of a sufficiently strong current in the circuit, may be made a bright flash, fusing the surfaces of the metals at the contact. This spark does not occur on making the circuit, nor is it perceptible unless the circuit includes a coil of many turns of wire, or else, if of comparatively few turns, contains a core of iron.

Faraday's Experiments on Self Induction

429. By means of the arrangement shown in Fig. 117, with a galvanometer substituted for the human body in the cross connection $A\,B$, but with the coil at C, Faraday made many interesting experiments on such currents.

The galvanometer needle was prevented by proper stops from moving in the direction in which it was urged by the steady current due to the battery, that is, a current from A to B. When the circuit $A\,B\,C$ was complete, and the battery circuit was broken at K_1 it was found that the needle showed a current in the direction from B to A.

The stops were then set so as to prevent the needle from returning to zero after deflection by the steady current from A to B, but at the same time leave it free to move still further in the direction of the deflection. When the circuit was completed it was found that the needle sustained a transient deflection.

A platinum wire substituted for the galvanometer glowed when the battery circuit was broken at K_1, although it did not glow under the steady current. Also by placing an electrolytic cell in $A\,B$, chemical decomposition was produced.

Faraday's Theory of Self Induction

430. It was thus shown that the induced currents caused in the circuit by putting in and throwing out the battery produced all the effects of ordinary currents. Faraday explained them by the magnetic induction through the circuit itself, produced by its own current induction. When the battery is thrown in, lines of magnetic induction are passed through the circuit and the number of these is greater the greater the number of turn of wire in the circuit, and is also increased by placing within the coil an iron core which becomes magnetized by the steady current.

When the circuit is broken these lines of induction disappear from the circuit. Thus an electromotive force in one direction is produced by the creation of the field, and an opposite one by the withdrawal of the field. In the former case the induced current caused is opposite to the steady current which is being set up, in the latter case it is in the same direction as the steady current which is being annulled.

The directions of these currents follow the rule given for currents of mutual induction by the law of Lenz. When induction is produced through a circuit the current produced is such as to oppose the action, approach of magnets or circuits (or creation of magnets by the starting of a current in the inducing circuit), and is therefore in the direction to diminish by its own induction that which is being produced by the external action.

Again when induction is being withdrawn the process may be regarded if we please as consisting in the insertion of opposite induction to the first so as to annul it. This is diminished by the induction due to the induced current.

This is precisely what took place in Faraday's experiments. At make a current opposite to the steady current beginning to flow was set up and caused a current to flow round the coil in the opposite direction, and therefore in the cross connection from A to B, the same direction as that in which the steady current would have flowed.

Henry's Experiments

431. Faraday's experiments were made in the year 1831, and as we have seen fully established the fundamental principles of the whole subject of current induction. Henry's investigations were made independently about ten years later, and his discoveries practically confirmed the conclusions of Faraday. Some curious points of apparent difference, however, existed between the results of these two pioneers of electrical discovery which have only been explained comparatively recently. These will be referred to in connection with some of Henry's more important experiments in the chapter on *The Experimental Verification of the Theory of Induction* in Vol. II. We shall here only

notice shortly[1] the points established by Henry in his earlier experiments.

These experiments were performed with coils of copper ribbon and helices of wire of various sizes and numbers of turns. For example to show effects of self induction, a circuit was made of a small battery and a flat coil of ribbon, by connecting the ends of the coil to mercury cups which formed the terminals of the battery. Then the experimenter, touching one of the battery terminals with one hand, broke the circuit by lifting one of the coil terminals out of the cup. Each time he did this he experienced a shock due to the induced current, and a flash took place at the mercury cup.

It was found that up to a certain limit when the length of the coil was increased the flash diminished in brilliance while the shock increased in intensity. When this limit was passed both flash and shock diminished.

432. These results were due to self-induction, and illustrated the combined effect of increase both of resistance and of inductive action. Lines of magnetic induction were thrown out of the coil by the alteration in the circuit, and an induced current flowed round the derived circuit formed by the human body and the coil as in Fig. 117. The diminution of the flash was due to the increased resistance of the circuit caused by the lengthening of the coil. The body, however, being of moderately great resistance, the gradual increase of resistance of the coil at first brought up the resistance of the circuit by an amount small in comparison with that by which the electromotive force was increased.

433. Experiments on mutual induction were also made by Henry. One of the most remarkable of these consisted in combining all his coils of ribbon into one large primary coil of about $5\frac{1}{2}$ feet in diameter, which was suspended vertically. A secondary four feet in diameter, made of a mile of copper wire $\frac{1}{16}$ inch in diameter of cross-section, was placed co-axially with the primary at a distance between them of a few feet. With a distance of three or four feet between the coils, and with a large-surface battery of eight elements, severe shocks were obtained by an experimenter placing his tongue between the terminals of the secondary, and breaking the primary. The shocks were quite perceptible when the coils were placed at much greater distances.

434. In connection with his mutual induction experiments, Henry made an important observation. By using a secondary of a few turns, and a primary of many turns in which was a battery of great electromotive force, he found that he could obtain an induced current of considerable amount but of low electromotive force. This result is that now achieved on a commercial scale in electric lighting operations by means of what is called a "step-down transformer."

Besides using physiological effects in order to detect induced currents Henry employed a horse-shoe of soft iron surrounded by a magnetizing coil, and tested the direction of the induced current by observing the nature of the magnetization produced. He found generally

[1] A more detailed account will be found in Fleming's *Alternate Current Transformer*, vol. i.

speaking that induced currents which gave slight shocks magnetized the soft iron and produced bright flashes, while those that gave severe shocks gave only slight flashes and feeble magnetization.

435. By means of the soft iron horse-shoe and experience of shocks, Henry experimented with currents produced in a tertiary coil, and even with induced currents of a higher order. The shocks experienced were true indications of the existence of such currents; the inference obtained from the direction of magnetization was apt to mislead. For one so-called tertiary current consisting, as we have seen, of two currents in opposite directions, it is obvious that the apparent direction as shown by the soft iron will depend on which of the currents is effective in producing the magnetization, and this depends, as we shall see later, on a variety of circumstances.

Application of Principle of Energy

436. A great step was taken in advance by v. Helmholtz, Lord Kelvin, and Joule in the study they made of the energetics of the voltaic circuit, and of electromagnetic action generally. Thus von Helmholtz and Lord Kelvin independently accounted for an induced electromotive force due to the motion of magnets or circuits by a reference to the theory of conservation of energy. Von Helmholtz's earliest expression of his views is contained in his famous essay on the "Conservation of Energy."[1] His discussion of the case of two circuits must however be regarded as imperfect, owing to neglect of the electrokinetic energy of the system. The correct solution was given by Lord Kelvin in the paper referred to in p. 342 below.

The foregoing sketch of the experimental basis of electromagnetic induction must suffice for the present. Many other investigations of great importance have been carried out; but most of these will be described in connection with their various subjects as these arise in the further discussion of electromagnetic theory.

437. It must now always be remembered that, according to Maxwell's theory and Hertz's experiments, all the phenomena of electric and magnetic induction are the results of the propagation of electric and magnetic induction at a finite speed through the medium filling the field. Among the investigations referred to above and to be described later, are those which have established this great generalisation. We shall see that under certain conditions the phenomena which present themselves are quite different from those we should expect to find if inductive effects were transmitted instantaneously, as it was long the habit to tacitly suppose. In the discussion of many ordinary phenomena however this supposition can generally be made with safety, and of course in many questions the element of time is without influence at all. It was this fact, no doubt, that prevented the earlier experimental verification of Maxwell's remarkable theory.

[1] *Die Erhaltung der Kraft.* Translated by Dr. Tyndall in Taylor's *Scientific Memoirs*, Part II., p. 114. Republished in Ostwald's *Classiker der exakten Wissenschaften.*

SECTION II.—*Dynamical Theory of Current Induction.*

438. We have found an expression [(33) of Art. 403 above] for the electrokinetic energy of a system of currents which fulfils the condition such an expression ought to fulfil, of giving as its rate of variation in a given direction, the observed force in that direction on a circuit or part of a circuit carrying a current. If this expression be really and truly the electrokinetic energy of the currents or that part of the energy on which the various phenomena of mutual action depend, it ought to fulfil the ordinary conditions of a material system, and be subject to the dynamical equations which hold in every case in which mutual action takes place between the bodies of such a system, that is, when the distribution of energy among the different bodies undergoes variation.

These dynamical equations come to the aid of the principle of conservation of energy, which (see Art. 237 *et seq.* above) is generally insufficient to enable us to account for the phenomena of a material system. They are the outcome of dynamical laws which themselves are the results of certain axioms, or postulated propositions the truth of which, or applicability of which for the formation of a system of dynamics depends on experience.

439. A well known and striking example of the inadequacy of the law of conservation of energy for the explanation of physical phenomena, and the necessity for having recourse to the results of experience to supplement it is to be found in the dynamical theory of heat. The first law of thermodynamics is simply the law of conservation of energy applied to a substance taking in and giving out heat, and doing work against external forces. But by itself this law does not give any properly so-called thermodynamic result, and we have to use in addition the famous second law, which has for its foundation a certain postulate the truth of which appears from experience.

So in electrical dynamics we have recourse to a dynamical theory established by Lagrange for the motions of molar matter, and by its power and exquisite flexibility admirably adapted for the subjugation to dynamical law of a new department of science. This application was first made fully and consistently by Clerk Maxwell, and there can be no doubt that this was one of the most important steps in advance ever made in electrical theory. Accordingly we have given in Chapter VII. above a sketch of Lagrange's dynamical method, and of some important investigations, by Lord Kelvin, Routh, Helmholtz, and others, connected with the application of the method to particular classes of problems which are analogous to those which are met with in general electromagnetic theory.

Electro-dynamics. Electrical Co-ordinates

440. The application of the dynamical theory of Lagrange to the solution of electrical problems and the interpretation of electrical phenomena is only possible in so far as we are able to follow the energy

changes of our system, for on this depends our power to build up an expression for the kinetic energy of the system which will give the proper applied or internal forces observed in various actual cases. The process consists in settling from the observed forces in different cases what terms must exist in T, and then deducing from the expression found not only the forces observed, which would not increase our information, but other forces which have not been observed. These can then be looked for, and if found increase our stock of knowledge of phenomena, and confirm the theory by which they were discovered.

441. In this application we have first to consider what terms enter into the expression for the total kinetic energy of a system of conductors in which currents are flowing. We shall have two sets of co-ordinates, one of electrical co-ordinates, the other independent entirely of the electrical or magnetic state of the system. These we shall denote by $\phi, \psi, \ldots., p, q, \ldots.$ respectively. Then T will be made up of three parts, one T_1 depending on the electrical co-ordinates alone, another T_2 depending on the co-ordinates $p, q, \ldots$ alone, and a third T_3 depending on these co-ordinates conjointly. Thus

$$\left.\begin{aligned} T_1 &= \tfrac{1}{2}\{(\phi, \phi)\dot{\phi}^2 + \ldots. + 2(\phi, \psi)\dot{\phi}\dot{\psi} + \ldots.\} \\ T_2 &= \tfrac{1}{2}\{(p, p)\dot{p}^2 + \ldots. + 2(p, q)\dot{p}\dot{q} + \ldots.\} \\ T_3 &= (\phi, p)\dot{\phi}\dot{p} + \ldots. \end{aligned}\right\} \quad . \quad . \quad (1)$$

There is no reason so far as we know to believe that T_3 exists, and in considering electrical phenomena we may disregard T_2, and confine our attention to T_1. Also T_1 is found to depend on only the velocities of co-ordinates, not the co-ordinates themselves.

The question now arises, what are to be considered electrical co-ordinates of a system of currents? In answering this question we are guided by the fact that the state of the system remains unchanged when the currents are all kept constant, and the arrangement and positions of the conductors in the field are unchanged. Thus we are led to take the currents in the several conductors as $\dot{\phi}, \dot{\psi}, \ldots.$ and to the conclusion that the co-efficients $(\phi, \phi), (\phi, \psi), \ldots$ depend only on the co-ordinates which fix the positions of the system.

442. In what follows we regard the magnetic energy as electro-kinetic energy, and we have already seen how it is measured. In some of our discussions we shall find it necessary to introduce the electric energy of the system if of sensible effect, and we may consider that as potential energy. (Whether we consider a quantity of energy as kinetic or potential altogether depends on our point of view and if a proper regard is paid to the signs of the expressions used, the same result will be reached on either hypothesis.) The tendency of scientific progress is to explain phenomena by the motion of matter, and the ordinary division of energy into potential and kinetic is likely sooner or later to be replaced by a more accurate classification. At present, potential energy is energy we are able to define by the position of

co-ordinates of the molar matter of our system: if we knew all about it we might be able to express it in terms of the velocities of particles of our system the co-ordinates of which are beyond our control or observation, the uncontrollable co-ordinates of Art. 263 above. From this point of view the ordinary transformation of potential into kinetic energy and *vice versa*, is only a process of re-distribution of kinetic energy between the different parts of the system.

So far as electrical phenomena are concerned we are quite unable to refer any portion of energy to the motions of particles of the system, that is to say we are ignorant of the connection between the generalised co-ordinates and those of the particles composing the system. Hence we may regard both kinds of energy, magnetic and electric, as kinetic if it suits our convenience.

It is very remarkable, however, that the generalised co-ordinates should, as the process of derivation above shows, be capable of connecting any properly known physical state with the motions of the particles of the system. To understand the nature of these connections we must first ascertain what are the equations (2) of Art. 240 above, by which the electrical co-ordinates are given in terms of the independent co-ordinates of the particles of matter composing the system. This in general we cannot do, and the problem may not be solved for a long time to come. Happily its solution is not necessary in order that we may intelligently use the dynamical method, by which to attain to a clearer understanding of the interrelations of observed phenomena.

Electrokinetic Energy. Current- or Electrokinetic Momentum

443. We have already seen that the electrokinetic energy is capable of being expressed in the form

$$T = \tfrac{1}{2}\{L_1\gamma_1^2 + 2M_{12}\gamma_1\gamma_2 + \ldots + L_2\gamma_2^2 + 2M_{23}\gamma_2\gamma_3 + \ldots\} \quad (2)$$

where $\gamma_1, \gamma_2, \ldots$ are the currents in the different circuits, and $L_1, L_2, \ldots$ $M_{12}, M_{23}, \ldots$ are their self and mutual inductances.

Differentiating with respect to the different currents we obtain the inductions through the different circuits. Denoting these by $N_1, N_2, \ldots, N_k, \ldots$ we obtain

$$\left.\begin{aligned}
N_1 &= \frac{\partial T}{\partial \gamma_1} = L_1\gamma_1 + M_{12}\gamma_2 + \ldots M_{1k}\gamma_k + \ldots \\
N_2 &= \frac{\partial T}{\partial \gamma_2} = M_{21}\gamma_1 + L_2\gamma_2 + \ldots M_{2k}\gamma_k + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
N_k &= \frac{\partial T}{\partial \gamma_k} = M_{k1}\gamma_1 + M_{k2}\gamma_2 + \ldots L_k\gamma_k + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right| \quad . \quad (3)$$

where $M_{kj} = M_{jk}$.

It is plain that $N_1, N_2, \ldots N_k, \ldots$ are the generalised components of momentum of the system, when current is regarded as a velocity. We shall call them the components of electrokinetic momentum for the sake of distinction from another quantity which perhaps is more properly called the magnetic momentum (see Art. 56 above).

444. To understand more fully the meanings of the quantities $L_1, L_2, \ldots M_{12}, M_{23}, \ldots$ let all the currents be zero except say γ_1. Then N_1 becomes $L_1\gamma_1$, N_2 becomes $M_{21}\gamma_1$, and so on. Thus L_1 is the magnetic induction produced through the first circuit by unit current in it and similarly for the others. On the other hand M_{21} is the magnetic induction through the second circuit (indicated by the suffix $_2$) produced by unit current in the first circuit, and in the same way M_{kj} is the induction through the k^{th} circuit produced by unit current in the j^{th}.

The expression for the kinetic energy shows that the induction through the j^{th} circuit due to the current in the k^{th} is equal to the induction through the k^{th} circuit produced by an equal current in the j^{th}, that is, that $M_{jk}=M_{kj}$. For the energy term which yields the induction is the same in both cases.

445. Applying now Lagrange's equations in the form (71) of Art. 262, taking E for the electric energy (energy of charged conductors) regarded as potential energy, and $y_1, y_2, \ldots$ as the electrical co-ordinates corresponding to $\gamma_1, \gamma_2, \ldots$, so that $\gamma_1=\dot{y}_1$, $\gamma_2=\dot{y}_2, \ldots$, remembering that T Art. 441 does not involve the electrical co-ordinates but only their velocities, we obtain for the equation of the k^{th} circuit

$$\frac{dN_k}{dt} + \frac{\partial \mathsf{E}}{\partial y_k} + \frac{\partial F}{\partial \gamma_k} = E_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where F denotes the dissipation function and E_k the proper impressed electromotive force. Now it is known from the experiments of Joule that the time-rate of dissipation of energy in any system of circuits has the value

$$R_1\gamma_1^2 + R_2\gamma_2^2 + \ldots + R_k\gamma_k^2 + \ldots$$

Hence the dissipation function is given by the equation

$$F = \tfrac{1}{2}(R_1\gamma_1^2 + R_2\gamma_2^2 + \ldots + R_k\gamma_k^2 + \ldots) \quad . \quad . \quad (5)$$

and (14) becomes

$$\frac{dN_k}{dt} + \frac{\partial \mathsf{E}}{\partial y_k} = E_k - R_k\gamma_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The quantities $R_1, R_2, \ldots R_k, \ldots$ are called the resistances of the circuits. They are in fact the rates of dissipation of energy in the different circuits per unit of the current flowing in each case.

The quantities $y_1, y_2, \ldots, y_k, \ldots$ are the charges of the conductors, and we have seen, Art. 186 above, that E is a homogeneous quadratic function of these quantities.

The force E_k is called the "electromotive force" in the circuit to which it belongs, and (6) asserts that if E be zero the rate of increase of electrokinetic momentum is equal to the excess of the electromotive force over that required to overcome the dissipative resisting force. In other words, if the electrokinetic momentum N_k is undergoing change an electromotive force acts in the circuit opposing the resisting force which causes dissipation of the electrokinetic energy.

It is to be observed that while we speak of the energy of a circuit, the energy referred to is part of the energy of the field. When the energy is dissipated in the circuit it must travel from the field into the conductor across its lateral surface. We shall see later how this flow takes place.

Case of two mutually influencing Circuits

446. Consider now in particular a system made up of two circuits. Let the currents in them be γ_1, γ_2, the inductances L_1, L_2, M, the electromotive forces E_1, E_2, and the resistances R_1, R_2, and let for the present E be zero. Then

$$T = \tfrac{1}{2}(L_1\gamma_1^2 + 2M\gamma_1\gamma_2 + L_2\gamma_2^2).$$

Applying (6) we have

$$\left.\begin{aligned} E_1 - \frac{d}{dt}(L_1\gamma_1 + M\gamma_2) &= R_1\gamma_1 \\ E_2 - \frac{d}{dt}(L_2\gamma_2 + M\gamma_1) &= R_2\gamma_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

where $L_1\gamma_1 + M\gamma_2$ replaces N_1, and $L_2\gamma_2 + M\gamma_1$ replaces N_2.

We see from these results that here, and also in the general case, dN_k/dt is the measure of the applied inductive electromotive force, which is employed in increasing the current momentum. The reaction against this force $-dN_k/dt$ is the actual inductive electromotive force given by the circuit itself, which therefore opposes the increase of current momentum.

This may be seen more clearly by supposing the circuits left without impressed electromotive force, that is by supposing $E_k = 0$. Then the inductive electromotive force will be $-dN_k/dt$, and will be equal to $R_k\gamma_k$.

Also the applied electromagnetic forces are $-\partial T/\partial x_1, -\partial T/\partial x_2$, etc., and therefore the electromagnetic forces these work against must be $\partial T/\partial x_1$, $\partial T/\partial x_2$, etc. The mutual electromagnetic forces between the different circuits are thus equal to the space rates of increase of the electrokinetic energy.

447. Suppose, for example, two circuits to be left to themselves in presence of one another. Let the circuits be rigid in form so that L_1, L_2 do not change, while of course M changes in consequence of the

displacement which takes place on account of their mutual action. Let dT be the change of T which takes place in a small interval of time dt; then

$$dT = L_1\gamma_1 d\gamma_1 + L_2\gamma_2 d\gamma_2 + M(\gamma_2 d\gamma_1 + \gamma_1 d\gamma_2) + \gamma_1\gamma_2 dM \quad (8)$$

If now a co-ordinate x be used to fix the relative positions of the circuits, a change dx in x will have taken place in the same time, and an amount of work done by mutual electromagnetic forces which has the value $\partial T/\partial x.dx$.

Calling this dW we see at once that

$$dW = \gamma_1\gamma_2 dM.$$

This work is spent in producing ordinary molar kinetic energy in the conductors, or if these are acted on by external forces in doing external work, or in both ways.

The impressed electromotive forces do work, over and above that dissipated, of amount

$$(E_1 - R_1\gamma_1)\gamma_1 dt + (E_2 - R_2\gamma_2)\gamma_2 dt$$

which by (7) has the value

$$L_1\gamma_1 d\gamma_1 + L_2\gamma_2 d\gamma_2 + M(\gamma_1 d\gamma_2 + \gamma_2 d\gamma_1) + 2\gamma_1\gamma_2 dM = dT + dW,$$

so that the whole of the energy is accounted for.

448. Analysing what has taken place, we see that the impressed electromotive forces working against the re-acting inductive electromotive forces in the circuits do an amount of work $dT+dW$. The first part is done in increasing the electrokinetic energy, and the second in displacing the circuits. The alteration of the electrokinetic energy is made up of two parts, namely the first four terms of (8), and the last term respectively. The first is the work done in increasing the currents, the second the increase of kinetic energy arising from the displacement. Thus the impressed electromotive forces furnish over and above that required for the changes in the currents, an amount of work $2\gamma_1\gamma_2 dM$, one-half of which goes to increase the electrokinetic energy, the other to furnish the work done against electromagnetic forces.

If the circuits move from rest to rest again, the changes in the currents are zero, since at the beginning and end of the changes $E_1 = R_1\gamma_1$, $E_2 = R_2\gamma_2$, and the *whole* energy furnished by the batteries is $2\gamma_1\gamma_2 dM$, of which one-half is accounted for in dT and the other in dW, the work done in moving the conductors against the external forces by which they are brought to rest.

This very important result was arrived at by Lord Kelvin in 1851. (*Electrostatics and Magnetism*, 2nd Ed., p. 446.)

449. Faraday's experiments and the conclusions of the dynamical theory are, as we have seen, in complete accordance. Both concur in giving inductive electromotive forces proportional to the rate of

variation of the induction through the circuit, whether the change is due to variation of the strength of the current or to displacement of the circuits. Hence we shall regard equations (7) as established, and proceed to develop their consequences. The agreement of these with the results of verifying experiments affords further proof of the theory.

Unit Electromotive Force and Unit Resistance—Volt, Ohm, Ampere, &c.

450. We now define unit electromotive force as that set up in a conductor when it cuts across one unit tube of magnetic induction per unit time; or supposing the magnetic induction reckoned in C.G.S. units as explained at Art. 23 above, the C.G.S. unit of electromotive force would exist in a conductor 1 centimetre long with its length perpendicular to the lines of induction in a uniform magnetic field of induction 1 C.G.S. unit, moving in a direction at right angles to itself and to the induction with a speed of 1 centimetre per second.

Or, alternatively, the C.G.S. unit of electromotive force will exist in a circuit the surface integral of magnetic induction (what has been called above the *total* induction), through which is altered per second by 1 C.G.S unit.

The electromotive force in the circuit of a Daniell's cell is about 5 per cent. more than 10^8 such units, and the practical unit of electromotive force is taken as 10^8 C.G.S. units, and is called 1 *volt*.

The unit current in this system of units (commonly called the electromagnetic system) has been defined in Art. 369 above. We define, then, by means of the relation connecting electromotive force, current, and resistance, unit resistance as the resistance of a circuit in which, when the electromotive force in it is unity, the current is also unity. The circuit thus has 1 C.G.S. unit of resistance when the electromotive force and the current in it are both unity.

This unit of resistance is small in comparison with the resistances which have to be numerically expressed in practice, and the multiple 10^9 of it is taken as practical unit of resistance, and called 1 *ohm*.

The C.G.S. unit of current also differs from the practical unit of current, which is the current in a circuit of resistance 1 ohm, and containing an electromotive force of 1 volt. It is called a current of 1 *ampere*.

If a circuit carries a current of γ C.G.S. units the energy of the current is $\frac{1}{2}L\gamma^2$ in ergs if there be no mutual inductance to be taken into account. If the current be unity, and L be 1 C.G.S. unit, the energy is half an erg. L then is 1 C.G.S. unit for a circuit through which a current of 1 C.G.S. unit produces a surface integral of magnetic induction of 1 C.G.S. unit.

Again let 1 C.G.S. unit of current in a circuit A produce a surface integral of magnetic induction amounting to 1 C.G.S. unit through another circuit B, then we know that the same current in B would produce just 1 C.G.S. unit of total induction through A. The mutual inductance of the two circuits is then 1 C.G.S. unit.

An inductance of 10^9 C.G.S. units of inductance is called 1 *henry*.

The quantity of electricity conveyed per second by a current of 1 ampere, is called 1 *coulomb*. The capacity of a conductor which is charged to a potential of 1 volt by 1 coulomb when all other conductors in the field are at zero potential is called 1 *farad*, a *microfarad* is $1/10^6$ of a farad.

These and other units will be further discussed in the chapter on *Dimensions of Units*.

Maxwell's Dynamical Illustration

451. A remarkable dynamical illustration of the equations of currents was given by Clerk Maxwell, and another somewhat similar has more recently been described by Lord Rayleigh. Maxwell's apparatus is shown in Fig. 118. On a horizontal axis is fixed a vertical bevel toothed wheel a. Parallel to this, but fixed to a loose sleeve on the small axle, is an equal bevel toothed wheel b. Between is a third wheel c equal to the others, and gearing with them. This wheel turns round a long bar, which is rigidly fixed to the axle of the wheel a. The bar thus turns with the wheel a, and the centre of the wheel c turns with the cross-bar, while the wheel itself may turn round the bar. The bar carries weights, C, C, which can be fixed at any chosen distance from the centre of the bar, so as to increase or diminish the moment of inertia of the bar, which is supposed to be great in comparison with that of any other part of the apparatus.

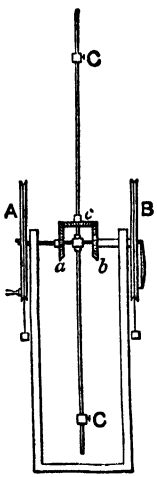


FIG. 118.

Two large pulleys A, B fitted with rope brakes are fixed A to the axle of a, and B to the sleeve carrying b. We shall speak of the "wheel A" as the whole system rigidly connected with the pulley A, with the exception of the cross-bar and weights, and of all rigidly connected with the sleeve as "the wheel B."

452. In order to find the equations of motion of this system according to Lagrange's method, we must first calculate the kinetic energy. Let a be the radius of each of the circles of contact of the wheels a, b, and c, ω_1, ω_2 angular velocities of a and b in the same direction. The tangential velocities of the two points of contact are $\omega_1 a, \omega_2 a$, and the arithmetic mean of these is the tangential velocity of the centre of the wheel c along the circle in which it moves. Further the angular velocity of the wheel c round the cross-bar is $\frac{1}{2}(\omega_1 - \omega_2)$, since the tangential velocity of one end of the diameter through the points of contact is $\frac{1}{2}(\omega_1 - \omega_2)a$ relatively to the other end.

We now denote the moments of inertia of the wheels a, b, and the cross-bar with all attachments, by $m_1 k_1^2, m_2 k_2^2, mk^2$, and that of the wheel

c round the cross-bar by $m_3k_3^2$. The kinetic energy T is then given by the equation

$$T = \tfrac{1}{2}\{(m_1k_1^2 + \tfrac{1}{4}m_3k_3^2 + \tfrac{1}{4}mk^2)\omega_1^2 + (m_2k_2^2 + \tfrac{1}{4}m_3k_3^2 + \tfrac{1}{4}mk^2)\omega_2^2 + \tfrac{1}{2}(mk^2 - m_3k_3^2)\omega_1\omega_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

found by calculating the kinetic energies of the different parts of the system and adding them together. This may be put in the form

$$T = \tfrac{1}{2}(L_1\omega_1^2 + 2M\omega_1\omega_2 + L_2\omega_2^2) \quad . \quad . \quad . \quad . \quad (10)$$

which is precisely that of the electrokinetic energy of a pair of mutually influencing circuits.

The kinetic energy does not depend on the angles through which the wheels have turned, but only on the angular velocities; and thus the machine forms a good example of a cyclic system (Art. 269 above) with three independent cyclic velocities $\omega_1, \omega_2, \omega_3$.

Here the wheels A and B correspond to the two circuits, while L_1, L_2, M, represent their self and mutual inductances. The rotating arms and attached masses (as well as the wheels A and B themselves), in which the energy T is stored, represent the medium in which the circuits are situated, through which their mutual action is propagated, and which is the vehicle and store of electrokinetic energy.

Let resisting forces be applied by the brakes to the wheels A and B proportional to the angular velocities ω_1, ω_2 respectively, and let the external couples applied to the wheels be Θ_1, Θ_2. Then Lagrange's equations for the two wheels, obtained by differentiating T, are

$$\left.\begin{aligned} \frac{d}{dt}(L_1\omega_1 + M\omega_2) &= \Theta_1 - R_1\omega_1 \\ \frac{d}{dt}(L_2\omega_2 + M\omega_1) &= \Theta_2 - R_2\omega_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

since the kinetic energy does not involve the positions of the wheels A, B. These equations are precisely the same as those of current induction for two circuits (7) above.

453. The forces in these equations have a simple interpretation. For example, $Md\omega_2/dt$ is the applied couple or generalised force on A, rendered necessary by the acceleration $d\omega_2/dt$ in B, that is, if B moves with this acceleration a force of this amount must be applied to A to keep it from moving.

The model may be made to illustrate the transient induced currents at make and break of the circuits. Take the case of two circuits which we may call a primary and a secondary, and let there be no applied electromotive force E_2 in the latter. Then, to correspond, Θ_2 must be made zero in the equations of the model. Let now a couple Θ_1 be

applied to A so as to start the system from rest. At the beginning ω_2 is zero, and the equation of the wheel B gives

$$L_2 \frac{d\omega_2}{dt} + M\frac{d\omega_1}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

That is, the angular acceleration of the wheel B is opposite to that of A. Thus ω_2 acquires a negative value and $-R_2\omega_2$ is positive. Therefore ω_2 increases in numerical value so long as $M d\omega_1/dt$ is greater than $R_2\omega_2$ in numerical value, and increases fastest at first, since then $\omega_1 = 0$, and $d\omega_1/dt$ has its greatest value.

454. The further changes can best be studied by integrating the equations, but this we shall do later for the electrical equations. We shall see that if Θ_1 be kept constant, ω_2 will rise to a negative maximum, then die away to zero, while ω_1 approaches a constant value. The wheel A will then rotate steadily, while B does not move.

All this is paralleled by the rise of the current in the primary, when an electromotive force is applied to that circuit, while none exists in the secondary. At the instant of making the primary an inverse current begins to flow in the secondary, rises to a maximum, and dies away again to zero, as the current on the primary approaches its steady value.

During the variable period the cross-bar and the wheel A are getting into rotation, and acquiring a store of electrokinetic energy. Similarly in the case of the circuits energy is being thrown out into the medium, and a store of electrokinetic energy is there accumulated, which can only be recovered in part or in whole by varying or stopping the current.

455. What takes place when the primary is broken can also be traced from the machine. After A has attained its steady state let it be retarded. The equations show that then $d\omega_1/dt$ being negative, and ω_2 zero, $d\omega_2/dt$ is positive, and B goes forward in the direction in which A is moving. Its forward velocity increases to a maximum, and then falls off as that of A dies away until finally both wheels are at rest.

So when the primary circuit is broken. There is an electromotive force in the secondary which is greater the greater the rate of variation of the current in the primary, that is the more sudden the break, and causes a current to flow in the direction of that in the primary. Further the electromotive force in the secondary depends on the arrangement of the circuits and their surroundings, so as to suggest a store of energy in the medium analogous to that of the motion of the cross and attachments.

456. Further if R_2 be so great that when ω_2 is small $R_2\omega_2$ is considerable, then by (12) when the couple Θ_1 is applied to A, the forward acceleration $d\omega_1/dt$ of A is small, since the negative acceleration $d\omega_2/dt$ is likewise small. The reason is obvious. The wheel B is retarded

by the brake, and the wheel A if it turns must carry with it the cross and attachments the moment of inertia of which is great.

On the contrary when the wheel A is suddenly stopped the stress applied to B will be very great and it will be moved forward with great acceleration.

This is the case of a high resistance or air-gap in the secondary. A spark is prevented when the circuit is completed, while a spark several inches in length may be produced if M be large enough by suddenly breaking the circuit of the primary. This is the action of the inductorium, or Ruhmkorff Coil (see below, Art. 489).

457. The mechanism can also be made to illustrate the action of a secondary circuit, which contains a Leyden jar. Let an arm attached to B bear upon a spring attached to the framework. As the wheel turns the spring is deflected, until at a certain deflection when the wheel has turned through a certain angle the spring is released. Thus if A has sufficient acceleration the wheel B will continue to deflect the spring until the latter slips and recoils, while B runs on, to come round and repeat the same operation as long as there is sufficient acceleration in A.

The charging of a Leyden jar is analogous to the bending of the spring, a spark to the slip. The capacity of the jar corresponds to the amount of bending. If the capacity of the jar is very great no spark may take place, but may discharge backwards through the secondary. This is precisely similar to the case in which the deflection the spring can take is so great as to prevent slipping. The wheel B gradually comes to rest, and then is brought back by the recoil of the spring.

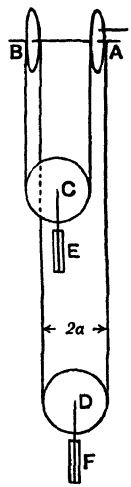


FIG. 119.

Lord Rayleigh's Dynamical Illustration

458. Lord Rayleigh's mechanical model is shown in Fig. 119. It consists of a pair of wheels A, B, loose on a horizontal spindle, over which is passed an endless cord, carrying two equal movable pulleys in the bights, with attached weights as shown. The weights are equal and so the system has no variable potential energy, since through whatever height one of the weights is raised, the other descends through the same.[1] Resisting forces can be applied to A and B as before.

The kinetic energy has the same form of expression as before, and the same analogies hold and are illustrated by the equations of motion. It will be a good exercise for the reader to form these equations and work out their consequences in the

[1] This ingenious apparatus was invented by Huyghens for driving clocks. Then one weight is heavier than the other, and one of the wheels, A say, driving the clock, the driving weight can be raised by pulling the smaller weight down by the cord passing over B. The driving power is not taken off the clock, and no "maintaining power" is needed.

motion of the weights. We shall deal with other dynamical illustrations of electromagnetic action when we come to general electromagnetic theory.[1]

459. In all these cases it has been assumed that the loss of energy from the circuits has been due to heat dissipation. We shall see later how energy can be spent in electrical circuits in doing useful work by the action of electromagnetic forces. But it ought to be borne in mind that in this theory no account is taken of radiation of energy, which undoubtedly takes place whenever a variation in an electric circuit takes place. For example, alternating machines in ordinary working generate electrical waves in the medium of 800 or 1,000 miles in length. Unless the frequency of alternation is very great the loss of energy from such causes is negligible. The whole subject will be discussed later.

Mutual Action of two Invariable Circuits

460. We shall now give in detail the solution of the problem of two mutually influencing circuits which are invariable in form and position. The results will be at once comparable with those of the dynamical problem just dealt with. The values are, since L_1, L_2, M are assumed constant,

$$\left.\begin{aligned} L_1\frac{d\gamma_1}{dt} + M\frac{d\gamma_2}{dt} + R_1\gamma_1 &= E_1 \\ L_2\frac{d\gamma_2}{dt} + M\frac{d\gamma_1}{dt} + R_2\gamma_2 &= E_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

To solve these we proceed in the ordinary way: we first eliminate γ_2, then γ_1, and solve the resulting equations for γ_1, γ_2. This is done most simply by separating the symbols, and grouping into one operator all that act on the same quantity. Thus we get the equations

$$(L_1\frac{d}{dt} + R_1)\gamma_1 + \quad M\frac{d}{dt}\gamma_2 - E_1 = 0$$

$$M\frac{d}{dt}\gamma_1 + (L_2\frac{d}{dt} + R_2)\gamma_2 - E_2 = 0.$$

Hence operating on the first of these by $L_2d/dt + R_2$, and on the second by Md/dt, and subtracting, we eliminate γ_2 and obtain

$$(L_1L_2 - M^2)\frac{d^2\gamma_1}{dt^2} + (L_2R_1 + L_1R_2)\frac{d\gamma_1}{dt} + R_2(R_1\gamma_1 - E_1) = 0 \quad (14)$$

Since E_1 is a constant we may write $\gamma_1 - E_1/R_1$ for γ_1 everywhere,

[1] A very interesting dynamical model has been invented by Boltzmann, and is described in his *Vorlesungen über Maxwell's Theorie, Zweite Vorlesung.*

and hence by the ordinary rule for the solution of linear differential equations with constant co-efficients we get

$$R_1\gamma_1 - E_1 = A_1e^{\alpha t} + B_1e^{\beta t} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where A_1, B_1 are constants, e the base of the Naperian system of logarithms, and α, β are the roots of the quadratic.

$$(L_1L_2 - M^2)x^2 + (L_2R_1 + L_1R_2)x + R_1R_2 = 0 \quad . \quad . \quad (16)$$

that is have the values

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = -\frac{1}{2(L_1L_2 - M^2)} \cdot \{L_2R_1 + L_1R_2 \pm \sqrt{(L_2R_1 + L_1R_2)^2 - 4R_1R_2(L_1L_2 - M^2)}\} \quad (17)$$

Similarly by eliminating γ_2 we get

$$R_2\gamma_2 - E_2 = A_2e^{\alpha t} + B_2e^{\beta t} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where α, β have the same values as before, and A_2, B_2 are other constants.

461. The values of γ_1, γ_2, given by these equations for any time t, depend on the constants A_1, B_1, A_2, B_2, which must be determined to suit the given circumstances of the case.

The quantities α, β are the same in all circumstances, depending as they do on the form and dimensions only of the circuits. That they are real may be seen at once thus:—The roots of the quadratic (27) are real if

$$(L_1R_2 + L_2R_1)^2 > 4(L_1L_2 - M^2)R_1R_2$$

that is if

$$(L_1R_2 - L_2R_1)^2 > -4M^2R_1R_2$$

which is obviously true, since the quantity on the left is positive, while that on the right is essentially negative.

It is necessary further, in order that it may be impossible for either current to become infinite that both α and β be negative. This involves the inequality

$$(L_1R_2 + L_2R_1)^2 > \{(L_1R_2 + L_2R_1)^2 - 4(L_1L_2 - M^2)R_1R_2\}$$

which is true if $L_1L_2 > M^2$, that is, if the mutual inductance of two circuits is less than the geometric mean of their self inductances. This is obvious from the energy equation, or from the lines of induction of a unit current flowing in either circuit. These lines all pass through their own circuit, but clearly do not all pass through the other. The other circuit however may consist of n turns, and hence $M < nL_1$. Again, a unit current flowing in the second circuit gives a total induction through it of L_2, and all these do not pass through the first, and, as we suppose, the second circuit has the greater effective area. Thus

$M < L_2$. But if L'_2 be the average inductance of a single turn of the second coil, $L_2 = n^2 L'_2$, since the lines due to each turn give an induction nL'_2, and there are n turns. But clearly also $M < nL'_2$, that is, $M < L_2/n$. Hence $M^2 < nL_1L_2/n$, that is, $M^2 < L_1L_2$.

462. Let now both circuits be closed at the same instant, which is taken as the zero of reckoning for t. Thus for $t = 0$, $\gamma_1 = 0$, $\gamma_2 = 0$, and we get from (15) and (18) $-E_1 = A_1 + B_1$, $-E_2 = A_2 + B_2$.

Hence equations (15) and (18) become

$$\left.\begin{aligned} R_1\gamma_1 &= E_1(1 - e^{\beta t}) + A_1(e^{\alpha t} - e^{\beta t}) \\ R_2\gamma_2 &= E_2(1 - e^{\beta t}) + A_2(e^{\alpha t} - e^{\beta t}) \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

The coefficients A_1, A_2 remaining in these equations could be determined by calculating $d\gamma_1/dt, d\gamma_2/dt$ for the epoch $t = 0$, and substituting in the differential equations. We shall find it more convenient however to determine the constants to suit the particular circumstances of the cases to which equations (7) are applied.

For example, consider a secondary in which there is no impressed electromotive force E_2, and let both circuits be closed at the same moment, or, which comes to the same thing, let the secondary circuit be kept closed, while the primary is made or broken. We easily get by the process described:

$$\left.\begin{aligned} A_1 &= \frac{E}{\alpha - \beta}\left\{\beta + \frac{L_2R_1}{L_1L_2 - M^2}\right\} \\ A_2 &= -\frac{E}{\alpha - \beta}\frac{R_2M}{L_1L_2 - M^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

463. The variation of the primary and secondary currents is illustrated in Fig. 120, which is a copy of a cut drawn to scale from an actual case,[1] in which an electromotive force of 100 volts is applied to a primary circuit of resistance 10 ohms and self-inductance ·05 henry, between which and a secondary of resistance 5 ohms and self-inductance ·4 henry, the mutual inductance is ·02 henry.

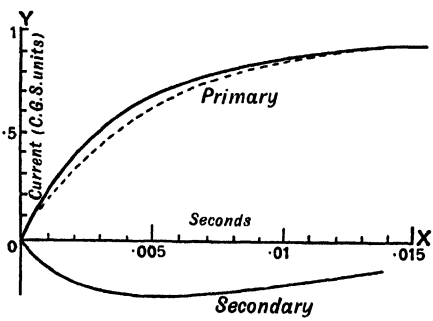


FIG. 120.

The primary current rising from zero to its steady value is shown above the line together with the induced current, that is the difference between the steady current and the actual current. The induced current in the secondary is shown below the line.

[1] From *Alternate-Current Working* by Alfred Hay, B.Sc. London, Biggs & Co.

It is to be noticed that while the current in the primary gradually increases towards its steady value the current in the secondary rises to its maximum and then falls off towards zero as the current in the primary becomes constant. The dotted curve in Fig. 120 is the curve of rise of current for this case on the supposition that there is no mutual inductance. It appears (and the reader may easily satisfy himself that it is so from the equations above) that the effect of mutual inductance is to make the rise of the current in the primary more rapid at first, and afterwards to retard the rise, as the dotted curve if continued would cross the full curve.

464. The time in which the current rises to its maximum can be calculated by finding the value of t for which $d\gamma_2/dt$ is zero. Differentiating the second of (19) and putting $d\gamma_2/dt = 0$, we find

$$t = \frac{1}{\alpha - \beta} \log \frac{\beta}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From the values of α, β given in equation (17) it is clear that t has its least value when $M^2 = L_1 L_2$. This is approximately the case when the primary and secondary coils are equal, and as nearly as possible coincident. Then $L_1 = L_2 = M$.

When the condition $M^2 = L_1 L_2$ is fulfilled the value of $d\gamma_2/dt$ when $t = 0$, that is $-EM/(L_1 L_2 - M^2)$, is infinite, and the current takes at once its maximum value. It is needless to say that this condition is never really fulfilled, as M^2 must always be sensibly less than $L_1 L_2$.

465. The march of the secondary current at break will be discussed presently. We shall first find the whole quantity of electricity which flows through the secondary at make. By equation (19) above we have

$$R_2 \int_0^\infty \gamma_2 dt = A_2 \int_0^\infty (e^{\alpha t} - e^{\beta t}) dt = A_2 \frac{\alpha - \beta}{\alpha\beta} \quad . \quad . \quad (22)$$

Using the value of A_2 given in (20) and noticing that, by the quadratic of which α, β are the roots, $\alpha\beta = R_1 R_2/(L_1 L_2 - M^2)$, we get

$$\int_0^\infty \gamma_2 dt = -\frac{E}{R_1} \frac{M}{R_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The current flows, theoretically, for an infinite time before it has become steady, but in any actual case the whole variable stage due to induction does not last sensibly beyond a fraction of a second.

This result has been obtained, by the process adopted here, as an example of the use of general integrals obtained for the case specified in which currents are started in both circuits at the same instant. But

it can be much more easily deduced from the second differential equation of (24). Thus integrating over the variable period we find

$$M\gamma_1 + L_2\int_0^\infty \frac{d\gamma_2}{dt}\,dt + R_2\int_0^\infty \gamma_2 dt = 0,$$

and the current γ_2 being zero both when $t = 0$ and when $t = \infty$ the first of these integrals is zero, and we have

$$\int_0^\infty \gamma_2 dt = -\frac{M}{R_2}\gamma_1 = -\frac{E}{R_1}\frac{M}{R_2} \quad . \quad . \quad . \quad . \quad (24)$$

the result already given above.

The same process gives at once for the total quantity of electricity which passes in the secondary circuit when the primary is broken the same result with opposite sign, viz.:

$$\int_0^\infty \gamma_2 dt = \frac{E}{R_1}\frac{M}{R_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

For the primary current is initially γ_1, and finally zero, while γ_2 is both initially and finally zero.

Thus exactly as much electricity passes through the secondary at break of the primary as passes at make, but the quantities are opposite in sign. This has been verified by direct experiment, and affords strong evidence of the correctness of the theory from which the result has been shown to flow.

466. It is interesting to study the march of the current in the secondary circuit (see Fig. 120). First suppose the secondary circuit to be kept closed, while the primary is broken. Let the variable stage of the primary current extend over a time τ, then this is called the duration of the break. Then integrating over this period the differential equation of the secondary circuit we get

$$-M\gamma_1 + L_2\gamma_2 + R_2\int_0^\tau \gamma_2 dt = 0$$

or

$$\gamma_2 = \frac{M}{L_2}\gamma_1 - \frac{R_2}{L_2}\int_0^\tau \gamma_2 dt \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If the time τ be very short the integral on the right will have a very small value, and may be neglected. Strictly speaking the break is

never instantaneous, but in a sharp break, we may say that the current in the secondary rises very quickly to the value $M\gamma_1/L_2$ nearly, and then gradually dies away. The mode of variation is illustrated by Fig. 121.

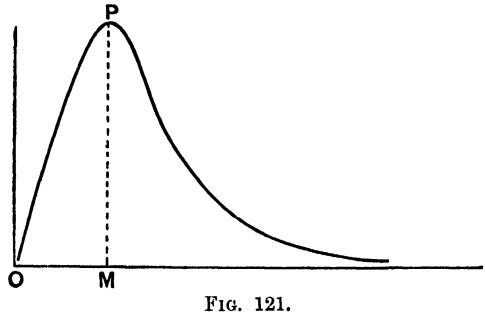


FIG. 121.

The duration τ of the break is OM, and the ordinate MP is approximately $M\gamma_1/L_2$.

The energy initially is $\frac{1}{2}L_2M^2\gamma_1^2/L_2^2 = \frac{1}{2}M^2\gamma_1^2/L_2$, and at any time when the current is γ_2 is $\frac{1}{2}L_2\gamma_2^2$. The rate at which the energy is dissipated in heat in the circuit is $R_2\gamma_2^2$. Thus we have

$$-\frac{d}{dt}(\tfrac{1}{2}L_2\gamma_2^2) = R_2\gamma_2^2$$

or

$$L_2\frac{d\gamma_2}{dt} + R_2\gamma_2 = 0.$$

Integrating this and remembering that when $t = 0, \gamma_2 = \gamma_1 M/L_2$ we find

$$\gamma_2 = \frac{ME}{L_2R_1}e^{-\frac{R_2t}{L_2}} \quad \ldots\ldots\ (27)$$

which shows how the current dies away in the secondary.

It is to be understood that if the primary or secondary circuit or both consist of coils surrounding iron cores, the march of the induced currents is very different from that studied and illustrated here. The inductances are no longer constant, but vary with the current. For results in such cases see a paper by T. Gray, *On the Measurement of the Magnetic Properties of Iron*, Phil. Trans. R.S. 184 (1893), A.

Single Circuit with Self-Inductance

467. We can easily investigate the theory of a single circuit with self-inductance. We have only to take one of equations (13), putting $M = 0$. Dropping suffixes we get for the equation of currents

$$L\frac{d\gamma}{dt} + R\gamma = E \quad \ldots\ldots\ (28)$$

Integrating we find

$$\gamma = \frac{E}{R} + Ae^{-\frac{R}{L}t}.$$

When $t = 0$, $\gamma = 0$, and therefore we must put $A = -E/R$. Thus the equation of current is

$$\gamma = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The rise of the current with time is shown in Fig. 122. The curve terminates in the Fig. at the value of the current after the lapse from the

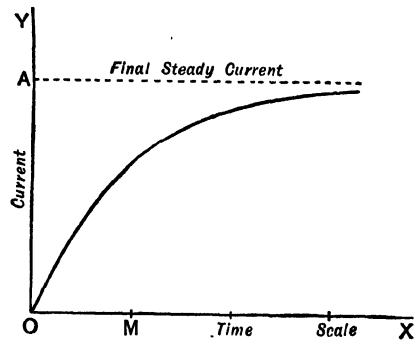


FIG. 122.

closing of the circuit of about three and a half times the time-constant L/R, which is represented by OM.

The part $Ee^{-Rt/L}/R$ is the extra or induced current, and dies away to zero, as the total current attains its steady value E/R. The whole quantity of electricity which passes in the induced current at make can be found at once by integration. Let q be this quantity, then

$$q = -\frac{E}{R}\int_0^\infty e^{-\frac{R}{L}t}\,dt = -\frac{EL}{R^2} \quad . \quad . \quad . \quad . \quad (30)$$

The quantity of electricity passing in the break is also easily ound. To render the problem definite let the battery be thrown out at a given instant, and an equivalent resistance be introduced. The current circulating is $\gamma_0 (= E/R)$ at the beginning, and at any subsequent stage in the variable state has a value γ. The rate of loss of electrokinetic energy is $-L\gamma d\gamma/dt$, and the rate of dissipation is $R\gamma^2$. Equating these two rates we get $Ld\gamma/dt + R\gamma = 0$, which of

course might have been obtained by putting $E=0$ in equation (28). Thus we have

$$\int_0^\infty \gamma dt = -\frac{L}{R}\int \frac{d\gamma}{dt}dt = \frac{L\gamma_0}{R} = \frac{EL}{R^2} \quad . \quad . \quad . \quad (30')$$

that is, the quantity passing at break in the induced current is the same in amount but opposite in sign to that which passes at make.

The equality of these quantities of electricity is independent of any want of uniformity of distribution of the current over the cross-section owing to rapidity of variation of the current. (See Chap. XI.)

The current at any instant after the removal of the electromotive force is given by the equation

$$\gamma = \frac{E}{R}e^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which is illustrated by Fig. 123, the ordinates in which show values of γ with successive values of t. OM is the value of L/R, the so-called time-

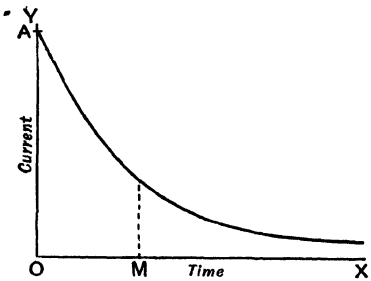


FIG. 123.

constant. This is the interval in which the current falls to $1/e$ of its initial value. From equations (29) and (31) it is evident that the curves in Figs. 122, 123 are the same, but differently placed with respect to the axes. The difference between OA and any ordinate of Fig. 122 is the corresponding ordinate of Fig. 123.

Theory of a Network of Conductors

468. As another example of the theory of current induction we may take a set of conductors, whether or not containing electromotive forces, joined so as to form a network. The dynamical equations are at once applicable to such a system, just in the same way as to a system of complete circuits, provided we use instead of resistances, inductances, and electromotive forces in circuits, the resistances, inductances, self and mutual, of the conductors, and the impressed differences of potential between their terminals.

The two fundamental principles stated in Art. 224 above, and

applied to the case of steady flow, are applicable also to this more general problem. The first, the principle of continuity, requires no modification, the statement of the second principle requires to be changed in the manner indicated below.

There is a difficulty, however, in deciding just what is the self-inductance of a conductor joining two points in a circuit, or the mutual inductance of two such conductors in the same or in different circuits. Happily, however, there is no real practical difficulty, as in most cases the conductors to be considered are *coils*, which may be regarded as each so many complete circuits given in position and dimensions by the turns of wire. The total magnetic induction through each such turn of wire is quite definite and can be calculated, and different methods lead to the same result.

469. The difficulty here alluded to is apparently avoided by the use of the cycle method of dealing with a network described in Art. 230 above. Any network of conductors is regarded as made up of a series of meshes or cells, as in the arrangement shown in Fig. 124, which consists of three distinct meshes *ADCA*, *ABDA*, *CDBC*. Each individual conductor is common to two meshes, except those conductors which form the outer edge of the network. Maxwell supposed a current to circulate round each mesh in the same direction, so that the actual current in each conductor was made up of the difference of the currents in two adjoining meshes. Each mesh is from this point of view regarded as a complete circuit with its own current flowing round it and the self and mutual inductances of the system are quite definite, being those of the distinct circuits formed by the meshes.

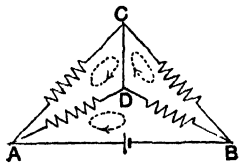


FIG 124.

470. There is no difficulty in writing down the electrokinetic energy and finding the equations of motion from either point of view. If in Maxwell's method we denote by $L_1, L_2, \ldots$ the self-inductances of the different meshes, each regarded as a separate circuit, in which flow currents $\gamma_1', \gamma_2', \ldots$ by $M_{12}, M_{23}, \ldots$ the mutual inductances of the pairs of meshes indicated by the suffices, the electrokinetic energy has the expression

$$T = \tfrac{1}{2}(L_1\gamma_1'^2 + 2M_{12}\gamma_1'\gamma_2' + \ldots) \quad . \quad . \quad . \quad (32)$$

If R_{jk} denote the resistance of a conductor common to two meshes j, k, one-half the rate of dissipation of energy in heat in the whole system, that is the dissipation function, is

$$F = \tfrac{1}{2}\Sigma R_{jk}(\gamma_j' - \gamma_k')^2 \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where the summation is extended to all the conductors of the system.

We can now write down the equations for the different meshes. They are of the type

$$\frac{d}{dt}\frac{\partial T}{\partial \gamma_j'} + \frac{\partial F}{\partial \gamma_j'} = E_j \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where E_j is the electromotive force in the circuit indicated by the suffix j.

There are advantages in this method of procedure, but it cannot be said to be of any practical service in inquiries concerning inductances. In such applications it is usual to write the electrokinetic energy in the form

$$T = \tfrac{1}{2}\Sigma\{L_{jk}(\gamma'_j - \gamma'_k)^2 + 2M_{(jk)(lm)}(\gamma'_j - \gamma'_k)(\gamma'_l - \gamma'_m)\} \quad (35)$$

where L_{jk} is the self-inductance of the conductor common to the two meshes j, k, and $M_{(jk)(lm)}$ is the mutual inductance of the two conductors common one to the meshes j, k, the other to the meshes l, m. But this is after all simply to return to the other method, to which at present we shall adhere. There is no difficulty really in writing down the equations for all the different conductors by this method, applying the principle of continuity at each meeting point. Further only one symbol is required for the current in each conductor, so that the formulæ are briefer.

471. If then we denote by $L_1, L_2 \ldots M_{12}, M_{23}, \ldots$ the self inductances of the conductors $1, 2 \ldots$ and the mutual inductances of the pairs of conductors $12, 23, \ldots$ the electrokinetic energy has the value

$$T = \tfrac{1}{2}(L_1\gamma_1^2 + 2M_{12}\gamma_1\gamma_2 + \ldots. + L_2\gamma_2^2 + 2M_{23}\gamma_2\gamma_3 + \ldots.) \quad (36)$$

The dissipation function is

$$F = \tfrac{1}{2}(R_1\gamma_1^2 + R_2\gamma_2^2 + \ldots.) \quad . \quad . \quad . \quad . \quad (37)$$

If there is electric energy E such as that of charged condensers situated in the conductors, the equations of the circuits are of the type

$$\frac{d}{dt}\frac{\partial T}{\partial \gamma_k} + \frac{\partial \mathsf{E}}{\partial y_k} + \frac{\partial F}{\partial \gamma_k} = E_k - V_k \quad . \quad . \quad . \quad . \quad (38)$$

where E_k is the *internal* electromotive force in the conductor, and V_k is the difference of potential between its terminals, taken with the negative sign, since we suppose E_k to act with the current, and V_k to oppose it.

Adding these equations for all the conductors forming a circuit, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \gamma_j} + \frac{\partial T}{\partial \gamma_{j+1}} + \ldots.\right) + \frac{\partial \mathsf{E}}{\partial y_j} + \frac{\partial \mathsf{E}}{\partial y_{j+1}} + \ldots. + \frac{\partial F}{\partial \gamma_j} + \frac{\partial F}{\partial \gamma_{j+1}} + \ldots. = E \quad (39)$$

where E is the total internal electromotive force in the circuit. The sum of the differences of potential between the terminals of the conductors is of course zero for every circuit.

By Art. 181 above if $C_j, C_{j+1}, \ldots.$ be the capacities in the succes-

sive conductors of the circuit and $y_j, y_{j+1}, \ldots$ denote the corresponding charges, we have

$$\mathsf{E} = \tfrac{1}{2}\left(\frac{y^2_j}{C_j} + \frac{y^2_{j+1}}{C_{j+1}} + \ldots.\right).$$

Hence (39) may be written in the form

$$\Sigma\left\{\frac{d}{dt}(L_{j\gamma j} + M_{jk\gamma k}) + R_{j\gamma j} + \frac{y_j}{C}\right\} = E \quad . \quad . \quad (40)$$

in which we shall generally use it. This may be taken as the most general form of the so-called "second law," given by Kirchhoff (Art. 230 above) for a system of linear conductors.

We shall find many examples of the use of this equation when we come to the measurement of inductances, though in most of these we shall have to use only the less general form of equation, which does not include terms depending on electrostatic energy. At present we shall consider a few problems of practical importance, taking first a case investigated experimentally by v. Helmholtz, whose method will be described in Vol. II., in the chapter on the *Experimental Verification of the Theory of Induction.*

Primary with Secondary as Derived Circuit

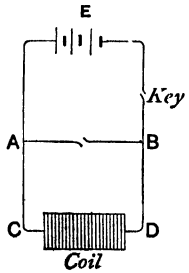


FIG. 125.

472. Let the same arrangement as that described in Art. 427 above be made, namely, a battery and coil in circuit with a cross-connection between them as shown in Fig. 125. If γ_1, γ_2 be the currents in the coil and the cross-connection, and γ the total current, r_1, r_2, r the resistances of the coil, cross-connection, and battery with connecting wires to AB, then we have by the principle of continuity

$$\gamma = \gamma_1 + \gamma_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

By equation (7) we have, since there is we suppose no mutual induction to be considered, and the only self-induction is in the coil

$$\left.\begin{aligned} r_2\gamma_2 + r\gamma &= E \\ L\frac{d\gamma_1}{dt} + r_1\gamma_1 + r\gamma &= E \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

for the two circuits $EABE, EACDBE$. These by (41) may be written

$$\left.\begin{aligned} r_2\gamma_2 + r(\gamma_1 + \gamma_2) &= E \\ L\frac{d\gamma_1}{dt} + (r + r_1)\gamma_1 + r\gamma_2 &= E \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (43)$$

The value of γ_2 derived from the first of these substituted in the second gives for γ the equation

$$L\frac{d\gamma_1}{dt} + \frac{rr_1 + r_1r_2 + r_2r}{r + r_2}\gamma_1 = \frac{r_2E}{r + r_2} \quad . \quad . \quad . \quad (44)$$

Writing Σrr_1 for $rr_1 + r_1r_2 + r_2r$ we get by integration

$$\gamma_1 = \frac{r_2E}{\Sigma rr_1}\left\{1 - e^{-\frac{\Sigma rr_1}{L(r + r_2)}t}\right\} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and hence for γ_2

$$\gamma_2 = \frac{E}{r + r_2}\left\{1 - \frac{rr_2}{\Sigma rr_1}\left(1 - e^{-\frac{rr_1}{L(r + r_2)}t}\right)\right\} \quad . \quad . \quad (46)$$

The quantity of electricity which flows through the coil during any interval τ reckoned from the closing of the circuit is thus

$$\frac{r_2E}{\Sigma rr_1}\int^{\tau}\left\{1 - e^{-\frac{\Sigma rr_1}{L(r + r_2)}t}\right\}dt = \frac{r_2E}{\Sigma rr_1}\left[\tau - \frac{L(r + r_2)}{\Sigma rr_1}\left\{1 - e^{-\frac{\Sigma rr_1}{L(r + r_2)}\tau}\right\}\right].$$

If after the lapse of the interval τ after the make the circuit be broken by raising the key, the quantity of electricity which flows through the coil is found as follows. The current flowing through the coil in the break satisfies the equation.

$$L\frac{d\gamma}{dt} + (r_1 + r_2)\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (47)$$

from which, remembering that the current at the beginning of the break has the value given in (45), we find

$$\gamma = \frac{r_2E}{\Sigma rr_1}\left(1 - e^{-\frac{\Sigma rr_1}{L(r + r_2)}\tau}\right)e^{-\frac{r_1 + r_2}{L}t} \quad . \quad . \quad . \quad (48)$$

The quantity which flows through the coil in the break is therefore

$$\frac{r_2LE}{(r_1 + r_2)\Sigma rr_1}\left(1 - e^{-\frac{\Sigma rr_1}{L(r + r_2)}\tau}\right).$$

Oscillatory and Non-Oscillatory Discharge of a Condenser

473. Later we shall discuss fully cases of primary and secondary circuits in which the electromotive forces acting are simple harmonic functions of the time; and the very important arrangement of primary and secondary circuits which we have in an ordinary make and break induction coil will be dealt with when its construction and mode of action are being considered. We shall next consider a very curious and important problem, which has played a great part in modern electrical discovery.

Let a condenser be charged to any given difference of potential, and let its plates be then connected by a wire of self-inductance T and resistance R. The condenser will discharge along the wire from one plate to the other. Let, at any time t, the difference of potential between the plates be V, the current γ, and the capacity C. Then the energy of the condenser is $\frac{1}{2}CV^2$ in virtue of its charge, and the current has electrokinetic energy $\frac{1}{2}L\gamma^2$. The total electrical energy is $\frac{1}{2}(CV^2+L\gamma^2)$. The total time rate of charge of this energy must be equal to the rate at which energy is being transformed into heat in the circuit, plus that at which energy is being radiated from the varying current system.

If we neglect the latter part we shall have

$$\tfrac{1}{2}\frac{d}{dt}(CV^2 + L\gamma^2) + R\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad (48')$$

But $\gamma = -CdV/dt$ so that the equation just found is

$$CL\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V = 0 \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Solving we get, putting $(R^2-4L/C)^{\frac{1}{2}}/2L = a$,

$$V = e^{-\frac{R}{2L}t}(Ae^{at} + Be^{-at}) \quad . \quad . \quad . \quad . \quad . \quad (50)$$

or as it may be written

$$V = e^{-\frac{R}{2L}t} D\cosh(at - \zeta)$$

where A and B or D and ζ are constants to be determined from the initial circumstances for any particular case.

If a is real this represents an ordinary discharge gradually approaching complete equalisation of potentials between the plates, and, theoretically, only reaching it in an infinite time.

If a is imaginary, which will be the case if $R^2 < 4L/C$, the solution is

$$V = e^{-\frac{R}{2L}t} A\cos\left\{\frac{1}{2L}\left(\frac{4L}{C} - R^2\right)^{\frac{1}{2}}t - \theta\right\} \quad . \quad . \quad (51)$$

where A and θ are constants to be determined from the given initial circumstances. This represents an oscillatory discharge with gradually diminishing range of potential. The period T is given by

$$T = \frac{4\pi L}{\sqrt{\frac{4L}{C} - R^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and the logarithmic decrement of the potential is $RT/4L$.

The discharging current is $-(C dV/dt)/R$, and is obtained from (50) for non-oscillatory and by (51) for oscillatory subsidence of the potential. The values of the current at different times for the two extreme cases of non-oscillatory subsidence are plotted in curves in Fig. 127.

474. The existence of an oscillatory discharge depends, as we have shown, on the relation of the resistance to the inductance of the discharging coil, and the capacity of the condenser. If the inductance is great enough in comparison with the resistance of the coil, electrical oscillations will take place, and there is no manner of doubt that many electrical discharges which appear mere single sparks are each a succession of backward and forward discharges caused by successive oscillations of the potential of the condenser.

The possibility of this form of discharge was suggested first apparently by v. Helmholtz, in his famous essay *Die Erhaltung der Kraft*, from certain unexplained phenomena of magnetization produced by passing Leyden jar discharges through a coil surrounding a bar of steel. The theory given here is practically that given by Lord Kelvin in a remarkable paper published in the *Phil. Mag.* for June, 1853.

475. The discharge of a condenser is thus similar to the motion of a deflected spring when resisted by a force proportional to the velocity of displacement. For the equation of motion (49) can be written

$$L\frac{d^2V}{dt^2} + R\frac{dV}{dt} + \frac{1}{C}V = 0 \quad . \quad . \quad . \quad . \quad . \quad (53)$$

which shows that L corresponds to the inertia of the spring, V to its displacement, $1/C$ to the return force per unit displacement (that is, C may be regarded as the modulus of yielding, or *permittance* as Heaviside calls it), and R to the resisting force per unit of the velocity of displacement. In such a case we know that if the inertia is very small and the resisting force has a large enough value, the spring will simply slip slowly back to its equilibrium position without oscillation about it, just as does a pendulum bob, of small inertia, deflected in a highly viscous fluid like treacle, and then left to itself. If, however, the spring has a certain amount of inertia it will get into motion, and as it nears the equilibrium position will move more quickly than before, overshooting that position and oscillating about it, if the inertia is sufficiently great. When the inertia is such that the spring is just brought to rest without passing the equilibrium position, and the slightest addition of mass would cause the spring to pass beyond that position before coming to rest, the motion of the spring is dead beat, and the condition $R^2C = 4L$ is fulfilled. Up to this limit obviously the addition of inertia diminishes the time of discharge. Hence the addition of the analogous quantity, self-inductance, to the discharging conductor will, in like circumstances, increase the rapidity of discharge of a condenser. Thus the self-inductance of a lightning-conductor may facilitate the discharge of a thundercloud. This is, of course, a con-

sequence of Lord Kelvin's theory, but it seems to have been first explicitly pointed out by W. E. Sumpner in a paper read to the Physical Society, Ap. 14, 1888 (*Phil. Mag.*, June, 1888).

The rate of discharge and amount of charge left in the condenser at different times are shown in Figs. 127, 128 below for the cases of (1) zero self-inductance, and (2) just as much as can exist without oscillatory discharge.

To make the matter as clear as possible we may trace the effect of adding self-inductance to the discharging conductor. If the coil possesses no inductance the equation reduces to

$$V = V_0 e^{-\frac{t}{RC}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

which gives the potential at time t in terms of the initial potential Vo and the value of t.

Since V and V_0 are of the same dimensions, it is clear that t/RC is a mere number, that is to say, RC is a time. It is the interval in which the potential is diminished from any value V_0 to V_0/e, and is called the time-constant of the condenser. V_0/e is the common ratio of the geometrical progression, the terms of which are the values of V at successive intervals each equal to RC.

476. The rate of falling off of the potential, and therefore of the charge in the condenser, is easily traced numerically. Since $e = 2{\cdot}71828 \ldots$, e^{10} is about 20000, therefore in an interval ten times RC the potential has fallen to about $\frac{1}{20000}$th of what it was before.

To trace the effect of adding inductance we go back to equation (50). The falling off of the potential now depends upon two exponentials, for which $1/(R/2L - a)$ and $1/(R/2L + a)$ may be regarded as the respective time-constants. If the roots of the auxiliary quadratic are real, both of these quantities must be positive, since in that case $a = R/2L.(1 - 4L/CR^2)^{\frac{1}{2}}$, and is real and less than $R/2L$.

The second of these time-constants is the smaller of the two, and since the term depending upon it is practically wiped out, while the other is still sensible, the time of discharge depends upon the larger time-constant.

If λ be written for L/R^2C the time-constants are

$$\tau_1 = \frac{2L}{R} \frac{1}{1 - \sqrt{1 - 4\lambda}}, \quad \tau_2 = \frac{2L}{R} \frac{1}{1 + \sqrt{1 - 4\lambda}} \quad . \quad (55)$$

and their product

$$\tau_1 \tau_2 = \frac{L^2}{R^2 \lambda} = \tau^2 \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where $\tau = RC$.

Plotting τ_1, τ_2 as the ordinates of a curve of which successive values

of λ are the abscissæ, we get the curve shown in Fig. 126, which is clearly a parabola with axis parallel to OX. The part of the curve drawn full is that of which the ordinates are the values of the larger time-constant; the dotted portion of the curve has for ordinates the values of the smaller time-constant.

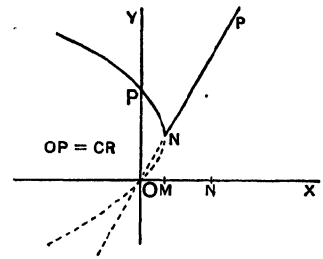


Fig. 126.

The time-constants are equal when $\lambda = 1/4$ (OM in Fig. 126),[1] and are then represented by the ordinate which touches the curve at the vertex. This value of λ gives $4L = R^2C$, that is, makes the roots of the auxiliary quadratic for (49) equal. This equality of roots marks the transition, with change of the relative values of L, R, C (in the present case L only is supposed to be varied from real to imaginary roots, that is, from non-oscillatory through dead-beat to oscillatory subsidence).

When $\lambda = 1/4$ the greater time-constant, on which the time of discharge depends, has its least value; hence the time of discharge is reduced to the least value which it can have for any value of L (including zero) just when oscillations are about to result from the increase of L. The theory of oscillatory discharge thus shows, as stated above, that the discharge of the condenser is actually hastened by the existence of self-induction.

Figs. 127, 128, taken from Prof. Lodge's paper (*loc. cit.*), illustrate the discharge in the limiting cases of (1) no self-inductance in the discharging conductor, (2) just as much self-inductance as brings the discharge to the verge of oscillation. Fig. 127 shows rate of discharge,

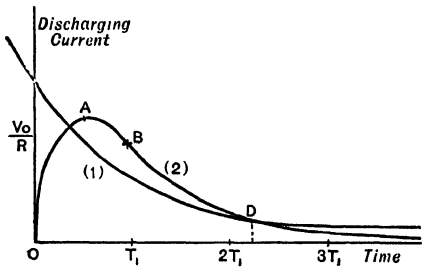


Fig. 127.

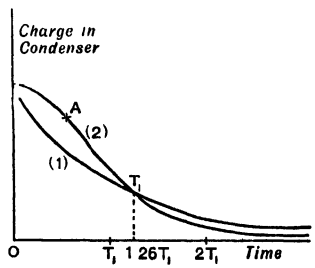


Fig. 128.

or discharging current, Fig. 128 the charge remaining at different times after the instant of contact. Curve (1) in each figure is drawn for the case of zero self-inductance, and corresponds to the point P of Fig. 127. Curve (2) is drawn for that of $L = R^2C/4$, and corresponds

[1] The graphical illustration shown in Fig. 126 is taken from a paper by O. J. Lodge, *Electrician*, May 18, 1888.

to the point N of Fig. 127. In each case the time T_1 has the value CR.

The rate of discharge in (2) Fig. 127 rises quickly to a maximum, which it reaches at time $t=\frac{1}{2}CR$, then falls off at first quickly, then more and more gradually. Curve (2) of Fig. 128 shows that at any time after the instant 1·26 T, the charge left in the condenser is less than that for the same time in the case of zero self-induction. The case illustrated by curve (2) is, as noticed above, that of most rapid non-oscillatory discharge.

477. If λ is greater than $\frac{1}{4}$, that is, if L is greater than $R^2C/4$, the ordinate drawn from the corresponding point N on the axis OX meets the curve in two imaginary points, and the discharge is oscillatory with amplitude diminishing at rate given by the factor $exp(-R/2L)t$. Thus the expression for the amplitude has a time-constant $2L/R$ or $2\tau\lambda$. The different values of this time-constant for different values of λ are the ordinates of the full part of the straight line OP in Fig. 126.

The mode of variation of V with t is shown in Fig. 129, in which

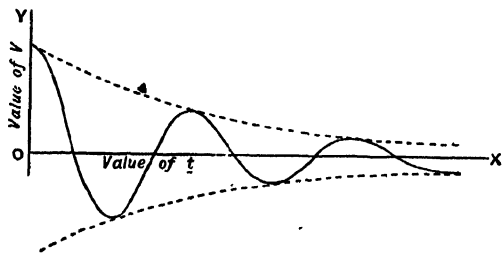


FIG 129.

the abscissæ are values of t, and the corresponding ordinates values of V. The successive maxima and minima lie on the curves of which $V=\pm Ae^{-Rt/2L}$ is the equation, and are separated by intervals of t each equal to $2L\pi/(4L/C-R^2)^{\frac{1}{2}}$. They are to be found of course by calculating the values of t for which dV/dt, derived from (51), vanishes, and substituting in (51).

The values of t for which V is zero are separated by the same interval, and each lies mid-way between two successive values of t for which dV/dt is zero.

Lord Kelvin suggested in his paper that oscillatory discharges might be observed by examining in a rotating-mirror the image of the spark in a Leyden jar discharge. This was done by Feddersen only two years later, and the theory fully verified.

A great deal of attention has within the last few years been directed to oscillatory discharges, with the result that the theory here discussed has been made the starting-point of a magnificent theory of electrical vibrations first elaborated by Maxwell in his paper on the

Electromagnetic Field, and again in his treatise on Electricity and Magnetism, and splendidly verified and extended by Hertz and many other electricians. Some account of these investigations will be given in the next and later chapters.

In some respects the theory of oscillatory discharge here given is only an approximation. The energy lost in radiation is neglected, as has been mentioned. But if the electrostatic capacity of the coil is considerable it ought to appear in the equations, and prevent the current from being the same throughout the coil at any one instant, just as in the process of signalling through a submarine cable the capacity of the cable renders the current different at different cross-sections at the same instant. If, however, the capacity of the condenser be great compared with that of the coil, this error will be insensible. In some experiments actually made it has been assumed that the theory found for a condenser was applicable to the coil alone, an assumption which it does not seem safe to make.

Condenser and Inductance-Coil in Series with Simple Harmonic Electromotive Force

478. We shall now consider a few examples in which there is electrostatic capacity as well as inductance to be taken account of, and in which the electromotive forces concerned are periodic. First take a simple circuit containing self-inductance L, resistance R, a condenser of capacity C, and an electromotive force varying as a simple harmonic function of the time, and therefore represented by $E_0 \sin nt$. The arrangement is that represented by Fig. 130 with the coil p, there shown joined in

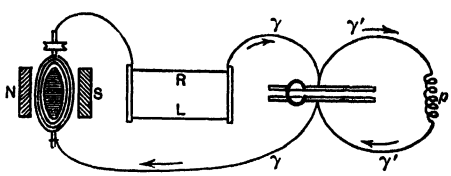


FIG. 130.

parallel with the condenser, removed. An alternating machine indicated on the left of the diagram produces the harmonic electromotive force in the circuit of the coil and condenser joined in series. If the current in the circuit be γ at any time the charge of the condenser is $\int \gamma dt$ taken from one convenient zero of reckoning of time to the actual instant considered. For the integral $\int \gamma dt$ taken from any chosen initial instant can only differ from that taken from another zero of time by a constant

which will not affect the solution of the problem of oscillation. Thus we obtain for the equation of motion, since E is now $E_0 \sin nt$,

$$L\frac{d\gamma}{dt} + R\gamma + \frac{1}{C}\int \gamma dt = E_0 \sin nt \quad . \quad . \quad . \quad . \quad (57)$$

which is equivalent to the equation

$$L\frac{d^2\gamma}{dt^2} + R\frac{d\gamma}{dt} + \frac{1}{C}\gamma = nE_0 \cos nt \quad . \quad . \quad . \quad . \quad (58)$$

The complete solution of this equation (which, it is to be observed, is the same as (49) with the term on the right added) is the solution for $E_0 = 0$, together with the particular solution found by assuming

$$\gamma = KE_0 \sin (nt - \theta),$$

and determining K and θ by substituting in the differential equation and remembering that the result is an identity which must hold for all values of t. This process gives

$$K = \frac{1}{\left\{R^2 + \left(nL - \frac{1}{Cn}\right)^2\right\}^{\frac{1}{2}}}, \quad \theta = \tan^{-1}\frac{nCR}{1 - n^2CL} - \frac{\pi}{2} \quad (59)$$

Thus, if α, β be the roots of the quadratic $CLx^2 + CRx + 1 = 0$, as given in (17) above, the complete solution of (58) is

$$\gamma = Ae^{\alpha t} + Be^{\beta t} + \frac{E_0}{\left\{R^2 + \left(nL - \frac{1}{Cn}\right)^2\right\}^{\frac{1}{2}}} \sin (nt - \theta) \quad (60)$$

The first part of the value of γ will ultimately become insensible whether α and β be real or complex, in the first case through non-oscillatory, in the second through oscillatory subsidence (the case of equal roots presents no exception) and therefore the value of γ will be given wholly by the equation

$$\gamma = \frac{E_0}{\left\{R^2 + \left(nL - \frac{1}{Cn}\right)^2\right\}^{\frac{1}{2}}} \sin (nt - \theta) \quad . \quad . \quad (61)$$

which is that of forced electrical oscillations.

Inductance Counteracted by Capacity. Arrangement for Maximum Current. Impedance, Effective Impedance. Graphical Representation of Results

479. The effect of the capacity C of the condenser is thus to counteract that of the self-inductance in diminishing the range of values of

the current, and to diminish the phase-angle θ. Thus if we choose L for given values of n, C, or C for given values of n, L, or n for given values of CL, so that $n^2CL = 1$, we have

$$\gamma = \frac{E_0}{R} \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and the current has the greatest amplitude that the arrangement admits of. This is the expression for the current that would flow if there were no inductance in the circuit, and no condenser were included, that is, if L were zero, and C were infinite.

When $C = \infty$ we have

$$\gamma = \frac{E_0}{(R^2 + n^2L^2)^{\frac{1}{2}}} \sin (nt - \theta) \quad . \quad . \quad . \quad . \quad (63)$$

where $\theta = -\tan^{-1}(R/nL) - \pi/2$. This is the expression for the current when the plates of the condenser are connected across by a short piece of thick wire. The amplitude of nL is much less than that given by (62), viz. E_0/R. It was noticed by Sir W. R. Grove (*Phil. Mag.* March, 1868) that when a magneto-electric machine was placed in the primary circuit of an induction coil a much greater effect on the secondary was produced when the primary circuit was kept open at the contact-breaker (Art. 489 below) than when the primary was kept closed. The explanation is obvious from the discussion above, and was given by Maxwell in a letter to Sir W. R. Grove, published in the *Phil. Mag.* for May, 1868. See also Art. 483 below.

The denominator in the expression for γ in (63) has been called by Heaviside the *impedance* of the circuit. It has sometimes been regarded as a resistance; but it is not a resistance in any true sense. For the activity in the circuit at any instant is

$$\gamma^2 R = \frac{RE_0^2}{R^2 + n^2L^2} \sin^2(nt - \theta)$$

and if $\sqrt{R^2 + n^2L^2}$ were properly the resistance of the circuit the activity would be $\gamma^2\sqrt{R^2 + n^2L^2}$. In the more general case in which there is capacity we may call $\{R^2 + (nL - 1/Cn)^2\}^{\frac{1}{2}}$ the *effective impedance.*

The relation of the impedance to the resistance and inductance is shown of course by a triangle ABC, right angled at B, in which the length of AB represents R, the length of BC represents nL, and that of the hypotenuse represents the impedance. A diagram is unnecessary.

The reader will find it convenient to remember that the more general equation (61) is obtained from (63) by substituting for L the expression $L - 1/Cn^2$. In the graphical representation just referred to, the side BC of the triangle is to be made to represent $nL - 1/Cn$ instead of nL.

480. The ordinates of the full curve in Fig. 131 show the values of the amplitude of the current in amperes, for different values of the inductance in henrys used as abscissae, when R is 100 ohms, C is one microfarad and n is 1,000. It will be seen that the ordinates rise from the amplitude for $C = \infty$, viz., one ampere, to a maximum of 10 amperes, which is reached when the inductance has been raised to one henry. The ordinates then fall off so that the curve is symmetrical about the maximum ordinate, and an amplitude of one ampere is reached again when the inductance has become two henrys. The dotted curve shows the lead which the current has in phase over the electromotive force, that is, its ordinates (measured from the chain dotted line) give in degrees as indicated on the right, the values of $\pi/2 - \tan^{-1} nCR/(1 - n^2CL)$ in degrees for the different values of L taken as abscissae. It will be noticed that the lead vanishes, in changing to a lag, when L is one henry.

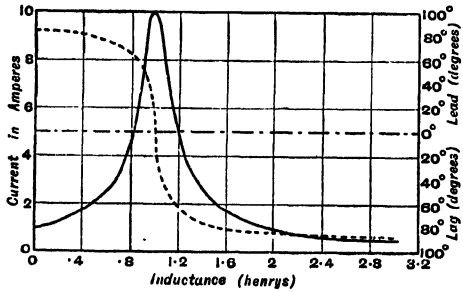


FIG. 131.

(This illustration is taken from a paper by Professor Perry and Mr. H. Bayly in the *Electrician* for July 21, 1893.)

In Fig. 132 curves are plotted as in Fig. 131 for each of a number of different values of R for a circuit in which the capacity of the condenser included and the frequency are kept constant while the self-inductance is varied. The curves are drawn as follows. First R is laid down as an ordinate OA, say, along the line OY, and a length OM to the left is laid off along the axis of abscissae equal to $1/Cn$. A value of $nL - 1/Cn$ is then laid off as an abscissa ON. It is clear that MN is equal to nL.

The line AN represents evidently what we have called the effective impedance $\sqrt{R^2 + (nL - 1/Cn)^2}$. An ordinate Na is drawn from N of length equal to AN and gives a point a on a curve passing through A. Thus a curve concave upwards and symmetrical about OY is obtained, the ordinates of which by (62) express the ratio to E_0, for the different values of nL, of the amplitudes of the electromotive forces required to produce a given current. Similarly curves are drawn for other values $OB, OC, \ldots$ of R.

Next, to obtain the curve of current from these curves of amplitude

of electromotive force OO' is taken equal to MN, and an equilateral hyperbola, having the lines $O'Y'$, $O'X$ as asymptotes, is laid down on the diagram, such that the product of any ordinate measured from $O'X$, and any abscissa measured from $O'Y'$ is E_0. Taking, then, any ordinate $N\alpha$ corresponding to nL of the first curve of amplitude of electromotive force, find the point on HH, which has the same ordinate. The corresponding abscissa from O' is the current-amplitude, which, laid down as an ordinate $N\alpha$, gives a point on the current-amplitude curve, and so on. Thus the various curves of Fig. 132 are drawn in which $A'\alpha$, $B'\beta$, $C'\gamma$, ... are the curves corresponding to the resistances OA, OB, OC, ... respectively.

In a similar way curves of amplitude of electromotive force for a given current, and of current-amplitude, can be drawn for each of different

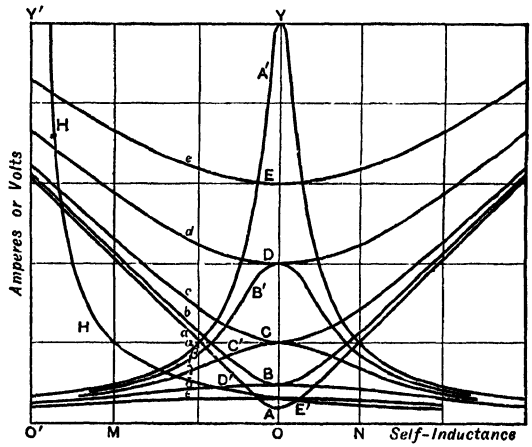


FIG. 132.

values of R, for L and n constant and C variable, and for L and C constant and n variable. (See a paper by W. E. Sumpner in the *Electrician* for July 28, 1893, from which Fig. 132 and the mode of constructing the curves are taken.)

It can be shown that if the resistance and self-inductance be wholly in a coil joined in parallel with the condenser instead of as here in series, the same diagrams will be available, but the current curves will become those of electromotive force, and *vice versa*, and the lengths $OA, OB, OC, \ldots$ instead of representing the values of R, will now represent those of $1/R$.

Effect of Inductance on Signalling through a Cable or in Telephony

481. It is interesting to observe that (61) shows that when the circuit contains a given condenser, and has a given resistance, the

addition of self-inductance up to a certain point increases the current, and that the maximum is obtained when $CLn^2=1$. This result is of importance in the theory of signalling through a cable. In that case the capacity is distributed along the cable, and the mathematical theory (which will be given in Part II.) is somewhat complicated, but the general result is the same as that just obtained.

It will be noticed from (61) that the effect of inductance is only serious when n is considerable, that is when the frequency of alternation is great. For slow signalling through a telegraph cable the inductance may be neglected; but it is a mistake to suppose, as is sometimes done, that it is necessarily deleterious, and that it should always be made as small as possible. In very rapid ordinary working, and especially in telephony, the presence of a certain amount of self-inductance improves the clearness of the signals.

From (59) it will be seen that for zero inductance the retardation of phase would have a value $\tan^{-1}(nCR)-\pi/2$ depending on the frequency of the vibrations. In telephony the corresponding effect is a retardation of phase of the signal at the receiving end behind that at the sending end, and this produces confusion of the signals, inasmuch as in a composite sound vibrations of one pitch have a different retardation from those of another pitch. Since in this case all the values of n are great, the value of $nCR/(1-n^2CL)$ is approximately $-CR/nCL$, or zero, and the existence of self-inductance L enables the value of θ to be nearly $-\pi/2$ for all the actual values of n, and so removes distortion. Of course, on the other hand, excessive self-inductance may, by (61), produce too great attenuation of the signals.

Difference of Potential between Terminals of Condenser. Resonance

482. By (57) the difference of potential V between the plates of the condenser is given by the equation

$$V = E_0 \sin nt - L\frac{d\gamma}{dt} - R\gamma,$$

or by (61)

$$V = \frac{E_0 \sin(nt-\theta)}{\{(1-n^2CL)^2 + n^2C^2R^2\}^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (63')$$

If the denominator of this expression be less than unity, the value of V will be greater than the numerator of (63'), and the amplitude of V will be greater than that of the impressed electromotive force. This curious result has been verified in practice by observations on the electric light mains carrying alternating currents between London and the Ferranti Company's generating station at Deptford. It was there found by Mr. Ferranti that the so-called "electrical pressure" (which may be taken as the square root of the mean of V^2) on the terminals

of the alternator, working at its normal speed with a certain exciting current, was increased by connecting the machine to the mains. This was no doubt due to a partial fulfilment of the conditions necessary for a small value of the denominator of the expression in (63′), that is, it was a case more or less of what has been called *resonance.*[1]

Maxwell's Dynamical Analogies of Inductance and Capacity. Resonance

483. The above explanation of the effect thus observed was pointed out at the time by Glazebrook,[2] Lodge,[3] and others; but the theory of the whole matter was really given first by Maxwell in the paper cited in Art. 479.[4]

In that paper Maxwell points out analogies to inductance, capacity and resistance. The inductance of the primary coil acts like inertia, resisting the starting and stopping of the current in the same way as the inertia of a body resists a push or pull tending to set it into motion, or to bring it to rest. The capacity of the condenser acts like a spring, which must be bent or compressed if the body is accelerated or retarded, as, for example, when railway-buffers resist the motion of a carriage towards an obstacle.

He then imagines as an example a boat floating in a viscous liquid, and kept in place by buffers at the bow and stern abutting against fixed obstacles. If there were no obstacles a long continued pull would move the boat, however great its mass. But on the other hand alternate pulls and pushes would produce very little motion. With the buffers in position, however, alternate pulls and pushes in, or nearly in, the period of the springs would produce in a short time considerable backward and forward motion of the boat. The work spent in each impulse is in great measure stored up in the springs in consequence of their resilience, and a very much greater motion results than could have been obtained with the boat free. This agreement of the period of the alternating impulses with the free period of the springs is called *resonance.*

When a circuit is closed without a condenser the current in it is like the motion of the boat when free in the viscous liquid, and under an alternating electromotive force the amplitude of the current produced is small. But when a condenser is applied, and the period of the alternating electromotive force is that of the discharge of the condenser, the relation between the inductance and the capacity is such as, for that value of the period, to give a very much greater alternating current.

484. The greatest amplitude of V is attained when n is so chosen as to make the quantity $(1-n^2CL)^2 + n^2C^2R^2$ as small as possible with the given values of C, L, R. For this purpose n^2 should have the value

[1] *Electrician*, Dec. 19, 1890.
[2] *Electrician*, Dec. 26, 1890.
[3] *Ibid*, April 24, 1891.
[4] Also *Collected Papers*, Vol. II. p. 12.

$(2L - CR^2)/2CL^2$. If R be so small that CR^2 is negligible in comparison with $2L$ this gives $n^2 = 1/CL$, that is, n is then 2π times the natural frequency of vibration of the condenser and coil as arranged. We have then

$$\frac{\text{amplitude of } V}{E_0} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

which will be much greater than unity, since CR^2 has been supposed negligibly small as compared with $2L$.

Primary and Secondary. Inductance and Capacity in Primary with Harmonic Electromotive Force

485. We now consider the problem of a primary circuit arranged as in the case just considered, with the addition of a secondary circuit containing no electromotive force. The equation of the primary is

$$L_1\frac{d\gamma_1}{dt} + M\frac{d\gamma_2}{dt} + R_1\gamma_1 + \frac{1}{C}\int\gamma_1 dt = E_0 \sin nt \quad . \quad . \quad (65)$$

Since we are concerned only with the forced electrical oscillations, and these will be simple harmonic and of the same period as the impressed electromotive force, we see that

$$\frac{1}{C}\int\gamma_1 dt = -\frac{1}{Cn^2}\frac{d\gamma_1}{dt}.$$

Therefore if $L'_1 = L_1 - 1/Cn^2$ the first of the above equations may be written in the form

$$L'_1\frac{d\gamma_1}{dt} + M\frac{d\gamma_2}{dt} + R_1\gamma_1 = E_0 \sin nt \quad . \quad . \quad . \quad (66)$$

The equation of the secondary is

$$M\frac{d\gamma_1}{dt} + L_2\frac{d\gamma_2}{dt} + R_2\gamma_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (67)$$

The problem is therefore the same as that of a primary without condenser and with self-inductance $L'_1 = L_1 - 1/Cn^2$, and a secondary without condenser and without electromotive force. If however the secondary contained a condenser of capacity C_2, we should only have to put for L_2 the expression $L_2 - 1/C_2n^2$ to take the condenser into account.

Eliminating γ_2 by operating on (66) by $R_2 + L_2 d/dt$ and on (67) by Md/dt we obtain

$$(L'_1L_2 - M^2)\frac{d^2\gamma_1}{dt^2} + (L'_1R_2 + L_2R_1)\frac{d\gamma_1}{dt} + R_1R_2\gamma_1 = E_0(R_2^2 + n^2L_2^2)^{\frac{1}{2}} \sin(nt - \epsilon) \quad . \quad (68)$$

where $\epsilon = \tan^{-1}(-nL_2/R_2)$. The solution of this equation is for forced oscillations

$$\gamma_1 = \frac{(R_2^2 + n^2L_2^2)^{\frac{1}{2}}E_0}{[\{R_1R_2 - n^2(L'_1L_2 - M^2)\}^2 + n^2(L'_1R_2 + L_2R_1)^2]^{\frac{1}{2}}} \sin(nt - \theta_1) \quad (69)$$

where

$$\theta_1 = \tan^{-1}\frac{-nL_2\{R_1R_2 - n^2(L'_1L_2 - M^2)\} + nR_2(L'_1R_2 + L_2R_1)}{R_1R_2^2 - n^2\{R_2(L'_1L_2 - M^2) - L_2(L'_1R_2 + L_2R_1)\}}. \quad (70)$$

Similarly for the forced oscillations in the secondary we obtain

$$\gamma_2 = \frac{-MnE_0}{[\{R_1R_2 - n^2(L'_1L_2 - M^2)\}^2 + n^2(L'_1R_2 + L_2R_1)^2]^{\frac{1}{2}}} \sin(nt - \theta_2) \quad (71)$$

in which

$$\theta_2 = \tan^{-1}\frac{n^2(L'_1L_2 - M^2) - R_1R_2}{n(L'_1R_2 + L_2R_1)} \quad . \quad . \quad . \quad (72)$$

486. It is easy to verify from these results that the phase of the primary current is in advance of the secondary by the angle $\pi/2 - \epsilon$.

Equation (69) can be written in the form

$$\gamma_1 = \frac{E_0 \sin(nt - \theta_1)}{\left\{\left(R_1 + n^2\frac{M^2R_2}{R_2^2 + n^2L_2^2}\right)^2 + n^2\left(L_1 - \frac{1}{Cn^2} - n^2\frac{M^2L_2}{R_2^2 + n^2L_2^2}\right)^2\right\}^{\frac{1}{2}}} \quad (73)$$

which shows that the effect of the secondary has been virtually to increase the resistance of the primary by $n^2M^2R_2/(R_2^2 + n^2L_2^2)$, and to diminish the inductance by $n^2M^2L_2/(R_2^2 + n^2L_2^2)$. Similarly the current in the secondary circuit can be seen to be the same as it would be if the circuit were independent, and contained a harmonic electromotive force of amplitude $E_0Mn/(R_1^2 + n^2L_1^2)^{\frac{1}{2}}$, and had a resistance $R_2 + n^2M^2R_1/(R_1^2 + n^2L_1^2)$ and a self inductance $L_2 - n^2M^2L'_1/(R_1^2 + n^2L_1^2)$. These results are due to Clerk Maxwell.[1]

The reader may easily verify by putting $R_2 = \infty$ that the results already established for a single circuit containing a condenser are obtained.

Conductors in Parallel, and Containing Resistance, Inductance, and Capacity. Equivalent Resistance and Inductance of Parallel Circuits

487. We may take next the important case of a number of conductors joined in parallel between two points A, B to which a simple harmonic difference of potential $V_0 \sin nt$ is applied. The arrangement

[1] *A Dynamical Theory of the Electromagnetic Field, Phil. Trans. R.S.* vol. clv. (1865), or Maxwell's *Collected Papers*, vol. i. p. 547.

shown in Fig. 130 is a case in point. Fig. 133 shows another. We suppose that in each conductor there is resistance, self-inductance, and capacity, of course generally different for the different conductors, and that there is no mutual inductance anywhere in the system. By what has been stated above the capacities need not appear in the equations, and we may work as if their values were infinite, provided we regard the L of any conductor as denoting for it *self-inductance* $-1/Cn^2$, where C is the capacity of the corresponding condenser. Distinguishing the quantities relating to the various conductors by suffixes, we obtain the equations

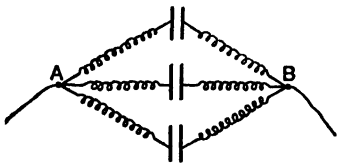


FIG. 133.

$$\left.\begin{aligned} L_1\frac{d\gamma_1}{dt} + R_1\gamma_1 &= V_0 \sin nt \\ L_2\frac{d\gamma_2}{dt} + R_2\gamma_2 &= V_0 \sin nt \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (74)$$

The typical solution is

where
$$\left.\begin{aligned} \gamma &= Ae^{-\frac{R}{L}t} + \frac{V_0}{(R^2 + n^2L^2)^{\frac{1}{2}}} \sin (nt - \eta) \\ \eta &= \tan^{-1}\frac{nL}{R} \end{aligned}\right\} \quad . \quad . \quad (75)$$

This solution applied to each of (75) gives the currents in the different parallel conductors. Adding these together (neglecting as usual the exponential terms) we find for the total current, Γ say, at any instant entering at one of the points and leaving at the other

$$\Gamma = \Sigma\gamma = V_0\Sigma\left\{\frac{1}{(R^2 + n^2L^2)^{\frac{1}{2}}} \sin (nt - \eta)\right\} \quad . \quad . \quad (76)$$

This may be written in the form

where
$$\left.\begin{aligned} \Gamma &= \frac{V_0}{(\mathsf{R}^2 + n^2\mathsf{L}^2)^{\frac{1}{2}}} \sin (nt - \phi) \\ \phi &= \tan^{-1}\frac{n\mathsf{L}}{\mathsf{R}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (77)$$

by writing R for $A/(A^2+n^2B^2)$, L for $B/(A^2+n^2B^2)$ in which A denotes $\Sigma R/(R^2+n^2L^2)$, B denotes $\Sigma L/(R^2+n^2L^2)$.

The total current is thus the same as if the points A, B were connected by a single conductor of resistance R and inductance L. These may be called respectively the effective resistance and inductance of

the system of parallel conductors. The angle ϕ is the lag in phase of the total current entering or leaving the parallel system at any instant behind that of the impressed difference of potential.

The effective capacity of the system of parallels is in general indeterminate, and is of no practical importance.

488. Now let the circuit of AB be completed by a single conductor of resistance R, inductance L, and containing an electromotive force $E_0 \sin(nt+\zeta)$, so that ζ is the lag of the difference of potential $V_0 \sin nt$ behind the electromotive force. The solution for the case in which this conductor also contains a condenser C will be obtained by putting $L - 1/Cn^2$ for L. The current in this conductor is Γ, so that we have for the differential equation

$$L\frac{d\Gamma}{dt} + R\Gamma + V_0 \sin nt = E_0 \sin(nt + \zeta).$$

Substituting the value of Γ already found, and remembering that the identity thereby obtained must hold for all values of t, we easily see that it gives the two equations

$$\left.\begin{aligned} \frac{V_0}{E_0} &= \frac{(\mathsf{R}^2 + n^2\mathsf{L}^2)^{\frac{1}{2}}}{\{(\mathsf{R}+R)^2 + n^2(\mathsf{L}+L)^2\}^{\frac{1}{2}}} \\ \zeta + \phi &= \tan^{-1}\frac{n(\mathsf{L}+L)}{\mathsf{R}+R} \end{aligned}\right\} \quad . \quad . \quad . \quad (78)$$

Instead of the first of (77) we therefore have

$$\Gamma = \frac{E_0}{\{(\mathsf{R}+R)^2 + n^2(\mathsf{L}+L)^2\}} \sin\{nt - (\zeta + \phi)\} \quad . \quad (79)$$

The first of these shows that the amplitude of the impressed difference of potential bears to that of the whole electromotive force in the circuit the ratio of the effective impedance of the system of parallels to the effective impedance of the whole circuit. The second equation shows that the lag in phase of the current behind the electromotive force is given by substituting, in the expression (75) already found for one of the parallels and the impressed difference of potential, the whole effective inductance and resistance of the circuit for the corresponding quantities relating to the single conductor. These results were to be anticipated.

As an example take the arrangement shown in Fig. 130. Let the inductance of the branch containing the coil p be zero, and R_1 be its resistance, L,R the inductance and resistance of the branch containing the magneto-electric machine. The inductance and resistance of the branch containing the condenser we shall suppose to be zero, and denote the capacity by C. We easily find

$$\mathsf{R} = \frac{R_1}{1 + n^2C^2R_1^2}, \quad \mathsf{L} = \frac{-CR_1^2}{1 + n^2C^2R_1^2},$$

from which the current can be found by (79) and the second of (78)

Action of Induction Coil, or Inductorium, with Condenser across Break in Primary. Case I. Secondary Closed

489. A very important practical case of a circuit containing a condenser is found in the ordinary induction coil, or inductorium, Fig. 134, for producing high electromotive forces in a secondary coil by breaking the circuit of a primary. The primary is a coil of comparatively few turns of thick copper wire wound round a cylinder enclosing a core of soft iron, preferably a bundle of thin iron wires, sufficiently insulated from one another to prevent sensible currents from being induced and

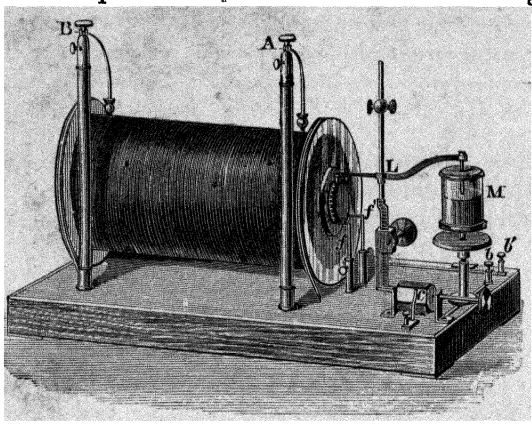


FIG. 134.

circulating in the iron. The secondary is a coil composed of a great length of highly insulated thin wire and is wound in a large number of turns on a cylinder outside the primary. To avoid damage to the insulation by internal sparks, the coil is wound in sections, so as to avoid placing close together parts which may, when the coil is in action, be at a great difference of potential. Owing to the large number of turns in the secondary, the change of total magnetic induction through it produced by stopping or starting the current is very great, and if the change is effected suddenly the electromotive force reaches a high value, though of course its amount varies during the passage of the secondary current. Coils have been constructed of such power as to give sparks of over 3 feet in length between the terminals of the secondary. Further practical details, and calculations regarding the mutual induction in actual cases are reserved for a later chapter.

The circuit of the primary is made and broken by a vibrator of one or other of the two forms shown, one in Fig. 134, the other in Fig. 135. In the former a brass arm mounted on a spring support carries at one end a piece of iron, at the other a platinum wire which the spring tends to make dip into a cup of mercury as indicated. The iron piece at the other end is placed just above the projecting end of the core or of an

iron armature attached to the coil. To hinder sparking the surface of the mercury is covered with an insulating layer of alcohol, from which the wire is not withdrawn when the break is in action. A battery of cells of low internal resistance is placed in one direction or the other in series with the primary by means of a commutator, and the circuit is completed by the contact of the platinum point with the mercury.

When a current flows in the primary and the core is magnetized, the piece of iron, being magnetized inductively, is pulled down and raises the platinum wire at the other end from the mercury, thus breaking the circuit. When the circuit is broken the iron core loses its magnetism, the spring is allowed to act, and the contact with the mercury is restored. As sparks do not take place so readily in alcohol as in air, on account of its greater dielectric strength or power to resist rupture by electric stress, the formation of a spark is nearly if not wholly prevented.

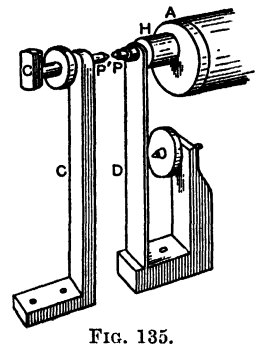


FIG. 135.

In another form of contact breaker more usually employed, the contact is made and broken by the action of an upright spring D (Fig. 135) and the attraction of the magnetized core on an iron piece, H, carried by the spring at its upper end. One face, P, of this contact piece, shod with platinum, is held by the spring against another piece of platinum, P′, mounted on one end of an adjusting screw, and so completes the circuit. Soon after the circuit is completed, however, it is interrupted by the magnetization of the core, and the current quickly ceases.

The secondary coil is for the most part used to give discharges across a spark-gap of adjustable width between points or knobs attached to its terminals; and these discharges are rendered unidirectional by occurring at the break, not at the make, of the primary, owing to the greater suddenness with which the break can be effected. The electromotive force, depending as it does on the rate of decay of the primary current, is enhanced by any arrangement which increases that rate. One of these is the immersion of the break in alcohol, another is the attachment of a small condenser, usually made of sheets of tinfoil separated by paraffined paper or silk, and contained in the wooden base of the instrument. One set of plates of this are connected to one side of the break, the other to the other side: for example one to the lower end of the metal piece C (Fig. 135), the other to the metal block supporting the spring D, and similarly in any other make and break arrangement.

490. The action of the condenser has been the subject of some discussion. The following explanation is due to Lord Rayleigh,[1] but it assumes a closed secondary in which currents are induced in the manner described above. In most cases of use of an induction-coil the

[1] *Phil. Mag.* June, 1870.

secondary cannot be regarded as closed in the ordinary sense except when a spark is passing, and the *régime* of the current is therefore very different from that here supposed. We shall return presently to this point.

We suppose then that the secondary is closed, that the discharge of the condenser which takes place at break of the circuit is oscillatory, and that the diminution of current due to resistance may be neglected, as only the first one or two oscillations are concerned. The equations of the primary and secondary are, if the variation of the inductances, produced by the iron core, be neglected,

$$\left.\begin{aligned} L_1\frac{d\gamma_1}{dt} + M\frac{d\gamma_2}{dt} + \frac{1}{C}\int\gamma_1 dt &= E \\ M\frac{d\gamma_1}{dt} + L_2\frac{d\gamma_2}{dt} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

These give by elimination of γ_2

$$(L_1L_2 - M^2)\frac{d\gamma_1}{dt} + \frac{L_2}{C}\int\gamma_1 dt = E \quad . \quad . \quad . \quad . \quad (80)$$

Also the second of (79) integrated is

$$M\gamma_1 + L_2\gamma_2 = M\gamma_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

where γ_0 is the current in the primary just before the break begins.

Equation (80) shows that the oscillation in the primary is the same as if the secondary did not exist and the self-inductance of the primary were changed from L_1 to $L_1 - M^2/L_2$. Since the period of oscillation is $2\pi\sqrt{LC}$ (Art. 473), where L is the effective inductance, the secondary diminishes the period of oscillation in the primary. It may be remarked here that if two coils be made up of wires coiled side by side so as to be nearly identical both in geometrical arrangement and in position L_1L_2 will be onl slightly greater than M^2 and the period will be exceedingly short. y

Equation (81) indicates that the two currents oscillate together, the current in the secondary being at its maximum when that in the primary is at its minimum and *vice versa*. After half a period has elapsed from the beginning γ_1 has changed to $-\gamma_0$ and we have

$$\gamma_2 = 2\frac{M}{L_2}\gamma_0,$$

and this is the largest value that γ_2 can have. It is double the maximum the secondary current would have in the case of a simple break in the primary however suddenly brought about.

It is thus the oscillatory discharge of the condenser in the primary that according to this theory gives rise to the enhanced value of the secondary current, and it is the first maximum of current, of approximate amount $2M/L_2\gamma_0$, which produces such effects as the

magnetization of needles when surrounded by the secondary coil. It was found by Lord Rayleigh (*loc. cit.*) that the magnetizing effect of the induced current in a secondary coil produced at break of the primary was greater the smaller the number of turns in the secondary, that is to say the smaller L_2, which is in accordance with the foregoing theoretical result.

It may easily be verified in practical cases whether the assumption made above that no serious damping has been brought about during the first few oscillations is well founded. It is only necessary that the period $2\pi\sqrt{LU}$ should be small in comparison with the time-constant L/R.

Action of Condenser in Induction Coil. Case II. Secondary Open

491. If the secondary be unclosed and have no sensible capacity very little current will be set up in it, and $d\gamma_2/dt$ will be always infinitesimal since there is no sudden change. Also R_2 is very great. The foregoing explanation is therefore hardly applicable, and we must look at the matter from another point of view. Now when there is no condenser in the primary the electromotive force of self-induction in the primary acting with the current brings about a difference of potential between the two surfaces of the contact breaker which starts a spark as soon as the surfaces of the contact breaker have, so to speak, begun to separate, and the discharge, owing to the breaking down of the dielectric, is kept up as the surfaces further separate. Thus, dropping suffixes for the primary, $d\gamma/dt$ never becomes great, inasmuch as the flow is maintained across the spark-gap until the initial energy $\frac{1}{2}L\gamma^2$ of the current is dissipated in the spark and in the wire of the circuit.

On the other hand, with a condenser the rise of potential is more gradual at first, as the capacity of the terminals is so much greater, and the spark, if it occurs at all, is delayed. The result is that the rise of potential at the terminals, falling behind that required to produce a spark, is wholly effective in reducing the current, so that, provided the capacity of the condenser is not taken greater than is just necessary to prevent the spark from occurring, the value of $d\gamma_1/dt$ is made very much greater, and there is a correspondingly greater electromotive force in the secondary.

The equation of flow of electricity in the primary is given in Art. 473 above. The greatest value of the electromotive force in the secondary can be deduced from this by calculating $d\gamma/dt$, and finding for what value of t this is a maximum. If the discharge of the condenser in the primary is non-oscillatory $d\gamma/dt$ will have its greatest value at the beginning of the break, and when, as in most cases, the discharge is oscillatory, the maximum, it is obvious without calculation, will take place about a quarter period after the beginning of the break. The reader may easily obtain an exact solution for himself.[1]

[1] See also a letter by Lodge in the *Electrician*, June 14, 1889.

Mechanical Illustration of Action of Induction Coil with Condenser

492. The action of the condenser may be mechanically illustrated as follows. Imagine a carriage on a railway. Its inertia is analogous to L and its velocity to γ, so that the kinetic energy is the analogue of $\frac{1}{2}L\gamma^2$. Now let there stand across the path of the carriage an obstacle, say a light carriage provided with spring buffers, and imagine this rendered incapable of motion along the rails in the direction to meet the first carriage, while left free to move the other way. Let this obstacle have connected to it at each end a series of cords running from it parallel to the rails in the direction opposite to that in which the carriage is moving, and passing in the same direction round pulleys on a horizontal shaft running in fixed bearings at right angles to the railway. The cords are so arranged that when the shaft is turned in the direction to tighten them they become taut in succession, first one cord at each side, then a little later another at each side, and so on, the cords already tightened causing their pulleys to slip on the shaft, so that they are not broken as the shaft goes on turning.

When the carriage touches the buffers let one shaft be started so as to tighten the cords. [This corresponds to the beginning of the break in the circuit, and the cords represent the interposed dielectric in the break]. If the spring is very stiff the force on the obstacle may be sufficient to break the first cord before the second has been tightened, then break the second and so on, the obstacle being pushed forward by the carriage. If the inertia of the obstacle and the resilience of each cord be small the carriage will not be very greatly retarded, that is the quantity corresponding to $d\gamma/dt$ will be small. The breaking of the cords represents the spark, and the stiff spring a condenser of so very slight capacity as to allow the spark to pass readily.

If however the spring be sufficiently long and weak the compression of the spring will have to be considerable to produce force enough on the obstacle sufficient to break a cord; and before that amount of compression has been reached the turning shaft will have moored the obstacle by a greater number of cords than one. The compressed spring however will push the carriage back, and so diminish its speed. The more quickly the shaft acts the stiffer is the spring that may be employed, and the greater is the recoil on the carriage. To diminish the velocity of the carriage at the greatest rate the shaft should run as quickly as possible, and the spring should be just weak enough to prevent the breaking of the cords. Obviously the greatest recoil force exerted by a given spring on the carriage is attained about a quarter period of the spring later than the beginning of the impact.

This action corresponds precisely to the prevention of the spark by the condenser, and the thereby enhanced value of $d\gamma/dt$, and therefore of the electromotive force in the secondary. The best arrangement is that in which the break is made as rapidly as possible, and the condenser has no more capacity than is sufficient to prevent the spark.

Dynamical Theory of Vibrating System, having only Kinetic Energy and Acted on by Dissipative Forces. Effective "Resistance" and "Inductance." Illustration by Mutually Influencing Circuits

493. The formation of the equations of current by the Lagrangian method from the expressions for the energy of the system leads, as has been seen, to an extension of the rule stated in Art. 225 above, which, with the principle of continuity, leads to the equations of steady currents. Indeed the rule, as already stated, suffices for currents in variable *régime*, if the electromotive forces of self-induction and mutual induction are reckoned along with the other electromotive forces and with their proper signs. Hence it has not been necessary to refer to the Lagrangian equations in the examples worked out above. We shall now obtain some important results by direct application of the Lagrangian method. Most of these are due to Lord Rayleigh.[1]

Some of these results as well as some of the examples worked out above are interesting examples of general dynamical theorems, due to Lord Kelvin and M. Bertrand, and analogous theorems due to Lord Rayleigh, but we have no space here to enter into the discussion. The reader may refer to Lord Rayleigh's papers and his *Theory of Sound*.

494. Consider a system in which a harmonic force Ψ_1 corresponding to the co-ordinate ψ_1 is applied to the system. Let the system have zero potential energy, but be acted on by dissipative forces given by a function F (Art. 262 above), which is a homogeneous quadratic function of the velocities $\dot{\psi}_1$, $\dot{\psi}_2$ . . . of the system. Let also the functions T and F contain no products of velocities except those in which $\dot{\psi}_1$ appears. We have

$$\left.\begin{aligned} T &= \tfrac{1}{2}(a_{11}\dot{\psi}_1^2 + a_{22}\dot{\psi}_2^2 + \ldots + 2a_{12}\dot{\psi}_1\dot{\psi}_2 + 2a_{13}\dot{\psi}_1\dot{\psi}_3 + \ldots) \\ F &= \tfrac{1}{2}(b_{11}\dot{\psi}_1^2 + b_{22}\dot{\psi}_2^2 + \ldots + 2b_{12}\dot{\psi}_1\dot{\psi}_2 + 2b_{13}\dot{\psi}_1\dot{\psi}_3 + \ldots) \end{aligned}\right\} \quad (82)$$

Hence by (71), Art. 262, the equations of motion are

$$\left.\begin{aligned} &a_{11}\dot{\psi}_1 + a_{12}\ddot{\psi}_2 + a_{13}\ddot{\psi}_3 + \ldots + b_{11}\dot{\psi}_1 + b_{12}\dot{\psi}_2 + \ldots = \Psi_1 \\ &a_{12}\ddot{\psi}_1 + a_{22}\ddot{\psi}_2 + b_{12}\dot{\psi}_1 + b_{22}\dot{\psi}_2 = 0 \\ &a_{13}\ddot{\psi}_1 + a_{33}\ddot{\psi}_3 + b_{13}\dot{\psi}_1 + b_{33}\dot{\psi}_3 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (83)$$

In response to the force Ψ_1 the motion of the system will be simple harmonic in the same period, $2\pi/n$ say. Thus representing any ψ by e^{int} where $i = \sqrt{-1}$ we get $\ddot{\psi} = in\dot{\psi}$ and therefore (83) becomes

$$\left.\begin{aligned} &(ina_{11} + b_{11})\dot{\psi}_1 + (ina_{12} + b_{12})\dot{\psi}_2 + (ina_{13} + b_{13})\dot{\psi}_3 + \ldots = \Psi_1 \\ &(ina_{12} + b_{12})\dot{\psi}_1 + (ina_{22} + b_{22})\dot{\psi}_2 = 0 \\ &(ina_{13} + b_{13})\dot{\psi}_1 + (ina_{33} + b_{33})\dot{\psi}_3 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (84)$$

[1] *Phil. Mag.* (Ser. 4) 49, March, 1875, and Ser. 5, 21, May, 1886; *Theory of Sound*, Vol. I. (2nd edition) p. 433.

Substituting in the first of (84) the values of $\dot{\psi}_2, \dot{\psi}_3, \ldots$ obtained in terms of $\dot{\psi}_1$ from the remaining equations, we get

$$\{ina_{11} + b_{11} - \frac{(ina_{12} + b_{12})^2}{ina_{22} + b_{22}} - \frac{(ina_{13} + b_{13})^2}{ina_{33} + b_{33}} - \ldots\}\dot{\psi}_1 = \Psi \quad (85)$$

The factor in brackets on the left is a complex quantity, and may therefore be written in the form $R' + inL'$, where R' and L' are real. We may call R' the resistance of the system, and L' its self inductance. Separating the real and imaginary parts of the factor in brackets in (85) we easily find, if there be m co-ordinates

$$\left.\begin{aligned} R' &= b_{11} - \sum_{h=2}^{h=m} \frac{b_{1h}^2}{b_{hh}} + n^2 \sum_{h=2}^{h=m} \frac{(a_{1h}b_{hh} - a_{hh}b_{1h})^2}{b_{hh}(b^2_{hh} + n^2 a^2_{hh})} \\ L' &= a_{11} - \sum_{h=2}^{h=m} \frac{a_{1h}^2}{a_{hh}} + \sum_{h=2}^{h=m} \frac{(a_{1h}b_{hh} - a_{hh}b_{1h})^2}{(a_{hh}(b^2_{hh} + n^2 a^2_{hh})} \end{aligned}\right\} \quad (86)$$

It is clear that as the frequency $n/2\pi$ increases R' increases and L' diminishes towards the limits

$$\left.\begin{aligned} R' &= b_{11} - \sum_{h=2}^{h=m} \frac{b_{1h}^2}{b_{hh}} + \sum_{h=2}^{h=m} \frac{(a_{1h}b_{hh} - a_{hh}b_{1h})^2}{b_{hh}a_{hh}^2} \\ L' &= a_{11} - \sum_{h=2}^{h=m} \frac{a_{1h}^2}{a_{hh}} \end{aligned}\right\} \quad (87)$$

Thus when n is great L' is independent of the b-coefficients, that is the resultant inductance is not affected by the dissipation.

It is to be observed also that if there is no kinetic energy of the system the third term in (87) does not appear and we have the same minimum of resistance as when the frequency is very small.

When n is very small

$$\left.\begin{aligned} R' &= b_{11} - \sum_{h=2}^{h=m} \frac{b_{1h}^2}{b_{hh}} \\ L' &= a_{11} - \sum_{h=2}^{h=m} \frac{a_{1h}^2}{a_{hh}} + \sum_{h=2}^{h=m} \frac{(a_{1h}b_{hh} - a_{hh}b_{1h})^2}{a_{hh}b^2_{hh}} \end{aligned}\right\} \quad (88)$$

that is R' is independent of the inertia coefficients, and L' has its maximum value.

495. These results find immediate illustration in electrical problems. For example take the case of a primary and secondary circuit already fully treated in Art. 485 above. We have with the notation there adopted $a_{11}=L_1$, $a_{12}=M$, $a_{22}=L_2$, $b_{11}=R_1$, $b_{12}=0$, $b_{22}=R_2$, and therefore the effective resistance and self-inductance are, for the primary,

$$\left.\begin{aligned} R'_1 &= R_1 + \frac{n^2M^2R_2}{R_2^2+n^2L_2^2} \\ L'_1 &= L_1 - \frac{n^2M^2L_2}{R_2^2+n^2L_2^2} \end{aligned}\right\} \quad \ldots\ldots \quad (89)$$

Thus if the frequency be very small the secondary has no effect on the primary. If the frequency be very great R'_1 approaches the limit $R_1+M^2R_2/L_2^2$ and L'_1 to L_1-M^2/L_2.

For the secondary we get as before

$$\left.\begin{aligned} R'_2 &= R_2 + \frac{n^2M^2R_1}{R_1^2+n^2L_1^2} \\ L'_2 &= L_2 - \frac{n^2M^2L_1}{R_1^2+n^2L_1^2} \end{aligned}\right\} \quad \ldots\ldots \quad (90)$$

the limiting values of which are R_2, L_2 for very small frequency, and $R_2+M^2R_1/L_1^2$, L_2-M^2/L_1 for great frequency.

496. The reader may work out for himself the case of a series of conductors forming primary, secondary, tertiary, &c., circuits, but such that no mutual induction exists except between the primary and secondary, the secondary and the tertiary, and so on.

For example consider four circuits. The current in the fourth is due to the inductive action of the third. The reaction of the fourth in the third causes the latter to have an effective resistance R'_3 and self-inductance L'_3 at once calculable from (89) by substituting for R_1, L_1, R_2, L_2, M the quantities R_3, L_3, R_4, L_4, M_{34}. Then if R'_3, L'_3 be used as the resistance and self-inductance of the third circuit the fourth may be ignored. The effective resistance and self-inductance of the second circuit due to the reaction of the third can be found in the same way, and the third then ignored. Finally the effective resistance and self-inductance of the primary can be found by another application of the same formula.

When the alternation is very rapid it will be found that the phases of the currents in the different circuits of the series depend in the case of very rapid alternation on the induction coefficients only, and differ successively by half a period.

497. Another of Lord Rayleigh's examples (*loc. cit.*) is the case of two parallel conductors, of resistances R_1, R_2, self-inductances L_1, L_2, and mutual inductance M, joining two parts of a circuit in which an alternating current is flowing. If γ be the total current the portion γ_1,

γ_2 into which it would divide itself between the parallel conductors were there no inductance would be $R_2\gamma/(R_1+R_2)$, $R_1\gamma/(R_1+R_2)$. We may take γ as the velocity for one co-ordinate, and for the other velocity a quantity γ' such that the current in the first conductor is $R_1\gamma/(R_1+R_2)+\gamma'$ and in the second $R_1\gamma/(R_1+R_2)-\gamma'$. We obtain easily

$$\left.\begin{aligned} T &= \tfrac{1}{2}\frac{L_1R_2^2+2MR_1R_2+L_2R_1^2}{(R_1+R_2)^2}\gamma^2+\frac{(L_1-M)R_2-(L_2-M)R_1}{R_1+R_2}\gamma\gamma' \\ &\qquad +\tfrac{1}{2}(L_1-2M+L_2)\gamma'^2 \\ F &= \tfrac{1}{2}\frac{R_1R_2}{R_1+R_2}\gamma^2+\tfrac{1}{2}(R_1+R_2)\gamma'^2 \end{aligned}\right\} \quad (91)$$

It is supposed that there is no energy of electric charge to be taken into account.

The literal coefficient of the first term in the value of T is a_{11}, that of the second term is a_{12}, and that of the third is a_{22}. Similarly the coefficients of the terms of F are b_{11}, b_{22} respectively, and $b_{12}=0$. Thus we get at once for the effective resistance and self-inductance of the parallel connection

$$\left.\begin{aligned} R' &= \frac{1}{R_1+R_2}\left\{R_1R_2+n^2\frac{\{(L_1-M)R_2-(L_2-M)R_1\}^2}{(R_1+R_2)^2+n^2(L_1-2M+L_2)^2}\right\} \\ L' &= \frac{1}{L_1-2M+L_2}\left\{L_1L_2-M^2+\frac{\{L_1-M)R_2-(L_2-M)R_1\}^2}{(R_1+R_2)^2+n^2(L_1-2M+L_2)^2}\right\} \end{aligned}\right\} \quad (92)$$

As noticed above $L_1L_2-M^2$ is necessarily positive, though it may be made very nearly zero by winding the two parallel conductors together. Also L_1-2M+L_2 is essentially positive as it is twice the kinetic energy of the system when the current in the first conductor is $+1$ and that in the other -1.

The reader will notice that when n is very small $R'=R_1R_2/(R_1+R_2)$, while that of $L'=(L_1R_2^2+2MR_1R_2+L_2R_1^2)/(R_1+R_2)^2$, and that when n is very great $R'=\{R_1(L_2-M)^2+R_2(L_1-M)^2\}/(L_1-2M+L_2)^2$, $L'=(L_1L_2-M^2)/(L_1-2M+L_2)$. He may also find from (91) the currents in the two conductors at any instant, and their phases with reference to the total current.

Another very interesting example is obtained in the Wheatstone Bridge arrangement of conductors, when inductances as well as resistances have to be taken into account. This will be fully dealt with, however, in the discussion in Vol. II. of methods of comparing inductances. (See also *Absolute Measurements*, Vol. II., Part II., Chapter VIII.).

CHAPTER XI

GENERAL ELECTROMAGNETIC THEORY

SECTION I.—*Electromagnetic Theory of Light*

Electromotive Intensity at Element of Moving Conductor. Total Electromotive Force

498. We have seen above (Arts. 420—425) that if an element of conducting material move with velocity $\dot{s}\,[=(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}]$ in a field in which the magnetic induction at the element is **B** (components a, b, c), an impressed electric intensity **e** is produced which is equal to the vector-product of $\dot{s}$ and **B**. For the components P, Q, R of **e** we obtain by the process of Art. 387 above

$$(P, Q, R) = (c\dot{y} - b\dot{z},\quad a\dot{z} - c\dot{x},\quad b\dot{x} - a\dot{y}) \quad . \quad . \quad (1)$$

The direction of **e** is found to be that in which a right-handed screw would advance if the handle were turned round from the direction of B to that of $\dot{s}$. (See Fig. 136.)

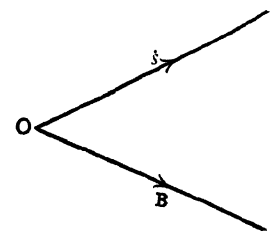


FIG. 136.

But at any point of the field the value of **B** may be changing with the time. The line-integral of the electromotive intensity just found taken round a linear conductor forming a circuit, gives what is sometimes called the *total electromotive force*, so far as that is due to motion of the circuit. To this there must be added the total electromotive force in the circuit due to that part of the time-rate of change of magnetic induction through the circuit which is independent of motion of the circuit. The total electromotive force from all sources in this circuit will be obtained by taking the line-integral of **E** round the circuit, where **E** is the resultant electric intensity and has the components

$$(P, Q, R) = \left(c\dot{y} - b\dot{z} - \frac{\partial F}{\partial t} - \frac{\partial \Psi}{\partial x},\ a\dot{z} - c\dot{x} - \frac{\partial G}{\partial t} - \frac{\partial \Psi}{\partial y},\ b\dot{x} - a\dot{y} - \frac{\partial H}{\partial t} - \frac{\partial \Psi}{\partial z}\right) \quad (2)$$

Here F, G, H are the components of vector-potential, and $\dot{x}, \dot{y}, \dot{z}$ are the components of the velocity of the element of conductor, considered as cutting across the tubes of induction. This velocity, it must be noticed, is the velocity of the conductor relatively to the tubes, and not with reference to any system of axes connected with moving matter. The function Ψ includes the electrostatic potential, and also any other terms giving components of electric force which, integrated round a closed circuit, give a zero result. For example, when the total electromotive force for a moving conductor is deduced from the time-rate of change of the line-integral of vector-potential round the circuit considered, there appear, besides the terms of the components in (2) depending on the vector potential, terms which are respectively

$$-\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(F\dot{x} + G\dot{y} + H\dot{z})$$

and which vanish when integrated round a closed circuit.

499. We have here considered electromotive force due to variation of the magnetic induction through a closed circuit, but we are now led by phenomena which have to be explained to take the view that whenever a portion of matter, whether conducting in any degree or not, is in motion in a magnetic field so that it cuts across tubes of magnetic induction, inductive electromotive intensity given by (1) is produced in it, and further that electromotive intensity is set up *wherever* magnetic induction is varying in value. In an insulator, however, no ordinary conduction current producing dissipation of energy can exist, but the substance is the seat of an electric field intensity, the value of which may be constant or may vary with the time. If the latter is the case, the time-rate of variation of the corresponding electric induction divided by 4π is taken as an electric current which with the conduction current constitutes a circuital system of currents producing the magnetic field, and is subject to all the laws of currents. The justification of this assumption lies in the agreement, so far as investigation has yet proceeded, of its results with observation. By its aid we regard all currents as closed; for example, when a conductor is being charged from an electric machine to which it is connected by a wire, the electric field surrounding the conductor is undergoing change, and a current exists in the dielectric which renders the charging current in the wire circuital.

The hypothesis is perfectly reasonable from the point of view of flow of energy already referred to in Arts. 145–148 and elsewhere above. Into the wire there flows across its lateral surface a certain amount of energy which is dissipated in the conducting substance, producing heat; into a portion of the dielectric in which the electric field intensity is being increased there flows an amount of energy which is not dissipated, if the medium is a perfect insulator, but remains stored as energy of electric strain.

Flow of Energy in the Insulating Medium, and Dissipation in a Conductor. Displacement Currents

500. If the wire or other conductor receiving energy from the medium were a perfect conductor, there would be no dissipation in it, but energy would enter it. Into an ordinary conductor energy can be conveyed in consequence of the fact that it has to some extent the properties of an insulator as regards supporting electric strain, temporarily at least, so that electrical strain penetrates its outer strata, and energy is dissipated in the matter in the interior in consequence of the ultimate breaking down of the strain which is there set up.

Thus the inward flow of electric energy upon any portion of the dielectric medium occasions a time-rate of variation of the electric induction, which, to a numerical factor, is related to the rate at which energy is received per unit of volume, as is the conduction current to the rate of dissipation of energy per unit of volume in the conductor. It is the flow of energy in the medium of which the magnetic field is an accompaniment, wherever it occurs: the ultimate disposal of the energy as increase of electrical strain and corresponding stress, and as increase of electrokinetic energy, or in the production of heat, does not itself have any effect except in so far as it indirectly reacts on the march of the phenomena.

The closing of the circuits of all currents in conductors, by means of displacement currents in the dielectric, constitutes the most remarkable feature of Maxwell's theory, and leads at once by a natural development to his great generalisation, the electromagnetic theory of light. This we shall now endeavour to explain, referring the reader for a most instructive discussion of electric displacement, and displacement to Heaviside's articles on *Electromagnetic Induction and its Propagation.*[1]

Impressed Electric Force. Illustration by Ideal Magneto-Electric Machine. Energy is received by System at Seat of Impressed Forces

501. Returning now for a little to a conductor moving in a magnetic field, take the case, described in Art. 425 above, in which a conductor, or a straight wire at right angles to the tubes of magnetic induction, is moved at a uniform speed in a direction at right angles at once to the wire and to the induction. The electric state of the wire remains unchanged while the motion continues. A difference of electric potential between its extremities is maintained which it is consistent with experience to take as the line-integral of **e** along the conductor. This difference of potential of course tends to produce a backward current in the conductor, and thus to annul itself, and is prevented from doing so by the inductive electromotive force. An electric field is thus produced, surrounding the moving conductor and maintained by the inductive action.

[1] *Electrical Papers*, Vol. I.

502. This is an example of *impressed* electric force at each point balancing an electric force which otherwise would set up a current, and involve redistribution and dissipation of energy. For although as the wire moves carrying its electrification with it, the electric displacement is being continually changed in position in the medium, once the steady state has been set up there is neither expenditure nor gain of energy in continuing the motion. The energy stored in the electric state is dissipated when the motion is stopped.

The impressed electric intensity is the electric intensity at every point, which balances that due to the electrification produced. Thus we have an illustration of the important fact, insisted on with great force by Heaviside, that in considering electric and magnetic intensity and the consequent flow of energy in the field it is necessary to take account of the impressed intensities, and make a sharp distinction between them and the intensities set up in consequence of changes of the medium produced by them.

503. If the wire we are considering touch at each end one of two parallel rails of conducting material, which are connected elsewhere, a current of amount γ will flow round the circuit as shown in Fig 116. The electromagnetic force on the sliding wire will tend to stop the motion which produces the current, and if l be the length of the slider its amount will be $\mathbf{B}l\gamma$. To keep the slider in motion therefore, work will have to be expended on it at rate $\mathbf{B}l\gamma v$. But $\mathbf{B}v$ is the impressed electric intensity, $\mathbf{e}$ say, along the slider at each point, and the time-rate at which energy enters the system, from without, per unit length of the slider is $\mathbf{e}\gamma$. When dissipation in the slider is taken into account, the actual rate of flow of energy to the medium per unit of length of the slider is $-\mathbf{E}\gamma$, where $\mathbf{E}$ here denotes the actually existing electric *field-intensity* parallel to the slider, and the rate of dissipation in the slider itself is $(\mathbf{e}+\mathbf{E})l\gamma$. It is to be remembered that $\mathbf{E}$, which is the *field-intensity*, is opposed to $\mathbf{e}$.

504. Thus in considering the delivery of energy to any system from without we have to bring the impressed forces properly into the account. So far we have dealt only with impressed electric intensity, but the same thing holds for magnetic intensity. If we had such a thing anywhere as a magnetic current G, and $\mathbf{h}$ denoted the impressed magnetic force at the place, the energy delivered there per unit time would be $\mathbf{h}G$. We shall consider the flow of energy later in the present chapter, and again in Vol II., when we shall have further to consider impressed forces, and to classify them. The reader should however refer to Heaviside's *Electromagnetic Theory*, Vol. I. Chapter II.

Complete Specification of Current in Imperfect Conductor or Imperfect Insulator. Fundamental Circuital Equations of Field.

505. The total current in a not perfectly conducting material is the sum of the ordinary conduction current, the displacement current, and a

current which we shall consider in a later section of this chapter due to moving matter carrying with it an electric charge. The total current in the direction of x at any point is thus

$$u = \kappa P + \frac{k}{4\pi}\dot{P} + e\dot{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where κ denotes the conductivity of the medium, k the dielectric inductivity, and e the electric charge per unit of volume of the moving matter at the point in question. The product $e\dot{x}$ may be taken as the definition of the measure of the convection current at any point.

Similar equations in $Q, \dot{Q}, \dot{y}, R, \dot{R}, \dot{z}$, hold for the components v, w of the total currents.

Applying the circuital theorem obtained in Art. 379 above, that is, assuming that the experimental results which have been found to hold for circuital conductional currents hold also for currents made circuital by displacement currents, and for convection currents with circuits however completed, we obtain for matter at rest relatively to the medium $(kp + 4\pi\kappa)\,\mathbf{E} = \text{curl}\,\mathbf{H}$, where p stands for $\partial/\partial t$.

506. Again we have another circuital theorem, namely, that the total electromotive force round a circuit is equal to the time-rate of diminution of magnetic induction through it. This for an infinitesimal circuit may be expressed by $-p\,\mathbf{B} = \text{curl}\,\mathbf{E}$. To complete the analogy between this equation and the one stated above Heaviside introduces a non-existent quantity $g\mathbf{H}$, which may be regarded as 4π times the magnetic current. Thus we obtain the two corresponding circuital equations

$$\left.\begin{aligned}(kp + 4\pi\kappa)\mathbf{E} &= \text{curl}\,\mathbf{H}\\ (\mu p + 4\pi g)\mathbf{H} &= -\text{curl}\,\mathbf{E}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these two relations Heaviside has derived his solutions of problems of wave propagation, and they are equivalent to a form into which Hertz has thrown Maxwell's equations and which he has used in his theoretical researches on electromagnetic radiation.

Since there is no divergence of the current anywhere, and the magnetic induction is also without divergence, we hav the additional two equations

$$\text{div.}\,\mathbf{E} = 0, \quad \text{div.}\,\mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the first of which holds at all points except where there is electrification, the second at every point, whether within or without magnets.

Here we suppose that there is no impressed force, magnetic or electric, to be taken into account, and that the state considered is one set up by an electromagnetic wave propagating itself. It may however be remarked that no polar electric or magnetic intensities, that is intensities derivable from potential functions, can enter these equations, inasmuch as the curl of every such intensity is zero.

Propagation of a Plane Electromagnetic Wave

507. Let the electromagnetic wave be a plane wave, propagated along the axis of z with wave front parallel to the axes of x, y. That the equations are consistent with such a disturbance it is easy to show by eliminating $\mathbf{E}$, say, from (4). Thus we obtain

$$(\mu p + 4\pi g)(kp + 4\pi\kappa)\mathbf{H} = \nabla^2\mathbf{H} \quad . \quad . \quad . \quad . \quad (6)$$

the well-known equation of wave propagation. Precisely the same equation could have been found for $\mathbf{E}$ by eliminating $\mathbf{H}$. It will be noticed that $\mathbf{H}$ and $\mathbf{E}$ are directed quantities at right angles to one another and both by the restriction imposed in the disturbance lying in the wave front.

Taking $g = 0$, we have for the periodic solution of (6) according as $\mathbf{E}$ or $\mathbf{H}$ is the quantity sought,

$$(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_0, \mathbf{H}_0) \text{ exp. } i(mz - nt) \quad . \quad . \quad . \quad . \quad (7)$$

where $i = \sqrt{-1}$, and $2\pi/n$ is the period of vibration. Substituting from this in (6) we find that the condition

$$m^2 - n^2k\mu - 4\pi\kappa\mu ni = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

must be satisfied. Thus since we suppose n to be real, m^2 is essentially complex, and therefore so also is m. Writing $m = q - ri$, where q and r are real; squaring and equating real and imaginary parts we obtain

$$q^2 - r^2 = n^2k\mu, \quad qr = -2\pi\kappa\mu n \quad . \quad . \quad . \quad . \quad (9)$$

Since r^2 must be a positive quantity since r is real, these equations give

$$\left.\begin{aligned} r &= -\frac{n\sqrt{\mu}}{\sqrt{2}}\left\{\sqrt{k^2 + 16\pi^2\kappa^2/n^2} - k\right\}^{\frac{1}{2}} \\ q &= \frac{n\sqrt{\mu}}{\sqrt{2}}\left\{\sqrt{k^2 + 16\pi^2\kappa^2/n^2} + k\right\}^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

in which the radicals in each case have the positive sign, and r is made negative for the reason that the disturbance is not to increase in amplitude as it travels out from the source. Thus q is a positive real quantity.

By (9) the solution becomes

$$(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_0, \mathbf{H}_0) \text{ exp. } rz \text{ exp. } i(qz - nt) \quad . \quad . \quad (11)$$

We are only concerned with the real part of the quantity of the right of (11) in the phenomenon observed. Hence we obtain finally

$$(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_0, \mathbf{H}_0) \text{ exp. } rz \,.\, \cos(qz - nt) \quad . \quad . \quad . \quad (12)$$

where the zero of time is so chosen that $\mathbf{E}_0$, $\mathbf{H}_0$ are the maximum values of $\mathbf{E}$, $\mathbf{H}$ in the plane $z = 0$.

This result gives for the wave length and velocity of propagation V the values

$$\left.\begin{aligned} \lambda &= \frac{2\pi}{q} = 2\pi/\left\{\frac{n\sqrt{\mu}}{\sqrt{2}}\left(\sqrt{k^2 + 16\pi^2\kappa^2/n^2} + k\right)^{\frac{1}{2}}\right\} \\ V &= \frac{n}{q} = 1/\left\{\frac{\sqrt{\mu}}{\sqrt{2}}\left(\sqrt{k^2 + 16\pi^2\kappa^2/n^2} + k\right)^{\frac{1}{2}}\right\} \end{aligned}\right\} \quad . \quad (13)$$

and for the ratio, at any time, of the amplitude in any plane $z = \lambda$, to that in the plane $z = 0$.

$$\text{exp. } r\lambda = \text{exp. } - \frac{2\pi\left(\sqrt{k^2 + 16\pi^2\kappa^2 n^2} - k\right)^{\frac{1}{2}}}{\left(\sqrt{k^2 + 16\pi^2\kappa^2 n^2} + k\right)^{\frac{1}{2}}} \quad . \quad . \quad (14)$$

Thus the wave length and velocity of propagation are less, and the rate of diminution of amplitude with distance from the source is greater the greater the conductivity, other things remaining the same.

If $\kappa = 0$, that is if the medium is an insulator, $r = 0$, and

$$q = n\sqrt{k\mu}, \quad V = \frac{1}{\sqrt{k\mu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Plane Wave in Æolotropic Dielectric, Principal Wave-Velocities

508. We shall now, following Maxwell, suppose the medium to be æolotropic as regards electric inductivity, but at the same time isotropic as regards magnetic inductivity. Taking f, g, h as before as components of electric induction, we have as in Art. 168 above

$$\left.\begin{aligned} f &= k_{11}P + k_{12}Q + k_{13}R \\ g &= k_{12}P + k_{22}Q + k_{23}R \\ h &= k_{13}P + k_{23}Q + k_{33}R \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If we suppose the induction taken in a direction in which it is parallel to the electric intensity, then, denoting the electric inductivity for that direction by k, putting kP, kQ, kR, for f, g, h in (16), and eliminating P, Q, R we get the determinantal equation

$$\left|\begin{matrix} k_{11} - k, & k_{12}, & k_{13} \\ k_{12}, & k_{22} - k, & k_{23} \\ k_{13} & k_{23} & k_{33} - k \end{matrix}\right| = 0 \quad . \quad . \quad . \quad . \quad (17)$$

which is a cubic in k, with three real roots k_1, k_2, k_3, say, which apply to three directions at right angles to one another. For if l, m, n

be the direction cosines of one of these we may write (16) in the form

$$k_1 l + k_{12} m + k_{13} n = kl$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

three equations from which by substitution of k_1, k_2, k_3 the three corresponding sets of values of l, m, n, namely l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3, can be found. It can be verified easily that $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, &c. These directions we call *principal directions.*

509. We now find the corresponding equations of wave propagation, taking components of induction in the principal directions. In so doing we shall suppose the conductivity κ of the medium to be zero. The circuital equations (4) above for the determination of the three components P, Q, R of electric intensity in the three principal directions yield three equations of the form

$$k_1 \mu \frac{\partial^2 P}{\partial t^2} = \nabla^2 P - \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \quad . \quad . \quad . \quad (18)$$

since $\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$ is not in this case zero.

Consider now a plane wave travelling at right angles to the plane $lx + my + nz = 0$, with velocity V. The disturbance may be expressed by the equation

$$(P, Q, R) = (P_0, Q_0, R_0) \exp. \frac{2i\pi}{\lambda}(lx + my + nz - Vt) \quad (19)$$

Substituting from (19) *in* (18) and writing v_1^2, v_2^2, v_3^2 respectively for $1/k_1\mu$, $1/k_2\mu$, $1/k_3\mu$ we obtain the conditions expressed by

$$\left.\begin{aligned} (V^2 - v_1^2)P + v_1^2 l(lP + mQ + nR) &= 0 \\ (V^2 - v_2^2)Q + v_2^2 m(lP + mQ + nR) &= 0 \\ (V^2 - v_3^2)R + v_3^2 n(lP + mQ + nR) &= 0 \end{aligned}\right\} \quad . \quad . \quad (20)$$

These by elimination of P, Q, R yield

$$\frac{v_1^2 l^2}{V^2 - v_1^2} + \frac{v_2^2 m^2}{V^2 - v_2^2} + \frac{v_3^2 n^2}{V^2 - v_3^2} = -1$$

which since $l^2 + m^2 + n^2 = 1$ may be written

$$\frac{l^2}{V^2 - v_1^2} + \frac{m^2}{V^2 - v_2^2} + \frac{n^2}{V^2 - v_3^2} = 0 \quad . \quad . \quad . \quad (21)$$

This is Fresnel's equation connecting the velocity of light in an æolotropic body with the direction of propagation, and v_1^2, v_2^2, v_3^2 are the squares of the principal velocities of propagation. These are the velocities of an ordinary wave in the three principal directions, that is the velocities in three different isotropic dielectrics characterised by the constants k_1, k_2, k_3.

510. Let l', m', n' be the direction cosines of the electric induction. The components are proportional to k_1P, k_2Q, k_3R, and these by (20), since $v_1^2 = 1/k_1\mu$, &c., are proportional to $l/(V^2-v_1^2)$, $m/(V^2-v_2^2)$, $n/(V^2-v_3^2)$. Hence $ll'+mm'+nn'=0$, or the displacement is in the wave front.

Take the direction of propagation as that of x, and the direction of the electric induction as that of y, then $n=1, l=m=0$, $m'=1, l'=n'=0$. But since $m' \propto m/(V^2-v_2^2)$, and $m=0$, we have $V=v_2$. Similarly when the electric induction is in the direction of x, $V=v_1$. In the former case we have an *ordinary* wave travelling along z. But by optical experiment it is known that an ordinary ray of light travelling parallel to a principal axis of a crystal, for example, at right angles to the optic axis of a uniaxal crystal such as Iceland spar, is polarized in a plane parallel to the axis and the direction of propagation. Hence the electromagnetic theory gives the direction of electric displacement as at right angles to the plane of polarization. This conclusion has been verified by the experiments of Hertz and of Trouton.

Question of Relation of Electrical Vibrations to Elastic Vibrations of Ether

511. The question as to whether the direction of the electric intensity, or that of the magnetic intensity, coincides with the direction of the vibratory motion attributed to the ether, in the ordinary dynamical theory of the luminiferous ether as an elastic substance, transmitting by its great rapidity waves of distortion at the velocity of light, is not one that admits of answer at present. There is first of all the question of what action in the matter composing the ether, if it is matter possessing inertia, corresponds to the electromagnetic vibrations, and on this investigation so far has thrown little if any light. It may be that there is no direction of vibration of the matter of the ether in the ordinary sense at all, and that the whole affair is electromagnetic, or it may be that the explanation is an affair of the motion of electrons, that is atoms of matter carrying electric charges. Whatever may turn out to be the final result in this inquiry, the relation of the electric vibration to the plane of polarisation is perfectly clearly made out, and is as stated above.

Determination of Velocity of Propagation—Ratio of Units

512. Clerk Maxwell regarded light as an electromagnetic disturbance in the ether, and put forward the theory of which a sketch has just been given as one of the propagation of light pure and simple. His views have since been strikingly confirmed by the experiments of Hertz and those who followed him, on the production and propagation of electromagnetic waves; and of these researches we shall give some account presently. But previous to this direct verification, one prediction of the theory had been found in striking agreement with actual fact.

For according to the theory the ratio of the numerical value of a quantity of electricity, measured in electrostatic units, to the numerical value of the same quantity when measured in electro-magnetic units, ought to be a definite velocity, coinciding, if the electro-magnetic theory of light is correct, with the velocity of light. This ratio has been determined and found to be almost exactly the same as the velocity of light as measured by the methods of Foucault and Fizeau.

513. To make this matter clear we shall use an illustration due to Maxwell, modifying however the mode of applying it to suit the ideas adopted in this book, according to which k and μ are fundamental physical qualities, depending on the nature and, in some cases no doubt, the state of the medium as regards electrification and magnetization.

It has been shown in Art. 387 that the electromagnetic force on an element ds of a conductor carrying a current γ in a magnetic field is $\mathbf{B}\gamma\sin\theta ds$ if $\mathbf{B}$ be the magnetic induction at the element, and θ the angle between the element and the direction of the magnetic induction.

If the field be produced by a current γ' in an infinitely long straight conductor parallel to ds at distance b from it, we get by integration of the expression $\gamma'\sin\theta' ds'/r^2$ the value $2\gamma'/b$ for the magnetic field intensity. Hence if μ be the permeability of the medium, the electromagnetic force on ds is $2\mu\gamma\gamma' ds/b$, and if the first conductor be straight the force on a length $b/2$ of it is $\mu\gamma\gamma'$.

The quantities of electricity conveyed by the two currents in time t are γt, $\gamma' t$. Let these be used to charge two conducting spheres whose centres are at a distance apart great in comparison with the radius of either. The electrostatic repulsion between the spheres would then be $\gamma\gamma' t^2/kr^2$. If r be chosen so that this force is the same as the attraction between the conductors exerted on a length equal to half the distance between them, we have $\mu\gamma\gamma' = \gamma\gamma' t^2/kr^2$ or

$$\frac{1}{\mu k} = \frac{r^2}{t^2} \quad \ldots\ldots\ldots \quad (22)$$

that is $1/\sqrt{\mu k}$ may be expressed as a velocity. This result holds whatever hypothesis as to dimensions is adopted for μ and k.

514. This is moreover a perfectly definite velocity. For if t^2/r^2 remain constant the repulsion between the spheres remains unchanged, while these charges are increased at the rates γ, γ' respectively. Therefore $1/\sqrt{\mu k}$ is equal to the velocity with which the spheres must be separated in order that their mutual repulsion may remain equal to the force of attraction on a length of either of the parallel conductors equal to half the distance between them. It has just been shown that $1/\sqrt{\mu k}$ is the velocity of propagation of an electro-magnetic wave in an isotropic perfectly insulating dielectric.

515. If now we denote by v the ratio of the electromagnetic to the electrostatic unit of quantity, the charges on the spheres expressed in ordinary electrostatic units are $v\gamma t$, $v\gamma' t$, if γ, γ' now denote the ordinary

electromagnetic measure of the currents. Hence the force between the two spheres is $v^2\gamma\gamma' t^2/Kr^2$, where K denotes the specific inductive capacity of the medium, defined in the ordinary way as the ratio of the electric inductivity of the medium to that of the medium of reference (air or vacuum, for example). But if ϖ denote the ordinary electromagnetic value of the permeability defined as in Art. 72 above, $\varpi\gamma\gamma' = v^2\gamma\gamma' t^2/Kr^2$ that is

$$v^2 = \varpi K \frac{r^2}{t^2} \quad .$$

or by (22)

$$v^2 = \frac{\varpi K}{\mu k} .$$

Now for air $K=1$, $\varpi=1$ and we get

$$v = \frac{1}{\sqrt{\mu k}} . \qquad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

that is v is equal to the velocity of propagation of an electromagnetic disturbance in air.

For a full account of measurements o v the reader may refer to the author's *Treatise on Absolute Measurements in Electricity and Magnetism*, Vol. II., Part II., Chapter XI. Some account of recent work on this subject will be found also in Volume II. of the present work.

Electromagnetic Radiation. Hertz's Solution of Maxwell's Equations. Vibrating Electric Doublets

516. The mathematical theory given above is not very well adapted for the complete investigation of the radiation in a field in which the changes are due to harmonically varying electric charges on a conductor or conductors which have been charged in a particular manner and then left to themselves. Such a case we have in the oscillatory discharge of a condenser discussed in Chapter X. above; though there, as was noticed, no account was taken of the radiation of energy in bringing the action to a stop. A somewhat different theoretical discussion was given by the late Professor Hertz[1] for the case of an electric doublet, that is two equal and opposite electric charges concentrated at two points infinitely near to one another, or, more properly, at a distance infinitely small in comparison with the distance from either charge of any point at which the electric or magnetic force is considered. We shall deal here with this simple arrangement, neglecting however at first the damping of the vibration: that is, we shall suppose the moment of the doublet, that is, the product of either charge into the

[1] Wiedemann's *Annalen*, Jan. 1889, and *Nature*, Feb. 21, 1889, also Hertz's *Electric Waves*, Jones's Translation (Macmillan & Co.).

distance between the two points, to vary as a simple harmonic function of the time, but to retain throughout the same maximum value.

517. Such a source of electromagnetic waves or electric vibrator, as it is called, may be regarded as physically realised, except for points at distances from it comparable with its dimensions. by two equal and oppositely charged spheres connected by a straight conductor (Fig. 137). Such was the vibrator employed by Hertz in some of his most important researches. The spheres were charged to opposite potentials by an induction coil, and then discharged into one another. The discharge was oscillatory, and set up electromagnetic waves in the surrounding medium, which were propagated outwards from the vibrator in all directions. The existence of the waves in the medium was detected by a simple receiver, or resonator, as it has been called, which in one form consisted of a circle of wire, complete with the exception of a very small spark-gap between two small knobs which tipped the ends of the wire, and was properly placed relatively to the vibrator.

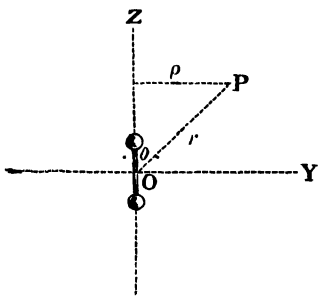


FIG. 137.

It is to be observed that the vibrations here set up were a rapidly damped out set of electrical oscillations, and that in consequence it was not necessary that the natural period of electrical oscillation of the receiver should be made very nearly equal to that of the exciter (see Art. 539).

518. We shall now give Hertz's solution of the equations of the electromagnetic field for an electric doublet, the moment of which is subjected to simple harmonic variation with the time. The axis of the doublet, that is the line joining the charges, will be supposed to lie along the axis of z, and the origin will be taken at the point midway between the two charges. Since everything is symmetrical round the axis, it is only necessary to consider the disturbance at any instant, at a point distant z from the origin, and ρ from the axis. Calling planes through the axis meridian planes, and a plane through the origin perpendicular to the axis an equatorial plane, we see that the electric intensity at every point lies in a meridian plane, and that the lines of magnetic intensity are circles round the axis. The foundation of the theory are the two circuital equations (4), with the relations (5).

Taking α, β as the components of magnetic intensity at any point, in a plane at right angles to the axis, we have $\gamma = 0$, and $\partial\alpha/\partial x + \partial\beta/\partial y = 0$. Thus $\alpha dy - \beta dx$ is a complete differential of some function of x, y. We take this as $\partial\Pi/\partial t$ so that

$$\alpha = \frac{\partial^2\Pi}{\partial t \partial y}, \quad \beta = -\frac{\partial^2\Pi}{\partial t \partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The first circuital relation gives for the components, P, Q, R, of electric intensity the equations

$$k\frac{\partial P}{\partial t} = \frac{\partial^3 \Pi}{\partial t \partial x \partial z}, \quad k\frac{\partial Q}{\partial t} = \frac{\partial^3 \Pi}{\partial t \partial y \partial z}, \quad k\frac{\partial R}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2}{y^2}\right)$$

These indicate that the values of the quantities

$$kP - \frac{\partial^2 \Pi}{\partial x \partial z}, \quad kQ - \frac{\partial^2 \Pi}{\partial y \partial z}, \quad kR + \frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2}$$

are independent of t, and cannot therefore have any influence on the propagation of waves. We therefore assume

$$kP = \frac{\partial^2 \Pi}{\partial x \partial z}, \quad kQ = \frac{\partial^2 \Pi}{\partial y \partial z}, \quad kR = -\left(\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2}\right) \quad . \quad (25)$$

The second circuital equation applied to these gives

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial^2 \Pi}{\partial t^2} - \frac{1}{k\mu}\nabla^2 \Pi\right) = 0 \\ \frac{\partial}{\partial y}\left(\frac{\partial^2 \Pi}{\partial t^2} - \frac{1}{k\mu}\nabla^2 \Pi\right) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Thus the quantity in brackets is a function of z, t only and we write

$$\frac{\partial^2 \Pi}{\partial t^2} - \frac{1}{k\mu}\nabla^2 \Pi = f(z, t) \quad . \quad . \quad . \quad . \quad (27)$$

We may put $f(z,t)=0$ without affecting the electric and magnetic fields. For let $\Pi = \phi(x, y, z, t)$ be a solution of (26). Since each component of electric or magnetic force involves differentiation with respect to x or y, they will be given also by the solution $\Pi = \phi(x, y, z, t) + \psi(z, t)$, where $\psi(z, t)$, is any function of z and t. This would give the differential equation (26) with the addition of a term $\chi(z, t)$ on the right arising from $\psi(z, t)$. We may choose $\chi(z, t)$ so that $f(z, t) + \chi(z, t) = 0$, and therefore putting $f(z, t) = 0$ in (27) will not affect the electric or magnetic intensity at any point. Thus we have finally for the equation of propagation

$$\frac{\partial^2 \Pi}{\partial t^2} = \frac{1}{k\mu}\nabla^2 \Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

If r denote $\sqrt{x^2+y^2+z^2}$ or $\sqrt{\rho^2+z^2}$, the general solution of this equation is

$$\Pi = \frac{1}{r}\{F_1(r - vt) + F_2(r + vt)\} \quad . \quad . \quad . \quad . \quad (29)$$

where $v = 1/\sqrt{k\mu}$, and F_1, F_2 are arbitrary functions.

A solution adapted to the vibrator imagined is

$$\Pi = \frac{\Phi}{r} \sin (mr - nt) \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $m = 2\pi/\lambda$ and $n/m = v = 1/\sqrt{k\mu}$. This satisfies the differential equation and is of the form (28). Moreover, if we take Φ as the maximum moment of the doublet, we see that in the immediate neighbourhood of the origin, that is, at any point whose distance from the doublet is a small fraction of the wave-length of the disturbance, the electric field should at each instant precisely correspond to that of a small magnetic doublet of the same moment numerically as that of the electric doublet at the instant in question. The lines of intensity for a magnetic doublet are shown in Arts. 29, 30 above, and the field is there derived from a potential $V = -\mathbf{m}\partial(1/r)/\partial x$. Writing $\Phi \sin nt/k$ for $\mathbf{m}$, we see that the electric intensity to correspond ought to be given by a potential

$$V = -\frac{1}{k}\Phi \sin nt \frac{\partial}{\partial z}\left(\frac{1}{r}\right).$$

But since for the region considered mr is a very small angle, we may write approximately for all points very near the doublet.

$$\Pi = -\Phi \frac{\sin nt}{r}$$

and by (25) the electric potential in the field near the vibrator is

$$V = -\frac{1}{k}\frac{\partial \Pi}{\partial z} = \frac{1}{k}\Phi \sin nt \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

which agrees exactly with the magnetic analogue.

So far the solution agrees with what we should expect to be the case. Again it gives electric and magnetic intensity everywhere zero at an infinite distance, which also must be the case on account of the divergence of the waves.

Electric and Magnetic Intensities in Field of Doublet

519. For the sake of a generalisation of this solution to come as a succeeding article the three components P, Q, R of electric intensity have so far been retained in the analysis. But in order to calculate the electric and magnetic fields of the doublet we now take account of the fact that the electric intensity is everywhere directed along a meridian plane, and is symmetrically distributed about the axis. It is sufficient therefore to use cylindrical co-ordinates z along the axis from the origin, and ρ at right angles to the axis. (See Fig. 137). Identifying ρ with y, we have Q, R for the components of $\mathbf{E}$ in the meridian plane at the

point considered, while the magnetic intensity there reduces to a. We can find these components from the solution (30) by the second and third of (25), and the first of (24), by putting ρ for y, $r = \sqrt{\rho^2 + z^2}$, and writing the third equation of (25) in the form proper to symmetry round the axis of z, namely,

$$R = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Pi}{\partial \rho}\right) \quad . \quad . \quad . \quad . \quad (31)$$

Thus we find

$$\left.\begin{aligned} kQ &= \frac{3\Phi}{r^3}\left\{\left(1 - \frac{m^2r^2}{3}\right)\sin(mr - nt) - mr\cos(mr - nt)\right\}\sin\theta\cos\theta \\ kR &= \frac{\Phi}{r^3}\Big[2\{\sin(mr - nt) - mr\cos(mr - nt)\} \\ &\qquad - \{3\sin(mr - nt) - 3mr\cos(mr - nt) \\ &\qquad - m^2r^2\sin(mr - nt)\}\sin^2\theta\Big] \\ a &= \frac{n\Phi}{r^2}\{mr\sin(mr - nt) + \cos(mr - nt)\}\sin\theta \end{aligned}\right\} . \quad (32)$$

For points very near the vibrator these equations become

$$\left.\begin{aligned} kQ &= \frac{3\Phi}{r^3}\sin(mr - nt)\sin\theta\cos\theta \\ kR &= -\frac{\Phi}{r^3}\sin(mr - nt)\sin^2\theta \\ a &= \frac{n\Phi}{r^2}\cos(mr - nt)\sin\theta \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (33)$$

520. The expression here given for a is that of the magnetic intensity which according to Art. 391 above would be produced by a current γ in an element of length ds such that $\gamma ds = n\Phi\cos(mr - nt)$. But $n\Phi\cos(mr - nt)$ is the actual current in the doublet at time t multiplied by the length of the element. For points very near the doublet therefore the theory leads to the expression stated in Art. 391, and hence to that expression for the magnetic intensity produced at *any* point r, θ by an element of a steady current.

When r is very great equations (32) become

$$\left.\begin{aligned} kQ &= -\frac{\Phi}{r}m^2\sin(mr - nt)\sin\theta\cos\theta \\ kR &= \frac{\Phi}{r}m^2\sin(mr - nt)\sin^2\theta \\ a &= \frac{n\Phi}{r}m\sin(mr - nt)\sin\theta \end{aligned}\right\} . \quad . \quad . \quad (34)$$

Direction of Vibration. "Longitudinal Light." Rate of Propagation in Different Directions and at Different Distances

521. From (34) we obtain some important conclusions. Since Q, R, α all involve $\sin(mr-nt)$ as one factor, with others which are numerical or dependent on θ and r, the electric and magnetic intensities are, at a great distance from the origin, propagated together with velocity n/m, and are in the same phase.

Again (34) gives $Q\sin\theta+R\cos\theta=0$, that is, the electric intensity $\mathbf{E}$ has no component along the radius vector r. The direction of electric vibration is therefore at a great distance from the origin perpendicular to the radius vector from the centre of disturbance to the point; that is, it is transverse to the direction of the ray. The magnetic intensity is in the same plane and perpendicular to the ray, and of course being perpendicular to the meridian plane is at right angles to $\mathbf{E}$.

For points very near the origin, however, the direction of the resultant electric intensity is not at right angles to the ray, but has a longitudinal component. To obtain so-called "longitudinal light" therefore it is not necessary to go outside of Maxwell's theory of the electromagnetic field. The longitudinal component of the electric induction along the axis is given by kR in (32) with θ put equal to zero. Thus we have

$$kR = \frac{2\Phi}{r^3}\{\sin(mr-nt) - mr\cos(mr-nt)\}$$

$$= \frac{2\Phi}{r^3}\sqrt{1+m^2r^2}\sin(mr-nt-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $\tan\epsilon=mr$. The transverse component kQ of the induction is here zero, so hat the vibration along the axis is wholly longitudinal. It will be observed that at points on the axis the value of α is zero.

On the axis of the vibrator therefore there is a very special state of things. There is vibration of the electric induction, but that is wholly longitudinal, and there is no magnetic induction whatever. If there are special phenomena which can be produced by light of longitudinal vibrations, and consisting of only one kind of vibration, they should be looked for along the axis of a vibrator. It will be noticed that as the amplitude of the longitudinal component varies as the inverse cube of r, its value will be very small in comparison with the induction elsewhere, as given by (32), except near the origin.

The velocity of propagation of electric intensity along the axis of the vibrator is to be found by calculating dr/dt from $mr-nt-\tan^{-1}mr=0$, and is therefore $n(1+m^2r^2)/m^3r^2$. It is very great when r is small and approaches the value n/m, or $1/\sqrt{k\mu}$, as r increases.

522. In the equatorial plane on the other hand the value of kQ, which is here the longitudinal component, is zero, and kR is the resultant electric induction. The directions of kR and α are at right

angles to one another, the former in, the latter at right angles to the meridian plane through the point. The values of these quantities are given by (32) with $\sin\theta = 1$. Hence

$$\left.\begin{aligned} kR &= \frac{\Phi}{r^3}\sqrt{1 - m^2r^2 + m^4r^4}\sin(mr - nt - \epsilon) \\ \alpha &= \frac{n\Phi}{r^2}\sqrt{1 + m^2r^2}\sin(mr - nt - \epsilon') \end{aligned}\right\} \quad . \quad (36)$$

where $\tan\epsilon = mr/(1 - m^2r^2)$, $\tan\epsilon' = -1/mr$.

The velocity of propagation of electric induction in the equatorial plane is thus $n(m^4r^4 - m^2r^2 + 1)/m^3r^2(m^2r^2 - 2)$. This is greater than the velocity along the axis, except of course for great values of r, where it is n/m as in the other case. Moreover, it is infinite when $r = 0$, and when $r^2 = 2/m^2$, or $r = \lambda/(\pi\sqrt{2})$, and is negative at intermediate points.

We obtain therefore the very remarkable result that the electric induction is propagated outwards and inwards in the equatorial plane from a point outside the vibrator. In the representation of the lines of electric intensity (and induction) in the electric field given below in Figs. 138—141, this point is the centre of the small circle seen on each side of the vibrator in Fig. 141. At this point the electric force attains any value which it there takes about ·12 of a period before the corresponding value is attained at the origin.

The magnetic induction and intensity are propagated in the equatorial plane with a velocity $n(1 + m^2r^2)/m^3r^2$, which is also infinite when r is zero, but diminishes as r is increased towards the limiting value.

523. The reader may verify that the interval in which a zero or maximum value of the magnetic intensity travels out in the equatorial plane from the origin to a great distance r is $rm/n - T/4$, and that a zero value of the electric intensity travelling out in the same plane from a point in the circle of radius $\lambda/(\pi\sqrt{2})$, round the vibrator, reaches a point, distant r from the origin, in the interval $rm/n - T/2$ after the instant at which the zero value reached the origin. When, however, this zero value reaches the origin, the current has its maximum value, and so has the magnetic intensity. In the succeeding interval, $rm/n - T/4$, this maximum travels out a distance r, but the zero value of the electric intensity has arrived earlier by $T/4$, so that at a great distance the electric intensity is a maximum at the same instant as the magnetic intensity, in accordance with equations (34).

The foregoing discussion of velocities in the equatorial plane is in its essential particulars taken from a paper by Trouton in the *Phil. Mag.* for March, 1890, to which the reader may refer for further information.

Electromagnetic Theory of the Blue Sky

524. The action of Hertzian vibrators, it may here be remarked, is very instructive in connection with the dynamical explanation given by Lord Rayleigh[1] of the blue colour of the sky. Let a beam of plane polarized light of different wave-lengths, as in white light, be propagated across a space in which there are a number of particles all very small in comparison with any wave-length of the light. On the supposition that there is a difference in the density of the ether in the two media, and not in its rigidity, it can be shown on the elastic solid theory of the ether that these particles will act as centres of disturbance from which light will be radiated, the amplitude of which will vary as the inverse square, and the intensity as the inverse fourth power, of the wave-length. The forces acting on the ether within the particle will be in the wave-front and in the direction of the vibration in the exciting ray, and the theory shows that the direction of the vibration in the scattered light lies in a plane through the particle containing the direction of the force, and is transverse to the direction of propagation of the scattered ray. The light is therefore polarized in a plane through the ray at right angles to the plane just specified.

Further, there is no light scattered in the direction of the force on the particle, that is in the direction of the axis of symmetry, while the maximum of intensity is found for rays in a plane through the particle perpendicular to the axis of symmetry.

When the exciting beam is not polarized, the light scattered by the particle in any direction parallel to the wave-front is wholly due to the component of the exciting vibration which is perpendicular to that direction, and is therefore completely polarized in a plane through the exciting ray and the scattered ray, which in this case are at right angles to one another.

These results are found to be in accordance with experiments on the light scattered from a space in which small particles are suspended and through which a beam of plane polarized light is passed, as in those of Tyndall on light passed along a glass tube filled with carbon disulphide vapour, and viewed from the sides of the tube. Also it is found that the light received from a part of the sky, distant 90° from the sun, is always polarized in a plane through the sun.

525. Returning to the Hertzian vibrations, we have found [(34) above] that at a great distance from the vibrator the amplitude of the electric vibration is inversely as the square of the wave-length (and the intensity therefore as the inverse fourth power), is a maximum in the equatorial plane and is there transverse to the ray. Each ray is, moreover, polarized in a plane through the ray at right angles to the meridian plane in which it lies.

On the other hand, along the axis of the vibrator there is no magnetic intensity, and the electric vibration there is comparatively

[1] See *Phil. Mag.* Feb. 1871, or *Wave Theory of Light, Encyc. Brit.* 9th edition.

small and is longitudinal. We may therefore account for the blue of the sky on the electromagnetic theory by supposing the particles to be Hertzian vibrators set into forced electrical oscillation by the electromagnetic waves passing across the space where they are situated.

It is to be observed that this result will only hold at a distance of many wave-lengths from the particles; in the vicinity of the vibrator the radiation is much more complex.

We see that here again, the vibrators being supposed to be electric, the direction of the electric induction is at right angles to the plane of polarization of the radiation. The real proof of the relation of the direction of the electric vibration to the plane of polarization will be given below. To show that the present result is not conclusive, it is only necessary to recall that, if the vibrators were magnetic, the magnetic and the electric intensities in the present investigation would be interchanged without alteration.

Propagation of Electric Potential

526. The solution here given is equivalent to one which may be obtained by the following process. If the medium is at rest the equations of electric intensity are by (2)

$$(P, Q, R) = \left(-\frac{\partial F}{\partial t} - \frac{\partial \Psi}{\partial x}, \quad -\frac{\partial G}{\partial t} - \frac{\partial \Psi}{\partial y}, \quad -\frac{\partial H}{\partial t} - \frac{\partial \Psi}{\partial z} \right) \quad (37)$$

and the condition of zero electrification at any point of the field, $\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z = 0$, gives

where
$$\left.\begin{aligned} \frac{\partial J}{\partial t} + \nabla^2 \Psi &= 0 \\ J &= \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \end{aligned}\right\} \quad \ldots\ldots \quad (38)$$

The value of Ψ cannot generally be taken as independent of the time, in the case in which there are varying electric charges. Let us then assume that $J = -k\mu \partial \Psi/\partial t$, and inquire what this assumption involves. In the first place we obtain by means of it the equation

and
$$\left.\begin{aligned} k\mu \frac{\partial^2 \Psi}{\partial t^2} &= \nabla^2 \Psi \\ k\mu \frac{\partial^2 J}{\partial t^2} &= \nabla^2 J \end{aligned}\right\} \quad \ldots\ldots \quad (39)$$

which are equivalent. The solution of either of these equations gives Ψ and J.

527. If a system of electric currents, of components u, v, w, exist in the field, it is easy to prove from the values of the vector potential, given in (21) Chap. IX. above, or the same modified for the case of propagation, that where there is zero divergence of the current (and this must be the case where there is no electrification), that the vector-potential fulfils the condition

$$J = 0.$$

This condition is fulfilled throughout the dielectric by the induction currents there existing. It does not hold, however, where there is varying electric charge, and this there always is at the origin of the disturbance if the waves are due to the oscillations of electric charges on conductors. Hence we may consider only the vector-potential due to the currents at the conductors at which the changes of electrification are proceeding.

Modifying the expressions for the vector-potential of the system of currents in the field to provide for propagation, with velocity $1/\sqrt{k\mu}$($= n/m$), of values of the vector-potential, in the case of simple harmonic time-variation of the currents, we obtain

$$(F, G, H) = \mu \int \frac{(u_0, v_0, w_0)}{r} \cos(mr - nt) d\varpi$$

where $(u_0, v_0, w_0) \cos nt$ are the components of current at the element $d\varpi$ of volume, and r is the distance of the element $d\varpi$ from the point at which (F, G, H) is to be found. The integral is taken throughout the whole space, including both conductors and field.

We suppose now that $e_0 \sin nt$ is the electric charge per unit volume at the element $d\varpi$ at time t. This of course is confined to the conductors at which the disturbance originates. We write then

$$\Psi = -\frac{1}{k} \int \frac{e_0}{r} \sin(mr - nt) d\varpi \quad \ldots \ldots \quad (40)$$

where the integral is taken throughout the space containing electric charge.

It can easily be proved that the differential equation (39) is satisfied by the solution (40). The quantity Ψ may be regarded as in a sense an electric potential due to the harmonically varying charges. To this potential each element of charge, as that at $d\varpi$, makes a contribution, which is propagated outwards in every direction from the element, with speed n/m. From this and F, G, H, as shown in (37), the values of P, Q, R are to be found.

Further, it can be verified that this value of Ψ with those of F, G, H satisfies the equation

$$k\mu \frac{\partial \Psi}{\partial t} + J = 0.$$

From the values of F, G, H the components of magnetic induction and intensity can be calculated and the field completely determined.

This mode of solution is applicable to any space distribution of (u_0, v_0, w_0), or of the corresponding amplitude electrification density e_0. Attention was directed to it by Professor FitzGerald in 1890.[1]

528. This solution agrees with that of Hertz for the electric vibrator. Remembering that here the only place where J is not zero is in the vibrator, and that there the current is along the axis of z, we have $F' = 0$, $G = 0$, and

$$H = \mu \frac{w_0 ds}{r} \cos(mr - nt)$$

where ds is the distance between the two point-charges of the doublet. Also we have

$$\Psi = -\frac{1}{k}\frac{\partial}{\partial z}\frac{e_0 ds}{r} \sin(mr - nt) \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Now by (24) above $\mu \partial \Pi/\partial t = H$, since in this case $\mu a = \partial H/\partial y$. But in this case also $J = \partial H/\partial z$, and hence we have

$$-\mu \frac{\partial^2 \Pi}{\partial t \partial z} + \frac{\partial H}{\partial z} = 0, \quad k\mu \frac{\partial \Psi}{\partial t} + \frac{\partial H}{\partial z} = 0$$

which give

$$k\Psi = -\frac{\partial \Pi}{\partial z}.$$

Thus we may write by (41)

$$\Pi = \frac{e_0 ds}{r} \sin(mr - nt) \quad . \quad . \quad . \quad . \quad (42)$$

which is the value of Π used by Hertz.

Graphical Representation of Field of Vibrator

529. Very instructive diagrams were given by Hertz in his paper already referred to, illustrating the successive changes which take place in the electric field of a vibrator in the course of a period. The lines of electric intensity (and induction) can be drawn from their equation, which is easily found to be

$$\rho \frac{\partial \Pi}{\partial \rho} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

For the differential equation of a line of intensity is easily found to be

$$\frac{\partial}{\partial z}\left(\rho \frac{\partial \Pi}{\partial \rho}\right) dx + \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Pi}{\partial \rho}\right) d\rho = 0$$

[1] *B. A. Rep.* 1890, also *Phil. Mag.*, Sept. 1896.

which integrated yields (43), in which substitution from (30) gives

$$\frac{\Phi}{r}\{\sin(mr - nt) - mr\cos(mr - nt)\}\sin^2\theta = c \quad . \quad (44)$$

where c is a constant for any particular line.

530. These lines are given in Figs. 138—141 as drawn by Hertz. Fig. 138 shows the electric field as it exists at the beginning of an oscillation when the vibrator is in the neutral state, the remaining figures of the series give it as it appears after the lapse of successive eighths of a complete period. [N.B. The λ marked on the curves is one-half of the wave length.]

In the immediate vicinity of the vibration the lines are not drawn, and those drawn are stopped at a circle surrounding the vibrator. The

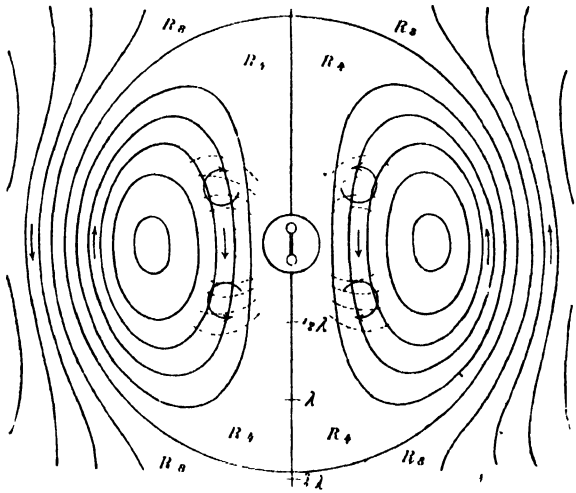


FIG. 138.

vibrator is of a dumb-bell shape, and its field in its own immediate vicinity must be very different from that of a doublet, though they will agree at some distance from the origin.

Fig. 139 shows the field after one-eighth of a period from the instant of zero electrification. The lines drawn are enclosed within the circle given by (44) for $t = \frac{1}{8}T$, and $c = 0$. This circle travels outwards with the velocity of propagation of the magnetic intensity.

In Fig. 140 another $\frac{1}{8}T$ has elapsed, and the lines and enclosing circle have spread out farther. After still another $\frac{1}{8}T$ (Fig. 141) a remarkable change has been set up in the vicinity of the vibrator; the lines of electric intensity have begun to contract inward on the source, while in the outer parts continuing their progress outwards.

The lines thus are throttled, so to speak, and break off at the neck into closed curves, which spring up first in the interior of the system as shown by the small circles inside the outer loop of the dotted curve.

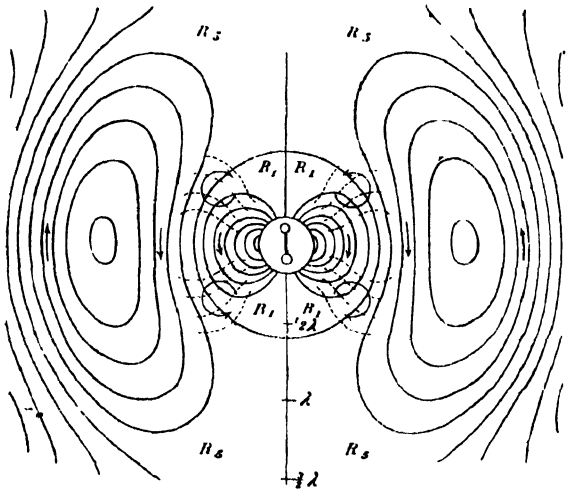


FIG. 139.

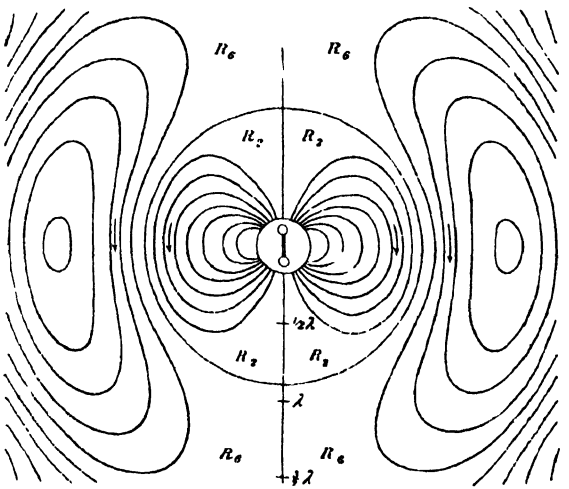


FIG. 140.

Just at these points the electric intensity takes any possible value before the corresponding value is reached at the origin. Thus in Fig. 141 the electric intensity has just become zero at the small circles. As t increases from $\frac{3}{5}T$ to $\frac{1}{2}T$ the curves break off successively from

within until they have all broken off into two groups of closed curves seen to right and left of the origin in the first of this series of figures. These are the cross-sections of what we may call an electric vortex which is produced at the points shown in Fig. 141 and remains symmetrically situated round the axis of the vibrator. The circular axis of this vortex travels out at first infinitely quickly, but ultimately slows down to the velocity of light.

As time now goes on from $t = \frac{1}{2}T$ to $t = \frac{5}{8}T$ the lines spread out from the source as in the first eighth of a period, except that they are now reversed in direction, and as they move force outwards the closed tubes which have broken off, rendering them more concave and more elongated,

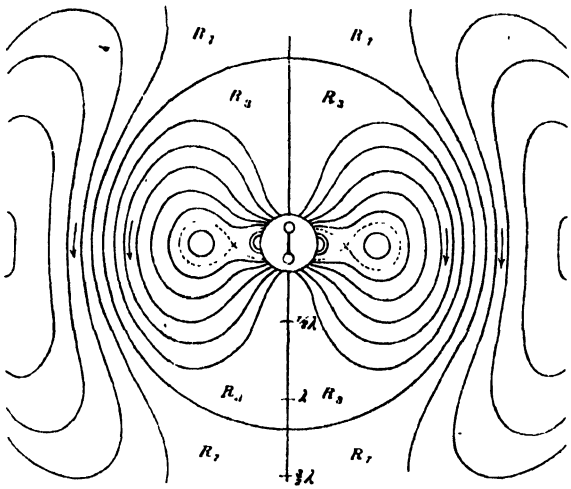


FIG. 141.

so that they approximate more and more at all points to lines transverse to the radius vector drawn from the origin. These successive groups of closed curves in which the direction of the electric intensity is alternately right and left handed are cross sections by a plane through the axis of the successive half-waves thrown off by the vibrator. The waves are thus each made up of two successive so-called electric vortex-rings, each consisting of a system of tubes of induction surrounding its circular axis.

The closed tubes of intensity and induction here considered travel out, carrying their energy with them, and this constitutes the radiation of energy which is continually going on. The energy radiated is at the cost of the energy supplied to the vibrator, just as the energy carried out by waves formed by a steamer is supplied by the fuel burnt in the furnaces. The vibrations are thus damped out at a rate much greater than that due to the dissipation of energy by the conduction current in

the rod connecting the two spheres and the spheres themselves. This rapid dissipation is the chief obstacle in the way of obtaining *maintained* electrical vibrations of great power. We shall return presently to the calculation of the rate of radiation of energy by electric waves, and to effects produced by the damping out of the vibrations.

Experimental Verification of Theory of Vibrator

531. Hertz verified the foregoing theory experimentally, by making direct observations on electromagnetic waves by means of a vibrator and receiver as described above. The arrangement of apparatus is shown in Fig. 142, except that the vibrator was modified by the substitution for the spheres of plates coplanar with the axis. The

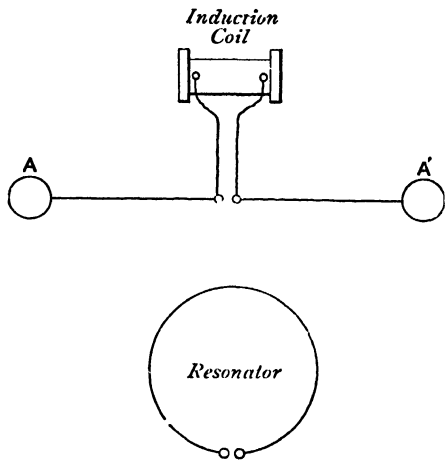


FIG. 142.

vibrator was charged initially by an induction coil, the terminals of which were connected to the two sides of the spark-gap. As soon as the difference of potential between the spheres A, A' had become great enough, a spark passed and electrical oscillations were set up, which depended for their period on the dimensions of the apparatus, but were enormously more rapid than the action of the coil. The oscillations had therefore time to be damped out by radiation and dissipation of energy long before they were renewed by the coil. There was therefore a succession of oscillatory discharges separated by intervals of inaction.

It was found that the receiver acted best when it was chosen of a particular size, though to get it to respond fairly well no very exact timing was required. Thus it acted to some extent as a resonator, though as will be seen later, in consequence of the rapid damping out, an impulse is given to the resonator, which starts it at first in forced

oscillation, but quickly dies away leaving the resonator finally vibrating in its own proper period with a much slower rate of subsidence.

The receiver was made of wire 2 mms. thick, and the diameter of the circle was 35 cms. The spark was produced between two small knobs, the distance between which was regulated by a fine screw, which moved one end of the wire. In some of the experiments a receiver in the form of a square 60 cms. inside, made of similar wire, was used. In this the spark-gap was at the middle of one of the sides.

Approximate Theory of Hertzian Receiver

532. The following sketch of a rough theory of the resonator is all we have here space for, but the action of both receiver and vibrator will be considered more fully in Vol. II. in the account there to be given of later work on Hertzian vibrations. Denoting by P the electric intensity, parallel to an element of the resonating circle, produced by the action and supposing that the electric intensity is oscillatory with period $2\pi/n$, and is a function of the distance s of the element from some point of the circle, say the centre of the spark-gap, taken as origin, we have

$$P = \phi(s) \cos nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

If $\phi(s)$ be a periodic function of S, we get by Fourier's series

$$\phi(s) = A + B \cos \frac{2\pi}{S} s + \ldots . + B' \sin \frac{2\pi}{S} s + \ldots . \qquad (46)$$

where S is the whole circumference of the circle. The terms here exhibited are a constant term and the two gravest simple harmonic components. The second of the two last must disappear also for the origin at the spark-gap. Hence at this origin and diametrically opposite we have respectively

$$\phi(s) = A + B, \quad \phi(s) = A - B.$$

533. Hertz took the view that the action of the vibrator was most effective on the portion of the resonator opposite the spark-gap, that is, that the vibration set up depended in the main on $A - B$. The vibration no doubt, consists in a backward and forward flow of electricity in the connecting wire from one knob to the other (which is of course only the manifestation at the conductor of a surging to and fro of the induction tubes in the ether), which gradually increases in amplitude, if the period of the receiver is nearly equal to that of the exciter, until the maximum difference of potential between the knobs reaches that required to produce a spark.

Let V denote the difference of potential between the knobs at any

time, then, on the supposition that the exciting vibrations are damped, those in the resonator not, we have

$$\frac{\partial^2 V}{\partial t^2} + p^2 V = A' e^{-\kappa t} \sin nt \quad . \quad . \quad . \quad . \quad . \quad (47)$$

if $2\pi/p$ be the natural period of free vibration. The solution for forced vibration is

$$V = -\frac{A' e^{-\kappa t}}{\sqrt{(n^2 - p^2 - \kappa^2)^2 + n^2\kappa^2}} \sin(nt - \epsilon) \quad . \quad . \quad (48)$$

where $\tan \epsilon = 2n\kappa/(n^2 - p^2 - \kappa^2)$.

If κ be very small, V will be very great if $p = n$, and will have the same sign as A' or the opposite, according as $p >$ or $< n$. But if $p = n$ $\tan \epsilon$ is very great and, $\epsilon = \pi/2$, approximately. Thus where resonance, is just attained there is a difference of phase of half a period. When κ is not small the forced vibration is a maximum when $n^2 = p^2 + \kappa^2/2$, and this maximum is smaller the greater κ.

The constant term in $\phi(s)$ [46] gives a term $A \cos nt$ in P. This has the same value at any instant all round the circle. It may be interpreted as the electric intensity set up by the exciter at each element of the conductor, due to variation of magnetic induction of the same value at every point, or if the magnetic induction is not uniform, it is that part of the electric force which is the same at each point of the circle.

If we denote by E the part of the electric intensity impressed on the resonator which does not depend on variation of the uniform part of the magnetic induction, and is due in part to other causes, for example the potential Ψ of Art. 526 above, by ψ the angle it makes with the plane of the circle, and by θ the inclination of its projection on the plance of the circle to the radius through the spark-gap, the tangential component at the spark-gap is $-E \cos \psi \sin \theta \cos nt$. Thus the amplitude of the electromotive intensity producing a spark is of the form $A + C \sin \theta$, where C represents $-E \cos \psi$.

Experiments with Different Positions of the Receiver

534. In the experiments it was sufficient of course to observe for positions of the centre of the receiver at different points of one of the four quadrants of the horizontal plane made by two rectangular horizontal lines, one the axis of the vibrator, the other through the centre of the spark-gap. The receiver was used with its plane (1) vertical, (2) horizontal.

In case (1) no sparks passed when the circle was placed with its plane vertical, and the diameter through the spark-gap horizontal. Sparks however passed with increasing intensity as the receiver was turned round in its own plane so as to bring this diameter nearer the vertical. Whatever the position of the plane of the circle was the

sparks passed most freely when the gap was at the highest or lowest point of the circle.

According to the theory given above it is clear that for any vertical position of the circle $A=0$. For a vertical position of the gap the action of C is equal and opposite in the two halves of the circle, for a horizontal position its action is most effective unless $\psi=\pi/2$.

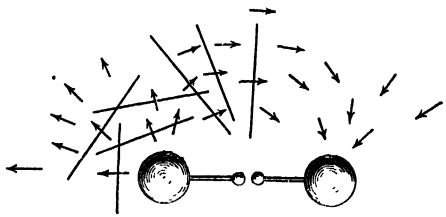


FIG. 143.

When the gap was at the top or bottom of the circle, if the receiver was turned round a vertical axis there were found two positions in which the sparks passed with maximum intensity, and two in which there was almost extinction of the spark. The positions of maximum were 180° apart, and the two positions of minimum were at the two points midway between these. For the former $\psi=0$, and $\theta=\pi/2$, for the element opposite the gap, for the zero positions $\psi=\pi/2$.

Fig. 143 shows a number of these positions. The longer lines are the positions of the spark-gap when the sparking was a minimum; the short arrow-pointed lines are the directions of the electric intensity and indicate very clearly lines of electric intensity, which near the vibrator resemble the lines in Fig. 139.

On the other hand, Fig. 144 shows the positions of the gap for experiments made with the plane of the circle horizontal. For position I and the gap at b_1, or b'_1, no spark was observed; with the spark-gap at a_1, a'_1, however, equal maxima of intensity of spark were found to exist. It is clear that in position I, the circle received a zero number on the whole of tubes of magnetic induction.

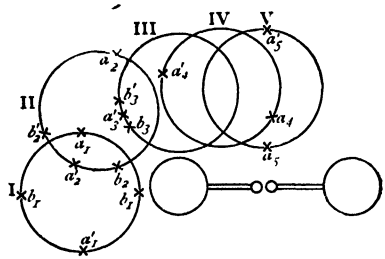


FIG. 144.

For position II the magnetic induction through the circle was no longer zero. Two positions of minimum sparking were found at b_2 and b'_2, and two maxima of unequal intensity at a_2 and a'_2. The line $a_2\,a'_2$ was at right angles to the electric intensity, and thus the action was represented by $A+B$ at one position and by $A-B$ at the other. The two effects, the electric and the magnetic inductive, conspired at a_2 and were opposed at a'_2. The spark lengths at a_2 and a'_2 were 3·5 mms. and 2·5 mms. respectively.

When the spark-gap was at b_2 or b'_2, the electric intensity, which was at right angles to $a_2\,a'_2$, was equally inclined to the circle at those

points, and gave components along the circle which neutralised the electric intensity due to magnetic induction.

In position III the two null points were found closed up nearer to a'_3, the smaller maximum, while the greater was at a_3 diametrically opposite. Over a considerable region opposite a_3 only a small effect was observed. The spark length at a_3 was 4 mms.

In the positions IV and V, no positions of extinction were found for the gap, but only a maximum and a minimum, at a_4, a'_4 in IV, and a_5, a'_5 in V. The line $a\,a'$, it will be noticed, has turned round through nearly $\pi/2$ in the passage from III to V, in order to keep always perpendicular to the electric intensity. The spark lengths were 5·5 mms. at a_4, 1·5 mm. at a'_4, 6 mms. at a_5, and 2·5 mms. at a'_5.

Other positions of the resonator gave results in accordance with theory. The circle was placed in position V of Fig. 144 with the gap at a_5, and turned round the diameter parallel to the vibrator so as to raise the gap. During the changes θ remained $\pi/2$, so that C remained constant, but A changed with the inclination of the plane of the circle to the horizontal. Thus the inclination of the plane of the circle to the horizontal being ϕ, and A_0 being the value of A for $\phi=0$, the value of $A+B$ changed from A_0+B through successive values of $A_0\cos\phi+B$ to $A\,B$ for the highest position of the gap. The spark-length varied from 6 mms. to 2 mms.

As the next quadrant was turned through the spark-length passed through zero and increased again to the smaller maximum 2·5 mms. at a'_5 when ϕ was 180°. The action was then represented by $-A_0+B$, and $-A_0$ preponderating gave the smaller (negative) maximum, while between $\phi=\pi/2$, and $\phi=\pi$, $A_0\cos\phi+B$ took the value 0.

As ϕ was changed from π to $3\pi/2$, $A_0\cos\phi+B$ changed from $-A_0+B$ to B again, and so the spark length changed through zero to 2 mms. once more. As the circle was turned through the last quadrant to its original position, the spark-length increased from 2 mms. to 6 mms.

Exploration of Electric Field by Receiver

535. The experiments just described were all made with the receiver near the vibrator. At distances of from 1 to 1·5 metres from the vibrator, Hertz found that the maximum and minimum positions were not clearly defined, except for certain positions, but became distinct again at distances exceeding 2 metres. Fig. 145 shows the electric field as roughly mapped by Hertz in a room of 14 metres by 12. From this he drew the following conclusions: (1) That at distances beyond 3 metres the electric intensity is parallel to the oscillation, and is due in the main to magnetic induction (that is it depends on the F, G, H terms in the electric intensity components in (2) above). (2) For distances less than 1 metre from the vibrator the electric intensity is almost wholly electrostatic (that is arises from the propagated poten-

tial Ψ). (3) The electric intensity is determinate at all points along the axis of the vibrator, and in the equatorial plane, but within a certain region, marked by the asterisks in the diagram, becomes indeterminate. The effect falls off much more rapidly with increase of distance along the axis than in the equatorial plane.

These results were somewhat puzzling when first observed, but are quite clearly explained by the theory worked out by Hertz. All the features of Fig. 145 are to be found in the diagrams of Figs. 139—141 above. The so-called electrostatic field near the vibrator, the region of indeterminacy beyond it, and, in the neighbourhood of the equatorial plane at least, the parallelism of the electric intensity to the vibrator where the sections of the rings are thrown off by the vibrator, are shown in the plane of the figures.

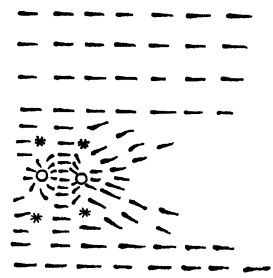


FIG. 145.

Hertz made a large number of experiments on the effect of placing conductors and insulators of various kinds in the field of his vibrator, and found that the distribution of intensities was disturbed in a manner to be expected. An account of these will be found in *Electric Waves*, p. 95, or in *Absolute Measurements*, Vol. II., p. 803 *et seq.*

Period of Vibrator. Determination of Wave Length and Velocity of Propagation

536. So far these experiments give no estimate of wave length, though their results are most instructive as a verification of the theory of the vibrator given above. A first approximation to the period can be obtained from the dimensions of the vibrator in the following manner. If we neglect the effect on the period of the damping due to radiation and to dissipation, we may take as the period $2\pi\sqrt{LC}$ where L is the self-inductance of the arrangement and C is its capacity. It is shown in Art. 408 above that if A, B be the ends of two wires at which currents enter, and A', B' be those at which the currents leave the mutual electrokinetic energy of the system of two wires, T, is given by

$$T = \mu\gamma_1\gamma_2 \iint \frac{\cos\epsilon}{r} ds_1 ds_2 + \frac{\mu\gamma_1\gamma_2}{2}(AB + A'B' - AB' - A'B)$$

in which the integrals are taken along the two wires.

Applying this to two parallel filaments of the conductor connecting the spheres in Hertz's vibrator, putting l for the length of each fila-

ment and x their distance apart, and supposing γ_1 γ_2 each unity, we obtain

$$\frac{T}{\mu} = 2l \log \frac{l + \sqrt{l^2 + x^2}}{x} + 3(x - \sqrt{l^2 + x^2})$$
$$= 2l \log \left(\frac{2l}{x} - \frac{3}{2}\right)$$

if x/l may be neglected.

This may be extended by integration over the cross-section of the conductor, if we know how the total current is distributed among the different filaments. We have simply to take instead of x the proper geometric mean distance[1] of the current-carrying section from itself.

If the current be only on the surface of the conductor the G.M.D. (geometric mean distance) is simply the radius, a, and we have

$$L = 2\mu l \left(\log \frac{2l}{a} - \frac{3}{2}\right) . . . . . (49)$$

If the current is uniform over the cross section log (G.M.D.) is $\log a - \frac{1}{4}$, so that

$$L = 2\mu l \left(\log \frac{2l}{a} - \frac{5}{4}\right) . . . . . (50)$$

Taking μ as unity, l as 100 cms., a as ·25 cm. as in Hertz's dumb-bell apparatus, we obtain $L = 1037$. The spheres were 15 cms. in radius, so that $C = 7{\cdot}5/v^2$ where v is the ratio of the units (Art. 512 above). Thus the period was $1{\cdot}85 \times 10^{-8}$ second.

Rate of Radiation of Energy from Vibrator

537. It may be interesting to calculate from the period the amount of energy radiated in the time of a single vibration, and hence to estimate the activity necessary to maintain the action of the exciter. The vibrator, the dimensions of which have just been given, had its spheres charged to a maximum difference of potential of 1 cm. The difference of potential between the spheres was thus about 60 C.G.S. electrostatic units, and the charge of each sphere was 60×15 C.G.S. units. Thus Φ for the vibrator was $60 \times 15 \times 100$ and the energy radiated in half a period was (see Art. 549 below) $60^2 \times 15^2 \times 100^2 \times 8\pi^4/3\lambda^3$, λ being taken in cms. If the velocity was that of light the wave length was about 550 cms., and hence in each half-period about 12000 ergs. passed from the vibrator to the surrounding medium. The whole energy of the vibrator when charged to the potential stated above

[1] If S_1, S_2 be two areas in the same plane and having any boundaries, and r_{12} be the distance between an element dS_1 in one and an element dS_2 in the other, then $G.M.D. = \{\iint \log r_{12}\, dS_1 dS_2\}/S_1 S_2$ where the integrals are taken over S_1, S_2. Of course when the areas coincide we have the $G.M.D.$ of an area from itself. The knowledge of geometric mean distances is of great importance in the calculation of inductances of parallel conductors and close parallel coils of great radius.

was $\frac{1}{2} \times 2 \times 60^2 \times 15$ ($= 54000$) ergs. Thus about $\frac{2}{11}$ of the whole energy was radiated in the first half-period, that is the amplitude of vibration suffered from radiation a diminution of roughly $\frac{1}{11}$. The period of vibration being $1{\cdot}85 \times 10^{-8}$ second, the average rate of radiation was therefore about $1{\cdot}34 \times 10^{12}$ ergs. per second, or, approximately, 179 horsepower. Very considerable power would therefore be necessary to maintain the vibrations of even such a small vibrator as that of Hertz.

Reflection of Waves in Air from Metallic Surfaces. Standing Waves

538. Hertz carried out a series of experiments in his lecture theatre, a room 15 metres long, 14 metres wide, and 6 metres high, on the reflection of electric waves from a large plate of zinc, 4 metres high by 2 metres broad, which covered the middle part of one end wall. The clear breadth of the room free from obstacles was only 8·5 metres, on account of a row of iron columns along each of the two sides. The

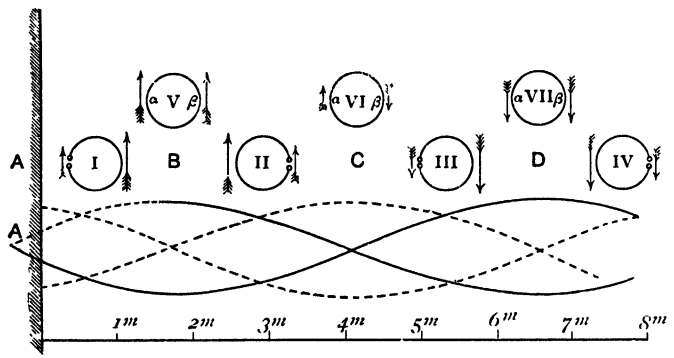


FIG. 146.

exciter (which was an instrument with plates instead of balls which had been used for experiments in propagation along wires) was placed with its axis vertical at the middle of the end remote from that covered with the zinc plate.

The waves generated were thus incident nearly normally on the conducting plate, with the electric vibration in the vertical plane through the axis of the vibrator. The same receiver, as before described, the circle of 35 cms. radius, was carried along the normal through the centre of the vibrator, and the positions of maximum and minimum sparking in the neighbourhood of the zinc plate observed. The positions I, II, III, IV of Fig. 146 were those of most intense sparking, and V, VI, VII those of least intense sparking. In the former set the spark gap was turned alternately in opposite directions, in the second set the sparking was the same for both right and left positions of the spark gap.

When the spark gap was at the top of the circle, so that the electric intensity could have but little effect, feeble sparking only was produced in position V, a maximum at VI, and a minimum again at VII. Thus the magnetic induction was evidently a minimum at V and VII, and a maximum at VI.

Clearly these results point to a standing wave of electric and magnetic induction produced as represented by the full and dotted curves in Fig. 146. The diagram shows that the electric intensity has its phase changed by half a period relatively to the magnetic intensity, so that in the standing vibration the nodes of one correspond to the loops of the other.

Apparently the node for the electric intensity was behind the wall-surface about ·68 metre, and the next loop but one, about 6·52 metres in front of it, so that the wave-length was about 9·6 metres. With the period 3×10^{-8} second, which was nearly that of the vibrator, this would give $3{\cdot}2 \times 10^{10}$ cms. per second as the velocity of propagation of the waves in air. This is very approximately the velocity of light.

Multiple Resonance

539. Hertz experimented on the rate of propagation of electric waves along wires, and found a velocity differing considerably from that obtained for waves in free space. A vast amount of work has since been done on this subject, and we defer its discussion to Vol. II. But the general nature of the explanation is as follows:

It has been suggested by Messrs. Sarasin and De la Rive that the wave-length observed in free space may depend to a great extent on the dimensions of the resonator, and may be connected with what has been called *multiple resonance*. It has been noticed by these experimenters, as well as by FitzGerald and Trouton,[1] that the exciter apparently gives rise neither to a single vibration of distinct period nor to a limited number of distinct vibrations, but rather to such a complex of vibrations as would give a wide band of continuous spectrum. Thus all vibrations, agreeing with possible modes of vibration of the resonator, would be reinforced. That this is not contained in the theory is true, but the theory is very incomplete. It is hard to believe that the vibrations can be perfectly simple.

The following explanation of multiple resonance has been proposed by Poincaré.[2] The logarithmic decrement of the vibrations of the exciter is probably much greater than that of the resonator, and so the vibrations of the exciter diminish in amplitude more quickly than those set up in the resonator. This is confirmed by experiments on the damping of the vibrations in the exciter and receiver made by V. Bjerknes.[3] Thus the resonator, being started by the exciter,

[1] *Nature*, vol. xxxix. (1889-9), p. 391. [2] *Électricité et Optique*, 2de Partie.
[3] *Wied. Ann.* 44 (1891), p. 74.

continues its own vibrations after those of the exciter have become insensible, but then vibrates in its own proper period, giving vibrations of longer period and of greater wave-length than those which excited it. The wave-length being determined by interference, and used with the too short period of the exciter, gives too great a velocity of propagation. With this explanation Hertz has expressed himself as practically in accord.[1] As he remarks, the oscillations of the exciter, represented graphically, do not give a curve of sines pure and simple, but a curve of sines the amplitude of which gradually diminishes. Such an oscillation causes all the resonators receiving it to vibrate, but those in tune with the exciter more violently than the others. This agrees with the theory given in Art. 533; and the fact that the apparent spectrum seems more extended when wires are connected to the vibrator than when the propagation takes place freely in air may be due to a greater damping effect in the former case.

Effect of Size of Reflecting Surface

540. It may be noted here that it has been found by Mr. Trouton[2] that the size of the reflecting sheet has a great deal to do with the distance of the nodes from the surface. Using long narrow strips held (1) so that the length was in the direction of the magnetic component, (2) in the direction at right angles to that component, he found that the node was in the former case shifted outwards from the reflecting surface very markedly. For example with waves 68 cms. long the distance of the magnetic node varied from 24·2 cms. for a strip 16 cms. wide to 17 cms. ($\frac{1}{4}$ wave-length) for a large sheet. This effect was due no doubt, as stated by Mr. Trouton, to the action of the charge periodically accumulated at the edges of the sheet.

Smallness of size in the magnetic direction carried the node in towards the surface; and this may very possibly have been the case in the experiments of Hertz, described above. The breadth of the sheet (in the direction of the magnetic force) was 2 metres, or about the same in effect as a strip 14 cms. broad used with Mr. Trouton's 68 cms. waves. This would give a sensible inward displacement of the node.

Reflection of Electric Waves by Mirrors. Refraction by Prisms

541. Experiments were also made by Hertz on the production of plane polarized waves, by means of a linear vibrator consisting of two cylinders placed in line with a spark-gap between their opposed ends. The cylinders were about 12 cms. long and 3 cms. in diameter each, and the ends of the spark-gap were well rounded. The vibrator was placed vertically in the focal line of a parabolic cylindrical reflector made of ordinary sheet zinc nailed on a wooden framework cut into proper parabolic shape. The cylinders were connected with an induction coil

[1] *Electric Waves*, p. 16.

[2] *Phil. Mag.*, July, 1881.

by insulated wires passing through holes in the zinc behind them. The mirror was about 2 metres in length, and about 70 cms. in depth along the axis of the parabolic figure, as shown in Fig. 147. The exciter thus placed produced waves of electric force, the direction of which near the source was parallel to its axis. These were received by the mirror, and reflected into a parallel beam which could be observed by means of a suitable receiver. In most of the experiments however the beam was received by a similar reflector facing the former so as to concentrate the radiation on its focal line which in some experiments was parallel in others at right angles to the former.

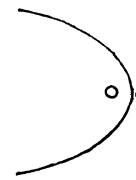


FIG. 147.

In the focal line of the other mirror was placed a receiver made of two pieces of thick wire each 50 cms. long, placed in line as shown in Fig. 148, with a gap of about 5 cms. between their ends, and completed by

FIG. 148.

two thin wires about 12 cms. long led out at right angles to the rods to the back of the mirror. These were tipped with a knob and point as shown, so as to form an adjustable spark-gap which could be conveniently observed from behind.

Polarization of Electromagnetic Beam. Relation of Plane of Polarization to Direction of Electric Vibration

542. It was found by this arrangement that electric radiation could be detected at a much greater distance from the source than with the ordinary vibrator and receiver used as described above without reflectors. In these as in all other experiments the knobs of the vibrator have to be repeatedly cleaned, and its spark-gap must be screened from the direct light of the spark in the induction coil.

Clearly a parallel beam of plane polarized light was thus obtained, and consisted, as the experiments showed, of electrical vibrations parallel to the vibrator accompanied by magnetic vibrations at right angles to the former and to the direction of propagation. Placing the axial planes of the mirrors in coincidence gave augmentation of the electric effect, crossing the mirrors extinguished the effect at the receiver in the second mirror.

Again, a grating of parallel copper wires placed between the mirrors entirely stopped the vibration when the wires were at right angles to the vibrator, but allowed it to pass freely when turned through 90° from the former position.

Also it was found, in a repetition of these experiments by Prof. FitzGerald and Mr. Trouton,[1] that the electromagnetic beam was reflected from a wall about three feet thick when the vibrator was at right angles to the plane of reflection, and not at all at the polarizing angle when the vibrator was in the plane of reflection. This result showed that the *electric* vibration is at right angles to the plane of polarization. This is a very important result as it settles the question as to the relation of the plane of polarization to the electric vibration. The question of the relation of this to the ether vibration is a distinct question to which no answer has yet been given. It may be that there is after all no vibration, in the ordinary sense, of the matter of the ether.

543. Hertz found that such an electromagnetic wave was not only reflected like a light wave, but is also refracted according to the same law of refraction. An immense prism of pitch, having an isosceles triangular section of 120 cms. side and a refracting angle of 30°, was made by melting pitch into a wooden supporting case. The prism was placed with its refracting edge vertical, at a distance of 2·6 metres from the vibrator (also vertical), and the beam was made incident on the face at an angle of 65°. The receiving mirror was estimated as 2·5 metres from the prism on the other side, and showed a radiation beginning, reaching a maximum, and falling off to zero, at the respective deviations 11°, 22°, 34°.

The experiments were repeated with the focal lines of the mirrors horizontal, and practically no difference in the results was observed.

The index of refraction for pitch given by the experiments was 1·69, which nearly agrees with the index 1·5 to 1·6 found for pitchy substances by optical experiments.

Prof. Oliver Lodge and Dr. Howard have made observations on the concentration of such vibrations by means of lenses.[2] Two enormous lenses of hyperbolic cylindrical figure were constructed of mineral pitch, and were placed with their axial planes coincident, and their plane faces, or bases, turned towards one another. These lenses were so proportioned that the beam produced by a linear exciter in the external focal line of one might emerge parallel from the plane face of that lens, and then be concentrated by the second lens on the corresponding focal line.

Transparency of Ordinarily Opaque Substances to Electromagnetic Waves

544. An interesting point noticed in many of these experiments is the perfect transparency to these vibrations of optically opaque substances. A stone wall three feet thick has been found to offer no obstacle to the passage of such waves. In fact in some experiments made by Prof. FitzGerald at Dublin the receiver was placed on a pillar in the garden

[1] *Nature, loc. cit.*, p. 417 above.

[2] *Phil. Mag.*, July, 1889.

outside while the exciter was in action in the laboratory. This is no doubt a phenomenon of similar character practically to that of the transparency of a thin film of metal to light, and is conditioned by the nature of the material and the relation of the wave-length to the thickness of the stratum. It has been found by Maxwell, *El. and Mag.* Vol. II., Chap. XIX., that the transparency of thin metallic films is greater than that given by the electromagnetic theory according to the conductivity of the material. [See also Wien, *Wied. Ann.* 35 (1888).] A form of radiation, which appears to consist of waves of length small in comparison with that of visible light, and to which some kinds of ordinarily opaque matter are quite transparent, has recently been discovered by Röntgen at Würzburg, and has attracted much attention.

Great additions to our knowledge of electrical radiation have been made during the last few years by an army of investigators in this field in this country and all over the world. Some account of their work will be given in Vol. II.

SECTION II.—*Flow of Energy in the Electromagnetic Field.*

Motion of Energy across Bounding Surface of a Closed Space. Poynting's Theorem

545. If we assume, as has been done above, the localisation of the electrokinetic and electric energy in the field, we obtain for the energy per unit volume at any point the value $\mu\mathbf{H}^2/8\pi + k\mathbf{E}^2/8\pi$. We neglect terms which theory shows must exist in the complete expression for the energy, as shown by the phenomena discovered by Hall, and other effects more or less small in amount which, if not yet discovered, must be held for theoretical reasons to have a real existence.

Now consider any closed space in the field, and let the total energy within it at any instant be $\mathsf{E} + T$, where E denotes the electric energy and T the electrokinetic energy. Supposing that k and μ do not vary with the time, we have, if $d\varpi$ be an element of volume at which the components of electric intensity are P, Q, R and those of magnetic intensity α, β, γ,

$$\frac{d(\mathsf{E} + T)}{dt} = \frac{1}{4\pi}\int\left\{k\left(P\frac{dP}{dt} + Q\frac{dQ}{dt} + R\frac{dR}{dt}\right) + \mu\left(\alpha\frac{d\alpha}{dt} + \beta\frac{d\beta}{dt} + \gamma\frac{d\gamma}{dt}\right)\right\}d\varpi \quad . \quad . \quad . \quad (51)$$

where the integration is taken throughout the closed space considered.

But if u, v, w be the components of the total current, and p, q, r those of the conduction current (the latter in the general case not merely generating heat in the conductor) we have for those of the displacement current

$$k\frac{dP}{dt} = u - p = \frac{1}{4\pi}\left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right) \quad p \quad . \quad . \quad . \quad (52)$$

with two similar equations. Also we may write the equations of electric intensity (2) above in the form

$$P, Q, R = P' + c\dot{y} - b\dot{z}, \quad Q' + a\dot{z} - c\dot{x}, \quad R' + b\dot{x} - a\dot{y} \quad (53)$$

From these we obtain by differentiation

$$\frac{d\alpha}{dt} = \frac{d}{dt}\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right) = \frac{\partial Q'}{\partial z} - \frac{\partial R'}{\partial y} \quad . \quad . \quad . \quad . \quad (54)$$

and two similar equations.

Substituting from these in (51) and using (53) we obtain on rearrangement

$$\frac{d(\mathsf{E} + T')}{dt} = \frac{1}{4\pi}\int\left\{\frac{\partial}{\partial x}(R'\beta - Q'\gamma) + \frac{\partial}{\partial y}(P'\gamma - R'\alpha) + \frac{\partial}{\partial z}(Q'\alpha - P'\beta)\right\}d\varpi$$
$$- \int\{(cv - bw)\dot{x} + (aw - cu)\dot{y} + (bu - av)\dot{z}\}d\varpi$$
$$- \int(Pp + Qq + Rr)d\varpi.$$

The components of electromagnetic force (not those of total mechanical force, which include forces due to electrification), on a part of the medium moving with velocity $(\dot{x}, \dot{y}, \dot{z})$, are X, Y, $Z = cv - bw$, $aw - cu$, $bu - av$. Hence we get by integration over the closed surface of the space and transposition

$$\frac{d(\mathsf{E} + T')}{dt} + \int(X\dot{x} + Y\dot{y} + Z\dot{z})d\varpi + \int(Pp + Qq + Rr)d\varpi$$
$$= \frac{1}{4\pi}\int\{l(R'\beta - Q'\gamma) + m(P'\gamma - R'\alpha) + n(Q'\alpha - P'\beta)\}dS \quad (55)$$

where l, m, n are the direction cosines of the normal to the surface element dS drawn outwards.

If l', m', n' be the direction cosines of a normal to the plane defined by the resultant $\mathbf{E}'$ of the component electric intensities P', Q', R', and the resultant magnetic intensity $\mathbf{H}$, and drawn in the direction in which a right-handed screw would move if the handle were turned round in the plane of $\mathbf{E}'$ and $\mathbf{H}$ from the direction of $\mathbf{H}$ to that of $\mathbf{E}'$ (Fig. 149) the element of the surface integral on the right has the value $\mathbf{HE}'\sin\theta(ll' + mm' + nn')/4\pi$, where θ is the angle between $\mathbf{H}$ and $\mathbf{E}$. The rate of flow of energy per unit of area is therefore represented in magnitude and direction by the vector-product of $\mathbf{H}$ and $\mathbf{E}'$ (that is the vector $\mathbf{HE}'\sin\theta$ at right angles to the plane of $\mathbf{H}$ and $\mathbf{E}'$) divided by 4π. The component of flow across unit of area at right angles to the direction (l', m', n') is thus $\mathbf{HE}'\sin\theta(ll' + mm' + nn')/4\pi$. The direction of flow is opposite to that of (l', m', n') as defined above. Thus in Fig. 149 the flow is in the direction xO.

546. The terms in equation (55) may be thus interpreted. The first term on the left is the time-rate of increase of the energy within the closed space, the second is the rate at which work is done by electromagnetic forces, and the third combines the rates at which energy is dissipated and is expended in producing chemical changes, and the equation asserts that the sum of these rates is equal to that of the flow of energy across the bounding surface of the space, as shown by the expression on the right-hand side. This theorem is due to Professor Poynting (*Phil. Trans. R. S.*, Part VI., 1885).

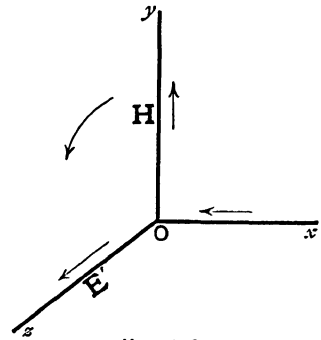


FIG. 149.

If energy is conveyed into the system by the action of impressed forces arising from another system, and producing energy within the space, terms must be added to (55) taking account of the energy so delivered. This point will be fully discussed and illustrated in Vol. II.

It is to be observed that the addition of any term $(l\phi+m\chi+n\psi)dS$, of proper dimensions, to the element of the integral would not alter the value of the integral over the closed surface provided ϕ, χ, ψ are functions of the co-ordinates fulfilling the condition $\partial\phi/\partial x+\partial\chi/\partial y+\partial\psi/\partial z=0$. It is thus not strictly demonstrated that the flow across an element of the closed surface is that stated above. The results obtained in the following examples (taken from Poynting's paper) agree with known facts, and so far confirm the theory.

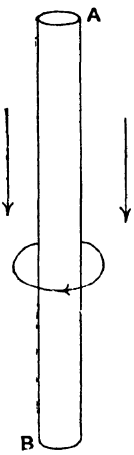


FIG. 150.

EXAMPLES—**1. Straight Wire with Steady Current. Energy Stream-Lines**

547. Take first the case, illustrated by Fig. 150, of a long straight wire of circular section in which a steady current of strength γ is flowing. The displacement does not vary, and there is therefore no displacement current. The magnetic intensity is tangential to a normal cross-section of the conductor and is of amount $2\gamma/r$. The electric intensity is parallel to the conductor and is equal to the current per unit of area of section divided by the conductivity of the conductor, that is to $\gamma/\pi r^2\kappa$. The rate of flow of energy across unit area is thus by the theorem $(2\gamma/r\times\gamma/\pi r^2\kappa)/4\pi=\gamma^2/2\pi^2r^3\kappa$, and the direction of flow is by the rule found above, inwards from the surrounding medium to the wire. The rate of flow of energy inward upon unit length of the wire is thus $\gamma^2/\pi r^2\kappa$, and across l units of

length is $\gamma^2 l/\pi r^2 \kappa$, or $\gamma^2 R$, if R is the resistance of the length l of the wire. This agrees with the result already several times used above.

Thus the conducting wire controls the manner of arrangement of the electric and magnetic equipotential surfaces in the field. The flow of energy is along the lines of intersection of such surfaces, which may therefore be called energy stream-lines, when the intensities are derived from potentials. In a metallic conductor there is dissipation of energy received from the medium; and if at any place electrical and magnetic energy are utilised in doing work, this by the theory does not come along the conductor, but from the surrounding medium, along paths perpendicular to the electric and magnetic intensities, the distribution of which is conditioned by the existence of the conductor.

2. Discharge of a Condenser

548. Let a charged condenser have its plates connected by wires to another pair of plates, so that the capacity is increased. The tubes of electric induction formerly existing for the most part in the portion of the medium between the plates of the original condenser, and entirely depending for their arrangement on these plates, move out sideways with their ends on the connecting wires, until the state of strain has been set up between the other pair of plates. While the motion of the tubes is proceeding, magnetic intensity exists in the field, but dies away with the motion of the tubes.

If the change proceed slowly, the intensities will be approximately derivable from potentials. There will then be a fall of potential along the wire connecting the insulated plates, and some of the equipotential surfaces must cut the wires, and so there is a flow of energy into the conducting wire, which is dissipated in heat according to the law of Joule.

This view of the transference of energy from the medium between the plates of one condenser to that in another, has already been insisted on in Chap. V. above. And the view seems eminently reasonable. It is very difficult to suppose that the energy has been transferred along the conductor to the other condenser, and there inserted between the plates. Step by step with the process of charging the other condenser has gone on the growth of electric strain in the dielectric between its plates, and since the energy is certainly stored up in the dielectric, it is natural to suppose that the strain has been propagated through the medium under the guidance of the conductors. These, as we have seen, localise the electric and magnetic equipotential surfaces, which exist in the case of slow change, and the intersections of which are the stream lines of energy.

Let the plates of the condenser be connected by a wire L, M, N, a in Fig. 151. The tubes of electric induction will pass out with their ends on the wire, and will shorten as they advance, being swallowed up at each end in the wire with dissipation of their energy. Finally,

somewhere about midway between the ends of the conductor the last fragment of the tube disappears laterally into the conductor. The magnetic tubes of force which encircle the conductor contract down upon it and disappear within it, giving up also their energy as they do so.

The curves in the figure shown intersecting the conductor are intended to represent the intersection of the equipotential surfaces for the time being with the plane of the diagram. The rate of flow is again along the lines of intersection of the magnetic and electric equipotential surfaces, if these can be said to exist. Strictly speaking,

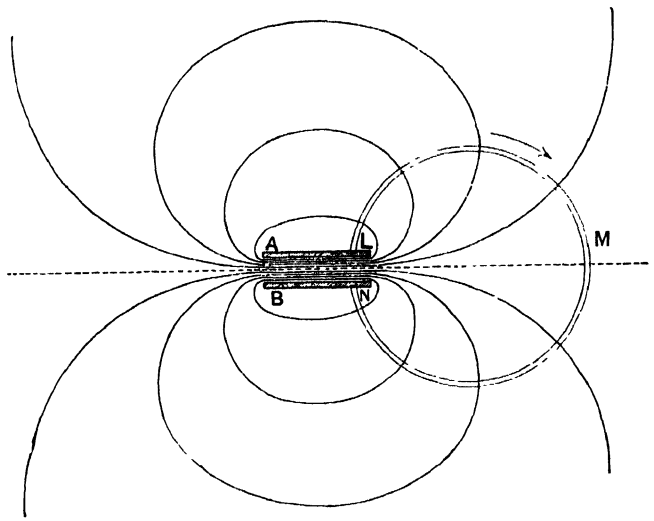


FIG. 151.

except in cases of slow discharge, the intensities are not derivable from potentials.

If the conductor is of such form that the total magnetic induction through it is very great, the electrokinetic energy of the field will become very great, and the flow of energy absorption in the conductor will be much altered. The tubes of electric induction will then move, with still a certain amount of absorption, though much less, in the conductor until their ends pass one another, when the tube returns reversed to the dielectric between the plates, and the condenser is charged the opposite way. Then this discharges with reversal as before, and so the electric oscillation goes on, with radiation and dissipation of energy, until all energy has disappeared from the system.

[See further on oscillatory discharge and radiation of energy in Vol. II. See also J. J. Thomson on "Faraday Tubes of Force," *Recent Researches in Electricity and Magnetism*, Chap. I.]

3. Radiation of Energy from a Hertzian Vibrator

549. Close to the vibrator, as we have seen above, the lines of electric intensity alter their arrangement in such a manner that there is flux and reflux of energy across a closed surface surrounding the vibrator. The flow outwards is on the whole greater than the flow inwards, and thus in every complete oscillation a certain balance of energy is radiated. We shall be able to estimate this easily by considering a closed spherical surface, having its centre at the centre of the vibrator, and of radius r containing a very large number of wave-lengths. Taking the expressions given in Art. 520 above for the intensities at a very great distance, and first finding the rate of flow of energy outwards across a zone of breadth $rd\theta$ surrounding the axis, the intensities $\mathbf{E}$ and $\mathbf{H}(=\alpha)$ are tangential to the surface, and at right angles to one another. Thus for the rate of flow across the zone we have

$$\tfrac{1}{2}r^2\,\mathbf{EH}\sin\theta d\theta = \tfrac{1}{2}\frac{\Phi}{k}m^3n\sin^2(nr - nt)\sin^3\theta d\theta.$$

Thus for the total outward flow across the sphere in half a period we obtain

$$\tfrac{1}{2}r^2\int_0^{T/2}\int_0^{\pi}\mathbf{EH}\sin\theta dt d\theta = \frac{\Phi^2}{k}\frac{8\pi^4}{3\lambda^3}$$

Since we take k in ordinary electrostatic units and the medium is air k is taken as unity. This is the result used in Art. 537 above.

Several other examples of the flow of energy will be found in Prof. Poynting's paper *loc. cit.* See also *Absolute Measurements*, Vol. II., p. 219.

Distribution of Current in Cross Section of Cylindrical Conductor

550. As a final example for the present of the results derivable from Maxwell's equations of the electromagnetic field, we give here an investigation of the distribution of the current over the cross section of a long straight cylindrical conductor carrying rapidly alternating currents, and of the resulting resistance and self inductance of the conductor. We follow here Lord Rayleigh's [1] mode of treatment.

Let the axis of the conductor be along z, the component w of current parallel to the axis is a function of the time and the distance ρ of the point from the axis. The components of vector-potential are

[1] *Phil. Mag.*, May, 1886. Another discussion will be found in *Absolute Measurements*, Vol. II., p. 331, and a third by the Bessel Function Analysis in Gray and Mathews, *Bessel Functions and their Applications to Physics*, p. 157.

$F = 0$, $G = 0$, and H, which must be a function of the same variables as w. Let

$$H = S + T + T_1\rho^2 + T_2\rho^4 + \ldots + T_n\rho^{2n} + \ldots \quad (56)$$

in which S, T, T_1, ... are functions of the time t.

Now by the circuital equations (4)

$$4\pi w = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}$$

$$\mu\alpha = \frac{\partial H}{\partial y}, \quad \mu\beta = -\frac{\partial H}{\partial x},$$

and therefore

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + 4\pi\mu w = 0$$

or

$$\frac{\partial^2 H}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H}{\partial \rho} + 4\pi\mu w = 0 \quad . \quad . \quad . \quad . \quad . \quad (57)$$

From this and (56) we obtain

$$-\pi\mu w = T_1 + 2^2 T_2\rho^2 + 3^2 T_3\rho^4 + \ldots + n^2 T_n\rho^{2n-2} + \ldots \quad (58)$$

If κ be the conductivity of the material, the component electromotive intensity at every point where the current is w is w/κ. Hence by (2)

$$\frac{w}{\kappa} = -\frac{\partial H}{\partial t} - \frac{\partial\Psi}{\partial z}$$

where Ψ is as before the potential corresponding to that part of the electromotive intensity which does not depend on induction. This by (56) is

$$\frac{w}{\kappa} = -\frac{\partial\Psi}{\partial z} - \frac{dS}{dt} - \frac{dT}{dt} - \rho^2\frac{dT_1}{dt} - \ldots \quad . \quad . \quad (59)$$

Equations (58) and (59) give

$$\frac{T_1}{\kappa} = \pi\mu\left(\frac{\partial\Psi}{\partial z} + \frac{dS}{dt} + \frac{dT}{dt}\right)$$

$$\frac{2^2 T_2}{\kappa} = \pi\mu\frac{dT_1}{dt}, \ldots, \quad n^2\frac{T_n}{\kappa} = \pi\mu\frac{dT_{n-1}}{dt}, \ldots$$

Putting $dS/dt = -\partial\Psi/\partial z$, which is here assumed to be a function of the time only, we obtain

$$T_1 = \pi\mu\kappa\frac{\partial T}{dt}, \quad T_2 = \frac{1}{2^2}\pi^2\mu^2\kappa^2\frac{d^2T}{dt^2}, \ldots, \quad T_n = \frac{1}{(n\,!)^2}\pi^n\mu^n\kappa^n\frac{d^nT}{dt^n}$$

and therefore

$$-\pi\mu w = \pi\mu\kappa\frac{dT}{dt} + \pi^2\mu^2\kappa^2\frac{d^2T}{dt^2}\rho^2 + \ldots . + \frac{1}{\{(n-1)!\}^2}\pi^n\mu^n\kappa^n\frac{d^nT}{dt^n}\rho^{2n-2} + \ldots . \quad (60)$$

If γ be the total current in the conductor

$$\gamma = 2\pi\int_0^a w\rho d\rho$$

where a is the radius of the wire. Writing σ for $\pi a^2\kappa$, the conductance of unit length of the wire, we obtain from (60)

$$-\mu\gamma = \mu\sigma\frac{dT}{dt} + \frac{2\mu^2\sigma^2}{1^2.2^2}\frac{d^2T}{dt^2} + \ldots . + \frac{n\mu^n\sigma^n}{(n!)^2}\frac{d^nT}{dt^n} + \ldots . \quad (61)$$

Outside the wire H does not depend on the distribution of the current in the wire, but only on the total current γ. Hence at the surface $H = A\gamma$, where A is a multiplier to be determined. Thus

$$A\gamma = S + T + T_1a^2 + \ldots . + T_na^{2n} + \ldots .$$

and therefore by the values of $T_1, T_2, \ldots .$ found above,

$$A\gamma - S = T + \mu\sigma\frac{dT}{dt} + \frac{\mu^2\sigma^2}{1^2.2^2}\frac{d^2T}{dt} + \ldots . + \frac{\mu^n\sigma^n}{(n!)^2}\frac{d^nT}{dt^n} + \ldots .$$

or if we write

$$\phi(x) = \sum_{n=0}^{n=\infty}\frac{x^n}{(n!)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$A\gamma - S = \phi\left(\mu\sigma\frac{d}{dt}\right)T$$

and therefore

$$\frac{dS}{dt} = A\frac{d\gamma}{dt} - \phi\left(\mu\sigma\frac{d}{dt}\right)\frac{dT}{dt}.$$

Equation (61) may also be written

$$\gamma = -\sigma\phi'\left(\mu\sigma\frac{d}{dt}\right)\frac{dT}{dt}.$$

Hence elimination of dT/dt between the two last equations gives

$$\frac{dS}{dt} = A\frac{d\gamma}{dt} + \frac{1}{\sigma}\frac{\phi\left(\mu\sigma\frac{d}{dt}\right)}{\phi'\left(\mu\sigma\frac{d}{dt}\right)}\gamma \quad . \quad . \quad . \quad . \quad . \quad (63)$$

But since $dS/dt = -\partial\Psi/\partial z$, dS/dt is the part of the electromotive intensity at each point which does not depend on the inductive action of the current. We have supposed this to be the same at every point, and hence, if E be its line integral along a length l of the conductor, $E = l dS/dt$. The last equation becomes

$$E = lA\frac{d\gamma}{dt} + \frac{l}{\sigma}\frac{\phi\left(\mu\sigma\frac{d}{dt}\right)}{\phi'\left(\mu\sigma\frac{d}{dt}\right)}\gamma \quad . \quad . \quad . \quad . \quad (64)$$

551. If the currents be simple harmonic with respect to the time, they are, to a constant factor, represented by the real part of e^{int} where $n = 2\pi/T$ (where T is now used to denote the period). We have then to replace in (64) d/dt by in, and we obtain

$$E = inlA\gamma + \frac{l}{\sigma}\frac{\phi(i\mu\sigma n)}{\phi'(i\mu\sigma n)}\gamma \quad . \quad . \quad . \quad . \quad . \quad (64')$$

If x be small

$$\frac{\phi(x)}{\phi'(x)} = 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{48}x^3 - \frac{1}{180}x^4 + \frac{13}{8640}x^5 - \ldots .$$

and therefore

$$\frac{\phi(i\sigma\mu n)}{\phi'(i\sigma\mu n)} = 1 + \frac{1}{12}\mu^2\sigma^2n^2 - \frac{1}{180}\mu^4\sigma^2n^4 - \ldots . + i\left(\frac{1}{2}\mu\sigma n - \frac{1}{48}\mu^3\sigma^3n^3 + \frac{13}{8640}\mu^5\sigma^5n^5 - \ldots .\right).$$

Thus we obtain

$$E = R\left(1 + \frac{1}{12}\frac{\mu^2l^2n^2}{R^2} - \frac{1}{180}\frac{\mu^4l^4n^4}{R^4} + \ldots .\right)\gamma + inl\left\{A + \mu\left(\frac{1}{2} - \frac{1}{48}\frac{\mu^2l^2n^2}{R^2} + \frac{13}{8640}\frac{\mu^4l^4n^4}{R^4} - \ldots .\right)\right\} \quad . \quad (65)$$

since $l/\sigma = R$, the resistance for steady currents.

This equation is of the form

$$E = R'\gamma + inL'\gamma = R'\gamma + L'\frac{d\gamma}{dt} \quad . \quad . \quad . \quad . \quad (66)$$

where

$$\left.\begin{aligned} R' &= R\left(1 + \frac{1}{12}\frac{\mu^2l^2n^2}{R^2} - \frac{1}{180}\frac{\mu^4l^4n^4}{R^4} + \ldots .\right) \\ L' &= \left\{lA + \mu\left(\frac{1}{2} - \frac{1}{48}\frac{\mu^2l^2n^2}{R^2} + \frac{13}{8640}\frac{\mu^4l^4n^4}{R^4} - \right)\right\}\end{aligned}\right\} \quad . \quad (67)$$

which are the effective resistance and self-inductance of the wire in consequence of the rapid variation of the current.

If the frequency of the alternation be very small, the resistance approximates to R, and the self-inductance to $l(A + \frac{1}{2}\mu)$, the values for steady currents. This gives the value of A, which depends on the situation of the return current. If the conductor be enclosed in a perfectly conducting co-axial sheath of internal radius b, $A = 2log\,(b/a)$.

With increasing frequency the resistance increases without limit, and the inductance diminishes towards the value lA. This result may be obtained from the analytical theorem of Bessel Functions that when x is very great $\phi(x) = e^{2\sqrt{x}}/(2\sqrt{\pi x})$ so that in this case $\phi(x)/\phi'(x) = \sqrt{x}$, and therefore $\phi(i\sigma\mu n)/\phi'(i\sigma\mu n) = \sqrt{\frac{1}{2}\sigma\mu n}(1 + i)$, which gives by (64′)

$$\left.\begin{aligned} R' &= \sqrt{\tfrac{1}{2}\mu l n R} \\ L' &= l\left(A + \sqrt{\frac{\mu R}{2nl}}\right) \end{aligned}\right\} \quad \ldots\ldots \quad (68)$$

552. As an example, take an iron wire ·4cm. in diameter, with conductivity $1/10^4$, and $\mu = 300$, we find that $\frac{1}{12}\mu^2 l^2 n^2/R^2$ is for a period of $\frac{1}{1000}$ sec. about 47, so that the resistance is vastly increased, and the self-inductance diminished by the rapid alternation.

For copper taking $\mu = 1$, and $\kappa = 1/1640$, this term is about $3 \times \pi^2 n^2 a^4/10^8$. A frequency of 100 gives therefore $\cdot 12a^4$. Thus the effect of alternation becomes very sensible when $a > 1$.

Taking the depth of the effective surface stratum of a conductor as that surface stratum which would offer the resistance R' to a steady current it has been found from these results that for copper, lead, and iron, its values for different frequencies are as in the following table—

Frequency of Alternation.	Copper.	Lead.	Iron ($\mu = 300$.)
80	·719cm.	2·49cm.	·0976cm.
120	·587 ,,	2·04 ,,	·0798 ,,
160	·509 ,,	1·76 ,,	·0691 ,,
200	·455 ,,	1·58 ,,	·0617 ,,

For further information the reader may consult *Absolute Measurements*, Vol. II.

Section III.—Moving Electric Charges.

Convection Currents

553. It was discovered by Professor H. A. Rowland in 1876[1] that an electrified ebonite disk turning about its own axis produced a magnetic field, deflecting for example a needle placed below or above it. We have thus to consider a moving charge of electricity as a current, and to inquire how the idea is to be introduced into our system. Imagine a point charge of amount q moving with speed v in a straight line in a uniform dielectric. If it were standing still the total electric induction through a circle, the axis of which is the line of motion, and distance of any element of which from the charge is r, would be

$$2\pi q \int_0^\theta \sin\theta d\theta = 2\pi q(1 - \cos\theta)$$

where θ is the angle r makes with the axis.

If then the charge be moving with velocity v towards the circle, and the corresponding change of displacement be supposed for the moment to take place instantaneously throughout the field, the value of the electric induction will be altered per unit time by an amount $2\pi q \sin\theta d\theta/dt$, and $d\theta/dt$ is obviously $v \sin\theta/r$. Therefore the rate of increase of the total electric induction through the circuit is $2\pi qv \sin^2\theta/r$. This then is the total rate of change of the electric induction through the circuit, and is therefore 4π times the measure of the total displacement current through the circuit per unit of time. Hence if **H** be the magnetic intensity produced at the circle we must have by the first circuital theorem of Art. 506

$$\mathbf{H} \cdot 2\pi r \sin\theta = 2\pi qv \frac{\sin^2\theta}{r}$$

or

$$\mathbf{H} = qv \frac{\sin\theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

But this clearly is the magnetic intensity that would be produced, by an element of a current of length ds, and carrying a current γ, such that $\gamma ds = qv$. The direction of the magnetic intensity is related to that of the current in the manner already specified several times above (*e.g.*, see Fig. 149, Art. 546).

554. The conclusion is suggested therefore that the moving point-charge should be regarded as a current-element of moment, so to speak, qv, and that wherever there is moving electrification, there should

[1] *Ber. d. Berl. Akad.* 1876, p. 211. See also Rowland and Hutchinson, *Phil. Mag.* [27] (1889).

be taken at each point components of current ρu, ρv, ρw, where ρ is the volume density of the electrification at the point. Of course in the actual case the changes of displacement in the field are not propagated with infinite speed, and their finite propagation, which we have found above to exist, must be taken account of. This we shall do in the following articles, following a method due to Oliver Heaviside.

The equations of propagation as derived from the circuital equations (4) become with the convection currents expressed

$$\left.\begin{aligned} k\left(\frac{\partial P}{\partial t}+4\pi\rho u,\ \frac{\partial Q}{\partial t}+4\pi\rho v,\ \frac{\partial R}{\partial t}+4\pi\rho w\right)&=\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\ \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\ \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\\ \mu_0\left(\frac{\partial\alpha}{\partial t},\ \frac{\partial\beta}{\partial t},\ \frac{\partial\gamma}{\partial t}\right)&=-\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\end{aligned}\right\} \quad (70)$$

The first of these gives

$$k\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+4\pi\left(\frac{\partial}{\partial y}\rho w-\frac{\partial}{\partial z}\rho v\right)=\nabla^2\alpha$$

which, if p be written for $\partial/\partial t$, and V^2 for $1/k\mu$, becomes by the second

$$\left(\frac{p^2}{V^2}-\nabla^2\right)\alpha=4\pi\left(\frac{\partial}{\partial y}\rho w-\frac{\partial}{\partial z}\rho v\right) \quad . \quad . \quad . \quad . \quad (71)$$

Similar equations can of course be written down for β, γ.

If F, G, H be the components of vector-potential from which α, β, γ are derived, we have

$$\left(\frac{p^2}{V^2}-\nabla^2\right)\left(\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}\right)=4\pi\left(\frac{\partial}{\partial y}\rho w-\frac{\partial}{\partial z}\rho v\right)$$

with two similar equations. These equations are satisfied by putting

$$\left.\begin{aligned}\left(\frac{p^2}{V^2}-\nabla^2\right)F&=4\pi\rho u\\ &\cdots\cdots\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Hence we obtain the symbolical solution

$$F=\frac{4\pi\rho u}{\frac{p^2}{V^2}-\nabla^2}=-\frac{\nabla^{-2}(4\pi\rho u)}{1-\frac{p^2}{V^2\nabla^2}} \quad . \quad . \quad . \quad . \quad (73)$$

with two others for G, H.

555. Now consider the equation

$$\nabla^2\phi=-4\pi\rho u.$$

A solution is

$$\phi=\frac{\rho u}{r}$$

where r is the distance of the point considered from the point at which ρu is situated at the instant in question. This may be written

$$\phi = -\nabla^{-2}(4\pi\rho u) = \frac{\rho u}{r}.$$

Therefore

$$\cdot\,(F, G, H) = \mu\left(1 - \frac{p^2}{V^2\nabla^2}\right)^{-1}\Sigma\left(\frac{\rho u}{r}, \frac{\rho v}{r}, \frac{\rho w}{r}\right) \quad . \quad . \quad (74)$$

The symbol of summation is used since the whole distribution of moving electricity is concerned in producing F, G, H, and not merely that at the point considered. These equations enable the magnetic intensity to be found, and hence the whole problem may be regarded as solved.

As a particular example, consider the case of a point-charge of amount q moving along the axis of z with velocity w. Then

$$F = 0, \quad G = 0$$

$$H = \mu q w\left(1 - \frac{p^2}{V^2\nabla^2}\right)^{-1}\frac{1}{r}.$$

556. It can be verified at once that $\nabla^2 r^{n+2} = (n + 2)(n + 3)r^n$, so that

$$\nabla^{-2}(r^n) = \frac{r^{n+2}}{(n + 2)(n + 3)}.$$

Thus

$$\nabla^{-2}\frac{1}{r} = \frac{r}{2\,!}, \quad \nabla^{-4}\frac{1}{r} = \frac{r^3}{4\,!}, \ldots, \quad \nabla^{-2n}\frac{1}{r} = \frac{r^{2n-1}}{(2n)\,!}.$$

Again, since the origin is at the moving charge,

$$p = -w\frac{\partial}{\partial z}.$$

Hence the solution becomes

$$H = \mu q w\left\{\frac{1}{r} + \frac{1}{2\,!}\frac{w^2}{V^2}\frac{\partial^2 r}{\partial z^2} + \frac{1}{4\,!}\frac{w^4}{V^4}\frac{\partial^4 r^3}{\partial z^4} + \ldots\right\} \quad . \quad (75)$$

It is easy to prove that

$$\frac{\partial^{2n}(r^{2n-1})}{\partial z^{2n}} = 1^2 \cdot 3^2 \ldots (2n - 1)^2\frac{\sin^{2n}\theta}{r}$$

where θ is the angle between the direction of motion and the line (of

length r) drawn from the moving charge to the point considered. Substituting from the last equation in the series for H, we find that

$$H = \mu \frac{qw}{r}\left\{1 + \frac{1}{2}\frac{w^2}{V^2}\sin^2\theta + \frac{3}{8}\frac{w^4}{V^4}\sin^4\theta + \frac{5}{16}\frac{w^6}{V^6}\sin^6\theta + \ldots\right\}$$
$$= \mu \frac{qw}{r}\left(1 - \frac{w^2}{V^2}\sin^2\theta\right)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (76)$$

557. It will now be convenient to use cylindrical co-ordinates, and to put $x = 0$, and $y = h$, the distance of the point considered from the axis of z, along which the charge is moving. The components of magnetic intensity are given by

$$\mu\alpha = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial h}, \quad \mu\beta = -\frac{\partial H}{\partial x}, \quad \gamma = 0$$

and, as in the suppositions just made, x does not enter in H, and therefore $\beta = 0$, the magnetic intensity is at right angles to the planes of v and h. Thus we have for its value

$$\alpha = \frac{1}{\mu}\frac{\partial H}{\partial h} = -\frac{qw}{r^2}\sin\theta \frac{1 - \frac{w^2}{V^2}}{\left(1 - \frac{w^2}{V^2}\sin^2\theta\right)^{\frac{3}{2}}} \quad . \quad . \quad (77)$$

558. Fig. 152 shows the direction of the magnetic intensity at P the point considered. The moving point charge is at O, and has velocity w in the direction of the arrow.

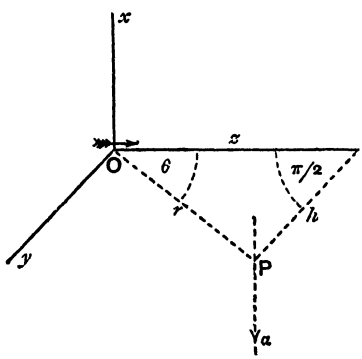


FIG. 152.

It will be observed that the direction of the magnetic intensity is the same as that of the intensity due to a current flowing in the positive direction of the axis of z, and that in both the lines of force are circles round that axis.

Further, if w be small in comparison with V, the magnetic intensity is precisely that which would be produced by a current element $\gamma ds = qw$ situated at the origin and directed along the axis of z.

On the other hand, if $w = V$, the magnetic intensity is zero everywhere except in the equatorial plane ($\theta = \pi/2$) through the moving charge, where it is infinite.

559. We have now to find the electric intensities. In going back for these to (70) we must modify the equations to suit the cylindrical

co-ordinates we have chosen. In the first place, since $\gamma = \beta = 0$, we have $P = O$; and since v, w are zero at the point considered, we have

$$k\frac{\partial Q}{\partial t} = \frac{\partial a}{\partial z}, \quad k\frac{\partial R}{\partial t} = -\frac{1}{h}\frac{\partial}{\partial h}(ha).$$

Since $\partial/\partial t = -w\partial/\partial z$, these equations may be written

$$-kw\frac{\partial Q}{\delta z} = \frac{\partial a}{\partial z}, \quad kw\frac{\partial R}{\partial z} = \frac{a}{h} + \frac{\partial a}{\partial h}.$$

The first of these gives at once

$$Q = -\frac{1}{kw}a,$$

or,

$$kQ = \frac{q\sin\theta}{r^2}\frac{1 - \frac{w^2}{V^2}}{\left(1 - \frac{w^2}{V^2}\sin^2\theta\right)^{\frac{3}{2}}} \quad \ldots\ldots (78)$$

It will be found that the equation for R is satisfied by the value

$$kR = \frac{q\cos\theta}{r^2}\frac{1 - \frac{w^2}{V^2}}{\left(1 - \frac{w^2}{V^2}\sin^2\theta\right)^{\frac{3}{2}}} \quad \ldots\ldots (79)$$

560. The electric intensity is therefore radial. Its intensity is least along the axis, and greatest in the equatorial plane. For very small values of w, however, the field is simply that of a stationary point-charge at O, multiplied by a correcting factor of value very nearly unity.

When however greater and greater values of w are considered the electric intensity becomes greater and greater in the direction outwards, and smaller and smaller in the direction along the axis. Finally for $w = V$, the electric intensity is still radial, but is zero everywhere except at points in the equatorial plane, and there it is infinite.

When $w > V$ the solution does not apply, and the case must be dealt with specially.

The values of the electric forces calculated above will be found, on integration over a spherical surface with its centre at O, to satisfy the condition that the integral of electric induction over the surface should be equal to $4\pi q$.

561. We give here two or three applications of the results found above. These applications are also due to Heaviside. The results are here merely stated with an indication of how they may be obtained. The working out in detail is left to the reader.

If the charge, instead of being a point-charge, is distributed uniformly along a line of finite length, lying along the axis of z and moving in that direction with the speed of light, and the density of the charge be q per unit of length, the direction of $\boldsymbol{a}$ is everywhere in circles round the axis, and the solution takes the form

$$a = \frac{2qV}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

while the electric force is radial and given by the equations

$$kQ = \frac{2q}{h}, \quad R = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

The field is entirely contained between the two infinite planes at right angles to the axis, and containing the extremities of the line, and at every point of that space is given by these equations.

562. If a perfectly conducting cylinder be placed round the line coaxially the field will be terminated radially by the cylinder, which will have on its inner surface and between the planes just specified a quantity of electricity uniformly distributed and equal and opposite to that on the line. As the line of electrification moves forward the opposite electrification which terminates the displacement tubes on the cylinder will move forward, keeping pace with it. If the tube and wire are parallel, not coaxial, the solution gives phenomena of just the same character. The distribution of forces in the field is in this latter case of course not the same as before.

This is the case of a plane wave moving along a wire surrounded by a conducting tube, on the hypothesis that there is no absorption of energy by the conductors and therefore no distortion of the wave. On the hypothesis stated this is the solution of the problem of a rudimentary telegraph circuit.

563. Again let a plane infinite in both directions be charged to uniform density q, and be made to move at right angles to itself with any speed. The solution shows that the magnetic intensity is zero everywhere whatever the speed, provided the plane be infinite both ways. The electric intensity is $2\pi q$, the electrostatic intensity which the plane would produce if there were no motion. The effects of the convection current are in fact balanced by those of the displacement current which exists at the same place as the former and practically neutralises it, so that there is no true current at all.

564. Next let a plane infinite in both directions move in its own plane with steady velocity w. The magnetic force is $2\pi qw$, at every point on one side of the plane, and $-2\pi qw$ at every point on the other side, in both cases being parallel to the plane and at right angles to the direction of motion. There is here no displacement current, since the displacement at no point changes. Thus the electric field is simply that of the charged plane.

All these results can be deduced from the solution given above for a point charge. Integration for a value of $w<V$ along a line distribution gives the total effect due to the distribution moving either along the line or transversely to it, or any direction compounded of these. Thence by integrating the effects of parallel narrow strips in a plane distribution results for a plane distribution in motion are obtained. The results enumerated above are the particular solutions for the circumstances specified.

Further researches on the subject are contained in Heaviside's *Collected Papers* and his *Electromagnetism.* See also a paper by C. F. G. Searle, *Phil. Trans.* R.S. 187, A. (1896), p. 675. The subject will be further dealt with in Vol. II.

Radiation in a Magnetic Field. The Zeemann Effect. Theory

565. It is important to notice that electromagnetic waves are generated by electric charges in periodic motion. Consider, for simplicity, a small point-charge, or an ion (an ultimate portion of matter with which is associated an electric charge) moving with definite period in a circular orbit. A periodic change of its electric and magnetic fields is set up which is propagated with speed $1/\sqrt{k\mu}$. This motion might be resolved into two rectilinear components of which the equations of motion would be

$$\ddot{x} + n^2x = 0, \quad \ddot{y} + n^2y = 0 \quad . \quad . \quad . \quad . \quad (82)$$

where, if the force towards the centre of the orbit vary as the displacement from that point, n is a constant independent of the radius of the orbit. To these might be superadded a vibration in the direction of the axis with equation

$$\ddot{z} + n^2z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

From the first two components will emanate waves of polarized light, that set up by the x-vibration being polarized in a plane through the y axis, and that produced by the y-vibration polarized in a plane through the x axis.

This circular motion may be imagined as that of, say, a pair of negative ions symmetrically placed in the x, y plane with respect to positive charges situated on the axis of z, supposed to be the axis of rotation. If the distance of the revolving ions from the axis be very small in comparison with their distance from the positive charges, and the electric repulsion between them and any other forces can be neglected in comparison with the force towards the axis due to the attraction exerted by the positive charges, the latter force will be of the proper amount to satisfy equations (82).

566. Now let a magnetic field directed parallel to the axis be impressed on the system. There will act, it is easy to see, on each ion

a force proportional to the current element to which it is equivalent, that is to the charge and to the velocity. These forces will on each moving ion be in the direction at right angles at once to its motion and to the magnetic field. Thus the equations of motion will become

$$\ddot{x} + \kappa\dot{y} + c^2x = 0, \quad \ddot{y} - \kappa\dot{x} + c^2y = 0 \quad . \quad . \quad . \quad (84)$$

which are precisely analogous to the equations obtained in Art. 255 above for the small motions of the bob of a gyrostatic pendulum. If $\mathbf{H}$ be the intensity of the field, m the effective mass, and ϵ the charge of the ion, it is seen at once that $\kappa = \mu\epsilon\mathbf{H}/m$.

The equations of motion found only hold strictly when the velocity of the ions is small compared with $1/\sqrt{k\mu}$ (see *Convection Currents*, Vol. II).

If we suppose that the motions are harmonic in period $2\pi/n$ and suppose the displacement to be proportional in each case to e^{int} we have the conditions

$$(c^2 - n^2)x + i\kappa ny = 0, \quad (c^2 - n^2)y - i\kappa nx = 0$$

which give at once the relation

$$c^2 - n^2 \pm \kappa n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Thus we obtain two values of n, and there are two modes of vibration, the period of which, if κ be small, are given by $n = c \pm \frac{1}{2}\kappa$.

The component of vibration in the direction of the magnetic field is not affected by the field, but remains of its original period.

567. The two linear vibrations of period given by $c + \frac{1}{2}\kappa$ are equivalent to motion in a circle in one direction, the others of period $c - \frac{1}{2}\kappa$ to motion in a circle in the opposite direction. Thus the radiation consists of a beam made up of two rays, from each of a complex of such rotating molecules, which, if received in a direction at right angles to the magnetic field, will be seen to be plane polarized in planes parallel to the field, and a third ray, the unmodified component of vibration in the direction of the field, which is plane polarized at right angles to the field. The period of this is midway between those of the modified components.

Thus, instead of the single line seen in the spectrum when the magnetic field is zero, a triplet of lines is obtained when a powerful field is applied. The polarization is tested in the usual way by means of a Nicol's prism.

If the beam is received in the direction of the axis the ray of mean period is not perceived since the vibration is end on, and, as explained above, there is no transverse component of vibration along the axis. The beam, however, will be analysed into the two circularly polarized rays specified above. The direction of polarization will indicate, as Zeemann has pointed out, whether the motion is that of a positive or negative charge.

Experimental Verification

568. The result thus theoretically obtained was discovered by Dr. Zeemann at Leyden[1] in 1897, and was first theoretically explained by Lorentz. To obtain the effect a sodium flame was placed between the poles of a large electromagnet, and the light emitted was examined by means of a diffraction grating of great power.

Many observers have since repeated and confirmed the results of Zeemann, and recently Mr. Thomas Preston of Dublin has succeeded in obtaining excellent photographs of doublets and triplets of lines produced by the action of an intense magnetic field.[2] In some cases, in the spectrum of iron for example, the line appeared to be quadrupled; but this result Mr. Preston sees reason to attribute to a reversal of the central line of the triplet.

The doubling of the middle line of a triplet has been observed for sodium, magnesium and cadmium by M. Cornu,[3] but he is very distinctly of opinion that the doubling is real, and not due to reversal. The two middle lines observed were found to be both polarized at right angles to the magnetic field. The production of a quadruplet of lines is not indicated by the dynamical theory given above.

569. Dr. Larmor[4] has recently discussed various interesting questions concerning the Zeemann effect, among them the phenomena to be expected if the moving point-charges are electrons, consisting of ultimate indivisible electric charges, without inertia except that depending on their charges. He has pointed out that the frequency intervals between the double lines and between the outside lines of triplets should be the same for all lines, and the same also for different spectra. This is a result that will no doubt soon be confirmed or disproved by experiment.

The order of magnitude of ϵ/m has been determined by Zeemann, and found to be such that m is about 1/1000 of the mass of the molecule of the radiating matter.

[1] *Phil. Mag.*, March and July, 1897.
[2] *Proc. R. S.* for Jan. 27, 1898.
[3] *L'Éclairage Électrique*, Jan. 29, 1898.
[4] *Phil. Mag.*, Dec. 1897.

CHAPTER XII

THE VOLTAIC CELL

Volta's Experiments on Contact Electricity

570. We have seen that a current of electricity is generated in a circuit by variation of the number of lines of magnetic induction passing through it, whether this variation is produced by the motion of the circuit in a magnetic field or by creating or annulling tubes of magnetic induction.

But it was discovered by Volta about the year 1793 that if a chain of different metals is formed the metals are electrified, apparently at least, to different potentials. For example, when zinc was put into contact with copper, the zinc apparently became electrified positively, the copper negatively; further, he obtained results of experiments showing that the difference of electric potential between the terminal metals of a series was equal to the difference which existed between these metals when put into direct contact, and therefore that when the terminal metals were the same they were at the same potential. The following table (given by Volta) indicates a series of metals arranged in such order that, if each were put into contact with the one next below it in the list, they would be respectively positively and negatively charged.

Zinc.	Iron.
Lead.	Copper.
Tin.	Silver.

571. It is unnecessary to go into details regarding Volta's method of experimenting. The following is a sketch of his procedure. A disk of one metal was soldered on another so as to form a double plate, for example, a plate of silver was soldered on one of zinc. Holding the double plate by the zinc, he touched the lower plate of his condensing electroscope with the silver, and, while this contact continued, touched the upper plate of the electroscope with his fingers. Breaking the second contact first he then removed the double plate, and lifted the

upper plate of the electroscope. The straws diverged, and were found to be negatively charged.

Volta then repeated the experiment, holding the double plate by the silver, and bringing the zinc into indirect contact with the lower plate of the electroscope, by pressing between them a piece of cloth or moist paper. It was found that the straws of the electroscope diverged with positive electricity.

Interpretation of Volta's Results

572. Volta's views as to the meaning of his results may be most conveniently expressed, in the language of modern science, by saying that he regarded the differences of potential produced between dissimilar *metals* in contact as the cause of the flow of electricity in a closed circuit, consisting of a series of metals with a liquid or liquids interposed between its terminals to complete the chain of contacts. On the other hand the current which in such cases flowed was held by Fabroni, and after him by other experimenters, to be due to the chemical action which, it was soon perceived, took place in the circuit, and chiefly manifested itself at the places of contact of the metals with the liquids.

Volta's views, on the other hand, were strongly defended by many of the most eminent physicists who followed him, and a modified voltaic theory, which ascribes the production of the current to the differences of potential produced by contact, and accounts for the energy evolved in the circuit by the chemical changes in the circuit, which have now for the greater part been quantitatively studied with more or less accuracy in different cases, is now held by several eminent authorities. With regard however to the *actual* amounts of these contact differences, and especially as to whether the contacts of metals with one another or of metals with liquids, are more intimately concerned in the phenomena, is still matter of considerable debate.

We give here a brief account of some of the chief investigations, and a short statement of the present position of the question.[1]

Objections to Volta's Method. His further Experiments

573. It was objected by the chemical theorists to Volta's first experiment that the effect produced was due to the contact of damp fingers with the metal, and to the second that the moist cloth or paper only was efficacious. He accordingly repeated the first experiment in the following manner. A large Leyden jar, of which the inner coating was composed of copper and the outer of tin, had its inner coating connected directly to the upper plate of the condensing electroscope, the outer

[1] For further information the reader should refer to a Report, *On the Seat of the Electromotive Forces in the Voltaic Cell*, by Professor Oliver J. Lodge, D.Sc., F.R.S., B.A. Report, 1884, p. 464. This Report has been of much assistance in the preparation of the present chapter.

coating was joined to the lower plate through the zinc-silver double plate. The plates of the electroscope being both composed of copper, the upper plate and the interior coating of the Leyden jar had equal and opposite charges, being of course uncharged before being thus joined up, but were at the same potential, V, say. On the other hand the difference of potential between the lower plate and the exterior coating depended on the contacts of the different conductors in that part of the arrangement. Thus denoting the difference of potential between copper and silver by Cu/Ag, between copper and zinc by Cu/Zn, and between zinc and tin by Zn/Sn, we get for the difference between the lower plate of the condenser and the exterior coating of the jar the value

$$Cu/Ag + Ag/Zn + Zn/Sn = Cu/Sn = V_2 - V_1 \quad . \quad . \quad (1)$$

if V_1, V_2 denote the potentials of these plates respectively. Hence if C_1 be the capacity of the electroscope-condenser and C_2 that of the jar, Q the charge of electricity on the upper plate of the condenser, and therefore $-Q$ that of the inner coating of the jar, we get approximately

$$C_1(V - V_1) = Q, \quad C_2(V_2 - V) = Q \quad . \quad . \quad . \quad . \quad (2)$$

These equations give

$$Cu/Sn = V_2 - V_1 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right),$$

or

$$Q = \frac{C_1}{1 + \frac{C_1}{C_2}} Cu/Sn \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus if C_2 is great in comparison with C_1, that is if the Leyden jar employed is very large, Q is simply the charge due to the difference of potential Cu/Sn, that is the difference is simply Cu/Sn. On the other hand if the condenser does not exist in any form C_2 is zero, and Q is also zero.

Thus Volta's second and improved form of the experiment told nothing about Zn/Cu, but indicated a difference between copper and tin in contact.

Result for Chain containing Liquid

574. It was found by Volta himself that the law that the difference of potential between the terminals of a chain of metals is the same as it would be if the terminals were in direct contact, does not hold if there are liquids interposed between the metals of the chain. Thus if the chain consist of copper, zinc, water, copper, the two terminal coppers are not at the same potential. This can be proved easily enough by experiments with an electroscope. It is only necessary to connect one

copper terminal to the lower plate of the condenser, and through it with the gold leaves of the instrument, and the other copper to the upper plate. When the upper plate is raised the gold leaves diverge with positive electricity if the lower plate is connected with the copper which is in contact with the water, and with negative electricity if the lower plate is connected with the other terminal.

A quadrant electrometer may be used instead of an electroscope, and the difference of potential between the terminals directly measured. In the case supposed a difference of potential of about ·75 volt is found. According to the ordinary contact theory this is held to be due to the contact difference of potential between copper and zinc, while the function of the water is taken to be that of simply bringing the copper and zinc plates immersed in it to the same potential.

This view is confirmed by the apparent differences of potential found by experiment and quantitatively measured by a great number of experimenters. We give here a short account of some of the methods used in these researches, and shall then consider the interpretation of the results.

Experiments of Kohlrausch

575. A number of valuable measurements were made by Kohlrausch,[1] who arranged parallel plates of the metals to be experimented on, so that they formed a condenser. The method consisted in bringing the plates close together, and connecting them for a moment by a wire, then separating them, and bringing one in contact with the indicator of a Dellman electrometer, the other with the earth. The deflection of the electrometer was noted. The experiment was repeated with a Daniell's cell interposed between the plates in the connecting wire, then with the cell reversed.

The theory of the experiment is as follows. By the first contact the two plates are brought to different potentials, and are correspondingly charged. They are then separated to a considerable distance, and the diminished capacity of each plate enables its potential to increase so that the Dellman electrometer can show a measurable deflection. When however the Daniell's cell is interposed, the difference of potential set up by the contact is increased by the difference which exists between the terminals of the cell, and a comparison of the readings enables the former difference to be determined in terms of the latter.

Thus calling M/M' the difference of potential between the metals when in direct contact, D that between the terminals of a Daniell's cell when they are formed of the same metal, and putting α, β, γ for the three deflections, we easily find

$$M/M' = k\alpha, \quad M/M' + D = k\beta, \quad M/M' - D = k\gamma$$

[1] *Pogg. Ann.* Vol. 82 (1851).

account being taken of the signs of α, β, γ. These give

$$M/M' = \frac{\alpha}{\beta - \alpha} D \text{ or } M/M' = \frac{\alpha}{\alpha - \gamma} D$$

so that there is a controlling equation for each measurement.

The Daniell's cell afforded a standard of comparison for the experiments in different pairs of metals, which could hardly be carried out always with the same initial distance between the plates.

The results obtained indicated the ratios

$$\frac{Zn/Pt}{D} = \frac{4{\cdot}49}{7{\cdot}48}, \quad \frac{Zn/Cu}{D} = \frac{3{\cdot}99}{7{\cdot}07}$$

and therefore

$$\frac{Zn/Pt}{Zn/Cu} = \frac{106{\cdot}4}{100}.$$

These it will be seen are considerably lower values than were obtained later for the contact differences of potential of the same metals.

Hankel's Experiments

576. The next experiments of note are those of Hankel[1] on the contact differences of potential between metals, and between metals and liquids. The apparatus is shown in Fig. 153. L is the liquid contained in a

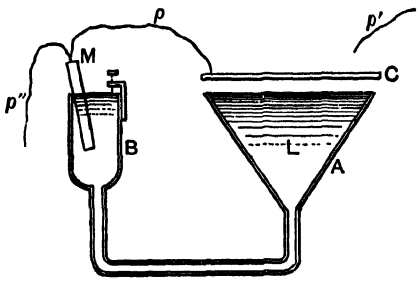


FIG. 153.

funnel connected by a tube with the vessel B, in which dips a strip of a metal M. A copper plate C rests a little way above and parallel to the liquid surface, and can be put into contact with M by a platinum wire p, and with an electrometer by another platinum wire p'. A third wire p'' kept up a permanent connection between M and the earth. The method of experimenting was as follows:—

(1) The funnel being full of liquid, the plates C and M were brought into contact by the wire p. The contact was then broken, and the plate C raised so as to come into contact with p'. This gave a deflection α, pro-

[1] *Pogg. Ann.* 115, 126, 131.

portional to the contact differences in the chain of substances. Thus k being a constant

$$Cu/Pt + Pt/M + M/L = k\alpha$$

or

$$Cu/M + M/L = k\alpha$$

by Volta's law.

(2) The funnel is emptied, and a plate of the metal M laid on its mouth. This plate is brought into contact with C and with the earth by platinum wires. Then the contact between C and the plate of M is broken, and C lifted and brought into contact with p' as before, giving a deflection β. This gives

$$Cu/M = k\beta.$$

(3) The plate of copper C is replaced by a plate of zinc, and the last experiment repeated. This gives

$$Zn/M = k\gamma$$

where γ is the deflection.

From these equations eliminating k we obtain

$$M/L = \frac{\alpha - \beta}{\beta - \gamma} Cu/Zn.$$

Also they give the relations

$$\frac{Cu/M}{Zn/M} = \frac{\beta}{\gamma},$$

$$Cu/M = \frac{\beta}{\beta - \gamma} Cu/Zn, \quad Zn/M = \frac{\gamma}{\beta - \gamma} Cu/Zn.$$

Hankel's results showed very different values for the contact difference between a metal and water according as the test was made immediately upon immersion or some minutes afterwards. Several metals, such as copper, platinum, silver, gold, iron, and tin, which showed a positive potential relatively to water, became just as strongly negative after immersion for from 10 to 30 minutes. He also found that the contact differences of metals varied with the state of their surfaces as to polish, and depended also to some extent upon whether the surfaces had, after having been polished, been exposed to the air.

Experiments confirmatory of Hankel's results were made a little later by Gerland in 1868 and 1869 (*Pogg. Ann.* 133, 137).

Lord Kelvin's Experiments

577. The next investigation of importance is that of Lord Kelvin, made about 1862. A charged metal arm was suspended horizontally, by a torsion thread, over the line of separation between two semicircular plates, one of copper, the other of zinc. The arm being positively,

charged, was deflected towards the copper when the metals were put in contact, showing that the zinc was positively electrified. When, however, a drop of water was made to bridge across the gap between the semicircles they were found to be at the same potential. This seemed to prove conclusively that the water had merely the effect of equalising the potentials of the two metals, and led Lord Kelvin to conclude "that two metals dipped into one electrolytic liquid will (when polarization is done away with) be at the same potential."

By his invention also of the quadrant electrometer, Lord Kelvin put into the hands of experimentalists an instrument immensely superior to any that had ever before been at their disposal, and thus greatly facilitated further investigation. This instrument he has himself applied to the measurement of contact differences, by the method of balancing the contact difference by a known fraction of the electromotive force of a Daniell's cell, from two points in a resistance connecting the terminals of the cell.

Lord Kelvin's Copper and Zinc Electric Machine

578. On the principle of his water-dropping electric machine, Lord Kelvin at this time constructed a machine which acted by aid of the difference of potential existing between copper and zinc in contact. Fig. 154 shows the arrangement. A copper funnel surrounded by a zinc cylinder contains a quantity of copper filings, which are allowed to trickle out through the mouth, and are received by a copper vessel below. This vessel rapidly acquires a negative charge.

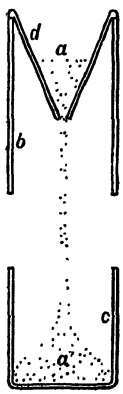


FIG. 154.

Consider a particle of copper just leaving the funnel, and therefore breaking away from the mass of filings above it. It is in the middle of a space surrounded by the zinc cylinder, which is positively electrified relative to the copper, and hence, since the particle has the potential of copper, it must be negatively electrified. It falls, carrying its negative charge with it, and being received in the interior of the vessel below gives up to the latter all its charge. Thus the negative potential of the receiver continually increases. By connecting the receiver and funnel by a copper wire, a current of negative electricity may be made to flow round the circuit, from the receiver to the funnel through the wire, and by convection along the stream of filings.

With regard to the source of the energy in this arrangement it is sufficient to notice that the particles fall against electric repulsion. Work is therefore done against electric forces by gravity, and the particles reach the receiver with smaller velocities than they would otherwise have, and the difference of energies is stored up in the electric distribution on the receiver.

Lord Kelvin's Induction Electric Machine, Founded on Contact Electrical Action

On the same principle Lord Kelvin constructed a revolving induction electric machine which is shown in Fig. 155. There are two connected brass inductors T of the shape shown, one of which is lined with one metal, the other with the other metal. A non-conducting wheel with carrier studs which are touched by springs $A\,A'$ at opposite ends of a diameter is rotated within the inductors, and the springs become oppositely charged. The disk is kept turning and the springs are connected to the terminals of a quadrant electrometer which measures the

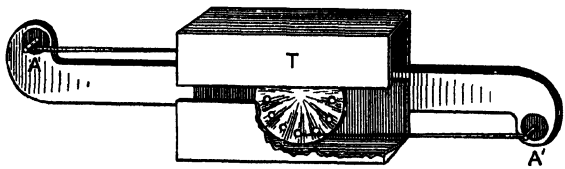


FIG. 155.

difference of potential produced. The absolute value of this is obtained by disconnecting the inductors from one another and removing their linings so as to make them of the same metal, and then connecting them with the terminals of a Daniell's cell and again rotating the carrier wheel, while the difference of potential of the springs is tested by means of the electrometer. Thus the contact difference between zinc and copper or between any pair of metals could be evaluated. Of course by keeping the metals in contact and applying the requisite fraction of the electromotive force of a Daniell's cell the method could be made a null one.

Experiments of Ayrton and Perry. Clifton's Experiments

579. A very large series of determinations of contact differences of potential was carried out by Ayrton and Perry in 1876.[1] A diagram of their apparatus is shown in Fig. 156. On a platform $A\,B$ were placed in contact the substances to be tested, for example, a metal and a liquid as shown at P and L. This platform could be turned round a pivot M through 180° on wheels running on a railway R. Two well insulated gilded plates 3, 4 were attached to an upper bar, which could be raised or lowered by a parallel ruler arrangement attached to the top of the case.

In an experiment these plates were lowered close to the substances to be tested, and were put in contact for a moment by a wire, then raised and connected to a quadrant electrometer. The platform $A\,B$ was now turned through 180° on its railway, the plates brought down,

[1] *Proc. R. S.* 1878, and *Phil. Trans. R. S.* 1880.

connected for a moment, raised and connected to the electrometer once more. The deflections were in opposite directions on the scale, and the double deflection could be taken as proportional to the difference of potential between the plates P, L.

To evaluate this difference of potential the plates P, L were replaced by brass plates, which were brought to various differences of potential by applying to them certain fractions of the electromotive force of a Daniell's cell produced by placing the terminals so as to include different parts of the total resistance in the circuit. Thus the deflections on the electrometer scale were evaluated, and the contact differences

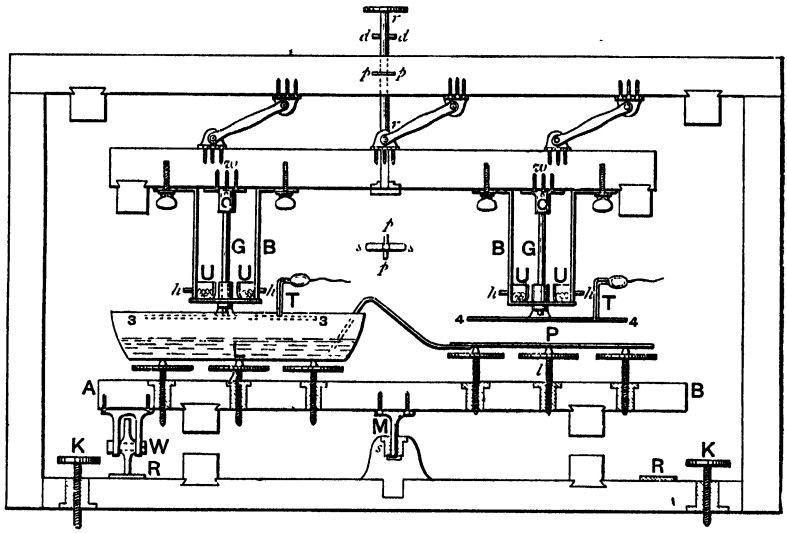


FIG. 156.

measured in volts. A table of results abridged from Everett's *Units and Physical Constants*, for which the table was specially prepared by the experimenters, is given at p. 457, as an appendix.

The results of these experiments were not published until 1878, owing to some delay in the communication of the paper to the Royal Society, and in the meantime a series of careful experiments had been made by Professor Clifton of Oxford, who measured with great care the contact differences of potential for substances ordinarily used in batteries.

Comparison of Results

580. Both Clifton and Ayrton and Perry found that the electromotive forces of different cells could be obtained by simply summing all the differences of potential at the surfaces of contact of dissimilar

substances in the circuit. Thus the table given below (extracted also from Everett's *Units*), gives the electromotive forces as calculated and as observed for two of the more well-known cells, viz., those of Daniell and Grove.

DANIELL'S CELL.

Substances in Contact.	Difference of Potential (Volts).
Copper and saturated copper sulphate	+ ·070
Saturated copper sulphate and saturated zinc sulphate	− ·095
Saturated zinc sulphate and zinc	+ ·430
Zinc and copper	0·750
Total . . .	+1·155
Observed difference	+ 1·10

GROVE'S CELL.

Substances in Contact.	Difference of Potential (Volts).
Copper and platinum	+ 0·238
Platinum and strong nitric acid	+ 0·672
Strong nitric acid and very weak sulphuric acid .	+ 0·078
Very weak sulphuric acid and zinc	+ 0·241
Zinc and copper	+ 0·750
	1·979
Electromotive force observed on open circuit . . .	1·9

The agreement of the numbers is very satisfactory, and other examples of the same kind strengthen the conclusion that the electromotive force of any cell can be built up in this way. It is to be observed, however, that this gives only the electromotive force with the plates in the condition in which they were when experimented on, and with no current flowing through the battery. When a current is made to flow the electromotive force in many cases, and to a small extent even in the so-called constant cells, falls off in amount owing to the deposition on the plates of the gaseous products of the decomposition of the liquids.

Pellat's Experiments

582. In an elaborate set of experiments made by Pellat,[1] and published in 1881, a plan of compensation, also previously devised by Lord Kelvin and used in some unpublished researches, was employed to convert the condenser method into a null method, requiring therefore only a very sensitive electrometer arranged to show a deflection for the

[1] *Journ. de Phys.* xvi. (1888).

smallest possible difference of potential. This plan consisted in applying to the plates an equal and opposite difference of potential to that produced by contact, and so annulling their electrification. This was obtained by using the arrangement shown in Fig. 157. A couple of Daniell's cells B are joined up in series with a rheostat R, and a resistance slide S, S'. The sliding piece, p, is connected with the earth

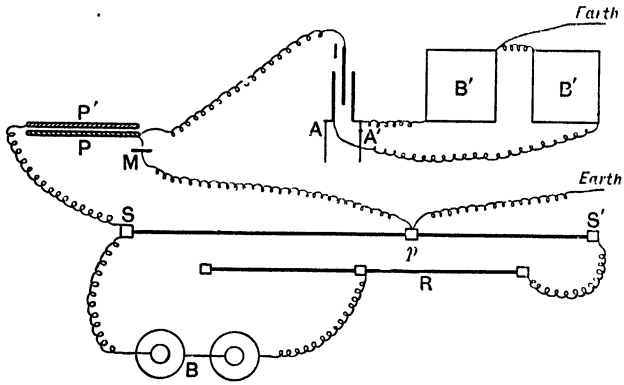


FIG. 157.

and with a wire by which contact can be made with one of the plates at M. The same plate (P) is connected with the indicator I of the electrometer, and the extremity S of the slide to the other plate P'. B' is a battery employed to charge the plates of the Bohnenberger electrometer which Pellat used.

The method of experimenting consisted in raising the plate P' (the contact between P and M having been previously broken), and observing the deflection produced by the rise of the potential of P. If no deflection took place then P was unelectrified, and the compensation had been complete.

The following are a few of Pellat's results—

Metal in Contact with Standard Gold.	Clean, almost unscratched surface.	Surface strongly scratched with emery.
Zinc	·85	1·08
Lead	·70	·77
Tin	·60	·73
Nickel	·38	·45
Bismuth	·36	·48
Iron	·29	·38
Brass	·29	·37
Copper	·14	·22
Platinum	−·03	+·06
Gold	−·04	+·07
Silver	−·06	+·04

Experiments on Metals Immersed in Different Gases

583. Pellat has also endeavoured to measure contact differences of metals immersed in different gases, and so far as his experiments go they corroborate a conclusion previously arrived at by Pfaff that the contact electric difference is practically unaffected by the gas by which the metals are surrounded, so long as no visible chemical action, such as tarnishing of the surfaces by oxidation, takes place.

Practically the same conclusion has been arrived at by Lord Kelvin and by Von Zahn and others. On the other hand, J. Brown of Belfast has found that copper is positive with respect to iron in sulphuretted hydrogen, while, as stated above, it is negative in air.

Later Views on Contact Electricity. Differences of Potential Inferred from Flow of Energy

584. It is impossible here to give a full account of experiments on this subject, and we shall therefore now try to state shortly theoretical views which at the present time appear to find a certain amount of favour.

On the experimental results stated above has been built a theory that the seat of the electromotive force of a voltaic cell is at the junctions of the dissimilar substances in contact, and that the differences of potential measured in the manner described are actually contact differences between the metals, which added together, with their proper signs, give the electromotive force of the cell. The electromotive force being thus accounted for, the energy consumed by the cell is further seen to be furnished by the chemical changes within the cell which accompany the flow of the current, and so the contact and chemical theories, which once were in severe conflict, are in a sense reconciled.

On the other hand it is held by several authorities that the contact difference of potential between two metals is only apparent, being the difference between the potential in the air near one metal and that in the air near the other metal, while the potentials of the metals themselves are one and the same. It is clear that all the experiments described above are consistent with this view. The films of air close to the plates adhere to them when they are separated, and the potentials are altered just as if each had itself a real difference of potential and a charge on its surface. Again the view holds for Lord Kelvin's copper and zinc ring experiment and for a quadrant electrometer, with zinc and copper quadrants. The needle is acted on by the charged air films just as if these were real charges of the plates.

According to this view the different effects found by Brown when sulphur became the active substance of the medium are to be regarded as also only apparent, and that the metals themselves are to be taken as really at the same potential.

Further it can be shown that though the actual values of the individual differences of potential between the metals may still be unknown, yet their sum for a non-metallic chain of substances must be the same as that obtained from measurements of the kind which have been described.

585. The view has been put forward with great skill and force by Dr. Lodge that the difference of potential between two dissimilar substances ought to be measured by the work done there per unit time on unit current flowing from one substance to the other. The total work spent per unit time in causing a current to flow in a circuit is, as we have seen $E\gamma$, (where E is the electromotive force and γ the current) in the case in which the current is produced in moving a conductor across the lines of force of a magnetic field. This is given out again in the circuit either in heat or in some other form of energy. In all cases the rate at which energy is *evolved* at any part of the circuit, apart from places at which chemical changes take place, is equal to the product of the current into the difference of potential down which the current flows.

Energy value of Electromotive Force of a Cell

586. Lord Kelvin pointed out in 1851 that the only source of energy in the circuit is the chemical potential energy used up, and he assumed the time-rate of consumption of this to be equal to $E\gamma$, the rate at which work is given out in the circuit. For, as he showed, any voltaic arrangement can be replaced by a magnetic electric generator giving the same electromotive force, and the rate at which energy is given out in the working part of the circuit can be made exactly the same as before. This theory, it is to be observed, is not quite exact, as the value of the electromotive force requires correction for thermodynamic reasons. The whole subject will be discussed in Vol. II.; but it may be stated here, and it is very easy indeed to prove the particular result, that, if E be the electromotive force of a cell, which varies in electromotive force with temperature only, and $\Sigma(J\theta\epsilon)$ be the dynamical value of the chemical changes which take place in the cell when a unit of electricity is passed through it [see Art. 589, (7)]

$$E - t\frac{dE}{dt} = \Sigma(J\theta\epsilon),$$

where t is the absolute temperature. (See also Art. 601 below.)

Hence it is only going a little further to say that wherever the current flows up a difference of potential whether in a voltaic cell or a voltameter at the junction of dissimilar substances, or in a wire moving across lines of force in a magnetic field, there energy is absorbed, and that the energy absorbed per unit current per second measures the difference of potential. It is in fact extending the view already well

established for the whole of the chemical action in a cell or voltameter to every element of the difference of potential, and associating with that element, as regards both its amount and its locality, the energy change which takes place in any time. The actual potential difference, at any rate when the cell is generating a current, can be found approximately when the heats of combination are known. [A fuller discussion of the calculation of electromotive forces from such data, with an examination of the limitations and corrections to which the theory is subject, will be found in Vol. II.]

Explanation of Volta Effects as Air Metal—Metal-Air Differences of Potential. Thermoelectric Measure of Difference of Potential

587. The differences of potential to which this theory leads are very much smaller in many cases than those obtained from direct observation of voltaic contact effects. The latter are however explicable by regarding the voltaic difference of potential A/B between two metals A and B as really air $/A+A/B+B/$ air, so that the true A/B may be really very different from this sum, which is the observed or apparent A/B. But it can be shown at once that this in no way interferes with the method of finding the electromotive force of a cell by adding up the contact differences in the circuit. Denote the apparent contact-difference between A and B by A'/B', and let there be substances $A, B, C, \ldots G$ arranged in order in the circuit. Then we have

$$\begin{aligned} A'/B' &= \text{air}/A + A/B + B/\text{air} \\ B'/C' &= \text{air}/B + B/C + C/\text{air} \\ &\cdots\cdots\cdots \\ F'/G' &= \text{air}/F + F/G + G/\text{air} \\ G'/A' &= \text{air}/G + G/A + A/\text{air}. \end{aligned}$$

Adding up we find

$$A'/B' + B'/C' + \ldots + F'/G' + G'A' = A/B + B/C + \ldots + F/G + G/A$$

since

$$\text{air}/A + B/\text{air} + \text{air}/B + \ldots + A/\text{air} = 0.$$

Thus the electromotive force has the same value in whatever way it is obtained. It is to be noticed that any one of the electromotive forces of contact between air and the substances in contact may be very great; there is no known method of finding its value.

Energy Criterion of Existence of an Electromotive Force

588. It must be admitted that the only electromotive force at the junction of a pair of dissimilar substances which has been determined without ambiguity is that measured by the amount of heat absorbed or evolved at the junction when unit current flows across it; that is, it

is equal to the coefficient Π of the Peltier effect, as explained in Art. 596 below. This was the view held by Clerk Maxwell, and set forth by him in his treatise on *Electricity and Magnetism*.[1]

This determination of the amount and locality of an electromotive force, however, depends on taking absorption or evolution of energy as the test of its existence in the manner just explained, and must stand or fall with the validity of that criterion. But this is the method adopted in all our calculations regarding the supply of energy to an electric system from without, and the evolution of energy in the system itself. Thus if we take as the system considered a conductor in which heat is being generated by a current, the rate at which heat is generated by unit current is the measure of the electromotive force which must be impressed on the conductor by the part of the electric system outside the conductor that the current may exist. Again, when a linear conductor is moved at right angles to itself and to the lines of induction of a magnetic field with velocity v, the dynamical force which must be applied to each element of it of length ds is precisely $\mathbf{B}\gamma ds$, where $\mathbf{B}$ is the magnetic induction and γ the current, and therefore the rate at which work is spent in the element is $\mathbf{B}\gamma v ds$. The rate at which work is done upon the conductor is then $\mathbf{B}v ds$, which we have seen in Chap X. above is the measure of the electromotive force in the element.

The question therefore resolves itself into whether this process can be accurately applied experimentally to each part of the heterogeneous circuit of a voltaic cell. If it can the question would seem to be settled. All that remains is to work the matter fully out by experiment, by making local energy determinations all round a heterogeneous circuit, a work undoubtedly of very great difficulty, though not perhaps in simple cases impossible. As has just been shown, experiments on contact electromotive force cannot be regarded as conclusive, and further investigation, by the method of the absorption and evolution of energy at different parts of the circuit must be awaited, before the conclusions of any theory are held to be definitely proved.

According to Poynting's theory energy is thrown out into the medium wherever in the circuit the current flows up a slope of potential under the action of an impressed electromotive force, and flows into the circuit from the medium where the current flows down a slope of potential. Thus, in the original paper in which this theory is set forth, a diagram of the flow of energy is given, according to which the greater part of the flow of energy to the medium takes place at the surface of contact of the zinc and acid. Dr. Lodge,[2] however, has pointed out that in ordinary arrangements the stream-lines of energy may be crowded together along the zinc, and then pass out into the medium at the copper-zinc junction, as apparently they do on the ordinary voltaic theory in which the electromotive force has its seat at the surface of contact of the two metals. The view that the energy evolved at different parts of the circuit cannot be transmitted along the wire

[1] Vol. I. p. 369 (3rd edition). *Electrician*, April 26, 1879. [2] *Phil. Mag.*, June 1885.

without its passage being accompanied by some physical manifestation of its presence, but that it travels through the medium guided by the conductor, has been sufficiently emphasised above.

Summary of Results of Later Theory

589. The following *resumé* of statements which are consistent with this theory, and of others which are not, is taken from Dr. Lodge's Report.

(1) Two metals in contact ordinarily acquire opposite charges; for instance clean zinc receives a positive charge by contact with copper, of such a magnitude as would be otherwise produced under the same circumstances by an E.M.F. of about ·8 volt.

(2) This apparent contact E.M.F. or "volta force" is independent of all other *metallic* contacts wheresoever arranged; hence the metals can be arranged in a numerical series such that the "contact force" of any two is equal to the difference of the numbers attached to them, whether the contact be direct or through intermediate metals. But whether this series changes when the atmosphere, or medium surrounding the metal, changes is an open question. It certainly changes when the free metallic surfaces are in the slightest degree oxidised or otherwise dirty. And in general this "volta force" is very dependent on all non-metallic contacts.

(3) In a closed chain of any substance whatever, the resultant E.M.F. is the algebraic sum of the volta forces measured electrostatically in air for every junction in the chain, neglecting magnetic or impressed E.M.F.

(4) The E.M.F. in any closed circuit is equal to the energy conferred on unit electricity as it flows round it.

[In the next four statements magnetic and other impressed E.M.F. is to be neglected.]

(5) At the junction of two metals any energy conferred on, or withdrawn from, the current must be in the form of heat. At the junction of any substance with an electrolyte, energy may be conveyed to or from the current at the expense of chemical action as well as of heat.

(6) In a metallic circuit of uniform temperature the sum of the E.M.F.'s is zero by the second law of thermodynamics (see Vol. II.); if the circuit is partly electrolytic, the sum of the E.M.F.'s is equal to the sum of the energies of chemical action going on per unit current per second.

(7) In any closed conducting circuit the total intrinsic E.M.F. is equal to the dynamical value of the sum of the chemical actions going on per unit electricity conveyed $\Sigma(J\theta\epsilon)$ where ϵ is the quantity of matter affected by chemical change in the passage of unit current for unit time, θ the heat evolved or absorbed in the change, and J the dynamical equivalent of a unit of heat), diminished by the energy expended in algebraically generating reversible heat.

(8) The locality of any E.M.F. may be detected, and its amount measured, by observing the reversible heat or other form of energy there produced or absorbed per unit current per second.

The following statements held to be true by many contact theorists are inconsistent with the theory.

(1) Two metals in air or water or dilute acid, but not in direct contact, are practically at the same potential.

(2) Two metals in contact are at seriously different potentials, [*i.e.* differences of potential greater than such milli-volts as are concerned in thermoelectricity].

(3) The contact force between a metal and a dielectric, or between a metal and an electrolyte such as water and dilute acid is small.

It is to be remembered that authorities are still divided in opinion on this subject, and that what has been given above is to be regarded only as an attempt to state the opposing views. The discussion will be resumed in Vol. II., where some account will be given of the behaviour of different cells, and of the thermodynamics of the subject.

APPENDIX TO CHAPTER XII

Abridged from Everett's *Units and Physical Constants.*

CONTACT DIFFERENCES OF POTENTIAL IN VOLTS (AYRTON AND PERRY)

Solids with Solids in Air.

	Carbon.	Copper.	Iron	Lead.	Platinum	Tin.	Zinc.	Amalgamated Zinc.	Brass.
Carbon	...	·370	·485*	·858	·113	·795	1·096	1·208*	·414*
Copper	−·370	...	·146	·542	−·238	·456	·750	·894	·087
Iron	−·485*	−·146	...	·401*	−·369	·318*	·600*	·744*	−·064
Lead	−·858	−·542	−·401*	...	−·771	−·090	·210	·357*	−·472
Platinum	−·113*	·238	·369	·771	.	·690	·981	1·125*	·287
Tin	−·795*	−·456	−·313*	·090	−·690	..	·281	·463	−·372
Zinc	−1·096	−·750	−·600*	−·210	−·981	−·281	...	·144	−·679
Amalgamated Zinc	−1·208*	−·894	−·744*	−·357*	−1·125*	−·463	−·144	...	−·822*
Brass	−·414	−·087	·064	·472	−·287	·372	·679	·822	..

The average temperature at the time of experimenting was about 18° C. The numbers without an asterisk were obtained directly by experiment, those with an asterisk by calculation, using the well-known assumption that in a compound circuit of metals, all at the same temperature, there is no electromotive force.

The numbers in a vertical column below the name of a substance are the differences of potential in volts between that substance and the substance in the same horizontal row as the number, the two substances being in contact. Thus, lead is positive to copper, the electromotive force of contact being ·542 volts.

The metals were those of commerce, and therefore only commercially pure.

Solids with Liquids in Air.

	Carbon.	Copper.	Iron.	Lead.	Platinum.	Tin.	Zinc.	Amalgamated Zinc.	Brass
Mercury	·092	·308	·502	..	·156	..	...	.	..
Distilled water	·01 to ·17	·269 to 1	·148	·171	·285 to ·345	·177	−·105 to +·156	·100	·231
Sea salt solution, sp. g. 1·18 at 20·5 C.	.	−·475	·605	− 267	−·856	−·334	−·565	...	−·435
Sal-ammoniac saturated at 15·5° C.		−·396	−·652	·189	·057	−·364	−·637	...	−·348
Concentrated sulphuric acid	·85 to ·55	1·113	..	·723 to 1·252	1·6 to 1·3	..	...	848	.
1 distilled water, 5 strong sulphuric acid by weight	·3 to ·01	..	.	−·120	..	−·256	..	..	·016

Solids with Liquids and Liquids with Liquids in Air.

	Copper.	Zinc.	Amalgamated Zinc.	Mercury.	Distilled water.	Copper sulphate solution saturated at 15° C.	Zinc sulphate solution, sp. g. 1·125 at 16·9° C.
Copper sulphate solution, sp. g. 1·087 at 16·6° C.	·103	...	...		..	..	·090
Copper sulphate saturated at 15° C.	·070	...	..	.	−·043	...	.
Zinc sulphate solution, sp. g. 1·125 at 16·9° C.	.	−·238	..	...	..	...	.
Zinc sulphate solution, saturated at 15·3° C.	.	−·430	−·284	.	−·200	−·095	..
1 distilled water mixed with 3 zinc sulphate saturated	..	−·444	..	...	...	−·102	
20 distilled water, 1 strong sulphuric acid	...	−·344	...	..	...	...	
5 distilled water, 1 strong sulphuric acid	..	...	−·420	..	...	...	
Distilled water with trace of sulphuric acid	..	−·241	...	...	...	..	...
Mercurous sulphate paste	...	...	...	·475	...	...	...

The average temperature at the time of experimenting was about 16° C.

All the liquids and salts employed were chemically pure; the solids, however, were only commercially pure.

CHAPTER XIII

THERMOELECTRICITY

Elementary Phenomena

590. It was discovered by Seebeck in 1822 that if a circuit be formed of two different metals, and the junctions be at different temperatures, a current flows round the circuit. This is illustrated by heating with a flame the junction of a bar of antimony with one of bismuth, as shown in Fig. 158.

To fix the ideas let the circuit be composed of a wire of bismuth and a wire of antimony. It is found that no current flows if the junctions of the bismuth and antimony are at the same temperature. If, however, one of the junctions is warmed a current flows round the circuit and passes across the hot junction from bismuth to antimony. Cooling the same junction below the temperature of the other also produces a current, but in the opposite direction. The interposition of solder, or even of a chain of one or more different metals in a junction (*e.g.* a wire, say, of copper joining, as in Fig. 158, the remote extremities of the bars of the two metals) will not affect the electromotive force in the circuit, if the junctions of the chain are all at one temperature.

The effect can be much increased by using a chain of two metals arranged alternately, and heating the alternate junctions, as shown in Fig. 159, which is the arrangement of course of the thermopile.

It will be seen later that these statements hold only for ordinary temperatures, and are not true in all circumstances. For example, if one junction be below, the other above a certain temperature, depending on the metals employed, no current, or a reverse current, may be produced by making the higher temperature sufficiently high. The intermediate temperature thus referred to will be seen to be such that if the temperature of one junction be lower by a small difference, and the other higher by the same small difference, no current flows, and is therefore called the neutral temperature for the metals concerned.

591. The elementary phenomena may be conveniently studied by making a circuit of two wires, say of copper and iron, by twisting or soldering the wires together at the junctions, and inserting a galvanometer in one of them, say the copper. One junction may be immersed in a beaker of water so as to keep its temperature constant, the other may be gradually heated by a spirit lamp or Bunsen flame (if it is not soldered). As the junction is heated the current indicated by the galvanometer will gradually increase, reach a maximum, diminish to

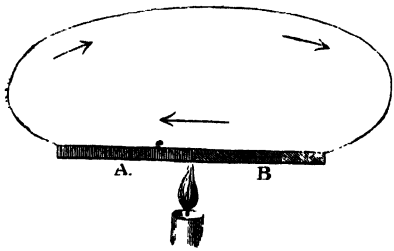


FIG. 158.

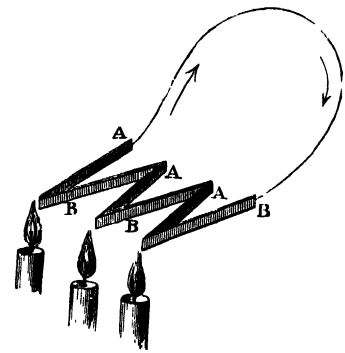


FIG. 159.

zero, and finally be reversed. The temperature of the hot junction when the current is a maximum is the neutral temperature, that at which the current changes sign is the temperature of inversion for the metals employed which correspond to the temperature of the colder junction.

Thermoelectric Inversion

592. The phenomenon of inversion was first observed in 1823 by Cumming, who found that the order of the metals in a thermoelectric series was not the same at all temperatures.

As we have seen in considering voltaic action no current is produced in a purely metallic circuit if the conductors are all at one temperature. To this may be added here the fact that in a homogeneous circuit or homogeneous part of a circuit no electromotive force is produced by inequalities of temperature provided that in the case of part of a circuit the extremities of the conductor are at the same temperature.

Thermoelectric series were formed by Seebeck and others, according to which any substance in the series if made into a circuit with any metal following it in the series, gives a current from the former to the

latter across the warmer junction. The following table gives a few of the substances in one of these series:—

Bismuth	Copper	Silver
Nickel	Mercury	Zinc
Cobalt	Lead	Iron
Palladium	Gold	Antimony.

Thermoelectric Power

593. It was found experimentally by E. Becquerel that the total electromotive force for any two temperatures of the junctions is the sum of the electromotive forces of the couple for any differences of temperature making up (not by mere addition, but by actual position in the temperature scale) the range of temperature between the junctions.

Let the temperature of one junction be $t-\frac{1}{2}dt$, that of the other $t+\frac{1}{2}dt$, where dt is a small difference of temperature, and let the electromotive force of the couple for these two temperatures be measured. The ratio of this electromotive force to the difference of temperatures dt is called the thermoelectric power of the couple at temperature t. We shall here denote it by P.

By Becquerel's experimental result the electromotive force E, when the temperature of one junction is t_0 and that of the other t, is given by

$$E = \int_{t_0}^{t} P dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P is the thermoelectric power at the intermediate temperature at which any elementary difference of temperature dt is taken. Thus we see that

$$P = \frac{dE}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

594. The thermoelectric power of a pair of metals can easily be found by making up a circuit of two wires of the metals in series with a high resistance galvanometer, and observing the current when one junction is kept at a constant low temperature, and the temperature of the other is varied by small steps until any required range has been covered. Any required temperature of the junctions can be produced by immersing them in baths of water or oil, of which the temperatures can be observed by means of thermometers. The current through the galvanometer obtained in absolute units and multiplied by the resistance in circuit gives the electromotive force for each observation.

The temperatures of the hot junction are now laid off as abscissæ, and the corresponding electromotive forces as ordinates of a curve. If

tangents then are drawn to the curves at different points, the inclinations of these tangents to the axis of abscissæ will measure the thermoelectric powers of the couple at the temperatures corresponding to the ordinates drawn to the same points.

If now a curve be plotted with thermoelectric powers of the couple as ordinates, and temperatures as abscissæ, the area of this curve taken between the axis of abscissæ and any pair of ordinates will measure the electromotive force of the couple when the junctions are at the temperatures corresponding to these ordinates. If one of these ordinates be at the point corresponding to the fixed temperature of the cold junction in the experiments referred to above, the areas of the curve of thermoelectric powers will for different positions of the other ordinate, of course be the ordinates of the curve of total electromotive force from which the second curve was derived. The curve of thermoelectric

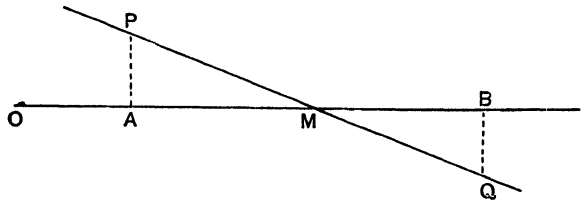


FIG. 160.

powers, drawn as described, is generally found to be a straight line, as represented in Fig. 160. Here OA denotes t_0, the lower of the two extreme temperatures, OB the higher t, while OM represents the neutral temperature.

The areas AMP and BMQ are to be taken with opposite signs, and when they are equal the electromotive force in the current is zero.

If $OB = t_1$, the temperature of inversion, and the curve of thermoelectric power be either as here represented a straight line, or be skew-symmetrical with respect to M, it is clear that $T - t_0 = t_1 - T$, or

$$T = \frac{t_1 + t_0}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In such cases the neutral temperature T is the mean of t_1 and t_0.

A very important experimental result with respect to thermoelectric power is the following: If at any temperature P_{AB} be the thermoelectric power of a couple composed of two metals A and B, and P_{BC} be the thermoelectric power of B and another C, then at the same temperature

$$P_{AC} = P_{AB} + P_{BC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In reckoning the sum here it is to be observed that the sign of the thermoelectric power is to be observed, and taken account of. Care is

to be taken to place the metals in the proper order, so that with respect to the hot and cold junctions the positions of AC, AB, BC may be the same in the verifying experiments. Hence if the curve of thermoelectric powers of the two metals AB be represented by the line QR, Fig. 161, and that of AC by SR, the thermoelectric powers of BC will be represented by the *difference* of the ordinates of these two curves, and their neutral temperature will be that corresponding to R, namely the temperature represented by OM.

A table of thermoelectric powers of different substances with respect to lead is given at the end of this chapter.

Total Electromotive Force

595. It follows from Fig. 161 that the curve drawn having temperatures as abscissæ and total electromotive forces as ordinates is a

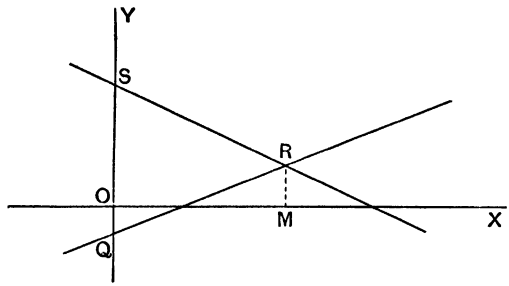


FIG. 161.

parabola, having its maximum ordinate at M and cutting the axes at A and B. For the area between PQ (Fig. 160) the axis, and the ordinates corresponding to temperatures t_0 and t, is

$$E = \int_{t_0}^{t} \frac{dE}{dt}\, dt = \tfrac{1}{2}(P_0 + P)(t - t_0)$$

where P_0, P are the thermoelectric powers at t_0 and t respectively. But, by Fig. 160, $P = P_0(T-t)/(T-t_0)$ and therefore

$$E = \frac{P_0}{2}\left\{t - t_0 + \frac{T - t}{T - t_0}(t - t_0)\right\} \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of a parabola of which t is an abscissa and E the corresponding ordinate. The observation that the curve of total electromotive force is generally a parabola was first made by Gaugain.

The curve of total electromotive force between any other initial temperature t'_0 and the corresponding temperature of inversion t'_1 is

obtained from the general curve by drawing a dotted line parallel to the axis of abscissæ through the top of the ordinate for the temperature t'_0. The part of the curve lying above the line so drawn is the curve of total electromotive force for the range of temperatures stated.

Peltier Effect

596. In 1834 Peltier discovered that if a current of electricity was sent from a battery through a circuit of two metals initially at the same (ordinary) temperature one junction is cooled and the other heated. The difference of temperature thus set up would by itself send a current in the opposite direction to that producing it. This heating and cooling effect is known as the Peltier thermal effect. It is reversible, inasmuch as it depends on the direction of the current whether the effect is a heating or a cooling.

Since one junction is heated and the other cooled, heat disappears at one junction and is evolved at the other, and as a result an electromotive force (called the Peltier electromotive force) opposing the current is developed.

These facts may be experimentally verified by joining up a battery with a galvanometer G_1 and wires of iron and copper as shown in the diagram Fig. 162. By means of a rocker the battery can be at any time cut out of the circuit and the galvanometer G_2 inserted, by withdrawing the contact piece *ab*, and inserting *cd*. It will then be found that a current through the galvanometer G_2 opposed to the steady current which flowed before will now be set up, and will gradually die away as the difference of temperature disappears.

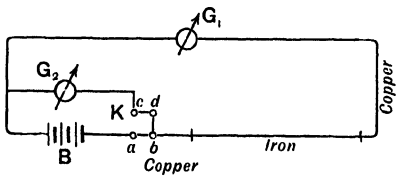


FIG. 162.

It is found by experiment that these heating and cooling effects are at any one temperature directly proportional to the current. Thus if a wire of resistance R contain a junction the rate at which heat is developed in it by a current γ is $R\gamma^2 + \Pi\gamma$ where Π is a quantity which may be positive or negative according to the direction of the current and the nature of the metals in contact at the junction. Hence if between the extremities of the portion of the circuit considered there be produced by a battery or otherwise an applied difference of potential V, the electrical activity in this part of the circuit is $V\gamma$ and we have

$$V\gamma = R\gamma^2 + \Pi\gamma$$

or the electromotive force actually available for working through the resistance R is

$$V - \Pi = R\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Π thus assists or opposes the applied electromotive force according as it is negative or positive, that is, according as heat is absorbed or given out at the junction.

Thus in a circuit of two metals there are in general two Peltier electromotive forces acting at the junctions. In an ordinary circuit (generally throughout nearly at one temperature) in which a current is kept flowing by a battery these do not cause any perceptible alteration of the current as they are equal and opposite, unless the current is so very great as to cause serious heating or cooling at the junctions.

The Peltier effect may obviously be evaluated by careful calorimetrical experiments on the heat evolved in a conductor containing a junction. The same current being sent in opposite directions through the conductor will give opposite thermal effects at the junction, and the difference between the heats generated per unit of time will be twice the value of $\Pi\gamma$.

Some values of Π are given in a table at the end of the present chapter.

Source of Energy in Thermoelectric Circuit. Peltier Effect is Zero at Neutral Temperature

597. We are led by the Peltier phenomenon at once to a partial answer to the question, What is the source of the energy expended in the circuit of a thermoelectric couple, when there is no battery or magnetic generator in the circuit? We have seen from Peltier's discovery that heat is taken in at the hot junction and given out at the other, when the current is produced by purely thermal action. The source of the work done in the circuit is thus, in part at least, the excess of the heat absorbed at the hot junction over that given out at the cold. We say *in part*, for, as we shall presently see, another thermal effect remains to be considered.

Thus if Π_1 be the Peltier electromotive force at the hot junction and Π_2 that at the other we have

$$\Pi_1\gamma - \Pi_2\gamma = R\gamma^2 + A \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where R is the total resistance of the circuit, and A is the activity other than Joulean generation of heat which goes on in the circuit.

598. It is found, as has been stated, that the thermoelectric power of two metals is zero at the neutral temperature, that is if one junction be slightly above that temperature and the other just as much below it, there is no electromotive force. Hence, according to the view of the matter just taken, there ought in this case to be neither absorption nor evolution of heat, and this is found to be the case as nearly as it can be tested by experiment. That there is no Peltier thermal effect at the neutral temperature has not, however, been quite conclusively settled by direct observation, as all experiments on this subject are greatly

complicated by effects of thermal conduction. A theory, however, of the thermoelectric circuit which assumes the vanishing of the Peltier effect at a junction at the neutral temperature will now be developed, and will be found to agree well with experimental results.

Thomson Effect—Electric Convection of Heat

599. Suppose one junction to be at temperature T (the neutral temperature) and the other at a lower temperature, there is neither absorption nor evolution of heat at the higher temperature, and there is evolution at the lower. Further there is Joulean evolution of heat throughout the circuit. Hence it is clear that energy must be obtained elsewhere than at the junctions, and it was proved by Lord Kelvin [1] that there is absorption or evolution of heat when a current flows in a conductor along which there is a gradient of temperature. For example, when a current flows down a gradient of temperature in a copper wire, it evolves heat, and absorbs heat when it flows up the gradient. The reverse is the case in an iron wire. Thus the source of energy in the case supposed is clear. The heat absorbed by the current in flowing round the circuit, along the unequally heated conductors, is greater than that evolved in the same process by an amount which is the equivalent of the energy electrically expended in the circuit in the same time.

Now consider a copper and iron circuit with junctions at temperatures t_0 and t, and suppose the current to flow as it does at ordinary temperatures when produced by pure thermal action, from iron to copper at the cold junction, and therefore from copper to iron at the hot. Let $\sigma_c \gamma dt$ denote the heat absorbed by the current in the copper in ascending through the difference of temperature dt, at temperature t, and $\sigma_i \gamma dt$ that absorbed in the iron in ascending through the same difference of temperature. Thus, since the current descends the gradient in iron, the total heat absorbed in the circuit per unit of time is

$$\Pi\gamma - \Pi_0\gamma + \gamma \int_{t_0}^{t} (\sigma_c - \sigma_i) dt$$

and as this must be equal to $E\gamma$, the whole electrical activity in the circuit, we have

$$E = \Pi - \Pi_0 + \int_{t_0}^{t} (\sigma_c - \sigma_i) dt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The quantities σ_c. σ_i were called by Lord Kelvin the "specific heats of electricity" in copper and iron respectively. They are of

[1] *Phil. Trans. R. S.*, 1855.

course not specific heats in the ordinary sense at all, and it might be better to use some name which did not seem to be based on the idea that electricity is a material substance, an idea which of course the choice of the name was not intended to convey. Clearly σ_i is negative by what has been stated above, and therefore heat is absorbed both in the copper and the iron, when the current flows round the circuit.

The effect of the absorption in copper is clearly to increase the gradient where the current flows from cold to hot, and to diminish the gradient where the current flows from hot to cold. The exact reverse is the case in iron.

600. Lord Kelvin investigated the alteration of gradient produced by a current by arranging a bar of metal with a thermometer in the centre, and two others symmetrically placed at each side. When the bar was heated at the middle and cooled at the ends, so that the two side thermometers showed equal temperatures, their equality was found to be disturbed by passing a current along the bar. The direction of the disturbance showed the sign of the effect.

The body of metal in the tested parts between the heater and the coolers was made as small as possible to enable the change of temperature to be as great as possible, if any should be found to be produced by the passage of the current.

The thermometers having been read on each side of the heater with no current in the conductor, the current was made to flow in one direction along the conductor, then, after flowing for some time to enable the effect, if any, to show itself, was reversed. It was found when the conductor was copper that the thermometer on the side of the heater adjacent to the end by which the current entered showed a fall of temperature, the thermometer on the other side a rise of temperature. When however the conductor was of iron, the effects were reversed; the first thermometer showed a rise, the second a fall of temperature. The effect in copper was considerably smaller than in iron.

These experiments established what Lord Kelvin then called the "electrical convection of heat," and he gave a list of metals with the signs of the "specific heats" of electricity in each, metals behaving like copper being regarded as giving a positive, metals behaving like iron a negative "specific heat." Some values of these so-called specific heats of electricity are given in the table at the end of the present chapter.

It is clear from equation (8) that the absorption and evolution of heat give rise to an electromotive force in the circuit. We shall regard this electromotive force as having its seat where the thermal effect takes place. Since in (8) we may suppose one junction, that at temperature t, to have the neutral temperature, we have $\Pi = 0$, and the energy given out in the circuit by the current is wholly drawn from the heat absorbed in its passage through the unequally heated conductors. Since in the copper-iron circuit heat is absorbed in both metals, we see that a current flowing from cold to hot in a piece of copper, or from hot to cold in iron, experiences an electromotive force aiding the flow.

Thermodynamic Theory

601. Lord Kelvin made in 1854 a most important application of the dynamical theory of heat to this subject.[1] Since heat is absorbed at the hot junction, and given out at the cold, and again heat is absorbed or evolved when a current flows in a conductor along a gradient of temperature, and these thermal effects are reversible with the current, we may regard the conductors with the current in them as a heat engine, in which the places of absorption of heat form a compound source, or rather a number of separate sources at different temperatures, while the places at which heat is given out form a number of refrigerators also at different temperatures. In this view of the subject, thermal conduction is neglected, and only the Peltier and Thomson effects, and the ordinary Joulean, or other electrical work done in the circuit, is taken account of.

By the first law of thermodynamics, the heat absorbed, *minus* that evolved at the junctions or elsewhere in any time, is equal to the whole work done in the circuit. But if E be the total electromotive force, and γ the current, the total electrical activity is $E\gamma$, and equating this to the total rate of absorption of heat we get again

$$E = \Pi - \Pi_0 + \int_{t_0}^{t} (\sigma - \sigma')dt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

if σ, σ' be the specific heats in the two metals forming the circuit.

Now the second law of thermodynamics affirms that if a quantity of heat dq is taken in by a heat engine at absolute thermodynamic temperature t, and this engine be put through any compound cycle (made up of any number of elementary Carnot cycles) in which heat given out is reckoned negative, then

$$\int \frac{dq}{t} = 0$$

for the cycle. Applying this to the case in hand we obtain, if t, t_0 now denote the absolute temperatures of the junctions,

$$\frac{\Pi}{t} - \frac{\Pi_0}{t_0} + \int_{t_0}^{t} \frac{\sigma - \sigma'}{t} dt = 0 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Differentiating this, we find

$$\frac{d}{dt}\left(\frac{\Pi}{t}\right) + \frac{\sigma - \sigma'}{t} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But (8) differentiated gives

$$\frac{dE}{dt} = \frac{d\Pi}{dt} + \sigma - \sigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

[1] See *Math. and Physical Papers*, Vol. I.

and we find by elimination of $\sigma - \sigma'$ between (10) and (11)

$$\frac{dE}{dt} = \frac{\Pi}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Thus, according to the thermodynamic theory, since the thermoelectric power is zero at the neutral temperature, so also is the Peltier effect.

The thermoelectric power for a pair of metals being in general a linear function of the temperature and by (11) having the value Π/t, we see that

$$\frac{d}{dt}\left(\frac{\Pi}{t}\right) = \text{constant.}$$

Hence (10) gives

$$\sigma - \sigma' = t \times \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

that is the difference between the specific heats of electricity in two metals is proportional to the absolute temperature. If then it could be shown that the specific heat in one metal is zero, the specific heat in all other metals would be proportional to the absolute temperature, on the supposition of course, just expressed, that the thermoelectric power of each metal with reference to the standard metal of specific heat zero varies uniformly with the temperature.

It has been found by Le Roux [1] that the specific heat of electricity in lead is nearly, if not quite, zero, and Professor Tait has constructed alloys of platinum and iridium which have the same property.

Thermoelectric Diagram

602. Very valuable information as to the various thermoelectric quantities is given by the form of thermoelectric diagram proposed and used first by Lord Kelvin, and afterwards applied by Professor Tait in his extensive researches on this subject. The nature of the diagram and its use will be seen from Fig. 163, and a complete diagram embracing a number of substances is given in Fig. 164, which is taken from Professor Tait's book on *Heat*.. The lines *EM*, *FM* represent the curves of thermoelectric power, each with respect to the same metal, and are supposed to be straight lines so far at least as the point *M*. [The remaining part of the diagram, in which the line *FM* is shown curved, will be dealt with later.] The temperatures of the junctions t_0 and t_1 are supposed to correspond to the ordinates *BC*, *AD*. Now *BC* denotes the thermoelectric power of the two metals at temperature t_0, and *AD* their thermoelectric power at the higher temperature t_1. But since the thermoelectric powers by the theory given above have been shown to be equal to Π_0/t_0, Π_1/t_1 respectively, we see that

$$\Pi_0 = BC \times t_0, \quad \Pi_1 = AD \times t_1$$

[1] *Ann. de Chim. et de Phys.* (4), 10.

that is if absolute temperatures are measured from the origin as zero

$$\Pi_0 = \text{area } BCcb, \quad \Pi_1 = \text{area } ADda \quad . \quad . \quad . \quad (14)$$

But Π_0 and Π_1 are the quantities of heat evolved and absorbed at the cold and hot junctions respectively by unit current in unit time. Thus these quantities of heat, in other words the Peltier electromotive forces at the junctions, are represented by these areas.

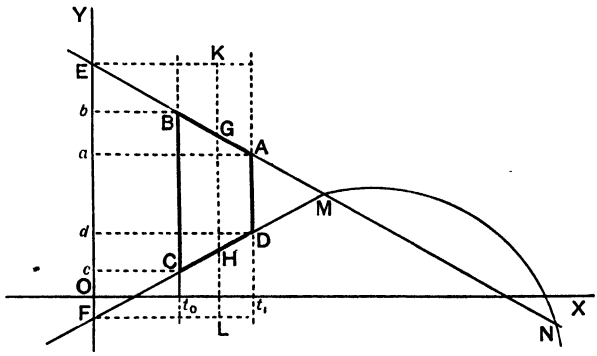


FIG. 163.

Further, the theory given has also shown that the difference $\sigma - \sigma'$ of the specific heats in the metals is given by

$$\sigma - \sigma' = -t\frac{d}{dt}\left(\frac{\Pi}{t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where t and Π denote any values of the temperature and the Peltier electromotive force intermediate between the values at the junctions. If GH be drawn in the diagram parallel to BC, $GH = \Pi/t$. Thus it is clear from the diagram that

$$\int_0^t (\sigma - \sigma')dt = \text{area } BAab + \text{area } DCcd \quad . \quad . \quad . \quad (16)$$

and this is the whole heat absorbed per unit of time by unit of current in consequence of the Thomson effect.

The whole heat absorbed is thus represented in the diagram by the area $BADCcb$, and the heat given out by $BCcb$. The excess of the former above the latter, namely, the area of the quadrilateral $ADCB$, is the amount utilised in electrical work in the circuit, and measures also the total electromotive force in the circuit.

The diagram also shows that if lines parallel to OX be drawn through E and F to meet GH produced both ways in K and L, KG

represents by its length the value of the specific heat of electricity in one metal, and LH that in the other. For clearly we may take $d(\Pi/t)/dt$ as the algebraic sum of the inclinations of the straight lines EM, FM, to the axis of X, and the specific heat of electricity in either

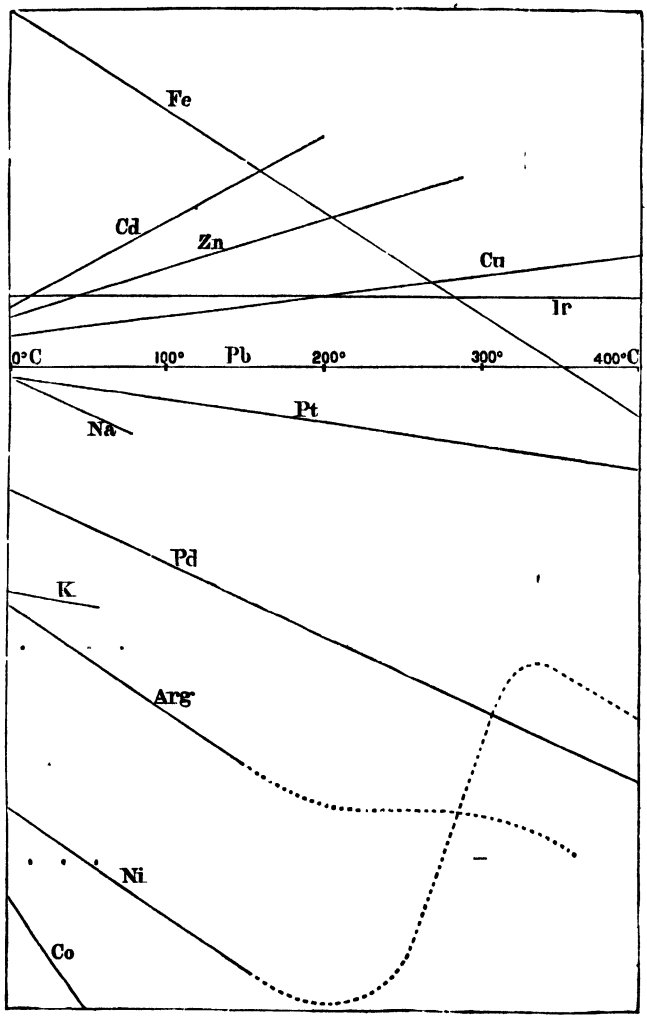


FIG. 164.

metal is the rate of variation with temperature of the thermoelectric power of that metal with reference to any other whatever, multiplied by the absolute temperature. Thus in the one case we get for the product KG, and in the other HL.

603. If as shown in Fig. 163 the curve of thermoelectric power of one metal meet that of the other in two or more points *M*, *N*, &c., then if the junctions be at the temperatures corresponding to two of these points, say *M*, *N*, there will be no Peltier absorption or evolution of heat, and no Peltier electromotive force; and the whole action will take place in consequence of the Thomson effect. Thus the whole heat absorbed by unit current in unit time, that is, the electromotive force in the circuit will be represented by the loop formed by the two curves between *M* and *N*.

A good example of this is formed by the diagram of nickel (see Fig. 164). According to Professor Tait, the specific heat of electricity in nickel changes sign about 200° C., and again near 320° C. Another is furnished by the diagram of iron and one of Professor Tait's alloys of platinum and iridium. The curves of thermoelectric powers of iron and the alloy intersect no less than three times at higher temperatures, showing therefore three neutral points. Hence, if the functions of a couple formed by iron and the alloy be at any of these two points, the

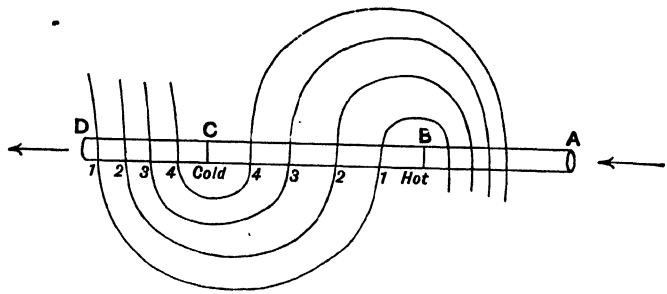


FIG. 165.

current must be almost entirely supported by the Thomson effect in iron, since there is little or no effect in the alloy.

In such a case as this, in which the Thomson effect is zero in one of the metals, there is only absorption of heat in the other metal, and no electromotive force can have its seat in that in which there is zero thermal effect.

604. Fig. 165 illustrates a case of the flow of energy, described by Poynting (*loc. cit.*, Art. 546). A circuit is formed of two metals of the lead type, joined by a metal of the iron type, in which a current flowing from hot to cold absorbs heat. *B* is a hot junction, *C* a cold, each supposed at the neutral temperature for the pair of metals there in contact. Thus if a current flow from a battery in the circuit, there is no convergence of equipotential surfaces upon *B* or *C*, and no thermal effects take place there. If the resistance of *BC* be sufficiently small, there will be a gradual rise of potential in the metal from *B* to *C*, and the gradient of electromotive force being opposed to the current, heat will

be absorbed in *BC*, will be transformed into electric energy, and carried out through the medium, to converge ultimately on the conductors in which it is utilised. Thus the equipotential surfaces which pass through *BC* will cut the circuit twice elsewhere (besides at the battery), once beyond *B* and once beyond *C*. In consequence, however, of the resistance of the conductor, there is a fall of potential superimposed on the rise produced by the Thomson effect, and a corresponding evolution of heat; and it is possible so to adjust matters that the rise shall just balance the fall of potential, and the evolution just balance the absorption at every point. Thus we should have the curious case of a homogeneous conductor carrying a current, and yet in mass throughout at one potential. It is to be observed, however, that there is a gradient of temperature along *BC*, so that there is no real paradox in the result.

605. From the theorem that the specific heat of electricity is proportional to the absolute temperature, the following result has been deduced by Professor Tait. Let $\sigma = ct$, $\sigma' = c't$, then instead of (10) we get

$$\frac{dE}{dt} = \frac{d\Pi}{dt} + (c - c')t \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and instead of (9)

$$\frac{d}{dt}\left(\frac{\Pi}{t}\right) + c - c' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Integrating the second of these we find

$$\frac{\Pi}{t} = (c - c')(T - t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

since Π is zero at the neutral temperature T.

Integrating now (17) from t_0 to t, and inserting the value of Π given in (19), we get

$$E = (c - c')(t - t_0)\left(T - \frac{t + t_0}{2}\right) \quad . \quad . \quad . \quad (20)$$

which agrees with (5) above if $P_0/(T - t_0) = c - c'$.

It should be noticed that the results stated in (19) and (20) are given at once by the thermoelectric diagram in Fig. 163.

APPENDIX TO CHAPTER XIII.

TABLE OF THERMOELECTRIC CONSTANTS.

	Thermoelectric Power with lead at mean temperature of 20° C. (microvolts).	Neutral Point with lead.[1] (Centigrade.)	Specific Heat of Electricity ÷ absolute temperature.	Peltier effect in therms per ampere per hour, the current flowing from copper to the substance named.
Aluminium	·68	195	·00039	...
Antimony, axial	−22·6	−156	·02221	4·8
,, equatorial	−26·4	...	..	..
Bismuth	89·0	−580	−·01073	19·1
Cadmium	−3·48	−62	·00425	0·46
Copper	−1·52	−143	·00094	...
Gold	−3·0	−277	·00101	...
Iron	−16·2	356	−·00481	2·5
Lead	0·0	...	·00000	..
Nickel (up to 100° C.)	22·8	−438	−·00507	4·362
Palladium	6·9	−174	−·00355	..
Platinum	8·82	−55	−·00109	·32
Silver	−2·41	−144	·00148	−·413
Tin	0·33	78	·00055	...
Zinc	−2·79	−98	·00235	·39

For further information see Smithsonian Physical Tables by T. Gray, pp. 248–250.

[1] Obtained by putting $dE/dt = A + Bt$, and calculating, from the observed value of dE/dt, the value of t $(-A/B)$ for which $dE/dt = 0$.

INDEX

Activity in network of wires, 174
Æolotropic medium, 113
Alternating current, 365
AMPÈRE, equivalence of current and magnetic shell, 278, 291
  electromagnetic theory, 284
  experiments, 317
Ampere, defined, 343
Attraction, electric, 101
Attraction and repulsion, magnetic, 8
AYRTON, 447

Barlow's wheel,
  used as dynamo, 331
Battery, arrangement of, 167
Bayly, H., 368
BIOT and SAVART, theorem of, 285
Blue sky, electromagnetic theory, 402
BOLTZMANN, 348
BRYAN, G. H., 211

Cable signalling with inductance, 369
Capacity, electrostatic, 120
  unit of, 344
Cavities, magnetic induction and intensity in, 43
Characteristic equation, 131
  for surface, 132
  for magnetic potential, 46
  function, 183
Charges, electric, subdivision of, 108
Circuit heterogeneous, 163
Circuits, reducible and irreducible, 223
Circuital equations, 388
Circulation, 221, 225
CLERK MAXWELL, 180
  dynamical model of current system, 344
  dynamical analogies, 371
CLIFTON, 448
Coercive force, 8
COLLET, Captain, 100
Compass, mariner's, 2
  heeling error of, 97
Compass, compensation of, 98
  semicircular and quadrantal errors, 99
Condenser, spherical, 114
Condenser, oscillatory discharge of, 359
Condensers or leydens, 120
Conductance, 169
Conductors and insulators, 105
Conductors, system of, energy of, 121, 127
  reciprocal relation, 123, 125
  capacities, 125
Conjugate conductors, 172
Conjugate functions, 230
Connectivity, 246
Constraint, forces of, 199
  work done by, 201
Contact electricity, 440
Continuity, principle of, 168
  equation of, 215, 231
Controllable and uncontrollable co-ordinates, 205
Convection currents, 431
  of heat, electric, 465, 467
CORNU, 439
COULOMB, magnetic experiments of, 8
  torsion balance of, 9
Coulomb, defined, 343
Curl, 259
  of magnetic force round current, 281, 287, 289
Current, electric, newer theory of, 102, 158
  analogue of, 160
  distribution of, in cross-section of conductor, 426
  steady, 160
  direction of, 161
  magnetic potential of, 286
  most general specification of, 296
  and magnetic distribution, energy of, 297
  current elements, 299, 309, 315
  straight, embedded in conducting medium, 299
  radial, magnetic action of, 301

Current, specification of, 388
unit of, 343
Currents, magnetic fields of, 276
equations of, 288, 340
energy of, 293
Current-induction, 324
dynamical theory of, 337, 381
Cyclic constants, 248
Cyclosis, 249

Damping of vibrations, 417
Daniell's cell, 162
E.M.F. of, 449
De Magnete, 4
Demagnetizing forces, 54
Dielectrics, 104
Dip, magnetic, 2
Dissipation function, 205
Displacement, electric, 112
currents, 310
Divergence, 216
Doublet, vibrating electric, 395
verification of theory, 409
Dynamical theory of current-induction, 337, 381

Efficiency of battery, 167
Electric attraction, phenomena of, 101
field, 109
energy of, 112
exploration of, 128
Electrified bodies, forces on, 106
Electromagnetic action, experimental illustrations of, 305
force, 295
Electrostatics, general problem of, 119
Ellipsoid, uniformly magnetized, 52, 54
Energy, electric, 112, 129
in æolotropic medium, 113
of system of conduction, 121
electrokinetic, 294, 339
flow of, 387
location and transfer of, 102
radiation of, 426
stream-lines of, 423
Equipotential curves, 19, 118
EULER, 214

FARADAY, ice-pail experiment, 107
experiments on current induction, 324
theory of current induction, 327
experiments on unipolar induction, 329
disk dynamo, 331
experiments on self-induction, 333
FELICI, experiments on current-induction, 325
Field, electric, 109
energy of, 112
exploration of, 128
Field, magnetic, 4
intensity of, 12
FITZGERALD, 417, 420
FLINDERS, Captain, 100
Flow of energy, 421
Fluid, hypothesis of incompressible, 111
nature of 212
acceleration of, 214
equations of motions of, 226
kinetic energy of, 228
stream-lines of, 228, 232
motion of, in two dimensions, 229
motion of minimum energy, 239
Force, electromotive, 162
in conductor, 163
theorem of in circuit of network, 168
unit of, 343
inductive, 331
energy value of, 452
criterion of existence of, 453
magnetic, lines of, 4
equations and graphical description of, 13, 14
for small magnets, 15
Freedom, degrees of, 181

GAUSS, theory of terrestrial magnetism, 56
GAUSSIN, 100
Generalized co-ordinates, 180
velocities, 180
momenta, 187
forces, 199
Generators, electric, arrangement of, 166
Gerland, 445
GILBERT, Dr., 4
GREEN, his problem, 139
his function, 141
for spherical conductor, 142
his theorem, 235
deductions from, 241
in multiply-connected space, 249
Grove's cell, E.M.F. of, 449
Gyrostatic terms in Lagrange's equations, 190
pendulum, 193

HAMILTON, Sir W. R., 180
characteristic function, 183
equations of motion, 203
HANKEL, 444
Heat, production of, by currents, 164
minimum dissipation of energy in, 175
HEAVISIDE, 304, 305, 432, 435
Heeling error of compass, 97
HELMHOLTZ, von, cyclic systems, 209
energy theory of current-induction, 336
HENRY, J., experiments on current-induction, 334

Henry, defined, 344
HERTZ, solution of Maxwell's equations, 395
electric doublet, 395
field of, 398, 405
receiver, 410
vibrator, 409
refraction of waves, 418
HOWARD, 420

Ice-pail experiment, 107
Ignoration of co-ordinates, 188, 251
Images, electric, 142
Impedance, 366
Induced distribution due to point charge, 142
derivation of from equilibrium distribution, 148
Inductance, 309
in network of conductors, 355
unit of, 344
and capacity, 366
in parallel conductors, 373
Induction, electric, 103
tubes of, 103, 115
specification of, 112
surface integral of, 112
tension along tubes of, 116
coefficients of, 125
magnetic, 35
solenoidal condition, 37
in cavities, 43
surface conditions for, 43
Induction, experiments on current, 324
coil, action of, 376
condenser in, 379
mechanical illustration of, 380
Inductivity, magnetic, 12, 41, 45
Infinite plane, induced distribution on, 144
Insulators and conductors, 105
Intensity, magnetic, 35
surface conditions for, 46
electric, 103
specification of, 112
magnetic, for currents, 304
motional electric, 385
impressed electric, 387
Inversion, geometrical, 147
electrical, 148
Irrotational motion of fluid, 215, 218
kinetic energy of, 245

JACOBI, C. G. J., 184
JOULE, law of, 164

Kaleidoscope, electric, 156
KELVIN, Lord, mariner's compass, 98
theory of magnetism, 10, 46
KELVIN, Lord, magnetic deflector, 100
electrical inversion, 147, 148
ignoration of co-ordinates, 191
energy theory of current-induction, 336
copper and zinc electric machine, 446
theorem of fluid motion, 225
induction electric machine, 447
KIRCHHOFF, principles of current flow, 168
theory of rectilineal vortices, 275
KOHLRAUSCH, 443

LAGRANGE, dynamical method of, 180, 186
theory of fluid motion, 214
Lagrangian function, modified, 189
Lamellar distribution of magnetism, 32
solenoidal distribution of magnetism, 34
LAPLACE, equation of, 46
LARMOR, 192, 439
Least action, 180
theorems of, 183
LENZ, his law, 332
Liquid, cyclic motion of, 193
kinetic energy of, 249
LODGE, O. J., 363, 420, 441, 452, 455
Lodestone, 1
LORENTZ, 439

Magnet, axis of, 22
couple on in magnetic field, 23
potential energy of, 24
Magnetic field of moving point-charge, 434
radiation in, 437
Magnetic matter, 10
potential, 29
shell, 31
Magnetism, permanent, 1
unit of, 11
elementary theory of, 29
surface distribution of, 21
lamellar distribution of, 32
terrestrial, 58
Magnetization by induction, 5
processes of, 7
particular cases of, 39
Magnets, action between, 27
Mariner's compass, 2
MAUPERTUIS principle of least action, 180
Medium, action of, 106
Model, dynamical, 344, 347
Moment, magnetic, 22
Momentum, electrokinetic, 339
Motional forces, 205

Network—*see* Wires
of conductors, 355
NORMAN, Robert, 2

Oersted, his experiment, 276
Ohm, his law, 161
Ohm, defined, 343
Orbis virtutis, 4

Parallel conductors, resistance and inductance of, 169
planes, induced charge on, 150
inversion of, 154
Pellat, 449
Peltier, 453, 463
effect, 463
at neutral temperature, 464
Perforated solids, motion of, in liquid, 251
Periphractic spaces, 233
Permeability, magnetic, 12
Perry, 368, 447
Poincaré, 417
Point charges, mutual energy of; apparent repulsion between, 40
Polarization of electromagnetic beam, 419
plane of, 419
Poles of magnet, 3
magnetic, 26
terrestrial magnetic, 59
Potential, electric, 117
coefficients of, 125, 135
propagation of, 403
magnetic, 17, 29
characteristic equation for, 46
of current, 286
Poynting, 421, 423, 471
Preston, Th., 439
Primary and secondary circuits, 348
alternating currents in, 372

Quadrantal error of compass, correction of, 99

Radiation, electromagnetic, 395, 415
Rayleigh, Lord, dissipation function, 205
dynamical model of current system, 347
distribution of currents, 426
Reciprocal action between two currents, 341, 348
Rectilineal vortices, 269
Reflection of electromagnetic waves, 416, 418
Refraction of electromagnetic waves, 418
Resistance, electric, 162, 164
Resistance, unit of, 343
Resonance, 370
multiple, 417
Routh, 192
Rowland, H. A., 431

Sarasin and De la Rive, 417
Searle, 437
Screen, conducting, 109
Seebeck, 456
Self-inductance, 353
Self-induction, Faraday's experiments on, 333
his theory of, 334
Semicircular error of compass, correction of, 99
Shell, magnetic, 31
Small magnet, lines of force and equipotential surfaces for, 15
Sphere, uniformly magnetized, 55
field of, 56
Spherical conductor, energy of charged, 114
induced distribution on, 142
electric image in, 142
Spherical conductors, force between two, 146
distribution on two in contact, 154
inversion of uniform, 150
Stationary action, 183
Sumpner, W. E., 363, 369
Superposition, principle of, 41
Susceptibility, magnetic, 45
Suspension for magnet, 3
System of bodies, dynamical theory of, 181

Tait, 466, 467, 469
Tangent galvanometer, 291
Terrella, 4
Terrestrial magnetism, 58
magnetic poles, 59
Thermodynamic relations, 207
Thermoelectric inversion, 457
power, 457
E.M.F., 459
circuit, energy in, 462
diagrams, 466
Thermoelectricity, 456
thermodynamic, theory of, 464
Thermokinetic principle, 207
Thomson, J. J., 211
Thomson and Tait, 211
Torsion balance, 9
Trouton, 417, 418, 420

Uniformly magnetized magnet, 11
Unipolar induction, 329

Vector potential, 48, 259
specification of, 50
Velocity potential, 215
permanence of, 219
of system of vortices, 263
Vibrator, 409
period of, 414
Volta, 440
experiments of, 441

Volt, defined, 343
Voltaic cell, 440
Vortex motion, 215, 218, 255
lines and tubes, 255
surface, 256
determination of velocities, 257
electromagnetic analogues, 265
vortex sheet, electromagnetic analogue of, 265
kinetic energy, 267
rectilineal vortices, 269
filament, 260
velocity due to, 262

Wave, electromagnetic, 390
in æolotropic dielectric, 391
velocity of propagation of, 393
reflection of, 416, 418
Wave, standing, 416
determination of length of, 417
WEBER, W., electrodynamic experiments, 321
experimental verification of Ampère's theory of electromagnetism, 322

Wires, network of, 168
bridge arrangement of, 170
analytical treatment of, 171
cycle method for, 174
activity in, 174
reduction to bridge arrangement, 176
effect of wire joining two points, 178

ZEEMANN, 437, 439
effect, 437

END OF VOL. I.

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