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THE CALCULUS OF EXTENSION

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THE CALCULUS OF EXTENSION

by

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including examples by

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To

MY PARENTS

PRINTED IN GREAT BRITAIN

PREFACE

This book had its origin when Professor Neville in 1929 passed on to me some papers left to the Mathematical Association by Professor Genese. These contained lecture notes and examples on Grassmann's methods. Though the notes themselves have been used but little, a large number of examples from this source have been incorporated in the earlier chapters (77, 46, 4, 6, 16 in Chapters I–V respectively). As I had been interested in this field for years, I thought it might be worth while to extend the work beyond the strictly elementary field covered in Genese's notes, and to give a coherent account of Grassmann's methods, with a number of applications sufficient to justify their use.

The emphasis on identities is my own, and my aim has been to express geometric theorems as identities, involving not co-ordinates but the geometric entities themselves which appear in the theorems.

It should be mentioned that the methods in the papers quoted are frequently very different from those in this book, and that my debt to Baker's *Principles* is far greater than that indicated by the references to that work.

Numerous accidents have delayed publication. When I left England in December 1933, the book was ready in the rough. The duties of a new and congenial post, and, it must be added, residence in a country where there are no mathematical libraries and where the external conditions of life, for the newcomer, severely hamper any sustained mental effort, together made revision a much longer task than I had anticipated, and printing did not begin till the spring of the fateful year 1939.

I desire to acknowledge my gratitude to the Library of University College, London, which for many years was my main link with scientific literature, to the Library of Hull University College, to the Library of the Mathematical Association and its Librarian, and to the Council of Auckland University College for the help they have given to the Library here, and for the interest they have shewn in the research work of members of

the College, to Professor Saddler for criticism of part of the MS, to my colleague Dr Bullen for valuable assistance with the proof-reading, to Professor Mehmke for off-prints of papers not accessible to me.

To Professor Neville my debt is unusually heavy. Not only was he, perhaps unwittingly, the instigator of the project, but his criticism of the first draft led me to make considerable changes in the text, and while the sheets were passing through the Press, he was untiring, in spite of the troubles of the times, in detecting errors and misprints, in suggesting amendments, in verifying and supplying references inaccessible to me here, and generally, in using his, in some respects, more fortunate geographical position, to soften my antipodean disadvantages.

Once again it is a pleasure to express gratitude to the Syndics of the University Press in undertaking to publish such a book and in such days as these, and to the Staff for the pains they have taken in the complicated task of printing it.

H. G. F.

September 1940

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Etsi omnis methodus licita est, tamen non omnis expedit

LEIBNIZ, quoted by GRASSMANN
as motto of his *Geometrische Analyse*

INTRODUCTION

Grassmann's Calculus of Extension is an abstract algebra with a wide range of applications; we shall be mainly concerned with its use in Geometry. In this field it has the same value that vector methods have in Physics. In the usual analytical geometry coordinates are used, and geometric entities are represented by equations involving these coordinates. As a rule, the use of these coordinates introduces geometric entities, such as axes of reference, which are strictly foreign to the purpose in hand. In the method of this book, we use equations involving the geometric entities themselves, such as points, lines, circles, or quadrics, and not their coordinates; to prove a geometric theorem is to prove such an equation, and as in most cases the equation turns out to be an identity, we have an automatic method for proving geometric theorems.

Before we can construct such equations we must introduce operations, which we call addition and multiplication, in the set of geometric entities, and must find the laws governing these operations. A simple case is furnished by the ordinary vector algebra in which vectors are added and multiplied scalarwise and vectorwise. The addition of points in our work is not unlike the addition of vectors in vector algebra, but it is more fundamental, and the vector algebra can be deduced from the point algebra, a vector being merely the difference of two points. It is the multiplication of points which introduces the essentially novel feature. The product $[ab]$ of two points a, b is interpreted as the vector tied down to the line ab, and having a magnitude equal to the length of the interval ab, and a direction from a to b. Calling a tied vector a 'rotor', it is necessary, before we can go further, to define the addition of rotors. The theory of this addition is essentially the theory of statics, these rotors being analogous to forces. The present algebra, unlike the vector algebra, distinguishes between vectors, which are differences of points, and rotors, which are products of points.

In the first chapter we consider geometry in the plane, in the second chapter geometry in space, and in setting up the laws of our algebra we use some simple geometric theorems which are stated as they are needed. Once the fundamental laws of calculation have been established, the algebra may be used to deduce further geometric propositions. Just as the product $[ab]$ of two points denotes an interval with a sense along the line ab, so the product $[abc]$ of three non-collinear points denotes an area with a sense on the plane abc, and it becomes evident that there is no need to restrict the work to three dimensions. These products $[ab]$, $[abc]$ are called 'progressive outer products' to distinguish them from products introduced later.

In dealing with metric and projective geometry we need another type of product, the 'inner product', of which the scalar product of vectors is the simplest case. If u, v be vectors, the inner product of u, v is denoted by $[u|v]$, and may be regarded as the vector product of u and $|v$ when $|v$ is a vector perpendicular to v, called the 'supplement' of v. The generalisation of this point of view leads to the definition of the supplement $|\alpha$ of any geometric entity α and to the extension of the notion of inner product. In metric geometry of three dimensions, the supplement of a vector is a 'bivector', a quantity analogous to the outer product of three points but not anchored to a special plane, but associated with any one of a set of parallel planes; in projective geometry of three dimensions, the supplement of a point can be interpreted as the polar plane of the point with respect to a fixed quadric.

The method of presentation is this: We first apply the algebra to metric and projective geometry; in dealing with metric geometry we assume a few propositions and so shew how the algebra must proceed. We build up the projective geometry from the bottom, running it parallel with the algebra, but at one or two places we use some work from the usual analytical geometry. Our aim at first is to shew the algebra at work, to illustrate its power and its range. When this has been done, the later chapters give a logical systematic account of the algebra in n dimensions; in that portion we can rely on

Grassmann's own work,* which was a general theory with practically no applications of the kind found in this book.

The calculus can be applied to certain parts of algebra and we illustrate its use in the theory of determinants. It can also be applied to differential geometry, but as there it practically coincides with the ordinary vector algebra we omit this application. Another important field in which it is of use, is the theory of Pfaffian differential forms; we do not treat these here.

In dealing with differentiation, in order to avoid the discussion of points not strictly related to our subject, an old-fashioned point of view is taken. The difficulties that remain are in no way different from those in ordinary analysis.

After the general theory has been developed we sketch the theory of matrices in the service of geometry, and apply the general theory to circles, ordinary and oriented, and to line geometry. Finally we introduce the notion of 'algebraic product' of geometric entities and illustrate its use.

The algebra of vectors created by Grassmann and Hamilton has at last won an established place in Physics. Grassmann's methods are of equal use in Geometry, but this application is less widely appreciated. It is hoped that this book will redress the balance.

* Particularly the second version of the *Ausdehnungslehre* (1862), quoted as A_2.

NOTATION

Entities of the same kind are denoted by letters of the same kind, as follows:

Scalars:	small roman letters, particularly k, l.
Points, vectors, extensives of step one:	small italics, a, b, v.
Lines, rotors, bivectors:	capital italics, A, L.
Planes, leaves:	small greek, α, π.
Transformations, matrices:	gothic $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$, $\mathfrak{b}$.

In order to avoid extra brackets, $\sqrt{[\,][\,]}$ is frequently printed instead of $\sqrt{\{[\,][\,]\}}$, where $[\,]$ stands for expressions in our algebra.

CHAPTER I

PLANE GEOMETRY

§ 1. *Addition of extensives and multiplication by scalars.*

The algebra we use in this book is due to Grassmann; it is an abstract algebra which furnishes useful methods for attacking geometric questions; it employs an operation called '*addition*', and several kinds of '*multiplication*'. In an interpretation of the algebra, the elements a, b, c, ... of the algebra may represent points, lines, oriented circles, conics, and so on. In our first interpretations, they represent points.

In the abstract development of the algebra, we shall call the elements '*extensives*'. The result of the '*addition*' of extensives a, b is denoted by $a+b$.

The fundamental assumptions for the operation of addition in the abstract algebra are:

1. If a, b be any extensives, then $a+b$ is an extensive which exists and is unique.

(This assumption is subject to a limitation later, § 4, p. 13. At present we ignore the limitation.)

2. If a, b be any extensives, then $a+b=b+a$.

3. If a, b, c be any extensives, then $(a+b)+c=a+(b+c)$.

Thus we may write $a+b+c$ for $(a+b)+c$, without confusion.

4. If a, b be any given extensives, there is a unique extensive c, such that $a+c=b$.

This extensive is denoted by $b-a$.

Thus $a+(b-a)=b$, $(b-a)+a=b$. If $a+c=a+d$, then $c=d$.

From these assumptions it follows that, if a, b be any extensives, then $a-a=b-b$. For, by 4, there are unique elements c, o_1, o_2 such that $a+o_1=a$, $b+o_2=b$, $a+c=b$. Then, using 2, 3, we have

$$b+o_1=a+c+o_1=a+o_1+c=a+c=b=b+o_2.$$

Hence by 4, $o_1=o_2$.

Thus by the definition in 4, $a-a=b-b$. It will cause no confusion if we denote $a-a$ by the same symbol 0, as is used for the arithmetic zero.

Then, for all extensives a, $a+0=0+a=a$.

We can '*multiply*' an extensive a by a scalar k, and write the product as ka.

This multiplication is to satisfy the following assumptions, where k, k_1, k_2 are scalars, and a, b extensives:

5. ka is an extensive.
6. $1a=a$.
7. $(k_1k_2)\,a=k_1(k_2a)$.
8. $(k_1+k_2)\,a=k_1a+k_2a$.
9. $k(a+b)=ka+kb$.

In 8 take $k_1=1$, $k_2=0$; thence by 4, $0a=a-a=0$.

With subtraction defined by 4, we may define $-a$ as $0-a$; it then follows that $(-1)\,a=-a$, $(-k)\,a=-ka$.

The above assumptions give the whole theory of addition and subtraction of extensives and of multiplication of extensives by scalars, and they exhibit the analogy with ordinary algebra.

By a 'scalar' we shall in the first instance mean a real number, later we shall include complex numbers among scalars.

§ 2. *Interpretation of extensives as points of a Euclidean plane.*

1. We give an interpretation of the algebra in which a point is represented by an extensive a, and where ka represents the same point with a weight k attached, as in statics. A point in the usual sense must then be regarded as having a unit weight attached to it, and if a is the point, we may write it either as a or as $1a$. The difference $a-b$ of two points of unit weight will be interpreted as the vector from b to a. The usual vector algebra, as far as it concerns addition of vectors, will be included as a case in this section. In order that it should be seen what properties are taken from geometry in this interpretation, it is best to begin, not with addition, but with subtraction.

2. If a, b be points of a Euclidean plane, $a-b$ is to represent the vector from b to a; its magnitude is given by the length of the interval between the points. If c, d be two points such that the interval dc is parallel to and equal in length to the interval

ba, and if the intervals (supposed directed) have the same sense, the vectors represented by $a-b$ and $c-d$ are identified.

Thus $a-b=c-d$ means that the directed intervals ab, cd are congruent, parallel, and in the same sense. If now we assume that the geometric relations of congruence and parallelism between intervals are symmetric and transitive, then the sign $=$ has the usual properties, namely:

If $a-b=c-d$, then $c-d=a-b$. (1)

If $a-b=c-d$, and $c-d=e-f$, then $a-b=e-f$. (2)

We now assume the fundamental theorems that if ab, cd be congruent parallel intervals in the same sense, so are ba and dc, and so are ac and bd. Symbolically:

If $a-b=c-d$, then $b-a=d-c$ and $a-c=b-d$. (3)

We further assume: if ab be any interval, and c any point, then there is just one interval cd from c, congruent to, parallel to, and in the same sense as ab. This gives:

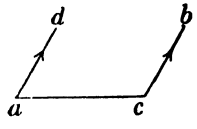
If a, b, c be points, there is just one point d, such that

$$a-b=c-d. \tag{4}$$

Thus
$$\left.\begin{aligned} &a-b=a-c, \text{ if and only if } b=c;\\ &a-b=d-b, \text{ if and only if } a=d.\end{aligned}\right\} \tag{5}$$

We use the letters u, v, w to represent vectors.

3. *Addition of a point and a vector.* If a is a point, u a vector, then $a+u$ and $u+a$ are defined as the point d such that $d-a=u$.



Thus
$$\left.\begin{aligned} &(a+u)-a=u,\\ &a+(d-a)=d,\\ &(a+u)-a=(b+u)-b.\end{aligned}\right\} \tag{6}$$

If $a+u=a+v$, then $u=v$. (7)

Take $u=b-c$; then the equations

$$d-a=b-c \quad \text{and} \quad d=a+(b-c)$$

are equivalent.

Also, by (6), (3),

$$(a+u)-(b+u)=a-b. \tag{8}$$

4. *Addition of vectors.* If u, v be vectors, a a point, then $a+u$ is a point, $(a+u)+v$ is a point.

Also $$[(a+u)+v]-a=[(b+u)+v]-b.$$

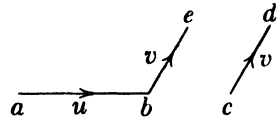
For, by (8),

$$[(a+u)+v]-[(b+u)+v]=(a+u)-(b+u)=a-b.$$

The result follows by (3).

Hence, for all points a, $[(a+u)+v]-a$ is the same vector. We denote this vector by $u+v$. Then, by this definition,

$$(a+u)+v=a+(u+v).$$



We may write these expressions as $a+u+v$.

If $u=b-a$, $v=d-c$, take e so that $d-c=e-b$, and we have

$$a+u=b, \quad a+u+v=b+v=b+(e-b)=e, \quad u+v=e-a.$$

Thus we have the usual parallelogram law for the addition of vectors.

Also $$(b-a)+(e-b)=e-a. \tag{9}$$

Take f so that $d-c=f-a$, then $a+v=f$, $a+v+u=f+u=e$, since $f-a=e-b$ gives $e-f=b-a=u$.

Thus $a+(u+v)=a+(v+u)$. Hence, by (7),

$$u+v=v+u. \tag{10}$$

Hence, by (9), $$(e-b)+(b-a)=e-a. \tag{11}$$

By (4), if u, v, w be any vectors, a any point, we can find points b, c, d such that $u=a-b$, $v=b-c$, $w=c-d$. Then (11) gives

$$(u+v)+w=(a-c)+(c-d)=a-d,$$

$$u+(v+w)=(a-b)+(b-d)=a-d.$$

Hence $(u+v)+w=u+(v+w)$; we write these expressions as $u+v+w$.

5. *The nul vector.* If a, b be any points, then $a-b=a-b$ gives, by (3), $a-a=b-b$. We call $a-a$ the nul vector, or zero vector, and denote it by o.

By (11), $$(a-b)+(b-a)=\text{o}.$$

If $u=a-b$, we denote $b-a$ by $-u$; thence $u+(-u)=\text{o}$.

We define $u-v$ as $u+(-v)$. Thus $u-u=\text{o}$.

6. We now assume that if k be any scalar (real number), and ab a directed interval, then there is just one point c on the line ab, such that $ac:ab = k$, where ab, ac here denote lengths, taking signs into account. We write, in our notation,

$$c-a = k(b-a).$$

7. *Sums of points.* If a, b be points, x, y scalars, $x+y \neq 0$, then $\dfrac{xa+yb}{x+y}$ is defined to be the point p such that

$$x(a-p)+y(b-p) = 0.$$

Thus p is the point on the line ab such that the ratio of its signed distances from a, b is $-y/x$; by 6, this point is existent and unique if $x+y \neq 0$. It is the 'mean centre' of weights x, y at a, b.

If a is a point, k a scalar, we regard ka as a 'weighted' point, a point in the usual sense with a weight attached. We call k the weight of ka. Then, if

$$\frac{xa+yb}{x+y} = p, \quad (x+y \neq 0),$$

we can define $xa+yb$ as $(x+y)\,p$. Hence $\dfrac{x}{x+y}\,a+\dfrac{y}{x+y}\,b = p$.

If u is a vector, a a point, k a scalar, we define $u+ka$, $ka+u$ as $k\left(a+\dfrac{u}{k}\right)$.

We can now form sums of vectors and weighted points. The assumptions of § 1 all hold.

It is now easily shewn that $(a-b)+c = a+c-b$, using our geometrical interpretations. Thus it is not now necessary to treat $(a-b)$ as a whole.

For example, from $a-b = c-d$, it follows that $a+d = b+c$, $\frac{1}{2}(a+d) = \frac{1}{2}(b+c)$, corresponding to the theorem that the diagonals of a parallelogram bisect each other.

8. If weights x, y, z be attached to points a, b, c, then, if $x+y \neq 0$, $x+y+z \neq 0$,

$$\frac{xa+yb}{x+y} = p, \quad \frac{(x+y)\,p+zc}{x+y+z} = q$$

give $\quad xa+yb = (x+y)\,p, \quad (x+y)\,p+zc = (x+y+z)\,q,$

$$xa+yb+zc = (x+y+z)\,q.$$

If $x+y+z \neq 0$, we can always multiply the weights x, y, z by a common scalar k, so that $kx+ky+kz=1$. This will not change the position of q, since that depends only on the ratios $x:y:z$. We can hence take

$$p = xa+yb+zc, \quad x+y+z=1. \qquad (12)$$

9. We may compare these equations with those used in analytical geometry.

(*a*) Take Cartesian axes, and let a, b, c, p have coordinates (k_1, l_1), (k_2, l_2), (k_3, l_3), (k, l), then (12) is replaced by the equations

$$(x+y+z)\,k = xk_1+yk_2+zk_3, \quad (x+y+z)\,l = xl_1+yl_2+zl_3,$$

and it may be regarded as a shorthand form of the last two equations.

(*b*) Take moments, round the line bc, of the three weights x, y, z, and of their resultant weight; it is easily found that x, y, z are proportional to the areal coordinates of p, the triangle abc being the triangle of reference.

If $x+y+z=1$, then x, y, z are the actual areal coordinates.

10. To restrict the work to the plane, we assume:

If a, b, c are distinct non-collinear points, any point p can be put in the form

$$p = xa+yb+zc, \quad x+y+z=1, \quad \text{for suitable } x, y, z.$$

11. If a, b, c be non-collinear points, and $k_1 a+k_2 b+k_3 c=0$, then $k_1 = k_2 = k_3 = 0$.

If $k_1+k_2+\ldots+k_n=0$, then $k_1 a_1+k_2 a_2+\ldots+k_n a_n$ is a vector, and conversely.

12. All this work can be extended to space of any dimensions. In ordinary space any point p may be written

$$p = xa+yb+zc+wd, \quad (x+y+z+w=1),$$

where a, b, c, d are independent points, that is, no three are collinear, and the four are not coplanar.

Examples on weighted points

1. *Def.* If a, b, ..., p are n points, then $(a+b+\ldots+p)/n$ is their '*mean centre*'. The three medians of a triangle abc meet in a point, one-third of the way along each median from its base; this point is the mean centre of a, b, c.

For, if d is the mid-point of bc, then $d = \frac{1}{2}(b+c)$; if g is a point on da such that, in the customary notation, $dg:ga = 1:2$, then

$$g = \tfrac{1}{3}(2d+a) = \tfrac{1}{3}(a+b+c).$$

2. If $abcd$ be a quadrilateral or tetrahedron, the joins of the mid-points of edges ab and cd, of bc and ad, of ca and bd meet in the mean centre of a, b, c, d and are there bisected.

For $\quad \frac{1}{4}(a+b+c+d) = \frac{1}{2}\{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\}$

$$= \tfrac{1}{2}\{\tfrac{1}{2}(b+c)+\tfrac{1}{2}(a+d)\} = \tfrac{1}{2}\{\tfrac{1}{2}(c+a)+\tfrac{1}{2}(b+d)\}.$$

3. *Menelaus' Theorem.* A line cuts the sides bc, ca, ab of triangle abc in points l, m, n. If

$$l = \mathrm{x}b+\mathrm{x}'c,\ m = \mathrm{y}c+\mathrm{y}'a,\ n = \mathrm{z}a+\mathrm{z}'b,\ \mathrm{x}+\mathrm{x}' = \mathrm{y}+\mathrm{y}' = \mathrm{z}+\mathrm{z}' = 1,$$

then $$\mathrm{xyz} = -\mathrm{x}'\mathrm{y}'\mathrm{z}'.$$

For the weighted point $\mathrm{y}l-\mathrm{x}'m = \mathrm{xy}b-\mathrm{x}'\mathrm{y}'a$ is in lm and ab, and hence must be the point n with some weight attached.* This will be the case if, and only if, $\mathrm{xy}:\mathrm{z}' = -\mathrm{x}'\mathrm{y}':\mathrm{z}$, that is, if $\mathrm{xyz} = -\mathrm{x}'\mathrm{y}'\mathrm{z}'$.

4. *Ceva's Theorem.* The lines ap, bp, cp cut the opposite sides of triangle abc in l, m, n respectively. If

$$l = \mathrm{x}b+\mathrm{x}'c, \quad m = \mathrm{y}c+\mathrm{y}'a, \quad n = \mathrm{z}a+\mathrm{z}'b,$$

then $$\mathrm{xyz} = \mathrm{x}'\mathrm{y}'\mathrm{z}'.$$

For let $p = \mathrm{k}_1a+\mathrm{k}_2b+\mathrm{k}_3c$, then $p-\mathrm{k}_1a = \mathrm{k}_2b+\mathrm{k}_3c$ is a weighted point on ap and on bc, and hence is the point l with some weight attached. Thence $(\mathrm{k}_2+\mathrm{k}_3)\,l = \mathrm{k}_2b+\mathrm{k}_3c$, whence $\mathrm{x}:\mathrm{x}' = \mathrm{k}_2:\mathrm{k}_3$.

Similarly $\quad \mathrm{y}:\mathrm{y}' = \mathrm{k}_3:\mathrm{k}_1, \quad \mathrm{z}:\mathrm{z}' = \mathrm{k}_1:\mathrm{k}_2; \quad$ hence $\quad \mathrm{xyz} = \mathrm{x}'\mathrm{y}'\mathrm{z}'$.

5. If d, e, f be the mid-points of the sides of triangle abc, p a point in plane abc, and if pd, pe, pf be divided at l, m, n in the same ratio $\mathrm{x}:\mathrm{y}$, then al, bm, cn are concurrent.

For let $\mathrm{x}+\mathrm{y} = 1$, then

$$l = \mathrm{y}p+\mathrm{x}\tfrac{1}{2}(b+c).$$

Hence $\frac{1}{2}\mathrm{x}.a+l = \mathrm{y}p+\frac{1}{2}\mathrm{x}(a+b+c)$. The result now follows by symmetry.

6. If d, e, f be the mid-points of the sides of triangle abc, and ap, bp, cp meet the opposite side-lines in l, m, n, then the joins of d, e, f to the mid-points of al, bm, cn are concurrent.

For, let $p = \mathrm{x}a+\mathrm{y}b+\mathrm{z}c$, then $(\mathrm{y}+\mathrm{z})\,l = \mathrm{y}b+\mathrm{z}c$. (Cf. Ex. 4.) Also $d = \frac{1}{2}(b+c)$. Thence

$$2\mathrm{x}(\mathrm{y}+\mathrm{z}).\tfrac{1}{2}(a+l)+2\mathrm{yz}.d = \mathrm{x}(\mathrm{y}+\mathrm{z}).a+\mathrm{y}(\mathrm{z}+\mathrm{x}).b+\mathrm{z}(\mathrm{x}+\mathrm{y}).c.$$

* This device is often of use.

7. If the incircle of triangle *abc* touches *bc* in *d*, then the incentre lies on the join of the mid-points of *bc* and *ad*.

8. *Centre of gravity of a quadrilateral area abcd.* We assume that the centre of gravity of a triangle is the mean centre of its vertices. Let the areas of triangles *abc*, *acd* be k_1, k_2 respectively, and let *g* be the centre of gravity required.

(i) $$\begin{aligned} 3(k_1+k_2)\,g &= k_1(a+b+c)+k_2(a+c+d) \\ &= (k_1+k_2)\,(a+b+c+d)-(k_1 d+k_2 b) \\ &= (k_1+k_2)\,(a+b+c+d-e), \end{aligned}$$

where $$(k_1+k_2)\,e = k_1 d+k_2 b.$$

Thus *e* is a point on *bd*; similarly it is on *ac*, and hence it is at the cut of the diagonals; thus we have the usual rule $3g = a+b+c+d-e$.

(ii) $3g = a+c+f$, where $b+d = e+f$. Interpret this. (Bérard's construction.)

(iii) Let g_1, g_2 be centres of gravity of the triangles *abc*, *adc*, and let g_1g_2 cut *ac* in *l*; in g_1g_2 take *m* so that $g_1 l$ and mg_2 are equal intervals in the same sense, then $m = g$. (Baltzer's construction.)

For $$\begin{aligned} 3(k_2 g_1+k_1 g_2) &= k_2(a+b+c)+k_1(a+d+c) \\ &= (k_1+k_2)\,(a+c+e), \end{aligned}$$

and these expressions represent a weighted point on g_1g_2 and *ac*; hence we have $a+c+e = 3l$. But $g_1+g_2-g = \frac{1}{3}(a+c+e)$; hence $g_1+g_2 = g+l$.

9. Given the centroids of the four triangles into which a quadrilateral $a_1a_2a_3a_4$ is divided by its diagonals, construct it.

$g_1 = \frac{1}{3}(a_2+a_3+a_4)$, and so on. Hence $a_1 = g_2+g_3+g_4-2g_1$.

10. *If abcd be any quadrilateral, plane or skew, and r, s divide ab, dc in equal ratios, and p, q divide ad, bc in equal ratios, then pq, rs cut in a point m such that pm : mq = ar : rb and rm : ms = ap : pd.*

For let $$r = xa+yb, \quad s = xd+yc, \quad (x+y = 1),$$

$$p = x'a+y'd, \quad q = x'b+y'c, \quad (x'+y' = 1).$$

Hence $x'r-xp = yq-y's$. The point represented by the equal expressions $xp+yq$, $x'r+y's$ is on both lines *pq*, *rs* which must therefore cut, even if the quadrilateral is skew. The rest follows at once.

11. *The mid-points a_1, ..., a_n of the sides of a polygon are given. If* n *is odd, they determine the polygon; if* n *is even, there is either no such polygon, or there is an infinite number.*

For let $p_1, \ldots, p_n$ be the vertices, then

$$p_1+p_2 = 2a_1,\ p_2+p_3 = 2a_2,\ \ldots,\ p_n+p_1 = 2a_n.$$

If n is odd, these give $p_1 = a_1 - a_2 + a_3 - \ldots + a_n$; thus p_1 and similarly all the p are determined. If n is even, and the equations are consistent, we must have $a_1 - a_2 + a_3 - \ldots - a_n = 0$. Hence the mean centre of $a_1, a_3, a_5, \ldots$ coincides with that of $a_2, a_4, a_6, \ldots$. If this condition holds, p_1 may be taken anywhere, and the other p found. The figure may be in space.

12. A triangle whose sides are to pass through three given points p, q, r and be divided there in given ratios $x:1, y:1, z:1$ is, in general, uniquely determined. But if $xyz = -1$, then either p, q, r are collinear, and the problem has an infinite number of solutions, or p, q, r are not collinear, and the problem has no solution.

Vectors and weighted points

13. If the point p be on plane abc, distinct from a, b, c, and the parallelograms $pblc$, $pcma$, $panb$ be completed, then the intervals al, bm, cn bisect each other.

For $\quad b+c = p+l, \quad c+a = p+m, \quad a+b = p+n;$

hence $\quad \frac{1}{2}(a+l) = \frac{1}{2}(b+m) = \frac{1}{2}(c+n).$

14. Points p, q, r lie on the side-lines of triangle abc, and

$$bp:pc = cq:qa = ar:rb;$$

then the centroid of triangle pqr is the centroid of triangle abc. If the parallelograms $qabl$, $racm$ be completed, then rl, qm are each parallel to the median through a.

For $\quad a-r = k(a-b),\ b-p = k(b-c),\ c-q = k(c-a)$; k scalar.

Hence $\quad a+b+c = p+q+r.$

Also
$$\begin{aligned} r-l &= (r-b)+(b-l) = (r-b)+(a-q) \\ &= (1-k)(a-b)+(1-k)(a-c) \\ &= 2(1-k)[a-\tfrac{1}{2}(b+c)]. \end{aligned}$$

15. *If triangles abc, a'b'c' have their corresponding sides parallel, the joins of corresponding vertices are concurrent, in the* 'centre of perspective', *or parallel.*

For $\quad b-c = k_1(b'-c'),\ c-a = k_2(c'-a'),\ a-b = k_3(a'-b')$;

k_1, k_2, k_3 scalars.

Hence $\quad (k_2-k\ \ a'+(k_3-k_1)\,b'+(k_1-k_2)\,c' = 0.$

Now a', b', c' are assumed to be non-collinear; hence the coefficients of the last equation vanish. Hence $k_1 = k_2 = k_3 = k$, say. The first equations now give $a - ka' = b - kb' = c - kc'$; this is a weighted point on aa', bb', cc', if $k \neq 1$, while if $k = 1$, we have the parallel case.

16. *If triangles abc, a'b'c', a''b''c'' have corresponding sides parallel, the centres of perspective of the triangles in pairs are collinear.*

For, if these centres be s, s', s'', then, by Ex. 15, we can let

$$(1-k)\,s = a - ka', \quad (1-k')\,s' = a' - k'a'', \quad (1-k'')\,s'' = a'' - k''a,$$

$$a - b = k(a' - b'), \quad a' - b' = k'(a'' - b''), \quad a'' - b'' = k''(a - b).$$

Hence $kk'k'' = 1$, and Menelaus' Theorem (Ex. 3) applied to triangle $aa'a''$ gives the result.

17. (*Bobillier.*) If *abcd* be any quadrilateral, and p, q be points on ad, bc such that $ap : pd = ab : dc = bq : qc$, then pq is equally inclined to ab, dc.

For if x, y be the lengths of sides ab, cd, then

$$(x+y)\,p = ya + xd, \quad (x+y)\,q = yb + xc.$$

Hence

$$(x+y)\,(p-q) = y(a-b) + x(d-c),$$

$$(x^{-1}+y^{-1})\,(p-q) = x^{-1}(a-b) + y^{-1}(d-c).$$

But the last two terms represent unit vectors along ba, cd respectively. Hence $p - q$ is a vector along the bisector of an angle between ba and cd.

Cor. A bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

18. *If the triangle pqr has its sides qr, rp, pq respectively parallel to da, db, dc, then there is a point f such that fp, fq, fr are respectively parallel to bc, ca, ab.*

For let $$d = xa + yb + zc, \quad x + y + z = 1,$$

then $$q - r = k_1(a - d), \quad r - p = k_2(b - d), \quad p - q = k_3(c - d);$$

k_1, k_2, k_3 scalars.

Hence $$0 = k_1 a + k_2 b + k_3 c - (k_1 + k_2 + k_3)\,(xa + yb + zc).$$

But as a, b, c are non-collinear, their coefficients, in this equation, vanish. Thus

$$k_1 = x(k_1 + k_2 + k_3), \quad k_2 = y(k_1 + k_2 + k_3), \quad k_3 = z(k_1 + k_2 + k_3).$$

Now consider $$f = xp + yq + zr, \quad x + y + z = 1.$$

We have $p-f=y(p-q)+z(p-r)$,

$$(k_1+k_2+k_3)(p-f)=k_2(p-q)+k_3(p-r)$$
$$=k_2k_3(c-d)-k_2k_3(b-d)=k_2k_3(c-b).$$

Hence pf is parallel to bc; and so on.

19. If x, y, z be lengths of chords of a triangle which meet inside the triangle and are parallel to the sides of lengths a, b, c, then

$$xa^{-1}+yb^{-1}+zc^{-1}=2.$$

20. Parallel lines are drawn through the vertices of a triangle; the intercepts on these parallels between a vertex and the opposite side-lines have lengths x, y, z. Then $x^{-1}+y^{-1}+z^{-1}=0$.

21. If a, b, c be non-collinear points, and

$$x_1a+y_1b+z_1c, \quad x_2a+y_2b+z_2c$$

be vectors (and hence $x_1+y_1+z_1=x_2+y_2+z_2=0$), then they are parallel if, and only if, $x_1x_2^{-1}=y_1y_2^{-1}=z_1z_2^{-1}$, and then these ratios equal the ratio of the magnitudes of the vectors.

For, let $b-a=u$, $c-a=v$, then $y_1u+z_1v=k(y_2u+z_2v)$ holds if, and only if, the given vectors are parallel. But then, as u is not a multiple of v, we have $k=y_1y_2^{-1}=z_1z_2^{-1}$.

22. Masses x, y, z at a, b, c are displaced to a', b', c'. We can choose x, y, z so that the centre of gravity is not changed.

For there are scalars k_1, k_2 such that $c-c'=k_1(a-a')+k_2(b-b')$. Hence $k_1a+k_2b-c=k_1a'+k_2b'-c'$.

23. *Desargues' Theorem. If abc, $a'b'c'$ be triangles with distinct vertices and aa', bb', cc' meet (in p), then the cuts q_1, q_2, q_3 of the corresponding sides are collinear, if these cuts exist.*

For scalars k_1, k_2, k_3 can be chosen so that

$$p=k_1a+(1-k_1)a'=k_2b+(1-k_2)b'=k_3c+(1-k_3)c'.$$

Thence $$k_2b-k_3c=(1-k_3)c'-(1-k_2)b'.$$

Thus each of these expressions equals $(k_2-k_3)q_1$, where q_1 is the cut of bc and $b'c'$, if it exists.

Hence $k_2b-k_3c=(k_2-k_3)q_1$, and so on. Adding these:

$$(k_2-k_3)q_1+(k_3-k_1)q_2+(k_1-k_2)q_3=0.$$

Hence either q_1, q_2, q_3 are collinear, or $k_1=k_2=k_3=k$, say. In the latter case, we have $k(b-c)=(k-1)(b'-c')$, and similar equations; the corresponding sides of the given triangles are then parallel.

Investigate the case when $k_2=k_3\neq k_1$.

§ 3. *Points at infinity.*

1. If a, b be points, and x, y positive scalars, $x \neq y$, then $xa - yb$ is a point, of weight $x - y$, dividing the interval ab externally in the ratio y:x. If this ratio tends to unity, the weighted point $xa - yb$ moves off to infinity, and its weight tends to zero. This suggests that we make our Euclidean plane projective and interpret $a - b$ as the '*point at infinity*' on the line ab. This interpretation will be used as well as that in the previous section.

As the weight of $a - b$ is zero, there is nothing in this interpretation which corresponds to the magnitude of a vector. Nevertheless, we shall distinguish between the symbols $k(a - b)$ and $a - b$, though both represent the same point at infinity, just as we distinguish between kp and p. The sum of two points at infinity is another such point; as all such points (in a plane) are sums of two such points with appropriate weights, we shall say they lie on a line, the '*line at infinity*'.

For if $a - b$, $a - c$, $a - d$ are distinct points at infinity, then a, c, d are non-collinear, and hence there are scalars x, y, z such that $b = xa + yc + zd$, $x + y + z = 1$. Then

$$a - b = (x + y + z)\, a - (xa + yc + zd) = y(a - c) + z(a - d).$$

2. *Congruence.* If a, b be any extensives, then $a \equiv b$ means there is a scalar k such that $ka = b$. We say a, b are '*congruent*'.

Two weighted points are congruent when they have the same position and differ, if at all, only in their weights. Two vectors are congruent, when their directions are parallel, and they differ, if at all, only in magnitude or sense, or both.

Any two non-zero scalars are congruent.

3. *Graphical theorems.* Such a theorem involves only the joining of points and the cutting and parallelism of lines (and planes), and not, for example, lengths or their ratios, or the measures of angles. In proving such theorems, it is useful to 'absorb weights' and denote a weighted point by a single letter. Thus in the proof of Desargues' Theorem, which is a graphical theorem, in Ex. 23 above, we may absorb weights k_1, $1 - k_1$ into the points

a, a', that is, we may replace $k_1 a$ by a, and so on. The proof then runs as follows:

abc, $a'b'c'$ are triangles with distinct vertices; aa', bb', cc' meet in p. Hence $p = a + a' = b + b' = c + c'$. Hence $b - c = c' - b'$. This is therefore a weighted point q_1 at the cut of bc and $b'c'$. We have then $q_1 = b - c$, and similarly $q_2 = c - a$, $q_3 = a - b$. Hence $q_1 + q_2 + q_3 = 0$, and so q_1, q_2, q_3 are collinear.

By thus absorbing weights the line at infinity is not distinguished from any other line. The method is extensively used in Baker's *Principles of Geometry*.

Example 24. Let $a_1, \ldots, a_n$ be points in a plane or space of any dimensions. Let b_{12} be any point on line $a_1 a_2$, b_{23} on $a_2 a_3$, ..., $b_{n-1,n}$ on $a_{n-1} a_n$. Let c_{123} be the cut of $a_1 b_{23}$ and $b_{12} a_3$; c_{234} be the cut of $a_2 b_{34}$ and $b_{23} a_4$, ..., then $a_1 c_{234}$, $b_{12} b_{34}$, $c_{123} a_4$ are concurrent, and so on.

For absorb weights, so that $b_{12} = a_1 + a_2$, ..., $b_{n-1,n} = a_{n-1} + a_n$. Thence $c_{123} = a_1 + a_2 + a_3$, since this point is on $b_{12} a_3$ and $a_1 b_{23}$; $c_{234} = a_2 + a_3 + a_4$. The lines mentioned therefore meet in the point $a_1 + a_2 + a_3 + a_4$. (Mehmke.)

§ 4. *Outer multiplication of two extensives of step one.*

1. We now introduce the distinctive feature of Grassmann's algebra. The '*outer product*' of two extensives a, b is of a different nature from the extensives themselves, in the sense that the product cannot be added to a or to b. We first proceed abstractly, and subject the notion of 'outer product' to the following laws:

If k, k_1, k_2 be scalars, and a, b, c any extensives, then

1. The '*outer product*' of a and b, taken in this order, exists, and is unique, but is of a different nature from a, b in the sense that it cannot be added to them, though it may be added to similar products. This restriction on addition is the one mentioned in § 1. The outer product of a, b, in this order, is denoted by $[ab]$ or $[a.b]$.

2. $[a.kb] = k[ab]$.
3. $[ka.b] = k[ab]$.
4. $[a(b+c)] = [ab] + [ac]$.
5. $[(b+c)\,a] = [ba] + [ca]$.
6. $[aa] = 0$.
7. If $a \neq 0$, then $[ab] = [ac]$ if, and only if, $b = c$.

From 6, writing $a+b$ for a, we have $[(a+b)\,(a+b)] = 0$.

Hence, by 4, 5, $[aa]+[ab]+[ba]+[bb]=0$, whence, by 6, we have

8. $[ab]=-[ba]$.

If 8 be assumed, then only one of 4, 5 need be assumed.

By 4, 5, 6, $\quad [a(a+b)]=[ab], \quad [(a+b)\,a]=[ba]$.

By 2, 3, $\quad [k_1 a . k_2 b]=k_1 k_2 [ab]$.

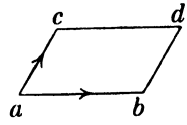
If $p=k_1 a+k_2 b+\ldots$, then $[pp]=0$.

Except for 6, 8 and their consequences, these laws are analogous to the laws of multiplication in ordinary algebra. The fundamental extensives, with which we began, are said to be of the '*first step*'. An outer product $[ab]$ of two of these is an extensive of the '*second step*'.

Only extensives of the same step can be added together.

2. *Interpretation when a, b, ... represent points.* In this case, the product $[ab]$ will be interpreted as an interval anywhere on the line ab, but not elsewhere, with a length equal to that of the interval ab, and with a direction from a to b.

Thus $[ab]=[cd]$ shall mean that a, b, c, d are collinear, the intervals ab, cd are of the same length, and the direction from a to b is the same as that from c to d.



If a, b, c, d are collinear, we shall interpret $[ab]+[ac]$ as $[ad]$, where the length of the interval ad, taking signs into account, is the sum of those of ab and ac. If a, b, c are not collinear, we interpret $[ab]+[ac]$ as $[ad]$, where $abdc$ is a parallelogram.

Putting it shortly, $[ab]$ is interpreted as a vector, *anchored to a line*, with the parallelogram law of addition. Vectors, anchored to lines, we call '*rotors*'.

It is now necessary to see that, with these interpretations, the assumed laws of outer products are obeyed.

Of these laws, 2, 3 are definitional, necessary because our points may be weighted.

Let $abdc$ be a parallelogram, p the cut of its diagonals. Since the diagonals of a parallelogram bisect each other, $[ap]=\frac{1}{2}[ad]$, $[ab]+[ac]=[ad]$, by our definitions. But $\frac{1}{2}(b+c)=p$, hence

$$[ap]=[a.\tfrac{1}{2}(b+c)]=\tfrac{1}{2}[a(b+c)].$$

Comparing these, we find that 4 is satisfied in this case. When a, b, c are collinear, it is simpler, taking p as the mid-point of the interval bc.

A slightly more complicated argument shews that, from our interpretation,

$$[a(k_1 b + k_2 c)] = (k_1 + k_2)\,[ap],$$

where $(k_1 + k_2)p = k_1 b + k_2 c$, and k_1, k_2 are scalars with $k_1 + k_2 \neq 0$. The case when $k_1 = -k_2$ is considered later.

The laws 6, 7 are clearly satisfied in our interpretation. We can see 8 independently; if $[ba] = [ac]$, then a is the mid-point of bc, hence $[ab] + [ac] = 0$, whence 8.

That the addition of those products, which have a first or a last factor common, satisfies the associative and commutative laws, follows from 4, 5 and the associative and commutative laws for extensives of the first step. Or again, it may be verified in the interpretation, since these laws have been shewn for the addition of vectors, and hence hold for the addition of rotors which proceed from the same point. The addition of rotors in general is considered later.

It is a common practice in geometry, in dealing with intervals in the same line, to take one direction as positive, and if ab is an interval to write $ab = -ba$. We go further, and regard ab as a product of a and b, and set up laws for the addition of such products.

3. The method obviously has applications in statics, if we interpret $[ab]$ as a force, represented in magnitude, direction, *and line of application* by the interval ab. If cd is an equal, parallel interval with the same sense as ab but not on the line ab, then the product $[cd]$ does not represent the same force as $[ab]$, but an equal and parallel force; for $[ab]$ represents a *rotor*, tied down to a line. The untied vector is represented by $b - a$. In ordinary vector analysis, the vector and rotor are often not distinguished in the symbolism, and the existence of a simple notation for each is one of the advantages of the present method.

4. $[ab] + [bc] = [ac]$ if, and only if, a, b, c are collinear. In particular a, b, c need not all be distinct.

The equation can be written

$$[ab] + [bc] + [ca] = 0.$$

5. *Note on notation.* As at present we are using only outer multiplication of extensives, in addition to their multiplication by scalars, we can often omit the bracket [ ] without confusion, and write *ab* instead of [*ab*]. Later, when other types of multiplication are introduced, it will be necessary to restore the bracket.

Examples. 25. *Leibniz' Theorem.* Let

$$k_1 a_1 + k_2 a_2 + \ldots + k_n a_n = (k_1 + k_2 + \ldots + k_n)\, g,$$

where the k are scalars. Multiply by any point p:

$$k_1[a_1 p] + k_2[a_2 p] + \ldots + k_n[a_n p] = (k_1 + k_2 + \ldots + k_n)\,[gp].$$

Hence the sum of the rotors $k_1[a_1 p]$, $k_2[a_2 p]$, ..., $k_n[a_n p]$ is the rotor $(k_1 + k_2 + \ldots + k_n)\,[gp]$, where g is the centre of gravity of weights k_1 at a_1, k_2 at a_2, ..., k_n at a_n.

26. If *abpq* be any quadrilateral, and *l*, *m* the mid-points of *ap* and *bq*, then

$$(a+p)\,(b+q) = 2l.2m = 4lm.$$

27. *The mid-points of the diagonals of a complete quadrilateral abpq are collinear*, on the '*diameter*' of the quadrilateral. (Cf. Ex. 30.)

For, let *aq*, *bp* cut in *c*, *ab*, *pq* cut in *r*. Let

$$2l = a+p, \quad 2m = b+q, \quad 2n = c+r.$$

Then

$$ab + aq + pb + pq = (a+p)\,(b+q) = 4lm,$$
$$br + qc + bc + qr = (b+q)\,(c+r) = 4mn,$$
$$ra + ca + cp + rp = (c+r)\,(a+p) = 4nl.$$

Add these equations, note that $ab + br + ra = 0$, since *a*, *b*, *r* are collinear, and the similar equations. Thence $lm + mn + nl = 0$, hence *l*, *m*, *n* are collinear.

28. If in the previous example, the parallelograms *abdq*, *arec* be completed, then the mid-points of *ap*, *ad*, *ae* are collinear. Shew this directly.

§ 5. *Outer products of three extensives of step one.*

1. In the abstract algebra we form '*outer products*' obeying the following laws:

If *a*, *b*, *c* be extensives of step one, and k_1, k_2, k_3 scalars, then

1. The '*outer product*' of *a*, *b*, *c*, in this order, exists, is unique, and is denoted by [*abc*] or [*a.b.c*].

2. This outer product is also the '*outer product*' of $[ab]$ and c, also of a and $[bc]$; in symbols

$$[abc] = [[ab]\, c] = [a[bc]], \quad \text{or} \quad [abc] = [ab.c] = [a.bc].$$

3. $$[k_1 a.k_2 b.k_3 c] = k_1 k_2 k_3 [abc].$$

4. $$[ab(c+d)] = [abc] + [abd].$$

Since $[ab] = -[ba]$, we can deduce, from 2, 3, $[abc] = -[bac]$, for $[abc] = [[ab]\, c] = [-[ba]\, c] = [[(-b)\, a]\, c] = -[bac]$.

Similarly $[abc] = -[acb]$. Hence we can deduce

5. $$[abc] = [bca] = [cab] = -[bac] = -[cba] = -[acb].$$

In particular, if two or all of a, b, c are congruent (§ 3·2), then $[abc] = 0$.

From 4, 5, we deduce

6. $$[a(b+c)\, d] = [abd] + [acd], \quad [(a+b)\, cd] = [acd] + [bcd].$$

We also assume

7. $$[a(bc+df)] = [abc] + [adf].$$

2. *Interpretation when $a, b, c, \ldots$ are points in one plane.* When we are dealing with points in one plane, we interpret $[abc]$, $[[ab]\, c]$ and $[a[bc]]$, each as *twice* the area of the triangle abc, and take it as positive or negative, according as the order of a, b, c round the triangle is counter-clockwise or clockwise. Thus $[abc]$ is a scalar. The geometric fact which makes this interpretation possible is that in any triangle the product of the altitude and the base is independent of which line is chosen as the base.

We must now verify that the above laws hold with the present interpretation. Of these 3 is definitional, necessary because we use weighted points.

To verify 4. Let $\frac{1}{2}(c+d) = m$, and let p_1, p_2, p_3 be the lengths of the perpendiculars from m, c, d on to ab, each with its sign; then $p_1 = \frac{1}{2}(p_2 + p_3)$. If l be the length of ab, we have

$$[abc] = p_2 l, \quad [abd] = p_3 l, \quad [abm] = p_1 l.$$

Hence 4.

By a similar argument, we can verify, if $k_1 \neq k_2$,

$$[ab(k_1 c + k_2 d)] = k_1 [abc] + k_2 [abd].$$

Our conventions on sign give 5 at once, and 6 follows. As those products are scalars, their addition needs no definition. We consider 7 below.

3. If our geometric interpretation is valid, then $[abc]=0$, if, and only if, a, b, c are collinear. We can deduce this from the general rules, if we assume that some products of three extensives do not vanish.

For if a, b, c be collinear, there are scalars k_1, k_2 such that $c=k_1a+k_2b$, and then $[abc]=0$, by 3, 4, 5.

Conversely, if $[abc]=0$, and $[abd]\neq 0$, and hence d is a point not on the line ab, then there are scalars k_1, k_2, k_3 such that $c=k_1a+k_2b+k_3d$. Hence, by 3, 4,

$$[abc]=k_1[aba]+k_2[abb]+k_3[abd]=k_3[abd].$$

But $[abc]=0$, $[abd]\neq 0$, hence $k_3=0$, so that c is on the line ab.

As a corollary, note that if $[pab]=0$ for all p, then $a\equiv b$, $[ab]=0$.

4. We have $[pab]=[pcd]$ for all p if, and only if, $[ab]=[cd]$.

For if $[pab]=[pcd]$ for all p, then putting $p=a$, we have $[acd]=0$. Hence a, c, d, and similarly b, c, d, are collinear. Hence either $a\equiv b$, or there are scalars k_1, ..., k_4, such that

$$c=k_1a+k_2b, \quad d=k_3a+k_4b,$$

whence

$$[cd]=(k_1k_4-k_2k_3)\,[ab], \quad [pcd]=(k_1k_4-k_2k_3)\,[pab].$$

Hence, *either* $k_1k_4-k_2k_3=1$ and then $[ab]=[cd]$, *or* $[pab]=0$, for all p, and $[pcd]=0$ for all p. The latter alternative gives $[ab]=0$, $[cd]=0$.

Conversely, if $[ab]=[cd]$, then $[pab]=[pcd]$, for all p.

5. In the statical interpretation where $[ab]$ is a force or rotor, $[abc]$ is its 'moment' round c, and the laws represent well-known theorems on moments in statics.

6. If a, b, c, d be four (coplanar) points, then

$$[abc]\,d-[bcd]\,a+[cda]\,b-[dab]\,c=0.$$

For, if all four points are collinear, all the terms vanish, and if, say a, b, c are not collinear, then there are scalars k_1, k_2, k_3 such that $d = k_1 a + k_2 b + k_3 c$. Then

$$[dbc] = k_1[abc], \quad [dca] = k_2[bca] = k_2[abc],$$

$$[dab] = k_3[cab] = k_3[abc].$$

Hence $$[abc]\, d = [dbc]\, a + [dca]\, b + [dab]\, c,$$

which is equivalent to, and an important variant of, the formula to be proved. It can be regarded as a statement of the relation between weights and areal coordinates mentioned in § 2·9.

7. Multiplying the last formula, outer multiplication, by $[pd]$, we have

$$[dbc]\,[pad] + [dca]\,[pbd] + [dab]\,[pcd] = 0,$$

connecting any five coplanar points.

8. In the last formula of 6 the sum of the coefficients of a, b, c must be $[abc]$, the coefficient of d. Hence for any four coplanar points,

$$[abc] = [dbc] + [dca] + [dab].$$

This is easily shewn directly, in the geometrical interpretation.

9. We now consider law 7; in our interpretation, if we suppose the rotors bc, df lie along lines with a common point p, we can write them as pq, pr; and we have to consider

$$[a(pq + pr)] = [apq] + [apr].$$

Let $e = \frac{1}{2}(q + r)$, then since $[aee] = 0$,

$$2e = q + r, \quad [aqe] = [aer], \quad [pqe] = [per].$$

By 8,

$$[pqe] = [apq] + [aqe] + [aep] = [apq] - [ape] + [aqe],$$

$$[per] = [arp] + [ape] + [aer] = [ape] - [apr] + [aer].$$

Hence $$[apq] + [apr] = 2[ape] = [a.2pe] = [a(pq + pr)].$$

For the case when the rotors are parallel, see § 6·11.

Notation. We shall *at present* often write abc for $[abc]$.

Examples. 29. *Menelaus' Theorem.* Using the notation of Ex. 3, p. 7, we have $lmn = 0$, hence

$$(xb + x'c)\,(yc + y'a)\,(za + z'b) = (xyz + x'y'z')\,[abc] = 0,$$

$$xyz + x'y'z' = 0.$$

30. *The mid-points of the diagonals of a complete quadrilateral are collinear.*

For, using the notation of Ex. 27, p. 16,

$$abr = pqr = aqc = pbc = 0.$$

Hence $$8[lmn] = [(a+p)\,(b+q)\,(c+r)] = abc + aqr + pbr + pqc$$
$$= (abc - qpc) - (arq - pbr) = 0,$$

since $abc - qpc$ and $arq - pbr$ each equals twice the area of the quadrilateral $abpq$.

31. The mean centre of b, c, q, r, and of similar sets, lies on the diameter of the quadrilateral.

For $$(b+q)\,(c+r)\,(b+q+c+r) = 0.$$

32. If s is any point on the diameter of the quadrilateral $abpq$, then $sab + spq$ equals the area of the quadrilateral.

For $4lm = (a+p)\,(b+q)$, $slm = 0$. Hence $sab + spq = sqa + sbp$, and the sum of these is twice the area of the quadrilateral.

33. *If $ab'ca'bc'$ is a (plane) hexagon, the diameters of the pairs of quadrilaterals $(c'ab'ca', ca'bc'a)$, $(ab'ca'b, a'bc'ab')$, $(b'ca'bc', bc'ab'c)$ meet in three points which lie on a line.* (The quadrilateral $c'ab'ca'$ is the one with sides $c'a$, ab', $b'c$, ca'.)

For let $c'a$, $a'c$ meet in d, then the diameters of $c'ab'ca'$, $ca'bc'a$ are respectively

$$(b' + d)\,(c + a) = -ab' + b'c + da + dc,$$

$$(a' + c')\,(b + d) = a'b - bc' + c'd + a'd.$$

If the point p is on both these diameters, then by addition:

$$[p(b'c + c'a + a'b - bc' - ca' - ab')] = 0.$$

Assume that the sum of rotors in the last expression can be reduced to a single rotor. The underlying theory is given shortly. Hence, by symmetry, the cuts of the diameters in question lie on the line

$$b'c + c'a + a'b - bc' - ca' - ab'.$$

34. The points d, e, f are the mid-points of the sides of triangle abc, any line through a cuts de in n and df in m. Then bm is parallel to cn, and bn, cm, ef are concurrent.

35. From any point p in bc parallels are drawn to ab, ac forming a parallelogram $pqar$. If pq, pr cut the median through a in m, n, then bm, cn, qr are parallel, when q is on ac, r on ab.

36. If $[p(ca+ab)] = [abc]$, then p lies on bc.

37. If d, e, f be the mid-points of the sides of triangle abc, and p be any point in its plane, then $pad+pbe+pcf = 0$.

38. The triangles abc, $a_1b_1c_1$ are coplanar. From a point p vectors are drawn:

$$u = a_1 - a = a_2 - p, \quad v = b_1 - b = b_2 - p, \quad w = c_1 - c = c_2 - p.$$

Then $a_2b_2c_2 = a_1b_1c_1 + abc +$ twice the area of $a_1cb_1ac_1b$. (Mehmke.)

For take p at a, then

$$\begin{aligned} a_2b_2c_2 &= (a+a_1-a)\,(a+b_1-b)\,(a+c_1-c) \\ &= a_1b_1c_1 + (a_1bc + a_1ca + a_1ab) + (a_1cb_1 + a_1b_1a + a_1ac_1 + a_1c_1b). \end{aligned}$$

39. If d, e, f be points on bc, ca, ab dividing these intervals in the ratio $k:1$, then $[def]:[abc] = k^3+1:[k+1]^3$.

If d_1, e_1, f_1 be analogous points with $k_1:1$ for the ratio, and if l, m, n be the cuts of ef and e_1f_1, of fd and f_1d_1, of de and d_1e_1, then al, bm, cn are concurrent if, and only if, $kk_1 = 1$.

40. If a_1, b_1, c_1 be variable points on the side-lines of triangle abc, forming a triangle of constant area, then if aa_1, bb_1, cc_1 be divided at a_2, b_2, c_2 in the same ratio, the triangle $a_2b_2c_2$ will have constant area. (Mehmke.)

41. If ad, be, cf are chords of a triangle abc cutting in pairs in p, q, r, then, denoting lengths by bars,

(i) $def:abc = (\overline{bd}.\overline{ce}.\overline{af} + \overline{cd}.\overline{ae}.\overline{bf}) : \overline{bc}.\overline{ca}.\overline{ab}$.

(ii) $pqr:abc = (\overline{bd}.\overline{ce}.\overline{af} - \overline{cd}.\overline{ae}.\overline{bf})^2 : (\overline{ca}.\overline{ab} - \overline{ae}.\overline{af}) \times (\overline{ab}.\overline{bc} - \overline{bf}.\overline{bd})\,(\overline{bc}.\overline{ca} - \overline{cd}.\overline{ce})$.

Use $\overline{bc}.d = \overline{dc}.b + \overline{bd}.c$, and so on. Ceva's Theorem is a special case of (i).

42. If $abcd$ be any quadrilateral, and ab, cd cut in f, and bc, ad cut in e, then $\overline{fb}.\overline{fd} : \overline{fa}.\overline{fc} = \overline{eb}.\overline{ed} : \overline{ea}.\overline{ec}$.

For we can find scalars x, y, z, w such that

$$xa + yb + zc + wd = 0.$$

Then $$f = (xa+yb)/(x+y) = (zc+wd)/(z+w),$$

$$[fb] = x(x+y)^{-1}\,[ab], \quad [fd] = z(z+w)^{-1}\,[cd].$$

43. Area of pedal triangle of *abc*. If A, B, C be the angles of the triangle, the foot of the altitude from *a* is

$$(\cot \mathrm{B}.c+\cot \mathrm{C}.b)/(\cot \mathrm{B}+\cot \mathrm{C}).$$

Outer multiplication of this and two similar expressions gives the result

$$2 \cos \mathrm{A} \cos \mathrm{B} \cos \mathrm{C}.[abc].$$

44. If r be the in-radius, h_1, h_2, h_3 the altitudes of a triangle, then

$$r^{-1} = h_1^{-1} + h_2^{-1} + h_3^{-1}.$$

45. *Tangential coordinates.* If *aa'*, *bb'*, *cc'* be parallel intervals of lengths l_1, l_2, l_3, and

$$p = xa + yb + zc, \quad q = xa' + yb' + zc', \quad x + y + z = 1,$$

then $p - q = (l_1 x + l_2 y + l_3 z)\, v$, where v is the unit vector parallel to *aa'*.

Hence, if l_1, l_2, l_3 be the distances of *a*, *b*, *c* from any line, then the distance of *p* from that line is $l_1 x + l_2 y + l_3 z$. Thus *p* with areal coordinates (x, y, z) is on the line if $l_1 x + l_2 y + l_3 z = 0$. The ratios of the scalars l_1, l_2, l_3 are the '*tangential coordinates*' of the line.

Hence the tangential coordinates of any line (the point-coordinates being areals) are the ratios of the perpendiculars from *a*, *b*, *c* to the line.

46. *Resultant of forces.* The resultant of the forces $l_1[bc]$, $l_2[ca]$, $l_3[ab]$ is along the line whose equation, in areal coordinates, is

$$l_1 x + l_2 y + l_3 z = 0.$$

For the moment of the forces round *p*, $= xa + yb + zc$, is

$$l_1[pbc] + l_2[pca] + l_3[pab] = (l_1 x + l_2 y + l_3 z)\,[abc],$$

and this moment vanishes if, and only if, *p* is on the line of action of the resultant. The case when $l_1 = l_2 = l_3$ is considered later.

47. *Determinants.* If

$$\left.\begin{aligned} p &= x_1 a + y_1 b, \quad q = x_2 a + y_2 b, \\ a &= k_1 c + l_1 d, \quad b = k_2 c + l_2 d, \end{aligned}\right\} \qquad \text{(i)}$$

then

$$\left.\begin{aligned} p &= (x_1 k_1 + y_1 k_2)\, c + (x_1 l_1 + y_1 l_2)\, d, \\ q &= (x_2 k_1 + y_2 k_2)\, c + (x_2 l_1 + y_2 l_2)\, d. \end{aligned}\right\} \qquad \text{(ii)}$$

From (i),

$$[pq] = (x_1 y_2 - x_2 y_1)\,[ab] = (x_1 y_2 - x_2 y_1)(k_1 l_2 - k_2 l_1)\,[cd].$$

From (ii),

$$[pq] = ((x_1 k_1 + y_1 k_2)(x_2 l_1 + y_2 l_2) - (x_1 l_1 + y_1 l_2)(x_2 k_1 + y_2 k_2))\,[cd].$$

Comparing these, we have the usual rule for the multiplication of two-rowed determinants.

48. If

$$p = x_1 a + y_1 b + z_1 c, \quad q = x_2 a + y_2 b + z_2 c, \quad r = x_3 a + y_3 b + z_3 c,$$

then
$$[pqr] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [abc].$$

An obvious extension of Ex. 47 gives the rule for the multiplication of three-rowed determinants. Also note

$$[pq] = (y_1 z_2 - y_2 z_1)\,[bc] + (z_1 x_2 - z_2 x_1)\,[ca] + (x_1 y_2 - x_2 y_1)\,[ab].$$

49. If $a_1, a_2, \ldots, a_n$ be any points, not necessarily of unit weight, and $c_1, c_2, \ldots, c_n$ be points of unit weight, and

$$(k_1 + \ldots + k_n)\, a = k_1 a_1 + \ldots + k_n a_n, \quad \text{(k scalars)},$$

then
$$\Sigma k_1 k_2 k_3 [a_1 a_2 a_3]\,[c_1 c_2 c_3] = \Sigma k_i \Sigma k_1 k_2 [a a_1 a_2]\,[a_1 c_2].$$

If also
$$(k_1 + \ldots + k_n)\, c = k_1 c_1 + \ldots + k_n c_n,$$

then
$$\Sigma k_1 k_2 [a a_1 a_2]\,[a c_1 c_2] = \Sigma k_1 k_2 [c a_1 a_2]\,[c c_1 c_2]. \quad \text{(Neuberg.)}$$

50. Prove that, if a, b, c, p, q, r be points, then

$$[abc]\,[pqr] = [abq]\,[crp] + [abr]\,[cpq] + [acp]\,[bqr] + [bcq]\,[arp] + [bcr]\,[apq],$$

and interpret it as a theorem on determinants. (Use § 5·6.)

§ 6. *Outer products which involve vectors.*

So far we have only considered outer products of points; we now consider outer products which involve vectors, thus filling gaps left in §§ 4, 5.

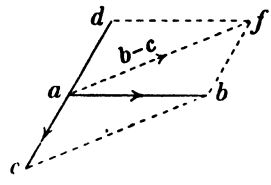
1. *Outer product of a point and a vector.* If a, b, c be points (of unit weight), then $b - c$ is a vector; we define $[a(b-c)]$ by means of the equation

$$[a(b-c)] = [ab] - [ac]. \qquad (1)$$

If
$$[ad] = -[ac],$$

then
$$[a(b-c)] = [ab] + [ad]$$

and the figure shews that $[a(b-c)]$ must be interpreted as the rotor through a, whose vector is the vector $b-c$. Note that, if f be the point such that $f - a = b - c$, then $[a(f-a)] = [af]$, and hence



$$[a(b-c)] + [ac] = [a(f-a)] + [ac] = [af] + [ac]$$
$$= [a(f+c)] = [a(a+b)] = [ab].$$

Our definition gives us therefore the usual relations between sums and differences.

We define $[(b-c)\,a] = [ba]-[ca]$.

Hence, if v is a vector, a a point, then $[av] = -[va]$.

The law, corresponding to § 4 (7): if $[p(a-b)] = [p(c-d)]$, then $a-b = c-d$, holds here.

If a is a point, v a vector, k a scalar, then $[a.\mathrm{k}v] = \mathrm{k}[av]$.

If p is a point, u, v vectors, then $[p(u+v)] = [pu]+[pv]$. For, we can find points a, b, c such that $u = a-b$, $v = b-c$. Thus $u+v = a-c$, and the result follows at once.

Thus, if k, k_1 be scalars, then

$$[p(\mathrm{k}u+\mathrm{k}_1 v)] = \mathrm{k}[pu]+\mathrm{k}_1[pv].$$

2. *Outer product of two vectors.* If a, b, p, q be points (of unit weight), we define the outer product of vectors $a-b$, $p-q$ by the equation

$$[(a-b)\,(p-q)] = [a(p-q)]-[b(p-q)], \qquad (2)$$

where the right-hand side is the difference of two equal rotors, parallel and in the same sense. This difference has not yet been introduced; it is a new type of magnitude called a '*bivector*'. Like the rotor, it is of step two.

There is a point c such that $p-q = a-c$, then

$$[(a-b)\,(a-c)] = [a(a-c)]-[b(a-c)] = -[ac]-[ba]+[bc].$$

Hence $$[(a-b)\,(a-c)] = [bc]+[ca]+[ab]. \qquad (3)$$

3. If $a-b$, $p-q$ be parallel vectors, then a, b, c are collinear. Hence, by (2), (3) and § 4·4, $[(a-b)\,(p-q)] = 0$.

The outer product of two parallel vectors is zero.

Conversely, if $[(a-b)\,(a-c)] = 0$, multiply the right-hand side of (3) by a, and we have $[abc] = 0$; hence a, b, c are collinear. Hence, *if the outer product of two vectors is zero, they are parallel.*

4. If u, v be any vectors, $[uv] = -[vu]$, by (3), since $[pq] = -[qp]$ for points.

Also, if k is a scalar,

$$[u.\mathrm{k}v] = [\mathrm{k}u.v] = \mathrm{k}[uv].$$

Also, since $ku+k'v$ is a vector, we have

$$\begin{aligned}[(a-b)(ku+k'v)] &= [a(ku+k'v)]-[b(ku+k'v)]\\ &= k[au]-k[bu]+k'[av]-k'[bv]\\ &= k[(a-b)u]+k'[(a-b)v].\end{aligned}$$

Thus for any vectors u, v, w,

$$[w(ku+k'v)] = k[wu]+k'[wv].$$

5. *Outer product of a rotor* $[ab]$ *and a vector* v. Since $[bv]$ is a rotor, the product $[a.bv]$ is already defined as twice the area of the triangle whose vertex is a and whose base is given by $[bv]$. The geometric interpretation gives at once

$$[a.bv] = -[b.av].$$

We define the product of ab and v by the equation

$$[ab.v] = [a.bv].$$

Then, since $[a.bv] = -[b.av]$, we have $[ab.v] = -[ba.v]$.

6. *Outer product of a point and a bivector.* Using (3) and § 5·8, we have

$$\begin{aligned}[p.(a-b)(a-c)] &= [p.(bc+ca+ab)]\\ &= [pbc]+[pca]+[pab] = [abc].\end{aligned}$$

This scalar, independent of p, is called the '*moment*' or '*magnitude*' of the bivector $[(a-b)(a-c)]$.

7. *Equality of bivectors.* Two bivectors R_1, R_2 will be regarded as equal (and each can be replaced by the other in any equation) if $[pR_1]=[pR_2]$, for all points p.

Thus if the moment of a bivector is zero, the bivector is the outer product of any two parallel vectors.

If $abdc$ is a parallelogram, its area equals the moment of $[(b-a)(c-a)]$. Thus a non-zero bivector can be represented by any parallelogram in the plane, with a suitable area and sense. Thus $[abc]=[apq]$ and $[(b-a)(c-a)]=[(p-a)(q-a)]$ imply each other.

If ω is a bivector of unit moment, any bivector is of form $k\omega$, where k is a scalar, its moment. Thus though a bivector is not a scalar, any equation between bivectors can be replaced by one between scalars, by omitting ω.

8. *The sum of two bivectors.* If ω is the unit bivector, the two bivectors can be written $k_1\omega$, $k_2\omega$ (k_1, k_2 scalars). Their sum is $(k_1+k_2)\omega$.

9. *The sum of a bivector and a rotor.* If $[ab]$ be the rotor, then by suitable choice of c the bivector can be put in the form

$$[(c-a)(b-a)] = [c(b-a)] - [a(b-a)],$$

while $[ab] = [a(b-a)]$. Hence we take the sum to be a rotor, through c, equal and parallel to $[ab]$.

10. *The sum of two parallel rotors.* If parallel vectors ku, $k'u$, $(k+k' \neq 0)$, act at p, q respectively, then since

$$[pu] = [(p-r)u] + [ru],$$

the rotors $k[pu]$, $k'[qu]$ can be written

$$k[(p-r)u] + k[ru] \quad \text{and} \quad k'[(q-r)u] + k'[ru],$$

where r is any point.

$$[p.ku] + [q.k'u] = k[(p-r)u] + k'[(q-r)u] + k[ru] + k'[ru],$$

by 1, 4,
$$= [(k(p-r) + k'(q-r))u] + [r(ku+k'u)].$$

Take r so that $(k+k')r = kp+k'q$, then

$$[p.ku] + [q.k'u] = [r.(k+k')u],$$

which corresponds to the rule in statics for adding parallel rotors, and gives at once

$$[p.ku] + [q.k'u] = [(kp+k'q)u],$$

which is a case of the distributive law, so far missing.

11. If a be a point, R_1, R_2 parallel rotors, then

$$[a(R_1+R_2)] = [aR_1] + [aR_2]. \qquad \text{(i)}$$

For if S be a rotor not parallel to R_1, R_2, then $R_1 - S$, $R_2 + S$ are not parallel; hence, by § 5·9,

$$[a(R_1+R_2)] = [a(R_1 - S + S + R_2)] = [a(R_1 - S)] + [a(S+R_2)]$$
$$= [aR_1] - [aS] + [aS] + [aR_2] = [aR_1] + [aR_2].$$

Hence (i) holds for all R_1, R_2.

12. *Sum of any number of rotors.* If R_1, R_2, ..., R_n be any number of rotors, we can find points p_i and vectors v_i such that $R_i = [p_i v_i]$, $(i = 1, \ldots, n)$. Then if p be any point

$$R_i = [pv_i] + [(p_i - p)v_i].$$

Now $[(p_i - p)\, v_i]$ is a bivector and hence equals $k_i \omega$ for some scalar k_i where ω is the unit bivector. Hence the sum of the rotors is

$$[pv_1] + \ldots + [pv_n] + k_1 \omega + \ldots + k_n \omega$$
$$= [p(v_1 + \ldots + v_n)] + (k_1 + \ldots + k_n)\, \omega.$$

The first term is a rotor through p, or zero. If it is not zero, we use 9 and find that the total sum is a rotor. If it is zero, the sum is the second term, i.e. a bivector, or zero.

13. If we adopt the interpretation in which $b - a$ represents not a vector, but a point at infinity, then $b - a$, $d - c$ will represent the same point at infinity, if and only if ab and cd are parallel. From this point of view, it is clear why their product should be zero then. If the lines ab, cd be not parallel, then $[(b - a)\,(d - c)]$ will represent a rotor along the line at infinity. Each such rotor is a scalar multiple of any fixed rotor along that line.

14. If i, j be two fixed independent vectors in a plane, and u, v any vectors therein, we can find scalars x_1, x_2, y_1, y_2 such that

$$u = x_1 i + y_1 j, \quad v = x_2 i + y_2 j,$$

and then
$$[uv] = (x_1 y_2 - x_2 y_1)\, [ij].$$

This analytical expression corresponds to the representation of a bivector by an area. The moment of $[uv]$ is the product of the magnitudes of u, v and the sine of the angle from u to v.

15. Since $[(b - a)\,(c - a)] = [b(c - a)] - [a(c - a)]$, and the two terms of the last expression represent equal and opposite parallel rotors, therefore *bivectors correspond to couples in statics, their moments or magnitudes to the moments of couples.*

Examples. 51. The area of the quadrilateral $abcd$ is $[(a - c)\,(b - d)]$.

52. If $ab(c + d) = cd(a + b)$, then ad, bc are parallel.

$$[(b + d)\,(c - b)\,(a - d)] = 0.$$

53. The parallelograms $apqr$, $abcd$ are of equal area; p is on ab, r is on ad; pq, cd cut in s; cb, rq cut in t. Then a, s, t are collinear.

For $(s-a)(t-a) = ((s-p)+(p-a))((t-b)+(b-a))$
$= (s-p)(t-b)+(p-a)(b-a)$
$+(s-p)(b-a)+(p-a)(t-b)$

the first two terms vanish; the last two, with the hypothesis, give the result.

54. *If abcdef is a hexagon such that ab, cd, ef can be displaced along themselves so as to form a triangle, the same is true of bc, de, fa.* (Hamilton.)

For, if p is any point, then

$$ab+bc+cd+de+ef+fa$$
$$= (a-p)(b-p)+(b-p)(c-p)+\ldots+(f-p)(a-p).$$

The right-hand side is clearly a bivector; so is $ab+cd+ef$ by hypothesis; hence so is $bc+de+fa$.

55. If $abc\ldots qa$ is any closed (plane) polygon, then $ab+bc+\ldots+qa$ is a bivector whose moment is twice the area of the polygon. (If the polygon crosses itself, some areas are, of course, reckoned as negative, and some positive, according to the sense in which the perimeters are described.)

56. The vertices of two triangles abc, def lie on parallel lines ad, be, cf. Then $abc+2def = aef+bfd+cde$.

57. What inferences can be drawn from the equations?

$$\text{(i) } p(a-b)=cd, \quad \text{(ii) } p(a+b)=cd,$$
$$\text{(iii) } (p-a)(p-b)=(c-a)(c-b).$$

58. Through a fixed point a line is drawn to meet two parallel lines in p, q. Through p, q parallels are drawn to given lines. Then the locus of their cuts is a line.

59. A line drawn through the vertex a of a parallelogram $abcd$ cuts cb, cd in p, q; a line through c cuts ab, ad in r, s. Then pr, qs are parallel.

For let $b-a=u, \quad d-a=v,$

$$p-b=\mathrm{k}v, \quad q-d=\mathrm{l}u, \quad r-b=\mathrm{k}_1 u, \quad s-d=\mathrm{l}_1 v.$$

Then

$$0=(p-a)(q-a)=(u+\mathrm{k}v)(v+\mathrm{l}u)=uv+\mathrm{kl}.vu=(1-\mathrm{kl})[uv].$$

Hence $\mathrm{kl}=1$. Similarly, $\mathrm{k}_1\mathrm{l}_1=1$.

$$(p-r)(q-s)=(\mathrm{k}v-\mathrm{k}_1 u)(\mathrm{l}u-\mathrm{l}_1 v)=\mathrm{kl}[vu]+\mathrm{k}_1\mathrm{l}_1[uv]=0.$$

60. *The side-lines ab, cd of a quadrilateral abcd meet in e; ad, bc meet in f. If the bisectors of the angles at e, f are parallel, they are parallel to the bisectors of the angles between ac and bd.*

61. *The parallel case of Pappus' Theorem.* If a, b, c, a', b', c' be any points, then

$$(a-b')(a'-b)+(b-c')(b'-c)+(c-a')(c'-a)$$
$$=(b'c'+c'a'+a'b')-(bc+ca+ab).$$

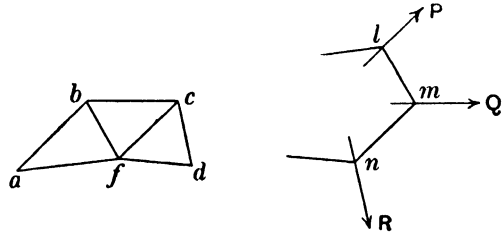
Now, if a, b, c be collinear, and a', b', c' be collinear, both terms on the right-hand side vanish; hence if bc' is parallel to $b'c$, and ca' parallel to $c'a$, then ab' is parallel to $a'b$.

62. *To find the resultant of any number of coplanar forces.*

A rotor is equivalent to a parallel rotor through any arbitrary point and a bivector. For

$$ab=p(b-a)+(ab+bp+pa).$$

Let P, Q, R be the forces, and let $b-a$, $c-b$, $d-c$ be vectors parallel to the vectors of the forces. Let f be any point, and l any point on the line of action of P. Draw the line lm parallel to $b-f$ to cut the line of Q in m, say; draw mn parallel to $c-f$ to cut the line of R in n, say. Through l, n draw parallels to $a-f$ and $d-f$ respectively.



Let $\quad a-f=u_1, \quad b-f=u_2, \quad c-f=u_3, \quad d-f=u_4.$

First, suppose lu_1, nu_4 cut in s. Since $P=l(b-a)$, and so on,

$$P+Q+R=l(b-a)+m(c-b)+n(d-c)$$
$$=l(u_2-u_1)+m(u_3-u_2)+n(u_4-u_3)$$
$$=(l-m)\,u_2+(m-n)\,u_3+nu_4-lu_1=nu_4-lu_1,$$

since $l-m$ is parallel to u_2, and $m-n$ to u_3.

Hence $P+Q+R=s(u_4-u_1)$. Thus s is a point on the resultant. The magnitude and direction of the resultant is given by

$$u_4-u_1=d-a.$$

Next, if u_1, u_4 be parallel, then nu_4-lu_1 is a bivector (or couple). Similarly for any number of forces.

63. Define a zigzag line as a broken line having alternate parts parallel; if two parallel zigzags be inscribed in the same given angle, then their intersections lie on another zigzag inscribed in an angle with the same vertex.

64. Construct $F = 2bc + 3ca - 4ab$; i.e. find the resultant of given forces along the sides of a triangle. Let $7l = 3c + 4b$, $3m = c + 2a$, then $F = 2bc + 7la = 6bm + 3ca$. Since l is on bc, and m on ca, therefore F is a rotor through l and m. In fact $2F = 21lm$.

16. If u, v, w be any vectors in a plane, then

$$[vw]\,u + [wu]\,v + [uv]\,w = 0. \qquad (1)$$

This is to be interpreted thus: the bivectors $[vw]$, $[wu]$, $[uv]$ are multiples, $l_1\omega$, $l_2\omega$, $l_3\omega$ of the unit bivector ω. Then the equation means $l_1u + l_2v + l_3w = 0$.

Proof. If u, v be not parallel, we can find scalars k_1, k_2 such that $w = k_1u + k_2v$. Thence

$$[wu] = k_2[vu] = -k_2[uv], \quad [vw] = -k_1[uv].$$

These give equation (1), for the left-hand side vanishes when they are substituted.

If u, v be parallel, then $[uv] = 0$, $u = kv$ for some scalar k. Then

$$[vw]\,u + [wu]\,v = [vw]\,kv + k[wv]\,v = 0,$$

since $[vw] = -[wv]$.

17. For four coplanar vectors u, v, w, r we have from (1)

$$[vw]\,[ur] + [wu]\,[vr] + [uv]\,[wr] = 0. \qquad (2)$$

If they be all of unit length, and $\phi = \widehat{u, v}$; $\theta = \widehat{v, w}$; $\psi = \widehat{w, r}$, then by the end of 14, p. 27, we have, from (2),

$$\sin\phi \sin\psi = \sin(\theta + \phi) \sin(\theta + \psi) - \sin\theta \sin(\theta + \phi + \psi).$$

If $\theta = \pi/2$, this agrees with the addition formula for the cosine.

If $u = u_1i + u_2j$, (u_1, u_2 scalars), and so on, then

$$[uv] = (u_1v_2 - u_2v_1)\,[ij]$$

and (2) gives the identity in scalars:

$$(u_1v_2 - u_2v_1)(w_1r_2 - w_2r_1) + (v_1w_2 - v_2w_1)(u_1r_2 - u_2r_1) + (w_1u_2 - w_2u_1)(v_1r_2 - v_2r_1) = 0.$$

Replacing u_1/u_2 by u, and so on, this becomes the identity in scalars:

$$(u - v)(w - r) + (v - w)(u - r) + (w - u)(v - r) = 0.$$

§ 7. *Supplements of vectors in a plane.*

1. If u be a vector in a fixed plane, we define its '*supplement*' in that plane as the vector obtained by rotating it with unchanged magnitude, through one right angle in the positive direction, and we denote this supplement by $|u$.

2. Let i, j be two unit vectors, and $|i = j$; then, if $||i$ means the supplement of $|i$, we have*

$$|j = -i, \quad ||i = -i, \quad ||j = -j. \tag{1}$$

If $u = \mathrm{x}_1 i + \mathrm{x}_2 j$, easy geometry gives

$$|u = \mathrm{x}_1 |i + \mathrm{x}_2 |j = -\mathrm{x}_2 i + \mathrm{x}_1 j.$$

Thence if v_1, v_2 be any vectors, $\mathrm{k}_1, \mathrm{k}_2$ any scalars, we have

$$\left.\begin{aligned} |(\mathrm{k}_1 v_1 + \mathrm{k}_2 v_2) &= \mathrm{k}_1 |v_1 + \mathrm{k}_2 |v_2, \\ |(v_1 + v_2) = |v_1 + |v_2, \quad |(v_1 - v_2) &= |v_1 - |v_2. \end{aligned}\right\} \tag{2}$$

With u as above,

$$||u = -\mathrm{x}_2 |i + \mathrm{x}_1 |j = -\mathrm{x}_1 i - \mathrm{x}_2 j = -u.$$

3. If u, v are vectors, so are $|u$ and $|v$; if a, b be points, we write the product of the vectors $a - b$ and $|u$ in the form $[(a-b)\,|u]$, and we write the product of the point a and the vector $|u$ in the form $[a\,|u]$. Then, by § 6,

$$[(a-b)\,|u] = [a\,|u] - [b\,|u], \quad [a\,|(u-v)] = [a\,|u] - [a\,|v],$$
$$[a\,|(\mathrm{k}_1 u + \mathrm{k}_2 v)] = \mathrm{k}_1 [a\,|u] + \mathrm{k}_2 [a\,|v].$$

We have not defined $|p$ when p is a point, and so *we cannot say* $|(p-q) = |p - |q$, when p, q are points, since the right-hand side is not defined.

4. *Inner products of vectors.* With vectors i, j as above, $[ij]$ is the unit bivector; in any equation which involves vectors only, it may be replaced by unity.

If u, v be vectors, the outer product $[u.\,|v]$ of the vectors u and $|v$, which we shall write $[u\,|v]$, may be regarded as arising from the vectors u and v; we say it is the '*inner product*' of u

* Thus taking the supplement is analogous to multiplying by $\sqrt{-1}$ in the complex plane. For many interesting and ingenious applications of the complex variable to geometry see F. Morley and F. V. Morley, *Inversive Geometry* (1933).

and v. Since it is the outer product of vectors u and $|v$, it is a bivector, but as we shall take $[ij]=1$, it will be found that we can treat $[u|v]$ as a scalar, in all equations in which it enters.

5. The laws on outer products shew that the inner multiplication of vectors is distributive over addition:

$$[u|(v+w)] = [u|v]+[u|w];$$

also $[i|i] = [ij] = 1$, $[j|j] = -[ji] = 1$, $[i|j] = [j|i] = 0$.

6. Inner multiplication of vectors is commutative,

$$[u|v] = [v|u].$$

For, if $u = u_1 i + u_2 j$, $v = v_1 i + v_2 j$, then

$$\begin{aligned}[u|v] &= [(u_1 i+u_2 j)|(v_1 i+v_2 j)] \\ &= [(u_1 i+u_2 j)(v_1 j-v_2 i)] = u_1 v_1 + u_2 v_2.\end{aligned}$$

Thus this type of multiplication has two of the important properties of ordinary algebraic multiplication. But it has not all the properties of the latter, since $[i|j]=0$, although neither i nor j is zero.

7. If u, v denote the magnitudes of u, v, and θ, ϕ their inclinations to i, then

$$u = u\cos\theta.i + u\sin\theta.j, \quad v = v\cos\phi.i + v\sin\phi.j,$$
$$[u|v] = uv(\cos\theta\cos\phi+\sin\theta\sin\phi) = uv\cos(\phi-\theta).$$

The inner product of u, v is the product of the magnitudes of u, v and the cosine of the angle between them.

The magnitude of a vector is taken always as positive. If $u=-u'$, we have $[u|v] = -[u'|v]$, agreeing with the fact that the angles $\widehat{u,v}$ and $\widehat{u',v}$ are supplementary.

Compare the inner product with the outer (§ 6·14):

$$[uv] = uv(\cos\theta\sin\phi-\sin\theta\cos\phi)[ij] = uv\sin(\phi-\theta).$$

The inner product vanishes when the vectors are perpendicular, the outer when they are parallel; and only in these cases, if the magnitude of neither vector is zero. Also $[|u.|v] = [uv]$.

8. We write u^2 for $[u|u]$; then u^2 is the square of the magnitude of u. Thus if a, b are points, then $(a-b)^2$ means

$$[(a-b)|(a-b)].$$

If u, v are vectors, then

$$(u-v)^2 = [(u-v)\,|\,(u-v)] = [u\,|\,(u-v)] - [v\,|\,(u-v)]$$
$$= u^2 - [u\,|\,v] - [v\,|\,u] + v^2 = u^2 - 2[u\,|\,v] + v^2.$$

Thus if a, b, c are points, we have the extension of Pythagoras' Theorem:

$$(b-c)^2 = (a-b)^2 + (a-c)^2 - 2[(a-b)\,|\,(a-c)].$$

§ 8. *Identities in vectors.*

Since inner multiplication is distributive and commutative, all algebraic identities which involve only two factors in each term are true here.

1. If a, b, c, d are vectors, then

$$[(a-b)\,|\,(c-d)] + [(b-c)\,|\,(a-d)] + [(c-a)\,|\,(b-d)] = 0.$$

As this involves only the differences of vectors, it will also be true when a, b, c, d are points; for take any fixed point p, then $a-b = (p-b)-(p-a)$, and we can apply the formula to the vectors $p-a$, $p-b$, $p-c$, $p-d$.

To interpret the formula, take a, b, c, d as the vertices of a quadrilateral; then, if bars denote lengths,

$$\overline{ab}.\overline{cd}\cos(ab, cd) + \overline{bc}.\overline{ad}\cos(bc, ad) + \overline{ca}.\overline{bd}\cos(ca, bd) = 0.$$

(Carnot.)

In particular, if ab is perpendicular to cd, and bc to ad, so is ca to bd.

2. If $m_1, m_2, \ldots, m_n$ be scalars, and $a_1, a_2, \ldots, a_n, p$ be points, and

$$g = (m_1 a_1 + m_2 a_2 + \ldots + m_n a_n)/m,$$

where

$$m = m_1 + m_2 + \ldots + m_n \neq 0,$$

then

$$m_1(p-a_1)^2 + \ldots + m_n(p-a_n)^2$$
$$= m_1(g-a_1)^2 + \ldots + m_n(g-a_n)^2 + m(p-g)^2. \quad (1)$$

This is the theorem of parallel axes for moments of inertia of coplanar particles. The formula is easily verified when p, g, a_1, $\ldots$, a_n are vectors, and as it only involves their differences, it is true when they are points.

Since $m \neq 0$, the right-hand side of (1) is not independent of p, and therefore the locus of a point p which moves so that the left-hand side is constant is a circle, centre g.

If $m=0$, but $\Sigma m_i a_i \neq 0$, the locus is a line, for then

$$\sum_{i=1}^{n} m_i(p-a_i)^2 = \sum_{i=1}^{n-1} m_i((p-a_i)^2-(p-a_n)^2)$$
$$= 2\sum_{i=1}^{n-1} m_i[(p-\tfrac{1}{2}(a_i+a_n))\,|\,(a_n-a_i)].$$

If $\Sigma m_i a_i = 0$, and hence $\Sigma m_i = 0$, then $\Sigma m_i(p-a_i)^2$ is independent of p.

3. $2[(a-b)\,|\,(c-d)] = (a-d)^2+(b-c)^2-(a-c)^2-(b-d)^2$,

where a, b, c, d are points or vectors. Hence, for points,

$$2\overline{ab}.\overline{cd}\cos(ab, cd) = \overline{ad}^2+\overline{bc}^2-\overline{ac}^2-\overline{bd}^2.$$

4. $((a-d)+(b-c))^2-((a-b)+(c-d))^2 = 4[(a-d)\,|\,(b-c)]$,

where a, b, c, d are points or vectors. Hence, if a, b, c, d be a tetrad of orthocentric points, then

$$(a+b-c-d)^2 = (a-b+c-d)^2 = (-a+b+c-d)^2;$$

so that, if $n = \frac{1}{4}(a+b+c+d)$, then

$$(n-\tfrac{1}{2}(a+d))^2 = (n-\tfrac{1}{2}(b+d))^2$$
$$= (n-\tfrac{1}{2}(c+d))^2 = (n-\tfrac{1}{2}(b+c))^2 = \ldots,$$

and the mid-points of the six joins of a, b, c, d lie on a circle, centre n, the nine-point circle.

Since $(a-d)+(b-c) = 2[\frac{1}{2}(a+b)-\frac{1}{2}(c+d)]$, the first formula gives: if the joins of the mid-points of the opposite sides of a quadrilateral are equal, the diagonals are perpendicular, and conversely.

5. $[(a+b-c-d)\,|\,(b+d-a-c)] = (b-c)^2-(a-d)^2$.

If the joins of the mid-points of opposite sides of a quadrilateral are perpendicular, the diagonals are equal, and conversely.

It is easy to give the general interpretations of 4, 5. Those of 6, ..., 9 are clear.

6. If $e = \frac{1}{2}(a+c)$, $f = \frac{1}{2}(b+d)$, then

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2$$
$$= (a-c)^2+(b-d)^2+4(e-f)^2.$$

7. $(a+b+c+d)^2+(a+b-c-d)^2$
$+(a-b+c-d)^2+(-a+b+c-d)^2$
$=(-a+b+c+d)^2+(a-b+c+d)^2$
$+(a+b-c+d)^2+(a+b+c-d)^2$
$=4(a^2+b^2+c^2+d^2).$

8. $(a-b)^2+(b-c)^2+(c-a)^2+(a-d)^2+(b-d)^2+(c-d)^2$
$=(-a+b+c-d)^2+(a-b+c-d)^2+(a+b-c-d)^2.$

9. $4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2$
$=(a-b)^2+(b-c)^2+(c-a)^2$
$+(a-d)^2+(b-d)^2+(c-d)^2.$

The last has an interpretation only for vectors.

10. If a, b, c be collinear points, then

$$(a-p)^2(b-c)+(b-p)^2(c-a)+(c-p)^2(a-b)$$

is a vector along ab which is independent of the position of p. (Stewart.)

11. With the notation of 2, put $p=a_1, a_2, \ldots, a_n$ in turn in equation (1), and add the resulting equations after multiplication by $m_1, m_2, \ldots, m_n$ respectively, then

$$m\sum_{i=1}^{n} m_i(g-a_i)^2 = \sum_{r,s=1}^{n} m_r m_s(a_r-a_s)^2, \quad (r<s). \qquad (2)$$

Also, by addition of the resulting equations (1):

$$\sum_{r,s}(m_r+m_s)(a_r-a_s)^2 = n\sum_t m_t(g-a_t)^2+m\sum_t(g-a_t)^2, \quad (r<s).$$

This with (2) gives

$$m^2\sum_r(g-a_r)^2 = m\sum_{r,s}(m_r+m_s)(a_r-a_s)^2$$
$$-n\sum_{r,s} m_r m_s(a_r-a_s)^2, \quad (r<s). \quad \text{(Lagrange.)}$$

In particular, for n points of unit weight,

$$n\sum_r(g-a_r)^2 = \sum_{r,s}(a_r-a_s)^2, \quad (r<s).$$

12. If a, b, c, d be points, then (§ 5·6)

$$[bcd]\,a+[cad]\,b+[abd]\,c-[abc]\,d=0.$$

The sum of the weights $[bcd]$, $[cad]$, $[abd]$, $-[abc]$ is zero. Hence 11 (2) gives

$$[dca]\,[dab]\,(b-c)^2+[dab]\,[dbc]\,(c-a)^2+[dbc]\,[dca]\,(a-b)^2 = [abc]\,([dbc]\,(a-d)^2+[dca]\,(b-d)^2+[dab]\,(c-d)^2).$$

13. $$[bcd]\,(a-p)^2+[cad]\,(b-p)^2+[abd]\,(c-p)^2 = [bcd]\,(a-d)^2+[cad]\,(b-d)^2 + [abd]\,(c-d)^2+[abc]\,(d-p)^2.$$

14. If u is a unit vector, i.e. if $u^2=1$, and v is any vector, then v can be resolved into a vector $[u\,|v]\,u$ along u, and a vector $[uv]\,|u$ perpendicular to u.

For by 6·16 (1), $[vw]\,u+[wu]\,v+[uv]\,w=0$.

Put $|u$ for w, then

$$[uv]\,|u=[u\,|u]\,v-[v\,|u]\,u=v-[v\,|u]\,u;$$

for since u, $|u$ are both vectors, we have

$$[|u.u]=-[u.\,|u]=-[u\,|u].$$

Since $[uv]$ may be regarded as a scalar, we may write $[uv\,|u]$ for $[uv]\,|u$. Thence

$$v=[v\,|u]\,u+[uv\,|u], \quad \text{if } u^2=1.$$

15. If u, v, w, r be any four (coplanar) vectors, then similarly

$$[u\,|v]\,w+[uv\,|w]=[u\,|w]\,v+[uw|v],$$

$$[u\,|w]\,[v\,|r]-[uw]\,[vr]=[u\,|r]\,[v\,|w]-[ur]\,[vw], \quad \text{Cf. 6·17 (2)}$$

$$(u^2+v^2)\,(w^2+r^2)=([u\,|w]-[v\,|r])^2+([uw]+[vr])^2 + ([u\,|r]-[v\,|w])^2+([ur]+[vw])^2.$$

If in the last we put

$$u=ai+bj, \quad v=ci+dj, \quad w=a_1i+b_1j, \quad r=c_1i+d_1j,$$

where a, b, c, d are scalars, we have a formula which expresses $(a^2+b^2+c^2+d^2)\,(a_1^2+b_1^2+c_1^2+d_1^2)$ as the sum of four squares.

Examples. 65. The angle in a semicircle is a right angle.

For $$[(p-a)\,|(p-b)]=(p-\tfrac{1}{2}(a+b))^2-(\tfrac{1}{2}(a-b))^2.$$

66. If $[u\,|(u_1+u_2)]=[u_1\,|u_2]\pm x_1x_2$, where u, u_1, u_2 are vectors and x_1, x_2 the magnitudes of u_1, u_2, then u bisects the angle between u_1 and u_2. If all the vectors proceed from one point, their ends are collinear. The internal bisector corresponds to the upper sign.

67. A parallel to the base bc of a triangle abc cuts ac in e and ab in f; then the radical axis of the circles on diameters be, cf is an altitude of the triangle.

For, if p be either cut of the circles, and

$$(c-e) = \mathrm{k}(c-a), \quad (b-f) = \mathrm{k}(b-a),$$

then $$[(p-b)\,|\,(p-e)] = 0, \quad [(p-c)\,|\,(p-f)] = 0.$$

Hence $$[(p-b)\,|\,(p-c)] + [(p-b)\,|\,(c-e)] = 0,$$

$$[(p-c)\,|\,(p-b)] + [(p-c)\,|\,(b-f)] = 0.$$

Hence $$[(p-b)\,|\,(c-e)] = [(p-c)\,|\,(b-f)],$$

$$[(p-b)\,|\,(c-a)] + [(p-c)\,|\,(a-b)] = 0.$$

Hence, by 8·1, $$[(p-a)\,|\,(b-c)] = 0.$$

68. *Bobillier's Theorem.* Ex. 17, p. 10. We have

$$(1+\mathrm{k})\,p = a + \mathrm{k}d, \quad (1+\mathrm{k})\,q = b + \mathrm{k}c$$

for some scalar k. Let u, v be unit vectors along ab and dc, then

$$\begin{aligned}(1+\mathrm{k})\,[(p-q)\,(u-v)] &= [(a-b)\,(u-v)] + \mathrm{k}[(d-c)\,(u-v)] \\ &= -[(a-b)\,v] + \mathrm{k}[(d-c)\,u].\end{aligned}$$

Now pq will be equally inclined to ab and dc if the first expression vanishes, that is, by the last expression, if $\overline{ab} = \mathrm{k}.\overline{dc}$.

69. If g, h, s be the centroid, orthocentre and circumcentre of a triangle abc, then g, h, s are collinear.

(i) $(b-s)^2 = (c-s)^2$, (ii) $[(a-h)\,|\,(b-c)] = 0$, (iii) $3g = a+b+c$.

From (i), $[(b+c-2s)\,|\,(b-c)] = 0$. Hence (ii), (iii) give:

$$[(a+b+c-h-2s)\,|\,(b-c)] = 0, \quad [(3g-h-2s)\,|\,(b-c)] = 0.$$

Similarly $$[(3g-h-2s)\,|\,(c-a)] = 0, \quad [(3g-h-2s)\,|\,(a-b)] = 0.$$

But as a non-zero vector cannot be perpendicular to three (or even to two) non-parallel (coplanar) non-zero vectors, we must have

$$3g = h + 2s.$$

The join of s, g, h is the '*Euler line*' of the triangle.

70. *If four lines be such that the Euler line of the triangle formed by the first three is parallel to the fourth, the same is true for any three of the lines and the other line.* (Zeeman).

For let u, v, w be the vectors $b-c$, $c-a$, $a-b$; then $u+v+w = 0$.

Now $$[w\,|\,u]\,[u\,|\,v]\,a + [u\,|\,v]\,[v\,|\,w]\,b + [v\,|\,w]\,[w\,|\,u]\,c$$

is a weighted point at h, say. We first shew that h is the orthocentre of triangle abc.

For $h-a \equiv [u|v]\,[v|w]\,(b-a)+[v|w]\,[w|u]\,(c-a)$,

$$[(h-a)\,|(b-c)] \equiv -[u|v]\,[v|w]\,[w|u]+[v|w]\,[w|u]\,[v|u] = 0.$$

Similarly $[(h-b)\,|(c-a)] = 0$, $[(h-c)\,|(a-b)] = 0$.

Let l be the vector parallel to the Euler line of the triangle abc; consider the rotor

$$L = [v|w]\,[u|l]\,[bc]+[w|u]\,[v|l]\,[ca]+[u|v]\,[w|l]\,[ab].$$

Then, using the value of h above,

$$[Lh] \equiv ([u|l]+[v|l]+[w|l])\,[abc] = [(u+v+w)|l]\,[abc] = 0.$$

Thus L goes through h, and it will be the Euler line of abc, if it also goes through $a+b+c$, that is, if

$$[v|w]\,[u|l]+[w|u]\,[v|l]+[u|v]\,[w|l] = 0.$$

Now this condition is symmetric in u, v, w, l; for instance, it can be written

$$[u|l]\,[v|w]+[l|v]\,[u|w]+[v|u]\,[l|w] = 0.$$

Hence the statement.

71. *Circles of Apollonius.* If p moves so that the distances of p from fixed points a, b are in a fixed ratio, then p describes a line or circle.

For, if $s = \frac{1}{2}(a+b)$, $a-s = u$, $p-s = v$, $(p-b)^2 = \mathrm{k}(p-a)^2$, k scalar, then

$$(u+v)^2 = \mathrm{k}(v-u)^2, \quad (v+\mathrm{l}u)^2 = (\mathrm{l}^2-1)\,u^2 = \text{constant},$$

where $\mathrm{l} = (1+\mathrm{k})\,(1-\mathrm{k})^{-1}$. Hence $(p-(s-\mathrm{l}u))^2$ is constant.

72. *The circles whose diameters are the diagonals of a complete quadrilateral are coaxal.*

For let the sides ab, cd of the quadrilateral $abdc$ cut in e, and let ac, bd cut in f. There are scalars x, y, z, w such that

$$\mathrm{x}a+\mathrm{y}b+zc+\mathrm{w}d = 0, \quad \mathrm{x}+\mathrm{y}+z+\mathrm{w} = 0.$$

Hence $\mathrm{x}a+\mathrm{y}b = -zc-\mathrm{w}d$ is a weighted point on ab and cd, that is, at e. Hence $\mathrm{x}a+\mathrm{y}b = (\mathrm{x}+\mathrm{y})\,e$. Similarly $\mathrm{x}a+zc = (\mathrm{x}+z)\,f$.

Let s be either cut of the circles on diameters bc, ad, and let $a_1 = a-s$, $b_1 = b-s$, and so on. Then

$$[b_1|c_1] = 0, \quad [a_1|d_1] = 0, \quad \mathrm{x}a_1+\mathrm{y}b_1+zc_1+\mathrm{w}d_1 = 0.$$

$$[e_1|f_1] \equiv [(\mathrm{x}a_1+\mathrm{y}b_1)\,|(\mathrm{x}a_1+zc_1)] = [\mathrm{x}a_1\,|(\mathrm{x}a_1+\mathrm{y}b_1+zc_1)]$$
$$= -[\mathrm{x}a_1\,|\mathrm{w}d_1] = 0.$$

Cor. If through a point d in a plane abc perpendiculars are drawn to da, db, dc cutting bc, ca, ab in p, q, r, then p, q, r are collinear.

73. If a, b, c, d be the vectors represented by the sides of a quadrilateral, taken in order, and $[a|c]=[b|d]$, then the diagonals are perpendicular, and conversely.

For $$[(a+b)\,|(a+d)] = [a\,|(a+b+d)]+[b|d]$$
$$=[a\,|(a+b+c+d)] = 0.$$

Similarities (74–82)

74. Directly similar triangles are described on the sides of a polygon, taken in order; then the mean centre of the new vertices coincides with that of the vertices of the polygon. (Laisant.)

For, if a_1, a_2, ..., a_n be vertices of the original polygon, the new vertices are

$$b_1 = \mathrm{x}a_1+\mathrm{y}a_2+\mathrm{z}\,|(a_2-a_1), \quad b_2 = \mathrm{x}a_2+\mathrm{y}a_3+\mathrm{z}\,|(a_3-a_2), \quad \ldots,$$
$$b_n = \mathrm{x}a_n+\mathrm{y}a_1+\mathrm{z}\,|(a_1-a_n),$$

where x, y, z are scalars, and $\mathrm{x}+\mathrm{y}=1$.

Hence $$b_1+b_2+\ldots+b_n = (\mathrm{x}+\mathrm{y})\,(a_1+a_2+\ldots+a_n)$$
$$=a_1+a_2+\ldots+a_n.$$

Cor. $b_1-a_1 = \mathrm{x}a_1+\mathrm{y}a_2-a_1+\mathrm{z}\,|(a_2-a_1) = \mathrm{y}(a_2-a_1)+\mathrm{z}\,|(a_2-a_1)$.

We can write the right side as $(a_2-a_1)\,\mathfrak{A}$, where $\mathfrak{A}$ is an operator, which, acting on a vector u, produces $\mathrm{y}u+\mathrm{z}\,|u$, to be denoted by $u\mathfrak{A}$.

Clearly $(u+v)\,\mathfrak{A} = u\mathfrak{A}+v\mathfrak{A}$, that is, the operator is distributive over addition.

An operator like $\mathfrak{A}$, which turns a vector through a fixed angle, and multiplies its magnitude by a fixed scalar, is called a '*similarity*'.

75. If *pbc*, *qca*, *rab* be directly similar triangles described externally on the sides of triangle *abc*, then $[ap]+[bq]+[cr]$ is a bivector (or zero).

For $$p = \mathrm{x}b+\mathrm{y}c+\mathrm{z}\,|(c-b), \quad q = \mathrm{x}c+\mathrm{y}a+\mathrm{z}\,|(a-c),$$
$$r = \mathrm{x}a+\mathrm{y}b+\mathrm{z}\,|(b-a),$$

where x, y, z are scalars. Take any point s, and let $a_1 = a-s$, $b_1 = b-s$, and so on, then

$$p = (\mathrm{x}+\mathrm{y})\,s+\mathrm{x}b_1+\mathrm{y}c_1+\mathrm{z}\,|c_1-\mathrm{z}\,|b_1, \quad [sa] = [sa_1] = -[a_1 s].$$

Expand $[ap] = [(s+a_1)\,p]$, and similar expressions, then

$$[ap]+[bq]+[cr] = (\mathrm{x}+\mathrm{y})\,[sa_1+sb_1+sc_1+a_1s+b_1s+c_1s]$$
$$+(\mathrm{x}-\mathrm{y})\,(b_1c_1+c_1a_1+a_1b_1)$$
$$=(\mathrm{x}-\mathrm{y})\,[b_1c_1+c_1a_1+a_1b_1].$$

76. If $u = b-c$, $v = c-a$, $w = a-b$, then $v^2+[v|w]+w^2$ equals the expressions obtained by cycling u, v, w. Hence the centres of the equilateral triangles described externally on the sides of any triangle are the vertices of an equilateral triangle.

77. If the diagonals of *abcd* are equal and perpendicular, the vertices of directly similar triangles described on the sides of the quadrilateral form a quadrilateral of the same kind. (van Aubel.)

For $a-c = |(b-d)$, $|(a-c) = d-b$. The vertices of the new quadrilateral are $a+(a-b)\,\mathfrak{A}$, $b+(b-c)\,\mathfrak{A}$, and so on, where $\mathfrak{A}$ is a similarity. The joins of opposite vertices are parallel to the vectors

$$u = a-c+(a-b-c+d)\,\mathfrak{A}, \quad v = b-d+(b-c-d+a)\,\mathfrak{A}.$$

Thence

$$|v = |(b-d)+|\{(b-c-d+a)\,\mathfrak{A}\} = (a-c)+\{|(b-d)+|(a-c)\}\,\mathfrak{A}$$
$$= a-c+(a-b-c+d)\,\mathfrak{A} = u.$$

We have assumed that, for any vector w, $(|w)\,\mathfrak{A} = |(w\mathfrak{A})$. This is easily seen to be true. (See § 10, p. 51.)

78. *If p, q, r, s be the centres of squares described externally on the sides of any quadrilateral abcd, then the intervals pr, qs are perpendicular and equal in length.* (van Aubel.)

For $p = \frac{1}{2}(a+b)+\frac{1}{2}|(a-b)$, and so on.* Hence, using $||v = -v$,

$$p-r = \tfrac{1}{2}(a+b-c-d)+\tfrac{1}{2}|(a-b-c+d),$$
$$q-s = \tfrac{1}{2}(b+c-d-a)+\tfrac{1}{2}|(b-c-d+a) = |(p-r).$$

79. *If d, e, f be the centres of squares described externally on the sides of triangle abc, then the mid-points of the sides of abc are the centres of squares described internally on the sides of def.* (Neuberg.)

For $\quad d = \frac{1}{2}(a+b)+\frac{1}{2}|(a-b), \quad e = \frac{1}{2}(b+c)+\frac{1}{2}|(b-c).$

Hence $\quad (d+e)-|(d-e) = a+c.$

80. If *abc* be a triangle and perpendiculars *ap*, *cq* be drawn to *ab*, *cb*, so that $\widehat{bap} = -\widehat{bcq}$, $ap:ab = cq:cb$, then *cp*, *aq* meet on the altitude of the triangle from *b*.

81. If $a_1b_1c_1$, $a_2b_2c_2$, ..., $a_nb_nc_n$ be directly similar triangles, and p, q, r be the mean centres of $a_1, \ldots, a_n$, of $b_1, \ldots, b_n$ and of $c_1, \ldots, c_n$, then triangle pqr is similar to $a_1b_1c_1$.

* We might of course have $-\frac{1}{2}|$ instead of $+\frac{1}{2}|$ throughout. One sign corresponds to externally, the other to internally described squares; which to which depends on sign of [*abc*].

82. If a and a_1 traverse similar systems, then $a_2 = p + (a - a_1)$ traverses a system similar to them. Consider the following proof.

If a_1, b_1 correspond to a, b, then

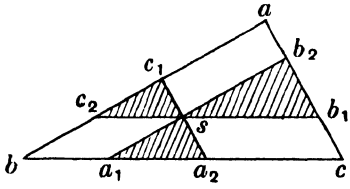
$$(a - a_1) - (b - b_1) = (a - b) - (a_1 - b_1) = (a - b)(\mathfrak{J} - \mathfrak{S}),$$

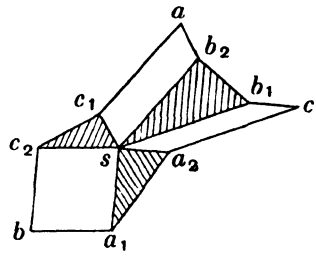
where $\mathfrak{J} - \mathfrak{S}$ is an operation on vectors w, such that

$$w(\mathfrak{J} - \mathfrak{S}) = w - w\mathfrak{S}.$$

If p moves on a line (or circle), and s is a fixed point, spq a triangle of fixed species, then q moves on a line (or circle).

83. *Fundamental theorem on three-bar-motion.* (Cayley and Clifford.) The figure is made up of rods jointed at the joins. In the first position a_1, s, b_2 lie on a straight line parallel to ab, on which c_1, c_2 lie. Similarly for the other rods. Then we prove that, during the deformation, abc remains similar to a fixed triangle.





For, let $\mathfrak{A}$ be a similarity which turns $c_1 - c_2$ into $c_1 - s$, then

$$\begin{aligned}
(b_2 - s)\,\mathfrak{A} &= b_2 - b_1, \quad (s - a_1)\,\mathfrak{A} = s - a_2, \\
(a - b)\,\mathfrak{A} &= (a - c_1 + c_1 - c_2 + c_2 - b)\,\mathfrak{A} \\
&= (a - c_1)\,\mathfrak{A} + (c_1 - c_2)\,\mathfrak{A} + (c_2 - b)\,\mathfrak{A} \\
&= (b_2 - s)\,\mathfrak{A} + (c_1 - c_2)\,\mathfrak{A} + (s - a_1)\,\mathfrak{A} \\
&= b_2 - b_1 + c_1 - s + s - a_2 \\
&= b_2 - b_1 + a - b_2 + b_1 - c = a - c.
\end{aligned}$$

Hence, if a, b be fixed in the second figure, then, though s can move, c will be fixed. Now, if ac_1, c_1c_2, c_2b be three bars jointed at c_1, c_2, and if a, b be fixed points, then any point, such as s, rigidly attached to c_1c_2 describes '*three-bar-motion*'. The above shews that the same motion is given by the bars ab_2, b_2b_1, b_1c, and by the bars ba_1, a_1a_2, a_2c.

Cor. The triangle ab_1c_2 remains similar to a fixed triangle.

84. Forces at the vertices of triangle abc perpendicular to, and proportional to, the opposite sides balance.

For let s be any point, and put* $a_1 = a - s$, $b_1 = b - s$, ..., and so on, then $b - c = b_1 - c_1$,

$$|(b-c) = |b_1 - |c_1,$$

$$a\,|(b_1 - c_1) + b\,|(c_1 - a_1) + a\,|(b_1 - c_1) = (s - a_1)\,|(b_1 - c_1) + \ldots + \ldots = 0.$$

Cor. Similarly for polygons of $2n+1$ sides, and, if we draw two perpendiculars from each vertex, for polygons of $2n$ sides.

85. If *al*, *bm*, *cn* be ordinates to some line *lmn*, then forces at *a*, *b*, *c* parallel and proportional to *mn*, *nl*, *lm* are equivalent to a couple of moment [*abc*].

86. Forces at the mid-points of the sides of a polygon perpendicular to, and proportional to them, balance. Forces at the vertices of a polygon perpendicular to, and proportional to the diagonals joining adjacent vertices, balance.

87. *Orthologic triangles. If the perpendiculars from the vertices of triangle abc on to the sides of triangle pqr meet (in w), then the perpendiculars from the vertices of triangle pqr on to the sides of triangle abc are concurrent.*

For take any origin s, and let $a - s = a_1$, and so on, then

$$[(a-w)\,|(q-r)] = 0, \quad [(b-w)\,|(r-p)] = 0, \quad [(c-w)\,|(p-q)] = 0.$$

By addition $\quad [a\,|(q-r)] + [b\,|(r-p)] + [c\,|(p-q)] = 0.$

Hence $\quad [a_1\,|(q_1 - r_1)] + [b_1\,|(r_1 - p_1)] + [c_1\,|(p_1 - q_1)] = 0,$

which can be written

$$[p_1\,|(b_1 - c_1)] + [q_1\,|(c_1 - a_1)] + [r_1\,|(a_1 - b_1)] = 0,$$

whence $\quad [p\,|(b-c)] + [q\,|(c-a)] + [r\,|(a-b)] = 0.$

Pairs of triangles in this relation are called '*orthologic*'. The way in which the vertices correspond is relevant.

88. If triangles $a_1a_2a_3$ and $b_1b_2b_3$ be orthologic, and c_1, c_2, c_3 divide a_1b_1, a_2b_2, a_3b_3 in the same ratio, then $c_1c_2c_3$ is orthologic to $a_1a_2a_3$ and $b_1b_2b_3$, and if d_1, d_2, d_3 divide a_1b_1, a_2b_2, a_3b_3 in the same ratio, then $c_1c_2c_3$ is orthologic to $d_1d_2d_3$. Two triangles inversely similar are orthologic. (Casey.)

89. If a', b', c', ..., q' be the feet of perpendiculars from p to the sides *ab*, *bc*, *cd*, ... of a polygon *abcd* ..., then the locus of p such that the area of $a'b'$ ... q' is constant is a circle, or line.

For $\quad [a'b'p] + [a'b'b] = 2[a'b'm]$, (where $2m = p + b$)

$$= 2[m(m-a')\,(m-b')]. \qquad \text{(i)}$$

* This method is of frequent use. We say we are taking s as origin.

Now $$(m-a')^2 = (m-b')^2 = (m-b)^2.$$

Hence the magnitude of (i) is $\frac{1}{2}(b-p)^2 \sin 2B$, where B is the angle at b. Hence

$$4 \text{ area } a'b'c' \ldots - 2 \text{ area } abc \ldots$$
$$= \tfrac{1}{2} \sin 2A.(a-p)^2 + \tfrac{1}{2} \sin 2B.(b-p)^2 + \ldots.$$

Hence the locus of p is a circle, or line. (§ 8·2, p. 33.)

90. If a, b, c, d be four points, the perpendicular from the mid-point of ab to cd cuts that from the mid-point of bc to ad in a point collinear with the centroid g of a, b, c, d and the circumcentre of abc.

For $$[(a+b)\,|\,(c-d)] - [(b+c)\,|\,(a-d)]$$
$$- [(a+b)\,|\,(a-b)] - [(b+c)\,|\,(b-c)] = 4[g\,|\,(c-a)].$$

91. Through a fixed point o inside a circle, centre c, radius a, two perpendicular lines are drawn to cut the circle in p, q; if the rectangle $oprq$ be completed, then the locus of r is a circle.

For let $p_1 = p - o$, and so on. Then, if u_1 be any variable vector, starting from o, the equation of the circle is

$$u_1^2 - 2[u_1\,|\,c_1] = a^2 - c_1^2.$$

Hence $$p_1^2 - 2[p_1\,|\,c_1] = q_1^2 - 2[q_1\,|\,c_1] = a^2 - c_1^2.$$

But $[p_1\,|\,q_1] = 0$. Hence

$$(p_1+q_1)^2 - 2[(p_1+q_1)\,|\,c_1] + c_1^2 = 2a^2 - c_1^2; \quad (r_1 - c_1)^2 = 2a^2 - c_1^2.$$

92. If p, q be any points on a circle, s a fixed point of the plane, and the parallelogram $psqr$ be completed, then the perpendicular from r to pq goes through a fixed point.

For take the centre o as origin, and let $s_1 = s - o$, and so on. Let s' be the reflection of s in o, then

$$s_1' = -s_1, \quad r_1 + s_1 = p_1 + q_1, \quad [(r_1+s_1)\,|\,(p_1-q_1)] = p_1^2 - q_1^2 = 0,$$
$$[(r_1 - s_1')\,|\,(p_1 - q_1)] = 0.$$

93. If p, q be points on a circle abc such that

$$(p-a)^2 + (p-b)^2 + (p-c)^2 = (q-a)^2 + (q-b)^2 + (q-c)^2,$$

then pq is perpendicular to the radius through the centroid of abc.

94. If $2(p-a)^2 + (p-b)^2 + (p-c)^2 = (a-b)^2 + (a-c)^2$, and p lies on bc, then p is either the mid-point of bc, or the foot of the altitude from a.

95. If h is the orthocentre of triangle abc, and u, v, w be vectors from h to a, b, c, then $[v|w]=[w|u]=[u|v]=\mathrm{k}$, say. If r be a variable vector from h, the equations of the circumcircle and nine-point circle are

$$r^2-[r|(u+v+w)]+2\mathrm{k}=0$$

and

$$r^2-\tfrac{1}{2}[r|(u+v+w)]+\tfrac{1}{2}\mathrm{k}=0.$$

If g is the centroid of the triangle, the circle on diameter gh, the circumcircle, and the nine-point circle are coaxal.

96. If o is the circumcentre of triangle abc, and u, v, w vectors from o to a, b, c, then $u+v+w$ is the vector of the orthocentre.

97. If a_1, a_2, a_3, a_4 be points on a circle, h_1, ..., h_4 orthocentres of $a_2a_3a_4$, ..., $a_1a_2a_3$, then the quadrilaterals $h_1 \ldots h_4$ and $a_1 \ldots a_4$ are congruent.

98. *Distance between points in areal coordinates.*

If

$$p=\mathrm{x}_1a+\mathrm{y}_1b+\mathrm{z}_1c, \quad q=\mathrm{x}_2a+\mathrm{y}_2b+\mathrm{z}_2c,$$

$$\mathrm{x}_1+\mathrm{y}_1+\mathrm{z}_1=\mathrm{x}_2+\mathrm{y}_2+\mathrm{z}_2=1,$$

then

$$p-q=(\mathrm{x}_1-\mathrm{x}_2)\,a+(\mathrm{y}_1-\mathrm{y}_2)\,b+(\mathrm{z}_1-\mathrm{z}_2)\,c$$

$$=(\mathrm{y}_1-\mathrm{y}_2)\,(b-a)+(\mathrm{z}_1-\mathrm{z}_2)\,(c-a),$$

$$(p-q)^2=(\mathrm{y}_1-\mathrm{y}_2)^2\,(b-a)^2+(\mathrm{z}_1-\mathrm{z}_2)^2\,(c-a)^2$$

$$+2(\mathrm{y}_1-\mathrm{y}_2)\,(\mathrm{z}_1-\mathrm{z}_2)\,[(b-a)\,|(c-a)]$$

$$=(\mathrm{y}_1-\mathrm{y}_2)^2\,\mathrm{c}^2+(\mathrm{z}_1-\mathrm{z}_2)^2\,\mathrm{b}^2+2(\mathrm{y}_1-\mathrm{y}_2)\,(\mathrm{z}_1-\mathrm{z}_2)\,\mathrm{bc}\cos\mathrm{A}$$

$$=-\mathrm{a}^2(\mathrm{y}_1-\mathrm{y}_2)\,(\mathrm{z}_1-\mathrm{z}_2)-\mathrm{b}^2(\mathrm{z}_1-\mathrm{z}_2)\,(\mathrm{x}_1-\mathrm{x}_2)$$

$$-\mathrm{c}^2(\mathrm{x}_1-\mathrm{x}_2)\,(\mathrm{y}_1-\mathrm{y}_2),$$

where a, b, c are the lengths of the sides of the triangle.

99. Forces whose vectors are lu, mv, nw act along the sides of triangle abc, in order, where u, v, w are unit vectors along the sides. To find the magnitude of the resultant R.

$$R^2=(lu+mv+nw)^2$$

$$=\mathrm{l}^2+\mathrm{m}^2+\mathrm{n}^2+2\mathrm{mn}\cos(\pi-\mathrm{A})$$

$$+2\mathrm{nl}\cos(\pi-\mathrm{B})+2\mathrm{lm}\cos(\pi-\mathrm{C}),$$

where A, B, C are the angles of the triangle.

In *trilinear* coordinates, the line of action of the resultant is

$$\mathrm{l}x+\mathrm{m}y+\mathrm{n}z=0. \qquad \text{(Cf. Ex. 46, p. 22.)}$$

If (x, y, z) be the trilinear coordinates of any point q, the moment of the resultant round q is $\mathrm{l}x_1+\mathrm{m}y_1+\mathrm{n}z_1$. This gives at once the formula for the distance of q from the line $\mathrm{l}x+\mathrm{m}y+\mathrm{n}z=0$.

100. If u, v be vectors, and θ the angle from u to v, then

$$[uv] = \bar{u}\bar{v}\sin\theta, \quad [u\,|\,v] = \bar{u}\bar{v}\cos\theta,$$

where $\bar{u}$, $\bar{v}$ are the magnitudes of u, v. Hence $\cot\theta = \dfrac{[u\,|\,v]}{[uv]}$.

If v is changed in sign, the right-hand side is not changed in value; neither is the left-hand side, since θ is increased by $\pm\pi$. Thus it is best to regard θ as the measure of the *cross* between the lines of the vectors. Interchange of u and v alters the signs of both sides.

101. If $a-o = u$, $b-o = v$, then the circle aob is described by the end of the vector r from o which satisfies

$$r^2 - [(u+v)\,|\,r] = \mathrm{k}[(b-a)\,r], \quad \mathrm{k} = \cot\measuredangle aob.$$

For, if p be on the circle, then $\measuredangle boa = \measuredangle bpa$, where $\measuredangle$ denotes the cross

$$\frac{[(u-r)\,|\,(v-r)]}{[(u-r)\,(v-r)]} = \mathrm{k} = \frac{[u\,|\,v]}{[uv]}.$$

102. If o, a be fixed points, and v is a variable unit vector from o, then

$a - [(a-o)\,|\,v]\,v$ lies on the circle on diameter oa.

103. (i) In a triangle abc with angles A, B, C,

$$|(c-b) = \cot \mathrm{C}.(a-b) + \cot \mathrm{B}.(a-c), \quad \cot \mathrm{A} = \frac{[(a-c)\,|\,(a-b)]}{[abc]}.$$

For, if ap is the altitude from a, and k, y, x be the lengths of ap, bp, cp, then

$$|(c-b) = (\mathrm{x}+\mathrm{y})\,\mathrm{k}^{-1}\,(a-p), \quad (\mathrm{x}+\mathrm{y})\,(a-p) = \mathrm{x}(a-b) + \mathrm{y}(a-c).$$

(ii) Hence, with areal coordinates x, y, z, the line whose tangential coordinates are $((\mathrm{m}-\mathrm{n})\cot \mathrm{A}$, $(\mathrm{n}-\mathrm{l})\cot \mathrm{B}$, $(\mathrm{l}-\mathrm{m})\cot \mathrm{C})$ is perpendicular to the line whose coordinates are (l, m, n). For the condition that q is on the former line is

$$[q((\mathrm{m}-\mathrm{n})\cot \mathrm{A}.(c-b) + (\mathrm{n}-\mathrm{l})\cot \mathrm{B}.(a-c) + (\mathrm{l}-\mathrm{m})\cot \mathrm{C}.(b-a))] = 0.$$

104. (i) If $b-a = u$, $c-a = v$, then $|(c-b) = \mathrm{k}_1 u + \mathrm{k}_2 v$, where

$$\mathrm{k}_1 = \frac{[u\,|\,v] - v^2}{[uv]}, \quad \mathrm{k}_2 = \frac{[u\,|\,v] - u^2}{[uv]}.$$

(ii) If $d = \mathrm{k}_1 b + \mathrm{k}_2 c$, $\mathrm{k}_1 + \mathrm{k}_2 = 1$, then denoting crosses by $\measuredangle$,

$$\cot\measuredangle adc = \mathrm{k}_1 \cot\measuredangle dac + \mathrm{k}_2 \cot\measuredangle dab = \mathrm{k}_1 \cot\measuredangle abc + \mathrm{k}_2 \cot\measuredangle acb.$$

For if $w = k_1 u + k_2 v = d - a$, then since $\measuredangle abc = \measuredangle(u, v-u)$,

$$\cot\measuredangle abc = \frac{[u\,|\,(v-u)]}{[uv]}, \quad \cot\measuredangle acb = \frac{[v\,|\,(u-v)]}{[vu]},$$

$$k_1 \cot\measuredangle abc + k_2 \cot\measuredangle acb = \frac{[w\,|\,(v-u)]}{[uv]} = \cot\measuredangle adc.$$

(iii) The join of $x_1 a + y_1 b + z_1 c$ and $x_2 a + y_2 b + z_2 c$ makes with bc an angle θ, where

$$\cot\theta = ((z_2 - z_1)\cot C - (y_2 - y_1)\cot B)\,(x_2 - x_1)^{-1}.$$

105. *Pothenot's problem.* If the sides of triangle abc subtend angles θ, ϕ, ψ at $p = xa + yb + zc$, $(x + y + z = 1)$, then, taking

$$[abc] = 1, \quad u = c - b, \quad v = a - c, \quad w = b - a,$$

we have $$x(\cot A - \cot\theta) = yzu^2 + zxv^2 + xyw^2.$$

This and the two analogous equations will determine x, y, z.

For $$[uv] = [vw] = [wu] = 1, \quad [bc] + [ca] + [ab] = 1,$$

$$\begin{aligned}[(p-b)\,(p-c)] &= [pb] - [pc] + [bc] \\ &= x[ab] - z[bc] - x[ac] - y[bc] + [bc] \\ &= x([bc] + [ca] + [ab]) = x,\end{aligned}$$

$$\begin{aligned}-[(p-b)\,|\,(p-c)] &= -[(zu - xw)\,|\,(xv - yu)] \\ &= yz.u^2 + zx.v^2 + xy.w^2 + x[w\,|\,v],\end{aligned}$$

since $u + v + w = 0$,

$$\cot A = -[w\,|\,v], \quad \cot\theta = -x^{-1}(yz.u^2 + zx.v^2 + xy.w^2) - [w\,|\,v].$$

106. Find the locus of points at which two given collinear intervals subtend (i) equal angles, (ii) supplementary angles.

107. Chords of a triangle abc are drawn through the isogonal of the centroid, $(b-c)^2.a + (c-a)^2.b + (a-b)^2.c$, $= s$, parallel to the sides. Their ends then lie on a circle. If p, q, r be the feet of the perpendiculars from s to the sides, then $s \equiv p + q + r$.

108. If $x(b-c) + y(c-a) + z(a-b)$ is antiparallel to bc, then

$$(b-a)^2\,(z-x) = (a-c)^2\,(y-x).$$

If $x_1 a + y_1 b + z_1 c$ and $x_2 a + y_2 b + z_2 c$ are isogonal conjugates, then

$$x_1 x_2/(b-c)^2 = y_1 y_2/(c-a)^2 = z_1 z_2/(a-b)^2.$$

109. *Brocard points.* If $p \equiv \frac{1}{(c-a)^2}.a + \frac{1}{(a-b)^2}.b + \frac{1}{(b-c)^2}.c$, then

$$\measuredangle pab = pbc = pca = \cot^{-1}(\cot A + \cot B + \cot C).$$

For $$\cot\measuredangle pab = [(b-a)\,|\,(p-a)] \div [abp].$$

Let $\quad K = 1/(c-a)^2 + 1/(a-b)^2 + 1/(b-c)^2,$

$$K[(b-a)\,|(p-a)] = \frac{1}{(a-b)^2}(b-a)^2 + \frac{1}{(b-c)^2}[(c-a)\,|(b-a)]$$

$$= \frac{1}{2(b-c)^2}((b-c)^2 + (c-a)^2 + (a-b)^2)$$

$$= \frac{1}{(b-c)^2}([(c-a)\,|(a-b)] + [(a-b)\,|(b-c)] + [(b-c)\,|(c-a)]),$$

$$K[abp] = \frac{1}{(b-c)^2}[abc].$$

Hence $\quad \cot \measuredangle pab = \cot A + \cot B + \cot C.$

110. *A generalisation of Simson's line.* Through a variable point p of plane abc parallels are drawn to oa, ob, oc, where o is a fixed point; these cut bc, ca, ab in d, e, f. To find the locus of p when d, e, f are collinear.

Take o as the origin of vectors, and let

$$a-o=u,\quad b-o=v,\quad c-o=w,\quad d=p+k_1u,\quad e=p+k_2v,\quad f=p+k_3w,$$

then $\quad [def] = [(p+k_1u)\,(p+k_2v)\,(p+k_3w)] = 0.$

But $[uvw]=0$, hence

$$k_1^{-1}[vw] + k_2^{-1}[wu] + k_3^{-1}[uv] = 0.$$

Now $\quad d = p + k_1u = xa + yb + zc + k_1(a-o), \quad (x+y+z=1),$

is independent of a.

Hence if $\quad o = x_1a + y_1b + z_1c, \quad (x_1+y_1+z_1=1),$

then $k_1 = x(x_1-1)^{-1} = -x(y_1+z_1)^{-1}$. But $[vw]=x_1$, ...

Hence the equation of the locus is

$$x_1(y_1+z_1)\,x^{-1} + y_1(z_1+x_1)\,y^{-1} + z_1(x_1+y_1)\,z^{-1} = 0.$$

In particular, if o be the orthocentre,

$$x_1 : y_1 : z_1 = \tan A : \tan B : \tan C,$$

and the equation becomes that of the circumcircle.

111. If a line through the centroid of triangle abc meets ab, ac in p, q, then $\overline{bp}/\overline{pa} + \overline{cq}/\overline{qa} = 1$.

112. If through any point p parallels to ag, bg, cg (where g is the centroid of abc) meet bc, ca, ab in d, e, f, then

$$\overline{pd}/\overline{ag} + \overline{pe}/\overline{bg} + \overline{pf}/\overline{cg} = 3/2.$$

113. If d, e, f be the mid-points of bc, ca, ab, and p is any point on ef, while bp, fd meet in q, and cp, de in r, then qr goes through the centroid of abc.

§ 9. *Identities connected with circles.*

1. For any points in a plane, (§ 5·6·8),

$$[mbc]\,(a-q)+[mca]\,(b-q)+[mab]\,(c-q)=[abc]\,(m-q).$$

Take $q=\frac{1}{2}(p+o)$, and multiply by $p-o$, inner multiplication, then

$$\begin{aligned}((a-p)^2-(a-o)^2)\,[mbc]+((b-p)^2-(b-o)^2)\,[mca]\\ +((c-p)^2-(c-o)^2)\,[mab]\\ =((m-p)^2-(m-o)^2)\,[abc].\end{aligned}$$

If a, b, c are on a circle, centre o, and $(a-o)^2=\mathrm{r}^2$, then

$$\begin{aligned}(a-p)^2\,[mbc]+(b-p)^2\,[mca]+(c-p)^2\,[mab]\\ =((m-p)^2-(m-o)^2+\mathrm{r}^2)\,[abc]\\ =((m-p)^2-\mathrm{k})\,[abc],\end{aligned}\qquad(1)$$

where k is the power of m in the circle abc.

If m is also on this circle, then

$$\begin{aligned}(a-p)^2\,[mbc]+(b-p)^2\,[mca]+(c-p)^2\,[mab]\\ =(m-p)^2\,[abc],\end{aligned}\qquad(2)$$

connecting four concyclic points a, b, c, m and any point p.

In (1), take $m=p$, then, if k is the power of m in the circle abc,

$$\begin{aligned}\mathrm{k}[abc]+(a-m)^2\,[mbc]\\ +(b-m)^2\,[mca]+(c-m)^2\,[mab]=0.\end{aligned}\qquad(3)$$

In particular, if m, a, b, c be concyclic,

$$(a-m)^2\,[mbc]+(b-m)^2\,[mca]+(c-m)^2\,[mab]=0.\qquad(4)$$

2. We can write this in terms of the vectors

$$a-m=u,\quad b-m=v,\quad c-m=w.$$

Since $[mbc]$ = moment of bivector $[(b-m)\,(c-m)]$, we have

$$u^2[vw]+v^2[wu]+w^2[uv]=0,\qquad(5)$$

so that (4) states that the moment of a certain sum of bivectors is zero.

3. We can also deduce (5) from § 6·16:

$$[vw]\,u+[wu]\,v+[uv]\,w=0.\qquad(6)$$

Let $m'-m=d$, where m' is the point on the circle opposite to m, then $[u\,|d]=u^2$, since

$$[u\,|(u-d)]=[(a-m)\,|(a-m')]=0.\quad\text{(Ex. 65, p. 36.)}$$

Multiply (6) by $|d$, and then (5) follows.

4. *Interpretations of* (4), (6). If we turn u, v, w in (6) through a right angle,

$$[vw]\,|u+[wu]\,|v+[uv]\,|w=0.$$

Hence, by (5),

$$[uv]\,w^2+[wu]\,v^2=-[vw]\,u^2=[wu]\,[u\,|v]+[uv]\,[u\,|w].$$

Hence

$$\frac{[v\,|(v-u)]}{[uv]}=\frac{[w\,|(w-u)]}{[uw]},\quad \frac{[v\,|(v-u)]}{[v(v-u)]}=\frac{[w\,|(w-u)]}{[w(w-u)]}.$$

Thus $\sphericalangle mba=\sphericalangle mca$, the fundamental angle property of a circle, expressed in terms of crosses.

Again, if m, a, b, c be in order on a circle, (4) gives

$$\overline{mb}\,.\,\overline{mc}\,.\sin bmc\,.\,\overline{ma}^2-\overline{ma}\,.\,\overline{mc}\,.\sin amc\,.\,\overline{mb}^2 + \overline{ma}\,.\,\overline{mb}\sin amb\,.\,\overline{mc}^2=0.$$

But $\sin bmc=\overline{bc}$: diameter, by the angle-property; hence if we divide the last formula by $\overline{ma}\,.\,\overline{mb}\,.\,\overline{mc}$, we obtain Ptolemy's Theorem.

5. From (4) and § 8·12, we have: if m, a, b, c be concyclic points,

$$[mca]\,[mab]\,.(b-c)^2+[mab]\,[mbc]\,.(c-a)^2 + [mbc]\,[mca]\,.(a-b)^2=0.$$

If $m=\mathrm{x}a+\mathrm{y}b+\mathrm{z}c$, this gives the usual equation for the circumcircle:

$$\mathrm{x}^{-1}(b-c)^2+\mathrm{y}^{-1}(c-a)^2+\mathrm{z}^{-1}(a-b)^2=0.$$

Examples. 114. If p is any point on the circle abc, then

$$(p-a)^2\sin 2\mathrm{A}+\ldots+\ldots$$

is constant.

115. If the vectors of a, b, c from any origin be u, v, w, then the vector of the circumcentre of abc is

$$\tfrac{1}{2}[(v^2-w^2)\,|u+(w^2-u^2)\,|v+(u^2-v^2)\,|w],$$

if $[abc]=1$.

The equation of the circumcircle is, if r is a variable vector,

$$r^2-(u^2[r(v-w)]+v^2[r(w-u)]+w^2[r(u-v)]) = u^2[vw]+v^2[wu]+w^2[uv].$$

116. If a_1, a_2, a_3, a_4 be four points in a plane and A_1 the power of a_1 for the circles $a_2a_3a_4$, and so on, then

$$A_1[a_2a_3a_4] = -A_2[a_1a_3a_4] = A_3[a_1a_2a_4] = -A_4[a_1a_2a_3].$$

117. With the same notation $A_1^{-1} + A_2^{-1} + A_3^{-1} + A_4^{-1} = 0$.

118. Circles with centres a, b, c, d cut a given circle orthogonally; P_1, P_2, P_3, P are the powers of any point for the circles, then

$$P_1[bcd] + P_2[cad] + P_3[abd] = P[abc].$$

§ 10. *Transformations of vectors.*

1. If $\mathfrak{S}$ is any operation which turns vectors into vectors, and u is any vector, we may denote by $\mathfrak{S}u$, or by $u\mathfrak{S}$, the vector produced by $\mathfrak{S}$. Thus the operation of taking the supplement, denoted by $|$, gives $|u$ when performed on u; the operation denoted by $\mathfrak{A}$ on p. 39 turned a vector through a constant angle and multiplied its magnitude by a scalar constant. Whether we write $\mathfrak{S}u$ or $u\mathfrak{S}$ is a matter of convenience. The latter is the form we shall usually prefer.

2. If $k.u\mathfrak{S} = ku.\mathfrak{S}$, for *all* vectors u and scalars k, and $u\mathfrak{S} + v\mathfrak{S} = (u+v)\mathfrak{S}$, for *all* vectors u, v, then $\mathfrak{S}$ is a '*linear operator*', or '*linear transformation*'.

3. If $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{R}$ be operators such that $u\mathfrak{S} + u\mathfrak{T} = u\mathfrak{R}$ for *all* vectors u, we say $\mathfrak{R}$ is the '*sum*' of $\mathfrak{S}$, $\mathfrak{T}$, and write $\mathfrak{S} + \mathfrak{T} = \mathfrak{R}$.

Clearly $\mathfrak{S} + \mathfrak{T} = \mathfrak{T} + \mathfrak{S}$, $(\mathfrak{S} + \mathfrak{T}) + \mathfrak{U} = \mathfrak{S} + (\mathfrak{T} + \mathfrak{U})$.

If for *all* vectors u, $(u\mathfrak{S})\,\mathfrak{T} = u\mathfrak{R}$, we say $\mathfrak{R}$ is the '*sequence-product*' of $\mathfrak{S}$, $\mathfrak{T}$ in that order, and write $\mathfrak{S}\mathfrak{T} = \mathfrak{R}$.

Clearly $(\mathfrak{S}\mathfrak{T})\,\mathfrak{U} = \mathfrak{S}(\mathfrak{T}\mathfrak{U})$, but it is not necessarily true that $\mathfrak{S}\mathfrak{T} = \mathfrak{T}\mathfrak{S}$.

We have $\mathfrak{S}(\mathfrak{T} + \mathfrak{U}) = \mathfrak{S}\mathfrak{T} + \mathfrak{S}\mathfrak{U}$, $(\mathfrak{T} + \mathfrak{U})\,\mathfrak{S} = \mathfrak{T}\mathfrak{S} + \mathfrak{U}\mathfrak{S}$.

4. A '*direct similarity*' $\mathfrak{S}$ is a transformation which turns figures into directly similar figures. It is always of form such that $u\mathfrak{S} = k_1 u + k_2 |u$, where k_1, k_2 are scalars. It is determined when the transform of one vector u is known.

If $\mathfrak{T}$ is another direct similarity, and $u\mathfrak{T} = l_1 u + l_2 |u$, l_1, l_2 scalars, then $u(\mathfrak{S} + \mathfrak{T}) = (k_1 + l_1)\,u + (k_2 + l_2)\,|u$.

Hence $\mathfrak{S} + \mathfrak{T}$ is a direct similarity.

The similarity $\mathfrak{S}$, which is such that $u\mathfrak{S} = u$ for all vectors u, is called the identity, and is denoted by $\mathfrak{J}$; then $\mathfrak{S}\mathfrak{J} = \mathfrak{J}\mathfrak{S} = \mathfrak{S}$ for all $\mathfrak{S}$.

If $\mathfrak{S}$, $\mathfrak{T}$ be as before, then

$$|(u\mathfrak{S}) = k_1|u + k_2||u = k_1|u - k_2 u,$$

and

$$|u.\mathfrak{S} = k_1|u + k_2||u = k_1|u - k_2 u.$$

Hence $|(u\mathfrak{S}) = |u.\mathfrak{S}$.

Also

$$\begin{aligned}(u\mathfrak{S})\,\mathfrak{T} &= (k_1 u + k_2|u)\,\mathfrak{T} = k_1 u\mathfrak{T} + k_2(|u)\,\mathfrak{T} = k_1 u\mathfrak{T} + k_2|(u\mathfrak{T}) \\ &= k_1 l_1 u + k_1 l_2|u + k_2|(l_1 u + l_2|u) \\ &= (k_1 l_1 - k_2 l_2)\,u + (k_1 l_2 + k_2 l_1)\,|u. \qquad (1)\end{aligned}$$

But, by definition, $(u\mathfrak{S})\,\mathfrak{T} = u\mathfrak{S}\mathfrak{T}$. Hence $\mathfrak{S}\mathfrak{T}$ is a similarity.

5. A set of operations which is such that the identity $\mathfrak{J}$ is in the set, and the product of any two operations is in the set, is called a '*group*', when

(*a*) For any operations $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$ of the set, $(\mathfrak{S}\mathfrak{T})\,\mathfrak{U} = \mathfrak{S}(\mathfrak{T}\mathfrak{U})$.

(*b*) If $\mathfrak{S}$ is any operation of the set, there is an operation $\mathfrak{S}^{-1}$ in the set, such that $\mathfrak{S}\mathfrak{S}^{-1} = \mathfrak{S}^{-1}\mathfrak{S} = \mathfrak{J}$.

$\mathfrak{S}^{-1}$ is called the '*inverse*' of $\mathfrak{S}$. If for all operations $\mathfrak{S}$, $\mathfrak{T}$ of the set, $\mathfrak{S}\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, the group is '*abelian*'.*

6. *The set of direct similarities forms an abelian group.* The necessary conditions for this all follow at once from (1), except (*b*), which we shew thus: if in (1) we assume that $\mathfrak{S}$ and hence k_1, k_2 are given, and we take l_1, l_2 to satisfy

$$l_1(k_1^2 + k_2^2) = k_1, \quad l_2(k_1^2 + k_2^2) = -k_2,$$

then

$$k_1 l_1 - k_2 l_2 = 1, \quad k_1 l_2 + k_2 l_1 = 0.$$

Hence $u\mathfrak{S}\mathfrak{T} = u$ for all u; that is, $\mathfrak{S}\mathfrak{T} = \mathfrak{J}$.

7. The operation of taking the supplement is clearly a similarity, and if we write $u\mathfrak{P}$ instead of $|u$, the equation $|u.\mathfrak{S} = |(u\mathfrak{S})$ of 4 becomes $(u\mathfrak{P})\,\mathfrak{S} = (u\mathfrak{S})\,\mathfrak{P}$; hence $\mathfrak{P}\mathfrak{S} = \mathfrak{S}\mathfrak{P}$, which is a special case of 6.

* Subsections 2, 3, 5 are clearly general, and can be taken to refer to transformations applied to any extensives.

8. If $\mathfrak{S}$ is a similarity, and $v = u\mathfrak{S}$, then $\mathfrak{S}$ is fully fixed by the vectors v, u, and the fact that $\mathfrak{S}$ transforms v into u. An obvious notation is to write

$$\mathfrak{S} = \frac{v}{u}.$$

Then $\dfrac{c-a}{b-a} = \dfrac{c'-a'}{b'-a'}$ means that abc, $a'b'c'$ are directly similar triangles; for this is the case if, and only if, both sides of the equation represent the same similarity.

9. If $\dfrac{c-a}{b-a} = \dfrac{c'-a'}{b'-a'}$, then $\dfrac{c-a}{c'-a'} = \dfrac{b-a}{b'-a'}$.

For, if $c-a = (b-a)\,\mathfrak{S}$, $(c'-a')\,\mathfrak{T} = c-a$, then

$$c'-a' = (b'-a')\,\mathfrak{S};$$

hence $\quad b-a = (c-a)\,\mathfrak{S}^{-1} = (c'-a')\,\mathfrak{T}\mathfrak{S}^{-1} = (b'-a')\,\mathfrak{S}\mathfrak{T}\mathfrak{S}^{-1}$.

But $\mathfrak{S}\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, by 6; hence

$$\mathfrak{S}\mathfrak{T}\mathfrak{S}^{-1} = \mathfrak{T}\mathfrak{S}\mathfrak{S}^{-1} = \mathfrak{T}, \quad (b-a) = (b'-a')\,\mathfrak{T}.$$

10. If u_1, v_1, u_2, v_2 be vectors, and $\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2}$, then each of these quotients equals $\dfrac{\mathrm{k}_1 u_1 + \mathrm{k}_2 u_2}{\mathrm{k}_1 v_1 + \mathrm{k}_2 v_2}$, $(\mathrm{k}_1, \mathrm{k}_2$ scalars).

For, if $u_1 = v_1\,\mathfrak{S}$, $u_2 = v_2\,\mathfrak{S}$, then

$$\mathrm{k}_1 u_1 + \mathrm{k}_2 u_2 = (\mathrm{k}_1 v_1 + \mathrm{k}_2 v_2)\,\mathfrak{S}.$$

11. The product of the similarities $\dfrac{u_1}{u_2}$ and $\dfrac{u_2}{u_3}$ is $\dfrac{u_1}{u_3}$. Hence $\dfrac{u_1}{u_2}.\dfrac{u_2}{u_3} = \dfrac{u_1}{u_3}$. Thus the symbols for similarities can be multiplied like ordinary scalars.

If $\dfrac{u}{v} = \mathfrak{S}$, then $\dfrac{v}{u} = \mathfrak{S}^{-1}$. If $u\mathfrak{A} = v\mathfrak{B}$, then $\dfrac{u}{v} = \mathfrak{B}\mathfrak{A}^{-1}$.

12. If a, b, a', b' be points, the point c such that the triangles cab, $ca'b'$ are directly similar is $c = a + (b-a)\,\mathfrak{S}$, where

$$\mathfrak{S} = \frac{a'-a}{(b-a)-(b'-a')}.$$

For, if there is a similarity $\mathfrak{S}$ such that

$$c-a=(b-a)\,\mathfrak{S}, \quad c-a'=(b'-a')\,\mathfrak{S},$$

then $\quad a'-a=((b-a)-(b'-a'))\,\mathfrak{S}, \quad c=a+(b-a)\,\mathfrak{S}.$

The first of these gives $\mathfrak{S}$. We also have

$$c=a+(a'-a)\,\mathfrak{S}', \quad \text{where} \quad \mathfrak{S}'=\frac{b-a}{(b-a)-(b'-a')}.$$

For, take $u=(b-a)-(b'-a')$, then $u\mathfrak{S}=a'-a$, $u\mathfrak{S}'=b-a$, and the formulae for c agree, if $(b-a)\,\mathfrak{S}=(a'-a)\,\mathfrak{S}'$, that is, if $u\mathfrak{S}'\mathfrak{S}=u\mathfrak{S}\mathfrak{S}'$, which is true by 6.

13. So far, we have considered similarity transformations on vectors only. If a, b, a', b' be given points in a plane, then the correspondence $\mathfrak{S}$ in which the fixed points a, b correspond to a', b' respectively, and p corresponds to p', is a '*direct similarity*' if, and only if, the triangles pab, $p'a'b'$ are directly similar. We write $a\mathfrak{S}=a'$, and so on.

Then, if q, r correspond to q', r', we have, using 10, 11,

$$\frac{a-p}{a-b}=\frac{a'-p'}{a'-b'}, \quad \frac{a-q}{a-b}=\frac{a'-q'}{a'-b'};$$

$$\frac{p-q}{a-b}=\frac{p'-q'}{a'-b'}, \quad \frac{p-r}{a-b}=\frac{p'-r'}{a'-b'}, \quad \frac{p-q}{p-r}=\frac{p'-q'}{p'-r'}.$$

Thence the triangles pqr, $p'q'r'$ are directly similar.

The point c, such that $c\mathfrak{S}=c$, is the '*self-corresponding*' point of $\mathfrak{S}$; it is given by 12.

14. *If a_1, a_2 describe directly similar systems, and $a_1a_2a_3$ be directly similar to a fixed triangle, then a_3 describes a similar system, and the three systems have in pairs the same self-corresponding point.* (Petersen-Schoute.)

For, let a_1, b_1, c_1 be points of the first system, a_2, b_2, c_2 corresponding points of the second system, and let $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ be directly similar triangles.

Let $\quad a_3-a_1=(a_2-a_1)\,\mathfrak{S}_1,$

then $\quad b_3-b_1=(b_2-b_1)\,\mathfrak{S}_1, \quad c_3-c_1=(c_2-c_1)\,\mathfrak{S}_1.$

Let $c_1-a_1=(b_1-a_1)\,\mathfrak{S}$, then $c_2-a_2=(b_2-a_2)\,\mathfrak{S}$,

by 13, since the first and second systems are similar.

Then

$$\begin{aligned} c_3-a_3 &= c_1+(c_2-c_1)\,\mathfrak{S}_1-a_1-(a_2-a_1)\,\mathfrak{S}_1 \\ &= c_1-a_1+(c_2-a_2)\,\mathfrak{S}_1-(c_1-a_1)\,\mathfrak{S}_1 \\ &= (b_1-a_1)\,\mathfrak{S}+(b_2-a_2)\,\mathfrak{S}\mathfrak{S}_1-(b_1-a_1)\,\mathfrak{S}\mathfrak{S}_1 \\ &= (b_1-a_1)\,(\mathfrak{S}-\mathfrak{S}\mathfrak{S}_1)+(b_2-a_2)\,\mathfrak{S}\mathfrak{S}_1, \\ b_3-a_3 &= (b_1-a_1)+(b_2-b_1)\,\mathfrak{S}_1-(a_2-a_1)\,\mathfrak{S}_1 \\ &= (b_1-a_1)\,(\mathfrak{I}-\mathfrak{S}_1)+(b_2-a_2)\,\mathfrak{S}_1. \end{aligned}$$

Hence $(b_3-a_3)\,\mathfrak{S}=c_3-a_3$, since $\mathfrak{S}\mathfrak{S}_1=\mathfrak{S}_1\mathfrak{S}$, by 6.

This proves the first part.

The self-corresponding point for the first two systems is

$$\begin{aligned} p &= a_1+(a_2-a_1)\,\frac{b_1-a_1}{(b_1-a_1)-(b_2-a_2)} \\ &= a_2+(a_1-a_2)\,\frac{b_2-a_2}{(b_2-a_2)-(b_1-a_1)}. \end{aligned}$$

The self-corresponding point for the first and third systems is

$$a_1+(a_3-a_1)\,\frac{b_1-a_1}{(b_1-a_1)-(b_3-a_3)}.$$

This is the same as p, by 11, since

$$\begin{aligned} &\frac{b_1-a_1}{(b_1-a_1)-(b_2-a_2)}\cdot\frac{(b_1-a_1)-(b_3-a_3)}{b_1-a_1} \\ &\qquad=\frac{(a_3-a_1)-(b_3-b_1)}{((a_3-a_1)-(b_3-b_1))\,\mathfrak{S}_1^{-1}}=\mathfrak{S}_1=\frac{a_3-a_1}{a_2-a_1}. \end{aligned}$$

The self-corresponding point for the second and third systems is

$$a_2+(a_3-a_2)\,\frac{b_2-a_2}{(b_2-a_2)-(b_3-a_3)},$$

and this can easily be shewn to equal the second expression for p.

§ 11. *The regressive product of two rotors in a plane.*

1. Let the lines ab, cd meet in p, then $[abp]=[cdp]=0$. Let $p=k_1c+k_2d$, then $k_1[abc]+k_2[abd]=0$, and this gives the ratio $k_1:k_2$, and hence the position of p. Note the geometrical interpretation of this result.

The '*regressive product*' of any two rotors $[ab]$, $[cd]$ is denoted by $[ab.cd]$, and is defined as follows:

$$[ab.cd]=[abd]\,c-[abc]\,d. \qquad (1)$$

Thus if the lines of the rotors meet, this product is their point of meeting, with a weight attached equal to

$[abd]-[abc]=$ twice 'area' of quadrilateral $acbd=[(b-a)(d-c)]$.

If the lines are parallel, then $[abd]=[abc]$, and the product is a vector.

Earlier outer products will be distinguished from this type of outer product by being called '*progressive*'. The dot on the left-hand side of (1) is the symbol for multiplication.

2. By § 5·6,

$$[abc]\,d=[bcd]\,a+[cad]\,b+[abd]\,c. \tag{2}$$

Hence, by (1), $$[ab.cd]=[acd]\,b-[bcd]\,a. \tag{3}$$

This and (1) are the fundamental formulae, and the only ones that need be remembered. From (1),

$$[cd.ab]=[cdb]\,a-[cda]\,b=[bcd]\,a-[acd]\,b.$$

Hence, by (3), $$[cd.ab]=-[ab.cd]. \tag{4}$$

Since $[ab]=-[ba]$, we have, by (1),

$$[ab.cd]=-[ba.cd]=[ba.dc]. \tag{5}$$

3. If A, B be rotors, their regressive product is $[A.B]$. We omit the dot when no ambiguity results, and write $[AB]$. If k is a scalar,

$$\left.\begin{aligned}[\mathrm{k}A.B]=[A.\mathrm{k}B]=\mathrm{k}[AB],\\ [AB]=-[BA],\quad [AA]=0.\end{aligned}\right\} \tag{6}$$

If A, B, C be rotors, then, by (1), (3),

$$\left.\begin{aligned}[(A+B)\,C]=[AC]+[BC],\\ [C(A+B)]=[CA]+[CB].\end{aligned}\right\} \tag{7}$$

4. An important special case of (1) is

$$[ab.ac]=[abc]\,a. \tag{8}$$

If we assume this and (6), and the formulae derived by cyclic permutation, for one set of non-collinear points, and further

that regressive multiplication is distributive over addition, we can deduce (1). For let $d = \mathrm{x}a + \mathrm{y}b + zc$. Then

$$[cd] = \mathrm{x}[ca] + \mathrm{y}[cb], \quad [abd] = z[abc],$$

$$\begin{aligned}[ab.cd] &= \mathrm{x}[ab.ca] + \mathrm{y}[ab.cb] = -\mathrm{x}[ab.ac] - \mathrm{y}[bc.ba] \\ &= -\mathrm{x}[abc]\, a - \mathrm{y}[abc]\, b = -[abc]\,(\mathrm{x}a + \mathrm{y}b) \\ &= [abc]\,(zc - d) = [abd]\, c - [abc]\, d.\end{aligned}$$

5. If we multiply (2), which is an equation between weighted points, by a,

$$\begin{aligned}[abc]\,[da] &= [cad]\,[ba] + [abd]\,[ca] \\ &= [dac]\,[ab] - [dab]\,[ac].\end{aligned}$$

Write $[ab] = L$, $[ac] = M$. Then, since by (8),

$$[LM] = [abc]\, a, \quad [d.LM] = [abc]\,[da],$$

we have
$$[d.LM] = [dM]\, L - [dL]\, M. \tag{9}$$

As two rotors whose lines meet can always be written in form $[ab]$, $[ac]$, therefore (9) holds for such rotors.

If $L = [ab]$, $M = [pq]$ be parallel rotors, then, by (1), $[LM]$ is a vector $[abp]\,(p - q)$; and $[d.LM] = [abp]\,[d(p-q)]$ is a rotor through d parallel to L and M and of magnitude $[abp]\,\overline{pq}$, where bars denote lengths.

We can shew (9) in this case also.

Let N be a rotor whose line cuts the lines of L and M, and let $P = M - N$. The lines of P, L cut.

Then $[LN] + [LP] = [LM]$, $[dN] + [dP] = [dM]$,

$$\begin{aligned}[d.LM] &= [d.LN] + [d.LP] \\ &= [dN.L] - [dL.N] + [dP.L] - [dL.P] \\ &= [dM.L] - [dL.M].\end{aligned}$$

6. Note that if a, b are points, and L a rotor, then $[ab.L]$ and $[a.bL]$ are not equal, for the first is a weighted point at the cut of $[ab]$ and L, and the second a weighted point at a, since $[bL]$ is scalar. Hence *the associative law for multiplication need not hold for products which involve both progressive and regressive multiplication.*

7. $$[bc.pa]+[ca.pb]+[ab.pc]=2[abc]\,p. \qquad (10)$$

This follows from $[bc.pa]=[abc]\,p-[pbc]\,a$, and similar formulae, and

$$[abc]\,p=[pbc]\,a+[pca]\,b+[pab]\,c.$$

8. $$[[ab.bc].ca]=[ab.[bc.ca]]=[abc]^2. \qquad (11)$$

For $[ab.bc]=[abc]\,b$, $[[ab.bc].ca]=[abc][b.ca]=[abc]^2$,

$$[bc.ca]=[bca]\,c=[abc]\,c, \quad [ab.[bc.ca]]=[abc]^2.$$

We can hence write $[ab.bc.ca]$ for each of these expressions.

9. If any of the letters in (1) represent vectors, we assume this equation as the definition of regressive multiplication in this case. For example, if L is a bivector, we can write (1) as $[L.cd]=[Ld]\,c-[Lc]\,d$. Then, since $[Lc]=[Ld]$, we find that $[L.cd]$ is a vector. If L, M are bivectors, $[LM]=0$. (Cf. § 14·1.)

10. If L, M, N be rotors or bivectors, then

$$[LM.N]=[L.MN].$$

A product which involves only regressive multiplication of extensives of step two is associative.

For, if the lines of L, M, N do not have a common point (finite or at infinity), this follows by (11); and if they have such a common point p, both products vanish, for then we can find points l, m, n such that $L=[pl]$, $M=[pm]$, $N=[pn]$. Then

$$[LM]=[plm]\,p, \quad [LM.N]=0,$$
$$[MN]=[pmn]\,p, \quad [L.MN]=0.$$

Conversely, if $[LMN]=0$, then the lines L, M, N have a common point; for if they met in three distinct non-collinear points, then (11) shews $[LMN]\neq 0$.

$$[LMN]=-[MLN]=[NLM]=\ldots. \qquad (12)$$

11. The 'dual' formulae to (8), (11) are: if A, B, C be extensives of step two,

$$\left.\begin{aligned}[AB.AC]&=[ABC]\,A,\\ [AB.BC.CA]&=[ABC]^2.\end{aligned}\right\} \qquad (13)$$

To prove these, let $A=[bc]$, $B=[ca]$, $C=[ab]$, then

$$[AB] = [bc.ca] = [abc]\, c,$$
$$[AC] = [bc.ab] = -[ab.bc] = -[abc]\, b,$$
$$[AB.AC] = [abc]^2\, [bc].$$

And, by (11), $$[ABC] = [abc]^2.$$

This proves the first one of (13), the second follows like (11). If A, B, C are concurrent, both sides of both formulae vanish.

12. We can now write (2) in the form

$$[ABC]\, d = [dA]\, [BC] + [dB]\, [CA] + [dC]\, [AB].$$

The dual to (2) is

$$[ABC]\, D = [DBC]\, A + [DCA]\, B + [DAB]\, C, \quad (14)$$

from which follows, if $A = [bc]$, and so on,

$$[abc]\, D = [Da]\, [bc] + [Db]\, [ca] + [Dc]\, [ab]. \quad (15)$$

With this compare § 5, Ex. 46. We can deduce (14) by taking $D = k_1 A + k_2 B + k_3 C$, as in § 5·6 for the case of points.

13. From (8) we were able to prove (1). Hence, from (13), we can deduce $$[AB.CD] = [ABD]\, C - [ABC]\, D \quad (16)$$

for any four rotors (or bivectors) in a plane. Thus each formula has its dual.

14. If a, b be points, $L=[cd]$, we can write (3) in the form

$$[ab.L] = [aL]\, b - [bL]\, a. \quad (17)$$

The dual is $$[AB.l] = [Al]\, B - [Bl]\, A. \quad \text{(Cf. (9).)} \quad (18)$$

If M is of step two, then, since $[ab]$ is of step two, we have, by (12), $$[[ab.L]\, M] = [ab.LM],$$

and, by (17), $$[ab.LM] = [aL]\, [bM] - [bL]\, [aM]. \quad (19)$$

This is self-dual.

15. From (15),

$$[abc]\, L = [aL]\, [bc] + [bL]\, [ca] + [cL]\, [ab],$$

$$\begin{aligned}[abc]\, [LMN] &= [aL]\, [bc.MN] + [bL]\, [ca.MN] + [cL]\, [ab.MN] \\ &= [aL]\, ([bM]\, [cN] - [bN]\, [cM]) + \ldots + \ldots \\ &= \begin{vmatrix} [aL], & [aM], & [aN] \\ [bL], & [bM], & [bN] \\ [cL], & [cM], & [cN] \end{vmatrix}. \end{aligned} \quad (20)$$

Examples. 119. If a, b be points, u, v vectors, and the lines au, bv cut in c, then

$$c = a + \frac{[(a-b)\,v]}{[vu]}.u = b + \frac{[(b-a)\,u]}{[uv]}.v.$$

For $[au.bv] = [auv]\,b - [aub]\,v = [auv]\,b + [a(b-a)\,u]\,v$,

of weight $[uv]$. Cf. § 6·6.

120. The centre of gravity g of the quadrilateral $abcd$ is

$$[(ab+dc)\,(ad+bc)] + [ac.bd].$$

121. If a, b, c be the lengths of the sides of triangle abc, then

$$[bc.ca] = \text{ab} \sin\gamma.c. \quad (\gamma = \angle acb.)$$

122. If $abcd$ be a cyclic quadrilateral, and P_1, Q_1, P_2, Q_2 be unit forces along ab, dc, da, cb, then P_1+Q_1, P_2+Q_2 act along the bisectors X_1, X_2 of the interior angles between the opposite sides produced.

If A, B, C, D be the angles of the quadrilateral,

$$\begin{aligned}[X_1X_2] &\equiv [(P_1+Q_1)\,(P_2+Q_2)]\\ &= [P_1P_2] + [P_1Q_2] + [Q_1Q_2] + [Q_1P_2]\\ &= \sin \text{A}.a - \sin \text{B}.b + \sin \text{C}.c - \sin \text{D}.d\\ &= 2 \sin \text{A}.\tfrac{1}{2}(a+c) - 2 \sin \text{B}.\tfrac{1}{2}(b+d).\end{aligned}$$

Hence *the bisectors meet on the diameter of the quadrilateral. They cut at right angles*, for if p_1 is the vector of P_1, and so on,

$$[(p_1+q_1)\,|\,(p_2+q_2)] = [p_1\,|\,p_2] + [p_1\,|\,q_2] + [q_1\,|\,q_2] + [q_1\,|\,p_2] = 0.$$

123. If we have two inversely similar polygons, and the joins of corresponding vertices be divided in the ratio of corresponding sides, the points of division are collinear.

124. In general $[ab.cd.ef]$ and $[abc]\,[def]$ are not equal. If, however, they are equal, and b, e lie on cd, then $bcde$ is a harmonic range.

If $[ab.cd.ef] = [a.bcd.ef]$, then the points $[ab.cd]$, b, a, $[ab.ef]$ form a harmonic range.

§ 12. *Desargues' Theorem, Pappus' Theorem and related theorems.*

When we deal with graphical theorems (p. 12), we shall make no distinction between elements at infinity (vectors, bivectors) and other elements. In any metric interpretation, the distinction must be restored.

1. $[LMN]\,[bc.L..ca.M..ab.N]$
$$= [abc]\,[a.MN..b.NL..c.LM]. \quad (21)$$

As always, capitals represent extensives of step two, small letters extensives of step one, the repeated dots serve both as a sign for multiplication, and as brackets.

For, by (17), $\quad [bc.L] = [bL]\,c - [cL]\,b.$

Hence the left side (21) equals

$$[LMN]\,[bL.cM.aN - cL.aM.bN]\,[abc].$$

By (9), $\quad [a.MN] = [aN]\,M - [aM]\,N.$

Hence the right side of (21) equals

$$[abc]\,[aN.bL.cM - aM.bN.cL]\,[LMN].$$

Hence (21) follows. We can rewrite it, by putting

$$L = [b'c'], \quad M = [c'a'], \quad N = [a'b'],$$

then $$[LMN] = [a'b'c']^2,$$

and $$[bc.b'c'..ca.c'a'..ab.a'b']$$
$$= [abc]\,[a'b'c']\,[aa'.bb'.cc']. \quad (22)$$

If one side vanishes so does the other; hence Desargues' Theorem: *if the corresponding sides of two triangles cut in collinear points, then the joins of corresponding vertices are concurrent, and conversely.*

The presence of the factor $[abc]\,[a'b'c']$, on one side, shews what modifications are needed, if we merely take two *triads* of points. The direct theorem has to be restricted, but not the converse theorem. The identity (22) has an interpretation when the two sides do not vanish. This is of a metric nature, easily obtained from (8), (11) and Ex. 121, p. 59.

2. A necessary and sufficient condition for *triangles abc, a'b'c'* to be in perspective, with a, b, c corresponding to a', b', c', is, putting $L = [b'c']$, and so on,

$$[aN.bL.cM] = [aM.bN.cL].$$

If they are also in perspective with a, b, c corresponding to b', c', a', then $\quad [aL.bM.cN] = [aN.bL.cM].$

Hence, then, $\quad [aL.bM.cN] = [aM.bN.cL].$

Thus the triangles are then also in perspective with a, b, c corresponding to c', a', b'.

Hence *triangles doubly in perspective in this cyclic way are triply in perspective.*

3. The last fact also follows from the identity:

$$[aa'.bb'.cc'] + [ab'.bc'.ca'] + [ac'.ba'.cb'] = 0. \quad (23)$$

To prove the latter:

$$\begin{aligned}[aa'.bb'.cc'] &= [aa'b'][bcc'] - [aa'b][b'cc'] \\ &= [aa'b'][bcc'] - [cb'c'][aba'].\end{aligned}$$

Similarly, or by cycling a, b, c and a', c', b',

$$[bc'.ca'.ab'] = [bc'a'][cab'] - [aa'b'][bcc'],$$
$$[cb'.ac'.ba'] = [cb'c'][aba'] - [bc'a'][cab'].$$

Adding these equations, we have (23).

If we had cycled a', b', c' in the first of the three, we should have obtained an expression which does not obviously vanish, though by (23) it must do so.

4. We can also deduce (23) from the following, by cycling a', b', c' and adding:

$$\begin{aligned}&[abc][aa'.bb'.cc'] \\ &\quad = [a'ca][b'ab][c'bc] - [a'ab][b'bc][c'ca]. \quad (24)\end{aligned}$$

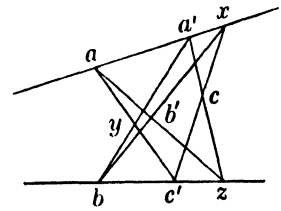
To prove (24):

$$[abc]\,a' = [a'bc]\,a + [a'ca]\,b + [a'ab]\,c,$$
$$[abc][aa'] = [a'ca][ab] + [a'ab][ac].$$

Write down the formulae obtained by cycling a, b, c; then the outer product of their left sides, and of their right sides, gives (24) if $[abc] \neq 0$. When

$$[abc] = 0,$$

(24) is easily shewn.



As the figure shews, (23) gives Pappus' Theorem: *if aa', bb', cc' concur and ab', bc', ca' concur, then ac', ba', cb' concur.*

So enunciated, the theorem involves only six points explicitly.

5. If we enunciate the theorem so as to bring in all the nine points, we have, changing their names: *if a, b, c be collinear, and a', b', c' be collinear, and all distinct, then the cuts of ab', $a'b$, of bc', $b'c$, of ca', $c'a$ are collinear.*

This follows from the formula

$$[bc'.b'c..ca'.c'a..ab'.a'b]$$
$$= [aa'b'][bb'c'][cc'a'][abc] - [abb'][bcc'][caa'][a'b'c']. \quad (25)$$

To shew this, use $[bc'.b'c] = [bb'c']\,c - [bcc']\,b'$, and similar formulae; multiplying these together, we get the left-hand side of (25), and the right-hand side together with the following on the right-hand side:

$$([aa'b'][aa'c][bb'c'][cc'b] - [bb'c'][bb'a][cc'a'][aa'c])$$
$$+ \text{two similar expressions.}$$

This sum cancels out.

Brianchon's Theorem for the corresponding special case is of course given by the dual formulae, such as

$$[AA'.BB'.CC'] + [AB'.BC'.CA']$$
$$+ [AC'.BA'.CB'] = 0. \quad (25')$$

6. We now consider the right-hand side of (25) in detail. The sign changes when a', b are interchanged, or b', c are interchanged, or c', a are interchanged.

If a, b be interchanged, the sum of the old and new expressions is

$$([aa'b'][bb'c'] - [ba'b'][ab'c'])[cc'a'][abc]$$
$$+ ([cac'][bca'] - [bcc'][caa'])[abb'][a'b'c'].$$

Now $$[aa'b']\,b + [a'bb']\,a + [bab']\,a' = [aa'b]\,b'.$$

Hence $$[aa'b'][bb'c'] + [a'bb'][ab'c'] + [bab'][a'b'c'] = 0.$$

Hence the bracket ( ) in the first term equals $[abb'][a'b'c']$. Similarly, the bracket ( ) in the second term equals $-[cc'a'][abc]$.

Thus the expression vanishes; and so (25) merely changes sign when a, b are interchanged. Similarly if b, c or c, a be interchanged. Thus, in all, if any two letters are interchanged, and hence if any permutation of the letters is made, (25) changes at most in sign. Hence Busche's Theorem: *if a, b, c, a', b', c' be*

any six points in the plane, then $[bc'.b'c..ca'.c'a..ab'.a'b]$ *is at most changed in sign when the points undergo any permutation.*

We can easily deduce Pappus' Theorem again from this. For if we interchange b, c', the left-hand side of (25) becomes

$$[c'b.b'c..ca'.ba..ab'.a'c'];$$

now if a, b, c are collinear, and a', b', c' are collinear, this is congruent to $[c'b.b'c..c..b'] = 0$. Hence the original expression then vanishes.*

The metric interpretation of Busche's Theorem is: if $abca'b'c'$ be a simple hexagon, then the product of the lengths of the sides, the sines of the three angles at which the opposite sides cut, and the area of the triangle whose vertices are at these cuts, is independent of the order in which the points $a, \ldots, c'$ are taken.† (Cf. Ex. 121, p. 59.)

7. We could have proceeded as follows: Let

$$a' = x_1 a + y_1 b + z_1 c, \quad b' = x_2 a + y_2 b + z_2 c,$$
$$c' = x_3 a + y_3 b + z_3 c, \quad [abc] = 1.$$

Then $$[b'c.bc'] = x_2 x_3 . a + x_3 y_2 . b + x_2 z_3 . c = x_2 x_3 . p_1,$$

where $$p_1 = a + x_2^{-1} y_2 . b + x_3^{-1} z_3 . c.$$

This and similar equations give

$$[b'c.bc'..c'a.ca'..a'b.ab'] = kd,$$

where

$$k = x_1 x_2 x_3 y_1 y_2 y_3 z_1 z_2 z_3,$$
$$d = \begin{vmatrix} x_1^{-1} & x_2^{-1} & x_3^{-1} \\ y_1^{-1} & y_2^{-1} & y_3^{-1} \\ z_1^{-1} & z_2^{-1} & z_3^{-1} \end{vmatrix}.$$

A permutation of a', b', c' will at most change the sign of d.

8. Since $[aa'.bc] = [aa'c]\, b - [aa'b]\, c$, and so on, we have

$$[aa'.bc..bb'.ca..cc'.ab]$$
$$= [aa'c]\,[bb'a]\,[cc'b]\,[abc] - [aa'b]\,[bb'c]\,[cc'a]\,[abc]. \quad (26)$$

Dually, we have a similar formula,

$$[AA'.B'C'..BB'.C'A'..CC'.A'B']$$
$$= [AA'C']\,[BB'A']\,[CC'B']\,[A'B'C']$$
$$- [AA'B']\,[BB'C']\,[CC'A']\,[A'B'C'].$$

* H. Liebmann, *Synthetische Geometrie* (1934).
† Mehmke, *Prace Mat. Fiz.* 29 (1918), p. 67.

Now let $A=[bc]$, $A'=[b'c']$, and so on, then

$$[AA'C'] = [bc.b'c'.a'b'] = [bb'c]\,[a'b'c'],$$
$$[AA'B'] = [bc.b'c'.c'a'] = [cc'b]\,[a'b'c'].$$

Hence from these and similar formulae,

$$[AA'.B'C'..BB'.C'A'..CC'.A'B'] = ([bb'c]\,[cc'a]\,[aa'b] - [cc'b]\,[bb'a]\,[aa'c])\,[a'b'c']^5.$$

By (26), this gives us

$$[abc]\,[AA'.B'C'..BB'.C'A'..CC'.A'B'] = -[aa'.bc..bb'.ca..cc'.ab]\,[a'b'c']^5.$$

Hence Bricard's Theorem: *if the cuts of aa', bb', cc' with the opposite sides of triangle abc are collinear, then the joins of the points AA', BB', CC' with the opposite vertices of the triangle $a'b'c'$ concur.*

9. $$[pa'.bc..pb'.ca..pc'.ab]\,[a'b'c'] = [pa.b'c'..pb.c'a'..pc.a'b']\,[abc]. \quad (27)$$

For

$$[pa'.bc..pb'.ca..pc'.ab] = ([pa'c]\,b - [pa'b]\,c)\,([pb'a]\,c - [pb'c]\,a)\,([pc'b]\,a - [pc'a]\,b)$$
$$= ([pa'c]\,[pb'a]\,[pc'b] - [pa'b]\,[pb'c]\,[pc'a])\,[abc],$$

and the first factor is not changed in value when dashed and undashed letters are interchanged.

If abc, $a'b'c'$ be coplanar triangles, and p a point such that pa', pb', pc' cut bc, ca, ab respectively in collinear points, then pa, pb, pc cut $b'c'$, $c'a'$, $a'b'$ respectively in collinear points.

10. *If pa', pb', pc' cut bc, ca, ab respectively in collinear points, and also cut ca, ab, bc respectively in collinear points, they cut ab, bc, ca respectively in collinear points.*

For if we put $pa'=A'$, $bc=A$, ..., then (25′) shews the theorem.

11. $$[abc]^2\,[pa.b'c'..a'...pb.c'a'..b'...pc.a'b'..c'] + [a'b'c']^2\,[pa'.bc..a...pb'.ca..b...pc'.ab..c] = 0.$$

For $$[pa.b'c'..a'] = -[pac']\,[a'b'] - [pab']\,[c'a'],$$

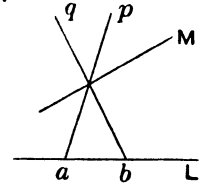
$$[pa.b'c'..a'...pb.c'a'..b'...pc.a'b'..c'] = [pb'c.pc'a.pa'b - pbc'.pca'.pab']\,[a'b'c']^2.$$

Bricard. *If the cuts of pa, pb, pc with the sides b'c', c'a', a'b' of triangle a'b'c' when joined to the opposite vertices give concurrent lines, then the cuts of pa', pb', pc' with the sides bc, ca, ab of triangle abc when joined to the opposite vertices give concurrent lines.*

§ 13. *Sextuply perspective triangles.*

1. *Convention for multiplication.* A continued multiplication like $[apMqL]$ is always to be interpreted as follows: the first factor is multiplied by the second, the product by the third factor, the product so resulting by the fourth factor, and so on. The associative law need not be true.

2. *Projection in a plane.* Points on a line L are projected from p as centre on to the line M, and back again from the point q as centre on to L. If a is a point on L, it is projected into $[apM]$ on M, and back again to $[apMqL]$ on L.



$$[apM] = [aM]\,p - [pM]\,a,$$
$$[apMq] = [aM]\,[pq] - [pM]\,[aq],$$
$$[apMqL] = [aM]\,[pqL] - [pM]\,[aqL]$$
$$= [aM]\,[pqL] + [pM]\,[qL]\,a,$$

since $[aL] = 0$.

Now $[pqL]$ is a weighted point r, say, at the cut of $[pq]$ and L, and $[pM]\,[qL]$ is a scalar k, say.

Hence $b = [apMqL] = [aM]\,r + ka$; and if a, a_1 become b, b_1, then $la + l_1 a_1$ becomes a point congruent to $lb + l_1 b_1$.

3. If, by the above process, a becomes b, b becomes c, and c becomes a, then

$$b = [aM]\,r + ka, \quad c = [bM]\,r + kb, \quad a = [cM]\,r + kc, \quad (1)$$
$$a = [cM]\,r + k[bM]\,r + k^2[aM]\,r + k^3 a.$$

Hence $k^3 = 1$; by (1), if $k = 1$, then r is a vector, or point at infinity. We have always that $c + kb + k^2 a$ is on M, where $k^3 = 1$.

4. *Sextuply perspective triangles.** We have seen that triangles may be triply perspective, and that if triangles *abc* and *a'b'c'* are in perspective (corresponding vertices are always written in corresponding places), then, omitting bracket [ ],

$$aa'b'.bb'c'.cc'a' = ac'a'.ba'b'.cb'c'. \quad (2)$$

* We assume our field of scalars is complex.

If also bac and $a'b'c'$ are in perspective, then

$$ba'b' \,.\, ab'c' \,.\, cc'a' = bc'a' \,.\, aa'b' \,.\, cb'c'.$$

Multiplying together the right-hand sides, and the left-hand sides of these, we have an equation which can be written:

$$[cc'a']^3/ac'a' \,.\, bc'a' \,.\, cc'a' = [cb'c']^3/ab'c' \,.\, bb'c' \,.\, cb'c'. \quad (3)$$

If also cba and $a'b'c'$ are in perspective, in which case so are acb and $a'b'c'$ (§ 12·2), then each function equals

$$[ca'b']^3/aa'b' \,.\, ba'b' \,.\, ca'b'. \quad (4)$$

If also bca and $a'b'c'$ are in perspective, then so are cab and $a'b'c'$, and from (3), (4), and similar equations,

$$\begin{aligned}[cb'c']^3 : [cc'a']^3 : [ca'b']^3 &= [ab'c']^3 : [ac'a']^3 : [aa'b']^3 \\ &= [bb'c']^3 : [bc'a']^3 : [ba'b']^3.\end{aligned}$$

Take $c = a' + b' + c'$, and $[a'b'c'] = 1$, then we find $[cb'c']^3 = 1$, and so on. Hence

$$[ab'c']^3 = [ac'a']^3 = [aa'b']^3; \quad [bb'c']^3 = [bc'a']^3 = [ba'b']^3.$$

Let $\quad a = k_1 a' + k_2 b' + k_3 c', \quad b = l_1 a' + l_2 b' + l_3 c',$

then $\qquad k_1^3 = k_2^3 = k_3^3 = l_1^3 = l_2^3 = l_3^3 = 1.$

Since a, b, c are distinct points, if we take $k_3 = l_3 = 1$, which can be secured merely by adjusting weights, we have

$$k_1 = \epsilon, \; k_2 = \epsilon^2, \; l_1 = \epsilon^2, \; l_2 = \epsilon, \quad \text{where} \quad \epsilon^3 = 1, \; \epsilon \neq 1.$$

Hence we have the standard case

$$a = \epsilon a' + \epsilon^2 b' + c', \quad b = \epsilon^2 a' + \epsilon b' + c', \quad c = a' + b' + c'.$$

Examples. 125. The triangle whose vertices have areal coordinates (x, y, z), (y, z, x), (z, x, y) is triply perspective with the reference triangle abc.

For $\quad [a(xa + yb + zc)] + [b(ya + zb + xc)] + [c(za + xb + yc)] = 0.$

126. If $p_t = [b_2 a_k \,.\, b_3 a_l] = [b_3 a_l \,.\, b_1 a_j] = [b_1 a_j \,.\, b_2 a_k]$ be the perspective centres of two triply perspective triangles $a_1 a_2 a_3$ and $b_1 b_2 b_3$, when i, k, j take the values 1, 2, 3; 2, 3, 1; 3, 1, 2, then $p_1 p_2 p_3$ is triply perspective to both.*

* Jahnke, *Crelles Journ.* 123 (1901), p. 42.

§ 14. *Regressive products involving bivectors in a plane.*

1. As the regressive product of rotors in a plane has been defined, the regressive product of bivectors has been implicitly defined, since a bivector is the difference of two rotors, and multiplication is distributive. But as each bivector is a multiple of the unit bivector ω, and as $[\omega\omega] = 0$, the regressive product of two bivectors is always zero.

2. If ω be the unit bivector, a, b, c any points of unit weight, then

$$[\omega a] = 1, \quad [\omega b] = 1, \quad [\omega(a-b)] = 0,$$

$$[\omega . ab] = [\omega b]\, a - [\omega a]\, b = a - b.$$

Hence *if ω be multiplied by any point, it gives its weight; if it be multiplied by any vector, it gives zero; if by any rotor, it gives the vector of that rotor.*

3. We now consider equations involving vectors and their products only, we can hence replace ω by 1. Then if u, v be vectors, $[uv]$ is to be treated as a scalar. We define the supplement of a scalar as itself, and hence write $|[uv] = [uv]$. Since $|u$ is the vector u turned through a positive right angle, we have

$$[|u\,.\,|v] = [uv] = |[uv]. \tag{1}$$

4. In the spread of vectors, a regressive product is of type $[u_1 v_1 . u_2 v_2]$, and each factor $[u_1 v_1]$, $[u_2 v_2]$ is a scalar. The product hence equals $[u_1 v_1]\,|[u_2 v_2]$, which we usually write as $[u_1 v_1\,|u_2 v_2]$ and by (1) it can be regarded as obtained by multiplying $[u_1 v_1]$ by $|u_2$ and the result by $|v_2$.

$$[uv]^2 = u^2 v^2 - [u\,|v]^2, \tag{2}$$

$$[u_1 v_1\,|u_2 v_2] = [u_1\,|u_2]\,[v_1\,|v_2] - [v_1\,|u_2]\,[u_1\,|v_2]. \tag{3}$$

For, by § 6·16,

$$[u_1 v_1]\, w + [v_1 w]\, u_1 + [w u_1]\, v_1 = 0.$$

Take $w = |u_2$, then since $[|u_2 . u_1] = -[u_1\,|u_2]$, we have

$$[u_1 v_1]\,|u_2 = [u_1\,|u_2]\, v_1 - [v_1\,|u_2]\, u_1. \tag{4}$$

Multiply this by $|v_2$, and we have (3), of which (2) is a special case.

Examples. 127. If w is a variable vector, u a fixed vector, and $[w|u]=u^2$, then all points $o+w$, where o is a fixed point, lie on a line perpendicular to u.

128. To find the cut of the lines given by $[w|u]=u^2$, and $[w|v]=v^2$, where u, v are known vectors from o. The vector w_1 to the cut is determined by

$$[|u|v]\,w_1=[w_1|v].|u-[w_1|u].|v=v^2.|u-u^2.|v.$$

129. If a, b be points, u, v unit vectors, k_1, k_2 scalars, to find the cut of $(a+k_1u)|u$ and $(b+k_2v)|v$.

If ω is the unit bivector, this cut is, since $[\omega\omega]=0$,

$$[(a|u+k_1\omega)\,(b|v+k_2\omega)]=[a|u.\,b|v]+k_1[\omega.b|v]+k_2[a|u.\omega].$$

Now $[a|u.b|v]$ is a point c with weight $[|u.|v]=[uv]=\sin(u,v)$. The second and third terms reduce to $-k_1|v$ and $k_2|u$.

Hence the cut is $\quad [uv]\,c-k_1|v+k_2|u.$

130. If squares be described on the sides of any triangle abc exterior to the triangle, and if their sides which are parallel to the sides of the triangle be produced to form a triangle def, then ad, be, cf are concurrent.

For the side opposite to ab is

$$[(a+|(a-b))\,(b+|(a-b))]=[ab]+(a-b)^2\,\omega.$$

This side cuts the side opposite to ac in

$$d=[abc]\,a+(a-c)^2\,(a-b)+(a-b)^2\,(a-c).$$

Thence $\quad [da]=(a-c)^2\,[ab]+(a-b)^2\,[ac],$

and this line goes through

$$(b-c)^2\,a+(c-a)^2\,b+(a-b)^2\,c.$$ (Cf. Ex. 107, p. 46.)

§ 15. *Supplements of rotors in a plane.*

1. We have already defined the supplements of vectors in a plane. We define the supplement of a rotor as the supplement of the corresponding vector, but confusion is saved if, for this new supplement, we introduce a new sign $\downarrow$.

$$\downarrow[ab]=|(b-a).$$

If L, M, ... be rotors, then

$$\downarrow(L+M+\ldots)=\downarrow L+\downarrow M+\ldots.$$

We shall not at present define the supplement of a point, so that $[LM]$ has no supplement, unless L, M are parallel, in which case $[LM]$ is a vector, and has a supplement in the old sense.

2. $$[L \downarrow M] = [M \downarrow L]. \qquad (1)$$

For, if $L=[ab]$, $M=[cd]$, $\theta=(\widehat{ab, cd})$, and v be a vector perpendicular to and equal to $d-c$, we have, if bars denote magnitudes,

$$[L \downarrow M] = [L\,|(d-c)] = [abv]$$
$$= \overline{ab}.\bar{v} \sin (ab, v) = \overline{ab}.\overline{cd} \cos \theta.$$

If a, b, c, d be points, then $\downarrow[cd]$ is a point at infinity, or vector; hence

$$[a[b \downarrow cd]] = [ab \downarrow cd]. \qquad (2)$$

3. $$a \downarrow [bc] + b \downarrow [ca] + c \downarrow [ab] = 0. \qquad (3)$$

For this is equivalent to Ex. 84, p. 41.

If L, M, N be rotors, and so $\downarrow L$, $\downarrow M$, $\downarrow N$ are points at infinity, or vectors, then

$$[MN \downarrow L] = [M \downarrow L]\, N - [N \downarrow L]\, M. \qquad (4)$$

We can write Ex. 72, p. 38, as the formula

$$[d \downarrow ad.bc] + [d \downarrow bd.ca] + [d \downarrow cd.ab] = 0. \qquad (5)$$

To shew this, since $[d \downarrow ad]$ is a line, we have, using (2),

$$[d \downarrow ad.bc] = [cd \downarrow ad]\, b - [bd \downarrow ad]\, c.$$

4. *Orthopoles.** If A, B, C, L be rotors, then $[BC \downarrow L]$ is a rotor from the cut of B, C perpendicular to L; hence $[BC \downarrow L.L]$ is at the foot of that perpendicular on L:

$$[BC \downarrow L.L. \downarrow A] + [CA \downarrow L.L. \downarrow B]$$
$$+ [AB \downarrow L.L. \downarrow C] = 0. \qquad (6)$$

If we put $[BC]=a$, and so on, (6) says: *if p, q, r be the feet of perpendiculars from the vertices a, b, c of a triangle to any line L, then the perpendiculars from p, q, r to bc, ca, ab respectively are concurrent* (*or parallel*). The point of concurrence is the '*orthopole*' of L for *abc*.

* For this treatment, cf. Daniëls, *Ens. Math.* 20 (1918), p. 97.

To shew (6),

$$[BC \downarrow L] = [B \downarrow L]\, C - [C \downarrow L]\, B.$$

Hence $[BC \downarrow L.L] = [B \downarrow L]\,[CL] - [C \downarrow L]\,[BL]$,

$$\begin{aligned}[BC \downarrow L.L. \downarrow A] &= [B \downarrow L]\,[CL \downarrow A] - [C \downarrow L]\,[BL \downarrow A] \\ &= ([B \downarrow L]\,[C \downarrow A] - [C \downarrow L]\,[B \downarrow A])\, L \\ &\quad + [C \downarrow L]\,[A \downarrow L]\, B - [A \downarrow L]\,[B \downarrow L]\, C,\end{aligned}$$

from which (6) follows at once.

The orthopole itself is the regressive product of any two of the lines, namely (omitting obvious brackets),

$$\begin{aligned}&[A \downarrow L.B \downarrow L.C \downarrow L]\,([A \downarrow L]\,[BC] + \ldots + \ldots) \\ &\quad + [B \downarrow A.C \downarrow L - C \downarrow A.B \downarrow L].[B \downarrow L.C \downarrow L].[AL] + \ldots + \ldots.\end{aligned}$$

$A \downarrow L$, $B \downarrow L$, $C \downarrow L$ are linearly dependent, and thus L occurs effectively three times in each term. Thus if the orthopole is given, there are three corresponding lines.

5. Let a, b, c, a', b', c' be points, and $[bc] = A$, $[b'c'] = A'$, and so on.

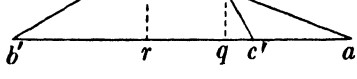
The identity of § 12·3 (23) holds between any six points or vectors. Thence

$$\begin{aligned}&[a \downarrow A'.b \downarrow B'.c \downarrow C'] + [b \downarrow A'.c \downarrow B'.a \downarrow C'] \\ &\qquad + [c \downarrow A'.a \downarrow B'.b \downarrow C'] = 0. \quad (7)\end{aligned}$$

Hence *if $a'b'c'$ be orthologic to abc and to bca, it is so to cab.*

6. Since $\downarrow b'c'$, $\downarrow c'a'$, $\downarrow a'b'$ are points on the line at infinity, therefore, if a, b, c are collinear, the identity § 12·3 (25) gives

$$[a \downarrow c'a'.b \downarrow b'c'..b \downarrow a'b'.c \downarrow c'a'..c \downarrow b'c'.a \downarrow a'b'] = 0. \quad (8)$$



Let a, b, c be collinear points on the sides $b'c'$, $c'a'$, $a'b'$ of triangle $a'b'c'$. Then $[a \downarrow c'a'.b \downarrow b'c']$ is the orthocentre of triangle abc'. Thus (8) shews that the orthocentres of triangles abc', bca', cab' are collinear. Similarly, the orthocentre of $a'b'c'$ is on this line.

If $q=b\downarrow b'c'.b'c'$, $r=c\downarrow b'c'.b'c'$, then $[q\downarrow a'b'.r\downarrow c'a']$ is the orthopole of the line $ab'c'$ for $a'bc$. Since $[qra]=0$, we have an equation similar to (8):

$$[a\downarrow c'a'.q\downarrow b'c'..a\downarrow a'b'.r\downarrow b'c'..q\downarrow a'b'.r\downarrow c'a']=0.$$

But $$[q\downarrow b'c']=[b\downarrow b'c'],$$

thence $[a\downarrow c'a'.q\downarrow b'c']$ is the orthocentre of abc';

$$[r\downarrow b'c']=[c\downarrow b'c'],$$

thence $[a\downarrow a'b'.r\downarrow b'c']$ is the orthocentre of $ab'c$; $[q\downarrow a'b'.r\downarrow c'a']$ is the orthopole of $ab'c'$ for $a'bc$.

Hence *the orthopole of $ab'c'$ for the triangle $a'bc$, and similarly the orthopole of each line for the triangle formed by the other three, lies on the line of orthocentres.*

7. Through any point p draw perpendiculars P_i, P_i' to the sides of triangles abc, $a'b'c'$. Through the vertices of the triangles draw lines Q_i, Q_i' perpendicular to the corresponding P_i, P_i'; then if the Q_i concur, so do the Q_i'. (Daniëls.)

Such questions as the following are best treated by means of the space representation of circles considered later.

Examples. 131. If a line goes through the circumcentre of a triangle, its orthopole is on the nine-point circle.

132. The orthopole of L for abc has the same power in all pedal circles of points on L with respect to abc.

CHAPTER II

GEOMETRY IN SPACE

§ 16. *Weighted points in space, their sums and products.*

1. The considerations of § 2, to the place § 2·10, where we restricted the work to one plane, hold in space of any dimensions. To restrict the work to three-dimensional space, we define:

Def. The points a, b, c, d are '*independent*' if they are distinct and not coplanar. (Hence no three of them are collinear.)

We then assume (§ 2·12):

If a, b, c, d be independent points, and p any point, we can find scalars x, y, z, w such that

$$(x+y+z+w)p = xa+yb+zc+wd, \quad x+y+z+w \neq 0. \quad (1)$$

Then p is the centre of gravity of weights x, y, z, w at a, b, c, d respectively; further x, y, z, w are proportional to the tetrahedral coordinates of p with respect to the tetrahedron $abcd$.

It follows that, if $a_1, \ldots, a_5$ be any five points, we can find scalars $k_1, \ldots, k_5$, not all zero, such that $k_1 a_1 + \ldots + k_5 a_5 = 0$.

Def. Two weighted points are '*independent*', when they differ in position; three weighted points are '*independent*', when they are not collinear.

2. *Outer products of two and of three points.* These products are to obey the laws of §§ 4, 5. They represent extensives of steps two and three. Sums of extensives of the same step are to obey the laws of § 1. Extensives of different steps are never added.

The outer product $[ab]$ of two points a, b of unit weight, in this order, is an extensive '*of step two*', and shall represent the rotor along the line ab in the direction from a to b, and of magnitude equal to the length of ab. Thus $[ab]=[cd]$ if, and only if, a, b, c, d are collinear, the sense from a to b is the same as the sense from c to d, and the intervals ab, cd are equal in length.

If k be scalar, then $[ka.b] = [a.kb] = k[ab]$, and each represents the rotor $[ac]$, where a, b, c are collinear, the sense from a to b is the same as that from a to c, and the length of the interval ac is k times that of the interval ab.

The work in § 4 on the addition of rotors which proceed from the same point requires no change; the sum of two is defined by the parallelogram law, the fundamental laws of addition are then satisfied, and as before

$$[a(b+c)] = [ab]+[ac], \quad [(b+c)\,a] = [ba]+[ca],$$
$$[ab] = -[ba], \qquad [aa] = 0.$$

Sums of rotors, whose lines do not meet, are considered later.

If a, b, c be non-collinear, and of unit weight, their outer product $[abc]$, in this order, is an extensive '*of step three*', and shall represent a '*leaf*', that is, a portion of the plane abc situated anywhere in the plane, with a sense of rotation attached, the sense being given by the circuit of triangle abc, when its sides are traversed in the order bc, ca, ab; the magnitude of the portion of the plane shall be *twice* the area of triangle abc, and will be called the '*magnitude of the leaf*'.

If a, b, c be collinear, we define $[abc]$ to be zero. Conversely, if $[abc] = 0$, then a, b, c are not independent.

Thus, if a, b, c be non-collinear, then $[abc] = [def]$ if, and only if, the six points involved are coplanar, the areas of triangles abc, def equal, and the senses of description the same.

If k_1, k_2, k_3 be scalar, then $[k_1 a.k_2 b.k_3 c] = k_1 k_2 k_3 [abc]$.

We regard $[abc]$ also as the outer product of $[ab]$ and c, and as the outer product of a and $[bc]$. Since $[ab] = -[ba]$, this leads to

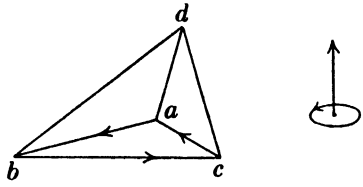
$$[abc] = [bca] = [cab] = -[bac] = -[cba] = -[acb], \quad (2)$$

in agreement with our conventions connecting sign and sense of description.

We denote leaves by Greek letters α, β, ...

3. *Sense of a tetrahedron.* Let $abcd$ be a tetrahedron. Consider the sense of description when the sides of triangle abc are traversed in order bc, ca, ab, and consider the sense of direction from the plane abc to the vertex d. If these two senses are

related like the rotation and translation of a right-handed screw, we say the volume of the tetrahedron is positive; if in the contrary sense, negative. Thus, for example, *abcd* and *cbad* are to be regarded as tetraheda with equal and opposite volumes.



4. *The outer product* $[abcd]$ *of four distinct points* a, b, c, d of unit weight is defined to be the scalar equal to *six* times the volume (taken with its sign) of the tetrahedron *abcd*.

Hence, if any two letters of $[abcd]$ be interchanged, the sign of the product is reversed; if three be cyclically permuted, the product is not altered; if four be cyclically permuted one place, the sign is reversed.

For example:

$$\left.\begin{aligned} &[abcd] = -[bacd] = -[abdc] = -[acbd] = -[dbca], \\ &[abcd] = [bcad] = [cabd] = -[cbad] \\ &\qquad\qquad = -[bacd] = -[acbd], \\ &[abcd] = -[bcda] = [cdab] = -[dabc]. \end{aligned}\right\} \tag{3}$$

The second line should be compared with (2).

Of the twenty-four permutations of the letters in $[abcd]$, twelve give a product equal to $[abcd]$, and twelve a product equal to $-[abcd]$. The first set of permutations result from an *even* number of interchanges, the second from an *odd* number.

If a, b, c, d are coplanar, we define $[abcd]$ to be zero. In particular, this is the case, if any of the points coincide, or any three are collinear.

Conversely, if $[abcd] = 0$, then a, b, c, d are not independent.

5. If one of a, b, c, d be multiplied by a scalar k, then we assume that $[abcd]$ is, in consequence, multiplied by k.

6. ***Distributive laws.*** For any points

$$[abc(d+e)] = [abcd] + [abce].$$

For if p_1, p_2, p be the lengths of perpendiculars from d, e, $\frac{1}{2}(d+e)$ to plane abc, then $2p = p_1 + p_2$. But

$$[abcd], \quad [abce], \quad [abc.\tfrac{1}{2}(d+e)]$$

are respectively the products of twice the area of triangle abc by p_1, p_2, p. Hence, by considering volumes, the result follows.

Similarly $[abc.k_1 d] + [abc.k_2 e] = (k_1 + k_2)\,[abcp]$,

where $k_1 d + k_2 e = (k_1 + k_2)\,p$.

Thus,
$$\begin{aligned}[ab(e+f)\,d] &= -[abd(e+f)] = -[abde] - [abdf] \\ &= [abed] + [abfd],\end{aligned}$$
$$[a(b+f)\,cd] = [abcd] + [afcd],$$
$$[(a+f)\,bcd] = [abcd] + [fbcd].$$

7. The associative law is to hold for our products. Thus

$$[abcd] = [a.bcd] = [ab.cd] = [abc.d] = [a[bc]\,d].$$

This may be taken as the definition of such products as $[ab.cd]$, $[a.bcd]$.

That this is possible is a consequence of our sign conventions, which allow us to treat, say, $[ab]$ and $[cd]$ as wholes, and the distributive laws.

8. If $[abc] = [def]$, then $[abcp] = [defp]$ for all points p. This follows from our geometric interpretation.

Conversely, if $[abcp] = [defp]$ for all points p, then $[abc] = [def]$.

For take $p = d$, then $[abcd] = [defd] = 0$. Hence d, and similarly e, f, are on the plane abc. The geometric interpretation (volumes and areas) then gives the conclusion.

9. *The sum of two leaves whose planes cut, or coincide.* Let the planes have the line ab common, then since the leaves may be represented by any area of the right size and sense in their planes, we may take them to be $[abc]$ and $[abd]$ for suitable points c, d.

If then $p = \frac{1}{2}(c+d)$, we have *for all points* q,

$$[abcq] + [abdq] = 2[abpq],$$

and we define the sum of $[abc]$, $[abd]$ as $2[abp]$. Thus we have the distributive law

$$[abc] + [abd] = [ab(c+d)].$$

Similarly, if $k_1 c + k_2 d = (k_1 + k_2) p$, $k_1 \neq -k_2$, then *for all points* q, $[ab.k_1 cq] + [ab.k_2 dq] = (k_1 + k_2)\,[abpq]$,

and we write

$$[ab.k_1 c] + [ab.k_2 d] = (k_1 + k_2)\,[abp].$$

10. If $n < 5$, the points $a_1, \ldots, a_n$ are not independent if, and only if, there are scalars $k_1, \ldots, k_n$, not all zero, such that

$$k_1 a_1 + \ldots + k_n a_n = 0, \quad k_1 + \ldots + k_n = 0.$$

For, take $n = 4$: if $k_1 a_1 + \ldots + k_4 a_4 = 0$ and, say, $k_1 \neq 0$, multiply by $[a_2 a_3 a_4]$, then $k_1 [a_1 a_2 a_3 a_4] = 0$. Thence $[a_1 a_2 a_3 a_4] = 0$, the four points are coplanar.

If a_1, a_2, a_3 be independent, and a_4 depend on them, then for some scalars k, $(k_1 + k_2 + k_3)\, a_4 = k_1 a_1 + k_2 a_2 + k_3 a_3$. Take $k_4 = -(k_1 + k_2 + k_3)$.

If a_1, a_2, a_3 be not independent, take $k_4 = 0$. Treat the cases $n = 2$, 3 like $n = 4$.

If $n \geqq 5$, the points are never independent.

11. The geometric definition of our products gives, for any points p, a, b, c, d,

$$[pbcd] - [pacd] + [pabd] - [pabc] = [abcd] \qquad (4)$$

or
$$[pbcd] + [apcd] + [abpd] + [abcp] = [abcd]. \qquad (5)$$

Also
$$[abcd]\,p = [pbcd]\,a + [apcd]\,b + [abpd]\,c + [abcp]\,d, \qquad (6)$$

$$[abcd]\,p + [bcdp]\,a + [cdpa]\,b + [dpab]\,c + [pabc]\,d = 0.$$

We can deduce these without invoking geometry afresh:

If a, b, c, d be independent, we have (1) for some scalars x, y, z, w.

Multiply (1) in turn by $[bcd]$, $[acd]$, $[abd]$, $[abc]$, then, if

$$\left.\begin{aligned} s &= x + y + z + w, \\ s[pbcd] &= x[abcd], \\ s[pacd] &= y[bacd] = -y[abcd], \\ s[pabd] &= z[cabd] = z[abcd], \\ s[pabc] &= w[dabc] = -w[abcd]. \end{aligned}\right\} \qquad (7)$$

Hence $s([pbcd] - [pacd] + [pabd] - [pabc]) = s[abcd]$.

If $s \neq 0$, this gives (4).

If $s = 0$, then $[abcd] = 0$. If then, say, $w \neq 0$, multiply the left-hand side of (4) by w:

$$\begin{aligned}[(pbc - pac + pab)(-xa - yb - zc)] - w[pabc] \\ = -x[pabc] - y[pabc] - z[pabc] - w[pabc] \\ = -s[pabc] = 0.\end{aligned}$$

Hence (4) holds generally.

To deduce (6), multiply equations (7) by a, $-b$, c, $-d$ respectively, and add. Then (6) follows if $s \neq 0$. The case $s = 0$ can be treated as before.

From (6) we deduce, on multiplication by p, $[pq]$, $[pqr]$,

$$[pbcd]\,[ap] + [apcd]\,[bp] + [abpd]\,[cp] + [abcp]\,[dp] = 0, \qquad (8)$$

$$[pbcd]\,[apq] + [apcd]\,[bpq] + [abpd]\,[cpq] + [abcp]\,[dpq] = 0, \qquad (9)$$

$$[pbcd]\,[apqr] + [apcd]\,[bpqr] + [abpd]\,[cpqr] + [abcp]\,[dpqr] = 0. \qquad (10)$$

If $d = q$, we have the oft-used formulae:

$$[bcpq]\,[apqr] + [capq]\,[bpqr] + [abpq]\,[cpqr] = 0, \qquad (11)$$

$$[bcpq]\,[apq] + [capq]\,[bpq] + [abpq]\,[cpq] = 0. \qquad (12)$$

Examples. 1. The incentre i of a tetrahedron $abcd$ is on the join of the centroids g, g_1 of the tetrahedron and of its surface.

For, if x, y, z, w be the areas of the faces, and $s = x + y + z + w$, then

$$4g = a + b + c + d, \quad si = xa + yb + zc + wd,$$

$$sg_1 = \tfrac{1}{3}x(b + c + d) + \tfrac{1}{3}y(c + d + a) + \tfrac{1}{3}z(d + a + b) + \tfrac{1}{3}w(a + b + c).$$

Hence

$$4g - 3g_1 = i.$$

2. The lines $[(a + b)(c + d)]$, $[(a + c)(b + d)]$, $[a(b + c + d)]$ are concurrent.

For they can be written

$$[(a + b)(a + b + c + d)], \quad [(a + c)(a + b + c + d)], \quad [a(a + b + c + d)].$$

Thus the joins of the mid-points of opposite edges of a tetrahedron, and the joins of each vertex to the centroid of the opposite face, all go through the centroid of the tetrahedron.

3. The line through $xa+yb+zc+wd$ which cuts ac and bd is

$$[(xa+zc)\,(yb+wd)].$$

4. If $[p_1q_1]+[p_2q_2]+[p_3q_3]=0$, then the lines of $[p_1q_1]$, $[p_2q_2]$, $[p_3q_3]$ are coplanar, and either concurrent or parallel.

We have not yet defined the sum of rotors whose lines do not meet. At present we treat them formally. Multiply by $[p_1p_2]$, then $[p_1p_2p_3q_3]=0$. Hence q_3, and similarly q_1, q_2, lie in plane $p_1p_2p_3$. If p_1q_1 and p_2q_2 cut in a, then multiplication by a gives $[ap_3q_3]=0$. Hence p_3q_3 goes through a.

5. If any plane through the centroid g of a tetrahedron $abcd$ cuts the edges ad, bd, cd in p, q, r, then $\overline{ad}.\overline{pd}^{-1}+\overline{bd}.\overline{qd}^{-1}+\overline{cd}.\overline{rd}^{-1}=4$. (Bars denote lengths.)

For, let $(1+x)\,p=xd+a$, $(1+y)\,q=yd+b$, $(1+z)\,r=zd+c$.

Then, since $[pqrg]=0$, we have

$$[(xd+a)\,(yd+b)\,(zd+c)\,(a+b+c+d)]=0,$$

whence $(1-x-y-z)\,[abcd]=0$, $x+y+z=1$.

Multiply $(1+x)\,p=xd+a$ by d, then $(1+x)\,[pd]=[ad]$.

6. Exhibit the leaf $2[bcd]-4[cad]-2[abd]+[abc]$ as the product of three points, and find where its plane cuts the plane of bcd.

The leaf is

$$[(a+2d)\,(b-4d)\,(c-2d)]=\tfrac{1}{2}[(b-2c)\,(c-2d)\,(b+2a)].$$

7. *If a, b, c, are collinear, and a', b', c' are collinear, then the diameters of the quadrilaterals $aa'bb'$, $bb'cc'$, $cc'aa'$ meet in a point.*

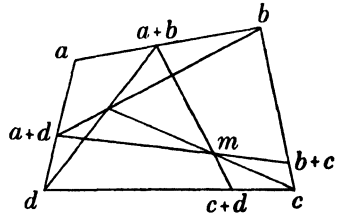
For $[(a+b')\,(a'+b)]+[(b+c')\,(b'+c)]+[(c+a')\,(c'+a)]$

$$=[bc]+[ca]+[ab]-[b'c']-[c'a']-[a'b'],$$

and this is zero on account of the collinearity.

8. If $abcd$ be a skew quadrilateral, and p, q, r, s points on its sides, such that

$$p=xa+x'b,\quad q=yb+y'c,$$
$$r=zc+z'd,\quad s=wd+w'a,$$



then

$$[pqrs]=(x'y'z'w'-xyzw)\,[abcd].$$

Hence pq, rs meet, if and only if

$$xyzw=x'y'z'w',$$

and this depends only on the ratios in which p, q, r, s divide the sides.

If we absorb weights suitably, we can take $p=a+b$, $q=b+c$, $r=c+d$, $s=d+a$, the point of meeting m is then $a+b+c+d$.

The lines $[(a+b)\,d]$, $[b(d+a)]$, $[mc]$

meet in the point $a+b+d$.

This gives a theorem on the generators of a hyperboloid of one sheet, and proves that a certain triad of Pascal lines meet in a point. (Baker, *Principles*, III, p. 44.)

9. If p, q move uniformly along two lines, and r divides the interval pq in a fixed ratio, then the locus of r is a line, or a fixed point. The points which divide in a fixed ratio, the joins of any two points, one on each of two fixed lines, lie in a plane.

10. Any plane through the mid-points p, r of the opposite edges ab, cd of a tetrahedron $abcd$ divides any other pair of edges in equal ratios, and bisects the volume of the tetrahedron.

If the plane cuts bc and ad in q, s, and $q = x_1 b + x_2 c$, $s = y_1 d + y_2 a$, $(x_1 + x_2 = y_1 + y_2 = 1)$, then, as in Ex. 8, we have $x_1 y_1 = x_2 y_2$, hence $x_1 = y_2$, $x_2 = y_1$.

The volume on one side of the plane is

$$[apqs] + [asqr] + [aqcr] = \tfrac{1}{2}(x_2 y_1 + x_1 y_1 + x_1)\,[abcd] = \tfrac{1}{2}[abcd].$$

§ 17. *Vectors, bivectors, and trivectors in space.*

1. We have already considered in § 2 the equality, and the addition, of vectors. As in the plane, if a, b, p are points of unit weight, and therefore $a-b$ is a vector, then $[p(a-b)]$ is the rotor through p parallel to the vector.

If $[p(a-b)] = [p(c-d)]$ for all p, then $a-b = c-d$.

2. If $u_1, \ldots, u_4$ be any vectors, we can find scalars $k_1, \ldots, k_4$, not all zero, such that $k_1 u_1 + \ldots + k_4 u_4 = 0$.

For we can write $u_1 = a - b_1$, $u_2 = a - b_2$, $u_3 = a - b_3$, $u_4 = a - b_4$, where a is any fixed point, and the b are taken suitably. If the b are independent, then $(k_1 + \ldots + k_4)\,a = k_1 b_1 + \ldots + k_4 b_4$, for suitable scalars $k_1, \ldots, k_4$.

If the b are not independent, we can find k such that

$$k_1 b_1 + k_2 b_2 + k_3 b_3 + k_4 b_4 = 0, \quad k_1 + k_2 + k_3 + k_4 = 0.$$

Hence $k_1 u_1 + \ldots + k_4 u_4 = 0$ in both cases.

3. *The outer product of two vectors u, v* is a '*bivector*' $[uv]$. As u, v each determine a set of parallel lines, they together fix a set of parallel planes, and $[uv]$ may be considered to lie on any of them. We define the equality of bivectors as in § 6·7:

If $[(b-a)(c-a)] = [(p-a)(q-a)]$, then $[abc] = [apq]$, and conversely. We may accordingly represent the bivector $[(b-a)(c-a)]$ by any area, in one of its parallel planes, whose magnitude and sense are the same as those of the area representing $[abc]$.

If α be any extensive or combination of extensives for which a magnitude is defined, we denote that magnitude by mag α.

Thus if u, v be vectors and $\bar{u} = \text{mag } u$, $\bar{v} = \text{mag } v$, then mag $[uv] = \bar{u}\bar{v} \sin (u, v)$, where (u, v) is the angle from u to v.

As in the plane $[uv] = -[vu]$; $[uv] = 0$ if, and only if, u, v are parallel or $u = 0$, or $v = 0$.

The difference of two equal and parallel rotors au, bu is a bivector $[(a-b)u]$.

The sum of two bivectors is a bivector. For if u be any vector, we can draw the planes of the bivectors through u, and find vectors v_1, v_2 in these planes such that the bivectors are $[uv_1]$, $[uv_2]$. Their sum can be defined as $[u(v_1+v_2)]$.

4. *The outer product of three vectors u, v, w is a 'trivector'* $[uvw]$.

$[(b-a)(c-a)(d-a)]$ and $[(p-a)(q-a)(r-a)]$

are defined to be equal, if $[abcd] = [apqr]$.

Hence, if $[uvw] = [u_1 v_1 w_1]$, the tetrahedron with vertex at any point a and edges from a equal and parallel to the vectors u, v, w is equal in volume and sense to the tetrahedron with vertex at a and edges from a equal and parallel to the vectors u_1, v_1, w_1. This volume, multiplied by 6, with its proper sign, is the '*magnitude*' of the trivector.

$[uvw] = -[vuw] = [wuv]$, and so on, as for points in a plane.

We may also consider $[uvw]$ as the product, not defined so far, of u and $[vw]$ or of $[uv]$ and w;

$$[uvw] = [u.vw] = [uv.w] = [w.uv].$$

If $\alpha' = (wu, uv)$, $\beta = (w, u)$, $\gamma = (u, v)$ be signed angles, then geometric formulae for areas and volumes give us

$$\text{mag } [uvw] = \text{mag } u . \text{mag } v . \text{mag } w . \sin\beta \sin\gamma \sin\alpha'.$$

5. *The product of a point and a trivector.* The '*moment*' of $[(b-a)(c-a)(d-a)]$ round p is defined to be

$$[p(b-a)(c-a)(d-a)].$$

It is independent of p, for assuming the distributive law, it equals

$$[pbcd]+[apcd]+[abpd]+[abcp]$$
$$=[abcd]=\text{magnitude of trivector.}$$

If $v=p-q$ is a vector, Γ a trivector, we define

$$[v\Gamma]=[(p-q)\,\Gamma]=[p\Gamma]-[q\Gamma].$$

The outer product of a vector and a trivector is therefore zero. The outer product of any four vectors is zero.

6. If a, b, c, d be independent points, and $b-a=u$, $c-a=v$, $d-a=w$, then any vector r is of the form $r=\mathrm{k}_1u+\mathrm{k}_2v+\mathrm{k}_3w$, the k scalars.

For r can be written as $f-a$, where

$$f=\mathrm{x}_1a+\ldots+\mathrm{x}_4d,\quad(\mathrm{x}_1+\ldots+\mathrm{x}_4=1).$$

Hence $$r=\mathrm{x}_2(b-a)+\mathrm{x}_3(c-a)+\mathrm{x}_4(d-a),$$

as required.

Hence *any bivector* $[rs]$ *is a linear combination of* $[vw]$, $[wu]$, $[uv]$, *and any trivector* $[rst]$ *is a multiple of* $[uvw]$.

7. Let Ω be a trivector of unit moment, the '*unit trivector*'. Then any trivector is a scalar multiple of Ω. Thence ***two trivectors of equal magnitude are identical.***

Hence if $$[abcd]=[ab'c'd'],$$

then $$[(b-a)\,(c-a)\,(d-a)]=[(b'-a)\,(c'-a)\,(d'-a)].$$

In any equation which involves only vectors and their products, a trivector may be replaced by its magnitude, that is, may be treated as a scalar.

If u, v, w are vectors, we define

$$\sin[uvw]=\frac{\text{mag}\,[uvw]}{\sqrt{u^2}\sqrt{v^2}\sqrt{w^2}}.$$

8. *The product of a point p and a bivector U* is a leaf whose plane goes through the point and is parallel to the planes of the bivector, and whose magnitude equals that of the bivector.

For U can be written $[(a-p)\,(b-p)]$, by suitable choice of a, b. Hence $[pU]=[pab]$.

If a vector u be parallel to the planes of a bivector U, then $[uU]=0$.

The product of a rotor and a vector is a leaf whose plane goes through the line of the rotor and is parallel to the lines of the vector. For its magnitude, see 10 below.

9. The definition in 3 gives the distributive law

$$[u(v+w)] = [uv]+[uw].$$

If U be a bivector, an argument, similar to that in § 5·2, shews

$$[U(v+w)] = [Uv]+[Uw].$$

10. *Difference of leaves.* $[abc]-[abd]$ (cf. § 16·9) is interpreted as $[ab(c-d)]$; let p be a point such that $c-d=b-p$, then

$$[ab(c-d)] = [ab(b-p)] = -[abp],$$

$$[abc]+[abp] = [ab(c+p)] = [ab(b+d)] = [abd].$$

Thus the usual relations between sums and differences hold.

11. *Sum of two parallel leaves.* Two parallel leaves can be written in the form $k_1[auv]$, $k_2[buv]$, where k_1, k_2 are scalars, a, b points, u, v vectors. If $k_1+k_2\neq 0$, their sum is the parallel leaf $[(k_1a+k_2b)\,uv]$; if $k_1+k_2=0$, their sum is the trivector $k_1[wuv]$, where $w=a-b$.

12. Since $[abc]+[dba]+[dcb]+[dac]$ can be written as $[(a-b)\,(b-c)\,(c-d)]$, it is a trivector. If the perimeters of the faces of the tetrahedron $abcd$ be each described in the positive sense given by the terms $[abc]$, ..., then each edge is described once in each of its two directions.

If the outer products of a sum of leaves by three independent vectors vanish, the sum is a trivector or zero.

13. For any vectors u_1, u_2, u_3, u_4,

$$[u_2u_3u_4]\,u_1-[u_1u_3u_4]\,u_2+[u_1u_2u_4]\,u_3-[u_1u_2u_3]\,u_4=0.$$

For we can find scalars k_1, ..., k_4, not all zero, such that

$$k_1u_1+\ldots+k_4u_4=0;$$

hence

$$k_1[u_1u_3u_4]+k_2[u_2u_3u_4]=0.$$

This and similar equations give the ratios of the k.

Thus, if vectors $u_1, \ldots, u_4$ are in equilibrium, then

$$u_1 + \ldots + u_4 = 0$$

gives

$$\frac{\text{mag } u_1}{\sin [u_2 u_3 u_4]} = -\frac{\text{mag } u_2}{\sin [u_1 u_3 u_4]} = \frac{\text{mag } u_3}{\sin [u_1 u_2 u_4]} = -\frac{\text{mag } u_4}{\sin [u_1 u_2 u_3]}.$$

(Möbius.)

Examples. 11. If an ordinary polyhedron be taken, and leaves be placed on its face-planes whose magnitudes are the areas of the faces, and whose senses are so adjusted that when the perimeters of the faces are described in the positive sense, each edge is described once in each direction, then the sum of the leaves is a trivector. This adjustment of the senses is always possible when the polyhedron is two-sided, as the term is used in topology, and only in that case.

12. If *oa*, *ob*, *oc* be edges of a parallelepiped, then its diagonal *od* cuts the triangle *abc* at its centroid, and is trisected there.

13. If a line through the centroid g of a tetrahedron *abcd* cuts the planes *bcd*, *acd*, *abd*, *abc* in p, q, r, s, and if the lengths of gp, gq, gr, gs, taking account of signs, be x, y, z, w, then

$$\mathrm{x}^{-1} + \mathrm{y}^{-1} + \mathrm{z}^{-1} + \mathrm{w}^{-1} = 0.$$

For, if v be a unit vector along the line, then $p = g + \mathrm{x}v$ (x scalar),

$$[pbcd] = 0, \quad [gbcd] + \mathrm{x}[vbcd],$$

$$[vbcd] = -\mathrm{x}^{-1}[gbcd] = -\tfrac{1}{4}\mathrm{x}^{-1}[abcd],$$

and so on. Then

$$[v(bcd - acd + abd - abc)] = 0$$

gives the result.

14. The points a, b, c and d, $= \mathrm{x}a + \mathrm{y}b + \mathrm{z}c$, are projected from p to a', b', c', d', where $d' = \mathrm{x}'a' + \mathrm{y}'b' + \mathrm{z}'c'$. Then $\mathrm{lx}'\mathrm{x}^{-1} = \mathrm{my}'\mathrm{y}^{-1} = \mathrm{nz}'\mathrm{z}^{-1}$, where l, m, n are scalars depending only on the position of the planes *abc*, *a'b'c'*.

15. If through the mid-points of the edges *bc*, *ca*, *ab* of a tetrahedron *abcd* parallels be drawn to the opposite edges, they meet in a point on the join of d to the centroid of *abcd*.

16. If $v = p - o$, where o is a fixed point, p a variable point, u a fixed vector, $[vu]$ a fixed bivector U, then the locus of p is a line parallel to u.

If the lines corresponding to $[vu_1] = U_1$ and $[vu_2] = U_2$ cut, then

$$[u_1 U_2] + [u_2 U_1] = 0.$$

17. If u, v be unit vectors along two lines, in plane α, which intersect and are equally inclined to plane β, then $[Uu]=[Uv]$, where U is a bivector parallel to β. The vector $u-v$ is along the cut of α and β.

18. If w be a variable vector, u, v fixed vectors, all from a fixed point, and if $w=u+\mathrm{k}v$, then the end-point of w describes a line as the scalar k varies.

The shortest distance between two lines represented by w, $=u_1+\mathrm{k}v_1$, and w, $=u_2+\mathrm{k}v_2$, has a length

$$\text{mag } [(u_1-u_2)\,v_1v_2]/\text{mag } [v_1v_2].$$

For if the line of the shortest distance cuts the lines in points p, q which correspond to the following values of w:

$$w_1=u_1+\mathrm{k}_1v_1, \quad w_2=u_2+\mathrm{k}_2v_2,$$

then
$$p-q=w_1-w_2=u_1-u_2+\mathrm{k}_1v_1-\mathrm{k}_2v_2;$$
$$[(p-q)\,v_1v_2]=[(u_1-u_2)\,v_1v_2];$$
$$\text{mag } [(p-q)\,v_1v_2]=\text{mag } [v_1v_2]\,\mathrm{l},$$

where l is the length of the perpendicular from p on to the plane through q parallel to the plane of $[v_1v_2]$.

19. *If any transversal cuts the faces of the tetrahedron abcd in points a', b', c', d' on the faces opposite a, b, c, d, then the mid-points of the intervals aa', bb', cc', dd' are coplanar, on the 'median plane'.*

For, let $u=\mathrm{x}a+\mathrm{y}b+\mathrm{z}c+\mathrm{w}d$, $(\mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{w}=0)$, be a unit vector along the transversal, and

$$p=\mathrm{x}_1a+\mathrm{y}_1b+\mathrm{z}_1c+\mathrm{w}_1d, \quad (\mathrm{x}_1+\mathrm{y}_1+\mathrm{z}_1+\mathrm{w}_1=1)$$

be any point on the transversal. If the transversal cuts *bcd* in

$$a'=p+\mathrm{k}u,$$

then
$$[a'bcd]=0, \quad \mathrm{k}=-[pbcd]/[ubcd]=-\mathrm{x}_1\mathrm{x}^{-1},$$
$$\mathrm{x}(a+a')=\mathrm{x}a+\mathrm{x}p-\mathrm{x}_1u.$$

Similarly for $\mathrm{y}(b+b')$ and so on. Hence

$$\mathrm{x}(a+a')+\ldots+\ldots+\mathrm{w}(d+d')$$
$$=(\mathrm{x}a+\ldots+\mathrm{w}d)+(\mathrm{x}+\ldots+\mathrm{w})\,p-(\mathrm{x}_1+\ldots+\mathrm{w}_1)\,u=0.$$

20. *From a line and five planes we get five median planes for the five tetrahedra; these planes meet in a line. From a line and six planes we get fifteen median planes; these meet in a point.*

(Serret, *Géométrie de direction*, p. 237.)

21. Parallel lines through the vertices a, b, c, d of a tetrahedron cut the opposite faces in $a'b'c'd'$; then $[a'b'c'd'] = -3[abcd]$.

For let $a' - a = u_1, \quad b' - b = u_2,$

and so on. Then $[u_1 u_2] = 0$ (parallel vectors).

$$[a'b'c'd'] = abcd + u_1 bcd - u_2 acd + u_3 abd - u_4 abc.$$

But a' lies on bcd, hence

$$[(a + u_1)\, bcd] = 0, \quad [u_1 bcd] = -[abcd].$$

Similarly, $[u_2 acd] = [abcd]$, and so on.

§ 18. *The supplements of vectors in space.*

1. Let i, j, k be mutually perpendicular unit vectors, such that $[ijk]$ is the unit trivector Ω. In dealing with equations which involve only vectors and their products, we can replace Ω by unity.

Any vector u is of form $u = li + mj + nk$; l, m, n scalars.

2. If u be any vector, we denote its '*supplement*' by $|u$, and define it as follows:

$$|i = [jk], \quad |j = [ki], \quad |k = [ij].$$

If $u = li + mj + nk$, then $|u = l|i + m|j + n|k.$

3. We write $I = [jk], \quad J = [ki], \quad K = [ij].$

If U be any bivector, we denote its '*supplement*' by $|U$, and define it as follows:

$$|I = i, \quad |J = j, \quad |K = k.$$

If $U = lI + mJ + nK$, then $|U = l|I + m|J + n|K.$ (17·6)

The supplement of a vector is a bivector; of a bivector, a vector. If we take the supplement of the supplement, we have $||i = |I = i$, $||I = |i = I$, and so on; hence $||u = u$, $||U = U$. This is different in sign from the corresponding result in the plane (§ 7·2); the significance of the difference will be seen later.

$$|(u + v) = |u + |v, \quad |(U + V) = |U + |V.$$

4. If $u = li + mj + nk,$

then $|u = l[jk] + m[ki] + n[ij] = l^{-1}[(lj - mi)\,(lk - ni)].$

Now $li + mj$ is the projection of u on the ij plane, $lj - mi$ is perpendicular to and equal to this projection, and it lies in the

ij plane; $li+nk$ is the projection of u on the ik plane, $lk-ni$ is perpendicular to and equal to this projection, and it lies in the ik plane. Thus $lj-mi$ and $lk-ni$ are both perpendicular to u.

Hence the planes of $|u$ are perpendicular to u; dually, the lines of $|U$ are perpendicular to U.

5. The outer product $[u.|v]$ of u and $|v$ is written $[u|v]$ and is called the '*inner product*' of u and v. Similarly for $[U|V]$. We have

$$[i|i] = [ijk] = 1, \quad [j|j] = 1, \quad [k|k] = 1.$$

$$[I|I] = [J|J] = [K|K] = 1.$$

The inner product of two distinct elements of i, j, k is zero. Similarly for I, J, K.

$$\begin{aligned}[u|u] &= [(li+mj+nk)\,|(li+mj+nk)] \\ &= [(li+mj+nk)\,(l[jk]+m[ki]+n[ij])] \\ &= l^2+m^2+n^2.\end{aligned}$$

Hence, by elementary geometry, $[u|u]$ is the square of the length of the interval from the origin which represents u. Hence $\sqrt{[u|u]} = \text{mag } u$, the square root being taken with positive sign.

By our definitions, u and $|u$ are related in sense like the translation and rotation of a right-handed screw. This with the end remark of 4 shews that *the result of the operation of taking the supplement of u is independent of i, j, k.*

6. If $\quad u = li+mj+nk, \quad v = l'i+m'j+n'k,$

then $$[u|v] = ll'+mm'+nn',$$

whence $[u|v] = [v|u]$. Similarly $[U|V] = [V|U]$, for bivectors.

Invoking the formula in solid geometry for the angle between two directed lines, we have

$$[u|v] = \text{mag } u.\text{mag } v.\cos(u, v),$$

$[u|v] = 0$, if, and only if, u, v are perpendicular, or $u = 0$, or $v = 0$.

7. If v is any vector, mag v = mag $|v$. Hence if $|v = V$, then mag V = mag $|V$.

For if u be a vector perpendicular to the planes of the bivector $[u_1u_2]$, and in such a direction that $[uu_1u_2]$ is positive, then, by § 17·4, mag $[uu_1u_2]$ is the product of mag u and mag $[u_1u_2]$.

Now if u is a unit vector, then $[u\,|u] = 1$. But $[u\,|u]$ is the outer product of the vector u and the bivector $|u$, and u is perpendicular to the planes of $|u$. Hence in this case mag $|u = 1$.

Hence in general $\quad \text{mag } u = \text{mag } |u, \quad \text{mag k}u = \text{mag k}\,|u.$

We often write $\quad u^2$ for $[u\,|u], \quad U^2$ for $[U\,|U]$.

8. If $\quad v_1 = x_1 i + y_1 j + z_1 k, \quad v_2 = x_2 i + y_2 j + z_2 k,$

$$v_3 = x_3 i + y_3 j + z_3 k,$$

then

$$[v_1 v_2] = (y_1 z_2 - y_2 z_1)\,[jk] + (z_1 x_2 - z_2 x_1)\,[ki] + (x_1 y_2 - x_2 y_1)\,[ij],$$

$$|[v_1 v_2] = (y_1 z_2 - y_2 z_1)\,i \quad + (z_1 x_2 - z_2 x_1)\,j \quad + (x_1 y_2 - x_2 y_1)\,k.$$

Hence mag $|[v_1 v_2]$, and hence mag $[v_1 v_2]$, equals the square root of

$$(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_1)^2.$$

If U, V be bivectors, then $[U\,|V] = \text{mag } U \text{ mag } V \cos (U, V)$.

9. The outer product of $[v_1 v_2]$ and $v_3 = x_3 i + y_3 j + z_3 k$ is

$$[v_1 v_2 v_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

10. If V is a bivector, and p_1, p_2, p_3, o points, and bars denote lengths, then

$$[V(p_3 - o)] = \text{mag } V . \overline{op_3} . \cos (op_3, |V),$$

$$[(p_1 - o)\,(p_2 - o)\,(p_3 - o)]$$
$$= \overline{op_1} . \overline{op_2} \sin (op_1, op_2) . \overline{op_3} \sin (op_3, op_1 p_2).$$

§ 19. *Regressive products of bivectors in space.*

1. There is a correspondence between many of the formulae so far developed and those for points in a plane. This is exemplified in the table:

Point a	Vector u
Rotor $[ab]$	Bivector $[uv]$
Leaf $[abc]$	Trivector $[uvw]$

The reason for the correspondence is obvious when vectors in space are replaced by points on the plane at infinity. Later on, we shall find something in the plane corresponding to the *inner* product of vectors.

To the formula for points in a plane,

$$[bcd]\,a-[cda]\,b+[dab]\,c-[abc]\,d=0,$$

corresponds that for vectors in space,

$$[vwr]\,u-[wru]\,v+[ruv]\,w-[uvw]\,r=0.$$

2. To extend this analogy, we define the '*regressive product*' of two bivectors *in the spread of vectors*. Corresponding to the formula in a plane:

$$[ab.cd]=[abd]\,c-[abc]\,d;$$

we define the '*regressive product*' of the bivectors $[uv]$, $[wr]$ as follows:

$$[uv.wr]=[uvr]\,w-[uvw]\,r. \qquad (1)$$

Thence

$$[uv.wr]=[uwr]\,v-[vwr]\,u. \qquad (2)$$

Thus $[uv.wr]$ is a vector parallel to the planes of $[uv]$ and to the planes of $[wr]$. Hence *it is a vector parallel to the cuts of any two planes of the bivectors.*

If U, V be bivectors, then

$$[UV]=-[VU], \quad [UU]=0. \qquad (3)$$

$$[U.wr]=[Ur]\,w-[Uw]\,r; \quad [wr.U]=[wU]\,r-[rU]\,w. \qquad (4)$$

3.

$$|[uv]=[|u.|v].$$

For if u, v be parallel, the planes of $|u$, $|v$ are parallel and both sides vanish.

If u, v be not parallel, then any two planes of $|u$, $|v$ respectively meet in a line, and we can find vectors u_1, u_2, w such that $|u=[wu_1]$, $|v=[wu_2]$, where u_1, u_2 are in a plane of $[uv]$. Then the angles (u, v) and (u_1, u_2) are equal in magnitude and sign, and w is perpendicular to u_1, u_2, u, v. We can take w to be a unit vector.

Then

$$\text{mag } u=\text{mag } [wu_1]=\text{mag } u_1,$$

$$\text{mag } v=\text{mag } u_2, \quad |[uv]=\text{k}w,$$

where

$$\text{k}=\text{mag } [uv]=\text{mag } u.\text{mag } v.\sin(u, v)$$

$$=\text{mag } u_1.\text{mag } u_2.\sin(u_1, u_2).$$

But $$[|u.|v] = [wu_1.wu_2] = [wu_1u_2]\,w,$$
$$[wu_1u_2] = \text{mag } u_1.\text{mag } u_2.\sin(u_1, u_2).$$

Hence $|[uv] = [|u.|v]$.

4. If U, V be bivectors, then $|[UV] = [|U.|V]$.

For if $U = |u$, $V = |v$, then, by 3,

$$[UV] = [|u.|v] = |[uv].$$

Take supplements, then

$$|[UV] = [uv] = [|U.|V],$$

since $u = |U$, $v = |V$.

5. If U, V be bivectors and $|U = u$, $|V = v$, then, by § 18·7,

$$\begin{aligned}\text{mag } [UV] &= \text{mag } |[UV] = \text{mag } [|U.|V] \\ &= \text{mag } [uv] = \text{mag } u.\text{mag } v.\sin(u, v).\end{aligned}$$

But $\sin(u, v) = \sin(U, V)$. Hence

$$\text{mag } [UV] = \text{mag } U.\text{mag } V.\sin(U, V).$$

The derivation shews how the angle (U, V) must be chosen.

6. The outer product of a vector w and a bivector W is the scalar $[w|w_1]$, where $|W = w_1$, $W = |w_1$.

If U, V, W be bivectors, and u_1, v_1, w_1 their supplements, then $[UV]$ is a vector w, say, and the outer product of $[UV]$ and W is the scalar $[wW]$. Hence

$$\begin{aligned}[UV.W] &= [w|w_1] = [w_1|w] = [|w.w_1] \\ &= [|[UV].w_1] = [[|U.|V]\,w_1] \\ &= [u_1v_1.w_1] = [u_1v_1w_1].\end{aligned}$$

Similarly $$[U.VW] = [u_1v_1w_1].$$

Hence $[UV.W] = [U.VW]$, and each may be written $[UVW]$.

7. We define the '*supplement*' of a scalar as the scalar itself. Then $|[wW] = [|w.|W]$. For, if $|W = w_1$, we have

$$|[wW] = [wW] = [w|w_1] = [w_1|w] = [|w.w_1] = [|w.|W].$$

Hence also

$$|[uvw] = [|u.|[vw]] = [|u|v|w], \quad |[UVW] = [|U|V|W].$$

Note that while (§ 17·4) we have $[u|v] = [|v.u]$, we have (§ 17·3) $[u|V] = -[|V.u]$.

8. If u, v, w, r be vectors, then with the usual convention on multiplication (§ 13·1), $[uv|w.|r]$ means the result of multiplying $[uv]$ by $|w$, and the result by $|r$, both multiplications being outer. Since the associative law holds for bivectors by 6, the result is the product of $[uv]$ and $[|w.|v]$, that is, of $[uv]$ and $|[wr]$.

We denote this product by $[uv|wr]$. It is a scalar. Hence, by 7,

$$[uv|wr] = |[uv|wr] = [|uv.||wr] = [|uv.wr] = [wr|uv].$$

By 5, this gives

$$\text{mag}\,[uv|wr] = \text{mag}\,uv.\text{mag}\,wr.\cos(uv, wr).$$

9. If u, v, w be vectors, not necessarily coplanar, then

$$[vw|u] = [v|u]\,w - [w|u]\,v. \tag{5}$$

For, let $U = |u$, then, by (4),

$$[vw.U] = [vU]\,w - [wU]\,v.$$

Take supplements, then, since

$$|[vw|u] = [|vw.u] = -[u|vw],$$

we have

$$[u|vw] = [w|u]\,|v - [v|u]\,|w = [u|w]\,|v - [u|v]\,|w. \tag{6}$$

Multiply (5) by $|r$, where r is any vector, then

$$[vw|ur] = [v|u]\,[w|r] - [w|u]\,[v|r]. \tag{7}$$

In particular, writing $[vw]^2$ for $[vw|vw]$, we have

$$[vw]^2 = v^2w^2 - [v|w]^2. \tag{8}$$

By (5), (6), (7)

$$\left.\begin{aligned} [u|vw] + [v|wu] + [w|uv] &= 0, \\ [vw|u] + [wu|v] + [uv|w] &= 0, \\ [vw|ur] + [wu|vr] + [uv|wr] &= 0. \end{aligned}\right\} \tag{9}$$

10. Also

$$[r|u]\,|vw + [r|v]\,|wu + [r|w]\,|uv = [uvw]\,r. \tag{10}$$

For, if u, v, w be independent, there are scalars k, not all zero, such that

$$[uvw]\,r = \mathrm{k}_1\,|[vw] + \mathrm{k}_2\,|[wu] + \mathrm{k}_3\,|[uv].$$

Since $[uvw]$ is scalar, this gives

$$[uvw]\,[r\,|u] = k_1\,|[vw]\;|u = k_1\,|[vwu] = k_1[vwu] = k_1[uvw].$$

If u, v, w be not independent, their outer product vanishes, and the proof is easy.

From (10), (8) we have

$$[uvw]^2 = \begin{vmatrix} [u\,|u], & [u\,|v], & [u\,|w] \\ [v\,|u], & [v\,|v], & [v\,|w] \\ [w\,|u], & [w\,|v], & [w\,|w] \end{vmatrix}. \qquad (11)$$

Also, if u, v, w be vectors, U, V, W bivectors, then

$$[uvw]\,[UVW] = \begin{vmatrix} [uU], & [uV], & [uW] \\ [vU], & [vV], & [vW] \\ [wU], & [wV], & [wW] \end{vmatrix}. \qquad (12)$$

For any vectors u, v,

$$v = \frac{1}{u^2}\,[v\,|u]\,u + \frac{1}{u^2}\,[uv\,|u]. \quad \text{(Cf. § 8·14.)}$$

11. Since $[u\,|v] = \bar{u}\bar{v}\cos(u, v)$, where bars denote magnitudes, (11) gives

$$[uvw]^2 = \bar{u}^2\bar{v}^2\bar{w}^2(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma),$$

where $\alpha = (\widehat{v, w})$, $\beta = (\widehat{w, u})$, $\gamma = (\widehat{u, v})$. From (7), (9),

$$\sin(v, w)\sin(u, r)\cos(vw, ur) = \cos(v, u)\cos(w, r) - \cos(w, u)\cos(v, r), \qquad (13)$$

$$\sin(u, v)\sin(v, w)\cos(uv, vw) = \cos(u, v)\cos(v, w) - \cos(u, w), \qquad (14)$$

$$\sin(v, w)\sin(u, r)\cos(vw, ur) + \sin(w, u)\sin(v, r)\cos(wu, vr) + \sin(u, v)\sin(w, r)\cos(uv, wr) = 0.$$

12. Many formulae in a plane which involve only vectors and their inner products hold when these vectors are in space, and we have in each case a dual formula obtained by replacing each vector by a bivector. For example,

$$[(a-b)\,|(c-d)] + [(b-c)\,|(a-d)] + [(c-a)\,|(b-d)] = 0.$$

If a, b, c, d be vectors, this is shewn in § 8·1, and as only the differences of vectors enter, it holds also when a, b, c, d are

points. Hence if *ab* is perpendicular to *cd*, and *bc* to *ad*, then so is *ca* to *bd*. The identities of § 8·1 to § 8·11 have immediate applications to space figures. Thus from (8·4): if in a tetrahedron *bc* is perpendicular to *ad*, then the join of the mid-points of *ca* and *bd* has the same length as the join of the mid-points of *ab* and *cd*, and conversely.

Again

$$[((a+b)-(c+d))\,|((b-c)-(a-d))] = (b-c)^2-(a-d)^2$$

gives us: *if bc, ad are of equal length, the join of the mid-points of ab and cd is equally inclined to them.* (Cf. chap. I, Exs. 17 and 68.) From this and Ex. 9, p. 79, we have a theorem of later use:

If a line is displaced in space, the mid-points of the joins of corresponding points lie on a line equally inclined to the two positions of the given line.

13. *If* p, p_1 *be any points and* u, u_1, v, v_1 *any unit vectors, and* $[u\,|v]=[u_1\,|v_1]$, *then the plane through the mid-points of the joins*

$$pp_1,\quad (p+\mathrm{k}u)\,(p_1+\mathrm{k}u_1),\quad (p+\mathrm{l}v)\,(p_1+\mathrm{l}v_1),\quad (\mathrm{k},\,\mathrm{l}\text{ any scalars})$$

is equally inclined to the planes

$$[p(p+\mathrm{k}u)\,(p+\mathrm{l}v)] \quad\text{and}\quad [p_1(p_1+\mathrm{k}u_1)\,(p_1+\mathrm{l}v_1)].$$

For, by (7)

$$\begin{aligned}&[(u+u_1)\,(v+v_1)\,|(uv-u_1v_1)]\\ &\quad=[(u+u_1)\,|u]\,[(v+v_1)\,|v]-[(u+u_1)\,|v]\,[(v+v_1)\,|u]\\ &\qquad-[(u+u_1)\,|u_1]\,[(v+v_1)\,|v_1]+[(u+u_1)\,|v_1]\,[(v+v_1)\,|u_1]=0,\end{aligned}$$

for the terms cancel, by hypothesis, on multiplying out.

Cor. If a plane pqr be moved into position $p_1q_1r_1$, *the joins of corresponding points have their mid-points on a plane equally inclined to pqr and* $p_1q_1r_1$. (Cf. § 103.)

There is an exceptional case when $u=-u_1$, $v=-v_1$, the mid-points then all coincide.

Examples. 22. (i) If g is the centroid, o the circumcentre of the tetrahedron *abcd*, and p a point such that g is the mid-point of op, then p is the cut of planes through the mid-points of the edges *abcd* perpendicular to the opposite edges.

For let $a_1 = a - o$, and so on; then

$$a_1^2 = b_1^2 = c_1^2 = d_1^2, \quad g_1 = \tfrac{1}{4}(a_1 + b_1 + c_1 + d_1), \quad p_1 = 2g_1.$$

$$[(\tfrac{1}{2}(a_1 + b_1) - \tfrac{1}{2}(a_1 + b_1 + c_1 + d_1))\,|\,(c_1 - d_1)] = -\tfrac{1}{2}[(c_1 + d_1)\,|\,(c_1 - d_1)]$$
$$= \tfrac{1}{2}(d_1^2 - c_1^2) = 0.$$

Hence the join of p to the mid-point of ab is perpendicular to cd.

(ii) If q is one-third of the way from g along gp, then q is the centre of a sphere through the centroids of the faces of the tetrahedron and through the points one-third of the way from p to the vertices.

For $$q_1 = \tfrac{1}{3}(2g_1 + p_1) = \tfrac{1}{3}(a_1 + b_1 + c_1 + d_1),$$

whence $$(q_1 - f_1)^2 = (q_1 - l_1)^2,$$

where $$f_1 = \tfrac{1}{3}(b_1 + c_1 + d_1), \quad l_1 = \tfrac{1}{3}(2p_1 + d_1).$$

23. If the opposite edges of the tetrahedron $abcd$ be at right angles, then its altitudes meet, in h say. Take h for origin, $a_1 = a - h$, and so on. Then the six products $[a_1\,|\,b_1]$, $[a_1\,|\,c_1]$, ... are all equal. Let k be their common value, then

$$[b_1 c_1 d_1]\, a_1 = \mathrm{k}\,|([b_1 c_1] + [c_1 d_1] + [d_1 b_1]).$$

The point h is the cut of the shortest intervals between the edges.

The equation of the circumsphere is (r being a variable vector from h as origin)

$$r^2 - 4[r\,|\,g_1] + 3\mathrm{k} = 0.$$

The square of its radius is $\frac{1}{4}(a_1^2 + \ldots + d_1^2)$; its centre is $2g - h$.

The mid-points of the edges and the feet of the perpendiculars from h to the edges lie on the sphere (with r as before),

$$r^2 - 2[r\,|\,g_1] + \mathrm{k} = 0.$$

The sphere of Ex. 22 now goes through the orthocentres of the faces and its equation is

$$3r^2 - 4[r\,|\,g_1] + \mathrm{k} = 0$$

and the radius is one-third of the circumradius. This sphere, the circumsphere, and the sphere on diameter hg have a common radical plane $[r\,|\,g_1] = \mathrm{k}$.

24. If the opposite edges of the tetrahedron $abcd$ are equal, the centroid and circumcentre coincide, and conversely.

For take the circumcentre as origin of vectors, then $a_1^2 = b_1^2 = c_1^2 = d_1^2$ and $(a_1 - b_1)^2 = (c_1 - d_1)^2$, and so on, give

$$[a_1\,|\,b_1] = [c_1\,|\,d_1], \quad [b_1\,|\,c_1] = [a_1\,|\,d_1], \quad [c_1\,|\,a_1] = [b_1\,|\,d_1].$$

Hence $$(a_1 + b_1 + c_1 + d_1)^2 = 4a_1^2 + 4([a_1\,|\,b_1] + [a_1\,|\,c_1] + [a_1\,|\,d_1])$$
$$= 4[a_1\,|\,(a_1 + b_1 + c_1 + d_1)].$$

Thus the vector g_1 is perpendicular to $g_1 - a_1$, where

$$4g_1 = a_1 + b_1 + c_1 + d_1.$$

Similarly, g_1 is perpendicular to $g_1 - b_1$, $g_1 - c_1$, $g_1 - d_1$; hence $g_1 = 0$.

Conversely, if $g_1 = 0$, then $a_1 + b_1 = -c_1 - d_1$; squaring, we have $[a_1 | b_1] = [c_1 | d_1]$, and hence $(a_1 - b_1)^2 = (c_1 - d_1)^2$, and so on.

25. We can generalise Ex. 22 to higher spreads. *If* $a_1, \ldots, a_n$ *be independent points in a spread of step* n, *and* g *their centroid, then a hypersphere goes through the centroids of the faces of the simplex* $a_1, \ldots, a_n$, *and it is homothetic to the circumsphere, the centre of the homothety being* g, *and its ratio* $1 : n-1$.

For $a_1 - g = (n-1)(g - (a_2 + \ldots + a_n)/(n-1))$.

If through the centroid of a boundary simplex of step $n-2$, *we draw a prime perpendicular to the opposite edge, all these perpendiculars go through a point* m *and* $(n-2)(g-o) = 2(g-m)$, *where* o *is the circumcentre of the simplex.*

If the simplex $a_1, \ldots, a_n$ *is such that its* n *altitudes cut in a point* h, *then* $m = h$, *and the sphere through the centroids of the faces goes through the feet of the altitudes and divides the intervals from* h *to the vertices in the ratio* $1 : n-2$. (Mehmke.*)

26. If at the vertex d of a tetrahedron $abcd$ we draw perpendicular planes to ad, bd, cd, they cut the opposite edges in collinear points.

27. If u is perpendicular to $U = [vw] \neq 0$, it is perpendicular to v and w.

For $0 = [vw|u] = [v|u]\, w - [w|u]\, v$, and v, w are independent.

28. If oa, ob, oc be mutually perpendicular lines, then the foot h of the perpendicular from o to the plane abc is the orthocentre of triangle abc.

For, take o as origin, and let $h_1 = h - o$, and so on. Then $[a_1 b_1 | h_1] = 0$. Hence, as in Ex. 27, $[a_1 | h_1] = [b_1 | h_1] = 0$. But

$$[a_1 | b_1] = [b_1 | c_1] = [c_1 | a_1] = 0.$$

Hence $[(b_1 - c_1) | (h_1 - a_1)] = 0$, and so on.

29. If v is equally inclined to the coplanar vectors u_1, u_2, u_3, it is perpendicular to their plane.

For
$$[v|u_1] = [v|u_2] = [v|u_3]$$

gives
$$[v|(u_1 - u_2)] = [v|(u_2 - u_3)] = 0.$$
$$[v|(u_1 - u_2)\,(u_2 - u_3)] = 0.$$

* *Sitz. Heidelberger Akad. der Wiss.* (1931), 10 Abh.

30. If u_1, u_2, u_3, u_4 be four vectors from a point on a sphere to four other points on the sphere, then

$$[u_2u_3u_4]\,u_1^2-[u_3u_4u_1]\,u_2^2+[u_4u_1u_2]\,u_3^2-[u_1u_2u_3]\,u_4^2=0.$$

(Cf. § 9 (5).)

31. Through a fixed point o inside a sphere three perpendicular intervals op, oq, or are drawn to the sphere, and the box $opqrs$ is completed, where s is the vertex opposite to o. Then the locus of s is a sphere concentric to that given. (Cf. Ex. 91, p. 43.)

32. If through a point of space go perpendicular planes α and α_1, and perpendicular planes β and β_1, then the plane through $[\alpha\beta]$ and $[\alpha_1\beta_1]$ is perpendicular to that through $[\alpha\beta_1]$ and $[\alpha_1\beta]$. (Analogue of Hesse's Theorem, § 26·6.)

33. If α, β be intersecting planes, p any point on α, and the perpendicular line to α at p cuts β in q, and the perpendicular to β at q cuts α in r, then qr is perpendicular to the cut of α, β.

For, if u is the vector of the cut, then $[(q-p)\,|\,u]=0$, $[(r-q)\,|\,u]=0$. Hence $[(p-r)\,|\,u]=0$.

34. If $abcd$ be a tetrahedron, such that altitudes from b, c intersect, then bc is perpendicular to ad.

35. If op is perpendicular to oab, and oc to ab, then the planes opc, oab are perpendicular.

For, take o as origin, and let $p_1=p-o$, and so on, then

$$[a_1\,|\,p_1]=[b_1\,|\,p_1]=0,\quad [c_1\,|\,(a_1-b_1)]=0.$$

Hence
$$\begin{aligned}[p_1c_1\,|\,(a_1b_1+b_1p_1+p_1a_1)] &= -p_1^2[b_1\,|\,c_1]+p_1^2[a_1\,|\,c_1]\\ &= p_1^2[(a_1-b_1)\,|\,c_1]=0.\end{aligned}$$

36. If $oabc$ be a tetrahedron and faces boc, coa, aob be mutually perpendicular, then $[b_1c_1]^2+[c_1a_1]^2+[a_1b_1]^2$ is the square of the area of abc, where $a_1=a-o$, and so on. (Square $[b_1c_1]+\ldots+\ldots$, and use $[c_1a_1\,|\,a_1b_1]=0$, ...)

37. If p, q be the centroids of the faces bcd, cda of the tetrahedron $abcd$, then the radical plane of the spheres on diameters ap, bq bisects cd and is perpendicular to ab.

For let o be a common point of the spheres, and hence on their radical plane. Take o as origin, then $[p_1\,|\,a_1]=[q_1\,|\,b_1]=0$; hence

$$[(b_1+c_1+d_1)\,|\,a_1]=0,\quad [(c_1+d_1+a_1)\,|\,b_1]=0,$$
$$[(c_1+d_1)\,|\,(a_1-b_1)]=0.$$

But c_1+d_1 is a vector from o through the mid-point of cd, while $|(a-b)$ gives a plane perpendicular to ab.

38. If the opposite edges of a tetrahedron be equal, then (i) the faces are congruent triangles, (ii) the opposite dihedral angles are equal, (iii) each edge is equally inclined to its opposite edge, (iv) the four solid angles are congruent.

For, take the circumcentre as origin, then $(b_1-c_1)^2=(a_1-d_1)^2$ gives $[b_1\,|\,c_1]=[a_1\,|\,d_1]$.

Similarly $[c_1\,|\,a_1]=[b_1\,|\,d_1]$, $[a_1\,|\,b_1]=[c_1\,|\,d_1]$. Hence

(i) $(b_1c_1+c_1a_1+a_1b_1)^2=(b_1c_1+c_1d_1+d_1b_1)^2$.

(ii) $[b_1c_1d_1\,|\,a_1c_1d_1]=[c_1a_1b_1\,|\,d_1a_1b_1]$.

(iii) $[ab.cd]=[bc.ad]$.

(iv) $[(d_1-a_1)(d_1-b_1)(d_1-c_1)]=-[(a_1-b_1)(a_1-c_1)(a_1-d_1)]$.

39. If a_1, a_2, a_3, a_4 be arbitrary points, and k_1, ..., k_4 scalars, then $\Sigma k_i(a_i-p)^2$ is constant as p varies, if, and only if, a_1, ..., a_4 are coplanar, and k_1, ..., k_4 are proportional to

$$[a_2a_3a_4],\quad -[a_1a_3a_4],\quad [a_1a_2a_4],\quad -[a_1a_2a_3].$$

40. If $p=xa+yb+zc+wd$, $x+y+z+w=1$, then

$$x\overline{pa}^2+\ldots+w\overline{pd}^2=\Sigma xy\overline{ab}^2=-\text{power of } p \text{ for sphere } abcd.$$

41. If u, v, w be vectors, the vectors equally inclined to their lines are multiples of $\overline{vw}.u+\overline{wu}.v+\overline{uv}.w$, where $\overline{uv}=\text{mag }[uv]$, and so on.

42. If $abcd$ be a tetrahedron, the foot p of the perpendicular from d to the plane abc is given by $V^2d-[abcd]\,|\,V$, where

$$V=[bc]+[ca]+[ab].$$

For, since $|\,V$ is a vector perpendicular to plane abc, we can let $d-p=k\,|\,V$ (k scalar); then $[V(d-p)]=kV^2$. But

$$\begin{aligned}[V(d-p)]&=[(bc+ca+ab)(d-p)]\\&=[bcd-acd+abd-abc]=[abcd]\,\Omega.\end{aligned}$$

43. If u_1, u_2 be vectors, V a bivector, k a scalar, then if $[vu_1]=V$, $o+v$ will describe a line, and if $[v\,|\,u_2]=k$, then $o+v$ will describe a plane. To find where the line and plane cut, we have v given by

$$[V\,|\,u_2]=[vu_1\,|\,u_2]=[v\,|\,u_2]\,u_1-[u_1\,|\,u_2]\,v=ku_1-[u_1\,|\,u_2]\,v.$$

44. Planes given in a similar way by $[v\,|\,u_i]=k_i$ $(i=1,2,3)$ meet in the point $o+w$, where

$$[u_1u_2u_3]\,|\,w=k_1[u_2u_3]+k_2[u_3u_1]+k_3[u_1u_2].$$

45. If $[vu_1]=V_1$, $[vu_2]=V_2$ give two skew lines (as in Ex. 43), and a line through o meets them in p, q, then

$$q-p=\left(\frac{1}{[V_1u_2]}+\frac{1}{[V_2u_1]}\right)[V_1V_2].$$

For the direction of pq is given by $[V_1 V_2]$. Thence

$$p-o = \mathrm{k}[V_1 V_2], \quad q-o = \mathrm{k}'[V_1 V_2],$$
$$\mathrm{k}[V_1 V_2 . u_1] = [(p-o)\, u_1] = V_1, \quad \mathrm{k}'[V_1 V_2 . u_2] = V_2.$$

But $$[V_1 V_2 . u_1] = [V_1 u_1]\, V_2 - [V_2 u_1]\, V_1 = -[V_2 u_1]\, V_1.$$

46. The condition that $p = o + \mathrm{x}i + \mathrm{y}j + zk$ should lie on the plane $[(o+\mathrm{l}i)\,(o+\mathrm{m}j)\,(o+\mathrm{n}k)]$ is $\mathrm{xl}^{-1} + \mathrm{ym}^{-1} + \mathrm{zn}^{-1} = 1$.

47. The line $[o(li+mj+nk)]$ cuts the plane

$$[(o+\mathrm{l}_1 i)\,(o+\mathrm{m}_1 j)\,(o+\mathrm{n}_1 k)]$$

in p, where $$(\mathrm{ll}_1^{-1} + \mathrm{mm}_1^{-1} + \mathrm{nn}_1^{-1})\,(p-o) = \mathrm{l}i + \mathrm{m}j + \mathrm{n}k.$$

48. If k is the magnitude of the rotor

$$\mathrm{l}[bc] + \mathrm{m}[ca] + \mathrm{n}[ab] + \mathrm{p}[ad] + \mathrm{q}[bd] + \mathrm{r}[cd],$$

then $$-\mathrm{k}^2 = (\mathrm{n}-\mathrm{l}-\mathrm{q})\,(\mathrm{l}-\mathrm{m}-\mathrm{r})\,(b-c)^2 + \ldots + \ldots$$
$$+ (\mathrm{p}+\mathrm{q}+\mathrm{r})\,\{(\mathrm{m}-\mathrm{n}-\mathrm{p})\,(d-a)^2 + \ldots + \ldots\}.$$

49. The centre of the sphere through the points a, b, c, d is

$$d + \frac{1}{2[a_1 b_1 c_1]}\{a_1^2 |[b_1 c_1] + b_1^2 |[c_1 a_1] + c_1^2 |[a_1 b_1]\},$$

where $$a_1 = a-d, \quad b_1 = b-d, \quad c_1 = c-d.$$

For, if o is the centre, and $o_1 = o-d$, then

$$(o_1 - a_1)^2 = (o-d)^2 = o_1^2,$$

thence $$a_1^2 = 2[a_1 | o].$$

Now $|[b_1 c_1]$, $|[c_1 a_1]$, $|[a_1 b_1]$ are not coplanar, hence

$$o_1 = \mathrm{k}|[b_1 c_1] + \mathrm{k}'|[c_1 a_1] + \mathrm{k}''|[a_1 b_1], \quad a_1^2 = 2[a_1 | o_1] = 2\mathrm{k}[a_1 b_1 c_1].$$

50. If V_1, V_2, V_3, V_4 be the bivectors of the faces of a tetrahedron, taken with senses as in Ex. 11, p. 83, then

$$V_1^2 + V_2^2 + 2[V_1 | V_2] = V_3^2 + V_4^2 + 2[V_3 | V_4],$$
$$V_1^2 = V_2^2 + V_3^2 + V_4^2 + 2[V_3 | V_4] + 2[V_4 | V_2] + 2[V_2 | V_3].$$

The determinant, whose term in the (r, s) place is $[V_r | V_s]$, vanishes.

§ 20. *Geometry of vectors from a point, and geometry on a sphere.*

1. If we take a sphere, centre o, radius unity, we can associate each vector from o with the point on the sphere where its line cuts the sphere, and give to the point a weight equal to the magnitude of the vector; thus interpreting the geometry of vectors as a geometry of points on a sphere.

If then a, b be points of unit weight on the sphere, and $a_1 = a - o$, $b_1 = b - o$, the point $a+b$ *on the sphere* will have a weight whose square is

$$a_1^2 + b_1^2 + 2[a_1 | b_1] = 2(1 + [a_1 | b_1]) = 4 \cos^2 \tfrac{1}{2}(a_1, b_1).$$

We take as the angle (a_1, b_1) that of the minor arc on the sphere between a and b.

Hence the weight, taken positive, of $a+b$ is not 2, but $2 \cos \frac{1}{2}(a, b)$, and the position of $a+b$ is at the mid-point of the minor arc ab. This assumes that a, b are not antipodal; if they are antipodal, then $a+b=0$, $a=-b$.

Similarly, if $a \neq -b$, then the weight of $a-b$ is $2 \sin \frac{1}{2}(a, b)$ and its position is the mid-point of the arc complementary to ba.

2. The outer product $[ab]$ *on the sphere*, of two points a, b of the sphere, of any weight, is to correspond to the outer product of the corresponding vectors. Hence we represent $[ab]$ by the great circle through a, b, with a sense and magnitude equal to the sense and magnitude of the bivector.

3. If u, v, w be any independent vectors, then $[uvw] \neq 0$.

Let
$$u' = \frac{|[vw]}{[uvw]}, \quad v' = \frac{|[wu]}{[uvw]}, \quad w' = \frac{|[uv]}{[uvw]}. \tag{1}$$

Then
$$[u|u'] = [v|v'] = [w|w'] = 1. \tag{2}$$

$$[uvw]^2 \, |[v'w'] = [wu.uv] = [uvw]\, u.$$

Hence
$$[v'w'] = \frac{|u}{[uvw]}, \quad [uvw]\,[u'v'w'] = 1. \tag{3}$$

$$u = \frac{|[v'w']}{[u'v'w']}, \quad v = \frac{|[w'u']}{[u'v'w']}, \quad w = \frac{|[u'v']}{[u'v'w']}. \tag{4}$$

If u, v, w now represent points on the sphere, then $|u$, $|v$, $|w$ will represent the sides of the polar triangle of uvw. Let α, β, γ be the angles between u, v, w; α', β', γ' those between u', v', w'; then, by § 17·4, denoting magnitudes by bars,

$$[uvw] = \bar{u}\bar{v}\bar{w} \sin\beta \sin\gamma \sin\alpha'.$$

Similarly, $[u'v'w'] = \bar{u}'\bar{v}'\bar{w}' \sin\beta' \sin\gamma' \sin\alpha$.

Hence

$$\left.\begin{aligned}\frac{\sin\alpha'}{\sin\alpha}=\frac{\sin\beta'}{\sin\beta}=\frac{\sin\gamma'}{\sin\gamma}=\frac{[uvw]}{\bar{u}\bar{v}\bar{w}}\cdot\frac{1}{\sin\alpha\sin\beta\sin\gamma}=\mathrm{m},\ \text{say},\\ \frac{\sin\alpha}{\sin\alpha'}=\frac{\sin\beta}{\sin\beta'}=\frac{\sin\gamma}{\sin\gamma'}=\frac{[u'v'w']}{\bar{u}'\bar{v}'\bar{w}'}\cdot\frac{1}{\sin\alpha'\sin\beta'\sin\gamma'}=\mathrm{m}^{-1}.\end{aligned}\right\}\quad(5)$$

Hence $$\frac{\sin^2\alpha'}{\sin^2\alpha}=\frac{[uvw]}{[u'v'w']}\cdot\frac{\bar{u}'\bar{v}'\bar{w}'}{\bar{u}\bar{v}\bar{w}}\cdot\frac{\sin\alpha'\sin\beta'\sin\gamma'}{\sin\alpha\sin\beta\sin\gamma}.$$

The last factor on the right-hand side is $\sin^3\alpha'/\sin^3\alpha$, hence

$$\frac{\sin\alpha'}{\sin\alpha}=\frac{[u'v'w']}{[uvw]}\cdot\frac{\bar{u}\bar{v}\bar{w}}{\bar{u}'\bar{v}'\bar{w}'}.\quad(6)$$

4. Since $$[vu\,|\,uw]=[u\,|\,v]\,[u\,|\,w]-u^2[v\,|\,w],$$

we have, as in § 19,

$$\sin\beta\sin\gamma\cos\alpha'=\cos\beta\cos\gamma-\cos\alpha.$$

Thus
$$\left.\begin{aligned}\cos\alpha&=\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos\alpha',\\ \cos\beta&=\cos\gamma\cos\alpha-\sin\gamma\sin\alpha\cos\beta',\\ \cos\gamma&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\gamma'.\end{aligned}\right\}\quad(7)$$

Dually
$$\left.\begin{aligned}\cos\alpha'&=\cos\beta'\cos\gamma'-\sin\beta'\sin\gamma'\cos\alpha,\\ \cos\beta'&=\cos\gamma'\cos\alpha'-\sin\gamma'\sin\alpha'\cos\beta,\\ \cos\gamma'&=\cos\alpha'\cos\beta'-\sin\alpha'\sin\beta'\cos\gamma.\end{aligned}\right\}\quad(8)$$

In comparing these formulae with those given in most text-books on spherical trigonometry, we must note that α' is an exterior angle of the trihedron formed by u, v, w.* We shall shew that all formulae of spherical trigonometry can be deduced from (7).

Examples. 51. The bisector of the vertical angle of a spherical triangle divides the base into intervals whose sines are as the sines of adjacent sides.

For, as the magnitudes of the bivectors $[ab]$, $[ac]$ are $\sin(a, b)$, $\sin(a, c)$, the bisector of angle bac is $\sin\beta\,.\,[ac]+\sin\gamma\,.\,[ab]$, and this goes through the point $\sin\beta\,.\,c+\sin\gamma\,.\,b$.

52. Since $[a\,|\,bc]+[b\,|\,ca]+[c\,|\,ab]=0$, the altitudes of a spherical triangle abc concur, in h, say. Those of the polar triangle $a'b'c'$ also concur in h, where $|[bc]\equiv a'$, and so on.

The points $[bc\,.\,b'c']$, $[ca\,.\,c'a']$, $[ab\,.\,a'b']$ lie on a great circle. Since $|[a\,|\,bc]=[|a\,.\,bc]=[b'c'\,.\,bc]$, this great circle is along $|h$. (Cf. § 26·6.)

* This change in usage was proposed by Grassmann.

53. The analogues of Ceva's and Menelaus' Theorems involve the sines of arcs, instead of the lengths of intervals.

54. $[abc]^2 = [ab]^2\,[ac]^2 - [ab\,|\,ac]^2$.

$[abc] = \sin\alpha.\sin \mathrm{p}$, where p is the arc-length of the altitude of abc from a.

$[ab\,|\,ac] = \sin\beta\,\sin\gamma\,\cos\alpha'$.

Hence $\sin^2\alpha\,\sin^2 \mathrm{p} = \sin^2\beta\,\sin^2\gamma - (\sin\beta\,\sin\gamma\,\cos\alpha')^2$
$= \sin^2\beta\,\sin^2\gamma\,\sin^2\alpha'$.

55. If a, b, c be on a great circle, and p any point (on the sphere), then $\sin bc.\cos ap + \sin ca.\cos bp + \sin ab.\cos cp = 0$.
(Stewart.)

56. $[b\,|\,ca.c\,|\,ab] = [a\,|\,b]\,[c\,|\,a]\,|\,bc + [b\,|\,c]\,[a\,|\,b]\,|\,ca + [c\,|\,a]\,[b\,|\,c]\,|\,ab$.

Hence $[b\,|\,ca.c\,|\,ab] = [c\,|\,ab.a\,|\,bc] = [a\,|\,bc.b\,|\,ca]$.

Hence if abc be a spherical triangle, and $a'b'c'$ its polar triangle, then

$$\sin\beta\,\sin\gamma\,[bb'.cc'] = \cos\beta\,\cos\gamma\,\sin\alpha.a' + \ldots + \ldots.$$

But aa' is perpendicular to bc, bb' to ca, and $[a\,|\,bc.b\,|\,ca]$ represents the orthocentre of the triangle, which is therefore at

$$\tan\alpha.a' + \tan\beta.b' + \tan\gamma.c'.$$

57. If two diagonals of a complete quadrilateral be quadrants, so is the third.

58. If the diagonals of a spherical quadrilateral bisect each other in s, then the cuts of opposite sides are distant a quadrant from s; if the diagonals are also equal, these cuts are distant a quadrant from one another.

59. If the sides ab, ac of a spherical triangle have a total length equal to half the circumference, then the median through a is a quadrant.

60. If d, e, f be the mid-points of the sides of traingle abc, and bc cuts fe in a', then the arc $a'd$ is a quadrant; if ed is a quadrant, so are ef and fd.

For $[ef] \equiv [(a+b)\,(a+c)] = [ba] + [ac] + [bc]$,

$a' \equiv [bc.ef] \equiv [abc]\,(b-c)$, $[a'\,|\,d] \equiv [(b+c)\,|\,(b-c)] = 0$.

If also $[d\,|\,e] = 0$, then $[e\,|\,f] = 0 = [d\,|\,f]$; for

$$[(b+c)\,|\,(c+a)] = [(c+a)\,|\,(a+b)].$$

61. If d, e, f be on the sides bc, ca, ab of a triangle abc, and ad, be, cf are concurrent, then $[ef.bc]$, $[fd.ca]$, $[de.ab]$ are on a great circle, and b, d, c, $[ef.bc]$ are harmonic.

62. The polar circles $|a$, $|b$, $|c$ of a, b, c cut bc, ca, ab respectively in points on the great circle

$$[c|a]\,[a|b]\,[bc]+[a|b]\,[b|c]\,[ca]+[b|c]\,[c|a]\,[ab]=0.$$

63. If a_1, a_2, a_3, ... be fixed points on a sphere, k_1, k_2, ... constant scalars, then the locus of p, such that $k_1\cos a_1p+k_2\cos a_2p+\ldots$ is constant, is a circle.

64. If aa', bb', cc' be chordal arcs of a spherical triangle abc meeting in p, then

$$\frac{\sin pa'.\cos pa}{\sin aa'}+\ldots+\ldots=1.$$

65. To find the circumcentre of triangle abc.

$$\begin{aligned}[(b+c)\,|(bc+ca+ab)] &= [(b+c)\,|(b-c)\,(b-a)]\\ &= [(b+c)\,|(b-a)]\,.\,|(b-c)-[(b+c)\,|(b-c)]\,.\,|(b-a)\\ &= [(b+c)\,|(b-a)]\,.\,|(b-c)\equiv|(b-c).\end{aligned}$$

Hence the circumcentre is at the point $|(bc+ca+ab)$.

It is obvious that many formulae, theorems, and proofs in the plane, apply without change to the sphere, when 'line' is replaced by 'great circle'. For instance Desargues' Theorem and Pappus' Theorem and most of § 12 apply.

Furthermore, pole and polar, for a conic in a plane, correspond on the sphere to point and great circle distant a quadrant from the point. From these interpretations a wealth of theorems can be deduced from those on conics, but it is not worth while actually to state them.

Readers familiar with elliptic geometry in the plane can also translate the above exercises into this field.

§ 21. *Deduction of formulae in spherical trigonometry.**

1. All quantities in this section are scalar.

We deduce all formulae from § 20 (7), in particular we can obtain (5), (6), (8), and also Napier's and Delambre's analogies: (9), (10), (11) below.

$$\left.\begin{aligned}\cos\alpha &= \cos\beta\cos\gamma-\sin\beta\sin\gamma\cos\alpha',\\ \cos\beta &= \cos\gamma\cos\alpha-\sin\gamma\sin\alpha\cos\beta',\\ \cos\gamma &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\gamma'.\end{aligned}\right\}\qquad(7)$$

* Study, *Sphärische Trigonometrie* (1893).

$$\tan \tfrac{1}{2}(\alpha'+\beta') \cot \tfrac{1}{2}\gamma' = -\cos \tfrac{1}{2}(\alpha-\beta)/\cos \tfrac{1}{2}(\alpha+\beta). \quad (9)$$

$$\tan \tfrac{1}{2}(\alpha'-\beta') \cot \tfrac{1}{2}\gamma' = -\sin \tfrac{1}{2}(\alpha-\beta)/\sin \tfrac{1}{2}(\alpha+\beta). \quad (10)$$

$$\left.\begin{aligned} \frac{\sin \frac{1}{2}(\beta'+\gamma')}{\sin \frac{1}{2}\alpha'} &= \frac{\cos \frac{1}{2}(\beta-\gamma)}{\cos \frac{1}{2}\alpha}, & \frac{\sin \frac{1}{2}(\beta'-\gamma')}{\sin \frac{1}{2}\alpha'} &= -\frac{\sin \frac{1}{2}(\beta-\gamma)}{\sin \frac{1}{2}\alpha}, \\ \frac{\cos \frac{1}{2}(\beta'+\gamma')}{\cos \frac{1}{2}\alpha'} &= -\frac{\cos \frac{1}{2}(\beta+\gamma)}{\cos \frac{1}{2}\alpha}, & \frac{\cos \frac{1}{2}(\beta'-\gamma')}{\cos \frac{1}{2}\alpha'} &= \frac{\sin \frac{1}{2}(\beta+\gamma)}{\sin \frac{1}{2}\alpha}. \end{aligned}\right\} \quad (11)$$

For in (7) substitute

$$\left.\begin{aligned} \cot \tfrac{1}{2}\alpha = l_1, \quad \cot \tfrac{1}{2}\beta = l_2, \quad \cot \tfrac{1}{2}\gamma = l_3, \\ \cot \tfrac{1}{2}\alpha' = L_1, \quad \cot \tfrac{1}{2}\beta' = L_2, \quad \cot \tfrac{1}{2}\gamma' = L_3. \end{aligned}\right\} \quad (11')$$

Then $\cos \alpha = (l_1^2-1)/(l_1^2+1)$, $\sin \alpha = 2l_1/(l_1^2+1)$, and so on.

Put

$$\left.\begin{aligned} \frac{n_1}{n_0} = l_2 l_3, \quad \frac{n_2}{n_0} = l_3 l_1, \quad \frac{n_3}{n_0} = l_1 l_2; \\ \frac{N_1}{N_0} = L_2 L_3, \quad \frac{N_2}{N_0} = L_3 L_1, \quad \frac{N_3}{N_0} = L_1 L_2, \end{aligned}\right\} \quad (11'')$$

where n_0, N_0 are arbitrary non-zero quantities. Then (9), (10) and the similar formulae correspond to

$$\frac{n_0-n_3}{n_0+n_3} = -\frac{N_0-N_3}{N_1+N_2}, \quad \frac{n_2-n_3}{n_2+n_3} = -\frac{N_2-N_3}{N_0+N_1}, \quad (12)$$

and the formulae obtained by cycling the subscripts 1, 2, 3, while (11) corresponds to the facts that

$$\frac{n_0+n_1}{N_2+N_3}, \quad -\frac{n_2-n_3}{N_2-N_3}, \quad -\frac{n_1-n_0}{N_1-N_0}, \quad \frac{n_2+n_3}{N_0+N_1} \quad (13)$$

each equal

$$\frac{n_0}{N_0} \frac{\sin \frac{1}{2}\beta' \sin \frac{1}{2}\gamma'}{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma} \cdot \frac{\cos \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha'}. \quad (14)$$

We wish to shew these, and the dual formulae, from (7). Now in the present notation, the first equation of (7) is

$$\frac{l_1^2-1}{l_1^2+1} = \frac{(l_2^2-1)(l_3^2-1)}{(l_2^2+1)(l_3^2+1)} - \frac{4l_2l_3}{(l_2^2+1)(l_3^2+1)} \cdot \frac{L_1^2-1}{L_1^2+1}.$$

Hence

$$\frac{L_1^2-1}{L_1^2+1} = \frac{1}{2l_2l_3} \cdot \frac{1+l_2^2l_3^2-l_3^2l_1^2-l_1^2l_2^2}{l_1^2+1} = \frac{n_0^2+n_1^2-n_2^2-n_3^2}{2(n_0n_1+n_2n_3)},$$

$$L_1^2 = \frac{(n_0+n_1-n_2+n_3)(n_0+n_1+n_2-n_3)}{(-n_0+n_1+n_2+n_3)(n_0-n_1+n_2+n_3)}.$$

The second and third formulae of (7) give similar equations for L_2^2 and L_3^2. Multiplying these together in pairs, and selecting a square root,* we get, since $L_2L_3 = N_1/N_0$:

$$\frac{N_1}{N_0} = \frac{n_0 - n_1 + n_2 + n_3}{-n_0 + n_1 + n_2 + n_3}, \tag{15}$$

and the equations obtained by cycling the subscripts 1, 2, 3. From these we easily obtain (12), and so (9), (10).

So far N_0, n_0 have been arbitrary non-zero numbers. We now take

$$\left.\begin{aligned} 2N_0 &= -n_0 + n_1 + n_2 + n_3, \\ \text{thence, by (15),}\quad 2N_1 &= n_0 - n_1 + n_2 + n_3. \\ \text{Similarly}\quad 2N_2 &= n_0 + n_1 - n_2 + n_3, \\ 2N_3 &= n_0 + n_1 + n_2 - n_3. \end{aligned}\right\} \tag{16}$$

Solving these we have the 'dual' formulae:

$$\left.\begin{aligned} 2n_0 &= -N_0 + N_1 + N_2 + N_3, \\ 2n_1 &= N_0 - N_1 + N_2 + N_3, \\ 2n_2 &= N_0 + N_1 - N_2 + N_3, \\ 2n_3 &= N_0 + N_1 + N_2 - N_3. \end{aligned}\right\} \tag{17}$$

Further, $n_0 + n_1 = N_2 + N_3$, $\quad n_3 - n_2 = N_2 - N_3$,

$$n_0 - n_1 = N_1 - N_0, \quad n_2 + n_3 = N_0 + N_1.$$

Hence the fractions in (13) equal unity. As the duals of (7), (9), (10), (11) have the same connection with (17), as those formulae themselves have with (16), there is no need to consider the duals separately, and it only remains to shew (5) and to prove that the fractions in (13) do equal (14), that is, that (14) equals unity.

To shew (5), we have

$$\frac{\sin\alpha'}{\sin\alpha} = \frac{L_1(l_1^2+1)}{l_1(L_1^2+1)} = \frac{n_0n_1 + n_2n_3}{N_0N_1 + N_2N_3} \cdot \frac{N_0^2}{n_0^2} \cdot \frac{L_1L_2L_3}{l_1l_2l_3},$$

and since $n_0n_1 + n_2n_3 = N_0N_1 + N_2N_3$, we have

$$\frac{\sin\alpha'}{\sin\alpha} = \frac{\sin\beta'}{\sin\beta} = \frac{\sin\gamma'}{\sin\gamma} = \frac{N_0^2}{n_0^2} \cdot \frac{L_1L_2L_3}{l_1l_2l_3}, \tag{18}$$

which is (5), except that we have not yet found the value of m.

* This selection gives the formulae for the classical spherical triangles, in which all sides and interior angles have magnitudes between 0 and π. Cf. Study, *l.c.*

To shew that the fractions (14) equal unity; the coefficient of n_0/N_0 in (14) equals the values obtained by cycling α, β, γ and α', β', γ', by (18). Hence its square equals

$$\frac{\sin\frac{1}{2}\alpha'\sin\frac{1}{2}\beta'\sin\frac{1}{2}\gamma'}{\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}\cdot\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha'\cos\frac{1}{2}\beta'}\cdot\frac{\sin\frac{1}{2}\gamma'}{\sin\frac{1}{2}\gamma}$$

$$=\frac{l_1l_2l_3}{L_1L_2L_3}\frac{\sin\gamma'}{\sin\gamma}=\frac{N_0^2}{n_0^2},$$

by (18). Hence the fractions (14) equal unity. The fixing of the sign requires considerations into which we cannot enter.*

2. From (7), we deduce

$$(\sin\alpha'\sin\beta\sin\gamma)^2=\sin^2\beta\sin^2\gamma-(\cos\beta\cos\gamma-\cos\alpha)^2$$
$$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma$$
$$+2\cos\alpha\cos\beta\cos\gamma. \quad (19)$$

But by § 19·11

$$[uvw]/\overline{uvw}$$
$$=\sqrt{(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)}=n,$$

say. Dually

$$[u'v'w']/\overline{u'}\,\overline{v'}\,\overline{w'}$$
$$=\sqrt{(1-\cos^2\alpha'-\cos^2\beta'-\cos^2\gamma'+2\cos\alpha'\cos\beta'\cos\gamma')}=n',$$

say.

From (19) and its dual we have

$$\sin\alpha'\sin\beta\sin\gamma=n, \quad \sin\alpha\sin\beta\sin\gamma'=n',$$

and hence, as in § 20, we get (5).

§ 22. *Regressive multiplication in space.*†

1. We write the '*regressive product*' of the leaf $[pqr]$ and the rotor $[ab]$ as

$$[pqr.ab] \quad \text{or} \quad [ab.pqr],$$

and define it by the equation

$$[pqr.ab]=[ab.pqr]=[pqrb]\,a-[pqra]\,b. \quad (1)$$

If the rotor and leaf intersect, the right-hand side represents a weighted point at the cut, for it is clearly on $[ab]$; and it is on

* Study, *l.c.*

† This multiplication must be distinguished from regressive multiplication of bivectors in vector-spreads, § 19.

$[pqr]$, for if the right-hand side be multiplied by $[pqr]$, the result is zero.

From the identity, § 16·11 (6),

$$[pqrb]\,a-[pqra]\,b=[abqr]\,p+[abrp]\,q+[abpq]\,r,$$

we deduce

$$[pqr.ab]=[ab.pqr]=[abqr]\,p+[abrp]\,q+[abpq]\,r. \quad (2)$$

2. If A is a rotor, α a leaf, (1), (2) give

$$[\alpha.ab]=[ab.\alpha]=[a\alpha]\,b-[b\alpha]\,a=[\alpha b]\,a-[\alpha a]\,b, \quad (3)$$

$$[pqr.A]=[A.pqr]=[Aqr]\,p+[Arp]\,q+[Apq]\,r. \quad (4)$$

3. Writing A for $[pq]$ in (1), we have

$$[Ar.ab]=[ab.Ar]=[Arb]\,a-[Ara]\,b=-[Abr]\,a-[Ara]\,b.$$

Substitute p, q for a, b in this, and add to (4), then

$$[A.pqr]+[pq.Ar]=[Apq]\,r,$$

or, writing B for $[pq]$

$$[A.Br]+[B.Ar]=[AB]\,r. \quad (5)$$

4. A special case of (1) is $[ab.aqr]=[abqr]\,a$.

Let $[ab]=A$, $[aqr]=\alpha$, then $[A\alpha]=[abqr]\,a$.

Hence $\quad \text{mag}\,[A\alpha]=\text{mag}\,A.\text{mag}\,\alpha.\sin(A,\alpha)$.

$[A\alpha]=0$ if, and only if, either $A=0$ or $\alpha=0$ or A lies on α.

5. If A, B be rotors, α, β leaves, k scalar, then (3), (4) give

$$[A(\alpha+\beta)]=[A\alpha]+[A\beta], \quad [(A+B)\alpha]=[A\alpha]+[B\alpha], \quad (6)$$

$$[kA.\alpha]=[A.k\alpha]=k[A\alpha], \quad [A\alpha]=[\alpha A]. \quad (7)$$

6. If $[ab]$ and $[pqr]$ be parallel, then $[apqr]=[bpqr]$; hence

$$[ab.pqr]=[apqr]\,(b-a),$$

which is a vector.

If $[ab]$ lies on $[pqr]$, their regressive product is zero.

7. The '*regressive product*' of two leaves $[abc]$, $[pqr]$, in this order, is written $[abc.pqr]$, and is defined thus:

$$[abc.pqr]=[abcp]\,[qr]+[abcq]\,[rp]+[abcr]\,[pq]. \quad (8)$$

If the leaves cut, the product represents a rotor along their cut, for it is clearly on the plane pqr, and if we multiply the

right-hand side by $[abc]$ regressively, and use (2), we get as the coefficient of a,

$$[abcp]\,[bcqr]+[abcq]\,[bcrp]+[abcr]\,[bcpq]=0, \quad \text{by § 16 (11)}.$$

Similarly, the coefficients of b, c in the product vanish. Thus by (3) the expression (8) represents a rotor in plane abc.

If abc and pqr be parallel, then $[abcp]=[abcq]=[abcr]$, and the product $[abc.pqr]$ is a bivector, being a multiple of

$$[qr]+[rp]+[pq].$$

8. A special case of (8) is

$$[abc.abd]=[abcd]\,[ab].$$

Hence, if α, β be two intersecting leaves, then, since they can always be put in the form $[abc]$, $[abd]$, we have, if α, β intersect,

$$\text{mag}\,[\alpha\beta]=\text{mag}\,\alpha.\text{mag}\,\beta.\sin(\alpha,\beta).$$

9.
$$\left.\begin{aligned}[\alpha(\beta+\gamma)]&=[\alpha\beta]+[\alpha\gamma],\\ [k\alpha.\beta]=[\alpha.k\beta]&=k[\alpha\beta],\quad [\alpha\beta]=-[\beta\alpha].\end{aligned}\right\} \qquad (9)$$

10. We have deduced all our results from (1), (8). Now, if we assume the positions of $[pqr.ab]$ and $[pqr.\alpha]$ are at the cuts mentioned, we can deduce (1), (8) from the following:

Genese's rule. *If l, m, n be points, or rotors, or leaves, provided they are not all rotors, and provided $[ln]$ is a scalar, we have*

$$[l.mn]=[ln]\,m-[lm.n]. \qquad (10)$$

To deduce (1). Let $[ab.pqr]=ka+k'b$ (k, k′ scalar), then

$$[a..ab.pqr]=k'[ab].$$

Now, by (10) $\quad [a..ab.pqr]=[apqr]\,[ab].$

Hence $k'=[apqr]$. Similarly, $k=[pqrb]$. Hence (1) follows.

To deduce (8). Let

$$[pqr.\alpha]=k_1[qr]+k_2[rp]+k_3[pq], \quad (k_1, k_2, k_3 \text{ scalars}),$$

then $\quad [p..pqr.\alpha]=k_1[pqr].$

By (10) $\quad [p..pqr.\alpha]=[p\alpha]\,[pqr].$

Hence $k_1=[p\alpha]$. Similarly we can find k_2, k_3 and deduce an equation equivalent to (8).

11. The formulae (3), (5) are special cases of Genese's rule Dual to them, we have

$$[\alpha\beta.a] = [\alpha a]\,\beta - [\beta a]\,\alpha, \tag{11}$$

$$[A.B\alpha] + [B.A\alpha] = [AB]\,\alpha. \tag{12}$$

We deduce these, of course, without use of the rule:

For (11). Take $\alpha = [pqr]$, $\beta = [pqs]$, then by (8)

$$[\alpha\beta] = [pqrs]\,[pq].$$

Using § 16 (12), we find

$$\begin{aligned}[\alpha\beta.a] &= [pqrs]\,[pqa] = -[pqar]\,[pqs] - [pqsa]\,[pqr] \\ &= [\alpha a]\,\beta - [\beta a]\,\alpha.\end{aligned}$$

For (12). Let A, B cut α in a, b. We can write

$$A = [ap], \quad B = [bq], \quad \alpha = [abc].$$

Then

$$[A\alpha] = [ap.abc] = -[pabc]\,a,$$

$$[B\alpha] = [bq.abc] = -[qabc]\,b,$$

$$\begin{aligned}[A.B\alpha] + [B.A\alpha] &= -[qabc]\,[apb] - [pabc]\,[bqa] \\ &= [abqc]\,[abp] + [abcp]\,[abq] \\ &= -[abpq]\,[abc] = [AB]\,\alpha.\end{aligned}$$

Cases when parallelism occurs are treated the same way, using vectors or bivectors when necessary.

12. All cases of (10), it will be found, have now been shewn from (1), (8), except

$$[a.B\gamma] = [a\gamma]\,B - [aB.\gamma].$$

Since $[a.B\gamma] = -[B\gamma.a]$, since a, $[B\gamma]$ are points, we can write this

$$[aB.\gamma] = [a\gamma]\,B + [B\gamma.a]. \tag{13}$$

To prove (13), let $B = [pq]$ cut $\gamma = [pbc]$ in p, then

$$\begin{aligned}[aB.\gamma] = [apq.pbc] &= [apbc]\,[pq] + [qpbc]\,[ap] \\ &= [a\gamma]\,B + [qpbc]\,[ap],\end{aligned}$$

$$[B\gamma.a] = [pq.pbc..a] = [pqbc]\,[pa] = [qpbc]\,[ap].$$

13. $[L\alpha.\beta] = [L.\alpha\beta]$. (Genese's rule does not apply, since $[L\beta]$ is not scalar.)

For let L cut α in p, and β in q, and let $L = [pq]$. Let α, β cut in ab, then we can take a, b so that $\alpha = [pab]$, $\beta = \mathrm{k}[qab]$.

Then $[L\alpha.\beta] = \mathrm{k}[pq.pab..qab]$

by (1) $= \mathrm{k}[pabq]\,[pqab] = \mathrm{k}[pqab]^2$,

$[L.\alpha\beta] = \mathrm{k}[pq..pab.qab]$

by (8) $= \mathrm{k}[pq.pabq.ab] = \mathrm{k}[pqab]^2$.

Similarly $[\alpha L.\beta] = [\alpha.L\beta]$.

14. $$[\alpha.\beta\gamma] = [\alpha\beta.\gamma]. \qquad (14)$$

Each product can therefore be denoted by $[\alpha\beta\gamma]$; it represents a weighted point at the cut of the three planes.

15. $$[abc.\alpha\beta\gamma] = \begin{vmatrix} [a\alpha], & [a\beta], & [a\gamma] \\ [b\alpha], & [b\beta], & [b\gamma] \\ [c\alpha], & [c\beta], & [c\gamma] \end{vmatrix}, \qquad (15)$$

$$[AB]\,[abcd] = [Abc]\,[Bad] + [Acd]\,[Bab] + [Adb]\,[Bac] + [Bbc]\,[Aad] + [Bcd]\,[Aab] + [Bdb]\,[Aac], \qquad (16)$$

$$[abc.abd.acd.bcd] = [abcd]\,[ab.acd.bcd] = [abcd]^3. \qquad (17)$$

General formulae of this nature will be proved later.

16. *Multiplication by the unit trivector Ω annihilates a vector or a bivector, reduces a point to its weight, a rotor to its vector, a leaf to its bivector.*

For Ω is the difference of two parallel equal leaves. It may not be put equal to unity, except in a formula which involves only vectors and their products. Hence

$$[ab.\Omega] = [a\Omega]\,b - [b\Omega]\,a = b - a, \quad \text{since} \quad [b\Omega] = [a\Omega] = 1.$$

$$[abc.\Omega] = [a\Omega]\,[bc] + [b\Omega]\,[ca] + [c\Omega]\,[ab] = [bc] + [ca] + [ab].$$

Examples.

66. $[bc.aqr] + [ca.bqr] + [ab.cqr] = 2[abcq]\,r - 2[abcr]\,q$.

67. $[abp.cq] - [abq.cp] + [pqb.ac] - [pqa.bc] = 3[pqab]\,c$.

68. *If two tetrahedra be in perspective, that is, if the joins of corresponding vertices concur, then the lines of intersection of corresponding face-planes are coplanar.*

For if *abcd* be one tetrahedron, p the centre of perspective, then the vertices of the other will be points a', b', c', d' given by

$$(1 + \mathrm{k}_1)\,a' = p + \mathrm{k}_1 a, \ \ldots, \ (1 + \mathrm{k}_4)\,d' = p + \mathrm{k}_4 d,$$

where $\mathrm{k}_1, \ldots, \mathrm{k}_4$ are scalars. Hence

$$[b'c'd'] \equiv [p(\mathrm{k}_3\mathrm{k}_4[cd] + \mathrm{k}_4\mathrm{k}_2[db] + \mathrm{k}_2\mathrm{k}_3[bc])] + \mathrm{k}_2\mathrm{k}_3\mathrm{k}_4[bcd].$$

This plane cuts $[bcd]$ in the line

$$k_3k_4[cd]+k_4k_2[db]+k_2k_3[bc].$$

There are four such lines, and they all lie in the plane

$$k_2k_3k_4[bcd]-k_1k_3k_4[acd]+k_1k_2k_4[abd]-k_1k_2k_3[abc].$$

It is clear that, if we had absorbed weights, we could have omitted the scalars k. This absorbing of weights is often advantageous in a descriptive theorem, i.e. one not involving metric notions. Thus we have, if a, b, c have suitable weights,

$$a'=p+a,\quad b'=p+b,\quad c'=p+c,\quad d'=p+d,$$
$$[abc.a'b'c'] = [abc.(pbc+pca+pab+abc)]$$
$$= [abcp]\,([bc]+[ca]+[ab]),$$

which lies in the plane

$$[bcd]-[acd]+[abd]-[abc].$$

69. *If triangles abc, $a'b'c'$ are in perspective from d, then the cuts of $ab'c'$, $a'bc$ and of $bc'a'$, $b'ca$, and of $ca'b'$, $c'ab$, and of abc, $a'b'c'$ are coplanar.*

For adjust weights so that $a'=a+d$, $b'=b+d$, $c'=c+d$, then

$$[a'bc.ab'c'] = [a'ab'c']\,[bc]+[bab'c']\,[ca']+[cab'c']\,[a'b]$$
$$= [dabc]\,[bc]+[badc]\,[ca]+[cabd]\,[ab]$$
$$+[badc]\,[cd]+[cabd]\,[db]$$
$$= [abcd]\,(-[bc]+[ca]+[ab]+[cd]-[bd]).$$

Similarly,

$$[b'ca.bc'a'] = [abcd]\,(-[ca]+[ab]+[bc]+[ad]-[cd]),$$
$$[c'ab.ca'b'] = [abcd]\,(-[ab]+[bc]+[ca]+[bd]-[ad]).$$

The sum of these is

$$[abcd]\,([bc]+[ca]+[ab]) = [abc.a'b'c'].$$

70. *If $[aa'.bcd]$, $[bb'.cad]$, $[cc'.abd]$, $[dd'.abc]$ are coplanar, then $[a'..bcd.b'c'd']$, $[b'..cad.c'a'd']$, $[c'..abd.a'b'd']$, $[d'..abc.a'b'c']$ are concurrent.* (Fontené.)

71. A bisector of a dihedral angle of a tetrahedron divides the opposite edge in the ratio of the areas of the faces containing the angle.

The plane bisecting the angle along bc is

$$\alpha \equiv k_1[abc]+k_2[dbc],$$

where $k_1:k_2$ is the ratio of the areas of $[dbc]$, $[abc]$. It cuts ad in

$$p \equiv [\alpha.ad] = [abcd]\,(k_1a+k_2d).$$

72. If a, b be points, u, v vectors, then

$$[a|u.b|v] = [a|uv] + [(b-a)|v].|u.$$

For
$$\begin{aligned}[a|u.b|v] &= [a|u.(a+b-a)|v] \\ &= [a|u.a|v] + [a|u.(b-a)|v] \\ &= [a|uv] + [(b-a)|v]\ |u,\end{aligned}$$

since the product of a point and a trivector, $[a.(b-a)|v]$, is a scalar $[(b-a)|v]$, and the product of four vectors is zero.

Or thus: Since $[a|u.b|v]$ is the cut of planes $[a|u]$, $[b|v]$ it is in direction $|[uv]$. Hence $[a|u.b|v]$ is obtained from $[a|[uv]]$ by adding to it a bivector which is a multiple of $|u$:

$$[a|u.b|u] = [a|uv] + \mathrm{k}.|u = [b|uv] + \mathrm{k}'.|v.$$

Hence
$$\mathrm{k}|u = [(b-a)|uv] + \mathrm{k}'|v,$$

$$\mathrm{k}|[uv] = [(b-a)|uv]\,|v = [(b-a)|v]\,|[uv],$$

since
$$[|uv.|v] = [|uvv] = 0.$$

Hence
$$\mathrm{k} = [(b-a)|v].$$

Cor. If V_1, V_2 are bivectors, then

$$[aV_1.bV_2] = [a.V_1V_2] + V_1[(b-a)\ V_2].$$

73. *If five edges of a tetrahedron abcd meet (or are perpendicular to) five corresponding edges of a second tetrahedron (ab corresponding to a'b', and so on), then the same is true for the sixth pair of edges.*

74. *If abcd and a'b'c'd' are tetrahedra, and perpendiculars from the vertices of the first to the corresponding faces of the second (a corresponding to b'c'd', and so on) are concurrent, then perpendiculars from vertices of the second to corresponding faces of the first are concurrent and the plane through ab perpendicular to a'b', and the similar planes are concurrent.*

CHAPTER III

APPLICATIONS TO PROJECTIVE GEOMETRY

§ 23. *Foundations. Cross-ratio.*

1. Projective geometry in the strict sense knows nothing of the *lengths* of intervals or of the *size* of angles, nor of order relations such as those indicated by such words as '*between*', '*inside*', neither does it distinguish between parallel and non-parallel lines, for any two coplanar lines are assumed to meet.

It is possible to build up projective geometry abstractly from our point of view.

The elements with which we begin are '*weighted points*', and we allow as an operation the multiplication of a weighted point a by a scalar k. The result of this operation is written ka. We assume that the scalars k constitute a field, that is, they obey the laws of addition, subtraction, multiplication, and division of ordinary algebra. In particular, this field may be the field of real numbers, or the field of complex numbers.

We assume the laws of § 1.

Elements which differ only by a scalar factor are called '*congruent*'; we use the sign $\equiv$ for '*is congruent to*'. Two congruent extensives are said to have '*the same position*'. If a, b are non-congruent weighted points, elements of form $ka+k'b$ (k, k′ scalars) constitute a '*line*'. Thus a, b, c are in the same line if, and only if, there are scalars k, k′, k″, not all zero, such that $ka+k'b+k''c=0$.

Thus two non-congruent points fix just one line on which they lie, and a line is fixed by any two of its non-congruent points.

If a, b, c be not on the same line, elements of form $ka+k'b+k''c$ constitute a '*plane*'. Thus a, b, c, d are on the same plane if, and only if, there are scalars k, k′, k″, k‴ not all zero, such that $ka+k'b+k''c+k'''d=0$.

Thus three non-collinear points fix just one plane on which they lie, and a plane is fixed by any three non-collinear points

on it. If two points lie on a plane, so do all points of the line through them.

If the field of scalars k is the real field, we have a '*real*' plane; if it is the complex field, a '*complex*' or '*imaginary*' plane. If a, b, c be non-collinear points of a complex plane, and k, k′, k″ be restricted to be real, the set of points $\mathrm{k}a+\mathrm{k}'b+\mathrm{k}''c$ constitute a real plane, which is a portion of the complex plane.

In precisely the same way, we can introduce spreads of higher dimension.

2. *Outer products.* We cannot now define $[ab]$ as in the previous chapters, because we are not to use the notion of length. The only distinction we have drawn so far between pairs of points is to divide them into congruent and non-congruent pairs.

If a, b are congruent, we define $[ab]$ as zero.

Thus $\quad [aa]=0, \quad [a.\mathrm{k}a]=0, \quad [\mathrm{k}a.a]=0.$

We assume the distributive laws:

$$[a(\mathrm{k}b+\mathrm{l}c)]=\mathrm{k}[ab]+\mathrm{l}[ac],$$

$$[(\mathrm{k}b+\mathrm{l}c)\,a]=\mathrm{k}[ba]+\mathrm{l}[ca].$$

These have not an immediate interpretation in the usual projective geometry, but as they give

$$[(a+b)\,(a+b)]=[aa]+[ba]+[ab]+[bb],$$

and since the left-hand side is zero, and $[aa]=[bb]=0$, we have $[ab]=-[ba]$.

We assume, k being a scalar,

$$[\mathrm{k}a.b]=[a.\mathrm{k}b]=\mathrm{k}[ab].$$

We assume the outer product of three points is associative, and since $[aa]=0$, we assume $[aab]=0$ for all points b.

We assume that, if any one of a, b, c be multiplied by a scalar k, then $[abc]$ is so multiplied.

These, with $[ab]=-[ba]$, shew that $[aba]$, $[baa]$ vanish for all points b. Also $[abc]=-[bac]=[bca]=\ldots$ as before.

Finally, we assume the distributive law:

$$[(\mathrm{k}a+\mathrm{k}'b+\mathrm{k}''c)\,pq]=\mathrm{k}[apq]+\mathrm{k}'[bpq]+\mathrm{k}''[cpq].$$

Now if a, b, c are collinear, there are scalars k, k', k'', not all zero, such that $\mathrm{k}a+\mathrm{k}'b+\mathrm{k}''c=0$. Suppose $\mathrm{k}\neq 0$. Then

$$0=[(\mathrm{k}a+\mathrm{k}'b+\mathrm{k}''c)\,bc]=\mathrm{k}[abc].$$

Hence $[abc]=0$.

Conversely, as in § 5·3, if we assume some products of three points do not vanish, we can shew that if $[abc]=0$, then a, b, c are collinear.

3. *Regressive products.* We wish the regressive product of two congruent lines to be zero, and of two non-congruent lines to be a weighted point on both. We secure this if we assume

$$[ab.ac]=[abc]\,a.$$

The proof in § 11·4 of the general formulae for regressive products there given is valid here, if we assume that regressive multiplication is distributive over addition.

There should now be no difficulty in extending the work to space.

Although the interpretations of sums of weighted points in terms of centres of gravity, of outer products in terms of lengths and areas are foreign to projective geometry, yet as our formulae are capable of these interpretations, we shall on occasion mention the metric interpretation of our formulae.

4. *Def.* If a, b be any non-congruent extensives of the same step, the extensives $\mathrm{k}a+\mathrm{k}'b$, where k, k' are any scalars, form a '*pencil*' or '*range*'.

[It is usual to speak of 'pencils' of lines, and of 'ranges' of points.]

If we do not distinguish between congruent extensives, we can denote elements of a pencil or range by $\mathrm{k}a+b$, where k is a varying scalar.

The range $\mathrm{k}a+\mathrm{k}'b$ includes the extensive a as one of its values. In order that $\mathrm{k}a+\mathrm{k}'b$ and $\mathrm{k}a+b$ should have the same set of values, we must therefore assume that $\mathrm{k}a+b$ can take the value a. We make this assumption as an explicit convention, and say $\mathrm{k}a+b\equiv a$, when $\mathrm{k}=\infty$.

If p_1, p_2, p_3, p_4 are in a pencil, say $p_i=\mathrm{k}_i a+b$, then

$$[p_1p_2]=(\mathrm{k}_1a+b)\,(\mathrm{k}_2a+b)=(\mathrm{k}_1-\mathrm{k}_2)\,[ab].$$

Def. The '*cross-ratio*' of the extensives p_1, p_2, p_3, p_4 in a pencil, taken in this order, is denoted by $R(p_1, p_2, p_3, p_4)$, and defined to be

$$[p_1p_2]\,[p_3p_4]/[p_2p_3]\,[p_4p_1],$$

or

$$(k_1-k_2)\,(k_3-k_4)/(k_2-k_3)\,(k_4-k_1).$$

Thus the cross-ratio of the coplanar lines $[pa]$, $[pb]$, $[pc]$, $[pd]$ is

$$[pab]\,[pcd]/[pbc]\,[pda].$$

If the cross-ratio is -1, the extensives form a '*harmonic*' set.

If $p_i = k_i a + l_i b$ $(i = 1, \ldots, 4)$, the definition gives at once

$$R(p_1 p_2 p_3 p_4) = \frac{(k_1 l_2 - k_2 l_1)\,(k_3 l_4 - k_4 l_3)}{(k_2 l_3 - k_3 l_2)\,(k_4 l_1 - k_1 l_4)},$$

$$R(ap_1\, bp_2) = \frac{l_1 k_2}{l_2 k_1}.$$

5. These cross-ratios are clearly not changed when any of the extensives involved are replaced by congruent extensives; they depend only on the positions of the extensives. Thus, by absorbing weights, we can take four extensives in a pencil to be a, b, q_1, q_2, where $q_1 = a+b$, $q_2 = a+kb$. Then $R(aq_1\, bq_2) = k^{-1}$.

In particular, a harmonic set can always be written a, $a+b$, b, $a-b$.

6. *Def.* Two ranges of extensives are in '*projective correspondence*' or are '*projective ranges*', when they can be put into one-to-one correspondence, such that, if a, b, p_1, p_2, p_3, ... in one range correspond to a', b', p_1', p_2', p_3', ... in the other, the extensives can be weighted (i.e. replaced by congruent extensives) so that then, if $p_i = a + l_i b$, we have $p_i' = a' + l_i b'$.

Then the cross-ratio of any four extensives in one range equals that of the corresponding extensives in the other; and if

$$k_1 p_1 + k_2 p_2 + k_3 p_3 = 0, \quad \text{then} \quad k_1 p_1' + k_2 p_2' + k_3 p_3' = 0.$$

Since any three extensives a, b, c in a range can be weighted so that $a+b=c$, and then, if $d = a+kb$ is any extensive in the range, k is fixed by the position of d. It follows that a projective correspondence between two ranges is determined when we know the *positions* of three elements of the second range which correspond to three in the first.

7. *Double points, or self-corresponding points*, of two projective ranges of points on the same line.

Let the ranges be such that $a+\mathrm{k}b$ corresponds to $c+\mathrm{k}d$ as k varies.

Let $c = a+\mathrm{x}b$, $d = a+\mathrm{y}b$, then $c+\mathrm{k}d = (1+\mathrm{k})\,a+(\mathrm{x}+\mathrm{ky})\,b$.

This is congruent to $a+\mathrm{k}b$ if $\mathrm{k}^2+(1-\mathrm{y})\,\mathrm{k}-\mathrm{x} = 0$.

Thus for the case of the complex line, we have two self-corresponding points, which may coincide. If the line is real, there may or may not be self-corresponding points.

8. If $a, b, p_1, p_2, \ldots$ and $a', b', p_1', p_2', \ldots$ be projective ranges, and if $b \equiv b'$, the ranges are in '*perspective*'; we can absorb weights so that $b+b' = 0$, and $p+p' = a+a'$, where p, p' are corresponding extensives suitably weighted.

For if $p = a+\mathrm{k}b$, then $p' = a'+\mathrm{k}b' = a'-\mathrm{k}b$.

Thus, for example, if the ranges are projective ranges of points on two lines, the lines joining corresponding points are concurrent; if they are perspective pencils of lines

$$(L = A+\mathrm{k}B,\ L' = A'+\mathrm{k}B',\ B \equiv B'),$$

the cuts of corresponding lines are collinear.

9. If the cross-ratio of the extensives a, b, c, d in one range equals the cross-ratio of the extensives a', b', c', d' in another range, and we absorb weights so that

$$a+b = c, \quad a'+b' = c',$$

then if $d = a+\mathrm{k}b$, we have $d' = a'+\mathrm{k}b'$.

Thus, if two ranges are in one-to-one correspondence, and the cross-ratio of any four extensives of one range equals that of the corresponding extensives of the other range, the ranges are projective.

10. Hence, if the ranges of extensives $R_1, R_2, \ldots, R_n$ be such that each is projective to the next following, then R_1 and R_n are projective.

11. *If the planes α, β, γ, δ meet in a common line, their cross-ratio is the cross-ratio of their cuts with any transversal.*

For, by absorbing weights, we can let $\beta = \alpha+\gamma$, $\delta = \alpha+\mathrm{k}\gamma$, then $R(\alpha\beta\gamma\delta) = \mathrm{k}^{-1}$.

Let the transversal $[pq]$ cut the planes in p_1, p_2, p_3, p_4, then

$$p_2 \equiv [pq.(\alpha+\gamma)] = [pq.\alpha]+[pq.\gamma] = p_1+p_3,$$

$$p_4 \equiv p_1+\mathrm{k}p_3, \quad R(p_1p_2p_3p_4) = \mathrm{k}^{-1}.$$

Hence cross-ratio is unchanged by projection and section.

12. *v. Staudt's Theorem. If abcd is a tetrahedron, $\alpha, \beta, \gamma, \delta$ the planes opposite to a, b, c, d and L any line, then*

$$R(La, Lb, Lc, Ld) = R(L\alpha, L\beta, L\gamma, L\delta).$$

For there are scalars $\mathrm{k}_1, \mathrm{l}_1, \mathrm{k}_2, \mathrm{l}_2$ such that

$$[Lb] = \mathrm{k}_1[La]+\mathrm{l}_1[Lc], \quad [Ld] = \mathrm{k}_2[La]+\mathrm{l}_2[Lc].$$

Hence $$\mathrm{l}_1[Lca] = [Lba], \quad \mathrm{k}_1[Lac] = [Lbc],$$

$$\mathrm{l}_1\mathrm{k}_1^{-1} = [Lab]/[Lbc], \quad \mathrm{l}_2\mathrm{k}_2^{-1} = [Lda]/[Lcd].$$

Hence $$R(La, Lb, Lc, Ld) = [Lab]\,[Lcd]/[Lbc]\,[Lda],$$

$$R(L\alpha, L\beta, L\gamma, L\delta) = [L\alpha\beta]\,[L\gamma\delta]/[L\beta\gamma]\,[L\delta\alpha].$$

But $$[L\alpha\beta] = [L.bcd.cda] = -[abcd]\,[Lcd]. \quad (\S\,22{\cdot}13.)$$

This and the analogous formulae give the theorem at once.

Example. 1. The joins of d to the vertices of a triangle abc meet the opposite side-lines in p, q, r; if $pbp'c$, $qcq'a$, $rar'b$ are harmonic ranges, then p', q', r' are collinear.

For we can absorb weights so that $a+b+c = d$, then $b+c = d-a$ is a weighted point at p. Hence we can put $b+c = p$, thence $b-c = p'$. Similarly by suitable weighting $c-a = q'$, $a-b = r'$. Hence

$$p'+q'+r' = 0.$$

§ 24. *Conics.*

1. If $L = A+\mathrm{k}B$, $L' = A'+\mathrm{k}B'$ be projective pencils of lines in a plane, given by varying k, then L, L' cut in a point p which satisfies $$[pA]+\mathrm{k}[pB] = 0, \quad [pA']+\mathrm{k}[pB'] = 0.$$

Eliminating k, we have

$$[pA]\,[pB']-[pB]\,[pA'] = 0. \qquad (1)$$

If $A \equiv A'$, the pencils are perspective, and p lies either on A, or on a line through the cut of B and B'. If $A-A' \not\equiv B-B'$, the locus of p is a '*non-degenerate conic*'.

Note, in passing, the *metric* interpretation of (1). Since $[pA]$ is the distance from p to A multiplied by a scalar depending on

the weight of p and the magnitude of A, the equation says that the product of the distances from p to A and B' is in a constant ratio to the product of the distances from p to A' and B.

2. The equation of the locus can also be found as follows:

$$p \equiv [(A + \mathrm{k}B)\,(A' + \mathrm{k}B')] = a + 2\mathrm{k}b + \mathrm{k}^2 c, \qquad (2)$$

where $a = [AA'] \neq 0$, $2b = [BA'] + [AB']$, $c = [BB'] \neq 0$.

The points a, c are on the curve.

Conversely, if $[abc] \neq 0$, a point p satisfying (2) describes a non-degenerate conic. For write b for $2b$, for convenience, and let $A = [bc]$, $B = [ca]$, $C = [ab]$.

Then (2) gives $[ap] \equiv C - \mathrm{k}B$, $[cp] \equiv B - \mathrm{k}A$; pencils $[ap]$, $[cp]$ as k varies are projective.

A line L cuts the locus (2) in points p such that k satisfies

$$[aL] + 2\mathrm{k}[bL] + \mathrm{k}^2[cL] = 0,$$

that is, in two points, if the field of k is complex, and two points at most, if it is real. These points coincide, and L is a '*tangent*' to the locus, if

$$[bL]^2 = [aL]\,[cL].$$

3. From equation (1), it appears that the points $[AB]$, $[A'B']$, $[AA']$, $[BB']$ are on the curve. For example, if $p = [AB]$, then $[pA] = [pB] = 0$. Denote these points by a, a', b, b', then

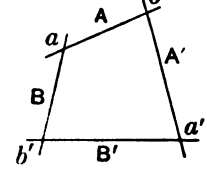
$$A \equiv [ab], \quad B \equiv [ab'], \quad A' \equiv [a'b], \quad B' \equiv [a'b'],$$

and (1) becomes

$$[pab]\,[pa'b'] = \mathrm{x}[pab']\,[pa'b], \quad \mathrm{x} \text{ scalar}. \qquad (3)$$

Hence, by § 23·4, *p moves so that $R(pa, pb, pa', pb')$ is constant.*

If the curve goes also through c', we can find x by substituting $p = c'$ in (3).



Then

$$[c'ab]\,[c'a'b'] = \mathrm{x}[c'ab']\,[c'a'b]. \qquad (3')$$

Using this value of x, we have from (3)

$$[pab]\,[pa'b']\,[ab'c']\,[a'bc'] - [pab']\,[pa'b]\,[abc']\,[a'b'c'] = 0. \qquad (4)$$

Thus the conic is fixed by the five points a, b, a', b', c' on it, but it must be noticed that these points have been introduced in different ways.

4. Consider equations (3), (4), disregarding the method by which they were obtained. If three of the points a, b, a', b', c' are collinear, for instance if $[ab'c']=0$, a term vanishes, in this case the first. The second term must then vanish, and then either p is on $[ab']$ or on $[a'b]$, or b is on $[ac']\equiv[ab']$, or a' is on $[b'c']\equiv[ab']$. The six points involved thus lie on two lines.

This clearly happens when, in (3'), $k=0$ or ∞. If $k=1$, then since (§ 5·7)

$$[pab]\,[pa'b'] = [pab']\,[pa'b] + [paa']\,[pbb'],$$

equation (3) becomes

$$[paa']\,[pbb'] = 0,$$

and again represents a line pair. But, in general, equation (3) represents a non-degenerate conic, since (1) does so.

5. The point p is on the conic fixed by a, b, a', b', c' and given by equation (4). It was shewn in § 12 that the left hand of (4) changes sign at most, when two letters are interchanged. Hence the five points define the same conic, whatever be the order in which they are introduced. Further, a is on the conic defined by p, b, a', b', c', because of this property of interchange of letters. Thus any one of the defining points may be replaced by any other point of the conic; thus all five may be replaced by any such five. We have assumed that the left hand of (4) is not identically zero, and that the conic is not degenerate.

Hence *a non-degenerate conic is uniquely defined by any five of its points*. Hence, by (3), *the cross-ratio of the joins of a variable point of a non-degenerate conic to any fixed four of its points is constant*.

Any five points in a plane, no three collinear, define a non-degenerate conic. For, if a, b, c, d, e be the points, then, by § 23·6, the projective correspondence between the pencils of lines through a and through e is fixed when we make $[ab]$, $[ac]$, $[ad]$ correspond to $[eb]$, $[ec]$, $[ed]$ respectively.

6. *Pascal's Theorem*. We have (4) for any six points on a conic. Write c for p, then

$$[cab]\,[ca'b']\,[ab'c']\,[a'bc'] - [cab']\,[ca'b]\,[abc']\,[a'b'c'] = 0.$$

We may change a, b, c into b, c, a, then this becomes

$$[abc][aa'b'][bb'c'][cc'a'] - [abb'][caa'][bcc'][a'b'c'] = 0,$$

or, by § 12 (25),

$$[bc'.b'c..ca'.c'a..ab'.a'b] = 0. \qquad (5)$$

Hence this holds if, and only if, a, b, c, a', b', c' lie on a conic, degenerate or not. This is Pascal's Theorem.

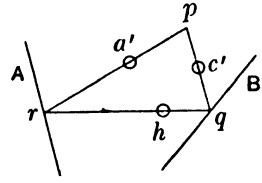
7. Write p for b in (5) and we have

$$[pc'.b'c..ca'.c'a..ab'.a'p] = 0.$$

Let $[b'c] = B$, $[ca'.c'a] = h$, $[ab'] = A$, then

$$[pc'B..h..A.a'p] = 0.$$

The three main factors represent points, and hence their product is associative. Hence $[pc'Bh..A.a'p] = 0$. But $[pc'Bh]$, A, $[a'p]$ are lines, and hence their product is associative (§ 11·10) and may be written $[pc'BhA.a'p]$. Now here $[pc'BhA]$, a', p are points, and so finally $[pc'BhAa'p] = 0$.



Hence if a', c', h are fixed points, and A, B lines, and the join of p and c' cuts B in $[pc'B] = q$, say, and $[qh]$ cuts A in $[qhA] = r$, say, then, if $[ra']$ goes through p, the locus of p is a conic. (Note the convention of § 13·1.)

8. *Conic envelopes.* Dually to 1, the envelope of lines joining corresponding points $a + \mathrm{k}b$, $a' + \mathrm{k}b'$ of two projective ranges of points, not in perspective, on distinct lines is a (non-degenerate) conic envelope. If L be such a join, then

$$[La] + \mathrm{k}[Lb] = 0, \quad [La'] + \mathrm{k}[Lb'] = 0.$$

Eliminate k, then

$$[La][Lb'] - [Lb][La'] = 0.$$

The *metric* interpretation of this is that L moves so that the product of its distances from a and b' is in a constant ratio to the product of its distances from b and a'.

It is clear that the development dual to the above can be carried out in detail. In particular Brianchon's Theorem, the dual of Pascal's Theorem, follows: The lines A, B, C, A', B', C' touch a conic envelope, if, and only if,

$$[BC'.B'C..CA'.C'A..AB'.A'B] = 0. \qquad (6)$$

9. By (2), any point p on a non-degenerate conic locus through a, c can be written $p \equiv a + 2kb + k^2c$, where b is a suitably chosen fixed point.

The line

$$L = [p(b+kc)] = [(a+2kb+k^2c)\,(b+kc)]$$
$$= [ab] + k[ac] + k^2[bc]$$

goes through p, and

$$[La] = k^2[abc], \quad [Lb] = -k[abc], \quad [Lc] = [abc].$$

Hence
$$[Lb]^2 = [La]\,[Lc].$$

Hence, by 2, L is a tangent at p.

Hence *at each point of a non-degenerate conic locus there is a tangent. This tangent is unique.* For any line L is of form

$$k_1[ab] + k_2[ac] + k_3[bc].$$

By 2, this is a tangent only if $k_2^2 = k_1 k_3$. Hence then

$$L \equiv [ab] + k'[ac] + k'^2[bc].$$

If this goes through $p = a + 2kb + k^2c$, then

$$0 = [Lp] = k^2[abc] + 2kk'[acb] + k'^2[abc],$$

hence
$$(k' - k)^2 = 0, \quad k' = k.$$

10. *The tangents to a non-degenerate conic locus give a conic envelope.*

Take points on the locus in form $p = a + 2kb + k^2c$. Tangents are then of form $L \equiv [ab] + k[ac] + k^2[bc]$, and $[ab]$, $[bc]$ in particular satisfy the condition for being tangents. Now the dual to 2 shews that all lines of a conic envelope are of form $L = C + kB + k^2A$, where C, A are lines of the envelope and B a suitably chosen line, and that conversely, if $[ABC] \neq 0$, then a line L satisfying this equation describes a conic envelope.

11. *If the vertices of triangles abc, a'b'c' lie on a conic, their sides touch another conic.*

For by § 12 (25), since

$$[bc'.b'c..ca'.c'a..ab'.a'b] = 0,$$

we have

$$[aa'b']\,[bb'c']\,[cc'a']\,[abc] = [abb']\,[bcc']\,[caa']\,[a'b'c'].$$

Let $$a=[BC],\quad b=[CA],\quad c=[AB],$$
$$a'=[B'C'],\quad b'=[C'A'],\quad c'=[A'B'].$$

Hence $[aa'b'] = [BC.B'C'.C'A'] = [A'B'C'][BCC']$,

and so on, whence

$$[BCC'][CAA'][ABB'][A'B'C'] = [CC'A'][AA'B'][BB'C'][ABC].$$

Hence $$[BC'.B'C..CA'.C'A..AB'.A'B] = 0.$$

Examples. 2. The points a, b, c, p, q, r are on a conic if

$$[pab][prc]/[pbc][pra] = [qab][qrc]/[qbc][qra].$$

Let $$p = x_1a + y_1b + z_1c,\quad q = x_2a + y_2b + z_2c,$$
$$r = x_3a + y_3b + z_3c,$$

then this gives
$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0.$$

3. If $p_1p_2'p_3p_1'p_2p_3'$ be a hexagon inscribed in a conic, and

$$[p_2p_3'] = A_1,\quad [p_3p_1'] = A_2,\quad [p_1p_2'] = A_3,$$
$$[p_2'p_3] = A_1',\quad [p_3'p_1] = A_2',\quad [p_1'p_2] = A_3',$$

then $$[A_1A_1'.A_2A_2'.A_3A_3'] + [A_2A_1'.A_3A_2'.A_1A_3'] + [A_3A_1'.A_1A_2'.A_2A_3'] = 0$$

is an identity, § 12 (25′). The first term vanishes by Pascal's Theorem. The others give

$$[p_3p_1'p_2'][p_1p_2'p_3'][p_2p_3'p_1'][p_1p_2p_3] = [p_3'p_1p_2][p_1'p_2p_3][p_2'p_3p_1][p_1'p_2'p_3'].$$

Cycle p_1, p_2, p_3, then from the last equation and that so obtained, we have, on division,

$$\frac{[p_1'p_2p_3][p_1p_2'p_3][p_1p_2p_3']}{[p_1'p_3p_1][p_2p_2'p_1][p_2p_3p_3']} = \frac{[p_1p_2'p_3'][p_2p_3'p_1'][p_3p_1'p_2']}{[p_2p_2'p_3'][p_3p_3'p_1'][p_1p_1'p_2']},$$

which gives *Carnot's Theorem* on the ratios of the intervals cut off by a conic on the sides of a triangle.

4. Two variable points p, q moving on a fixed line L are connected by the relation $q = psL_1s_1L$, where s, s_1 are fixed points, and L_1 is a fixed line. Prove that these ranges are projective.

The relation can be written

$$q = pL_1 . ss_1 L + sL_1 . s_1 L . p.$$

Also, if in any one instance, when $p \neq q$, we have $p = qsL_1 s_1 L$, then $sL . s_1 L_1 + sL_1 . s_1 L = 0$. (Univ. of Wales.)

5. If $a, b, c, d, a_1, b_1, c_1, d_1$ be on a conic, then

$$[ab . a_1 b_1 . . cd . c_1 d_1], \quad [bc . b_1 c_1 . . ad . a_1 d_1], \quad [ca . c_1 a_1 . . bd . b_1 d_1]$$

are concurrent.

§ 25. *Canonical equation of a conic.*

1. We found, for a point p on a non-degenerate conic through a, c

$$p \equiv a + 2kb + k^2 c, \tag{7}$$

where b is a suitably chosen point.

Take $\frac{1}{2}(a+c) = e_1$, $b = e_2$, $\frac{1}{2}(a-c) = e_3$, then:

If

$$p \equiv x_1 e_1 + x_2 e_2 + x_3 e_3, \tag{8}$$

we have $p \equiv (x_1 + x_3) a + 2x_2 b + (x_1 - x_3) c$, and p is on the conic, if

$$x_2^2 = (x_1 + x_3)(x_1 - x_3), \quad \text{that is,} \quad -x_1^2 + x_2^2 + x_3^2 = 0.$$

This will serve as the canonical equation of a conic whether the field of scalars is real or imaginary. If the field is imaginary, we can make the equation more symmetrical by absorbing the weight $\sqrt{(-1)}$ into e_1, still keeping (8). The last equation then becomes

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

2. If the field of scalars is imaginary, any line pq cuts a conic in two points (coincident or not).

For if $p = k_1 e_1 + k_2 e_2 + k_3 e_3$, $p' = k_1' e_1 + k_2' e_2 + k_3' e_3$, then $p + lp'$ is on the conic $x_1^2 + x_2^2 + x_3^2 = 0$, if

$$(k_1 + lk_1')^2 + (k_2 + lk_2')^2 + (k_3 + lk_3')^2 = 0,$$

a quadratic to determine l.

Whether the field of scalars be real or complex, if

$$p = x_1 e_1 + x_2 e_2 + x_3 e_3$$

be on the conic $\pm x_1^2 + x_2^2 + x_3^2 = 0$, the usual argument in analytical geometry shews that $y_1 e_1 + y_2 e_2 + y_3 e_3$ is on the tangent at p if $\pm x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$.

3. *Def.* If the field of scalars is imaginary, two points, not on a conic, are '*conjugate*' with respect to the conic, if their join cuts the conic in two points separating them harmonically; a point on the conic is '*conjugate*' to any point on the tangent to the conic at the point.

If $p_1 = x_1 e_1 + x_2 e_2 + x_3 e_3$, $p_2 = y_1 e_1 + y_2 e_2 + y_3 e_3$ be the points, then $p_1 + k p_2$ is on the conic $\pm x_1^2 + x_2^2 + x_3^2 = 0$ if

$$\pm (x_1 + k y_1)^2 + (x_2 + k y_2)^2 + (x_3 + k y_3)^2 = 0. \qquad (8')$$

Now by § 23·4, if k_1, k_2 (not zero) be roots of this equation, then p_1, $p_1 + k_1 p_2$, p_2, $p_1 + k_2 p_2$ are a harmonic range if $k_1 k_2^{-1} = -1$, or $k_1 + k_2 = 0$. Hence the coefficient of k in (8') vanishes:

$$\pm x_1 y_1 + x_2 y_2 + x_3 y_3 = 0. \qquad (8'')$$

If p_1 be on the conic, we get the same condition, by 2.

If p_1 be fixed, equation (8'') shews that p_2 describes a line, the '*polar*' of p_1; p_1 is the '*pole*' of this line.

If the field of scalars is real, some lines through p_1 may not cut the conic; the points conjugate to p_1, with the above definition, would then only take up part of the line. We call the whole of the line the 'polar' of p_1, and extend the meaning of 'conjugate' points so that all points of this line are conjugate to p_1.

The equation (8'') shews that if p_2 lies in the polar of p_1, then p_1 lies on the polar of p_2, and that each point has a unique polar. Thus if p, q be points on a given line, their polars are fixed, and the pole of $[pq]$ lies on them, since p, q are on $[pq]$. Thus each line has a pole, which is unique.

§ 26. *Supplements with respect to a non-degenerate conic.*

1. The theory of supplements, so far developed, contains a gap; the supplement of a point was not defined, unless the point were at infinity, and then it was another point at infinity. We now introduce the notion of supplement differently; this and the previous notion must not be used together, or at least *the two must not be denoted by the same sign in the same investigation.*

2. Consider the conic whose equation referred to reference points e_1, e_2, e_3 is $\pm x_1^2 + x_2^2 + x_3^2 = 0$.

Let $E_1 = [e_2 e_3]$, $E_2 = [e_3 e_1]$, $E_3 = [e_1 e_2]$ and take the weights of e_i so that $[e_1 e_2 e_3] = 1$; then $[E_1 E_2 E_3] = 1$.

If $p = x_1 e_1 + x_2 e_2 + x_3 e_3$, then $q = y_1 e_1 + y_2 e_2 + y_3 e_3$ is on the polar of p if $\pm x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$. Now the polar is

$$\pm x_1 E_1 + x_2 E_2 + x_3 E_3, = P, \text{ say,}$$

since $$[qP] = \pm x_1 y_1 + x_2 y_2 + x_3 y_3.$$

In particular p lies on its own polar, if and only if

$$\pm x_1^2 + x_2^2 + x_3^2 = 0.$$

If a line through any point p meets the conic in q, r, then the tangents at q, r being polars of q, r meet on the polar of p. This is the starting-point of the usual theory.

3. If $p = x_1 e_1 + x_2 e_2 + x_3 e_3$, $P = \pm x_1 E_1 + x_2 E_2 + x_3 E_3$, the signs being taken according to those in the equation $\pm x_1^2 + x_2^2 + x_3^2 = 0$, we write $p = |P$, $P = |p$, and speak of supplements.

Thus $$||p = |P = p, \quad ||P = p. \tag{9}$$

Thus p and P are pole and polar, and as these were defined geometrically in terms of harmonic section, our definition of supplement, apart from the weight factor, is independent of the canonical equation adopted for the conic.

In particular $|e_1 = \pm E_1$, $|e_2 = E_2$, $|e_3 = E_3$.

Since $[e_1 e_2 e_3] = 1$, we have

$$[e_1 | e_1] = \pm 1, \quad [e_2 | e_2] = 1, \quad [e_3 | e_3] = 1,$$

and, of course, $$[e_i | e_j] = 0, \quad i \neq j.$$

Also $$|(x_1 e_1 + x_2 e_2 + x_3 e_3) = |p = \pm x_1 E_1 + x_2 E_2 + x_3 E_3$$
$$= x_1 |e_1 + x_2 |e_2 + x_3 |e_3.$$

Hence, if p, q be any points, k_1, k_2 scalars, then

$$|(k_1 p + k_2 q) = k_1 |p + k_2 |q. \tag{10}$$

For, if $p = x_1 e_1 + x_2 e_2 + x_3 e_3$, $q = y_1 e_1 + y_2 e_2 + y_3 e_3$, then

$$|(k_1 p + k_2 q) = |((k_1 x_1 + k_2 y_1) e_1 + \ldots) = (k_1 x_1 + k_2 y_1) |e_1 + \ldots$$
$$= k_1 (x_1 |e_1 + \ldots) + k_2 (y_1 |e_1 + \ldots) = k_1 |p + k_2 |q.$$

Equation (10) expresses the fact that the polar of a point on $[pq]$ goes through the cut of the polars of p and q. It shews also that the cross-ratio of four collinear points equals that of their (concurrent) polars.

Dually, for lines

$$|(k_1 L + k_2 M) = k_1 |L + k_2 |M. \tag{11}$$

4. *Inner products* are defined as before, namely, $[p\,|q]$ is the outer product of p and $|q$.

Now take $x_1^2+x_2^2+x_3^2=0$ as the equation of the conic.

If $\quad p=x_1e_1+x_2e_2+x_3e_3, \quad q=y_1e_1+y_2e_2+y_3e_3,$

then $|q=y_1E_1+y_2E_2+y_3E_3$, and so

$$[p\,|q]=x_1y_1+x_2y_2+x_3y_3. \tag{12}$$

Thence $$[p\,|q]=[q\,|p]. \tag{13}$$

This product vanishes, if, and only if, p, q are conjugate.

Dually $$[L\,|M]=[M\,|L]. \tag{14}$$

This product vanishes if, and only if, L, M are conjugate.

$$[pq]=(x_2y_3-x_3y_2)\,E_1+(x_3y_1-x_1y_3)\,E_2+(x_1y_2-x_2y_1)\,E_3,$$

$$[|p\,.\,|q]=(x_2y_3-x_3y_2)\,e_1+(x_3y_1-x_1y_3)\,e_2+(x_1y_2-x_2y_1)\,e_3,$$

since $[E_2E_3]=[e_1e_2e_3]\,e_1=e_1$, and so on.

Thence $$|[pq]=[|p\,.\,|q]. \tag{15}$$

Dually $$|[LM]=[|L\,.\,|M]. \tag{16}$$

5. We define the supplement of a scalar as the scalar itself.

Then, if $L=y_1E_1+y_2E_2+y_3E_3$, we have

$$[pL]=x_1y_1+x_2y_2+x_3y_3, \quad \text{and} \quad [|p\,.\,|L]=x_1y_1+x_2y_2+x_3y_3.$$

Hence $$|[pL]=[|p\,.\,|L]. \tag{17}$$

The only one of these formulae which is different from those in the earlier interpretation (§ 7) is (9), though we have several extra formulae. The reason for the difference will be made plain later.

6. We write $[a\,|bc]$ for $[a\,.\,|bc]$, and $[ab\,|cd]$ for $[[ab]\,|[cd]]$.

Hence $$[ab\,|cd]=[cd\,|ab].$$

Thus $[a\,|bc]$ is a line, the join of a to the pole of bc; it is quite distinct from $[a\,|b]\,c$, which is a point, and from $[a\,|b]\,|c$, which is a line.

If $|a=A$, then

$$[bc\,|a]=[bcA]=[bA]\,c-[cA]\,b.$$

Hence $$[bc\,|a]=[b\,|a]\,c-[c\,|a]\,b. \tag{18}$$

Take supplements. Since bc and $|a$ are lines,

$$|[bc\,|a] = [|bc.\,||a] = [|bc.a].$$

But $|[bc]$ and a are points, hence $[|bc.a] = -[a\,|bc]$. Hence (18) gives

$$[a\,|bc] = [a\,|c]\;|b - [a\,|b]\;|c. \qquad (19)$$

From (18),

$$[bc\,|a] + [ca\,|b] + [ab\,|c] = 0. \qquad (20)$$

Let $|a = [b'c']$, and so on, then $[bc\,|a]$ is the cut of lines bc, $b'c'$. Hence

Hesse's Theorem. The triangle abc is in perspective with its polar triangle, that is, with the triangle whose sides are the polars of a, b, c.

The dual theorem

$$[BC\,|A] + [CA\,|B] + [AB\,|C] = 0$$

gives the same statement.

The axis of perspective of the two triangles is (cf. § 20, Exs. 52, 56),

$$([b\,|a]\;c - [c\,|a]\;b)\;([c\,|b]\;a - [a\,|b]\;c)$$

or

$$[a\,|b]\;[a\,|c]\;[bc] + [b\,|c]\;[b\,|a]\;[ca] + [c\,|a]\;[c\,|b]\;[ab].$$

7. Since $[bc]$, $|a$, $|d$ are lines, their outer multiplication is associative. Hence

$$[bc\,|a.\,|d] = [bc.\,|a\,|d] = [bc\,|ad].$$

Multiply (20) on the right by $|d$, and we have

$$[bc\,|ad] + [ca\,|bd] + [ab\,|cd] = 0.$$

Hence *if abcd be any quadrangle, and bc be conjugate to ad, and ca to bd, for any conic, then ab is conjugate to cd.*

8. *If pqr is in perspective with the polar triangle of abc, then abc is in perspective with the polar triangle of pqr.*

For, weighting the points suitably, we have

$$[p\,|bc] + [q\,|ca] + [r\,|ab] = 0,$$

and hence

$$[bc\,|p] + [ca\,|q] + [ab\,|r] = 0.$$

This also follows from a consequence of Desargues' Theorem (§ 12 (21)):

$$[pqr]\;[bc\,|p.ca\,|q.ab\,|r] + [abc]\;[qr\,|a.rp\,|b.pq|\,c] = 0.$$

9. Since $[bc\,|\,p] = [b\,|\,p]\,c - [c\,|\,p]\,b$, we have

$$[bc\,|\,p\,.\,ca\,|\,q\,.\,ab\,|\,r] = ([a\,|\,r]\,[b\,|\,p]\,[c\,|\,q] - [a\,|\,q]\,[b\,|\,r]\,[c\,|\,p])\,[abc].$$

Hence $$[a\,|\,r]\,[b\,|\,p]\,[c\,|\,q] = [a\,|\,q]\,[b\,|\,r]\,[c\,|\,p]$$

is the condition that *abc* is in perspective with the polar triangle of *pqr*.

Hence if *abc* and *pqr*, and also *abc* and *qrp* are in this relation, so are *abc* and *rpq*.

10. The formulae of § 15 can now be reinterpreted. They are all valid here, since all the properties used in deducing them hold here.

Thus § 15 (5) says: *The joins of d to the poles of ad, bd, cd cut bc, ca, ab respectively in collinear points.*

And (6) says: *if the joins of a, b, c to the pole of L cut L in p, q, r respectively, then the joins of p, q, r to the poles of bc, ca, ab concur.*

11. If $x_1, x_2, y_1, y_2, z_1, z_2, x_1', \ldots, z_2'$ are scalars, then the condition that

$$[a'b'c']\,[(x_1 b + x_2 c)\,(y_1 c + y_2 a)\,(z_1 a + z_2 b)] + [abc]\,[(x_1' b' + x_2' c')\,(y_1' c' + y_2' a')\,(z_1' a' + z_2' b')] = 0$$

is $$x_1 y_1 z_1 + x_2 y_2 z_2 + x_1' y_1' z_1' + x_2' y_2' z_2' = 0.$$*

In the dual formula, which holds under a like condition, put $[ab]$ for C, $[a'b']$ for C', and so on, then the left-hand side equals

$$[A'B'C']\,[(x_1\,.\,ca + x_2\,.\,ab)\,(y_1\,.\,ab + y_2\,.\,bc)\,(z_1\,.\,bc + z_2\,.\,ca)] + [ABC]\,[(x_1'\,.\,c'a' + x_2'\,.\,a'b')\,(y_1'\,.\,a'b' + y_2'\,.\,b'c')\,(z_1'\,.\,b'c' + z_2'\,.\,c'a')].$$

The condition will be satisfied, if we take

$$\begin{aligned} x_1 &= [b\,|\,a'], & y_1 &= [c\,|\,b'], & z_1 &= [a\,|\,c'], \\ x_2 &= [c\,|\,a'], & y_2 &= [a\,|\,b'], & z_2 &= [b\,|\,c'], \\ x_1' &= -[b'\,|\,a], & y_1' &= -[c'\,|\,b], & z_1' &= -[a'\,|\,c], \\ x_2' &= -[c'\,|\,a], & y_2' &= -[a'\,|\,b], & z_2' &= -[b'\,|\,c]. \end{aligned}$$

We apply this to the dual formula. We have, for example,

$$x_1 c - x_2 b = [b\,|\,a']\,c - [c\,|\,a']\,b = [bc\,|\,a'],$$

$$x_1[ca] + x_2[ab] = [bc\,|\,a']\,a,$$

thence $$[a'b'c']^2\,([bc\,|\,a']\,a\,.\,[ca\,|\,b']\,b\,.\,[ab\,|\,c']\,c) - [abc]^2\,([b'c'\,|\,a]\,a'\,.\,[c'a'\,|\,b]\,b'\,.\,[a'b'\,|\,c]\,c') = 0.$$

* Daniëls, *Ens. Math.* 20 (1918), p. 97.

Hence, *if the polars of a', b', c' cut bc, ca, ab in p, q, r respectively, and the polars of a, b, c cut $b'c'$, $c'a'$, $a'b'$ in p', q', r' respectively, and if pa, qb, rc concur, then $p'a'$, $q'b'$, $r'c'$ concur.*

In the earlier interpretation of supplements, this formula has no place. The dual formula, however, has an interpretation in the earlier sense, and gives:

If the perpendiculars from a, b, c to $b'c'$, $c'a'$, $a'b'$ cut bc, ca, ab respectively in collinear points, then the perpendiculars from a', b', c' to bc, ca, ab cut $b'c'$, $c'a'$, $a'b'$ in collinear points.

Orthologic triangles in § 15 correspond here to triangles each in perspective with the polar triangle of the other. The orthocentre of a triangle corresponds to the centre of perspective of the triangle and its polar triangle. It is now easy to re-interpret the rest of § 15.

12. *If abc, $a'b'c'$ be self-polar triangles for a conic, then the six vertices lie on a conic, and the six sides of the triangles touch a conic.*

For $\mathrm{k}_1 a = |[bc]$, $\mathrm{k}'_1 a' = |[b'c']$, and so on.

Hence $\mathrm{k}_1\mathrm{k}_2\mathrm{k}_3[abc] = |[bc.ca.ab] = [abc]^2$.

Thus $[abc] = \mathrm{k}_1\mathrm{k}_2\mathrm{k}_3$. Similarly, $[a'b'c'] = \mathrm{k}'_1\mathrm{k}'_2\mathrm{k}'_3$.

$$\begin{aligned}[bc'.b'c..ca'.c'a..ab'.a'b] &= aa'b'.bb'c'.cc'a'.abc - abb'.bcc'.caa'.a'b'c' \qquad (\S\,12\ (25)) \\ &= \mathrm{k}'_1\mathrm{k}'_2\mathrm{k}'_3[a|c'][b|a'][c|b'][abc] \\ &\quad - \mathrm{k}_1\mathrm{k}_2\mathrm{k}_3[c|b'][a|c'][b|a'][a'b'c'] = 0.\end{aligned}$$

Hence the first part follows by Pascal's Theorem, the second dually, or by § 24·11.

13. *Def.* The extensives a, a_1; b, b_1; c, c_1; ... on the same range are pairs '*in involution*' if $aa_1bb_1cc_1\ldots$ is projective with $a_1ab_1bc_1c\ldots$.

If there is a projective correspondence between extensives of a range and extensives of the same range in which a, a_1, b, c, ... correspond to a_1, a, b_1, c_1, ..., then b_1 corresponds to b, and c_1 to c, ... For the correspondence is fixed by the fact that a, a_1, b correspond to a_1, a, b_1, and we have

$$R(aa_1bb_1) = R(a_1ab_1b).$$

Desargues' Theorem. If a line L cuts a conic through a, b, c, d in points p, q, then

$[L.ab]$, $[L.cd]$; $[L.bc]$, $[L.ad]$; $[L.ca]$, $[L.bd]$; p, q *are pairs in an involution.*

For
$$R([L.ab], p, [L.bc], q)$$
$$= R([abp]\,q - [abq]\,p,\ p,\ [bcp]\,q - [bcq]\,p,\ q)$$
$$= \frac{[abp]\,[bcq]}{[bcp]\,[abq]}. \qquad (1)$$

Now, by § 24 (3), $\dfrac{[abp]\,[cdp]}{[bcp]\,[dap]} = \dfrac{[abq]\,[cdq]}{[bcq]\,[daq]}$.

Hence (1) equals $\dfrac{[cdq]\,[dap]}{[daq]\,[cdp]}$.

Hence $R([L.ab], p, [L.bc], q) = R([L.cd], q, [L.da], p)$.

Hence $[L.ab]$, $[L.cd]$; $[L.bc]$, $[L.da]$; p, q are pairs in an involution on L.

§ 27. *The regulus.**

1. *Def.* A '*quadric cone*' is the set of points on lines joining points of a conic to a point not in the plane of the conic.

Def. A '*regulus*' is the set of lines joining corresponding points of two projective ranges of points on skew lines.

If a, b, c and a', b', c' be any corresponding points on the two ranges, we may absorb weights, so that $c = a + b$, $c' = a' + b'$, then the points $p = a + \mathrm{k}b$, $p' = a' + \mathrm{k}b'$ correspond. Their join is $H = [pp'] = [aa'] + \mathrm{k}([ba'] + [ab']) + \mathrm{k}^2[bb']$.

Let $A = [aa']$, $B = [bb']$, $C = [cc']$, then

$$C - A - B = [ba'] + [ab'].$$

Hence $$H = (1 - \mathrm{k})\,A + (\mathrm{k}^2 - \mathrm{k})\,B + \mathrm{k}C.$$

Hence *the lines of the regulus are linear combinations of any three lines of the system.*

Hence *any line F, which cuts three lines of the regulus, cuts them all.* The lines of a regulus are called its '*generators*'.

* For a treatment of the regulus, twisted cubic, quadric, and cubic surface by Grassmann's methods, see Mehmke, *Vorlesungen über Punkt und Vektor-Rechnung* (Leipzig, 1913).

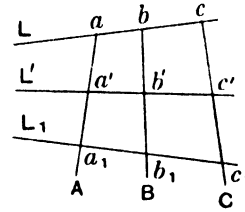
For, if $[FA] = [FB] = [FC] = 0$, then $[FH] = 0$.

Such a line is a '*directrix*' of the regulus.

Any point p on a line of the regulus is of the form

$$p = a + \mathrm{k}b + \mathrm{k}_1(a' + \mathrm{k}b') = a + \mathrm{k}b + \mathrm{k}_1 a' + \mathrm{k}_1 \mathrm{k}b'.$$

2. Let $L = [ab]$, $L' = [a'b']$, then $L_1 = [a_1 BCa_1]$ is a line cutting B, C, and if we take a_1 anywhere on A, then L_1 cuts A, B, C, and hence cuts all lines of the regulus. *The lines of the regulus cut L_1 in a range of points projective with the ranges on L, L'.*



For we can adjust weights so that $d = a + \mathrm{k}b$ and $d' = a' + \mathrm{k}b'$ are the cuts of D (any line of the regulus) with L and L', and so that $a_1 = a + \mathrm{k}_1 a'$, $b_1 = b + \mathrm{k}_1' b'$ are the cuts of L_1 and A, B respectively. Let L_1 cut D in d_1. Now since $[a_1 b_1 dd'] = 0$, we have $\mathrm{k}_1 = \mathrm{k}_1'$ (cf. § 16, Ex. 8). Hence $d + \mathrm{k}_1 d' = a_1 + \mathrm{k}b_1$. Thus this point is on both D and L_1, and hence is at d_1.

Accordingly $d_1 \equiv a_1 + \mathrm{k}b_1$, when $d = a + \mathrm{k}b$. Hence the statement.

3. The argument in 2 shews that lines which meet three given skew lines cut them in projective ranges; hence such lines constitute a regulus. Thus the directrices of a regulus form a regulus, the '*opposite*' regulus to the given.

4. Lines which are linear combinations of three skew lines form a regulus, for they meet all lines which meet the given lines.

5. With the above notation, let $\alpha = [L_1 a]$, $\beta = [L_1 b]$, $\gamma = [L_1 c]$.

Then
$$[H\gamma] = (1 - \mathrm{k})\,[A\gamma] + (\mathrm{k}^2 - \mathrm{k})\,[B\gamma],$$
$$[Hc_1] = (1 - \mathrm{k})\,[Ac_1] + (\mathrm{k}^2 - \mathrm{k})\,[Bc_1].$$

Hence, as H describes the regulus, the point $[H\gamma]$ describes a range on L_1, which is projective to the pencil of planes through L_1 described by $[Hc_1]$. That is, the cuts of H and L_1 form a range projective to the planes through H and L_1. Similarly, if L_2 be another line cutting A, B, C, the cuts of H and L_2 form a range projective to the planes through H and L_2. But the

cuts of H with L_1 and L_2 form projective ranges. Hence the pencil of planes through H and L_1 is projective to the pencil of planes through H and L_2, as H varies.

Hence *the regulus is the cut of projective pencils of planes through any two directrices* L_1 *and* L_2.

If H, L cut in p, the plane $\pi \equiv [pL'L_1L]$ goes through H, L and is the '*tangent plane*' at p. Also $p \equiv [\pi L'L_1L]$. Thus as p moves along a generator, p and π describe projective pencils.

If p is any point on any generator, $[pL'L_1Lp] = 0$.

Examples. 6. The bisectors of the inner angles of a skew quadrilateral are on a regulus; so are the bisectors of two interior and two exterior angles. The bisectors of the interior angles of a skew pentagon or hexagon are linearly dependent.*

For if S_1, S_2, ... be rotors along the sides of an n-gon, then

$$(S_1 - S_2) + (S_2 - S_3) + \ldots + (S_n - S_1) = 0.$$

Take all the rotors of the same magnitude, and we have the results.

7. If the lines L_1, L_2, L_3, L_4 lie on a regulus, we can weight them so that $L_1 + L_2 + L_3 + L_4 = 0$. Then

$$[L_1L_2] = [L_3L_4], \quad [L_1L_2] + [L_2L_3] + [L_3L_1] = 0.$$

Hence $\sqrt{[L_1L_2]}\,\sqrt{[L_3L_4]} \pm \sqrt{[L_2L_3]}\,\sqrt{[L_1L_4]} \pm \sqrt{[L_3L_1]}\,\sqrt{[L_2L_4]} = 0.$

To save brackets, we always write $\sqrt{[\,]\,[\,]}$ for $\sqrt{([\,]\,[\,])}$.

The last condition is independent of the weight of the lines. We shall see later, that, unlike the first, it is not sufficient to ensure that the lines, when skew, lie on a regulus.

8. From a variable point p, two fixed lines L, M are projected on to two fixed planes α, β. If the projections cut each other, the locus of p is given by $[pL.\alpha..pM.\beta] = 0$.

If L, M, $[\alpha\beta]$ are parallel to one plane, the locus contains a line at infinity.

9. *The four altitudes of a tetrahedron are generators of a regulus, the four perpendiculars to the faces at the orthocentres are generators of the opposite regulus.*

$$a\,|[bc+cd+db] - b\,|[cd+da+ac] + c\,|[da+ab+bd] - d\,|[ab+bc+ca] = 0.$$

10. If a series of parallel planes cuts four generators of a regulus, the areas of the triangles so formed are in a constant ratio.

* Mehmke, *Jahresb. Deut. Math. Ver.* 21 (1912), p. 58.

11. *If p be on the hyperboloid through the altitudes of a tetrahedron $a_1a_2a_3a_4$, then a_4 is on that of $pa_1a_2a_3$.* (Zeeman.)

For, let u_1, u_2, u_3, u_4 be unit vectors on the altitudes from a_1, a_2, a_3, a_4 respectively, and let v be the vector of a transversal of these altitudes, and p a point on this transversal. Then

$$p - \mathrm{k}_1 v = a_1 + \mathrm{l}_1 u_1, \quad (\mathrm{k}_1, \mathrm{l}_1 \text{ scalars}).$$

Thus $[(p-a_1)\,u_1 v] = 0, \quad [(p-a_2)\,u_2 v] = 0, \quad [(p-a_3)\,u_3 v] = 0.$

Hence $$[(p-a_1)\,u_1 . (p-a_2)\,u_2 . (p-a_3)\,u_3] = 0.$$

Multiply by $[u_1u_2u_3]$, and express the product in determinant form by § 19 (12), then

$$\mathrm{h}_{12}\mathrm{h}_{23}\mathrm{h}_{31} = -\mathrm{h}_{13}\mathrm{h}_{32}\mathrm{h}_{21}, \quad \text{where} \quad \mathrm{h}_{ij} = [(p-a_i)\,u_i u_j].$$

Take $[a_1a_2a_3a_4] = 1$.

Now $[u_1u_2] \equiv |(a_3-a_4)$, and so on, the constant multiplier depending on the sine of the dehedral angle along a_3a_4 and the length of that side.

Hence $$\mathrm{h}_{12} = [(p-a_1)\,u_1u_2] \equiv [(p-a_1)|(a_3-a_4)],$$

$$[(p-a_1)|(a_3-a_4)]\,[(p-a_2)|(a_1-a_4)]\,[(p-a_3)|(a_2-a_4)]$$
$$= -[(p-a_1)|(a_2-a_4)]\,[(p-a_3)|(a_1-a_4)]\,[(p-a_2)|(a_3-a_4)].$$

If we interchange p, a_4 in this, the equation is not altered.

12. *If $pqr\ldots$ and $p'q'r'\ldots$ be related ranges on skew lines, any line L meeting one pair of cross-joins meets an infinite number of other pairs.*

Suppose the corresponding points of the ranges are $a+\mathrm{k}b$, $c+\mathrm{k}d$, for the same k, and let

$$[L(a+\mathrm{k}_1 b)\,(c+\mathrm{k}_2 d)] = 0, \quad [L(a+\mathrm{k}_2 b)\,(c+\mathrm{k}_1 d)] = 0.$$

Then $$[Lac] + \mathrm{k}_1[Lbc] + \mathrm{k}_2[Lad] + \mathrm{k}_1\mathrm{k}_2[Lbd] = 0,$$
$$[Lac] + \mathrm{k}_2[Lbc] + \mathrm{k}_1[Lad] + \mathrm{k}_1\mathrm{k}_2[Lbd] = 0.$$

Since $\mathrm{k}_1 \neq \mathrm{k}_2$, we have, therefore, $[L(bc-ad)] = 0$.

If then L meets $[(a+\mathrm{k}_3 b)\,(c+\mathrm{k}_4 d)]$, it meets $[(a+\mathrm{k}_4 b)\,(c+\mathrm{k}_3 d)]$.

§ 28. *The twisted cubic.*

Let $\xi = \alpha + \mathrm{k}\beta$, $\xi' = \alpha' + \mathrm{k}\beta'$, $\xi'' = \alpha'' + \mathrm{k}\beta''$ be corresponding planes in three projective pencils of planes. These planes cut in the point

$$p = [\xi\xi'\xi''] = a + \mathrm{k}b + \mathrm{k}^2 c + \mathrm{k}^3 d, \qquad (1)$$

where $$a = [\alpha\alpha'\alpha''], \quad b = [\beta\alpha'\alpha''] + [\alpha\beta'\alpha''] + [\alpha\alpha'\beta''],$$
$$c = [\alpha\beta'\beta''] + [\beta\alpha'\beta''] + [\beta\beta'\alpha''], \quad d = [\beta\beta'\beta''].$$

In the general case, none of these vanish, and p describes a curve which cuts a general plane π in points at which the values of k are given by

$$[a\pi]+k[b\pi]+k^2[c\pi]+k^3[d\pi]=0,$$

that is, in *three* points. The curve is a '*twisted cubic*'. On it lie the points a, d. Since β, β', β'' can be taken as any corresponding planes in the three pencils, d can be taken anywhere on the curve. Similarly so can the point a.

If we project the points of the curve from d on to any plane γ, we have

$$q=[pd\gamma]=[ad\gamma]+k[bd\gamma]+k^2[cd\gamma]=a_1+kb_1+k^2c_1, \text{ say.}$$

Hence, as p varies on the cubic, q describes a conic; *the projection of the curve from any point on it is a quadric cone.* Taking two points on the curve, we therefore have the curve as the *cut of two quadric cones which have a common generator*, the vertices of the cones being on the curve, and the common generator their join.

By (1), $$k^{-1}[pad]=[bad]+k[cad].$$

Compare this with $\xi=\alpha+k\beta$. Then, as k varies, that is, as p moves on the curve, $[pad]$ describes a pencil of planes projective with the pencil $\alpha+k\beta$. The line $[ad]$ joining two points on the curve is a '*bisecant*'. Thence the planes from a variable point p of the curve to any fixed bisecants form projective pencils, because all are projective to $\alpha+k\beta$.

§ 29. *Quadrics.**

1. Two spreads of step two are '*projective*' when they can be put into one-to-one correspondence, such that, if $a, b, c, p_1, \ldots$ in one spread correspond to $a', b', c', p_1', \ldots$ in the other, these extensives can be so weighted that if $p=a+k_1b+k_2c$, then $p'=a'+k_1b'+k_2c'$.

Then if p_1', p_2', p_3', p_4' correspond to p_1, p_2, p_3, p_4, and

$$k_1p_1+k_2p_2+k_3p_3+k_4p_4=0,$$

then $$k_1p_1'+k_2p_2'+k_3p_3'+k_4p_4'=0.$$

* Cf. Mehmke, *Vorlesungen*, for the argument in 3, 4.

If p, q, r, s be any four independent extensives in the first spread, the projectivity is fixed when the positions of the corresponding p', q', r', s' are known. For we can absorb weights so that $p+q+r=s$, $p'+q'+r'=s'$; the extensive $p'+k_1q'+k_2r'$ which corresponds to $p+k_1q+k_2r$ has then a definite position.

2. Let A, B, C be three lines, not coplanar, but meeting in d; then as k_1, k_2 vary, $A+k_1B+k_2C$ can represent any line L through d. We have a '*bundle*' of lines, centre d.

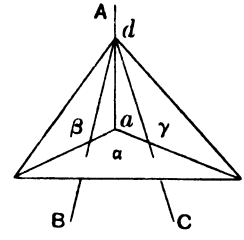
Let α, β, γ be three planes, not collinear, but meeting in a; then as k_1, k_2 vary, $\alpha+k_1\beta+k_2\gamma$ can represent any plane π through a. We have a '*bundle*' of planes, centre a.

The two bundles are projective when a line of the first is made to correspond to the plane of the second with the same k_1, k_2. The projectivity is fixed when four corresponding pairs of independent elements are known.

3. Thus we can assume that A goes through a, and that β, γ go through d and hence through $[ad] \equiv A$.

The locus of the cuts p of corresponding lines and planes is a '*quadric*'.

$$p = [(A+k_1B+k_2C)(\alpha+k_1\beta+k_2\gamma)].$$



Now we can replace C by $B+kC$, if we replace γ by $\beta+k\gamma$. After this substitution, the coefficient of k_1k_2 becomes

$$2[B\beta]+k([B\gamma]+[C\beta]),$$

and since $[B\beta]$, $[B\gamma]$, $[C\beta]$ are all at d, we can, by choice of k, make this expression vanish. We may therefore assume that C, γ are chosen in the first instance, so that the coefficient of k_1k_2 in the expression for p vanishes.

Then $$p = a+k_1b+k_2c+(k_1^2l_1+k_2^2l_2)\,d,$$

where $a=[A\alpha]$, $b=[B\alpha]$, $c=[C\alpha]$, $l_1d=[B\beta]$, $l_2d=[C\gamma]$,

where l_1, l_2 are *fixed* scalars, and k_1, k_2 as before, varying scalars.

If $k_1=k_2=0$, then $p=a$; if $k_1=k_2=\infty$, then $p\equiv d$. Hence a, d are on the quadric.

4. We next shew that the centres of the bundles can be taken at any two distinct points of the quadric.

Let A, B, C be *any* independent lines of the bundle of lines, α, β, γ the corresponding planes. Take β_1, γ_1, δ, so that

$$A = [\beta_1\gamma_1], \quad B = [\beta_1\delta], \quad C = [\delta\gamma_1].$$

Let $L = A + k_1 B + k_2 C$, $\pi_1 = \beta_1 + k_2\delta$, $\pi_2 = \gamma_1 + k_1\delta$,

then $$L = [(\beta_1 + k_2\delta)(\gamma_1 + k_1\delta)] = [\pi_1\pi_2].$$

Let $\pi = \alpha + k_1\beta + k_2\gamma$,

$$\begin{aligned}\pi' &= x_1\pi_1 + x_2\pi_2 + \pi \\ &= \alpha + x_1\beta_1 + x_2\gamma_1 + k_1(x_2\delta + \beta) + k_2(x_1\delta + \gamma).\end{aligned}$$

Now, if we fix x_1, x_2, then the planes

$$\alpha + x_1\beta_1 + x_2\gamma_1, \quad x_2\delta + \beta, \quad x_1\delta + \gamma$$

are fixed and π' describes a bundle as k_1, k_2 vary, projective to the bundle described by L. Since

$$[L\pi'] = [\pi_1\pi_2\pi'] = [\pi_1\pi_2\pi] = [L\pi],$$

the pair of bundles of π', L give the same quadric as the pair of bundles of π, L.

Hence the quadric goes through the centre of the bundle of π', that is, through the point

$$\begin{aligned}q &= [(\alpha + x_1\beta_1 + x_2\gamma_1)(x_2\delta + \beta)(x_1\delta + \gamma)] \\ &= [(\alpha + x_1\beta_1 + x_2\gamma_1)([\beta\gamma] + x_1[\beta\delta] + x_2[\delta\gamma])],\end{aligned}$$

which must accordingly be on the quadric for all x_1, x_2.

Now the plane $\alpha + x_1\beta_1 + x_2\gamma_1$ and the line

$$[\beta\gamma] + x_1[\beta\delta] + x_2[\delta\gamma]$$

describe projective bundles as x_1, x_2 vary, and the cut q of corresponding elements lies, as we have seen, on the quadric. Hence so does the centre $[\alpha\beta_1\gamma_1] = [\alpha A]$ of the bundle of planes. But A is any line of the bundle of lines, hence $[\alpha A]$ may be taken anywhere on the quadric. Thus the centre of the bundle of planes may be taken anywhere on the quadric.

Since we can interchange lines and planes in the argument, we can also replace the bundle of lines by one whose centre is any point of the quadric.

Thus in the original formulation we may take the bundle of lines (instead of planes) through a, and the bundle of planes through d. Or we may take both formulations together, so that

if lines A, B, C correspond to planes α, β, γ, then the planes through BC, CA, AB respectively correspond to the lines $[\beta\gamma]$, $[\gamma\alpha]$, $[\alpha\beta]$.

5. Conversely, any point p which moves so that

$$p \equiv a + k_1 b + k_2 c + (k_1^2 l_1 + k_2^2 l_2)\, d,$$

where l_1, l_2 are fixed scalars, and a, b, c, d independent points, describes a quadric as k_1, k_2 vary.

For, let $\alpha \equiv [abc]$, $[ad] \equiv A$, $[bd] \equiv B$, $[cd] \equiv C$, and let β, γ be distinct planes through ad. We can take weights so that

$$a = [A\alpha], \quad b = [B\alpha], \quad c = [C\alpha], \quad [B\gamma] + [C\beta] = 0;$$

then $$p \equiv [(A + k_1 B + k_2 C)\,(\alpha + k_1 \beta + k_2 \gamma)].$$

For the right-hand side

$$= a + k_1 b + k_2 c + k_1^2 [B\beta] + k_2^2 [C\gamma]; \quad [B\beta] \equiv [C\gamma] \equiv d.$$

6. Any point on a generator of a regulus is of form

$$p = a + k_1 b + k_2 c + k_1 k_2 d,$$

and this can be written

$$p = a + \tfrac{1}{2}(k_1 + k_2)\,(b + c) + \tfrac{1}{2}(k_1 - k_2)\,(b - c) + \tfrac{1}{4}[(k_1 + k_2)^2 - (k_1 - k_2)^2]\, d.$$

Hence *the points on the generators of a regulus lie on a quadric.* If the field of scalars is real, the quadric is a hyperboloid of one sheet, unless one of its generators is a line at infinity, in which case it is a hyperbolic paraboloid. This case occurs when the ranges on the generators are similar ranges.

7. *Any plane π which cuts the quadric, cuts it in a conic.* For, if a, d be two points on the plane and quadric, we may take them as the centres of the bundles. The pencil of lines in π, which is part of the bundle of lines through a, is projective to the pencil of planes through that line through d which corresponds to π considered as one of the planes through a (cf. end of 4). This pencil of planes cuts π in a pencil of lines projective to the pencil in π through a. These two projective pencils of lines cut in a conic (degenerate or not) which lies on the quadric.

8. If the field of scalars is the field of complex numbers, a line in general cuts a quadric in two points, since a plane through

the line and a point of the quadric cuts the quadric in a conic. If a line meets a quadric in three points, the section of the quadric by a plane through it, being a conic, must be a pair of lines; hence all points of these lines lie on the quadric.

9. A set of n lines, or rotors, $L_1, \ldots, L_n$ is '*linearly dependent*', or '*dependent*' if scalars $k_1, \ldots, k_n$, not all zero, can be found so that

$$k_1 L_1 + \ldots + k_n L_n = 0.$$

If no such set of scalars exist, the lines, or rotors, are '*independent*'.

If $L_1, \ldots, L_4$ be lines, skew and independent, the lines which meet L_1, L_2, L_3 lie on a regulus. If the field of scalars is the complex field, L_4 cuts the corresponding quadric in two points (which may coincide). The lines through these points meeting L_1, L_2, L_3 are the two lines (which may coincide) meeting the four given independent lines.

10. A line L which is a linear combination of four independent lines $L_1, \ldots, L_4$ must meet the lines M which meet $L_1, \ldots, L_4$, since $[ML_1] = \ldots = [ML_4] = 0$.

Hence if the field of scalars is complex, these lines L meet, in general, just two lines, and just one line L in the set of linear combinations of $L_1, \ldots, L_4$ goes through a general point of space. Such a set of lines is called a '*linear congruence*'; the two lines met by all lines of the congruence are the '*directrices*' of the congruence.

Examples. 13. A right angle moves about its vertex so that its plane goes through a fixed line through its vertex, and an arm moves in a given plane. Then the other arm describes a quadric cone.

14. The hyperboloid which contains ab, $a'b'$, aa', bb', cc', where c is on $[ab]$, c' on $[a'b']$, has a centre which is the point of concurrence of diameters of the quadrilaterals $aa'bb'$, $bb'cc'$, $cc'aa'$. (Cf. § 16, Ex. 7, p. 78.)

15. If L_1, L_2, M_1, M_2, N be given skew lines, and

$$[pL_1 L_2 . pM_1 M_2 . pN] = 0,$$

then p describes a twisted cubic which is the cut of the quadrics

$$[pL_1 L_2 . pN] = 0 \quad \text{and} \quad [pM_1 M_2 . pN] = 0.$$

These quadrics have a common generator. Similarly, each of

$$[pL_1L_2\dots L_{2l}\cdot pM_1M_2\dots M_{2m}\cdot pN_1N_2\dots N_{2n+1}]=0,$$
$$[pL_1L_2\dots L_{2l+1}\cdot pM_1M_2\dots M_{2m+1}\cdot pN_1N_2\dots N_{2n+1}\cdot G]=0$$

gives a twisted cubic* for the locus of p.

To see this, for example, for the last equation, let the first three factors, which are planes, cut in q. Then

$$[qL_{2l+1}\dots L_1\cdot qM_{2m+1}\dots M_1\cdot qN_{2n+1}\dots N_1]\equiv p,$$

and $[qG]=0$.

Thus $\quad qL_{2l+1}\dots L_1,\quad qM_{2m+1}\dots M_1,\quad qN_{2n+1}\dots N_1$

are projective pencils of planes, each projective with the same range on G.

If the sides of a polygon go through fixed points, and all vertices but one move on fixed lines, the last vertex describes a twisted cubic.

16. If p, a, b, c, a', b', c' be on a twisted cubic, then since the projection of the curve from p is a quadric cone, therefore the lines

$$[pb'c.pbc'],\quad [pc'a.pca'],\quad [pa'b.pab']$$

are coplanar, by Pascal's Theorem. Let

$$[pb'c.bc']=l,\quad [pc'a.ca']=m,\quad [pa'b.ab']=n,$$
$$[pbc'.b'c]=l',\quad [pca'.c'a]=m',\quad [pab'.a'b]=n',$$

then $[ll']$ is a line through p cutting $b'c$ and bc'. Hence Pascal's Theorem gives ll', mm', nn' are concurrent and coplanar. Hence

$$[pb'c.bc'..pc'a.ca'..pa'b.ab'..p]=0. \qquad \text{(Chasles.)}$$

As in the proof of Pascal's Theorem (§ 12 (25)) the equation may be written

$$[paa'b'][pbb'c'][pcc'a'][pabc]-[pabb'][pbcc'][pcaa'][pa'b'c']=0$$

or

$$[pab'c'][pbc'a'][pca'b'][pabc]-[pa'bc][pb'ca][pc'ab][pa'b'c']=0.$$

The equation represents Weddle's quartic surface, the locus of vertices of quadric cones through a, b, c, a', b', c'. The fifteen joins of a, ..., c' lie on the surface. The ten cuts of pairs of planes through the points, such as abc, $a'b'c'$, are also on the surface.†

17. If a variable polygon $pp_1p_2\dots p_np$ moves in space so that its sides pp_1, p_1p_2, ..., p_np meet fixed lines, and the vertices p_1, p_2, ..., p_n lie on fixed lines, then p describes a quadric.‡

* Caspary, *Crelles Journ.* 100 (1887), p. 405.

† The second of these equations for Weddle's surface is due to Caspary, *Bull. Sc. Math.* (2), 11 (1887), 1 re. partie, p. 222.

‡ Grassmann, *Crelles Journ.* 49 (1855), p. 170.

§ 30. *Supplements with respect to a non-degenerate quadric.*

1. We have seen that the points of a (non-degenerate) quadric can be expressed in the form

$$p = a + \mathrm{k}_1 b + \mathrm{k}_2 c + \mathrm{k}_3 d \quad \text{when} \quad \mathrm{k}_3 = \mathrm{k}_1^2 \mathrm{l}_1 + \mathrm{k}_2^2 \mathrm{l}_2,$$

and hence by

$$p = \mathrm{y}_1 b + \mathrm{y}_2 c + \mathrm{y}_3 (a+d) + \mathrm{y}_4 (a-d),$$

where
$$\mathrm{l}_1 \mathrm{y}_1^2 + \mathrm{l}_2 \mathrm{y}_2^2 = \mathrm{y}_3^2 - \mathrm{y}_4^2. \tag{1}$$

As new base-points, take

$$e_1 = \mathrm{l}_1^{-\frac{1}{2}} b, \quad e_2 = \mathrm{l}_2^{-\frac{1}{2}} c, \quad e_3 = a+d, \quad e_4 = a-d.$$

Then
$$p \equiv \mathrm{x}_1 e_1 + \mathrm{x}_2 e_2 + \mathrm{x}_3 e_3 + \mathrm{x}_4 e_4,$$

where
$$\mathrm{x}_1^2 + \mathrm{x}_2^2 - \mathrm{x}_3^2 + \mathrm{x}_4^2 = 0. \tag{2}$$

If the field of scalars is the real field, the above assumes that l_1, l_2 are positive. If either or both be negative, replace the negative l by $-\mathrm{l}$, then the signs of one or both of x_1^2, x_2^2 in (2) become negative.

If the field of scalars is the complex field, we can ignore this complication, and also replace e_3 by $\sqrt{(-1)}\, e_3$, then (2) becomes

$$\mathrm{x}_1^2 + \mathrm{x}_2^2 + \mathrm{x}_3^2 + \mathrm{x}_4^2 = 0. \tag{3}$$

For brevity we take this as our equation. The various cases in real space are obtained when appropriate signs are changed, as in the work on conics.

2. This paragraph uses the methods of ordinary analytical geometry. A line is a '*generator*' of a quadric, if all its points lie on the quadric. A line is a '*tangent*' to a quadric at a point p of the quadric if it is either a generator or has no point save p on the quadric.

If $p = \mathrm{y}_1 e_1 + \ldots + \mathrm{y}_4 e_4$ is on the quadric (3) and pq is a tangent at p, where $q = \mathrm{z}_1 e_1 + \ldots + \mathrm{z}_4 e_4$, then

$$\mathrm{y}_1^2 + \ldots + \mathrm{y}_4^2 = 0,$$

and $(\mathrm{y}_1 + \mathrm{k}\mathrm{z}_1)^2 + \ldots + (\mathrm{y}_4 + \mathrm{k}\mathrm{z}_4)^2 = 0$ is either an identity in k (then pq is a generator), or has no root save $\mathrm{k} = 0$.

Hence $\quad 2\mathrm{k}(\mathrm{y}_1 \mathrm{z}_1 + \ldots + \mathrm{y}_4 \mathrm{z}_4) + \mathrm{k}^2 (\mathrm{z}_1^2 + \ldots + \mathrm{z}_4^2) = 0$

is either an identity in k, or has no root save $\mathrm{k} = 0$.

In both cases $$y_1z_1+\ldots+y_4z_4=0, \qquad (4)$$

and in the first case, we have also

$$z_1^2+\ldots+z_4^2=0, \quad (y_1+kz_1)^2+\ldots+(y_4+kz_4)^2=0.$$

Thus (4) is the condition that pq is tangent at p.

From (4) it follows that all tangents at p to the quadric lie on a plane, the '*tangent plane*' at p, and all lines through p in this plane are tangent lines.

Def. Two points not on the quadric are '*conjugate*' with respect to the quadric, the field of scalars being complex, if their join cuts the quadric in two points separating them harmonically.

If $p=y_1e_1+\ldots+y_4e_4$, $q=z_1e_1+\ldots+z_4e_4$ be the points, then as in § 25·3, p, q are conjugate, if and only if

$$y_1z_1+\ldots+y_4z_4=0. \qquad (5)$$

Def. If a point p is on the quadric, any point on a tangent line at p is '*conjugate*' to p. Hence (5) is the general condition for conjugate points.

By (5), if p be fixed, the points conjugate to p lie in a plane, the '*polar plane*' of p; p is the '*pole*' of this plane. *Each point has a unique polar plane.*

In real space, as some lines through p may not cut the quadric, the points conjugate to p may only take up part of a plane. We call the whole of the plane the '*polar plane*' of p, and extend the meaning of '*conjugate*' so that all points of this plane are conjugate to p.

By (5): *if the polar plane of p goes through q, then the polar plane of q goes through p.* Hence if p, q, r be on the polar plane of s, then s is on the polar planes of p, q, r.

3. Let $$[e_1e_2e_3e_4]=1,$$

$$\epsilon_1=[e_2e_3e_4], \quad \epsilon_2=-[e_1e_3e_4], \quad \epsilon_3=[e_1e_2e_4], \quad \epsilon_4=-[e_1e_2e_3],$$

then $$[\epsilon_1\epsilon_2\epsilon_3\epsilon_4]=1.$$

If $p=y_1e_1+\ldots+y_4e_4$, then $q=z_1e_1+\ldots+z_4e_4$ is on the polar plane of p, if (5) holds. Now this polar plane is

$$y_1\epsilon_1+\ldots+y_4\epsilon_4=\pi, \text{ say,}$$

for $$[q\pi]=y_1z_1+\ldots+y_4z_4=0.$$

Hence if π, π_1, π_2 be the polar planes of p, p_1, p_2, then if $p = k_1 p_1 + k_2 p_2$, we have $\pi = k_1 \pi_1 + k_2 \pi_2$, and conversely. Hence if p, q, r be independent points on the polar plane of s, then the polar planes of p, q, r (on which s lies) are independent. Thus they have a single common point; the polar plane of any point on $[pqr]$ goes through this common point. Hence *each plane has a unique pole.* Further, our first statement shews that *the polar planes of points on a line, go through another line.* The relation between the lines is symmetrical, each is the '*polar*' of the other.

4. If $p = y_1 e_1 + \ldots + y_4 e_4$, $\pi = y_1 \epsilon_1 + \ldots + y_4 \epsilon_4$, and thus p, π are pole and polar plane, we write $\pi = |p$ and speak of '*supplements*'. In particular,

$$|e_1 = \epsilon_1, \ldots, |e_4 = \epsilon_4,$$

$$|(y_1 e_1 + \ldots + y_4 e_4) = |p = y_1 \epsilon_1 + \ldots + y_4 \epsilon_4 = y_1 |e_1 + \ldots + y_4 |e_4.$$

Hence if p, q be any points, k_1, k_2 scalars, then

$$|(k_1 p + k_2 q) = k_1 |p + k_2 |q.$$

As $\pi = |p$ expresses that π is the polar plane of p, the last equation states the fact, proved above, that the polar planes of points on a line go through a line.

5. We define the '*supplement*' of $[pq]$ as $[|p . |q]$, and write it $|[pq]$. By the last theorem, the supplement of a line is congruent to its polar line.

$$|[e_2 e_3] = [\epsilon_2 \epsilon_3] = -[e_1 e_3 e_4 . e_1 e_2 e_4] = -[e_1 e_2 e_3 e_4] [e_4 e_1] = [e_1 e_4].$$

Similarly,

$$|[e_2 e_3] = [e_1 e_4], \quad |[e_3 e_1] = [e_2 e_4], \quad |[e_1 e_2] = [e_3 e_4],$$
$$|[e_1 e_4] = [e_2 e_3], \quad |[e_2 e_4] = [e_3 e_1], \quad |[e_3 e_4] = [e_1 e_2].$$

Thus the repeated supplement

$$||[e_i e_j] = [e_i e_j].$$

Hence $||[pq] = [pq]$, $||L = L$ for any line L.

If $p = y_1 e_1 + \ldots + y_4 e_4$, $q = z_1 e_1 + \ldots + z_4 e_4$, then

$$\begin{aligned}[pq] = {} & (y_2 z_3 - y_3 z_2) [e_2 e_3] + (y_3 z_1 - y_1 z_3) [e_3 e_1] \\ & + (y_1 z_2 - y_2 z_1) [e_1 e_2] + (y_1 z_4 - y_4 z_1) [e_1 e_4] \\ & + (y_2 z_4 - y_4 z_2) [e_2 e_4] + (y_3 z_4 - y_4 z_3) [e_3 e_4].\end{aligned}$$

We obtain $|[pq]$ from $[pq]$ by interchanging $[e_2e_3]$ with $[e_1e_4]$, $[e_3e_1]$ with $[e_2e_4]$, $[e_1e_2]$ with $[e_3e_4]$.

6. We define the '*supplement*' of $[pqr]$ as $[|p.|q.|r]$ and write it $||[pqr]$. Thus $||[pqr]$ is the pole of $[pqr]$, as the cut of the polar planes of p, q, r:

$$|\epsilon_1 = |[e_2e_3e_4] = [\epsilon_2\epsilon_3\epsilon_4] = -e_1, \quad |\epsilon_2 = -e_2, \quad |\epsilon_3 = -e_3, \quad |\epsilon_4 = -e_4.$$

$$||e_i = -e_i, \quad ||\epsilon_i = -\epsilon_i, \quad (i = 1, 2, 3, 4).$$

7. From the definitions of $|[pq]$ and $||[pqr]$, and the fact that $[pq]$ is a linear combination of $[e_1e_2]$, $[e_2e_3]$, ..., and so on, and that $[pqr]$ is a linear combination of ϵ_1, ..., ϵ_4, we have

$$|(k_1L + k_2M) = k_1|L + k_2|M \qquad \text{for any lines } L, M,$$

$$|(k_1\alpha + k_2\beta) = k_1|\alpha + k_2|\beta \qquad \text{for any planes } \alpha, \beta.$$

For any point p, plane π,

$$||p = -p, \quad ||\pi = -\pi.$$

8. We define the '*supplement*' of a scalar as the scalar itself. Thus $|[abcd] = [abcd]$. Since

$$[|e_1.|\epsilon_1] = -[\epsilon_1.e_1] = [e_1\epsilon_1], \quad [|e_1.|\epsilon_2] = 0,$$

we have $$[|a.|\pi] = [a\pi] = |[a\pi].$$

Similarly $$|[LM] = [|L.|M].$$

9. If a, b are points, π_1, π_2 planes, L, M lines, then

$$[a.|b], \quad [\pi_1.|\pi_2], \quad [L.|M]$$

are called '*inner products*' and are written

$$[a|b], \quad [\pi_1|\pi_2], \quad [L|M].$$

All are scalars.

Similarly, $[a|\pi]$ means $[a.|\pi]$, and so on.

If $|\pi = b$, then

$$||[a|\pi] = |[ab] = [|a.|b] = -[|a.\pi] = [\pi|a].$$

If $[a|\pi] = 0$, then a, π are pole and polar plane, for then $a \equiv |\pi$.

If $[p|q] = 0$, then p is on the polar plane $|q$ of q.

We write p^2 for $[p|p]$, α^2 for $[\alpha|\alpha]$. *If* $p^2 = 0$, *then* p *is on our quadric. If* $\alpha^2 = 0$, *then the pole of* α *is on* α.

$$[L|M] = |[L|M] = [|L.||M] = [|L.M] = [M|L].$$

If $[L|M]=0$, then L, M are '*conjugate*' lines, that is, each meets the polar of the other. In particular,

$$[e_i|e_i]=1, \quad [e_i|e_j]=0, \quad [e_i|e_i]=1, \quad [e_i|e_j]=0, \quad (i\neq j;\ i, j=1, \ldots, 4).$$

$[E_i|E_i]=1$, where E_i is any of $[e_1e_2]$, $[e_2e_3]$, ..., $[e_3e_4]$.

$[E_i|E_j]=0$, $(i\neq j)$.

Examples. 18. *If in a tetrahedron, two pairs of opposite edges are conjugate, so is the third pair.*

For $[bc|ad]+[ca|bd]+[ab|cd]=0$, as in § 19 (9).

Def. Two tetrahedra are '*polar*' to one another, if the vertices of one (and therefore of each) are poles of the corresponding faces of the other.

19. *The joins of corresponding vertices of two polar tetrahedra lie in general on a regulus, so do the cuts of corresponding faces.*

The first part follows from

$$[a|bcd]-[b|cda]+[c|dab]-[d|abc]=0. \qquad (6)$$

The second part is the dual, obtained by taking supplements.

To shew (6), let $|a=\alpha$, then

$$|[a|bcd]=[bcd|a]=[bcd.\alpha]=[b\alpha]\,[cd]+[c\alpha]\,[db]+[d\alpha]\,[bc]$$
$$=[b|a]\,[cd]+[c|a]\,[db]+[d|a]\,[bc].$$

This and similar formulae give (6).

20. For convenience, put $|[bcd]=a'$, $|[acd]=-b'$, and so on, then (6) can be written $[aa']+[bb']+[cc']+[dd']=0$.

If three of these lines meet in a point, all do. In particular, this happens when the opposite edges of the tetrahedron *abcd* are conjugate. For then

$$[a'b']=-|[bcd.cda]=[abcd]|[cd],$$
$$[aa'.bb']\equiv[ab.a'b']\equiv[ab|cd]=0, \text{ and so on.}$$

21. The investigation dual to § 29 leads to the conclusion: if $k_1L_1+k_2L_2+k_3L_3$ be a bundle of lines projective to a plane of points $k_1p_1+k_2p_2+k_3p_3$, (L_1, L_2, L_3 not coplanar, p_1, p_2, p_3 not collinear), then the planes which join corresponding line and point envelope a surface whose equation can be put in the form (the field of scalars being the complex field):

$$x_1^2+x_2^2+x_3^2+x_4^2=0, \quad \text{where} \quad \alpha=x_1e_1+x_2e_2+x_3e_3+x_4e_4$$

is a plane of the system. Thus α is a plane of the system if $\alpha^2=0$. If β is any plane, $|\beta$ is its pole for the quadric; hence, if α touches the

quadric (i.e. is a plane of the system) its point of contact p satisfies $p^2 = 0$. Thus *the points on a (non-degenerate) quadric envelope form a quadric locus.*

22. *If a, b, c, d be points on a quadric, and p, q be the cuts of tangent planes at c, d with $[ab]$, and p', q' be the cuts of the tangent planes at a, b with $[cd]$, then $apbq$ and $cp'dq'$ are projective.*

For $p = [ab\,|\,c] = [a\,|\,c]\,b - [b\,|\,c]\,a, \quad q = [a\,|\,d]\,b - [b\,|\,d]\,a,$

$$R(apbq) = [a\,|\,c]\,[b\,|\,d] \div [b\,|\,c]\,[a\,|\,d].$$

This gives the result, which is a projective generalisation of the theorem: *if a sphere inscribed in a tetrahedron touches plane bcd in a_1, and so on, then the angles between $a_1 b$, $a_1 c$, $a_1 d$ equal the corresponding angles in the other faces.*

10. *If the polar lines of qr, rp, pq cut any plane abc in a', b', c' respectively, and aa', bb', cc' are concurrent, then pp', qq', rr' are concurrent, where p', q', r' are cuts of $[pqr]$ and the polar lines of bc, ca, ab.*

For $[abc.de] = [bcde]\,a + [cade]\,b + [abde]\,c,$

whence, writing $|[qr]$ for $[de]$, and noting that $a' = [abc\,|\,qr]$, we have

$$a' = [abc\,|\,qr] = [bc\,|\,qr]\,a + [ca\,|\,qr]\,b + [ab\,|\,qr]\,c.$$

$$[aa'] = [ca\,|\,qr]\,[ab] - [ab\,|\,qr]\,[ca],$$

$$[bb'] = [ab\,|\,rp]\,[bc] - [bc\,|\,rp]\,[ab],$$

$$[cc'] = [bc\,|\,pq]\,[ca] - [ca\,|\,pq]\,[bc].$$

As these lines are concurrent, we have

$$[ab\,|\,rp]\,[bc\,|\,pq]\,[ca\,|\,qr] = [ab\,|\,qr]\,[bc\,|\,rp]\,[ca\,|\,pq],$$

and the form of this proves the theorem.

11. *If the tetrahedron $pqrs$ is in perspective with the polar tetrahedron of $abcd$, then $abcd$ is in perspective with the polar tetrahedron of $pqrs$.*

For $[qrs\,|\,a] = [q\,|\,a]\,[rs] + [r\,|\,a]\,[sq] + [s\,|\,a]\,[qr],$

$$[rsp\,|\,b] = [r\,|\,b]\,[sp] + [s\,|\,b]\,[pr] + [p\,|\,b]\,[rs].$$

These cut if $([r\,|\,a]\,[s\,|\,b] - [s\,|\,a]\,[r\,|\,b])\,[pqrs] = 0,$

or since $[pqrs] \neq 0$, if $[rs\,|\,ab] = 0$.

Hence the 'opposite' edges of the two tetrahedra are conjugate lines, and the theorem follows by symmetry.

12. *If a, b, c, d be coplanar points on the quadric $p^2=0$, then*

$$\sqrt{[a\,|b]\,[c\,|d]} \pm \sqrt{[b\,|c]\,[a\,|d]} \pm \sqrt{[c\,|a]\,[b\,|d]} = 0.$$

If three of the points are collinear, one factor in each term is zero. In general

$$a^2 = b^2 = c^2 = d^2 = 0, \quad k_1 a + k_2 b + k_3 c + k_4 d = 0,$$
$$(k_1, \ldots, k_4 \text{ scalars, none zero}).$$

Hence
$$k_1 a + k_2 b = -k_3 c - k_4 d,$$
$$k_1 k_2 [a\,|b] = k_3 k_4 [c\,|d] = \pm \sqrt{k_1 k_2 k_3 k_4}\, \sqrt{[a\,|b]\,[c\,|d]},$$
$$k_3 k_4 [c\,|d] + k_1 k_4 [a\,|d] + k_2 k_4 [b\,|d]$$
$$= k_4 [(k_1 a + k_2 b + k_3 c)\,|d] = k_4 [-k_4 d\,|d] = 0.$$

The first expression in the preceding equation equals

$$\sqrt{k_1 k_2 k_3 k_4}\,(\pm \sqrt{[a\,|b]\,[c\,|d]} \pm \sqrt{[b\,|c]\,[a\,|d]} \pm \sqrt{[c\,|a]\,[b\,|d]}).$$

13. *If the sides ab, bc, cd, da of a skew quadrilateral cut a quadric (which may be a cone) in the pairs of points a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2 respectively, and if a_1, b_1, c_1, d_1 are coplanar, then a_2, b_2, c_2, d_2 are coplanar.*

For using supplements with respect to the quadric, $a + kb$ is on the quadric if $(a+kb)^2 = 0$.

Hence, if $a + k_1 b = a_1$, $a + k_1' b = a_2$, then $k_1 k_1' = [a\,|a] \div [b\,|b]$.

Hence, if

$$b_1 = b + k_2 c, \quad c_1 = c + k_3 d, \quad d_1 = d + k_4 a,$$
$$b_2 = b + k_2' c, \quad c_2 = c + k_3' d, \quad d_2 = d + k_4' a,$$

then
$$k_1 k_1' k_2 k_2' k_3 k_3' k_4 k_4' = 1.$$

But a_1, b_1, c_1, d_1 are coplanar, hence $k_1 k_2 k_3 k_4 = 1$. Hence $k_1' k_2' k_3' k_4' = 1$, and a_2, b_2, c_2, d_2 are coplanar.

Cor. If the sides of the quadrilateral touch the quadric, and are not generators, either the points of contact are coplanar, or the plane through three of them cuts the fourth side in the harmonic conjugate of the point of contact with respect to the ends of that side.

For the condition that ab touches the quadric is that the roots of the quadratic in k are equal. Hence $[a\,|b]^2 = a^2 b^2$. The point of contact is $a + k_1 b$, where $k_1 = -a^2 \div [a\,|b]$. If $b + k_2 c$, $c + k_3 d$, $d + k_4 a$ be the other points of contact, the outer product of all four is $[abcd]\,(1 - k_1 k_2 k_3 k_4)$. The conditions of contact give $k_1 k_2 k_3 k_4 = +1$ or -1, corresponding to the two cases.

When the quadric is a cone only the first case is possible. For consider a conic as a degenerate quadric, and let *ab*, *bc*, *cd*, *da* each meet it in a point. Draw any quadric $\mathcal{Q}$ with the sides of *abcd* as generators and passing through a fifth point of the conic. Then $\mathcal{Q}$ contains the conic; and the four planes, each through a side of *abcd* and the tangent to the conic where the side meets it, all touch $\mathcal{Q}$ and pass through the pole with respect to $\mathcal{Q}$, of the plane of the conic. Hence if the sides of a quadrilateral meet a conic, then the tangent planes through them to the conic meet in a point. Dually, *if the sides of a quadrilateral touch a cone, the points of contact lie on a plane.*

14. *A quadric cuts the edges of a tetrahedron abcd in points such as* $\mathrm{k}_{12}a+b$, $a+\mathrm{k}_{21}b$; *the points* $\mathrm{k}_{12}a+b$, $\mathrm{k}_{13}a+c$, $\mathrm{k}_{14}a+d$ *are joined by a plane which cuts* $[bcd]$ *in a line. The four such lines are on a regulus.*

For
$$[(\mathrm{k}_{12}a+b)(\mathrm{k}_{13}a+c)(\mathrm{k}_{14}a+d).bcd] \equiv \mathrm{k}_{12}[cd]+\mathrm{k}_{13}[db]+\mathrm{k}_{14}[bc] = L_1, \text{ say.}$$

Similarly
$$L_2 \equiv \mathrm{k}_{23}[da]+\mathrm{k}_{24}[ac]+\mathrm{k}_{21}[cd],$$
$$L_3 \equiv \mathrm{k}_{34}[ab]+\mathrm{k}_{31}[bd]+\mathrm{k}_{32}[da],$$
$$L_4 \equiv \mathrm{k}_{41}[bc]+\mathrm{k}_{42}[ca]+\mathrm{k}_{43}[ab].$$

These lines are dependent, if we can find scalars $\mathrm{k}_1, \ldots, \mathrm{k}_4$ such that
$$\mathrm{k}_1\mathrm{k}_{12} = -\mathrm{k}_2\mathrm{k}_{21}, \quad \mathrm{k}_2\mathrm{k}_{23} = -\mathrm{k}_3\mathrm{k}_{32}, \quad \mathrm{k}_3\mathrm{k}_{31} = +\mathrm{k}_1\mathrm{k}_{13},$$
$$\mathrm{k}_1\mathrm{k}_{14} = -\mathrm{k}_4\mathrm{k}_{41}, \quad \mathrm{k}_2\mathrm{k}_{24} = +\mathrm{k}_4\mathrm{k}_{42}, \quad \mathrm{k}_3\mathrm{k}_{34} = -\mathrm{k}_4\mathrm{k}_{43},$$

that is, such that
$$\mathrm{k}_1\mathrm{k}_2^{-1} = -\mathrm{k}_{21}\mathrm{k}_{12}^{-1}, \quad \mathrm{k}_2\mathrm{k}_3^{-1} = -\mathrm{k}_{32}\mathrm{k}_{23}^{-1}, \quad \mathrm{k}_3\mathrm{k}_1^{-1} = +\mathrm{k}_{13}\mathrm{k}_{31}^{-1},$$
$$\mathrm{k}_1\mathrm{k}_4^{-1} = -\mathrm{k}_{41}\mathrm{k}_{14}^{-1}, \quad \mathrm{k}_2\mathrm{k}_4^{-1} = +\mathrm{k}_{42}\mathrm{k}_{24}^{-1}, \quad \mathrm{k}_3\mathrm{k}_4^{-1} = -\mathrm{k}_{43}\mathrm{k}_{34}^{-1}.$$

These are compatible if $\mathrm{k}_{12}\mathrm{k}_{23}\mathrm{k}_{31} = \mathrm{k}_{13}\mathrm{k}_{32}\mathrm{k}_{21}$, and so on for each cycle of three subscripts. The argument in 13 applied to the triangles *abc*, *bcd*, *cda*, *dab* proves these relations.

15. Dually, if tangent planes be drawn through the edges of tetrahedron *abcd* to a quadric, and if $\mathrm{k}_{12}\alpha+\beta$, $\alpha+\mathrm{k}_{21}\beta$ be such a pair, α, β, γ, δ being the planes of the tetrahedron, then the

join of a to the point $[(k_{12}\alpha+\beta)(k_{13}\alpha+\gamma)(k_{14}\alpha+\delta)]$, and similar joins, lie on a regulus.

We find the *same* compatibility conditions as in 14.

Hence, in 14, the *twelve planes*

$$k_{34}[abc]+[abd],\ [abc]+k_{43}[abd],\ \ldots$$

touch a quadric.

§ 31. *The Grassmann cross-ratio of four lines.*

1. The Grassmann cross-ratio of four lines A, B, C, D in space is defined as

$$R(ABCD) = [AB]\,[CD] \div [BC]\,[DA].$$

If A, B, C, D have two transversals, distinct, cutting them in a, b, c, d and a', b', c', d', then

$$R(ABCD) = [aba'b']\,[cdc'd'] \div [bcb'c']\,[dad'a'] = R, \text{ say.}$$

Now

$$[aba'b'] = \text{mag } ab \,.\, \text{mag } a'b' \,.\, \text{dist } (ab, a'b') \,.\, \sin (ab, a'b'),$$

where a, b, a', b' are supposed to be of unit weight, and dist $(ab, a'b')$ denotes the distance between the lines ab, $a'b'$.

From this and similar formulae, we get $R = rr'$, where

$$r = [ab]\,[cd] \div [bc]\,[da], \quad r' = [a'b']\,[c'd'] \div [b'c']\,[d'a'].$$

Cor. 1. The Grassmann cross-ratio of four lines in a regulus is the square of the cross-ratio of their cuts with any transversal.

Cor. 2. Four lines A, B, C, D of a regulus cut any transversal harmonically if $[AB]\,[CD] = [BC]\,[DA]$. We say B, D separate A, C harmonically.

2. If L_1, L_2, L_3, L_4 either lie on a regulus or have a *unique* common transversal, then

$$\sqrt{[L_1L_2]\,[L_3L_4]} \pm \sqrt{[L_2L_3]\,[L_1L_4]} \pm \sqrt{[L_3L_1]\,[L_2L_4]} = 0.$$

For first suppose L_1, L_2, L_3, L_4 have a unique transversal M; consider the regulus through L_1, L_2, L_3, and let N be the line of this regulus through the cut of L_4 and M. The tangent plane to the regulus at this point is the plane through N and M. Now L_4 lies in this plane, for it must touch the quadric of the regulus,

since if it met it again, $L_1, \ldots, L_4$ would have *two* distinct common transversals. Hence L_4 is dependent on M and N, and since N is dependent on L_1, L_2, L_3, it follows that $L_1, \ldots, L_4, M$ are dependent.

If on the other hand $L_1, \ldots, L_4$ lie on a regulus, they are dependent.

Hence, in both cases, there are scalars $\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4$ such that $\mathrm{k}_1 L_1 + \mathrm{k}_2 L_2 + \mathrm{k}_3 L_3 + \mathrm{k}_4 L_4$ is either zero, or a rotor A whose line meets $L_1, \ldots, L_4$, and no k vanishes.

Then $$\mathrm{k}_1 L_1 + \mathrm{k}_2 L_2 = -\mathrm{k}_3 L_3 - \mathrm{k}_4 L_4 + A.$$

Multiply each side outerwise by itself, then

$$\mathrm{k}_1 \mathrm{k}_2 [L_1 L_2] = \mathrm{k}_3 \mathrm{k}_4 [L_3 L_4].$$

Similarly,

$$\mathrm{k}_2 \mathrm{k}_3 [L_2 L_3] = \mathrm{k}_1 \mathrm{k}_4 [L_1 L_4], \quad \mathrm{k}_3 \mathrm{k}_1 [L_3 L_1] = \mathrm{k}_2 \mathrm{k}_4 [L_2 L_4].$$

Similarly, from

$$\mathrm{k}_1 L_1 + \mathrm{k}_2 L_2 + \mathrm{k}_3 L_3 = -\mathrm{k}_4 L_4 + A,$$

we have $$\mathrm{k}_1 \mathrm{k}_2 [L_1 L_2] + \mathrm{k}_2 \mathrm{k}_3 [L_2 L_3] + \mathrm{k}_3 \mathrm{k}_1 [L_3 L_1] = 0.$$

Hence

$$\sqrt{[L_1 L_2]}\,\sqrt{[L_3 L_4]} \pm \sqrt{[L_2 L_3]}\,\sqrt{[L_1 L_4]} \pm \sqrt{[L_3 L_1]}\,\sqrt{[L_2 L_4]} = 0.$$

This result and method may be compared with 12 of the preceding section. It is equivalent to the vanishing of the 4-rowed determinant whose ij-element is $[L_i L_j]$.

In this form, it easily follows that the determinant equals that obtained by replacing $L_1, \ldots, L_4$ by their polar lines for our standard quadric.

The two cases are distinguished by the fact that if the lines lie on a regulus they are dependent, otherwise not.

3. *If $abcd$, $a'b'c'd'$ be two tetrahedra, then $R(aa', bb', cc', dd')$ equals the cross-ratio of the cuts of corresponding planes.*

For let $\alpha = [bcd]$, $\beta = -[acd]$, $\gamma = [abd]$, $\delta = -[abc]$, with similar meanings for $\alpha', \beta', \gamma', \delta'$. Then

$$[\alpha\alpha'] = [\alpha . b'c'd'] = [\alpha b'][c'd'] + [\alpha c'][d'b'] + [\alpha d'][b'c'],$$

$$[\beta\beta'] = [\beta . a'c'd'] = [\beta c'][d'a'] + [\beta d'][a'c'] + [\beta a'][c'd'],$$

$$\begin{aligned}[\alpha\alpha' . \beta\beta'] &= ([\alpha c'][\beta d'] - [\alpha d'][\beta c'])\,[a'b'c'd'] \\ &= [\alpha\beta . c'd']\,[a'b'c'd'].\end{aligned}$$

Hence the cross-ratio of the cuts of the planes is equal to

$$[\alpha\beta.c'd'][\gamma\delta.a'b'] \div [\beta\gamma.d'a'][\delta\alpha.b'c'].$$

Since $[\alpha\beta] \equiv [cd]$, and the scalar factors involved cancel when the substitution is made in the last fraction, this cross-ratio equals

$$[cd.c'd'][ab.a'b'] \div [da.d'a'][bc.b'c'],$$

which is the cross-ratio of the joins of corresponding vertices.

4. *If abcd, a'b'c'd' be any tetrahedra, then the cross-ratio of the joins of a, b, c, d to the poles of the faces b'c'd', c'd'a', d'a'b', a'b'c' of a'b'c'd' for any quadric equals the cross-ratio of the joins of a', b', c', d' to the poles of the faces bcd, cda, dab, abc of abcd.*

For let $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ be as in 3, then

$$R(a|\alpha', b|\beta', c|\gamma', d|\delta') = R(a'|\alpha, b'|\beta, c'|\gamma, d'|\delta).$$

For since

$$[a|\alpha'.b|\beta'] = -[ab|\alpha'\beta'] = -[ab|c'd'][a'b'c'd'],$$

the left-hand side equals

$$[ab|c'd'][cd|a'b'] \div [bc|d'a'][da|b'c'],$$

while the right-hand side equals the expression obtained from this by the interchange of dashed and undashed letters: but this interchange does not affect the value of the expression.

5. *If abcd and a'b'c'd' be two tetrahedra, and the joins of a, b, c, d to the poles of the faces b'c'd', c'd'a', d'a'b', a'b'c' of a'b'c'd' for any quadric lie on a regulus, then the joins of a', b', c', d' to the poles of the faces bcd, cda, dab, abc of abcd lie on a regulus.*

For, if the joins of a, b, c, d to the poles of $b'c'd', c'd'a', d'a'b', a'b'c'$ lie on a regulus, then $b'c'd'|a, c'd'a'|b, d'a'b'|c, a'b'c'|d$ are dependent lines.

But $$[b'c'd'|a] = [b'|a][c'd'] + [c'|a][d'b'] + [d'|a][b'c'],$$

and three similar relations. We can therefore find scalars k_1, k_2, k_3, k_4 such that

$$k_1[d'|a] = k_4[a'|d], \quad k_1[c'|a] = k_3[a'|c],$$
$$k_1[b'|a] = k_2[a'|b], \quad k_2[c'|b] = k_3[b'|c],$$
$$k_3[d'|c] = k_4[c'|d], \quad k_4[b'|d] = k_2[d'|b].$$

These give the relations, necessary for consistency,

$$[a'|c]\,[b'|a]\,[c'|b] = [a'|b]\,[b'|c]\,[c'|a],$$

and those obtained by cycling a, b, c, d and a', b', c', d'. But these are unchanged if we interchange the dashed and undashed letters.

Cor. If two tetrahedra be such that perpendiculars from the vertices of one on to the faces of the other belong to a regulus, the relation is reciprocal.

Note. We have a similar theorem when the four lines have a unique transversal.

6. *Def.* If $f(p)$ is an extensive, function of extensive p, we say it is a '*linear homogeneous function*' when

$$f(\mathrm{k}_1 p + \mathrm{k}_2 q) = \mathrm{k}_1 f(p) + \mathrm{k}_2 f(q),$$

where k_1, k_2 are scalars, and q any extensive of the same step as p.

If $f(a, b, c, d)$ be a linear homogeneous function of extensives a, b, c, d which vanishes when any two of these extensives are equal, then it is merely multiplied by a scalar when a is replaced by a linear combination $p = \mathrm{k}_1 a + \mathrm{k}_2 b + \mathrm{k}_3 c + \mathrm{k}_4 d$, of a, b, c, d.

For
$$\begin{aligned} f(p, b, c, d) &= \mathrm{k}_1 f(a, b, c, d) + \mathrm{k}_2 f(b, b, c, d) + \dots \\ &= \mathrm{k}_1 f(a, b, c, d). \end{aligned}$$

*If G, H be polar lines for a quadric, $abcd$ a tetrahedron, and $[aG]$ meets H in a', and so on for b', c', d', then the joins of a', b', c', d' to the vertices of the polar tetrahedron of $abcd$ lie on a regulus.**

For let $[aG.H] = a'$, and so on;

$$a'' = |[bcd], \quad b'' = -|[acd], \quad c'' = |[abd], \quad d'' = -|[abc],$$

then
$$S = [a'a''] + [b'b''] + [c'c''] + [d'd'']$$

is a homogeneous linear function of a, b, c, d, which vanishes when two of the points coincide. For example, if $a = b$, then

$$a' = b', \quad a'' = -b'', \quad c'' = d'' = 0.$$

Hence S is unchanged, save for a scalar factor, when a is replaced by any point, and similarly when a, b, c, d are replaced by any points.

* Mehmke, *Archiv Math. Phys.* (3) 18, (1911), p. 370. The method used here is Mehmke's. It gives another proof of (6), p. 143 and of all such formulae. (Take *abcd* as a polar tetrahedron.)

Now take a, b on G, then $a'=b'=0$; take c, d on H, but so that a, b, c, d are independent and $[c|d]=0$; then $c'\equiv c$, $c''\equiv c$, $d'\equiv d$, $d''\equiv d$, since now ab, cd are polar lines.

But in this case $S=0$. Hence $S=0$ always.

Cor. If a', b', c', d' be the projections of the vertices of the tetrahedron $abcd$ on to a line, then the perpendiculars from a', b', c', d' to bcd, cda, dab, abc respectively lie on a regulus. (Neuberg.)

7. If $$[aa']+[bb']+[cc']+[dd']=0,$$

and $$\alpha=[bcd],\quad \beta=-[acd],\quad \gamma=[abd],\quad \delta=-[abc],$$

with similar meanings for α', β', γ', δ', then

$$S=[\alpha\alpha']+[\beta\beta']+[\gamma\gamma']+[\delta\delta']=0.$$

For take $[abcd]=1$, then the coefficient of $[bc]$ in

$$S'=[aa']+[bb']+[cc']+[dd']$$

is $$[S'ad]=[bb'ad]+[cc'ad]=-[\gamma b']+[\beta c'];$$

the coefficient of $[\beta'\gamma']$ in S is

$$[S\alpha'\delta']=-[\gamma b']+[\beta c'].$$

But $S'=0$ means that all the coefficients vanish. Hence then $S=0$.

Hence *if the joins of corresponding vertices of two tetrahedra lie on a regulus, the cuts of their corresponding faces lie on a regulus, and conversely.**

Note. We have a similar theorem when the joins have a unique transversal.

8. If $A'=[\beta'\gamma']$, $B'=[\gamma'\alpha']$, $C'=[\alpha'\beta']$, we say abc, $\alpha'\beta'\gamma'$ are '*in perspective*' when aA', bB', cC' go through a line. Let $A=[bc]$, $B=[ca]$, $C=[ab]$, then: *if abc, $\alpha'\beta'\gamma'$ are in perspective, then $A\alpha'$, $B\beta'$, $C\gamma'$ lie on a line.*

For, let

$$\sigma=[aA']+[bB']+[cC'],\quad s=[\alpha'A]+[\beta'B]+[\gamma'C].$$

Then σ contains the cut d' of α', β', γ'; express σ as a linear homogeneous function of α', β', γ'. Further s lies in the plane abc; express s as a linear homogeneous function of a, b, c.

We find that the coefficients in the two functions are the same.

Hence, if $\sigma=0$, then $s=0$, and conversely.

* Mehmke, *Math. Zeit.* 35 (1932), p. 618, for the method for 7, 8, 9.

9. *If*, in the sense of 8, *bcd, cda, dab, abc are in perspective with* $\beta'\gamma'\delta'$, $\gamma'\delta'\alpha'$, $\delta'\alpha'\beta'$, $\alpha'\beta'\gamma'$, *where* α', β', γ', δ' *are the faces of* $a'b'c'd'$, *then the joins of the corresponding vertices of abcd and* $a'b'c'd'$ *are dependent.*

For, if $S = [aa'] + [bb'] + [cc'] + [dd']$,

and if $\alpha' = [b'c'd']$, $\beta' = -[c'd'a']$, ...,

then $[Sd'] = [aa'd'] + [bb'd'] + [cc'd']$

$$\equiv [a\beta'\gamma'] + [b\gamma'\alpha'] + [c\alpha'\beta'] = 0.$$

Similarly, $[Sa']$, $[Sb']$, $[Sc']$ vanish. Hence $S = 0$.

The converse is true.

10. *The tetrahedron abcd is inscribed in a quadric. From any point p on the section of the quadric by the plane abc, lines are drawn to the poles* q_1, q_2, q_3 *of the planes bcd, cad, abd. Then these lines cut the quadric again in three points coplanar with d.*

(Müller.*)

For take $x^2 = 0$ as the quadric. If $[pq]$ cut $x^2 = 0$ again in $p + kq$, we have

$$p^2 = 0, \quad (p + kq)^2 = 0, \quad kq^2 + 2[p|q] = 0.$$

Hence the cuts in question are

$$q_i^2 . p - 2[p|q_i]\, q_i \quad (i = 1, 2, 3);$$

the plane through them is

$$q_1^2[p|q_2]\,[p|q_3]\,[pq_2q_3] + \ldots + \ldots - 2[p|q_1]\,[p|q_2]\,[p|q_3]\,[q_1q_2q_3]. \quad \text{(i)}$$

Now $q_1 = |[dbc]$, $q_1^2 = [dbc]^2 = 2[d|b]\,[b|c]\,[c|d]$,

and if $p = x_1 a + x_2 b + x_3 c$,

then $-[p|q_1] = [pdbc] = x_1[abcd]$,

$-[p|q_2] = [pdca] = x_2[abcd]$,...

$$[pq_2q_3] = [p.\,|dca.\,|dab] = -[abcd]\,[p|da],$$

$$[q_1q_2q_3] \equiv |d, \quad [q_1q_2q_3d] = 0, \quad [dp|da] = -[a|d]\,[p|d].$$

Hence the outer product of (i) and d is the sum of three terms like

$$q_1^2[p|q_2]\,[p|q_3]\,[pq_2q_3d] = -2[abcd]^3\,[p|d]\,[a|d]\,[b|d]\,[c|d].x_2x_3[b|c],$$

and it vanishes, since

$$0 = p^2 = 2(x_2x_3[b|c] + x_3x_1[c|a] + x_1x_2[a|b]).$$

* *Crelles Journ.* 122 (1900), p. 30. The proof there is geometrical.

§ 32. *Associated points.**

1. If p be a point on a quadric through the lines L, M, N, then $[pLMNp]=0$.

If $L=[ab]$, $M=[cd]$, $N=[ef]$, then

$$[pLMNp] = [pab.cd.efp]$$
$$= [pabd]\,[cefp]-[pabc]\,[defp].$$

Hence the first equation may be written,

$$[Ldp]\,[Ncp] = [Lcp]\,[Ndp].$$

2. If a, b, c, a', b', c' be general points, let $V=0$ be the quadric through the lines aa', bb', cc', $W=0$ that through bc', ca', ab', and $U=0$ that through $b'c$, $c'a$, $a'b$, then (§ 22·13)

$$V = [paa'.bb'.cc'p] = [paa'b']\,[bcc'p]-[paa'b]\,[b'cc'p]$$
$$= [paa'b']\,[bcc'p]-[pcb'c']\,[aba'p].$$

Similarly, or by cycling a, b, c and a', c', b',

$$W = [pbc'.ca'.ab'p] = [pbc'a']\,[cab'p]-[paa'b']\,[bcc'p],$$
$$U = [pcb'.ac'.ba'p] = [pcb'c']\,[aba'p]-[pbc'a']\,[cab'p].$$

Hence $U+V+W=0$. Compare this with the proof of Pascal's Theorem in § 12·3.

Omitting p is equivalent to projecting from a point p of U, V, W on to a plane not through p, and working in that plane.

3. Nine general points determine a quadric†. As a quadric is given by an equation of the second degree, three quadrics $Q_1=0$, $Q_2=0$, $Q_3=0$, in general position cut in eight points $p_1, \ldots, p_8$. A quadric through seven of these points is of the form $x_1Q_1+x_2Q_2+x_3Q_3=0$, and hence goes through the eighth point.

All quadrics through seven of the points go through all the eight points. Such points are '*associated points*'.

4. If $a, b, c, d, a', b', c', d'$ be associated points, there is a quadric through a, b, c, d with generators $L=[a'b']$, $M=[c'd']$. For a quadric through a, b with generators L, M has equation, by 1, $[Lap]\,[Mbp] = k[Lbp]\,[Map]$, (k scalar).

* Turnbull, *Journal Indian Math. Soc.* 17 (1928). *Proc. Cambridge Phil. Soc.* 22 (1925), p. 481.

† We take this from analytical geometry, since a general quadratic expression in four variables has ten coefficients. That a general quadratic equation in four variables represents a quadric follows from the fact, proved later, that such an expression can be reduced to a 'sum of squares'.

If it goes through c, then

$$[Lac]\,[Mbc] = k[Lbc]\,[Mac].$$

Now a, b, c and three points on each of L, M are nine points determining a quadric through L, M. Its equation is, therefore:

$$[Lbc]\,[Lap]\,[Mca]\,[Mbp] - [Lca]\,[Lbp]\,[Mbc]\,[Map] = 0.$$

But as $a, b, c, d, a', b', c', d'$ are associated points, the quadric goes through d. Hence we have a relation connecting these associated points:

$$[Lbc]\,[Lad]\,[Mca]\,[Mbd] - [Lca]\,[Lbd]\,[Mbc]\,[Mad] = 0. \quad (1)$$

5. This is equivalent to

$$[bc.Ld..ca.Md..La.Mb] = 0. \quad (2)$$

For $[bc.Ld]$, $[ca.Md]$ are points p, q, say; $[La]$, $[Mb]$ are planes α, β, and $[pq.\alpha\beta] = [p\alpha]\,[q\beta] - [p\beta]\,[q\alpha]$,

$$[p\alpha] = ([Lbd]\,c - [Lcd]\,b)\,[La] = [Lbd]\,[Lca] + [Lcd]\,[Lab] = -[Lbc]\,[Lad], \quad \text{by § 16 (11).}$$

$$[q\beta] = -[Mca]\,[Mbd], \quad [p\beta] = -[Lbd]\,[Mbc],$$

$$[q\alpha] = -[Mad]\,[Lca].$$

Hence the left-hand side of (2) is equal to the left-hand side of (1). Equation (2) expresses a theorem of Hesse.

6. To the left-hand side of (1) add the expression obtained by interchanging a, c, and we have, omitting brackets,

$$\begin{aligned} &Lbc.Lad.Mca.Mbd - Lca.Lbd.Mbc.Mad \\ &\quad + Lba.Lcd.Mac.Mbd - Lac.Lbd.Mba.Mcd \\ &= (Lbc.Lad + Lab.Lcd)\,Mca.Mbd \\ &\quad - Lca.Lbd\,(Mbc.Mad + Mab.Mcd) \\ &= -Lca.Lbd.Mca.Mbd + Lca.Lbd.Mca.Mbd = 0. \end{aligned}$$

Hence the left-hand side of (1), and hence of (2), changes sign when a, c and similarly when any two of a, b, c, d are interchanged.

7. The left-hand side of (1) can be written

$$[a'b'bc]\,[a'b'ad]\,[c'd'ca]\,[c'd'bd] - [a'b'ca]\,[a'b'bd]\,[c'd'bc]\,[c'd'ad].$$

Cycle c', d', a and add, then

$$[a'b'bc]\,[c'd'ca]\,([a'b'ad]\,[c'd'bd]+[a'b'c'd]\,[d'abd]+[a'b'd'd]\,[ac'bd])$$
$$-[a'b'bd]\,[c'd'ad]\,([a'b'ca]\,[c'd'bc]+[a'b'cc']\,[d'abc]+[a'b'cd']\,[ac'bc])=0,$$

since each round bracket vanishes (§ 16 (11)).

Thus three expressions of type (1) are connected by an identity.

8. Since there is a quadric through L, M, and a, b, c, d, we have

$$R(La, Lb, Lc, Ld)=R(Ma, Mb, Mc, Md),$$
$$R(a, b, Lc.ab, Ld.ab)=R(Ma.cd, Mb.cd, c, d).$$

Hence $[Lc.abc]$, $[Ld.abd]$, $[Ma.cda]$, $[Mb.cdb]$ *are dependent*, since they join corresponding points of two projective ranges. (Zeuthen.*)

9. *If* $a, b, c, d, a', b', c', d'$ *be associated points, the lines*

$$[abc.a'b'c'],\quad [bcd.b'c'd'],\quad [cda'.c'd'a],\quad [da'b'.d'ab]$$

are dependent. (Weddle.)

For let these lines be A, B, C, D. It suffices to shew that on each of the eight planes $[abc]$, $[a'b'c']$, ... lies a line cutting A, B, C, D.

Now, for example, $[abc]$ cuts D, B, C in the points

$$[ab.da'b'],\quad [bc.c'd'b'],\quad [da'c.c'd'a.abc],$$

and these are collinear, since, by (2),

$$[ab.da'b'..bc.c'd'b'..da'c.c'd'a]=0.$$

* *Acta Math.* 12 (1889), p. 362.

CHAPTER IV

ROTORS IN SPACE, THE SCREW AND THE LINEAR COMPLEX

§ 33. *Screws.*

1. A sum of rotors is called a '*screw*'. Thus a screw is of form

$$S = [p_1 v_1] + [p_2 v_2] + \ldots + [p_n v_n],$$

where $p_1, \ldots, p_n$ are points or vectors, and $v_1, \ldots, v_n$ are vectors. *It can be reduced to a rotor through a given point p, and another rotor.* For

$$\begin{aligned} S &= [p(v_1 + v_2 + \ldots + v_n)] + [(p_1 - p)\, v_1] + \ldots + [(p_n - p)\, v_n] \\ &= [pv] + U, \end{aligned}$$

where $v = v_1 + v_2 + \ldots + v_n$ is a vector,

and $U = [(p_1 - p)\, v_1] + \ldots + [(p_n - p)\, v_n]$ is a bivector.

Now U can be written in the form $[(p - q)\, w]$, where q is a suitable point, and w a suitable vector. Then

$$S = [p(v + w)] - [qw].$$

Hence S is the sum of a rotor through p and a rotor $-[qw]$.

2. If $A + B = C + D$, then $[AB] = [CD]$;

$$[(A + B)\, L] = [(C + D)\, L]$$

for any rotor L.

For $$[(A + B)\,(A + B)] = [(C + D)\,(C + D)],$$

$$[AA] = 0, \quad [AB] = [BA],$$

and so on, giving $[AB] = [CD]$.

These theorems are easily interpreted in terms of volumes.

3. *S is a rotor or bivector, if and only if $[SS] = 0$; if also $[S\Omega] \neq 0$, it is a rotor. A necessary and sufficient condition that S is a bivector is $[S\Omega] = 0$.* (Ω is the unit bivector.)

For, if S is a rotor or bivector, then $[SS] = 0$ at once. Conversely, since S can always be put in the form $A + B$, and $[SS]$ is

then $2[AB]$, therefore if $[SS]=0$, then A, B must be concurrent or parallel, and $A+B$ is a rotor or bivector (a couple in statics).

If S is a bivector, then $[S\Omega]=0$ (§ 22·16); and if $S=A+B$, and $[S\Omega]=0$, then the vectors of A, B are equal and opposite (§ 22·16), and S is a bivector.

4. If S is a screw, L a rotor of unit magnitude, then $[SL]$ is the '*moment*' of the screw S round the line of L. This moment equals the sum of the moments round L of rotors whose sum is S.

If S, S' be two screws, then $[SS']$ is their '*mutual moment*'.

If $$S = \mathrm{l}[ad]+\mathrm{m}[bd]+\mathrm{n}[cd]+\mathrm{l}'[bc]+\mathrm{m}'[ca]+\mathrm{n}'[ab],$$

then $$[SS] = 2(\mathrm{ll}'+\mathrm{mm}'+\mathrm{nn}')\,[abcd].$$

Hence, if a, b, c, d are independent, then $\mathrm{ll}'+\mathrm{mm}'+\mathrm{nn}'=0$ is a necessary and sufficient condition that S is a rotor or bivector.

The coefficients l, m, n, l′, m′, n′ are the 'Plücker* coordinates' of S with respect to a, b, c, d.

If the rotors $[pq]$ and $[rs]$ have Plücker coordinates l_1, m_1, n_1, l'_1, m'_1, n'_1 and l_2, m_2, n_2, l'_2, m'_2, n'_2 respectively, then

$$[pqrs] = ((\mathrm{l}_1\mathrm{l}'_2+\mathrm{l}_2\mathrm{l}'_1)+(\mathrm{m}_1\mathrm{m}'_2+\mathrm{m}_2\mathrm{m}'_1)+(\mathrm{n}_1\mathrm{n}'_2+\mathrm{n}_2\mathrm{n}'_1))\,[abcd].$$

Hence a necessary and sufficient condition that $[pq]$ and $[rs]$ are coplanar, is that the coefficient of $[abcd]$ vanishes. If $\mathrm{l}_1+\mathrm{l}_2=\mathrm{l}$, $\mathrm{l}'_1+\mathrm{l}'_2=\mathrm{l}'$, and so on, this condition becomes

$$\mathrm{ll}'+\mathrm{mm}'+\mathrm{nn}'=0,$$

and in this form it merely means

$$[(pq+rs)\,(pq+rs)]=0.$$

5. If a, b, c, d be independent, then any rotor is a linear combination of $[ad]$, $[bd]$, $[cd]$, $[bc]$, $[ca]$, $[ab]$.

For, if $[pq]$ be the rotor, then p, q are linear combinations of a, b, c, d, and if we multiply out the expression obtained for $[pq]$ when these combinations are substituted for p, q, we have the result.

These six rotors $[ad]$, and so on, are independent. Hence, if *any* six independent rotors be given, as they are linear combinations of $[ad]$, ..., $[ab]$, any rotor is also a linear combination of *them*. (Cf. § 53·11.)

* Originally introduced, in a more general form, by Grassmann.

6. If A, B be rotors, p a point, then, by § 22·3,

$$[A.Bp]+[B.Ap]=[AB]\,p, \quad [A.Ap]=0.$$

Hence, if $S=C+D$, where C, D are rotors, then

$$\begin{aligned}[S.Sp] &= [(C+D).(C+D)\,p] = [C.Dp]+[D.Cp] \\ &= [CD]\,p = \tfrac{1}{2}[SS]\,p.\end{aligned}$$

Hence $[S.Sp]=\frac{1}{2}[SS]\,p$. Similarly, $[S.S\alpha]=\frac{1}{2}[SS]\,\alpha$.

7. The following can be shewn in the same way:

$$[abc.S]=[S.abc]=[bcS]\,a+[caS]\,b+[abS]\,c,$$

$$[Sa.bc]=[Sac]\,b-[Sab]\,c,$$

$$[abc.dS]=[adS]\,[bc]+[bdS]\,[ca]+[cdS]\,[ab].$$

Examples. 1. *If four forces along skew lines be in equilibrium, the lines are generators of a regulus.*

2. If p, a, b, c, d be fixed points, and q, r, t vary so that

$$[pq]+[rt]=[bc]+[ad],$$

then q moves in a fixed plane.

For $[bc]+[ad]-[pq]$ is a rotor, R, say. Then $[RR]=0$ gives

$$[bcad]=[(bc+ad)\,pq]=[(bcp+adp)\,q].$$

But $[bcp]+[adp]$ is a fixed leaf.

3. If i_1 is the incentre of the face of the tetrahedron $a_1a_2a_3a_4$ opposite to a_1, and p_1 is the perimeter of that face, and i_2, p_2, ... have similar meanings, then the system of forces

$$p_1.a_1i_1+p_2.a_2i_2+p_3.a_3i_3+p_4.a_4i_4$$

is in equilibrium.

4. If the lines

$$l[ad]+\ldots+l'[bc]+\ldots \quad \text{and} \quad l_1[ad]+\ldots+l_1'[bc]+\ldots$$

intersect, their point of intersection is

$$(l_1l'+mm_1'+nn_1')\,a+(l'm_1-l_1'm)\,b+(l'n_1-l_1'n)\,c+(mn_1-m_1n)\,d.$$

5. The most general screw S such that the points $[S.a_1a_2a_3]$ and $[S.a_1a_2a_4]$ are respectively the centroids of the triangles $a_1a_2a_3$ and $a_1a_2a_4$ can be written as the sum of multiples of a_1a_2 and

$$-a_2a_3+a_3a_1+(a_1+a_2+a_3)\,a_4.$$

For such a system the join of the points $[S.a_1a_3a_4]$ and $[S.a_2a_3a_4]$ is parallel to the edge a_1a_2. (Univ. of Wales.)

§ 34. *The linear complex.*

We now consider the properties of screws in projective geometry, and work in projective space, the field of scalars in this section being either the real or the complex field.

1. *If two projective pencils of lines in distinct planes, with distinct centres, have three pairs of corresponding lines which meet, then all pairs of corresponding lines meet.*

For let $L = A + \mathrm{k}B$, $L' = A' + \mathrm{k}B'$ be corresponding lines in the two pencils. We can suppose the pairs A, A'; B, B'; $A + \mathrm{k}_1 B$, $A' + \mathrm{k}_1 B'$ meet; then

$$[AA'] = [BB'] = 0, \quad [(A + \mathrm{k}_1 B)(A' + \mathrm{k}_1 B')] = 0.$$

Hence $[AB'] + [A'B] = 0$, $[(A + \mathrm{k}B)(A' + \mathrm{k}B')] = 0$ for all scalars k.

2. If two projective pencils of lines which lie in distinct planes have their centres at distinct points of the cut of the planes, and have that cut as self-corresponding line, then the set of lines meeting distinct corresponding lines is called a '*linear complex*'.

(Other cases of projective pencils besides those in 1, 2 are considered later.)

Let $X = A + \mathrm{k}B$, $X' = A' + \mathrm{k}B'$ be corresponding lines of the pencils, and let $B \equiv B'$ be along the self-corresponding line. We can adjust the weights absorbed in X', A', B', so that $B + B' = 0$, while the first two equations still hold. Then $X + X' = A + A'$. Hence any line L which cuts X and X' satisfies $[L(A + A')] = 0$.

Conversely, suppose $[L(A + A')] = 0$. We can find a line Y, in the first pencil, which meets L; if $Y = A + \mathrm{k}_1 B$, then

$$[LA] + \mathrm{k}_1 [LB] = 0, \quad \text{hence} \quad [LA'] + \mathrm{k}_1 [LB'] = 0,$$

hence the corresponding line Y' meets L. Hence L is in the complex.

Writing S for $A + A'$, we have: *the linear complex consists of all lines L such that $[LS] = 0$, and only of these.*

3. Now let S be any screw; consider lines L such that $[LS] = 0$, and let $B = [ab]$ be such a line. We can write S in the form $A_1 + A_2$, where A_1 is a rotor through a; then since $[BS] = 0$, we have

$$[abA_1] + [abA_2] = 0.$$

But $$[abA_1] = 0,$$

hence $$[abA_2] = 0; \quad [BA_1] = [BA_2] = 0.$$

Consider the pencils

$$L_1 = A_1 + \mathrm{k}B_1, \quad L_2 = A_2 + \mathrm{k}B_2,$$

where $$B = B_1 = -B_2.$$

All lines L which satisfy $[LS] = 0$ satisfy $[L(A_1 + A_2)] = 0$. Hence by 2, they are lines which meet corresponding lines of the two pencils; the set is thus a linear complex, which we shall say '*corresponds*' to the screw S.

4. The linear complex of lines L such that $[LS] = 0$, where S is a screw, is the set of lines round which S has zero moment. These lines are the '*nul lines*' of S.

If S is a rotor, or bivector (considered as a line at infinity), the lines L such that $[LS] = 0$, are the set which meet S. This set is a '*special*' complex. It arises from the case in 1, where the pencils are '*in perspective*', and the line of S is the join of their centres.

5. *Any screw S can be reduced to a rotor along any given line which is not a nul line of S, together with a rotor along another line, the 'conjugate' of the given line with respect to the screw or corresponding complex.*

For, if L is any rotor along the given line, and we suppose

$$S = \mathrm{k}L + M,$$

where k is a scalar, and M a rotor, the theorem will be proved if we can determine k and M. Now

$$0 = [MM] = [(S - \mathrm{k}L)\,(S - \mathrm{k}L)] = [SS] - 2\mathrm{k}[SL].$$

Hence $$\mathrm{k} = [SS] \div 2[SL], \quad M = S - \mathrm{k}L.$$

The solution exists and is unique, whenever $[SL] \neq 0$, that is, whenever L is not a nul line of S.

6. If S is a screw, not a rotor, and $S = A + B$, where A, B are rotors, then the nul lines of S which go through any point p lie in a plane, the '*nul plane*' of p for S (or for the corresponding complex); p is the '*nul point*' of the plane.

For, if $L=[pq]$ is a nul line of S, then $[Ap]+[Bp]$ is a plane α through p, such that

$$[\alpha q] = [(Ap+Bp)\,q] = [(A+B)\,pq] = [Spq] = 0.$$

Hence q, and hence L, lies on the plane α.

Conversely, every line in the plane through the nul point is a nul line.

Thus the nul plane of p for S is $[Sp]$.

7. *If q is on the nul plane of p, then p is on the nul plane of q.*

For $[Sp.q] = -[Sq.p]$; if q is on $[Sp]$, then $[Sp.q]=0$, whence $[Sq.p]=0$, and p is on the nul plane of q.

Cor. All the nul lines which lie in a given plane go through the nul point of the plane.

8. *Six lines* in a linear complex are dependent*; that is, there is a linear combination of the lines which vanishes.

Otherwise there would be six independent lines $L_1, \ldots, L_6$ such that $[L_1 S]=0, \ldots, [L_6 S]=0$, and since any line is a linear combination of the independent lines $L_1, \ldots, L_6$, we have $[LS]=0$ for all lines L. Let $S=A+B$ (A, B rotors), then $[AS]=0$ gives $[AB]=0$, hence A, B are coplanar. Hence S is *a rotor which meets all lines whatever*, which is impossible.

9. If $\alpha=[Sp]$, then $[S\alpha]=[S.Sp]=\frac{1}{2}[SS]\,p$.

Hence, if S is not a rotor, and $\alpha=[Sp]$, then $p\equiv[S\alpha]$.

Also $$[S(k_1 p_1+k_2 p_2)] = k_1[Sp_1]+k_2[Sp_2].$$

Hence, as p moves along a line $[p_1 p_2]$, its nul plane traverses the pencil determined by the nul planes of p_1 and p_2.

10. If $[p_1 p_2]$ is not a nul line for a general screw S, and hence $[Sp_1 p_2]\neq 0$, then $[p_1 p_2]$ and $[Sp_1 . Sp_2]$ are conjugate lines for S.

For, by 5, we can express S in the form $S=k_1 L+k_2 M$, where $L=[p_1 p_2]$, and $k_1, k_2\neq 0$. Then

$$[Sp_1] = k_2[Mp_1], \quad [Sp_2] = k_2[Mp_2],$$

hence $$M\equiv[Sp_1 . Sp_2].$$

* There is hardly any need in the present investigation to distinguish between 'line' and 'rotor'. Lines are dependent when, if any non-zero rotors be placed along them, the rotors are dependent.

Also $[Sp_1 . Sp_2] \neq 0$, for if $[Sp_1] \equiv [Sp_2]$, then

$$[Mp_1] \equiv [Mp_2], \quad [Sp_1 p_2] \equiv [Mp_1 p_2] = 0,$$

contrary to the hypothesis.

We say the lines L, M are '*conjugate*' lines for the complex as well as for the screw.

11. If L, M be conjugate lines for S, and L_1, M_1 be also conjugate for S, then L, M, L_1, M_1 lie on a regulus.

For, as there are scalars k, k', k_1, k_1', none zero, such that $S = kL + k'M = k_1 L_1 + k_1' M_1$, it follows that L, M, L_1, M_1 are dependent.

§ 35. *Screws and linear complexes in involution.*

In this section we take our field of scalars to be the field of complex numbers.

1. As the scalars k_1, k_2 vary, the screws $k_1 S_1 + k_2 S_2$ form a '*pencil*' of screws given by the screws S_1, S_2. In this pencil there are just two rotors, when we ignore magnitudes; their lines may be real, coincident, or imaginary.

For, if S_1, S_2 be not rotors, then

$$[(k_1 S_1 + k_2 S_2)(k_1 S_1 + k_2 S_2)] = 0$$

is a quadratic equation for $k_1 : k_2$. If these roots are distinct, and hence the lines of the rotors P, Q in the pencil are distinct, then each S in the pencil is a linear combination of P, Q.

2. In the pencil of screws, we can introduce cross-ratio and harmonic separation (§ 23).

If $$S_1 = k_1 P + l_1 Q, \quad S_2 = k_2 P + l_2 Q,$$

where P, Q are rotors, then the condition that P, Q separate S_1, S_2 harmonically is $k_1 l_2 + k_2 l_1 = 0$. If this holds, then

$$[S_1 S_2] = (k_1 l_2 + k_2 l_1)[PQ] = 0.$$

If $[S_1 S_2] = 0$, we say S_1, S_2 are '*in involution*'.

Two rotors are in involution, if and only if their lines meet.

If two screws are in involution, we say the corresponding linear complexes are '*in involution*'.

3. *If S, S' are screws in involution, the nul lines of S are conjugate in pairs for S'.*

For, if L is a line, and $[LS'] \neq 0$, the conjugate of L for S' is

$$S' - \frac{1}{2}\frac{[S'S']}{[LS']} L = L', \text{ say.} \qquad (\S 34{\cdot}5)$$

If then $$[LS] = 0,$$

we have $$[L'S] = [S'S] = 0.$$

4. The lines which are common to all complexes corresponding to the screws $k_1 S_1 + k_2 S_2$ are those which cut both rotors of the pencil of screws, for they must be nul lines for all screws of the pencil. They thus constitute a congruence of lines (§ 29·10). Through a general point of space goes just one line of the congruence.

5. Three independent screws S_1, S_2, S_3 give a spread of screws $k_1 S_1 + k_2 S_2 + k_3 S_3$ of step three, or a '*bundle*' of screws. The rotors in the bundle are given by the values of k which satisfy $[(k_1 S_1 + k_2 S_2 + k_3 S_3)(k_1 S_1 + k_2 S_2 + k_3 S_3)] = 0$.

This gives a quadratic equation in k_1, k_2, k_3, and thus there is an infinite number of rotors in the bundle. If all were linear combinations of two of them, then S_1, S_2, S_3 would also be linear combinations of these two, and so would not be independent. Hence at least three of the rotors in the bundle are independent; if these be L_1, L_2, L_3, then all screws in the bundle are linear combinations of L_1, L_2, L_3 (since S_1, S_2, S_3 are so) and hence all rotors in the bundle of screws are on the regulus through L_1, L_2, L_3; hence all nul lines of the bundle, or all lines common to the complexes determined by S_1, S_2, S_3, are on the opposite regulus.

6. Since any screw is a linear combination of rotors along the edges of a tetrahedron, and a general linear combination of these rotors is not zero, therefore screws form a spread of step six. Any screw can be put in the form $k_1 S_1 + \ldots + k_6 S_6$, where S_1, ..., S_6 are six given independent screws. Similarly for linear complexes.

§ 36. *The simultaneous use of supplements for vectors and rotors.*

1. We return to the consideration of real space. In dealing with points and vectors in space together, if we defined $|[ab]$ and $|[abc]$ as the supplements of the corresponding vector $b-a$ and of the bivector $[bc]+[ca]+[ab]$, we should have, if we assume that the operation of taking the supplement is distributive over addition,

$$|[abc] = |([bc]+[ca]+[ab]) = |((c-b)+(a-c)+(b-a)) = 0,$$

which is barren.

Accordingly we define the operation of taking the supplement of a rotor or a leaf independently, and introduce a fresh sign.

We define $\downarrow[abc]$ as the vector v, say, perpendicular to $[abc]$, with a magnitude equal to mag $[abc]$, and a sense such that $[abcv]$ is positive, and we define $\downarrow[ab]$ as the bivector U, say, perpendicular to ab with a magnitude equal to mag $[ab]$, and sense such that $[abU]$ is positive.

From this it easily follows that, if α, β are intersecting leaves,

$$\downarrow(\alpha+\beta) = \downarrow\alpha+\downarrow\beta,$$

and that if L, M are intersecting rotors,

$$\downarrow(L+M) = \downarrow L+\downarrow M.$$

If we assume these laws generally, it follows that if α, β be parallel leaves of equal and opposite magnitudes, so that $\alpha+\beta$ is a trivector, then $\downarrow(\alpha+\beta)=0$; and if L, M be parallel rotors of equal and opposite magnitude, so that $L+M$ is a bivector, then $\downarrow(L+M)=0$. To preserve the distributive law, we must therefore take the supplement $\downarrow$ of a trivector and of a bivector to be zero, and we shall assume the supplement $\downarrow$ of a vector to be zero.

For the old supplement $|$, we have the same rules as before.

2. If Ω is the unit trivector and P a rotor, α a leaf, then

$$\downarrow P = |[\Omega P], \quad \downarrow\alpha = |[\Omega\alpha], \quad \downarrow\downarrow P = 0, \quad \downarrow\downarrow\alpha = 0. \quad (1)$$

If P, Q be rotors, α, β leaves, then

$$[P\downarrow Q] = \text{mag } P \text{ mag } Q \cos(P, Q),$$
$$[\alpha\downarrow\beta] = \text{mag } \alpha \text{ mag } \beta \cos(\alpha, \beta),$$
$$[P\downarrow Q] = [Q\downarrow P], \quad [\alpha\downarrow\beta] = [\beta\downarrow\alpha]. \quad (2)$$

3. *A screw can be reduced to a rotor and a bivector in a plane perpendicular to the rotor*; either or both of the rotor and bivector may vanish.*

For, suppose $$S = A + \mathrm{k}\,{\downarrow}A, \tag{3}$$

where A is a rotor, and k a scalar to be determined. By (1), (3), ${\downarrow}S = {\downarrow}A$. Hence, if k were known, then A would be uniquely fixed by $$A = S - \mathrm{k}\,{\downarrow}S. \tag{4}$$

Now, since A is a rotor, $[AA] = 0$, hence

$$0 = [(S - \mathrm{k}\,{\downarrow}S)\,(S - \mathrm{k}\,{\downarrow}S)] = [SS] - 2\mathrm{k}[S\,{\downarrow}S]. \tag{5}$$

Hence, if $[S\,{\downarrow}S] \neq 0$, then k is uniquely fixed, and hence so is A. k is the '*pitch*' of the screw, A its '*axis*'.

If $[SS] = 0$, and $[S\,{\downarrow}S] \neq 0$, then S is a rotor.

If $[S\,{\downarrow}S] = 0$, then S is not of form (3), unless $S = 0$, because (5), (4), (3) would give $[SS] = 0$, $[AS] = 0$, $[A\,{\downarrow}A] = 0$, and hence $A = 0$.

If $[S\,{\downarrow}S] = 0$, and $S \neq 0$, then S is a bivector.

4. We have $$\mathrm{k} = \frac{[AS]}{[A\,{\downarrow}S]} = \frac{[SS]}{2[S\,{\downarrow}S]}.$$

For $$[S\,{\downarrow}S] = [S\,{\downarrow}A] = [A\,{\downarrow}S] = [A\,{\downarrow}A],$$

since $${\downarrow}S = {\downarrow}A.$$

And $$[SS] = [SA] + \mathrm{k}[S\,{\downarrow}A] = [SA] + \mathrm{k}[A\,{\downarrow}A] = 2[SA],$$

since $$[SA] = [AS] = [AA] + \mathrm{k}[A\,{\downarrow}A] = \mathrm{k}[A\,{\downarrow}A].$$

5. If A, B, C be three perpendicular concurrent rotors, then $A + {\downarrow}A$, $B + {\downarrow}B$, $C + {\downarrow}C$, $A - {\downarrow}A$, $B - {\downarrow}B$, $C - {\downarrow}C$ are independent screws, mutually in involution, for $[AB] = [A\,{\downarrow}B] = 0$, and so on. Hence six independent screws mutually in involution, exist.

Examples. 6. Forces acting at the vertices of a tetrahedron *abcd*, perpendicular to and proportional to the opposite faces, balance.

For, take d as the origin of vectors, writing $a_1 = a - d$, ... and so on, then the sum of the forces from a, b, c is

$$\begin{aligned}&[a\,|\,b_1 c_1] + [b\,|\,c_1 a_1] + [c\,|\,a_1 b_1]\\ &\quad = [(d + a_1)\,|\,b_1 c_1] + [(d + b_1)\,|\,c_1 a_1] + [(d + c_1)\,|\,a_1 b_1]\\ &\quad = [d\,|\,b_1 c_1] + [d\,|\,c_1 a_1] + [d\,|\,c_1 b_1] = [d\,|\,(b - a)\,(c - a)].\end{aligned}$$

* This proof of Poinsot's theorem is given by Lotze, *Die Grundgleichungen der Mechanik* (1922). This tract uses Grassmann's methods.

7. Forces acting at the centroids of the faces of *abcd* perpendicular to and proportional to the faces, all inwards, balance.

For, taking d as the origin of vectors, the sum of the forces is one-third of

$$\begin{aligned}&[(a+b+c)\,|\,(b_1c_1+c_1a_1+a_1b_1)]+[(b+c+d)\,|\,-b_1c_1]\\&\quad+[(c+a+d)\,|\,-c_1a_1]+[(a+b+d)\,|\,-a_1b_1]\\&=[a\,|\,b_1c_1]+[b\,|\,c_1a_1]+[c\,|\,a_1b_1]-[d\,|\,(b_1c_1+c_1a_1+a_1b_1)]=0,\end{aligned}$$

as in Ex. 6.

8. If forces act along the sides of a triangle *abc* taken in order, and are proportional to the distances of the opposite vertices from a plane, then their resultant acts along the cut of the planes.

For $$[abc.\alpha]=[a\alpha]\,[bc]+[b\alpha]\,[ca]+[c\alpha]\,[ab].$$

§ 37. *The mutual moment of two screws.*

1. If $S_1=A+l_1\downarrow A$, $S_2=B+l_2\downarrow B$, then since (§ 17·5) $[\downarrow A.\downarrow B]=0$, we have $[S_1S_2]=[AB]+(l_1+l_2)\,[A\downarrow B]$.

If k_1, k_2 be the magnitudes of A and B, and if ϕ is the angle between them, and d the distance between them, then

$$[S_1S_2]=k_1k_2d\sin\phi+(l_1+l_2)\,k_1k_2\cos\phi.$$

2. If $S=A+l\downarrow A=P+Q$, where P, Q are rotors, let C be a rotor perpendicular to P, Q and meeting both, then

$$[C\downarrow P]=[C\downarrow Q]=[CP]=[CQ]=0.$$

Hence using $\downarrow\downarrow A=0$, we have $[C\downarrow A]=0$, $[CA]=0$.

Hence the common perpendicular of two conjugate lines of S cuts the axis of S at right angles.

3. If $S=A+l\downarrow A$, and L is a nul line of S, then $[SL]=0$. Hence if d is the distance between A and L, then

$$l=-\frac{[AL]}{[A\downarrow L]}=-d\tan(A,L). \qquad (6)$$

Thus to draw a line of the complex which corresponds to S, through a given point p, we find the foot q of the perpendicular from p to A and through p draw a line L perpendicular to pq at an angle (A, L) to A given by (6). All lines through p in the plane of pq and L are in the complex. This could have been taken as the (metric) definition of the complex.

4. More generally, let $S = kP + k'Q$, where P, Q are rotors, k, k′ scalars. Then if L be a line of the complex, we have

$$k : k' = -[QL] : [PL].$$

Hence *the moments of a force along a line of the complex about two conjugate lines are in a ratio which depends only on the conjugate lines chosen.*

§ 38. *The cylindroid.*

1. Consider the pencil of screws $S = k_1 S_1 + k_2 S_2$, where S, S_1, S_2 are expressed in standard form:

$$S = A + x \downarrow A, \quad S_1 = A_1 + x_1 \downarrow A_1, \quad S_2 = A_2 + x_2 \downarrow A_2,$$

where $[A_1 \downarrow A_1] = [A_2 \downarrow A_2] = 1$, and suppose $[S_1 S_1]$, $[S_2 S_2]$, $[S_1 \downarrow S_2]$ are not zero. Then

$$[S_1 S_1] = 2x_1, \qquad [S_2 S_2] = 2x_2,$$
$$[S_1 \downarrow S_1] = [S_2 \downarrow S_2] = 1, \quad [S_1 \downarrow S_2] = [A_1 \downarrow A_2] = \cos\phi,$$

ϕ being the angle between the axes of S_1 and S_2.

Then
$$x = \frac{[SS]}{2[S \downarrow S]} = \frac{[(k_1 S_1 + k_2 S_2)\,(k_1 S_1 + k_2 S_2)]}{2[(k_1 S_1 + k_2 S_2) \downarrow (k_1 S_1 + k_2 S_2)]}$$
$$= \frac{x_1 k_1^2 + k_1 k_2 [S_1 S_2] + x_2 k_2^2}{k_1^2 + 2k_1 k_2 \cos\phi + k_2^2}$$
$$= \frac{x_1 k_1^2 + k_1 k_2 d \sin\phi + k_1 k_2 (x_1 + x_2)\cos\phi + x_2 k_2^2}{k_1^2 + 2k_1 k_2 \cos\phi + k_2^2}.$$

2. As $k_1 : k_2$ varies, the locus of the axis A of S is given by

$$A = S - x \downarrow S \equiv k_1 S_1 + k_2 S_2 - x(k_1 \downarrow S_1 + k_2 \downarrow S_2),$$

with the value of x just found, and is called a '*cylindroid*'.

3. Let P be the rotor which cuts A_1, A_2 at right angles, then

$$[A_1 P] = [A_2 P] = 0, \quad [A_1 \downarrow P] = [A_2 \downarrow P] = 0.$$

Hence $[PS_1] = [PS_2] = 0$, $[P \downarrow S_1] = [P \downarrow S_2] = 0$.

Hence for each axis A of screws of the pencil,

$$[PA] = 0, \quad [P \downarrow A] = 0.$$

All generators of the cylindroid cut P at right angles.

4. Suppose G, G' be two perpendicular intersecting generators of the cylindroid, if such exist. Then

$$[GG'] = 0, \quad [G \downarrow G'] = 0.$$

Let
$$Y = G + 1 \downarrow G = S_1 + yS_2,$$
$$Y' = G' + 1' \downarrow G' = S_1 + y'S_2.$$

Then
$$[Y \downarrow Y'] = 0 = [YY'].$$

$$\left.\begin{aligned} [S_1 \downarrow S_1] + (y + y')\,[S_1 \downarrow S_2] + yy'[S_2 \downarrow S_2] &= 0, \\ [S_1 S_1] + (y + y')\,[S_1 S_2] + yy'[S_2 S_2] &= 0. \end{aligned}\right\} \quad (7)$$

These equations determine y, y′ uniquely in the general case. Hence, in general, there is one pair of intersecting perpendicular generators.

5. To shew that these generators are real, we must prove that $y - y'$ is real. Write the first equation of (7) as

$$a + 2bx + cx^2 = cz,$$

where
$$y + y' = 2x, \quad yy' = x^2 - z,$$
$$a = [S_1 \downarrow S_1], \quad b = [S_1 \downarrow S_2], \quad c = [S_2 \downarrow S_2].$$

Then $(y - y')^2 = 4z$, and we have to shew that z is positive. This follows from

$$c^2 z = (cx + b)^2 + ac - b^2,$$
$$ac - b^2 = [S_1 \downarrow S_1]\,[S_2 \downarrow S_2] - [S_1 \downarrow S_2]^2$$
$$= [S_1 S_2 \downarrow S_1 S_2] \geqq 0.$$

6. In the pencil of screws, let L_1, L_2 be the perpendicular intersecting axes; let them cut in e, and let e_1, e_2 be unit vectors along L_1, L_2, and e_3 a unit vector perpendicular to them, with $[e_1 e_2 e_3] = 1$.

We may suppose, without loss of generality, that L_1, L_2 are of unit magnitude, and

$$S_1 = L_1 + k_1 \downarrow L_1, \quad S_2 = L_2 + k_2 \downarrow L_2, \quad S = l_1 S_1 + l_2 S_2.$$

Let $S = L + k \downarrow L$. Then L cuts $[ee_3]$ at $e + ze_3$, say.

Let
$$L = [(e + ze_3)\,(xe_1 + ye_2)].$$

Then, since
$$[e_3 e_1] = \downarrow [ee_2] = \downarrow L_2,$$
$$[e_2 e_3] = \downarrow [ee_1] = \downarrow L_1,$$

we have $$L = xL_1 + yL_2 + xz\downarrow L_2 - yz\downarrow L_1,$$

$$S = xL_1 + yL_2 + (kx - yz)\downarrow L_1 + (ky + xz)\downarrow L_2 = l_1 S_1 + l_2 S_2.$$

Hence $$x = l_1, \quad kx - yz = k_1 l_1 = k_1 x,$$

$$y = l_2, \quad ky + xz = k_2 l_2 = k_2 y.$$

These give $$(k - k_1)\, x = yz, \quad (k_2 - k)\, y = xz,$$

whence we have the equation of the cylindroid in rectangular cartesian coordinates,

$$(x^2 + y^2)\, z = (k_2 - k_1)\, xy. \tag{10}$$

Also $$k(x^2 + y^2) = k_1 x^2 + k_2 y^2,$$

hence $$k(l_1^2 + l_2^2) = k_1 l_1^2 + k_2 l_2^2,$$

which gives k in terms of l_1, l_2, k_1, k_2.

It should be noted that according to the definition, given in 2, of the cylindroid, points of the z-axis lie on the cylindroid only when they lie on generators; in real space such points fill an interval only, whereas each point of the z-axis satisfies equation (10).

Examples. 9. If v_1, v_2, v_3, v_4 be vectors, and p a point, then

$$\begin{aligned}&[p\,|v_1.(p+v_2)\,(p+v_3)\,(p+v_4)]\\ -&[p\,|v_2.(p+v_1)\,(p+v_3)\,(p+v_4)]\\ +&[p\,|v_3.(p+v_1)\,(p+v_2)\,(p+v_4)]\\ -&[p\,|v_4.(p+v_1)\,(p+v_2)\,(p+v_3)] = 0.\end{aligned}$$

For, $$\begin{aligned}&[(p+v_2)\,(p+v_3)\,(p+v_4)]\\ &\quad = [p(v_3v_4 + v_4v_2 + v_2v_3)] + [v_2v_3v_4]\\ &\quad = pV_1 + k_1\Omega, \text{ say. } (\S\,17{\cdot}7.)\end{aligned}$$

Now $[p\,|v_1.pV_1]$, or $p[|v_1.V_1]$, is the product of p and a vector, and hence is a rotor. Assume p to be of unit weight.

Also $$[p\,|v_1.k_1\Omega] = k_1\,|v_1, \quad \text{where} \quad k_1 = [v_2v_3v_4],$$

and $$[|v_1.V_1] = -[V_1\,|v_1] = -[(v_3v_4 + v_4v_2 + v_2v_3)\,|v_1],$$

$$[v_3v_4\,|v_1] = [v_3\,|v_1]\, v_4 - [v_4\,|v_1]\, v_3.$$

Hence the sum of terms like $[|v_1.V_1]$ has, when expanded, as the coefficient of v_4, the expression

$$[(v_2 - v_3)\,|v_1] + [(v_3 - v_1)\,|v_2] + [(v_1 - v_2)\,|v_3] = 0.$$

Similarly, the other coefficients vanish, and the sum is zero.

Also $[v_2v_3v_4]\,|v_1-[v_1v_3v_4]\,|v_2+[v_1v_2v_4]\,|v_3-[v_1v_2v_3]\,|v_4=0.$

Hence, *if p be any point, abcd a tetrahedron, then the planes through p perpendicular to the joins of p to a, b, c, d cut the opposite faces in lines on a regulus.*

10. A line L cuts the faces of *abcd* opposite to a, b, c, d in p, q, r, s; then the lines ap, bq, cr, ds, L are dependent.

11. Screws of pitch k in the spread $\mathrm{k}_1S_1+\mathrm{k}_2S_2+\mathrm{k}_3S_3$, ($S_1, S_2, S_3$ being screws) have axes which lie on a regulus.

For, if $R_1, \ldots, R_4$ be four such screws, and $R_i=A_i+\mathrm{k}\downarrow A_i$, then there are scalars $\mathrm{x}_1, \ldots, \mathrm{x}_4$ such that $\mathrm{x}_1R_1+\ldots+\mathrm{x}_4R_4=0$. Take supplements.

Thence $\mathrm{x}_1\downarrow A_1+\ldots+\mathrm{x}_4\downarrow A_4=0$, hence $\mathrm{x}_1A_1+\ldots+\mathrm{x}_4A_4=0$.

In particular, the lines of rotors of the system lie on a regulus.

12. If A, B, C be rotors, and

$$[A(B-C)]=0, \quad [A\downarrow(B-C)]=0,$$

then the axis of $B-C$ cuts the line of A at right angles.

Hence deduce the theorem of Petersen and Morley:

If A, B, C be skew lines, and A', B', C' be the common perpendiculars of the pairs $(B, C), (C, A), (A, B)$ respectively, then the common perpendiculars of the pairs $(A, A'), (B, B'), (C, C')$ have a common perpendicular.

§ 39. *Möbius tetrahedra.*

1. *An identity.* If a, b, c, a', b', c' be any points, then*

$$[bca'.cab'.abc'.a'b'c']+[b'c'a.c'a'b.a'b'c.abc]=0. \quad (11)$$

For $\quad [abc'.a'b'c']=[abc'a']\,[b'c']+[abc'b']\,[c'a'],$

$$[cab'.abc'.a'b'c']=[abc'a']\,[cab'c']\,b'+[abc'b']\,[cab'a']\,c' -[abc'b']\,[cab'c']\,a',$$

$$\begin{aligned}[bca'.cab'.abc'.a'b'c'] &= [abc'a']\,[cab'c']\,[bca'b'] \\ &\quad +[abc'b']\,[cab'a']\,[bca'c'] \\ &= [abc'a']\,[cab'c']\,[bca'b'] \\ &\quad -[abb'c']\,[caa'b']\,[bcc'a'].\end{aligned}$$

Similarly $\quad [b'c'a.c'a'b.a'b'c.abc]=[a'b'ca]\,[c'a'bc]\,[b'c'ab] -[a'b'bc]\,[c'a'ab]\,[b'c'ca].$

Adding these last two equations, we have (11).

The dual equation, of course, holds for planes.

* We call this the Möbius identity, although it does not occur in Möbius' works. Nor have I been able to find it stated elsewhere.

2. If the first term is zero, then bca', cab', abc' meet in a point, d, say, on $a'b'c'$. The last term is then zero, and the planes $b'c'a$, $c'a'b$, $a'b'c$ meet in a point d' on abc.

Hence if $abcd$ and $a'b'c'd'$ be tetrahedra, and a', b', c' points on the planes bcd, cda, dab, and a, b, c, d be points on planes $b'c'd'$, $c'd'a'$, $d'a'b'$, $a'b'c'$, then d' is on abc. Each tetrahedron has its vertices on the faces of the other. Such tetrahedra are called '*Möbius tetrahedra*'.

3. If $abcd$ and $a'b'c'd'$ are Möbius tetrahedra, then so are $ab'c'd$ and $a'bcd'$, also $a'b'cd$ and $abc'd'$, also $a'bc'd$ and $ab'cd'$.

4. Suppose a', b', c', d' are on the faces of a, b, c, d, then

$$\begin{aligned} k_1 a' &= \quad . \quad + k_{12} b + k_{13} c + k_{14} d, \\ k_2 b' &= k_{21} a \quad . \quad + k_{23} c + k_{24} d, \\ k_3 c' &= k_{31} a + k_{32} b \quad . \quad + k_{34} d, \\ k_4 d' &= k_{41} a + k_{42} b + k_{43} c \quad . \quad . \end{aligned}$$

If $[ab'c'd']$, $[ba'c'd']$, $[ca'b'd']$ vanish, we have

$$k_{23} k_{34} k_{42} = -k_{24} k_{43} k_{32}, \quad k_{34} k_{41} k_{13} = -k_{31} k_{14} k_{43},$$
$$k_{41} k_{12} k_{24} = -k_{42} k_{21} k_{14}.$$

These give $k_{12} k_{23} k_{31} = -k_{13} k_{32} k_{21}$, hence $[da'b'c'] = 0$.

For let $k_{ij}/k_{ji} = h_{ij}$, then

$$h_{23} h_{34} h_{42} = h_{31} h_{14} h_{43} = h_{41} h_{12} h_{24} = -1.$$

Since $h_{ij} h_{ji} = 1$, the product of these expressions gives $h_{12} h_{23} h_{31} = -1$. Thus we have another proof of the existence of Möbius tetrahedra.

5. Now adjust the weights of b', c', d' so that

$$h_{12} = h_{13} = h_{14} = -1,$$

then $h_{34} = h_{24} = -1$, and hence $h_{23} = -1$.

The determinant of the k_{ij} *is then skew.*

Conversely, *if that determinant so modified is skew, then abcd and* $a'b'c'd'$ *are Möbius tetrahedra.*

6. *If* aa', bb', cc' *be non-coplanar and* uv, $u'v'$ *cut them harmonically in* u, v, w *and* u', v', w', *then the planes* abc, $ab'c'$, $a'bc'$, $a'b'c$ *meet in a point, and hence, by* (11), $a'b'c'$, $a'bc$, $ab'c$, abc' *meet in a point.* (Steiner.)

For we can weight the points, so that $a+a'$, $b+b'$, $c+c'$ are at u, v, w, and $a-a'$, $b-b'$, $c-c'$ are at u', v', w'. Then, since $[uvw]=0$, $[u'v'w']=0$, we have

$$[(a+a')(b+b')(c+c')]=0, \quad [(a-a')(b-b')(c-c')]=0.$$

Hence
$$[abc]+[ab'c']+[a'bc']+[a'b'c]=0$$
$$[a'b'c']+[a'bc]+[ab'c]+[abc']=0.$$

If $[a'bc.ab'c.abc']=d$, and $[ab'c'.a'bc'.a'b'c]=d'$, then $abcd$ and $a'b'c'd'$ are Möbius tetrahedra.

7. If a line meets aa', bb', cc' in u, v, w, and if u' is the harmonic conjugate of u with respect to a, a', and if v', w' have similar meanings, then $[u'v'w']$ goes through d and d' defined in 6.

For, weighting points as in 6, $[uvw]$ and $[u'v'w']$ are respectively:

$$([abc]+[ab'c']+[a'bc']+[a'b'c]) + ([a'b'c']+[a'bc]+[ab'c]+[abc']),$$
$$([abc]+[ab'c']+[a'bc']+[a'b'c]) - ([a'b'c']+[a'bc]+[ab'c]+[abc']).$$

Since $[uvw]=0$, therefore $[u'v'w']$ is congruent to each of the expressions in the round brackets, and hence the plane goes through d and d'.

8. If d, d' be defined as in 6, and a line meets aa', bb', cc', dd' in u, v, w, p, and if u', v', w' be defined as in 7, and p' similarly, then these four points are collinear. If also a, b, c, d are collinear, and a', b', c', d' collinear, then the generators $[uv]$ and $[u'v']$ separate harmonically the generators ab, $a'b'$ on the hyperboloid through them (§ 31·1).

9. If the joins of corresponding vertices of two tetrahedra lie on a regulus, then this regulus contains the cuts of corresponding faces if, and only if, the tetrahedra are Möbius tetrahedra. (Cf. § 31·7.)

We leave the details of 8, 9 to the reader.

10. If S is a screw, L a line, p a point, then

$$[SL]\,p=[Sp.L]+[Lp.S]. \qquad (12)$$

For, S is a sum of rotors, and (12) is true when S is replaced by a rotor (§ 22·3). Hence (12) holds generally, by addition.

If S' is a screw and $S' = L_1 + L_2 + \ldots$, then adding equations obtained from (12) by substituting L_1, L_2, ... for L, we get

$$[SS']\,p = [Sp.S'] + [S'p.S]. \qquad (13)$$

In particular (cf. § 33·6)

$$\tfrac{1}{2}[SS]\,p = [Sp.S] = [S.Sp]. \qquad (14)$$

11. If L, M be rotors, p, q points, we have, if $L = [ab]$, $M = [cd]$,

$$\begin{aligned}[Lp.Mq] + [Mp.Lq] &= [Lpq]\,M + [Lpc]\,[dq] + [Lpd]\,[qc] \\ &\quad + [Mpq]\,L + [Mpa]\,[bq] + [Mpb]\,[qa] \\ &= [Lpq]\,M + [Mpq]\,L - [LM]\,[pq],\end{aligned}$$

by § 16 (6).

Applying this to a screw S, the sum of rotors $L + M$, we find

$$[Sp.Sq] = [pS.qS] = [Spq]\,S - \tfrac{1}{2}[SS]\,[pq]. \qquad (15)$$

If in this we write $S + \mathrm{k}S'$ for S, and expand, the coefficients of k give

$$\begin{aligned}&[Sp.S'q] + [S'p.Sq] \\ &\qquad = [S'pq]\,S + [Spq]\,S' - [SS']\,[pq]. \qquad (16)\end{aligned}$$

12. From (15),

$$\begin{aligned}[Sp.Sq.Sr] &= [Spq]\,[S.Sr] - \tfrac{1}{2}[SS]\,[pq.Sr] \\ &= \tfrac{1}{2}[Spq]\,[SS]\,r - \tfrac{1}{2}[SS]\,([Spq]\,r - [S.pqr]) \\ &\qquad\qquad \text{(by (13), (14))} \\ &= \tfrac{1}{2}[SS]\,[S.pqr]. \qquad (17)\end{aligned}$$

We have used $[pq.Sr] = [Spq]\,r - [S.pqr]$, which follows from (13), by putting r for p and $[pq]$ for S'. Formula (17) is the symbolic expression of the fact that the nul planes of three points with respect to a screw meet at the nul point of the plane joining the three points.

13. From (17),

$$\begin{aligned}[Sp.Sq.Sr.St] &= \tfrac{1}{2}[SS]\,[S.pqr..St] \\ &= -\tfrac{1}{2}[SS]\,[St..S.pqr] \\ &= -\tfrac{1}{2}[SS]\,[St.S..pqr].\end{aligned}$$

The associative law which has been assumed in the last step is obvious geometrically when S is a rotor, and hence generally, by addition. It is a special case of a rule which will be shewn later from the algebra.

Now $$[St.S] = \tfrac{1}{2}[SS]\, t.$$

Hence $$[Sp.Sq.Sr.St] = -\tfrac{1}{4}[SS]^2\,[tpqr] = \tfrac{1}{4}[SS]^2\,[pqrt].$$

All these formulae have duals, when points are replaced by planes.

14. The dual of (15) is

$$[S\alpha.S\beta] = [\alpha S.\beta S] = [S\alpha\beta]\, S - \tfrac{1}{2}[SS]\,[\alpha\beta]. \quad (18)$$

Let $\alpha = bS_1$, $\beta = cS_1$. These are nul planes of b, c for S_1.

Let $$b' = [bS_1 S] = [\alpha S], \quad c' = [cS_1 S] = [\beta S].$$

Then (18) gives

$$[b'c'] = [S.bS_1.cS_1]\, S - \tfrac{1}{2}[SS]\,[bS_1.cS_1].$$

By (15), $$[bS_1.cS_1] = [S_1 bc]\, S_1 - \tfrac{1}{2}[S_1 S_1]\,[bc].$$

Hence
$$[b'c'] = [bcS_1]\,[SS_1]\, S - \tfrac{1}{2}[S_1 S_1]\,[bcS]\, S - \tfrac{1}{2}[SS]\,[bcS_1]\, S_1 + \tfrac{1}{4}[SS]\,[S_1 S_1]\,[bc],$$

$$[ab'c'] = [SS_1]\,[bcS_1]\,[aS] - \tfrac{1}{2}[S_1 S_1]\,[bcS]\,[aS] - \tfrac{1}{2}[SS]\,[bcS_1]\,[aS_1] + \tfrac{1}{4}[SS]\,[S_1 S_1]\,[abc].$$

Cycle a, b, c and add, using the dual of (14),

$$[Sabc.S] = \tfrac{1}{2}[SS]\,[abc],$$

and the easily shewn theorem:

$$[bcS_1]\, a + [caS_1]\, b + [abS_1]\, c = [abc.S_1],$$

and we have*

$$[ab'c'] + [bc'a'] + [ca'b'] = \tfrac{1}{4}[SS]\,[S_1 S_1]\,[abc] + [SS_1]\,[abc.S_1.S]. \quad (19)$$

Hence if $[SS_1] = 0$, that is if S, S_1 are in involution, then

$$[ab'c'.a'bc'.a'b'c.abc] = 0.$$

15. The nul point of $[pS_1]$ with respect to S is $[pS_1 S] = q$, say. Clearly $[pq]$ cuts the axes of both rotors of the pencil $kS + k_1 S_1$.

If $q' = [p'S_1 S]$, then

$$xq + x'q' = [(xp + x'p')\, S_1 S].$$

* E. Müller, *Monatshefte Math. Phys.* 14 (1903), p. 182.

Hence, as p traverses any line, q traverses a line. Hence the cross-ratio of p, q and the cuts of $[pq]$ with the rotors of the pencil is constant. When S, S_1 are in involution, this cross-ratio is -1. Hence we have the result of 6 again. The identity (19) includes this theorem.

§ 40. *Screws and quadrics.**

1. If ab be a generator of a regulus, and we denote supplements for the quadric through the regulus by the stroke, then

$$a^2 = b^2 = 0, \quad (a + \mathrm{k}b)^2 = 0 \text{ for all k.}$$

Hence
$$[a|b] = 0.$$

If ap is a generator of the opposite regulus, then abp is the tangent plane to the quadric at a, since $[a|p]=0$, $[a|abp]=0$, and so $|a \equiv [abp]$.

We can take p on ap and weight b, p so that $|a=[abp]$ and $[b|p]=1$.

Then
$$|[ab] = [abp|b] = [p|b]\,[ab] = [ab],$$
$$|[ap] = [abp|p] = [b|p]\,[pa] = -[ap].$$

Thus we have a distinction between generators of opposite reguli. We can take weights so that if L is a generator of the regulus containing $[ab]$, then $|L=L$, and it then follows that if L_1 is on the opposite regulus, then $|L_1 = -L_1$.

2. If two lines L_1, L_2 of a regulus be conjugate for a screw S, then the conjugate for the screw of each line of the regulus is on the regulus.

For if X be any line, $|X$ is its polar line for the quadric. There are scalars x_1, x_2 such that $S = \mathrm{x}_1 L_1 + \mathrm{x}_2 L_2$, and we may assume $|L_1 = L_1$, $|L_2 = L_2$, thence $|S = S$.

Then the conjugate for S of any other line L of the regulus is

$$M = S - \tfrac{1}{2}\frac{[SS]}{[LS]}L.$$

Since $|L=L$, and $|S=S$, we have $|M=M$.

* See Whitehead, *Universal Algebra*, § 174.

3. If $S \not\equiv \pm |S$, then the rotors in the pencil of screws defined by S and $|S$ are of form $xS+y|S$, where

$$[(xS+y|S)(xS+y|S)] = 0,$$

that is,
$$(x^2+y^2)[SS]+2xy[S|S] = 0.$$

If the solutions of this equation be

$$x:y = h:k \text{ and } x:y = k:h,$$

and be distinct, these rotors are $hS+k|S = L$, say, and $kS+h|S = |L$. Thus their lines are polar for the quadric, and they are conjugate for the screw S, since

$$hL-k|L = (h^2-k^2)\,S.$$

4. Hence if $S \not\equiv \pm |S$, then S is of form $L+k|L$ in general, where L is a rotor. This form is unique, for if

$$S = M+k|M, \quad \text{then} \quad |S = kM+|M,$$

and M, $|M$ are the two rotors in the pencil of screws defined by S and $|S$.

The only self-polar screws in this pencil are $S \pm |S$. For, if

$$S' = k_1S+k_2|S = \pm|S', \quad \text{then} \quad k_1|S+k_2S = \pm(k_1S+k_2|S);$$

hence $k_1 = \pm k_2$, $S' \equiv S \pm |S$, since $S \not\equiv \pm |S$.

5. If the solutions of the equation in x/y coincide, then

$$[SS] = \pm[S|S],$$

the pencil of screws is '*parabolic*'; it contains only one rotor, and that is self-polar for the quadric, and hence a generator.

6. If a pencil of screws has as its rotors two generators L_1, L_2 of a regulus on the quadric, then all lines of form $k_1L_1+k_2L_2$ are self-polar for the quadric, and the pencil cannot be defined by a screw and its supplement.

7. In general a linear complex has two lines on a regulus, for if *ab* and *cd* be the two lines L, $|L$ of 4, conjugate for the corresponding screw S, and polar for the quadric through the regulus, and if they meet the quadric in *a*, *b*, *c*, *d*, then *ac*, *ad* and *bc*, *bd* are generators of the quadric and nul lines of the screw.

If the pencil $S+|S$ is parabolic, the two lines coincide in a generator of the quadric.

If $S \equiv |S$, the complex contains all lines of one of the reguli and two of the opposite regulus.

CHAPTER V

DIFFERENTIATION AND MOTION

§ 41. *Differentiation of an extensive with respect to a scalar.*

1. If a point p moves along a curve, we can regard p as a function of the arc s (a scalar) of the curve, measured from a fixed point of the curve.

If ds is the arc-length between p and p_1, then $(p_1 - p)/\mathrm{ds}$ is a vector along pp_1. If we keep p fixed and make p_1 move along the curve so as to tend to p, we assume that this vector tends to a vector along the tangent at p, which is assumed to exist and to be unique.

The assumptions made are tantamount to certain assumptions on the continuity of the spread in which we are working, and on the type of curve with which we are dealing. As it is no part of our plan to analyse such assumptions, we shall always make, without necessarily mentioning them, the assumptions necessary to secure the existence and uniqueness of the limits which we need.

The limit of $(p_1 - p)/\mathrm{ds}$, as p_1 tends to p, will be written dp/ds and called the derivative of p for s.

This derivative is a unit vector, for as p_1 tends to p, $\mathrm{mag}\,(p_1 - p)$ differs from ds by an amount of order not less than ds^2.

2. *Example.* If o be a fixed point, and i, j perpendicular unit vectors, and $p = o + \mathrm{a}\cos\phi.i + \mathrm{b}\sin\phi.j$, then p describes an ellipse, centre o, as ϕ varies.

$$\begin{aligned} dp/\mathrm{ds} &= (\mathrm{a}\cos(\phi + \pi/2).i + \mathrm{b}\sin(\phi + \pi/2).j) \div \mathrm{ds}/\mathrm{d}\phi \\ &= (q - o) \div \mathrm{ds}/\mathrm{d}\phi, \end{aligned}$$

where q is the point on the ellipse corresponding to $\phi + \pi/2$.

Now since dp/ds is a unit vector along the tangent at p, it follows that $\mathrm{ds}/\mathrm{d}\phi$ is the length of the radius vector $q - o$, and this vector is parallel to the tangent at p.

Similarly $p = o + \mathrm{a}\cosh\mathrm{u}.i + \mathrm{b}\sinh\mathrm{u}.j$, (u scalar)

represents one branch of a hyperbola, centre o.

3. If A be any extensive, which is a function $f(\tau)$ of a *scalar* τ, and if there be an extensive C of the same step as A, such that, if h be a scalar, then

$$f(\tau+\mathrm{h})-f(\tau)=\mathrm{h}(C+D),$$

where D is an extensive of the same step as A, whose magnitude tends to zero when h tends to zero, then we write, as in the ordinary differential calculus,

$$C=dA/\mathrm{d}\tau, \quad dA=dA/\mathrm{d}\tau.\mathrm{d}\tau,$$

and dA is of the same step as A.

If A, B be any extensives, functions of the scalar τ, and $A.B$ be their outer *or* inner product, then if τ be changed to $\tau+\mathrm{h}$, $A.B$ becomes, omitting terms of higher orders,

$$(A+dA).(B+dB)=A.B+dA.B+A.dB+dA.dB.$$

Hence
$$d(A.B)=dA.B+A.dB+dA.dB$$
$$=\left(\frac{dA}{\mathrm{d}\tau}.B+A.\frac{dB}{\mathrm{d}\tau}\right)\mathrm{d}\tau+dA.dB.$$

If the magnitudes of dA, dB tend to zero, so do the magnitudes of their outer, and inner, products. Hence

$$\frac{d}{\mathrm{d}\tau}(A.B)=\frac{dA}{\mathrm{d}\tau}.B+A.\frac{dB}{\mathrm{d}\tau}.$$

Thus for outer products,

$$\frac{d}{\mathrm{d}\tau}[AB]=\left[\frac{dA}{\mathrm{d}\tau}.B\right]+\left[A.\frac{dB}{\mathrm{d}\tau}\right].$$

For inner products,

$$\frac{d}{\mathrm{d}\tau}[A|B]=\left[\frac{dA}{\mathrm{d}\tau}|B\right]+\left[A|\frac{dB}{\mathrm{d}\tau}\right]=\left[A|\frac{dB}{\mathrm{d}\tau}\right]+\left[B|\frac{dA}{\mathrm{d}\tau}\right].$$

Also
$$d[AB]=[A.dB]+[dA.B]=[A.dB]-[B.dA],$$
$$d[A|B]=[dA|B]+[A|dB],$$
$$\frac{d}{\mathrm{d}\tau}[ABC]=\left[\frac{dA}{\mathrm{d}\tau}BC\right]+\left[A\frac{dB}{\mathrm{d}\tau}C\right]+\left[AB\frac{dC}{\mathrm{d}\tau}\right].$$

4. Hence, if v be a vector, and, as usual, v^2 means $[v|v]$, we have $\mathrm{d}v^2=[v|dv]+[dv|v]=2[v|dv]$, (not $2[vdv]$).

If v^2 is constant, then $[v|dv]=0$, hence v and dv are perpendicular; thus the tangent to a circle is perpendicular to its radius.

It is easily shewn geometrically, or by the use of i, j, that

$$d|v = |dv.$$

Then, since $v^2 = -[|v.v]$, we have

$$\mathrm{d}v^2 = -[d|v.v] - [|v.dv] = -[|dv.v] - [|v.dv]$$
$$= [v|dv] + [dv|v] = 2[v|dv], \text{ as before.}$$

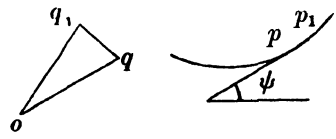
5. *Notation.* If an extensive p depends on a scalar τ, we denote differentiation for τ by dots (τ may be regarded as the time); if p be a moving point, we denote differentiation with respect to arc-length s by dashes. Thus

$$\dot{p} = dp/\mathrm{d}\tau, \quad \ddot{p} = d\dot{p}/\mathrm{d}\tau = d^2p/\mathrm{d}\tau^2,$$
$$p' = dp/\mathrm{ds}, \quad p'' = dp'/\mathrm{ds} = d^2p/\mathrm{ds}^2.$$

§ 42. *Plane curves, curvature, acceleration.*

1. Let $t = q - o$ be a unit vector parallel to the tangent at p, then $p' = t$; let $q_1 - o$ be a unit vector parallel to the tangent at p_1, then $p'' = \lim (q_1 - q)/\mathrm{ds}$, as p_1 tends to p.

Let ds_1 be the arc-length qq_1 on the locus of q, which is a circle. The figure shews that $\mathrm{ds}_1 = \mathrm{d}\psi$, where ψ is the angle the tangent at p makes with any fixed direction. Hence



$$p'' = \lim \frac{q_1 - q}{\mathrm{ds}_1} \cdot \frac{\mathrm{d}\psi}{\mathrm{ds}}.$$

We define the '*radius of curvature*' of the locus of p at p to be

$$\rho = \mathrm{ds}/\mathrm{d}\psi.$$

Now $\lim (q_1 - q)/\mathrm{ds}_1$ is a unit vector parallel to the tangent at q to the locus of q, and hence is the unit vector perpendicular to $q - o$. Hence

$$\lim (q_1 - q)/\mathrm{ds}_1 = |(q - o) = |p',$$

$$t' = p'' = \frac{1}{\rho} |p' = \frac{1}{\rho} |t.$$

The magnitude of t' is ρ^{-1}.

Then, if $p-o = \mathrm{x}i+\mathrm{y}j$ (x, y scalar), we have

$$p'' = \frac{\mathrm{d}^2\mathrm{x}}{\mathrm{ds}^2}.i+\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{ds}^2}.j, \quad \frac{1}{\rho^2} = \left(\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{ds}^2}\right)^2 + \left(\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{ds}^2}\right)^2.$$

Example. An angle at which two curves cut equals the supplement of the angle between their inverses at the corresponding point of intersection. (The theorem is best stated in terms of crosses.)

If r, s be vectors from o to corresponding points on the curves, then $[r\,|\,s] = \mathrm{k}^2$, $[rs] = 0$. Hence

$$[\dot{r}\,|\,s]+[r\,|\,\dot{s}] = 0, \quad [\dot{r}s]+[r\dot{s}] = 0,$$
$$[\dot{r}\,|\,s] \div [\dot{r}s] = -[\dot{s}\,|\,r] \div [\dot{s}r], \quad \cot(\dot{r}, s) = -\cot(\dot{s}, r).$$

2. If we regard τ as the measure of the time taken to describe the arc, then $\dot{\mathrm{s}}$ is a scalar representing the speed v, say. The velocity is

$$\dot{p} = \dot{\mathrm{s}}p' = \mathrm{v}p' = \mathrm{v}t,$$

where t is the unit vector along the tangent at p.

Since $$t = dt/\mathrm{d}\tau = dt/\mathrm{ds}.\mathrm{ds}/\mathrm{d}\tau = t'\mathrm{v} = \mathrm{v}\rho^{-1}\,|\,t,$$

we have for the acceleration,

$$\ddot{p} = \dot{\mathrm{v}}t+\mathrm{v}\dot{t} = \dot{\mathrm{v}}t+\mathrm{v}^2\rho^{-1}\,|\,t.$$

Thus the tangential acceleration is of magnitude $\dot{\mathrm{v}}$, the normal acceleration is of magnitude $\mathrm{v}^2\rho^{-1}$.

3. The angle $\mathrm{d}\psi$ between oq and oq_1 is that between $\dot{p}$ and $\dot{p}+\ddot{p}\mathrm{d}\tau$, hence

$$\sin \mathrm{d}\psi = \frac{[\dot{p}(\dot{p}+\ddot{p}\mathrm{d}\tau)]}{\mathrm{mag}\,\dot{p}.\mathrm{mag}\,(\dot{p}+\ddot{p}\mathrm{d}\tau)} \quad \text{or} \quad \mathrm{d}\psi = \frac{[\dot{p}\ddot{p}]\,\mathrm{d}\tau}{(\mathrm{mag}\,\dot{p})^2},$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}\psi} = \frac{[\dot{p}\,|\,\dot{p}]}{[\dot{p}\ddot{p}]}.$$

Now $\mathrm{ds} = \mathrm{d}\tau.\mathrm{mag}\,\dot{p}$, hence $\rho = \mathrm{ds}/\mathrm{d}\psi = [\dot{p}\,|\,\dot{p}]^{\frac{3}{2}} \div [\dot{p}\ddot{p}]$.

If we take $\tau = \mathrm{s}$, then $\dot{p} = p' = t$, $t^2 = 1$, $\rho^{-1} = [p'p'']$.

4. If $p-o = \mathrm{r}u$ (r scalar, u unit vector), τ any parameter, and θ be the angle between u and any fixed direction, then

$$\dot{u} = \dot{\theta}\,|\,u, \quad |\dot{u} = -\dot{\theta}u, \quad \dot{p} = \dot{\mathrm{r}}u+\mathrm{r}\dot{u} = \dot{\mathrm{r}}u+\mathrm{r}\dot{\theta}\,|\,u,$$

$$\ddot{p} = \ddot{\mathrm{r}}u+\dot{\mathrm{r}}\dot{u}+\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{r}\dot{\theta}).\,|\,u+\mathrm{r}\dot{\theta}.\frac{\mathrm{d}}{\mathrm{d}\tau}(|u)$$
$$= (\ddot{\mathrm{r}}-\mathrm{r}\dot{\theta}^2)\,u+\frac{1}{\mathrm{r}}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{r}^2\dot{\theta})\,|\,u.$$

Hence, if the acceleration of p be resolved into components along the radius vector, and perpendicular thereto, these are

$$\ddot{\mathrm{r}}-\mathrm{r}\dot{\theta}^2 \quad \text{and} \quad \frac{1}{\mathrm{r}}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{r}^2\dot{\theta}).$$

5. *Intersection of normals to a curve.* If a, b be points, u, v vectors, then the cut of the rotors $[au]$, $[bv]$ is

$$c = a+\frac{[(a-b)\,v]}{[vu]}.u = b+\frac{[(b-a)\,u]}{[uv]}.v.$$

(Ex. 119, p. 59.)

Hence, if p be a point, and $[pu]$, $[(p+dp)\,(u+du)]$ cut in c, then

$$[u(u+du)]\,c = [u(u+du)]\,(p+dp)+[dp.u]\,(u+du),$$

$$[u.du]\,c = [u.du]\,p-[u.dp]\,u.$$

This gives the centre of curvature c at p. For take for u the normal at p, $u=|\dot{p}$, then

$$c = p-\frac{[u.dp]}{[u.du]}u = p-\frac{[u\dot{p}]}{[u\dot{u}]}u = p-\frac{[|\dot{p}.\dot{p}]}{[|\dot{p}.|\ddot{p}]}u = p+\frac{[\dot{p}\,|\dot{p}]}{[\dot{p}\ddot{p}]}|\dot{p}.$$

Examples. 1. *Cartesian formulae for ρ.* Take $p = o+\mathrm{x}i+\mathrm{y}j$, $\tau=\mathrm{x}$, then

$$\dot{p} = i+\mathrm{dy/dx}.j, \quad \ddot{p} = \mathrm{d^2y/dx^2}.j, \quad \dot{p}^2 = 1+(\mathrm{dy/dx})^2, \quad [\dot{p}\ddot{p}] = \mathrm{d^2y/dx^2}.$$

2. *Polar formula for ρ.* Take $p = o+\mathrm{r}u$ (r scalar, u unit vector), $\tau=\theta$, then

$$\dot{p} = \frac{dp}{\mathrm{d}\theta} = \frac{\mathrm{dr}}{\mathrm{d}\theta}u+\mathrm{r}\frac{du}{\mathrm{d}\theta} = \frac{\mathrm{dr}}{\mathrm{d}\theta}.u+\mathrm{r}|u, \quad \text{since} \quad \frac{du}{\mathrm{d}\theta} = |u,$$

$$\dot{p}^2 = \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\right)^2+\mathrm{r}^2, \quad \ddot{p} = \frac{\mathrm{d^2r}}{\mathrm{d}\theta^2}.u+2\frac{\mathrm{dr}}{\mathrm{d}\theta}.|u-\mathrm{r}u,$$

since

$$\frac{d|u}{\mathrm{d}\theta} = \left|\frac{du}{\mathrm{d}\theta}\right. = -u.$$

Also

$$[\dot{p}\ddot{p}] = \frac{\mathrm{dr}}{\mathrm{d}\theta}\left(2\frac{\mathrm{dr}}{\mathrm{d}\theta}\right)-\mathrm{r}\left(\frac{\mathrm{d^2r}}{\mathrm{d}\theta^2}-\mathrm{r}\right). \quad \text{Use 3.}$$

3. *Centre of curvature of an ellipse.* Use the notation of 5, with $\phi=\tau$, hence $\dot{\phi}=1$.

$$p-o = \mathrm{a}\cos\phi.i+\mathrm{b}\sin\phi.j, \quad u = |\dot{p} = -\mathrm{b}\cos\phi.i-\mathrm{a}\sin\phi.j,$$

$$dp = (-\mathrm{a}\sin\phi.i+\mathrm{b}\cos\phi.j)\,\mathrm{d}\phi, \quad du = (\mathrm{b}\sin\phi.i-\mathrm{a}\cos\phi.j)\,\mathrm{d}\phi,$$

$$u^2 = \mathrm{a}^2\sin^2\phi+\mathrm{b}^2\cos^2\phi = \mathrm{b}_1^2, \text{ say},$$

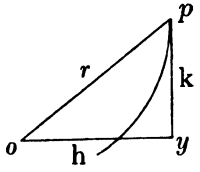
$$[udu] = (\mathrm{ab}\sin^2\phi+\mathrm{ab}\cos^2\phi)\,\mathrm{d}\phi = \mathrm{ab}.\mathrm{d}\phi, \quad [u\dot{u}] = \mathrm{ab}.$$

Substituting, we get $c = p + b_1^2/ab \,.\, u.$

If we put this equal to $o + (x_1 i + y_1 j)$, and substitute for p, u, we get $x_1 = a^{-1}(a^2 - b^2)\cos^3\phi$, $y_1 = b^{-1}(b^2 - a^2)\sin^3\phi$,

$$\rho = \sqrt{(p-c)^2} = b_1^2/ab \,.\, \sqrt{u^2} = b_1^3/ab.$$

4. If y is the foot of the perpendicular from o on to the tangent at p to any curve, then the locus of y touches the circle opy.



For $[(p-y)\,|\,(y-o)] = 0.$

Hence

$$[(p'-y')\,|\,(y-o)] + [(p-y)\,|\,y'] = 0.$$

But p' (the derivative) is a vector along the tangent at p. Hence

$$[p'\,|\,(y-o)] = 0, \quad [-y'\,|\,(y-o)] + [y'\,|\,(p-y)] = 0,$$
$$[y'\,|\,(\tfrac{1}{2}(p+o)-y)] = 0.$$

5. If in the figure of the previous example, h, k be the lengths of oy, yp, then $dh/d\psi = k$, where ψ is the angle between yp and a fixed direction.

For $p - o = ku - h\,|\,u$, where u is the unit vector along yp.

Hence

$$\frac{d}{d\psi}(p-o) = \frac{d}{d\psi}(ku - h\,|\,u), \quad \frac{d}{d\psi}(p-o) = \frac{d}{ds}(p-o)\frac{ds}{d\psi} = u\frac{ds}{d\psi}.$$

Thus

$$u\frac{ds}{d\psi} = \frac{dk}{d\psi}.u + k\frac{du}{d\psi} - \frac{dh}{d\psi}.|u - h\frac{d}{d\psi}|u = \frac{dk}{d\psi}.u + k\,|u - \frac{dh}{d\psi}.|u + hu,$$

since $\dfrac{du}{d\psi} = |u$, $\dfrac{d}{d\psi}|u = -u$. Outer multiplication by u gives

$$0 = ku^2 - \frac{dh}{d\psi}u^2, \quad \frac{dh}{d\psi} = k.$$

6. Prove $\sqrt{v^2}.d\sqrt{v^2} = [v\,|\,dv]$.

7. If a, b be fixed points, p a varying point, $p - a = v_1$, $p - b = v_2$, then

(i) If $v_1^2 + v_2^2$ is constant, we have $[(v_1 + v_2)\,|\,\dot{p}] = 0$, where $\dot{p} = \dot{v}_1 = \dot{v}_2$;

(ii) If $\sqrt{v_1^2} + \sqrt{v_2^2}$ is constant, then $\dot{p}$ bisects the angle between v_1 and $-v_2$;

(iii) If v_1^2/v_2^2 is constant, then $[\dot{p}\,|\,(v_2^2.v_1 - v_1^2.v_2)] = 0$.

Interpret these.

8. *Fermat's problem.* If v_1, v_2, v_3 be vectors from a point p to the vertices of a triangle abc, and l_1, l_2, l_3 their magnitudes, and if $d(l_1+l_2+l_3)=0$, then $\Sigma l_1^{-1}[v_1|dv_1]=0$, and hence $\Sigma l_1^{-1}v_1=0$; the angles between the vectors each equal $2\pi/3$. Distinguish the cases when all angles of the triangle are less than $2\pi/3$, and when one is not.

This solves the problem: to find a point p such that the sum of its distances from three points is a minimum.

9. If $p=ka+(1-k)\,b$, $q=kb+(1-k)\,c$, where k is a variable scalar, and $[abc]\neq 0$, then the point of contact of pq with its envelope is $r=kp+(1-k)\,q$.

For r lies on

$$[pq]=k^2[ab]+k(1-k)\,[ac]+(1-k)^2\,[bc],$$

and on the line obtained by differentiating this for k:

$$2k[ab]+(1-2k)\,[ac]-2(1-k)\,[bc].$$

The regressive product of these lines is congruent to

$$k^2a+2k(1-k)\,b+(1-k)^2\,c=kp+(1-k)\,q.$$

10. If lmn is a transversal of triangle abc, and its rotor is

$$x.bc+y.ca+z.ab,$$

then the intervals mn, nl, lm are in the ratios

$$(y-z)\,x:(z-x)\,y:(x-y)\,z.$$

If these ratios are constant, then lmn envelopes a parabola.

11. If p is any point on a rectangular hyperbola, centre c, and the perpendicular at c to cp meets the normal at p in r, then $\overline{pr}$ is minus the radius of curvature at p. If pr is produced an equal distance rs, then s lies on the curve.

12. If pq be one of a set of parallel chords of a rectangular hyperbola, centre c, then $cp^2+cq^2-pq^2$ is constant.

13. Using the formula $[\dot{p}^2]^{\frac{3}{2}}/[\dot{p}\ddot{p}]$ for the radius of curvature, prove that the length of the normal to a parabola at a point p intercepted by the directrix is half the radius of curvature at p.

14. If p move so that the sum of the squares of its distances pq, pr from two lines is constant, then the normal at p to the locus bisects qr.

15. If $r=c\cos^3\theta.i+c\sin^3\theta.j$, then $[\dot{r}\ddot{r}]=-\dot{r}^2$. The normal makes an angle θ with j. The length of the tangent intercepted by the axes is c.

16. A curve is such that if a radius cq is parallel to the tangent at p, then cp is parallel to the tangent at q. Prove that the area of triangle cpq is constant.

17. If $poqr$ is a variable parallelogram of constant area with the angle o fixed in position, then the tangent to the locus of r is parallel to pq.

18. The rod pq slides with its ends on two straight lines through o, the parallelogram $poqr$ is completed, and s is the foot of the perpendicular from r to pq. Then s is the point of contact of pq with its envelope.

19. If a constant distance pq be taken along the tangent to a curve at p, the normal to the locus of q goes through the centre of curvature at p. If pq is drawn making a constant angle with the tangent at p, the same conclusion follows. (Bertrand.)

20. If p, q be corresponding points on a curve and its involute, and m be the mid-point of pq, then the normal to the locus of m is parallel to the join of q to the centre of curvature at p.

21. If x is the length of the perpendicular from the origin to the tangent at p and $p - o = u$, then

$$\frac{\mathrm{dx}}{\mathrm{ds}} = \frac{[u'u''][u|u']}{[u'^2]^{\frac{3}{2}}}.$$

§ 43. *Tangents to a twisted cubic.**

If $p = a + 3\mathrm{k}b + 3\mathrm{k}^2c + \mathrm{k}^3q$ represents points on the twisted cubic as k varies, then denoting derivatives for k by dots

$$\tfrac{1}{3}\dot{p} = b + 2\mathrm{k}c + \mathrm{k}^2q, \quad \tfrac{1}{6}\ddot{p} = c + \mathrm{k}q.$$

The tangent T at p is $[p\dot{p}]$, that is

$$[ab] + 2\mathrm{k}[ac] + \mathrm{k}^2[aq + 3bc] + 2\mathrm{k}^3[bq] + \mathrm{k}^4[cq].$$

The tangents at a, q respectively are $[ab]$ and $[cq]$. The '*rank*' of the curve, that is, the number of tangents which cut a general line L, is four. For $[TL] = 0$ gives four values for k.

* A great deal of the classical theory of the differential geometry of twisted curves and surfaces can be worked out very simply by the present methods. But as these methods in this field do not differ greatly from the usual vector analysis, we omit this work and refer the reader to Weatherburn, *Differential Geometry* (Cambridge, 1927, 1930) and Blaschke, *Differentialgeometrie*, I (3rd ed. Berlin, 1930). Also Fehr, *Application de la Méthode Vectorielle de Grassmann à la Géométrie infinitésimale* (Paris, 1896; Geneva, 1907).

The '*osculating plane*' at p may be defined as $[p\dot{p}\ddot{p}]$; it is

$$\pi \equiv [abc] + k[abq] + k^2[acq] + k^3[bcq],$$

or, say,

$$\delta - k\gamma + k^2\beta - k^3\alpha.$$

The osculating planes at a, q are respectively δ and α. The '*class*' of the curve, that is, the number of osculating planes through a general point, is three, and the curve is determined by the tetrahedron *abcq* and one other point.

Since π can be written as $\delta + \frac{1}{3}[apq] - k^3\alpha$, the osculating planes at p, a, q meet on $[apq]$. *The osculating planes at three points of a twisted cubic meet on the plane joining the three points*

Examples. 22. The tangents cut each osculating plane in the points of a conic touching $[bc]$, $[cq]$ at b, q respectively.

23. The condition

$$\sqrt[4]{[L_1L_2]}\,[L_3L_4] + \sqrt[4]{[L_2L_3]}\,[L_1L_4] + \sqrt[4]{[L_3L_1]}\,[L_2L_4] = 0$$

is a necessary and sufficient condition that L_1, L_2, L_3, L_4 touch one and hence an infinite number of twisted cubics. The Grassmann cross-ratio of $L_1, \ldots, L_4$ is the fourth power of the ordinary cross-ratio of the points of contact of the tangent cubics.

§ 44. *Central motion.*

1. A point p moves so that its acceleration to a fixed point o is along po and is a function of the distance po. The vector of the acceleration is $\ddot{p}$, its rotor goes through p and is $[p\ddot{p}]$; for if v is any vector, the rotor through p with this vector is $[pv]$.

If $\text{mag}\,(p-o) = r$, and $F = fr$ is the magnitude of the force, then

$$\ddot{p} = f.(o-p), \quad [p\ddot{p}o] = 0,$$

$$\frac{d}{d\tau}[p\dot{p}o] = [\dot{p}\dot{p}o] + [p\ddot{p}o] = [p\ddot{p}o] = 0.$$

Hence $[p\dot{p}o]$ is a constant scalar h, say.

Since $[p\dot{p}]$ is the rotor of the velocity, $[p\dot{p}o]$ is its moment round o, which is accordingly constant. If θ is the vectorial angle,

$$\sin d\theta = \frac{[p(p+dp)\,o]}{\text{mag}\,(p-o).\text{mag}\,(p-o+dp)} = \frac{[p.dp.o]}{(p-o)^2}.$$

Hence

$$(p-o)^2\,\dot{\theta} = [p\dot{p}o],$$

$$h = r^2\dot{\theta}.$$

2. If P be the length of the perpendicular from o on to the tangent, then

$$[p\dot{p}o] = \mathrm{P}\dot{p} = \mathrm{h}, \quad \dot{p}^2 = \frac{\mathrm{h}^2}{\mathrm{P}^2}, \quad 2[\dot{p}\,|\,\ddot{p}] = -\frac{2\mathrm{h}^2}{\mathrm{P}^3}\,\dot{\mathrm{P}},$$

by differentiating.

Hence $\mathrm{f}[\dot{p}\,|\,(p-o)] = \dfrac{\mathrm{h}^2}{\mathrm{P}^3}\,\dot{\mathrm{P}}$, but $[\dot{p}\,|\,(p-o)] = \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}\tau}(p-o)^2$.

Hence we have the scalar equation

$$\mathrm{F} = \mathrm{fr} = \mathrm{h}^2\mathrm{P}^{-3}\,\mathrm{dP}/\mathrm{dr}.$$

3. *The inverse square law.* Suppose $\mathrm{F} = \mathrm{kr}^{-2}$, where k is a constant scalar, r the magnitude of the radius vector, then

$$\ddot{p} = -\mathrm{kr}^{-3}(p-o).$$

(The factor r^{-3} appears because $p-o$ has magnitude r.)

But $$\mathrm{h} = [p\dot{p}o] = [(p-o)\,\dot{p}],$$

if we replace the unit bivector by 1.

Hence
$$\begin{aligned}\mathrm{h}\,|\,\ddot{p} &= -\mathrm{kr}^{-3}\,[(p-o)\,\dot{p}]\,|\,(p-o)\\ &= -\mathrm{kr}^{-3}\,(\mathrm{r}^2.\dot{p} - [\dot{p}\,|\,(p-o)].(p-o)).\end{aligned}$$ (§ 14 (4).)

Take supplements, and use $[\dot{p}\,|\,(p-o)] = \mathrm{r}\dot{\mathrm{r}}$, then

$$\mathrm{h}\dot{p} = \mathrm{kr}^{-1}\,|\,\dot{p} - \mathrm{kr}^{-3}[\dot{p}\,|\,(p-o)]\,|\,(p-o) = \mathrm{kr}^{-1}\,|\,\dot{p} - \mathrm{kr}^{-2}\dot{\mathrm{r}}\,|\,(p-o).$$

Integrate, then

$$\mathrm{hk}^{-1}\dot{p} = \mathrm{r}^{-1}\,|\,(p-o) + \text{const. bivector} \qquad (1)$$

$$= \mathrm{r}^{-1}\,|\,(p-o) + \mathrm{e}\,|\,u, \text{ say,} \qquad (2)$$

where e is a constant scalar, and u a unit vector.

Now $\mathrm{h} = [(p-o)\,\dot{p}]$.

Hence
$$\begin{aligned}\mathrm{h}^2\mathrm{k}^{-1} &= \mathrm{r}^{-1}[(p-o)\,|\,(p-o)] + \mathrm{e}[(p-o)\,|\,u]\\ &= \mathrm{r}(1 + \mathrm{e}\cos\theta),\end{aligned}$$

where
$$[(p-o)\,|\,u] = \mathrm{r}\cos\theta.$$

Hence *the path is a conic, with axis u, the centre of force is at a focus, the latus rectum is* $2\mathrm{h}^2\mathrm{k}^{-1}$. By (1), the hodograph, that is, the locus of $\dot{p}+o$, is a circle; by (2) the velocity is the sum of components kh^{-1} perpendicular to op and ekh^{-1} perpendicular to the axis.

4. *The inverse cube law.* Here we have

$$\ddot{p} = -\mathrm{k}\mathrm{r}^{-4}(p-o). \tag{3}$$

Hence
$$[\dot{p}\,|\,\ddot{p}] = -\mathrm{k}\mathrm{r}^{-4}[\dot{p}\,|\,(p-o)]. \tag{4}$$

But $\mathrm{d}\,\dfrac{1}{(p-o)^2} = \dfrac{1}{(p+dp-o)^2} - \dfrac{1}{(p-o)^2} = -\dfrac{2[(p-o)\,|\,dp]}{\mathrm{r}^4}$.

Hence, integrating (4), we have

$$\dot{p}^2 = \mathrm{l} + \mathrm{k}\mathrm{r}^{-2}, \text{ when l, k are scalar constants.}$$

By (3)
$$[(p-o)\,|\,\ddot{p}] = -\mathrm{k}\mathrm{r}^{-2}.$$

Hence
$$[(p-o)\,|\,\ddot{p}] + \dot{p}^2 = \mathrm{l}.$$

Integrating, we have $[(p-o)\,|\,\dot{p}] = \mathrm{l}\tau$, where τ is the time.

Integrating again, $(p-o)^2 = \mathrm{l}\tau^2 + \mathrm{m}$.

Since also $\dot{p}^2 = \mathrm{h}^2\mathrm{p}^{-2}$, by 2, we have for the orbit

$$\mathrm{h}^2\mathrm{p}^{-2} - \mathrm{k}\mathrm{r}^{-2} = \mathrm{l}. \qquad \text{(Cotes' spirals.)}$$

If ϕ is the angle between the tangent and the radius vector,

$$\cot\phi = \frac{[(p-o)\,|\,\dot{p}]}{[(p-o)\,\dot{p}]} = \frac{\mathrm{l}\tau}{\mathrm{h}}.$$

If θ is the angle between the radius vector and a fixed direction, then

$$(p-o)^2\,\dot{\theta} = \mathrm{h}, \quad \dot{\theta} = \mathrm{h}(\mathrm{l}\tau^2 + \mathrm{m})^{-1}, \quad \theta = \int \mathrm{h}(\mathrm{l}\tau^2 + \mathrm{m})^{-1} d\tau.$$

These equations fully determine the character of possible orbits.

Examples. 24. A point describes a spiral $\mathrm{r} = \exp\theta$ with constant angular velocity $\dot{\theta}$ about the origin. Prove that the acceleration is perpendicular to the radius vector and of magnitude $2r\dot{\theta}^2$.

25. If in central elliptic motion, p is a vector to a point, q the vector along the conjugate diameter, ϕ the eccentric angle of p, then the acceleration is $-p\dot{\phi}^2 + q\ddot{\phi}$.

26. An ellipse is freely described under a force to a pole o; prove that the attraction at p varies as op/pm^3, where pm is the perpendicular from p on to the polar of o. If x, y be the coordinates of o referred to principal axes, then

$$\mathrm{ct} = \mathrm{y}\mathrm{b}^{-1}(1 - \cos\phi) + \mathrm{x}\mathrm{a}^{-1}\sin\phi - \phi,$$

where t is the time in the orbit, $\mathrm{c} = \mathrm{h}/\mathrm{ab}$, and ϕ is the eccentric angle.

§ 45. *General motion of one plane over a fixed plane of reference.*

1. Our differentiations will all be taken with reference to the fixed plane, so that, if p be a point, $\dot{p}$ is its velocity over the fixed plane; if a be another point of the moving plane, then $p = a + \mathrm{k}u$, where k is a constant scalar, and u a variable unit vector. Then

$$\dot{p} = \dot{a} + \mathrm{k}\dot{u} = \dot{a} + \mathrm{k}\omega\,|\,u = \dot{a} + \omega\,|\,(p-a),$$

where ω is the angular velocity of u referred to the fixed plane.

If p_1 is the point on the moving plane which is instantaneously at rest, then $\dot{p}_1 = 0, \quad \dot{a} + \omega\,|\,(p_1 - a) = 0,$

$$\dot{p} = -\omega\,|\,(p_1 - a) + \omega\,|\,(p-a) = \omega\,|\,(p - p_1),$$

$$|\dot{p} = \omega(p_1 - p), \quad p_1 = p + \omega^{-1}\,|\,\dot{p}.$$

p_1 is called the '*instantaneous centre of rotation*'.

Let c be the point in the fixed plane, which corresponds at each instant to that point p_1 of the moving plane which is instantaneously at rest.

Though $\dot{p}_1 = 0$, yet in general we must expect $\dot{c} \neq 0$, because c varies in the fixed plane.

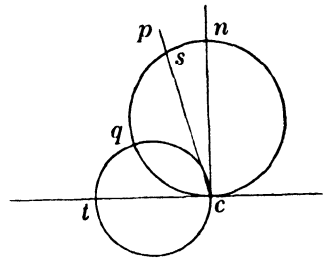
2. At each instant $\dot{p} = \omega\,|\,(p-c)$,

$$\ddot{p} = \dot{\omega}\,|\,(p-c) + \omega\,|\,(\dot{p} - \dot{c})$$
$$= \dot{\omega}\,|\,(p-c) - \omega^2(p-c) - \omega\,|\,\dot{c}.$$

Thus $\ddot{p}$ may be resolved into

$$-\omega^2(p-c)$$

due to rotation round c as a fixed point, $\dot{\omega}\,|\,(p-c)$ perpendicular to cp due to change of ω, and $-\omega\,|\,\dot{c}$ perpendicular to the direction of $\dot{c}$.



3. The normal acceleration of p, that is, the acceleration along pc, is

$$-\omega^2(p-c) - \omega\frac{[\,|\dot{c}\,|\,(p-c)]}{(p-c)^2}.(p-c)$$
$$= -\omega^2(p-c) - \omega\frac{[\dot{c}(p-c)]}{(p-c)^2}.(p-c).$$

This vanishes when $\omega = -\dfrac{[\dot{c}(p-c)]}{(p-c)^2}$.

Take n so that $n-c = -\omega^{-1}|\dot{c}$. Then the normal acceleration of p vanishes if $[(n-c)|(p-c)] = (p-c)^2$,

that is, if p is on the circle on diameter cn.

Then $\dot{p}$ is in the direction np, and p is at a point of inflexion, or '*flex*', on its path.

The locus of points which are at flexes on their paths is a circle, 'the circle of flexes'.

4. The tangential acceleration of p is

$$\dot{\omega}|(p-c) - \frac{\omega[(p-c)|\dot{c}]}{(p-c)^2}|(p-c).$$

This vanishes when

$$\dot{\omega}(p-c)^2 = \omega[(p-c)|\dot{c}].$$

Take t so that $t-c = \omega\dot{\omega}^{-1}\dot{c}$,

then the tangential acceleration of p vanishes when

$$[(p-c)|(t-c)] = (p-c)^2,$$

that is, when p is on the circle on diameter ct.

The locus of points which have zero tangential acceleration is a circle. The points are at cusps on their paths.

5. The point q where these two circles cut is the '*centre of acceleration*'.

At q the acceleration is zero. Let $\measuredangle ncq = \alpha$, then, since n, q, t are collinear,

$$\tan\alpha = \frac{|(n-c)}{t-c} = \frac{\omega^{-1}\dot{c}}{\omega\dot{\omega}^{-1}\dot{c}} = \frac{\dot{\omega}}{\omega^2}.$$

Since $\ddot{p} = \dot{\omega}|(p-c) - \omega^2(p-c) - \omega|\dot{c},$

$$\ddot{q} = \dot{\omega}|(q-c) - \omega^2(q-c) - \omega|\dot{c},$$

and $\ddot{q} = 0$, we have

$$\ddot{p} = \dot{\omega}|(p-q) - \omega^2(p-q) = (\dot{\omega}| - \omega^2)(p-q).$$

The acceleration of a general point p is composed of $-\omega^2(p-q)$ along pq and $\dot{\omega}|(p-q)$ perpendicular to qp, the angle between these accelerations being $\tan^{-1}\dot{\omega}/\omega^2 = \alpha$.

Since $n-c = -\omega^{-1}|\dot{c}$, we have

$$\ddot{p} = \dot{\omega}|(p-c) - \omega^2(p-c) + \omega^2(n-c) = \dot{\omega}|(p-c) + \omega^2(n-p),$$

expressing the acceleration of p as the sum of an acceleration along the path and one along pn.

6. Since $\ddot{p} = (p-q)\,\mathfrak{A}$, where $\mathfrak{A} = \dot{\omega}\,|\,-\omega^2$, we have

Burmester's Theorem. If p_1, p_2, p_3 be three points, and r_1, r_2, r_3 are chosen so that $r_i - p_i = \ddot{p}_i$, then the triangles $p_1p_2p_3$ and $r_1r_2r_3$ are similar.

Since $\dot{p} = \omega\,|\,(p-c)$, the corresponding theorem holds for velocities also.

7. *The curvature ρ^{-1} of the locus of p* is the magnitude of the normal acceleration divided by $\dot{p}^2$, that is

$$\left(-\omega^2 - \frac{\omega[\dot{c}(p-c)]}{(p-c)^2}\right)\sqrt{(p-c)^2} \div \omega^2(p-c)^2$$
$$= \left(-1 + \frac{[(n-c)\,|\,(p-c)]}{(p-c)^2}\right)\frac{1}{\sqrt{(p-c)^2}}$$
$$= \frac{-(p-c)^2 + [(n-c)\,|\,(p-c)]}{((p-c)^2)^{\frac{1}{2}}}.$$

The numerator equals

$$[(n-p)\,|\,(p-c)] = [(s-p)\,|\,(p-c)],$$

where s is the point where cp cuts the circle of flexes.

Hence
$$\frac{1}{\rho} = \frac{\text{mag}\,(s-p)}{(p-c)^2}.$$

Examples. 27. If k be the centre of curvature, and x, y, z the lengths of sc, pc, ck, then
$$\mathrm{x}^{-1} = \mathrm{y}^{-1} + \mathrm{z}^{-1}.$$

Deduce *Grübler's construction for k* when p, s, c are known: namely, take any point a; let $r \equiv [pa.c(a-s)]$, then $k \equiv [ps.r(a-c)]$.

28. *Mannheim's construction for the centre of curvature of an ellipse at p.* Let the normal at p meet the major axis in q; let the perpendicular at q to the normal cut op in r, where o is the centre of the ellipse; then the perpendicular from r to the major axis cuts the normal at p in the centre of curvature.

29. If two vertices of a triangle move on fixed lines, the third describes an ellipse.

30. Investigate the envelope of a curve carried by the motion in a plane.

§ 46. *Displacements in space.**

1. A '*displacement*' in space is a transformation of points into points which preserves distances and the sense of tetrahedra unchanged.

Hence it preserves angles, turns lines into lines, planes into planes, and thus is a linear transformation.

2. Consider a displacement which turns the point o into o_1 and vectors a, b, c into a_1, b_1, c_1. Suppose h is a vector not changed by the displacement, and let $u = a_1 - a$, $v = b_1 - b$, $w = c_1 - c$. Then since angles and magnitudes of vectors are preserved, we have

$$[h|u] = [h|v] = [h|w] = 0.$$

Hence
$$[uvw] = 0.$$

Hence there are scalars k_1, k_2, k_3 such that $k_1 u + k_2 v + k_3 w = 0$, and they satisfy

$$k_1 : k_2 : k_3 = \text{mag } [vw] : \text{mag } [wu] : \text{mag } [uv].$$

(We are assuming that u, v, w are not all parallel; if they are, we merely have a translation.)

Also $h \equiv |[vw] \equiv |[wu] \equiv |[uv] \equiv |[vw + wu + uv]$;

h is the vector of the '*axis*' of the displacement. We shall see later that such a vector always exists.

3. If $o = o_1$ is fixed, then each point $o + xh$ (x scalar) is fixed, and the displacement is a rotation round h of angle

$$\sin^{-1} (\text{mag } [h'ww_1] \div \text{mag } [h'w] \text{ mag } [h'w_1]), \qquad (1)$$

where h' is a unit vector along h, and w, w_1 are initial and final positions of any vector.

4. If $o \neq o_1$, then the point $p = o + a$ becomes $p_1 = o_1 + a_1$,

$$p_1 = p + (o_1 - o) + (a_1 - a) = p + u_1,$$

where
$$u_1 = (o_1 - o) + (a_1 - a) = (o_1 - o) + u.$$

Similarly $q = o + b$, $r = o + c$ become $q + v_1$, $r + w_1$, with similar definitions. Denote these points by q_1, r_1.

* For a treatment of kinematics and dynamics by Grassmann's methods, see Lotze, *Die Grundgleichungen der Mechanik* (1922).

Thus $h \equiv |[vw+wu+uv]| = |[v_1w_1+w_1u_1+u_1v_1]|$.

Hence *Chasles' construction for the axis.* Draw op', oq', or' parallel and equal to pp_1, qq_1, rr_1, then the axis is perpendicular to the plane $[p'q'r']$.

For, as $p' = o+u_1$, $q' = o+v_1$, $r' = o+w_1$, we have

$$[p'q'r'] = [o(v_1w_1+w_1u_1+u_1v_1)]+[u_1v_1w_1],$$

and this is parallel to $[v_1w_1+w_1u_1+u_1v_1]$, and hence perpendicular to h.

If p, q, r and p_1, q_1, r_1 be projected on to $[p'q'r']$, the axis goes through the point which is the centre of the rotation that turns the projections of p, q, r into those of p_1, q_1, r_1.

5. If s be any point, the rotor sh becomes an equal parallel rotor s_1h. If these are collinear, then $s-s_1$ is parallel to h, $s-s_1 = \mathrm{x}h$ (x scalar).

Let s be a point with this property, in plane $[pqr]$; let $s = \mathrm{k}_1p+\mathrm{k}_2q+\mathrm{k}_3r$, then $s_1 = \mathrm{k}_1p_1+\mathrm{k}_2q_1+\mathrm{k}_3r_1$ (since distances are preserved), where s_1, p_1, q_1, r_1 are the displaced positions of s, p, q, r.

Thence $\mathrm{k}_1:\mathrm{k}_2:\mathrm{k}_3 = \text{mag }[vwh]:\text{mag }[wuh]:\text{mag }[uvh]$, (2)

where $u = p_1-p$, $v = q_1-q$, $w = r_1-r$.

Thus s is a definite point of $[pqr]$, and the displacement is composed of a translation parallel to h and a rotation round the axis sh. The angle of the rotation is given by (1); the magnitude of the translation is mag $[uvw]$:mag h.

Hence follows *Mehmke's construction for the axis.* If p', q', r' be constructed as before, and oh, which is perpendicular to $[p'q'r']$, cuts this plane in h', then by (2), $p'q'r'h'$ is a figure similar to $pqrs$ and to $p_1q_1r_1s_1$. Hence s can be found and thus a point on the axis is known.

§ 47. *Motions in space.**

1. Since a displacement in space is a combination of a rotation round an axis L and a translation parallel to L, it follows that a motion is a combination of a rotation and a translation

* Lotze, *loc. cit.* The geometry of motion is treated by different methods from the present in the following books: Schoenflies-Stäckel, *La Géométrie du mouvement* (Paris, 1893); Mannheim, *Géométrie cinématique* (Paris, 1894).

(or rather of a rotation-velocity and a translation-velocity); the first is given by an axis and an angular velocity, the second by a vector r.

If we place a rotor R along the axis with a magnitude equal to that of the angular velocity, the velocity of p due to the rotation is $\downarrow[Rp]$, and the velocity due to the translation is $\downarrow[Vp]$, where V is a suitable constant bivector.

The velocity due to both is $\dot{p} = \downarrow[Sp]$, where $S, = R + V$, is a screw.

Velocities are compounded by adding the corresponding screws.

2. The velocity system which corresponds to S satisfies $[\dot{p} \downarrow Sp] = 0$, hence *the direction of the motion of p is perpendicular to the plane* $[Sp]$, *that is, to the nul plane of p for S.*

3. *Acceleration.* Since $\dot{p} = \downarrow[Sp]$,

$$\ddot{p} = \downarrow[\dot{S}p] + \downarrow[S\dot{p}] = \downarrow[\dot{S}p] + \downarrow[S \downarrow Sp].$$

If $\ddot{p} = 0$, then $[\dot{S}p] + [S\dot{p}]$ is a trivector or zero, and for this it is necessary and sufficient that its outer product with three independent vectors u, v, w vanishes:—

$$[u\dot{S}p] + [uS \downarrow Sp] = [p(S \downarrow Su - \dot{S}u)] = 0,$$

and similar equations for v, w.

Hence p is the cut of the planes

$$[S \downarrow Su] - [\dot{S}u], \quad [S \downarrow Sv] - [\dot{S}v], \quad [S \downarrow Sw] - [\dot{S}w].$$

If these are independent, there is only one point of zero acceleration.

4. If the motion is the resultant of a translation represented by the vector r and a rotation represented by ωR (ω scalar, R unit rotor, parallel to r), then

$$\dot{p} = r + \omega \downarrow[Rp], \quad \ddot{p} = \dot{r} + \omega \downarrow[\dot{R}p] + \omega \downarrow[R\dot{p}] + \dot{\omega} \downarrow[Rp],$$

$$\ddot{p} = \dot{r} + \omega \downarrow[\dot{R}p] + \dot{\omega} \downarrow[Rp] + \omega^2 \downarrow[R \downarrow Rp]. \qquad (3)$$

Let o be any other point of the body, then forming the equation for o similar to (3), we have

$$\ddot{p} - \ddot{o} = \omega \downarrow[\dot{R}(p-o)] + \dot{\omega} \downarrow[R(p-o)] + \omega^2 \downarrow[R \downarrow R(p-o)].$$

If q is the point at which the acceleration vanishes, then

$$\ddot{p} = \omega \downarrow[\dot{R}(p-q)] + \dot{\omega} \downarrow[R(p-q)] + \omega^2 \downarrow[R \downarrow R(p-q)].$$

Hence the body moves as if the point of zero acceleration were instantaneously fixed in the body.

Of the terms in (3), $\omega \downarrow[\dot{R}p]$ depends on the change of position of the axis; $\dot{\omega} \downarrow[Rp]$ is in the direction $\dot{p}-r$, or in the direction of the motion due to the rotation only, and it is of magnitude $\dot{\omega}$d where $\mathrm{d} = \mathrm{dist}\,(R, p)$; $\omega^2 \downarrow[R \downarrow Rp]$ is a vector of magnitude ω^2d parallel to the perpendicular from p to R.

5. As each motion is represented by a screw, therefore, if a body has freedom two, its possible motions are represented by a pencil of screws $\mathrm{k}_1 S_1 + \mathrm{k}_2 S_2$, and each screw of the pencil is a linear combination of two rotors L, M, real or imaginary. If we suppose them real, then each possible motion is the result of rotations round these lines, and the velocity of any point p is perpendicular to $[pLM]$. (Schönemann.)

Thus the normals to the surface of possible positions of p all cut L, M, and so constitute a linear congruence.

If a body has freedom three, an arbitrary general point can be moved by an infinitesimal motion to any adjacent place, but if the direction of motion is fixed for one point it is fixed for all. The rotors of the system of corresponding screws lie in a regulus, a point on these rotors moves instantaneously only in a certain plane. (Cf. § 35.)

Examples. 31. *The points p_1, p_2, p_3 of a straight rod move on planes obc, oca, oab respectively; p is a point on line $p_1p_2p_3$. Then the locus of p is an ellipsoid.*

We have $p_1 = p + \mathrm{k}_1 u$, $p_2 = p + \mathrm{k}_2 u$, $p_3 = p + \mathrm{k}_3 u$,

where k_1, k_2, k_3 are fixed scalars, and u is a unit variable vector.

$$[oabc]\, u = -[uobc]\,(a-o) - [uoca]\,(b-o) - [uoab]\,(c-o).$$

$$[pobc] = -\mathrm{k}_1[uobc],\quad [poca] = -\mathrm{k}_2[uoca],\quad [poab] = -\mathrm{k}_3[uoab]. \quad (4)$$

Hence
$$[oabc]\, u = \mathrm{k}_1^{-1}[pobc]\,(a-o) + \mathrm{k}_2^{-1}[poca]\,(b-o) + \mathrm{k}_3^{-1}[poab]\,(c-o), \quad (5)$$

$$[oabc]^2 = \mathrm{k}_1^{-2}[pobc]^2\,(a-o)^2 + 2\mathrm{k}_2^{-1}\mathrm{k}_3^{-1}[poca]\,[poab]\,[(b-o)|(c-o)] + \ldots.$$

Hence the locus of p is an ellipsoid,* with its centre at the cut of the planes.

It is an ellipsoid and not another type of quadric, since p cannot go off to infinity.

Cor. 1. If the given planes are perpendicular, the principal axes of the ellipsoid are their cuts.

Cor. 2. If the given planes are parallel to a line, p describes a plane.

Cor. 3. From equation (5), when $[pobc]:[poca]:[poab]$ are given, the direction of u is fixed.

32. *If in Ex.* 31, *a point* p_4 *on the line* p_1p_2 *also moves in a fixed plane abc, then the locus of p is an ellipse.*

For $p_4 = p + k_4 u$, $[uobc] + [uoca] + [uoab] = [(u-o)\,abc]$.

Now we have (4) and

$$[pabc] = -k_4[uabc],$$

hence

$$k_1^{-1}[pobc] + k_2^{-1}[poca] + k_3^{-1}[poab] - k_4^{-1}[pabc] = [oabc].$$

Thus p lies on a plane, and hence by Ex. 31 on an ellipse.

33. If the four normals to the four planes in Ex. 32 at the points where the line meets them lie on a regulus, motion is impossible.

34. The centres of the various elliptic loci in Ex. 32, for the various points of a given line, lie on a line H on which the planes cut off equal intervals; the angle between H and the moving line is constant.

35. If a line moves so that one point of it always moves perpendicular to the line, then this is so for all points of the line.

36. If a triangle moves so that its inclination to a fixed plane is constant, and its vertices move on planes, then any fixed point of the triangle describes an ellipse.

37. If a solid moves so that each point describes a plane curve, the motion is generated by the rolling of a circular cylinder inside a fixed circular cylinder of twice the radius, and sliding along it, so that some point of the solid stays in a fixed plane. All paths are ellipses, which for the points of one line degenerate into lines.

38. When a rod moves in a plane so that two of its points move on fixed lines, the ellipses described by points of the rod are all of equal area. When a rod moves in space so that three of its points move on fixed planes, the ellipsoids described by points of the rod all have equal volumes.

* This is easily seen by comparison with Cartesian equations.

39. If $a, b, c, \ldots$ and $a', b', c', \ldots$ be corresponding points of a line in a displacement, then the right-bisector planes of $aa', bb', cc', \ldots$ meet in a line equally inclined to ab and $a'b'$.

40. If a plane α undergoes a motion in which five of its points describe spheres whose centres are on a plane β, there is a sixth point of α which describes another such sphere.

41. (i) If a line moves so that all its points move on fixed spheres, their centres lie on a twisted cubic, or conic, or line.

(ii) Given five positions of a line, there are in general four points of the line each of whose five positions lie on a sphere.

(iii) If five points of a line stay on fixed spheres, all do.*

(iv) If four points of a line stay on fixed spheres whose centres are on a plane, any other point of the line stays on such a sphere.

(v) If three points of a line stay on fixed spheres whose centres are on a line, any other point of the line stays on such a sphere.

42. If two particles move under one another's influence, the force between them being along their join, but otherwise arbitrary, then the tangents to their paths cut any fixed plane in two points whose join always goes through a fixed point, and the joins of the tangents to any fixed point cut in a fixed plane.

For three such particles the instantaneous accelerations are concurrent.

For four such particles, if the forces are any powers of the distances, the instantaneous accelerations are concurrent if

$$r_{12}r_{34} = r_{23}r_{14} = r_{31}r_{24}.$$

* Duporcq, *Liouville's Journ.* (5), 4 (1897), p. 121.

CHAPTER VI

PROJECTIVE TRANSFORMATIONS ON THE LINE, PLANE, AND SPACE

§ 48. *Projectivities on a spread of step two.**

1. In chap. I, § 13·2, we considered a transformation of points on a line which was produced by 'projection and section'. The result of a finite number of such transformations is a projective transformation or projectivity (cf. chap. III, § 23), for if the transformation carries a_1, a_2 into b_1, b_2, it carries $k_1 a_1 + k_2 a_2$ into $k_1 b_1 + k_2 b_2$, and as this is the case for one transformation, it is so for the result of a finite number.

Thus a projective transformation leaves cross-ratio unchanged, and is determined when the positions of three transformed points are known (§ 23), or when the transforms, *with weights*, of *two* weighted points are known.

We now consider these projectivities from the point of view of linear transformations (§ 10), that is, of transformations which change $k_1 a_1 + k_2 a_2$ into $k_1 b_1 + k_2 b_2$ when they change a_1 into b_1, and a_2 into b_2. The work may then be interpreted also on linear transformations of vectors in a plane.

If $\mathfrak{P}$ be a projectivity, and $a = b\mathfrak{P}$, then $ka = kb.\mathfrak{P}$. We define the transformation $k\mathfrak{P}$ so that $ka = b.k\mathfrak{P}$.

If $\mathfrak{P}$, $\mathfrak{Q}$ be projectivities, then the equation $a.k_1\mathfrak{P} + a.k_2\mathfrak{Q} = b$ gives a projectivity between the ranges of a and b. If this projectivity is $\mathfrak{R}$, we write $\mathfrak{R} = k_1\mathfrak{P} + k_2\mathfrak{Q}$.

Then $\mathfrak{A}(\mathfrak{B} + \mathfrak{C}) = \mathfrak{A}\mathfrak{B} + \mathfrak{A}\mathfrak{C}$, $(\mathfrak{B} + \mathfrak{C})\mathfrak{A} = \mathfrak{B}\mathfrak{A} + \mathfrak{C}\mathfrak{A}$.

2. The projectivity which changes a_1, a_2 into points congruent with a_2, a_1, and changes p into q, will change q to a point congruent with p. We are assuming $a_1 \not\equiv a_2$. (Cf. § 26·13.)

For if $p = l_1 a_1 + l_2 a_2$, and a_1, a_2 become $k_2 a_2$, $k_1 a_1$, then

$$q = l_1 k_2 a_2 + l_2 k_1 a_1,$$

and q becomes $\quad k_1 k_2 (l_1 a_1 + l_2 a_2) \equiv p.$

* H. Grassmann, d.J., *Projektive Geometrie der Ebene*, Band I (1913).

Thus a projectivity $\mathfrak{S}$ which interchanges the positions of *one* pair of points, is such that for all a, if $a\mathfrak{S} \equiv b$, then $b\mathfrak{S} \equiv a$.

We can write this as $a\mathfrak{S}^2 \equiv a$, or $\mathfrak{S}^2 = \mathrm{k}\mathfrak{J}$, where k is a scalar, and $\mathfrak{J}$ the identical transformation.

We call $\mathfrak{S}$ an '*involution*', if $\mathfrak{S}^2 = \mathrm{k}\mathfrak{J}$, though this name is usually reserved for the case when $\mathrm{k} = 1$. We assume that $\mathfrak{S}$ is not a multiple of $\mathfrak{J}$.

3. If a, b be any points in our line, $\mathfrak{P}$, $\mathfrak{Q}$ any projectivities, then we define:

$$[ab.\mathfrak{P}\mathfrak{Q}] = \tfrac{1}{2}([a\mathfrak{P}.b\mathfrak{Q}] + [a\mathfrak{Q}.b\mathfrak{P}]).$$

Herein $[a\mathfrak{P}.b\mathfrak{Q}]$ is an ordinary outer product of the points $a\mathfrak{P}$, $b\mathfrak{Q}$.

Then
$$[\mathrm{k}.ab.\mathfrak{P}\mathfrak{Q}] = \mathrm{k}[ab.\mathfrak{P}\mathfrak{Q}];$$
$$[(a_1 + a_2)\, b.\mathfrak{P}\mathfrak{Q}] = [a_1 b.\mathfrak{P}\mathfrak{Q}] + [a_2 b.\mathfrak{P}\mathfrak{Q}].$$

We may therefore regard $[ab.\mathfrak{P}\mathfrak{Q}]$ as a product of $[ab]$ and $\mathfrak{P}\mathfrak{Q}$.

To justify this, we observe that $[ab.\mathfrak{P}\mathfrak{Q}] \div [ab]$ is a scalar, independent of a, b. For, if we replace these by $\mathrm{k}_1 a + \mathrm{l}_1 b$ and $\mathrm{k}_2 a + \mathrm{l}_2 b$, then $[ab]$ becomes $(\mathrm{k}_1 \mathrm{l}_2 - \mathrm{l}_1 \mathrm{k}_2)\,[ab]$. Hence, if we regard $[ab.\mathfrak{P}\mathfrak{Q}]$ as a product of $[ab]$ and $[\mathfrak{P}\mathfrak{Q}]$, the latter depends on $\mathfrak{P}$, $\mathfrak{Q}$ only.

We denote the *scalar* $[ab.\mathfrak{P}\mathfrak{Q}] \div [ab]$ by $[\mathfrak{P}\mathfrak{Q}]$, and call it the '*outer product*' of $\mathfrak{P}$ and $\mathfrak{Q}$.

But here we have the commutative law: $[\mathfrak{P}\mathfrak{Q}] = [\mathfrak{Q}\mathfrak{P}]$.

Also $[\mathfrak{P}(\mathfrak{Q} + \mathfrak{R})] = [\mathfrak{P}\mathfrak{Q}] + [\mathfrak{P}\mathfrak{R}]$, $[\mathfrak{J}\mathfrak{J}] = 1$.

We may write $[\mathfrak{P}^2]$ instead of $[\mathfrak{P}\mathfrak{P}]$, the square brackets being necessary to distinguish it from the sequence product (§ 10), for which no brackets are used.

4. If $\mathfrak{S}$ is an involution, then $[\mathfrak{J}\mathfrak{S}] = 0$, and conversely.

For if $a\mathfrak{S} \equiv b$, then $b\mathfrak{S} \equiv a$,

$$2[ab]\,[\mathfrak{J}\mathfrak{S}] = [a.b\mathfrak{S}] + [a\mathfrak{S}.b] = 0.$$

The converse follows by retracing steps.

5. *Pencils of projectivities.** If $\mathfrak{P} \not\equiv \mathfrak{Q}$, then $\mathrm{k}_1 \mathfrak{P} + \mathrm{k}_2 \mathfrak{Q}$ describes a '*pencil*' of projectivities as the scalars k_1, k_2 vary.

* In accordance with the general use of the sign $\equiv$, $\mathfrak{A} \equiv \mathfrak{B}$ means there is a scalar k not zero such that $\mathfrak{A} \equiv \mathrm{k}\mathfrak{B}$.

If $\mathfrak{P}$, $\mathfrak{Q}$ are both involutions, so is each projectivity of the pencil, since $[\mathfrak{J}\mathfrak{P}]=[\mathfrak{J}\mathfrak{Q}]=0$.

In the contrary case, there is still just one involution in the pencil (ignoring the weight factor in the involution), and it is given by

$$k_1[\mathfrak{J}\mathfrak{P}]+k_2[\mathfrak{J}\mathfrak{Q}]=0,$$

and hence is

$$[\mathfrak{J}\mathfrak{Q}]\,\mathfrak{P}-[\mathfrak{J}\mathfrak{P}]\,\mathfrak{Q}.$$

If we take $\mathfrak{Q}=\mathfrak{J}$, we have $\mathfrak{P}-[\mathfrak{J}\mathfrak{P}]\,\mathfrak{J}$; we call this the '*double point involution*' of $\mathfrak{P}$.

6. Let

$$\mathfrak{S}=\mathfrak{P}-[\mathfrak{J}\mathfrak{P}]\,\mathfrak{J},$$

then since $[\mathfrak{J}\mathfrak{P}]$ is a scalar, say k, we have:

Any projectivity can be written in the form $\mathfrak{P}=\mathfrak{S}+k\mathfrak{J}$, where $\mathfrak{S}$ is the double-point involution of $\mathfrak{P}$.

7. If $\mathfrak{P}=\mathfrak{S}+k\mathfrak{J}$, then since $[\mathfrak{J}\mathfrak{S}]=[\mathfrak{S}\mathfrak{J}]=0$, we have $[\mathfrak{P}^2]=[\mathfrak{S}^2]+k^2$.

Let $\mathfrak{Q}=\mathfrak{S}-k\mathfrak{J}$, then $[\mathfrak{Q}^2]=[\mathfrak{S}^2]+k^2$, $[\mathfrak{P}\mathfrak{Q}]=[\mathfrak{S}^2]-k^2$.

If a be any point, then $a\mathfrak{P}=a\mathfrak{S}+ka$; also $a\mathfrak{P}^{-1}\equiv a\mathfrak{S}-ka$, since

$$a(\mathfrak{S}\mathfrak{P}-k\mathfrak{P})=a(\mathfrak{S}-k\mathfrak{J})\,(\mathfrak{S}+k\mathfrak{J})=a(\mathfrak{S}^2-k^2)\equiv a.$$

(The products in the last line are sequence products.)

Hence $a\mathfrak{P}$, $a\mathfrak{P}^{-1}$ *separate* a *and* $a\mathfrak{S}$ *harmonically.*

Also if a is a double point of $\mathfrak{P}$, that is, if $a\equiv a\mathfrak{P}$, it is a double point of $\mathfrak{S}$. Hence the name 'double point involution'.

8. The projectivities $\mathfrak{P}$, $\mathfrak{Q}$ are '*harmonic*' if $[\mathfrak{P}\mathfrak{Q}]=0$.

Let $\quad e_1\mathfrak{P}=a_1, \quad e_2\mathfrak{P}=a_2, \quad e_1\mathfrak{Q}=b_1, \quad e_2\mathfrak{Q}=b_2;$

then $[\mathfrak{P}\mathfrak{Q}]=0$ gives $\quad [a_2b_1]=[a_1b_2]$.

(i) Let $\mathfrak{S}$ be the projectivity which interchanges the positions of a_1, b_1, and turns a_2 into b_2; it is then an involution, and it turns b_2 into a point in position a_2. Hence $\mathfrak{Q}\equiv\mathfrak{P}\mathfrak{S}$, $\mathfrak{Q}\mathfrak{S}\equiv\mathfrak{P}$.

(ii) Conversely, if $\mathfrak{Q}\equiv\mathfrak{P}\mathfrak{S}$, $\mathfrak{S}^2\equiv\mathfrak{J}$, then $[\mathfrak{P}\mathfrak{Q}]=0$.

For, then $\mathfrak{S}$ interchanges a_1, b_1 and also a_2, b_2 in position; and if $b_1=a_1\mathfrak{S}$, $b_2=a_2\mathfrak{S}$, we have

$$\begin{aligned}2[e_1e_2.\,\mathfrak{P}\mathfrak{Q}]&=[e_1\mathfrak{P}.e_2\mathfrak{Q}]+[e_1\mathfrak{Q}.e_2\mathfrak{P}]=[a_1b_2]+[b_1a_2]\\&=[a_1.a_2\mathfrak{S}]+[a_1\mathfrak{S}.a_2]=2[a_1a_2]\,[\mathfrak{J}\mathfrak{S}]=0.\end{aligned}$$

9. If the involution $\mathfrak{T}$ is harmonic to $\mathfrak{P} = \mathfrak{S} + k\mathfrak{J}$, then $0 = [\mathfrak{P}\mathfrak{T}] = [\mathfrak{S}\mathfrak{T}]$, since $[\mathfrak{J}\mathfrak{T}] = 0$. Hence $\mathfrak{T}$ is also harmonic to $\mathfrak{S}$. The converse is true. Hence the involutions $\mathfrak{T}$ harmonic to $\mathfrak{P}$ are all and only those harmonic to the double-point involution of $\mathfrak{P}$; and if $\mathfrak{T}$ is harmonic to the involution $\mathfrak{S}$, it is harmonic to $\mathfrak{S} + k\mathfrak{J}$.

10. If $\mathfrak{P}$ has distinct double points, then an involution $\mathfrak{T}$, which interchanges them, is harmonic to $\mathfrak{P}$.

For let

$$d_1\mathfrak{P} \equiv d_1, \quad d_2\mathfrak{P} \equiv d_2, \quad d_1 \neq d_2, \quad d_1\mathfrak{T} \equiv d_2, \quad d_2\mathfrak{T} \equiv d_1.$$

Then

$$2[d_1 d_2]\,[\mathfrak{T}\mathfrak{P}] = [d_1\mathfrak{T} . d_2\mathfrak{P}] + [d_1\mathfrak{P} . d_2\mathfrak{T}] = 0, \quad [\mathfrak{T}\mathfrak{P}] = 0.$$

11. By 8, if $\mathfrak{P} = \mathfrak{S}\mathfrak{S}'$, where $\mathfrak{S}$, $\mathfrak{S}'$ are involutions, then $[\mathfrak{S}\mathfrak{P}] = 0$. We now shew $[\mathfrak{S}'\mathfrak{P}] = 0$. We have, for any points e_1, e_2 on the line,

$$2[e_1 e_2]\,[\mathfrak{S}'\mathfrak{P}] = [e_1\mathfrak{S}' . e_2\mathfrak{S}\mathfrak{S}'] + [e_1\mathfrak{S}\mathfrak{S}' . e_2\mathfrak{S}'].$$

Choose for e_1, e_2 points whose positions are interchanged by $\mathfrak{S}$, then the last expression is zero, and hence $[\mathfrak{S}'\mathfrak{P}] = 0$.

It will be shewn later, from a general theorem (§ 101·2), that every projectivity on a line is the product of two involutions.

12. *If* $\mathfrak{P}$, $\mathfrak{Q}$ *are commutative, that is, if* $\mathfrak{P}\mathfrak{Q} = \mathfrak{Q}\mathfrak{P}$ (sequence product), *they have the same double-point involution.*

For let $\mathfrak{P} = \mathfrak{S} + k_1\mathfrak{J}$, $\mathfrak{Q} = \mathfrak{T} + k_2\mathfrak{J}$, then the condition gives $\mathfrak{S}\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, where $\mathfrak{S}$, $\mathfrak{T}$ are the double point involutions of $\mathfrak{P}$, $\mathfrak{Q}$.

Let the positions of a, b be interchanged by $\mathfrak{S}$; $a\mathfrak{S} = l_1 b$, $b\mathfrak{S} = l_2 a$. Then $a\mathfrak{S}^2 = l_1 l_2 a$, $\mathfrak{S}^2 = l_1 l_2 \mathfrak{J}$, by 2.

$$[ab]\,[\mathfrak{S}^2] = [a\mathfrak{S} . b\mathfrak{S}] = -l_1 l_2 [ab] = -[ab] . \mathfrak{S}^2.$$

As this holds for all such a, b, we have, whenever $\mathfrak{S}$ is an involution, $[\mathfrak{S}^2] = -\mathfrak{S}^2$.

(Since $\mathfrak{S}^2$ is a *transformation*, a multiple of $\mathfrak{J}$, and $[\mathfrak{S}^2]$ is a *scalar*, we ought to write $[\mathfrak{S}^2]\,\mathfrak{J} = -\mathfrak{S}^2$; we omit the $\mathfrak{J}$ for simplicity.)

Thus $\mathfrak{S}\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, $[\mathfrak{S}^2] = -\mathfrak{S}^2$, $[\mathfrak{T}^2] = -\mathfrak{T}^2$.

By 5, $\mathfrak{S} + \mathfrak{T}$ is an involution, hence $[(\mathfrak{S} + \mathfrak{T})^2] = -(\mathfrak{S} + \mathfrak{T})^2$.

Hence $[\mathfrak{S}\mathfrak{T}]+[\mathfrak{T}\mathfrak{S}]=-\mathfrak{S}\mathfrak{T}-\mathfrak{T}\mathfrak{S}$; each product commutes.

Hence $[\mathfrak{S}\mathfrak{T}]=-\mathfrak{S}\mathfrak{T}$; thus $\mathfrak{S}\mathfrak{T}=k\mathfrak{J}$, where k is a scalar.

Hence $\mathfrak{S}^2\mathfrak{T}=k\mathfrak{S}$, $l_1l_2\mathfrak{T}=k\mathfrak{S}$, $\mathfrak{T}\equiv\mathfrak{S}$.

Cor. Two involutions are commutative, if and only if they are congruent.

13. If $\mathfrak{S}\mathfrak{T}=k\mathfrak{T}\mathfrak{S}$ and $\mathfrak{S}^2\equiv\mathfrak{T}^2\equiv\mathfrak{J}$, then $(\mathfrak{S}\mathfrak{T})^2\equiv\mathfrak{J}$, so that $\mathfrak{S}\mathfrak{T}$ is an involution, $\mathfrak{U}$, say. Thence $\mathfrak{T}\mathfrak{U}\equiv\mathfrak{U}\mathfrak{T}\equiv\mathfrak{S}$, $\mathfrak{S}\mathfrak{U}\equiv\mathfrak{U}\mathfrak{S}\equiv\mathfrak{T}$.

If we only regard the positions of points, and therefore ignore the weights of transformations, if two involutions are commutative, their product is an involution.

Conversely, if $\mathfrak{S}\mathfrak{T}\equiv\mathfrak{U}$, $\mathfrak{S}^2\equiv\mathfrak{T}^2\equiv\mathfrak{U}^2\equiv\mathfrak{J}$, then

$$\mathfrak{S}\mathfrak{T}\mathfrak{U}\equiv\mathfrak{J},\quad \mathfrak{U}\mathfrak{T}\equiv\mathfrak{T}\mathfrak{U}\equiv\mathfrak{S},\quad \mathfrak{U}\mathfrak{S}\mathfrak{T}\equiv\mathfrak{J},\quad \mathfrak{U}\mathfrak{S}\equiv\mathfrak{S}\mathfrak{U}\equiv\mathfrak{T}.$$

By 8 (ii) *any pair of* $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$ *is harmonic*. Each interchanges the double points of each of the others.

14. If p, q be not double points of the involution $\mathfrak{S}$, and they become x_1p+y_1q, x_2p+y_2q by $\mathfrak{S}$, then $x_1=-y_2$.

For
$$p\mathfrak{S}^2=(x_1^2+x_2y_1)\,p+y_1(x_1+y_2)\,q\equiv p,$$
$$q\mathfrak{S}^2=x_2(x_1+y_2)\,p+(y_2^2+x_2y_1)\,q\equiv q.$$

Hence if $y_1\neq0$ or $x_2\neq0$, we have $x_1=-y_2$. If $y_1=x_2=0$, then p, q would be double points of $\mathfrak{S}$.

Thus if $p\mathfrak{S}=p_1$, $q\mathfrak{S}=q_1$, $2z=y_1+x_2$, $2w=y_1-x_2$,

then $$p_1=x_1p+(z+w)\,q,\quad q_1=(z-w)\,p-x_1q.$$

Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ be the involutions which change p, q into $p,-q$; q,p; $q,-p$, respectively. Then the last formulae give

$$\mathfrak{S}=x_1\mathfrak{H}_1+z\mathfrak{H}_2+w\mathfrak{H}_3.$$

Hence every involution is a linear combination of $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$. It is easily shewn that these involutions are commutative, apart from weights, and hence harmonic in pairs, and that

$$\mathfrak{H}_1\mathfrak{H}_2=\mathfrak{H}_3,\quad \mathfrak{H}_2\mathfrak{H}_3=-\mathfrak{H}_1,\quad \mathfrak{H}_3\mathfrak{H}_1=-\mathfrak{H}_2,$$
$$\mathfrak{H}_2\mathfrak{H}_1=-\mathfrak{H}_3,\quad \mathfrak{H}_3\mathfrak{H}_2=\mathfrak{H}_1,\quad \mathfrak{H}_1\mathfrak{H}_3=\mathfrak{H}_2,$$
$$\mathfrak{H}_1^2=\mathfrak{H}_2^2=\mathfrak{J},\quad \mathfrak{H}_3^2=-\mathfrak{J}.$$

Thus
$$\mathfrak{S}^2=(x_1^2+z^2-w^2)\,\mathfrak{J}.$$

Also $x_1\mathfrak{H}_1+z_1\mathfrak{H}_2+w_1\mathfrak{H}_3$, $x_2\mathfrak{H}_1+z_2\mathfrak{H}_2+w_2\mathfrak{H}_3$ are harmonic if, and only if, $x_1x_2+z_1z_2-w_1w_2=0$.

§ 49. *Collineations in a plane.**

1. *Def.* A '*collineation*' in a spread of points of any step, or between two spreads of the same step, is a transformation of points into points such that if the points (with any absorbed weights) a_1, a_2 become b_1, b_2, then $\mathrm{k}_1 a_1 + \mathrm{k}_2 a_2$ becomes $\mathrm{k}_1 b_1 + \mathrm{k}_2 b_2$, that is, it is a linear transformation (§ 10·2). Collinear points become collinear points.

2. A plane collineation is determined by the fate of three independent *weighted* points. For if a_1, a_2, a_3 become b_1, b_2, b_3, then $\mathrm{k}_1 a_1 + \mathrm{k}_2 a_2 + \mathrm{k}_3 a_3$ becomes $\mathrm{k}_1 b_1 + \mathrm{k}_2 b_2 + \mathrm{k}_3 b_3$.

Collineations will here be denoted by small German letters; then if $\mathfrak{p}$ is the collineation just mentioned, $a_1 \mathfrak{p} = b_1$.

3. If we ignore weights, and consider positions only, a collineation is determined by the fate of four independent points. For if e_1, e_2, e_3, e become a_1, a_2, a_3, a in position, we can adjust the absorbed weights so that

$$e = e_1 + e_2 + e_3, \quad a = a_1 + a_2 + a_3.$$

Then, if $e_1 \mathfrak{p} = a_1$, $e_2 \mathfrak{p} = a_2$, $e_3 \mathfrak{p} = a_3$, we have $e \mathfrak{p} = a$, and if

$$a_1 = \mathrm{a}_{11} e_1 + \mathrm{a}_{12} e_2 + \mathrm{a}_{13} e_3, \quad a_2 = \mathrm{a}_{21} e_1 + \mathrm{a}_{22} e_2 + \mathrm{a}_{23} e_3,$$
$$a_3 = \mathrm{a}_{31} e_1 + \mathrm{a}_{32} e_2 + \mathrm{a}_{33} e_3,$$

the a_{ij} are fixed by $\mathfrak{p}$ apart from a common scalar factor; any other point of position $p \equiv \mathrm{x}_1 e_1 + \mathrm{x}_2 e_2 + \mathrm{x}_3 e_3$ becomes

$$p \mathfrak{p} \equiv \mathrm{x}_1 e_1 \mathfrak{p} + \mathrm{x}_2 e_2 \mathfrak{p} + \mathrm{x}_3 e_3 \mathfrak{p} = \mathrm{x}_1 a_1 + \mathrm{x}_2 a_2 + \mathrm{x}_3 a_3,$$

and is fixed in position.

4. A collineation in a plane gives a linear transformation of lines into lines. We denote these transformations here by German capitals. But $\mathfrak{I}$ shall always denote identity.

If $\mathfrak{p}$ transforms e_1, e_2, e_3 into a_1, a_2, a_3, and $\mathfrak{q}$ transforms e_1, e_2, e_3 into b_1, b_2, b_3, then we define $[\mathfrak{p}\mathfrak{q}]$, the '*outer product*' of $\mathfrak{p}, \mathfrak{q}$, as follows:

$[\mathfrak{p}\mathfrak{q}]$ is the transformation which turns $[e_2 e_3], [e_3 e_1], [e_1 e_2]$ into

$$\tfrac{1}{2}([a_2 b_3] - [a_3 b_2]), \quad \tfrac{1}{2}([a_3 b_1] - [a_1 b_3]), \quad \tfrac{1}{2}([a_1 b_2] - [a_2 b_1]).$$

The product is commutative, and the bracket is necessary to distinguish it from the sequence product.

* H. Grassmann, d.J., *loc. cit.* Bände II, III.

5. Thus $2[ab]\,[\mathfrak{p}\mathfrak{q}] = [a\mathfrak{p}.b\mathfrak{q}]+[a\mathfrak{q}.b\mathfrak{p}]$.

For let $a = k_1e_1+k_2e_2+k_3e_3$, $b = l_1e_1+l_2e_2+l_3e_3$,

then $[ab] = (k_2l_3-k_3l_2)\,[e_2e_3] + (k_3l_1-k_1l_3)\,[e_3e_1]+(k_1l_2-k_2l_1)\,[e_1e_2]$,

$$\begin{aligned}2[ab]\,[\mathfrak{p}\mathfrak{q}] &= (k_2l_3-k_3l_2)\,([a_2b_3]-[a_3b_2])+\ldots+\ldots\\ &= [(k_1a_1+k_2a_2+k_3a_3)\,(l_1b_1+l_2b_2+l_3b_3)]\\ &\quad -[(l_1a_1+l_2a_2+l_3a_3)\,(k_1b_1+k_2b_2+k_3b_3)]\\ &= [a\mathfrak{p}.b\mathfrak{q}]-[b\mathfrak{p}.a\mathfrak{q}] = [a\mathfrak{p}.b\mathfrak{q}]+[a\mathfrak{q}.b\mathfrak{p}].\end{aligned}$$

6. We define the '*outer product*' $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]$ of $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ as follows:

For any points a, b, c,

$$[a\mathfrak{p}.b\mathfrak{q}.c\mathfrak{r}]+[a\mathfrak{q}.b\mathfrak{r}.c\mathfrak{p}]+[a\mathfrak{r}.b\mathfrak{p}.c\mathfrak{q}] +[a\mathfrak{q}.b\mathfrak{p}.c\mathfrak{r}]+[a\mathfrak{r}.b\mathfrak{q}.c\mathfrak{p}]+[a\mathfrak{p}.b\mathfrak{r}.c\mathfrak{q}]$$

when divided by $[abc]$ is a scalar independent of a, b, c.

One-sixth of the quotient is $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]$.

$6[\mathfrak{p}\mathfrak{q}\mathfrak{r}]\,[abc] = \Sigma[a\mathfrak{p}.b\mathfrak{q}.c\mathfrak{r}]$, the sum being over all permutations of $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$.

7. We write $[\mathfrak{p}^2]$, $[\mathfrak{p}^3]$ for $[\mathfrak{p}\mathfrak{p}]$, $[\mathfrak{p}\mathfrak{p}\mathfrak{p}]$.

If $e_i\mathfrak{p} = a_i$ $(i = 1, 2, 3)$, then $[\mathfrak{p}^3] = [a_1a_2a_3]$.

If $A_1 = [a_2a_3]$, $A_2 = [a_3a_1]$, $A_3 = [a_1a_2]$,

$E_1 = [e_2e_3]$, $E_2 = [e_3e_1]$, $E_3 = [e_1e_2]$,

then $E_1[\mathfrak{p}^2] = [e_2\mathfrak{p}.e_3\mathfrak{p}]$, $E_2[\mathfrak{p}^2] = [e_3\mathfrak{p}.e_1\mathfrak{p}]$, $E_3[\mathfrak{p}^2] = [e_1\mathfrak{p}.e_2\mathfrak{p}]$.

Hence $[\mathfrak{p}^2]$ changes E_1, E_2, E_3 into A_1, A_2, A_3.

8. Dually we can define $[\mathfrak{P}\mathfrak{Q}]$, $[\mathfrak{P}\mathfrak{Q}\mathfrak{R}]$.

For example, $[\mathfrak{P}^2]$ is a transformation $\mathfrak{p}$ of points, and we find $[\mathfrak{P}^3] = [\mathfrak{p}^3]^2$.

9. *Pasch collineations*: $[\mathfrak{p}\mathfrak{J}\mathfrak{J}] = 0$.

If $\mathfrak{p}$ be a collineation such that there is one triangle abc with $a\mathfrak{p}$, $b\mathfrak{p}$, $c\mathfrak{p}$ on bc, ca, ab respectively, then there are ∞^4 such triangles. (Pasch.*)

For, by hypothesis,

$$[a\mathfrak{p}.bc]+[b\mathfrak{p}.ca]+[c\mathfrak{p}.ab] = 0. \qquad \text{(i)}$$

But the left-hand side is $3[abc]\,[\mathfrak{p}\mathfrak{J}\mathfrak{J}]$, hence $[\mathfrak{p}\mathfrak{J}\mathfrak{J}] = 0$.

* *Math. Ann.* 23 (1884), p. 419; 26 (1886), p. 211; Kraus, 29 (1887), p. 234.

Hence (i) holds when any points p, q, r are taken for a, b, c. Take p, q arbitrary—there are ∞^4 such pairs of positions—and let r be the cut of $[p\mathfrak{p}.q]$ and $[q\mathfrak{p}.p]$; then $[p\mathfrak{p}.qr]=0$, $[q\mathfrak{p}.pr]=0$.

Hence by (i), $[r\mathfrak{p}.pq]=0$, $r\mathfrak{p}$ is on $[pq]$.

10. *If $\mathfrak{p}$ is a Pasch collineation, then $a\mathfrak{p}^3$ is on $[a.a\mathfrak{p}]$, $a\mathfrak{p}^4$ is on $[a\mathfrak{p}.a\mathfrak{p}^2]$, $a\mathfrak{p}^5$ is on $[a\mathfrak{p}^2.a\mathfrak{p}^3]$, and so on, $\mathfrak{p}^2$, $\mathfrak{p}^3$, ... being sequence powers.* (Gordan.)

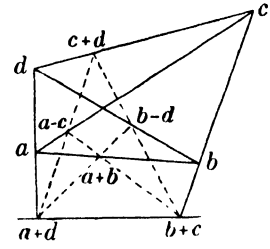
For, let $b=a\mathfrak{p}$, $c=b\mathfrak{p}=a\mathfrak{p}^2$, then $[a\mathfrak{p}.bc]=0$, $[b\mathfrak{p}.ca]=0$.

Hence $[c\mathfrak{p}.ab]=0$, and $c\mathfrak{p}$, $=a\mathfrak{p}^3$, is on $[a.a\mathfrak{p}]$.

The rest is similar.

11. *If five of the six vertices of a quadrilateral are on five of six sides of a quadrangle, the sixth vertex is on the sixth side.*

In the plane this is merely a version of Desargues' Theorem on perspective triangles. It holds in space, as the proof shews, though we only use the plane case. Let five of the vertices of the quadrilateral be on the sides bc, cd, da, bd, ac of the quadrangle. By absorbing weights, we can take the points on bc, cd, da to be $b+c$, $c+d$, $d+a$. The vertex on bd is then $b-d$, that on ac is $a-c$. The join of $b+c$ and $a-c$ cuts ab in $a+b$, so does the join of $a+d$ and $b-d$.



12. *If $\mathfrak{p}$ be a Pasch collineation, and a, b, c, d be any four independent points, there is a quadrilateral whose sides A, B, C, D go respectively through a', b', c', d', and whose vertices $[BC]$, $[CA]$, $[AB]$, $[AD]$, $[BD]$, $[CD]$ lie respectively on $[ad]$, $[bd]$, $[cd]$, $[bc]$, $[ca]$, $[ab]$, where $a'=a\mathfrak{p}$, ..., $d'=d\mathfrak{p}$.*

For we can draw* a triangle whose sides A, B, C pass through a', b', c' and whose vertices are on $[ad]$, $[bd]$, $[cd]$. We can weight these latter points as $a+d$, $b-d$, $c+d$. Then

$$[(b-d)(c+d)a']=0, \quad [(c+d)(d+a)b']=0,$$
$$[(d+a)(b-d)c']=0.$$

* Start with any point p on $[ad]$. Construct $q=[pc'.bd]$, $r=[qa'.cd]$, $p'=[rb'.ad]$. Then p, p' describe projective ranges on $[ad]$, because those points are connected by a series of projections and sections. Now take p at a double point of these ranges, and the triangle can be constructed by starting with p.

Add these together, and use $[pqr']+[qrp']+[rpq']=0$, for all points, then

$$\begin{aligned}0 &= [bca'+bda'+cda']+[cab'+dab'+cdb'] \\ &\qquad +[abc'+dbc'+dac'] \\ &= -[abd']-[acd']+[cdb']+[dbc'] \\ &= -[abd']-[acd']-[bcd'] = -[(a+b)(b+c)d'].\end{aligned}$$

Hence d' lies on the line $D\equiv[(a+b)(b+c)]$. Take this as the fourth side. The rest of the Theorem follows as in 11.

13. The case when $[\mathfrak{p}\mathfrak{p}\mathfrak{F}]=0$ is dual to the case just considered; for if $[\mathfrak{p}^2]=\mathfrak{P}$, we have $[\mathfrak{P}\mathfrak{F}\mathfrak{F}]=0$.

14. *Collineations* $\mathfrak{p}$, $\mathfrak{q}$ *with* $[\mathfrak{p}\mathfrak{q}\mathfrak{F}]=0$. *If* a, b, c *be points and* $\mathfrak{p}$, $\mathfrak{q}$ *collineations such that there is one hexagon with vertices* $b\mathfrak{p}$, $c\mathfrak{q}$, $a\mathfrak{p}$, $b\mathfrak{q}$, $c\mathfrak{p}$, $a\mathfrak{q}$ *whose opposite sides cut in the points* a, b, c, *assumed non-collinear, then there are* ∞^1 *triads* a, b, c, *with this property, and* $[\mathfrak{p}\mathfrak{q}\mathfrak{F}]=0$.

For if $[abc]\,[\mathfrak{p}\mathfrak{q}\mathfrak{F}]$ be expanded, the hypothesis secures that each term vanishes, and since $[abc]\neq 0$, we have $[\mathfrak{p}\mathfrak{q}\mathfrak{F}]=0$. There are ∞^6 triangles in the plane, and hence ∞^1 triangles fgh such that five terms of the expansion of $[fgh]\,[\mathfrak{p}\mathfrak{q}\mathfrak{F}]$ vanish. The sixth term then vanishes.

15. *If* a, b, c, a', b', c' *be given points, the locus of a point* p, *such that* $[pa]$, $[pa']$, $[pb]$, $[pb']$, $[pc]$, $[pc']$ *are pairs in involution, is a cubic curve.*

If $[abc]=[a'b'c']=0$, *the curve breaks up into the lines* ab, $a'b'$ *and the Pascal line of the hexagon* $ab'ca'bc'$.

For $R([pa],[pb],[pc],[pa'])=R([pa'],[pb'],[pc'],[pa])$,

$$[pab]\,[pca']\div[pbc]\,[pa'a]=[pa'b']\,[pc'a]\div[pb'c']\,[paa'].$$

If $[paa']\neq 0$, we have

$$[pb'c']\,[pca']\,[pab]+[pbc]\,[pc'a]\,[pa'b']=0;$$

hence p describes a cubic locus.

If $$[abc]=[a'b'c']=0,$$

we can absorb weights so that

$$a+b+c=a'+b'+c'=0.$$

The equation then becomes

$$[pab]\,[pa'b']\,([pab']+[pa'b])=0,$$

giving the triad of lines $[ab]$, $[a'b']$ and $[ab']+[a'b]$. The latter can be written $[bc']+[b'c]$, or $[ca']+[c'a]$, and hence is the Pascal line.

16. If a, b, c be any points, and $a\mathfrak{p}=a'$, $a\mathfrak{q}=a''$, and b', b'', c', c'' are similarly defined, we can define projective ranges on $b'c'$ and $b''c''$ by making $\mathrm{h}b'+\mathrm{k}c'$ and $\mathrm{h}b''+\mathrm{k}c''$ correspond. (Weights are now relevant.) These give, by 15, the Pascal line $[b'c'']+[b''c']=L$, say.

Similarly, we obtain the Pascal lines:

$$M=[c'a'']+[c''a'],\quad N=[a'b'']+[a''b'].$$

Then

$$\begin{aligned}[La]+[Mb]+[Nc] &= [b\mathfrak{p}.c\mathfrak{q}.a]+[b\mathfrak{q}.c\mathfrak{p}.a]\\ &\quad +\text{terms obtained by cycling } \mathfrak{p},\ \mathfrak{q},\ \mathfrak{R}\\ &= [abc]\,[\mathfrak{p}\mathfrak{q}\mathfrak{R}].\end{aligned}$$

Hence if $[\mathfrak{p}\mathfrak{q}\mathfrak{R}]=0$ and a be on L, b on M, then c is on N. There are ∞^4 triangles abc such that abc is inscribed in the triangle LMN which corresponds to abc. For take a, b arbitrarily, then a', b', a'', b'' are determined, and c can be found from $[La]=0$, $[Mb]=0$.

We call LMN an '*involution triangle*' of $\mathfrak{p}$, $\mathfrak{q}$.

17. If a, b, c, d be four general points, and we weight them so that $a+b+c+d=0$, and their transforms for $\mathfrak{p}$, $\mathfrak{q}$ be a', b', c', d' and a'', b'', c'', d'' respectively, we get projective ranges on corresponding sides of the quadrangles $a'b'c'd'$ and $a''b''c''d''$, if we make $\mathrm{k}a'+\mathrm{l}b'$ correspond to $\mathrm{k}a''+\mathrm{l}b''$. Hence we derive six Pascal lines:

$$L=[b'c'']+[b''c'],\quad M=[c'a'']+[c''a'],\quad N=[a'b'']+[a''b'],$$
$$P=[a'd'']+[a''d'],\quad Q=[b'd'']+[b''d'],\quad R=[c'd'']+[c''d'].$$

Since $a'+b'+c'+d'=a''+b''+c''+d''=0$, these six lines are the joins of four points; for we have, from these equations:

$$P-M+N=0,\quad Q-N+L=0,\quad R-L+M=0,\quad P+Q+R=0.$$

Now let $\mathfrak{p}'$ be the collineation which turns the points a, b, c into $[MN]$, $[NL]$, $[LM]$, which call a_1, b_1, c_1.

Now, by 16, if $[\mathfrak{p}\mathfrak{q}\mathfrak{J}]=0$, and if a be on L, b on M, then c is on N. Hence $[\mathfrak{p}'\mathfrak{p}'\mathfrak{J}]=0$, by 9, 13.

$$\text{Also} \quad d\mathfrak{p}' = -(a+b+c)\,\mathfrak{p}' \equiv [MN]+[NL]+[LM]$$
$$= [QR] = d_1, \text{ say.}$$

Hence $\mathfrak{p}'$ transforms a, b, c, d into $[MN]$, $[NL]$, $[LM]$, $[QR]$.

But $[\mathfrak{p}'\mathfrak{p}'\mathfrak{J}]=0$. Hence *there is a quadrilateral ABCD whose sides pass respectively through a, b, c, d and whose vertices* $[BC]$, $[CA]$, $[AB]$, $[AD]$, $[BD]$, $[CD]$ *lie respectively on* a_1d_1, b_1d_1, c_1d_1, b_1c_1, c_1a_1, a_1b_1, *that is, on P, Q, R, L, M, N.* We call $a_1b_1c_1d_1$ an '*involution quadrangle*' of $\mathfrak{p}$, $\mathfrak{q}$.

18. Let $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ be collineations such that $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$; let

$$a\mathfrak{p}=a', \quad a\mathfrak{q}=a'', \quad a\mathfrak{r}=a''',$$

and so on. Then there are ∞^1 triads of points a, b, c such that the hexagon $ab'ca'bc'$ has its opposite sides cutting in a'', b'', c''. Conversely, if this is so for one hexagon, then $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$.

If $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$, there are ∞^4 triads a, b, c which are transformed by $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ into triads each inscribed in the involution triangle of the other two. Conversely, the existence of one such triad entails $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$.

If $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$, and a, b, c, d be any four weighted points, then $a'b'c'd'$, $a''b''c''d''$, $a'''b'''c'''d'''$ are such that each corresponds to a quadrilateral whose sides go through the vertices and whose vertices lie on the sides of the involution quadrangle of the other two, in the order given in 17. Conversely, one such set of four points entails $[\mathfrak{p}\mathfrak{q}\mathfrak{r}]=0$.*

§ 50. *Collineations in space.*†

1. Collineations in general were defined in § 49. In space a collineation turns collinear, or coplanar, points, into such.

A collineation is fixed by the fate of four weighted independent points, or if we regard positions only, by the transformed positions of five independent points.

* Kraus, *Math. Ann.* 29. Pasch, *Math. Ann.* 26, by other methods.
† Muth, *Math. Ann.* 33 (1889), p. 493, by other methods.

2. If $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ be collineations, we define $[\mathfrak{pq}]$, $[\mathfrak{pqr}]$, $[\mathfrak{pqrs}]$ by the following equations, where a, b, c, d are any four independent points. It is easily shewn that the expressions $[\mathfrak{pq}]$, $[\mathfrak{pqr}]$, $[\mathfrak{pqrs}]$, so defined, are independent of a, b, c, d.

$$2[ab]\,[\mathfrak{pq}] = [a\mathfrak{p}.b\mathfrak{q}] + [a\mathfrak{q}.b\mathfrak{p}].$$

$$6[abc]\,[\mathfrak{pqr}] = [a\mathfrak{p}.b\mathfrak{q}.c\mathfrak{r}] + \text{terms obtained by permuting } \mathfrak{p}, \mathfrak{q}, \mathfrak{r}.$$

$$24[abcd]\,[\mathfrak{pqrs}] = [a\mathfrak{p}.b\mathfrak{q}.c\mathfrak{r}.d\mathfrak{s}] + \text{terms obtained by permuting } \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}.$$

We write $[\mathfrak{p}^2]$, $[\mathfrak{p}^3]$, $[\mathfrak{p}^4]$ for $[\mathfrak{pp}]$, $[\mathfrak{ppp}]$, $[\mathfrak{pppp}]$.

When points are transformed by $\mathfrak{p}$, lines are transformed by $[\mathfrak{p}^2]$, planes by $[\mathfrak{p}^3]$.

For, if $$a\mathfrak{p} = a', \quad b\mathfrak{p} = b', \quad c\mathfrak{p} = c',$$

then $$[ab]\,[\mathfrak{p}^2] = [a'b'], \quad [abc]\,[\mathfrak{p}^3] = [a'b'c'].$$

If also $d\mathfrak{p} = d'$, then $[abcd]\,[\mathfrak{p}^4] = [a'b'c'd']$. Hence if four independent points are always transformed by $\mathfrak{p}$ into independent points, then $[\mathfrak{p}^4] \neq 0$; and conversely.

Thence if $[\mathfrak{p}^4] \neq 0$, then $\mathfrak{p}$ has an inverse, and $\mathfrak{p}$ is '*non-singular*'.

3. If $[\mathfrak{p}\mathfrak{J}^3] = 0$, there are ∞^9 tetrahedra $abcd$ such that, if $a' = a\mathfrak{p}, \ldots, d' = d\mathfrak{p}$, then a', b', c', d' are on bcd, cda, dab, abc.

If $[\mathfrak{p}^3\mathfrak{J}] = 0$, there are ∞^9 tetrahedra $abcd$ such that a, b, c, d are on $b'c'd'$, $c'd'a'$, $d'a'b'$, $a'b'c'$.

These can be shewn as in § 49·9.

If $[\mathfrak{p}\mathfrak{J}^3] = [\mathfrak{p}^3\mathfrak{J}] = 0$, there are ∞^7 tetrahedra each in Möbius relation to the corresponding tetrahedron.

4. If $[\mathfrak{p}\mathfrak{J}^3] = 0$, and

$$a' = a\mathfrak{p} = b, \quad a'' = b\mathfrak{p} = b' = c, \quad a''' = c\mathfrak{p} = c' = d, \quad a^{\mathrm{iv}} = d\mathfrak{p} = d',$$

then $$0 = 6[abcd]\,[\mathfrak{p}\mathfrak{J}^3] = [a'bcd] + [ab'cd] + [abc'd] + [abcd'] = [abcd'].$$

Hence, if a be any point, then

$$[aa'a''a^{\mathrm{iv}}] = 0.$$

Similarly, if $[\mathfrak{p}^3\mathfrak{J}] = 0$, then

$$[aa''a'''a^{\mathrm{iv}}] = 0.$$

5. If there is a collineation $\mathfrak{p}$ between distinct planes α, α', there is a third plane β such that if α' be projected on to α from any point of β, there results a Pasch collineation (§ 49·9) on α.

For, let $[abc]=\alpha$, $[a'b'c']=\alpha'$, and let a, b, c, d (no three collinear) and a', b', c', d' correspond by the collineation, where d is on α, and hence d' on α'.

Consider $p=[a'bc.ab'c.abc']$. If we project the plane α' on to α from p, then, since $[pa'bc]=0$, and so on, a', b', c' are projected to points a'', b'', c'' on bc, ca, ab respectively; the projection gives a collineation on α, in which a, b, c correspond to a'', b'', c'', and which is a Pasch collineation (§ 49·9).

Similarly we get Pasch collineations by projecting from any of the points:

$$p_1=[b'cd.bc'd.bcd'],$$
$$p_2=[c'ad.ca'd.cad'],$$
$$p_3=[a'bd.ab'd.abd'].$$

Further, if the points q_1, q_2, q_3 give Pasch collineations, so does $k_1q_1+k_2q_2+k_3q_3$, for the conditions are $[q_1a'bc]=0$, and so on.

It remains to shew that p, p_1, p_2, p_3 are coplanar. Take weights so that $d=a+b+c$. Then

$$d'=a'+b'+c';$$
$$[b'cd]+[bc'd]+[bcd']=[a'bc]+[ab'c]+[abc'].$$

The last equation and similar ones shew that p, p_1, p_2, p_3 are all on the plane $[a'bc]+[ab'c]+[abc']$, which is accordingly the plane β required.

6. If a, b, c, d be four points on a plane α, we can weight them so that $a+b+c+d=0$; then, if $\mathfrak{p}$, $\mathfrak{q}$ be collineations in space which transform α into α', α'', and a into a', a'', and so on, we have

$$a'+b'+c'+d'=0, \quad a''+b''+c''+d''=0.$$

Let $\gamma=[a\mathfrak{J}.b\mathfrak{p}.c\mathfrak{q}]+$terms obtained by cycling $\mathfrak{J}$, $\mathfrak{p}$, $\mathfrak{q}$. Then γ is called the '*Z-plane*' of α, α', α''.

If we project a', b', c', a'', b'', c'' from any point p on to α, we shall have in α two collineations $\mathfrak{p}_1$, $\mathfrak{q}_1$, connecting a, b, c with the projections of a', b', c', and with the projections of a'', b'', c'' respectively (the weights being relevant).

If $a_1', b_1', \ldots, c_1''$ be the projections on α, then

$$a_1' = [\alpha . pa'] = [\alpha a']\, p - [\alpha p]\, a', \ \ldots, \ c_1'' = [\alpha c'']\, p - [\alpha p]\, c'',$$

$$[ab_1'c_1''p] = [\alpha p]^2\, [ab'c''p].$$

If p is taken on the Z-plane, then

$$[p(ab_1'c_1'' + ab_1''c_1' + a'b_1c_1'' + a'b_1''c_1 + a''b_1c_1' + a''b_1'c_1)] = 0,$$

and hence with this position of p, we have *on the plane* α, $[\mathfrak{S}\mathfrak{p}_1\mathfrak{q}_1] = 0$.

7. *To construct the Z-plane.* Let

$$f' = b' + c', \quad g' = c' + a', \quad h' = a' + b',$$

$$f'' = b'' + c'', \quad g'' = c'' + a'', \quad h'' = a'' + b''.$$

Join a to the lines $b'c''$, $b''c'$, $b'f''$, $b''f'$, $c'f''$, $c''f'$, and we have planes cutting in lines

$$[ab'c''.ab''c'], \quad [ab'f''.ab''f'], \quad [ac'f''.ac''f'].$$

The second of these is

$$\begin{aligned}&[(ab'b'' + ab'c'')\,(ab''b' + ab''c')]\\ &\quad = [ab'b''.ab''c'] + [ab'c''.ab''b'] + [ab'c''.ab''c']\\ &\quad = [ab'b''(ab''c' + ab'c'')] + [ab'c''.ab''c'].\end{aligned}$$

Hence it lies on the plane $ab'c'' + ab''c'$. So does the third line, and the first line.

Similarly, from b, c we obtain the planes $a'bc'' + a''bc'$ and $a'b''c + a''b'c$. These three planes meet in a point on the Z-plane:

$$ab'c'' + ab''c' + a'bc'' + a''bc' + a'b''c + a''b'c.$$

If we use the point $d\ (= -a - b - c)$ instead of a or b or c, we get three other points on the Z-plane.

8. *Degenerate or singular collineations.* If $[\mathfrak{p}^4] = 0$, and a, b, c, d be (independent) points, then $[a\mathfrak{p}.b\mathfrak{p}.c\mathfrak{p}.d\mathfrak{p}] = 0$, hence their transforms lie in a plane.

If $[\mathfrak{p}^3] = 0$, their transforms lie on a line; if $[\mathfrak{p}^2] = 0$, they coincide. Such collineations are '*singular*' or '*degenerate*'.

9. *Correlations and polarities.* A one-to-one correspondence or transformation in space between points and planes, such that if p, q correspond to α, β, then $k_1p + k_2q$ corresponds to $k_1\alpha + k_2\beta$, is a '*correlation*'.

A correlation turns points of a line into planes of a pencil; points of a plane into planes of a bundle. It is fixed when we know the fate, with weights, of four weighted points, or the positions of the transforms of five points.

An instance of a correlation is a '*polarity*', which makes a point correspond to its polar plane with respect to a quadric.

If $\mathfrak{P}$, $\mathfrak{Q}$ be correlations, then $\mathfrak{P}\mathfrak{Q}$ (sequence product) is a collineation.

The corresponding definitions and results in a plane are clear.

§ 51. *The tetrahedral complex.*

1. Let $L = A + \mathrm{k}B$, $L' = A' + \mathrm{k}B'$, where A, B are fixed lines in one plane, and A', B' are fixed lines in another plane. As k varies, L and L' describe two projective pencils. Consider all lines which cut any two corresponding lines of the pencils.

If the cut of the planes in which the pencils lie is self-corresponding, we know that in this way we obtain a linear complex (§ 34). Suppose now that the cut of the planes is not self-corresponding and also that the pencils are not in perspective (§ 34).

Let $A = [ma]$, $B = [mb]$, $A' = [m'a']$, $B' = [m'b']$.

Any point p on L is of the form $a + \mathrm{k}b + \mathrm{x}m$; any point p' on L' is of the form $a' + \mathrm{k}b' + \mathrm{y}m'$. The line pp' which cuts the corresponding lines can be considered as the join of the points $p + p'$ and $p - p'$, that is, of

$$(a + a') + \mathrm{k}(b + b') + \mathrm{x}m + \mathrm{y}m'$$

and

$$(a - a') + \mathrm{k}(b - b') + \mathrm{x}m + \mathrm{y}(-m').$$

It is therefore the join of corresponding points in the collineation in which the weighted points $a + a'$, $b + b'$, m, m' correspond to $a - a'$, $b - b'$, m, $-m'$.

2. A '*tetrahedral complex*' of lines is the set of lines joining corresponding points of a (non-singular) collineation which has just four distinct self-corresponding points,* and hence just four distinct self-corresponding planes (those joining the self-corresponding points).

* The question of the number of self-corresponding points in a collineation will be treated generally later.

The tetrahedron whose vertices are the self-corresponding points is the '*fundamental tetrahedron*' of the complex.

If $\mathfrak{r}$ be the collineation, the lines in the complex are of form $L=[p.p\mathfrak{r}]$.

Let $p'=p\mathfrak{r}$, $[p'.p'\mathfrak{r}]=L_1$, then

$$L[\mathfrak{r}^2]=[p\mathfrak{r}.p\mathfrak{r}\mathfrak{r}]=[p'.p'\mathfrak{r}]=L_1.$$

Now p' is on L and L_1; hence L and $L[\mathfrak{r}^2]$ meet.

Conversely, if $[LL_1]=0$, where $L_1=L[\mathfrak{r}^2]$, and if $p'=p\mathfrak{r}$ be the point where L, L_1 meet, then

$$0=[p'L_1]=[p\mathfrak{r}.L[\mathfrak{r}^2]]=[pL]\,[\mathfrak{r}^3].$$

But $[\mathfrak{r}^3]\neq 0$, hence $[pL]=0$. Hence L goes through p and p', and so is a line of the complex.

Hence *the complex consists of all lines L which cut the lines $L[\mathfrak{r}^2]$ which correspond to them in the collineation.*

3. Dually, *the complex is the set of cuts of corresponding planes π, $\pi[\mathfrak{r}^3]$.*

4. All lines through a self-corresponding point, all lines on a self-corresponding plane are regarded as being in the complex.

The complex is thus the set of lines L such that, if $[Lp]=0$, then $[L.p\mathfrak{r}]=0$.

5. *Fundamental theorem. The planes joining a variable line of the complex to the four self-corresponding points have a constant cross-ratio. The lines of the complex are cut by the self-corresponding planes in a constant cross-ratio.*

For, let $\mathfrak{R}=[\mathfrak{r}^3]$, and α, β be any planes; then $A=[\alpha.\alpha\mathfrak{R}]$, $B=[\beta.\beta\mathfrak{R}]$ are lines of the complex, if $\alpha\neq\alpha\mathfrak{R}$, $\beta\neq\beta\mathfrak{R}$.

Let $L=[\alpha\beta]=[qs]$, $M=[\alpha\mathfrak{R}.\beta\mathfrak{R}]$. Then

$$M\equiv[q\mathfrak{r}.s\mathfrak{r}]\equiv[qs]\,[\mathfrak{r}^2]\equiv[\alpha\beta]\,[\mathfrak{R}^2].$$

Suppose A, B, L, M do not lie in a self-corresponding plane, or go through a self-corresponding point. We have

$$M\equiv L[\mathfrak{R}^2]\equiv L[\mathfrak{r}^2].$$

The lines L, M are axes of two projective pencils of planes, $[Lp]$, $[Mp_1]$, where $p_1=p\mathfrak{r}$. For

$$[Mp_1]\equiv[L\mathfrak{r}^2.p\mathfrak{r}]=[Lp]\,[\mathfrak{r}^3]=[Lp]\,\mathfrak{R}.$$

Let *abcd* be the fundamental tetrahedron.

The cuts of corresponding planes $[Lp]$, $[M.p\mathfrak{r}]$ (L, M being fixed), lie on a regulus through L, M which goes through a, b, c, d. The regulus also goes through A, B. For we can take p on A, so that $p\mathfrak{r}$ is on A, then $[Lp]\,\mathfrak{R} \equiv [Mp_1]$, where $p_1 = p\mathfrak{r}$; and $[Lp]$, $[Mp_1]$ cut in A, since L, A are coplanar, and M, A are coplanar. Similarly for B.

Thus A, B lie on a regulus through a, b, c, d; hence

$$R(Aa, Ab, Ac, Ad) = R(Ba, Bb, Bc, Bd).$$

This is the first part of the theorem. The second follows dually or by § 23·12.

6. *If the quadrics corresponding to the polarities* $\mathfrak{P}$, $\mathfrak{P}_1$ *have just one common self-polar tetrahedron, and* p *is any point, then* $[p\mathfrak{P}.p\mathfrak{P}_1]$ *describes a tetrahedral complex.*

For, let $p\mathfrak{P} = \alpha$, then $p\mathfrak{P}_1 = \alpha\mathfrak{P}^{-1}\mathfrak{P}_1$; hence $[\alpha.\alpha\mathfrak{P}^{-1}\mathfrak{P}_1]$, that is $[p\mathfrak{P}.p\mathfrak{P}_1]$, describes a tetrahedral complex, provided that the collineation $\mathfrak{P}^{-1}\mathfrak{P}_1$ has just four self-corresponding points, and hence just four self-corresponding planes.

Now, if $\pi\mathfrak{P}^{-1}\mathfrak{P}_1 \equiv \pi$, then $\pi\mathfrak{P}^{-1} \equiv \pi\mathfrak{P}_1^{-1}$, hence π has the same pole for the two quadrics. The theorem now follows from the hypotheses.

Cor. The complex consists of the joins of the poles of a variable plane for the two quadrics, and of the lines whose polar lines for the two quadrics cut.

7. *If the points of a plane* α *be in projective correspondence with a bundle of lines whose centre is not on* α*, then the joins of points on* α *to all points on corresponding lines form a tetrahedral complex.*

8. *Lie's Theorem. A tetrahedron abcd is inscribed in a quadric* $\mathfrak{Q}$*; the generators through* a, b, c, d *of a regulus* $\mathfrak{R}$ *of* $\mathfrak{Q}$ *are* aa', bb', cc', dd'*; then a generator* L *of the opposite regulus* $\mathfrak{R}_1$ *cuts the faces of the tetrahedron and these generators in points of an involution.*

For, project from a on to $[bcd] = \alpha$. Let e be the cut of α and the generator of $\mathfrak{R}_1$ through a. Since bb', cc', dd' meet $[ae]$, their projections are $[be]$, $[ce]$, $[de]$. Let L' be the projection of L on α. Then by Desargues' Theorem (§ 26·13), L' cuts $[bc]$, $[de]$; $[cd]$, $[be]$; $[db]$, $[ce]$ in pairs of an involution. Hence L cuts $[abc]$, $[dd']$; $[acd]$, $[bb']$; $[abd]$, $[cc']$ in pairs of an involution.

Cor. 1. The generators of $\mathscr{R}_1$ are cut by the faces of *abcd* in projective sets of points, for they are cut in projective sets by the generators of $\mathscr{R}$.

Cor. 2. If $\mathscr{Q}$ is a cone, vertex p, and a, b, c, d points on the cone distinct from p, and no two be on the same generator, then the generators of $\mathscr{Q}$ are cut in projective sets by the faces of the tetrahedron *abcd*.

For, if L is any generator, then $R(La, Lb, Lc, Ld)$ is constant; hence by § 23·12, L is cut by the faces of *abcd* in a constant cross-ratio. Hence:

Cor. 3. If L_1, L_2 be two generators of the cone, their plane cuts the faces of the inscribed tetrahedron in four lines which with L_1, L_2 touch a conic envelope.

Cor. 4. Dually, if a non-degenerate conic envelope c be inscribed in a tetrahedron *abcd*, and does not lie in any face-plane, nor touch any edge of the tetrahedron, and if p be any point on the plane of c, not on c nor on any face-plane of the tetrahedron, then pa, pb, pc, pd and the tangents from p to c lie on a quadric cone.

Cor. 5. *If abcd be a tetrahedron inscribed in a quadric, the generators of the quadric are contained in a tetrahedral complex whose fundamental tetrahedron is abcd.*

§ 52. *Congruences of lines defined by planar fields.**

1. Let the distinct planes *abc*, $a'b'c'$ cutting in L be related by a collineation so that $p = a + k_1 b + k_2 c$ and $p' = a' + k_1 b' + k_2 c'$ are corresponding points for all k_1, k_2, and suppose the planes are not perspective for this collineation. Consider the lines $[pp']$.

Let π be any general plane through the corresponding points p, p'. Then $[p\pi] = 0$, $[p'\pi] = 0$, give two equations for k_1, k_2. Hence *on a general plane lies just one line joining two corresponding points.*

2. First, suppose that the planes, though not in perspective, are such that points on L correspond always to points on L. If q be any point, denote the corresponding point always by q'.

Take a, b, c not on L, and weight so that $a-b$, $b-c$, and hence $c-a$, are on L. Then so are $a'-b'$, $b'-c'$, $c'-a'$.

* Mehmke, *Vorlesungen über Punkt und Vektor-Rechnung* (1913).

Let $A = [aa']$, $B = [bb']$, $C = [cc']$.

Then $D = [pp'] = A + \mathrm{k}_1^2 B + \mathrm{k}_2^2 C + \mathrm{k}_1([ab'] + [ba'])$
$+ \mathrm{k}_2([ac'] + [ca']) + \mathrm{k}_1\mathrm{k}_2([bc'] + [cb'])$.

$A + B - [ab'] - [ba'] = [(a-b)(a'-b')] \equiv L$,

$B + C - [bc'] - [cb'] = [(b-c)(b'-c')] \equiv L$,

$C + A - [ca'] - [ac'] = [(c-a)(c'-a')] \equiv L$.

$D = A + \mathrm{k}_1^2 B + \mathrm{k}_2^2 C + \mathrm{k}_1(A+B)$
$+ \mathrm{k}_1\mathrm{k}_2(B+C) + \mathrm{k}_2(C+A) - \mathrm{k}L$ (k some scalar)
$= (1 + \mathrm{k}_1 + \mathrm{k}_2)(A + \mathrm{k}_1 B + \mathrm{k}_2 C) - \mathrm{k}L$.

Hence *any line of the set is a linear combination of A, B, C, L, and hence of any four independent lines of the set.*

The set is a *linear congruence* (§ 29·10).

3. Next, suppose points on L do not always correspond to points on L. If $\alpha = [abc]$, $\alpha' = [a'b'c']$, and t be the collineation, then, as p describes α, $p' = p\mathrm{t}$ describes α', and $q = p + \mathrm{k}p'$, where k is a fixed scalar, describes a plane β, since four positions of p on α, being dependent, give four dependent positions of q. Further, p on α and q on β correspond by a collineation. Hence there are ∞^1 planes, each pair of which gives the same set of lines as α, α' give. Such a plane is of form

$$[(a + \mathrm{k}a')(b + \mathrm{k}b')(c + \mathrm{k}c')],$$

and its envelope, as k varies, is a '*cubic developable*'.

4. With the assumptions of 3, we have, by 1, that in a general plane lies just one line of the set. If two lines L_1, L_2 of the set lie on a plane π, the plane is '*singular*'. Since the lines L_1, L_2 cut α, α' in corresponding points, therefore each point of the line $[\pi\alpha]$ corresponds to a point of the line $[\pi\alpha']$, the points describe projective ranges on these lines, and their joins envelope a conic on the plane π.

In particular, α, α' are singular planes; for the lines of the set through any point p of L are $[p.p\mathrm{t}^{-1}]$, which lies on α, and $[p.p\mathrm{t}]$, which lies on α'.

As α, α' may be replaced by any pair of planes which give the same set of lines, therefore, by 3, the singular planes envelope a cubic developable.

5. Dually to 3, consider two projective bundles of planes, with distinct centres,

$$\pi = \alpha + k_1\beta + k_2\gamma, \quad \pi' = \alpha' + k_1\beta' + k_2\gamma',$$

a plane through the two centres not always corresponding to such a plane. Through a general point goes one line which is the cut of corresponding planes. The set of lines is a '*congruence*', but not a *linear* congruence. If two lines go through a point, the point is '*singular*', and through it go ∞^1 lines, and they lie on a quadric cone.

There are ∞^1 bundles of planes each pair of which gives the same congruence. The centres of the bundles are singular points, and are of form $[(\alpha + k\alpha')(\beta + k\beta')(\gamma + k\gamma')]$, and their locus is accordingly a twisted cubic, of which $[\alpha\alpha']$, $[\beta\beta']$, $[\gamma\gamma']$ are bisecants.

6. Conversely, if a twisted cubic is given by

$$p = a + kb + k^2c + k^3d,$$

then $k^{-1}[ap] = [ab] + k[ac] + k^2[ad] = A + kB + k^2C$, say,

$$[dp] = [da] + k[db] + k^2[dc] = A' + kB' + k^2C', \text{ say.}$$

Thus we have two projective bundles of lines, and two corresponding lines meet on the cubic. Hence any twisted cubic is the locus of singular points of two general projective bundles of planes.

Example. If we have three projective bundles of planes, such that three corresponding planes do not always meet in a line, and such that the join of the centres of the bundles is not a self-corresponding plane for each bundle, then the cuts of corresponding planes lie on a cubic surface. The three centres of the bundles can be taken anywhere on the surface. (Grassmann.)

CHAPTER VII

THE GENERAL THEORY*

We now build up the general theory of Grassmann's algebra from its foundations.

§ 53. *Linear dependence.*

1. We begin with any number of elements, which we call '*extensives of the same step*'; we assume that they can be multiplied by scalars, and can be added to one another, and that these processes give definite extensives of the same step as before. The reader may think of points or vectors of the earlier chapters.

We assume that the addition of these extensives satisfies the laws given in § 1, in particular, the associative law, and the two distributive laws with respect to multiplication by scalars; the same extensive results when any extensive is multiplied by the scalar 0, and no confusion arises if we denote the resulting extensive by 0, though it is a different entity from the numerical zero.

2. *Def.* If $a_1, \ldots, a_n$ be any extensives of the same step, and if $k_1, \ldots, k_n$ be any scalars, not all zero, we call $k_1 a_1 + \ldots + k_n a_n$ a '*linear combination*' of $a_1, \ldots, a_n$.

Def. The set of extensives $a_1, \ldots, a_n$ is '*linearly dependent*' if, and only if, there are scalars $k_1, \ldots, k_n$ not all zero, such that

$$k_1 a_1 + \ldots + k_n a_n = 0. \qquad (1)$$

If there are no such scalars, the set is '*linearly independent*'. We shall usually omit the word 'linearly'. Any single extensive, not zero, will be called, for convenience, '*independent*'. This is, in fact, involved in the definition above.

3. *If* $r < n$, *and* r *of the extensives* $a_1, \ldots, a_n$ *be dependent, so is the complete set. If the complete set be independent, so is any sub-set.*

* This theory is the subject of the first portion of the *Ausdehnungslehre* of 1862, referred to as A_2.

4. If the set $a_1, \ldots, a_n$ be dependent, at least one of these extensives is a linear combination of the others, and conversely.

For we have equation (1) with at least one coefficient, say k_i, not zero. Divide by k_i, and take a_i to the other side of the equation.

The converse is clear.

5. If the set $a_1, \ldots, a_n$ be independent, but $a_1, \ldots, a_n, b$ dependent, then b is a linear combination of $a_1, \ldots, a_n$.

For there are scalars k such that $k_1 a_1 + \ldots + k_n a_n + kb = 0$, and k cannot be zero, otherwise the set $a_1, \ldots, a_n$ would be dependent. Hence as in 4.

6. If the set $a_1, \ldots, a_n$ be independent, and b is not a linear combination of $a_1, \ldots, a_n$, then the set $a_1, \ldots, a_n, b$ is independent. This follows from 5.

7. *If the set $a_1, \ldots, a_n$ be dependent, there is a sub-set, which is an independent set, and such that each of the remaining a is a linear combination of the extensives of this sub-set.*

8. If b is a linear combination of $a_1, \ldots, a_n$, then every linear combination of $b, a_2, \ldots, a_n$ can be expressed as a linear combination of $a_1, \ldots, a_n$. Conversely, if b is a linear combination of $a_1, \ldots, a_n$, in which the coefficient of a_1 is not zero, then every linear combination of $a_1, \ldots, a_n$ can be expressed as a linear combination of $b, a_2, \ldots, a_n$.

For, if

$$b = k_1 a_1 + \ldots + k_n a_n, \quad c = l_1 b + l_2 a_2 + \ldots + l_n a_n, \quad \text{(k, l scalars)},$$

then $$c = l_1 k_1 a_1 + (l_2 + l_1 k_2) a_2 + (l_3 + l_1 k_3) a_3 + \ldots + (l_n + l_1 k_n) a_n.$$

And if $k_1 \neq 0$, then

$$a_1 = k_1^{-1} b - k_2 k_1^{-1} a_2 - \ldots - k_n k_1^{-1} a_n.$$

Hence, if $d = m_1 a_1 + \ldots + m_n a_n$, then as in the first part, d can be expressed as a linear combination of $b, a_2, \ldots, a_n$.

9. *Def.* If $a_1, \ldots, a_n$ be extensives, then the extensives $k_1 a_1 + \ldots + k_n a_n$, as the k traverse all scalars, constitute the '*spread spanned*' by $a_1, \ldots, a_n$. If $a_1, \ldots, a_n$ be independent, then n is the '*step*' of the spread. We denote the spread spanned by $a_1, \ldots, a_n$ by $\mathscr{S}(a_1, \ldots, a_n)$.

10. *If the set* $b_1, \ldots, b_m$ *be independent, and the* b *be linear combinations of* $a_1, \ldots, a_n$, *then we can select* $n-m$ *of the* a, *so that the spread spanned by* $b_1, \ldots, b_m$ *and the selected* a *is that spanned by* $a_1, \ldots, a_n$.*

For, b_1 is a linear combination of $a_1, \ldots, a_n$, and we may suppose, renumbering if necessary, that the coefficient of a_1 is not zero. Then, by 8,

$$\mathscr{S}(b_1, a_2, \ldots, a_n) = \mathscr{S}(a_1, \ldots, a_n).$$

We use induction, and suppose, with $r < m$, that

$$\mathscr{S}(b_1, \ldots, b_r, a_{r+1}, \ldots, a_n) = \mathscr{S}(a_1, \ldots, a_n).$$

Now b_{r+1} is a linear combination of $a_1, \ldots, a_n$, and hence of $b_1, \ldots, b_r, a_{r+1}, \ldots, a_n$. Suppose

$$b_{r+1} = k_1 b_1 + \ldots + k_r b_r + l_{r+1} a_{r+1} + \ldots + l_n a_n, \quad \text{(k, l scalars)}.$$

Since the b are independent, not all the l vanish. Let $l_{r+1} \neq 0$, then, by 8,

$$\mathscr{S}(b_1, \ldots, b_r, a_{r+1}, \ldots, a_n) = \mathscr{S}(b_1, \ldots, b_{r+1}, a_{r+2}, \ldots, a_n),$$

for a_{r+1} is a linear combination of $b_1, \ldots, b_{r+1}, a_{r+2}, \ldots, a_n$.

We can so proceed till $r = m$.

Cor. 1. If $n = m$, then $\mathscr{S}(a_1, \ldots, a_n) = \mathscr{S}(b_1, \ldots, b_n)$.

Cor. 2. If m extensives $b_1, \ldots, b_m$ be linear combinations of n extensives $a_1, \ldots, a_n$, where $n < m$, then $b_1, \ldots, b_m$ are dependent.

For, if $b_1, \ldots, b_n$ be dependent, this is true; and if the latter set be independent, then $\mathscr{S}(b_1, \ldots, b_n) = \mathscr{S}(a_1, \ldots, a_n)$, and hence, as $b_{n+1}, \ldots, b_m$ are linear combinations of $a_1, \ldots, a_n$, they are such of $b_1, \ldots, b_n$.

11. *All extensives of a spread of step* n *are linear combinations of any* n *independent extensives of the spread. If a spread of step* n *is spanned by* $b_1, \ldots, b_n$ *these are independent.* (By 10, Cor. 1, 7, 9.)

12. *Defs.* Any n independent extensives $a_1, \ldots, a_n$ in a spread of step n constitute a '*basis*' or '*frame*' of the spread. If $a = k_1 a_1 + \ldots + k_n a_n$, then the scalars $k_1, \ldots, k_n$ are the '*co-ordinates*' of a with respect to the basis $a_1, \ldots, a_n$.

* This theorem and its applications below are due to Grassmann (A_2, § 20). In recent literature it is usually quoted as Steinitz' Austauschsatz (*Crelles Journ.* 143 (1913)).

13. If A, B be spreads of steps m, n respectively, with bases $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, and if A, B be sub-spreads of a spread, and have a common spread of step r, then there is a set of just r independent extensives in the common spread, say $c_1, \ldots, c_r$.

Hence, by 10, we may suppose $c_1, \ldots, c_r, a_{r+1}, \ldots, a_m$ is a basis of A, and $c_1, \ldots, c_r, b_{r+1}, \ldots, b_n$ is a basis of B.

Now $c_1, \ldots, c_r, a_{r+1}, \ldots, a_m, b_{r+1}, \ldots, b_n$ are independent. For if we had a relation $c+a+b=0$, wherein c depends on $c_1, \ldots, c_r$, and a depends on $a_{r+1}, \ldots, a_m$, and b depends on $b_{r+1}, \ldots, b_n$, then would $b+c \neq 0$, $a+c \neq 0$ (since these are linear combinations of the bases of B, A respectively), and hence $a \neq 0$, $b \neq 0$.

Now a is in A, $-b-c$ is in B; these equal extensives must therefore be in the common spread; on the other hand, since a is a linear combination of $a_{r+1}, \ldots, a_m$, and since $a_{r+1}, \ldots, a_m$, $c_1, \ldots, c_r$ are independent, therefore a is not a linear combination of $c_1, \ldots, c_r$ and hence a is *not* in the spread mentioned.

Thus we can have no relation of the type $c+a+b=0$.

The spread spanned by the independent extensives $c_1, \ldots, c_r$, $a_{r+1}, \ldots, a_m, b_{r+1}, \ldots, b_n$ contains all extensives which are linear combinations of extensives in A or in B or in both.

It is called the '*join*' of A and B, and its step is $s = m+n-r$. Hence:

If spreads of steps m, n *have a common spread of step* r, *and a join of step* s, *then* $m+n=r+s$.

The common spread of two or more spreads is their '*cut*'; the spread of lowest step containing them is their '*join*'.

14. *Def.* If $A=\mathscr{S}(a_1, \ldots, a_n)$ and $B=\mathscr{S}(b_1, \ldots, b_m)$ have no common extensive, save zero, then the spread

$$\mathscr{S}(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

is the '*direct sum*' of A and B.

Def. If $$a = k_1 a_1 + k_2 a_2 + \ldots + k_n a_n,$$

then $$k_1 a_1 + \ldots + k_r a_r \quad (r<n)$$

is the '*projection*' of a on $\mathscr{S}(a_1, \ldots, a_r)$ from $\mathscr{S}(a_{r+1}, \ldots, a_n)$.

15. If $R_1, R_2, \ldots$, spreads of steps $r_1, r_2, \ldots$, cut in a spread S of step s, and have R of step r as join, then the spread T of step $r-s$, such that R is the join of S, T, cuts $R_1, R_2, \ldots$ in spreads whose join is of step $r-s$.

For the cut T_i of T and R_i is of step $r_i - s$. If the T_i were all sub-spreads of a spread T' of step $r-s-1$, then as S, T' have no common points, their join is of step $r-1$, and this join contains S and the T_i and hence all the R_i, contrary to the hypothesis.

16. If the r spreads $R_1, R_2, \ldots, R_r$ be each of step r', and their join be R of step r, and their cut be S of step $r'-1$, then in an infinite number of ways we can find points $p_1, \ldots, p_r$ which span R, where p_i is in R_i.

For, if $p_1, \ldots, p_m$ be independent points, not in S, p_i in R_i, $(i = 1, \ldots, m < r)$, then $r'-1$ independent points in S together with p_i span R_i. If all points of $R_{m+1}, \ldots, R_r$ were dependent on $p_1, \ldots, p_m$, these spreads would be in $\mathscr{S}(p_1, \ldots, p_m)$, which therefore would contain S, and hence $\mathscr{S}(p_1, \ldots, p_m)$ would contain $R_1, \ldots, R_m$ as well as $R_{m+1}, \ldots, R_r$. Hence $m = r$.

17. If $R_1, \ldots, R_r$ be each of step r', and each pair cuts in a spread of step $r'-1$, then either they all cut in one such spread, or are all contained in a spread of step $r'+1$.

For, if R_1, R_2 cut in S, and R_3 does not contain S, then the cuts S_1, S_2 (of step $r'-1$) of R_3 with R_1, R_2 are distinct, and hence their join is R_3, for if they were in a sub-space of R_3, they would coincide. Hence the join of R_1, R_2 of step $r'+1$ contains R_3; similarly it contains R_4, for as R_4 does not cut R_1, R_2, R_3 in the same spread of step $r'-1$, it cuts at least two of them in distinct spreads of step $r'-1$; and so on.

§ 54. *Outer products.**

1. We work throughout in a spread of step n. A product of several factors is to be interpreted in accordance with the general convention of § 13·1.

We assume the distributive laws for all products:

$$a(b+c)\,d = abd + acd,$$

where a, b, c, d may be any expressions. A combination not satisfying these laws would not be called a product.

* A_2, §§ 37–76.

2. If $e_1, \ldots, e_n$ be independent extensives in our spread, their linear combinations are said to be '*extensives of step one*'. If b, c be of step one, we assume that $[bc]$ is a unique '*outer product*' of b and c; it is called an '*extensive of step two*'.

If $r \leqq n$, we assume that $a_1, \ldots, a_r$ combine to form an '*outer product* $[a_1 a_2 \ldots a_r]$ *of step* r'.

We assume that if one of the factors a_i is multiplied by a scalar k, the product is so multiplied. Extensives of step greater than one will be denoted by capitals.

Our assumptions give us, for example,

$$[AbcD] + [AadD] = [A(bc + ad)\, D].$$

If $\mathscr{S}$ is a spread of step n, we assume there is a set of n extensives spanning the spread whose outer product is not zero.

3. We assume there is a linear relation between the products $[e_i e_j]$, $(i, j = 1, \ldots, n)$, which has the same form whatever independent extensives, instead of $e_1, \ldots, e_n$, be taken as the basis of the spread, or in whatever order they are taken.

Suppose this relation is

$$\sum_{ij} k_{ij} e_i e_j = 0, \quad (i, j = 1, \ldots, n).$$

Change the sign of e_1 and subtract, then we have

$$\sum_{j} k_{1j} e_1 e_j + \sum_{l} k_{l1} e_l e_1 = 0, \quad (j, l > 1).$$

In this, change the sign of e_2 and subtract, then

$$k_{12} e_1 e_2 + k_{21} e_2 e_1 = 0.$$

Interchange e_1, e_2, then

$$k_{12} e_2 e_1 + k_{21} e_1 e_2 = 0.$$

Hence $(k_{12} - k_{21})(e_1 e_2 - e_2 e_1) = 0$. Either $k_{12} = k_{21}$ or $e_1 e_2 = e_2 e_1$.

But $k_{12} = k_{21}$ gives $e_1 e_2 + e_2 e_1 = 0$, unless $k_{12} = 0$. If $k_{12} = k_{21} = 0$, we proceed similarly with $e_3, e_4, \ldots, e_n$. Now we shall assume that not all the k vanish. But the relation between the $e_i e_j$ has to hold when the e_i are permuted in any way.

Hence, either $\quad e_r e_s = e_s e_r \quad$ for all r, s

or $\quad e_r e_s + e_s e_r = 0 \quad$ for all r, s.

In the latter case, replace e_s by $e_r + e_s$, then

$$e_r e_r + e_r e_s + e_r e_r + e_s e_r = 0, \quad \text{hence } e_r e_r = 0.$$

We take the latter alternative for outer multiplication, and accordingly assume

$$[ab] = -[ba], \quad [aa] = 0, \quad \text{for all } a, b.$$

Thus $$[AbcD] + [AcbD] = [A(bc+cb)\,D] = 0.$$

4. If A be a product of r extensives of step one, then $[\ldots aAb\ldots]$ can be changed to $[\ldots abA\ldots]$ as a result of r interchanges, one corresponding to each factor of A. As each interchange changes the sign of the product, we have

$$[\ldots aAb\ldots] = (-1)^r\,[\ldots abA\ldots].$$

Similarly $$[\ldots abA\ldots] = (-1)^{r+1}\,[\ldots bAa\ldots].$$

Hence $$[\ldots aAb\ldots] = -[\ldots bAa\ldots].$$

5. Thus *the interchange of two extensives in a product changes its sign.*

6. *If a product contains a factor twice, the product vanishes.*

7. *If A, B, C be products of* q, r, s *simple factors respectively, then*

$$[ABC] = (-1)^{(q+r)s}\,[CAB]$$
$$= (-1)^{(q+r)s+qr}[CBA] = (-1)^{qs+sr+qr}\,[CBA].$$

This follows from 5 by simple counting.

8. *If $b_1, \ldots, b_r$ be linear combinations of $a_1, \ldots, a_r$, then*

$$[b_1 \ldots b_r] = \mathrm{D}[a_1 \ldots a_r],$$

where D *is the determinant of the coefficients of the a in the expressions for the b.*

For, if $$b_1 = k_{11}a_1 + k_{12}a_2 + \ldots + k_{1r}a_r, \quad \ldots,$$
$$b_r = k_{r1}a_1 + k_{r2}a_2 + \ldots + k_{rr}a_r,$$

then $$[b_1 \ldots b_r] = [(k_{11}a_1 + k_{12}a_2 + \ldots + k_{1r}a_r)\ldots$$
$$(k_{r1}a_1 + k_{r2}a_2 + \ldots + k_{rr}a_r)],$$

and if the right-hand side be multiplied out, we obtain $[a_1 a_2 \ldots a_r]$ multiplied by a combination of the k constructed in the same way as the expansion of the determinant

$$\begin{vmatrix} k_{11}, & k_{12}, & \ldots, & k_{1r} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ k_{r1}, & k_{r2}, & \ldots, & k_{rr} \end{vmatrix}.$$

9. *If* $a_1, \ldots, a_m$ *be dependent, then* $[a_1 \ldots a_m] = 0$, *and conversely.*

For, if they be dependent, one can be expressed as a linear combination of the others; substitute the expression for it in $[a_1 \ldots a_m]$ and expand, using the distributive law; each term has a repeated factor and so vanishes.

Conversely, if $[a_1 \ldots a_m] = 0$, and it were possible that $a_1, \ldots, a_m$ were independent, we could adjoin extensives $a_{m+1}, a_{m+2}, \ldots, a_n$ so that $a_1, \ldots, a_n$ are independent. Now we have assumed that there is a set of n extensives, say $e_1, \ldots, e_n$, which span the spread, and whose outer product is not zero. But $e_1, \ldots, e_n$ are linear combinations of $a_1, \ldots, a_n$. Hence, by 8,

$$[e_1 \ldots e_n] = k[a_1 \ldots a_n] \quad \text{for some scalar k.}$$

But $[e_1 \ldots e_n] \neq 0$, $[a_1 \ldots a_n] = 0$. This gives a contradiction.

10. *If* $a_1, \ldots, a_m$ *be independent, so is the set formed of all products* $A, B, \ldots$ *of* r *of these.*

For, if $k_1 A + k_2 B + k_3 C + \ldots = 0$, and X be the product of those a which are not factors of A, then one or more of these occur in B, C, ..., and so on. Multiply by X, then, using 9,

$$k_1[AX] + k_2[BX] + k_3[CX] + \ldots = 0,$$
$$[BX] = [CX] = \ldots = 0, \quad [AX] \neq 0.$$

Hence $k_1 = 0$, similarly $k_2 = 0$, ... and so on.

Def. If A, B be any extensives, not zero, then $A \equiv B$ means that there is a scalar k such that $A = kB$. We call A, B '*congruent*'.

11. *If* $a_1, \ldots, a_m$ *are independent, then* $[a_1 \ldots a_m] \equiv [b_1 \ldots b_m]$, *if, and only if,* $\mathscr{S}(a_1, \ldots, a_m) = \mathscr{S}(b_1, \ldots, b_m)$.

Hence the first notation can replace the second.

For, if the spreads coincide, then by 8, $[a_1 \ldots a_m] \equiv [b_1 \ldots b_m]$.

Conversely, if $[a_1 \ldots a_m] = k[b_1 \ldots b_m]$, where k is a non-zero scalar, then $[a_1 \ldots a_m b_1] = k[b_1 \ldots b_m b_1] = 0$, by 6.

Hence $a_1, \ldots, a_m, b_1$ are dependent, by 9; hence b_1 is a linear combination of the a; similarly, so are $b_2, \ldots, b_m$. In the same way the a are linear combinations of the b.

12. *Def.* An '*elementary transformation*' is one which, applied to an ordered set of extensives of step one, adds to them a scalar multiple of one immediately before or after, that is, it changes

a set $(\dots p, q, \dots)$ into $(\dots p+\mathrm{k}q, q, \dots)$, or into $(\dots p, q+\mathrm{k}p, \dots)$, where k is some scalar.

The result of a series of elementary transformations is an '*affine transformation*'.

13. *If* $[abc\dots] = [a_1 b_1 c_1 \dots] \neq 0$, *then* a_1, b_1, c_1, ... *can be derived from* $a, b, c, \dots$ *by an affine transformation. Conversely, if the sets are so related, the outer products are equal.*

The converse is immediate. To shew the direct theorem we need some lemmas.

·1. By a series of elementary transformations, we can change $(\dots p, q, \dots)$ to $(\dots q, -p, \dots)$. For, denoting change by $\rightarrow$, we can secure

$$\begin{aligned}(p, q) &\rightarrow (p+q, q) \rightarrow (p+q, q-(p+q)) \\ &= (p+q, -p) \rightarrow (p+q-p, -p) \rightarrow (q, -p).\end{aligned}$$

·2. Thus we can interchange any two elements of a set, with a possible change of sign of one of them, for this interchange can be brought about by a succession of interchanges of neighbouring elements.

·3. We can change $(\dots p, \dots, q \dots)$ to $(\dots \mathrm{k}p, \dots, q/\mathrm{k} \dots)$, where k is any non-zero scalar, and $\mathrm{k}p$, q/k occupy the same places in the second bracket as p, q do in the first.

For, first, if p, q be adjacent, we can secure

$$\begin{aligned}(\dots p, q, \dots) &\rightarrow (\dots p, q+(\mathrm{k}-1)p, \dots) \\ &\rightarrow (\dots p+q+(\mathrm{k}-1)p, q+(\mathrm{k}-1)p, \dots) \\ &= (\dots \mathrm{k}p+q, q+(\mathrm{k}-1)p, \dots) \\ &\rightarrow (\dots \mathrm{k}p+q, q+(\mathrm{k}-1)p-\mathrm{k}^{-1}(\mathrm{k}-1)(\mathrm{k}p+q), \dots) \\ &= (\dots \mathrm{k}p+q, q/\mathrm{k}, \dots) \rightarrow (\dots \mathrm{k}p+q-q, q/\mathrm{k}, \dots) \\ &= (\dots \mathrm{k}p, q/\mathrm{k}, \dots).\end{aligned}$$

If p, q be not neighbours, we can secure

$$\begin{aligned}(\dots p, r, s, \dots, q, \dots) &\rightarrow (\dots \mathrm{k}p, \mathrm{k}^{-1}r, s, \dots, q, \dots) \\ &\rightarrow (\dots \mathrm{k}p, r, \mathrm{k}^{-1}s, \dots, q, \dots),\end{aligned}$$

and so on.

·4. If the set $(a, b, c, \dots)$ is the same as the set $(a', b', c', \dots)$ apart from arrangement, then we can change $(a, b, c, \dots)$ to $(\pm a', b', c', \dots)$ by a series of elementary transformations, where $+$ or $-$ is taken according as $[abc\dots] = \pm [a'b'c'\dots]$. This follows by ·2.

·5. The change from $(\ldots p, \ldots, q, \ldots)$ to $(\ldots p+\mathrm{k}q, \ldots, q, \ldots)$ or to $(\ldots p, \ldots, q+\mathrm{k}p, \ldots)$ (the terms written out being in the same positions in all brackets) can be brought about by a series of elementary transformations. This follows by 12 and ·4, since the outer products of the elements in each set are equal.

We can now shew the theorem. If

$$[abc\ldots lp] = [a_1 b_1 c_1 \ldots l_1 p_1] \neq 0,$$

then, by 11, $a_1, b_1, \ldots, p_1$ are linear combinations of $a, b, \ldots, l, p$. At least one combination has a non-zero coefficient of a; otherwise if r is the number of extensives in set $a, b, \ldots, p$, then $\mathscr{S}(a_1, b_1, \ldots, p_1)$ would be of step $\mathrm{r}-1$, and the outer product would vanish. Let a' be one of $a_1, b_1, \ldots$ with

$$a' = \mathrm{k}_1 a + \mathrm{x}_1 b + \ldots, \quad (\mathrm{k}_1 \neq 0).$$

Then, by elementary transformations, we can secure

$$(a, b, c, \ldots, p) \to (\mathrm{k}_1 a, b, c, \ldots, \mathrm{k}_1^{-1} p)$$
$$\to ((\mathrm{k}_1 a + \mathrm{x}_1 b + \ldots), b, c, \ldots, \mathrm{k}_1^{-1} p) = (a', b, c, \ldots, \mathrm{k}_1^{-1} p).$$

Thence, using 11,

$$\mathscr{S}(a_1, b_1, c_1, \ldots, p_1) = \mathscr{S}(a, b, c, \ldots, p) = \mathscr{S}(a', b, c, \ldots, \mathrm{k}_1^{-1} p).$$

Thus $a_1, b_1, c_1, \ldots, p_1$ are linear combinations of $a', b, c, \ldots, \mathrm{k}_1^{-1} p$, and at least one combination, say b', has a non-zero coefficient of b.

Suppose $\quad b' = \mathrm{y}_1 a' + \mathrm{k}_2 b + \ldots, \quad (\mathrm{k}_2 \neq 0).$

Then $b' \neq a'$, and by elementary transformations we can secure, as before, $\quad (a', b, c, \ldots, \mathrm{k}_1^{-1} p) \to (a', b', c, \ldots, \mathrm{k}_1^{-1} \mathrm{k}_2^{-1} p).$

So proceeding, we find that, by elementary transformations, we can secure

$$(a, b, c, \ldots, l, p) \to (a', b', c', \ldots, l', \mathrm{k}_1^{-1} \mathrm{k}_2^{-1} \ldots \mathrm{k}_{\mathrm{r}-1}^{-1} p), \quad \text{(i)}$$

where $a', b', \ldots, l'$ are $\mathrm{r}-1$ of $a_1, b_1, \ldots, p_1$ in some order. The outer products of the sets in (i) must be equal, and since $[abc\ldots p]$ is not zero, $a', b', \ldots, l'$ must be distinct elements of the set $(a_1, b_1, \ldots, l_1, p_1)$. Let p' be the remaining element of the set, and let

$$p = \mathrm{k}_1' a' + \mathrm{k}_2' b' + \ldots + \mathrm{k}' p'.$$

Then $\quad \mathrm{k}_1 \mathrm{k}_2 \ldots \mathrm{k}_{\mathrm{r}-1} [ab\ldots lp] = \mathrm{k}' [a'b'\ldots l'p']$, by (i).

Also $\quad [a'b'\ldots p'] = \pm [a_1 b_1 \ldots p_1] = \pm [ab\ldots p].$

Hence in (i) we may put $\pm \mathrm{k}'^{-1}$ for $\mathrm{k}_1^{-1} \mathrm{k}_2^{-1} \ldots \mathrm{k}_{\mathrm{r}-1}^{-1}$.

Now, by ·5, we can change $(a', b', \ldots, l', k'^{-1}p)$ by elementary transformations to $(a', b', c', \ldots, p')$, and as the latter set is $(a_1, b_1, \ldots, p_1)$ in some order, we can change it to $(a_1, b_1, \ldots, \pm p_1)$.

Hence, by elementary transformations, we can proceed from $(a, b, \ldots, p)$ to $(a_1, b_1, \ldots, \pm p_1)$, and since $[ab\ldots p] = [a_1 b_1 \ldots p_1]$, the sign before p_1 in the last set is positive.

Thus the theorem is proved.

§ 55. *Outer multiplication of spreads of higher step.**

1. *Def.* If $m < r \leqq n$, where n is the step of the basic spread, the '*outer product*' of $[a_1 a_2 \ldots a_m]$ and $[a_{m+1} \ldots a_r]$ is $[a_1 \ldots a_r]$.

2. *Def.* A '*simple extensive*' is one which can be expressed as the outer product of extensives of step one. A '*compound extensive*' is one which can only be expressed as a *sum* of such products.

For example, $[a_1 \ldots a_r]$ is a simple extensive.

(In ordinary space, a rotor is a simple extensive of step two, a general screw a compound extensive.)

3. *If* $A = [a_1 \ldots a_m] \neq 0$, $B = [b_1 \ldots b_r]$, *and all* $a_1, \ldots, a_m$ *are in* $\mathscr{S}(b_1, \ldots, b_r)$, *then there is a simple extensive* C *such that* $B = [AC]$.

For $a_1, \ldots, a_m$ are linear combinations of $b_1, \ldots, b_r$ and are independent. Thence we can adjoin $a_{m+1}, \ldots, a_r$ so that

$$[a_1 \ldots a_m a_{m+1} \ldots a_r] = [b_1 \ldots b_r].$$

Then take $$C = [a_{m+1} \ldots a_r].$$

Cor. If A, B are simple extensives of steps s_1, s_2 and $s_1 + s_2 = n + r$, then A, B have a common spread C of step r at least.

Thence there are simple extensives A_1, B_1 of steps $s_1 - r$, $s_2 - r$ respectively, such that $A = [CA_1]$, $B = [CB_1]$.

4. *If the sum of the steps of* A, B, C *be not greater than* n, *then*

$$[A.BC] = [ABC].$$

(The right-hand side, of course, means $[AB.C]$ (§ 13·1).)

For simple extensives, this follows from definitions, thence for compound extensives by the distributive law.

* A_2, §§ 78–85.

5. *If* $[AB]=0$, *and* $a_1, \ldots, a_m, b_1, \ldots, b_k$ *be independent, and* A *be of step* r_1 *in* $\mathscr{S}(a_1, \ldots, a_m)$ *and* B *of step* r_2 *in* $\mathscr{S}(b_1, \ldots, b_k)$, *where* $r_1 \leqq m$, $r_2 \leqq k$, *then* $A=0$ *or* $B=0$.

For $[AB]$ is the sum of certain weighted products of the a, b of step r_1+r_2; these products are independent (§ 54·10); hence as $[AB]=0$, their coefficients vanish. But these coefficients are of form $x_i y_j$, where x_i, y_j traverse all coefficients in the expressions for A, B respectively. If $B \neq 0$, then some y is not zero, hence all the x vanish, and $A=0$.

6. *If* S *be any extensive and* $[aS]=0$, $a \neq 0$, *then there is an extensive* P, *such that* $S=[aP]$.

For adjoin $a_2, \ldots, a_n$ to a such that $\mathscr{S}(a, a_2, \ldots, a_n)$ is our basic spread, then $S = k_1[aB_1]+k_2[aB_2]+\ldots+l_1 C_1+l_2 C_2+\ldots$, where the C do not contain a, and $[aB_1]$, $[aB_2]$, $\ldots$, C_1, C_2, $\ldots$ are independent.

Since $[aS]=0$, we have $l_1[aC_1]+l_2[aC_2]+\ldots=0$.

But the $[aC]$ are independent, hence $l_1=l_2=\ldots=0$.

Hence $S=[aP]$, where $P=k_1 B_1+k_2 B_2+\ldots$.

Cor. 1. *If* S *be any extensive, and*

$$[a_1 S]=[a_2 S]=\ldots=[a_m S]=0,$$

where $a_1, \ldots, a_m$ *are independent, then there is an extensive* S_m *such that* $S=[a_1 a_2 \ldots a_m S_m]$. *If* S *is of step* m, *then*

$$S \equiv [a_1 a_2 \ldots a_m].$$

Cor. 2. *If* S *is of step* m, *and* $[a_1 S]=[a_2 S]=\ldots=[a_{m+1} S]=0$, *then* $S=0$, *or* $[a_1 a_2 \ldots a_{m+1}]=0$.

§ 56. *Supplements and regressive products.**

1. As before, we work in a spread of step n, but we now assume it is spanned by extensives $e_1, \ldots, e_n$ *and that* $[e_1 e_2 \ldots e_n]=1$.

These e will be called '*basic unities*'; an outer product of r of them will be called a '*unity of step* r'.

2. *Def.* If E be any unity of any step, we define the '*supplement*' of E as $|E=[EE']\,E'$, where the factors of E' are all the $e_1, \ldots, e_n$ which do not appear in E, each taken just once.

* A_2, §§ 86–126.

The supplement of a scalar is defined as itself.

Thus

$$|e_1 = [e_2 \dots e_n], \quad |e_2 = -[e_1 e_3 \dots e_n], \quad |e_3 = [e_1 e_2 e_4 \dots e_n],$$
$$[e_i | e_i] = 1, \quad [e_i | e_j] = 0, \quad (i, j = 1, \dots, n, i \neq j).$$

3. We note that $[EE'] = \pm 1$, the sign depending on the order of the factors e; if the factors of E' are rearranged, then the signs of $[EE']$ and E' are either both preserved or both changed. We have $$[E|E] = [EE'][EE'] = 1.$$

If E, F are distinct unities of the same step, $[E|F] = 0$.

If E is of step r, then $|E$ is of step $n - r$.

4. *Def.* If $A = k_1 E_1 + k_2 E_2 + \dots + k_r E_r$, where $E_1, \dots, E_r$ are unities of the same step, then $|A = k_1 |E_1 + \dots + k_r |E_r$. (The distributive law for $|$.)

Hence if

$$A = k_1 E_1 + k_2 E_2 + \dots + k_r E_r, \quad B = k'_1 E_1 + k'_2 E_2 + \dots + k'_r E_r,$$

then
$$[A|B] = k_1 k'_1 + k_2 k'_2 + \dots + k_r k'_r,$$
$$[A|B] = [B|A].$$

5. *If A is of step* r, *then* $||A = (-1)^{r(n-r)} A$; *thus* $||A = A$ *or* $(-1)^r A$ *according as* n *is odd or even.*

[This explains why for vectors in a plane ($n = 2$), $||v = -v$, while in space $||v = v$.]

By the distributive law, it suffices to shew the first part for the unities E. Let $F = |E$, then $[EF] = 1$, $[FE] = (-1)^{r(n-r)}$,

$$|F = [FE]\, E = (-1)^{r(n-r)} E.$$

Hence $||E = (-1)^{r(n-r)} E$. Hence the first part.

The second part follows, since if n be odd, then one of r, $n - r$ is even; but if n be even, then $r(n - r)$ and r have the same parity.

6. If E_1, E_2 be unities of steps r_1, r_2, and $r_1 + r_2 > n$, then their outer product as already defined is zero; we define their '*regressive outer product*', denoted by $[E_1 E_2]$, as follows:

Def. If E_1, E_2 be unities of steps r_1, r_2, and $r_1 + r_2 > n$, then $[E_1 E_2]$ is such that $$|[E_1 E_2] = [|E_1 |E_2].$$

The earlier outer products will now be called '*progressive outer products*'.

Since $|E_1$, $|E_2$ are of steps $n-r_1$, $n-r_2$ whose sum is less than n, therefore $[|E_1|E_2]$ is a progressive product, and our definition is significant. The step of $[|E_1|E_2]$ is $2n-r_1-r_2$ (the extensive itself may be zero). The step of $[E_1E_2]$ is hence $(r_1+r_2)-n$. Thus we have 7.

7. *If A, B be any extensives of steps* r_1, r_2, *the step of* $[AB]$ *is* r_1+r_2 *or* $(r_1+r_2)-n$, *whichever of these two is not greater than* n.

8. *If A, B be any extensives of steps* r_1, r_2, *then*

$$|[AB]=[|A.|B].$$

(1) For, if $r_1+r_2>n$, this is true for unities, by definition, and hence generally, by the distributive law.

(2) If $r_1+r_2=n$, it is again sufficient to prove the theorem for unities.

(2·1) If E_1, E_2 contain a common basic unity e, there must be a basic unity which is in neither E_1 nor E_2, and hence which is in both $|E_1$ and $|E_2$. Now the sum of the steps of $|E_1$ and $|E_2$ is n. Thus $[E_1E_2]$ and $[|E_1|E_2]$ both vanish, and hence are equal.

(2·2) If E_1, E_2 contain no common basic unity, then

$$|E_1=[E_1E_2]\,E_2, \quad |E_2=[E_2E_1]\,E_1, \quad [E_1E_2]=\pm 1,$$

hence
$$[|E_1|E_2]=[E_1E_2]=|[E_1E_2].$$

(3) Lastly, if $r_1+r_2<n$, let $|A=A'$, $|B=B'$, then, by 5,

$$|A'=(-1)^{r_1(n-r_1)}\,A, \quad |B'=(-1)^{r_2(n-r_2)}\,B.$$

The sum of the steps of A', B' is greater than n, hence

$$|[A'B']=[|A'.|B']=(-1)^{r_1(n-r_1)+r_2(n-r_2)}\,[AB].$$

The step of $[A'B']$ is $n-r_1+n-r_2-n=n-r_1-r_2$, hence, by 5,
$$||[A'B']=(-1)^{(n-r_1-r_2)(r_1+r_2)}\,[A'B'].$$
Hence

$$(-1)^{r_1(n-r_1)+r_2(n-r_2)}\,|[AB]=(-1)^{(n-r_1-r_2)(r_1+r_2)}\,[|A|B],$$

and since

$$r_1(n-r_1)+r_2(n-r_2)-(n-r_1-r_2)\,(r_1+r_2)=2r_1r_2,$$

this gives
$$|[AB]=[|A|B].$$

9. *If E, F, G be unities, and the sum of their steps is* n, *then*

$$[EF.EG] = [EFG]\,E.$$

(1) If F, G contain a common basic unity, then $[EFG] = 0$, and there is at least one unity, e, say, not in any of E, F, or G; if $[EF] = |Q$, $[EG] = |R$, then e is in Q and R, hence

$$[QR] = 0, \quad [EF.EG] = [|Q.|R] = |[QR] = 0.$$

(2) If no two of E, F, G contain any common basic unity, the set contains each basic unity just once; then by 2,

$$[GEF] = \pm 1, \quad |G = [GEF]\,[EF],$$
$$|F = [FEG]\,[EG], \quad |[GF] = [GFE]\,E.$$

Hence $$[EF] = [GEF]\,|G, \quad [EG] = [FEG]\,|F,$$

$$\begin{aligned}[EF.EG] &= [GEF]\,[FEG]\,[|G|F] = [GEF]\,[FEG]\,|[GF] \\ &= [GEF]\,[FEG]\,[GFE]\,E \\ &= [GEF]\,[EFG]\,[GEF]\,E, \quad \text{(since } [EF] = \pm [FE]\text{)}, \\ &= [EFG]\,E.\end{aligned}$$

10. *Rule of a repeated factor. If A, B, C be any simple extensives, the sum of whose steps is* n, *then*

$$[AB.AC] = [ABC]\,A.$$

For let $a_1, \ldots, a_n$ be any extensives of step one, not necessarily distinct or independent; first assume the theorem is true when A, B, C have these as simple factors, arranged in any way; we shew that then it holds when a_1 is replaced by

$$a' = k_1 a_1 + \ldots + k_n a_n,$$

where $k_1, \ldots, k_n$ are any scalars.

(1) If a_1 is in A, let $A = [a_1 D]$. First change a_1 to $k_1 a_1 + k_2 a_2$, then A becomes

$$A' = k_1[a_1 D] + k_2[a_2 D] = k_1 A + k_2[a_2 D].$$

(1·1) If a_2 is in D, then $[a_2 D] = 0$, and

$$\begin{aligned}[A'B.A'C] &= k_1^2[AB.AC] = k_1^2[ABC]\,A \\ &= [k_1 ABC]\,k_1 A = [A'BC]\,A'.\end{aligned}$$

(1·2) If a_2 is in B or C, say in B, let $B = [a_2 G]$, then $[a_2 DB] = 0$,

$$[A'B] = k_1[AB], \quad [A'C] = k_1[AC] + k_2[a_2 DC],$$
$$[A'B.A'C] = k_1^2[AB.AC] + k_1 k_2[AB.a_2 DC].$$

But $[AB.a_2DC] = [a_1Da_2G.a_2DC] = -[a_2Da_1G.a_2DC]$
$= -[a_2Da_1GC]\,[a_2D]$, by hypothesis,

and $-[a_2Da_1GC] = [a_1Da_2GC] = [ABC]$ (§ 55·4).

Hence $[A'B.A'C] = k_1^2[ABC]\,A + k_1k_2[ABC]\,[a_2D]$
$= k_1[ABC]\,(k_1A + k_2[a_2D])$
$= k_1[ABC]\,A' = [A'BC]\,A'$.

Hence, if a_1 is in A, the hypothesis continues to hold when a_1 is changed to $k_1a_1 + k_2a_2$, and hence when it is changed to $(k_1a_1 + k_2a_2) + k_3a_3$, and so on.

(2) If a_1 is in B, and a_1 be replaced by $a' = k_1a_1 + \ldots + k_na_n$, let $B = a_1D$, $B' = a'D$, then

$$[AB'.AC] = [A\,\Sigma k_ia_iD.AC] = \Sigma k_i[Aa_iD.AC]$$
$$= \Sigma k_i[Aa_iDC]\,A, \text{ by hypothesis.}$$

But (§ 55·4)

$$\Sigma k_i[Aa_iDC]\,A = [Aa'DC]\,A = [AB'C]\,A.$$

(3) If a_1 is in C, the proof is similar to that of the last case.

Hence in all cases, $a_1, \ldots, a_n$ may be replaced by $b_1, \ldots, b_n$, if these are linear combinations of them. For of these b a certain number (perhaps all) will be independent, and the rest dependent on them. Suppose $b_1, \ldots, b_m$ independent, and $b_{m+1}, \ldots, b_n$ linear combinations of them. Then we can select m of the a, so that $[a_1 \ldots a_m] \equiv [b_1 \ldots b_m]$.

Now $\mathscr{S}(a_1, \ldots, a_n) = \mathscr{S}(b_1, a_2, \ldots, a_n) = \mathscr{S}(b_1, b_2, a_3, \ldots, a_n)$
$= \ldots = \mathscr{S}(b_1, \ldots, b_m, a_{m+1}, \ldots, a_n)$.

Thus the replacements corresponding to these equations can be made, and then finally we can replace $b_1, \ldots, b_m, a_{m+1}, \ldots, a_n$ by $b_1, \ldots, b_n$.

Now the theorem holds by 9, when $a_1, \ldots, a_n$ are any basic unities. Hence it holds generally.

11. *Def.* A spread S_1 is a '*subspread*' of a spread S_2, when each extensive of step one, contained in S_1, is also contained in S_2.

Then (§ 55·6, Cor. 1), there is an extensive S_3 with $S_2 = [S_1S_3]$.

12. *If A be simple, B, C simple or compound, and if the sum of the steps of A, B, C be* n *or* 2n, *then* $[AB.AC] = [ABC]\,A$.

For, when the sum of steps is n, this follows from 10, and the distributive law. When the sum is 2n, let A', B', C' be the

supplements of A, B, C. The sum of the steps of A', B', C' is n, hence $[A'B'.A'C'] = [A'B'C']\,A'$. Taking supplements of both sides, we have the equation stated.

Cor. Under the same conditions, $[AB.BC] = [ABC]\,B$, $[AC.BC] = [ABC]\,C$.

Note. The theorem is not, in general, true, if A is compound. For example, in step four, let $A = [ad] + [bc]$, $[abcd] \neq 0$, then

$$[Ab.Ac] = [adb.adc] = [adbc]\,[ad],$$
$$[Abc]\,A = [adbc]\,([ad] + [bc]).$$

13. *If A, C be simple, and the sum of their steps is* n, *and B be a subspread of A, and hence simple, then**

$$[A.BC] = [AC]\,B, \quad [CB.A] = [CA]\,B.$$

For there is an extensive D, with $A = [BD]$.

Hence $[A.BC] = [BD.BC] = [BDC]\,B = [AC]\,B$.

The same result follows if A is a subspread of B.

14. *If Q, R be simple extensives, not zero, of steps* q, r, *and*

(1) $q + r \leqq n$, *then* $[QR] \neq 0$, *if, and only if,* $q + r$ *is the step of the join of Q, R,*

(2) $q + r > n$, *then* $[QR] \neq 0$, *if, and only if,* $q + r - n$ *is the step of the cut of Q, R.*

For (1). In the first case, the product is progressive, and hence vanishes if, and only if, the factors of step one in Q, R are dependent, and then one is a linear combination of the rest. Hence if $[QR] = 0$, the join of Q, R has a step less than $q + r$.

For (2). If $q + r = n + p$, $p > 0$, then Q, R have common an extensive of step p at least; hence there is an extensive A of step p, and extensives B, C such that $Q = [AB]$, $R = [AC]$, the sum of the steps of A, B, C being n.

Hence $[QR] = [AB.AC] = [ABC]\,A = [QC]\,A$.

Hence $[QR]$ can vanish, if and only if $[QC] = 0$, and as the sum of the steps of Q and C is n, this only happens if the step of the join of Q and C is less than n; but A is a subspread of Q; hence the join of Q and $[AC]$, that is, of Q and R, has a step less than n; thus the cut of Q, R has a step greater than $q + r - n$.

* For brevity we shall often speak of the spread A, where A is a product, meaning thereby the spread represented by A.

15. *Generalisation of theory.* The above theory holds if, instead of beginning with $e_1, \ldots, e_n$, we begin with any n independent extensives $a_1, \ldots, a_n$ of step one, such that $[a_1 \ldots a_n] = 1$, and define supplement as follows, (where $\ddagger$ is the sign of the supplement):

$[\ddagger A] = [AA']\, A'$, where A is any product of a set of the a, and A' the product of the remaining a in any order.

We assume the distributive law, and take the supplement of a scalar to be the scalar itself.

For the theory will be shewn when we have proved $\ddagger[AB] = [\ddagger A . \ddagger B]$, where A, B are products of simple factors, which are elements of the set $a_1, \ldots, a_n$, and the sum of the steps of A and B is greater than n. This is easily shewn from the theorems above, in particular from 12 Cor.

Thus, in particular, 12 is independent of the elements used in defining regressive multiplication. But $|A$ and $\ddagger A$ need not be equal.

16. *If* $[PP'] = [AA'] = [BB'] = \ldots = [a_1 \ldots a_n] = 1$, *and if* P, P', $A, A', B, B', \ldots$ *are products of simple factors, these being elements of the set* $a_1, \ldots, a_n$, *and if* $P = [ABC \ldots]$, *then* $P' = [A'B'C' \ldots]$.

For define supplements by means of the a, then

$$\ddagger P = [PP']\, P' = P', \text{ and so on.}$$

$$\ddagger P = [\ddagger A . \ddagger B \ldots] = [A'B' \ldots].$$

17. *If* $\beta_r = [a_1 \ldots a_{r-1} a_{r+1} \ldots a_n]$, $[a_1 \ldots a_n] = K$,

then $[\beta_n \beta_{n-1} \ldots \beta_{m+1}] = [a_1 \ldots a_m]\, K^{n-m-1} \quad (m < n)$.

For
$$\begin{aligned}[a_1 \ldots a_r \beta_r] &= [a_1 \ldots a_r (a_1 \ldots a_{r-1} a_{r+1} \ldots a_n)] \\ &= [a_1 \ldots a_r a_{r+1} \ldots a_n]\, [a_1 \ldots a_{r-1}], \text{ by 13,} \\ &= [a_1 \ldots a_{r-1}]\, K.\end{aligned}$$

Hence $[\beta_n \beta_{n-1}] = [a_1 \ldots a_{n-1} \beta_{n-1}] = [a_1 \ldots a_{n-2}]\, K$,

$$[\beta_n \beta_{n-1} \beta_{n-2}] = [a_1 \ldots a_{n-2} \beta_{n-2}]\, K = [a_1 \ldots a_{n-3}]\, K^2,$$

and so on, to

$$[\beta_n \beta_{n-1} \ldots \beta_2] = a_1 K^{n-2},$$

$$[\beta_n \beta_{n-1} \ldots \beta_1] = K^{n-1}.$$

Cor. $[\beta_1 \ldots \beta_r] = [a_{r+1} \ldots a_n]\, K^{r-1}$.

18. *Def.* A product of extensives which involves only progressive or only regressive multiplication of the factors is '*pure*', the product being taken in accordance with § 13·1.

A product $[ABC...L]$ of step zero, in which AB, $AB.C$, $ABC.D$, ... are regressive, and the multiplication by L is the only progressive multiplication, will also be called pure regressive. An example of this is $[ab.ac.bc]$ in step three.

A product of two factors is always pure.

A product which is not pure is '*mixed*'.

A product of m extensives is thus pure progressive if the sum of their steps is not greater than n; it is pure regressive if that sum is not less than $n(m-1)$, for then the supplements of the factors will have the sum of steps not greater than

$$nm - n(m-1) = n,$$

and hence the product of the supplements will be progressive.

Pure progressive products, if not vanishing, represent the join of the spreads represented by their factors; pure regressive products, if not vanishing, represent the common cut of the spreads represented by their factors.

19. *Pure products obey the associative law, and permutation of factors at most changes the sign of the product.*

For their factors, if simple, can be resolved into factors of step one or $n-1$, according as the products are progressive or regressive. For factors which are compound, the result now follows by the distributive law.

20. *A pure progressive product* $[ABC...]$ *vanishes if, and only if, at least one pair of the spreads* A, B, C, ... *has a cut of step greater than zero; a pure regressive product* $[ABC...]$ *vanishes if, and only if, at least one pair of spreads* A, B, C, ... *has a join of step less than* n.

21. *If* $[ABC]$ *is a mixed product, and* A, B, C *are simple, not zero, then* $[ABC]$ *can only vanish, if either* $[AB]=0$, *or* A, B, C *have a join of step less than* n, *or a cut of step greater than zero.*

The last two cases arise when $[AB]$ is a progressive and a regressive product respectively.

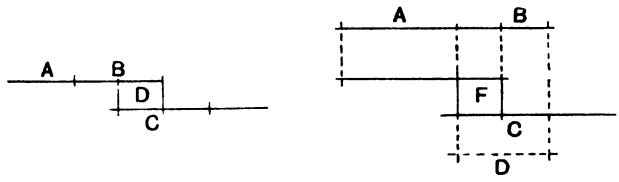
22. *Conditions that* $[ABC] \equiv [ACB]$, *where* A, B, C *are simple and not zero.*

This is the case (1) for pure products and (2) when both sides vanish, which happens when $[AB]$ and $[AC]$ vanish, or when the products are mixed and the spreads represented by A, B, C have a join of step less than n or a cut of step greater than zero.

The case remains (3) when the product is mixed and not zero; let then p, q, r be the steps of A, B, C, then

$$n < p+q+r < 2n.$$

If $p+q<n$, $p+r<n$, then $[AB]$, $[AC]$ are progressive, $[AB.C]$ is regressive. Now since $[AB] \neq 0$, therefore A, B have



no common spread; neither have A, C, but $[AB]$ and C have a common spread, of step $p+q+r-n$, and hence they have a common factor D, say, which must be a subspread of B and C of step $p+q+r-n$. Then $[AB.C] \equiv D$, $[AC.B] \equiv D$.

If $p+q>n$, $p+r>n$, take supplements; we find that B, C have a join of step $p+q+r-n$.

If $p+q \leqq n$, $p+r \geqq n$, then $[AB]$, C have a common spread D and $[ABC] \equiv D$. Also A, C have a common spread F, $[AC] \equiv F$, $[ACB] \equiv [FB]$. Hence, if $[ABC] \equiv [ACB]$, then $[FB] \equiv D$; thus B is a subspread of D, and hence of C. (Actually in this case, $[ABC]$, $[ACB]$ are equal and not merely congruent.)

Hence, *if* $[ABC]$ *is a mixed product, and not zero, then* $[ABC] \equiv [ACB]$, *if, and only if,* B, C *have a cut or join of step* $p+q+r-n$, *or one of* B, C *is contained in the other.*

23. *The conditions for the weak associative law* $[BAC] \equiv [B.AC]$ *for simple extensives* A, B, C *are the same as in* 22.

(The law is 'weak' because it concerns congruence and not equality.)

For, since $[BA] \equiv [AB]$, therefore if $[ABC] \equiv [ACB]$, (22) then

$$[BAC] \equiv [ABC] \equiv [AC.B] \equiv [B.AC];$$

the last step follows, by 19, since the last two expressions are products of two factors $[AC]$ and B only, and such a product is always pure. Conversely, (i) follows from $[BAC] \equiv [B.AC]$.

24. *If* $[A_1 A_2 \dots A_n]$ *is of step zero, then*

$$[A_1 A_2 \dots A_n] \equiv [A_1 . A_n A_{n-1} \dots A_2] \equiv [A_1 A_2 \dots A_{n-r-1} . A_n A_{n-1} \dots A_{n-r}].$$

For, let $$[A_1 A_2 \dots A_{n-2}] = P,$$

then $$[A_1 A_2 \dots A_n] = [P A_{n-1} A_n].$$

The step of the last product is zero, hence the sum of the steps of P, A_{n-1}, A_n is divisible by n, and hence is n or 2n; thus the product of P, A_{n-1}, A_n is pure by 18. Hence, by 19,

$$[P A_{n-1} A_n] \equiv [P A_n A_{n-1}] \equiv [P . A_n A_{n-1}],$$

$$[A_1 A_2 \dots A_n] \equiv [A_1 A_2 \dots A_{n-2} . A_n A_{n-1}].$$

Considering the last expression as the product of $[A_1 A_2 \dots A_{n-3}]$, A_{n-2}, and $[A_n A_{n-1}]$, we find similarly that it is congruent to $[A_1 A_2 \dots A_{n-3} . A_n A_{n-1} A_{n-2}]$, and so on.

Def. If $a_1, a_2, \dots, a_n$ be any extensives, their '*multiplicative combinations*' of step r are the outer products of r of them.

25. *If* $a_1, \dots, a_n$ *be independent extensives of step one, and*

$$K = [a_1 a_2 \dots a_n] \neq 0, \quad B = [b_1 \dots b_r] \neq 0,$$

and $A_1, \dots, A_s$ *be the* $s = \binom{n}{r}$ *multiplicative combinations of* $a_1, \dots, a_n$ *of step* r, *and* $D_1, \dots, D_s$ *be such that* $[A_i D_i] = K$, *then*

$$KB = \sum_{i=1}^{i=s} [BD_i] A_i.$$

For $B = k_1 A_1 + \dots + k_s A_s$ for some k.

Hence $[BD_i] = \sum_{j=1}^{s} k_j [A_j D_i]$. But $[A_j D_i] = 0$, if $j \neq i$.

Hence $[BD_i] = k_i [A_i D_i] = k_i K$, $KB = \sum_{i=1}^{i=s} [BD_i] A_i$.

Cor. If E_i be such that

$$[E_i A_i] = [a_1 \dots a_n],$$

then
$$KB = \Sigma [E_i B]\, A_i.$$

Def. The scalars $K^{-1}[BD_i]$ are the '*coordinates*' of B in the frame $A_1, \dots, A_s$.

26. *If A be of step* p, *and* $B = [b_1 \dots b_q] \neq 0$ *be of step* q, *and* $p+q = n+r$, $(r < p, q)$, *and if* $C_1, C_2, \dots, C_s$ *be all the multiplicative combinations of* r *of the* b, *and* D_k *be the product of the* $b_1, \dots, b_q$ *not in* C_k, *the factors being arranged so that* $[C_k D_k] = B$, *then* $[AB] = \sum_1^s [AD_k]\, C_k$.

For, since $b_1, \dots, b_q$ are independent, we can adjoin $b_{q+1}, \dots, b_n$ so that $\mathscr{S}(b_1, \dots, b_n)$ is the fundamental spread. Now A is a linear combination of the multiplicative combinations of $b_1, \dots, b_n$ of step p.

Let $A = \Sigma x_i F_i + \Sigma y_j G_j$, where F_i, G_j are simple extensives of step p, and F_i contains just r of the factors $b_1, \dots, b_q$, namely those in C_i, while G_j contains more than r of these, and hence has in common with each D_i (which contains $q-r$ of these b) at least one of $b_1, \dots, b_q$.

Thus $[G_j D_i] = 0$, for each i, j. As B has in common with each G_j a spread of step greater than r, we have $[G_j B] = 0$ for each j, by 14.

Hence
$$[AB] = \Sigma x_i [F_i B] = \Sigma x_i [F_i . C_i D_i].$$

Now F_i contains all the factors of the set $b_1, \dots, b_q$ which appear in C_i; hence C_i is a subspread of F_i. Also if $i \neq k$, then $[F_i D_k] = 0$, since then $[C_i D_k] = 0$.

Hence, by 13,

$$[F_i . C_i D_i] = [F_i D_i]\, C_i = \sum_k [F_i D_k]\, C_k,$$

$$[AB] = \sum_{ik} x_i [F_i D_k]\, C_k = \sum_{ijk} [(x_i F_i + y_j G_j) D_k]\, C_k = \sum_k [AD_k]\, C_k.$$

27. *If* $A = [a_1 \dots a_r] \neq 0$, $B = [\beta_1 \dots \beta_s]$, *where* $\beta_1, \dots, \beta_s$ *are of step* $n-1$, *and* $r > s$, *then* B *is of step* $n-s$, *and*

$$[AB] = [A_1 B]\, D_1 + \dots + [A_p B]\, D_p,$$

where $A_1, \ldots, A_p$ *are all the* $p = \binom{r}{s}$ *multiplicative combinations of* $a_1, \ldots, a_r$ *of step* s, *and* $D_1, \ldots, D_p$ *are such that*

$$[A_1 D_1] = [A_2 D_2] = \ldots = A.$$

For adjoin $a_{r+1}, \ldots, a_n$ so that $a_1, \ldots, a_n$ span the fundamental spread; and form all multiplicative combinations $B_1, \ldots, B_t$ of $a_1, \ldots, a_n$ of step $n-s$. Then there are scalars k such that

$$B = k_1 B_1 + k_2 B_2 + \ldots + k_t B_t. \qquad (1)$$

A B_k has at least $r-s$ factors of step one common with A. Let $B_1, \ldots, B_p$, where $p = \binom{r}{s}$, be those B which have just $r-s$ such factors. Then $[AB_i] = 0$, if $i > p$, by 14. Hence

$$[AB] = k_1[AB_1] + \ldots + k_p[AB_p].$$

With A_1, D_i as in hypothesis, we have $B_i = [D_i C_i]$, where C_i consists of those $n-r$ factors of B_i which do not appear in A. Then, using 12, Cor.,

$$[AB_i] = [A_i D_i . D_i C_i] = [A_i D_i C_i]\, D_i = [AC_i]\, D_i,$$

$$[AB] = \sum_{i=1}^{p} k_i [AC_i]\, D_i. \qquad (2)$$

Multiply (1) by A_i. The sum of the steps of A_i, B_k is n, hence the only one of the B_k which gives a value not zero is that which contains only factors absent from A_i, and this is $B_i = [D_i C_i]$.

Hence $$[A_i B] = k_i [A_i D_i C_i] = k_i [AC_i].$$

Then, by (2), $$[AB] = \sum_{i=1}^{p} [A_i B]\, D_i.$$

28. *If* $A = [a_1 \ldots a_r] \neq 0$, $C = [\gamma_1 \ldots \gamma_r]$, *where the* γ *are of step* $n-1$, *then*

$$[AC] = [a_1 \ldots a_r . \gamma_1 \ldots \gamma_r] = \begin{vmatrix} [a_1\gamma_1], & [a_1\gamma_2], & \ldots, & [a_1\gamma_r] \\ [a_2\gamma_1], & [a_2\gamma_2], & \ldots, & [a_2\gamma_r] \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ [a_r\gamma_1], & [a_r\gamma_2], & \ldots, & [a_r\gamma_r] \end{vmatrix}.$$

For, take the outer products β_i of $a_1, \ldots, a_n$ of step $n-1$, such that $[a_i \beta_i] = [a_1 \ldots a_n]$, then for some scalars k_{ij} we have

$$\gamma_i = \sum_j k_{ij} \beta_j. \qquad (3)$$

$[\gamma_1 \dots \gamma_r] = \det |k_{ij}| \, [\beta_1 \dots \beta_r]$ + terms involving other β than $\beta_1, \dots, \beta_r$, where $\det |k_{ij}|$ means the determinant with elements k_{ij}, as i, j traverse 1, ..., r.

Hence
$$[AC] = [a_1 \dots a_r . \gamma_1 \dots \gamma_r] = \det |k_{ij}| \, [a_1 \dots a_r . \beta_1 \dots \beta_r]. \quad (4)$$

By (3),
$$[a_s \gamma_i] = \sum_j k_{ij} [a_s \beta_j] = k_{is} [a_s \beta_s] = k_{is} [a_1 \dots a_n].$$

Put $[a_1 \dots a_n] = K$.

Hence
$$k_{ij} = [a_j \gamma_i] \div [a_1 \dots a_n] = [a_j \gamma_i] \div K.$$

Substitute this in (4), and we have the determinant of the theorem, multiplied by $[a_1 \dots a_r . \beta_1 \dots \beta_r]$ and divided by K^r.

But the two latter quantities are equal, for by 17, Cor.
$$[\beta_1 \dots \beta_r] = [a_1 \dots a_n]^{r-1} \, [a_{r+1} \dots a_n].$$

Hence
$$[a_1 \dots a_r . \beta_1 \dots \beta_r] = K^r.$$

Thus the formula follows.

29. *Example. If $R_1, \dots, R_m$ be independent spreads of steps $r_1, \dots, r_m$, and $n = r_1 + \dots + r_m$, and p be any point of S, the join of all the R, outside the joins of any $m-1$ of the R, then through p goes just one spread of step m which meets each R in just one point.*

For, we can find independent points
$$a'_1, a'_2, \dots, a'_{r_1} \text{ in } R_1, \quad a''_1, a''_2, \dots, a''_{r_2} \text{ in } R_2, \quad \dots,$$
$$a^{(m)}_1, a^{(m)}_2, \dots, a^{(m)}_{r_m} \text{ in } R_m,$$
and can take
$$p = x'_1 a'_1 + x'_2 a'_2 + \dots + x'_{r_1} a'_{r_1} + x''_1 a''_1 + \dots + x''_{r_2} a''_{r_2} + \dots + x^{(m)}_1 a^{(m)}_1 + \dots + x^{(m)}_{r_m} a^{(m)}_{r_m}.$$

Then
$$T = [(x'_1 a'_1 + \dots + x'_{r_1} a'_{r_1}) (x''_1 a''_1 + \dots + x''_{r_2} a''_{r_2}) \dots (x^{(m)}_1 a^{(m)}_1 + \dots + x^{(m)}_{r_m} a^{(m)}_{r_m})]$$
is of step m; it goes through p and contains one point of each of $R_1, \dots, R_m$, viz. those given by its factors. It cannot contain more than one point of R_1, for if, for example, it goes through
$$q = y_1 a'_1 + \dots + y_{r_1} a'_{r_1},$$
then $[Tq] = 0$ would give a relation between spreads of step $m+1$, which are independent.

Nor can any other spread of step m which cuts each R_i in just one point pass through p, for if it did, we should again have a dependence, by expanding an outer product, between spreads of step $m+1$.

We have assumed $T \neq 0$. If $T=0$, the factors of T are dependent, and hence p is on the join of $m-1$ of the R_i, contrary to the hypothesis.

§ 57. *Condition that an extensive should be simple.* Extensives of step two.*

1. *A necessary and sufficient condition that an extensive A, of step* m, *in the spread $\mathscr{S}(e_1, \ldots, e_n)$ is simple is that $[AB.A]=0$ for all simple extensives B of step* $n-m+1$.

Necessity. $[AB]$ is regressive of step one, and if A is simple, $[AB]$ is in A. Hence $[AB.A]=0$.

Sufficiency. We need only assume

$$[AE.A] = 0, \tag{5}$$

for all unities E of step $n-m+1$; then $[AB.A]$ will vanish for all simple extensives B of that step. Let $[e_1 \ldots e_n]=1$.

We may assume that A contains a term in $[e_1 \ldots e_m]$. Let

$$A = k[e_1 \ldots e_m] + \sum_{r=1}^{m} \sum_{s=1}^{n-m} k_{rs}[e_1 \ldots e_{r-1} e_{r+1} \ldots e_m e_{m+s}] + \sum_i l_i U_i,$$

where $k \neq 0$, and each U_i is a unity of step m which contains at most $m-2$ of $e_1, \ldots, e_m$. Then if $1 \leqq r \leqq m$,

$$[A.e_r e_{m+1} \ldots e_n] = k e_r + (-1)^{m-r} \sum_{s=1}^{n-m} k_{rs} e_{m+s} = a_r, \text{ say.}$$

Since $e_r e_{m+1} \ldots e_n$ is a unity of step $n-m+1$, the last equation and (5) give $[a_r A]=0$, $r=1, \ldots, m$.

Since $k \neq 0$, therefore $a_1, \ldots, a_m$ are independent, and since $[a_r A]=0$, they are elements in A, and as this is of step m, we have $A=[a_1 \ldots a_m]$, and A is simple.

Example. If $n=5$ and $m=2$, take $[e_1 \ldots e_5]=1$. A general extensive of step two is of form

$$A = \sum_{i,j=1}^{5} a_{ij}[e_i e_j], \quad (a_{ij} = -a_{ji}).$$

* See the notes by Study to the edition of the second *Ausdehnungslehre* in Grassmann's *Ges. Werke.*

Take $B=[e_1 \dots e_4]$, then

$$[AB] = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4,$$
$$[AB.A] = (a_{23}a_{15} + a_{31}a_{25} + a_{12}a_{35})[e_1e_2e_3] + \dots.$$

The condition that A is simple is that the four coefficients of this expression and of the similar expressions vanish.

2. *A general extensive of step two in a spread of step* n *can be expressed as the sum of at most* $s=[\frac{1}{2}n]$ *rotors.**

This is true when $n=2, 3, 4$, as we know; we assume it for all $n \leqq 2m-1$, and then prove it for $n=2m, 2m+1$.

(1) If $n=2m$, then a sum of $m+1$ rotors can be reduced to a sum of m rotors, or fewer.

For let $S = R_1 + \dots + R_{m+1}$, $R_i = [a_i b_i]$, $(i=1, \dots, m+1)$, where a_i, b_i are points (extensives of step one).

If $a_1, b_1, \dots, a_m, b_m$ be dependent, their joining spread is of step $2m-1$ at most, and hence, by hypothesis, $R_1 + \dots + R_m$ can be expressed as the sum of at most $m-1$ rotors, and hence S can be expressed as the sum of m rotors or fewer.

If $a_1, b_1, \dots, a_m, b_m$ be independent, and, say,

$$b_{m+1} = k_1 a_1 + l_1 b_1 + \dots + k_m a_m + l_m b_m, \quad (k, l \text{ scalars}),$$

then S is the sum of extensives of the form

$$[a_i b_i] + k_i[a_{m+1} a_i] + l_i[a_{m+1} b_i], \quad (i=1, \dots, m).$$

But each of these is in a spread of step $\leqq 3$, and hence gives a rotor, or zero. Hence S is the sum of m rotors, or fewer.

(2) If $n = 2m+1$, then a sum of $m+1$ rotors can be replaced by a sum of m rotors, or fewer.

For, let

$$S = R_1 + \dots + R_{m+1}, \quad R_i = [a_i b_i], \quad (i=1, \dots, m+1).$$

As before, if $a_1, \dots, b_m$ be dependent, then $R_1 + \dots + R_m$ is the sum of fewer than m rotors, and hence S is the sum of at most m rotors.

If $a_1, \dots, b_m$ be independent, and a_{m+1}, b_{m+1} are in the spread defined by them, so is R_{m+1}; by the first part, these $m+1$ rotors in a spread of step $2m$ can be replaced by a sum of at most m

* Rothe, *Archiv Math. Phys.* Ser. 3, 20, (1913), p. 210. It is the algebraic form of a theorem on Pfaffian differential expressions.

rotors. But if a_{m+1} is not in the spread $[a_1 \dots b_m]$, then $a_1, \dots, b_m, a_{m+1}$ are independent, and we may take

$$b_{m+1} = k_1 a_1 + l_1 b_1 + \dots + k_m a_m + l_m b_m + k_{m+1} a_{m+1}.$$

Then

$$R_{m+1} = k_1[a_{m+1} a_1] + l_1[a_{m+1} b_1] + \dots + k_m[a_{m+1} a_m] + l_m[a_{m+1} b_m],$$

and S is the sum of extensives of the form

$$[a_i b_i] + k_i[a_{m+1} a_i] + l_i[a_{m+1} b_i], \quad (i = 1, \dots, m).$$

Each of these is a rotor, or zero, and hence S is the sum of m rotors, or fewer.

3. *If S can be expressed as a sum of* r *rotors, then* $[S^{r+1}] = 0$, *where* $[S^{r+1}]$ *means the outer product of* $r+1$ *factors* S. *If S cannot be expressed as a sum of fewer than* r *rotors, then* $[S^r] \neq 0$.

For $S = R_1 + \dots + R_r$ gives

$$[S^2] = 2([R_1 R_2] + [R_1 R_3] + \dots + [R_{r-1} R_r]),$$
$$[S^3] = 6([R_1 R_2 R_3] + [R_1 R_2 R_4] + \dots + [R_{r-2} R_{r-1} R_r]),$$
$$\dots\dots$$
$$[S^r] = r![R_1 R_2 \dots R_r],$$
$$[S^{r+1}] = 0.$$

If $[S^r] = 0$, and $R_i = [a_i b_i]$, then b_r is in $[a_1 b_1 a_2 b_2 \dots a_r]$; hence S is in a spread of step $2r-1$, and hence is the sum of at most $r-1$ rotors.

4. *Def.* An extensive S of step two is of '*rank*' r, if it is a sum of r rotors, and cannot be expressed as a sum of fewer. The condition for this is $[S^{r+1}] = 0$, $[S^r] \neq 0$.

5. *To express S, of rank* r, *as a sum of* r *rotors.*

Let A_1 be a rotor of rank r, and suppose $S = k_1 A_1 + S_1$, where S_1 is of rank $r-1$, then $[S_1^r] = 0$, $[(S - k_1 A_1)^r] = 0$.

Hence $$[S^r] - r k_1 [S^{r-1} A_1] = 0.$$

Now $[S^r]$ and $[S^{r-1} A_1]$ are simple, and of step $2r$ in the same spread of step $2r$. Hence k_1 is determined by the last equation, provided $[S^{r-1} A_1] \neq 0$, i.e. if A_1 is not in the 'complex' in $[S^r]$ such that $[S^{r-1} A_1] = 0$. As S_1 is of rank $r-1$, we can proceed with the reduction, and express S in the form $S = A_1 + \dots + A_r$, where $S_i = S - (A_1 + \dots + A_i)$ is of rank $r-i$, and $[S_{i-1}^{r-i} A_i] \neq 0$.

§ 58. *The spread of step five.* Segre's figure.*

Let R be a spread of step five.

1. If A, B, C, D be rotors in R, then

$$[AB.CD] = [ABCD] + [ABDC].$$

For let $C = [pq]$, $D = [rs]$, then

$$\begin{aligned}[AB.CD] = [AB.pqrs] &= [ABs]\,[pqr] - [ABp]\,[qrs] \\ &\quad + [ABq]\,[rsp] - [ABr]\,[spq] \\ &= [ABs]\,[rC] - [ABp]\,[qD] \\ &\quad + [ABq]\,[pD] - [ABr]\,[sC],\end{aligned}$$

$$[ABC] = [AB.pq] = [ABq]\,p - [ABp]\,q,$$

$$[ABD] = [AB.rs] = [ABs]\,r - [ABr]\,s.$$

These shew the theorem.

2. In particular $[AB.CA] = [ABCA]$, since $[ABA] = 0$, and hence $[ABAC] = 0$.

Hence $[AB.AC] = [ABCA]$, since $[CA] = [AC]$,

$$[AB.AC.AD] = [ABCA.AD] = [ABCA.DA], \qquad (\S\,56{\cdot}19).$$

Now $[ABC]$ is a point p, say; $[ABCA] = [pA]$.

Hence $[ABCA.DA] = [pA.DA] = [pDA]\,A = [ABCDA]\,A$.

Thus $[AB.AC.AD] = [ABCDA]\,A$.

Similarly $[AC.AB.AD] = [ACBDA]\,A$.

But $[AB.AC.AD] = -[AC.AB.AD]$.

Hence $[ABCDA] = -[ACBDA]$

$$= -[ABDCA], \text{ similarly.}$$

Hence if any two of B, C, D be congruent, then $[ABCDA] = 0$.

We note also that

$$[ABCDE] = [BACDE], \quad [ABCDE] = [ABCED],$$

the latter since $[pDE] = [pED]$ for any point p.

* E. Müller, 'Beiträge zur Grassmannschen Ausdehnungslehre', *Wiener Ber.* II A, 118 (1909), p. 1047.

3. *Associated lines.* Let $A_1, \ldots, A_4$ be four independent rotors in R and let

$$\Gamma_{ik} = A_i A_k = A_k A_i = \Gamma_{ki}, \quad (i, k = 1 \ldots 4).$$

Let
$$B_1 = [\Gamma_{12}\Gamma_{13}\Gamma_{14}], \quad B_2 = [\Gamma_{23}\Gamma_{24}\Gamma_{21}],$$
$$B_3 = [\Gamma_{34}\Gamma_{31}\Gamma_{32}], \quad B_4 = [\Gamma_{41}\Gamma_{42}\Gamma_{43}],$$
$$C_1 = [\Gamma_{23}\Gamma_{34}\Gamma_{42}], \quad C_2 = [\Gamma_{34}\Gamma_{41}\Gamma_{13}],$$
$$C_3 = [\Gamma_{41}\Gamma_{12}\Gamma_{24}], \quad C_4 = [\Gamma_{12}\Gamma_{23}\Gamma_{31}],$$
$$\mathrm{k}_1 = [A_1A_2A_3A_4A_1], \quad \mathrm{k}_2 = -[A_2A_3A_4A_1A_2],$$
$$\mathrm{k}_3 = [A_3A_4A_1A_2A_3], \quad \mathrm{k}_4 = -[A_4A_1A_2A_3A_4].$$

Then, by 2, $B_1 = \mathrm{k}_1 A_1, \ldots, B_4 = \mathrm{k}_4 A_4,$

$$\begin{aligned}[B_1C_1] &= [\Gamma_{12}\Gamma_{13}\Gamma_{14}.\Gamma_{23}\Gamma_{34}\Gamma_{42}] \\ &= [\Gamma_{13}\Gamma_{14}\Gamma_{23}\Gamma_{34}\Gamma_{42}]\,\Gamma_{12} + [\Gamma_{14}\Gamma_{12}\Gamma_{23}\Gamma_{34}\Gamma_{42}]\,\Gamma_{13} \\ &\qquad + [\Gamma_{12}\Gamma_{13}\Gamma_{23}\Gamma_{34}\Gamma_{42}]\,\Gamma_{14} \\ &= -\mathrm{k}_{12}\Gamma_{12} - \mathrm{k}_{31}\Gamma_{31} + \mathrm{k}_{14}\Gamma_{14},\end{aligned}$$

where k_{ij} is that scalar which is the outer product of Γ_{12}, Γ_{23}, Γ_{31}, Γ_{14}, Γ_{24}, Γ_{34}, in that order, omitting Γ_{ij}. In interchanging the Γ, we note that, since they are of step four, they behave like extensives of step one.

Similarly,
$$[B_2C_2] = -\mathrm{k}_{12}\Gamma_{12} + \mathrm{k}_{23}\Gamma_{23} - \mathrm{k}_{24}\Gamma_{24},$$
$$[B_3C_3] = \mathrm{k}_{23}\Gamma_{23} - \mathrm{k}_{31}\Gamma_{31} + \mathrm{k}_{34}\Gamma_{34},$$
$$[B_4C_4] = \mathrm{k}_{14}\Gamma_{14} - \mathrm{k}_{24}\Gamma_{24} + \mathrm{k}_{34}\Gamma_{34}.$$

Hence

$$\tfrac{1}{2}([B_1C_1] + [B_2C_2] + [B_3C_3] + [B_4C_4])$$
$$= -\mathrm{k}_{12}\Gamma_{12} + \mathrm{k}_{23}\Gamma_{23} - \mathrm{k}_{31}\Gamma_{31} + \mathrm{k}_{14}\Gamma_{14} - \mathrm{k}_{24}\Gamma_{24} + \mathrm{k}_{34}\Gamma_{34}.$$

Now the last expression vanishes by § 56·25. For Γ_{23}, Γ_{31}, Γ_{14}, Γ_{24}, Γ_{34} are independent extensives of step four (§ 54·10), and Γ_{12} is a linear combination of them, whose coefficients are determined as in § 56·25.

Hence
$$[B_1C_1] + \ldots + [B_4C_4] = 0,$$

or
$$\mathrm{k}_1[A_1C_1] - \mathrm{k}_2[A_2C_2] + \mathrm{k}_3[A_3C_3] - \mathrm{k}_4[A_4C_4] = 0.$$

Hence *if* $A_1, \ldots, A_4$ *be any four independent rotors in* R, *and* C_1 *is the intersection of solids through the pairs* A_2A_3, A_3A_4, A_4A_2, *and similarly for* C_2, C_3, C_4, *then the solids* A_1C_1, A_2C_2, A_3C_3, A_4C_4 *meet in a line.* (A 'solid' is a spread of step four.)

This new line is called the '*associate*' of $A_1, \ldots, A_4$.

4. We have

$$[\Gamma_{23}\Gamma_{31}] = [A_2A_3 . A_3A_1] = [A_3A_2A_1A_3],$$
$$[\Gamma_{14}\Gamma_{24}\Gamma_{34}] = -\mathrm{k}_4 A_4.$$

Thus $\mathrm{k}_{12} = -\mathrm{k}_4[A_3A_2A_1A_3A_4] = -\mathrm{k}_4[A_3A_2A_1A_4A_3]$
$= \mathrm{k}_4[A_3A_4A_1A_2A_3] = \mathrm{k}_3\mathrm{k}_4.$

Similarly

$$[\Gamma_{31}\Gamma_{12}] = [A_1A_3A_2A_1], \quad [\Gamma_{12}\Gamma_{23}] = [A_2A_1A_3A_2],$$
$$[\Gamma_{14}\Gamma_{24}] = [A_4A_1A_4A_2], \quad [\Gamma_{23}\Gamma_{24}] = [A_3A_2A_4A_2],$$
$$[\Gamma_{31}\Gamma_{14}] = [A_1A_3A_4A_1],$$

and we find

$$\mathrm{k}_{23} = -\mathrm{k}_1\mathrm{k}_4, \ \mathrm{k}_{31} = \mathrm{k}_2\mathrm{k}_4, \ \mathrm{k}_{14} = -\mathrm{k}_2\mathrm{k}_3, \ \mathrm{k}_{24} = \mathrm{k}_3\mathrm{k}_1, \ \mathrm{k}_{34} = -\mathrm{k}_1\mathrm{k}_2.$$

Hence, by 3,

$$\mathrm{k}_3\mathrm{k}_4\Gamma_{12} + \mathrm{k}_1\mathrm{k}_4\Gamma_{23} + \mathrm{k}_2\mathrm{k}_4\Gamma_{31} + \mathrm{k}_2\mathrm{k}_3\Gamma_{14} + \mathrm{k}_3\mathrm{k}_1\Gamma_{24} + \mathrm{k}_1\mathrm{k}_2\Gamma_{34} = 0.$$

Let $$L_1 = \mathrm{k}_1^{-1}A_1, \quad L_2 = \mathrm{k}_2^{-1}A_2,$$
$$L_3 = \mathrm{k}_3^{-1}A_3, \quad L_4 = \mathrm{k}_4^{-1}A_4;$$

then $$[L_1L_2] + [L_2L_3] + [L_3L_1] + [L_1L_4] + [L_2L_4] + [L_3L_4] = 0,$$

or $$[(L_1 + L_2 + L_3 + L_4)^2] = 0.$$

Hence $L_1 + L_2 + L_3 + L_4$ is a rotor. Call it $-L_5$.

Then $$L_2 + L_3 + L_4 = -L_1 - L_5,$$

hence $$[L_2L_3] + [L_3L_4] + [L_4L_2] = [L_1L_5].$$

But C_1 is the cut of $[L_2L_3]$, $[L_3L_4]$, $[L_4L_2]$. Hence C_1 is in solid $[L_1L_5]$, and L_5 is in solid $[C_1L_1]$. Similarly, L_5 is in $[C_2L_2]$, $[C_3L_3]$, $[C_4L_4]$. Hence L_5 is the associate line.

Thus *if five lines are dependent, but each set of four of them independent, then each is the associate line of the other four; conversely, the associate line of four independent lines is a linear combination of them.*

Clearly *any plane which meets four of the lines meets the fifth.*

For the plane α and line L meet, if, and only if, $[\alpha L] = 0$.

5. If $A = [bc']$, $B = [ca']$, $C = [ab']$ be independent rotors, we can weight the points so that $a + b + c + a' + b' + c' = 0$.

There is only one line which cuts A, B, C, and it is given by each of the equal expressions

$$[(b + c')(c + a')], \quad [(c + a')(a + b')], \quad [(a + b')(b + c')].$$

For example, this line cuts $[bc']$, since $[bc'(b+c')(c+a')] = 0$.

Now
$$[BCA] = [ca'.ab'.bc'] = [ca'ab'.bc']$$
$$= [ca'ab'c']\, b - [ca'ab'b]\, c'$$
$$= [abca'b'](b+c').$$

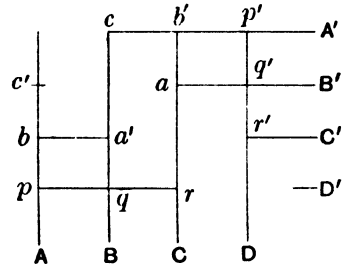
Similarly
$$[CAB] = [abcb'c'](c+a'),$$
$$[ABC] = [abcc'a'](a+b').$$

Hence
$$[BCA]+[CAB]+[ABC] = 0.$$

The cuts of A, B, C and the common transversal are $b+c'$, $c+a'$, $a+b'$.

The transversal is $[BCA.CAB]$ or $[AB.BC.CA]$.

6. Let A, B, C, D be four independent rotors,* and let the transversals A', B', C' of the triads BCD, CAD, ABD respectively cut them in points c, b', p'; a, c', q'; b, a', r' respectively.



Then we can take
$$A = [bc'], \quad B = [ca'], \quad C = [ab'], \quad D = [q'r'],$$
$$A' = [cb'], \quad B' = [ac'], \quad C' = [ba'],$$

and can take weights so that
$$p' = b'+c, \quad q' = c'+a, \quad r' = a'+b. \qquad (1)$$

Let
$$p = b+c', \quad q = c+a', \quad r = a+b',$$
$$l = a+a', \quad m = b+b', \quad n = c+c',$$
$$L = [aa'], \quad M = [bb'], \quad N = [cc'],$$
$$D' = [(c+a')(a+b')], \quad E = [(a+a')(b+b')].$$

* Baker, *Principles*, vol. IV. We have adopted the notation there with unessential changes.

Then

$$A+B+C+D-E = bc'+ca'+ab' \\ +(c'+a)(a'+b)-(a+a')(b+b') \\ =(a+b+c+b'+c')a', \qquad (2)$$

$$A'+B'+C'-D'+E = a(b+c+a'+b'+c'). \qquad (3)$$

So far we have only fixed weights so that (1) is true. We can adjust them still further so that

$$p'+q'+r'=0, \quad \text{or} \quad a+b+c+a'+b'+c'=0.$$

Then $$p+q+r=0, \quad l+m+n=0,$$

and D' is the transversal of A, B, C,

$$L+M+N-D+D' = (a+b+c+a'+c')b' = 0, \qquad (4)$$

and the right-hand sides of (2), (3) vanish.

Let $$P=[pp'], \quad Q=[qq'], \quad R=[rr'].$$

Then $$P=[(a+a')(b+c')],$$

$$Q=[(b+b')(c+a')], \quad R=[(c+c')(a+b')],$$

$$A+A'+L-R+Q = bc'+cb'+aa'-(a+b')(a'+b) \\ +(b+b')(c+a') \\ =b(c+a+a'+b'+c')=0.$$

These and similar equations, and (2), (3), (4), shew that the rows in the following table are associate lines among the fifteen lines

.	E	D	A	B	C
E	.	D'	A'	B'	C'
D	D'	.	L	M	N
A	A'	L	.	R	Q
B	B'	M	R	.	P
C	C'	N	Q	P	.

We have also such formulae as

$$AA'+BB'+CC' = DD',$$

$$A..BC.CD.DB-B..CD.DA.AC \\ +C..DA.AB.BD-D..AB.BC.CA=0.$$

Examples. 1. Prove generally

$$[(b+c')(b'+c)]+[(c+a')(c'+a)]+[(a+b')(a'+b)]$$
$$=[bc]+[ca]+[ab]-[b'c']-[c'a']-[a'b'].$$

The right-hand side vanishes when a, b, c are collinear, and a', b', c' are collinear. Hence

(i) If the lines $[ab]$, $[a'b']$ are coplanar, we can weight the points so that $a+b' \equiv a'+b$, $b+c' \equiv b'+c$, and then $c+a' \equiv c'+a$. Pappus' Theorem follows.

(ii) If the lines are not coplanar, and through a point p be drawn lines pu, pv, pw to cut $(bc', b'c)$, $(ca', c'a)$, $(ab', a'b)$ respectively, then pu, pv, pw are coplanar.

(iii) If we take points of unit weight and a, b, c are collinear, and a', b', c' are collinear, then the joins of the mid-points of ab' and $a'b$, of bc' and $b'c$, of ca' and $c'a$ are coplanar.

2. Generally

$$[(b+c')(b'+c)\,aa']+[(c+a')(c'+a)\,bb']+[(a+b')(a'+b)\,cc']$$
$$=[abc(a'+b'+c')]+[a'b'c'(a+b+c)].$$

Hence, if $$a+b+c+a'+b'+c'=0,$$

$$b+c'=p,\ c+a'=q,\ a+b'=r,\ b'+c=p',\ c'+a=q',\ a'+b=r',$$

then $$[pp'aa']+[qq'bb']+[rr'cc']=0.$$

Hence if, in step five, p, q, r be collinear points on bc', ca', ab', and p', q', r' be collinear points on $b'c$, $c'a$, $a'b$, and pp' meets aa', and qq' meets bb', then rr' meets cc'.

3. $$[(b+c')(a+b')]+[(b'+c)(c'+a)]=[aa']+[bb']+[cc'].$$

Hence the transversals of bc', ab', of $b'c$, $c'a$, and the diagonals aa', bb', cc' are associated lines.

(Baker, *Principles*, IV, p. 117, Ex. 2.)

CHAPTER VIII

APPLICATION OF THE GENERAL THEORY TO SYSTEMS OF LINEAR EQUATIONS AND DETERMINANTS

§ 59. *Matrices and linear equations.*

1. *Def.* A '*matrix*' is a set of scalars a_{ik}, functions of integers i, k, where $i = 1, \ldots, n$; $k = 1, \ldots, m$, and n, m are fixed for a given matrix. Matrices will be added and multiplied in accordance with rules given later. If $m = n$, we have a '*square matrix*'.

The scalars a_{ik} may be arranged in an array

$$\begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1m} \\ a_{21}, & a_{22}, & \ldots, & a_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nm} \end{pmatrix}.$$

We often indicate the matrix by the notation (a_{ij}) with a specification of the range of the suffixes i, j.

If (a_{ik}), $(i = 1, \ldots, n; k = 1, \ldots, m)$, is a matrix, the matrix (a_{ki}), $(k = 1, \ldots, m; i = 1, \ldots, n)$, obtained by interchanging rows and columns in the array of the original matrix is called the '*transposed*' matrix.

2. Consider the set of linear equations for $x_1, \ldots, x_n$:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= b_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m &= b_n, \end{aligned}\right\} \qquad (1)$$

where m, n may have any integral values, and all letters denote scalars. Denote the matrix of the coefficients by (a_{ij}).

Introduce extensive unities $e_1, \ldots, e_n$, and let a_k, b be extensives defined by

$$a_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{nk}e_n, \quad (k = 1, \ldots, m),$$

$$b = b_1e_1 + b_2e_2 + \ldots + b_ne_n.$$

Then we may replace the n algebraic equations by the equation

$$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = b \qquad (2)$$

between the extensives $a_1, \ldots, a_m$, b in the spread $\mathscr{S}(e_1, \ldots, e_n)$.

If equation (2) can be satisfied by some $x_1, \ldots, x_m$, then b is a linear combination of $a_1, \ldots, a_m$ and hence is in $\mathscr{S}(a_1, \ldots, a_m)$.

Conversely, if b is in this spread, the equations can be solved.

Hence, *a necessary and sufficient condition that* (1) *can be solved is*

$$\mathscr{S}(a_1, \ldots, a_m) = \mathscr{S}(a_1, \ldots, a_m, b).$$

3. *Defs.* We call the a_k the '*column-extensives*' of the matrix a_{ij}.

The '*rank*' of the matrix is the maximum number of independent column-extensives. It is clearly not greater than m, the number of these extensives.

Hence, *a necessary and sufficient condition that* (1) *can be solved is that the ranks of the matrix* a_{ij} *and of the matrix formed from* a_{ij} *and* b_i *are the same.*

4. If $b_1 = \ldots = b_n = 0$ in equation (1), we have n '*homogeneous linear*' equations. To solve these we must find scalars $x_1, \ldots, x_m$, such that

$$x_1 a_1 + \ldots + x_m a_m = 0.$$

Let the rank of the matrix (a_{ij}) be $r < m$, and let $a_1, \ldots, a_r$ be linearly independent, then $a_{r+1}, \ldots, a_m$ are linear combinations of $a_1, \ldots, a_r$; hence so is $x_{r+1} a_{r+1} + \ldots + x_m a_m$ if one at least of $x_{r+1}, \ldots, x_m$ is not zero.

Thus there are definite scalars $x_1, \ldots, x_r$ such that

$$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = 0.$$

Hence, *if the rank of the matrix* (a_{ij}) *is* $r < m$, *we can take* $x_{r+1}, \ldots, x_m$ *arbitrary, not all zero, and then these and certain definite values of* $x_1, \ldots, x_r$ *constitute a solution of the homogeneous linear equations.*

5. If $m > n$, we can adjoin unities $e_{n+1}, \ldots, e_m$ to $e_1, \ldots, e_n$.

In any case if $x_1, \ldots, x_m$ constitute a solution of the homogeneous or non-homogeneous equations, we call the extensive $x = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m$ a '*solution-extensive*'.

If x, y be solution-extensives of the homogeneous equations, so are $k_1 x + k_2 y$, (k_1, k_2 scalars). Thus the solution-extensives of

these equations fill a spread, and since $m-r$, and no more, of the $x_1, \ldots, x_m$ can be taken arbitrarily, the step of the spread is $m-r$.

Since the order in which the equations are taken is irrelevant, the rank of the matrix (a_{ij}) is not changed by permutation of its rows.

6. Return to the non-homogeneous equation in $x_1, \ldots, x_m$:

$$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = b, \qquad (2)$$

and suppose it solvable; suppose also the corresponding homogeneous equation is solvable, and let $y = y_1 e_1 + y_2 e_2 + \ldots + y_m e_m$ be a solution-extensive of the latter, so that

$$y_1 a_1 + y_2 a_2 + \ldots + y_m a_m = 0.$$

Then if x is a solution-extensive of (2), so is $x+y$. Conversely, if u, v be any solution-extensives of (2), then $u-v$ is a solution-extensive of the homogeneous equation.

Thus *we get all solution-extensives of* (2), *if we add to any one of them all the solution-extensives of the corresponding homogeneous equation.*

Also by 2, the spread of the solution-extensives of (2) has the same step as the spread of the solution-extensives of the corresponding homogeneous equation.

7. In the spread $\mathscr{S}(e_1, \ldots, e_m)$, let $E_i = |e_i$, $(i = 1, \ldots, m)$, then the '*row-extensives*' of the matrix (a_{ij}) are $A_1, \ldots, A_n$, where

$$\left.\begin{aligned} a_{11}E_1 + a_{12}E_2 + \ldots + a_{1m}E_m &= A_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}E_1 + a_{n2}E_2 + \ldots + a_{nm}E_m &= A_n. \end{aligned}\right\} \qquad (3)$$

If, as above, $x = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m$ is a solution-extensive of the homogeneous equations, these may be written, if $[e_1 e_2 \ldots e_m] = 1$, in the form

$$[A_1 x] = 0, \ \ldots, \ [A_n x] = 0. \qquad (4)$$

Suppose just $s \leqq n$ of the extensives $A_1, \ldots, A_n$ are independent, and take them to be $A_1, \ldots, A_s$, then each x which satisfies $[A_1 x] = 0, \ \ldots, \ [A_s x] = 0$ satisfies all equations (4).

Now the step of the spread of solution-extensives is $m-r$, hence the rank of the matrix of the first s equations of (4) is r.

But the *column*-extensives of the matrix of the first s equations of (4) have s coordinates, therefore they lie in a spread of step $\leqq s$, hence $s+1$ of them must be dependent. Hence the maximum number r of linearly independent column-extensives of the matrix is $\leqq s$. Hence $r \leqq s$.

The same argument applied to the transposed matrix gives $s \leqq r$. Hence $r = s$.

Hence *the rank of a matrix*, which was defined as the number of independent column-extensives, *is also the number of independent row-extensives.*

The rank of a matrix is hence the maximum order of the non-vanishing determinants of its sub-matrices. But it will be noticed that so far we have not needed the theory of determinants.

8. From 4, if we have n homogeneous linear equations in n unknowns, and these are solvable by values of the variables, not all zero, then the determinant of the coefficients of the equations vanishes.

Conversely, if this determinant vanishes, at least one solution exists, in which the values of the variables are not all zero. If the determinant does not vanish, the only solution is that in which the values of the variables are all zero.

If we have r equations in any number of unknowns, and the rank of the matrix of the coefficients is r, we say the equations are '*independent*'.

A system of n independent non-homogeneous linear equations in n unknowns has a unique solution. The equations are independent if, and only if, the determinant of the coefficients of the variables is not zero.

9. To solve $$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = b, \qquad (5)$$ when the condition for solvability is satisfied, and the notation of 2 is used.

From (5) we deduce by outer multiplications in $\mathscr{S}(e_1, \ldots, e_m)$:

$$\left.\begin{aligned} x_1[a_1 a_2 \ldots a_m] &= [b a_2 a_3 \ldots a_m], \\ x_2[a_1 a_2 \ldots a_m] &= [a_1 b a_3 \ldots a_m], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m[a_1 a_2 \ldots a_m] &= [a_1 a_2 \ldots a_{m-1} b]. \end{aligned}\right\} \qquad (6)$$

First, suppose $[a_1 \dots a_m] \neq 0$; then $a_1, \dots, a_m$ are independent, and as the condition of solvability is

$$\mathscr{S}(a_1, \dots, a_m) = \mathscr{S}(a_1, \dots, a_m, b),$$

there are scalars $y_1, \dots, y_m$ such that

$$b = y_1 a_1 + y_2 a_2 + \dots + y_m a_m.$$

Substituting this in (6), we find $x_i = y_i$, $(i = 1, \dots, m)$. Hence the values of x_i given by (6) satisfy (5).

Next, if the rank of the matrix (a_{ij}) is $r < m$, then r of $a_1, \dots, a_m$ are independent, say $a_1, \dots, a_r$, and $[a_1 \dots a_m] = 0$, while $a_{r+1}, \dots, a_m$ depend on $a_1, \dots, a_r$.

By 4, we can take $x_{r+1}, \dots, x_m$ arbitrary, then (5) gives

$$x_1 a_1 + \dots + x_r a_r = c, \quad \text{where} \quad c = b - x_{r+1} a_{r+1} - \dots - x_m a_m.$$

Thence

$$[a_1 \dots a_r] x_1 = [c a_2 a_3 \dots a_r], \quad \dots, \quad [a_1 \dots a_r] x_r = [a_1 \dots a_{r-1} c],$$

and, as before, these give values of the x which satisfy (5).

Also when $x_{r+1}, \dots, x_m$ are given, then $x_1, \dots, x_r$ are unique; for if $x_1', \dots, x_r'$ be any set satisfying the equations, then

$$(x_1 - x_1') a_1 + \dots + (x_r - x_r') a_r = 0,$$

and because $a_1, a_2, \dots, a_r$ are independent, each coefficient vanishes.

§60. *Determinants. General theorems.**

1. If $e_1, e_2, \dots, e_n$ be unit-extensives, with $[e_1 e_2 \dots e_n] = 1$, and $a, b, \dots, l$ be n extensives in $\mathscr{S}(e_1, e_2, \dots, e_n)$, where

$$a = \sum_{i=1}^{n} a_i e_i, \quad b = \sum_{i=1}^{n} b_i e_i, \quad \dots, \quad l = \sum_{i=1}^{n} l_i e_i,$$

$$(a_1, b_1, \dots, l_1 \text{ scalars}), \quad (1)$$

then $[ab \dots l]$ is that determinant which in the usual notation is denoted by its diagonal terms thus: $|a_1 b_2 \dots l_n|$.

The *minors* of the first row $a_1, a_2, \dots, a_n$ of this determinant are, in our notation,

$$[e_1 bc \dots l], \ [be_2 c \dots l], \ [bce_3 \dots l], \ \dots, \ [bcd \dots le_n],$$

and, in the ordinary notation,

$$|b_2 c_3 \dots l_n|, \ |b_1 c_3 \dots l_n|, \ |b_1 c_2 d_4 \dots l_n|, \ \dots, \ |b_1 c_2 d_3 \dots l_{n-1}|.$$

* The sections on determinants are to be regarded as illustrative only.

The co-factors of $a_1, a_2, \ldots, a_n$, i.e. the minors with the signs with which they occur in the ordinary expansion of the determinant, are

$$[e_1 bc \ldots l],\ [e_2 bc \ldots l],\ [e_3 bc \ldots l],\ \ldots,\ [e_n bc \ldots l].$$

Thus, for example,

$$[e_2 bc \ldots l] = [e_2(b_1 e_1 + \ldots + b_n e_n)(c_1 e_1 + \ldots + c_n e_n) \ldots (l_1 e_1 + \ldots + l_n e_n)]$$

$$= \begin{vmatrix} b_1, & b_3, & \ldots, & b_n \\ c_1, & c_3, & \ldots, & c_n \\ \ldots\ldots & & & \\ l_1, & l_3, & \ldots, & l_n \end{vmatrix} [e_2 e_1 e_3 \ldots e_n] = -\begin{vmatrix} b_1, & b_3, & \ldots, & b_n \\ c_1, & c_3, & \ldots, & c_n \\ \ldots\ldots & & & \\ l_1, & l_3, & \ldots, & l_n \end{vmatrix}.$$

The minor obtained when, for example, the rows b, d, f, ... are retained and the columns, 1, 3, 4, ... are struck out is

$$[e_1 be_3 e_4 df \ldots].$$

Again, for example,

$$[e_1 bce_4 f \ldots] = |\, b_2 c_3 f_5 \ldots\,|.$$

(To avoid confusion we omit e from the sequence of letters $a, b, c, d, f, \ldots$.)

Example. If a determinant vanishes, then the co-factors of the elements of any one row are proportional to the co-factors of the elements of any other row. If all minors of order $m+1$ vanish, then all m-lined minors formed from any set of m rows are proportional to the corresponding minors formed from any other set of m rows.

2. The rule for adding determinants is simply the distributive law: $[abc \ldots l] + [a'bc \ldots l] = [(a+a')bc \ldots l]$.

3. *Laplace's expansion and Sylvester's Theorem.* Consider $[abcd]$, where a, b, c, d are in the spread $\mathscr{S}(e_1, e_2, e_3, e_4)$ and given by (1). By § 56·25 *Cor.*, we have

$$[e_1 e_2 e_3 e_4][bcd] = [e_1 bcd][e_2 e_3 e_4] - [e_2 bcd][e_1 e_3 e_4] + [e_3 bcd][e_1 e_2 e_4] - [e_4 bcd][e_1 e_2 e_3].$$

Hence

$$[e_1 e_2 e_3 e_4][abcd] = [e_1 bcd][ae_2 e_3 e_4] + [e_2 bcd][e_1 ae_3 e_4] + [e_3 bcd][e_1 e_2 ae_4] + [e_4 bcd][e_1 e_2 e_3 a]. \quad (2)$$

If now we take $[e_1e_2e_3e_4] = 1$, then $[e_1bcd]$, $[e_2bcd]$, ... are the co-factors $A_1, \ldots, A_4$ of $a_1, \ldots, a_4$, while

$$[ae_2e_3e_4] = a_1, \quad [e_1ae_3e_4] = a_2, \quad \ldots.$$

Hence $$[abcd] = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4,$$

the usual expansion.

Next,

$$\begin{aligned}[e_1e_2e_3e_4][cd] = {} & [e_1e_4cd][e_2e_3] + [e_2e_4cd][e_3e_1] \\ & + [e_3e_4cd][e_1e_2] + [e_2e_3cd][e_1e_4] \\ & + [e_3e_1cd][e_2e_4] + [e_1e_2cd][e_3e_4]. \quad (3)\end{aligned}$$

This is a special case of § 56·25. Multiply by $[ab]$, then we have

$$[e_1e_2e_3e_4][abcd] = [e_1e_4cd][abe_2e_3] + \ldots. \quad (4)$$

Now take $$[e_1e_2e_3e_4] = 1,$$

then $$[e_1e_4cd] = c_2d_3 - c_3d_2.$$

Do this for each term of (4), and we have

$$\begin{aligned}[abcd] = {} & (c_2d_3 - c_3d_2)(a_1b_4 - a_4b_1) + (c_3d_1 - c_1d_3)(a_2b_4 - a_4b_2) \\ & + (c_1d_2 - c_2d_1)(a_3b_4 - a_4b_3) + (c_1d_4 - c_4d_1)(a_2b_3 - a_3b_2) \\ & + (c_2d_4 - c_4d_2)(a_3b_1 - a_1b_3) + (c_3d_4 - c_4d_3)(a_1b_2 - a_2b_1).\end{aligned}$$

This is a simple case of Laplace's expansion. The general formula follows in the same way from § 56·25.

We could, however, interpret each outer product in (2) or (4) as a determinant, writing a_i for e_i. We then have a case of Sylvester's Theorem on the product of two determinants. The general theorem is given below.

4. *Determinants of minors.** In § 56·17 take $n = 4$, then if $[abcd] = K$, we have

$$[abc.abd.acd.bcd] = K^3, \text{ or } [bcd.acd.abd.abc] = K^3. \quad (5)$$

Now if A_1, A_2, A_3, A_4 be the minors of the first row of $[abcd]$, then

$$[bcd] = A_1[e_2e_3e_4] + A_2[e_1e_3e_4] + A_3[e_1e_2e_4] + A_4[e_1e_2e_3].$$

As there are similar results for the other factors of the left-hand side of (5), that expression is the determinant of minors. Thus this determinant equals K^3.

* Turnbull, *Determinants and Matrices.*

Similarly from

$$[abc.abd.acd] = aK^2, \quad [abc.abd] = [ab]K,$$

we also derive theorems on the minors of the determinant $[abcd]$. (Jacobi.)

5. *Müller's generalisation of Laplace's expansion and of Sylvester's product theorem.**

Let $a_1^t, a_2^t, \ldots, a_n^t$, $(t = 1, \ldots, p)$, be p sets of n extensives of step r_t, in a domain of step n. Let $\Delta^{(t)} = [a_1^t a_2^t \ldots a_n^t]$.

Let $A_1^t, \ldots, A_s^t$ be the $s = \binom{n}{r_t}$ multiplicative combinations of $a_1^t, a_2^t, \ldots, a_n^t$ of step r_t, and $D_1^t, D_2^t, \ldots, D_s^t$ be such that

$$[D_i^t A_i^t] = \Delta^{(t)}.$$

Let B^t be a non-zero extensive of step $r_t < n$, then, by §56·25,

$$\Delta^{(t)} B^t = \sum_i [D_i^t B^t] A_i^t, \quad \text{(i runs from 1 to s)}.$$

Let B^{p+1} be a non-zero extensive of step $n-(r_1+r_2+\ldots+r_p)$, then $[B^1 B^2 \ldots B^{p+1}] = \Delta$, say, a scalar.

Hence
$$\Delta^{(1)}\Delta^{(2)} \ldots \Delta^{(p)}\Delta = \Sigma[D_i^1 B^1][D_j^2 B^2] \ldots [D_k^p B^p][A_i^1 A_j^2 \ldots A_k^p B^{p+1}],$$

where the sum is over all combinations of values of i, j, ..., k, and each bracket represents a determinant.

Hence we have *a generalisation of Sylvester's theorem*:

If $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(p)}, \Delta$ *be* $p+1$ *determinants of order* n, *and* Δ *be split into* $p+1$ *sets of say* $r_1, r_2, \ldots, r_p, r_{p+1}$ *rows, and we write the first* p *of these sets with* $n-r_1, n-r_2, \ldots, n-r_p$ *rows of* $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(p)}$ *respectively, to form a determinant, and write the remaining rows of these determinants with the remaining rows of* Δ *to form a determinant, and then multiply together the determinants so formed, and add all such possible products, we get* $\Delta^{(1)}\Delta^{(2)} \ldots \Delta^{(p)}\Delta$. (The signs of the determinants must be properly chosen.)

A special case. Let

$$\Delta^{(1)} = \Delta^{(2)} = \ldots = \Delta^{(p)} = [e_1 e_2 \ldots e_n] = 1,$$

then $\Delta = \Sigma[D_i B^1][D_j B^2] \ldots [D_k B^p][A_i A_j \ldots A_k B^{p+1}]$.

* E. Müller, 'Beweis einiger Determinantensätze', *Zeit. Math. Phys.* 44, (1899), p. 28. The superscripts are merely distinguishing marks.

But now the D, A are products of the unities,

$$[D_i B^1],\ [D_j B^2],\ \ldots,\ [D_k B^p],\ [A_i A_j \ldots A_k B^{p+1}]$$

are minors of Δ taken from $r_1, r_2, \ldots, r_p, r_{p+1}$ rows.

The last outer product is zero unless some of $e_1, \ldots, e_n$ appear twice in it, i.e. unless $[D_i B^1], \ldots, [D_k B^p]$ are taken from distinct columns. This generalises Laplace's expansion.

Examples. 1. In determinant notation,

$$|a_1|a_2b_3c_4| - a_2|a_1b_3c_4| = |a_1c_2|.|a_3b_4| - |a_1b_2|.|a_3c_4|.$$

(M.M.* § 180.)

In our notation this is

$$[ae_2e_3e_4][e_1abc] + [e_1ae_3e_4][e_2abc] = [ace_3e_4][e_1e_2ab] - [abe_3e_4][e_1e_2ac],$$

and this follows from

$$[e_2abc]e_1 + [abce_1]e_2 + [bce_1e_2]a + [ce_1e_2a]b + [e_1e_2ab]c = 0,$$

on outer multiplication by $[ae_3e_4]$.

2. If $|a_1b_2c_3d_4| = 0$, then

$$|a_1b_3|.|a_1c_2d_4| = |a_1b_2|.|a_1c_3d_4| + |a_1b_4|.|a_1c_2d_3|.$$

(M.M. § 180.)

3. $$\begin{vmatrix} |a_1b_2|, & |a_2b_3|, & |a_3b_4| \\ |a_1c_2|, & |a_2c_3|, & |a_3c_4| \\ |a_1d_2|, & |a_2d_3|, & |a_3d_4| \end{vmatrix} = a_2a_3|a_1b_2c_3d_4|.$$

(M.M. §§ 77, 188.)

The left-hand side, if $[e_1e_2e_3e_4] = 1$, equals

$$\begin{vmatrix} [abe_3e_4], & [e_1abe_4], & [e_1e_2ab] \\ [ace_3e_4], & [e_1ace_4], & [e_1e_2ac] \\ [ade_3e_4], & [e_1ade_4], & [e_1e_2ad] \end{vmatrix},$$

and by § 56·28, this equals $[bcd.\gamma_1\gamma_2\gamma_3]$, where

$$\gamma_1 = [ae_3e_4], \quad \gamma_2 = [e_1ae_4], \quad \gamma_3 = [e_1e_2a],$$

and hence $$[\gamma_1\gamma_2\gamma_3] = [ae_3e_4e_1][ae_4e_1e_2]a.$$

Thus the original determinant equals

$$[e_1ae_3e_4][e_1e_2ae_4][abcd] = a_2a_3|a_1b_2c_3d_4|.$$

4. If, in Ex. 3, we write e_1, e_2, e_3, e_4 for a, b, c, d, and vice versa, we obtain

$$\begin{vmatrix} [e_1e_2cd], & [e_1e_2ad], & [e_1e_2ab] \\ [e_1e_3cd], & [e_1e_3ad], & [e_1e_3ab] \\ [e_1e_4cd], & [e_1e_4ad], & [e_1e_4ab] \end{vmatrix} = -[e_1acd][e_1abd][e_1e_2e_3e_4], \quad (6)$$

* M.M. refers to Muir, *Treatise on the Theory of Determinants*, revised by Metzler, privately published, New York (1931).

or, if $[e_1e_2e_3e_4] = 1$, then

$$\begin{vmatrix} |c_3d_4|, & |a_3d_4|, & |a_3b_4| \\ |c_2d_4|, & |a_2d_4|, & |a_2b_4| \\ |c_2d_3|, & |a_2d_3|, & |a_2b_3| \end{vmatrix} = |a_2c_3d_4| \,.\, |a_2b_3d_4|. \qquad \text{(M.M. § 180.)}$$

A formula obtained in this way from another is called its '*complementary*'. For example, the complementary of

$$[abe_3e_4][acde_1] + [ace_3e_4][adbe_1] + [ade_3e_4][abce_1] = [ae_1e_3e_4][abcd]$$

is $$[e_1e_2cd][e_1e_3e_4a] + [e_1e_3cd][e_1e_4e_2a] + [e_1e_4cd][e_1e_2e_3a] = [e_1acd][e_1e_2e_3e_4] \quad (7)$$

and each of these can be expressed in determinant form.

But we can also replace $e_1, \ldots, e_4$ *by any extensives, and thus obtain a formula which includes both as special cases.*

5. The other general law for formulae involving determinants is the '*law of extensible minors*'.

Consider any of the above formulae which involves scalars only, and suppose it relates, for example, to step 4. Consider $\mathscr{S}(e_1, \ldots, e_n)$, of any step n. The formula will hold in the sub-spread $\mathscr{S}(e_1, \ldots, e_4)$. Thus taking (7), we can prolong the outer products by multiplying by any extensives $fg \ldots l$, till they become of step n. Hence, if A be any extensive of step $n-4$,

$$[e_1e_2cdA][e_1e_3e_4aA] + [e_1e_3cdA][e_1e_4e_2aA] + [e_1e_4cdA][e_1e_2e_3aA] = [e_1acdA][e_1e_2e_3e_4A].$$

Again, (6) is an extension of

$$\begin{vmatrix} [e_2cd], & [e_2ad], & [e_2ab] \\ [e_3cd], & [e_3ad], & [e_3ab] \\ [e_4cd], & [e_4ad], & [e_4ab] \end{vmatrix} = -[acd][abd][e_2e_3e_4], \qquad (8)$$

and this follows from § 56·28, since in step 3,

$$[e_2e_3e_4\,..cd.ad.ab] = -[e_2e_3e_4][cda][dab].$$

If we put $b = c$ in (8), and change the notation, we get the extension, (M.M. § 188.)

$$\begin{vmatrix} |a_2b_3c_4d_5|, & |a_2b_3c_4d_6|, & |a_2b_3c_4d_7| \\ |a_1b_3c_4d_5|, & |a_1b_3c_4d_6|, & |a_1b_3c_4d_7| \\ |a_1b_2c_4d_5|, & |a_1b_2c_4d_6|, & |a_1b_2c_4d_7| \end{vmatrix} = -|a_1b_2c_3d_4|^2 |a_4b_5c_6d_7|.$$

6. $[a_1a_2a_3a_4\ldots]$
$= [a_1a_2][a_3a_4\ldots] - [a_1a_3][a_2a_4\ldots] + [a_1a_4][a_2a_3\ldots],$

$[pq][pqa_1a_2a_3a_4\ldots]$
$= [pqa_1a_2][pqa_3a_4\ldots] - [pqa_1a_3][pqa_2a_4\ldots]$
$+ [pqa_1a_4][pqa_2a_3\ldots],$

$[pq][pa_1a_2a_3] = [pqa_1a_2][pa_3] - [pqa_1a_3][pa_2] + [pqa_2a_3][pa_1].$*

7. $\begin{vmatrix} [abe_3e_4], & [bce_3e_4], & [cde_3e_4] \\ [abe_4e_2], & [bce_4e_2], & [cde_4e_2] \\ [abe_2e_3], & [bce_2e_3], & [cde_2e_3] \end{vmatrix} = [abcd][be_2e_3e_4][ce_2e_3e_4].$

(M.M. §§ 77, 180.)

For, in step 4,

$[cd][be_2e_3e_4] = [bcde_4][e_2e_3] + [bcde_2][e_3e_4] + [bcde_3][e_4e_2]$
$+$ terms involving $[be_i]$ $(i = 1, \ldots, 4)$.

Hence $[abcd][be_2e_3e_4]$
$= [bcde_4][abe_2e_3] + [bcde_2][abe_3e_4] + [bcde_3][abe_4e_2].$ (9)

Now $[bcde_4]e_2 + [cde_4e_2]b + [de_4e_2b]c + [e_4e_2bc]d + [e_2bcd]e_4 = 0.$

Hence $[bcde_4][ce_2e_3e_4] - [cde_4e_2][bce_3e_4]$
$+ [e_4e_2bc][cde_3e_4] = 0.$ (10)

Multiply (9) by $[ce_2e_3e_4]$, use (10) and similar formulae, and we obtain the result.

8. If capital letters denote co-factors, and we use the ordinary determinant notation†:

$|a_1A_2A_3.b_2B_3B_1.c_3C_1C_2| = -|a_1b_2c_3|.|a_2a_3.b_3b_1.c_1c_2|^2,$

$|A_1a_2a_3.B_2b_3b_1.C_3c_1c_2| = -|a_2a_3.b_3b_1.c_1c_2|^2,$

$|a_1A_1.b_2B_2.c_3C_3| = |a_1b_2c_3|\{|a_1b_2c_3|^2 - |a_2a_3.b_3b_1.c_1c_2|\},$

$|B_1C_1.C_2A_2.A_3B_3| = -|a_1b_2c_3|^2.|b_1c_1.c_2a_2.a_3b_3|,$

$|A_1^2B_2^2C_3^2| - |A_1A_2.B_2B_3.C_3C_1|$
$= |a_1b_2c_3|^2\{|a_1^2b_2^2c_3^2| - |a_1a_2.b_2b_3.c_3c_1|\}.$

6. *A general theorem on determinants whose elements are determinants.*‡

Let $E_1, \ldots, E_s$ be the $s = \binom{n}{r}$ outer products, in any order, of step r of the original unities $e_1, \ldots, e_n$. These unities of step r may be regarded as unities of a spread $\mathscr{S}$ of step s. Let $a_1, \ldots, a_n$ be

* H. F. Baker, *Proc. Lond. Math. Soc.* (i), 29, p. 141; *Am. Journ. Math.* 20, (1898), p. 360.
† Muir, *Contributions to the History of Determinants*, pp. 307, 346, 357.
‡ E. Müller, *loc. cit.*

extensives of $\mathscr{S}(e_1, \ldots, e_n)$, and let $A_1, \ldots, A_s$ be multiplicative combinations of the a formed in the same way as $E_1, \ldots, E_s$ are formed from the e. Let $E'_1, \ldots, E'_s$ be combinations of $e_1, \ldots, e_n$ of step $n-r$ such that $[E_i E'_i] = 1$. Then $A_j = \sum_k [A_j E'_k] E_k$.

If $A_1 = [a_{i_1} a_{i_2} \ldots a_{i_r}]$, and $E'_k = [e_{k_1} e_{k_2} \ldots e_{k_{n-r}}]$,

and $\Delta = [a_1 \ldots a_n]$, then $[A_i E'_k]$ is the cofactor of Δ derived from rows $i_1, i_2, \ldots, i_r$ by suppressing columns $k_1, k_2, \ldots, k_{n-r}$.

Consider outer multiplication in spread $\mathscr{G}$ and denote it by $[\,]_{\mathscr{G}}$, and let $[E_1 \ldots E_s]_{\mathscr{G}} = 1$.

Then $[A_1 A_2 \ldots A_s]_{\mathscr{G}} = \mathrm{Det}\,[A_j E'_k]$, $(j = 1, \ldots, s;\ k = 1, \ldots, s)$,

where Det[ ] denotes the determinant of the outer products $[A_j E'_k]$ formed in $\mathscr{S}(e_1, \ldots, e_n)$. Thus *this determinant*, Δ_1 say, *has for its elements the minors of* Δ *of order* r.

Let
$$b = x_1 a_1 + \ldots + x_n a_n, \tag{11}$$

fix i, and denote the multiplicative combinations of $a_1, a_2, \ldots, a_{i-1}$, $b, a_{i+1}, \ldots, a_n$ of step r corresponding to $E_1, \ldots, E_s$ by $B_1, \ldots, B_s$. Of these, $t = \binom{n-1}{r-1}$ contain b, and if in them we substitute for b, then such a B_k becomes a sum of extensives A_j in which A_i appears with coefficient x_i. If we form the product $[B_1 \ldots B_s]_{\mathscr{G}}$, then from B_k we can omit all terms A_j except A_k. The B which do not contain b equal the corresponding A. Hence

$$[B_1 \ldots B_s]_{\mathscr{G}} = x_i^t [A_1 \ldots A_s]_{\mathscr{G}}.$$

But, by (11),

$$[a_1 \ldots a_n]\, x_i = [a_1 \ldots a_{i-1} b a_{i+1} \ldots a_n] = \Delta', \text{ say.}$$

Hence $x_i = \Delta'/\Delta$, $[B_1 \ldots B_s]_{\mathscr{G}} \div \Delta'^t = [A_1 \ldots A_s]_{\mathscr{G}} \div \Delta^t$.

Hence *this equation holds when all the a are replaced by any independent extensives in* $\mathscr{S}(a_1, \ldots, a_n)$, *if* Δ' *is their outer product.*

In particular, take $e_1, \ldots, e_n$ for $a_1, \ldots, a_n$, then the fraction equals unity. Hence
$$[A_1 \ldots A_s]_{\mathscr{G}} = \Delta^t.$$

Hence $\Delta_1 = \Delta^t$ where $t = \binom{n-1}{r-1}$, and this case is *Sylvester's theorem on compound determinants.*

7. We call Δ_1 above the rth compound determinant of Δ. Then *a minor of the* k*th order of* Δ_1 *is the product of the cofactor of the corresponding minor in the* $n-r$*th compound determinant and* Δ^p, *where* $p = \binom{n-1}{m}$. (M.M. § 185.)

The m*th compound of a product of two determinants equals the product of the* m*th compounds of the two factors, element by element.*

8. *Bazin-Reiss-Picquet. If* D, D′ *be determinants of the same order* n, *and every set of* q *columns of* D *be replaced in turn by every set of* q *columns of* D′, *and* Δ *be the determinant of the square array thus formed, then*

$$\Delta = D^r D'^s \quad \text{where} \quad r = \binom{n-1}{q}, \; s = \binom{n-1}{q-1}.$$

(M.M. § 202.)

For example,*

$$\begin{vmatrix} [b_1 a_2 \dots a_n], & [a_1 b_1 a_3 \dots a_n], & \dots, & [a_1 a_2 \dots a_{r-1} b_1 a_{r+1} \dots a_n] \\ [b_2 a_2 \dots a_n], & [a_1 b_2 a_3 \dots a_n], & \dots, & [a_1 a_2 \dots a_{r-1} b_2 a_{r+1} \dots a_n] \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ [b_r a_2 \dots a_n], & [a_1 b_r a_3 \dots a_n], & \dots, & [a_1 a_2 \dots a_{r-1} b_r a_{1+1} \dots a_n] \end{vmatrix}$$
$$= [a_1 \dots a_n]^{r-1} [b_1 \dots b_r a_{r+1} \dots a_n].$$

This theorem, of which the last theorem in Ex. 5 is a case, is an extension of a simple case of our general theorem:

$$\begin{vmatrix} [b_1 a_2 \dots a_r], & [a_1 b_1 a_3 \dots a_r], & \dots, & [a_1 a_2 \dots a_{r-1} b_1] \\ [b_2 a_2 \dots a_r], & [a_1 b_2 a_3 \dots a_r], & \dots, & [a_1 a_2 \dots a_{1-1} b_2] \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ [b_r a_2 \dots a_r], & [a_1 b_r a_3 \dots a_r], & \dots, & [a_1 a_2 \dots a_{r-1} b_r] \end{vmatrix}$$
$$= [a_1 \dots a_r]^{r-1} [b_1 \dots b_r]$$

and, if we take $a_1, \dots, a_r$ as unities, $e_1, \dots, e_r$ in step r, this is obvious.

§ 61. *Arrays of* m *rows and* n *columns*, $n > m$.

1. The array may be denoted by m row-extensives $a_1, \dots, a_m$ in $\mathscr{S}(e_1, \dots, e_n)$ or by n column-extensives $b_1, \dots, b_n$ in $\mathscr{S}(e_1, \dots, e_m)$.

2. *If in a given array a certain* p*-rowed determinant is not zero, but all* $p+h$*-rowed determinants which contain it, vanish, then all* $p+h$*-rowed determinants of the array vanish.* (M.M. § 234.)

* See Muir, *History of Determinants*, 2, p. 206.

Consider the case $h=1$. Take row-extensives $a_1, \ldots, a_m$ in $\mathscr{S}(e_1, \ldots, e_n)$. We can suppose that $[a_1 \ldots a_p e_{p+1} \ldots e_n]$ is the non-vanishing determinant. By hypothesis, a_i, $(i=p+1, \ldots, n)$, is on each spread

$$[a_1 \ldots a_p e_{p+2} \ldots e_n],\ [a_1 \ldots a_p e_{p+1} e_{p+3} \ldots e_n],\ \ldots,\ [a_1 \ldots a_p e_{p+1} \ldots e_{n-1}],$$

that is, on $n-p$ spreads of step $n-1$. These cut in the spread $[a_1 \ldots a_p]$, hence $a_{p+1}, \ldots, a_n$ are in the latter spread, and thus every product of $p+1$ of the $a_1, \ldots, a_n$ vanishes.

The general theorem is proved by an extension of the argument.

3. *If in an array of* n *rows,* $n+1$ *columns,* $n-p$ *of the* n*-lined minors vanish, and at least one does not, then the array common to the said* $n-p$ *minors has all its minors of order* $n-p$ *vanishing.* (M.M. §238.)

Take row-extensives $a_1, \ldots, a_n$ in $\mathscr{S}(e_1, \ldots, e_{n+1})$, and let

$$A_1 = [e_1 a_1 \ldots a_n],\ A_2 = [e_2 a_1 \ldots a_n],\ \ldots,\ A_{n+1} = [e_{n+1} a_1 \ldots a_n].$$

Suppose $A_1, A_2, \ldots, A_{n-p} = 0$, $A_{n-p+1} \neq 0$, then $[a_1 \ldots a_n] \neq 0$, and $e_1, e_2, \ldots, e_{n-p}$ are in $[a_1 \ldots a_n]$.

Hence $[e_1 \ldots e_{n-p}]$ when multiplied by any p of the $a_1, \ldots, a_n$ vanishes.

But these products are the minors of order $n-p$ of the common array.

For example, if $n = 3$, and

$$[e_i a_1 a_2 a_3] = 0 \ (i=1, 2, 3), \quad [e_4 a_1 a_2 a_3] \neq 0,$$

then e_1, e_2, e_3 are in $[a_1 a_2 a_3]$, and $[a_1 a_2 a_3] \neq 0$.

Hence $[e_1 e_2 e_3] \equiv [a_1 a_2 a_3]$, and the e_4-coordinates of a_1, a_2, a_3 vanish.

4. *If in an array of* n *rows and* $n+1$ *columns, the sum of the elements in each row is zero, then the principal minors are equal, apart from sign.* (M.M. §250.)

For take column-extensives $b_1, \ldots, b_{n+1}$ in $\mathscr{S}(e_1, \ldots, e_n)$, then $b_1 + b_2 + \ldots + b_{n+1} = 0$; thence $[b_1 b_3 \ldots b_{n+1}] = -[b_2 b_3 \ldots b_{n+1}]$.

5. *Relations between the* m*-rowed minors of an array of* m *rows and* n *columns* $(n > m)$. (M.M. §240.)

Take column-extensives $b_1, \ldots, b_n$ in $\mathscr{S}(e_1, \ldots, e_m)$; these must be dependent. Consider the spreads of step m formed from them,

such as $[b_1 \dots b_m]$. Between these there will be relations, and these will be replaced by relations, and not lost, when we take $b_1, \dots, b_m$ as $e_1, \dots, e_m$. By doing this we fix m^2 of the coordinates of $b_1, \dots, b_n$ and leave $m(n-m)$ unfixed. These coordinates give the $\binom{n}{m} - 1$ ratios of the minors in question. Hence between these minors there are at most

$$\binom{n}{m} - 1 - m(n-m)$$

relations.

For example, in step 3, consider the array of three rows and six columns corresponding to the column-extensives a, b, c, f, g, h. There are ten relations.

Nine of these are

$$[abc][agh] + [acg][abh] + [agb][ach] = 0$$

(which is an extension of the relation in step 2:

$$[bc][gh] + [cg][bh] + [gb][ch] = 0),$$

and those obtained by cycling a, b, c and cycling f, g, h.

The tenth relation is

$$[fgh][abc]^2 = \begin{vmatrix} [fbc], & [fca], & [fab] \\ [gbc], & [gca], & [gab] \\ [hbc], & [hca], & [hab] \end{vmatrix}.$$

Example 9. The following determinant appears frequently in later work, writing ij for a_{ij}:

$$\begin{vmatrix} 0 & 12 & 13 & 14 \\ 21 & 0 & 23 & 24 \\ 31 & 32 & 0 & 34 \\ 41 & 42 & 43 & 0 \end{vmatrix}$$

$$= (\sqrt{12.34} + \sqrt{23.14} + \sqrt{31.24})(+ + -)(+ - +)(- + +),$$

where the last three factors contain the same terms as the first factor, with the signs indicated.

CHAPTER IX

TRANSFORMATIONS AND SQUARE MATRICES WITH APPLICATIONS TO CENTRAL QUADRICS

§62. *General notation.*

1. Let $e_1, \ldots, e_n$ span a spread of step n, and let $a_1, \ldots, a_n$ be n extensives of step one in the spread. Let

$$\begin{aligned} a_1 &= a_{11} e_1 + a_{12} e_2 + \ldots + a_{1n} e_n, \\ a_2 &= a_{21} e_1 + a_{22} e_2 + \ldots + a_{2n} e_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n &= a_{n1} e_1 + a_{n2} e_2 + \ldots + a_{nn} e_n. \end{aligned} \qquad (1)$$

These can be regarded as the equations of a linear transformation in the spread which changes $e_1, \ldots, e_n$ into $a_1, \ldots, a_n$. We say a_{rs} is the '*coordinate*' of the transformation in the rs place.

The transformation, being linear (§ 10·2), replaces

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

by $$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = a, \text{ say.}$$

We denote the transformation by $\mathfrak{A}$, and if $\mathfrak{A}$ transforms x into a, we write

$$a = x\mathfrak{A}.$$

We have

$$x\mathfrak{A} = \sum_{i=1}^{n} x_i a_i = \sum_{i,j=1}^{n} x_i a_{ij} e_j, \quad e_i \mathfrak{A} = \sum_{j=1}^{n} a_{ij} e_j = a_i.$$

Hence the coordinates of $x\mathfrak{A}$ in the frame $e_1, \ldots, e_n$ are

$$\sum_i x_i a_{ij}, \quad (j = 1, \ldots, n).$$

We may write $(a_{ij})_n$ for the matrix* of equations (1), and omit n if no confusion results.

We write $\det \mathfrak{A}$ for the determinant whose terms are a_{ij}.

2. Since our transformation is linear, we have, if x, y be any extensives of step one, $(x+y)\mathfrak{A} = x\mathfrak{A} + y\mathfrak{A}$.

* Throughout the rest of this book all matrices are *square* matrices.

If $\mathfrak{A}$, $\mathfrak{B}$ be linear transformations, and

$$x\mathfrak{A} = \sum_{i,j} x_i a_{ij} e_j, \quad x\mathfrak{B} = \sum_{i,j} x_i b_{ij} e_j,$$

then
$$x\mathfrak{A} + x\mathfrak{B} = \sum_{i,j} x_i (a_{ij} + b_{ij}) e_j.$$

Let
$$x\mathfrak{C} = \sum_{i,j} x_i c_{ij} e_j, \quad c_{ij} = a_{ij} + b_{ij},$$

then
$$x\mathfrak{A} + x\mathfrak{B} = x\mathfrak{C}.$$

As this holds for all x, we can define $\mathfrak{A}+\mathfrak{B}$ to mean $\mathfrak{C}$, when the coordinate of $\mathfrak{C}$ in the rs place is the sum of the corresponding coordinates of $\mathfrak{A}$, $\mathfrak{B}$. Then $x(\mathfrak{A}+\mathfrak{B}) = x\mathfrak{A} + x\mathfrak{B}$.

3. A '*zero transformation*' is one all of whose coordinates vanish. We denote it by $\mathfrak{O}$. Then $x\mathfrak{O} = 0$ for all extensives x (of step one); and if $x\mathfrak{A} = 0$ for all x, then $\mathfrak{A} = \mathfrak{O}$. We may have $x\mathfrak{A} = 0$ for some x, and yet $\mathfrak{A} \neq \mathfrak{O}$. If $x = 0$, then $x\mathfrak{A} = 0$ for all $\mathfrak{A}$.

$$\mathfrak{A} + \mathfrak{O} = \mathfrak{O} + \mathfrak{A} = \mathfrak{A}.$$

4. If k is scalar, we define $k\mathfrak{A}$ so that $x.k\mathfrak{A} = k.x\mathfrak{A}$ for all extensives x, and we define $\mathfrak{A}k$ so that $x.\mathfrak{A}k = k.x\mathfrak{A}$ for all extensives x. Hence if $\mathfrak{A} = (a_{ij})$, then $k\mathfrak{A} = \mathfrak{A}k = (ka_{ij})$.

5. If x, y, z be extensives of step one and $y = x\mathfrak{A}$, $z = y\mathfrak{B}$, then $z = (x\mathfrak{A})\mathfrak{B}$. We naturally wish to write $z = x.(\mathfrak{A}\mathfrak{B})$.

Using components, let $y_j = \sum_i x_i a_{ij}$, $z_k = \sum_j y_j b_{jk}$, then $z_k = \sum_{i,j} x_i a_{ij} b_{jk}$. Hence $z_k = \sum_i x_i c_{ik}$, provided $c_{ik} = \sum_j a_{ij} b_{jk}$.

Hence we define the product $\mathfrak{C}$ of matrices $\mathfrak{A} = (a_{ij})_n$, $\mathfrak{B} = (b_{ij})_n$ as $(c_{ij})_n$, where $c_{ik} = \sum_j a_{ij} b_{jk}$, and write $\mathfrak{C} = \mathfrak{A}\mathfrak{B}$.

Then if $y = x\mathfrak{A}$, $z = y\mathfrak{B}$ and $\mathfrak{A}\mathfrak{B} = \mathfrak{C}$, we have

$$z = x\mathfrak{A}.\mathfrak{B} = x.\mathfrak{A}\mathfrak{B} = x\mathfrak{C}.$$

Also
$$\mathfrak{A}.\mathfrak{B}\mathfrak{D} = \mathfrak{A}\mathfrak{B}.\mathfrak{D}, \quad \mathfrak{A}(\mathfrak{B}+\mathfrak{D}) = \mathfrak{A}\mathfrak{B} + \mathfrak{A}\mathfrak{D},$$
$$(\mathfrak{A}+\mathfrak{B})\mathfrak{D} = \mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{D}, \quad \mathfrak{A}\mathfrak{O} = \mathfrak{O}\mathfrak{A} = \mathfrak{O}.$$

But $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}$ need not be true.

We write $\mathfrak{A}^2$ for $\mathfrak{A}\mathfrak{A}$, and $\mathfrak{A}^n$ for $\mathfrak{A}^{n-1}.\mathfrak{A}$.

The rule for multiplying determinants gives

$$\det(\mathfrak{A}\mathfrak{B}) = \det\mathfrak{A}.\det\mathfrak{B}.$$

We call the present type of product of transformations or matrices, a '*sequence product*'.

6. The '*identical transformation*' is the transformation, denoted by $\mathfrak{J}$, which satisfies $x\mathfrak{J} = x$ for all extensives x of step one.

If a_{ij} be its components, then $a_{ij} = 0$ if $i \neq j$, $a_{ij} = 1$ if $i = j$. For all transformations $\mathfrak{A}$, $\mathfrak{A}\mathfrak{J} = \mathfrak{J}\mathfrak{A} = \mathfrak{A}$.

7. A '*non-singular*' transformation or matrix $\mathfrak{A}$ of order n is one such that its equations (1) have rank n, that is, $\det \mathfrak{A} \neq 0$. For such transformations, the equation $y = x\mathfrak{A}$ has a unique solution x when y is given. We denote it by $x = y\mathfrak{A}^{-1}$ and call $\mathfrak{A}^{-1}$ the transformation or matrix '*inverse*' of $\mathfrak{A}$.

Since $y = x\mathfrak{A}$, $x = y\mathfrak{A}^{-1}$, we have, for all x, y; $x = x\mathfrak{A}\mathfrak{A}^{-1}$, $y = y\mathfrak{A}^{-1}\mathfrak{A}$. Hence

$$\mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{A}^{-1}\mathfrak{A} = \mathfrak{J}, \quad (\mathfrak{A}^{-1})^{-1} = \mathfrak{A}, \quad (\mathfrak{A}\mathfrak{B})^{-1} = \mathfrak{B}^{-1}\mathfrak{A}^{-1}.$$

The last follows since

$$\mathfrak{A}\mathfrak{B}.\mathfrak{B}^{-1}\mathfrak{A}^{-1} = \mathfrak{A}.\mathfrak{B}\mathfrak{B}^{-1}.\mathfrak{A}^{-1} = \mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{J}.$$

If $x\mathfrak{A} = 0$ for some $x \neq 0$, then $\det \mathfrak{A} = 0$ (§ 59·8).

If $\mathfrak{A}$ is non-singular, and $\mathfrak{A}\mathfrak{B} = \mathfrak{O}$, then $\mathfrak{B} = \mathfrak{O}$; if $\mathfrak{B}$ is non-singular, and $\mathfrak{A}\mathfrak{B} = \mathfrak{O}$, then $\mathfrak{A} = \mathfrak{O}$. These follow from the definition of multiplication and § 59·8.

8. When the frame $e_1, \ldots, e_n$ is given, the transformation $\mathfrak{A}$ is determined by the matrix of its coordinates, and if we keep the frame fixed throughout, there is little need to distinguish between the transformation and the matrix which represents it. When frames are changed, the distinction must be considered.

9. If $\mathfrak{A} = (a_{ij})_n$ be a matrix, the '*transposed*' matrix (§ 59·1) will always be denoted by $\mathfrak{A}^* = (a^*_{ij})_n$. Thus $a^*_{ij} = a_{ji}$.

$$(\mathfrak{A}^*)^* = \mathfrak{A}, \quad (\mathfrak{A}\mathfrak{B})^* = \mathfrak{B}^*\mathfrak{A}^*, \quad \det \mathfrak{A}^* = \det \mathfrak{A}.$$

$$\text{If} \quad x = \sum_{i=1}^{n} x_i e_i, \quad \mathfrak{A} = (a_{ij})_n, \quad a_i = \sum_{j=1}^{n} a_{ij} e_j, \quad e_i \mathfrak{A} = a_i,$$

$$\text{then} \quad x\mathfrak{A} = \sum_i x_i a_i = \sum_{i,j} x_i a_{ij} e_j = \sum_{i,j} x_j a_{ji} e_i = \sum_{i,j} a^*_{ij} x_j e_i.$$

Now as $x\mathfrak{A}$ represents the extensive whose i-th coordinate is $\sum_j x_j a_{ji}$, we naturally let $\mathfrak{B}x$ represent the extensive whose i-th coordinate is $\sum_j b_{ij} x_j$, where the b_{ij} are the coordinates of $\mathfrak{B}$

Notice that in each case the adjacent suffixes are the same.

Hence $$x\mathfrak{A} = \mathfrak{A}^* x, \quad x\mathfrak{A}^* = \mathfrak{A}x.$$

The operation of transposing a matrix is evidently relevant to a frame; transformations which in one frame correspond to transposed matrices do not necessarily do so in another.

10. In working with matrices of fixed order n, we denote by $\mathfrak{E}_{ij}$ the matrix whose coordinate in the ij place is 1, and all other coordinates zero.

$$\mathfrak{E}_{ij}\mathfrak{E}_{jk} = \mathfrak{E}_{ik}, \quad \mathfrak{E}_{ij}\mathfrak{E}_{lk} = \mathfrak{O} \quad \text{if } l \neq j.$$

Any matrix $\mathfrak{A}$ of order n can be written in form $\sum_{ij} a_{ij}\mathfrak{E}_{ij}$, when $\mathfrak{A} = (a_{ij})$.

A matrix whose coordinates vanish except those on the diagonal is a '*diagonal matrix*'. If these elements are $a_1, \ldots, a_n$, we may denote the matrix by $\operatorname{diag}(a_1, \ldots, a_n)$. If $a_i \neq 0$ $(i = 1 \ldots n)$, its inverse is $\operatorname{diag}(a_1^{-1}, \ldots, a_n^{-1})$.

§ 63. *Elementary properties of matrices of order* n.

1. *The rank of* $\mathfrak{AB}$ *is not greater than the rank of* $\mathfrak{A}$ *or* $\mathfrak{B}$.

For, let the row-extensives of $\mathfrak{A}$ in spread $\mathscr{S}(e_1, \ldots, e_n)$ be $a_1, \ldots, a_n$, and let the column-extensives of $\mathfrak{B}$ in the spread $\mathscr{S}(E_1, \ldots, E_n)$ be $B_1, \ldots, B_n$, where $E_i = |e_i$, $(i = 1, \ldots, n)$, and $[e_1 \ldots e_n] = 1$.

Then the element of $\mathfrak{AB}$ in the ik place is $[a_i B_k]$. Hence any t-rowed minor of $\det(\mathfrak{AB})$ is of form

$$\begin{vmatrix} [a_{i_1}B_{k_1}], & \ldots, & [a_{i_1}B_{k_t}] \\ [a_{i_t}B_{k_1}], & \ldots, & [a_{i_t}B_{k_t}] \end{vmatrix} = [a_{i_1}a_{i_2}\ldots a_{i_t}.B_{k_1}B_{k_2}\ldots B_{k_t}],$$

where $i_1, i_2, \ldots, i_t$ and $k_1, k_2, \ldots, k_t$ are each a set of t distinct numbers drawn from the set $1 \ldots n$ (§ 56·28).

Now if the rank of $\mathfrak{A}$ is $t-1$, then $[a_{i_1}a_{i_2}\ldots a_{i_t}] = 0$, and hence any t-rowed minor of $\det \mathfrak{AB}$ vanishes. Hence the rank of $\mathfrak{AB}$ is not greater than the rank of $\mathfrak{A}$ or $\mathfrak{B}$.

Cor. If $\mathfrak{A}$ *is non-singular, then* $\mathfrak{AB}$ *and* $\mathfrak{BA}$ *have the same rank as* $\mathfrak{B}$.

For, if the rank of $\mathfrak{B}$ is r, that of $\mathfrak{AB}$ is $\leqq r$. Let $\mathfrak{AB}$ have rank s; then since $\mathfrak{B} = \mathfrak{A}^{-1}.\mathfrak{AB}$, we have $r \leqq s$. Hence $r = s$.

2. *The secular equation.*

If $\mathfrak{A}$ be a matrix, and $k_0, k_1, \ldots k_r$ be scalars, then

$$k_0 + k_1\mathfrak{A} + k_2\mathfrak{A}^2 + \ldots + k_r\mathfrak{A}^r$$

is a '*polynomial*' in $\mathfrak{A}$, where $\mathfrak{A}^2, \mathfrak{A}^3, \ldots$ are sequence-powers. These polynomials can be multiplied together like ordinary polynomials in a scalar x, and, since $\mathfrak{A}^m\mathfrak{A}^s = \mathfrak{A}^s\mathfrak{A}^m$, they obey the commutative law of multiplication, as well as the other formal laws of the algebra of scalars.

Let x be an extensive in $\mathscr{S}(e_1, \ldots, e_n)$, and $\mathfrak{A}$ be a linear transformation in this spread, with $\det\mathfrak{A} \neq 0$. If x be such that $x\mathfrak{A} \equiv x$, $x \neq 0$, we say x is '*latent*' for $\mathfrak{A}$.

If such a latent extensive exists and $x\mathfrak{A} = \lambda x$, ($\lambda$ scalar), then $\det(\mathfrak{A} - \lambda\mathfrak{J}) = 0$.

We call $\mathfrak{A} - \lambda\mathfrak{J}$ the '*secular matrix*' of $\mathfrak{A}$; $\det(\mathfrak{A} - \lambda\mathfrak{J})$ is the '*secular polynomial*'; $\det(\mathfrak{A} - \lambda\mathfrak{J}) = 0$ is the '*secular equation*', it is of order n, the order of $\mathfrak{A}$; a root of the equation is a '*characteristic root*' of $\mathfrak{A}$.

If λ is a characteristic root of $\mathfrak{A}$, and $f(\mathfrak{A})$ a polynomial in $\mathfrak{A}$, then $f(\lambda)$ is a characteristic root of $f(\mathfrak{A})$, and it corresponds to the same latent extensive x.

For, if $x\mathfrak{A} = \lambda x$, then $x\mathfrak{A}^2 = \lambda x\mathfrak{A} = \lambda^2 x, \ldots, x\mathfrak{A}^n = \lambda^n x$.

This is true, also, if $\lambda = 0$. Hence $x.f(\mathfrak{A}) = f(\lambda)x$.

Cor. If $\lambda \neq 0$, then λ^{-1} is a characteristic root of $\mathfrak{A}^{-1}$, and λ^{-n} of $\mathfrak{A}^{-n}$, provided $\mathfrak{A}$ is non-singular.

For operate on $x\mathfrak{A} = \lambda x$ by $\mathfrak{A}^{-1}$; then $x = \lambda x\mathfrak{A}^{-1}$, $\lambda^{-1}x = x\mathfrak{A}^{-1}$.

To each characteristic root corresponds at least one latent extensive, which may be imaginary if the root is so.

For, if λ is a characteristic root, $(\mathfrak{A} - \lambda\mathfrak{J})x = 0$ has a solution $x \neq 0$, since $\det(\mathfrak{A} - \lambda\mathfrak{J}) = 0$ (§ 57·8).

3. *Similar matrices.*

If $\mathfrak{B}$ is a non-singular matrix, we say $\mathfrak{A}$ and $\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1}$ are '*similar matrices*'. If $\mathfrak{A}$ is also non-singular, so is $\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1}$.

Similar matrices have the same secular equation, and hence the same characteristic roots.

For $$\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1} - \lambda\mathfrak{J} = \mathfrak{B}(\mathfrak{A} - \lambda\mathfrak{J})\mathfrak{B}^{-1},$$

$$\det(\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1} - \lambda\mathfrak{J}) = \det\mathfrak{B}.\det(\mathfrak{A} - \lambda\mathfrak{J}).\det\mathfrak{B}^{-1} = \det(\mathfrak{A} - \lambda\mathfrak{J}).$$

4. *Symmetric matrices.*

If $x = \Sigma x_i e_i$, $y = \Sigma y_i e_i$ be extensives in $\mathscr{S}(e_1, \ldots, e_n)$, where $[e_1 e_2 \ldots e_n] = 1$, and $\mathfrak{A} = (a_{ij})$, then

$$[x\mathfrak{A}\,|\,y] = [\sum_{i,j} x_i a_{ij} e_j \,|\, \sum_k y_k e_k] = \sum_{i,j} x_i a_{ij} y_j, \qquad (\S 56{\cdot}2),$$

$$[y\mathfrak{A}^*\,|\,x] = [\sum_{i,j} y_i a_{ji} e_j \,|\, \sum_k x_k e_k] = \sum_{i,j} y_i a_{ji} x_j = \sum_{i,j} x_j a_{ji} y_i.$$

Hence $[x\mathfrak{A}\,|\,y] = [y\mathfrak{A}^*\,|\,x] = [x\,|\,y\mathfrak{A}^*]$, by § 56·4.

Thus $[x\mathfrak{A}\,|\,y] = [x\,|\,y\mathfrak{A}^*]$; similarly

$$[x\,|\,y\mathfrak{A}] = [x\mathfrak{A}^*\,|\,y]. \qquad (2)$$

If π be a spread of step $n-1$, and $\pi = |\,y$, we have

$$[x\mathfrak{A}\,.\,\pi] = [x\,.\,\pi\mathfrak{A}^*],$$

where $\pi\mathfrak{A}^*$ is the same function of $|e_1, |e_2, \ldots, |e_n$ that $x\mathfrak{A}^*$ is of $e_1, e_2, \ldots, e_n$. Hence also $[\pi\,.\,x\mathfrak{A}] = [\pi\mathfrak{A}^*\,.\,x]$.

A matrix $\mathfrak{A}$ is '*symmetric*' if $\mathfrak{A} = \mathfrak{A}^*$; then $a_{ij} = a_{ji}$, for each ij, and $[x\mathfrak{A}\,|\,y] = [x\,|\,y\mathfrak{A}]$ for all extensives x, y.

A matrix $\mathfrak{A}$ is '*skew-symmetric*' if $\mathfrak{A} = -\mathfrak{A}^*$.

5. *The characteristic roots of a symmetric matrix (with real coordinates) are all real.*

For let $\mathfrak{A}$ be a symmetric matrix representing a transformation in $\mathscr{S}(e_1, \ldots, e_n)$; if k be a latent root, then there is an extensive $x \neq 0$ such that $x\mathfrak{A} = kx$. Let $\bar{x}$ be the extensive whose coordinates in the frame $(e_1, \ldots, e_n)$ are conjugate complex to those of x. (Hence if x is real, then $x = \bar{x}$.) Let $\bar{k}$ be the scalar, conjugate complex to k. Then $\bar{x}\mathfrak{A} = \bar{k}\bar{x}$, since the coordinates of $\mathfrak{A}$ are real.

Hence $\quad [x\mathfrak{A}\,|\,\bar{x}] = k[x\,|\,\bar{x}], \quad [\bar{x}\mathfrak{A}\,|\,x] = \bar{k}[\bar{x}\,|\,x].$

Now $\mathfrak{A}$ is symmetric.

Hence, by 4, $[x\mathfrak{A}\,|\,\bar{x}] = [\bar{x}\mathfrak{A}\,|\,x]$, hence $k = \bar{k}$. Hence k is real.

6. *Outer products of matrices.*

We generalise the work in §§ 48–50. If $\mathfrak{A}$ is a linear transformation or matrix, in $\mathscr{S}(e_1, \ldots, e_n)$, and $p' = p\mathfrak{A}$, $q' = q\mathfrak{A}$, $r' = r\mathfrak{A}$, where p, q, r are any extensives in the spread, then $[p'q'] = [p\mathfrak{A}\,.\,q\mathfrak{A}]$ is a transformation of $[pq]$. We write

$$[p'q'] = [pq][\mathfrak{A}^2].$$

Also $[p'q'r'] = [p\mathfrak{A}\,.\,q\mathfrak{A}\,.\,r\mathfrak{A}]$ is a transformation of $[pqr]$. We write

$$[p'q'r'] = [pqr][\mathfrak{A}^3].$$

Similarly, we can introduce $[\mathfrak{A}^r]$, $(r = 1, \ldots, n)$. We call $[\mathfrak{A}^2]$, $[\mathfrak{A}^3], \ldots$ '*outer powers*' of $\mathfrak{A}$. They are *linear transformations* in the spreads traversed by $[pq], [pqr], \ldots$ as p, q, r, ... traverse $\mathscr{S}(e_1, \ldots, e_n)$.

For example, if L, M, N be rotors in ordinary space, and $L+M=N$, then there are points p, q, r, s such that

$$L = [pq], \quad M = [pr], \quad N = [ps], \quad s = q+r.$$

Thence

$$\begin{aligned}[pq][\mathfrak{A}^2]+[pr][\mathfrak{A}^2] &= [p\mathfrak{A}.q\mathfrak{A}]+[p\mathfrak{A}.r\mathfrak{A}] = [p\mathfrak{A}.(q\mathfrak{A}+r\mathfrak{A})] \\ &= [p\mathfrak{A}.(q+r)\mathfrak{A}] = [p\mathfrak{A}.s\mathfrak{A}] = [ps][\mathfrak{A}^2].\end{aligned}$$

The matrix $[\mathfrak{A}^r]$ is of order $\binom{n}{r}$.

If $p, q, \ldots, w$ be n independent extensives in $\mathscr{S}(e_1, \ldots, e_n)$, and $p' = p\mathfrak{A}, \ldots, w' = w\mathfrak{A}$, then $[pq \ldots w]$ and $[p'q' \ldots w']$ are scalars; hence we must regard $[\mathfrak{A}^n]$ as a scalar, whereas $[\mathfrak{A}^2], [\mathfrak{A}^3], \ldots, [\mathfrak{A}^r]$ with $r < n$ are not scalars but linear transformations.

If $a_1 = e_1\mathfrak{A}, \ldots, a_n = e_n\mathfrak{A}$, then

$$[a_1 \ldots a_n] = [e_1 \ldots e_n][\mathfrak{A}^n] = [\mathfrak{A}^n], \quad \text{if } [e_1 \ldots e_n] = 1.$$

Hence $[\mathfrak{A}^n]$ is the outer product of the transforms of $e_1, \ldots, e_n$ by $\mathfrak{A}$.

If $\mathfrak{A} = (a_{ij})$, then $a_i = \Sigma a_{ij} e_j$, hence $[\mathfrak{A}^n] = \det \mathfrak{A}$.

If $$c_1 = b_1\mathfrak{A}, \quad c_2 = b_2\mathfrak{A}, \quad \ldots, \quad c_n = b_n\mathfrak{A},$$

then $$[c_1 c_2 \ldots c_n] = [b_1 b_2 \ldots b_n][\mathfrak{A}^n].$$

Hence, if $\det \mathfrak{A} = 1$ then the outer product of n extensives is not altered by the transformation $\mathfrak{A}$.

As in §§ 48, 49 we can define the outer product $[\mathfrak{A}\mathfrak{B}]$, and show

$$[\mathfrak{A}(\mathfrak{B}+\mathfrak{C})] = [\mathfrak{A}\mathfrak{B}]+[\mathfrak{A}\mathfrak{C}], \quad [\mathfrak{A}\mathfrak{B}] = [\mathfrak{B}\mathfrak{A}].$$

The formal laws of ordinary algebra thus hold, but we must distinguish between $\mathfrak{A}$ and $[\mathfrak{A}\mathfrak{J}]$.

7. If $\mathfrak{A} = (a_{ij})$, the '*adjugate matrix*' of $\mathfrak{A}$ is $\mathfrak{B} = (b_{ij})$, where b_{ij} is the cofactor of a_{ji} (not a_{ij}) in $\det \mathfrak{A}$.

This matrix exists, of course, even if $\mathfrak{A}$ is singular, and we always have $\mathfrak{A}\mathfrak{B} = \mathfrak{J}.\det \mathfrak{A}$, $\mathfrak{B}\mathfrak{A} = \mathfrak{J}.\det \mathfrak{A}$. For $\sum_j a_{ij} b_{jk}$ equals $\det \mathfrak{A}$ or 0, according as $i = k$ or $i \neq k$.

If $\det \mathfrak{A} \neq 0$, then $\mathfrak{B} = \mathfrak{A}^{-1}.\det \mathfrak{A}$.

If $\mathfrak{C}$ be the adjugate of $\mathfrak{A}-\lambda\mathfrak{J}$, then an element of $\mathfrak{C}$ will be a polynomial in λ of degree not greater than $n-1$, with coefficients which are sums of products of elements of $\mathfrak{A}$.

Hence $$\mathfrak{C} = \mathfrak{C}_0 + \mathfrak{C}_1\lambda + \mathfrak{C}_2\lambda^2 + \ldots + \mathfrak{C}_{n-1}\lambda^{n-1},$$

and $$(\mathfrak{A}-\lambda\mathfrak{J})\,\mathfrak{C} = \mathfrak{J}\det(\mathfrak{A}-\lambda\mathfrak{J}) = \mathfrak{J}.\phi(\lambda),$$

where $\phi(\lambda)$ is the characteristic polynomial of $\mathfrak{A}$.

Hence if $$\phi(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n,$$

then

$$\mathfrak{A}(\mathfrak{C}_0 + \mathfrak{C}_1\lambda + \ldots + \mathfrak{C}_{n-1}\lambda^{n-1}) - \lambda(\mathfrak{C}_0 + \mathfrak{C}_1\lambda + \ldots + \mathfrak{C}_{n-1}\lambda^{n-1}) = (a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n)\mathfrak{J}.$$

Equating coefficients of powers of λ,

$$a_0\mathfrak{J} = -\mathfrak{C}_{n-1}, \quad a_1\mathfrak{J} = \mathfrak{A}\mathfrak{C}_{n-1} - \mathfrak{C}_{n-2}, \quad a_2\mathfrak{J} = \mathfrak{A}\mathfrak{C}_{n-2} - \mathfrak{C}_{n-3}, \quad \ldots, \quad a_n\mathfrak{J} = \mathfrak{A}\mathfrak{C}_0.$$

Hence $$a_0\mathfrak{A}^n + a_1\mathfrak{A}^{n-1} + a_2\mathfrak{A}^{n-2} + \ldots + a_n\mathfrak{J} = -\mathfrak{A}^n\mathfrak{C}_{n-1} + \mathfrak{A}^n\mathfrak{C}_{n-1} - \mathfrak{A}^{n-1}\mathfrak{C}_{n-2} + \mathfrak{A}^{n-1}\mathfrak{C}_{n-2} - \ldots + \mathfrak{A}^2\mathfrak{C}_1 - \mathfrak{A}\mathfrak{C}_0 + \mathfrak{A}\mathfrak{C}_0 = 0.$$

Hence $\mathfrak{A}$ *satisfies its secular equation.* (Hamilton-Cayley.)

8. *Change of frame.* So far we have kept our frame fixed. If we take a new frame $e'_1, \ldots, e'_n$ so that the old basic unities referred to this frame are

$$e_i = \sum_{j=1}^{n} b_{ij}e'_j, \quad (i = 1, \ldots, n), \quad \det b_{ij} \neq 0,$$

then the extensive $x = \Sigma x_i e_i$ acquires the form $\sum_{ij} x_i b_{ij} e'_j$, and has coordinates $\sum_i x_i b_{ij}$, $(j = 1, \ldots, n)$ in the new frame.

If $\mathfrak{B}$ is the matrix of the b_{ij}, we can say that the coordinates x_i of x in the old frame are replaced in the new by coordinates of $x\mathfrak{B}$, namely $\Sigma x_i b_{ij}$, $(j = 1, \ldots, n)$.

Then if, in the old frame, the extensives x, y were connected by $y = x\mathfrak{A}$, then, in the new, they will be connected by $y\mathfrak{B} = (x\mathfrak{B})\,\mathfrak{A}$, or by $y = x\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1}$, since $\det\mathfrak{B} \neq 0$.

Thus *the same transformation is represented in different frames by similar matrices.* Hence, by 3, *the secular equation is independent of the frame.*

9. If $a_1, \ldots, a_n$ be extensives satisfying (1) of § 62, namely,

$$a_1 = a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n,$$
$$a_2 = a_{21}e_1 + a_{22}e_2 + \ldots + a_{2n}e_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n = a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nn}e_n,$$

and if

$$A_1 = [a_2 \ldots a_n],\ A_2 = -[a_1 a_3 \ldots a_n],\ \ldots,\ A_n = (-1)^{n-1}[a_1 a_2 \ldots a_{n-1}],$$
$$E_1 = [e_2 \ldots e_n],\ E_2 = -[e_1 e_3 \ldots e_n],\ \ldots,\ E_n = (-1)^{n-1}[e_1 e_2 \ldots e_{n-1}],$$

then $A_1, \ldots, A_n$ are the transforms of $E_1, \ldots, E_n$ by $[\mathfrak{A}^{n-1}]$.

But if b_{ij} is the co-factor of a_{ij} in $\mathfrak{A}$, then

$$\begin{aligned} A_1 &= [a_2 \ldots a_n] \\ &= b_{11}[e_2 \ldots e_n] - b_{12}[e_1 e_3 \ldots e_n] + b_{13}[e_1 e_2 e_4 \ldots e_n] - \ldots \\ &= b_{11}E_1 + b_{12}E_2 + \ldots + b_{1n}E_n, \end{aligned}$$

$$\begin{aligned} A_2 &= -[a_1 a_3 \ldots a_n] \\ &= b_{21}E + b_{22}E_2 + \ldots + b_{2n}E_n. \end{aligned}$$

Thus the element of $[\mathfrak{A}^{n-1}]$ in the ij place is the co-factor of a_{ij}.

If $[\mathfrak{A}^{n-1}]$ be regarded simply as a matrix, it is thus identical with the matrix $\mathfrak{A}^{*-1} \det \mathfrak{A}$.

The case when $\mathfrak{A}$ is symmetric is specially important, the matrices $[\mathfrak{A}^{n-1}]$ and $\mathfrak{A}^{-1} \det \mathfrak{A}$ are then the same. Hence then, if $[p\pi] = 0$, where p is of step one, π of step $n-1$, then

$$[p\mathfrak{A} . \pi\mathfrak{A}^{-1}] = 0.$$

10. If $\mathfrak{A}$ is symmetrical, and $[p\mathfrak{A}\,|\,p] = 0$ for all p, then $[(p+q)\mathfrak{A}\,|\,(p+q)] = 0$ for all p, q. Hence $[p\mathfrak{A}\,|\,q] = 0$ for all p, q and so $p\mathfrak{A} = 0$ for all p. Hence $\mathfrak{A} = \mathfrak{O}$.

If $\mathfrak{A}$ be any matrix, not $\mathfrak{O}$, and $[p\mathfrak{A}\,|\,p] = 0$ for all p, then since $[p\mathfrak{A}\,|\,p] = [p\mathfrak{A}^*\,|\,p]$, we have $[p(\mathfrak{A}+\mathfrak{A}^*)\,|\,p] = 0$ for all p. Hence $\mathfrak{A}+\mathfrak{A}^* = \mathfrak{O}$, and $\mathfrak{A}$ is skew symmetric.

11. *Def.* The matrix $\mathfrak{A}$ is '*orthogonal*' if

$$\mathfrak{A}^*\mathfrak{A} = \mathfrak{J}. \qquad (3)$$

Then $\det \mathfrak{A}^* \det \mathfrak{A} = \det \mathfrak{J} = 1$, and since $\det \mathfrak{A} = \det \mathfrak{A}^*$, we have $\det \mathfrak{A} = \pm 1$. According as the upper or lower sign holds, $\mathfrak{A}$ is a '*direct*' or '*indirect*' orthogonal matrix.

Since $\det \mathfrak{A} \neq 0$, therefore $\mathfrak{A}^{-1}$ exists, and (3) gives

$$\mathfrak{A}^* = \mathfrak{A}^{-1}, \quad \mathfrak{A}\mathfrak{A}^* = \mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{J}.$$

Hence, by 4, equation (2),

$$[x\mathfrak{A} \,|\, y\mathfrak{A}] = [x\mathfrak{A}\mathfrak{A}^* \,|\, y] = [x \,|\, y]. \tag{4}$$

Inner products are not changed by an orthogonal matrix.

If $\mathfrak{A}$ is an orthogonal matrix with real components, its real latent roots are ± 1.

For, if $\quad x\mathfrak{A} = \lambda x, \quad$ then $\quad [x\mathfrak{A} \,|\, x\mathfrak{A}] = \lambda^2 [x \,|\, x]$.

Hence (4) gives $\lambda = \pm 1$.

If $\mathfrak{A}$, $\mathfrak{B}$ be orthogonal, so is $\mathfrak{A}\mathfrak{B}$. For $\mathfrak{A}^* = \mathfrak{A}^{-1}$, $\mathfrak{B}^* = \mathfrak{B}^{-1}$ gives

$$(\mathfrak{A}\mathfrak{B})^* = \mathfrak{B}^*\mathfrak{A}^* = \mathfrak{B}^{-1}\mathfrak{A}^{-1} = (\mathfrak{A}\mathfrak{B})^{-1}.$$

If $\mathfrak{A} = (a_{ij})_n$ is orthogonal, then $\Sigma a_{ij}a_{jk} = \delta_{ik} = 0$ or 1, according as $i \neq k$ or $i = k$.

The relations $\mathfrak{A}\mathfrak{A}^* = \mathfrak{J}$, $\mathfrak{A}^*\mathfrak{A} = \mathfrak{J}$ can be thus interpreted: If $a_1, a_2, \ldots, a_n$ be the column- (or row-) extensives of an orthogonal matrix, then

$$a_1^2 = a_2^2 = \ldots = a_n^2 = 1, \quad [a_i \,|\, a_j] = 0, \quad (i, j = 1, \ldots, n;\ i \neq j).$$

Hence $\quad [a_i a_j]^2 = 1, \quad [a_i a_j a_k]^2 = 1, \quad \ldots,$

$$(i, j, k = 1, \ldots, n;\ i, j, k \ldots \text{unequal}).$$

Hence $[\mathfrak{A}^2], [\mathfrak{A}^3], \ldots$ are orthogonal matrices.

§64. *Linear transformations of vectors in three-dimensional Euclidean space.*

The vectors are supposed to be all drawn from the same origin.

1. If u, v, w be three independent vectors, and hence $[uvw] \neq 0$, and if $\mathfrak{A}$ be a linear transformation on them, they become $u\mathfrak{A}$, $v\mathfrak{A}$, $w\mathfrak{A}$, and $[uvw]$ becomes

$$[u\mathfrak{A} . v\mathfrak{A} . w\mathfrak{A}] = [uvw][\mathfrak{A}^3], \quad \text{where } [\mathfrak{A}^3] = \det \mathfrak{A}.$$

We call these linear transformations '*strains*' and denote $\det \mathfrak{A}$ by K, and suppose $K \neq 0$. Then three non-coplanar vectors remain non-coplanar. The sign of $[uvw]$ is preserved or changed according as $K > 0$ or $K < 0$, and we call the transformation '*direct*' or '*indirect*' according as $K > 0$ or $K < 0$. As a result of the strain, the magnitude of any trivector is multiplied by K.

2. If x be any vector, and u, v, w be independent vectors, then, by § 17·13, $[uvw]\,x = [vwx]\,u + [wux]\,v + [uvx]\,w$.

Hence $[uvw]\,x\mathfrak{A} = [vwx]\,u\mathfrak{A} + [wux]\,v\mathfrak{A} + [uvx]\,w\mathfrak{A}$.

Denote $u\mathfrak{A}$, $v\mathfrak{A}$, $w\mathfrak{A}$, $x\mathfrak{A}$ by u', v', w', x'.

If u, v, w be unit orthogonal vectors, then

$$x' = [u\,|x]\,u' + [v\,|x]\,v' + [w\,|x]\,w'. \qquad (5)$$

3. *Relation to matrices.* Write, for a moment, e_1, e_2, e_3 for u, v, w, and let $\mathfrak{A} = (a_{ij})_3$, $x = x_1 e_1 + x_2 e_2 + x_3 e_3$;

then $x\mathfrak{A} = \sum_{i,j} x_i a_{ij} e_j$, $e_j\mathfrak{A} = \sum_k a_{jk} e_k$.

If e_1, e_2, e_3 be unit orthogonal vectors, then

$x_i = [e_i\,|x]$, $x\mathfrak{A} = \sum_i [e_i|x]\,e_i\mathfrak{A}$, corresponding to (5).

Also $x\mathfrak{A}^* = \sum_{j,k} x_k a_{jk} e_j$, $[e_j\mathfrak{A}\,|x] = \sum_k a_{jk}[e_k\,|x] = \sum_k a_{jk} x_k$.

Hence $x\mathfrak{A}^* = \sum_j [e_j\mathfrak{A}\,|x]\,e_j$.

Translating this into the notation of 2, we have

$$\begin{aligned} x\mathfrak{A}^* &= [e_1\mathfrak{A}\,|x]\,e_1 + [e_2\mathfrak{A}\,|x]\,e_2 + [e_3\mathfrak{A}\,|x]\,e_3 \\ &= [u'\,|x]\,u + [v'\,|x]\,v + [w'\,|x]\,w. \end{aligned}$$

The strain, thus defined, and denoted by $\mathfrak{A}^*$, we call the '*transposed*' strain of $\mathfrak{A}$.

Since $\det\mathfrak{A} = \det\mathfrak{A}^*$, a strain and its transposed strain multiply volumes in the same ratio.

4. $$\begin{aligned} x(\mathfrak{A} - \mathfrak{A}^*) &= ([u\,|x]\,u' - [u'\,|x]\,u) + ([v\,|x]\,v' - [v'\,|x]\,v) \\ &\qquad + ([w\,|x]\,w' - [w'\,|x]\,w) \\ &= [uu'\,|x] + [vv'\,|x] + [ww'\,|x] \\ &= [(uu' + vv' + ww')\,|\,x]. \end{aligned}$$

5. If the vector x is unchanged in direction by the strain $\mathfrak{A}$, then $x(\mathfrak{A} - \lambda) = 0$ for some scalar λ.

Hence $\det(\mathfrak{A} - \lambda) = 0$, which is a cubic in λ, and therefore always has one real root. Hence there is at least *one invariant direction* (§ 63·2).

6. A '*pure*' or '*non-rotational*' strain is one which leaves (at least) three mutually perpendicular directions invariant.

If u, v, w be unit vectors in these directions, then $u\mathfrak{A} \equiv u$, $v\mathfrak{A} \equiv v$, $w\mathfrak{A} \equiv w$. Hence, by 4, $x(\mathfrak{A} - \mathfrak{A}^*) = 0$ for all x; hence $\mathfrak{A} = \mathfrak{A}^*$.

A necessary and sufficient condition for a non-singular $\mathfrak{A}$ *to be a pure strain is* $\mathfrak{A} = \mathfrak{A}^*$. For the necessity has just been shown. To show sufficiency, let $\mathfrak{A} = \mathfrak{A}^*$, then the secular equation $\det(\mathfrak{A} - \lambda\mathfrak{J}) = 0$ has all its roots real (§ 63·5).

Let them be k_1, k_2, k_3, and let u_1, u_2, u_3 be the corresponding invariant directions (§ 63·2), so that

$$u_i\mathfrak{A} = k_i u_i, \quad (i = 1, 2, 3).$$

Then
$$k_1 k_2[u_1 | u_2] = [k_1 u_1 | k_2 u_2] = [u_1\mathfrak{A} | u_2\mathfrak{A}] = [u_1\mathfrak{A}\mathfrak{A}^* | u_2]$$
$$= [u_1\mathfrak{A}^2 | u_2] = [k_1 u_1\mathfrak{A} | u_2] = k_1^2[u_1 | u_2].$$

Hence, either $k_1 = 0$, or $k_1 = k_2$, or $[u_1 | u_2] = 0$.

If $$k_1 = 0, \quad \text{then} \quad u_1\mathfrak{A} = 0,$$

and even if u_1, v, w be independent vectors, we have

$$[u_1\mathfrak{A} . v\mathfrak{A} . w\mathfrak{A}] = 0.$$

Hence $$[u_1 vw][\mathfrak{A}^3] = 0, \quad [\mathfrak{A}^3] = 0,$$

and $\mathfrak{A}$ is singular. We exclude this case.

Hence, if $k_1 \neq k_2$, then $[u_1 | u_2] = 0$, and hence $u_1 \perp u_2$.

Similarly, *if* k_1, k_2, k_3 *be all unequal, there is a triad of mutually perpendicular invariant directions.*

If $k_1 = k_2 \neq k_3$, then $[u_1 | u_3] = 0 = [u_2 | u_3]$. Hence u_3 is perpendicular to the plane of u_1 and u_2; all vectors in this plane have invariant directions, for

$$(x_1 u_1 + x_2 u_2)\mathfrak{A} = k_1(x_1 u_1 + x_2 u_2), \quad \text{since} \quad k_1 = k_2.$$

If $k_1 = k_2 = k_3 = k$, say, then all vectors have invariant directions, since $u\mathfrak{A} = ku$.

Hence in every case, if $\mathfrak{A} = \mathfrak{A}^*$, there are (at least) three mutually perpendicular invariant directions, and $\mathfrak{A}$ is a pure strain.

Cor. If k_1, k_2, k_3 are unequal, there are only three invariant directions, for a fourth would be perpendicular to at least two of the three.

7. A '*dilatation*' from the origin is a transformation which multiplies each vector by the same scalar k, and leaves its direction invariant. It is represented by the matrix $k\mathfrak{J}$.

If $[\mathfrak{A}^3] = K \neq 0$, and $\mathfrak{A} = K\mathfrak{B}$, then $\det \mathfrak{B} = 1$.

Hence any strain $\mathfrak{A}$ is the product of a dilatation from the origin and a strain of determinant unity.

8. *If three mutually perpendicular unit vectors become such by a strain $\mathfrak{A}$, then any three mutually perpendicular unit vectors become mutually perpendicular unit vectors, and $\mathfrak{A}^{-1} = \mathfrak{A}^*$, that is, $\mathfrak{A}$ is orthogonal.*

For, if u, v, w and u', v', w', their transforms by $\mathfrak{A}$, be each a mutually perpendicular set of unit vectors, then, by 3,

$$u'\mathfrak{A}^* = [u'\,|\,u']\,u + [v'\,|\,u']\,v + [w'\,|\,u']\,w = u,$$

$$v'\mathfrak{A}^* = v, \quad w'\mathfrak{A}^* = w.$$

Since $[uvw] \neq 0$ therefore $\mathfrak{A}$ is non-singular.

But, as $u' = u\mathfrak{A}$, we have $u = u'\mathfrak{A}^{-1}$; similarly

$$v = v'\mathfrak{A}^{-1}, \quad w = w'\mathfrak{A}^{-1}.$$

Hence for all vectors x,

$$x(\mathfrak{A}^* - \mathfrak{A}^{-1}) = 0,$$

hence

$$\mathfrak{A}^* = \mathfrak{A}^{-1}.$$

Conversely, if $\mathfrak{A}$ is orthogonal, it turns perpendicular vectors into such.

For, if $[x\,|\,y] = 0$, $\mathfrak{A}^* = \mathfrak{A}^{-1}$,

then $[x\mathfrak{A}\,|\,y\mathfrak{A}] = [x\mathfrak{A}\mathfrak{A}^*\,|\,y] = [x\mathfrak{A}\mathfrak{A}^{-1}\,|\,y] = [x\,|\,y] = 0.$

9. *Orthogonal strains leave the magnitudes of vectors unchanged.*

For, by 8, putting $x = y$, we have

$$[x\mathfrak{A}\,|\,x\mathfrak{A}] = [x\,|\,x].$$

They leave the absolute magnitudes of trivectors unchanged, since $\det \mathfrak{A} = \pm 1$.

10. Direct orthogonal strains are called '*rotations*'. They satisfy $\mathfrak{R}\mathfrak{R}^* = \mathfrak{J}$, $[\mathfrak{R}^3] = 1$.

A rotation, which is not identity, leaves just one direction invariant. For, let the rotation $\mathfrak{R}$ change the three unit orthogonal

vectors u, v, w into u', v', w', which, by 8, 9, will also be unit orthogonal vectors:

$$[uvw] = 1, \quad [\mathfrak{R}^3] = 1, \quad [u'v'w'] = 1,$$

$$\begin{aligned}[uvw][(\mathfrak{R}-\lambda)^3] &= [u(\mathfrak{R}-\lambda).v(\mathfrak{R}-\lambda).w(\mathfrak{R}-\lambda)] \\ &= 1-\lambda([uv'w']+[u'vw']+[u'v'w]) \\ &\qquad +\lambda^2([u'vw]+[uv'w]+[uvw'])-\lambda^3 \\ &= 1-\lambda([u\,|u']+[v\,|v']+[w\,|w']) \\ &\qquad +\lambda^2([u\,|u']+[v\,|v']+[w\,|w'])-\lambda^3 \\ &= (1-\lambda)\{\lambda^2+\lambda+1 \\ &\qquad -\lambda([u\,|u']+[v\,|v']+[w\,|w'])\}. \qquad (6)\end{aligned}$$

Hence the characteristic roots of $\mathfrak{R}$ are $\lambda = 1$, and the roots of the quadratic factor. Now since $\mathfrak{R}$ does not change the magnitude of any vector, the roots of the quadratic factor, if real, must be $+1$ or -1.

If $\lambda = 1$ is a root, then since $[u\,|u']$, $[v\,|v']$, $[w\,|w']$ cannot exceed 1 and their sum is 3, each equals 1, and u, v, w coincide with u', v', w'.

If $\lambda = -1$ is a root, the other root is also -1 (since the product of the roots is $+1$), then u, v, w become $+u$, $-v$, $-w$, if u corresponds to the root of the linear factor. The vector u remains fixed, and all vectors $k_1 v + k_2 w$ perpendicular to it are reversed.

If the roots of the quadratic factor are imaginary, $\lambda = 1$ is the only real root of the secular equation, and there is only one vector of invariant direction.

11. *Indirect orthogonal transformations.*

If $\mathfrak{A}$ be a strain of this kind, then $K = -1$, $\mathfrak{R} = -\mathfrak{A}$ is a rotation. Hence *these transformations are rotations preceded or followed by a reversal of the vector, that is, by a reflection in the origin.*

If the orthogonal unit vectors u, v, w become u', v', w' by $\mathfrak{A}$, then $|u' = -[v'w']$, and so on, since $[u'v'w'] = -1$. Hence (6) is replaced by

$$\begin{aligned}-1+\lambda([u\,|u']+[v\,|v']+[w\,|w']) \\ +\lambda^2([u\,|u']+[v\,|v']+[w\,|w'])-\lambda^3 = 0.\end{aligned}$$

One root is $\lambda = -1$, the others satisfy

$$\lambda^2-\lambda-\lambda([u\,|u']+[v\,|v']+[w\,|w'])+1 = 0.$$

If these be real, they are both -1, or both $+1$, and hence

$$[u\,|u']+[v\,|v']+[w\,|w'] = -3 \text{ or } 1.$$

The first case, ($\lambda = -1$), gives $[u\,|u'] = [v\,|v'] = [w\,|w'] = -1$, and $\mathfrak{A}$ is a reflection in the origin. If $\lambda = +1$ is one root of the quadratic, the other is $+1$, and the three characteristic roots are $-1, 1, 1$. As in 10, it follows that one vector is reversed, and all those perpendicular to it are invariant. Thus $\mathfrak{A}$ is a reflection in a plane.

If $\lambda = -1$ is the only real root, no vector is invariant, but one is merely reversed. Hence *there is always one vector which is reversed, and either no vector is invariant or there is a plane of invariant vectors.*

§65. *Central quadrics.*†

1. If $\mathfrak{A}$ is a pure strain, ($\mathfrak{A} = \mathfrak{A}^*$), the corresponding '*strain quadric*' is the locus of ends of vectors v such that $[v\mathfrak{A}\,|v] = 1$.

Differentiating, we have

$$[d(v\mathfrak{A})\,|v] + [v\mathfrak{A}\,|dv] = 0.$$

Now since $\mathfrak{A}$ is a fixed matrix,

$$d(v\mathfrak{A}) = (v + dv)\,\mathfrak{A} - v\mathfrak{A} = dv\,.\,\mathfrak{A}.$$

Since $\mathfrak{A} = \mathfrak{A}^*$, we have

$$[d(v\mathfrak{A})\,|v] = [(dv\,.\,\mathfrak{A})\,|v] = [dv\,|v\mathfrak{A}] = [v\mathfrak{A}\,|dv].$$

Hence, if $[v\mathfrak{A}\,|v] = 1$, then $[v\mathfrak{A}\,|dv] = 0$.

But dv is a vector in the tangent plane to the quadric at the point v. (We shall often say 'the point v' for the point at the end of the vector v.) Hence

The tangent plane to the strain quadric at the point v is perpendicular to $v\mathfrak{A}$.

Thus the strain quadric is such that the vector from the origin to any point p of the quadric is turned into a position perpendicular to the tangent plane at p.

Let q be the vector (from the origin) to the foot of the perpendicular to the tangent plane at the point v.

Then $q = \mathrm{k}\,.\,v\mathfrak{A}$, (k scalar).

Hence

$$1 = [v\,|v\mathfrak{A}] = [(v-q)\,|v\mathfrak{A}] + [q\,|v\mathfrak{A}] = [q\,|v\mathfrak{A}] = \mathrm{k}[v\mathfrak{A}]^2,$$

$$\mathrm{k} = 1/[v\mathfrak{A}]^2, \quad q^2 = 1/[v\mathfrak{A}]^2.$$

† We recall, § 63·4, that if $\mathfrak{A} = \mathfrak{A}^*$, then $[x\mathfrak{A}\,|y] = [x\,|y\mathfrak{A}]$.

2. *The mutually perpendicular invariant directions of* $\mathfrak{A}$ (§ 64·6) *are along the 'principal axes' of the quadric.*

For let u be a vector to a point of a quadric whose magnitude is stationary with respect to nearby vectors. Then we say u is along a '*principal axis*'. A plane perpendicular to such an axis is a '*principal plane*'. $[u\mathfrak{A}\,|u] = 1$, u^2 is stationary.

Hence $[u\,|du] = 0$, whenever du satisfies $[u\mathfrak{A}\,|du] = 0$.

Hence $u\mathfrak{A} = ku$, (k scalar),

$$1 = [u\mathfrak{A}\,|u] = ku^2, \quad k = 1/u^2, \quad u(\mathfrak{A} - 1/u^2.\mathfrak{J}) = 0.$$

Hence $$\det(\mathfrak{A} - 1/u^2.\mathfrak{J}) = 0.$$

The roots of the secular equation for $\mathfrak{A}$ *are* $1/u^2$, *where* $\sqrt{u^2}$ *are the lengths of the principal semi-axes.*

The results of this paragraph and the preceding clearly hold *for space of any dimensions.*

3. *To find the lengths of the axes of sections made in a direction perpendicular to a vector* v.

If u is the vector along such an axis, then

$$[u\mathfrak{A}\,|u] = 1, \quad [u\,|v] = 0, \quad u^2 \text{ is stationary.}$$

Hence $[u\,|du] = 0$, whenever u satisfies $[v\,|du] = 0$, $[u\mathfrak{A}\,|du] = 0$.

Hence u, v, $u\mathfrak{A}$ are coplanar vectors, and the directions of the axes are given by $[u.u\mathfrak{A}.v] = 0$, as an equation in u.

Further, $u\mathfrak{A} = k_1 v + k_2 u$, for some scalars k_1, k_2.

Hence $$1 = [u\,|u\mathfrak{A}] = k_2 u^2, \quad u(\mathfrak{A} - 1/u^2.\mathfrak{J}) = k_1 v,$$
$$u = k_1 v(\mathfrak{A} - 1/u^2.\mathfrak{J})^{-1}.$$

Hence $$[v\,|v(\mathfrak{A} - 1/u^2.\mathfrak{J})^{-1}] = 0$$

is the equation in $\sqrt{u^2}$ for the lengths of the axes of the section perpendicular to v.

To compare with the usual Cartesian results, we can take the equation of our quadric referred to its principal axes as

$$ax^2 + by^2 + cz^2 = 1.$$

If u be a vector whose components are (x, y, z), and $[u\mathfrak{A}\,|u] = 1$, then $u\mathfrak{A}$ has components (ax, by, cz); hence

$$\mathfrak{A} = \operatorname{diag}(a, b, c), \quad (\mathfrak{A} - 1/u^2.\mathfrak{J}) = \operatorname{diag}(a - r^{-2}, b - r^{-2}, c - r^{-2}),$$

where r is the length of the vector u.

The inverse of a diagonal matrix is obtained by inverting its elements; hence if l, m, n be the direction cosines of v, we have

$$l^2(a-r^{-2})^{-1}+m^2(b-r^{-2})^{-1}+n^2(c-r^{-2})^{-1}=0.$$

But of course our work holds for the *general* equation of the quadric.

4. *The locus of the mid-points of the chords of a quadric parallel to a given vector u is a plane through the origin with bivector $|u\mathfrak{A}$.*

The locus of course goes through the origin; let v be the vector to one of the mid-points, then for some scalar k

$$[(v+ku)\mathfrak{A}\,|(v+ku)]=1, \quad [(v-ku)\mathfrak{A}\,|(v-ku)]=1.$$

Hence $$[v\mathfrak{A}\,|v]+k[u\mathfrak{A}\,|v]+k[v\mathfrak{A}\,|u]+k^2[u\mathfrak{A}\,|u]=1,$$

$$[v\mathfrak{A}\,|v]-k[u\mathfrak{A}\,|v]-k[v\mathfrak{A}\,|u]+k^2[u\mathfrak{A}\,|u]=1.$$

Thus $[u\mathfrak{A}\,|v]+[v\mathfrak{A}\,|u]=0$, but as $\mathfrak{A}=\mathfrak{A}^*$, $[u\mathfrak{A}\,|v]=[v\mathfrak{A}\,|u]$.

Hence $$[u\mathfrak{A}\,|v]=0, \quad [v\mathfrak{A}\,|u]=0.$$

Thus v is on $|u\mathfrak{A}$, and, further, the locus of mid-points of chords parallel to v is $|v\mathfrak{A}$ and contains u. The directions of u and v are '*conjugate*' for $\mathfrak{A}$; the vector u and bivector $|u\mathfrak{A}$ are '*conjugate*'.

If $[u\mathfrak{A}.v\mathfrak{A}]=|w$, then $[v\mathfrak{A}\,|w]=[u\mathfrak{A}\,|w]=0$.

We may define the *points* u and v to be conjugate when $[u\,|v\mathfrak{A}]=1$, and we may define the locus of points conjugate to the point u as the '*polar*' of u. The usual properties of poles and polars then easily follow.

5. *If the normal to a central quadric at a point p, not on a principal plane, meets the principal planes in p_1, p_2, p_3, then the intervals pp_1, pp_2, pp_3 are in constant ratios.*

For let k_1, k_2, k_3 be the characteristic roots of $\mathfrak{A}$ and $\alpha_1, \alpha_2, \alpha_3$ the corresponding principal planes on which p_1, p_2, p_3 are supposed to lie. If v_1, v_2, v_3 be the principal axes perpendicular to $\alpha_1, \alpha_2, \alpha_3$ respectively, then

$$v_1\mathfrak{A}=k_1v_1, \quad [p_1\,|v_1]=0.$$

Also $$c_1(p-p_1)=p\mathfrak{A}, \quad c_1 \text{ scalar.}$$

Hence $$c_1[p\,|v_1]=[p\mathfrak{A}\,|v_1]=[v_1\mathfrak{A}\,|p]=k_1[v_1\,|p].$$

But $[p\,|v_1]\neq 0$. Hence $c_1=k_1$. Similarly $c_2=k_2$, $c_3=k_3$, where c_2, c_3 are defined like c_1. Hence the lengths $\bar{pp}_1$, $\bar{pp}_2$, $\bar{pp}_3$ are in ratios $k_1^{-1}:k_2^{-1}:k_3^{-1}$.

6. *Quadric envelopes.* If $|v\mathfrak{A} = \tau$, then τ is the bivector of the tangent plane at the point v of the quadric $\mathfrak{A}$.

The equation $[v\,|v\mathfrak{A}] = 1$ may be written $[v\tau] = 1$.

Hence $[v\mathfrak{A}.\tau[\mathfrak{A}^2]] = [v\tau]\,[\mathfrak{A}^3] = [\mathfrak{A}^3] = \mathrm{A}$, say.

Let $[\mathfrak{A}^2] = \mathrm{A}\mathfrak{B}$, then $[\tau\mathfrak{B}\,|\tau] = 1$. (1)

This is the equation of the quadric regarded as an envelope of planes; $\mathfrak{B}$ is an operator which transforms bivectors into bivectors, whereas $\mathfrak{A}$ and $\mathfrak{A}^{-1}$ turn vectors into vectors.

When $\mathfrak{B}$ and $\mathfrak{A}^{-1}$ are represented by matrices, their corresponding coordinates are equal, and no harm arises if we write $\mathfrak{A}^{-1}$ for $\mathfrak{B}$.

If λ is a characteristic root of $\mathfrak{A}$, then λ^{-1} is a characteristic root of $\mathfrak{B}$ (§ 63·2). The characteristic roots of $\mathfrak{B}$ are hence the squares of the lengths of the principal semi-axes of the quadric $\mathfrak{B}$.

We can write (1) as $[\tau\mathfrak{B}.v\mathfrak{A}] = 1$. Comparing with $[v\,|v\mathfrak{A}] = 1$, we have

$$|v = \tau\mathfrak{B}, \quad v = |\tau\mathfrak{B}.$$

Hence the point of contact of τ with $\mathfrak{B}$ is $|\tau\mathfrak{B}$.

Also $[v\tau] = 1$ gives

$$[\tau\mathfrak{B}.v[\mathfrak{B}^2]] = [\mathfrak{B}^3], \quad \text{or} \quad [v[\mathfrak{B}^2]\,|v] = [\mathfrak{B}^3].$$

7. *Confocal quadrics.* If $\mathfrak{A}$ is a pure strain and $\mathfrak{B} = \mathfrak{A}^{-1}$ (cf. 6), then the quadrics whose envelope equations are

$$[\tau\,|\tau(\mathfrak{B} - \mathrm{k}\mathfrak{J})] = 1, \quad (\mathrm{k\ scalar})$$

are '*confocal*' to $\mathfrak{A}$.

If k is a root of the secular equation for $\mathfrak{B}$, the quadric becomes a '*focal conic*'; for if x is any root of that equation, then $\mathrm{x} - \mathrm{k}$ is a root of the secular equation of $\mathfrak{B} - \mathrm{k}\mathfrak{J}$, and, when $\mathrm{x} = \mathrm{k}$, the corresponding semi-axis vanishes, by 2.

The locus equation of the confocal is $[v(\mathfrak{B} - \mathrm{k}\mathfrak{J})^{-1}\,|v] = 1$.

8. *Two confocals have perpendicular tangent planes at any common point.*

For the envelope equations of the confocals can be taken as

$$[\tau(\mathfrak{B} - \mathrm{k}_1\mathfrak{J})\,|\tau] = 1, \quad [\tau(\mathfrak{B} - \mathrm{k}_2\mathfrak{J})\,|\tau] = 1, \quad \mathrm{k}_1 \neq \mathrm{k}_2.$$

The point of contact of the tangent-plane α to the first is $|\alpha(\mathfrak{B} - \mathrm{k}_1\mathfrak{J})$; the point of contact of the tangent plane β to the second is $|\beta(\mathfrak{B} - \mathrm{k}_2\mathfrak{J})$.

These points coincide if

$$\alpha(\mathfrak{B}-k_1\mathfrak{J}) = \beta(\mathfrak{B}-k_2\mathfrak{J}).$$

But $$[\alpha(\mathfrak{B}-k_1\mathfrak{J})\,|\alpha] = 1, \quad [\beta(\mathfrak{B}-k_2\mathfrak{J})\,|\beta] = 1,$$

$$[\beta\mathfrak{B}\,|\alpha] = [\alpha\mathfrak{B}\,|\beta].$$

Hence $$[\beta(\mathfrak{B}-k_2\mathfrak{J})\,|\alpha] = 1 = [\alpha(\mathfrak{B}-k_1\mathfrak{J})\,|\beta].$$

This gives $(k_1-k_2)[\alpha\,|\beta] = 0$, and so $[\alpha\,|\beta] = 0$.

This theorem and proof hold for spreads of any step.

9. *There are just three confocals through any point p, not the centre.*

For, if $\mathfrak{B}-k\mathfrak{J} = \mathfrak{C}$ goes through p, then $[p[\mathfrak{C}^2]\,|p] = [\mathfrak{C}^3]$ by 6, and this is a cubic equation for k.

Since at least one root is real, at least one quadric is real. The reality of the other two follows from the next paragraph.

10. *If u is any point on a quadric locus $\mathfrak{A}$, the principal axes of the central section which is parallel to the tangent plane at u are parallel to the normals at u to the other confocals through u.*

For let w be the end of a principal axis of the section. This section is perpendicular to $u\mathfrak{A}$. Hence, by 3, w, $w\mathfrak{A}$, $u\mathfrak{A}$ are coplanar.

$$w = k_1 u\mathfrak{A} + k_2 w\mathfrak{A}, \quad (k_1, k_2 \text{ scalars}) \tag{1}$$

$$[w\mathfrak{A}\,|w] = 1, \quad [u\mathfrak{A}\,|w] = 0. \tag{2}$$

Hence

$$w^2 = k_2, \quad w\mathfrak{A}^{-1} = k_1 u + k_2 w, \quad w(\mathfrak{A}^{-1}-k_2\mathfrak{J}) = k_1 u. \tag{3}$$

Let $\mathfrak{B} = \mathfrak{A}^{-1}$, then $[v(\mathfrak{B}-k_2\mathfrak{J})^{-1}\,|v] = 1$ is the locus equation of a confocal to $\mathfrak{A}$.

Now $[u\mathfrak{A}\,|u] = 1$; $[w\mathfrak{A}\,|u] = 0$, by (2). Hence $[w\,|u] = k_1$, by (1).

But, by (3), $w = k_1 u(\mathfrak{B}-k_2\mathfrak{J})^{-1}$, hence $[u(\mathfrak{B}-k_2\mathfrak{J})^{-1}\,|u] = 1$.

Thus the confocal last-mentioned goes through u, and hence $u(\mathfrak{B}-k_2\mathfrak{J})^{-1}$, or w, is parallel to the normal at u to a confocal through u.

11. *The vectors of $\mathfrak{A}$ parallel to the normals of $(\mathfrak{B}-k\mathfrak{J})^{-1}$, at points where these quadrics cut, have a constant length if $\mathfrak{B} = \mathfrak{A}^{-1}$.*

For, if u be a point of the cut, then $u(\mathfrak{B}-k\mathfrak{J})^{-1}$ is the vector of the normal there. If v is the parallel vector of $\mathfrak{A}$, then

$$[v\mathfrak{A}\,|v] = 1, \quad v = xu(\mathfrak{B}-k\mathfrak{J})^{-1}, \quad (x \text{ scalar}).$$

Now $$[u\mathfrak{A}\,|v] = 0,$$

since v is perpendicular to $u\mathfrak{A}$, by 8 and 1.

Hence $$v(\mathfrak{B}-k\mathfrak{J}) = xu, \quad v(\mathfrak{B}-k\mathfrak{J})\,\mathfrak{A} = x.u\mathfrak{A},$$

$$v-k.v\mathfrak{A} = x.u\mathfrak{A}, \quad v^2-k[v\mathfrak{A}\,|v] = x[u\mathfrak{A}\,|v] = 0, \quad v^2 = k.$$

12. *The centres of curvature at a point on a quadric.*†

Let v be on the quadric $\mathfrak{C}$. The point $p = v+xv\mathfrak{C}$, (x scalar), is on the normal at v to the quadric locus $\mathfrak{C}$. Take x so that p is at a centre of curvature. This means that a direction dv can be found on the quadric so that $dp = 0$, that is, so that

$$dv(\mathfrak{J}+x\mathfrak{C})+v\mathfrak{C}.dx = 0, \quad \text{or} \quad dv+v(\mathfrak{C}^{-1}+x\mathfrak{J})^{-1}dx = 0.$$

But $$[v\mathfrak{C}\,|dv] = 0, \quad \text{hence} \quad [v\mathfrak{C}\,|v(\mathfrak{C}^{-1}+x\mathfrak{J})^{-1}] = 0.$$

Put $\mathfrak{C} = (\mathfrak{B}-k\mathfrak{J})^{-1}$, then the last equation can be written

$$[v(\mathfrak{B}-k\mathfrak{J})^{-1}\,|v(\mathfrak{B}-(k-x)\,\mathfrak{J})^{-1}] = 0.$$

Thus, by 8, $(\mathfrak{B}-(k-x)\,\mathfrak{J})^{-1}$ is a confocal to $\mathfrak{C}$ through v. Hence, if $(\mathfrak{B}-k_1\mathfrak{J})^{-1}$, $(\mathfrak{B}-k_2\mathfrak{J})^{-1}$ be the two confocals through v, then $x = k-k_i$ or $k-k_2$.

Hence the centres of curvature of $\mathfrak{C}$ at v are

$$\begin{aligned} v+xv\mathfrak{C} &= v(1+(k-k_i)\,(\mathfrak{B}-k\mathfrak{J})^{-1}) \\ &= v(\mathfrak{B}-k_i\mathfrak{J})\,(\mathfrak{B}-k\mathfrak{J})^{-1}, \quad (i=1,2). \end{aligned}$$

Now $$(\mathfrak{B}-k_i\mathfrak{J})\,(\mathfrak{B}-k\mathfrak{J}) = (\mathfrak{B}-k\mathfrak{J})\,(\mathfrak{B}-k_i\mathfrak{J}).$$

Hence $$(\mathfrak{B}-k_i\mathfrak{J})\,(\mathfrak{B}-k\mathfrak{J})^{-1} = (\mathfrak{B}-k\mathfrak{J})^{-1}\,(\mathfrak{B}-k_i\mathfrak{J}).$$

Thus $$v+xv\mathfrak{C} = |(|v(\mathfrak{B}-k\mathfrak{J})^{-1})\,(\mathfrak{B}-k_i\mathfrak{J}),$$

or a centre of curvature at v is the pole for $(\mathfrak{B}-k_i\mathfrak{J})^{-1}$ of the tangent plane at v to $(\mathfrak{B}-k\mathfrak{J})^{-1}$.

This result extends to spreads of any step.

13. A vector w with imaginary coordinates, such that $w^2 = 0$, is called '*isotropic*'. A square root of a matrix $\mathfrak{F}$ is a matrix $\mathfrak{G}$, such that $\mathfrak{G}^2 = \mathfrak{F}$. We write $\mathfrak{G} = \mathfrak{F}^{\frac{1}{2}}$. They will be investigated later, and are now introduced tentatively. It will be shewn later that matrices of the form $(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}$ for various values of k commute in multiplication.

† Joly, *Manual of Quaternions* (1905), p. 122.

The point where the confocals of the quadric envelope $\mathfrak{B}$, corresponding to k_1, k_2, k_3, cut is

$$v = w[(\mathfrak{B}-k_1\mathfrak{J})(\mathfrak{B}-k_2\mathfrak{J})(\mathfrak{B}-k_3\mathfrak{J})]^{\frac{1}{2}} = w\mathfrak{D}^{\frac{1}{2}}, \text{ say,}$$

where $w^2 = 0, \quad [w\,|\,w\mathfrak{B}] = 0, \quad [w\,|\,w\mathfrak{B}^2] = 1.$

For $[v\,|\,v(\mathfrak{B}-k_1\mathfrak{J})^{-1}]$

$$= [w\mathfrak{D}^{\frac{1}{2}}\,|\,w\mathfrak{D}^{\frac{1}{2}}(\mathfrak{B}-k_1\mathfrak{J})^{-1}] = [w\,|\,w\mathfrak{D}^{\frac{1}{2}}(\mathfrak{B}-k_1\mathfrak{J})^{-1}\mathfrak{D}^{\frac{1}{2}}]$$
$$= [w\,|\,w\mathfrak{D}(\mathfrak{B}-k_1\mathfrak{J})^{-1}] = [w\,|\,w(\mathfrak{B}-k_2\mathfrak{J})(\mathfrak{B}-k_3\mathfrak{J})]$$
$$= k_2k_3w^2-(k_2+k_3)[w\,|\,w\mathfrak{B}]+[w\,|\,w\mathfrak{B}^2] = 1.$$

14. Combining 12 and 13, we have, since a centre of curvature at v of $\mathfrak{C}$ is $v(\mathfrak{B}-k\mathfrak{J})^{-1}(\mathfrak{B}-k_1\mathfrak{J})$, the surface of centres is

$$w(\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}},$$

as k_1, k_2 vary.

The corresponding work on conics gives the result:

If $$(\mathfrak{A}-k_1\mathfrak{D})^{-1}$$

is a confocal conic through the point v, the centre of curvature at v for $\mathfrak{A}$ is $v\mathfrak{A}^{-\frac{1}{2}}(\mathfrak{A}-k_1\mathfrak{J})^{\frac{1}{2}}$.

15. *Ivory's Theorem.* Consider the confocals

$$[v(\mathfrak{B}-k\mathfrak{J})^{-1}\,|\,v] = 0, \quad (k \text{ scalar}).$$

Let $$\mathfrak{C}^2 = \mathfrak{B}-k\mathfrak{J},$$

then $\mathfrak{C} = (\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}, \quad \mathfrak{C}^{-1} = (\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}, \quad \mathfrak{C} = \mathfrak{C}^*.$

We assume $(\mathfrak{B}-k\mathfrak{J})^{-1}$ represents an ellipsoid. If then it be referred to its principal axes, its matrix takes the form diag (a, b, c) where a, b, c are *positive* scalars. Then diag $(\sqrt{a}, \sqrt{b}, \sqrt{c})$, with positive square roots, is taken for $\mathfrak{C}^{-1}$, in that frame. Let r be a unit vector, then

$$1 = r^2 = [r(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}(\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}\,|\,r] = [r(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}\,|\,r(\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}].$$

Let $$p = r(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}},$$

then $$[p\,|\,p(\mathfrak{B}-k\mathfrak{J})^{-1}] = 1.$$

Hence p is on the ellipsoid $(\mathfrak{B}-k\mathfrak{J})^{-1}$, which call the ellipsoid k.

We call $p_1 = r(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}$, $p_2 = r(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}$ '*corresponding points*' on the ellipsoids k_1, k_2. If one is on a principal plane, so is the other.

Let $q_1 = r'(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}$, $q_2 = r'(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}$, $(r'^2 = 1)$

be another pair of corresponding points, then

$$\begin{aligned}[p_1\,|q_2] &= [r(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}\,|r'(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}]\\ &= [r(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}\,|r']\\ &= [r(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}\,|r']\\ &= [r(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}\,|r'(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}] = [p_2\,|q_1]. \qquad \text{(i)}\end{aligned}$$

(We have assumed again that $(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}$ and $(\mathfrak{B}-k_2\mathfrak{J})^{\frac{1}{2}}$ commute.)

Also $p_1^2 = [r(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}\,|r(\mathfrak{B}-k_1\mathfrak{J})^{\frac{1}{2}}] = [r(\mathfrak{B}-k_1\mathfrak{J})\,|r]$,

$$\begin{aligned}p_1^2-p_2^2 &= [r(\mathfrak{B}-k_1\mathfrak{J})\,|r]-[r(\mathfrak{B}-k_2\mathfrak{J})\,|r]\\ &= (k_2-k_1)r^2 = k_2-k_1. \qquad \text{(ii)}\end{aligned}$$

The differences of the squares of the distances of corresponding points from the centre is constant.

Thus $$q_1^2-q_2^2 = k_2-k_1. \qquad \text{(iii)}$$

Hence, by (i), (ii), (iii),

$$(p_1+q_2)^2 = (p_2+q_1)^2, \quad (p_1-q_2)^2 = (p_2-q_1)^2.$$

The last gives Ivory's Theorem: *The distance between two points one on each of two confocal ellipsoids equals the distance between the corresponding points.*

The other equation is also easily interpreted.

16. If r be a unit vector, then $p = r(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}$ is on the ellipsoid k.

Keep r constant, but let k vary; then p describes a curve, and the tangent at p to the curve is given by $dp = r\mathrm{d}(\mathfrak{B}-k\mathfrak{J})^{\frac{1}{2}}$, or, as in the ordinary differential calculus; $dp = -\frac{1}{2}r(\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}\mathrm{d}k$.

Hence $r(\mathfrak{B}-k\mathfrak{J})^{-\frac{1}{2}}$ or $p(\mathfrak{B}-k\mathfrak{J})^{-1}$ is a vector along the tangent at p to the curve.

Hence, as p is on the ellipsoid $[u(\mathfrak{B}-k\mathfrak{J})^{-1}\,|u] = 1$, the tangent to the curve at p is normal to this ellipsoid, by 1.

Thus the locus of points on the ellipsoids $(\mathfrak{B}-k\mathfrak{J})^{-1}$ *as* k *varies, which correspond to a fixed point on the ellipsoid* $\mathfrak{B}^{-1}$, *is normal to the confocals, and hence it is the cut of the two other confocals through the fixed point.*

The corresponding theorem on the hyperboloid of one sheet presents special features and is treated next.

17. *Smith-Durrande Theorem. If two points be taken on a generator of a hyperboloid of one sheet, and the corresponding points on a confocal hyperboloid of one sheet, then the distance between the first pair equals the distance between the second pair.*

For define corresponding points on two confocal hyperboloids of one sheet as they were defined for ellipsoids in 15; then the theorem and proof of 16 still hold. (The vector r has now one coordinate imaginary.)

If p, q be the first pair of points, then for any continuous change,

$$\frac{d}{d\tau}(p-q)^2 = 2[(p-q)\,|\,(\dot{p}-\dot{q})].$$

But, if, as the hyperboloid deforms along a series of confocal hyperboloids, p, q move along points corresponding to p, q, then, by 16, they move normally to the hyperboloid and hence

$$[(p-q)\,|\,\dot{p}] = 0, \quad [(p-q)\,|\,\dot{q}] = 0,$$

provided $[pq]$ is along a tangent plane to the hyperboloid, that is, provided $[pq]$ is a generator.

Hence $(p-q)^2$ is constant in this case.

If we consider confocal ellipsoids, the formulae of 15 give:

$$(p_1-q_1)^2-(p_2-q_2)^2 = 2(\mathrm{k}_2-\mathrm{k}_1)-2[p_1\,|\,q_1]+2[p_2\,|\,q_2],$$

$$[p_1\,|\,q_1] = [r(\mathfrak{B}-\mathrm{k}_1\mathfrak{J})^{\frac{1}{2}}\,|\,r'(\mathfrak{B}-\mathrm{k}_1\mathfrak{J})^{\frac{1}{2}}] = [r(\mathfrak{B}-\mathrm{k}_1\mathfrak{J})\,|\,r'],$$

$$[p_2\,|\,q_2] = [r(\mathfrak{B}-\mathrm{k}_2\mathfrak{J})\,|\,r'].$$

Hence $$(p_1-q_1)^2-(p_2-q_2)^2 = 2(\mathrm{k}_2-\mathrm{k}_1)(1-[r\,|\,r']).$$

But since r, r' are (real) unit vectors, this vanishes only when $\mathrm{k}_1 = \mathrm{k}_2$ or $r = r'$.

18. As the hyperboloid deforms, its points on a principal plane remain on that plane, since the instantaneous motion of a point is in the direction of the normal at the point. (The cuts of the generators with the principal planes describe conics.) Thus the lengths of the intervals cut on the generators by the principal planes are constant, and we have a case of the motion of §47, Ex. 31. The locus of a point on a generator as the hyperboloid deforms lies on the ellipsoid of that example.

Let p_1q_1 and p_2q_2 be two positions of a moving generator,

where p_1, p_2 are on one principal plane, and q_1, q_2 on another. Then p_1, q_1 on one hyperboloid correspond to p_2, q_2 on a confocal hyperboloid. As in 15, we have $p_1^2 - q_1^2 = p_2^2 - q_2^2$. Since also $(p_1 - q_1)^2 = (p_2 - q_2)^2$, it follows that the foot of the perpendicular from the centre to the generator maintains a fixed position on that generator.

Hence if a plane through a point fixed on the generator and perpendicular to it goes through o in one position, it does so in all. Hence† any such plane envelopes a sphere, centre o.

Also, by 15, $[p_1 \mid q_2] = [p_2 \mid q_1]$; whence the join of the mid-points of p_1p_2 and q_1q_2 is perpendicular to them.

19. From 5, it easily follows that *the normals at all points of a set of confocal quadrics are cut by the principal planes in points p, q, r such that the ratio $\overline{pq}:\overline{pr}$ is constant.*

Thus the cross-ratio of the cuts with the principal planes and the plane at infinity is constant; we have that case of the tetrahedral complex considered in the next section.

20. Let L, M be the generators, through p, of the deforming hyperboloid. As the hyperboloid deforms, p stays on the ellipsoid given by the motion of L (cf. 18 and §47, Ex. 31), and also on that given by the motion of M, and hence on their cut. These ellipsoids are concentric and co-axial, and through their intersection goes an infinite number of such ellipsoids, one of which, $\mathscr{C}$ say, touches at p the plane through p perpendicular to L.

Then L is the normal to $\mathscr{C}$ at p. Consider the normal to $\mathscr{C}$ at another point p_1 on the locus of p; this normal is divided by the principal planes in the same ratio as L (5), hence it is a new position of L, for it is easy to shew that these ratios and the given point p_1 fix the line.

Hence the planes touching $\mathscr{C}$ at points of the locus of p, being perpendicular to L in its various positions, at a fixed point on L, all touch a fixed sphere, centre o, by 18.

Thus *the locus of p is the locus of the points of contact with the ellipsoid $\mathscr{C}$ of a common tangent plane of $\mathscr{C}$ and a fixed sphere, that is, it is a 'polhode'.*

† For this and 20 see Mannheim, *Géométrie Cinématique* (1894), p. 191.

21. *If three perpendicular chords be drawn through a point v of a quadric $\mathfrak{A}$, then the plane through their ends goes through a fixed point.* (Frégier.)

For, if u_1, u_2, u_3 be three perpendicular unit vectors through v, then there are scalars k_1, k_2, k_3 such that

$$[(v+k_i u_i)\,\mathfrak{A}\,|\,(v+k_i u_i)] = 1, \quad (i=1,2,3).$$

Hence $k_i = -2[v\mathfrak{A}\,|\,u_i] \div [u_i\mathfrak{A}\,|\,u_i]$, since $[v\mathfrak{A}\,|\,v] = 1$.

The points $v+k_i u_i$ are on the quadric. The plane through them goes also through the point

$$(l_1(v+k_1u_1)+l_2(v+k_2u_2)+l_3(v+k_3u_3))\,(l_1+l_2+l_3)^{-1},$$

for any scalars l_1, l_2, l_3.

In particular, it goes through the point

$$v-2([v\mathfrak{A}\,|\,u_1]\,u_1+[v\mathfrak{A}\,|\,u_2]\,u_2+[v\mathfrak{A}\,|\,u_3]\,u_3) \div ([u_1\mathfrak{A}\,|\,u_1]+[u_2\mathfrak{A}\,|\,u_2]+[u_3\mathfrak{A}\,|\,u_3])$$

or

$$v-2v\mathfrak{A}\div k,$$

where

$$\begin{aligned} k &= [u_1\mathfrak{A}\,|\,u_1]+[u_2\mathfrak{A}\,|\,u_2]+[u_3\mathfrak{A}\,|\,u_3] \\ &= [u_1\mathfrak{A}.u_2u_3]+[u_2\mathfrak{A}.u_3u_1]+[u_3\mathfrak{A}.u_1u_2] \\ &= 3[u_1u_2u_3]\,[\mathfrak{A}\mathfrak{J}\mathfrak{J}] = 3[\mathfrak{A}\mathfrak{J}\mathfrak{J}], \end{aligned}$$

where we define $[\mathfrak{A}\mathfrak{J}\mathfrak{J}]$ as in the work in §49. It can be shewn that this outer product is independent of the frame. See p. 293.

§66. *Confocal quadrics from the projective standpoint.*

1. Consider the operation of taking poles and polars with respect to any non-degenerate quadric. In Chap. III, where we had only one quadric, this operation was treated in our symbolism as one of taking the supplement, and was denoted by $|$. In this section, we shall distinguish between the operation of taking the pole of a plane, and that of taking the polar plane of a point.

If α be a plane we denote its pole, for the quadric indicated by $\mathfrak{p}$, by the symbol $\alpha\mathfrak{p}$. Thus $\mathfrak{p}$ is a linear transformation which turns planes into points.

If $p = \alpha\mathfrak{p}$, then $\alpha = p\mathfrak{p}^{-1}$, so that $\mathfrak{p}^{-1}$ turns a point into its polar plane for the quadric indicated by $\mathfrak{p}$.

We need these symbols because we deal with several quadrics and because we consider perpendicular lines and planes. In the

metric work in Chap. IV the vector perpendicular to the plane α was denoted by $\downarrow\alpha$. Here we adopt a projective point of view, and so replace the vector by a point at infinity; instead of $\downarrow\alpha$, we accordingly write $\alpha\mathfrak{e}$. The symbol $\mathfrak{e}$ thus represents a degenerate linear transformation of planes into points, since all planes become points on the same plane, that at infinity. If ω is the plane at infinity $\omega\mathfrak{e} = 0$. For $\alpha\mathfrak{e}$ is on ω for all planes α, hence $[\alpha\mathfrak{e}.\omega] = 0$, $[\omega\mathfrak{e}.\alpha] = 0$ for all planes α. Hence $\omega\mathfrak{e} = 0$.

The transformations $\mathfrak{p}$, $\mathfrak{e}$ are polarities (§ 50·9). If α touches the quadric indicated by $\mathfrak{p}$, then $[\alpha\mathfrak{p}.\alpha] = 0$, and $\alpha\mathfrak{p}$ is the point of contact.

2. If $\mathfrak{p}$ is a polarity, then $\mathfrak{p}+k\mathfrak{e}$ is a '*confocal polarity*'. It is convenient to speak of the quadric $\mathfrak{p}$, meaning the quadric envelope of planes α which satisfy $[\alpha\mathfrak{p}.\alpha] = 0$. We may then speak of confocal quadrics $\mathfrak{p}$, $\mathfrak{p}+k\mathfrak{e}$.

If ω is the plane at infinity, then $\mathfrak{p}$ is a '*paraboloid*' if $[\omega\mathfrak{p}.\omega] = 0$, for then $\mathfrak{p}$ touches ω. Then $\mathfrak{p}+k\mathfrak{e}$ is also a paraboloid. For $\omega\mathfrak{e} = 0$, $[\omega\mathfrak{e}.\omega] = 0$.

If $\mathfrak{p}$ is a degenerate polarity, that is, if the planes α which satisfy $[\alpha\mathfrak{p}.\alpha] = 0$ are those touching a cone, then $\mathfrak{p}+k\mathfrak{e}$ is also degenerate.

3. If α is a tangent plane through p to $\mathfrak{p}+k\mathfrak{e}$, then

$$[\alpha p] = 0, \quad [\alpha(\mathfrak{p}+k\mathfrak{e}).\alpha] = 0.$$

Hence $[\alpha\mathfrak{p}\alpha]+k[\alpha\mathfrak{e}\alpha] = 0$. Thus α envelopes a quadric which must be a cone, since α always goes through p; and as k is changed, we get confocal cones.

When we speak of confocal quadrics, we shall assume they are not paraboloids, or degenerate quadrics.

4. If $\mathfrak{p}+k\mathfrak{e}$ is a quadric confocal to $\mathfrak{p}$ and through p, and β is its tangent plane at p, then $\beta(\mathfrak{p}+k\mathfrak{e}) = p$, $[\beta p] = 0$, whence $[\beta(\mathfrak{p}+k\mathfrak{e}).\beta] = 0$. Hence β repeated is one of the cones mentioned in 3.

5. *Reye axes.* The '*axis*' of a plane α for the confocals $\mathfrak{p}+k\mathfrak{e}$ is the line $[\alpha\mathfrak{p}.\alpha\mathfrak{e}]$.

The poles of α for these quadrics, namely $\alpha(\mathfrak{p}+k\mathfrak{e})$, *lie on the axis of α.* Since $[\alpha\mathfrak{p}.\alpha\mathfrak{e}]$ is $[\alpha\mathfrak{p}\downarrow\alpha]$ in the earlier notation, therefore

the axis of α is perpendicular to α. Or, we can reach this result thus: since $\alpha(\mathfrak{p}+k_1 e)-\alpha(\mathfrak{p}+k_2 e)\equiv \alpha e$, therefore the join of the two poles is perpendicular to α.

Since $\omega e = 0$, where ω is the plane at infinity, therefore any line through the centre $\omega\mathfrak{p}$ of the quadric should be regarded as an axis, for the point p is on the axis of α if $[p.\alpha\mathfrak{p}.\alpha e] = 0$, and this is true for all p when $\alpha = \omega$.

6. If L is an axis it is of form $L = [\alpha\mathfrak{p}.\alpha e]$; hence

$$[L.\alpha e] = 0, \quad [L.\alpha\mathfrak{p}] = 0.$$

Hence *a line through the pole of a plane with respect to one of the quadrics and perpendicular to the plane is an axis.*

7. The polarity $\mathfrak{p}$ turns a line L into $L[\mathfrak{p}^2]$, and a point p into $p[\mathfrak{p}^3]$.

If L is the axis of α, then $[Lp] = 0$ where $p = \alpha\mathfrak{p}$. Transform by $\mathfrak{p}$ and we have $[L[\mathfrak{p}^2].p[\mathfrak{p}^3]] = 0$, or $[L[\mathfrak{p}^2].\alpha] = 0$, since $p[\mathfrak{p}^3]\equiv p\mathfrak{p}^{-1} = \alpha$ (§ 63·9). But $[L.\alpha e] = 0$, or in the old notation, $[L \downarrow \alpha] = 0$, and $L[\mathfrak{p}^2]$ lies on α.

Hence L is perpendicular to $L[\mathfrak{p}^2]$. *An axis is perpendicular to its polar lines with respect to the quadrics.*

8. The axis of α goes through $q = \alpha(\mathfrak{p}+k e)$ for each k. Take $k = -[\alpha\mathfrak{p}\alpha]\div[\alpha e\alpha]$. Then $[q\alpha] = 0$, and hence α touches $\mathfrak{p}+k e$ at q, but since the axis is perpendicular to α and goes through q, it is normal to $\mathfrak{p}+k e$ at q.

Hence *each axis, not through the centre, is normal to just one quadric of the system. Each normal to a quadric of the system is an axis.*

The foot of the normal is the '*principal point*' of the axis. (When the normal is along a principal axis of the confocals, each of its points is a principal point.)

9. If p is the principal point on an axis L, not through the centre, then L touches two of the confocals through p, for it is normal to one confocal through p, and the confocals cut at right angles.

The normals through p to the two confocals just mentioned are also Reye axes with p as a principal point. Hence, since through each point, not the centre, go three confocals:

Through each point p, not the centre, goes a triad of perpendicular axes, of which p is the principal point.

10. The axes $[\alpha\mathfrak{p}.\alpha\mathfrak{e}]$ through any point p are those which satisfy $[p.\alpha\mathfrak{p}.\alpha\mathfrak{e}] = 0$, as the plane α varies. Put $x = \alpha\mathfrak{p}$, then $\alpha = x\mathfrak{p}^{-1}$, and the equation becomes $[px.x\mathfrak{p}^{-1}\mathfrak{e}] = 0$. Considered as a locus of points x, this is a quadric, and by construction, a cone, which goes through the centre c, by 5, or since

$$c\mathfrak{p}^{-1}\mathfrak{e} = \omega\mathfrak{e} = 0.$$

The Reye axes through a point p, not the centre, lie on a quadric cone through the centre.

11. In general, a plane touches only one confocal of the system $\mathfrak{p}+k\mathfrak{e}$, for in 8, $k = -[\alpha\mathfrak{p}\alpha] \div [\alpha\mathfrak{e}\alpha]$ is unique, unless $[\alpha\mathfrak{p}\alpha] = [\alpha\mathfrak{e}\alpha] = 0$. If α is real, then $[\alpha\mathfrak{e}\alpha] = 0$ only when $\alpha = \omega$. But if our field of scalars is the complex field, we can have imaginary planes α satisfying both equations; such a plane α touches all the confocals, the points of contact are $\alpha(\mathfrak{p}+k\mathfrak{e})$, and lie on the axis $[\alpha\mathfrak{p}.\alpha\mathfrak{e}]$. This axis lies on α, since $[\alpha..\alpha\mathfrak{p}.\alpha\mathfrak{e}] = 0$, and it is perpendicular to α, since $[\alpha\mathfrak{p}.\alpha\mathfrak{e}.\alpha\mathfrak{e}] = 0$. Conversely, an (imaginary) plane which touches two confocals touches all.

12. Dually to §51·6, if $\mathfrak{p}_1$, $\mathfrak{p}_2$ be two polarities, the lines $[\alpha\mathfrak{p}_1.\alpha\mathfrak{p}_2]$ constitute a tetrahedral complex. Our complex of axes is thus a special case of the tetrahedral complex, as was shewn, in another way, in §65·19.

Examples. 1. If L is not an axis, the axes of all planes through L lie on a regulus opposite to the regulus of lines polar to L for the confocals.

2. A line is an axis if its polar lines for two confocals are coplanar; the polar lines for all confocals are then coplanar.

3. The Reye complex includes all lines at infinity on the confocals, all lines in or perpendicular to the principal planes, the axes of all conics on any of the confocals, the join of the feet of any two intersecting normals to a confocal, and the polar of that join. The axes in a plane envelope a parabola.

4. Two planes conjugate for two of the confocals are conjugate for all and are perpendicular. Two perpendicular planes conjugate for one confocal are conjugate for all.

Def. A line L is a '*focal axis*' of a quadric $\mathfrak{p}$ if any two perpendicular planes which cut in L are conjugate for $\mathfrak{p}$.

A focal axis for $\mathfrak{p}$ is a focal axis for all $\mathfrak{p}+k\mathfrak{e}$.

5. Let α, β be perpendicular planes through a focal axis of $\mathfrak{p}$. We can find k so that $\mathfrak{p}+k\mathfrak{e}=\mathfrak{q}$ touches α. Hence $[\alpha\mathfrak{q}\alpha]=0$, $[\alpha\mathfrak{q}\beta]=0$. If l is any scalar, there is a scalar l′ such that $\alpha+l'\beta$ is perpendicular to $\alpha+l\beta$, and hence conjugate to it for $\mathfrak{q}$. Thus

$$[(\alpha+l\beta)\,\mathfrak{e}(\alpha+l'\beta)]=0, \quad [(\alpha+l\beta)\,\mathfrak{q}(\alpha+l'\beta)]=0.$$

These give $l\neq l'$, $[\beta\mathfrak{q}\beta]=0$. Hence β touches $\mathfrak{q}$.

Then $[\alpha\beta]$ is a generator of $\mathfrak{q}$, for $\alpha+x\beta$ touches $\mathfrak{q}$ for all x, since

$$[(\alpha+x\beta)\,\mathfrak{q}(\alpha+x\beta)]=0.$$

Thus *any focal axis is a generator of some quadric of the confocal system.*

6. Since two planes through L conjugate for a quadric separate tangent planes through L harmonically, and two perpendicular planes separate the tangent planes to the circle at infinity harmonically, the *complex of focal axes is a special case of the harmonic complex* of lines such that tangent planes through them to one given quadric separate harmonically those to another given quadric. This is investigated later, p. 472.

7. If $\mathfrak{A}$, $\mathfrak{B}$ be matrices of order three, then

$$\begin{aligned}\det(\mathfrak{A}-\lambda\mathfrak{B}) &= [(\mathfrak{A}-\lambda\mathfrak{B})^3]\\ &= [\mathfrak{A}^3]-3\lambda[\mathfrak{A}^2\mathfrak{B}]+3\lambda^2[\mathfrak{A}\mathfrak{B}^2]-\lambda^3[\mathfrak{B}^3].\end{aligned}$$

Thus $[\mathfrak{A}^2\mathfrak{B}]$, $[\mathfrak{A}\mathfrak{B}^2]$ are independent of the frame.

Similarly for matrices of any order.

CHAPTER X

GENERAL THEORY OF INNER PRODUCTS

§ 67. *Inner products of extensives of step one.*

1. If $a = a_1 e_1 + \ldots + a_n e_n$ and $b = b_1 e_1 + \ldots + b_n e_n$ be two extensives of step one in $\mathscr{S}(e_1, \ldots, e_n)$, then

$$[a\,|\,b] = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

We can regard $[a\,|\,b]$ as a kind of product of a and b, for the distributive law, with respect to addition, holds. We call $[a\,|\,b]$ the '*inner product*' of a and b.

Similarly, if E, F be extensives of any step $\leqq n$ in $\mathscr{S}(e_1, \ldots, e_n)$, then $[E\,|\,F]$ is their '*inner product*'.

Also $$[E\,|\,F] = [F\,|\,E].$$

2. *Def.* Two extensives a, b of step one are '*orthogonal*' if, and only if, $[a\,|\,b] = 0$.

Def. If $A = \mathscr{S}(a_1, \ldots, a_r)$, then an extensive b of step one is '*orthogonal*' to A if, and only if, it is orthogonal to each of $a_1, \ldots, a_r$; it is then orthogonal to all extensives of step one in A.

3. *If A is a spread of step* r, *and a subset of a spread R of step* n, *then the set of extensives of R which are orthogonal to A form a spread of step* $n - r$.

For, if $a_1, \ldots, a_r$ be a basis of A, then the equations

$$[a_1\,|\,x] = 0,\ [a_2\,|\,x] = 0,\ \ldots,\ [a_r\,|\,x] = 0 \qquad (1)$$

are independent, since $a_1, \ldots, a_r$ are independent.

Any extensive x which satisfies (1) is orthogonal to all extensives in A, and the extensives satisfying (1) form a spread of step $n - r$, (§ 59·5, ·8).

Cor. No extensive, save zero, in $\mathscr{S}(e_1, \ldots, e_n)$ is orthogonal to all $e_1, \ldots, e_n$.

4. *Def.* If $A = \mathscr{S}(a_1, \ldots, a_r), B = \mathscr{S}(b_1, \ldots, b_s)$, then A is '*orthogonal*' to B if, and only if, each a_i is orthogonal to each b_j; then each extensive in A is orthogonal to each in B.

Sometimes A is called '*completely orthogonal*' to B. It should be noticed that $[A\,|\,B] = 0$ is not a *sufficient* condition for this relation.

5. *If A, of step* r, *is a subspread of a spread of step* n, *then its extensives are just the solution-extensives of a set of* $n-r$ *homogeneous linear equations in* n *variables.*

For let $a_1, \ldots, a_r$ be a basis of the spread A, and $b_1, b_2, \ldots, b_{n-r}$ be a basis of spread B orthogonal to A.

Then $[b_j | a_i] = 0, \quad (i = 1, \ldots, r;\ j = 1, \ldots, n-r).$

The set of extensives orthogonal to B is the set of solution-extensives of the equations:

$$[b_1 | x] = 0, \ \ldots, \ [b_{n-r} | x] = 0. \qquad (2)$$

Hence $a_1, \ldots, a_r$ are in the set. But the set fills a spread C of step $n-(n-r) = r$, and hence $a_1, \ldots, a_r$ is a basis of C. Thus $C \equiv A$, and the extensives of A satisfy the $n-r$ equations (2).

6. *Def.* The extensives $a_1, \ldots, a_n$ of $\mathscr{S}(e_1, \ldots, e_n)$ form a '*normal frame*' if $a_i^2 = 1$ for each $i = 1, \ldots, n$, and $[a_i | a_j] = 0$ when $i \neq j$, $(i = 1, \ldots, n;\ j = 1, \ldots, n)$. Then $[a_i a_j]^2 = [a_i a_j a_k]^2 = \ldots = 1$.

For example, the frame $e_1, \ldots, e_n$ is normal.

7. *If $a_1, \ldots, a_n$ is a normal frame in $\mathscr{S}(e_1, \ldots, e_n)$, then $a_1, \ldots, a_n$ are independent and hence $\mathscr{S}(a_1, \ldots, a_n) = \mathscr{S}(e_1, \ldots, e_n)$.*

For, if $k_1 a_1 + \ldots + k_n a_n = 0$ ($k_1, \ldots, k_n$ scalars), then inner multiplication by a_1 gives $k_1 a_1^2 = 0$, and hence $k_1 = 0$. Similarly all the k vanish.

If $a_1, \ldots, a_n$ is a normal frame in $\mathscr{S}(e_1, \ldots, e_n)$ and x is orthogonal to each of $a_1, \ldots, a_r$, then x is in $\mathscr{S}(a_{r+1}, \ldots, a_n)$.

For, if $x = k_1 a_1 + \ldots + k_n a_n$, ($k_1, \ldots, k_n$ scalars), then $[x | a_i] = 0$ gives $k_i = 0$, $(i = 1, \ldots, r)$.

8. *Def.** An '*elementary circular transformation*' of $a_1, \ldots, a_n$ is one which changes two extensives a, b of the set into extensives of the form

$$xa + yb, \ \pm(xb - ya), \quad \text{where} \quad x^2 + y^2 = 1, \quad (x, y \text{ scalars}),$$

and leaves the other extensives of the set unaltered.

If $x = \cos\alpha$, $y = \sin\alpha$, we call α the '*angle*' of the transformation.

The change of a, b to $-a$, b is an elementary circular transformation of angle π, $(x = -1, y = 0)$.

* For 8–12, see A_2, §§ 155–163.

If, in the definition, we take the positive sign in the ambiguity, we have a '*direct*' transformation, if the negative, an '*indirect*' transformation. Transformations which differ in sign only, we call '*opposite*'.

Def. A '*circular*' transformation is one obtained by a succession of elementary circular transformations.

9. *By a circular transformation, normal frames* $(a, b, c, \ldots)$ *become such, the inner product and the square of the outer product of any two extensives of the frame are unchanged.*

We need only shew this for an elementary circular transformation.

Then $$(\mathrm{x}a+\mathrm{y}b)^2 = \mathrm{x}^2a^2+\mathrm{y}^2b^2 = 1,$$

if $$a^2 = b^2 = 1, \quad \text{and} \quad [a\,|b] = 0.$$

Also

$$(\mathrm{x}b-\mathrm{y}a)^2 = 1, \quad [(\mathrm{x}a+\mathrm{y}b)\,|(\mathrm{x}b-\mathrm{y}a)] = \mathrm{xy}(b^2-a^2) = 0.$$

If c be any other extensive of the normal frame, then

$$[(\mathrm{x}a+\mathrm{y}b)\,|c] = 0, \quad [(\mathrm{x}b-\mathrm{y}a)\,|c] = 0, \quad [(\mathrm{x}a+\mathrm{y}b)\,c]^2 = 1.$$

Also $[(\mathrm{x}a+\mathrm{y}b)(\mathrm{x}b-\mathrm{y}a)] = \mathrm{x}^2[ab]-\mathrm{y}^2[ba] = (\mathrm{x}^2+\mathrm{y}^2)\,[ab] = [ab]$.

Cor. By a circular transformation, the inner product and the square of the outer product of any two extensives are unchanged.

10. *If* $a_1, \ldots, a_n$ *be a normal frame, and* b *be in* $\mathscr{S}(a_1, \ldots, a_n)$ *and* $b^2 = 1$, *we can change* $a_1, \ldots, a_n$ *by a circular transformation into a set containing* b. *Our field of scalars is here assumed to be real.*

For, let $b = \mathrm{k}_1a_1+\ldots+\mathrm{k}_na_n$, and let, with positive square roots,

$$c_2 = \frac{\mathrm{k}_1a_1+\mathrm{k}_2a_2}{\sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2)}}, \quad c_3 = \frac{\sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2)}\,c_2+\mathrm{k}_3a_3}{\sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2+\mathrm{k}_3^2)}}, \quad \ldots,$$

$$c_n = \frac{\sqrt{(\mathrm{k}_1^2+\ldots+\mathrm{k}_{n-1}^2)}\,c_{n-1}+\mathrm{k}_na_n}{\sqrt{(\mathrm{k}_1^2+\ldots+\mathrm{k}_n^2)}}.$$

There is an elementary circular transformation on a_1, a_2 which changes a_1 into c_2; one on c_2, a_3 which changes c_2 into c_3, ..., one on c_{n-1}, a_n which changes c_{n-1} into c_n.

Also
$$\begin{aligned} b &= \sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2)}\,c_2+\mathrm{k}_3a_3+\ldots+\mathrm{k}_na_n \\ &= \sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2+\mathrm{k}_3^2)}\,c_3+\mathrm{k}_4a_4+\ldots+\mathrm{k}_na_n \\ &= \sqrt{(\mathrm{k}_1^2+\mathrm{k}_2^2+\ldots+\mathrm{k}_n^2)}\,c_n = c_n, \end{aligned}$$

since $b^2 = 1$, $c_n^2 = 1$, and hence $\mathrm{k}_1^2+\ldots+\mathrm{k}_n^2 = 1$. Thus $c_n = b$.

11. *If* $a, b, c, \ldots, p, q$ *and* $a_1, b_1, c_1, \ldots, p_1, q_1$ *be two normal frames of the same step, and the field of scalars be real, we can change one frame into the other by a circular transformation.*

For $a, b, c, \ldots$ are in $\mathscr{S}(a_1, b_1, c_1, \ldots, p_1, q_1)$. Hence by a circular transformation on $a_1, b_1, \ldots$ we can replace one of these by a. Suppose we thereby obtain the set $a, b_2, c_2, \ldots$; this is normal, and $b_1, c_1, \ldots$ are orthogonal to a, and hence are in $\mathscr{S}(b_2, c_2, \ldots)$. Hence by a circular transformation on $b_2, c_2, \ldots$ we can replace one of these by b. Suppose we thereby obtain the set $b, c_3, d_3, \ldots$; this is normal and $c, d, \ldots$ are in $\mathscr{S}(c_3, d_3, \ldots)$. So proceeding, we reach a normal set $a, b, c, \ldots, p, q'$ where q' is in the spread $\mathscr{S}(q)$, that is, where $q = \mathrm{k}q'$, (k scalar).

But $q^2 = 1$, $q'^2 = 1$, hence $\mathrm{k} = \pm 1$.

If $\mathrm{k} = -1$, we replace the last elementary transformation by its opposite.

12. *If* G *be any subspread of* $\mathscr{S}(e_1, \ldots, e_n)$ *of step* $m \leqq n$, *we can find a normal set* $a_1, \ldots, a_n$ *such that* $G = \mathscr{S}(a_1, \ldots, a_m)$, *the field of scalars being real.*

For, let a_1 be in G, with $a_1^2 = 1$; by a circular transformation change $e_1, \ldots, e_n$ into a normal set containing a_1. Then a_1 is orthogonal to the other extensives of this set, and hence to each extensive of the spread they span. This spread, of step $n-1$, cuts G in a spread G_1 of step $n-1+m-n = m-1$.

Let a_2 be in G_1, with $a_2^2 = 1$, then a_2 is orthogonal to a_1. Hence, by a circular transformation, we can change $e_1, \ldots, e_n$ to a normal set containing a_1, a_2. Hence a_1, a_2 are orthogonal to the other extensives of this set, and hence to each extensive of the set they span. This spread, of step $n-2$, cuts G in a spread G_2 of step $m-2$.

Let a_3 be in G_2, with $a_3^2 = 1$, then a_3 is orthogonal to a_1 and a_2, and so on. Thus in any subspread of $\mathscr{S}(e_1, \ldots, e_n)$, a normal set of unities can be found, to which can be adjoined other unities, so that the whole set is normal and spans $\mathscr{S}(e_1, \ldots, e_n)$.

Cor. In finding a normal frame for $G = \mathscr{S}(e_1, \ldots, e_m)$, we took a_1 in G (of step m), a_2 in G_1 (of step $m-1$), ..., a_m in G_m (of step 1); the freedom of choice is hence

$$(m-1)+(m-2)+\ldots+1 = \tfrac{1}{2}m(m-1),$$

since each extensive is to have inner square unity and this is one condition on the freedom.

13. *If $e_1, \ldots, e_n$ and $a_1, \ldots, a_n$ are normal frames, spanning the same spread, and the field of scalars is real, then the frames are connected by an orthogonal transformation.*

For the frames are connected by a circular transformation, and this is the product of elementary circular transformations; each of the latter is orthogonal, because the array of its coefficients satisfies the necessary conditions; and the product of orthogonal transformations is orthogonal.

14. *Orthogonal transformations, and these only, preserve inner products unchanged; they also preserve outer products of maximum step unchanged, apart possibly from sign.* (Cf. § 63·11.)

For, if $[x\,|\,y] = [x\mathfrak{A}\,|\,y\mathfrak{A}]$ for all x, y, then

$$0 = [x\,|\,y] - [x\mathfrak{A}\,|\,y\mathfrak{A}] = [x\,|\,y] - [x\mathfrak{A}\mathfrak{A}^*\,|\,y] = [(x - x\mathfrak{A}\mathfrak{A}^*)\,|\,y]$$
$$= [x(\mathfrak{J} - \mathfrak{A}\mathfrak{A}^*)\,|\,y], \quad \text{for all } x, y.$$

Hence $x(\mathfrak{J} - \mathfrak{A}\mathfrak{A}^*) = 0$ for all x. Hence $\mathfrak{A}\mathfrak{A}^* = \mathfrak{J}$.

The last part holds, since $\det \mathfrak{A} = \pm 1$.

15. *Any orthogonal transformation, if the field of scalars is the real field, can be expressed as the product of elementary circular transformations applied to mutually orthogonal subspreads.*

To prove this we extend the field of scalars to the complex field, by adjoining $i = \sqrt{-1}$.

Let k be a characteristic root of the orthogonal transformation $\mathfrak{A}$ (with real coordinates), and let x be the corresponding latent extensive. We know that if k is real, then $k = \pm 1$ and hence $x\mathfrak{A} = \pm x$ (§ 63·11).

Suppose k is complex, then x is complex, that is, if

$$x = x_1 e_1 + \ldots + x_n e_n,$$

then some or all of $x_1, \ldots, x_n$ are complex.

Let $\bar{x}_i$, $\bar{k}$ denote the numbers conjugate complex to x_i, k, and let $\bar{x} = \bar{x}_1 e_1 + \ldots + \bar{x}_n e_n$, then

$$x\mathfrak{A} = kx, \quad \bar{x}\mathfrak{A} = \bar{k}\bar{x}, \quad [x\mathfrak{A}\,|\,\bar{x}\mathfrak{A}] = k\bar{k}[x\,|\,\bar{x}],$$
$$[x\mathfrak{A}\,|\,\bar{x}\mathfrak{A}] = [x\mathfrak{A}\mathfrak{A}^*\,|\,\bar{x}] = [x\,|\,\bar{x}].$$

Hence $k\bar{k} = 1$, so that k is of form $e^{i\theta}$, $(i = \sqrt{-1})$.

But $$x^2 = [x\mathfrak{A}]^2 = (kx)^2 = k^2x^2, \quad k \neq \pm 1.$$

Hence $x^2 = 0$. But as x is complex, it does not follow from this that it vanishes.

Let $x = y + iz$, where y, z are real extensives, then

$$0 = x^2 = y^2 - z^2 + 2i[y\,|\,z].$$

Hence $$y^2 = z^2, \quad [y\,|\,z] = 0.$$

Thus y, z are real extensives, orthogonal and of equal magnitude, and we can take them to be of unit magnitude.

By a circular transformation we can replace $e_1, \ldots, e_n$ by a normal frame of which y, z are two members, and such that the spread A spanned by the other members is orthogonal to $\mathscr{S}(y, z)$. This follows from (11).

Now $\mathfrak{A}$ induces an orthogonal transformation in the spread A. If this transformation has a complex characteristic root, then in A we can find real extensives y_1, z_1 orthogonal and of unit magnitude, analogous to y, z above, and by a circular transformation of the unities in A, we can replace these unities by a normal set, two of whose members are y_1, z_1. Hence y, z, y_1, z_1 are mutually orthogonal.

We proceed thus until all the complex roots of $\mathfrak{A}$ are exhausted. If these are 2p in number, we have 2p mutually orthogonal extensives $y, z, y_1, z_1, \ldots, y_{p-1}, z_{p-1}$ of unit magnitude, such that the whole spread is spanned by these and $n-2p$ other (real) extensives, the whole set being normal, and the spread B spanned by the last $n-2p$ extensives is orthogonal to $\mathscr{S}(y, z, \ldots, y_{p-1}, z_{p-1})$.

Then $\mathfrak{A}$ induces in B an orthogonal transformation whose roots are all real, and hence are ± 1. In this spread B are $n-2p$ extensives w such that $w^2 = 1$, $w\mathfrak{A} = \pm w$. We take these as the remaining unities.

Then $\mathscr{S}(y, z)$, $\mathscr{S}(y_1, z_1)$, ..., B are mutually orthogonal. The transformation induced in each of these spreads by $\mathfrak{A}$ is an elementary circular transformation.

For example, $$(y + iz)\mathfrak{A} = (\cos\theta + i\sin\theta).(y + iz).$$

Hence
$$\left.\begin{aligned} y\mathfrak{A} &= \cos\theta.y - \sin\theta.z, \\ z\mathfrak{A} &= \sin\theta.y + \cos\theta.z. \end{aligned}\right\} \qquad (3)$$

Consider the transformation induced in the spread B. By arranging the unities, we can write this transformation as

$$w_i\mathfrak{A} = w_i \ (i = 1, \ldots, s), \quad w_j\mathfrak{A} = -w_j \ (j = s+1, \ldots, n-2p).$$

The unities of the last set are orthogonal to those of the first, since $[w_i|w_j] = -[w_i\mathfrak{A}|w_j\mathfrak{A}] = -[w_i\mathfrak{A}\mathfrak{A}^*|w_j] = -[w_i|w_j]$,

hence $$[w_i|w_j] = 0.$$

The first set gives a spread all of whose extensives are latent, the second set gives a spread all of whose extensives are multiplied by -1.

Hence, if the orthogonal transformation has $2p$ conjugate complex roots, and r roots equal to 1, r_1 roots equal to -1, $(2p+r+r_1 = n)$, then it is a product of p transformations of type (3), r identical transformations, and r_1 transformations which reverse the signs of all extensives of a certain spread of step r_1.

Cor. 1. *If* n *is odd*, the secular equation has at least one real root, and hence *there is at least one extensive which is latent or merely changed in sign by* $\mathfrak{A}$.

Cor. 2. *The imaginary roots of the secular equation of an orthogonal matrix of real elements are conjugate imaginary in pairs, and of form* $e^{\pm i\alpha}$.

This is incidentally shewn in the proof, since if k is an imaginary root of the secular equation, so is $\bar{k}$.

16. If we interpret the $e_1, \ldots, e_n$ as vectors in Euclidean space of dimension n, an orthogonal transformation, since it leaves inner products invariant, is a '*rotation*'. Hence

A rotation round a point in n *dimensions has an invariant line if* n *is odd, but not necessarily if* n *is even.*

When the secular equation has p pairs of conjugate complex roots, the rotation leaves latent p mutually orthogonal planes, and leaves latent or merely reverses a further set of $n-2p$ vectors which lie in a spread orthogonal to all these p planes.

With this interpretation, a '*reflection*' is an elementary circular transformation of angle π, or a transformation which is of this type for a suitable normal frame.

§68. *Inner products of extensives of higher steps.*

1. If A, B be spreads of steps r, s respectively, subspreads of a spread of step n, then the step of $|B$ is $n-s$, and hence that of $[A|B]$ is $$n+r-s \quad \text{or} \quad n+r-s-n = r-s$$

according as $$s > r \quad \text{or} \quad s \leqq r.$$

If r = s, then $[A\,|B]$ is a scalar.

If E, F be distinct unities of equal steps, then

$$[E\,|E] = 1, \quad [E\,|F] = 0.$$

We write E^2 for $[E\,|E]$, then, if $E_1, \ldots, E_r$ be unities,

$$(k_1 E_1 + \ldots + k_r E_r)^2 = k_1^2 + \ldots + k_r^2;$$

if E, F be unities of any steps, then $[E\,|F] = 0$, unless E, F have no common factor.

Further, if $[EF] \neq 0$, then

$$[EF\,|E] = F, \quad [F\,|EF] = E.$$

2. *If A, B be any extensives of equal steps, then* $[A\,|B] = [B\,|A]$, *but if they be of steps* r, s *and* r < s, *then*

$$[A\,|B] = (-1)^{r(s-1)}\,|[B\,|A];$$

thus $[A\,|B] = |[B\,|A]$, unless r is odd and s is even, when $[A\,|B] = -|[B\,|A]$.

We can thus always reduce an inner product to the case when the first factor has a higher step than the second; and this simplifies the formulae.

3. If A, B be of steps r, s and $r \geqq s$, and $A = [a_1 \ldots a_r]$, then

$$[A\,|B] = [A_1\,|B]\,D_1 + \ldots + [A_p\,|B]\,D_p,$$

where $A_1, A_2, \ldots, A_p$ are all the $p = \binom{r}{s}$ multiplicative combinations of step s of $a_1, \ldots, a_r$, and $D_1, \ldots, D_p$ are their completing products, so that

$$[A_1 D_1] = [A_2 D_2] = \ldots = [A_p D_p] = A.$$

This follows at once from § 56·27.

Cor. If r = s, and $A = [a_1 \ldots a_r]$, $B = [b_1 \ldots b_r]$, then

$$[A\,|B] = \begin{vmatrix} [a_1\,|b_1], & [a_1\,|b_2], & \ldots, & [a_1\,|b_r] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [a_r\,|b_1], & [a_r\,|b_2], & \ldots, & [a_r\,|b_r] \end{vmatrix} = \frac{[a_1' \ldots a_r']}{[b_1 \ldots b_r]},$$

where

$$a_i' = \sum_{j=1}^{r} [a_i\,|b_j]\,b_j.$$

Special cases are:

$$[ab\,|c] = [a\,|c]\,b - [b\,|c]\,a$$
$$[abc\,|d] = [a\,|d]\,[bc] + [b\,|d]\,[ca] + [c\,|d]\,[ab]$$
$$[abcd\,|e] = [a\,|e]\,[bcd] - [b\,|e]\,[acd] + [c\,|e]\,[abd] - [d\,|e]\,[abc].$$

4. *If* $a_1, a_2, \ldots, a_n$ *be extensives of step one, and* $A_1, A_2, \ldots$ *their multiplicative combinations of step* s, *and* $B_1, B_2, \ldots$ *be such that*

$$[A_1 B_1] = [A_2 B_2] = \ldots = [a_1 a_2 \ldots a_n],$$

then

$$[A_1\,|B_1] + [A_2\,|B_2] + \ldots = 0.* \qquad (1)$$

For first let $\quad s \geqq n-s, \quad A_1 = [a_{i_1} a_{i_2} \ldots a_{i_s}],$

where $i_1, i_2, \ldots, i_s$ is some selection of s of the numbers $1, \ldots, n$.

Let $B_1 = (-1)^p [a_{j_1} a_{j_2} \ldots a_{j_{n-s}}]$, where $[A_1 B_1] = [a_1 \ldots a_n]$.

We can suppose the factors in A_1 and in B_1 so arranged that the subscripts are in increasing order

$$i_1 < i_2 < \ldots < i_s, \quad j_1 < j_2 < \ldots < j_{n-s}.$$

$$[A_1\,|B_1] = (-1)^p [a_{i_1} a_{i_2} \ldots a_{i_s}\,|a_{j_1} a_{j_2} \ldots a_{j_{n-s}}].$$

$$[A_1\,|B_1] + [A_2\,|B_2] + \ldots = \Sigma(-1)^p [a_{i_1} a_{i_2} \ldots a_{i_s}\,|a_{j_1} a_{j_2} \ldots a_{j_{n-s}}]$$
$$= \Sigma(-1)^p [a_{i_1} a_{i_2} \ldots a_{i_{n-s}}|a_{j_1} a_{j_2} \ldots a_{j_{n-s}}]\,[a_{i_{n-s+1}} \ldots a_{i_s}]. \qquad (2)$$

The first factor

$$[a_{i_1} a_{i_2} \ldots a_{i_{n-s}}\,|a_{j_1} a_{j_2} \ldots a_{j_{n-s}}]$$

is a determinant, the sum of terms like

$$(-1)^q [a_{l_1}\,|a_{j_1}]\,[a_{l_2}\,|a_{j_2}] \ldots [a_{l_{n-s}}\,|a_{j_{n-s}}], \qquad (3)$$

where $l_1, l_2, \ldots, l_{n-s}$ is some permutation of $i_1, \ldots, i_{n-s}$.

Since $l_1, l_2, \ldots, l_{n-s}, j_1, j_2, \ldots, j_{n-s}$ are all distinct, one is least; suppose it is l_1 or j_1. If we interchange these in (3) and change the sign, we get a term

$$(-1)^{q+1} [a_{j_1}\,|a_{l_1}]\,[a_{l_2}\,|a_{j_2}] \ldots [a_{l_{n-s}}\,|a_{j_{n-s}}], \qquad (4)$$

which we shall shew appears as a coefficient of some other term in (2) when the first factor is expanded. If $l_1 < j_2$, then

$$l_1 < j_2 < j_3 < \ldots < j_{n-s},$$

and this is clear. If $l_1 > j_2$, then in order to get $[a_{j_1}\,|a_{l_1}]$ into its normal position so that the subscripts $l_1, j_2, j_3, \ldots, j_{n-s}$ of the

* See A_2, § 183.

second factors in (4) are in increasing order, we may have to move it, say, r places to the right. After this change, the suffixes mentioned are all in increasing order.

Now, if in

$$[A_1 B_1] = [a_{i_1} a_{i_2} \dots a_{i_s} a_{j_1} a_{j_2} \dots a_{j_{n-s}}]$$

we interchange two a, and then move both r places to the right, the expression obtained equals $-[a_1 \dots a_n]$. Hence (4) satisfies the conditions on coefficients of terms in (2), and hence is such a coefficient; and it cancels the coefficient (3).

Thus each term pairs with another and the total sum is zero.

Secondly, if $s < n-s$, put $s(n-s-1) = t$, then

$$[A_1 | B_1] + [A_2 | B_2] + \dots = (-1)^t | ([B_1 | A_1] + [B_2 | A_2] + \dots) = 0,$$

by the first part.

Cor. If from $a_1, a_2, \dots, a_{4m}$ of step one, we form all multiplicative combinations $A_1, A_2, \dots$ of step 2m which contain a_1, and let $B_1, B_2, \dots$ be the complementary combinations, then

$$[A_1 | B_1] + [A_2 | B_2] + \dots = 0.$$

For, as the steps of A_1, B_1 are even, we have $[A_1 B_1] = [B_1 A_1]$. Further $B_1, B_2, \dots$ are all the multiplicative combinations of step 2m which do not contain a_1.

Hence A_1 is the combination complementary to B_1. Hence, in the last theorem $[A_1 | B_1] = [B_1 | A_1]$ and we need take only half the terms.

Examples:

$$[bc | a] + [ca | b] + [ab | c] = 0,$$

$$[ab | cd] + [bc | ad] + [ca | bd] = 0,$$

$$[abc | d] - [bcd | a] + [cda | b] - [dab | c] = 0.$$

§69. *Applications to geometry.*

1. If $E_1, \dots, E_r$ be unities of any step, and A an extensive of that step and the field of scalars be *real*, let $A = k_1 E_1 + \dots + k_r E_r$, then $A^2 = k_1^2 + \dots + k_r^2 \geqq 0$; and $A^2 = 0$ if, and only if, $A = 0$.

But if the field of scalars be *imaginary*, these conclusions need not follow.

2. If, for any step, $A^2 = 0$ only when $A = 0$, and in all other cases is positive, we say the inner products are '*definite*' for that step.

3. If A, B be of equal step, and not zero, and A^2, $B^2 > 0$, $\sqrt{A^2} = a$, $\sqrt{B^2} = b$, then we define the angle $\widehat{AB}$ as the angle in the interval 0 to π, satisfying $\cos \widehat{AB} = \dfrac{[A\,|\,B]}{a.b}$.

4. If $a, b, \ldots$ be of step one, and a^2, $b^2 > 0$, $\sqrt{a^2} = a$, $\sqrt{b^2} = b$, $a, b, \ldots > 0$, then we define $\sin(abc\ldots) = \dfrac{[abc\ldots]}{abc\ldots}$.

5. *If the inner products for the step concerned are definite, then $\widehat{AB}$ is real.*

Let $A \not\equiv B$, then if k be a real scalar, $(kA - B)^2 > 0$, hence

$$k^2A^2 - 2k[A\,|\,B] + B^2 > 0, \quad \text{for all k.}$$

Hence $\quad A^2.B^2 - [A\,|\,B]^2 > 0, \quad -1 < \cos \widehat{AB} < 1.$

6. If a, b be of step one, $\sin[ab] = \sin \widehat{ab}$.

For $\quad \sin^2[ab] = \dfrac{[ab]^2}{a^2b^2} = 1 - \dfrac{[a\,|\,b]^2}{a^2b^2} = 1 - \cos^2 \widehat{ab} = \sin^2 \widehat{ab},$

and by definition, $\quad \sin[ab] \geqq 0, \quad 0 \leqq \widehat{ab} \leqq \pi.$

Hence $\qquad \sin[ab] = \sin \widehat{ab}.$

7. *If the inner products for step one are definite, and*

$$\sin(a_1 a_2 a_3 \ldots) = \sin \theta,$$

then θ is real.

For we can adjoin real weights to $a_1, a_2, \ldots$ so as to secure, in step n, $a_1^2 = a_2^2 = \ldots = a_n^2 = 1$. We then shew that $[a_1 a_2 \ldots a_n]^2 \leqq 1$.

We may assume $[a_1 a_2 \ldots a_n] \neq 0$; if $a_1, a_2, \ldots, a_n$ is a normal frame, then $[a_1 \ldots a_n]^2 = 1$.

If a_1 is not normal to all $a_2, \ldots, a_n$, we can find a normal frame $a_1, b_2, b_3, \ldots, b_n$, such that

$$\mathscr{S}(a_1, b_2, b_3, \ldots, b_n) = \mathscr{S}(a_1, a_2, \ldots, a_n),$$

$$a_k = x_k a_1 + x_{k2} b_2 + x_{k3} b_3 + \ldots + x_{kn} b_n = x_k a_1 + c_k,$$

$$(k = 2, \ldots, m;\ x_k \text{ not all zero}).$$

Then $\qquad 1 = a_k^2 = x_k^2 + \sum_{r=2}^{n} x_{kr}^2, \quad c_k^2 = \sum_{r=2}^{n} x_{kr}^2.$

Hence $c_2^2 c_3^2 \ldots c_n^2 < 1$.

Let $$c_r/\sqrt{c_r^2} = v_r,$$

then $$[a_1 a_2 \dots a_n]^2 = [a_1 c_2 c_3 \dots c_n]^2$$
$$= c_2^2 c_3^2 \dots c_n^2 [a_1 v_2 v_3 \dots v_n]^2 < [a_1 v_2 v_3 \dots v_n]^2.$$

Hence we have replaced $a_2, \dots, a_n$ by $v_2, \dots, v_n$, all normal to a_1 and of unit magnitude, and have found

$$[a_1 a_2 \dots a_n]^2 < [a_1 v_2 \dots v_n]^2.$$

Similarly, if $v_3, v_4, \dots, v_n$ be not all normal to v_2, we can replace them by extensives $v_3', v_4', \dots, v_n'$ in $\mathscr{S}(v_3, v_4, \dots, v_n)$, all normal to v_2, of unit magnitudes, and such that

$$[a_1 v_2 v_3 \dots v_n]^2 < [a_1 v_2 v_3' v_4' \dots v_n']^2.$$

Proceeding thus, we finally reach a normal frame $a_1, w_2, w_3, \dots, w_n$ such that $$[a_1 a_2 \dots a_n]^2 < [a_1 w_2 w_3 \dots w_n]^2 = 1.$$

Hence the theorem. (Hadamard.)

8. If $a, b, \dots$ be mutually orthogonal extensives of unit magnitude *of any step for which inner multiplication is definite*, and if $p = xa + yb + \dots$, then $xa^2 = [a\,|\,p]$, $yb^2 = [b\,|\,p]$, ...

Hence

$$x = [a\,|\,p], \quad y = [b\,|\,p], \quad \dots, \quad p = [a\,|\,p]\,a + [b\,|\,p]\,b + \dots.$$

If $$q = x'a + y'b + \dots,$$

then $$[p\,|\,q] = [a\,|\,p]\,[a\,|\,q] + [b\,|\,p]\,[b\,|\,q] + \dots.$$

Hence $$\cos \widehat{pq} = \cos \widehat{ap} \cos \widehat{aq} + \cos \widehat{bp} \cos \widehat{bq} + \dots.$$

In particular, $$1 = \cos^2 \widehat{ap} + \cos^2 \widehat{bp} + \dots.$$

9. If $a, b, c, d, \dots$ be extensives of step one, inner multiplication being definite, and

$$a + b + c + d + \dots = 0, \quad a' = [bcd\dots], \quad b' = -[acd\dots],$$

then $$[aa'] = [bb'] = \dots = [abcd\dots]$$

and $$\text{mag}\,a : \text{mag}\,b : \dots = \sin a' : \sin b' : \dots. \qquad \text{(i)}$$

And if p be any extensive of step one, then

$$\text{mag}\,a \,.\, \cos ap + \text{mag}\,b \,.\, \cos bp + \dots = 0, \qquad \text{(ii)}$$

$$\sin a' \,.\, \cos ap + \sin b' \,.\, \cos bp + \dots = 0. \qquad \text{(iii)}$$

For $[acd\ldots]+[bcd\ldots]=0$, hence $[acd\ldots]^2=[bcd\ldots]^2$,

$$a^2c^2d^2\ldots\sin^2[acd\ldots]=b^2c^2d^2\ldots\sin^2[bcd\ldots],$$

$$a^2\sin^2 b'=b^2\sin^2 a',$$

and as mag a, mag b, ..., $\sin a'$, $\sin b'$, ... are positive we have (i).

We get (ii) from $[a\,|\,p]+[b\,|\,p]+\ldots=0$, and (iii) from (i), (ii).

10. If A, P be of equal step, and B, Q of equal step, then, if inner multiplication be definite,

$$\sin\widehat{AB}\sin\widehat{PQ}\cos(AB,PQ)=\sum_r\cos A_rP\cos B_rQ\sin A_r\sin B_r\sin P\sin Q,$$

where A_r are of the same step as A, and are multiplicative combinations of the simple factors of A, and similarly for B_r.

If $A=[a_1a_2\ldots a_n]$, $B=[b_1b_2\ldots b_n]$, then $\sin A\sin B\cos AB$ equals the determinant whose (i, j) element is $\cos a_ib_j$.

We leave this as an exercise.

Examples. 1. If $a_1a_2a_3$ and $b_1b_2b_3$ be two triangles in a plane, and $d_{ij}^2=(a_i-b_j)^2$, then $-4[a_1a_2a_3][b_1b_2b_3]$ equals the determinant of the d_{ij}^2 bordered in the first row and column by (0, 1, 1, 1).

Similarly for any step.

If the triangles be in space, $[a_1a_2a_3][b_1b_2b_3]$ is replaced by

$$[a_1a_2a_3\downarrow b_1b_2b_3].$$

2. *Kronecker's identities. Muir's statement.**

If $\begin{vmatrix}1,&\ldots,&2m\\1,&\ldots,&2m\end{vmatrix}$ be a symmetric determinant, the minor

$$\begin{vmatrix}1,&2,&\ldots,&m\\m+1,&m+2,&\ldots,&2m\end{vmatrix}$$

equals the sum of all the others got from it by interchanging a lower index with the last upper index.

For example, for $m=3$,

$$\begin{vmatrix}123\\456\end{vmatrix}=\begin{vmatrix}124\\356\end{vmatrix}+\begin{vmatrix}125\\436\end{vmatrix}+\begin{vmatrix}126\\453\end{vmatrix}.$$

3. The determinant of 6^2 perpendiculars from six points to six lines is zero if either the six points lie on a conic or the six lines touch a conic. (Cwojdzinski.)

* *History of Determinants*, 4, p. 113. Cf. Mehmke, *Math. Ann.* 26, (1886), p. 209.

4. $(-2)^{n-1}$
$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & [a_1|b_1] & [a_1|b_2] & \dots & [a_1|b_n] \\ 1 & [a_2|b_1] & [a_2|b_2] & \dots & [a_2|b_n] \\ \dots & \dots & \dots & \dots & \dots \\ 1 & [a_n|b_1] & [a_n|b_2] & \dots & [a_n|b_n] \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & (a_1-b_1)^2 & (a_1-b_2)^2 & \dots & (a_1-b_n)^2 \\ 1 & (a_2-b_1)^2 & (a_2-b_2)^2 & \dots & (a_2-b_n)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (a_n-b_1)^2 & (a_n-b_2)^2 & \dots & (a_n-b_n)^2 \end{vmatrix}.$$

If $a_i = b_i (i = 1, \dots, n)$, and we denote minors of the second determinant by [ ], then for suitable choice of the signs of the square roots

$$[22]^{\frac{1}{2}} + [33]^{\frac{1}{2}} + \dots + [n+1\, n+1]^{\frac{1}{2}} = 0,$$

where [22], for example, denotes the minor of the (2, 2) element.

§ 70. *Quadrics in step* n, *and their generator spreads.*

If in a spread of step n, Ω be a linear transformation which turns points into primes, or spreads of step $n-1$, then the locus of points p such that $[p\Omega p] = 0$ is a '*quadric*', also denoted by Ω.

If $[\Omega^n] = 0$, and $a_1, \dots, a_n$ be independent points, then

$$[a_1 \dots a_n][\Omega^n] = [a_1\Omega . a_2\Omega \dots a_n\Omega] = 0.$$

Hence $a_1\Omega, \dots, a_n\Omega$ all meet in a point, and hence the transform of each point is a prime through this fixed point. The quadric is then '*singular*' or '*degenerate*'.

Let Ω be a non-singular quadric in a spread of step n—and denote supplements for Ω by the stroke.

Let p_1 be a point on the quadric. Take p_2 on the quadric and on $|p_1$; $p_2 \neq p_1$. Then $[p_1|p_2] = 0$. Since the polars of p_1, p_2 are of step $n-1$, they cut in a spread of step $n-2$. Let p_3 be on the quadric, and on this cut, then $[p_1|p_3] = [p_2|p_3] = 0$. Thus proceeding, we get points $p_1, \dots, p_r$, each on the polar of all the others, and on the quadric, and hence each on its own polar also. If these points are independent, so are their polars, and then these cut in a spread of step $n-r$, which contains the independent points $p_1, \dots, p_r$. Hence $n - r \geqq r$.

Hence the maximum value of r is the greatest integer in $\frac{1}{2}n$.

Every point in the spread defined by $p_1, \ldots, p_r$ is on the quadric, since $[p_i | p_j] = 0$ (i, j = 1, ..., r). The spread is a '*generator spread*'.

We now construct such spreads.

First, let n be even, and $r = \frac{1}{2}n$. Let $e_1, \ldots, e_n$ be a normal frame, $e_i^2 = 1$, (i = 1, ..., n). Let

$$e_1 + \sqrt{(-1)}\, e_2 = j_1, \quad e_3 + \sqrt{(-1)}\, e_4 = j_2, \quad \ldots, \quad e_{2r-1} + \sqrt{(-1)}\, e_{2r} = j_r,$$
$$e_1 - \sqrt{(-1)}\, e_2 = j'_1, \quad e_3 - \sqrt{(-1)}\, e_4 = j'_2, \quad \ldots, \quad e_{2r-1} - \sqrt{(-1)}\, e_{2r} = j'_r.$$

Then $j_s^2 = 0$, $j_s'^2 = 0$, $[j_s | j_t] = 0$, $[j'_s | j'_t] = 0$, $[j_s | j'_t] = 0$, $(s \neq t;\ s, t = 1, \ldots, r)$.

Hence $j_1, \ldots, j_{n/2}, j'_1, \ldots, j'_{n/2}$ are on the quadric, and if we select a set of any n/2 of these elements, not containing j_r if it contains j'_r, then the spread joining them is a generator spread. If S is any generator spread, so formed, and S' is the spread obtained by changing the j into their corresponding j', and the j' into their corresponding j, then S and S' are 'conjugate' generator spreads. They have no common element.

Secondly, if n is odd and $n = 2r + 1$, let $e_1, \ldots, e_n$ be again a normal frame, then the generator spreads are given by the same formulae as in the first case.

CHAPTER XI

CIRCLES*

§71. *Sums of circles. Inner products of circles.*

1. *Notation.* We consider lines and circles in the real Euclidean plane completed by the adjunction of a single point at infinity, as in inversion geometry. A line and a point are therefore regarded as special cases of circles, and we use small latin letters for any of these figures. At first we shall also have to speak of points in the old or usual sense, and we shall distinguish these by a dash. Thus, if p is a point-circle, that is, a circle of zero radius, then p' will denote its centre, that is, the ordinary point corresponding to p.

A '*proper circle*' is one which is not a line; it may be a point-circle.

We always denote proper circles by the letters c, c_i, their radii by r, r_i, the point-circles at their centres by o, o_i, the centres themselves by o', o'_i, and this notation will be used without special mention.

Our circles shall have either *positive real* or *purely imaginary* radii; we can regard a proper circle as a point with a real number attached, whose square root is the radius.

2. *Inner products of proper circles.* If c_1, c_2 be proper circles, we define their inner product by

$$[c_1\,|\,c_2] = [c_2\,|\,c_1] = \tfrac{1}{2}(r_1^2 + r_2^2 - (o'_1 - o'_2)^2), \qquad (1)$$

where $(o'_1 - o'_2)^2$ means, as usual, the square of the distance from o'_1 to o'_2.

In particular, if p is a point-circle, c a proper circle, then

$$[p\,|\,c] = [c\,|\,p] = \tfrac{1}{2}(r^2 - (p' - o')^2). \qquad (2)$$

This is the negative of half the '*power*' of p' for the circle c, and it vanishes if and only if p is on the circle.

* Circles were treated by Grassmann's methods by Cox, *Quart. Journ.* 19 (1883), p. 74 and 25 (1891), p. 1; Mehmke, 'Anwendung der Grassmannschen Ausdehnungslehre auf die Geometrie der Kreise in der Ebene', *Inaug. Diss.* Tübingen, 1880. See E. Müller, *Monatshefte Math. Phys.* 3 (1892), p. 365; 4 (1893), pp. 1, 93; 7 (1896), p. 77; 9 (1898), p. 269, for sphere geometry in general, and Study, *Math. Ann.* 49 (1897), p. 497.

If o_1, o_2 be point-circles, $[o_1 \,|\, o_2] = -\frac{1}{2}(o_1' - o_2')^2$.

The inner square $c^2 = [c \,|\, c]$ of a proper circle equals r^2.

If c be any circle, then kc (k scalar) is the same circle with weight k attached. The weight k is to be real.

We shall not absorb weights; the weight of c will thus always be unity, unless the contrary is expressly stated.

3. *Sums of circles.* If $\Sigma k_i [c_i \,|\, p] = m[s \,|\, p]$ for all point-circles p, where c_i, s are proper circles, we write $\Sigma k_i c_i = ms$.

If $\Sigma k_i [c_i \,|\, p] = 0$ for all point-circles p, where the c_i are proper circles, we write $\Sigma k_i c_i = 0$.

Thus $$[(k_1 c_1 + k_2 c_2 + \ldots + k_n c_n) \,|\, p] = k_1 [c_1 \,|\, p] + k_2 [c_2 \,|\, p] + \ldots + k_n [c_n \,|\, p].$$

By these definitions, we can add equations which involve sums and differences of proper circles, and multiply them by scalars.

For example, $k_1 c_1 + k_2 c_2 = 0$ implies $k_1 = -k_2$; hence if $k_1 \neq 0$, then $c_1 = c_2$.

4. Conditions that $\Sigma k_i c_i = 0$, $k_i \neq 0$. If we replace the points p', o_i' by vectors u, v_i from any fixed origin to these points, then if $\Sigma k_i c_i = 0$, we have, for all u,

$$\Sigma k_i (r_i^2 - (u - v_i)^2) = 0,$$

$$\Sigma k_i (r_i^2 - v_i^2) + 2[u \,|\, \Sigma k_i v_i] - u^2 \Sigma k_i = 0.$$

Hence $$\Sigma k_i (r_i^2 - v_i^2) = 0, \quad \Sigma k_i v_i = 0, \quad \Sigma k_i = 0. \qquad (3)$$

These are necessary and sufficient conditions for $\Sigma k_i c_i = 0$. The second condition implies the last, and gives $\Sigma k_i o_i' = 0$.

5. If $\Sigma k_i c_i = 0$, then $\Sigma k_i [c_i \,|\, c] = 0$, where c is any proper circle.

For if v_i be the vector from o' to o_i', we have, by (1), (3),

$$2\Sigma k_i [c_i \,|\, c] = \Sigma k_i (r_i^2 + r^2 - (o_i' - o')^2) = \Sigma k_i (r_i^2 - v_i^2) = 0.$$

If $\Sigma k_i c_i = ms$, then $\Sigma k_i [c_i \,|\, c] = m[s \,|\, c]$.

6. If $[c_1 \,|\, p] = [c_2 \,|\, p]$, then $r_1^2 - (p' - o_1')^2 = r_2^2 - (p' - o_2')^2$, and conversely. Hence

$$2[(p' - \tfrac{1}{2}(o_1' + o_2')) \,|\, (o_1' - o_2')] = r_2^2 - r_1^2,$$

so that p' is on a line perpendicular to $o_1'-o_2'$. As the powers of p' with respect to c_1 and c_2 are equal, this line is what is usually called the '*radical axis*' of c_1, c_2.

The radical axes of three proper circles c_1, c_2, c_3, taken in pairs, are concurrent or parallel.

7. If $k_1c_1+k_2c_2+k_3c_3=0$, $k_i\neq 0$, then $\Sigma k_i o_i'=0$. Hence the centres o_1', o_2', o_3' are collinear.

If also $[c_1\,|\,p]=[c_2\,|\,p]$, then $[c_1\,|\,p]=[c_3\,|\,p]$;

for
$$k_3[c_3\,|\,p]=-k_1[c_1\,|\,p]-k_2[c_2\,|\,p]$$
$$=-(k_1+k_2)[c_1\,|\,p]=k_3[c_1\,|\,p];$$

hence
$$[c_1\,|\,p]=[c_3\,|\,p].$$

Hence, by 6, c_1, c_2, c_3 have in pairs the same radical axis. Such circles are called '*coaxal*'. When k_1, k_2 vary, $k_1c_1+k_2c_2$ describes a coaxal system.

8. If $k_1c_1+k_2c_2+k_3c_3+k_4c_4=0$, $k_i\neq 0$, there is no restriction on the centres; but as in 7, if p has the same power for c_1, c_2, c_3, it has that power for c_4. Hence the radical axes of pairs selected from c_1, c_2, c_3, c_4 are concurrent or parallel.

9. Given $k_1c_1+k_2c_2=mc_3$ $(m\neq 0)$; to determine c_3 from the proper circles c_1, c_2.

By 4, $m=k_1+k_2$, $k_1o_1'+k_2o_2'=(k_1+k_2)o_3'$.

The centre of c_3 is thus known. It remains to find its radius.

Since
$$k_1[c_1\,|\,p]+k_2[c_2\,|\,p]=(k_1+k_2)[c_3\,|\,p],$$

we have

$$k_1(r_1^2-(p'-o_1')^2)+k_2(r_2^2-(p'-o_2')^2)=(k_1+k_2)(r_3^2-(p'-o_3')^2),$$
$$(k_1+k_2)r_3^2=k_1r_1^2+k_2r_2^2-k_1(p'-o_1')^2$$
$$-k_2(p'-o_2')^2+(k_1+k_2)(p'-o_3')^2.$$

But $(k_1+k_2)^2(p'-o_3')^2=(k_1(p'-o_1')+k_2(p'-o_2'))^2$.

Hence
$$(k_1+k_2)^2r_3^2=k_1^2r_1^2+k_2^2r_2^2+k_1k_2(r_1^2+r_2^2-(o_1'-o_2')^2), \quad (4)$$

and this determines r_3 since $k_1+k_2\neq 0$.

In particular, if o_1, o_2 be point-circles, then $o_1+o_2=2c_3$ where c_3 has radius r_3 given by $4r_3^2=-(o_1'-o_2')^2$; the radius is pure-imaginary.

We can write (4) in the form

$$m^2c_3^2 = k_1^2c_1^2+k_2^2c_2^2+2k_1k_2[c_1\,|\,c_2]$$

or $$(k_1c_1+k_2c_2)^2 = k_1^2c_1^2+k_2^2c_2^2+2k_1k_2[c_1\,|\,c_2],$$

and this could have been deduced as usual from the distributive law of 5.

§72. *Improper circles.*

1. We cannot have $c_1-c_2 = ks$, where $k\neq 0$, and c_1, c_2, s are proper circles.

Let $k_1 = -k_2+\epsilon$, then (4) becomes

$$\epsilon^2r_3^2 = \epsilon(\epsilon r_1^2-k_2(r_1^2-r_2^2))+k_2(k_2-\epsilon)(o_1'-o_2')^2.$$

Hence, as $\epsilon\to 0$, we have $r_3\to\infty$, $\epsilon^2r_3^2\to k_2^2(o_1'-o_2')^2$.

First suppose c_1, c_2 not concentric; then the last result suggests that we take c_1-c_2 as the limit of a circle as its radius becomes indefinitely great. We shall accordingly interpret c_1-c_2 as the radical axis of c_1, c_2, with a certain weight or magnitude. This will be legitimate if c_3-c_4 is a multiple of c_1-c_2 whenever c_3, c_4 are in the coaxal system defined by c_1, c_2. And this is the case, for if

$$c_3 = k_1c_1+k_2c_2, \quad c_4 = m_1c_1+m_2c_2, \quad k_1+k_2 = m_1+m_2 = 1,$$

then $$c_3-c_4 = (k_1-m_1)(c_1-c_2).$$

We call such a circle difference c_1-c_2, a '*rotor*'.

2. If $c_1-c_2 = l$, we define $[l\,|\,c]$ and $[c\,|\,l]$ as $[c\,|\,c_1]-[c\,|\,c_2]$. Hence, if p is a point-circle then $[l\,|\,p] = 0$, if and only if p' is on l.

If $c_3-c_4 = m$ is another rotor, we define $[l\,|\,m]$ as $[l\,|\,c_3]-[l\,|\,c_4]$. Thence

$$[l\,|\,m] = [c_1\,|\,c_3]-[c_2\,|\,c_3]-[c_1\,|\,c_4]+[c_2\,|\,c_4] = [m\,|\,l],$$

$$\begin{aligned} l^2 = [l\,|\,l] &= c_1^2-2[c_1\,|\,c_2]+c_2^2 \\ &= r_1^2+r_2^2-(r_1^2+r_2^2-(o_1'-o_2')^2) = (o_1'-o_2')^2. \end{aligned}$$

Thence the '*magnitude*' of the rotor $l = c_1-c_2$ is $\sqrt{(o_1'-o_2')^2}$.

We take the sense of l so that the rotation from $o_1'o_2'$ to l is positive.

3. If $\Sigma k_i[s_i\,|\,p] = m[s\,|\,p]$, for all point-circles p, where s_i, s are proper or improper circles, we write $\Sigma k_i s_i = ms$.

If $\Sigma k_i[s_i\,|\,p] = 0$ for all point-circles p, where s_i are proper or improper circles, we write $\Sigma k_i s_i = 0$.

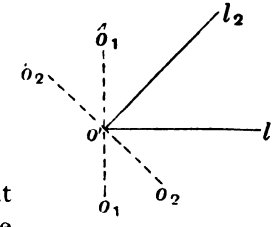
4. To add rotors l_1, l_2 whose lines meet in o'. We can take point circles o_1, $\hat{o}_1$, o_2, $\hat{o}_2$ so that $l_1 = o_1 - \hat{o}_1$, $l_2 = o_2 - \hat{o}_2$; l_1, l_2 are along the right bisectors of the joins of the respective centres, and their magnitudes are the distances between the centres.

Since

$$[o_1 \,|\, p] - [\hat{o}_1 \,|\, p] + [o_2 \,|\, p] - [\hat{o}_2 \,|\, p] = [(o_1 + o_2) \,|\, p] - [(\hat{o}_1 + \hat{o}_2) \,|\, p],$$

we have by 3,

$$l_1 + l_2 = (o_1 + o_2) - (\hat{o}_1 + \hat{o}_2).$$



But $o_1 + o_2$, $\hat{o}_1 + \hat{o}_2$ are circles of weight two, of equal radii, with centres p', q' the mid-points of $o'_1 o'_2$ and $\hat{o}'_1 \hat{o}'_2$.

Hence $(o_1 + o_2) - (\hat{o}_1 + \hat{o}_2)$ is a rotor through o' of magnitude $2p'q'$ perpendicular to $p'q'$; but this is the diagonal of the parallelogram $l_1 l_2$.

Hence concurrent rotors add by the parallelogram law, and we can deduce the usual laws for the addition of *parallel* rotors, if they be not equal and of opposite senses.

5. If the rotors l_1, l_2 be parallel, equal, and in opposite senses, then $l_1 + l_2$ is a new type of extensive, a '*bivector*'. As in the earlier work, two bivectors differ only by a weight factor. We denote the bivector of unit weight by θ.

If c_1, c_2 be concentric circles, c any non-concentric circle, then $c_1 - c$, $c_2 - c$ are parallel and equal rotors. Hence

$$c_1 - c_2 = (c_1 - c) - (c_2 - c)$$

is a multiple of θ.

Denoting signed lengths by bars, $c_1 - c$ is a rotor of magnitude $\overline{o'_1 o'}$ through the point a_1 where the radical axis of c, c_1 meets the line of centres, and $c_2 - c$ is a rotor of magnitude $\overline{o'_1 o'}$ through a_2 where the radical axis of c, c_2 meets the line of centres.

Now
$$r_1^2 - r^2 = \overline{o'_1 a_1}^2 - \overline{o' a_1}^2 = \overline{o'_1 o'}(\overline{o'_1 a_1} - \overline{a_1 o'}),$$
$$r_2^2 - r^2 = \overline{o'_1 a_2}^2 - \overline{o' a_2}^2 = \overline{o'_1 o'}(\overline{o'_1 a_2} - \overline{a_2 o'}).$$

Hence
$$r_1^2 - r_2^2 = \overline{o'_1 o'}(\overline{o'_1 a_1} - \overline{a_1 o'} - \overline{o'_1 a_2} + \overline{a_2 o'})$$
$$= 2\overline{o'_1 o'} \,.\, \overline{a_2 a_1}.$$

Thus, choosing a sign, we may take, when c_1, c_2 are concentric:

$$c_1 - c_2 = \tfrac{1}{2}(\mathrm{r}_1^2 - \mathrm{r}_2^2)\,\theta.$$

In particular, if o is the point-circle at the centre of c, then $c - o = \tfrac{1}{2}\mathrm{r}^2\theta$, which we may write as $c = o + \tfrac{1}{2}\mathrm{r}^2\theta$.

6. Thus the spread of circles is of step four, for the points or point-circles form a spread of step three, and θ can be regarded as an extra extensive unit. As we allow r^2 to take negative values, the spread of circles can be compared with other spreads of step four with real coordinates.

The general theory of Chap. VII shews that any circle is a linear combination of any four independent circles.

7. $2[(c_1 - c_2)\,|\,p] = 2[(p - \tfrac{1}{2}(o_1' + o_2'))\,|\,(o_1' - o_2')] + \mathrm{r}_1^2 - \mathrm{r}_2^2$

= twice the product of the distance between the centres and the distance of p' from the radical axis of c_1 and c_2.

The signs of the distances are taken as in the formula.

The first part follows at once from § 71·2, the second by the customary geometric argument.

Hence, if l is a rotor of unit magnitude, p a point-circle, then $[l\,|\,p]$ is the length, with a sign, of the perpendicular from p' to l.

8. If c_1, c_2 be concentric circles, $c_1 - c_2 = \tfrac{1}{2}(\mathrm{r}_1^2 - \mathrm{r}_2^2)\,\theta$, we define $[\theta\,|\,p]$ by means of $[c_1\,|\,p] - [c_2\,|\,p] = \tfrac{1}{2}(\mathrm{r}_1^2 - \mathrm{r}_2^2)\,[\theta\,|\,p]$; similarly for $[p\,|\,\theta]$. Thus $[\theta\,|\,p] = [p\,|\,\theta]$, and these are independent of the c used in their definitions.

If $c = o + \tfrac{1}{2}\mathrm{r}^2\theta$, then $[c\,|\,o] = o^2 + \tfrac{1}{2}\mathrm{r}^2[\theta\,|\,o]$, but $o^2 = 0$, $[c\,|\,o] = \tfrac{1}{2}\mathrm{r}^2$, by § 71·2.

Hence $[\theta\,|\,p] = 1$ for any point p of weight one.

If we form formally the inner square of $c = o + \tfrac{1}{2}\mathrm{r}^2\theta$, we have

$$c^2 = o^2 + \mathrm{r}^2[o\,|\,\theta] + \tfrac{1}{4}\mathrm{r}^4\theta^2 = \mathrm{r}^2 + \tfrac{1}{4}\mathrm{r}^4\theta^2.$$

Hence we must take $\theta^2 = 0$, and hence $[\theta\,|\,c] = [\theta\,|\,o] = 1$.

Hence $[\theta\,|\,c] = 1$ for any proper circle c of weight one.

If $c_1 - c_2 = l$, we define $[\theta\,|\,l]$ as $[\theta\,|\,c_1] - [\theta\,|\,c_2]$, thence $[\theta\,|\,l] = 0$ for any rotor l.

9. If $\Sigma \mathrm{k}_i s_i = \mathrm{m}s$ where s, s_i are any proper or improper circles, then $\Sigma \mathrm{k}_i[s_i\,|\,\theta] = \mathrm{m}[s\,|\,\theta]$.

If $\Sigma \mathrm{k}_i s_i = \mathrm{m}s$, then $\Sigma \mathrm{k}_i[s_i\,|\,\bar{s}] = \mathrm{m}[s\,|\,\bar{s}]$, where s, $\bar{s}$, s_i are any proper or improper circles.

10. If $c_1 - c_2 = c_3 - c_4$, and no two of c_1, c_2, c_3, c_4 be concentric, then the pairs c_1, c_2 and c_3, c_4 have the same radical axis. Also $c_1 + c_4 = c_2 + c_3$.

Suppose also $r_1 = r_3$ and $r_2 = r_4$. Then §§ 71·9 and 72·2 give

$$(o_1' - o_4')^2 = (o_2' - o_3')^2 \quad \text{and} \quad (o_1' - o_2')^2 = (o_3' - o_4')^2.$$

Hence in this case the centres are at the vertices of a rectangle.

11. If l is a rotor, c a proper circle, then

$$[l\,|\,c] = [l\,|\,o] = \text{moment of } l \text{ round } o'.$$

For, let $l = o_1 - o_2$, and by 10 we can take o_1', o_2', o' *collinear*. Then

$$\begin{aligned}[l\,|\,c] &= [l\,|\,(o + r^2\theta)] = [l\,|\,o] = [o_1\,|\,o] - [o_2\,|\,o] \\ &= -\tfrac{1}{2}[(o_1' - o')^2 - (o_2' - o')^2] \\ &= [(o' - \tfrac{1}{2}(o_1' + o_2'))\,|\,(o_1' - o_2')] \\ &= \text{product of distance of } l \text{ from the centre of } c, \\ &\qquad \text{and the magnitude of } l.\end{aligned}$$

12. If l, m are rotors, then $[m\,|\,l] = [m\,|\,(o_1 - o_2)] = k \,.\, o_1' o_2' \cos\alpha$, where $l = o_1 - o_2$, $\operatorname{mag} l = \overline{o_1' o_2'}$, $k = \operatorname{mag} m$, $\alpha = \widehat{l, m}$.

Since l is along a perpendicular to $o_1' o_2'$, $[m\,|\,l]$ has the same value as the inner product, in the earlier sense, of the corresponding vectors.

13. *Angles.* If c, c_1 be proper circles with *real* radii, or one or both be rotors, we define the '*angle*' α between them by

$$\cos\alpha = \frac{[c\,|\,c_1]}{\sqrt{c^2}\sqrt{c_1^2}}, \quad (0 \leqq \alpha \leqq \pi),$$

where here, and always, the positive square root is taken.

Since $$[c\,|\,c_1] = \tfrac{1}{2}(r^2 + r_1^2 - (o' - o_1')^2),$$

therefore $\cos\alpha$ will be between -1 and $+1$ (inclusive), and hence α real, if

$$(r + r_1)^2 \geqq (o' - o_1')^2 \quad \text{and} \quad (r - r_1)^2 \leqq (o' - o_1')^2;$$

the circles then cut or touch. In other cases, α is imaginary and the circles do not cut or touch.

If $\alpha = 0$, then $(r - r_1)^2 = (o' - o_1')^2$ and c, c_1 touch internally.

If $\alpha = \pi$, then $(r + r_1)^2 = (o' - o_1')^2$, and c, c_1 touch externally.

If c_1 be a proper circle, l a rotor, then l cuts c if $[c\,|\,l]^2 < c^2 \,.\, l^2$, and they touch if $[c\,|\,l]^2 = c^2 \,.\, l^2$.

In all cases the condition for tangency can be written

$$[c\,|\,c_1]^2 - c^2 c_1^2 = 0.$$

If c touches c_1, then c touches all circles $kc + k_1 c_1$.

If $\alpha = \frac{1}{2}\pi$, then $[c\,|\,c_1] = 0$, and c, c_1 cut orthogonally; the circles may be proper or not. We extend this relation to circles with imaginary radii; that is, if c, c_1 be any circles and $[c\,|\,c_1] = 0$, we say they cut orthogonally.

If p' is a point on c, then the point-circle p touches c *and* cuts it orthogonally.

14. If l, m be perpendicular unit rotors, meeting in o', then l, m, $o + \frac{1}{2}\theta$, $o - \frac{1}{2}\theta$ are a set of four mutually orthogonal circles; the last has an imaginary radius.

For $[l\,|\,m] = 0$, $[l\,|\,o] = 0$, $[l\,|\,\theta] = 0$, $[(o + \frac{1}{2}\theta)\,|\,(o - \frac{1}{2}\theta)] = 0$.

Their inner squares are $+1$ or -1. Any circle is a linear combination of these four.

15. Thus *as a basis of our spread of circles* (proper, improper, of radius real or purely imaginary) *we can take four mutually orthogonal circles of inner squares* ± 1.

Any circle orthogonal to $c_1, c_2, c_3, \ldots$ is orthogonal to

$$k_1 c_1 + k_2 c_2 + \ldots.$$

If c_1, c_2, c_3 be independent, there is just one circle orthogonal to them all.

If c_1, c_2, c_3, c_4 be linearly dependent, they have a common orthogonal circle; in particular, if p_1, p_2, p_3, p_4 be point-circles, linearly dependent, then p_1', p_2', p_3', p_4' are concyclic.

§73. *Outer products of circles.*

Taking as the basis of our spread of circles four mutually orthogonal circles of inner squares ± 1, the general theory of outer and inner products gives, the pencils and bundles being weighted:

1. If $c_1 \neq c_2$, the outer product $[c_1 c_2]$ of c_1 and c_2 is the pencil of circles $k_1 c_1 + k_2 c_2$. If $c_1 \equiv c_2$, then $[c_1 c_2] = 0$.

2. If $c_1 \neq c_2$, then $|[c_1 c_2]$ is the pencil of circles orthogonal to the pencil $[c_1 c_2]$.

3. If c_1, c_2, c_3 be independent circles, their outer product $[c_1c_2c_3]$ is the bundle of circles $k_1c_1+k_2c_2+k_3c_3$; if the circles be dependent, and hence coaxal, $[c_1c_2c_3] = 0$.

4. If c_1, c_2, c_3 be independent, then $|[c_1c_2c_3]$ is the circle, with a certain weight, cutting orthogonally all circles of bundle $[c_1c_2c_3]$. (§72·15.)

5. The supplement $|c$ of a circle c is the bundle of circles cutting c orthogonally.

6. The magnitude of $[c_1c_2]$ is $\sqrt{[c_1c_2]^2}$; the magnitude of $[c_1c_2c_3]$ is $\sqrt{[c_1c_2c_3]^2}$. (§68·3 Cor.)

We can express $[c_1c_2]^2$, $[c_1c_2c_3]^2$ by determinants.

If c_1, c_2, c_3 be proper circles which meet each other, then

$$[c_1c_2]^2 = c_1^2c_2^2 - [c_1|c_2]^2 = r_1^2r_2^2(1-\cos^2\alpha) = r_1^2r_2^2\sin^2\alpha,$$

where α is defined in §72·13.

Hence $\operatorname{mag}[c_1c_2] = r_1r_2\sin\alpha$.

Similarly, if θ_1, θ_2, θ_3 refer to the pairs c_2c_3, c_3c_1, c_1c_2, then

$$\operatorname{mag}[c_1c_2c_3] = r_1r_2r_3(1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3 + 2\cos\theta_1\cos\theta_2\cos\theta_3)^{\frac{1}{2}}.$$

7. $[c_1c_2c_3c_4]$ is a scalar, namely $\sqrt{[c_1c_2c_3c_4]^2}$; it can be expressed as a determinant. It vanishes when c_1, c_2, c_3, c_4 have a common orthogonal circle; in particular, if p_1, p_2, p_3, p_4 are point-circles, then $[p_1p_2p_3p_4] = 0$ is the condition that p_1', p_2', p_3', p_4' are concyclic.

8. If c is the orthogonal circle of c_1, c_2, c_3, then $c = k|[c_1c_2c_3]$, for some scalar k. Hence $c^2 = k^2[c_1c_2c_3]^2$. This gives the weight left undecided in 4.

9. If o_1, o_2 be point-circles, then

$$[o_1o_2]^2 = -[o_1|o_2]^2 = \tfrac{1}{4}((o_1'-o_2')^2)^2;$$
$$\operatorname{mag}[o_1o_2] = \tfrac{1}{2}(o_1'-o_2')^2.$$

10. If three independent circles c_1, c_2, c_3 meet in a point, their common orthogonal circle $|[c_1c_2c_3]$ is a point circle, and hence $[c_1c_2c_3]^2 = 0$. This expression also vanishes if c_1, c_2, c_3 are dependent, since then $[c_1c_2c_3] = 0$.

Conversely, if $[c_1c_2c_3]^2 = 0$, we have one of these cases.

By 6, if the circles meet, $[c_1c_2c_3]^2 = 0$ means

$$1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3+2\cos\theta_1\cos\theta_2\cos\theta_3 = 0,$$

which is equivalent to

$$\theta_1 \pm \theta_2 \pm \theta_3 = 0 \pmod{\pi}.$$

Examples. 1. If c_1, c_2, c_3 be independent proper circles, then

$$[c_2c_3]+[c_3c_1]+[c_1c_2] = [(c_1-c_2)(c_1-c_3)]$$

is the set of lines through the radical centre of c_1, c_2, c_3; the supplement of the spread is the set of circles whose centre is the radical centre.

2. Under the same conditions, $[(c_1-c_2)(c_1-c_3)]^2$ is four times the square of the area of the triangle formed by joining the centres.

3. Find the point-circles in the coaxal system $k_1c_1+k_2c_2$.

4. If c_1, c_2 be point-circles, then $k_1c_1+k_2c_2$ has a real or imaginary radius according as $k_1:k_2$ is negative or positive.

5. $k_1c_1+k_2c_2+k_3c_3$ is a point-circle if

$$k_1^2r_1^2+k_2^2r_2^2+k_3^2r_3^2+k_2k_3(r_2^2+r_3^2-d_1^2)$$
$$+k_3k_1(r_3^2+r_1^2-d_2^2)+k_1k_2(r_1^2+r_2^2-d_3^2) = 0,$$

that is, if

$$(k_1r_1^2+k_2r_2^2+k_3r_3^2)(k_1+k_2+k_3) = k_2k_3d_1^2+k_3k_1d_2^2+k_1k_2d_3^2,$$

where d_1 is the distance between o_2' and o_3', and so on.

Hence the equation may be regarded as the areal equation in k_1, k_2, k_3 of the circle orthogonal to c_1, c_2, c_3.

If the circles are point-circles, then

$$k_2k_3d_1^2+k_3k_1d_2^2+k_1k_2d_3^2 = 0;$$

this is therefore the equation of the circumcircle.

6. The circumcircle of $o_1'o_2'o_3'$ is, say,

$$k_1o_1+k_2o_2+k_3o_3+k\theta, \quad (k_1+k_2+k_3=1).$$

Hence
$$k_2d_3^2+k_3d_2^2 = k_3d_1^2+k_1d_3^2 = k_1d_2^2+k_2d_1^2 = 2k,$$
$$k_1 = 2kd_2^{-1}d_3^{-1}\cos(o_2'o_1'o_3'),$$
$$k_1+k_2+k_3 = 16kd_1^{-2}d_2^{-2}d_3^{-2}\Delta^2, \quad k = \tfrac{1}{16}\Delta^{-2}d_1^2d_2^2d_3^2.$$

The radius of the circumcircle is thus $\frac{1}{4}\Delta^{-1}d_1d_2d_3$.

7. If p_1', p_2', p_3', p_4' be concyclic points, then $[p_1p_2p_3p_4]^2 = 0$; hence (§ 30·12)

$$\sqrt{[p_1\,|\,p_2]\,[p_3\,|\,p_4]} \pm \sqrt{[p_2\,|\,p_3]\,[p_1\,|\,p_4]} \pm \sqrt{[p_3\,|\,p_1]\,[p_2\,|\,p_4]} = 0.$$

Deduce Ptolemy's Theorem, and interpret $[c_1c_2c_3c_4]^2 = 0$.

If c_1, c_2, c_3, c_4, c_5 be any circles, prove that the determinant whose (ij) term is $[c_i\,|\,c_j]$ vanishes. Interpret this.

8. For any four (coplanar) points p_1', ..., p_4', we have, for suitable k,

$$k_1p_1 + k_2p_2 + k_3p_3 + k_4p_4 + k_5\theta = 0.$$

The corresponding vanishing determinant connects the six distances of the four points. (Cf. p. 307, Ex. 4.)

9. If c_1, ..., c_4 be mutually orthogonal circles, and p' a point on c_4, and $p = k_1c_1 + k_2c_2 + k_3c_3$, then

$$k_1^2r_1^2 + k_2^2r_2^2 + k_3^2r_3^2 = 0, \quad [c_4\,|\,(c_4 - c_1)] = c_4^2.$$

Now $c_4 - c_1$ is along $o_2'o_3'$, and has magnitude equal to the length of $o_1'o_4'$, say l.

If p is the length of the perpendicular from o_4' to $o_2'o_3'$, then $pl = r_4^2$. Hence o_1' is the pole of $o_2'o_3'$ for c_4, and hence $o_1'o_2'o_3'$ is a self-polar triangle for c_4.

10. If c_1, c_2, c_3 be mutually orthogonal circles with centres o_1', o_2', o_3' and k_1, k_2, k_3 be the perpendiculars from o_1', o_2', o_3' to a transversal $p_1'p_2'p_3'$ of triangle $o_1'o_2'o_3'$, then $k_3c_2 - k_2c_3$ is a circle of weight $k_3 - k_2$, centre p_1', orthogonal to c_1. Hence, disregarding weights, $k_3(c_1 + c_2) - k_2(c_1 + c_3)$ is the circle on diameter $o_1'p_1'$. Hence the circle orthogonal to c_1, c_2, c_3, i.e. the polar circle of $o_1'o_2'o_3'$ (Ex. 9), is orthogonal to the circles on diameters $o_1'p_1'$, $o_2'p_2'$, $o_3'p_3'$.

11. If c_1, ..., c_4 be mutually orthogonal circles, then $c_1 + c_2$, $c_2 + c_3$, $c_3 + c_1$ are circles on diameters $o_1'o_2'$, $o_2'o_3'$, $o_3'o_1'$ and are orthogonal to c_4.

12. If p_1, ..., p_4 be point-circles, c_1 the circumcircle of $p_2'p_3'p_4'$, and so on, let $k_1p_1 + \ldots + k_4p_4 + \theta = 0$. Multiply by $|c_1$, and we have $1 + k_1[p_1\,|\,c_1] = 0$. Multiply by $|\theta$ and we have $k_1 + k_2 + k_3 + k_4 = 0$.

Hence $[p_1\,|\,c_1]^{-1} + [p_2\,|\,c_2]^{-1} + [p_3\,|\,c_3]^{-1} + [p_4\,|\,c_4]^{-1} = 0$.

Also $k_1p_1' + \ldots + k_4p_4' = 0$, hence

$$k_1 : k_2 : k_3 : k_4 = [p_2'p_3'p_4'] : -[p_1'p_3'p_4'] : [p_1'p_2'p_4'] : -[p_1'p_2'p_3'].$$

Hence $[p_2'p_3'p_4']\,[p_1\,|\,c_1] = -[p_1'p_3'p_4']\,[p_2\,|\,c_2]$

$$= [p_1'p_2'p_4']\,[p_3\,|\,c_3] = -[p_1'p_2'p_3']\,[p_4\,|\,c_4].$$

(v. Staudt.)

13. If $c_1, \ldots, c_4$ be mutually orthogonal circles, then

$$r_1^{-2} + \ldots + r_4^{-2} = 0.$$

For we can find k_i such that $k_1 c_1 + \ldots + k_4 c_4 = \theta$, since circles form a spread of step four, and $c_1, \ldots, c_4$ are independent (§ 67·7). Multiply the equation innerwise by $c_1, \ldots, c_4, \theta$ and we find

$$k_1 r_1^2 = 1, \ \ldots, \ k_1 + k_2 + k_3 + k_4 = 0.$$

14. If $o_1', \ldots, o_4'$ be the centres of four circles, and $t_1^2, \ldots, t_4^2$ the powers of an arbitrary point with respect to the circles, then

$$t_1^2[o_2' o_3' o_4'] - t_2^2[o_3' o_4' o_1'] + t_3^2[o_4' o_1' o_2'] - t_4^2[o_1' o_2' o_3'] = \text{const.}$$

If the four circles are orthogonal to a fifth, the constant is zero.

15. If p', q' be points such that all circles through them cut the circle c orthogonally, we say that the point-circles are '*inverse*' for c.

Then there is a linear relation between p, q, c, say

$$q \equiv k_1 p + k_2 c.$$

But
$$q^2 = p^2 = 0,$$

hence
$$2k_1[p\,|\,c] + k_2 c^2 = 0, \quad k_1(r^2 - d^2) + k_2 r^2 = 0,$$

where $d^2 = (p' - o')^2$.

Hence
$$q \equiv r^2 p + (d^2 - r^2) c, \quad q = -\frac{2[c\,|\,p]}{c^2} c + p.$$

16. The result of inverting the point-circle p in c and then the point-circle obtained in c_1 is

$$q = \frac{4[c\,|\,p]\,[c\,|\,c_1]}{c^2 c_1^2} c_1 - \frac{2[c_1\,|\,p]}{c_1^2} c_1 - \frac{2[c\,|\,p]}{c^2} c + p.$$

If $[c\,|\,c_1] = 0$, this is independent of the order in which the inversions are performed.

17. If c, c_1, k be circles, and $c_1 = -\frac{2[k\,|\,c]}{k^2} k + c$, then c, c_1 are inverse circles with respect to k. The centre of k is the inverse of θ with respect to k, namely $-\frac{2[k\,|\,\theta]}{k^2} k + \theta$.

18. If t_1^2, t_2^2, t_3^2 be powers of the vertices of a self-polar triangle for a circle of radius r, with respect to that circle, then, if Δ is the area of the triangle,

$$t_1^2 t_2^2 t_3^2 = -4\Delta^2 r^2, \quad t_1^{-2} + t_2^{-2} + t_3^{-2} = r^{-2}.$$

§74. *The six-circles theorem.*

The pairs of circles ab, bc, cd, da cut respectively in points p, p_1; q, q_1; r, r_1; s, s_1; *then if p, q, r, s be concyclic, so are* p_1, q_1, r_1, s_1.

For the circles $|[bcd]$, $|[cda]$ are orthogonal to b, c, d and to c, d, a respectively. Hence $|[(a+kb)cd]$ is orthogonal to c, d for each k.

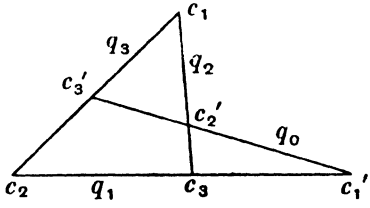
Let $|[bcd] = a_1$, $|[cda] = b_1$, $|[dab] = c_1$, $|[abc] = d_1$,

then $b_1 + ka_1$, $c_1 + k_1 b_1$, $d_1 + k_2 c_1$, $a_1 + k_3 d_1$ are orthogonal to the respective pairs c, d; d, a; a, b; b, c for each k. If now we take k so that $(b_1 + ka_1)^2 = 0$, these two values of k substituted in $b_1 + ka_1$ will give the pair of points in which c, d cut. The argument of §30·13 now proves the theorem, since four points are concyclic if, and only if, they are dependent.

A special case of the six-circles theorem is the Pivot Theorem: *If points d, e, f be taken on the sides bc, ca, ab of any triangle, then the circles aef, bfd, cde meet in a point.*

§75. *The sixteen-circles theorem.**

1. *If the circles* q_1, q_2, q_3, q_0 *are orthogonal to* c_0, *and* q_2, q_3 *to* c_1, *and* q_3, q_1 *to* c_2, *and* q_1, q_2 *to* c_3, *and* q_1, q_0 *to* c_1', *and* q_2, q_0 *to* c_2', *and* q_3, q_0 *to* c_3', *and also* c_1, c_2', c_3' *to* q_1' *and* c_1', c_2, c_3' *to* q_2', *and* c_1', c_2', c_3 *to* q_3' *and* c_1, c_2, c_3 *to* q_0', *then* q_0', q_1', q_2', q_3' *have a common orthogonal circle* c_0', *say.*



[We are not now using the convention that a dashed letter represents a point. The figure is schematic.]

For

$$\left.\begin{aligned} q_1 &\equiv |[c_1' c_2 c_3]\text{,}\quad q_2 \equiv |[c_1 c_2' c_3]\text{,}\quad q_3 \equiv |[c_1 c_2 c_3']\text{,}\quad q_0 \equiv |[c_1' c_2' c_3']\text{,}\\ q_1' &\equiv |[c_1 c_2' c_3']\text{,}\quad q_2' \equiv |[c_1' c_2 c_3']\text{,}\quad q_3' \equiv |[c_1' c_2' c_3]\text{,}\quad q_0' \equiv |[c_1 c_2 c_3]\text{,}\end{aligned}\right\} \quad (1)$$

* Cox, *loc. cit.* His proof is different, and it does not connect up with the 'Möbius identity'.

and since q_0, q_1, q_2, q_3 are orthogonal to the same circle, we have

$$[q_1 q_2 q_3 q_0] = 0.$$

Hence $$[c_1' c_2 c_3 . c_1 c_2' c_3 . c_1 c_2 c_3' . c_1' c_2' c_3'] = 0.$$

Hence, by § 39,

$$[c_1 c_2' c_3' . c_1' c_2 c_3' . c_1' c_2' c_3 . c_1 c_2 c_3] = 0.$$

Taking supplements, we have $[q_1' q_2' q_3' q_0'] = 0$, hence the theorem.

2. *If $c_0, c_1, c_2, c_3, c_1', c_2', c_3'$ be point-circles, so is c_0'.*

For, we can adjust weights so that

$$c_1' = x_1 c_0 + c_3 - c_2, \quad c_2' = x_2 c_0 + c_1 - c_3, \quad c_3' = x_3 c_0 + y c_2 - c_1,$$

and then, since c_0, c_1', c_2', c_3' are dependent, we have $y = 1$.

We can take the weights of the q, q' so that (1) are equations, and not merely congruences. Then

$$q_1 = x_1 | [c_0 c_2 c_3], \quad q_2 = x_2 | [c_0 c_3 c_1], \quad q_3 = x_3 | [c_0 c_1 c_2],$$

$|q_1 = -x_1 [c_0 c_2 c_3]$, and so on, by § 56·5.

$$\begin{aligned} [c_1 c_2' c_3'] &= [c_1 (x_2 c_0 - c_3)(x_3 c_0 + c_2)] \\ &= -x_2 [c_0 c_1 c_2] + x_3 [c_0 c_3 c_1] + [c_1 c_2 c_3] \\ &= x_2 x_3^{-1} | q_3 - x_3 x_2^{-1} | q_2 + [c_1 c_2 c_3], \qquad (2) \end{aligned}$$

$$c_0' = | [q_1' q_2' q_3'] = | [c_1 c_2' c_3' . c_1' c_2 c_3' . c_1' c_2' c_3].$$

By (2) and similar equations, this equals the supplement of

$$\begin{aligned} & x_1 x_2^{-1} x_3^{-1} (x_1 + x_2 + x_3) [c_1 c_2 c_3 | q_2 q_3] + \ldots + \ldots \\ &= x_1 (x_1 + x_2 + x_3) [c_1 c_2 c_3 . c_0 c_3 c_1 . c_0 c_1 c_2] + \ldots + \ldots \\ &\equiv x_1 c_1 + x_2 c_2 + x_3 c_3. \end{aligned}$$

Hence $c_0'^2 = 0$, if $(x_1 c_1 + x_2 c_2 + x_3 c_3)^2 = 0$, that is, if

$$\Sigma x_i x_j [c_i | c_j] = 0.$$

Now $$0 = c_1'^2 = 2x_1 [c_0 | (c_3 - c_2)] - 2[c_2 | c_3].$$

Hence $[c_2 | c_3] = x_1 [c_0 | (c_3 - c_2)]$, and so on.

$$\Sigma x_i x_j [c_i | c_j] = x_1 x_2 x_3 [c_0 | (c_3 - c_2 + c_2 - c_1 + c_1 - c_3)] = 0.$$

As a special case of this theorem in the Euclidean plane, we have:

The circumcircles of the four triangles formed by four general lines meet in a point. (Miquel.)

Examples. 19. Prove the existence of Simson's line from this theorem.

Take q_0, q_1, q_2, q_3 point-circles on circle c_0, and let c_1, c_2, c_3 be sides of triangle $q_1q_2q_3$. Let c_1', c_2', c_3' be circles on diameters q_0q_1, q_0q_2, q_0q_3. Then, since q_1' is orthogonal to c_1, c_2', c_3', it is the point-circle at the foot of the perpendicular from q_0 to q_2q_3. Similarly for q_2', q_3' while $q_0' = |[c_1c_2c_3]$ is the point-circle at infinity. Then $[q_1'q_2'q_3'q_0'] = 0$ shews the existence of Simson's line. (Cox.)

20. If q_0, q_1, q_2, q_3 be point-circles on circle c_0, and c_1, c_2, c_3, c_1', c_2', c_3' be circles on diameters q_2q_3, q_3q_1, q_1q_2, q_0q_1, q_0q_2, q_0q_3, then q_1' is the polar circle of $q_2q_3q_0$, and so on, q_0' that of $q_1q_2q_3$. Hence the four polar circles are orthogonal to the same circle. (Cox.)

21. *If a, b, c, d be four points on a plane and p a point on circle abc, then the inverses of p for the circles bcd, cda, dab lie on a circle through d.* Use § 31·10.

If we take d at infinity in the inversion-plane, we have essentially the theorem on the existence of Simson's line.

22. For that case of the sixteen-circles theorem where the c and the c' are all points, call c_i and c_i' '*opposite*' points. Then the inverses of a point c_i in the four circles through the opposite point c_i' lie on a circle through c_i. If we take c_0 at infinity, we have: *the centres of the circumcircles of the four triangles, formed by four general lines in the plane, lie on a circle through the cut of the circumcircles.*

The last can also be shewn from Ex. 1, p. 318 and the analogue of Ex. 65, p. 101.

3. We can state the theorems in this section as follows:

We know by the 'Möbius identity', § 39·1, that if

$$[c_1'c_2c_3 . c_1c_2'c_3 . c_1c_2c_3' . c_1'c_2'c_3'] = 0,$$

then

$$[c_1c_2'c_3' . c_1'c_2c_3' . c_1'c_2'c_3 . c_1c_2c_3] = 0.$$

By 2, we now have also that, if

$$c_i^2 = c_i'^2 = 0, \quad (i = 1, 2, 3)$$

and

$$[c_1'c_2c_3 . c_1c_2'c_3 . c_1c_2c_3']^2 = 0,$$

then

$$[c_1c_2'c_3' . c_1'c_2c_3' . c_1'c_2'c_3]^2 = 0.$$

As a theorem on quadrics, this says: *If seven of the vertices of two Möbius tetrahedra are on a quadric, so is the eighth vertex. The eight vertices of two Möbius tetrahedra are associated points.*

§ 76. *Some algebraic lemmas.*

1. If u, v, w be any three vectors in a spread of vectors of step three, denote the matrix

$$\begin{pmatrix} [u\,|u], & [u\,|v], & [u\,|w] \\ [v\,|u], & [v\,|v], & [v\,|w] \\ [w\,|u], & [w\,|v], & [w\,|w] \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{pmatrix}$$

and let a′, b′, ... be the normalised cofactors of this matrix, that is, the cofactors divided by the determinant Δ of the matrix. Let $\alpha = \widehat{vw}$, $\beta = \widehat{wu}$, $\gamma = \widehat{uv}$, these being signed angles.

Then $[uvw]^2 = \Delta$, $u^2 = \mathrm{a}$, $v^2 = \mathrm{b}$, $w^2 = \mathrm{c}$,

$$[v\,|w] = \mathrm{f}, \quad \cos\alpha = \mathrm{f}/\sqrt{\mathrm{bc}}.$$

Let u', v', w' be defined by

$$\sqrt{\Delta}.u' = |[vw], \quad \sqrt{\Delta}.v' = |[wu], \quad \sqrt{\Delta}.w' = |[uv],$$

and let $\alpha' = \widehat{v'w'}$, $\beta' = \widehat{w'u'}$, $\gamma' = \widehat{u'v'}$.

Then $\mathrm{a}' = (\mathrm{bc} - \mathrm{f}^2)/\Delta$, $[u'\,|u'] = [vw]^2/\Delta = \mathrm{a}'$,

$$\cos\alpha' = \mathrm{f}'/\sqrt{\mathrm{b}'\mathrm{c}'}, \quad [u'v'w'] = \Delta^{-1}.$$

Hence, by § 20·1 (6), $$\frac{\sin\alpha}{\sin\alpha'} = \frac{\Delta\sqrt{\mathrm{a}'\mathrm{b}'\mathrm{c}'}}{\sqrt{\mathrm{abc}}}. \qquad (1)$$

The roots $\sqrt{\mathrm{a}}$, $\sqrt{\mathrm{b}}$, $\sqrt{\mathrm{c}}$ must be related so that

$$(\sqrt{\mathrm{b}}\sqrt{\mathrm{c}} + \mathrm{f})(\sqrt{\mathrm{b}}\sqrt{\mathrm{c}} - \mathrm{f}) = \mathrm{bc} - \mathrm{f}^2,$$

and so on. We write $\sqrt{\mathrm{bc}}$ for $\sqrt{(\mathrm{bc})}$, and so on.

2. In § 21, we took n_0, N_0 as arbitrary non-zero scalars, and defined n_1, n_2, n_3, N_1, N_2, N_3 by

$$\mathrm{n}_1\mathrm{n}_0^{-1} = \cot\tfrac{1}{2}\beta\cot\tfrac{1}{2}\gamma,$$
$$\mathrm{N}_1\mathrm{N}_0^{-1} = \cot\tfrac{1}{2}\beta'\cot\tfrac{1}{2}\gamma',$$

and so on. We then fixed the value of N_0 by the formula

$$2\mathrm{N}_0 = -\mathrm{n}_0 + \mathrm{n}_1 + \mathrm{n}_2 + \mathrm{n}_3,$$

but still left n_0 arbitrary. (§ 21 (16), (17).)

We shall now define n_0 by the first of the following formulae; the rest then follow:

$$\left.\begin{aligned}
n_0 &= \sqrt{(2\Delta^{-1})}\,\sqrt{\{(\sqrt{bc}-f)(\sqrt{ca}-g)(\sqrt{ab}-h)\}}\\
&= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma,\\
n_1 &= \sqrt{(2\Delta^{-1})}\,\sqrt{\{(\sqrt{bc}-f)(\sqrt{ca}+g)(\sqrt{ab}+h)\}}\\
&= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma,\\
n_2 &= \sqrt{(2\Delta^{-1})}\,\sqrt{\{(\sqrt{bc}+f)(\sqrt{ca}-g)(\sqrt{ab}+h)\}}\\
&= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma,\\
n_3 &= \sqrt{(2\Delta^{-1})}\,\sqrt{\{(\sqrt{bc}+f)(\sqrt{ca}+g)(\sqrt{ab}-h)\}}\\
&= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma,
\end{aligned}\right\} \qquad (2)$$

where, for example, $\sqrt{bc}$ means $\sqrt{b}.\sqrt{c}$, with signs as determined above.

These formulae give at once:

$$\left.\begin{aligned}
n_0^2 &= aa'+bb'+cc'+2f'\sqrt{bc}+2g'\sqrt{ca}+2h'\sqrt{ab}-1,\\
n_1^2 &= aa'+bb'+cc'+2f'\sqrt{bc}-2g'\sqrt{ca}-2h'\sqrt{ab}-1,\\
n_2^2 &= aa'+bb'+cc'-2f'\sqrt{bc}+2g'\sqrt{ca}-2h'\sqrt{ab}-1,\\
n_3^2 &= aa'+bb'+cc'-2f'\sqrt{bc}-2g'\sqrt{ca}+2h'\sqrt{ab}-1.
\end{aligned}\right\} \qquad (3)$$

From (2) we have

$$\begin{aligned}
n_0+n_1 &= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}(\beta-\gamma)\\
&= 4\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\cos\tfrac{1}{2}\alpha'\sin\tfrac{1}{2}(\beta'+\gamma')\sin\alpha\operatorname{cosec}\alpha',
\end{aligned}$$

by § 21 (11).

These and the formulae derived similarly, together with (1) and § 21 (16), give us the formulae dual to (2) and (3):

$$\begin{aligned}
N_0 &= \sqrt{(2\Delta)}\,\sqrt{\{(b'c'-f')(\sqrt{c'a'}-g')(\sqrt{a'b'}-h')\}}\\
&= 4\sqrt{\Delta}\,.\sqrt{a'b'c'}\,.\sin\tfrac{1}{2}\alpha'\sin\tfrac{1}{2}\beta'\sin\tfrac{1}{2}\gamma',\\
N_0^2 &= aa'+bb'+cc'+2f\sqrt{b'c'}+2g\sqrt{c'a'}+2h\sqrt{a'b'}-1,
\end{aligned}$$

and so on.

Note that

$$\begin{aligned}
N_0 &= \tfrac{1}{2}(-n_0+n_1+n_2+n_3) = 2\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\sin\sigma,\\
N_1 &= \tfrac{1}{2}(n_0-n_1+n_2+n_3) = 2\sqrt{\Delta^{-1}}\,\sqrt{abc}\,.\sin(\sigma-\alpha),
\end{aligned}$$

and so on, from (2), § 21(16), where $2\sigma = \alpha+\beta+\gamma$.

Also $n_0n_1n_2n_3 = 4\Delta\, a'b'c'$, $N_0N_1N_2N_3 = 4\Delta^{-1}abc$,

$$\frac{\sin\alpha}{\sin\alpha'} = \sqrt{\frac{n_0n_1n_2n_3}{N_0N_1N_2N_3}}.$$

3. We deduce some formulae we need later. We first quote §21 (16), (17).

$$\left.\begin{aligned} N_0 &= \tfrac{1}{2}(-n_0+n_1+n_2+n_3), & n_0 &= \tfrac{1}{2}(-N_0+N_1+N_2+N_3),\\ N_1 &= \tfrac{1}{2}(n_0-n_1+n_2+n_3), & n_1 &= \tfrac{1}{2}(N_0-N_1+N_2+N_3),\\ N_2 &= \tfrac{1}{2}(n_0+n_1-n_2+n_3), & n_2 &= \tfrac{1}{2}(N_0+N_1-N_2+N_3),\\ N_3 &= \tfrac{1}{2}(n_0+n_1+n_2-n_3), & n_3 &= \tfrac{1}{2}(N_0+N_1+N_2-N_3). \end{aligned}\right\} \qquad (3')$$

These give

$$\left.\begin{aligned} n_0^2+n_1^2+n_2^2+n_3^2 &= N_0^2+N_1^2+N_2^2+N_3^2,\\ n_0^2+n_1^2-n_2^2-n_3^2 &= 2(N_2N_3-N_0N_1),\\ n_0^2-n_1^2+n_2^2-n_3^2 &= 2(N_3N_1-N_0N_2),\\ n_0^2-n_1^2-n_2^2+n_3^2 &= 2(N_1N_2-N_0N_3),\\ N_0^2+N_1^2-N_2^2-N_3^2 &= 2(n_2n_3-n_0n_1),\\ N_0^2-N_1^2+N_2^2-N_3^2 &= 2(n_3n_1-n_0n_2),\\ N_0^2-N_1^2-N_2^2+N_3^2 &= 2(n_1n_2-n_0n_3). \end{aligned}\right\} \qquad (3'')$$

$$n_0n_1+n_2n_3 = N_0N_1+N_2N_3,$$

and so on, cycling 1, 2, 3.

From (3) and the dual formulae, we get

$$\begin{aligned} n_0^2+n_1^2+n_2^2+n_3^2 &= -4+4(aa'+bb'+cc'),\\ n_0^2+n_1^2-n_2^2-n_3^2 &= 8f'\sqrt{bc},\\ n_0^2-n_1^2+n_2^2-n_3^2 &= 8g'\sqrt{ca},\\ n_0^2-n_1^2-n_2^2+n_3^2 &= 8h'\sqrt{ab},\\ N_0^2+N_1^2+N_2^2+N_3^2 &= -4+4(aa'+bb'+cc'),\\ N_0^2+N_1^2-N_2^2-N_3^2 &= 8f\sqrt{b'c'},\\ N_0^2-N_1^2+N_2^2-N_3^2 &= 8g\sqrt{c'a'},\\ N_0^2-N_1^2-N_2^2+N_3^2 &= 8h\sqrt{a'b'}. \end{aligned}$$

From §21 (11'), (11'') we easily find formulae for $\cos\alpha$, $\sin\alpha$, and thence can write the duals:

$$\left.\begin{aligned} \cos\alpha &= (n_2n_3-n_0n_1)(n_2n_3+n_0n_1)^{-1},\\ \sin\alpha &= 2(n_0n_1n_2n_3)^{\frac{1}{2}}(n_2n_3+n_0n_1)^{-1},\\ \cos\alpha' &= (N_2N_3-N_0N_1)(N_2N_3+N_0N_1)^{-1},\\ \sin\alpha' &= 2(N_0N_1N_2N_3)^{\frac{1}{2}}(N_2N_3+N_0N_1)^{-1}, \end{aligned}\right\} \qquad (3''')$$

and so on.

We deduce

$$-n_0^2+n_1^2+n_2^2+n_3^2 + 2n_2n_3\cos\alpha'+2n_3n_1\cos\beta'+2n_1n_2\cos\gamma'=4. \quad (4)$$

For, by (3′), (3″),

$$4n_2n_3 = N_0^2+N_1^2-N_2^2-N_3^2+2(N_0N_1+N_2N_3),$$

$$-n_0^2+n_1^2+n_2^2+n_3^2 = \tfrac{1}{2}\{N_0^2+N_1^2+N_2^2+N_3^2+2(N_0N_1-N_2N_3) + 2(N_0N_2-N_3N_1)+2(N_0N_3-N_1N_2)\}.$$

By (3‴) $\quad (N_0N_1+N_2N_3)\cos\alpha' = N_2N_3-N_0N_1.$

Hence the left-hand side of (4) equals

$$\tfrac{1}{2}(N_0^2+N_1^2+N_2^2+N_3^2)+\tfrac{1}{2}(N_0^2+N_1^2-N_2^2-N_3^2)\cos\alpha' + \tfrac{1}{2}(N_0^2-N_1^2+N_2^2-N_3^2)\cos\beta'+\tfrac{1}{2}(N_0^2-N_1^2-N_2^2+N_3^2)\cos\gamma'$$
$$= \tfrac{1}{2}(-4+4(aa'+bb'+cc'))+4(ff'+gg'+hh')$$
$$= 2(-1+aa'+hh'+gg'+hh'+bb'+ff'+gg'+ff'+cc')=4.$$

We also have

$$N_1(n_1n_2+n_0n_3)(n_3n_1+n_0n_2)+N_2(n_1n_2-n_0n_3)(n_3n_1+n_0n_2) + N_3(n_3n_1-n_0n_2)(n_1n_2+n_0n_3) = N_0((n_1n_2-n_0n_3)(n_1n_3-n_0n_2)+4n_0n_1n_2n_3). \quad (5)$$

For this formula is equivalent to

$$n_1^2n_2n_3(-N_0+N_1+N_2+N_3)-n_0^2n_2n_3(N_0-N_1+N_2+N_3) + n_0n_1n_3^2(N_0+N_1-N_2+N_3) + n_0n_1n_2^2(N_0+N_1+N_2-N_3) = 4n_0n_1n_2n_3N_0,$$

and the left-hand side of this equals (using 3′)

$$2n_0n_1n_2n_3(-n_0+n_1+n_2+n_3)=4n_0n_1n_2n_3N_0.$$

§77. *Theory of three circles.*

1. If c_1, c_2, c_3 be the circles, we shall denote $[c_i\,|\,c_j]$ by the letter in the ij place in the matrix

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}.$$

Thus $c_1^2 = r_1^2 = a$, $[c_1\,|\,c_2] = h$, and so on. We denote the normalised cofactors of the elements of the determinant of this matrix by a′, h′, ... and so on, the determinant itself by Δ.

Thus $[c_1c_2c_3]^2 = \Delta, \quad \Delta a' = bc - f^2 = [c_2c_3]^2.$

We take the signs of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ so that these quantities are r_1, r_2, r_3, and we define the angles between the circles, when these meet, by

$$\cos\alpha = f(bc)^{-\frac{1}{2}}, \quad \cos\beta = g(ca)^{-\frac{1}{2}}, \quad \cos\gamma = h(ab)^{-\frac{1}{2}}.$$

2. If c_1, c_2, c_3 be independent circles, s the circle orthogonal to them, first suppose s is a proper circle and assume

$$s = k_1c_1 + k_2c_2 + k_3c_3 + k\theta, \quad s^2 = r^2. \tag{1}$$

This assumption is permissible, since circles form a spread of step four, and c_1, c_2, c_3, θ are independent. For if

$$l_1c_1 + l_2c_2 + l_3c_3 = \theta,$$

then, substituting for θ in (1), and then multiplying scalarwise by s, we find $s^2 = 0$. Omit this case.

Multiply (1) scalarwise by c_1, c_2, c_3, s, θ respectively, and we have

$$-k = ak_1 + hk_2 + gk_3 = hk_1 + bk_2 + fk_3 = gk_1 + fk_2 + ck_3,$$
$$k = r^2, \quad 1 = k_1 + k_2 + k_3.$$

Hence

$$\left.\begin{aligned} -k(a' + h' + g') &= k_1, \\ -k(h' + b' + f') &= k_2, \\ -k(g' + f' + c') &= k_3, \\ -r^2(a' + b' + c' + 2f' + 2g' + 2h') &= 1, \\ -r^2 = ak_1^2 + bk_2^2 + ck_3^2 + 2fk_2k_3 + 2gk_3k_1 + 2hk_1k_2. & \end{aligned}\right\} \tag{2}$$

Hence

$$s = -r^2((a' + h' + g')c_1 + (h' + b' + f')c_2 + (g' + f' + c')c_3) + r^2\theta,$$

where r^2 is given by (2).

If the orthogonal circle s is a line, then

$$[\theta|s] \equiv k_1 + k_2 + k_3 = 0, \quad k_1(o_1' - o_3') + k_2(o_2' - o_3') = 0.$$

Thus $k_1(c_1 - c_3)$, $k_2(c_2 - c_3)$ are equal parallel rotors in opposite senses; their sum is a multiple of θ, and $s^2 = 0$, the case omitted.

We assume that $a' + h' + g'$, $h' + b' + f'$, $g' + f' + c'$ are all distinct from zero.

3. Introduce circles d_1, d_2, d_3 defined by the equations:

$$(a' + h' + g')d_1 = a'c_1 + h'c_2 + g'c_3,$$
$$(h' + b' + f')d_2 = h'c_1 + b'c_2 + f'c_3,$$
$$(g' + f' + c')d_3 = g'c_1 + f'c_2 + c'c_3.$$

Then $(a'+h'+g')[d_1|c_1] = a'a+h'h+g'g = 1,$

$(a'+h'+g')[d_1|c_2] = a'h+h'b+g'f = 0.$

Hence $[d_1|c_1] = -r^2k_1^{-1}, \quad [d_1|c_2] = 0, \quad [d_1|c_3] = 0.$

These and the similar equations shew that the circles d_1, d_2, d_3 in the bundle fixed by c_1, c_2, c_3 are each orthogonal to two of c_1, c_2, c_3, as well as to s, the orthogonal circle of c_1, c_2, c_3.

4. We have

$$(a'+h'+g')[d_1|d_1] = a'[c_1|d_1] = -a'r^2k_1^{-1},$$

$$(h'+b'+f')[d_1|d_2] = h'[c_1|d_1] = -h'r^2k_1^{-1},$$

so that $[d_1|d_1] = a'r^4k_1^{-2}, \quad [d_1|d_2] = h'r^4k_1^{-1}k_2^{-1}.$

5. Since $d_2 \equiv |[sc_3c_1]$, $d_3 \equiv |[sc_1c_2]$, the angle α' between d_2, d_3 is given by

$$\cos\alpha' = \frac{[sc_3c_1|sc_1c_2]}{\sqrt{[sc_3c_1]^2}\sqrt{[sc_1c_2]^2}} = \frac{\cos\beta\cos\gamma-\cos\alpha}{\sin\beta\sin\gamma}.$$

6. *Problem of Apollonius. Gergonne's construction.*

If the circle t touches the distinct real circles c_1, c_2, c_3, then

$$[tc_1]^2 = 0, \quad [t|c_1]^2 = t^2c_1^2.$$

For internal touching $[t|c_1] = \sqrt{t^2}\sqrt{c_1^2}$; for external touching $[t|c_1] = -\sqrt{t^2}\sqrt{c_1^2}$.

Hence $$\frac{[t|c_1]}{r_1} = \pm\frac{[t|c_2]}{r_2} = \pm\frac{[t|c_3]}{r_3}.$$

Now $r_2^{-1}c_2 - r_3^{-1}c_3$ is a circle whose centre is the external centre of similitude of c_2, c_3. If l is a rotor such that

$$[l|(r_2^{-1}c_2 - r_3^{-1}c_3)] = 0 \quad \text{and} \quad [l|(r_3^{-1}c_3 - r_1^{-1}c_1)] = 0,$$

then $$[l|(r_1^{-1}c_1 - r_2^{-1}c_2)] = 0.$$

Hence, by §72·2, the external centres of similitude of pairs of circles c_1, c_2, c_3 lie on the same line l, say, the axis of similitude.

For example, let t be the circle touching c_1, c_2, c_3 internally; let s be the circle orthogonal to c_1, c_2, c_3. Then

$$[t|(r_2^{-1}c_2 - r_3^{-1}c_3)] = 0, \quad [s|(r_2^{-1}c_2 - r_3^{-1}c_3)] = 0.$$

Hence t, s, l are dependent, since $c_2 \neq c_3$.

Let $t = s + \mathrm{k}l$, then $t - c_1 = s - c_1 + \mathrm{k}l$. But $t - c_1$ is the tangent at the point of contact of t and c_1, since it is their radical axis (§71·6). Hence this tangent goes through the cut of l and the radical axis $s - c_1$ of s and c_1.

Thus we have the construction: draw the common orthogonal circle s of c_1, c_2, c_3. Let the radical axis of s and c_1 cut the external axis of similitude in p, then the tangents from p to c_1 touch c_1 at points of contact of the tangent circle required.

7. *The tangent circles.* Let t_0 be the circle touching c_1, c_2, c_3 all internally, and let ρ_0 be its radius; let t_0' be the circle touching all externally and let ρ_0' be its radius. Let t_1 be the circle touching c_1 internally and c_2, c_3 externally and let ρ_1 be its radius. Let t_1' be the circle touching c_1 externally and c_2, c_3 internally and let ρ_1' be its radius. Similarly for t_2, t_2', t_3, t_3'.

Let s be, as before, the circle, of radius r, orthogonal to c_1, c_2, c_3.

Let t be any one of the tangent circles, and ρ its radius, and let

$$t = \mathrm{k}_1 c_1 + \mathrm{k}_2 c_2 + \mathrm{k}_3 c_3 + \mathrm{k}s. \tag{1}$$

This assumption is permissible if c_1, c_2, c_3, s be independent. If they are not, we shall have $\mathrm{l}_1 c_1 + \mathrm{l}_2 c_2 + \mathrm{l}_3 c_3 = s$ for some l not all zero. Inner multiplication by s gives $s^2 = 0$. Then if $\mathrm{l}_1 \neq 0$, we have

$$\mathrm{l}_1[c_1 c_2 c_3] = [s c_2 c_3], \quad \mathrm{l}_1^2[c_1 c_2 c_3]^2 = [s c_2 c_3]^2 = 0, \quad [c_1 c_2 c_3]^2 = 0.$$

Hence (§73·10) either c_1, c_2, c_3 are dependent or have a common point. We exclude these cases. By (1), $t^2 = \rho^2$, and

$$[t\,|\,c_1] = \epsilon_1 \rho \mathrm{r}_1, \quad [t\,|\,c_2] = \epsilon_2 \rho \mathrm{r}_2, \quad [t\,|\,c_3] = \epsilon_3 \rho \mathrm{r}_3,$$

where ϵ_i are ± 1, according to the nature of the tangency.

Inner multiplication of (1) by θ, s, c_1, c_2, c_3, t gives, since $\mathrm{r}_1^2 = \mathrm{a}$, and so on,

$$1 = \mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3 + \mathrm{k}, \quad [t\,|\,s] = \mathrm{k}\mathrm{r}^2. \tag{2}$$

$$\left.\begin{aligned} \epsilon_1 \rho \mathrm{r}_1 &= \mathrm{a}\mathrm{k}_1 + \mathrm{h}\mathrm{k}_2 + \mathrm{g}\mathrm{k}_3, \\ \epsilon_2 \rho \mathrm{r}_2 &= \mathrm{h}\mathrm{k}_1 + \mathrm{b}\mathrm{k}_2 + \mathrm{f}\mathrm{k}_3, \\ \epsilon_3 \rho \mathrm{r}_3 &= \mathrm{g}\mathrm{k}_1 + \mathrm{f}\mathrm{k}_2 + \mathrm{c}\mathrm{k}_3. \end{aligned}\right\} \tag{3}$$

$$\rho^2 = \rho(\epsilon_1 \mathrm{k}_1 \mathrm{r}_1 + \epsilon_2 \mathrm{k}_2 \mathrm{r}_2 + \epsilon_3 \mathrm{k}_3 \mathrm{r}_3) + \mathrm{k}^2 \mathrm{r}^2. \tag{4}$$

Hence

$$\left.\begin{aligned} \mathrm{k}_1 &= \rho(\epsilon_1 \mathrm{a}' \mathrm{r}_1 + \epsilon_2 \mathrm{h}' \mathrm{r}_2 + \epsilon_3 \mathrm{g}' \mathrm{r}_3), \\ \mathrm{k}_2 &= \rho(\epsilon_1 \mathrm{h}' \mathrm{r}_1 + \epsilon_2 \mathrm{b}' \mathrm{r}_2 + \epsilon_3 \mathrm{f}' \mathrm{r}_3), \\ \mathrm{k}_3 &= \rho(\epsilon_1 \mathrm{g}' \mathrm{r}_1 + \epsilon_2 \mathrm{f}' \mathrm{r}_2 + \epsilon_3 \mathrm{c}' \mathrm{r}_3). \end{aligned}\right\} \tag{5}$$

$$-k^2r^2 = \rho^2(aa'+bb'+cc' + 2\epsilon_2\epsilon_3 f'\sqrt{bc} + 2\epsilon_3\epsilon_1 g'\sqrt{ca} + 2\epsilon_1\epsilon_2 h'\sqrt{ab} - 1) = \rho^2 n_i^2, \quad (6)$$

say, since $a = r_1^2, \quad b = r_2^2, \quad c = r_3^2.$

From (5)

$$t = \rho\{\epsilon_1 r_1(a'c_1 + h'c_2 + g'c_3) + \epsilon_2 r_2(h'c_1 + b'c_2 + f'c_3) + \epsilon_3 r_3(g'c_1 + f'c_2 + c'c_3)\} + ks$$
$$= \rho\{\epsilon_1 r_1(a' + h' + g')\,d_1 + \epsilon_2 r_2(h' + b' + f')\,d_2 + \epsilon_3 r_3(g' + f' + c')\,d_3\} + ks. \quad (7)$$

The expression n_i in (6) is n_0 of § 76, if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$; it is n_1 if $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$; it is n_2 if $\epsilon_2 = 1$, $\epsilon_1 = \epsilon_3 = -1$; it is n_3 if $\epsilon_3 = 1$, $\epsilon_1 = \epsilon_2 = -1$.

For these values t is respectively t_0, t_1, t_2, t_3; if we change the signs of the ϵ throughout, we obtain t_0', t_1', t_2', t_3'.

Examples. 23. We have $\rho_0^{-1}t_0 + \rho_0'^{-1}t_0' \equiv s$. Hence the centre of s is the centre of similitude of t_0 and t_0'.

24. $\rho_0^{-1} + \rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1} = \rho_0'^{-1} + \rho_1'^{-1} + \rho_2'^{-1} + \rho_3'^{-1}$.
(Multiply (7) by θ.)

25. Absorb the ϵ into $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ so that $\sqrt{a}$ now stands for $\epsilon_1\sqrt{a}$ or r_1 which may now be of either sign, then (3), (4) give

$$\rho = k_1 r_1 + k_2 r_2 + k_3 r_3 + \rho^{-1}k^2r^2,$$
$$k_2(r_2 - r_1^{-1}h) + k_3(r_3 - r_1^{-1}g) + \rho^{-1}k^2r^2 = 0.$$

From these, and the equations similar to the last, eliminate k_1, k_2, k_3 and deduce

$$-\rho^{-2}k^2r^2\Delta = 2(\sqrt{bc} - f)(\sqrt{ca} - g)(\sqrt{ab} - h) = \Delta n_i^2.$$

26. Form the determinant of the inner products of c_1, c_2, c_3, t, s, and by expanding it, deduce the last equation of Ex. 25.

27. Discuss the conditions for the reality of the tangent circles.

8. *Hart circles.* Suppose a circle h of radius 1 can touch t_0 externally and t_1', t_2', t_3' internally, and let

$$1^{-1}h = x_1c_1 + x_2c_2 + x_3c_3 + xs. \quad (8)$$

Then $[h\,|\,t_0] = -\rho_0 1, \quad [h\,|\,t_i'] = \rho_i' 1 \quad (i = 1, 2, 3),$

$$[t_1'\,|\,c_1] = -\rho_1' r_1, \quad [t_1'\,|\,c_2] = \rho_1' r_2, \quad [t_1'\,|\,c_3] = \rho_1' r_3.$$

Also $[t_1' \mid s] = \sqrt{(-1)}\rho_1' n_1 r$. For by (1), (6), since $[c_1 \mid s] = 0$, we have $[t_1' \mid s] = kr^2 = \sqrt{(-1)}\rho_1' n_1 r$.

Similarly $[t_0 \mid s] = \sqrt{(-1)}\rho_0 n_0 r$.

By inner multiplication of (8) by θ, t_0, t_1', ..., we have

$$l^{-1} = x_1 + x_2 + x_3 + x,$$

$$(\rho_0 l)^{-1}[h \mid t_0] = x_1 r_1 + x_2 r_2 + x_3 r_3 + xr\sqrt{(-1)}\, n_0,$$

$$(\rho_1' l)^{-1}[h \mid t_1'] = -x_1 r_1 + x_2 r_2 + x_3 r_3 + xr\sqrt{(-1)}\, n_1,$$

and so on, giving

$$1 = -x_1 r_1 - x_2 r_2 - x_3 r_3 - xr\sqrt{(-1)}\, n_0,$$

$$1 = -x_1 r_1 + x_2 r_2 + x_3 r_3 + xr\sqrt{(-1)}\, n_1,$$

$$1 = x_1 r_1 - x_2 r_2 + x_3 r_3 + xr\sqrt{(-1)}\, n_2,$$

$$1 = x_1 r_1 + x_2 r_2 - x_3 r_3 + xr\sqrt{(-1)}\, n_3.$$

Using the definitions of N_0, N_1, N_2, N_3 in terms of n_0, n_1, n_2, n_3 (§ 76), these give

$$xr\sqrt{(-1)}\, N_0 = 2, \qquad xr\sqrt{(-1)}\, N_1 = -2x_1 r_1,$$

$$xr\sqrt{(-1)}\, N_2 = -2x_2 r_2, \qquad xr\sqrt{(-1)}\, N_3 = -2x_3 r_3.$$

Hence, from (8),

$$-l^{-1} N_0 h = r_1^{-1} N_1 . c_1 + r_2^{-1} N_2 . c_2 + r_3^{-1} N_3 . c_3 + 2\sqrt{(-1)}\, r^{-1} s. \tag{9}$$

Now $h^2 = l^2$, $[c_1 \mid s] = 0$, $c_1^2 = r_1^2$, $[c_2 \mid c_3] = r_2 r_3 \cos\alpha$, and so on.

Hence, squaring both sides of (9), we must have, for consistency,

$$-N_0^2 + N_1^2 + N_2^2 + N_3^2 + 2N_2 N_3 \cos\alpha + 2N_3 N_1 \cos\beta + 2N_1 N_2 \cos\gamma = 4, \tag{10}$$

and if this is true, then h exists satisfying the conditions set down.

But the dual of (10) was shewn in § 76. Hence (10) is true.*

Hence there is a circle touching t_0 externally and t_1', t_2', t_3' internally. Similarly there are eight circles touching four of the circles t_0, t_1, t_2, t_3, t_0', t_1', t_2', t_3' and not orthogonal to s—Hart systems *of the first kind.*

* If the circles do not cut, so that α, β, γ are imaginary, we can replace $\cos\alpha$, ..., ... by their algebraic equivalents in § 76. This remark has general application.

9. Let θ be the angle between h and c_1: multiply (9) by $|c_1$, then

$$-N_0\cos\theta = N_1 + N_2\cos\gamma + N_3\cos\beta$$

$$= N_1 + N_2\frac{n_1n_2 - n_0n_3}{n_1n_2 + n_0n_3} + N_3\frac{n_3n_1 - n_0n_2}{n_3n_1 + n_0n_2}$$

$$= N_0\frac{(n_1n_2 - n_0n_3)(n_1n_3 - n_0n_2) + 4n_0n_1n_2n_3}{(n_1n_2 + n_0n_3)(n_3n_1 + n_0n_2)},$$

by §76·3(5).

Hence $$-\cos\theta = \cos\beta\cos\gamma + \sin\beta\sin\gamma,$$

$\pi - \theta = \beta - \gamma$, and similar equations.

10. *Hart systems of second kind.* The circle

$$l'^{-1}h' = y_1c_1 + y_2c_2 + y_3c_3,$$

of radius l', is orthogonal to s. Suppose it touches t_0 externally and t_1' internally, then

$$\left.\begin{aligned} 1 = -y_1r_1 - y_2r_2 - y_3r_3, \quad 1 = -y_1r_1 + y_2r_2 + y_3r_3, \\ (r_1^2 = a,\ r_2^2 = b,\ r_3^2 = c). \end{aligned}\right\} \quad (11)$$

Hence $$-1 = y_1r_1, \quad 0 = y_2r_2 + y_3r_3,$$

$$1 = y_1^2a + y_2^2b + y_3^2c + 2y_2y_3f + 2y_3y_1g + 2y_1y_2h$$
$$= 1 + 2y_2y_3(f - r_2r_3) - 2r_1^{-1}(y_3g + y_2h),$$

$$f - r_2r_3 = gr_1^{-1}y_2^{-1} + hr_1^{-1}y_3^{-1} = (gr_2 - hr_3)r_1^{-1}r_2^{-1}y_2^{-1}$$
$$= -(gr_2 - hr_3)r_1^{-1}r_3^{-1}y_3^{-1}.$$

Hence $$l'^{-1}h' = -r_1^{-1}c_1 + \frac{gr_2 - hr_3}{f - r_2r_3}(r_1^{-1}r_2^{-1}c_2 - r_1^{-1}r_3^{-1}c_3).$$

This circle orthogonal to s and touching t_0, t_1' will also touch t_0', t_1 internally and externally respectively; for the condition for this is also (11). There are six circles of this kind.

Examples. 28. If we seek the circle which touches t_1' externally and t_0, t_2', t_3' internally, we get the equations

$$-1 = -x_1r_1 - x_2r_2 - x_3r_3 - \sqrt{(-1)}\,xrn_0,$$
$$-1 = -x_1r_1 + x_2r_2 + x_3r_3 + \sqrt{(-1)}\,xrn_1,$$
$$-1 = -x_1r_1 + x_2r_2 - x_3r_3 - \sqrt{(-1)}\,xrn_2,$$
$$-1 = -x_1r_1 - x_2r_2 + x_3r_3 - \sqrt{(-1)}\,xrn_3.$$

Adding the first two, and the second two, we have

$$x(n_1 - n_0) = x(-n_2 - n_3), \quad xN_0 = 0.$$

Hence in the general case, when $N_0 \neq 0$, we have

$$x = 0, \quad x_1 r_1 = 1, \quad x_2 = x_3 = 0.$$

Hence the circle sought is c_1, as it should be.

29. Investigate the meaning of $N_0 = 0$.

11. *Casey's criterion.* If t is any circle touching four independent circles $c_1, \ldots, c_4$, then $[t\,|c_i]^2 = t^2 c_i^2$.

Let $\quad k_{ij} = [c_i\,|c_j], \ (i, j = 1, \ldots, 4), \quad t^2 = r^2,$

then, absorbing the sign in r,

$$[t\,|c_i] = r\sqrt{k_{ii}}.$$

If $t = k_1 c_1 + k_2 c_2 + k_3 c_3 + k_4 c_4$, then inner multiplication by $t, c_1, \ldots, c_4$ gives

$$\left.\begin{aligned} r &= k_1\sqrt{k_{11}} + k_2\sqrt{k_{22}} + k_3\sqrt{k_{33}} + k_4\sqrt{k_{44}}, \\ r\sqrt{k_{11}} &= k_1 k_{11} + k_2 k_{12} + k_3 k_{13} + k_4 k_{14}, \end{aligned}\right\} \qquad (12)$$

and so on.

Eliminating the k_i, we get a determinant which must vanish, composed of the determinant of the k_{ij}, bordered in the first row and column by the elements $1, \sqrt{k_{11}}, \sqrt{k_{22}}, \sqrt{k_{33}}, \sqrt{k_{44}}$.

If we write $l_{ij} = k_{ij} - (k_{ii} k_{jj})^{\frac{1}{2}}$, it will be found that the determinant factors into

$$\{-(l_{12} l_{34})^{\frac{1}{2}} - (l_{23} l_{14})^{\frac{1}{2}} - (l_{31} l_{24})^{\frac{1}{2}}\}\{- + +\}\{+ - +\}\{+ + -\},$$

the terms in each factor being the same, with the signs as indicated.

Hence, if t touches $c_1, \ldots, c_4$, one of these factors vanishes.

In particular, if c_4 is a point-circle, then

$$(l_{12} l_{34})^{\frac{1}{2}} \pm (l_{23} l_{14})^{\frac{1}{2}} \pm (l_{31} l_{24})^{\frac{1}{2}} = 0$$

is satisfied by points on a circle touching c_1, c_2, c_3, and l_{14}, l_{24}, l_{34} are powers of the point in c_1, c_2, c_3.

We could also shew Casey's criterion, by deducing from (12), equations such as

$$(k_{12} - (k_{11} k_{22})^{\frac{1}{2}})\, k_2 + (k_{13} - (k_{11} k_{33})^{\frac{1}{2}})\, k_3 + (k_{14} - (k_{11} k_{44})^{\frac{1}{2}})\, k_4 = 0.$$

These give a four-rowed determinant which factors at once.

The resulting condition can be written

$$[c_1 c_2 c_3 c_4]^2 = (\sqrt{c_1^2}[c_2 c_3 c_4] - \sqrt{c_2^2}[c_1 c_3 c_4] + \sqrt{c_3^2}[c_1 c_2 c_4] - \sqrt{c_4^2}[c_1 c_2 c_3])^2.$$

12. *Circumcircles of arcual triangles.* Consider the circumcircles of the arcual triangles formed by the circles c_1, c_2, c_3, assuming that each two of these circles intersect in two distinct points, the six points so obtained being distinct.

Let $w = k_1 c_1 + k_2 c_2 + k_3 c_3 + ks$ be such a circumcircle, then since w, c_2, c_3 meet in a point, we have $[wc_2c_3]^2 = 0$ and similar equations (§ 73·10).

Now $$[wc_2c_3] = k_1[c_1c_2c_3] + k[sc_2c_3],$$

$$[c_1c_2c_3\,|\,sc_2c_3] = 0, \quad \text{since} \quad [s\,|\,c_1] = [s\,|\,c_2] = [s\,|\,c_3] = 0.$$

Hence $$0 = k_1^2[c_1c_2c_3]^2 + k^2[sc_2c_3]^2 = k_1^2 \Delta + k^2 \Delta a' r^2,$$

where $s^2 = r^2$.

Hence $k_1^2 + k^2 a' r^2 = 0$. From this and similar equations, we deduce $$k_1^2 a'^{-1} = k_2^2 b'^{-1} = k_3^2 c'^{-1} = -k^2 r^2.$$

Let $w^2 = w^2$, then we can put, for some k',

$$\frac{k'w}{w} = \sqrt{a'}\, c_1 + \sqrt{b'}\, c_2 + \sqrt{c'}\, c_3 + \sqrt{(-1)}\, r^{-1} s.$$

Thence

$$k'^2 = aa' + bb' + cc' + 2f\sqrt{b'c'} + 2g\sqrt{c'a'} + 2h\sqrt{a'b'} - 1 = N_0^2.$$

We may hence take $k' = N_0$, and obtain the eight circumcircles by choice of signs of $\sqrt{a'}$, $\sqrt{b'}$, $\sqrt{c'}$. We get nothing more geometrically, if we reverse the sign of $\sqrt{-1}$. Thus:

$$N_0 w_0^{-1} w_0 = \sqrt{a'}\, c_1 + \sqrt{b'}\, c_2 + \sqrt{c'}\, c_3 + \sqrt{(-1)}\, r^{-1} s,$$

$$N_1 w_1^{-1} w_1 = \sqrt{a'}\, c_1 - \sqrt{b'}\, c_2 - \sqrt{c'}\, c_3 + \sqrt{(-1)}\, r^{-1} s,$$

and so on; w_2, w_3 correspond to signs $-+-$ and $--+$ for the first three terms on the right-hand side; while w_0', w_1', w_2', w_3' correspond to signs $---$, $+++$, $+-+$, $++-$.

The radii are connected by the equation

$$w_0^{-1} + w_1^{-1} + w_2^{-1} + w_3^{-1} = w_0'^{-1} + w_1'^{-1} + w_2'^{-1} + w_3'^{-1}.$$

With the circles d_i as in 3, we have, for $w = w_0$,

$$[w\,|\,d_1] = wN_0^{-1}\sqrt{a'}.(a' + h' + g')^{-1},$$

$$[w\,|\,d_2] = wN_0^{-1}\sqrt{b'}.(h' + b' + f')^{-1},$$

$$[w\,|\,d_3] = wN_0^{-1}\sqrt{c'}.(g' + f' + c')^{-1},$$

$$[w\,|\,s] = wN_0^{-1}\sqrt{(-1)}\, r.$$

13. *Larmor circles.** We shew there is a circle l of radius l, say, touching w_0 externally and w_1, w_2, w_3 internally.

Suppose $\quad 1^{-1}l = m_1 d_1 + m_2 d_2 + m_3 d_3 + ms$

satisfies these conditions, then

$$-w_0 = 1^{-1}[l\,|w_0] = m_1[d_1\,|w_0] + m_2[d_2\,|w_0] + m_3[d_3\,|w_0] + m[s\,|w_0].$$

Hence

$$-N_0 = m_1\sqrt{a'}.(a'+h'+g')^{-1} + m_2\sqrt{b'}.(h'+b'+f')^{-1} + m_3\sqrt{c'}.(g'+f'+c')^{-1} + m\sqrt{(-1)}\,r.$$

Similarly

$$N_1 = m_1\sqrt{a'}.(a'+h'+g')^{-1} - m_2\sqrt{b'}.(h'+b'+f')^{-1} - m_3\sqrt{c'}.(g'+f'+c')^{-1} + m\sqrt{(-1)}\,r,$$

$$N_2 = -m_1\sqrt{a'}.(a'+h'+g')^{-1} + m_2\sqrt{b'}.(h'+b'+f')^{-1} - m_3\sqrt{c'}.(g'+f'+c')^{-1} + m\sqrt{(-1)}\,r,$$

$$N_3 = -m_1\sqrt{a'}.(a'+h'+g')^{-1} - m_2\sqrt{b'}.(h'+b'+f')^{-1} + m_3\sqrt{c'}.(g'+f'+c')^{-1} + m\sqrt{(-1)}\,r.$$

Hence $n_0 = 2m\sqrt{(-1)}\,r$, $n_1 = -2m_1\sqrt{a'}.(a'+h'+g')^{-1}$, ...

$$1^{-1}l = -\tfrac{1}{2}n_1 a'^{-\frac{1}{2}}(a'+h'+g')\,d_1 - \tfrac{1}{2}n_2 b'^{-\frac{1}{2}}(h'+b'+f')\,d_2 - \tfrac{1}{2}n_3 c'^{-\frac{1}{2}}(g'+f'+c')\,d_3 - \tfrac{1}{2}\sqrt{(-1)}\,r^{-1}n_0 s.$$

The circle l will exist if the inner square of one side of this equals that of the other, that is, if

$$1 = (m_1 d_1 + m_2 d_2 + m_3 d_3 + ms)^2.$$

Now, by 4, $\quad d_1^2 = a'(a'+h'+g')^{-2} = n_1^2/4m_1^2,$

$$s^2 = r^2, \quad [d_i\,|s] = 0 \quad (i = 1, 2, 3),$$

$$[d_1\,|d_2] = h'(a'+h'+g')^{-1}(h'+b'+f')^{-1} = h'n_1 n_2/4m_1 m_2\sqrt{a'b'}.$$

Hence the condition is

$$1 = \tfrac{1}{4}(n_1^2 + n_2^2 + n_3^2 - n_0^2 + 2f'(b'c')^{-\frac{1}{2}} n_2 n_3 + 2g'(c'a')^{-\frac{1}{2}} n_3 n_1 + 2h'(a'b')^{-\frac{1}{2}} n_1 n_2)$$

which is true by § 76 (4).

There are in all eight circles which touch w_0, w_1, w_2, w_3 and are not orthogonal to s. These correspond to Hart systems of the first kind. There are also systems corresponding to the six Hart circles of the second kind. We leave these to the reader.

* A. Larmor, *Proc. Lond. Math. Soc.* Ser. 1, **23**, (1892), p. 149.

14. *Application to quadrics.* Theorems such as those on the Hart and Larmor circles in the plane can be changed, by inversion, to theorems on circles on a sphere, and then by projection, to theorems on plane sections of a quadric.

In our method, this is secured at once by replacing circles by plane sections of a quadric, supplements being taken with respect to the quadric. Orthogonal circles are then replaced by conjugate plane sections, point-circles by tangent planes.

The theorem on Larmor circles becomes: *If three plane sections of a quadric meet in the pairs of points* p_1, p_1'; p_2, p_2'; p_3, p_3', *then the sections of the quadric by the planes* $p_1'p_2'p_3'$, $p_1'p_2p_3$, $p_1p_2'p_3$, $p_1p_2p_3'$ *touch a plane section; so do the sections by the planes* $p_1p_2p_3$, $p_1p_2'p_3'$, $p_1'p_2p_3'$, $p_1'p_2'p_3$.

We may of course put the hypothesis in the form that p_1p_1', p_2p_2', p_3p_3' are concurrent. The theorem is also true when the quadric is a cone. (We leave the consideration of supplements with respect to a degenerate quadric, such as a cone, as an exercise for the reader.) We could also adopt the dual interpretation, treating c as a point, and $|c$ as a plane section of a quadric.

Examples. 30. *A circle* w *makes equal 'angles' with* d_1, d_2, d_3.

For $\quad \mathrm{N}_0\mathrm{w}^{-1}[w\,|d_1] = \sqrt{\mathrm{a}'}\,[c_1\,|d_1] = \sqrt{\mathrm{a}'}\,(\mathrm{a}'+\mathrm{h}'+\mathrm{g}')^{-1} = \sqrt{d_1^2},$

$$\mathrm{N}_0\mathrm{w}^{-1}[w\,|s] = \sqrt{(-1)}\,\mathrm{r}.$$

Hence
$$\frac{[w\,|d_1]}{\sqrt{d_1^2}} = \frac{[w\,|s]}{\sqrt{s^2}\sqrt{-1}},$$

and so
$$\frac{[w\,|d_1]}{\sqrt{d_1^2}} = \frac{[w\,|d_2]}{\sqrt{d_2^2}} = \frac{[w\,|d_3]}{\sqrt{d_3^2}}.$$

Or we can shew the theorem directly as follows: Let c_1, c_2, c_3 cut in p_1', p_2', p_3' (c_2, c_3 cut in p_1' and so on). Then since s, d_1, p_1 are orthogonal to c_2, c_3, they are dependent.

Let $s = \mathrm{x}p_1 + \mathrm{y}d_1$. Multiply this in turn innerwise by w, p_1, s, then if w goes through p_1', p_2', p_3' we have

$$[w\,|s] = \mathrm{y}[w\,|d_1], \quad [s\,|p_1] = \mathrm{y}[d_1\,|p_1], \quad s^2 = \mathrm{x}[p_1\,|s],$$

(since $[s\,|d_1] = 0$).

Hence
$$s^2 = \mathrm{xy}[d_1\,|p_1].$$

But
$$s^2 = (\mathrm{x}p_1 + \mathrm{y}d_1)^2 = 2\mathrm{xy}[p_1\,|d_1] + \mathrm{y}^2 d_1^2.$$

Hence

$$s^2 = -y^2 d_1^2, \quad [w\,|\,d_1]^2 s^2 = -y^2 [w\,|\,d_1]^2 d_1^2 = -[w\,|\,s]^2 d_1^2,$$

as before.

31. If $t_0 = -c_1 - c_2 - c_3 + y_0 c, \quad t_1 = -c_1 + c_2 + c_3 + y_1 c,$

$t_2 = c_1 - c_2 + c_3 + y_2 c, \quad t_3 = c_1 + c_2 - c_3 + y_3 c,$

then $t_1 t_2 t_3 - t_0 t_2 t_3 + t_0 t_1 t_3 - t_0 t_1 t_2$

$$= 16 c_1 c_2 c_3 + 4 c_2 c_3 c(-y_0 - y_1 + y_2 + y_3) + 4 c_3 c_1 c(-y_0 + y_1 - y_2 + y_3) + 4 c_1 c_2 c(-y_0 + y_1 + y_2 - y_3).$$

§78. *Circles in normal form. Plücker's construction.*

If c is a proper circle of radius r, then $k = cr^{-1}$ is said to be in '*normal form*'. Then $k^2 = 1$. As we still allow pure imaginary radii, we must allow our circles to be multiplied by pure imaginary weight factors.

The circles $k_1 + k_2$, $k_1 - k_2$ are the '*inner*' and '*outer power circles*' of k_1, k_2. Since

$$[k_1\,|\,(k_1 - k_2)] = -[k_2\,|\,(k_1 - k_2)], \quad [k_1\,|\,(k_1 + k_2)] = [k_2\,|\,(k_1 + k_2)],$$

the power circles bisect the angles between k_1 and k_2.

Since $[(k_1 + k_2)\,|\,(k_1 - k_2)] = 0$, the power circles cut orthogonally.

If k_1, k_2 touch externally, then $[k_1\,|\,k_2] = -1$, and $k_1 + k_2$ is *the point-circle of contact*. For

$$(k_1 + k_2)^2 = k_1^2 + k_2^2 + 2[k_1\,|\,k_2] = 0,$$

$$[k_1\,|\,(k_1 + k_2)] = [k_2\,|\,(k_1 + k_2)] = 0.$$

If k_1, k_2, k_3 be any circles, in normal form, then the outer power circles of pairs of them are coaxal, for

$$(k_2 - k_3) + (k_3 - k_1) + (k_1 - k_2) = 0.$$

Let t be any circle touching k_1, k_2, k_3 with like contact; all circles of the coaxal system containing the outer power circles cut t orthogonally, for $[t\,|\,(k_2 - k_3)] = 0$, and so on.

In the coaxal system mentioned is a circle a_1, say, such that $[k_1\,|\,a_1] = 0$; then $[t\,|\,a_1] = 0$ and hence $[(k_1 \pm t)\,|\,a_1] = 0$. Hence $k_1 \pm t$, the point of contact of k_1 with t, is on a_1, the sign being + or − according as the contact is internal or external. Thus by

drawing a_1, the circle in the coaxal system of outer power circles which cuts k_1 orthogonally, we find the points of contact on k_1 of circles touching k_1, k_2, k_3. Similarly for the inner power circles.

The changes to be made in the earlier formulae, when the circles are taken in normal form, are obvious. For example if circles k_1, k_2 with real radii cut, then $\cos(k_1, k_2) = [k_1 | k_2]$.

Examples. 32. Four circles k_1, k_2, k_3, k_4, taken in cyclic order, are such that each touches its two neighbours externally. Then their points of contact are concyclic. (Cf. § 74 and § 30·13.)

For, taking the circles in normal form,

$$[k_1 | k_2] = [k_2 | k_3] = [k_3 | k_4] = [k_4 | k_1] = -1.$$

The points of contact are

$$k_1 + k_2, \quad k_2 + k_3, \quad k_3 + k_4, \quad k_4 + k_1.$$

These are concyclic, since

$$(k_1 + k_2) - (k_2 + k_3) + (k_3 + k_4) - (k_4 + k_1) = 0. \quad (\S 73{\cdot}7.)$$

33. If k_1, k_2, k_3, k_4 be circles (in normal form) of radii r_1, r_2, r_3, r_4, and each touches each externally, then $\Sigma r_i^{-2} = 2\Sigma r_i^{-1} r_j^{-1}$.

For we can find scalars $x_1, \ldots, x_4$ such that $x_1 k_1 + \ldots + x_4 k_4 = \theta$. Multiply innerwise by k_1, k_2, k_3, k_4, θ, and we find

(i) $x_1 - x_2 - x_3 - x_4 = [\theta | k_1] = r_1^{-1}$ and similar equations, and

(ii) $r_1^{-1} x_1 + \ldots + r_4^{-1} x_4 = 0$.

Hence
$$r_1^{-2} + r_2^{-2} + r_3^{-2} + r_4^{-2} = 4(x_1^2 + x_2^2 + x_3^2 + x_4^2),$$
$$\sum_{i,j} r_i^{-1} r_j^{-1} = 4 \Sigma x_i x_j.$$

From (i), (ii) we deduce $\Sigma x_i^2 = 2 \Sigma x_i x_j$. Hence the theorem.

34. If k, k' be two circles, and k_1, k_4 each touch both in like ways, and k_2, k_3 each touch both in unlike ways, and θ_{ij} be the angle between k_i and k_j, then

$$\theta_{12} \pm \theta_{13} \pm \theta_{24} \pm \theta_{34} = 2n\pi,$$

if the signs are suitably chosen.

For
$$[k | k_1] = [k' | k_1] = \epsilon_1, \quad [k | k_2] = -[k' | k_2] = \epsilon_2,$$
$$[k | k_3] = -[k' | k_3] = \epsilon_3, \quad [k | k_4] = [k' | k_4] = \epsilon_4,$$

where $\epsilon_1, \ldots, \epsilon_4$ are ± 1.

Then $k - \epsilon_1 k_1$, $k - \epsilon_2 k_2$, $k - \epsilon_3 k_3$, $k - \epsilon_4 k_4$ are concyclic; hence, by § 30·12,

$$\sqrt{[(k - \epsilon_1 k_1) | (k - \epsilon_2 k_2)]\,[(k - \epsilon_3 k_3) | (k - \epsilon_4 k_4)]} \pm \ldots \pm \ldots = 0.$$

This gives
$$\sqrt{(\epsilon_1\epsilon_2[k_1\,|k_2]-1)(\epsilon_3\epsilon_4[k_3\,|k_4]-1)}$$
$$\pm\sqrt{(\epsilon_2\epsilon_3[k_2\,|k_3]-1)(\epsilon_1\epsilon_4[k_1\,|k_4]-1)}$$
$$\pm\sqrt{(\epsilon_3\epsilon_1[k_3\,|k_1]-1)(\epsilon_2\epsilon_4[k_2\,|k_4]-1)}=0.$$

Similarly $k'-\epsilon_1k_1$, $k'+\epsilon_2k_2$, $k'+\epsilon_3k_3$, $k'-\epsilon_4k_4$ are concyclic, and we have a formula obtained from the last by changing the signs of ϵ_2, ϵ_3. Comparing the two, we find

$$\sqrt{(\epsilon_1\epsilon_2[k_1\,|k_2]-1)(\epsilon_3\epsilon_4[k_3\,|k_4]-1)}$$
$$\pm\sqrt{(\epsilon_3\epsilon_1[k_3\,|k_1]-1)(\epsilon_2\epsilon_4[k_2\,|k_4]-1)}$$
$$=\pm\sqrt{(\epsilon_1\epsilon_2[k_1\,|k_2]+1)(\epsilon_3\epsilon_4[k_3\,|k_4]+1)}$$
$$\pm\sqrt{(\epsilon_3\epsilon_1[k_3\,|k_1]+1)(\epsilon_2\epsilon_4[k_2\,|k_4]+1)}.$$

To fix ideas, take all the ϵ to be $+1$, then we have

$$\sin\tfrac{1}{2}\theta_{12}\sin\tfrac{1}{2}\theta_{34}\pm\sin\tfrac{1}{2}\theta_{31}\sin\tfrac{1}{2}\theta_{24}$$
$$=\pm\cos\tfrac{1}{2}\theta_{12}\cos\tfrac{1}{2}\theta_{34}\pm\cos\tfrac{1}{2}\theta_{31}\cos\tfrac{1}{2}\theta_{24},$$
$$\cos\tfrac{1}{2}(\theta_{12}\pm\theta_{34})=\pm\cos\tfrac{1}{2}(\theta_{31}\pm\theta_{24}),$$

whence the result.

35. The circles which touch one e-circle of a triangle internally and two externally meet in a point.

36. For circles in normal form, the inverse of k_1 in k is
$$k_1-2[k\,|k_1]\,k.$$

§79. *Steiner's closure theorem.**

1. Let a, b be real circles, and for convenience of representation, take a inside b. Suppose all circles are in normal form, and let the circle c touch a externally, and b internally. Then
$$[a\,|c]=-1,\quad [b\,|c]=1;\quad \text{hence}\quad [(a+b)\,|c]=0.$$

Let $a-b=\mathrm{x}d$, then
$$[(a+b)\,|d]=\mathrm{x}^{-1}[(a+b)\,|(a-b)]=0.$$

Hence c, d are in the bundle $|(a+b)$. (§73·5.)

Since $d^2=1$, we have
$$\mathrm{x}^2=a^2-2[a\,|b]+b^2=2(1-[a\,|b]). \qquad \text{(i)}$$

For all c which touch a, b in the required way,
$$\mathrm{x}[c\,|d]=[a\,|c]-[b\,|c]=-2.$$

* The treatment which follows is adapted from Mehmke's dissertation.

2. Let f be the circle in the pencil $[cd]$ orthogonal to d, then f lies in the bundle $|(a+b)$. Hence

$$[f\,|d] = 0, \quad [f\,|(a+b)] = 0 = [f\,|(a-b)];$$

hence

$$[f\,|a] = [f\,|b] = 0.$$

Thus as c varies, f describes a pencil of circles.

Let

$$c = x_1 d + y_1 f,$$

then

$$1 = c^2 = x_1^2 d^2 + y_1^2 f^2 + 2x_1 y_1 [d\,|f] = x_1^2 + y_1^2.$$

Also $[c\,|d] = x_1$. Hence $xx_1 = -2$, and by (i), x_1 and therefore y_1 have constant values as c and f vary.

3. Let $c_0, c_1, c_2, \ldots$ be successive positions of c, and $f_0, f_1, f_2, \ldots$ corresponding positions of f. Let each c touch its two neighbours externally, and we have

$$c_0 = x_1 d + y_1 f_0, \quad c_1 = x_1 d + y_1 f_1, \quad c_2 = x_1 d + y_1 f_2, \quad \ldots,$$

$$[c_0\,|c_1] = [c_1\,|c_2] = \ldots = [c_i\,|c_{i+1}] = \ldots = -1.$$

Hence $-1 = x_1^2 + y_1^2 [f_i\,|f_{i+1}]$. Thus $[f_i\,|f_{i+1}]$ is independent of i. Let it equal $\cos\alpha$.

Then

$$\sin^2 \tfrac{1}{2}\alpha = \tfrac{1}{2}(1-\cos\alpha) = \frac{1}{2}\left(1 + \frac{1+x_1^2}{1-x_1^2}\right)$$

$$= \frac{1}{1-x_1^2} = \frac{x^2}{x^2-4} = -\frac{1-[a\,|b]}{1+[a\,|b]} = \frac{(r_1-r_2)^2 - d^2}{(r_1+r_2)^2 - d^2},$$

where r_1, r_2 are the radii of a, b, and d the distance between their centres.

The last fraction is less than 1, hence α is real.

4. If $c_0 = c_n$, then $f_0 = f_n$, and as the f are in a pencil, and $[f_i\,|f_{i+1}] = \cos\alpha$, we must have $n\alpha = 2m\pi$ for some integer m. Conversely, if α be given by the last equation, the sequence of the f closes, whichever circle f_0 of the pencil of the f we begin with, and hence the sequence of the c closes, whichever circle touching a externally and b internally be taken for c_0. If π/α be irrational, the sequence does not close.

5. The points of contact of the c with one another lie on $a+b$; for since $[c_i\,|(a+b)] = 0$, $[c_{i+1}\,|(a+b)] = 0$, we have

$$[(c_i + c_{i+1})\,|(a+b)] = 0.$$

But $c_i + c_{i+1}$ is the point-circle of contact of c_i with c_{i+1}.

6. If p, q be the point-circles in the pencil $[ab]$, then every circle through p, q cuts a, b orthogonally. Through p, q draw g_j so that $[g_j|c_j] = 0$, then g_j goes through the points of contact of c_j with a, b. For

$$[g_j|a] = [g_j|b] = [g_j|c_j] = 0,$$

give

$$[g_j|(a+c_j)] = 0, \quad [g_j|(b-c_j)] = 0,$$
$$[g_j|d] = 0, \qquad [g_j|f_j] = 0.$$

The last equation shews that two circles g cut at the same angle as the corresponding f.

7. *Pappus' ancient theorem.* Let a, b touch internally, and let c_0, c_1, c_2, ... touch a, b in unlike ways; let a, b, c_0 have collinear centres on the line-circle l. Let each c touch its two neighbours externally. Then $[c_n|l] = 2n$. We leave this to the reader.

CHAPTER XII

ORIENTED CIRCLES AND SYSTEMS OF CIRCLES

§80. *Sums and differences of oriented circles.**

1. Consider the circles which lie in a fixed plane, all the elements of which we speak being real, and for convenience take the plane horizontal. To each circle attach an orientation or sense of description. When this is reversed, we have an '*opposite*' oriented circle. Then each oriented circle can be represented by the point on the vertical line through its centre, at a distance from the plane equal to its radius, the point being taken above the plane if the sense of description of the circle is widdershins, otherwise below the plane. If to a point on the plane we make correspond a '*point-circle*' with that point as centre, we have a one-to-one correspondence between our oriented circles and points of space.

By a 'circle' in this chapter we mean 'oriented circle'. A rotor of course always has an orientation. A rotor '*touches*', or is a '*tangent*' to, a circle when it has just one point common with the circle, *and the sense of description of the circle and the sense of description of the rotor, at the point of contact, are the same.*

Two circles '*touch*' or '*are tangent to one another*' when they have just one common point *and the senses of description of the circles at this point are the same.*

A circle or rotor '*touches*' any point-circle whose centre lies in it.

Two circles which do not touch have either *two* common tangents or *none.*

2. We denote circles by small letters; if o be a point-circle, we denote its centre by o'; thus o and o' coincide geometrically, but their natures are different. The point-circle at the centre o_i' of the circle c_i is denoted by o_i; if r_i be the radius of c_i, we write $c_i = o_i + r_i\phi$, where ϕ is a fourth unit extensive. The opposite circle is $c_i = o_i - r_i\phi$.

* E. Müller, *Monatshefte Math. Phys.* 9 (1898), p. 269.

In the space representation of 1, ϕ may be regarded as a unit vector perpendicular to the plane of the circles.

Circles form a spread of step four.

We say the circle $c = o + r\phi$ has unit weight. If k is a scalar, not zero, then $kc = ko + kr\phi$ is the same circle with weight k. We do not absorb weights. Our weights are real.

We add circles as follows: If $c_i = o_i + r_i\phi$, then

$$\begin{aligned} k_1c_1 + k_2c_2 + \ldots + k_nc_n &= (k_1o_1 + k_2o_2 + \ldots + k_no_n) \\ &\quad + (k_1r_1 + k_2r_2 + \ldots + k_nr_n)\phi \\ &= (k_1 + k_2 + \ldots + k_n)c, \end{aligned}$$

where

$$c = o + r\phi,$$

$$(k_1 + k_2 + \ldots + k_n)o' = k_1o_1' + k_2o_2' + \ldots + k_no_n',$$

$$(k_1 + k_2 + \ldots + k_n)r = k_1r_1 + k_2r_2 + \ldots + k_nr_n.$$

For example, $\frac{1}{2}(c_1 + c_2)$ is the circle of unit weight whose centre is mid-way between o_1' and o_2' and whose radius is $\frac{1}{2}(r_1 + r_2)$.

If we invoke geometry, it is obvious that, if c_1, c_2 have a pair of common tangents, then $(k_1c_1 + k_2c_2)/(k_1 + k_2)$ touches these tangents.

3. If circles $c_1, c_2, \ldots, c_n$ be independent, in which case $n \leqq 4$, then their linear combinations form a spread of step n.

As examples of spreads of step two, we have (i) circles of equal radii with centres on the same line, (ii) circles touching the same line at the same point, (iii) concentric circles.

As examples of spreads of step three, we have (i) all circles of equal radius, (ii) all point-circles, (iii) all circles with centres on a given line.

4. The formal '*difference*' of two circles c_1, c_2 is

$$c_1 - c_2 = o_1 - o_2 + (r_1 - r_2)\phi.$$

We call this a '*circle-vector*'; in the space representation it corresponds to an ordinary vector.

If

$$c_1 - c_2 = c_3 - c_4,$$

then

$$o_1' - o_2' = o_3' - o_4', \quad r_1 - r_2 = r_3 - r_4.$$

Hence, invoking geometry, the common tangents to c_1 and c_2, if they exist, are parallel to the common tangents to c_3 and c_4.

Hence we represent $c_1 - c_2$ by the two vectors of the common tangents of c_1 and c_2, from c_2 to c_1, when they exist. In the space representation, the vector corresponding to $c_1 - c_2$ always exists.

Circle-vectors form a spread of step three.

5. If o_1 be any point-circle, then $c - o_1$ is a circle-vector w, which can be represented by the vectors of the tangents from o_1' to c, if these tangents exist.

Then $c = o_1 + w$. We call o_1', or o_1, the '*vertex*' of the circle-vector.

If o_1, o_2 be point-circles, we define $o_1 - o_2$ as the vector $o_1' - o_2'$.

§81. *Inner products of circle-vectors.*

1. Since circles are of form $o + \mathrm{k}\phi$, therefore all circle-vectors are of form $v + \mathrm{k}\phi$, where v is an ordinary vector in our plane.

If v_1, v_2 be such ordinary vectors, $[v_1 | v_2]$ is taken to have its former meaning in the plane (chap. I). We define

$$[\phi | v] = [v | \phi] = 0$$

for all vectors v, and $\phi^2 = -1$.

Then, assuming the distributive law, we have for the inner products of circle-vectors,

$$[(v_1 + \mathrm{k}_1\phi) | (v_2 + \mathrm{k}_2\phi)] = [v_1 | v_2] - \mathrm{k}_1\mathrm{k}_2.$$

This inner product vanishes, if $[v_1 | v_2] = \mathrm{k}_1\mathrm{k}_2$.

2. If the circle-vectors $v_1 + \mathrm{k}_1\phi$, $v_2 + \mathrm{k}_2\phi$ can be represented by pairs of vectors whose lines touch a circle $c = o + \mathrm{r}\phi$, and the inner product of the circle-vectors vanishes, then the vertices of the circle-vectors are conjugate for c.

For, if p_1', p_2' be these vertices, then by §80·5,

$$c = p_1 + v_1 + \mathrm{k}_1\phi = p_2 + v_2 + \mathrm{k}_2\phi.$$

Hence $v_1 + \mathrm{k}_1\phi = o - p_1 + \mathrm{r}\phi$, $v_2 + \mathrm{k}_2\phi = o - p_2 + \mathrm{r}\phi$.

But $$[(v_1 + \mathrm{k}_1\phi) | (v_2 + \mathrm{k}_2\phi)] = 0.$$

Hence $[(o - p_1) | (o - p_2)] = \mathrm{r}^2$, $[(o' - p_1') | (o' - p_2')] = \mathrm{r}^2$.

The last equation is the condition that p_1', p_2' be conjugate for c.

3. If $c_1 = o_1 + r_1\phi$, $c_2 = o_2 + r_2\phi$, then

$$(c_1 - c_2)^2 = ((o_1 - o_2) + (r_1 - r_2)\phi)^2 = (o_1' - o_2')^2 - (r_1 - r_2)^2.$$

Hence $(c_1 - c_2)^2$ represents the square of the tangential interval of c_1 and c_2, if these circles have a real common tangent. It vanishes if the circles touch; it is negative, if they do not have a real common tangent.

4. Let the circles c_1, c_2, c_3, c_4 all touch the circle c, then

$$(c - c_1)^2 = (c - c_2)^2 = (c - c_3)^2 = (c - c_4)^2 = 0.$$

But as circle-vectors form a spread of step three, $c - c_1$, $c - c_2$, $c - c_3$, $c - c_4$ are dependent. Hence, by § 30·12,

$$\begin{aligned} &\sqrt{[(c - c_1)\,|\,(c - c_2)]\,[(c - c_3)\,|\,(c - c_4)]} \\ \pm &\sqrt{[(c - c_2)\,|\,(c - c_3)]\,[(c - c_1)\,|\,(c - c_4)]} \\ \pm &\sqrt{[(c - c_3)\,|\,(c - c_1)]\,[(c - c_2)\,|\,(c - c_4)]} = 0. \end{aligned}$$

But $(c_1 - c_2)^2 = [(c - c_2) - (c - c_1)]^2 = -2[(c - c_1)\,|\,(c - c_2)]$.

Hence $\sqrt{\{(c_1 - c_2)^2 (c_3 - c_4)^2\}} \pm \sqrt{\{(c_2 - c_3)^2 (c_1 - c_4)^2\}}$
$\pm \sqrt{\{(c_3 - c_1)^2 (c_2 - c_4)^2\}} = 0.$

This gives Casey's criterion, § 77·11.

5. If the circles c_1, c_2 touch, then the circle-vector $c_1 - c_2$ is their common tangent repeated. Call this type of circle-vector a '*univector*'. If $c_1 - c_2 = w$, then c_2 and $c_2 + w$ touch, hence

$$(c_2 + w - c_2)^2 = 0, \quad w^2 = 0.$$

The square of a univector vanishes.

Univectors form a spread of step three, but the spread is not linear; that is, if w_1, w_2, w_3 be univectors, $k_1 w_1 + k_2 w_2 + k_3 w_3$ need not be a univector.

The condition that it is a univector is that its square vanishes, and then

$$k_2 k_3 [w_2\,|\,w_3] + k_3 k_1 [w_3\,|\,w_1] + k_1 k_2 [w_1\,|\,w_2] = 0.$$

As this is a quadratic in the k, *the spread of univectors is quadratic.*

§ 82. *Outer products of circles.*

1. Working for a moment in a plane with ordinary points and rotors, we define the '*power*' of a unit rotor L for a circle, centre o', radius r, as $[o'L] \div r$.

If the circle cuts L at angle θ, this power equals $\cos\theta$. If L touches the circle, the power is 1. Circles for which a rotor L has equal powers form a spread of step three.

If $r_2 o_1' - r_1 o_2' \neq 0$, then each rotor through the point $r_2 o_1' - r_1 o_2'$ has equal powers in circles c_1, c_2. This point is the '*centre of similitude*' of the circles. If $r_1 = r_2$, the point is at infinity.

The rotor, or bivector $r_1[o_2' o_3'] + r_2[o_3' o_1'] + r_3[o_1' o_2']$, if not zero, has equal powers in circles c_1, c_2, c_3. It is their '*axis of similitude*'.

2. Returning to oriented circles, the outer product of three points* in the plane must be regarded not as numeric, since our spread is now of step four, but as a multiple of a unit leaf α. The outer product of four points in the plane vanishes. Thus c touches L if $[o'L] = r\alpha$.

The outer product $[\phi\alpha]$ of ϕ and α is taken as 1.

The outer products $[o_1 o_2]$, $[o_1 o_2 o_3]$, of point-circles, are defined as the outer products $[o_1' o_2']$, $[o_1' o_2' o_3']$ of the corresponding points.

The outer product of ϕ and a point or rotor is taken formally.

For the outer product of ϕ and point-circles, we take the definitions:

$$[\phi o] = [\phi o'], \quad [o\phi] = -[\phi o],$$

$$[\phi o_1 o_2] = [\phi o_1' o_2'], \quad [o_1 o_2 \phi] = [\phi o_1 o_2].$$

$$[\phi o_1 o_2 o_3] = [\phi o_1' o_2' o_3'] = -[o_1' o_2' o_3' \phi].$$

If $c_1 = o_1 + r_1\phi$, $c_2 = o_2 + r_2\phi$, $c_3 = o_3 + r_3\phi$, we then have

$$[c_1 c_2] = [o_1 o_2] + [\phi(r_1 o_2 - r_2 o_1)] = [o_1 o_2] + [\phi p],$$

where p is at the centre of similitude of c_1 and c_2.

$$\begin{aligned}[c_1 c_2 c_3] &= [o_1 o_2 o_3] + [\phi(r_1[o_2 o_3] + r_2[o_3 o_1] + r_3[o_1 o_2])] \\ &= [o_1 o_2 o_3] + [\phi M],\end{aligned}$$

where M is along the axis of similitude of c_1, c_2, c_3.

$$\begin{aligned}\text{Similarly} \quad [c_1 c_2 c_3 c_4] = [\phi(&r_1[o_2 o_3 o_4] - r_2[o_3 o_4 o_1] \\ &+ r_3[o_4 o_1 o_2] - r_4[o_1 o_2 o_3])].\end{aligned}$$

Thus $[c_1 c_2 c_3] = 0$ if, and only if, o_1, o_2, o_3 are collinear and $M = 0$.

* It is possible to construct the theory using point-circles throughout instead of points. The reader will be able easily to modify the above in this direction, if he so desires.

We interpret these: if $[c_1 c_2 c_3] = 0$, the corresponding points in the space representation are collinear, hence c_3 is a linear combination of c_1 and c_2; an easy calculation shews that c_1, c_2, c_3 *have in pairs the same centre of similitude.*

If $[c_1 c_2 c_3 c_4] = 0$, but the outer product of no three of the circles vanishes, the circles are in a spread of step three, and a similar argument shews that *they have in threes the same axis of similitude.* If the axis be a rotor and one of the circles cuts its line, all cut it at the same angle.

So far we have only used $[\phi p]$ and $[\phi M]$ formally, and thus have not interpreted $[c_1 c_2]$, $[c_1 c_2 c_3]$ themselves. From the general theory of outer products, $[c_1 c_2]$ means the set of circles c such that $[c_1 c_2 c] = 0$, that is, the set of circles which have in pairs the same centre of similitude as c_1, c_2; similarly $[c_1 c_2 c_3]$ is the set of circles which have in threes the same axis of similitude as c_1, c_2, c_3. We call a set of circles of step three a '*net*'.

§83. *Inner squares of nets of circles. Nul nets.*

1. If v, v_1, v_2 are ordinary vectors, then

$$(v + \mathrm{k}\phi)^2 = v^2 - \mathrm{k}^2,$$

$$\begin{aligned}[(v_1 + \mathrm{r}_1\phi)(v_2 + \mathrm{r}_2\phi)] &= [v_1 v_2] + [\phi(\mathrm{r}_1 v_2 - \mathrm{r}_2 v_1)] \\ &= [v_1 v_2] + [\phi u], \text{ say.}\end{aligned}$$

$$[\phi u]^2 = \phi^2 u^2 - [\phi\,|u]^2 = -u^2, \quad (\S 81{\cdot}1),$$

$$[\phi u\,|v_1 v_2] = [\phi\,|v_1]\,[u\,|v_2] - [\phi\,|v_2]\,[u\,|v_1] = 0.$$

Hence
$$\begin{aligned}[(v_1 + \mathrm{r}_1\phi)(v_2 + \mathrm{r}_2\phi)]^2 &= [v_1 v_2]^2 - u^2 \\ &= [v_1 v_2]^2 - (\mathrm{r}_1 v_2 - \mathrm{r}_2 v_1)^2.\end{aligned}$$

The square root of this is the '*magnitude*' of

$$[(v_1 + \mathrm{r}_1\phi)(v_2 + \mathrm{r}_2\phi)].$$

Define the magnitude of $[cc_1 c_2]$ as the magnitude of

$$[(c - c_1)(c - c_2)],$$

and denote its square by $[cc_1 c_2]^2$.

Then, if $o' - o'_1 = v_1$, $o' - o'_2 = v_2$, we have

$$\begin{aligned}[cc_1 c_2]^2 &= [(v_1 + (\mathrm{r} - \mathrm{r}_1)\phi)(v_2 + (\mathrm{r} - \mathrm{r}_2)\phi)]^2 \\ &= [v_1 v_2]^2 - [(\mathrm{r} - \mathrm{r}_1)v_2 - (\mathrm{r} - \mathrm{r}_2)v_1]^2 \\ &= [v_1 v_2]^2 - [\mathrm{r}(o'_1 - o'_2) + \mathrm{r}_1(o'_2 - o') + \mathrm{r}_2(o' - o'_1)]^2 \\ &= \mathrm{x}^2 - \mathrm{k}^2,\end{aligned}$$

where $x^2 = [oo_1o_2]^2$, and k is the magnitude of

$$M = r[o_1o_2] + r_1[o_2o] + r_2[oo_1].$$

But $[cc_1c_2] = x\alpha + k[\phi R]$, where α is the unit leaf, and R the unit rotor along M, the axis of similitude of c, c_1, c_2.

Hence we have the interpretations of x, k.

2. If $[c_1c_2c_3]^2 = 0$, the net $[c_1c_2c_3]$ is a '*nul net*'.

All the circles of a nul net touch the axis of similitude of the net or the opposite rotor. For the net, apart from its weight, is of form $\alpha \pm [\phi R]$, where R is a unit rotor along the axis of similitude.

Thus
$$[(\alpha \pm \phi R)(o + r\phi)] = \pm [\phi Ro] + r[\alpha\phi]$$
$$= [\phi(\pm Ro - r\alpha)].$$

If $o + r\phi$ is in the net, this product vanishes and hence $\pm [Ro] = r\alpha$; hence $o + r\phi$ touches R or its opposite rotor.

3. If V is the unit bivector, then $x\alpha + [\phi V]$ is a net. It contains $o + r\phi$ if $[(x\alpha + \phi V)(o + r\phi)] = 0$. Now $[Vo] = \alpha$. Hence

$$[(x\alpha + \phi V)(o + r\phi)] = [\phi Vo] + xr[\alpha\phi] = (xr - 1)[\alpha\phi].$$

Hence $o + r\phi$ is in the net $r^{-1}\alpha + [\phi V]$.

The set of circles of equal radius constitutes a net whose axis of similitude is the bivector V.

As a net is a spread of step three, the above formula shews we can regard α as the net of all point-circles, instead of as a unit leaf.

4. If v is a vector, then $v + k\phi$ is a circle-vector whose outer product with $[\phi V]$ vanishes. For

$$[\phi V(v + k\phi)] = [\phi Vv] = 0, \quad \text{since} \quad [Vv] = 0.$$

Hence extending the meaning of 'net', we may say, *the totality of circle-vectors constitutes the net* $[\phi V]$.

5. Any net is now of form $\mathcal{A} = x\alpha + k[\phi R]$. Hence the totality of nets constitutes a spread of step four, and we can take four independent nets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ as its basis. In the space representation, each net corresponds to a plane, the nul nets correspond to planes inclined at 45° to the horizontal, since for these $x = \pm k$, if R is a unit vector.

If $k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + k_3\mathcal{A}_3 + k_4\mathcal{A}_4$ is a nul net, and $\mathcal{A}_i = x_i\alpha + [\phi R_i]$, then

$$0 = (k_1\mathcal{A}_1 + \ldots + k_4\mathcal{A}_4)^2$$
$$= (k_1x_1 + \ldots + k_4x_4)^2 - (k_1v_1 + \ldots + k_4v_4)^2,$$

where v_i is the vector of R_i, and we may assume $v_i^2 = 1$. Thence

$$k_1^2(1-x_1^2)+\ldots+k_4^2(1-x_4^2)+2k_1k_2([v_1|v_2]-x_1x_2)+\ldots=0.$$

The spread of nul nets is hence a quadratic spread. In the space representation, the corresponding planes envelope a (real) circle at infinity; the quadratic envelope is hence degenerate. We can verify this latter fact from the equation thus:

Take four independent nul nets, with $x_i = 1$ $(i=1,\ldots,4)$, as the basis of the spread. Then

$$\sum_{i,j=1}^{4} k_ik_j([v_i|v_j]-1)=0. \qquad (1)$$

Now the vectors v in a plane constitute a spread of step three; hence there are scalars $k_1', \ldots, k_4'$, not all zero, such that

$$\sum_1^4 k_i'v_i=0, \quad \sum_1^4 k_i'=0.$$

Hence
$$\sum_1 k_i'([v_i|v_j]-1)=0 \quad (j=1,\ldots,4), \qquad (2)$$

and hence the discriminant of the quadratic (1) is zero, for this discriminant is the determinant of the equations (2) for the k_i'.

6. Since a nul net is a set of circles touching a fixed rotor, two independent nul nets, in general, 'meet' in a set of circles touching the two rotors which define the nets. There is an exceptional case when these rotors are parallel* and hence differ by a bivector V, for then the circles must also be in the net $[\phi V]$, and hence, by 4, would be circle-vectors.

Three independent rotors, no two parallel, define three nul nets, which 'meet' in the circle touching the three rotors.

Four rotors touch a circle when the corresponding nul nets are dependent; conversely, if four nul nets, given by rotors, no two of which are parallel, are dependent, the rotors touch a circle.

If four nul nets are dependent, and the rotors giving two of them are parallel, so are the rotors which give the other two.

If c_1, c_2 be circles with common tangents, there are two nul nets which include all circles of form $c_1 + kc_2$. The rotors corresponding

* That is parallel and in the same sense. This is intended throughout.

to them are the common tangents of c_1, c_2; the two nul nets are the nets, each given by one of these two common tangents. (If c_1, c_2 do not have real common tangents, the nul nets spoken of are 'imaginary'. We do not consider this case.)

The meets of nul nets are represented by the outer products of the nets.

§84. *Theorems on circles.*

1. *If c_1, c_2, c_3, c_4 be circles, and one of the nul nets through each of the four pairs c_1c_2, c_2c_3, c_3c_4, c_4c_1 contains a common circle, so do the other four nul nets.*

This corresponds to the six-circles theorem, §74, and it can be expressed as follows:

If T_{ij}, T'_{ij} be the common tangents to c_i, c_j, and if T_{12}, T_{23}, T_{34}, T_{41} touch a circle, then T'_{12}, T'_{23}, T'_{34}, T'_{41} touch a circle, (which may be of zero radius), *or they are two pairs of parallel lines.*

To prove the theorem, denote the nets $[c_2c_3c_4]$, $[c_3c_4c_1]$, $[c_4c_1c_2]$, $[c_1c_2c_3]$ by A_1, A_2, A_3, A_4; the nul nets in question are then given by $A_1 \pm k_2A_2$, $A_2 \pm k_3A_3$, $A_3 \pm k_4A_4$, $A_4 \pm k_1A_1$, where the k are properly chosen. If four of these nul nets, one from each pair, be dependent, the other four are dependent, by §30·13, since the condition for a net to be nul is that its inner square vanishes.

2. If T_{12}, T_{23}, T_{34}, T_{41} are concurrent, we have a case of 1, and the same conclusion follows. Another case arises when c_4 is opposite to c_3:

If T_{12}, T_{13} meet on a common tangent to c_2 and the opposite of c_3, then T'_{12}, T'_{13} and the other common tangent to c_2 and the opposite of c_3 meet.

3. *If four circles be arranged in cyclic order, and each touches its two neighbours, then the tangents at the points of contact touch a circle.* (The cone case. §30·13.)

4. *If four circles c_1, c_2, c_3, c_4 be in the same net, and A, B, C, A', B', C' be common tangents to the respective pairs c_2c_3, c_3c_1, c_1c_2, c_1c_4, c_2c_4, c_3c_4, and no two of these tangents be parallel, then the four circles which touch the triads of rotors ABC, $AB'C'$, $A'BC'$, $A'B'C$ are in a net.*

For, denote by A, B, Γ, A′, B′, Γ′ the nul nets corresponding to the rotors A, B, C, A', B', C', then the circles c_1, c_2, c_3, c_4 are [A′BΓ], [AB′Γ], [ABΓ′], [A′B′Γ′], and as these are in the same net, we have

$$[A'B\Gamma . AB'\Gamma . AB\Gamma' . A'B'\Gamma'] = 0.$$

But as A, B, ... are extensives of step three in our spread of step four, we can apply the identity of Möbius' Theorem, § 39.

Thence $$[AB'\Gamma' . A'B\Gamma' . A'B'\Gamma . AB\Gamma] = 0.$$

Hence the result.

5. Since any four point-circles are in a net, we have:

If we orient in any way the joins of four points, the circles each touching three of the joins are in a net.

6. If, in 4, one of the nets is nul, so is the other, by § 75·3, that is:

If circles c_1, c_2, c_3, c_4 touch a line, and A, B, C, A', B', C' be the other common tangents to the respective pairs c_2c_3, c_3c_1, c_1c_2, c_1c_4, c_2c_4, c_3c_4, and no two of these tangents be parallel, then the circles which touch the triads ABC, $AB'C'$, $A'BC'$, $A'B'C$ touch a line.

§ 85. *The circle as envelope of circles.*

1. Just as in chap. XI, the theory of a spread of points in a plane was extended to a theory of a spread of circles as point loci, we may similarly extend the above theory; for the latter, when we note its space interpretation, is like a theory of a spread of points, the circles functioning as points.

If, in the notation of previous sections, we write $c = o + \mathrm{r}\phi$, we shall write $s = c + \frac{1}{2}\mathrm{d}^2\theta$; this is to represent *the 'system' of (oriented) circles whose tangential distance from c is* d. The square of the tangential distance between $c_1 = o_1 + \mathrm{r}_1\phi$ and $c_2 = o_2 + \mathrm{r}_2\phi$ is defined as $(o_1 - o_2)^2 - (\mathrm{r}_1 - \mathrm{r}_2)^2$. It may be negative. Thus d is real or pure imaginary.

Thus, if $\mathrm{d} = 0$, c will now be regarded as a special case of s, as that system of circles whose tangential distance from c, the circle in the old sense, is zero; that is, c is now regarded as the envelope of all circles touching it. To avoid confusion, we denote the circle in the old sense by c', in the new sense by c.

Since a line and a point are special cases of circles, our present work includes the theory of circles regarded as loci of points and of circles regarded as envelopes of lines.

Systems of circles constitute a spread of step five.

We use the notation $s_i = c_i + \frac{1}{2}d_i^2\theta$ for the system of circles whose tangential distance from the circle c_i' is d_i. We call d_i the '*parameter*' of the system.

2. *Addition of systems.* If c' is of weight one, we say s is of weight one; we define addition of systems as follows:

$$k_1 s_1 + k_2 s_2 = k_1 c_1 + k_2 c_2 + \tfrac{1}{2}(k_1 d_1^2 + k_2 d_2^2)\theta,$$

where $k_1 c_1 + k_2 c_2 = (k_1 + k_2) c_3$ if $k_1 c_1' + k_2 c_2' = (k_1 + k_2) c_3'$,

the latter addition being defined as in § 80.

3. *Inner product of systems.* We define the inner product of systems as in § 71·2, where c here corresponds to a point-circle there:

$$[s_1 | s_2] = [s_2 | s_1] = \tfrac{1}{2}(d_1^2 + d_2^2 - (c_1' - c_2')^2),$$

where $(c_1' - c_2')^2$ is the square of the tangential interval between c_1' and c_2' if the tangent exists, and otherwise is defined as in § 81·3.

When $d_1 = d_2 = 0$, we have

$$[c_1 | c_2] = [c_2 | c_1] = -\tfrac{1}{2}(c_1' - c_2')^2.$$

Thus $c^2 = 0$ for all c, and $[c_1 | c_2] = -\frac{1}{2}(c_1 - c_2)^2$.

Hence $[s_1 | s_2] = [c_1 | c_2] + \frac{1}{2}(d_1^2 + d_2^2)$. From this, if the distributive law is to hold, we find that, as in § 72, we must take $\theta^2 = 0$, $[\theta | s] = 1$ for all systems s so far introduced.

We have $s^2 = d^2$, $[s_1 | c] = \frac{1}{2}d_1^2 + [c | c_1]$.

Assume $[c | (k_1 s_1 + k_2 s_2)] = k_1 [c | s_1] + k_2 [c | s_2]$.

If $s^2 = 0$, we say s is a '*nul system*'; since then $d = 0$, a nul system is a system of circles touching a given circle c'.

The circle c_1' is in the system s, if $[s | c_1] = 0$, for the latter equation gives

$$0 = [(c + \tfrac{1}{2}d^2\theta) | c_1] = -\tfrac{1}{2}(c' - c_1')^2 + \tfrac{1}{2}d^2, \quad d^2 = (c' - c_1')^2.$$

In particular the circle c_1' is in the system c, if c_1', c' touch: hence the condition that c_1', c' touch is $[c | c_1] = 0$.

4. *Parameter of the sum of two systems.* If $k_1+k_2\neq 0$, then $k_1 s_1+k_2 s_2$ is a system of weight k_1+k_2. If then

$$(k_1+k_2)s_3 = k_1 s_1+k_2 s_2,$$

we can find the parameter d_3 of s_3.

Let c' be a point-circle, then

$$[s_1|c] = [c_1|c]+\tfrac{1}{2}d_1^2 = \tfrac{1}{2}(d_1^2-t_1^2),$$

where t_1^2 is the power of the centre of c' with respect to the circle c_1'.

Let t_2^2, t_3^2 be the powers of that point with respect to circles c_2', c_3' respectively. Then, since

$$k_1[s_1|c]+k_2[s_2|c] = (k_1+k_2)[s_3|c],$$

we have $k_1(d_1^2-t_1^2)+k_2(d_2^2-t_2^2) = (k_1+k_2)(d_3^2-t_3^2)$,

that is, $(k_1+k_2)d_3^2 = k_1 d_1^2+k_2 d_2^2-k_1 t_1^2-k_2 t_2^2+(k_1+k_2)t_3^2$.

Now $(k_1+k_2)(c-c_3) = k_1(c-c_1)+k_2(c-c_2)$.

Squaring,

$$(k_1+k_2)^2 t_3^2 = k_1^2 t_1^2+k_2^2 t_2^2+2k_1 k_2[(c-c_1)|(c-c_2)].$$

Hence

$$\begin{aligned}(k_1+k_2)^2 d_3^2 &= k_1^2 d_1^2+k_2^2 d_2^2+k_1 k_2(d_1^2+d_2^2)\\ &\qquad -k_1^2 t_1^2-k_2^2 t_2^2-k_1 k_2(t_1^2+t_2^2)+(k_1+k_2)^2 t_3^2\\ &= k_1^2 d_1^2+k_2^2 d_2^2+k_1 k_2(d_1^2+d_2^2+2[(c-c_1)|(c-c_2)]-t_1^2-t_2^2)\\ &= k_1^2 d_1^2+k_2^2 d_2^2+k_1 k_2(d_1^2+d_2^2-(c_1-c_2)^2),\end{aligned}$$

since

$$(c_1-c_2)^2 = ((c-c_1)-(c-c_2))^2 = t_1^2+t_2^2-2[(c-c_1)|(c-c_2)].$$

5. *Difference of two systems.* Now take $k_2 = -k_1+\epsilon$, then

$$\epsilon^2 d_3^2 = \epsilon^2 d_2^2+\epsilon k_1(d_1^2-d_2^2)+k_1^2(c_1-c_2)^2-\epsilon k_1(c_1-c_2)^2.$$

Let $\epsilon\to 0$ in such a way that ϵd_3 remains finite and tends to a finite quantity, then $\epsilon^2 d_3^2\to k_1^2(c_1-c_2)^2$.

Thus if we take $k_1 = 1$, we find

$$s_1-s_2 = c_1-c_2+\tfrac{1}{2}(d_1^2-d_2^2)\theta$$

is a system whose parameter is infinite.

If $l = s_1-s_2$ we define $[\theta|l] = [\theta|s_1]-[\theta|s_2]$. Hence $[\theta|l] = 0$.

Conversely, if $[\theta\,|s] = 0$, then s is the difference of two systems. We enlarge the notion of systems to include this case.

We call s a general system if $[\theta\,|s] \neq 0$; hence if s is a general system of weight unity, then $[\theta\,|s] = 1$.

6. We say c_1' is in $s_1 - s_2$, if $[c\,|s_1] = [c\,|s_2]$; then

$$(c'-c_1')^2-(c'-c_2')^2$$

is constant, as c' varies. Let f_1, f_2 be the distances of the centre of c' from the centres of c_1', c_2' and let t_1^2, t_2^2 be the powers of the centre of c' with respect to these circles. Then, if r, r_1, r_2 be the radii of c', c_1', c_2', we have:

$$f_1^2-(r-r_1)^2-f_2^2+(r-r_2)^2 = \text{const.}$$

Hence
$$t_1^2-t_2^2+2r(r_1-r_2) = 2k,$$

where k is a constant. But $t_1^2-t_2^2 = 2pf$, where f is the distance between the centres of c_1' and c_2' and p is the distance from the centre of c' to the radical axis of c_1' and c_2'. Hence $(p-kf^{-1})r^{-1}$ is constant.

Hence c' has a fixed power in a fixed line parallel to the radical axis of c_1' and c_2'. In particular, c_1-c_2 is the system of circles with fixed powers in the radical axis of c_1', c_2'.

7. *The locus of the centres of the point-circles of a general system* $s = o + r\phi + \frac{1}{2}d^2\theta$ *is a non-oriented circle* m, *concentric with* c *and of radius* r_0 *such that* $r_0^2 = r^2 + d^2$.

This is the '*mid-circle*'. *If* $d^2 > 0$, *it is cut at the same angle by all circles of* s. For, if r_1 be the radius of a circle n of the system, and d_1 the distance between the centres of m and n, then

$$2r_0 r_1 \cos\widehat{mn} = r_0^2 + r_1^2 - d_1^2, \quad d_1^2 = d^2 + (r-r_1)^2.$$

Hence
$$2r_0 r_1 \cos\widehat{mn} = r_0^2 + r_1^2 - d^2 - (r-r_1)^2 = 2rr_1,$$

$$\cos\widehat{mn} = r/r_0.$$

If $r = 0$, then the circles of the system cut m orthogonally.

8. *Normal systems.* Two systems s, s_1 are '*normal*' if $[s\,|s_1] = 0$. *Five mutually normal non-nul systems are independent.* For, if $s_1, \ldots, s_5$ be mutually normal, and non-nul, and $k_1 s_1 + \ldots + k_5 s_5 = 0$, then inner multiplication by s_1 gives $k_1 s_1^2 = 0$, $k_1 = 0$.

If $d_1, \ldots, d_5$ *be parameters of the mutually normal systems* $s_1, \ldots, s_5$, *then* $\sum_1^5 d_1^{-2} = 0$. For, we can take $s_1, \ldots, s_5$ as the basis of the spread of systems, then for some scalars $x_1, \ldots, x_5$, $x_1 s_1 + \ldots + x_5 s_5 = \theta$.

Inner multiplication by $s_1, \ldots, s_5, \theta$, in turn, gives

$$x_1 d_1^2 = 1, \ \ldots, \ x_5 d_5^2 = 1, \quad x_1 + \ldots + x_5 = 0.$$

The result follows from this.

Four circles in general position fix one system containing them. For if $s_1, \ldots, s_5$ be a basis of the spread of systems, and $c_1, \ldots, c_4$ be four circles in general position, then

$$\sum_1^5 k_i [s_i \,|\, c_j] = 0 \quad (j = 1, \ldots, 4)$$

will give the ratios of the k_1.

9. If $s_1, \ldots, s_5$ be mutually normal non-nul systems, then $\sum_1^5 k_i s_i$ will be nul if $\sum k_i^2 d_i^2 = 0$. As this is quadratic in k_1, *the spread of nul-systems is a quadratic spread.*

We can take supplements with respect to the set of mutually normal non-nul systems.

10. We can now apply the general theory of outer and inner products, and the situation is formally analogous to that in § 73 for circles.

If $s_1, \ldots, s_4$ be *any* four independent systems, the system normal to them is $|[s_1 \ldots s_4]$. If this is a nul system, then $[s_1 \ldots s_4]^2 = 0$.

Thus, for example, the condition that $c_1', \ldots, c_4'$ are touched by one circle, is $[c_1 \ldots c_4]^2 = 0$. Using $[c_1 \,|\, c_2] = -\frac{1}{2}(c_1' - c_2')^2$, we again get Casey's condition.

If $s_1, \ldots, s_6$ be any six systems, they are dependent, and we find, as usual, that the determinant, whose ij element is $[s_i \,|\, s_j]$, vanishes when the s_i are nul systems; this gives a relation between the tangent intervals of any six circles.

If $s_1, \ldots, s_5$ be dependent, then $[s_1 \ldots s_5] = 0$; when the s_i are nul systems, the corresponding circles are in the same system, and hence, by 7, cut a non-oriented circle at the same angle. The condition that five circles do this is therefore $[c_1 \ldots c_5]^2 = 0$.

If the circles $c'_1, \ldots, c'_4$ are in a net, the nul systems $c_1, \ldots, c_4, \theta$ are dependent. Hence $[c_1 \ldots c_4 \theta]^2 = 0$ is the condition that $c'_1, \ldots, c'_4$ are in a net. The condition can be written:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & k_{12} & k_{13} & k_{14} \\ 1 & k_{21} & 0 & k_{23} & k_{24} \\ 1 & k_{31} & k_{32} & 0 & k_{34} \\ 1 & k_{41} & k_{42} & k_{43} & 0 \end{vmatrix} = 0,$$

where k_{ij} is the square of the tangential distance between c'_i and c'_j.

§86. *Theorems on circles.**

1. Nul systems can be represented by points of a quadric spread $\mathfrak{Q}$ in a spread of step five. Represent supplements with respect to the quadric by a stroke. This operation of taking the supplement corresponds strictly to that above. If c be the point of $\mathfrak{Q}$ which represents the circle c', the tangent four-spread† at c to $\mathfrak{Q}$ cuts $\mathfrak{Q}$ in a three-spread cone (that is, an ordinary cone, which, like a plane, is a spread of step three). The ∞^2 points of this cone represent the circles touching c'. The circles which touch another circle also, are represented by points which lie on a plane section of this cone.

2. *The circles c'_1, c'_2, c'_3, c'_4 touch a circle c'; circles $c'_{12}, c'_{23}, c'_{34}, c'_{41}$ touch the respective pairs $c'_1c'_2, c'_2c'_3, c'_3c'_4, c'_4c'_1$; circles k'_1, k'_2, k'_3, k'_4 touch c' and the respective pairs $c'_{41}c'_{12}, c'_{12}c'_{23}, c'_{23}c'_{34}, c'_{34}c'_{41}$. Then if c'_1, c'_2, c'_3, c'_4 touch another circle besides c', then k'_1, k'_2, k'_3, k'_4 also touch another circle.*

For $c', c'_{12}, c'_{23}, c'_{34}, c'_{41}$ are represented by a five-point $cc_{12}c_{23}c_{34}c_{41}$ inscribed in $\mathfrak{Q}$. The tangent four-spread at c cuts $\mathfrak{Q}$ in a three-spread cone on which lie $c_1, c_2, c_3, c_4, k_1, k_2, k_3, k_4$. Since c'_1, c'_2, c'_3, c'_4 touch another circle besides c', the points c_1, c_2, c_3, c_4 lie on a plane section of this cone. Now c_1, k_1 lie on $|[cc_{41}c_{12}]$; c_2, k_2 lie on $|[cc_{12}c_{23}]$; hence $[c_1k_1]$ and $[c_2k_2]$ lie on $|c$ and on $|c_{12}$, and hence on the plane where these spreads meet. Hence $[c_1k_1]$ and

* For a large collection of theorems, see E. Muller, *Jahresber. Deutsch. Math. Ver.* 20 (1911), p. 168.

† Recall that a four-spread has three dimensions in the usual terminology.

$[c_2k_2]$ cut. Thus, similarly, $[c_1k_1]$, $[c_2k_2]$, $[c_3k_3]$, $[c_4k_4]$ form a skew quadrilateral, the sides of which cut the cone in points c_1, c_2, c_3, c_4 which are coplanar, hence the remaining cuts k_1, k_2, k_3, k_4 are coplanar (§31·12); the corresponding circles accordingly touch another circle.

3. If we take c'_{ij} as the point circle at a cut of c'_i, c'_j, we have:

If circles $c'_1, \ldots, c'_4$ touch c', and c'_{ij} be a cut of c'_i, c'_j, and if through c'_{12}, c'_{23} we draw a circle k'_2 to touch c', and so on; then, if $c'_1, \ldots, c'_4$ touch another circle besides c', so do $k'_1, \ldots, k'_4$.

4. If we take $c'_1, \ldots, c'_4$ to be lines touching c', then $[\theta\,|c_i] = 0$; this corresponds to the assumption that $c'_1, \ldots, c'_4$ touch another circle. Hence, if c'_{ij} be the point where the lines c'_i, c'_j meet, and k_2 be the circle through c'_{12}, c'_{23} touching c', and so on, then $k'_1, \ldots, k'_4$ touch another circle besides c'.

5. If we take $c'_1, \ldots, c'_4$ to be point-circles on c', they are circles touching c' and its opposite circle; take c'_{ij} as the join of c'_i, c'_j. Then if k'_2 be the circle touching c'_{12}, c'_{23}, c', and so on, then k'_1, k'_2, k'_3, k'_4 touch another circle besides c'.

6. *If c'_1, c'_2, c'_3, c' be circles not touching the same circle, and t'_1, u'_1 be circles touching c'_2, c'_3, c'; and similarly for t'_2, u'_2, t'_3, u'_3; and if $t'_{1'23}$ be a circle touching $u'_1t'_2t'_3$, while $t'_{12'3}$, $t'_{123'}$, $t'_{1'2'3'}$ respectively touch the triads $t'_1u'_2t'_3$, $t'_1t'_2u'_3$, $u'_1u'_2u'_3$, then $t'_{1'23}$, $t'_{12'3}$, $t'_{123'}$, $t'_{1'2'3'}$ all touch the same circle, and with a similar notation $t'_{12'3'}$, $t'_{1'23'}$, $t'_{1'2'3}$, t'_{123} all touch the same circle.*

For t'_1, u'_1, t'_2, u'_2 touch c'_3 and c'; hence the points t_1, u_1, t_2, u_2 lie on a conic on the cone $\mathscr{C}$ in which the tangent four-spread to $\mathscr{Q}$ at c cuts $\mathscr{Q}$. Thus $[t_1u_1]$, $[t_2u_2]$ are coplanar; so are $[t_2u_2]$ and $[t_3u_3]$; so are $[t_3u_3]$ and $[t_1u_1]$. But, in general, all three lines are not coplanar. Hence these three lines meet in a point, for they all lie in a four-spread.

Hence the sections of the cone $\mathscr{C}$ by the planes $u_1t_2t_3$, $t_1u_2t_3$, $t_1t_2u_3$, $u_1u_2u_3$ touch a section of the cone by a plane π, say. (§77·14.)

Now if α, β be planes which cut the cone in sections that touch, then the cut of α, β touches $\mathscr{Q}$, since it is a tangent to the cut of $\mathscr{Q}$ by the tangent-spread at c.

Thus $|\alpha$, $|\beta$, which are lines through c, lie in a plane which touches $\mathcal{Q}$.

Hence each of the lines $|u_1 t_2 t_3$, $|t_1 u_2 t_3$, $|t_1 t_2 u_3$, $|u_1 u_2 u_3$, when joined to $|\pi$, gives a plane touching $\mathcal{Q}$; hence each of $t_{1'23}$, $t_{12'3}$, $t_{123'}$, $t_{1'2'3'}$ when joined to $|\pi$ gives a plane touching $\mathcal{Q}$.

Hence these four points lie on a space touching $\mathcal{Q}$.

Hence $t'_{1'23}$, $t'_{12'3}$, $t'_{123'}$, $t'_{1'2'3'}$ all touch a circle.

The rest follows similarly.

7. As a special case of 6, we have: *If c'_1, c'_2, c'_3 be circles and t'_1, u'_1 tangent lines to c'_2, c'_3; t'_2, u'_2 tangent lines to c'_3, c'_1; and t'_3, u'_3 tangent lines to c'_1, c'_2, and if $t'_{1'23}$ be the circle touching the lines $u'_1 t'_2 t'_3$, and so on, then $t'_{1'23}$, $t'_{12'3}$, $t'_{123'}$, $t'_{1'2'3'}$ touch the same circle and $t'_{12'3'}$, $t'_{1'23'}$, $t'_{1'2'3}$, t'_{123} touch the same circle.*

8. *Four rotors, each three independent, in a plane fix four circles each of which touches three of them; the four centres of these circles lie on a circle.* (Steiner-Laguerre.)

For return to the space representation of circles, take a plane parallel to our horizontal plane, and take a circle on it. Let four planes which touch the circle meet in threes in points a, b, c, d. Apply § 51·8 Cor. 4. The tangents from the centre o of the circle to the circle, are lines through the circular points of both parallel planes; these lines together with oa, ob, oc, od touch a quadric cone, whose section by the horizontal plane is a circle.

Now let the parallel plane recede to infinity, and take o in the direction perpendicular to our horizontal plane. We then have four planes equally inclined to the horizontal, meeting in a, b, c, d and the projections of a, b, c, d on the horizontal plane lie on a circle.

Take the four planes through the four given rotors, and we have our theorem.

§ 87. *Extension to spreads of higher step.*

Consider a spread of points, of step four, e.g. ordinary space. Then spheres as point-loci are represented by $o + \frac{1}{2}r^2\theta$, where θ is a new unity. The theory of chap. XI easily extends to this case.

For example, if s_1, s_2, s_3, s_4 be spheres, $[s_1 s_2]$ represents the pencil of spheres coaxal with s_1, s_2; $[s_1 s_2 s_3]$ represents the set of

spheres through two points, real or imaginary; $|[s_1 s_2 s_3]$ the set of spheres orthogonal to these; $[s_1 s_2 s_3 s_4]$ the set of spheres with a common orthogonal sphere $|[s_1 s_2 s_3 s_4]$.

Similarly, we can extend the theory of this chapter. The expression $o + r\phi$ represents an oriented sphere, centre o, radius r. If s_1, s_2 be oriented spheres, $[s_1 s_2]$ is the set of spheres which have in pairs the same centre of similitude, and so on.

The expression $o + r\phi + \frac{1}{2}d^2\theta$ represents a system of spheres whose tangential distance from $o + r\phi$ is d.

The extension to higher spreads is a trivial exercise, and the only question of interest is what theorems are valid in particular dimensions only. This is considered later.

The geometry of oriented spheres in a spread of step four has an intimate relation to the geometry of the restricted relativity theory.

CHAPTER XIII

THE GENERAL THEORY OF MATRICES*

We shall develop the theory of matrices mainly with a view to geometrical applications, but it will be necessary to give some investigations whose interest is algebraical rather than geometrical. Our matrices will all be *square* matrices.

§88. *Matrices whose elements are integers.*

1. Let $\mathfrak{A} = (a_{ij})$ be a matrix of order n whose elements are integers, positive or negative.

An '*elementary transformation*' of such a matrix is any one of the following processes:

(1) The addition to one row of any multiple of another row; that is the replacement of a_{ij}, for $j = 1, \ldots, n$, by $a_{ij} + ka_{tj}$, where t, i are fixed, and k is a constant integer;

(2) The addition to one column of any multiple of another column;

(3) Change of the order in which the rows appear;

(4) Change of the order in which the columns appear;

(5) Change of sign of all elements of any one row or of any one column.

Consider all the matrices which can be obtained from $\mathfrak{A}$ by any number of these operations, and suppose the zero matrix is not one of them. Of the non-zero elements of these matrices, there will be one (or more) whose absolute value is not greater than the absolute value of any of the other non-zero elements of the matrices. Select a matrix containing such an element, and permute the rows and columns of this matrix until this element is brought to the (1, 1) place. Call the element b_{11}.

Then any other non-zero element of the first row of this matrix is divisible by b_{11}; for if one were not, then by adding to its

* See van der Waerden, *Moderne Algebra*, 2 (1931). Turnbull and Aitken, *Canonical Matrices* (1932). The following were published after this chapter had been written: MacDuffee, *Theory of Matrices* (Berlin, 1933); Wedderburn, *Lectures on Matrices* (New York, 1934). I have made use of the first in revising §§ 97·14, 100·3, 101.

column a suitable multiple of the first column (whereby we obtain a matrix of our set), we could reduce the absolute value of this element to less than $|b_{11}|$, contrary to the definition of b_{11}.

Accordingly, the other non-zero elements of the first row of our selected matrix are divisible by b_{11}; hence, by adding suitable multiples of the first column to the other columns, we can make all elements of the first row, except b_{11}, vanish. Similarly we can make all elements of the first column, except b_{11}, vanish.

Suppose this done.

Then b_{11} divides all non-zero elements b_{ij} of the selected matrix, thus transformed. For if $b_{ij} = b_{11}q + r$, and $r \neq 0$, $b_{ij} \neq 0$, we can assume $|r| < |b_{11}|$. Add the first column to the j-th column, the element in place $(1,j)$ then becomes b_{11}; then subtract q times the first row from the i-th row, the element in place (i,j) then becomes r. But $|r| < b_{11}$. As the new matrix is one derived from $\mathfrak{A}$, this contradicts the definition of b_{11}.

Thus by elementary transformations, we have brought $\mathfrak{A}$ to the form in which b_{11} is the only non-zero element of the first row and first column, and in which b_{11} divides all other elements of the transformed matrix.

Consider next the $(n-1)$-rowed matrix obtained from our transformed matrix by omitting its first row and column. From this matrix we obtain, as before, by elementary transformations, a set of matrices. *If the zero matrix is not in this set*, we can, by our elementary transformations, obtain a matrix whose only non-zero element in the first row and column is the first element. Call it b_{22}. As before, any other non-zero element of the new matrix is divisible by b_{22}. Also, as the non-zero elements of the old $(n-1)$-rowed matrix were all divisible by b_{11}, and this property is clearly not affected by our elementary transformations, therefore b_{11} divides b_{22}.

Consider next the $(n-2)$-rowed matrix obtained from the last by omitting its first row and column, and proceed as before.

The process will only be arrested if, in the set of matrices considered at any stage, the zero matrix occurs; if this happens, we select the zero matrix for the matrix at that stage; if it never happens, we finally obtain a matrix of one row and column, and we denote its element by b_{nn}.

When the process is ended, restore all the rows and columns removed at successive stages. The result is an n-rowed matrix, all of whose elements vanish except the first r elements of the main diagonal, where $r \leqq n$; if these elements be $b_{11}, \ldots, b_{rr}$, *the final matrix is* $\operatorname{diag}(b_{11}, b_{22}, \ldots, b_{rr}, 0, \ldots, 0)$, *and each* b_{ii} *divides the next.*

Finally, by transformation (5), make all the b_{ii} positive.

Now our processes do not change the rank of the matrix (§ 59), because, for instance, they do not alter the number of independent column-extensives, and since r is the rank of the final matrix, we have, writing b_i for b_{ii}:

The 'normal form' of a matrix of integers, of rank r, *is the diagonal matrix* $\operatorname{diag}(b_1, b_2, \ldots, b_r, 0, \ldots, 0)$, *where each* b_i *divides the next, and all are positive. This normal form is obtained by elementary transformations.*

2. As in the similar work in § 54·12, we could have taken, as our elementary transformations, the addition to one row of a multiple of an *adjacent* row, and similarly for columns, and the change of sign of all elements of a row or of a column.

3. The operation of adding to the i-th row, the k-th multiple of the j-th row is the same as the multiplication on the left by $\mathfrak{I} + k\mathfrak{E}_{ij}$. The corresponding operation on columns is the same as multiplication on the right by that matrix. $\mathfrak{E}_{ij}$ is defined in § 62·10.

The operation of changing all signs in the r-th row (column) is the same as multiplication on the left (right) by

$$\mathfrak{E}_{11} + \mathfrak{E}_{22} + \ldots - \mathfrak{E}_{rr} + \ldots + \mathfrak{E}_{nn}.$$

Now all these multiplying matrices have determinant ± 1. Hence

A matrix of integers can be reduced to normal form by multiplying on the left and right by suitable matrices of determinant ± 1.

4. If $\mathfrak{B}$ is the normal form of $\mathfrak{A}$, then, from the nature of the elementary transformations, we can conversely obtain $\mathfrak{A}$ from $\mathfrak{B}$ by such transformations. Now if one matrix is obtained from another by such transformations, the k-rowed minors of either are linear combinations of those of the other, the coefficients being integers, as is easily seen by considering row- and column-vectors.

Hence the k-rowed minors of the normal matrix $\mathfrak{B}$ and those of $\mathfrak{A}$ have the same highest common factor. Call this d_k. Then

$$d_k = d_{k-1} b_k \quad (k = 1, \ldots, r),$$

from the form of $\mathfrak{B}$.

Now the d are uniquely determined by $\mathfrak{A}$, hence so are the b.

The numbers $b_1, b_2, \ldots, b_r$ *are the 'invariant factors' of* $\mathfrak{A}$, *they are uniquely determined by* $\mathfrak{A}$: $b_k = d_k d_{k-1}^{-1}$, *where* d_j *is the highest common factor of all* j*-rowed minors of* $\mathfrak{A}$.

5. Two matrices of integers are '*equivalent*' if we can pass from one to the other by a series of elementary transformations. This equivalence is a transitive relation. Such matrices can be reduced by elementary transformations to the same normal form; and the normal form for each is unique, by 4. Hence:

Two matrices of integers are equivalent if, and only if, their normal forms coincide, and therefore if, and only if, they have the same rank, and same invariant factors.

Examples. 1. If the elementary operations which refer to rows are allowed, but not those which refer to columns, then any matrix of integers can be reduced to the form in which all elements below the main diagonal vanish.

2. A '*lattice*' in $\mathscr{S}(e_1, \ldots, e_n)$ is a set of extensives whose coordinates are integers and such that if a, b be in the set, so are $a+b$ and $a-b$. (If r is the maximum number of independent extensives in the lattice, this is usually said to be of '*rank*' r; we continue to use our word '*step*'.) The extensives $u_1, \ldots, u_r$ are a '*basis*' of the lattice, if they are independent, and all extensives of the lattice are linear combinations of these, with integral coefficients, positive, negative, or zero.

Then, if a lattice of step n be given in $\mathscr{S}(e_1, \ldots, e_n)$, it is possible to choose a basis*

$$u_i = \Sigma b_{ij} e_j, \quad (i, j = 1, \ldots, n),$$

such that

$$b_{ij} = 0 \quad \text{if} \quad i < j.$$

If we are allowed to change the frame $e_1, \ldots, e_n$ to $e'_1, \ldots, e'_n$, where $e'_1, \ldots, e'_n$ are any independent linear combinations of $e_1, \ldots, e_n$ with integral coefficients, then we can choose frame and basis so that $b_{ij} = 0$ if $i \neq j$.

* This is important in the theory of numbers. The connection of lattices and crystals is obvious.

§ 89. *Matrices whose elements are polynomials.*

1. When the argument of §88 is examined, it is found that not all the properties of integers have been used, but the following only:

(1) The class of integers is closed under addition, subtraction and multiplication, and these operations obey the laws of ordinary algebra.

(2) The formal properties of 0 and 1 for these operations, namely: $a+0=a, \quad a.0=0, \quad +0=-0, \quad a.1=a.$

(3) The relations of 'greater than' and 'less than' for positive integers and their properties.

(4) The absolute value of an integer, not zero, is a positive integer.

(5) If a, b be integers, not zero, such that $|a|>|b|$, then there are integers q, r such that $a=bq+r$, $|r|<|b|$.

2. If then we have matrices composed of any elements with these properties, the above investigation applies. Consider the set of all polynomials in a single variable λ. The variable λ and the coefficients of the polynomials may move either in the real or in the complex field. The set has the first property, since by addition, subtraction, or multiplication of polynomials in λ, we get polynomials in λ.

For the next property, we regard the numbers 0, 1 as special cases of polynomials in λ.

We define, for the present purpose, the '*absolute value*' of a non-vanishing polynomial in λ, as its degree increased by unity; thus a non-zero number, regarded as an instance of a polynomial in λ, has absolute value unity; we further define the absolute value of 0, regarded as a polynomial in λ, to be zero. The absolute value of any polynomial, $\neq 0$, in λ is now a positive integer. This is the translation of property (4) in the present investigation. Thus '*positive* integer' has still its usual meaning and (3) carries over unchanged.

The fifth property now becomes: If a, b be polynomials, not zero, in λ, and the degree of a be greater than that of b, then there are polynomials q, r in λ such that $a=bq+r$, where the degree of r is less than that of b.

This is an elementary algebraic property.

A 'multiple' of a polynomial in λ will be a product by any polynomial in λ. In §88 we were permitted to multiply a row or column by ± 1, that is, by an integer of absolute value unity. So here we may multiply any row or column by a polynomial of 'absolute value' unity, i.e. by any non-zero constant.

Similarly, a matrix whose determinant had absolute value unity, in §88, is now replaced by one whose determinant is a non-zero constant.

Thus the elementary transformations are:

(1) Addition to one row of the product of any other by a polynomial in λ.

(2) Addition to one column of the product of any other by a polynomial in λ.

(3) Change of the order in which the rows appear.

(4) Change of the order in which the columns appear.

(5) Multiplication of any row or column by a non-zero constant.

A matrix whose elements are polynomials in λ will be called a '*λ-matrix*'. Two such matrices are '*equivalent*' if one can be obtained from the other by a series of elementary transformations, as now defined.

The argument of §88 as now interpreted gives:

If $\mathfrak{A}$ *be any λ-matrix, then there are λ-matrices* $\mathfrak{B}$, $\mathfrak{C}$, *whose determinants are independent of* λ, *such that* $\mathfrak{BAC}$ *is in normal form* $\operatorname{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$, *where the* a *are polynomials in* λ, *each dividing the next, the coefficient of the highest power of* λ *being made unity. Two such matrices are equivalent if, and only if, they have the same rank and 'invariant factors'* $a_1, a_2, \ldots, a_r$. *If* $\mathfrak{A}$, $\mathfrak{D}$ *be equivalent, then the* k*-rowed minors of* $\mathfrak{A}$, *and those of* $\mathfrak{D}$, *have the same polynomial for highest common factor (the coefficient of the highest power of* λ *being taken as unity in each case), and if this polynomial be* d_k, *then* $d_k = d_{k-1} a_k$, $(k = 1, \ldots, r)$.

§90. *Matrices of real numbers or of complex numbers.*

We get another interpretation of §88, if we consider matrices of real numbers. For

(1) Real numbers are closed under addition, subtraction, and multiplication.

(2) The formal properties of 0, 1 hold.

(3) The relations of 'greater than' and 'less than' have their usual properties; but actually these are not now used.

(4) The absolute value of a real number, not zero, is a positive real number.

(5) If a, b be real numbers, and $b \neq 0$, we can find a real number q such that $a = bq$.

As we may now multiply a row (column) by any real number, before we add it to another row (column), and since any real is divisible by any non-zero real, the only relevant distinction which can be made between the diagonal elements of the normal form, is between those which are zero and those which are not. 'Equivalent' matrices are now simply those of the same rank, and as 'normal' form of a matrix we can take

$$\text{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0).$$

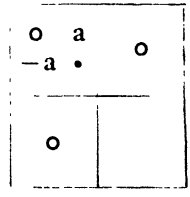
Then, if $\mathfrak{A}$ be any matrix, we can find matrices $\mathfrak{B}$, $\mathfrak{C}$ with non-zero determinant, such that $\mathfrak{BAC}$ is of normal form.

Similarly for matrices of complex numbers.

§91. *Normal forms of symmetric or skew-symmetric matrices of real or of complex numbers.*

If $\mathfrak{A}$ be either symmetric ($\mathfrak{A} = \mathfrak{A}^*$), or skew-symmetric ($\mathfrak{A} = -\mathfrak{A}^*$), and $\mathfrak{P}$ be any non-zero matrix, then $\mathfrak{PAP}^*$ is symmetric or skew-symmetric according as $\mathfrak{A}$ is so.

1. If $\mathfrak{A} = -\mathfrak{A}^*$, let the i-th row be the first which contains a non-zero element, and suppose this is in the ik place. Then $i \neq k$. Interchange the first and i-th rows, then the first and i-th columns, then the k-th and second rows, and then the k-th and second columns. By elementary transformations, we can reduce the other elements of the new first and second rows to zero (§90), and we then obtain a matrix of form shewn, the rectangles with only zero inside having all elements zero:



Now treat similarly the matrix of order $n-2$, which is obtained by omitting the first two rows and columns. Thus proceeding we finally reach, if n is even, a matrix of form

$$a(\mathfrak{E}_{12} - \mathfrak{E}_{21}) + b(\mathfrak{E}_{34} - \mathfrak{E}_{43}) + \ldots.$$

Hence there is a matrix $\mathfrak{P}$, with $\det \mathfrak{P} \neq 0$, such that $\mathfrak{PAP}^*$ has this form; for if $\Omega\mathfrak{A}$ is the result of any

elementary transformation on the rows of $\mathfrak{A}$, the same sort of elementary transformation applied to the columns of any matrix $\mathfrak{B}$ gives $\mathfrak{BQ}^*$.

We can also make each of a, b, ... equal to unity.

2. If $\mathfrak{A} = \mathfrak{A}^*$, the process of §90, when we treat rows and columns in the same way, and note the preceding remark, gives $\mathfrak{P}$ such that, for some n, $\mathfrak{PAP}^* = a_1\mathfrak{E}_{11} + a_2\mathfrak{E}_{22} + \ldots + a_n\mathfrak{E}_{nn}$, and we can also make $a_1, \ldots, a_n$ equal to ± 1. If the field is complex, we can make them all equal to $+1$.

Cor. By a linear transformation on the x, y, a skew-symmetric form $\Sigma a_{ij}x_iy_j$ (that is, one with $a_{ij} = -a_{ji}$), of even order, can be reduced to the form

$$(x_1y_2 - x_2y_1) + (x_3y_4 - x_4y_3) + \ldots.$$

Any symmetric form (that is, one with $a_{ij} = a_{ji}$) can be reduced, if the field is complex, to the form

$$x_1y_1 + x_2y_2 + \ldots.$$

For, if $x = x_1e_1 + x_2e_2 + \ldots$, $y = y_1e_1 + y_2e_2 + \ldots$, $\mathfrak{A} = (a_{ij})$, then

$$\Sigma a_{ij}x_iy_j = [x\mathfrak{A}\,|\,y],$$

and if x, y be transformed by $\mathfrak{P}$, then $[x\mathfrak{A}\,|\,y]$ becomes

$$[x\mathfrak{P}\mathfrak{A}\,|\,y\mathfrak{P}] = [x\mathfrak{P}\mathfrak{A}\mathfrak{P}^*\,|\,y],$$

and we can choose $\mathfrak{P}$ so that $\mathfrak{PAP}^*$ has the form desired.

If when $\Sigma a_{ij}x_iy_j$ is reduced to form $x_1y_1 + x_2y_2 + \ldots$, there are just r terms in this expression, the form is of '*rank*' r.

§92. *The minimum equation of a matrix of real or complex numbers.*

1. Since a matrix $\mathfrak{A}$ of order n is a linear combination of matrices $\mathfrak{E}_{ij}$, $(i, j = 1, \ldots, n)$, and since $\mathfrak{A}^s$, where s is any integer, is a linear combination of the same matrices, there is an integer r, such that $\mathfrak{J}, \mathfrak{A}, \mathfrak{A}^2, \ldots, \mathfrak{A}^r$ are dependent, that is, such that there are scalars $k_0, k_1, \ldots, k_r$ such that

$$k_r\mathfrak{A}^r + k_{r-1}\mathfrak{A}^{r-1} + \ldots + k_1\mathfrak{A} + k_0\mathfrak{J} = 0.$$

Thus there is an equation

$$k_r\lambda^r + k_{r-1}\lambda^{r-1} + \ldots + k_1\lambda + k_0 = 0$$

satisfied by $\mathfrak{A}$.

If $\psi(\lambda)$ is the polynomial of lowest degree, with leading coefficient unity, such that $\psi(\mathfrak{A}) = 0$, then $\psi(\lambda) = 0$ is the '*minimum equation*' of $\mathfrak{A}$.

If r is the degree of ψ, then r is the least number such that, whatever extensive p be taken in the spread of step n on which $\mathfrak{A}$ operates, the extensives $p, p\mathfrak{A}, p\mathfrak{A}^2, \ldots, p\mathfrak{A}^r$ are dependent.

2. *If $\psi(\lambda) = 0$ is the minimum equation of $\mathfrak{A}$, then ψ is the quotient of the secular polynomial $\phi(\lambda)$ of $\mathfrak{A}$ when divided by the highest common factor of the minors of $\mathfrak{A} - \lambda\mathfrak{J}$ of order $n-1$.*

For, first, if $f(\lambda)$ be any polynomial such that $f(\mathfrak{A}) = 0$, then $f(\lambda)$ is divisible by $\psi(\lambda)$. For, if $f(\lambda) = q(\lambda)\psi(\lambda) + r(\lambda)$, where the degree of $r(\lambda)$ is less than that of $\psi(\lambda)$, then since $f(\mathfrak{A}) = 0$, $\psi(\mathfrak{A}) = 0$, we have $r(\mathfrak{A}) = 0$, contrary to the definition of ψ.

Next, let $g(\lambda)$ be the highest common factor of the minors of $\mathfrak{A} - \lambda\mathfrak{J}$ of order $n-1$, and let $f(\lambda) = \phi(\lambda)/g(\lambda)$. We have to shew that $f(\lambda)$, $\psi(\lambda)$ differ at most by a constant factor.

Let $\mathfrak{B}$ be the adjugate of $\mathfrak{A} - \lambda\mathfrak{J}$, (§63·7); then $g(\lambda)$ is the highest common factor of its elements. Hence $\mathfrak{B} = g(\lambda)\mathfrak{M}$, where $\mathfrak{M}$ is a matrix of elements not having a polynomial in λ as a common factor. Now

$$\det(\mathfrak{A} - \lambda\mathfrak{J}) = \phi(\lambda).$$

Hence

$$(\mathfrak{A} - \lambda\mathfrak{J})\, g(\lambda)\mathfrak{M} = (\mathfrak{A} - \lambda\mathfrak{J})\mathfrak{B} = \phi(\lambda)\mathfrak{J} = g(\lambda)f(\lambda)\mathfrak{J},$$

$$(\mathfrak{A} - \lambda\mathfrak{J})\mathfrak{M} = f(\lambda)\mathfrak{J}. \qquad (1)$$

As this is an identity in λ, it remains true when λ is replaced by any matrix which commutes with $\mathfrak{A}$, for instance if λ is replaced by $\mathfrak{A}$ itself. Hence $f(\mathfrak{A}) = 0$, and by the first part of the argument, $f(\lambda)$ *is divisible by* $\psi(\lambda)$.

Next, $\psi(\lambda) - \psi(\mu) \equiv w(\lambda, \mu).(\lambda - \mu)$, where $w(\lambda, \mu)$ is a polynomial in λ, μ. Put $\mu = \mathfrak{A}$, then

$$\psi(\lambda)\mathfrak{J} = -w(\lambda, \mathfrak{A}).(\mathfrak{A} - \lambda\mathfrak{J}).$$

Hence, by (1)

$$(\mathfrak{A} - \lambda\mathfrak{J})\mathfrak{M}.\psi(\lambda) = \psi(\lambda)f(\lambda)\mathfrak{J} = -f(\lambda).w(\lambda, \mathfrak{A}).(\mathfrak{A} - \lambda\mathfrak{J}).$$

Now $\det(\mathfrak{A} - \lambda\mathfrak{J})$ is not identically zero, hence

$$\mathfrak{M}.\psi(\lambda) = -f(\lambda).w(\lambda, \mathfrak{A}).$$

The elements of $\mathfrak{M}$ have no common polynomial factor, hence $\psi(\lambda)$ *is divisible by* $f(\lambda)$.

Thus $f(\lambda)$ and $\psi(\lambda)$ differ at most by a constant factor.

Cor. $\mathfrak{A}$ satisfies its secular equation $\phi(\mathfrak{A}) = 0$. (Cf. §63·7.)

§93. *Canonical forms of transformations.**

In our work in chap. IX we usually kept the frame $(e_1, \ldots, e_n)$ fixed, and it was hardly necessary to distinguish between a transformation and the matrix which represented it. We now seek the frame in which a given transformation is represented by as simple a matrix as possible. We recall that a change of frame does not change the secular polynomial. (§63·8.)

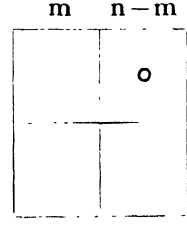
1. Let $\mathfrak{A}$ be a linear transformation in a spread R of step n. If p be an extensive in R, and $\mathfrak{A}$ is non-singular, then p, $p\mathfrak{A}$, $p\mathfrak{A}^2$, ..., $p\mathfrak{A}^n$, even if distinct, must be linearly dependent, and if, say,

$$p\mathfrak{A}^r = p\mathfrak{A}^{r+s}, \quad (r, r+s<n) \text{ for all } p, \text{ then } \mathfrak{A}^s = \mathfrak{J}.$$

Hence, in both cases, we can find m such that the extensives $p, p\mathfrak{A}, p\mathfrak{A}^2, \ldots, p\mathfrak{A}^m$ span a spread S such that each extensive of S is changed by $\mathfrak{A}$ into one of S. We say S is '*latent*' for $\mathfrak{A}$.

2. If m is the greatest number so that $p, p\mathfrak{A}, p\mathfrak{A}^2, \ldots, p\mathfrak{A}^{m-1}$ are independent, we can take these extensives as a basis of S, and call them $e_1, e_2, \ldots, e_m$, and then adjoin $e_{m+1}, \ldots, e_n$ so that $e_1, e_2, \ldots, e_n$ is a basis of R.

In the new frame, the transformation $\mathfrak{A}$ is represented by a matrix of the form shewn; where the sub-matrices to the top-left and bottom-right are square matrices of m rows and n−m rows respectively, and the rectangle in the top-right is filled with zeros. For a matrix of this form, and of this form only, represents a transformation leaving S latent.



For if $\mathfrak{A} = (a_{ij})$ is the matrix, it turns e_i into $\Sigma a_{ij} e_j$, (§62).

Since $e_1\mathfrak{A} = e_2$, $e_2\mathfrak{A} = e_3$, ..., $e_{m-1}\mathfrak{A} = e_m$, and $e_m\mathfrak{A}$ is a linear

* Weyl, *Mathematische Analyse des Raumproblems* (1923); A_2, §§ 388–390.

combination of $e_1, \ldots, e_m$, say $e_m \mathfrak{A} = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m$, (cf. § 59; $a_i = e_i \mathfrak{A}$), the top-left square has form

$$\mathfrak{E}_{12} + \mathfrak{E}_{23} + \ldots + \mathfrak{E}_{m-1,m} + x_1 \mathfrak{E}_{m,1} + x_2 \mathfrak{E}_{m,2} + \ldots + x_m \mathfrak{E}_{m,m}.$$

3. Let $S' = \mathscr{S}(e_{m+1}, e_{m+2}, \ldots, e_n)$. An important special case occurs when S' (as well as S) is latent for $\mathfrak{A}$. The matrix corresponding to the transformation then has a zero-matrix for the bottom-left sub-matrix.

We then say that the matrix is the '*direct sum*' of the square sub-matrices along the main diagonal, that is, of the top-left and bottom-right sub-matrices.

Similarly, if S or S' have latent sub-spreads, we can, by choice of frame, further simplify the matrix representing the transformation.

4. If the matrix $\mathfrak{M}$ is the direct sum of matrices $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_s$ (in which case $\mathfrak{M}$ is composed of sub-matrices $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_s$ arranged along the main diagonal), we write

$$\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \ldots \oplus \mathfrak{M}_s.$$

Then $\det \mathfrak{M} = \det \mathfrak{M}_1 \det \mathfrak{M}_2 \ldots \det \mathfrak{M}_s$, and the rank of $\mathfrak{M}$ is the sum of the ranks of $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_s$.

The first statement is obvious. For the second, consider $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$. Each non-zero minor of order q of $\mathfrak{M}$, which is not a minor of $\mathfrak{M}_1$ or $\mathfrak{M}_2$, is the product of a minor of $\mathfrak{M}_1$ of order $q - t$, say, and a minor of $\mathfrak{M}_2$ of order t. Thus, if r_1, r_2 be the ranks of $\mathfrak{M}_1, \mathfrak{M}_2$, then there is a minor of $\mathfrak{M}$ of order $r_1 + r_2$ which is not zero, but any minor of $\mathfrak{M}$ of higher order is zero.

5. *We now assume our field of scalars is the complex field*, so that the secular equation has roots in number equal to its degree.

Let α be a root of the secular polynomial $\phi(\lambda)$ of $\mathfrak{A}$. There is an extensive e_1 such that $e_1 \mathfrak{A} = \alpha e_1$, $e_1 \neq 0$. In R take a frame whose first extensive is e_1, the frame being $e_1, e_2', e_3', \ldots, e_n'$. Project R from e_1, that is, consider the subspread $\mathscr{S}(e_2', e_3', \ldots, e_n')$, and let $\phi_1(\lambda)$ be the secular polynomial of the transformation brought about by $\mathfrak{A}$ in this subspread.

Then the matrix of $\mathfrak{A}$, referred to the new frame, has all its elements in the first row zero, except that in place (1, 1) we have α. Hence

$$\phi(\lambda) = (\alpha - \lambda)\, \phi_1(\lambda).$$

If α is an l-fold root of $\phi(\lambda)$, and $l \geqq 2$, then $\phi_1(\alpha) = 0$, and we can find an extensive $e_2 \neq 0$, independent of e_1, such that $e_2\mathfrak{A} - \alpha e_2 = 0$ in the projected space, and hence, as an element of R, $e_2\mathfrak{A} - \alpha e_2$ is a multiple of e_1.

Similarly, if $l \geqq 3$, we can find e_3 such that $e_3\mathfrak{A} - \alpha e_3$ is in $\mathscr{S}(e_1, e_2)$.

Proceeding thus, we find e_l such that $e_l\mathfrak{A} - \alpha e_l$ is in

$$\mathscr{S}(e_1, e_2, \ldots, e_{l-1}).$$

Take $e_1, e_2, \ldots, e_l$ as elements of the frame of R, and adjoin any independent elements to form the frame $(e_1, e_2, \ldots, e_n)$ of R.

Since, if $k \leqq l$, then $e_k\mathfrak{A}$ depends on $e_1, \ldots, e_{k-1}$ only, therefore, for this frame, the matrix corresponding to the transformation $\mathfrak{A}$ has in the top-left sub-matrix of l rows, all elements zero above the main diagonal.

The spread $\mathscr{S}(e_1, e_2, \ldots, e_l)$ is latent for $\mathfrak{A}$. If p is in this spread, then $p(\mathfrak{A} - \alpha\mathfrak{J})^l = 0$; for, as is clear by induction, $e_r(\mathfrak{A} - \alpha\mathfrak{J})^r = 0$, if $r = 1, 2, \ldots, l$.

6. Now let $\phi(\lambda) = (\alpha_1 - \lambda)^{l_1}(\alpha_2 - \lambda)^{l_2} \ldots (\alpha_s - \lambda)^{l_s}$, ($\alpha_1, \alpha_2, \ldots, \alpha_s$ all distinct). We then get spreads $S_1, S_2, \ldots, S_s$ in R of steps $l_1, l_2, \ldots, l_s$ with secular polynomials $(\alpha_1 - \lambda)^{l_1}$, $(\alpha_2 - \lambda)^{l_2}$, $\ldots$, $(\alpha_s - \lambda)^{l_s}$ respectively.

If p_i is an extensive of S_i, then

$$p_i(\mathfrak{A} - \alpha_i\mathfrak{J})^{l_i} = 0, \quad (i = 1, \ldots, s). \tag{1}$$

The spreads $S_1, S_2, \ldots, S_s$ are each latent for $\mathfrak{A}$. They are '*unconnected*', that is, if $p = p_1 + p_2 + \ldots + p_s = 0$ where p_i is in S_i, $(i = 1, \ldots, s)$, then $p_1 = p_2 = \ldots = p_s = 0$.

To shew this, express $(\phi(\lambda))^{-1}$ in partial fractions:

$$(\phi(\lambda))^{-1} = g_1(\lambda)(\alpha_1 - \lambda)^{-l_1} + g_2(\lambda)(\alpha_2 - \lambda)^{-l_2} + \ldots + g_s(\lambda)(\alpha_s - \lambda)^{-l_s},$$

where $g_i(\lambda)$ is of degree $\leqq l_i - 1$.

Let $\quad f_i(\lambda) = g_i(\lambda)\,\phi(\lambda).(\alpha_i - \lambda)^{-l_i}$,

Then $\quad f_1(\lambda) + f_2(\lambda) + \ldots + f_s(\lambda) = 1$.

Hence $\quad f_1(\mathfrak{A}) + f_2(\mathfrak{A}) + \ldots + f_s(\mathfrak{A}) = \mathfrak{J}$.

Now $f_2(\lambda), \ldots, f_s(\lambda)$ all have $(\alpha_1 - \lambda)^{l_1}$ as a factor.

Hence, by (1), $p_1 f_i(\mathfrak{A}) = 0$, $(i = 2, \ldots, s)$. Hence $p_1 f_1(\mathfrak{A}) = p_1$.

Similarly, $p_i f_j(\mathfrak{A}) = 0$ or p_i, according as $i \neq j$ or $i = j$, $(i, j = 1, \ldots, s)$.

Thus if $p = p_1 + p_2 + \ldots + p_s = 0$, and we operate on p by $f_i(\mathfrak{A})$, we find $p_i = 0$ $(i = 1, \ldots, s)$.

Hence *each distinct root of the secular equation gives a latent sub-spread of R whose step equals the multiplicity of the root; these sub-spreads are unconnected. If the base extensives $e_1, \ldots, e_n$ are taken in sets such that each set spans one of these sub-spreads, the matrix of the transformation in this frame is the direct sum of the matrices of the transformations in the sub-spreads.*

In particular if all the roots of the secular equation are distinct, the sub-spreads are of step one, and the matrix of the transformation can be reduced to diagonal form.

7. We next consider one of the latent spreads S of step l, say, the secular polynomial being $\phi(\lambda) = (\alpha - \lambda)^l$. Let $\mathfrak{C}$ be the transformation in this spread induced by $\mathfrak{A}$.

Consider the transformation $\mathfrak{B} = \mathfrak{C} - \alpha\mathfrak{J}$; then the secular equation of $\mathfrak{B}$ is $\lambda^l = 0$, and each p of S satisfies $p\mathfrak{B}^l = 0$.

Let m be the least integer such that $p\mathfrak{B}^m = 0$ for all p of S, and let $T_0, T_1, \ldots, T_m$ be the sub-spreads of S such that $p\mathfrak{B}^m = 0$, $p\mathfrak{B}^{m-1} = 0, \ldots, p\mathfrak{B} = 0, p = 0$ respectively for all extensives p in them. Thus T_0 is S itself; while T_m contains 0 only.

For each $i = 0, \ldots, m-1$, T_i contains T_{i+1}. For if p is in T_{i+1}, then $p\mathfrak{B}^{m-i-1} = 0$; hence $p\mathfrak{B}^{m-i} = 0$, and p is in T_i.

We shall say that a set of extensives in T_i are independent, mod T_{i+1}, when no non-zero linear combination of them is an extensive in T_{i+1}.

Let r_i be the difference of steps of T_{i+1} and T_i; then r_i is the maximum number of extensives in T_i independent, mod T_{i+1}. Also

$$r_0 + r_1 + \ldots + r_{m-1} = l.$$

Take r_0 extensives a_k in T_0, which are independent, mod T_1. Let $a_k\mathfrak{B} = a'_k$. Then $a'_k\mathfrak{B}^{m-1} = a_k\mathfrak{B}^m = 0$. Hence a'_k is in T_1. Now these a'_k are independent, mod T_2; for, if not, then some non-zero linear combination y of them will lie in T_2. But any linear combination y of the a'_k corresponds to a linear combination x of the a_k, such that $x\mathfrak{B} = y$. Then if y lies in T_2, we have $y\mathfrak{B}^{m-2} = 0$; hence $x\mathfrak{B}^{m-1} = 0$, and x is in T_1. But

a non-zero linear combination of the a_k cannot lie in T_1, because these elements of T_0 are independent, mod T_1.

Hence the a'_k are independent, mod T_2. Hence to each set of extensives in T_0 independent, mod T_1, corresponds a set of as many extensives in T_1 independent, mod T_2. Hence $r_0 \leqq r_1$.

The r_0 extensives $a_k\mathfrak{B}$ are in T_1, and are among the set of r_1 extensives of T_1 independent, mod T_2. Adjoin $s_1 = r_1 - r_0$ extensives of T_1 to the $a_k\mathfrak{B}$ so that the set of r_1 extensives is independent, mod T_2. As before, we can shew that these r_1 extensives have transforms by $\mathfrak{B}$ which lie in T_2 and are linearly independent, mod T_3. Hence $r_1 \leqq r_2$.

So proceeding, we find $r_0 + r_1 + r_2 + \ldots + r_{m-1} = l$ extensives $e_1, \ldots, e_l$, such that

the first r_0 are in T_0, and are independent, mod T_1,
the next r_1 are in T_1, and are independent, mod T_2,
.....
the last r_{m-1} are in T_{m-1}, and are independent of one another.

Hence $e_1, \ldots, e_l$ are independent, and hence they span S.

If p_0 be any one of the first r_0 extensives, then $p_0, p_0\mathfrak{B}, p_0\mathfrak{B}^2, \ldots, p_0\mathfrak{B}^{m-1}$ lie respectively in $T_0, T_1, T_2, \ldots, T_{m-1}$. Besides the extensives $p_0\mathfrak{B}$ in T_1, there are s_1 new independent extensives p_1, say; and $p_1, p_1\mathfrak{B}, p_1\mathfrak{B}^2, \ldots, p_1\mathfrak{B}^{m-2}$ lie respectively in $T_1, T_2, \ldots, T_{m-1}$. Besides the extensives $p_0\mathfrak{B}^2, p_1\mathfrak{B}$ in T_2, there are s_2 new independent extensives p_2, say; and $p_2, p_2\mathfrak{B}, p_2\mathfrak{B}^2, \ldots, p_2\mathfrak{B}^{m-3}$ lie respectively in $T_2, T_3, \ldots, T_{m-1}$. If we call any set of extensives of form $p, p\mathfrak{B}, p\mathfrak{B}^2, \ldots$ a '*chain*', we have r_0 chains from T_0, s_1 from T_1, s_2 from T_2, ..., $s_{m-1} = r_{m-1}$ from T_{m-1}; thus the total number of chains is

$$r_0 + s_1 + s_2 + \ldots + s_{m-1} = r_0 + (r_1 - r_0) + (r_2 - r_1) + \ldots + r_{m-1} = r_{m-1}.$$

8. For example, if $l = 10$, and $r_0 = 2$, $r_1 = 3$, $r_2 = 5$, then $m = 3$, $s_1 = 1$, $s_2 = 2$, and the total number of chains is 5. If p_0, p'_0 be the r_0 extensives in T_0; p_1 the extra s_1 extensive in T_1, and p_2, p'_2 the extra s_2 extensives in T_2, then the chains are

$$p_0, p_0\mathfrak{B}, p_0\mathfrak{B}^2; \quad p'_0, p'_0\mathfrak{B}, p'_0\mathfrak{B}^2; \quad p_1, p_1\mathfrak{B}; \quad p_2; \quad p'_2.$$

Take the elements of these chains as the basis of our spread; the 10-rowed matrix of the transformation $\mathfrak{B}$ in the new frame is then the direct sum of five matrices, each of which corresponds to one chain. The matrix corresponding to the first chain is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For, if $p_0 = e_1$, $p_0\mathfrak{B} = e_2$, $p_0\mathfrak{B}^2 = e_3$, then $e_1\mathfrak{B} = e_2$, $e_2\mathfrak{B} = e_3$, $e_3\mathfrak{B} = 0$. The matrix corresponding to the next chain has the same form; that corresponding to the next chain p_1, $p_1\mathfrak{B}$ is obtained by omitting the last row and column from this matrix, and the matrices corresponding to the two last chains each consist of a single zero.

9. Return now to the original matrix $\mathfrak{C}$, which we left in 7. Since $\mathfrak{C} = \mathfrak{B} + \alpha\mathfrak{J}$, we obtain $\mathfrak{C}$ from $\mathfrak{B}$ in our frame, by adding α to each term of the main diagonal. In the example of 8 this gives $\mathfrak{C} = \alpha\mathfrak{J} + \mathfrak{E}_{12} + \mathfrak{E}_{23} + \mathfrak{E}_{45}$, where $\mathfrak{J}$ is of order 10. Similarly, in general, $\mathfrak{C}$ is reduced to the direct sum of matrices of the form $\alpha\mathfrak{J} + \mathfrak{E}_{12} + \mathfrak{E}_{23} + \ldots + \mathfrak{E}_{m-1,m}$, and of diagonal-matrices of the first order, where α is the characteristic root. Such matrices are called '*simple canonical matrices*'. The secular polynomials of these matrices composing the direct sum are the '*elementary divisors*' of $\mathfrak{C} - \lambda\mathfrak{J}$. Thus, in the example of 8, the elementary divisors of the corresponding $\mathfrak{C} - \lambda\mathfrak{J}$ are

$$(\alpha - \lambda)^3, \quad (\alpha - \lambda)^3, \quad (\alpha - \lambda)^2, \quad \alpha - \lambda, \quad \alpha - \lambda.$$

10. By 6 the original matrix was transformed to a direct sum of matrices, each corresponding to a different root of the secular polynomial of the matrix; these single matrices of the direct sum have now been transformed to direct sums of simple canonical matrices. Thus any matrix in the complex field can, by a change of frame, be reduced to the direct sum of simple canonical matrices. We call such a direct sum a '*canonical matrix*'. The method of derivation shews that this reduced form is unique.

If $\mathfrak{A}$ be any matrix of complex numbers, we can find a non-singular matrix $\mathfrak{P}$ such that $\mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1}$ is canonical. (§ 63·8.)

It is convenient to write

$$\mathfrak{U}_r = \mathfrak{E}_{12} + \mathfrak{E}_{23} + \mathfrak{E}_{34} + \ldots + \mathfrak{E}_{r-2,r-1} + \mathfrak{E}_{r-1,r}.$$

Then
$$\mathfrak{U}_r^2 = \mathfrak{E}_{13} + \mathfrak{E}_{24} + \mathfrak{E}_{35} + \ldots + \mathfrak{E}_{r-2,r},$$
$$\mathfrak{U}_r^3 = \mathfrak{E}_{14} + \mathfrak{E}_{25} + \ldots + \mathfrak{E}_{r-3,1},$$
$$\ldots\ldots\ldots\ldots$$
$$\mathfrak{U}_r^{r-1} = \mathfrak{E}_{1r}, \quad \mathfrak{U}_r^r = 0.$$

If we omit the last row and first column from the matrix $\mathfrak{C} = \alpha\mathfrak{J} + \mathfrak{U}_r$, we obtain a matrix of determinant unity. Hence the highest common factor of the first minors of $\mathfrak{C} - \lambda\mathfrak{J}$ is 1. Now the secular determinant of $\mathfrak{C} - \lambda\mathfrak{J}$ is clearly $(\alpha - \lambda)^r$. Hence this is the sole invariant factor of $\mathfrak{C} - \lambda\mathfrak{J}$, apart from unities, and it is the only elementary divisor.

Hence for simple canonical matrices $\mathfrak{C}$, and hence for direct sums of such, *invariant factors and elementary divisors of* $\mathfrak{C} - \lambda\mathfrak{J}$ *determine each other*. And since invariant factors are not altered by a change of frame, this is therefore true for all matrices.

11. The geometrical meaning of a transformation whose matrix is $\alpha\mathfrak{J} + \mathfrak{U}_r$ should be noted. If we take a spread of step three, the extensives e_1, e_2, e_3, which we will interpret as vectors, become respectively $\alpha e_1 + e_2$, $\alpha e_2 + e_3$, αe_3. The only vector whose direction is conserved is e_3. We can perform the transformation in two stages, if $\alpha \neq 0$, first transforming e_1, e_2, e_3 to $e_1 + \frac{1}{\alpha} e_2$, $e_2 + \frac{1}{\alpha} e_3$, e_3, and then multiplying all vectors by α. The first transformation is a '*shear*'. If $\alpha = 0$, the transformation is of course singular. Omitting this case, we see that $\alpha\mathfrak{J} + \mathfrak{U}_r$ represents a shear in a spread whose step is the order of the matrix, together with a transformation multiplying all vectors by α. It must not be forgotten that α may be complex.

12. *A necessary and sufficient condition that two matrices of (real or) complex numbers be similar, is that they have the same canonical form.*

Necessity. Two similar matrices (§ 63·3) represent the same transformation in different frames, by § 63·8, and the canonical form of the matrix of a transformation is unique, apart, of course, from the order of the direct sums.

Sufficiency. If matrices $\mathfrak{A}$, $\mathfrak{B}$ have the same canonical form $\mathfrak{C}$, there are non-singular matrices $\mathfrak{P}$, $\mathfrak{Q}$ such that $\mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1} = \mathfrak{C}$, $\mathfrak{Q}\mathfrak{B}\mathfrak{Q}^{-1} = \mathfrak{C}$. Hence

$$\mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1} = \mathfrak{Q}\mathfrak{B}\mathfrak{Q}^{-1}, \quad \mathfrak{A} = \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{B}\mathfrak{Q}\ {}^{1}\mathfrak{P} = \mathfrak{R}\mathfrak{B}\mathfrak{R}^{-1},$$

where $$\mathfrak{R} = \mathfrak{P}^{-1}\mathfrak{Q}.$$

13. *A necessary and sufficient condition that two matrices* $\mathfrak{A}$, $\mathfrak{B}$ *of (real or) complex numbers be similar is that* $\mathfrak{A} - \lambda\mathfrak{J}$, $\mathfrak{B} - \lambda\mathfrak{J}$ *have the same invariant factors.*

For, if $\mathfrak{A}$, $\mathfrak{B}$ be similar, there is a non-singular matrix $\mathfrak{P}$ such that $\mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1} = \mathfrak{B}$; hence

$$\mathfrak{P}(\mathfrak{A} - \lambda\mathfrak{J})\mathfrak{P}^{-1} = \mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1} - \lambda\mathfrak{J} = \mathfrak{B} - \lambda\mathfrak{J};$$

hence (§ 89) $\mathfrak{A} - \lambda\mathfrak{J}$, $\mathfrak{B} - \lambda\mathfrak{J}$ have the same invariant factors.

And if $\mathfrak{A} - \lambda\mathfrak{J}$, $\mathfrak{B} - \lambda\mathfrak{J}$ have the same invariant factors, the canonical forms of $\mathfrak{A}$, $\mathfrak{B}$ are the same, and hence $\mathfrak{A}$, $\mathfrak{B}$ are similar.

14. If $\mathfrak{B}$ be non-singular, we define the '*elementary divisors*' of $\mathfrak{A} - \lambda\mathfrak{B}$ to be those of $\mathfrak{A}\mathfrak{B}^{-1} - \lambda\mathfrak{J}$.

15. A '*multiplication*' in a spread is a transformation whose matrix is similar to a diagonal matrix.

The transformation $\mathfrak{A}$ in a spread of step n is a multiplication if, and only if, it has just n latent extensives of step one.

§ 94. *Geometric applications.*

1. If $\mathfrak{A}$ is a collineation in a spread of step n, and p an extensive of step one in that spread, and $p\mathfrak{A}$ lies in a 'prime' π (that is, in a spread of step $n-1$), then $[\pi . p\mathfrak{A}] = 0$, and we say p, π are '*conjugate*' for $\mathfrak{A}$.

Since $[\pi . p\mathfrak{A}] = 0$, we have $[\pi . \mathfrak{A}^* p] = 0$, $[\pi\mathfrak{A}^* . p] = 0$, (§§ 62·9, 63·4).

Now $\mathfrak{A}$, $\mathfrak{A}^*$ have the same secular equation. Interpreting the results of § 93·6 for the transformation $\mathfrak{A}^*$ on primes, we have:

To each distinct characteristic root α_i of $\mathfrak{A}$ corresponds a '*star*' of latent primes, whose step equals the multiplicity l_i of the characteristic root α_i, that is, a set of primes which have a common spread of step $n - l_i$ and none of higher step. This common spread is called the '*support*' of the star. The support is of course latent for $\mathfrak{A}$.

If p is an extensive of step one, not in a latent spread corresponding to α_i, so that $p\mathfrak{A}-\alpha_i p\neq 0$, then $p\mathfrak{A}-\alpha_i p$ lies on the support of the star corresponding to α_i. For it lies on any prime π which satisfies $\pi\mathfrak{A}^* = \alpha_i\pi$, for

$$[\pi(p\mathfrak{A}-\alpha_i p)] = [\pi.p\mathfrak{A}]-\alpha_i[\pi p]$$
$$= [\pi\mathfrak{A}^*.p]-\alpha_i[\pi p] = [(\pi\mathfrak{A}^*-\alpha_i\pi)p] = 0.$$

The ranges formed by p, $p\mathfrak{A}$, and the cuts of $[p.p\mathfrak{A}]$ with the supports of the latent primes are projective to one another as p varies, and also projective to the pencils formed by π, $\pi\mathfrak{A}^*$, and the joins of $[\pi.\pi\mathfrak{A}^*]$ to the latent spaces.

For the range of points mentioned is

$$p,\ p\mathfrak{A},\ p\mathfrak{A}-\alpha_1 p,\ p\mathfrak{A}-\alpha_2 p,\ \ldots,$$

and the pencil of primes is

$$\pi,\ \pi\mathfrak{A}^*,\ \pi\mathfrak{A}^*-\alpha_1\pi,\ \pi\mathfrak{A}^*-\alpha_2\pi,\ \ldots.$$

This generalises v. Staudt's theorem (§ 23·11), since any n independent points in a spread of step n can be taken as the latent points of a collineation. The primes through $n-1$ of the points are the latent primes.

We have, of course, corresponding theorems for correlations.

2. *Involutions.* A collineation $\mathfrak{A}$, which satisfies $\mathfrak{A}^2 = \mathfrak{J}$, is an '*involution*'. Since a matrix of form $\mathfrak{J}_n + \mathfrak{U}_n$ never has $\mathfrak{J}$ for its square, therefore, if we reduce $\mathfrak{A}$ to its canonical form, this form will be $\operatorname{diag}(k_1, k_2, \ldots, k_n)$.

Then $\mathfrak{A}^2 = \operatorname{diag}(k_1^2, k_2^2, \ldots, k_n^2) = \mathfrak{J}$ gives each k as ± 1.

$$\mathfrak{A} = \operatorname{diag}(\pm 1, \pm 1, \ldots, \pm 1).$$

3. For example, in space of step four, we have, besides identity, the involutions

$$\mathfrak{J}_1 = \operatorname{diag}(-1, 1, 1, 1), \quad \mathfrak{J}_2 = \operatorname{diag}(1, -1, 1, 1),$$
$$\mathfrak{J}_{12} = \operatorname{diag}(-1, -1, 1, 1),$$

and so on, eight in all, including $\mathfrak{J}$.

If e_1, e_2, e_3, e_4 be the vertices of the tetrahedron of reference, then if b is in $e_2e_3e_4$, the point $ke_1 + b$ is transformed by $\mathfrak{J}_1$ to the point $-ke_1 + b$. Hence $\mathfrak{J}_1$ transforms any point on the join $[e_1 b]$ into its harmonic conjugate with respect to e_1, b.

This collineation $\mathcal{J}_1$ is a '*central involution*'. It leaves e_1 and all points of $[e_2e_3e_4]$ latent; it leaves all planes through e_1 latent as a whole. There are no other latent points or planes.

Next consider $\mathcal{J}_{12}$. If p is on e_1e_2, and q on e_3e_4, then $\mathcal{J}_{12}$ changes $p+kq$ into $-p+kq$; hence it transforms any point on the join of points of e_1e_2, e_3e_4 into its harmonic conjugate with respect to these latter points.

This collineation is a '*biaxial involution*'. It leaves all points of $[e_1e_2]$ and of $[e_3e_4]$ latent. The other edges of the tetrahedron are latent as a whole; the points on them, other than e_1, e_2, e_3, e_4, are not latent. The lines $[e_1e_2]$, $[e_3e_4]$ are the '*axes*' of the involution.

Since $\mathcal{J}_r\mathcal{J}_s = \mathcal{J}_{rs}$ $(r, s = 1, \ldots, 4)$, as is clear from the diagonal forms, the eight involutions (four central, three biaxial, and identity) can be expressed in terms of $\mathcal{J}_1, \ldots, \mathcal{J}_4$. The relations between them, such as $\mathcal{J}_{12}\mathcal{J}_{23} = \mathcal{J}_{13}$, $\mathcal{J}_{12}\mathcal{J}_{34} = -\mathfrak{J}$, all follow from

$$\mathcal{J}_r^2 = \mathfrak{J}, \quad \mathcal{J}_r\mathcal{J}_s = \mathcal{J}_s\mathcal{J}_r, \quad \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3\mathcal{J}_4 = -\mathfrak{J}.$$

The minus sign in the last formula has no relevance when we consider only the transformations of *positions* of points.

4. If $\mathfrak{A}^2 = \mathfrak{J}$, $\mathfrak{B}^2 = \mathfrak{J}$, $\mathfrak{AB} = \mathfrak{BA} = \mathfrak{C}$,

then $\mathfrak{C}^2 = \mathfrak{J}$, $\mathfrak{AC} = \mathfrak{CA} = \mathfrak{B}$, $\mathfrak{BC} = \mathfrak{CB} = \mathfrak{A}$, $\mathfrak{ABC} = \mathfrak{J}$,

and we have three mutually commutative involutions.

For $\mathfrak{C}^2 = \mathfrak{AB}.\mathfrak{BA} = \mathfrak{A}^2 = \mathfrak{J}$, $\mathfrak{AC} = \mathfrak{A}^2\mathfrak{B} = \mathfrak{B}$,

and so on.

If $\mathfrak{A}^2 = \mathfrak{B}^2 = \mathfrak{C}^2 = \mathfrak{J}$, $\mathfrak{AB} = \mathfrak{C}$, then $\mathfrak{BA} = \mathfrak{C}$.

For since $\mathfrak{AB} = \mathfrak{C}$, we have

$$\mathfrak{A} = \mathfrak{AB}^2 = \mathfrak{CB}, \quad \mathfrak{J} = \mathfrak{AA} = \mathfrak{CBA}, \quad \mathfrak{C} = \mathfrak{C}^2\mathfrak{BA} = \mathfrak{BA}.$$

5. If two distinct central involutions $\mathfrak{A}$, $\mathfrak{B}$ are commutative, the centre of each is on the latent plane of the other.

For take the frame $e_1, \ldots, e_4$ so that $\mathfrak{A}$ is in canonical form $\mathrm{diag}(-1, 1, 1, 1)$. Then the equation $\mathfrak{AB} = \mathfrak{BA}$ shews that the first line and first column of $\mathfrak{B}$ contain zero only, except for the element in the $(1, 1)$ place. Hence $\mathfrak{B}$ leaves e_1 and $[e_2e_3e_4]$ latent.

Hence the centre of $\mathfrak{B}$ is either at e_1, the centre of $\mathfrak{A}$, and then $\mathfrak{A}$, $\mathfrak{B}$ are identical, or it lies in the latent plane of $\mathfrak{A}$. In the last case, the centre of $\mathfrak{A}$ lies in the latent plane of $\mathfrak{B}$.

6. If two biaxial involutions $\mathfrak{K}_1$, $\mathfrak{K}_2$ with axes (A, B) and (C, D) respectively are commutative, then *either* the pair A, B meet the pair C, D, *or* these lie on the same regulus and the pairs separate one another harmonically (§ 31·1).

For $A\mathfrak{K}_2 = A\mathfrak{K}_1\mathfrak{K}_2 = A\mathfrak{K}_2\mathfrak{K}_1$. Hence $A\mathfrak{K}_2$ is latent for $\mathfrak{K}_1$; hence *either* $A\mathfrak{K}_2 = B$ and then $B\mathfrak{K}_2 = A$, *or* $A\mathfrak{K}_2 = A$ and then, similarly, $B\mathfrak{K}_2 = B$.

In the latter case, the pair A, B cut the pair C, D, and our biaxial involutions are derived from the tetrahedron which has A, B and C, D as pairs of opposite edges.

In the first case, let p be any point on A, then $p\mathfrak{K}_2$ is on B. Thus $[p.p\mathfrak{K}_2]$ cuts A, B, and by the definition of $\mathfrak{K}_2$ it cuts C, D. Hence A, B, C, D are dependent and cut any transversal harmonically.

Then $\mathfrak{K}_1\mathfrak{K}_2$ is an involution whose axes lie on the regulus through A, B, C, D and separate both A, B and C, D harmonically.

If A_1, B_1, C_1, D_1 be lines on the opposite regulus separating one another harmonically, then (A_1, B_1) and (C_1, D_1) are axes of involutions $\mathfrak{K}_3$, $\mathfrak{K}_4$, say, and $\mathfrak{K}_3\mathfrak{K}_4$ is an involution; $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_1\mathfrak{K}_2$ commute with $\mathfrak{K}_3$, $\mathfrak{K}_4$, $\mathfrak{K}_3\mathfrak{K}_4$. In all, multiplying our involutions together, we get, including $\mathfrak{I}$, sixteen commutative involutions which leave the quadric of the regulus invariant.

If $\mathfrak{P}$ is the polarity given by that quadric, its products with the involutions give six nul systems and ten polarities (including $\mathfrak{P}$), in all thirty-two commutative transformations which leave the quadric latent.

§ 95. *Equivalent pairs of matrices.*

Def. If $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ be matrices whose elements are complex numbers, the pair $\mathfrak{A}_1$, $\mathfrak{A}_2$ is '*equivalent*' to the pair $\mathfrak{B}_1$, $\mathfrak{B}_2$, if there are matrices $\mathfrak{P}$, $\mathfrak{Q}$ whose elements are complex numbers such that

$$\mathfrak{B}_1 = \mathfrak{P}\mathfrak{A}_1\mathfrak{Q}, \quad \mathfrak{B}_2 = \mathfrak{P}\mathfrak{A}_2\mathfrak{Q}.$$

Theorem. If $\mathfrak{A}_2$, $\mathfrak{B}_2$ *be non-singular, a necessary and sufficient condition that* $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be equivalent to* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *is that* $\mathfrak{A}_1 - \lambda\mathfrak{A}_2$, $\mathfrak{B}_1 - \lambda\mathfrak{B}_2$ *are equivalent as* λ*-matrices in the sense of* § 89.

For let $\mathfrak{A} = \mathfrak{A}_1 - \lambda\mathfrak{A}_2$, $\mathfrak{B} = \mathfrak{B}_1 - \lambda\mathfrak{B}_2$. Then, if the pair $\mathfrak{A}_1$, $\mathfrak{A}_2$ be equivalent to $\mathfrak{B}_1$, $\mathfrak{B}_2$, the $\mathfrak{P}$, $\mathfrak{Q}$ of the definition may be regarded as simple cases of λ-matrices with determinants in-

dependent of λ, and we have $\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$. Hence the condition is necessary.

To shew sufficiency, let $\mathfrak{A}$, $\mathfrak{B}$ be equivalent as λ-matrices, then they have the same invariant factors and there are λ-matrices $\mathfrak{L}$, $\mathfrak{M}$ with determinants independent of λ, such that $\mathfrak{B} = \mathfrak{L}\mathfrak{A}\mathfrak{M}$. We have to shew that there are matrices $\mathfrak{P}$, $\mathfrak{Q}$ whose *elements* are independent of λ, such that $\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, for this gives

$$\mathfrak{B}_1 = \mathfrak{P}\mathfrak{A}_1\mathfrak{Q}, \quad \mathfrak{B}_2 = \mathfrak{P}\mathfrak{A}_2\mathfrak{Q}.$$

Let $$\mathfrak{A}_1\mathfrak{A}_2^{-1} = \mathfrak{C}, \quad \mathfrak{B}_1\mathfrak{B}_2^{-1} = \mathfrak{D}.$$

Then the equivalent matrices $\mathfrak{C} - \lambda\mathfrak{J}$ and

$$(\mathfrak{C} - \lambda\mathfrak{J})\mathfrak{A}_2 = \mathfrak{A}_1 - \lambda\mathfrak{A}_2 = \mathfrak{A}$$

have the same invariant factors, and so have the equivalent matrices $\mathfrak{D} - \lambda\mathfrak{J}$ and $(\mathfrak{D} - \lambda\mathfrak{J})\mathfrak{B}_2 = \mathfrak{B}_1 - \lambda\mathfrak{B}_2 = \mathfrak{B}$. But $\mathfrak{A}$, $\mathfrak{B}$ have the same invariant factors, therefore so have $\mathfrak{C} - \lambda\mathfrak{J}$ and $\mathfrak{D} - \lambda\mathfrak{J}$.

Hence $\mathfrak{C}$, $\mathfrak{D}$ are similar (§ 93·13), and there is a non-singular matrix $\mathfrak{P}$, whose elements are scalars, such that $\mathfrak{D} = \mathfrak{P}\mathfrak{C}\mathfrak{P}^{-1}$. Let $\mathfrak{Q} = \mathfrak{A}_2^{-1}\mathfrak{P}^{-1}\mathfrak{B}_2$. Then

$$\mathfrak{P}\mathfrak{A}_1\mathfrak{Q} = \mathfrak{P}\mathfrak{A}_1\mathfrak{A}_2^{-1}\mathfrak{P}^{-1}\mathfrak{B}_2 = \mathfrak{P}\mathfrak{C}\mathfrak{P}^{-1}\mathfrak{B}_2 = \mathfrak{D}\mathfrak{B}_2 = \mathfrak{B}_1,$$

$$\mathfrak{P}\mathfrak{A}_2\mathfrak{Q} = \mathfrak{P}\mathfrak{A}_2\mathfrak{A}_2^{-1}\mathfrak{P}^{-1}\mathfrak{B}_2 = \mathfrak{B}_2.$$

Note. The investigation of § 93, on which the present depends, assumed essentially the fundamental theorem of algebra. This was not assumed in §§ 88, 89. It is possible to deduce the results of the present section from those in §§ 88, 89, without use of the theory of similar matrices. The theorem is then shewn for matrices of real numbers also.

§ 96. *The square root of a matrix.*

1. *Lemma.* If f(x) is a polynomial of degree $n > 0$ which does not vanish when $x = 0$, then there is a polynomial g(x) of degree less than n, such that $(g(x))^2 - x$ is divisible by f(x).

For let $$f(x) = (x - a_1)^{r_1}(x - a_2)^{r_2} \dots (x - a_s)^{r_s},$$

where $a_1, a_2, \dots, a_s$ are distinct, and $r_1 + r_2 + \dots + r_s = n$.

Let $$f_i(x) = f(x).(x - a_i)^{-r_i}.$$

Let $g_i(x)$ be any polynomial of degree less than r_i.

Then $g_1(x)f_1(x)+g_2(x)f_2(x)+\ldots+g_s(x)f_s(x)=g(x)$, say,

is a polynomial of degree less than n, and the terms on the left-hand side, after the first, are all divisible by $(x-a_1)^{r_1}$.

Hence $(g(x))^2-x$ will be divisible by $(x-a_1)^{r_1}$ if, and only if, $(g_1(x)f_1(x))^2-x$ is so divisible.

Thus if g(x) is to satisfy our conditions, $(g_i(x)f_i(x))^2-x$ must be divisible by $(x-a_i)^{r_i}$. It remains to choose $g_1(x), g_2(x), \ldots, g_s(x)$.

Now $\sqrt{x} = \sqrt{a}[1+a^{-1}(x-a)]^{\frac{1}{2}}$

$= c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_{r-1}(x-a)^{r-1}+\ldots,$

where the c depend on a only. Let h(x) be the sum of the first r terms of this series. Then h(x) is a polynomial in x of degree less than r, and $x-(h(x))^2$ is divisible by $(x-a)^r$.

Let $h_i(x)$ be derived in this way from $x-a_i$.

Let $$\frac{h_i(x)}{f(x)} = \frac{k_i(x)}{(x-a_i)^{r_i}} + \frac{(x-a_i)^{r_i}l(x)}{f(x)}.$$

This is obtained by the usual partial fraction decomposition; the degree of k_i is less than r_i. Then $h_i(x)-f_i(x)k_i(x)$ is divisible by $(x-a_i)^{r_i}$.

Hence so is $$(h_i(x))^2-(f_i(x)k_i(x))^2,$$

and also $$x-(f_i(x)k_i(x))^2.$$

Take $k_i(x)$ for $g_i(x)$ above, then g(x) satisfies the required conditions.

2. *If* $\mathfrak{A}$ *is a non-singular matrix of order* n, *there is a matrix* $\mathfrak{B}$, *which is a polynomial in* $\mathfrak{A}$, *of order less than* n, *such that* $\mathfrak{B}^2=\mathfrak{A}$.

$\mathfrak{B}$ of course will be non-singular, but its elements may be complex numbers, even if those of $\mathfrak{A}$ are real.

For, let $f(\lambda)$ be the characteristic polynomial of $\mathfrak{A}$; then, by the lemma, there is a polynomial $g(\lambda)$, of order less than n, such that $(g(\lambda))^2-\lambda$ is divisible by $f(\lambda)$. For, since $\mathfrak{A}$ is non-singular, $f(\lambda)$ does not vanish when $\lambda=0$.

Now $f(\mathfrak{A})=0$, hence $(g(\mathfrak{A}))^2-\mathfrak{A}=0$. Hence $\mathfrak{B}, = g(\mathfrak{A})$, satisfies the conditions. We call $\mathfrak{B}$ the '*square root*' of $\mathfrak{A}$.

Cor. If $\mathfrak{B}, \mathfrak{B}_1$ be the square roots of $\mathfrak{A}, \mathfrak{A}_1$, and $\mathfrak{A}\mathfrak{A}_1=\mathfrak{A}_1\mathfrak{A}$, then $\mathfrak{B}\mathfrak{B}_1=\mathfrak{B}_1\mathfrak{B}$. For $\mathfrak{B}$ and $\mathfrak{B}_1$ are polynomials in $\mathfrak{A}, \mathfrak{A}_1$.

§97. *Correlations, polarities, congruent matrices.*

1. Let $\mathfrak{A}$ be a non-singular correlation, that is, a linear transformation which turns points into primes; and let $\mathfrak{A}$ be represented by the matrix (a_{ij}) in the frame $(e_1, \ldots, e_n)$. Let

$$|e_i = E_i \quad (i = 1, \ldots, n).$$

Let
$$p = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n,$$
$$q = y_1 e_1 + y_2 e_2 + \ldots + y_n e_n,$$

then
$$p\mathfrak{A} = \sum_{ij} x_i a_{ij} E_j, \quad [p\mathfrak{A}q] = \sum_{ij} x_i a_{ij} y_j [E_r e_r].$$

Hence $[p\mathfrak{A}q]$ is a bilinear form in p, q, or perhaps we should say, in $x_1, \ldots, x_n; y_1, \ldots, y_n$.

Also
$$[p\mathfrak{A}p] = [p\mathfrak{A}^* p];$$
$$[p\mathfrak{A}q] = \tfrac{1}{2}[p(\mathfrak{A} + \mathfrak{A}^*)q] + \tfrac{1}{2}[p(\mathfrak{A} - \mathfrak{A}^*)q].$$

We can regard $[p\mathfrak{A}q]$ either as the product of $p\mathfrak{A}$ and q, or as the product of p and $\mathfrak{A}q$. $[p\mathfrak{A}p]$ is a quadratic form, and there is no loss, if we assume there that $\mathfrak{A} = \mathfrak{A}^*$.

2. If $p\mathfrak{A} = q\mathfrak{A}^*$, then p, q describe projective figures. For $p = q\mathfrak{P}$, where $\mathfrak{P} = \mathfrak{A}^*\mathfrak{A}^{-1}$.

The '*involutory*' points of $\mathfrak{A}$ are the points p such that $p\mathfrak{A} \equiv p\mathfrak{A}^*$; $\mathfrak{A}$ itself is '*involutory*' if all points are involutory points of $\mathfrak{A}$.

In this case, $\mathfrak{A} = k\mathfrak{A}^*$ for some scalar k; hence

$$\mathfrak{A}^* = k\mathfrak{A}, \quad \mathfrak{A} = k^2\mathfrak{A}, \quad k = \pm 1, \quad \mathfrak{A} = \pm\mathfrak{A}^*.$$

If $\mathfrak{A} = \mathfrak{A}^*$, we have a '*polarity*'; if $\mathfrak{A} = -\mathfrak{A}^*$, a '*nul system*'.

3. If $\mathfrak{A}$ be a nul system, and $\pi = p\mathfrak{A}$, then

$$[\pi p] = [p\mathfrak{A}p] = [p\mathfrak{A}^* p] = -[p\mathfrak{A}p] = -[\pi p].$$

Hence $[\pi p] = 0$, *each point is on its corresponding prime.*

Conversely, if each point p is on its corresponding prime $p\mathfrak{A}$, then $[p\mathfrak{A}p] = 0$ for all p.

Hence $[p\mathfrak{A}^* p] = 0$, $[p(\mathfrak{A} + \mathfrak{A}^*)p] = 0$ for all p. But $\mathfrak{A} + \mathfrak{A}^*$ is symmetrical, therefore the last equation gives $\mathfrak{A} + \mathfrak{A}^* = \mathfrak{O}$ (§63·11).

If $\mathfrak{A}$ be non-singular, and $\mathfrak{A} = -\mathfrak{A}^*$, then $\mathfrak{A}$ is of even order.

For
$$[\mathfrak{A}^n] = (-1)^n [\mathfrak{A}^{*n}] = (-1)^n [\mathfrak{A}^n].$$

The nul planes of a screw and their nul points (§34) are connected by a correlation which is a nul system.

4. If $\mathfrak{A}$ is not a nul system, then the latent points of $\mathfrak{P} = \mathfrak{A}^*\mathfrak{A}^{-1}$ coincide with the involutory points of $\mathfrak{A}$, and satisfy $p\mathfrak{A} = kp\mathfrak{A}^*$ for some k.

Hence $(1-k)[p\mathfrak{A}p] = 0$; hence if $k \neq 1$, these points lie on the quadric $[p\mathfrak{A}p] = 0$.

Thus *any latent spread of* $\mathfrak{P}$, *not corresponding to latent root* 1, *lies on this quadric.*

5. If p, q be transformed by a collineation $\mathfrak{R}$, then $[p\mathfrak{A}q]$ becomes

$$[p\mathfrak{R}.\mathfrak{A}.q\mathfrak{R}] = [p\mathfrak{R}\mathfrak{A}\mathfrak{R}^*p].$$

Thus our correlation $\mathfrak{A}$ is replaced by $\mathfrak{R}\mathfrak{A}\mathfrak{R}^*$ (our frame has not been changed).

The matrices representing $\mathfrak{A}$ and $\mathfrak{R}\mathfrak{A}\mathfrak{R}^*$ are called '*congruent*'†. The relation of congruence is transitive. Thus the matrix which represents a bilinear form is changed into a congruent matrix by a collineation, or linear transformation in the spread, whereas a matrix which represents a linear transformation is changed into a similar matrix (§ 63·8).

6. *If* $\mathfrak{A}$, $\mathfrak{B}$ *be equivalent and symmetric matrices whose elements are complex numbers, then* $\mathfrak{A}$, $\mathfrak{B}$ *are congruent. In fact, if* $\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, *then there is a non-singular matrix* $\mathfrak{R}$ *which depends on* $\mathfrak{P}$, $\mathfrak{Q}$ *only, such that* $\mathfrak{B} = \mathfrak{R}\mathfrak{A}\mathfrak{R}^*$.

For $$\mathfrak{P}\mathfrak{A}\mathfrak{Q} = (\mathfrak{P}\mathfrak{A}\mathfrak{Q})^* = \mathfrak{Q}^*\mathfrak{A}^*\mathfrak{P}^* = \mathfrak{Q}^*\mathfrak{A}\mathfrak{P}^*,$$

$$(\mathfrak{Q}^{*-1}\mathfrak{P})\mathfrak{A} = \mathfrak{A}(\mathfrak{P}^*\mathfrak{Q}^{-1}).$$

Let $\mathfrak{Q}^{*-1}\mathfrak{P} = \mathfrak{C}$, then

$$\mathfrak{P}^*\mathfrak{Q}^{-1} = \mathfrak{C}^*, \quad \mathfrak{C}\mathfrak{A} = \mathfrak{A}\mathfrak{C}^*.$$

Hence $$\mathfrak{C}^2\mathfrak{A} = \mathfrak{C}\mathfrak{A}\mathfrak{C}^* = \mathfrak{A}\mathfrak{C}^*\mathfrak{C}^* = \mathfrak{A}\mathfrak{C}^{*2}.$$

Similarly, $$\mathfrak{C}^3\mathfrak{A} = \mathfrak{A}\mathfrak{C}^{*3}, \ldots, \mathfrak{C}^n\mathfrak{A} = \mathfrak{A}\mathfrak{C}^{*n}.$$

Hence, if $f(x)$ be any polynomial, then $f(\mathfrak{C})\mathfrak{A} = \mathfrak{A}f(\mathfrak{C}^*)$. Choose for $f(x)$ that polynomial such that $f(\mathfrak{C}) = \mathfrak{D}$, $\mathfrak{D}^2 = \mathfrak{C}$, $\mathfrak{D}$ non-singular (§ 96).

Then $$\mathfrak{D}^* = f(\mathfrak{C}^*), \quad \mathfrak{D}\mathfrak{A} = \mathfrak{A}\mathfrak{D}^*, \quad \mathfrak{A} = \mathfrak{D}^{-1}\mathfrak{A}\mathfrak{D}^*,$$

$$\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{P}\mathfrak{D}^{-1}\mathfrak{A}\mathfrak{D}^*\mathfrak{Q}.$$

† This is the usual term. It has of course no connection with the use in this book of the term congruence applied to extensives.

Now $\mathfrak{Q}^{*-1}\mathfrak{P} = \mathfrak{C} = \mathfrak{D}^2$, hence $\mathfrak{Q}^*\mathfrak{D} = \mathfrak{P}\mathfrak{D}^{-1} = \mathfrak{R}$, say; $\mathfrak{D}^*\mathfrak{Q} = \mathfrak{R}^*$.

Hence $$\mathfrak{B} = \mathfrak{R}\mathfrak{A}\mathfrak{R}^*.$$

Also $\mathfrak{R}$ depends on $\mathfrak{P}$, $\mathfrak{Q}$ only, since $\mathfrak{R} = \mathfrak{Q}^*\mathfrak{D}$, $\mathfrak{D}^2 = \mathfrak{Q}^{*-1}\mathfrak{P}$.

7. Similarly, *if* $\mathfrak{A}$, $\mathfrak{B}$ *be equivalent and skew-symmetric matrices whose elements are complex numbers, and* $\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, *then there is a non-singular matrix* $\mathfrak{R}$ *depending on* $\mathfrak{P}$, $\mathfrak{Q}$ *only, such that* $\mathfrak{B} = \mathfrak{R}\mathfrak{A}\mathfrak{R}^*$.

8. *If* $\mathfrak{A}$, $\mathfrak{B}$ *be both symmetric, or both skew-symmetric, and* $\mathfrak{A}_1$, $\mathfrak{B}_1$ *be both symmetric, or both skew-symmetric* (*the elements being complex numbers*), *and the pair* $\mathfrak{A}$, $\mathfrak{A}_1$ *be equivalent to the pair* $\mathfrak{B}$, $\mathfrak{B}_1$, *then there is a non-singular matrix* $\mathfrak{R}$ *such that*

$$\mathfrak{R}\mathfrak{A}\mathfrak{R}^* = \mathfrak{B}, \quad \mathfrak{R}\mathfrak{A}_1\mathfrak{R}^* = \mathfrak{B}_1.$$

For, by §95, there are matrices, non-singular, $\mathfrak{P}$, $\mathfrak{Q}$ such that $\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{B}$, $\mathfrak{P}\mathfrak{A}_1\mathfrak{Q} = \mathfrak{B}_1$ and the $\mathfrak{R}$ we need depends on $\mathfrak{P}$, $\mathfrak{Q}$ only, and in the same way in the two cases.

If, further, $\mathfrak{A}_1$, $\mathfrak{B}_1$ *be non-singular, then the pair* $\mathfrak{A}$, $\mathfrak{A}_1$ *is congruent to the pair* $\mathfrak{B}$, $\mathfrak{B}_1$ *if, and only if,* $\mathfrak{A} - \lambda\mathfrak{A}_1$, $\mathfrak{B} - \lambda\mathfrak{B}_1$ *have the same invariant factors* (7 and §95).

9. *If* $\mathfrak{A}$, $\mathfrak{B}$ *be symmetric or skew-symmetric matrices of complex numbers, then there is an orthogonal matrix* $\mathfrak{R}$ *such that* $\mathfrak{R}\mathfrak{A}\mathfrak{R}^* = \mathfrak{B}$, *if, and only if,* $\mathfrak{A} - \lambda\mathfrak{J}$, $\mathfrak{B} - \lambda\mathfrak{J}$ *have the same invariant factors, that is, if and only if,* $\mathfrak{A}$, $\mathfrak{B}$ *be similar.*

For, in 8, take $\mathfrak{A}_1 = \mathfrak{B}_1 = \mathfrak{J}$, then $\mathfrak{R}\mathfrak{R}^* = \mathfrak{J}$.

10. *If* $\mathfrak{A}$ *be a symmetric matrix of complex numbers, there is an orthogonal matrix* $\mathfrak{R}$ *such that*

$$\mathfrak{R}\mathfrak{A}\mathfrak{R}^* = a_1\mathfrak{E}_{11} + a_2\mathfrak{E}_{22} + \ldots + a_r\mathfrak{E}_{rr} = \operatorname{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0),$$

for some r, a_1, a_2, ..., a_r. *Here* r *is the rank of* $\mathfrak{A}$, *and* a_1, ..., a_r *its non-zero characteristic roots.*

For the canonical form of $\mathfrak{A}$ is similar to $\mathfrak{A}$, and hence is symmetric, and hence is a diagonal matrix, since $\alpha\mathfrak{J} + \mathfrak{U}$ is not symmetric. Let the canonical form be $\mathfrak{B} = \operatorname{diag}(a_1, a_2, \ldots, a_r, \ldots)$. The theorem now follows by 9.

11. *If $\mathfrak{A}$ be a symmetric matrix of real elements, there is an orthogonal matrix $\mathfrak{R}$ of real elements such that $\mathfrak{RAR}^*$ is diagonal.*

For $\mathfrak{A}$ has its characteristic roots all real (§63·5), and since $\mathfrak{R}^* = \mathfrak{R}^{-1}$ we have with the notation of 10, $\mathfrak{RAR}^{-1} = \mathfrak{B}$. Thus if we consider $\mathfrak{A}$ as a matrix of a linear transformation, it can be changed to $\mathfrak{B}$ by a change of frame, and since $\mathfrak{A}$, $\mathfrak{B}$ have real elements, the coefficients of the transformation which gives the change of frame are real; thus $\mathfrak{R}$ has real elements.

12. *The matrix of a quadric in vector space can, by an orthogonal transformation, be brought to the diagonal form. Two quadrics can be transformed into one another by an orthogonal transformation if, and only if, their secular matrices have the same invariant factors. If the quadric is real, the orthogonal transformation is real.*

This is only a translation of 9, 10, 11. It is a refinement of §91, which merely asserted that there was a diagonal matrix congruent to a given symmetric matrix.

13. *Simultaneous reduction of two quadratic forms to sums of squares in the complex field.*

If $[x\mathfrak{A}x]$, $[x\mathfrak{B}x]$ be the quadratic forms, $\mathfrak{B}$ non-singular, a necessary and sufficient condition that we can reduce both to sums of square terms (so that $\mathfrak{A}$, $\mathfrak{B}$ are replaced by diagonal matrices) by a collineation, or change of frame, is that all the elementary divisors of $\mathfrak{A} - \lambda\mathfrak{B}$ are of the first degree.

Necessity. If, by a collineation, $\mathfrak{A}$, $\mathfrak{B}$ become respectively

$$\mathfrak{C} = \operatorname{diag}(c_1, c_2, \ldots, c_n), \quad \mathfrak{D} = \operatorname{diag}(d_1, d_2, \ldots, d_n),$$

then $d_1, d_2, \ldots, d_n \neq 0$, since $\mathfrak{B}$ is non-singular, and the elementary divisors of $\mathfrak{C} - \lambda\mathfrak{D}$, which are $\lambda - c_1 d_1^{-1}, \ldots, \lambda - c_n d_n^{-1}$ are not affected by a collineation.

Sufficiency. By a collineation we can change $\mathfrak{A}$ into a diagonal matrix $\mathfrak{C} = \operatorname{diag}(c_1, c_2, \ldots, c_n)$. Then if $\lambda - k_1, \ldots, \lambda - k_n$ are the elementary divisors of $\mathfrak{A} - \lambda\mathfrak{B}$, they are also those of $\mathfrak{C} - \lambda\mathfrak{D}$, where $\mathfrak{D} = \operatorname{diag}(c_1 k_1^{-1}, \ldots, c_n k_n^{-1})$. Hence the pairs $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, $\mathfrak{D}$ are equivalent, (§95), and hence, by a collineation we can change $\mathfrak{A}$, $\mathfrak{B}$ into $\mathfrak{C}$, $\mathfrak{D}$, for by 8 the pairs are congruent.

Cor. The forms $[x\mathfrak{A}x]$, $[x\mathfrak{B}x]$ can be simultaneously reduced to sums of square terms if, and only if, $\mathfrak{AB} = \mathfrak{BA}$.

14. For symmetric, or skew-symmetric matrices, similarity and congruence imply each other by 9. The following is for general non-singular matrices.†

A necessary and sufficient condition that the non-singular matrices $\mathfrak{A}$, $\mathfrak{B}$ *in the complex field be congruent, is that* $\lambda\mathfrak{A}+\mathfrak{A}^*$, $\lambda\mathfrak{B}+\mathfrak{B}^*$ *have the same invariant factors.*

Necessity. If $\mathfrak{B} = \mathfrak{R}\mathfrak{A}\mathfrak{R}^*$, then

$$\mathfrak{B}^* = \mathfrak{R}\mathfrak{A}^*\mathfrak{R}^*, \quad \lambda\mathfrak{B}+\mathfrak{B}^* = \mathfrak{R}(\lambda\mathfrak{A}+\mathfrak{A}^*)\mathfrak{R}^*.$$

Thence $\lambda\mathfrak{A}+\mathfrak{A}^*$, $\lambda\mathfrak{B}+\mathfrak{B}^*$ are equivalent, and so have the same invariant factors.

Sufficiency. If $\lambda\mathfrak{A}+\mathfrak{A}^*$, $\lambda\mathfrak{B}+\mathfrak{B}^*$ have the same invariant factors, then (§95) the pair $(\mathfrak{A}, \mathfrak{A}^*)$ is equivalent to the pair $(\mathfrak{B}, \mathfrak{B}^*)$, that is, there are matrices, with constant elements, $\mathfrak{P}$, $\mathfrak{Q}$ such that $\mathfrak{B} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, $\mathfrak{B}^* = \mathfrak{P}\mathfrak{A}^*\mathfrak{Q}$, and hence

$$\mathfrak{B}+\mathfrak{B}^* = \mathfrak{P}(\mathfrak{A}+\mathfrak{A}^*)\mathfrak{Q}, \quad \mathfrak{B}-\mathfrak{B}^* = \mathfrak{P}(\mathfrak{A}-\mathfrak{A}^*)\mathfrak{Q}.$$

But $\mathfrak{A}+\mathfrak{A}^*$, $\mathfrak{B}+\mathfrak{B}^*$ are symmetric, and $\mathfrak{A}-\mathfrak{A}^*$, $\mathfrak{B}-\mathfrak{B}^*$ are skew-symmetric (if either of these vanish, the theorem is already proved). Hence, by 8, there is a non-singular matrix $\mathfrak{R}$ such that

$$\mathfrak{A}+\mathfrak{A}^* = \mathfrak{R}(\mathfrak{B}+\mathfrak{B}^*)\mathfrak{R}^*, \quad \mathfrak{A}-\mathfrak{A}^* = \mathfrak{R}(\mathfrak{B}-\mathfrak{B}^*)\mathfrak{R}^*.$$

Hence adding, $\mathfrak{A} = \mathfrak{R}\mathfrak{B}\mathfrak{R}^*$.

§98. *Collineations as products of polarities.*

Every collineation is the product of two polarities, one at least non-singular.

This will be shewn, if we prove:

Every matrix $\mathfrak{A}$ *is the product of two symmetric matrices, one at least non-singular.*

Our field of scalars is the complex field.

Let $\mathfrak{C}$ be the canonical form of $\mathfrak{A}$, then $\mathfrak{A} = \mathfrak{P}\mathfrak{C}\mathfrak{P}^{-1}$ for some non-singular $\mathfrak{P}$, $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3 \oplus \ldots \oplus \mathfrak{C}_s$, where each $\mathfrak{C}_i$ is a simple canonical matrix. $\mathfrak{C}_i = k\mathfrak{J}_r + \mathfrak{E}_{12} + \mathfrak{E}_{23} + \ldots + \mathfrak{E}_{r-1,r}$ for some r, k, where $\mathfrak{J}_r$ denotes the unit matrix of order r.

For each $\mathfrak{C}_i$ take a corresponding symmetric matrix $\mathfrak{D}_i$, where $\mathfrak{D}_i = \mathfrak{E}_{1,r} + \mathfrak{E}_{2,r-1} + \ldots + \mathfrak{E}_{r,1}$, and let $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus \ldots \oplus \mathfrak{D}_s$. The matrices $\mathfrak{C}_i$, $\mathfrak{D}_i$ each have order r, depending on i.

† The theorem also holds if $\mathfrak{A}$ is singular. Muth, *Theorie und Anwendungen der Elementartheiler* (1899), p. 143.

Then by the meaning of direct sum,

$$(\mathfrak{D}_1\oplus\mathfrak{D}_2\oplus\ldots\oplus\mathfrak{D}_s)(\mathfrak{C}_1\oplus\mathfrak{C}_2\oplus\ldots\oplus\mathfrak{C}_s) = \mathfrak{D}_1\mathfrak{C}_1\oplus\mathfrak{D}_2\mathfrak{C}_2\oplus\ldots\oplus\mathfrak{D}_s\mathfrak{C}_s.$$

Now

$$\begin{aligned}\mathfrak{D}_i\mathfrak{C}_i &= k(\mathfrak{E}_{1,r}+\mathfrak{E}_{2,r-1}+\ldots+\mathfrak{E}_{r,1}) \\ &\quad +\mathfrak{E}_{2,r-1}\mathfrak{E}_{r-1,r}+\mathfrak{E}_{3,r-2}\mathfrak{E}_{r-2,r-1}+\ldots+\mathfrak{E}_{r,1}\mathfrak{E}_{1,2} \\ &= k(\mathfrak{E}_{1,r}+\mathfrak{E}_{2,r-1}+\ldots+\mathfrak{E}_{r,1})+\mathfrak{E}_{2,r}+\mathfrak{E}_{3,r-1}+\ldots+\mathfrak{E}_{r,2},\end{aligned}$$

and since $\mathfrak{E}^*_{ij} = \mathfrak{E}_{ji}$, therefore $\mathfrak{D}_i\mathfrak{C}_i$ is symmetric, and $\mathfrak{D}\mathfrak{C}$, being the direct sum of symmetric matrices, is symmetric, $(\mathfrak{D}\mathfrak{C})^* = \mathfrak{D}\mathfrak{C}$.

Let $\mathfrak{Q} = \mathfrak{P}^{*-1}\mathfrak{D}\mathfrak{P}^{-1}$, then since $\mathfrak{D}^* = \mathfrak{D}$, we have

$$\mathfrak{Q}^{-1} = \mathfrak{P}\mathfrak{D}^{-1}\mathfrak{P}^*, \quad (\mathfrak{Q}^{-1})^* = \mathfrak{P}\mathfrak{D}^{*-1}\mathfrak{P}^* = \mathfrak{P}\mathfrak{D}^{-1}\mathfrak{P}^* = \mathfrak{Q}^{-1},$$

$$\mathfrak{D} = \mathfrak{P}^*\mathfrak{Q}\mathfrak{P}.$$

Hence

$$\mathfrak{D}\mathfrak{C} = \mathfrak{P}^*\mathfrak{Q}\mathfrak{P}\mathfrak{C} = \mathfrak{P}^*(\mathfrak{Q}\mathfrak{P}\mathfrak{C}\mathfrak{P}^{-1})\mathfrak{P} = \mathfrak{P}^*(\mathfrak{Q}\mathfrak{A})\mathfrak{P},$$

$$(\mathfrak{D}\mathfrak{C})^* = \mathfrak{P}^*(\mathfrak{Q}\mathfrak{A})^*\mathfrak{P}.$$

But $\mathfrak{D}\mathfrak{C} = (\mathfrak{D}\mathfrak{C})^*$. Hence $\mathfrak{Q}\mathfrak{A} = (\mathfrak{Q}\mathfrak{A})^*$. Thus $\mathfrak{A} = \mathfrak{Q}^{-1}.\mathfrak{Q}\mathfrak{A}$ is the product of symmetric matrices $\mathfrak{Q}^{-1}$ and $\mathfrak{Q}\mathfrak{A}$, and since $\mathfrak{D}$ is non-singular, so is $\mathfrak{Q}^{-1}$ since this is congruent to $\mathfrak{D}^{-1}$.

§ 99. *Reciprocal of one quadric with respect to another.*

1. In the complex field, let $\mathfrak{P}$ be a non-singular polarity which turns points into primes. In a fixed frame we can represent $\mathfrak{P}$ by a matrix, which we also denote by $\mathfrak{P}$. The point p is on the quadric given by the polarity if $[p\mathfrak{P}p] = 0$. This is the equation of the quadric as locus. The tangent prime at p is $p\mathfrak{P} = \pi$, say; thence $p = \pi\mathfrak{P}^{-1}$, and $[\pi.\pi\mathfrak{P}^{-1}] = 0$, or $[\pi\mathfrak{P}^{-1}\pi] = 0$. This is the equation of the quadric as envelope.

Let $\mathfrak{Q}$ be the matrix, in the same fixed frame, representing another non-singular quadric. If p is on $\mathfrak{P}$, and α is its polar prime for $\mathfrak{Q}$, then $[p.p\mathfrak{P}] = 0$ gives $[p\mathfrak{Q}.p\mathfrak{P}\mathfrak{Q}^{-1}] = 0$, (§ 63·9). But $p\mathfrak{Q} = \alpha$, hence $p = \alpha\mathfrak{Q}^{-1}$, $[\alpha.\alpha\mathfrak{Q}^{-1}\mathfrak{P}\mathfrak{Q}^{-1}] = 0$.

Hence the envelope of α is a quadric envelope with matrix $\mathfrak{R}^{-1} = \mathfrak{Q}^{-1}\mathfrak{P}\mathfrak{Q}^{-1}$. Regarded as a locus, the quadric has matrix $\mathfrak{R} = \mathfrak{Q}\mathfrak{P}^{-1}\mathfrak{Q}$.

2. If $\mathfrak{P}$, $\mathfrak{R}$ be given, the problem of finding a reciprocator $\mathfrak{Q}$ is essentially that of finding the square root of a matrix.

For $\mathfrak{P}^{-1}\mathfrak{R} = (\mathfrak{P}^{-1}\mathfrak{Q})^2$; and we can find $\mathfrak{S}$, a polynomial in $\mathfrak{P}^{-1}\mathfrak{R}$ such that $\mathfrak{S}^2 = \mathfrak{P}^{-1}\mathfrak{R}$ (§ 96·2), and then take $\mathfrak{Q} = \mathfrak{P}\mathfrak{S}$.

We must shew that the matrix $\mathfrak{Q}$ is symmetric, otherwise it will not represent a quadric.

Let $\mathfrak{S} = k_0 + k_1 \mathfrak{P}^{-1}\mathfrak{R} + k_2(\mathfrak{P}^{-1}\mathfrak{R})^2 + \ldots + k_n(\mathfrak{P}^{-1}\mathfrak{R})^n$.

Since $(\mathfrak{P}^{-1}\mathfrak{R})^* = \mathfrak{R}^*\mathfrak{P}^{*-1} = \mathfrak{R}\mathfrak{P}^{-1}$, we have

$$\mathfrak{S}^* = k_0 + k_1 \mathfrak{R}\mathfrak{P}^{-1} + k_2(\mathfrak{R}\mathfrak{P}^{-1})^2 + \ldots + k_n(\mathfrak{R}\mathfrak{P}^{-1})^n,$$

$$\mathfrak{P}\mathfrak{S} = k_0\mathfrak{P} + k_1\mathfrak{R} + k_2(\mathfrak{R}\mathfrak{P}^{-1})\mathfrak{R} + \ldots + k_n(\mathfrak{R}\mathfrak{P}^{-1})^{n-1}\mathfrak{R} = \mathfrak{S}^*\mathfrak{P},$$

$(\mathfrak{P}\mathfrak{S})^* = (\mathfrak{S}^*\mathfrak{P})^* = \mathfrak{P}^*\mathfrak{S} = \mathfrak{P}\mathfrak{S}$.

Hence $\mathfrak{Q}^* = \mathfrak{Q}$.

3. If $\mathfrak{P}$, $\mathfrak{Q}$ can be brought simultaneously to diagonal form, the change of frame which does this, brings $\mathfrak{R}$ to diagonal form.

For, if $\mathfrak{P} = \operatorname{diag}(a_1, a_2, \ldots, a_n)$, $\mathfrak{Q} = \operatorname{diag}(b_1, b_2, \ldots, b_n)$,

then $\mathfrak{R} = \operatorname{diag}(a_1^{-1}b_1^2,\ a_2^{-1}b_2^2, \ldots, a_n^{-1}b_n^2) = \operatorname{diag}(c_1, c_2, \ldots, c_n)$,

where $a_1 c_1 = b_1^2, \ldots, a_n c_n = b_n^2$.

4. If $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ be related so that $\mathfrak{Q}\mathfrak{R}^{-1}\mathfrak{Q} = \mathfrak{P}$, $\mathfrak{R}\mathfrak{P}^{-1}\mathfrak{R} = \mathfrak{Q}$,

then $\mathfrak{P}\mathfrak{Q}^{-1} = \mathfrak{Q}\mathfrak{R}^{-1} = \mathfrak{R}\mathfrak{P}^{-1}$, $\mathfrak{P}\mathfrak{Q}^{-1}\mathfrak{P} = \mathfrak{R} = \mathfrak{Q}\mathfrak{P}^{-1}\mathfrak{Q}$,

$$(\mathfrak{P}\mathfrak{Q}^{-1})^3 = \mathfrak{P}\mathfrak{Q}^{-1}.\mathfrak{Q}\mathfrak{R}^{-1}.\mathfrak{R}\mathfrak{P}^{-1} = \mathfrak{J}, \quad \text{or} \quad \mathfrak{S}^3 = \mathfrak{J},$$

when $\mathfrak{S} = \mathfrak{P}\mathfrak{Q}^{-1}$.

If $\det(\mathfrak{S} - \mathfrak{J}) \neq 0$, then $\mathfrak{S}^2 + \mathfrak{S} + \mathfrak{J} = \mathfrak{Q}$, (§ 62·7),

$$\mathfrak{S} + \mathfrak{S}^{-1} + \mathfrak{J} = \mathfrak{Q}, \quad \mathfrak{P}\mathfrak{Q}^{-1} + \mathfrak{Q}\mathfrak{P}^{-1} + \mathfrak{J} = \mathfrak{Q}, \quad \mathfrak{P} + \mathfrak{Q} + \mathfrak{R} = \mathfrak{Q}.$$

The geometrical interpretation of these is considered later (§ 118·9, ·10).

§ 100. *Self-equivalence of matrices of complex numbers.*

1. *A necessary and sufficient condition that* $\mathfrak{A} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, *for some* $\mathfrak{A}$, *where* $\mathfrak{A}$, $\mathfrak{P}$, $\mathfrak{Q}$ *are non-singular, is that the elementary divisors of the secular matrices of* $\mathfrak{P}$, $\mathfrak{Q}$ *can be put into one-to-one correspondence so that the corresponding divisors vanish for reciprocal values of* λ.

For, first, if $\mathfrak{A}$ be non-singular, and $\mathfrak{A} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$, then

$$\mathfrak{J} - \lambda\mathfrak{Q} = \mathfrak{A}^{-1}(\mathfrak{P} - \lambda\mathfrak{J})\mathfrak{A}\mathfrak{Q} \text{ for all } \lambda.$$

Hence the pairs $\mathfrak{J}$, $\mathfrak{Q}$ and $\mathfrak{P}$, $\mathfrak{J}$ are equivalent.

Conversely, if these pairs be equivalent, then there are non-singular matrices $\mathfrak{A}$, $\mathfrak{B}$ of constant elements, such that

$$\mathfrak{P} = \mathfrak{A}\mathfrak{J}\mathfrak{B}, \quad \mathfrak{J} = \mathfrak{A}\mathfrak{Q}\mathfrak{B}.$$

Hence $\mathfrak{P} = \mathfrak{A}\mathfrak{B}$, $\mathfrak{P} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}\mathfrak{B}$, and $\mathfrak{A} = \mathfrak{P}\mathfrak{A}\mathfrak{Q}$.

Now these pairs are equivalent, if, and only if, the matrices $\mathfrak{P}-\lambda\mathfrak{J}$, $\mathfrak{J}-\lambda\mathfrak{Q}$ have the same elementary divisors; that is, if and only if $\mathfrak{P}-\lambda\mathfrak{J}$, $\mathfrak{Q}-\lambda^{-1}\mathfrak{J}$ have the same elementary divisors.

Hence the theorem if $\mathfrak{A}$ be non-singular.

We next consider the case when $\mathfrak{A}$ is singular.

If $\mathfrak{A}$ be singular of rank r, *and $\mathfrak{P}\mathfrak{A}\mathfrak{Q}=\mathfrak{A}$, where $\mathfrak{P}$, $\mathfrak{Q}$ are non-singular, then $\mathfrak{P}$ has at least* r *characteristic roots which are reciprocals of characteristic roots of $\mathfrak{Q}$.*

For let
$$\mathfrak{J}_1 = \operatorname{diag}(1, 1, \ldots, 1, 0, \ldots, 0),$$
$$\mathfrak{J}_2 = \operatorname{diag}(0, 0, \ldots, 0, 1, \ldots, 1),$$
where there are r unities and $n-r$ zeros in $\mathfrak{J}_1$, and $n-r$ unities and r zeros in $\mathfrak{J}_2$.

Then $\mathfrak{J}_1^2=\mathfrak{J}_1$, $\mathfrak{J}_2^2=\mathfrak{J}_2$, $\mathfrak{J}_1\mathfrak{J}_2=\mathfrak{J}_2\mathfrak{J}_1=\mathfrak{O}$.

By hypothesis and § 90 there are non-singular matrices $\mathfrak{B}$, $\mathfrak{C}$ such that $\mathfrak{B}\mathfrak{A}\mathfrak{C}=\mathfrak{J}_1$.

Let $\mathfrak{P}_0=\mathfrak{B}\mathfrak{P}\mathfrak{B}^{-1}$, $\mathfrak{Q}_0=\mathfrak{C}^{-1}\mathfrak{Q}\mathfrak{C}$, then
$$\mathfrak{P}_0\mathfrak{J}_1\mathfrak{Q}_0=\mathfrak{B}\mathfrak{P}\mathfrak{B}^{-1}.\mathfrak{B}\mathfrak{A}\mathfrak{C}.\mathfrak{C}^{-1}\mathfrak{Q}\mathfrak{C}=\mathfrak{B}.\mathfrak{P}\mathfrak{A}\mathfrak{Q}.\mathfrak{C}=\mathfrak{B}\mathfrak{A}\mathfrak{C}=\mathfrak{J}_1.$$

Let $\mathfrak{P}_{ij}=\mathfrak{J}_i\mathfrak{P}_0\mathfrak{J}_j$, $\mathfrak{Q}_{ij}=\mathfrak{J}_i\mathfrak{Q}_0\mathfrak{J}_j$, then
$$\mathfrak{P}_{11}\mathfrak{Q}_{11}=\mathfrak{J}_1\mathfrak{P}_0\mathfrak{J}_1.\mathfrak{J}_1\mathfrak{Q}_0\mathfrak{J}_1=\mathfrak{J}_1.$$
$$\mathfrak{P}_{21}\mathfrak{Q}_0=\mathfrak{J}_2.\mathfrak{P}_0\mathfrak{J}_1\mathfrak{Q}_0=\mathfrak{J}_2\mathfrak{J}_1=\mathfrak{O};$$
but $\mathfrak{Q}_0$ is non-singular, hence $\mathfrak{P}_{21}=\mathfrak{O}$.
$$\mathfrak{P}_0\mathfrak{Q}_{12}=\mathfrak{P}_0\mathfrak{J}_1\mathfrak{Q}_0.\mathfrak{J}_2=\mathfrak{J}_1\mathfrak{J}_2=\mathfrak{O};$$
but $\mathfrak{P}_0$ is non-singular, hence $\mathfrak{Q}_{12}=\mathfrak{O}$.

Hence
$$\det(\mathfrak{P}-\lambda\mathfrak{J})=\det(\mathfrak{P}_{11}-\lambda\mathfrak{J}_1).\det(\mathfrak{P}_{22}-\lambda\mathfrak{J}_2),$$
$$\det(\mathfrak{Q}-\lambda\mathfrak{J})=\det(\mathfrak{Q}_{11}-\lambda\mathfrak{J}_1).\det(\mathfrak{Q}_{22}-\lambda\mathfrak{J}_2).$$

Also the sub-matrices composed of the first r rows and columns of $\mathfrak{P}_{11}$, $\mathfrak{Q}_{11}$ are inverses of one another. Hence if $c_1, c_2, \ldots, c_r$ be the characteristic roots of $\mathfrak{P}_{11}$, then $c_1^{-1}, c_2^{-1}, \ldots, c_r^{-1}$ are those of $\mathfrak{Q}_{11}$.

2. If $\mathfrak{A}=\mathfrak{P}\mathfrak{A}\mathfrak{P}^*$ and $\mathfrak{A}$, $\mathfrak{P}$ be non-singular, then by 1, $\mathfrak{J}-\lambda\mathfrak{P}^*$ and $\mathfrak{P}-\lambda\mathfrak{J}$ have the same elementary divisors. Hence so have $\mathfrak{J}-\lambda\mathfrak{P}$ and $\mathfrak{P}-\lambda\mathfrak{J}$.

Hence *a necessary and sufficient condition that $\mathfrak{A}=\mathfrak{P}\mathfrak{A}\mathfrak{P}^*$ ($\mathfrak{P}$ non-singular), for some non-singular $\mathfrak{A}$, is that the elementary*

divisors of $\mathfrak{P}-\lambda\mathfrak{J}$ *and* $\mathfrak{J}-\lambda\mathfrak{P}$ *are in pairs of equal degree, and vanish for reciprocal values of* λ, *except perhaps for those divisors which are powers of* $\lambda \pm 1$.

The last proviso is necessary, because we only know that the roots of $\det(\mathfrak{P}-\lambda\mathfrak{J})$ and of $\det(\mathfrak{P}-\lambda^{-1}\mathfrak{J})$ are the same; so that $+1$ or -1 might be solitary roots, while if $\alpha \neq 1$ is a root, then so is α^{-1}.

3. *A necessary and sufficient condition that* $\mathfrak{A} = \mathfrak{P}\mathfrak{A}\mathfrak{P}$ ($\mathfrak{P}$ *non-singular*), *for some non-singular* $\mathfrak{A}$, *is the same as in* 2.

This follows by 1. We can also shew the identity of conditions in 2, 3 as follows:

If there is a non-singular $\mathfrak{A}$ *such that* $\mathfrak{P}\mathfrak{A}\mathfrak{P} = \mathfrak{A}$, *then there is a non-singular* $\mathfrak{B}$ *such that* $\mathfrak{P}\mathfrak{B}\mathfrak{P}^* = \mathfrak{B}$, *and conversely.*

For, if $\mathfrak{C}$ be any matrix, then $\mathfrak{C}-\lambda\mathfrak{J}$, $\mathfrak{C}^*-\lambda\mathfrak{J}$ have the same elementary divisors. Hence there are matrices $\mathfrak{Q}$, $\mathfrak{R}$ with $\mathfrak{Q}\mathfrak{C}\mathfrak{R} = \mathfrak{C}^*$, $\mathfrak{Q}\mathfrak{R} = \mathfrak{J}$, and hence $\mathfrak{Q}\mathfrak{C}\mathfrak{Q}^{-1} = \mathfrak{C}^*$.

Take $\mathfrak{C} = \mathfrak{P}^{-1*}$, then, by hypothesis

$$\mathfrak{A}^{-1}\mathfrak{P}\mathfrak{A} = \mathfrak{P}^{-1} = \mathfrak{C}^* = \mathfrak{Q}\mathfrak{C}\mathfrak{Q}^{-1} = \mathfrak{Q}\mathfrak{P}^{-1*}\mathfrak{Q}^{-1},$$

$$\mathfrak{Q}^{-1}\mathfrak{A}^{-1}\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{P}^{-1*}, \quad \mathfrak{P}.\mathfrak{A}\mathfrak{Q}.\mathfrak{P}^* = \mathfrak{A}\mathfrak{Q}.$$

Hence $\mathfrak{B} = \mathfrak{A}\mathfrak{Q}$ satisfies the conditions.

4. *If* $\mathfrak{A}$ *be non-singular, a necessary and sufficient condition that non-singular matrices* $\mathfrak{P}$, $\mathfrak{Q}$ *satisfy* $\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{A}$, *where* $\det(\mathfrak{P}+\mathfrak{J})$, $\det(\mathfrak{Q}+\mathfrak{J}) \neq 0$, *is that there is a matrix* $\mathfrak{M}$ *such that*

$$\det(\mathfrak{A}+\mathfrak{M}),\ \det(\mathfrak{A}-\mathfrak{M}) \neq 0,$$

$$\mathfrak{P} = (\mathfrak{A}+\mathfrak{M})(\mathfrak{A}-\mathfrak{M})^{-1}, \quad \mathfrak{Q} = (\mathfrak{A}+\mathfrak{M})^{-1}(\mathfrak{A}-\mathfrak{M}).$$

Proof. Sufficiency. The conditions give

$$\mathfrak{P}(\mathfrak{A}-\mathfrak{M})\mathfrak{Q} = (\mathfrak{A}+\mathfrak{M})\mathfrak{Q} = \mathfrak{A}-\mathfrak{M},$$

$$\mathfrak{P}(\mathfrak{A}+\mathfrak{M})\mathfrak{Q} = \mathfrak{P}(\mathfrak{A}-\mathfrak{M}) = \mathfrak{A}+\mathfrak{M}.$$

Adding these $\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{A}$. That $\mathfrak{P}+\mathfrak{J}$, $\mathfrak{Q}+\mathfrak{J}$ are non-singular follows from the formulae of the next part of the proof.

Necessity. If $\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{A}$, and $\mathfrak{P}+\mathfrak{J}$, $\mathfrak{Q}+\mathfrak{J}$ be non-singular, take $\mathfrak{M} = (\mathfrak{P}+\mathfrak{J})^{-1}(\mathfrak{P}-\mathfrak{J})\mathfrak{A}$, then

$$\mathfrak{A}+\mathfrak{M} = (\mathfrak{P}+\mathfrak{J})^{-1}[(\mathfrak{P}+\mathfrak{J})+(\mathfrak{P}-\mathfrak{J})]\mathfrak{A} = 2(\mathfrak{P}+\mathfrak{J})^{-1}\mathfrak{P}\mathfrak{A},$$

$$\mathfrak{A}-\mathfrak{M} = (\mathfrak{P}+\mathfrak{J})^{-1}[(\mathfrak{P}+\mathfrak{J})-(\mathfrak{P}-\mathfrak{J})]\mathfrak{A} = 2(\mathfrak{P}+\mathfrak{J})^{-1}\mathfrak{A}.$$

Hence $\mathfrak{A}+\mathfrak{M}$, $\mathfrak{A}-\mathfrak{M}$ are non-singular.

Also $(\mathfrak{P}+\mathfrak{J})\mathfrak{M} = (\mathfrak{P}-\mathfrak{J})\mathfrak{A}$, $\mathfrak{P}(\mathfrak{A}-\mathfrak{M}) = \mathfrak{A}+\mathfrak{M}$,

$$\mathfrak{P} = (\mathfrak{A}+\mathfrak{M})(\mathfrak{A}-\mathfrak{M})^{-1}.$$

Since $\mathfrak{P}\mathfrak{A}\mathfrak{Q} = \mathfrak{A}$, we have

$$(\mathfrak{P}-\mathfrak{J})\mathfrak{A}(\mathfrak{Q}+\mathfrak{J}) = \mathfrak{A}-\mathfrak{A}\mathfrak{Q}+\mathfrak{P}\mathfrak{A}-\mathfrak{A} = (\mathfrak{P}+\mathfrak{J})\mathfrak{A}(\mathfrak{J}-\mathfrak{Q}),$$

$$(\mathfrak{P}+\mathfrak{J})^{-1}(\mathfrak{P}-\mathfrak{J})\mathfrak{A}(\mathfrak{Q}+\mathfrak{J}) = \mathfrak{A}(\mathfrak{J}-\mathfrak{Q}),$$

$$\mathfrak{M}(\mathfrak{Q}+\mathfrak{J}) = \mathfrak{A}(\mathfrak{J}-\mathfrak{Q}).$$

Hence $(\mathfrak{A}+\mathfrak{M})\mathfrak{Q} = \mathfrak{A}-\mathfrak{M}$, $\mathfrak{Q} = (\mathfrak{A}+\mathfrak{M})^{-1}(\mathfrak{A}-\mathfrak{M})$.

5. *If $\mathfrak{S}$ is symmetric and non-singular, a necessary and sufficient condition that there is a matrix $\mathfrak{P}$, with $\mathfrak{P}+\mathfrak{J}$ non-singular, such that $\mathfrak{P}\mathfrak{S}\mathfrak{P}^* = \mathfrak{S}$, is that there is a skew-symmetric matrix $\mathfrak{T}$, such that $\mathfrak{P} = (\mathfrak{S}+\mathfrak{T})(\mathfrak{S}-\mathfrak{T})^{-1}$, with $\mathfrak{S}+\mathfrak{T}$, $\mathfrak{S}-\mathfrak{T}$ non-singular.*

Proof. Sufficiency. The conditions give

$$\mathfrak{T} = -\mathfrak{T}^*, \quad \det(\mathfrak{S}+\mathfrak{T}) \neq 0, \quad \det(\mathfrak{S}-\mathfrak{T}) \neq 0,$$

$$\mathfrak{P}^* = (\mathfrak{S}-\mathfrak{T})^{*-1}(\mathfrak{S}+\mathfrak{T})^* = (\mathfrak{S}+\mathfrak{T})^{-1}(\mathfrak{S}-\mathfrak{T}).$$

Hence the result follows from 4 with $\mathfrak{P}^*$ for $\mathfrak{Q}$, $\mathfrak{T}$ for $\mathfrak{M}$.

Necessity. Take $\mathfrak{T} = (\mathfrak{P}+\mathfrak{J})^{-1}(\mathfrak{P}-\mathfrak{J})\mathfrak{S}$.

Then, as in 4, $\mathfrak{S}+\mathfrak{T}$, $\mathfrak{S}-\mathfrak{T}$ are non-singular.

Also $(\mathfrak{P}+\mathfrak{J})\mathfrak{T} = (\mathfrak{P}-\mathfrak{J})\mathfrak{S}$, $\mathfrak{P}\mathfrak{T}+\mathfrak{T} = \mathfrak{P}\mathfrak{S}-\mathfrak{S}$,

$$\mathfrak{P} = (\mathfrak{S}+\mathfrak{T})(\mathfrak{S}-\mathfrak{T})^{-1}.$$

Take $$\mathfrak{R} = (\mathfrak{P}+\mathfrak{J})\mathfrak{T}(\mathfrak{P}+\mathfrak{J})^*,$$

then $\mathfrak{R}$, $\mathfrak{T}$ are congruent, and

$$\begin{aligned}\mathfrak{R} &= (\mathfrak{P}-\mathfrak{J})\mathfrak{S}(\mathfrak{P}+\mathfrak{J})^* = (\mathfrak{P}\mathfrak{S}-\mathfrak{S})(\mathfrak{P}^*+\mathfrak{J}) \\ &= \mathfrak{P}\mathfrak{S}\mathfrak{P}^*-\mathfrak{S}\mathfrak{P}^*+\mathfrak{P}\mathfrak{S}-\mathfrak{S} \\ &= \mathfrak{S}-\mathfrak{S}\mathfrak{P}^*+\mathfrak{P}\mathfrak{S}-\mathfrak{S} = \mathfrak{P}\mathfrak{S}-\mathfrak{S}\mathfrak{P}^*,\end{aligned}$$

$$\mathfrak{R}^* = \mathfrak{S}\mathfrak{P}^*-\mathfrak{P}\mathfrak{S} = -\mathfrak{R}.$$

Hence $\mathfrak{T}^* = -\mathfrak{T}$, since $\mathfrak{R}$, $\mathfrak{T}$ are congruent.

Cor. If $\mathfrak{P}\mathfrak{T}\mathfrak{P}^* = \mathfrak{T}$, $\mathfrak{T}^* = -\mathfrak{T}$, $\det(\mathfrak{P}+\mathfrak{J}) \neq 0$, $\det\mathfrak{T} \neq 0$, then there is a symmetric matrix $\mathfrak{S}$ such that $\mathfrak{P} = (\mathfrak{S}+\mathfrak{T})(\mathfrak{S}-\mathfrak{T})^{-1}$.

6. *Geometric examples.*

A linear complex is self-transformed by a biaxial involution whose axes are any two lines of the complex.

If through a point on a quadric $\mathcal{Q}$ we draw four planes, the opposite edges of the complete four-face cut $\mathcal{Q}$ in three point-pairs of a biaxial involution whose axes are polar with respect to $\mathcal{Q}$.

Opposite edges of two Möbius tetrahedra inscribed in a quadric $\mathcal{Q}$ correspond in a fixed biaxial involution whose axes are polar with respect to $\mathcal{Q}$.

§ 101. *Matrices which are products of involutions.*

1. *If* $\mathfrak{P} = \mathfrak{A}\mathfrak{B}$, $\mathfrak{A}^2 = \mathfrak{B}^2 = \mathfrak{J}$, *then the secular equation of* $\mathfrak{P}$ *is reciprocal and* $\mathfrak{P}$ *satisfies the condition in* § 100·2*: conversely, if* $\mathfrak{P}$ *satisfies this condition, then* $\mathfrak{P}$ *is the product of two involutions.*

By an '*involution*' we mean a matrix whose square is identity; by a '*reciprocal equation*' we mean one such that if r is a root, then so is r^{-1}; thus roots occur in reciprocal pairs except that isolated roots $+1$ or -1 may occur any number of times.

For, if $\mathfrak{P} = \mathfrak{A}\mathfrak{B}$, $\mathfrak{A}^2 = \mathfrak{B}^2 = \mathfrak{J}$, then $\mathfrak{A}\mathfrak{P} = \mathfrak{B}$, $\mathfrak{P}\mathfrak{B} = \mathfrak{A}$; hence $\mathfrak{P}\mathfrak{A}\mathfrak{P} = \mathfrak{P}\mathfrak{B} = \mathfrak{A}$, $\det\mathfrak{P} = \det\mathfrak{A}.\det\mathfrak{B} = \pm 1$. Thus $\mathfrak{P}$ is non-singular. Hence $\mathfrak{P}$ satisfies the condition mentioned.

Conversely, if this condition holds, there is a non-singular matrix $\mathfrak{C}$ such that $\mathfrak{P}\mathfrak{C}\mathfrak{P} = \mathfrak{C}$, (§ 100·3).

Then $\mathfrak{P}\mathfrak{C}^2 = \mathfrak{P}\mathfrak{C}.\mathfrak{P}\mathfrak{C}\mathfrak{P} = \mathfrak{P}\mathfrak{C}\mathfrak{P}.\mathfrak{C}\mathfrak{P} = \mathfrak{C}.\mathfrak{C}\mathfrak{P} = \mathfrak{C}^2\mathfrak{P}$.

But if $\mathfrak{P}\mathfrak{D} = \mathfrak{D}\mathfrak{P}$, and if $f(x)$ be any polynomial in x, then $\mathfrak{P}f(\mathfrak{D}) = f(\mathfrak{D})\mathfrak{P}$.

For, if $\mathfrak{P}\mathfrak{D}^n = \mathfrak{D}^n\mathfrak{P}$, (n positive integer), then

$$\mathfrak{P}\mathfrak{D}^{n+1} = \mathfrak{D}^n\mathfrak{P}\mathfrak{D} = \mathfrak{D}^n\mathfrak{D}\mathfrak{P} = \mathfrak{D}^{n+1}\mathfrak{P}.$$

Hence, by induction, $\mathfrak{P}\mathfrak{D}^n = \mathfrak{D}^n\mathfrak{P}$ for all positive integers n, and hence, by addition, $\mathfrak{P}f(\mathfrak{D}) = f(\mathfrak{D})\mathfrak{P}$ for all polynomials f.

Now we had $\mathfrak{P}\mathfrak{C}^2 = \mathfrak{C}^2\mathfrak{P}$, hence $\mathfrak{P}f(\mathfrak{C}^2) = f(\mathfrak{C}^2)\mathfrak{P}$.

We can find f so that $f(\mathfrak{C}^2) = \mathfrak{F}$, and $\mathfrak{F}^2 = \mathfrak{C}^2$, (§96); $\mathfrak{F} \neq \mathfrak{C}$, by construction of f.

Then $\mathfrak{P}\mathfrak{F} = \mathfrak{F}\mathfrak{P}$, $\mathfrak{F}\mathfrak{C} = f(\mathfrak{C}^2)\mathfrak{C} = \mathfrak{C}f(\mathfrak{C}^2) = \mathfrak{C}\mathfrak{F}$,

$$\mathfrak{F}^{-1}\mathfrak{C} = \mathfrak{C}\mathfrak{F}^{-1}.$$

Take $\mathfrak{A} = \mathfrak{C}\mathfrak{F}^{-1}$, then

$$\mathfrak{A}^2 = \mathfrak{C}\mathfrak{F}^{-1}.\mathfrak{C}\mathfrak{F}^{-1} = \mathfrak{C}\mathfrak{C}\mathfrak{F}^{-1}\mathfrak{F}^{-1} = \mathfrak{C}^2\mathfrak{F}^{-2} = \mathfrak{J}.$$

Take $\mathfrak{B} = \mathfrak{A}\mathfrak{P}$, then $\mathfrak{A}\mathfrak{B} = \mathfrak{P}$.

It remains to shew that $\mathfrak{B}^2 = \mathfrak{J}$. Now $\mathfrak{P}\mathfrak{C}\mathfrak{P} = \mathfrak{C}$, $\mathfrak{P}\mathfrak{F}^{-1} = \mathfrak{F}^{-1}\mathfrak{P}$, hence

$$\begin{aligned}\mathfrak{B}^{-1} = (\mathfrak{A}\mathfrak{P})^{-1} &= \mathfrak{P}^{-1}\mathfrak{A}^{-1} = \mathfrak{P}^{-1}\mathfrak{A} = \mathfrak{P}^{-1}\mathfrak{C}\mathfrak{F}^{-1} \\ &= \mathfrak{C}\mathfrak{P}\mathfrak{F}^{-1} = \mathfrak{C}\mathfrak{F}^{-1}\mathfrak{P} = \mathfrak{A}\mathfrak{P} = \mathfrak{B}.\end{aligned}$$

2. *Cor.* In particular, a projective transformation on a line is given, as we know, by a two-rowed matrix, whose determinant may be taken as unity, if only positions, and not weights, of points are relevant. The secular equation of the transformation is then reciprocal. Hence the transformation is the product of two involutions.

§ 102. *Orthogonal transformations with real coefficients.*

1. *Any orthogonal transformation with real coefficients is the product of two involutions, not necessarily real.*

By the form of its secular equation (§ 67·15).

2. Since for an orthogonal transformation $\mathfrak{A}$, we have $\mathfrak{A}\mathfrak{A}^* = \mathfrak{J}$, then

$$\begin{aligned}\det\mathfrak{A}.\det(\mathfrak{A}+\mathfrak{J}) &= \det\mathfrak{A}.\det(\mathfrak{A}^*+\mathfrak{J}) = \det(\mathfrak{J}+\mathfrak{A}), \\ \det\mathfrak{A}.\det(\mathfrak{A}-\mathfrak{J}) &= \det\mathfrak{A}.\det(\mathfrak{A}^*-\mathfrak{J}) \\ &= \det(\mathfrak{J}-\mathfrak{A}) = (-1)^n\det(\mathfrak{A}-\mathfrak{J}),\end{aligned}$$

where n is the order of $\mathfrak{A}$.

Hence we have the cases:

For n even:

either $\det\mathfrak{A} = 1$, or $\det(\mathfrak{A}+\mathfrak{J}) = 0$ and $\det(\mathfrak{A}-\mathfrak{J}) = 0$.

For n odd:

either $\det\mathfrak{A} = 1$ or $\det(\mathfrak{A}+\mathfrak{J}) = 0$,

and either $\det\mathfrak{A} = -1$ or $\det(\mathfrak{A}-\mathfrak{J}) = 0$.

Hence for n even, an indirect $\mathfrak{A}$ has $+1$ and -1 as characteristic roots; for n odd, a direct $\mathfrak{A}$ has characteristic root $+1$, an indirect $\mathfrak{A}$ has characteristic root -1.

3. *A necessary and sufficient condition that $\mathfrak{A}$ is orthogonal, when $\mathfrak{A}+\mathfrak{J}$ is non-singular, is that there is a skew-symmetric matrix $\mathfrak{T}$ such that*

$$\mathfrak{A} = (\mathfrak{J}+\mathfrak{T})(\mathfrak{J}-\mathfrak{T})^{-1}, \quad \det(\mathfrak{J}+\mathfrak{T}) \neq 0, \quad \det(\mathfrak{J}-\mathfrak{T}) \neq 0.$$

(Cayley.)

By § 100·5: As we have assumed, $\det(\mathfrak{A}+\mathfrak{J}) \neq 0$, therefore $\mathfrak{A}$ is direct.

Any orthogonal matrix, direct or indirect, can be put in the form
$$\mathfrak{A} = \mathfrak{Q}(\mathfrak{J}+\mathfrak{T})(\mathfrak{J}-\mathfrak{T})^{-1},$$
where
$$\mathfrak{T} = -\mathfrak{T}^*, \quad \mathfrak{Q} = \mathfrak{Q}^* = \mathfrak{Q}^{-1}.$$

4. The elementary divisors of a real orthogonal matrix are all linear; any two such matrices are similar, or congruent, if and only if they have the same secular determinants, and then one can be transformed into the other by a real orthogonal transformation (§ 67).

5. *If p be a latent extensive for an orthogonal transformation $\mathfrak{A}$, which corresponds to a root $\alpha \neq \pm 1$ of the secular equation of $\mathfrak{A}$, then $p^2 = 0$.*

For $p\mathfrak{A} = \alpha p$, $\alpha^2 p^2 = [p\mathfrak{A} \,|\, p\mathfrak{A}] = [p\mathfrak{A}\mathfrak{A}^* \,|\, p] = [p \,|\, p] = p^2$.
Hence if $\alpha^2 \neq 1$, then $p^2 = 0$.

If p, q be latent extensives, not both corresponding to root $+1$, nor both to root -1, nor to reciprocal roots, then $[p \,|\, q] = 0$.

For $p\mathfrak{A} = \alpha p$, $q\mathfrak{A} = \beta q$, then
$$\alpha\beta[p \,|\, q] = [p\mathfrak{A} \,|\, q\mathfrak{A}] = [p\mathfrak{A}\mathfrak{A}^* \,|\, q] = [p \,|\, q],$$
and since $\alpha\beta \neq 1$, we have $[p \,|\, q] = 0$.

6. Geometrical treatment of 3.

Let $\mathfrak{A}$ be a direct orthogonal transformation, then $\mathfrak{A}\mathfrak{A}^* = \mathfrak{J}$, $\det(\mathfrak{A}+\mathfrak{J}) \neq 0$. Let p be any extensive, $p\mathfrak{A} = p'$, $\frac{1}{2}(p+p') = q$.

Then $p(\mathfrak{A}+\mathfrak{J}) = p+p' = 2q$, $p = q.2(\mathfrak{A}+\mathfrak{J})^{-1} = q(\mathfrak{J}-\mathfrak{T})$, say.

Then $q-p = q\mathfrak{T} = \frac{1}{2}(p+p')\mathfrak{T}$, $p'-p = (p+p')\mathfrak{T}$.

Hence $\mathfrak{A}-\mathfrak{J} = (\mathfrak{A}+\mathfrak{J})\mathfrak{T}$, $\mathfrak{A}(\mathfrak{J}-\mathfrak{T}) = \mathfrak{J}+\mathfrak{T}$,
$$\mathfrak{T} = (\mathfrak{A}+\mathfrak{J})^{-1}(\mathfrak{A}-\mathfrak{J}), \quad \mathfrak{A} = (\mathfrak{J}+\mathfrak{T})(\mathfrak{J}-\mathfrak{T})^{-1}.$$

It remains to shew that $\mathfrak{T}$ is skew-symmetric. We have
$$[(p+p')\mathfrak{T} \,|\, (p+p')] = [(p'-p) \,|\, (p'+p)] = 0,$$
since $p^2 = p'^2$. Hence $\mathfrak{T}$ is skew-symmetric (§ 63·10).

Note that $(\mathfrak{A}+\mathfrak{J})^{-1}(\mathfrak{A}-\mathfrak{J}) = (\mathfrak{A}-\mathfrak{J})(\mathfrak{A}+\mathfrak{J})^{-1}$,
$$(\mathfrak{J}+\mathfrak{T})(\mathfrak{J}-\mathfrak{T})^{-1} = (\mathfrak{J}-\mathfrak{T})^{-1}(\mathfrak{J}+\mathfrak{T}).$$

7. If $\mathfrak{A}$, $\mathfrak{B}$ be direct orthogonal matrices of order three, then $\mathfrak{A}+\mathfrak{B}$ is never of rank two. (Stieltjes.)

§ 103. *Displacements in space.*

If o be a fixed origin, p any point, which becomes p' by the displacement $\mathfrak{A}$, then

$$p'-o=(p-o)\mathfrak{A}+r, \qquad \text{(i)}$$

where r is a vector, and $\mathfrak{A}$ a direct orthogonal transformation.

When $\mathfrak{A}$ is an indirect orthogonal transformation, the transformation defined by (i) will be called an '*indirect displacement*'.

From (i), $\quad \frac{1}{2}(p+p')-o=\frac{1}{2}(p-o)(\mathfrak{A}+\mathfrak{I})+\frac{1}{2}r.$

If $\mathfrak{A}$ is direct, then (§ 64·10) either -1 is not a characteristic root or it is a double characteristic root. In the latter case, $\mathfrak{A}+\mathfrak{I}$ is of rank one, and as p varies, the points $\frac{1}{2}(p+p')$ are collinear.

If $\mathfrak{A}$ is indirect, then -1 is either a simple or a triple characteristic root (§ 64·11), and $\mathfrak{A}+\mathfrak{I}$ has respectively rank two or zero. Corresponding to these two cases, $\frac{1}{2}(p+p')$ lie in a plane or coincide for all p; for if $\mathfrak{A}=-\mathfrak{I}$, then $\frac{1}{2}(p+p')=o+\frac{1}{2}r$.

Hence, *for displacements in space, the mid-points of corresponding points either fill all space, or lie on a line; for indirect displacements, they either lie in a plane, or coincide.* (Cf. § 19·12, ·13.)

§ 104. *Inversions.**

1. Consider in a spread of vectors, of step n, the sub-set $\mathscr{S}$ of vectors whose inner squares are unity; all vectors in the present section are in $\mathscr{S}$. Consider the transformation of p to q given by

$$q=p-2[a\,|\,p]\,a, \qquad \text{(i)}$$

where a is any vector of the subspread. Since $a^2=1$, we have

$$q^2=p^2-4[a\,|\,p]^2+4[a\,|\,p]^2=p^2.$$

Hence the transformation is orthogonal and $\mathscr{S}$ is latent. The transformation is indirect, since, if $p_1, \ldots, p_n$ become $q_1, \ldots, q_n$, then $[p_1 \ldots p_n]=-[q_1 \ldots q_n]$.

We call this special transformation an '*inversion*' with '*centre*' a, and we denote the inversions with centres $a, b, \ldots$ respectively by $\mathfrak{A}, \mathfrak{B}, \ldots$.

* P. F. Smith, *Trans. Amer. Math. Soc.* 6 (1905), p. 1; Voss, *Math. Ann.* 13 (1878), p. 320.

If $q = p\mathfrak{A}$, then from (i) we have $a\mathfrak{A} = -a$, $[a\,|\,q] = -[a\,|\,p]$,

$$q\mathfrak{A} = q - 2[a\,|\,q]\,a = q + 2[a\,|\,p]\,a = p.$$

Hence $\mathfrak{A}^2 = \mathfrak{J}$, so that $\mathfrak{A}$ is an involution, and as it is orthogonal, therefore $\mathfrak{A}\mathfrak{A}^* = \mathfrak{J}$, and hence $\mathfrak{A} = \mathfrak{A}^*$.

2. *Products of distinct inversions.*

If $p_1 = p\mathfrak{A} = p - 2[a\,|\,p]\,a$, $p_2 = p_1\mathfrak{B} = p_1 - 2[b\,|\,p_1]\,b$,

then
$$[b\,|\,p_1] = [b\,|\,p] - 2[a\,|\,p]\,[a\,|\,b],$$

$$p\mathfrak{A}\mathfrak{B} = p_2 = p - 2[a\,|\,p]\,a - 2[b\,|\,p]\,b + 4[a\,|\,b]\,[a\,|\,p]\,b. \quad \text{(ii)}$$

If $\mathfrak{A} \neq \mathfrak{B}$, we have $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}$, if and only if $[a\,|\,b] = 0$, and then

$$p\mathfrak{A}\mathfrak{B} = p - 2[a\,|\,p]\,a - 2[b\,|\,p]\,b.$$

If we have n commutative inversions $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, with centres $a_1, \ldots, a_n$, then $[a_i\,|\,a_j] = \delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$. If p be any point, and $a_1, \ldots, a_n$ be independent, we have, from (ii), and
$$p = [a_1\,|\,p]\,a_1 + [a_2\,|\,p]\,a_2 + \ldots + [a_n\,|\,p]\,a_n,$$

$$p\mathfrak{A}_1 \ldots \mathfrak{A}_n = p - 2[a_1\,|\,p]\,a_1 - 2[a_2\,|\,p]\,a_2 - \ldots - 2[a_n\,|\,p]\,a_n = -p.$$

The product of n *commutative inversions in step* n *with independent centres merely reverses the weight of each extensive.*

3. If $\mathfrak{A}\mathfrak{B} = \mathfrak{C}\mathfrak{D}$, then

$$[ab] \equiv [cd], \quad [a\,|\,b] = \pm [c\,|\,d], \quad [a\,|\,c] = \pm [b\,|\,d].$$

For we have for all p, by (ii),

$$[a\,|\,p]\,a + [b\,|\,p]\,b - 2[a\,|\,b]\,[a\,|\,p]\,b = [c\,|\,p]\,c + [d\,|\,p]\,d - 2[c\,|\,d]\,[c\,|\,p]\,d. \quad \text{(iii)}$$

Multiply the sides of this outerwise by the corresponding sides of the equation obtained by putting q for p, then we have for all p, q,

$$[ab\,|\,pq]\,[ab] = [cd\,|\,pq]\,[cd]. \quad \text{(iv)}$$

Take $p = a$ in (iii), and multiply innerwise by d; again take $p = c$, and multiply innerwise by b. Thence

$$[a\,|\,b]\,[b\,|\,d] = [a\,|\,c]\,[c\,|\,d], \quad [c\,|\,d]\,[b\,|\,d] = [a\,|\,c]\,[a\,|\,b]. \quad \text{(v)}$$

Hence if $[a\,|\,b] \neq 0$, $[c\,|\,d] \neq 0$, then $[a\,|\,c] = \pm [b\,|\,d]$,

and if $[a\,|\,c] \neq 0$, $[b\,|\,d] \neq 0$, then $[a\,|\,b] = \pm [c\,|\,d]$.

And if, for example, $[a\,|\,b] = 0$, then either $[c\,|\,d] = 0$ or $[a\,|\,c] = 0$. In the latter case, since $[a\,|\,b] = 0$, and $a^2 = b^2 = c^2 = 1$,

and a, b, c are dependent, by (iv), we have $b = \pm c$, and then by (v), $[c\,|d] = 0$.

Again, if $[a\,|b] = [c\,|d] = 0$, putting $p = a, d$ in turn in (iii), we have

$$a = [a\,|c]\,c + [a\,|d]\,d, \quad d = [a\,|d]\,a + [b\,|d]\,b.$$

Hence $$[a\,|c]^2 + [a\,|d]^2 = [a\,|d]^2 + [b\,|d]^2 = 1,$$

and these give $[a\,|c] = \pm [b\,|d]$.

Thus we have $[a\,|c] = \pm [b\,|d]$ in all cases, and since $\mathfrak{AB} = \mathfrak{CD}$ implies $\mathfrak{CA} = \mathfrak{DB}$, therefore similarly $[a\,|b] = \pm [c\,|d]$.

We can, by choice of sign of d, take the positive sign in

$$[a\,|b] = \pm [c\,|d],$$

and then by (v), the sign is positive in the other formula.

Conversely, if $[ab] \equiv [cd]$, $[a\,|b] = \pm [c\,|d]$, then $\mathfrak{AB} = \mathfrak{CD}$.

4. If we take $c = \mathrm{k}_1 a + \mathrm{k}_2 b$, where

$$c^2 = \mathrm{k}_1^2 a^2 + 2\mathrm{k}_1\mathrm{k}_2 [a\,|b] + \mathrm{k}_2^2 b^2 = \mathrm{k}_1^2 + \mathrm{k}_2^2 + 2\mathrm{k}_1\mathrm{k}_2 [a\,|b] = 1,$$

and $$d = -\mathrm{k}_2 a + (\mathrm{k}_1 + 2\mathrm{k}_2 [a\,|b])\, b,$$

then $[a\,|b] = [c\,|d]$, $d^2 = 1$, and hence $\mathfrak{AB} = \mathfrak{CD}$.

Hence we can take c as any vector, of unit square, in the spread $[ab]$ and then d is fixed; similarly we can take d as any vector in the spread $[ab]$ and then c is fixed.

The product of three inversions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, *whose centres are dependent, is an inversion.* For take d in $[ab]$ so that $\mathfrak{BC} = \mathfrak{AD}$, then $\mathfrak{ABC} = \mathfrak{AAD} = \mathfrak{D}$.

5. If a, b, c be independent, we can replace $\mathfrak{ABC}$ by $\mathfrak{DEF}$, where d is any vector in the spread $[abc]$, and $[ef]$ is a fixed bivector when d is fixed. (We work in the spread $[abc]$.)

For, let $[da.bc] \equiv a_1$, then $\mathfrak{DA} = \mathfrak{D}_1\mathfrak{A}_1$, where d_1 is on $[da]$, then $\mathfrak{DABC} = \mathfrak{D}_1\mathfrak{A}_1\mathfrak{BC}$, where $\mathfrak{A}_1\mathfrak{BC}$ is a single inversion $\mathfrak{G}$, say, so that

$$\mathfrak{DABC} = \mathfrak{D}_1\mathfrak{G}, \quad \mathfrak{ABC} = \mathfrak{DD}_1\mathfrak{G}.$$

Further, if $\mathfrak{DEF} = \mathfrak{DE}_1\mathfrak{F}_1$, then $\mathfrak{EF} = \mathfrak{E}_1\mathfrak{F}_1$, $[ef] \equiv [e_1 f_1]$.

We can take e as any vector in the spread $[ef]$ and then f is fixed.

6. If $\mathfrak{DEF} = \mathfrak{D}_1\mathfrak{E}_1\mathfrak{F}_1 = \mathfrak{D}_2\mathfrak{E}_2\mathfrak{F}_2$ and d, d_1, d_2 are dependent, all points being in $[def]$, then $[ef]$, $[e_1 f_1]$, $[e_2 f_2]$ are dependent.

For, let $[ef.e_1f_1] \equiv g$, then $\mathfrak{D}\mathfrak{E}\mathfrak{F} = \mathfrak{D}\mathfrak{E}'\mathfrak{G}$, $\mathfrak{D}_1\mathfrak{E}_1\mathfrak{F}_1 = \mathfrak{D}_1\mathfrak{E}_1'\mathfrak{G}$, where e' is on $[ef]$, and e_1' is on $[e_1f_1]$.

Hence $\mathfrak{D}\mathfrak{E}'\mathfrak{G} = \mathfrak{D}_1\mathfrak{E}_1'\mathfrak{G}$, $\mathfrak{D}\mathfrak{E}' = \mathfrak{D}_1\mathfrak{E}_1'$;

d, d_1, e', e_1' are dependent on two of them, by 3.

Let $\mathfrak{D}\mathfrak{E}' = \mathfrak{D}_2\mathfrak{E}_2'$, as we may, since d, d_1, d_2 are dependent, and hence d, d_2, e' are dependent; then $\mathfrak{D}_2\mathfrak{E}_2\mathfrak{F}_2 = \mathfrak{D}\mathfrak{E}'\mathfrak{G} = \mathfrak{D}_2\mathfrak{E}_2'\mathfrak{G}$. Hence $[e_2f_2]$ goes through g. Thus we have in $[def]$ a linear transformation of vectors into bivectors, such that if one vector be taken as the first centre of three inversions giving $\mathfrak{D}\mathfrak{E}\mathfrak{F}$, the other two centres lie on the bivector corresponding to the first vector.

Since $\mathfrak{D}\mathfrak{E}\mathfrak{F} = \mathfrak{E}\mathfrak{D}_0\mathfrak{F} = \mathfrak{E}\mathfrak{F}\mathfrak{D}_0'$ for suitable $\mathfrak{D}_0$, $\mathfrak{D}_0'$, the third centre $\mathfrak{F}$ lies on the bivector corresponding to the second centre $\mathfrak{E}$.

7. The following can now be shewn by induction:

If $\mathfrak{A}_1\mathfrak{A}_2 \dots \mathfrak{A}_r$ cannot be reduced to a product of fewer than r inversions, then $a_1, \dots, a_r$ are independent, and the product equals $\mathfrak{B}_1\mathfrak{B}_2 \dots \mathfrak{B}_r$, where b_1 can be taken anywhere in $[a_1 \dots a_r]$, b_2 anywhere in the spread of step $r-1$ which corresponds to b_1 in a certain linear transformation of vectors into spreads of step $r-1$; b_3 can be taken anywhere in the spread of step $r-2$, which is the cut of the last spread and the spread which corresponds to b_2 in the transformation; and so on. Finally b_r is fixed.

8. *Any product of more than* r *inversions, whose centres are in a spread of step* r, *can be reduced to a product of* r *inversions or fewer.*

For, if $\mathfrak{A}_1\mathfrak{A}_2 \dots \mathfrak{A}_r\mathfrak{A}_{r+1} \dots \mathfrak{A}_{r+s}$ be a product of inversions, then, by 7, $\mathfrak{A}_2\mathfrak{A}_3 \dots \mathfrak{A}_{r+1} = \mathfrak{A}_1\mathfrak{A}_2' \dots \mathfrak{A}_r'$ for suitable $a_2', \dots, a_r'$, if the product $\mathfrak{A}_2\mathfrak{A}_3 \dots \mathfrak{A}_{r+1}$ does not itself reduce.

Hence $\mathfrak{A}_1\mathfrak{A}_2 \dots \mathfrak{A}_r\mathfrak{A}_{r+1} \dots \mathfrak{A}_{r+s} = \mathfrak{A}_2' \dots \mathfrak{A}_r'\mathfrak{A}_{r+2} \dots \mathfrak{A}_{r+s}$,

and so on.

9. An orthogonal transformation in step n can be reduced to a product of inversions and an orthogonal transformation in step $n-1$. The weights are supposed to be real.

For, let $\mathfrak{T} = (a_{ij})$ be the matrix of an orthogonal transformation. Since the sum of the squares of the elements in any row is unity, we have $a_{nt} \neq 0$ for some t in 1, ..., n. Take the frame so that $a_{nn} \neq 0$.

Consider the inversion $\mathfrak{R}$ in the centre $\sin \frac{1}{2}\theta . e_r + \cos \frac{1}{2}\theta . e_n$, for some $r \neq n$, the frame being normal.

If $p = p_1 e_1 + p_2 e_2 + \ldots + p_n e_n$, we have

$$\begin{aligned} p\mathfrak{R} &= p - 2[p\,|\sin\tfrac{1}{2}\theta.e_r + \cos\tfrac{1}{2}\theta.e_n](\sin\tfrac{1}{2}\theta.e_r + \cos\tfrac{1}{2}\theta.e_n) \\ &= p - 2(\sin^2\tfrac{1}{2}\theta.p_r.e_r + \cos^2\tfrac{1}{2}\theta.p_n.e_n \\ &\qquad + \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta(p_r.e_n + p_n.e_r)) \\ &= p' = p'_1 e_1 + p'_2 e_2 + \ldots + p'_n e_n, \text{ say.} \end{aligned}$$

Then $p'_i = p_i$, if $i \neq r, n$;

$$p'_r = \cos\theta.p_r - \sin\theta.p_n, \quad p'_n = -\sin\theta.p_r - \cos\theta.p_n.$$

Hence $\mathfrak{T}\mathfrak{R} = (b_{ij})$, where $b_{nj} = a_{nj}$ if $j \neq r, n$, while

$$b_{nr} = a_{nr}\cos\theta - a_{nn}\sin\theta, \quad b_{nn} = -a_{nr}\sin\theta - a_{nn}\cos\theta.$$

Take θ so that $b_{nr} = 0$. Then $b_{nn} \neq 0$, otherwise we should have $a_{nn}^2 + a_{nr}^2 = 0$, whereas $a_{nn} \neq 0$.

Hence, we have made $b_{nr} = 0$, and still have $b_{nn} \neq 0$, and the other elements of the last row of the matrix $\mathfrak{T}\mathfrak{R}$ are as in $\mathfrak{T}$.

Thus we can choose inversions $\mathfrak{R}_1, \ldots, \mathfrak{R}_s$ so that in $\mathfrak{T}\mathfrak{R}_1 \ldots \mathfrak{R}_s$ the last row of the matrix is $(0, 0, \ldots, c_{nn})$, where $c_{nn} = \pm 1$, since the matrix is orthogonal. For the same reason, the last column is $(0, 0, \ldots, c_{nn})$.

Hence $\mathfrak{T}\mathfrak{R}_1 \ldots \mathfrak{R}_s$ is an orthogonal transformation $\mathfrak{S}$ in $n-1$ variables.

Hence $\mathfrak{T} = \mathfrak{S}\mathfrak{R}_s \mathfrak{R}_{s-1} \ldots \mathfrak{R}_1$.

10. *An orthogonal transformation $\mathfrak{T}$ in step* n *can be factored into* n *inversions or fewer.*

For it is the product of an orthogonal transformation in step $n-1$ and inversions; and an orthogonal transformation in step two is either identity, an inversion, or the product of two inversions.

Hence, by induction, $\mathfrak{T}$ is the product of inversions, of which, by 8, at most n are required.

It should be noticed that, if n is even, and $\mathfrak{T}$ direct, it may be necessary also to reverse the signs of all the variables, after the product has been formed, if $\mathfrak{T}$ is to be obtained; for we have disregarded the weights, and in particular, the signs of our vectors.

Examples. 1. Erect a theory of inversions which preserve vectors in a spread of step n, whose inner squares are zero, and apply it to shew that every transformation of points into points, in a plane, which carries concyclic points into such, may be factored into four

or three inversions in circles, according as the transformation is direct or indirect. (Cf. § 73, Exs. 15, 16, 17.)

2. *Inversion in a system k of circles in a plane.* (Cf. § 85.)

If $c_1 = -\frac{2[c\,|k]}{k^2}k + c$, then the system c_1 is '*inverse*' to c with respect to k. Then $c_1^2 = c^2$. If c is a nul system, so is c_1. Also $[c\,|k] = -[c_1\,|k]$. Thus any circle in the system k is invariant for the inversion.

If c_2' is a circle of system k which touches c', then

$$[c_2\,|k] = 0 = [c_2\,|c].$$

Hence $[c_1\,|c_2] = 0$, therefore c_2' touches the circle c_1' inverse to c'.

If k is a set of circles orthogonal to a fixed circle, we have ordinary inversion.

If k is a nul net of circles, we have a '*Laguerre inversion*', a transformation of rotors into rotors, which reverses the sign of one rotor L, and turns rotors touching an oriented circle into rotors touching another oriented circle, the original and the transformed circle having L as radical axis. In the space representation this transformation corresponds to a reflection in a plane.

3. In the geometry of systems of circles, let c' be a circle in system k_1 and not in k_2; let c_1' be the inverse of c' in k_2; c_2' the inverse of c_1' in k_1; c_3' the inverse of c_2' in k_2; then c_3' is a linear combination of c', c_1', c_2'.

4. A transformation of rotors to rotors in a plane, which changes rotors touching an oriented circle into such, is the product of four or three Laguerre inversions according as it is direct or indirect.

5. A transformation of points in space into points, which turns cospherical points into such, is a product of four or five inversions in spheres according as it is direct or indirect.

6. We may define Laguerre inversion in space, oriented planes taking the place of rotors. A transformation which changes oriented planes into such, and those touching an oriented sphere into such, is the product of four or five Laguerre inversions.

7. A transformation of systems of spheres into systems of spheres is the product of six or seven inversions in systems of spheres; the corresponding numbers in the plane are four and five.

8. A collineation in space is the product of three biaxial involutions.

CHAPTER XIV

QUADRIC SPREADS IN SPREADS OF ANY STEP

§ 105. *Two lemmas on determinants.*

If
$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m = 0$$

be m equations for $x_1, x_2, \ldots, x_m$ and the matrix of the a_{ij} be of rank $m-1$, and hence $\det(a_{ij})$ vanishes, then the system has only one independent solution, say $x_1 = B_1, \ldots, x_m = B_m$ and any solution is of form $x_1 = kB_1, \ldots, x_m = kB_m$, where k is a scalar. Now the cofactors $A_{r1}, \ldots, A_{rm}$ of $a_{r1}, \ldots, a_{rm}$ give a solution where r is any one of $1, \ldots, m$. Hence there are scalars $A_1, \ldots, A_l$ such that

$$A_{r1} = A_r B_1, \quad A_{r2} = A_r B_2, \quad \ldots, \quad A_{rm} = A_r B_m \quad (r = 1, \ldots, m).$$

Hence

Lemma 1. *If* A *is a vanishing determinant of rank* $m-1$ *and order* m, *we can find* $A_1, \ldots, A_m, B_1, \ldots, B_m$ *such that the cofactors of elements of the determinant are of form* $A_{rs} = A_r B_s$, *where* $B_1, \ldots, B_m$ *is a solution of the linear equations above.*

Further, if A *is symmetrical, then since* $A_r B_s = A_s B_r$, *we can take* $A_i = B_i$ *and then* $A_{rs} = A_r A_s$.

Lemma 2. *If*
$$R = \begin{vmatrix} z & x_1 & x_2 & \ldots & x_m \\ y_1 & a_{11} & a_{12} & \ldots & a_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ y_m & a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix},$$

then
$$R = zA - \sum_{r,s=1}^{m} A_{rs} x_r y_s,$$

where $A = \det(a_{ij})$, *and* A_{ij} *is the cofactor of* a_{ij} *in* A.

If (a_{ij}) *be symmetrical, and of rank* $m-1$, then since we can find A_i such that $A_{rs} = A_r A_s$, *we have*

$$R = -\sum_{r,s=1}^{m} A_r A_s x_r y_s = -\sum_{r=1}^{m} A_r x_r \sum_{s=1}^{m} A_s y_s.$$

In particular, if $x_i = y_i$, $(i = 1, \ldots, m)$, we have

$$R = -\left(\sum_{r,s=1}^{m} A_r x_r\right)^2.$$

§ 106. *Quadrics in step* n.*

1. Let $x^2 = 0$ be the equation of a non-singular quadric $\mathcal{Q}$ in a normal frame; let $a_1, a_2, \ldots, a_n$ be independent points, where $a_2, \ldots, a_n$ are on the prime, (spread of step $n-1$), tangent to $\mathcal{Q}$ at p.

Then $[p\,|a_i] = 0$, $(i = 2, \ldots, n)$, $p^2 = 0$.

Suppose $p = k_2 a_2 + k_3 a_3 + \ldots + k_n a_n$, then $k_2, \ldots, k_n$ satisfy

$$k_2[a_2\,|a_2] + k_3[a_2\,|a_3] + \ldots + k_n[a_2\,|a_n] = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$k_2[a_n\,|a_2] + k_3[a_n\,|a_3] + \ldots + k_n[a_n\,|a_n] = 0.$$

These equations have a symmetric matrix $([a_i\,|a_j])$, whose determinant vanishes; for since $p \equiv |[a_2 a_3 \ldots a_n]$ and $[p a_2 \ldots a_n] = 0$, we have $[a_2 \ldots a_n]^2 = 0$.

If the determinant had rank $< n-2$, we could satisfy the equations by an independent set of k, and $[a_2 \ldots a_n]$ would touch $\mathcal{Q}$ in another point besides p.

We assume this is not the case.

Then by Lemma 1, we can find k_i so that $k_r k_s$ is the cofactor of $[a_r\,|a_s]$ in $[a_2 \ldots a_n]^2$.

Take the k in this way, absorbing weights in $a_1, a_2, \ldots, a_n$, if necessary, and then adjust the weight of a_1 so that $[a_1 a_2 \ldots a_n] = 1$.

2. Let q be any point on a tangent prime to $\mathcal{Q}$ through $a_2, \ldots, a_{n-1}$, then $[q a_2 \ldots a_{n-1}]^2 = 0$. Now $[q a_2 \ldots a_{n-1}]^2$ is the cofactor of a_n^2 in $[q a_2 \ldots a_n]^2$. Denote this last determinant by d; then since $[a_2 \ldots a_n]^2$ is of rank $n-2$, we can apply Lemma 2 to d.

The top line of d is q^2, $[q\,|a_2]$, $[q\,|a_3]$, ..., $[q\,|a_n]$ corresponding to z, $x_1, \ldots, x_m$ of Lemma 2; the A_i of the lemma are now k_i.

Hence $d = -[q\,|(k_2 a_2 + k_3 a_3 + \ldots + k_n a_n)]^2 = -[q\,|p]^2$.

Now $|p = k[a_2 \ldots a_n]$, where k is some scalar.

Hence $d = -k^2[q a_2 \ldots a_n]^2 = -k^2 d$.

* Baker, *Proc. Lond. Math. Soc.* (2), 9 (1911), p. 177.

If now the tangent prime through $a_2, \ldots, a_{n-1}$ is distinct from that which touches $\mathcal{Q}$ at p, we have $d \neq 0$, hence $k = \sqrt{(-1)}$.

$$|p = \sqrt{(-1)}\,[a_2 \ldots a_n], \quad [a_1|p] = \sqrt{(-1)}\,[a_1 a_2 \ldots a_n] = \sqrt{(-1)},$$
$$[a_r|p] = 0 \quad (r = 2, \ldots, n).$$

3. To find the other tangent prime to $\mathcal{Q}$ through

$$[a_2 a_3 \ldots a_{r-1} a_{r+1} \ldots a_n].$$

Let $q_n = |[a_1 a_2 \ldots a_{n-1}], \quad q_{n-1} = -|[a_1 a_2 \ldots a_{n-2} a_n], \ \ldots,$

$$q_r = (-1)^{n-r}|[a_1 a_2 \ldots a_{r-1} a_{r+1} \ldots a_n], \ \ldots,$$
$$q_2 = (-1)^{n-2}|[a_1 a_3 \ldots a_n], \quad (\S 56{\cdot}5).$$

Then $[a_r|q_r] = 1, \quad [a_r|q_s] = 0, \quad (r \neq s;\ r, s = 2, \ldots, n),$

$$[a_1|q_r] = 0, \quad [p|q_r] = k_r[a_r|q_r] = k_r, \quad (r = 2, \ldots, n).$$

Let $$a'_r = q_r - \tfrac{1}{2}k_r^{-1}q_r^2 p \quad (r = 2, \ldots, n).$$

Then $$a'^2_r = q_r^2 - k_r^{-1}q_r^2[p|q_r] = 0, \quad [a'_r|p] = k_r.$$

Hence a'_r is on the quadric $\mathcal{Q}$. It is also on the tangent prime through the a_s with $r \neq s$, since

$$[a'_r|a_r] = 1, \quad [a'_r|a_s] = 0, \quad (r \neq s;\ r, s = 2, \ldots, n).$$

Thus a'_r is the point of contact of the tangent prime, other than $|p$, through

$$[a_2 a_3 \ldots a_{r-1} a_{r+1} \ldots a_n].$$

4. Denote

$$[a_2 \ldots a_{r-1} a_{r+1} \ldots a_n] \quad \text{and} \quad [a_2 \ldots a_{s-1} a_{s+1} \ldots a_{r-1} a_{r+1} \ldots a_n]$$

by π_r and π_{rs} $(= \pi_{sr})$ respectively; $(r, s = 2, \ldots, n$, here and below).

We have $\pi_r = x\,|[pa'_r]$, x scalar, by the last statement of 3.

Thence $$1 = [a_1 \ldots a_n] = (-1)^r[a_1 a_r \pi_r]$$
$$= (-1)^r x[a_1 a_r | pa'_r].$$

But $$[a_1 a_r | pa'_r] = [a_1|p][a_r|a'_r] - [a_r|p][a_1|a'_r] = \sqrt{(-1)},$$

since $$[a_r|p] = 0, \quad [a_r|a'_r] = 1, \quad [a_1|p] = \sqrt{(-1)}.$$

Hence $$(-1)^r\sqrt{(-1)}.x = 1, \quad (-1)^r\sqrt{(-1)}.\pi_r = |[pa'_r],$$
$$(-1)^r\sqrt{(-1)}.[a'_s\pi_r] = [a'_s|pa'_r] = [a'_s|a'_r]\,|p - [a'_s|p]\,|a'_r,$$
$$[a'_s\pi_r]^2 = 2[a'_s|a'_r][a'_s|p][p|a'_r] = 2k_r k_s[a'_s|a'_r].$$

But $$[a'_s \pi_r] = \pm [a'_s a_s \pi_{rs}],$$

hence $$[a'_s a_s \pi_{rs}]^2 = 2k_r k_s [a'_s | a'_r].$$

Now let $a_2, \ldots, a_n$ *be on the quadric* $\mathcal{Q}$, *as well as* $a'_2, \ldots, a'_n$. Then the first row and first column of $[a'_s a_s \pi_{rs}]^2$ are 0, $[a_s | a'_s]$, 0, 0, ..., 0, and the second row and second column are $[a_s | a'_s]$, 0, $[a_s | a_2]$, ...

Hence $$[a'_s a_s \pi_{rs}]^2 = -[a_s | a'_s]^2 \pi_{rs}^2 = -\pi_{rs}^2.$$

Hence $$2k_r k_s [a'_r | a'_s] = -\pi_{rs}^2, \quad (r, s = 2, \ldots, n).$$

§ 107. *The double-six theorem.*

1. If $\mathcal{Q}$ be a non-singular quadric, in a spread of step six, and $a_2, \ldots, a_6$ be on $\mathcal{Q}$, and on the tangent prime to $\mathcal{Q}$ at p, and if $a'_2, \ldots, a'_6$ be the points of contact, other than p, of tangent primes from $[a_3 a_4 a_5 a_6], \ldots, [a_2 a_3 a_4 a_5]$ respectively, then the prime $[a'_2 \ldots a'_6]$ touches $\mathcal{Q}$.

For $[a'_2 \ldots a'_6]^2$ is a symmetrical determinant of order five, whose main-diagonal elements vanish. We have to shew that this determinant vanishes.

Consider any such determinant Δ of order five, and, for brevity, denote its ij-element by ij. Let Δ' be the determinant whose ij-element is half the cofactor of a_{ii}, a_{jj} in Δ, then

$$\Delta' = \begin{vmatrix} 0, & 34\cdot45\cdot53, & 24\cdot45\cdot52, & 23\cdot35\cdot52, & 23\cdot34\cdot42 \\ 34\cdot45\cdot53, & 0, & 14\cdot45\cdot51, & 13\cdot35\cdot51, & 13\cdot34\cdot41 \\ 24\cdot45\cdot52, & 14\cdot45\cdot51, & 0, & 12\cdot25\cdot51, & 12\cdot24\cdot41 \\ 23\cdot35\cdot52, & 13\cdot35\cdot51, & 12\cdot25\cdot51, & 0, & 12\cdot23\cdot31 \\ 23\cdot34\cdot42, & 13\cdot34\cdot41, & 12\cdot24\cdot41, & 12\cdot23\cdot31, & 0 \end{vmatrix}.$$

Multiply the columns by $12.13.14.15$, $21.23.24.25$, $31.32.34.35$, and so on; then divide the rows respectively by $23.24.25.34.35.45$, $13.14.15.34.35.45$, ..., and so on.

Thence we find $\Delta' = P.\Delta$, where P is the product of all the ij.

Now apply this to our determinant $[a'_2 \ldots a'_6]^2$. By § 106, we have $2k_2 k_3 [a'_2 | a'_3] = -\pi_{23}^2 = -[a_4 a_5 a_6]^2$, and so on.

Hence the ij-element of $[a'_2 \ldots a'_6]^2$ is the cofactor of the elements in places (ii) and (jj) of $[a_2 \ldots a_6]^2$, apart from a factor $-\frac{1}{2} k_i^{-1} k_j^{-1}$.

Hence $[a'_2 \ldots a'_6]^2$ is a multiple of $[a_2 \ldots a_6]^2$, and since the latter vanishes, by hypothesis, so does the former.

2. The theorem of 1 is peculiar to step six.* If $n \neq 6$, then for general positions of the a_r, on the cut of the quadric $\mathcal{Q}$ and its tangent prime at p, we have $[a'_2 \dots a'_n]^2 \neq 0$.

If $|[a'_2 \dots a'_n] = p'$, let $[p'a_r]$ cut $\mathcal{Q}$ again in a''_r, and let $[pp']$ cut $\mathcal{Q}$ again in q''.

Then $$|[a'_2 \dots a'_{r-1} a'_{r+1} \dots a'_n] \equiv [p'a_r] \equiv [p'a''_r].$$

Let $$|[a''_2 \dots a''_n] = p''.$$

We have $p''^2 = 0$, for $[pa_r p']$ cuts $\mathcal{Q}$ in a conic composed of lines $[pa_r]$, $[q''a''_r]$.

Hence the lines $[q''a''_r]$ $(r = 2, \dots, n)$ lie on $\mathcal{Q}$. Hence $[a''_2 a''_3 \dots a''_n]$ touches $\mathcal{Q}$ at q''. Hence $p'' = q''$.

3. If a non-vanishing symmetric determinant of order *six* has its main-diagonal elements zero, and no others zero, and no principal minor of order four zero, then if five of its principal first minors vanish, so does the sixth.

4. If in a non-vanishing symmetric determinant of order *four*, the main-diagonal elements are all zero, and also the minors of three of them, then the minor of the remaining element is not zero, if in any row the elements outside the main diagonal are all non-zero.

5. If in a non-vanishing symmetric determinant of order *five*, the main-diagonal elements are all zero, and also the minors of four of them, then the elements not on the main diagonal are all non-zero. If the elements of the determinant are real, the minor of the remaining element in the main diagonal is not zero.

We leave the proofs of 3, 4, 5 and their geometric interpretation to the reader.†

§ 108. *Screws as elements in a spread of step six.*

1. Any screw, in a space of step four, is of form

$$S = x_1 S_1 + \dots + x_6 S_6,$$

where x_i are scalars and $S_1, \dots, S_6$ suitably chosen screws. We can take $S_1, \dots, S_6$ as a system in involution (§ 36·5). Then S is a rotor, if, and only if,

$$x_1^2 + x_2^2 + \dots + x_6^2 = 0. \qquad (1)$$

* Barbilian, *Bull. Math. Phys. l'École polytech. Bucarest*, 1 (1929), p. 12; Tzitzeica, *ibid.* p. 16.

† W. V. Parker, *Amer. Math. Monthly*, 41 (1934), p. 174.

If $S = x_1 S_1 + \ldots + x_6 S_6$ and $T = y_1 S_1 + \ldots + y_6 S_6$,

then S, T are in involution, if, and only if,

$$x_1 y_1 + \ldots + x_6 y_6 = 0. \qquad (2)$$

2. Now regard $S_1, \ldots, S_6$ as unities in a spread of step six, and work in this spread. Then (1) is the equation of a quadric $\mathcal{Q}$ in the spread. We replace $S_1, \ldots, S_6$ by $e_1, \ldots, e_6$, then as $S_1, \ldots, S_6$ form a normal system, so do $e_1, \ldots, e_6$. We denote the operation of taking the supplement in this frame by a stroke.

To avoid confusion we denote the elements of the spread of step six by small letters, and say that S is a rotor if $s^2 = 0$; Q, R are in involution if $[q\,|r] = 0$.

The outer product of p, q is the line $[pq]$; it represents the spread of screws $x_1 P + x_2 Q$, and it meets $\mathcal{Q}$ in points representing the rotors in this spread of screws.

If p, q, r be independent elements, their outer product $[pqr]$ is a plane which cuts $\mathcal{Q}$ in a conic representing the system of rotors in the 3-spread of screws $x_1 P + x_2 Q + x_3 R$.

3. The polar 5-spread of an element s, namely $|s$, cuts $\mathcal{Q}$ in points which represent the nul lines of the screw S, or the linear complex given by S; for if p be such a point, then $p^2 = 0$, $[p\,|s] = 0$. (If P be the corresponding nul line, then in the space of step four, $[PP] = 0$, $[PS] = 0$.)

The polar 4-spread of the line $[qr]$, namely $|[qr]$, cuts $\mathcal{Q}$ in points which represent the nul lines of the 2-spread of screws $x_1 Q + x_2 R$, or the linear congruence given by Q, R.

The polar 3-spread of the plane $[pqr]$, namely $|[pqr]$, cuts $\mathcal{Q}$ in points of a conic which represent the nul lines of the 3-spread of screws $x_1 P + x_2 Q + x_3 R$.

Two rotors which intersect are in involution, and hence represent two conjugate points on $\mathcal{Q}$. Hence two opposite reguli in space of step four represent two sections of $\mathcal{Q}$ cut out by conjugate planes. Hence the rotors and the nul lines of a 3-spread of screws represent the points of such sections.

Three rotors which are dependent, and hence meet in a point, or lie on a plane, represent three points on a generator spread of $\mathcal{Q}$ (§ 70).

4. As linear complexes represent planes, we could consider the inner products of planes. The inner product of two planes vanishes when the corresponding linear complexes are in involution; the inner product of the poles of the planes then vanishes; these poles are represented by the screws which give the complexes; these screws are in involution.

5. The two rotors in the pencil $P+\mathrm{k}Q$ coincide when $[pq]$ touches $\mathit{2}$, that is, when

$$[pq]^2 = 0, \quad \text{or} \quad [p\,|q]^2 = p^2q^2.$$

The rotors in the spread $P+\mathrm{k}_1Q+\mathrm{k}_2R$ are given by values of k_1, k_2 which satisfy $(p+\mathrm{k}_1q+\mathrm{k}_2r)^2 = 0$. This is a product of two linear factors when $[pqr]^2 = 0$, that is, when $[pqr]$ is a generator-plane of $\mathit{2}$. The regulus of rotors contained in $P+\mathrm{k}_1Q+\mathrm{k}_2R$ then becomes a bundle of rotors or a coplanar set of rotors.

If p, q, r, s be independent, but $[pqrs]^2 = 0$, then $[|\,pqrs]^2 = 0$; hence the line $|[pqrs]$ touches $\mathit{2}$. The two rotors in the corresponding system of screws cut.

In particular, if p, q, r, s be on $\mathit{2}$, and hence P, Q, R, S be independent rotors, then $[pqrs]^2 = 0$ is the condition that just one rotor cuts P, Q, R, S.

If p, q, r, s, t be independent, but $[pqrst]^2 = 0$, that is, if $[pqrst]$ touches $\mathit{2}$, then the screw conjugate to the screws P, Q, R, S, T is a rotor; hence the screws have a common nul line.

In particular, if P, Q, R, S, T be independent rotors, then $[pqrst]^2 = 0$ is the condition that they have a common transversal; the five rotors then determine a special complex containing them.

By § 108 we now have the double-six theorem in its usual form: *if P, Q, R, S, T be general lines cutting a fixed line, and if P' is the other transversal of Q, R, S, T, and Q' is the other transversal of P, R, S, T, and so on for R', S', T', then P', Q', R', S', T' have a common transversal.*

If p_1, p_2, ..., p_6 satisfy $[p_1p_2\dots p_6]^2 = 0$, then $[p_1p_2\dots p_6]$, being scalar, vanishes. The screws P_1, ..., P_6 are dependent; if they are rotors, they are in the same linear complex.

Hence the condition that rotors P_1, ..., P_6 be in the same linear complex is that $\det[P_i\,|P_j]$ vanishes.

If p_1, ..., p_7 be any extensives, then $\det[p_i\,|p_j]$ vanishes.

6. *If $l_1, l_2, \ldots, l_5$ be five points on $\mathcal{Q}$, and $[l_1 \ldots l_5]$ does not touch $\mathcal{Q}$, but each of $[l_2 l_3 l_4 l_5]$, $[l_1 l_3 l_4 l_5]$, $[l_1 l_2 l_4 l_5]$, $[l_1 l_2 l_3 l_5]$, $[l_1 l_2 l_3 l_4]$ does so, then the solids which join any four of the points of contact touch $\mathcal{Q}$.*

For let $\mathrm{d} = [l_1 \ldots l_5]^2$, then $\mathrm{d} \neq 0$. Let c_{ij} be the cofactor of $[l_i | l_j]$ in $[l_1 \ldots l_5]^2$, and let

$$m_r = \mathrm{c}_{r1} l_1 + \mathrm{c}_{r2} l_2 + \ldots + \mathrm{c}_{r5} l_5.$$

Then $$[m_r | l_s] = \mathrm{c}_{r1}[l_1 | l_s] + \mathrm{c}_{r2}[l_2 | l_s] + \ldots + \mathrm{c}_{r5}[l_5 | l_s] = 0,$$

if $r \neq s$, by the properties of determinants.

Hence $$\begin{aligned} m_r^2 &= \mathrm{c}_{r1}[l_1 | m_r] + \mathrm{c}_{r2}[l_2 | m_r] + \ldots + \mathrm{c}_{r5}[l_5 | m_r] \\ &= \mathrm{c}_{rr}[l_r | m_r]. \end{aligned}$$

But since, for example, $[l_2 l_3 l_4 l_5]$ touches $\mathcal{Q}$, we have

$$[l_2 l_3 l_4 l_5]^2 = 0,$$

or $\mathrm{c}_{11} = 0$. Similarly $\mathrm{c}_{rr} = 0$, $(r = 1, \ldots, 5)$. Hence $m_r^2 = 0$. The points of contact are hence $m_1, \ldots, m_5$.

But $$[m_1 \ldots m_5]^2 = (\det(\mathrm{c}_{ij}))^2 . [l_1 \ldots l_5]^2 \neq 0.$$

The minors $[m_2 m_3 m_4 m_5]^2$, $[m_1 m_3 m_4 m_5]^2$, ... of $[m_1 \ldots m_5]^2$ are proportional to the corresponding minors of $[l_1 \ldots l_5]^2$ and hence vanish.

Hence, *if in a spread of step four, the lines $L_1, \ldots, L_5$ are not met by a common line, but each four are met by just one transversal (instead of the usual two), then each four of these five transversals are met by just one transversal.* (Weitzenböck,* Segre.)

§ 109. *Spheres in spreads of any step.*

1. *The distance formula.* Let $e_1, \ldots, e_n$ be the base-extensives in a spread of step n, and $u_1, \ldots, u_n$ be vectors to them from any point; and suppose we can form inner products of vectors.

Let $$p = \sum_1^n \mathrm{x}_i e_i, \quad q = \sum_1^n \mathrm{y}_i e_i, \quad (\mathrm{x}_i, \mathrm{y}_i \text{ scalars}),$$

$$\Sigma \mathrm{x}_i = 1, \quad \Sigma \mathrm{y}_i = 1;$$

then $$(p-q)^2 = (\Sigma(\mathrm{x}_i - \mathrm{y}_i)u_i)^2 = \sum_{i,j} (\mathrm{x}_i - \mathrm{y}_i)(\mathrm{x}_j - \mathrm{y}_j)[u_i | u_j].$$

* *Amsterdam Proceedings*, 31 (1928), p. 133.

Let $l_{ij}^2 = (e_i - e_j)^2$, then $l_{ij}^2 = u_i^2 + u_j^2 - 2[u_i | u_j]$,

$$2(p-q)^2 = \sum_{i,j} (x_i - y_i)(x_j - y_j)(u_i^2 + u_j^2 - l_{ij}^2)$$
$$= 2\sum_i (x_i - y_i) \sum_j (x_j - y_j) u_j^2 - \sum_{i,j} (x_i - y_i)(x_j - y_j) l_{ij}^2.$$

The first term of the preceding expression vanishes, since $\Sigma x_i = \Sigma y_i = 1$.

Hence $$(p-q)^2 = -\tfrac{1}{2} \sum_{i,j} (x_i - y_i)(x_j - y_j) l_{ij}^2.$$

(Cf. § 8, Ex. 98.)

Denote the matrix $-\frac{1}{2}(l_{ij}^2)$, $(i, j = 1, \ldots, n)$ by $\mathfrak{D}$, and regard it as representing a polarity in the spread of points, then

$$(p-q)^2 = [(p-q)\mathfrak{D}(p-q)].$$

In our fixed frame $e_1, \ldots, e_n$, $[p\mathfrak{D}q]$ has a meaning when p, q are points. We have $[e_i \mathfrak{D} e_j] = -\frac{1}{2} l_{ij}^2$, $[e_i \mathfrak{D} e_i] = 0$.

2. *Circumspheres.* If $c = c_1 e_1 + \ldots + c_n e_n$, $(\Sigma c_i = 1)$, be the circumcentre of the simplex $e_1, \ldots, e_n$, and if $p = x_1 e_1 + \ldots + x_n e_n$, $(\Sigma x_i = 1)$, be any point on the circumsphere of the simplex, and r be the radius, we have

$$r^2 = [(p-c)\mathfrak{D}(p-c)].$$

The sphere goes through e_1, hence

$$r^2 = [(e_1 - c)\mathfrak{D}(e_1 - c)] = [c\mathfrak{D}c] - 2[e_1 \mathfrak{D} c].$$

From this and similar equations, we have

$$[e_1 \mathfrak{D} c] = [e_2 \mathfrak{D} c] = \ldots = [e_n \mathfrak{D} c].$$

Hence $$[c\mathfrak{D}c] = [(c_1 e_1 + \ldots + c_n e_n)\mathfrak{D}c]$$
$$= (c_1 + c_2 + \ldots + c_n)[e_1 \mathfrak{D} c] = [e_1 \mathfrak{D} c],$$
$$r^2 = -[e_1 \mathfrak{D} c] = -[e_2 \mathfrak{D} c] = \ldots = -[e_n \mathfrak{D} c] = -[c\mathfrak{D}c],$$
$$[p\mathfrak{D}c] = [(x_1 e_1 + \ldots + x_n e_n)\mathfrak{D}c] = (x_1 + \ldots + x_n)[e_1 \mathfrak{D} c] = -r^2.$$

Hence the equation of the circumsphere is

$$[p\mathfrak{D}p] - 2[c\mathfrak{D}p] + [c\mathfrak{D}c] = r^2, \quad \text{or} \quad [p\mathfrak{D}p] = 0.$$

3. *Haskell's Pivot Theorem.**

If points be taken on the edges of a simplex and spheres be drawn through each vertex and the points on those edges which meet there, then these spheres go through a point.

* M. W. Haskell, *Archiv. Math. Phys.* (Ser. 3), 5, (1903), p. 278.

For since the equation of the circumsphere is $[p\mathfrak{D}p] = 0$, the equation of any other sphere is of the form $[p\mathfrak{D}p] - [p\pi] = 0$, where π is some prime.

This sphere goes through e_i, if $[e_i\pi] = 0$.

It goes through

$$p_{ij} = k_{ij}e_j + k_{ji}e_i, \quad (k_{ij} + k_{ji} = 1),$$

the point on the edge e_ie_j as well, if

$$[(k_{ij}e_j + k_{ji}e_i)\,\mathfrak{D}(k_{ij}e_j + k_{ji}e_i)] - [(k_{ij}e_j + k_{ji}e_i)\,\pi] = 0,$$

that is, if $$k_{ij}k_{ji}l_{ij}^2 + k_{ij}[e_j\pi] = 0,$$

or since $k_{ij} \neq 0$, if $$l_{ij}^2 k_{ji} + [e_j\pi] = 0.$$

Hence, if it goes through all p_{ij} as $j = 1, \ldots, n$, $(i \neq j)$, then

$$\sum_j l_{ij}^2 k_{ji} p_j + [p\pi] = 0,$$

where $$p = p_1e_1 + \ldots + p_ne_n, \quad (p_1 + p_2 + \ldots + p_n = 1).$$

Hence the sphere through e_i has equation

$$[p\mathfrak{D}p] + \sum_j l_{ij}^2 k_{ji} p_j = 0. \qquad \text{(i)}$$

Now all such spheres go through $q = q_1e_1 + \ldots + q_ne_n$, where the q_i satisfy

$$\sum_j l_{ij}^2 k_{j1} q_j = \sum_j l_{2j}^2 k_{j2} q_j = \sum_j l_{3j}^2 k_{j3} q_j = \ldots; \quad \sum_i q_i = 1. \qquad \text{(ii)}$$

For $$-2[q\mathfrak{D}q] = \sum_{i,j} q_i q_j l_{ij}^2 = \sum_{i,j} (k_{ij}q_i q_j + k_{ji}q_i q_j) l_{ij}^2$$
$$= 2\sum_j l_{ij}^2 k_{ji} q_j,$$

by (ii), since $l_{ij}^2 = l_{ji}^2$.

Hence q satisfies (i).

Cor. The sphere through $p_1, \ldots, p_n$ has equation

$$[p\mathfrak{D}p][p_1 \ldots p_n] = [p_1\mathfrak{D}p_1][pp_2 \ldots p_n] + [p_2\mathfrak{D}p_2][p_1pp_3 \ldots p_n] + \ldots + [p_n\mathfrak{D}p_n][p_1p_2 \ldots p_{n-1}p].$$

§ 110. *On spreads of even step and odd step.**

1. *If $p_1, \ldots, p_6$ be dependent extensives in a spread of step six, and $q_1, \ldots, q_6$ be extensives in the spread such that $[p_1 \ldots p_6]$ still vanishes when any two of the p_i are replaced by the corresponding q_i, then $[q_1q_2 \ldots q_6] = 0$.*

* See Baker, *Principles*, 4, p. 60.

For weight the p so that $p_1+p_2+\ldots+p_6=0$. (i)

Suppose the q not dependent on $p_1, p_2, \ldots, p_6$.

We can let $$q_i=\sum_{j=1}^{6} a_{ij}p_j-v, \qquad \text{(ii)}$$

where v is an extensive independent of $p_1, \ldots, p_6$ and $a_{ii}=0$.

Then
$$\begin{aligned}[p_1p_2p_3p_4q_5] &= [p_1\ldots p_4(a_{56}p_6-v)]\\ &= a_{56}[p_1\ldots p_4p_6]-[p_1\ldots p_4v]\\ &= -a_{56}[p_1\ldots p_4p_5]-[p_1\ldots p_4v].\end{aligned}$$

By hypothesis, $[p_1p_2p_3p_4q_5q_6]=0$.

Hence
$$\begin{aligned}0 &= a_{56}[p_1\ldots p_5v]-[p_1\ldots p_4v.a_{65}p_5]\\ &= (a_{56}+a_{65})[p_1\ldots p_5v].\end{aligned}$$

Hence $a_{56}=-a_{65}$. Similarly
$$a_{ij}=-a_{ji}, \quad (i,j=1,\ldots,6).$$

Thus the coefficients of $p_1, \ldots, p_6, v$ in the expressions
$$\sum_{j=1}^{6} a_{ij}p_j-v, \quad (i=1,\ldots,6), \quad p_1+\ldots+p_6$$
form a skew-symmetric determinant Δ, say. Let A_{ij} be the cofactor of a_{ij} in this determinant, then by (i), (ii)
$$A_{11}q_1+A_{21}q_2+\ldots+A_{61}q_6=\Delta p_1=0,$$
since Δ, as a skew-symmetric determinant of odd order, vanishes. Hence $[q_1\ldots q_6]=0$.

2. For a spread of step four, the analogous theorem gives:

If the six joins of p_1, p_2, p_3, p_4 meet the corresponding six joins of q_1, q_2, q_3, q_4 (where, for example, p_1p_2 meets q_3q_4), then if the p are coplanar so are the q. This includes essentially the theorem on the existence of Möbius tetrahedra.

That there is a theorem corresponding to that in 1 for all spreads of even step and that it cannot be extended to spreads of odd step, is clear, because it depends on $\Delta=0$.

3. If five of the pairs p_i, q_i in 1 be conjugate for a quadric, so is the sixth pair.

For denote supplements for the quadric by the stroke, then
$$[p_1|q_1]+[p_2|q_2]+\ldots+[p_6|q_6]=0.$$

4. If π be any prime, and p_i, q_i be as in 1, then

$$[p_i\pi][q_i\pi] = \sum_j a_{ij}[p_i\pi][p_j\pi] - [p_i\pi][v\pi], \quad (a_{ij} = -a_{ji}).$$

$$\sum_i [p_i\pi][q_i\pi] = -\sum_i [p_i\pi][v\pi]$$
$$= -[(p_1+p_2+\ldots+p_6)\pi][v\pi] = 0.$$

Conversely, if we assume $\sum_i [p_i\pi][q_i\pi]$ vanishes for all π, then *either* $[p_1 q_2 \ldots q_6] = 0$ *or* the following all hold: $[q_1 \ldots q_6] = 0$, and $[p_1 q_2 \ldots q_6]$ vanishes when any one q_i $(i > 1)$ is replaced by the corresponding p_i.

Hence, *if* $[q_1 \ldots q_6]$ *never vanishes when any one* q *is replaced by the corresponding* p, *then* $[q_1 \ldots q_6]$ *vanishes when any two* q *are replaced by the corresponding* p.

5. *If* $\alpha_1, \ldots, \alpha_6$ *be primes in a spread of step six which go through a point, and on each line in which four cut we take an arbitrary point, for example* p_{1234} *on the cut of* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, *then the five points arising from five of the* α *determine a prime by their join; for example, let* β_6 *be so determined by* $\alpha_1, \ldots, \alpha_5$, *then* $\beta_1, \ldots, \beta_6$ *meet in a point.**

For
$$\beta_6 \equiv [p_{2345}\cdot p_{1345}\cdot p_{1245}\cdot p_{1235}\cdot p_{1234}],$$
$$\beta_5 \equiv [p_{2346}\cdot p_{1346}\cdot p_{1246}\cdot p_{1236}\cdot p_{1234}].$$

Thus $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_5, \beta_6$ meet in p_{1234}; and so on, so that we have the situation described on the p, q in 1. For $[\alpha_1 \ldots \alpha_6] = 0$, and the product also vanishes when any two α are replaced by the corresponding β. Hence $[\beta_1 \ldots \beta_6] = 0$.

6. We now interpret the primes α of 5 (or points in the dual theorem) as spheres (of step four) in a spread of step five. (Cf. chap. XI.) Six dependent spheres have a common orthogonal sphere, corresponding to the point in which six dependent primes cut. Four spheres have a pencil of orthogonal spheres, corresponding to the line in which four primes cut. Hence the theorem of 5 now runs:

If $p_1, \ldots, p_6$ be six spheres (of step four) in a spread of step five, and all orthogonal to a given sphere, and we take any sphere orthogonal to each set of four, for instance p_{1234} orthogonal to p_1, p_2, p_3, p_4, then the five spheres so associated with five of the

* Baker, *loc. cit.*

p_i have a sphere orthogonal to them; for example, let $p_1, \ldots, p_5$ fix the sphere q_6 in this way; then the spheres $q_1, \ldots, q_6$ have a common orthogonal sphere.

In particular, if we take for our spheres $p_1, \ldots, p_6$, primes in step five, they are orthogonal to the point at infinity, when considered as special cases of spheres in the 'inversion geometry' of this step. As the sphere p_{1234} take the point-sphere where $p_1, \ldots, p_4$ meet, and so on. The sphere q_6 orthogonal to the five point-spheres associated with $p_1, \ldots, p_5$ is then the circumsphere of the simplex $p_1, \ldots, p_5$. Hence the six circumspheres of the six simplexes obtained from $p_1, \ldots, p_6$ by omitting each p in turn are orthogonal to a sphere. But as in § 75·2, this sphere turns out to be a point-sphere. For by Haskell's Theorem, any five of the circumspheres meet in a point. Thus *either* all six meet in a point, *or* we have a figure which is the inverse of a simplex and its circumsphere, and this is not the case. Hence:

Kuhne's Theorem: If six general primes be given in a spread of step five, the six simplexes obtained by omitting each in turn have circumspheres which meet in a point.

A similar theorem holds for any *odd* step.

7. Returning to 5, Kuhne's Theorem shews that if a quadric goes through the point where $\alpha_1, \ldots, \alpha_6$ meet, and through all such points as p_{1234}, then it goes through the point where the β meet.

This is seen by stating 6 symbolically and interpreting for 5.

§ 111. *Comparison of geometries.*

1. We have considered, amongst others, geometries of the following figures:

(i) Circles c in a plane. These form a spread of step four wherein $c^2 = 0$, if c is a point-circle; and $[c_1 | c_2] = 0$, if c_1, c_2 be orthogonal.

(ii) Systems c of oriented circles in a plane. These form a spread of step five, wherein $c^2 = 0$, if c is a nul system, that is, a set of oriented circles touching a given oriented circle; if c_1, c_2 be nul systems, then $[c_1 | c_2] = 0$ if the corresponding oriented circles touch.

(iii) Spheres c in space. These form a spread of step five, wherein $c^2 = 0$, if c is a point-sphere; and $[c_1 | c_2] = 0$, if c_1, c_2 be orthogonal.

(iv) Systems c of oriented spheres in space. These form a spread of step six, wherein $c^2 = 0$, if c is a nul system, that is, a set of oriented spheres touching a given oriented sphere; if c_1, c_2 be nul systems, then $[c_1 | c_2] = 0$ if the corresponding oriented spheres touch.

(v) Screws s in space. These form a spread of step six, wherein $s^2 = 0$, if s is a rotor; and $[s | s_1] = 0$ if s, s_1 are in involution, which if both are rotors means that they cut.

2. Thus, if reality conditions be ignored, geometries (iv) and (v) are equivalent, oriented spheres corresponding to rotors, tangent oriented spheres to intersecting rotors, as in the Lie sphere-line transformation.

Each geometry (iv), (v) is also the geometry of a quadric $\mathfrak{Q}_6$ in a spread of step six, and we can regard $\mathfrak{Q}_6$ as a sphere. A linear complex is the set of lines dependent on five lines; in the spread of step six this corresponds to the cut of $\mathfrak{Q}_6$ and a spread depending on five points, that is, to a quadric (or sphere) $\mathfrak{Q}_5$ in the spread of step five.

Thus the geometry of lines in a linear complex is the geometry on $\mathfrak{Q}_5$ in a spread of step five; hence it is equivalent to geometries (ii) and (iii), and to the geometry of oriented spheres linearly dependent on five oriented spheres.

The lines of a linear congruence are represented by points on the cut of $\mathfrak{Q}_6$ and a spread of step four, that is, by points on a quadric $\mathfrak{Q}_4$. Their geometry is hence equivalent to geometry (i).

3. For example: *Miquel's Theorem*, § 75·2, gives: If four reguli whose lines are in a congruence contain a common line, then any two of them have another common line; thus six lines are obtained. These in threes determine four new reguli; then these reguli have a common line in the congruence.

4. *Kuhne's Theorem* interpreted in geometry (v) gives: If six linear complexes $s_1, \ldots, s_6$ have a common line, any four have another common line; if we omit each complex in turn from the five complexes $s_1, \ldots, s_5$, we thus obtain five lines, through which

goes a complex s'_6. From the six complexes $s_1, \ldots, s_6$, by omitting each in turn, we thus derive six new complexes $s'_1, \ldots, s'_6$. These have a common line.

From § 110·4, the sets of complexes obtained from $s_1, \ldots, s_6$ by replacing two s by the corresponding s' have a common line, and those obtained from $s'_1, \ldots, s'_6$ by replacing two s' by the corresponding s have a common line.

5. *The double-six theorem* gives in geometry (iv) for oriented spheres: If five spheres touch a sphere, any four have another tangent sphere, the five spheres so obtained touch a sphere.

In particular: *the in-spheres of the five tetrahedra formed by five oriented planes touch a sphere.* For five oriented planes touch the point-sphere at infinity.

6. If we combine Kuhne's Theorem interpreted in 4 with the double-six theorem we have: *Grace's Theorem.* If six general lines $l_1, \ldots, l_6$ have a common transversal, any four have another common transversal; if we omit each line in turn from $l_1, \ldots, l_5$ we then obtain five lines, through which goes a complex s'_6 which, by the double-six theorem, is a special complex. The six special complexes s'_6 so obtained have a common line, that is, their six axes meet a line. Thus the six lines obtained from $l_1, \ldots, l_6$ (if these have a common transversal), by the double-six theorem, have themselves a common transversal.

This can be translated into a theorem on six oriented spheres which touch a given oriented sphere.

7. If in a spread of step n, $a_1 \ldots a_n$ be a simple n-gon, and $a_r a_s$ cuts $x^2 = 0$ at $a_r - k_{rs} a_s$, then

$$a_r^2 - 2k_{rs}[a_r|a_s] + k_{rs}^2 a_s^2 = 0;$$

the product of the two roots k_{rs}, k'_{rs} of this equation is a_r^2/a_s^2.

Hence $(k_{12}k_{23}k_{34}\ldots k_{n1})(k'_{12}k'_{23}k'_{34}\ldots k'_{n1}) = 1$.

Hence, *if* n *of these cuts, one on each side of the* n*-gon, lie in a spread of step* $n-1$, *so do the other* n; for the condition for this is $k_{12}k_{23}\ldots k_{n1} = (-1)^n$.

8. Dually, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an n-face, that is, n spreads of step $n-1$, then the adjacent α cut in n spreads $\alpha_r\alpha_{r+1}$ of step $n-2$. Through these draw pairs of tangent primes to the quadric

$x^2 = 0$; *if* n *of these, one from each pair, meet in a point, so do the other* n.

As $n-1$ of the α meet in a point, we can regard the n-face as arising from an n-gon by joining each set of $n-1$ vertices.

9. From 7: *If* $s_1, \ldots, s_5$ *be spheres in a spread of step four, taken in cyclic order, and we consider the point-pairs in the pencil given by two neighbouring spheres, then if five of these points, one from each pair, lie on a sphere, so do the remaining five.*

Again, if we regard the five spheres as extensives of step one, they fix a five-face, in which two faces meet in a bundle of spheres given by three neighbouring spheres. Through such a bundle will pass two nul spreads of step four, corresponding to the two points in which the three spheres meet. Hence 8 gives:

If $s_1, \ldots, s_5$ *be spheres in a spread of step four, taken in cyclic order, the cuts of three neighbours give five point-pairs; if five points, one from each pair, lie on a sphere, so do the other five.*

This is a generalisation* of the six-circles theorem, in a different direction from Haskell's.

10. Note that some theorems, such as Haskell's Theorem, hold in spreads of any step; others, such as Kuhne's Theorem, depend on the parity of the step; others, such as the double-six theorem or Hart's Theorem, are restricted to a single step. This gives a new and useful classification of geometric theorems. The most difficult are those in the third class.

Examples. 1. Four lines, with a point on each, give four other points by the pivot theorem (§ 74). These four points are concyclic if the original four are.

2. In a plane, let $a_1, \ldots, a_n$ be points; L_i a line through a_i; $c_{ij} \equiv [L_i L_j]$; a_{ij} the centre of the circle through a_i, a_j, c_{ij}; a_{ijk} the centre of the circle through a_{ij}, a_{jk}, a_{ki}. Then points a_{ijk}, a_{jkl}, a_{kli}, a_{lij} lie on a circle with centre a_{ijkl}, say. Also the points a_{ijkl}, a_{jklm}, a_{klmi}, a_{lmij}, a_{mijk} lie on a circle with centre a_{ijklm}, say. And so on. The final centres lie on a circle. (Richmond.)

3. In space of odd step, the circumspheres in Kuhne's Theorem have their centres on a sphere. (Cf. Ex. 22, p. 323.)

* E. Müller, *Monatshefte Math. Phys.* 3 (1892).

4. Five planes in space with a point on the cut of each pair give five pivot points. These lie on a sphere. Six planes give six such spheres. These go through a point. And so on. There is no analogue to these theorems in a plane.

§ 112. *Circles in space.**

1. If spheres in a spread of step four be regarded as point-loci, they are represented by expressions of form $o+\frac{1}{2}r^2\theta$, where o is a point, θ an additional unity. Hence they form a spread of step five.

If s_1, s_2 be two spheres, then $[s_1 s_2]$ has previously been taken to represent the pencil of spheres coaxal with them; it is now more convenient to interpret it to represent the circle, real or imaginary, in which s_1, s_2 intersect. Then, if the spheres be represented on a point-spread of step five, and we denote the point corresponding to s by s', then $[s'_1 s'_2]$ is the line in the spread of step five which represents the circle in which s_1, s_2 cut. Similarly, the outer product of three spheres may be interpreted as the point-pair at which they meet; this point-pair corresponds to a plane in the spread of step five.

We use dashed letters to represent figures in the point-spread of step five.

2. Corresponding to point-spheres s, we have in our point-spread of step five, points s' such that $s'^2 = 0$; these points s' thus lie on a quadric $\mathcal{Q}$. If s_1, s_2 be orthogonal spheres, then $[s_1 | s_2] = 0$; s'_1, s'_2 are conjugate points for $\mathcal{Q}$.

A circle $C = [ab]$ is '*orthogonal*' to a sphere s, if and only if the spheres $a+kb$ are orthogonal to s for all k, that is, if

$$[a|s] = [b|s] = 0.$$

A circle $C = [ab]$ is a '*point-circle*' if $C^2 = 0$; then $[a|b]^2 = a^2b^2$ and the spheres a, b touch internally or externally.

For any circle C, we have $[CC] = 0$; if C_1, C_2 be circles, then $k_1 C_1 + k_2 C_2$ is a circle, if and only if $[C_1 C_2] = 0$. Two circles C_1, C_2 are on the same sphere, s say, if and only if they are of form $C_1 = [ss_1]$, $C_2 = [ss_2]$, and then $[C_1 C_2] = 0$. When $[C_1 C_2] = 0$, the corresponding lines C'_1, C'_2 in the point-spread of step five, cut.

* See Coolidge, *Circle and Sphere.*

3. If three circles $C_1 = [a_1 b_1]$, $C_2 = [a_2 b_2]$, $C_3 = [a_3 b_3]$ are orthogonal to the same sphere, then a_1, b_1, a_2, b_2, a_3, b_3 are orthogonal to that sphere. Now the set of spheres orthogonal to the same sphere is of step four; hence a_3, b_3 are linear combinations of a_1, b_1, a_2, b_2. Thus C_1', C_2', C_3' are lines in the same spread of step four. Thus the geometry of circles orthogonal to the same sphere is the geometry of lines in ordinary space.

4. If the circle $C, = [ab]$, contains a point-pair cospherical with C_1, with C_2 and with C_3, then the point-pair is $[a_1 a_2 a_3]$, where the spheres a_i, b_i are such that $C_1 = [a_1 b_1]$, $C_2 = [a_2 b_2]$, $C_3 = [a_3 b_3]$. Also $[a_1 a_2 a_3 a] = [a_1 a_2 a_3 b] = 0$. Thence C' lies in a plane which meets C_1', C_2', C_3'.

5. Four circles are '*in general position*', if no two are cospherical, no three are orthogonal to the same sphere, and none contains a point-pair cospherical with each of the other three.

We can now interpret the results of § 58 in the spread of spheres.

Four circles in general position fix a fifth circle 'associate' *to them; it is a linear combination of the four.**

If $C_1, = [a_1 b_1]$, is orthogonal to a sphere a_2 through $C_2, = [a_2 b_2]$, then $[a_1 | a_2] = [b_1 | a_2] = 0$. Hence $[C_1 | C_2] = 0$, and C_2 is orthogonal to a sphere through C_1. Such circles are said to be '*in involution*'.

Now in the spread of step five, any plane which meets four general lines meets the associated line; in the spread of spheres an extensive of form $|[s_1 s_2]$ corresponds to a plane.

Hence *any circle in involution to four circles in general position is in involution to the fifth associated circle.*

If C_1, C_2, C_3, C_4 be four circles in general position, then $[C_1 C_2 . C_2 C_3 . C_3 C_1] = D_4$, say, is a circle cospherical with C_1, C_2, C_3. Define D_1, D_2, D_3 similarly. Then $[C_1 D_1]$, ..., $[C_4 D_4]$ are systems of spheres which have a common circle, that associated with C_1, C_2, C_3, C_4. (§ 58.)

6. If C_1, C_2 be circles on the same sphere a and any sphere through C_1 cuts orthogonally the sphere b_2 through C_2 which is orthogonal to a, we can write $C_1 = [ab_1]$, $C_2 = [ab_2]$.

* The five constitute a 'Stephanos pentacycle'.

Then $$[C_1 | C_2] = a^2[b_1 | b_2] - [a | b_1][a | b_2],$$
$$[(a + kb_1) | b_2] = 0 \text{ for all } k.$$

Thus $[a | b_2] = [b_1 | b_2] = 0$, $[C_1 | C_2] = 0$, and the relation is symmetrical.

Such circles are called '*orthogonal*'.

7. The '*foci*' of a circle $[ab]$ are the point-spheres in the pencil $a + kb$; hence $a + kb$ is a focus, if $a^2 + 2k[a | b] + k^2b^2 = 0$. The two foci are distinct, unless $[ab]^2 = 0$, that is, unless the circle is a point-circle.

If a, b are (real) spheres, then $[ab]^2 \geqq 0$, hence $a^2b^2 - [a | b]^2 \geqq 0$; thus the foci are imaginary, unless they coincide.

If $[ab]$ be a circle with distinct foci c, d, then $[ab] \equiv [cd]$.

8. Two circles, not point-circles, are '*in bi-involution*', if each sphere through one cuts orthogonally each sphere through the other.

Then, if $[ab]$, $[cd]$ be the circles, we have

$$[a | c] = [a | d] = [b | c] = [b | d] = 0.$$

Each circle goes through the foci of the other.

Conversely, if $C_1 = [ab]$ goes through the foci $c + k_1 d$, $c + k_2 d$ of $C_2 = [cd]$, then $[ab | c] = 0$, $[ab | d] = 0$, and hence the circles are in bi-involution.

9. In the representation on the point-spread of step five, the foci of the circle $[ab]$ correspond to the cuts of $[a'b']$ and $\mathcal{Q}$. Two circles are *in involution* if the corresponding lines are such that each cuts the polar plane of the other with respect to $\mathcal{Q}$; they are in *bi-involution*, if each of the corresponding lines lies in the polar plane of the other; they are *orthogonal*, if the corresponding lines cut, and each cuts the polar plane of the other.

10. *The common orthogonal circle of two circles.* Any two circles $[ab]$, $[cd]$ have a common orthogonal sphere $|[abcd]$, and we may suppose that a, b and c, d are the foci of the respective circles. The common orthogonal circle is cospherical with each circle.

Since spheres through circles $[ab]$, $[cd]$, respectively, are of form $a + kb$, $c + ld$, we may suppose these latter meet in the required circle C. Now $a - kb$ is a sphere through $[ab]$ orthogonal to $a + kb$, since $[(a + kb) | (a - kb)] = a^2 - k^2b^2 = 0$, since $a^2 = b^2 = 0$.

And $c-\mathrm{l}d$ is a sphere through $[cd]$ orthogonal to $c+\mathrm{l}d$. Any sphere through our required circle C has to cut $a-\mathrm{k}b$, $c-\mathrm{l}d$ orthogonally, by definition.

Hence $[(a-\mathrm{k}b)\,|\,(c+\mathrm{l}d)] = 0, \quad [(a+\mathrm{k}b)\,|\,(c-\mathrm{l}d)] = 0.$

These give

$$[a\,|\,c]-\mathrm{kl}[b\,|\,d] = 0, \quad \mathrm{k}[b\,|\,c]-\mathrm{l}[a\,|\,d] = 0.$$

If $\mathrm{l}\neq 0$, thence

$$[a\,|\,c]\,[a\,|\,d] = \mathrm{k}^2[b\,|\,c]\,[b\,|\,d].$$

If $\mathrm{k}\neq 0$, thence

$$[a\,|\,c]\,[b\,|\,c] = \mathrm{l}^2[a\,|\,d]\,[b\,|\,d].$$

If $\mathrm{l} = 0$ or $\mathrm{k} = 0$, then $[a\,|\,c] = 0$. First suppose none of $[a\,|\,c]$, $[b\,|\,d]$, $[b\,|\,c]$, $[a\,|\,d]$ vanish, then we have equal and opposite values of k, and corresponding equal and opposite values of l. Thus we get two circles $[(a+\mathrm{k}b)(c+\mathrm{l}d)]$ and $[(a-\mathrm{k}b)(c-\mathrm{l}d]$, which may be real or imaginary, orthogonal to the given circles $[ab]$, $[cd]$. The circles found are in bi-involution (8) since, for example,

$$[(a+\mathrm{k}b)\,|\,(a-\mathrm{k}b)] = 0, \quad [(a+\mathrm{k}b)\,|\,(c-\mathrm{l}d)] = 0.$$

If $[a\,|\,c] = 0$, then $[ac\,|\,ab] = 0$, $[ac\,|\,cd] = 0$, since a, c are point-circles (foci). Hence $[ac]$ is then the common orthogonal circle, if $a\neq c$. But then $[ac]^2 = 0$, hence $[ac]$ is a point-circle. If $[a\,|\,c] = 0$, but $[b\,|\,d]\neq 0$, then either $\mathrm{k} = 0$, $[a\,|\,d] = 0$ or $\mathrm{l} = 0$, $[b\,|\,c] = 0$ or $\mathrm{k} = \mathrm{l} = 0$; in the first case c, d have a as common orthogonal point-sphere; in the second case, a, b have c as common orthogonal point-sphere. We exclude these cases.

If $[a\,|\,c] = [b\,|\,d] = 0$, but $[b\,|\,c]$, $[a\,|\,d]\neq 0$, only the ratio $\mathrm{k}:\mathrm{l}$ is determinate and there is a single infinity of circles orthogonal to $[ab]$ and $[cd]$. The latter circles are then called '*paratactic*'.

Finally, if $[a\,|\,c] = [b\,|\,d] = [a\,|\,d] = [b\,|\,c] = 0$, then the circles are in bi-involution.

11. In dealing with circles orthogonal to two given circles, the circles and spheres used were all orthogonal to the sphere $|[abcd]$. Now spheres orthogonal to a given sphere form a spread of step four, and this spread can be represented on a point-spread of step four, or ordinary (not necessarily Euclidean) space. Lines

in the point-spread represent circles orthogonal to the given sphere, and there is a quadric $\mathcal{Q}$ in the point-spread such that, if circles C_1, C_2 are in involution, then the corresponding lines are conjugate for $\mathcal{Q}$, that is, each meets the polar of the other.

Since any two circles, in general position, are orthogonal to a sphere, we can replace the work in 10 by work on the point-spread in step four, containing a non-degenerate quadric $\mathcal{Q}$. This we now do, and for convenience, we drop the dashes from the letters.

12. *Lines in a point-spread of step four.*

Denote supplements for the quadric $\mathcal{Q}$ by the stroke. If ab, cd be two lines in the spread, there are, in general, just two lines, which cut ab, cd, $|ab$, $|cd$, and they are polar lines for $\mathcal{Q}$. To prove the latter fact, let L be one of the lines. Then $[Lab]$, $[Lcd]$, $[L\,|ab]$, $[L\,|cd]$ all vanish. Hence so do $[|L.|ab]$, $[|L.|cd]$, $[|L.ab]$, $[|L.cd]$. Therefore $|L$ is the other transversal.

Further, L and $|L$ are conjugate to the four given lines, since they cut their polar lines. The lines L, $|L$ correspond to the orthogonal circles of the circles which correspond to ab and cd.

Now consider the case when ab, cd, $|ab$, $|cd$ are dependent, but no two meet. There is then a single infinity of lines meeting all the four.

Let $$\mathrm{x}.ab+\mathrm{y}.cd+\mathrm{x}'\,|ab+\mathrm{y}'\,|cd=0,$$

then $$\mathrm{x}\,|ab+\mathrm{y}\,|cd+\mathrm{x}'.ab+\mathrm{y}'.cd=0.$$

Since no three of the lines are dependent, otherwise they would concur, we have

$$\mathrm{x}'\mathrm{x}^{-1}=\mathrm{y}'\mathrm{y}^{-1}=\mathrm{x}\mathrm{x}'^{-1}=\mathrm{y}\mathrm{y}'^{-1};$$

hence $\mathrm{x}'=\mathrm{x}$, $\mathrm{y}'=\mathrm{y}$ or $\mathrm{x}'=-\mathrm{x}$, $\mathrm{y}'=-\mathrm{y}$.

Hence, either $$\mathrm{x}(ab+|ab)+\mathrm{y}(cd+|cd)=0$$

or $$\mathrm{x}(ab-|ab)+\mathrm{y}(cd-|cd)=0.$$

Hence, either $$ab+|ab\equiv cd+|cd$$

or $$ab-|ab\equiv cd-|cd.$$

We say the lines ab, cd are '*right-paratactic*' in the first case, '*left-paratactic*' in the second. It is assumed that they are not polar lines.

If p is any point, there is a line through p right-paratactic to ab and one through p left-paratactic to ab. For, if $S = ab + |ab$, the first line mentioned through p is $[pq]$ where $q \equiv |[pS]$. For $pq + |pq \equiv ab + |ab$ gives

$$pS \equiv [p\,|\,pq] = [p\,|\,q]\,|\,p - p^2.|\,q, \quad [p\,|\,pS] \equiv [pq].$$

Two paratactic lines, having an infinite set of transversals conjugate to both, correspond to paratactic circles. All lines right-, (left-), paratactic to a given line have a single infinity of common transversals.

13. If ab, cd be paratactic lines, let ac, bd be conjugate to both, but be not polars of one another, and suppose none of a, b, c, d is on Ω.

Then $[ab\,|\,ac] = [ab\,|\,bd] = [ac\,|\,cd] = [bd\,|\,cd] = 0.$

Hence
$$\left.\begin{aligned} a^2[b\,|\,c] &= [a\,|\,b]\,[a\,|\,c], \quad b^2[a\,|\,d] = [a\,|\,b]\,[b\,|\,d], \\ c^2[a\,|\,d] &= [a\,|\,c]\,[c\,|\,d], \quad d^2[b\,|\,c] = [b\,|\,d]\,[c\,|\,d]. \end{aligned}\right\} \quad \text{(i)}$$

If $[a\,|\,b] = 0$, then $[b\,|\,c] = [a\,|\,d] = 0$, and either

$$[a\,|\,c] = [b\,|\,d] = 0 \quad \text{or} \quad [c\,|\,d] = 0.$$

Such cases being excluded by hypothesis, we have

$$\frac{[a\,|\,c]}{\sqrt{a^2}\sqrt{c^2}} = \frac{[b\,|\,d]}{\sqrt{b^2}\sqrt{d^2}}, \qquad \frac{[a\,|\,b]}{\sqrt{a^2}\sqrt{b^2}} = \frac{[c\,|\,d]}{\sqrt{c^2}\sqrt{d^2}}$$

or
$$\cos \widehat{ac} = \cos \widehat{bd}, \quad \cos \widehat{ab} = \cos \widehat{cd}.$$

If we call points $a + k_1 b$, $c + k_2 d$ on $[ab]$, $[cd]$ '*corresponding*', when $k_1 : k_2 = [a\,|\,d] : [b\,|\,c]$, then the joins of such points are conjugate to ab and cd. For

$$\begin{aligned} &[ab\,|\,(a + k_1 b)(c + k_2 d)] \\ &\quad = [ab\,|\,ac + k_1 bc + k_2 ad + k_1 k_2 bd] \\ &\quad = k_1[ab\,|\,bc] + k_2[ab\,|\,ad] \qquad \text{(since } [ab\,|\,ac] = [ab\,|\,bd] = 0\text{)} \\ &\quad = k_1([a\,|\,b]\,[b\,|\,c] - b^2[a\,|\,c]) + k_2(a^2[b\,|\,d] - [a\,|\,b]\,[a\,|\,d]) = 0, \end{aligned}$$

by (i), and the value of $k_1 : k_2$.

These results can be used to give a theory of Clifford parallels in elliptic space of step four.

CHAPTER XV

ALGEBRAIC PRODUCTS

§ 113. *The algebraic product of two points and of two rotors in a plane.**

1. As the basis of our spread we take e_1, e_2, e_3 with $[e_1 e_2 e_3] = 1$, and let $E_1 = [e_2 e_3], \quad E_2 = [e_3 e_1], \quad E_3 = [e_1 e_2].$

All products introduced are assumed to be *distributive over addition.*

Def. The '*algebraic product*' of two points a, b (of any weight) is the point-pair a, b. This product is to be independent of the order of the factors, and is written $\{a, b\}$ or $\{b, a\}$.

We write $\{a^2\}$ for $\{a, a\}$. When no confusion arises, we may omit the brackets.

2. We introduce formally such sums as $\{a, b\} + \{c, d\} + \ldots$.

The equation $\{a, b\} + \{c, d\} + \ldots = 0$

is to mean that, for all lines L in the plane, we have

$$[aL][bL] + [cL][dL] + \ldots = 0.$$

The equation

$$\{a, b\} + \{c, d\} + \ldots = \{a_1, b_1\} + \{c_1, d_1\} + \ldots$$

is to mean that, for all lines L in the plane, we have

$$[aL][bL] + [cL][dL] + \ldots = [a_1 L][b_1 L] + [c_1 L][d_1 L] + \ldots.$$

Hence we have, k being a scalar,

$$\{ka, b\} = \{a, kb\} = k\{a, b\};$$

$$\{a, b + c\} = \{a, b\} + \{a, c\}; \quad \{a + b, c\} = \{a, c\} + \{b, c\}.$$

If $\{a, b\} = \{a_1, b_1\}$, then $a \equiv a_1$, $b \equiv b_1$, or $a \equiv b_1$, $a_1 \equiv b$, since any line through a must go either through a_1 or through b_1.

If $\mathfrak{a}$, $\mathfrak{b}$ be any sums of algebraic products, then $\mathfrak{a} = \mathfrak{b}$ and $\mathfrak{a} - \mathfrak{b} = 0$ are equivalent equations.

* E. Müller, *Wiener Ber.* 2 A (1922, 1924).

3. If

$$a = k_1 e_1 + k_2 e_2 + k_3 e_3, \quad b = l_1 e_1 + l_2 e_2 + l_3 e_3, \quad \text{(k, l scalars)},$$

then $\{a, b\} = k_1 l_1 \{e_1^2\} + \ldots + \ldots + (k_2 l_3 + k_3 l_2)\{e_2, e_3\} + \ldots + \ldots.$

Hence any sum $\mathfrak{a}$ of products such as $\{a, b\}$ is a linear combination of

$$\{e_1^2\}, \quad \{e_2^2\}, \quad \{e_3^2\}, \quad \{e_2, e_3\}, \quad \{e_3, e_1\}, \quad \{e_1, e_2\},$$

and may be written

$$\mathfrak{a} = \sum_{i,j=1}^{3} k_{ij}\{e_i, e_j\}, \quad k_{ij} = k_{ji}.$$

Let

$$\sum_{j=1}^{3} k_{ij} e_j = a_i, \quad (i = 1, 2, 3),$$

then

$$\sum_{i,j=1}^{3} k_{ij}\{e_i, e_j\} = \sum_i \{e_i, \sum_j k_{ij} e_j\} = \sum_i \{e_i, a_i\}.$$

Hence any sum $\mathfrak{a}$ of products such as $\{a, b\}$ can be put in the '*normal form*'

$$\{a_1, e_1\} + \{a_2, e_2\} + \{a_3, e_3\},$$

where $\sum_{j=1}^{3} k_{ij} e_j = a_i$, for some scalars $k_{ij} = k_{ji}$.

Since

$$2\{a_i, e_i\} = \{(a_i + e_i)^2 - (a_i - e_i)^2\},$$

any such sum can be written as the sum of a certain number of positive and negative squares.

4. To each non-zero sum of algebraic products of points, we associate an *envelope*. We illustrate this by examples. With $\{a, b\}$ is associated the envelope of lines L such that $[aL]\,[bL] = 0$; the envelope is hence the pair of points a, b. With $\{a, b\} + \{c, d\}$ is associated the envelope of lines L such that

$$[aL]\,[bL] + [cL]\,[dL] = 0.$$

We call this '*the envelope* $\{a, b\} + \{c, d\}$'. And so on.

5. Dually, the '*algebraic product*' of two rotors L, M is the rotor-pair L, M; it is denoted by $\{L, M\}$ or $\{M, L\}$. Sums of such products are introduced formally.

The equation

$$\{L, M\} + \{P, Q\} + \ldots = 0$$

is to mean that, for all points p of the plane, we have

$$[Lp]\,[Mp] + [Pp]\,[Qp] + \ldots = 0.$$

The equation

$$\{L, M\}+\{P, Q\}+\ldots = \{L_1, M_1\}+\{P_1, Q_1\}+\ldots$$

is to mean that, for all points p of the plane, we have

$$[Lp]\,[Mp]+[Pp]\,[Qp]+\ldots = [L_1p]\,[M_1p]+[P_1p]\,[Q_1p]+\ldots.$$

All the above work has its dual analogue. In particular with a non-zero sum $\{L, M\}+\{P, Q\}+\ldots$ is associated the locus of points p such that

$$[Lp]\,[Mp]+[Pp]\,[Qp]+\ldots = 0.$$

§ 114. *The outer product of algebraic products.*

1. *Def.* The '*outer product*' of $\{a^2\}$, $\{b^2\}$ is denoted by $[a^2b^2]$, and is defined by

$$[a^2b^2] = \{[ab]^2\}, \qquad (1)$$

that is, it is the algebraic square of the rotor $[ab]$.

This is a reasonable definition, for $\{a^2\}$ is a repeated, $\{b^2\}$ is b repeated, and the outer product of the repeated points is defined to be the rotor $[ab]$ repeated.

2. *Def.* If $[bc] = L$, the '*outer product*' of $\{a^2\}$ and $\{L^2\}$ is denoted by $[a^2.L^2]$, and is defined by

$$[a^2.L^2] = [aL]^2. \qquad (2)$$

This is a scalar, since $[aL]$ is a scalar.

Similarly, we define $[L^2.a^2] = [La]^2$.

Thence $[a^2.[bc]^2] = [abc]^2$, $[[ab]^2.c^2] = [abc]^2$,

or, by the previous paragraph,

$$[a^2.b^2c^2] = [a^2b^2.c^2] = [abc]^2. \qquad (3)$$

This suggests the following definition:

Def. The '*outer product*' of $\{a^2\}$, $\{b^2\}$, $\{c^2\}$, in any order, is written $[a^2b^2c^2]$, and is defined to be $[abc]^2$.

3. Dually, if A, B, C be rotors, we define

$$[A^2B^2] = \{[AB]^2\}, \quad [A^2B^2C^2] = [ABC]^2. \qquad (4)$$

The first is the algebraic square of a point, the second is a scalar.

4. We have, by the distributive law, k and l being scalars,

$$\left.\begin{aligned}\{(a+\mathrm{k}b)^2\} &= \{a^2\}+2\mathrm{k}\{a,b\}+\mathrm{k}^2\{b^2\},\\ \{(c+\mathrm{l}d)^2\} &= \{c^2\}+2\mathrm{l}\{c,d\}+\mathrm{l}^2\{d^2\}.\end{aligned}\right\} \tag{5}$$

But, since $a+\mathrm{k}b$, $c+\mathrm{l}d$ are points, the outer product of the left-hand sides is

$$\{[(a+\mathrm{k}b)(c+\mathrm{l}d)]^2\} = \{([ac]+\mathrm{k}[bc]+\mathrm{l}[ad]+\mathrm{kl}[bd])^2\}.$$

Square out the last expression, and compare the coefficient of kl with that in the outer product of the right-hand sides of (5). This gives

$$2[\{a,b\}\{c,d\}] = \{[ac],[bd]\}+\{[bc],[ad]\}, \tag{6}$$

which we could have taken as a definition. If we equate other coefficients, we get instances of this. In particular,

$$[\{a,b\}\{a,b\}] = -\tfrac{1}{2}\{[ab]^2\}. \tag{7}$$

5. Similarly, since

$$[(a+\mathrm{k}b)(L+\mathrm{l}M)] = [aL]+\mathrm{k}[bL]+\mathrm{l}[aM]+\mathrm{kl}[bM],$$

and $[(a+\mathrm{k}b)^2(L+\mathrm{l}M)^2]$

$$= [(\{a^2\}+2\mathrm{k}\{a,b\}+\mathrm{k}^2\{b^2\})(\{L^2\}+2\mathrm{l}\{L,M\}+\mathrm{l}^2\{M^2\})],$$

we have, from (2), by equating coefficients of kl,

$$2[\{a,b\}\{L,M\}] = [aL][bM]+[bL][aM]. \tag{8}$$

This is a scalar; the dual argument shews that it equals $2[\{L,M\}\{a,b\}]$. In particular,

$$[\{a,b\}\{L^2\}] = [aL][bL]. \tag{9}$$

6. $$[\{a^2+b^2+c^2\}.\{L^2\}] = [aL]^2+[bL]^2+[cL]^2.$$

We often write $\{a^2+b^2+c^2\}$ instead of $\{a^2\}+\{b^2\}+\{c^2\}$.

$$[(\{a,b\}+\{c,d\}).\{L^2\}] = [aL][bL]+[cL][dL].$$

Hence the equations of the envelopes associated with $\{a^2+b^2+c^2\}$, and with $\{a,b\}+\{c,d\}$, are given by the vanishing of the outer products of these expressions and $\{L^2\}$. This holds generally, and the dual also.

7. The '*outer product*' of $\{a^2\}$ and a rotor L is thus defined:

Def. $$[a^2L] = [aL]\,a. \quad \text{Dually, } [L^2a] = [La]\,L. \tag{10}$$

As in the preceding section, we can deduce:

$$[\{a,b\}L] = \tfrac{1}{2}([aL]\,b+[bL]\,a). \tag{11}$$

By § 113·3, we now have a meaning for $[\mathfrak{a}L]$, where $\mathfrak{a}$ is a sum of algebraic products of points, if we assume the distributive law. Dually, if $\mathfrak{A}$ is a sum of algebraic products of rotors, we have a meaning for $[\mathfrak{A}p]$.

If $\mathfrak{A}$ is a line-pair with double point p, then $[\mathfrak{A}p] = 0$.

8. By (8), (11),

the outer product of $[\{a, b\} L]$ and M is $[\{a, b\} \{L, M\}]$,

that is, the outer product of $\{a, b\}$ and $\{L, M\}$. (12)

The outer product of $[\mathfrak{a}L]$ and M is $[\mathfrak{a}\{L, M\}]$; the outer product of $[\mathfrak{A}p]$ and q is $[\mathfrak{A}\{p, q\}]$.

§ 115. *Interpretations.*

1. Let a, b, ... be any points, L, M, ... any rotors.

Let $\mathfrak{a} = a^2 + b^2 + \ldots$, then $\mathfrak{a}$ is the name of the associated envelope, and $\mathfrak{a} = \sum_{i,j=1}^{3} k_{ij}\{e_i, e_j\}$, for some scalars $k_{ij} = k_{ji}$.

$$[\mathfrak{a}L^2] = \Sigma k_{ij}[\{e_i, e_j\} L^2] = \Sigma k_{ij}[e_i L][e_j L] = \Sigma k_{ij} l_i l_j,$$

where $L = l_1 E_1 + l_2 E_2 + l_3 E_3$, $E_1 = [e_2 e_3]$, and so on; l_1, l_2, l_3 scalars.

Thus $[\mathfrak{a}L^2] = 0$ is the equation of the envelope associated with $\mathfrak{a}$. It is a conic envelope.

Dually $[\mathfrak{A}p^2] = 0$ is a conic locus associated with a sum $\mathfrak{A}$ of algebraic products of rotors.

2. If p, q be points such that $[\mathfrak{A}\{p, q\}] = 0$, then the equation

$$[\mathfrak{A}(p + kq)^2] = 0, \quad \text{or} \quad [\mathfrak{A}p^2] + 2k[\mathfrak{A}\{p, q\}] + k^2[\mathfrak{A}q^2] = 0,$$

has equal and opposite roots in k. If these be $\pm k_1$, then $p \pm k_1 q$ are the points in which the line $[pq]$ meets the locus $\mathfrak{A}$. These points separate p, q harmonically.

Hence, *if* $[\mathfrak{A}\{p, q\}] = 0$, *then* p, q *are conjugate points for* $\mathfrak{A}$.

By § 114·8, the outer product of $[\mathfrak{A}p]$ and q is $[\mathfrak{A}\{p, q\}]$. Hence if p, q are conjugate points, then q is on the line $[\mathfrak{A}p]$.

Hence $[\mathfrak{A}p]$ *is the polar of* p *with respect to* $\mathfrak{A}$.

Dually, if $\mathfrak{a}$ is a conic envelope, L, M rotors, then L, M are conjugate if $[\mathfrak{a}\{L, M\}] = 0$; and $[\mathfrak{a}L]$ is the pole of L.

3. Since $[\mathfrak{a}L^2] = 0$, if, and only if, L touches $\mathfrak{a}$, therefore $[\mathfrak{a}[pq]^2] = 0$, if, and only if, q is on a tangent from p. Now $[\mathfrak{a}[pq]^2] = [\mathfrak{a}p^2 . q^2]$. Hence $[\mathfrak{a}p^2]$ is the pair of tangents from p, regarded as a (degenerate) conic locus.

Dually $[\mathfrak{A}L^2]$ is the cut of $\mathfrak{A}$ and L regarded as a conic envelope.

Examples. 1. If L, M be conjugate for $\{a, b\} + \{c, d\}$, then

$$[aL][bM] + [aM][bL] + [cL][dM] + [cM][dL] = 0.$$

Give the geometric interpretation of this.

2. If $a^2 + b^2 = 0$, then $\{a, a\} = -\{b, b\}$; hence $a \equiv b$, and $a = b = 0$.

3. If $a^2 + b^2 + c^2 = 0$, and points are allowed complex weights, then $\{(a + ib), (a - ib)\} = -c^2$. Hence, *either* $a^2 + b^2 = 0$, $c^2 = 0$, *or* $a + ib \equiv c$, $a - ib \equiv c$. All cases give $a \equiv b \equiv c$.

4. If $a^2 + b^2 = c^2 + d^2$, then

$$\{(a + c), (a - c)\} = \{(d + b), (d - b)\}.$$

Hence, *either* $a \pm c = 0$, $d \pm b = 0$ *or* $a - c \equiv d \pm b$, $a + c \equiv d \mp b$.

If $\{a, b\} L = 0$ and $a \not\equiv b$, then $L \equiv [ab]$.

5. If $a^2 + b^2 = \{c, d\}$, then c, d are collinear with a, b and separate them harmonically.

If $[\{a, b\}\{c, d\}] = 0$, the same conclusion follows.

If $p + k_1 q$, $p + k_2 q$ separate $p + l_1 q$, $p + l_2 q$ harmonically where k_1, k_2 and l_1, l_2 are respectively roots of

$$ax^2 + 2hx + c = 0, \quad a'x^2 + 2h'x + c' = 0,$$

then

$$ac' + a'c = 2hh'.$$

6. If $a^2 + b^2 = c^2 + d^2 = e^2 + f^2$, then a, b; c, d; e, f are pairs in involution on a line. Three dependent point-pairs are in involution on a line.

7. If

$$\mathfrak{a} = k_1 a^2 + k_2 b^2 + k_3 c^2 \not\equiv 0,$$

then

$$\mathfrak{a}[bc] = k_1[a^2[bc]] = k_1[abc]\, a.$$

Hence if $[abc] \not\equiv 0$, then a is the pole of bc for $\mathfrak{a}$. Thus abc is a self-polar triangle of $\mathfrak{a}$.

Since every non-degenerate conic has a self-polar triangle, therefore by absorbing weights, *any conic envelope can be put in the form* $a^2 + b^2 + c^2$; and this is also true for a degenerate conic envelope consisting of points p, q, since $\{p, q\} = \frac{1}{4}(p + q)^2 - \frac{1}{4}(p - q)^2$, the points a, b, c may have absorbed complex weights.

8. If $$\mathfrak{a} = a^2 + b^2 + c^2,$$

then $$\tfrac{1}{2}[\mathfrak{a}^2] = [ab]^2 + [bc]^2 + [ca]^2, \quad \tfrac{1}{6}[\mathfrak{a}^3] = [abc]^3.$$

Hence if $[\mathfrak{a}^3] = 0$, then $[abc] = 0$, and $c = k_1 a + k_2 b$, (k_1, k_2 scalars), so that $\mathfrak{a}$ is a linear combination of a^2, b^2 and $\{a, b\}$, and hence, as in ordinary algebra, is the product of two linear combinations of a and b, that is, of two points.

Conversely, if $\mathfrak{a} = \{p, q\}$, then clearly $[\mathfrak{a}^3] = 0$. Hence $[\mathfrak{a}^3] = 0$ *is a necessary and sufficient condition that* $\mathfrak{a}$ *is degenerate.*

9. If $[\mathfrak{a}L] = 0$ for some L, then $\mathfrak{a}$ is degenerate. For take $\mathfrak{a} = a^2 + b^2 + c^2$, then $[\mathfrak{a}L] = [aL]\,a + [bL]\,b + [cL]\,c$. Thus a, b, c are dependent. The conic is a line-pair, one line being L.

10. If $$\mathfrak{a} = k_1 a^2 + k_2 b^2 + k_3 c^2 + k_4 d^2 \neq 0,$$

then $$\mathfrak{a}[cd] = k_1 [acd]\,a + k_2 [bcd]\,b.$$

Hence, by (12), $[\mathfrak{a}\{[cd], [ab]\}] = 0$. Thus $[ab]$, $[cd]$ are conjugate for $\mathfrak{a}$. *Hence* $abcd$ *is a polar quadrangle for* $\mathfrak{a}$, that is, each pair of opposite joins is a conjugate pair.

11. If $\mathfrak{a} = k_1 a^2 + k_2 b^2 + k_3 c^2 \neq 0$, we have a case of Ex. 10, with $k_4 = 0$ for all d. Hence *if* abc *is a self-polar triangle, then* $abcd$ *is a polar quadrangle, where* d *is any point.*

If $a_1 b_1 c_1$ be not a self-polar triangle of $\mathfrak{a}$, there is just one point d such that $a_1 b_1 c_1 d$ is a polar quadrangle, namely,

$$d = [\mathfrak{a}[b_1 c_1]\,a_1 . \mathfrak{a}[c_1 a_1]\,b_1].$$

12. If $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$, then algebraic multiplication of each side by itself gives

$$[bc]^2 + [ca]^2 + [ab]^2 = [ef]^2 + [fd]^2 + [de]^2.$$

The first equation says that abc, def are self-polar triangles for the same conic. Now if another conic locus $\mathfrak{C}$ goes through a, then $[\mathfrak{C}a^2] = 0$. Hence if $\mathfrak{C}$ goes through a, b, c, d, e, then it goes through f. Similarly, from the second equation, if a conic envelope touches $[bc]$, $[ca]$, $[ab]$, $[ef]$, $[fd]$, then it touches $[de]$.

Hence, *if* abc, def *be self-polar triangles for a conic, their six vertices lie on a conic, and their six sides touch another.*

A necessary and sufficient condition that points a, b, c, d, e, f *lie on a conic is that, after absorbing weights,*

$$a^2 + b^2 + c^2 = d^2 + e^2 + f^2.$$

Hence *if* a, b, c, d, e, f *be on a conic, the sides of triangles* abc, def *touch another conic and the triangles are self-polar for a third conic.*

13. If $abcd$ and $abpq$ be polar quadrangles for a conic, then the six points a, b, c, d, p, q lie on a conic.

14. If $\mathfrak{a} = \{a, b\} - \{c, d\} \neq 0$, then

$$[\mathfrak{a}L^2] = [aL]\,[bL] - [cL]\,[dL].$$

Hence the envelope $\mathfrak{a}$ touches ac, bc, ad, bd, since $[\mathfrak{a}[ac]^2] = 0$, and so on. (Cf. § 24·8.)

15. If a, b separate c, d harmonically on the conic $\mathfrak{c}$, then

$$\mathfrak{c} = [\{a, b\}.\{c, d\}].$$

16. If $\mathfrak{a}$ is a conic envelope, and $[\mathfrak{a}p^2] = 0$, then $\mathfrak{a}$ is a point-pair, one of whose points is p.

17. If
$$\mathfrak{a} = \{a_1, e_1\} + \{a_2, e_2\} + \{a_3, e_3\},$$
$$a_i = k_{i1}e_1 + k_{i2}e_2 + k_{i3}e_3, \quad (i = 1, 2, 3),$$
then $[a_1e_1] + [a_2e_2] + [a_3e_3] = 0$ is equivalent to
$$k_{ij} = k_{ji}, \quad (i, j = 1, 2, 3).$$

18. If two triangles are sextuply perspective, any vertex is the polar of the opposite side for the conic through the remaining five points. (Wieleitner.)

By § 13·4 we can take the vertices of the triangles in the form

$$a, b, c; \quad a+b+c, \quad \omega a + \omega^2 b + c, \quad \omega^2 a + \omega b + c,$$

where
$$\omega^3 = 1, \quad \omega \neq 1.$$

Let $\mathfrak{C}$ be the conic through the last five points, then

$$\begin{aligned}\mathfrak{C}\{b^2\} = \mathfrak{C}\{c^2\} &= \mathfrak{C}\{(a+b+c)^2\} = \mathfrak{C}\{(\omega a + \omega^2 b + c)^2\} \\ &= \mathfrak{C}\{(\omega^2 a + \omega b + c)^2\} = 0, \qquad \text{(i)}\end{aligned}$$

and we have to shew

$$\mathfrak{C}\{a, b\} = 0, \quad \mathfrak{C}\{a, c\} = 0.$$

The last four equations of (i), when multiplied out, give $\mathfrak{C}\{a, b\} = 0$. Multiply the fourth and fifth by ω^2, ω and add to the first, then $\mathfrak{C}\{a, c\} = 0$.

§ 116. *Outer products of conics.**

1. The normal form of any conic envelope $\mathfrak{a}$ is

$$\{a_1, e_1\} + \{a_2, e_2\} + \{a_3, e_3\},$$

where $a_i = \Sigma k_{ij} e_j$, $k_{ij} = k_{ji}$. (§ 113·3).

* There is evidently a relation between the outer products of conics treated here, and the outer products of symmetrical matrices. If the definitions are compared, it will be found that there are different scalar factors involved. In each instance these have been chosen for convenience simply.

Now, if b, c be any points, then

$$\sum_{i=1}^{3}\{[a_i b],[e_i c]\} = \sum_{i,j=1}^{3} k_{ij}\{[e_j b],[e_i c]\},$$

$$\sum_{i=1}^{3}\{[a_i c],[e_i b]\} = \sum_{i,j=1}^{3} k_{ij}\{[e_j c],[e_i b]\} = \sum_{i,j=1}^{3} k_{ij}\{[e_j b],[e_i c]\},$$

since $k_{ij} = k_{ji}$. Thus the two expressions are equal.

Hence
$$\sum_{i=1}^{3}[\{a_i, e_i\}\{b, c\}] = \tfrac{1}{2}\sum_{i=1}^{3}(\{[a_i b],[e_i c]\}+\{[a_i c],[e_i b]\})$$
$$= \sum_{i=1}^{3}\{[a_i b],[e_i c]\}.$$

Hence

$$[(\{a_1,e_1\}+\{a_2,e_2\}+\{a_3,e_3\})(\{b_1,e_1\}+\{b_2,e_2\}+\{b_3,e_3\})]$$
$$= \{[a_1 b_2],[e_1 e_2]\}+\ldots+(\{[a_2 b_3],[e_2 e_3]\}+\{[a_3 b_2],[e_3 e_2]\})+\ldots$$
$$= \{([a_2 b_3]-[a_3 b_2]),[e_2 e_3]\}+\ldots+\ldots. \qquad (13)$$

From this, using (12) and (8), and dropping brackets which may be understood:

$$[\{a_1, e_1+a_2, e_2+a_3, e_3\}\{b_1, e_1+b_2, e_2+b_3, e_3\}\{c_1, e_1+c_2, e_2+c_3, e_3\}]$$
$$= [a_1 b_2 c_3]-[a_2 b_1 c_3]+[a_2 b_3 c_1]-[a_3 b_2 c_1]+[a_3 b_1 c_2]-[a_1 b_3 c_2], \qquad (14)$$

where, as always, $[e_1 e_2 e_3] = 1$, and the law of the expression is the same as for determinants.

In shewing this formula, we need

$$\sum_{i=1}^{3}\{[L_i b],[E_i c]\} = \sum_{i=1}^{3}\{[L_i c],[E_i b]\},$$

which is proved like the corresponding one at the beginning of the paragraph, when $\Sigma\{L_i, E_i\}$ is a normal form.

2. *The outer product of two conic loci is a conic envelope. Dually, the outer product of two conic envelopes is a conic locus. The outer product of three conic loci or of three conic envelopes is a scalar. These outer products are commutative.*

3. We can deduce from (6), by the method in 1,

$$4[\{a, b\}\{c, d\}\{e, f\}]$$
$$= [ace][bdf]+[bce][adf]+[ade][bcf]+[bde][acf].$$

Examples. 19. If $\mathfrak{a} = \{a_1, e_1\} + \{a_2, e_2\} + \{a_3, e_3\}$, in normal form, then

$$\left.\begin{aligned} \tfrac{1}{2}[\mathfrak{a}^2] &= \{[a_2 a_3], E_1\} + \{[a_3 a_1], E_2\} + \{[a_1 a_2], E_3\}, \\ \tfrac{1}{6}[\mathfrak{a}^3] &= [a_1 a_2 a_3]. \end{aligned}\right\} \quad (15)$$

If $\mathfrak{A} = [\mathfrak{a}^2]$, then

$$\begin{aligned} \tfrac{1}{4}[\mathfrak{A}^2] &= 2(\{[a_3 a_1 . a_1 a_2], e_1\} + \ldots + \ldots) \\ &= 2[a_1 a_2 a_3](\{a_1, e_1\} + \{a_2, e_2\} + \{a_3, e_3\}) = 2[a_1 a_2 a_3]\,\mathfrak{a}. \end{aligned}$$

Hence

$$[\mathfrak{A}^2] = 8[a_1 a_2 a_3]\,\mathfrak{a} = \tfrac{4}{3}[\mathfrak{a}^3]\,\mathfrak{a},$$

$$[\mathfrak{A}^3] = \tfrac{4}{3}[\mathfrak{a}^3][\mathfrak{a}\mathfrak{A}] = \tfrac{4}{3}[\mathfrak{a}^3]^2.$$

20. If $\mathfrak{a} = k_1\{b, c\} + k_2\{c, a\} + k_3\{a, b\}$, k_1, k_2, k_3 scalars, then

$$[\mathfrak{a}[bc]] = \tfrac{1}{2}[abc](k_2 c + k_3 b), \quad [\mathfrak{a}[bc]^2] = 0.$$

Hence $\mathfrak{a}$ touches $[bc]$, and similarly $[ca]$ and $[ab]$.

$$[\mathfrak{a}[bc + ca + ab]] = \tfrac{1}{2}[abc]((k_2 + k_3)a + (k_3 + k_1)b + (k_1 + k_2)c),$$

$$[\mathfrak{a}\{[bc + ca + ab]^2\}] = [abc]^2(k_1 + k_2 + k_3).$$

Hence $\mathfrak{a}$ touches $bc + ca + ab$, if $k_1 + k_2 + k_3 = 0$.

$$\mathfrak{a}^2 = -\tfrac{1}{2}k_1^2[bc]^2 - \ldots - \ldots + k_2 k_3\{[ca], [ab]\} + \ldots; \quad \mathfrak{a}^3 = \tfrac{3}{2}[abc]^2 k_1 k_2 k_3.$$

21. The equation of the envelope $\mathfrak{a}$ is $[\mathfrak{a}L^2] = 0$. The points on this envelope lie on the locus $[\mathfrak{a}^2]$, whose equation is $[\mathfrak{A}p^2] = 0$, where $[\mathfrak{a}^2] = \mathfrak{A}$, provided $[\mathfrak{a}^3] \neq 0$. For, let $\mathfrak{a} = a^2 + b^2 + c^2$, then

$$[\mathfrak{a}L] = [aL]\,a + [bL]\,b + [cL]\,c = q, \text{ say.}$$

$$\begin{aligned} \tfrac{1}{2}[\mathfrak{A}q] &= [([bc]^2 + [ca]^2 + [ab]^2)\,q] \\ &= [bcq][bc] + [caq][ca] + [abq][ab] \\ &= [abc]([bc][aL] + [ca][bL] + [ab][cL]) = [abc]^2 L. \end{aligned}$$

Hence if $[abc] \neq 0$, then if q is the pole of L for $\mathfrak{a}$, L is the polar of q for $\mathfrak{A}$. Whence $\mathfrak{a}$, $\mathfrak{A} = [\mathfrak{a}^2]$ *represent the same conic as envelope and as locus respectively.*

22. If $[\mathfrak{a}^3] = 0$, we know $\mathfrak{a}$ is degenerate. If $[\mathfrak{a}^2] = 0$, then $\mathfrak{a}$ represents a point taken twice.

§ 117. *Formulae.*

1. Consider the scalar $[Lc][Ld][Xc][Xd]$, where L, X are lines, c, d points.

Let $\quad c = c_1 e_1 + c_2 e_2 + c_3 e_3, \quad d = d_1 e_1 + d_2 e_2 + d_3 e_3,$

$$L = l_1 E_1 + l_2 E_2 + l_3 E_3, \quad X = x_1 E_1 + x_2 E_2 + x_3 E_3.$$

Then the scalar is

$$\sum_{i,j,k,m=1}^{3} l_i c_i l_j d_j x_k c_k x_m d_m \quad \text{or} \quad \sum_{i,j,k,m=1}^{3} x_k c_k c_i l_i l_j d_j d_m x_m. \tag{16}$$

Now any conic envelope is of the form $\mathfrak{c} = c^2 + c'^2 + \ldots$; any conic locus is of form $\mathfrak{L} = L^2 + L'^2 + \ldots$. Let $\mathfrak{d} = d^2 + d'^2 + \ldots$ be another conic envelope.

$$\left.\begin{aligned} \mathfrak{c} &= c^2 + c'^2 + \ldots = \sum_{i,j=1}^{3} c_i c_j \{e_i, e_j\} + \sum_{i,j=1}^{3} c'_i c'_j \{e_i, e_j\} + \ldots . \\ \mathfrak{L} &= L^2 + L'^2 + \ldots = \sum_{i,j=1}^{3} l_i l_j \{E_i, E_j\} + \sum_{i,j=1}^{3} l'_i l'_j \{E_i, E_j\} + \ldots . \end{aligned}\right\} \tag{i}$$

Add together all the expressions obtained from (16) when c is replaced by $c, c', c'', \ldots$; L by $L, L', L'', \ldots$, and d by $d, d', d'', \ldots$. The sum is

$$\sum_{k,m=1}^{3} x_k (\mathfrak{c}\mathfrak{L}\mathfrak{d})_{km} x_m, \tag{17}$$

where $(\mathfrak{c}\mathfrak{L}\mathfrak{d})_{km}$ is the (km) element in the matrix $\mathfrak{c}\mathfrak{L}\mathfrak{d}$ formed as the matrix product of the matrices of the coefficients of $\mathfrak{c}$, $\mathfrak{L}$, $\mathfrak{d}$, written in the form (i). The matrix $\frac{1}{2}(\mathfrak{c}\mathfrak{L}\mathfrak{d} + (\mathfrak{c}\mathfrak{L}\mathfrak{d})^*)$ is symmetric, and hence it represents a conic; we denote the corresponding conic envelope by $(\mathfrak{c}\mathfrak{L}\mathfrak{d})$. Then (17) can be written

$$[(\mathfrak{c}\mathfrak{L}\mathfrak{d})\{X^2\}] \quad \text{or} \quad [(\mathfrak{c}\mathfrak{L}\mathfrak{d})X^2].$$

The poles of a line X for $\mathfrak{c}$, $\mathfrak{d}$ are respectively $\Sigma[Xc]c$ and $\Sigma[Xd]d$, and these are conjugate for $\mathfrak{L}$ if

$$\Sigma[Xc][cL][Xd][dL] = 0,$$

the sum being taken over $c, c', \ldots, L, L', \ldots, d, d', \ldots$. But this condition is $[(\mathfrak{c}\mathfrak{L}\mathfrak{d})X^2] = 0$.

Hence $(\mathfrak{c}\mathfrak{L}\mathfrak{d})$ *represents the conic envelope of lines whose poles for* $\mathfrak{c}$, $\mathfrak{d}$ *are conjugate for* $\mathfrak{L}$.

In particular, since $(\mathfrak{c}\mathfrak{L}\mathfrak{c})$ is the envelope of lines whose poles for $\mathfrak{c}$ are on $\mathfrak{L}$, it is *the reciprocal of* $\mathfrak{L}$ *for* $\mathfrak{c}$.

Dually, $(\mathfrak{A}\mathfrak{c}\mathfrak{B})$ *is the locus of points whose polars for* $\mathfrak{A}$, $\mathfrak{B}$ *are conjugate for* $\mathfrak{c}$; $(\mathfrak{A}\mathfrak{c}\mathfrak{A})$ *is the reciprocal of* $\mathfrak{c}$ *for* $\mathfrak{A}$.

If $\mathfrak{L} = \{L, L'\}$, then

$$(\mathfrak{c}\mathfrak{L}\mathfrak{d}) = \tfrac{1}{2}(\{[\mathfrak{c}L], [\mathfrak{d}L']\} + \{[\mathfrak{c}L'], [\mathfrak{d}L]\}).$$

2. If a, b, c, d be points, then

$$[ab.cd] = [abd]\,c - [abc]\,d.$$

Hence $$\{[ab.cd]^2\} = [abd]^2\{c^2\} + [abc]^2\{d^2\} - 2[abd][abc]\{c, d\}. \quad (18)$$

Now, by (1), (4),

$$[a^2b^2] = \{[ab]^2\}, \quad \{[ab.cd]^2\} = [[ab]^2, [cd]^2],$$

and, by (2), $$[abd]^2 = [a^2b^2d^2].$$

Hence, if $\mathfrak{a} = a^2 + a'^2 + a''^2 + \ldots$, $\mathfrak{b} = b^2 + b'^2 + b''^2 + \ldots$, and similarly for $\mathfrak{c}$, $\mathfrak{d}$, and we substitute $a^2 + a'^2 + a''^2 + \ldots$ for a^2, and $b^2 + b'^2 + b''^2 + \ldots$ for b^2, and so on, in the first two formulae, and use the distributive law, we get

$$\begin{aligned}[\mathfrak{ab}] &= [(a^2 + a'^2 + \ldots)(b^2 + b'^2 + \ldots)] \\ &= [ab]^2 + [ab']^2 + [a'b]^2 + \ldots, \\ [\mathfrak{ab}.\mathfrak{cd}] &= [[ab]^2 + [ab']^2 + \ldots, [cd]^2 + [cd']^2 + \ldots] \\ &= \{[ab.cd]^2\} + \ldots.\end{aligned}$$

Similarly $[\mathfrak{abd}]\,\mathfrak{c}$, $[\mathfrak{abc}]\,\mathfrak{d}$ are respectively the sums of terms like $[abd]^2\{c^2\}$ and $[abc]^2\{d^2\}$.

Let $[ab] = L$, then

$$[abc][abd]\{c, d\} = [Lc][Ld]\{c, d\}.$$

The sum of such terms as this is $(\mathfrak{c}\mathfrak{L}\mathfrak{d})$, where

$$\mathfrak{L} = [ab]^2 + [ab']^2 + [a'b]^2 + \ldots;$$

for if $[Lc][Ld]\{c, d\}$ be multiplied by X^2, it gives

$$[Lc][Ld][cX][dX].$$

Hence, finally,

$$[\mathfrak{ab}.\mathfrak{cd}] = [\mathfrak{abd}]\,\mathfrak{c} + [\mathfrak{abc}]\,\mathfrak{d} - 2(\mathfrak{c}[\mathfrak{ab}]\,\mathfrak{d}). \quad (19)$$

Similarly, $$[\mathfrak{ab}.\mathfrak{cd}] = [\mathfrak{bcd}]\,\mathfrak{a} + [\mathfrak{acd}]\,\mathfrak{b} - 2(\mathfrak{a}[\mathfrak{cd}]\,\mathfrak{b}). \quad (20)$$

3. If we write $\mathfrak{L}$ for $[\mathfrak{cd}]$, this gives an equation for $(\mathfrak{a}\mathfrak{L}\mathfrak{b})$:

$$2(\mathfrak{a}\mathfrak{L}\mathfrak{b}) = [\mathfrak{b}\mathfrak{L}]\,\mathfrak{a} + [\mathfrak{a}\mathfrak{L}]\,\mathfrak{b} - [\mathfrak{ab}\mathfrak{L}]. \quad (21)$$

In particular, for the reciprocal of $\mathfrak{L}$ for $\mathfrak{a}$, we have

$$(\mathfrak{a}\mathfrak{L}\mathfrak{a}) = [\mathfrak{a}\mathfrak{L}]\,\mathfrak{a} - \tfrac{1}{2}[\mathfrak{a}^2\mathfrak{L}]. \quad (22)$$

4. By (19), (20), we have

$$[\mathfrak{bcd}]\,\mathfrak{a} + [\mathfrak{acd}]\,\mathfrak{b} - [\mathfrak{dab}]\,\mathfrak{c} - [\mathfrak{abc}]\,\mathfrak{d} = 2(\mathfrak{a}[\mathfrak{cd}]\,\mathfrak{b}) - 2(\mathfrak{c}[\mathfrak{ab}]\,\mathfrak{d}). \quad (23)$$

From (20), with $\mathfrak{c} = \mathfrak{b} = \mathfrak{a}$, $[\mathfrak{a}^2] = \mathfrak{A}$, we have

$$2(\mathfrak{a}\mathfrak{A}\mathfrak{b}) = [\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{a} + [\mathfrak{a}^3]\,\mathfrak{b} - [\mathfrak{a}^2.\mathfrak{a}\mathfrak{b}]. \qquad (24)$$

5. The method used to obtain (19) is quite general. To illustrate it again, take the theorem on points:

$$[abc]\,d = [bcd]\,a + [cad]\,b + [abd]\,c.$$

Squaring

$$[abc]^2\{d^2\} = [bcd]^2\{a^2\} + [cad]^2\{b^2\} + [abd]^2\{c^2\} - 2[adb][adc]\{b,c\} - 2[bdc][bda]\{c,a\} - 2[cda][cdb]\{a,b\}.$$

Let $\mathfrak{a} = a^2 + a'^2 + \ldots$, $\mathfrak{b} = b^2 + b'^2 + \ldots$,

$\mathfrak{c} = c^2 + c'^2 + \ldots$, $\mathfrak{d} = d^2 + d'^2 + \ldots$,

then $[\mathfrak{a}\mathfrak{b}\mathfrak{c}]$ is the sum of expressions like $[a^2b^2c^2]$ or $[abc]^2$.

Proceeding as before, we find

$$[\mathfrak{a}\mathfrak{b}\mathfrak{c}]\,\mathfrak{d} = [\mathfrak{b}\mathfrak{c}\mathfrak{d}]\,\mathfrak{a} + [\mathfrak{c}\mathfrak{a}\mathfrak{d}]\,\mathfrak{b} + [\mathfrak{a}\mathfrak{b}\mathfrak{d}]\,\mathfrak{c} - 2(\mathfrak{b}[\mathfrak{a}\mathfrak{d}]\,\mathfrak{c}) - 2(\mathfrak{c}[\mathfrak{b}\mathfrak{d}]\,\mathfrak{a}) - 2(\mathfrak{a}[\mathfrak{c}\mathfrak{d}]\,\mathfrak{b}). \qquad (25)$$

In particular, $(\mathfrak{a}[\mathfrak{a}\mathfrak{d}]\,\mathfrak{a}) = \frac{1}{2}[\mathfrak{a}^2\mathfrak{d}]\,\mathfrak{a} - \frac{1}{6}[\mathfrak{a}^3]\,\mathfrak{d}. \qquad (26)$

From (25), (20), we have

$$[\mathfrak{a}\mathfrak{b}.\mathfrak{c}\mathfrak{d}] + [\mathfrak{b}\mathfrak{c}.\mathfrak{a}\mathfrak{d}] + [\mathfrak{c}\mathfrak{a}.\mathfrak{b}\mathfrak{d}] = [\mathfrak{b}\mathfrak{c}\mathfrak{d}]\,\mathfrak{a} + [\mathfrak{c}\mathfrak{a}\mathfrak{d}]\,\mathfrak{b} + [\mathfrak{a}\mathfrak{b}\mathfrak{d}]\,\mathfrak{c} + [\mathfrak{a}\mathfrak{b}\mathfrak{c}]\,\mathfrak{d}. \qquad (27)$$

In particular

$$[\mathfrak{a}^2.\mathfrak{a}\mathfrak{b}] = [\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{a} + \tfrac{1}{3}[\mathfrak{a}^3]\,\mathfrak{b}, \quad [\mathfrak{a}^2.\mathfrak{a}^2] = \tfrac{4}{3}[\mathfrak{a}^3]\,\mathfrak{a}. \qquad (28)$$

Hence (24) gives $(\mathfrak{a}\mathfrak{A}\mathfrak{b}) = \frac{1}{3}[\mathfrak{a}^3]\,\mathfrak{b}. \qquad (29)$

6. From $[abc]\,L = [aL][bc] + [bL][ca] + [cL][ab]$,

we obtain $[\mathfrak{a}\mathfrak{b}\mathfrak{c}]\,L = [\mathfrak{a}L.\mathfrak{b}\mathfrak{c}] + [\mathfrak{b}L.\mathfrak{c}\mathfrak{a}] + [\mathfrak{c}L.\mathfrak{a}\mathfrak{b}]. \qquad (30)$

7. From $[ab.cd] = [acd]\,b - [bcd]\,a = [abd]\,c - [abc]\,d$,

we obtain

$$[\mathfrak{a}\mathfrak{b}.\mathfrak{c}\mathfrak{d}] = (\mathfrak{b}[\mathfrak{a}\mathfrak{d}]\,\mathfrak{c}) + (\mathfrak{a}[\mathfrak{b}\mathfrak{c}]\,\mathfrak{d}) + (\mathfrak{c}[\mathfrak{b}\mathfrak{d}]\,\mathfrak{a}) + (\mathfrak{b}[\mathfrak{c}\mathfrak{a}]\,\mathfrak{b}), \qquad (31)$$

which can also be deduced from (20) and (27).

In particular,

$$[\mathfrak{a}^2.\mathfrak{c}^2] = 4(\mathfrak{a}[\mathfrak{a}\mathfrak{c}]\,\mathfrak{c}), \quad [\mathfrak{a}^2.\mathfrak{a}^2] = 4(\mathfrak{a}[\mathfrak{a}^2]\,\mathfrak{a}). \qquad (32)$$

8. By (27),

$$[\mathfrak{a}\mathfrak{b}.\mathfrak{a}\mathfrak{b}] = [\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{b} + [\mathfrak{a}\mathfrak{b}^2]\,\mathfrak{a} - \tfrac{1}{2}[\mathfrak{a}^2.\mathfrak{b}^2]. \qquad (33)$$

By (19),

$$[\mathfrak{a}^2.\mathfrak{b}^2] = 2[\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{b} - 2(\mathfrak{b}\mathfrak{A}\mathfrak{b}) = 2[\mathfrak{a}\mathfrak{b}^2]\,\mathfrak{a} - 2(\mathfrak{a}\mathfrak{B}\mathfrak{a}), \quad (34)$$

where $\mathfrak{A} = [\mathfrak{a}^2], \quad \mathfrak{B} = [\mathfrak{b}^2].$

From these and (32), (33),

$$\begin{aligned}[\mathfrak{a}\mathfrak{b}.\mathfrak{a}\mathfrak{b}] &= [\mathfrak{a}\mathfrak{b}^2]\,\mathfrak{a} + (\mathfrak{b}\mathfrak{A}\mathfrak{b}) = [\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{b} + (\mathfrak{a}\mathfrak{B}\mathfrak{a}) \\ &= [\mathfrak{a}\mathfrak{b}^2]\,\mathfrak{a} + [\mathfrak{a}^2\mathfrak{b}]\,\mathfrak{b} - 2(\mathfrak{a}[\mathfrak{a}\mathfrak{b}]\,\mathfrak{b}). \quad (35)\end{aligned}$$

Again, from (20),

$$[\mathfrak{a}\mathfrak{b}.\mathfrak{c}^2] = [\mathfrak{a}\mathfrak{c}^2]\,\mathfrak{b} + [\mathfrak{b}\mathfrak{c}^2]\,\mathfrak{a} - 2(\mathfrak{a}\mathfrak{C}\mathfrak{b}).$$

By (27),

$$2[\mathfrak{a}\mathfrak{c}.\mathfrak{b}\mathfrak{c}] = -[\mathfrak{a}\mathfrak{b}.\mathfrak{c}^2] + 2[\mathfrak{a}\mathfrak{b}\mathfrak{c}]\,\mathfrak{c} + [\mathfrak{b}\mathfrak{c}^2]\,\mathfrak{a} + [\mathfrak{a}\mathfrak{c}^2]\,\mathfrak{b}. \quad (36)$$

Hence $$[\mathfrak{a}\mathfrak{c}.\mathfrak{b}\mathfrak{c}] = [\mathfrak{a}\mathfrak{b}\mathfrak{c}]\,\mathfrak{c} + (\mathfrak{a}\mathfrak{C}\mathfrak{b}). \quad (37)$$

9. If A, B be lines, c a point, then

$$[AB.c] = [Ac]\,B - [Bc]\,A.$$

Hence $$[AB.c]^2 = [Ac]^2\,B^2 + [Bc]^2\,A^2 - 2[Ac]\,[Bc]\,\{A, B\}.$$

Let $[AB] = p$, then

$$[AB.c]^2 = [pc]^2 = [p^2.c^2] = [A^2B^2.c^2]; \quad [Ac]^2 = [A^2c^2].$$

Taking $\mathfrak{a} = a^2 + a'^2 + \ldots$, and so on, and noting that the sum of such terms as $[Ac]\,[Bc]\,\{A, B\}$ is $(\mathfrak{A}\mathfrak{c}\mathfrak{B})$, we have

$$[\mathfrak{A}\mathfrak{B}.\mathfrak{c}] = [\mathfrak{A}\mathfrak{c}]\,\mathfrak{B} + [\mathfrak{B}\mathfrak{c}]\,\mathfrak{A} - 2(\mathfrak{A}\mathfrak{c}\mathfrak{B}). \quad (38)$$

Similarly, from $[AB.cd] = [Ac]\,[Bd] - [Bc]\,[Ad]$, we have

$$[\mathfrak{A}\mathfrak{B}.\mathfrak{c}\mathfrak{d}] = [\mathfrak{A}\mathfrak{c}]\,[\mathfrak{B}\mathfrak{d}] + [\mathfrak{B}\mathfrak{c}]\,[\mathfrak{A}\mathfrak{d}] - 2[(\mathfrak{A}\mathfrak{c}\mathfrak{B})\,\mathfrak{d}]. \quad (39)$$

Thence $$[(\mathfrak{A}\mathfrak{c}\mathfrak{B})\,\mathfrak{d}] = [(\mathfrak{A}\mathfrak{d}\mathfrak{B})\,\mathfrak{c}]. \quad (40)$$

10. If a, b be points, L, M lines, and $[LM] = p$, then

$$[ab.LM] = [aL]\,[bM] - [bL]\,[aM],$$

$$\begin{aligned}[a^2b^2.LM] &= [[ab]^2.LM] = [[ab]^2.p] = [abp]\,[ab], \text{ by (10)}, \\ &= ([aL]\,[bM] - [bL]\,[aM])\,[ab].\end{aligned}$$

But $$[a^2L] = [aL]\,a, \quad [b^2M] = [bM]\,b.$$

Hence $$[a^2L.b^2M] = [aL]\,[bM]\,[ab],$$

$$[a^2b^2.LM] = [a^2L.b^2M] - [a^2M.b^2L].$$

Summing for a^2, a'^2, ..., b^2, b'^2, ... as usual, we find

$$[\mathfrak{a}\mathfrak{b}.LM] = [\mathfrak{a}L.\mathfrak{b}M] - [\mathfrak{a}M.\mathfrak{b}L]. \quad (41)$$

Similarly,

$$[\mathfrak{ab}.[LM]^2] = [\mathfrak{ab}.LM][LM]$$
$$= [\mathfrak{a}L^2][\mathfrak{b}M^2] + [\mathfrak{a}M^2][\mathfrak{b}L^2]$$
$$- 2[\mathfrak{a}\{L, M\}][\mathfrak{b}\{L, M\}]. \quad (42)$$

Dually $$[\mathfrak{AB}.pq] = [\mathfrak{A}p.\mathfrak{B}q] - [\mathfrak{A}q.\mathfrak{B}p], \quad (43)$$

$$[\mathfrak{AB}.[pq]^2] = [\mathfrak{A}p^2][\mathfrak{B}q^2] + [\mathfrak{A}q^2][\mathfrak{B}p^2]$$
$$- 2[\mathfrak{A}\{p, q\}][\mathfrak{B}\{p, q\}]. \quad (44)$$

§ 118. *Interpretations.*

1. As the scalar k varies, $\mathfrak{A}+k\mathfrak{B}$ gives the curves of a pencil of loci. One of these goes through an arbitrary point p, for $[(\mathfrak{A}+k\mathfrak{B})p^2] = 0$ gives $[\mathfrak{A}p^2]+k[\mathfrak{B}p^2] = 0$, and this determines k. For all k, $\mathfrak{A}+k\mathfrak{B}$ goes through the common points of $\mathfrak{A}$ and $\mathfrak{B}$.

2. In general, two curves of the pencil touch an arbitrary line, for the equation

$$[(\mathfrak{A}+k\mathfrak{B})^2 L^2] = [\mathfrak{A}^2L^2]+2k[\mathfrak{AB}L^2]+k^2[\mathfrak{B}^2L^2] = 0$$

has two roots in k.

3. *Desargues' Theorem.* Since the cuts $p+hq$ of $[pq]$ and $\mathfrak{C}$ are given by

$$[\mathfrak{C}p^2]+2h[\mathfrak{C}\{p, q\}]+h^2[\mathfrak{C}q^2] = 0, \quad (45)$$

the cuts of $[pq]$ and the loci $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}+k\mathfrak{B}$ are linearly dependent point-pairs and hence are in involution (§ 115, Ex. 6).

4. The cuts of $[pq]$ and $\mathfrak{A}$, $\mathfrak{B}$ are given by equations similar to (45). These cuts separate one another harmonically, by § 115, Ex. 5, if

$$[\mathfrak{A}p^2][\mathfrak{B}q^2] + [\mathfrak{A}q^2][\mathfrak{B}p^2] - 2[\mathfrak{A}\{p, q\}][\mathfrak{B}\{p, q\}] = 0,$$

and hence, by (44), if $[\mathfrak{AB}.[pq]^2] = 0$, that is, if $[pq]$ touches $[\mathfrak{AB}]$.

The envelope of lines which are cut harmonically by $\mathfrak{A}$, $\mathfrak{B}$ is thus the conic $[\mathfrak{AB}]$, called the '*harmonic conic envelope*' of $\mathfrak{A}$, $\mathfrak{B}$.

Dually, the locus of points from which tangents to $\mathfrak{a}$, $\mathfrak{b}$ separate harmonically is the '*harmonic conic locus*' $[\mathfrak{ab}]$ of $\mathfrak{a}$, $\mathfrak{b}$.

5. *Equation of harmonic locus of* $\mathfrak{a}$, $\mathfrak{b}$.

If $\mathfrak{a} = \Sigma a_{ik}\{e_i, e_k\}$, $\mathfrak{b} = \Sigma b_{ik}\{e_i, e_k\}$, then

$$[\mathfrak{ab}] = \Sigma a_{ik} b_{jl}[\{e_i, e_k\}\{e_j, e_l\}]$$
$$= (a_{22}b_{33} + a_{33}b_{22} - 2a_{23}b_{23})[e_2e_3]^2 + \ldots + \ldots$$
$$+ 2(a_{23}b_{31} - a_{33}b_{12} - a_{12}b_{33} + a_{31}b_{23})[e_2e_3][e_3e_1] + \ldots + \ldots.$$

6. *The harmonic envelopes of* $\mathfrak{c}$ *and the conics of pencil* $\mathfrak{a} + k\mathfrak{b}$ *are in a pencil.*

7. If A, B, C be lines, p, q, r points, then

$$[ABC][pqr] = \begin{vmatrix} [Ap], & [Aq], & [Ar] \\ [Bp], & [Bq], & [Br] \\ [Cp], & [Cq], & [Cr] \end{vmatrix}.$$

Hence $[A^2B^2C^2][pqr]$ is this determinant multiplied by $[ABC]$; expanding the determinant, thus multiplied, the term

$$[Ap][Bq][Cr][ABC],$$

for example, gives $[A^2p][B^2q][C^2r]$. Hence we obtain another determinant, with A^2, B^2, C^2 substituted for A, B, C, in the above. The usual summing process then gives, expanding as in (14),

$$[\mathfrak{ABC}][pqr] = \begin{vmatrix} [\mathfrak{A}p], & [\mathfrak{A}q], & [\mathfrak{A}r] \\ [\mathfrak{B}p], & [\mathfrak{B}q], & [\mathfrak{B}r] \\ [\mathfrak{C}p], & [\mathfrak{C}q], & [\mathfrak{C}r] \end{vmatrix}. \qquad (46)$$

Dually, we have an expression for $[\mathfrak{abc}][LMN]$.

In particular,

$$\tfrac{1}{2}[\mathfrak{a}^2\mathfrak{b}][LMN] = [\mathfrak{a}L.\mathfrak{a}M.\mathfrak{b}N] + [\mathfrak{a}M.\mathfrak{a}N.\mathfrak{b}L] + [\mathfrak{a}N.\mathfrak{a}L.\mathfrak{b}M].$$

Let $\mathfrak{a}L = p$, $\mathfrak{a}M = q$, $\mathfrak{a}N = r$, $\mathfrak{b}L = p'$, $\mathfrak{b}M = q'$, $\mathfrak{b}N = r'$.

Then, if $[\mathfrak{a}^2\mathfrak{b}] = 0$, we have

$$[qrp'] + [rpq'] + [pqr'] = 0. \qquad (47)$$

Now, if LMN be a self-polar triangle for $\mathfrak{a}$, we have, if L, M, N be given suitable weights, $[qr] = L$, $[rp] = M$, $[pq] = N$.

Hence (47) gives

$$[L.\mathfrak{b}L] + [M.\mathfrak{b}M] + [N.\mathfrak{b}N] = 0$$

or

$$[\mathfrak{b}(\{L^2\} + \{M^2\} + \{N^2\})] = 0.$$

Hence, *if* $[\mathfrak{a}^2\mathfrak{b}] = 0$, *and a self-polar triangle of* $\mathfrak{a}$ *has two sides touching* $\mathfrak{b}$, *the third side touches* $\mathfrak{b}$. *Conversely, if there is one such triangle, then* $[\mathfrak{a}^2\mathfrak{b}] = 0$, *and there are* ∞^1 *such triangles.*

If $[\mathfrak{a}^2\mathfrak{b}] = 0$, we say $\mathfrak{a}^2$, $\mathfrak{b}$ are '*apolar*'.

Again, by (47), *if* $[\mathfrak{a}^2\mathfrak{b}] = 0$, *and LMN be any triangle, the triangle whose vertices are the poles of L, M, N for* $\mathfrak{b}$ *is perspective with the triangle whose sides are the polars of* $[MN]$, $[NL]$, $[LM]$ *for* $\mathfrak{a}$.

If $[\mathfrak{A}^2\mathfrak{B}] = 0$, we say $\mathfrak{A}^2$, $\mathfrak{B}$ are '*apolar*'. The duals of the above theorems hold.

If $\mathfrak{A} = [\mathfrak{a}^2]$, $\mathfrak{B} = [\mathfrak{b}^2]$, and the conics are non-degenerate, then $[\mathfrak{a}^2\mathfrak{B}] = 0$ is equivalent to $[\mathfrak{a}\mathfrak{b}^2] = 0$. Hence, *if a self-polar triangle of* $\mathfrak{a}$ *can be circumscribed to* $\mathfrak{b}$, *then any self-polar triangle of* $\mathfrak{b}$ *with two vertices on* $\mathfrak{a}$ *is inscribed to* $\mathfrak{a}$.

8. By (34), if $[\mathfrak{a}^2\mathfrak{b}] = 0$, then $[\mathfrak{a}^2\mathfrak{b}^2] = -2(\mathfrak{b}\mathfrak{A}\mathfrak{b})$.

Hence, as a numeric multiplier is merely a weight:

If $\mathfrak{a}^2$, $\mathfrak{b}$ *are apolar, then the reciprocal of* $\mathfrak{A}$ *with respect to* $\mathfrak{b}$ *is the harmonic conic of* $\mathfrak{A}$, $\mathfrak{B}$.

9. *If the conditions* $[\mathfrak{a}^2\mathfrak{b}] = 0$ *and* $[\mathfrak{a}\mathfrak{b}^2] = 0$ *both hold, and the conics are not degenerate,* we have the following results.

(i) By (33), $[\mathfrak{a}\mathfrak{b}]^2 = -\frac{1}{2}[\mathfrak{a}^2\mathfrak{b}^2]$. Hence *the harmonic conic of the loci* $\mathfrak{a}$, $\mathfrak{b}$ *is the harmonic conic of the envelopes* $\mathfrak{A}$, $\mathfrak{B}$; *it is the reciprocal of* $\mathfrak{B}$ *with respect to* $\mathfrak{a}$, *and of* $\mathfrak{A}$ *with respect to* $\mathfrak{b}$ (by 8).

(ii) Write $[\mathfrak{c}^2] = \mathfrak{C} = [\mathfrak{a}\mathfrak{b}]$, then by (26), $(\mathfrak{a}\mathfrak{C}\mathfrak{a}) = -\frac{1}{6}[\mathfrak{a}^3]\mathfrak{b}$; hence the reciprocal of $\mathfrak{C}$ with respect to $\mathfrak{a}$ is $\mathfrak{b}$. *The conics* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *are such that each is the reciprocal of any other with respect to the third.*

(iii) Since $[\mathfrak{a}\mathfrak{b}] = \mathfrak{C} = [\mathfrak{c}^2]$, we have $[\mathfrak{b}\mathfrak{c}^2] = [\mathfrak{a}\mathfrak{b}^2] = 0$; similarly $[\mathfrak{a}\mathfrak{c}^2] = 0$.

By (i), $\mathfrak{c} \equiv [\mathfrak{a}^2\mathfrak{b}^2]$, hence $[\mathfrak{c}\mathfrak{a}^2] \equiv [\mathfrak{a}\mathfrak{b}^2] = 0$; similarly $[\mathfrak{c}\mathfrak{b}^2] = 0$.

By (34), $[\mathfrak{a}^2\mathfrak{c}^2] = -2(\mathfrak{c}\mathfrak{A}\mathfrak{c}) = -2(\mathfrak{a}\mathfrak{C}\mathfrak{a}) \equiv \mathfrak{b}$; $[\mathfrak{a}\mathfrak{c}] \equiv \mathfrak{B}$; similarly $[\mathfrak{b}\mathfrak{c}] \equiv \mathfrak{A}$.

Hence *each of* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *is the harmonic envelope of the other two as loci, each is the harmonic locus of the other two as envelopes; each as locus (as envelope) is apolar to each as envelope (as locus).*

10. *Examples.* (i). If

$$\mathfrak{a} = \mathrm{k}_1\{b, c\} + \mathrm{k}_2\{c, a\} + \mathrm{k}_3\{a, b\},$$

$$\mathfrak{B} = \mathrm{k}'_1\{B, C\} + \mathrm{k}'_2\{C, A\} + \mathrm{k}'_3\{A, B\}$$

and

$$[bc] = A, \quad [ca] = B, \quad [ab] = C,$$

then $\mathfrak{a}$ touches A, B, C at k_2c+k_3b, k_3a+k_1c, k_1b+k_2a respectively, and $\mathfrak{B}$ goes through a, b, c.

$$[\mathfrak{a}\mathfrak{B}] = \tfrac{1}{2}(k_1k_1'+k_2k_2'+k_3k_3')[abc]^2,$$
$$[\mathfrak{a}^2\mathfrak{B}^2] = \tfrac{1}{4}(k_1k_1'+k_2k_2'+k_3k_3')^2[abc]^4.$$

The joins of the vertices to the points of contact of $\mathfrak{a}$ with the opposite sides meet at $k_2k_3a+k_3k_1b+k_1k_2c$. If this point lies on $\mathfrak{B}$, then $k_1k_1'+k_2k_2'+k_3k_3' = 0$, and hence then $[\mathfrak{a}\mathfrak{B}] = 0$, $[\mathfrak{A}\mathfrak{b}] = 0$ and we have the case considered in 9.

(ii) If $\mathfrak{a} = l_1\{a^2\}+l_2\{b^2\}+l_3\{c^2\}$, $\mathfrak{b} = l_1'\{a^2\}+l_2'\{b^2\}+l_3'\{c^2\}$,

then
$$\tfrac{1}{2}[\mathfrak{a}^2\mathfrak{b}] = (l_1'l_2l_3+l_1l_2'l_3+l_1l_2l_3')[abc]^2,$$
$$\tfrac{1}{2}[\mathfrak{a}\mathfrak{b}^2] = (l_1l_2'l_3'+l_1'l_2l_3'+l_1'l_2'l_3)[abc]^2.$$

If now $[\mathfrak{a}^2\mathfrak{b}] = [\mathfrak{a}\mathfrak{b}^2] = 0$, and we take $l_1 = l_1'$, we find

$$l_2' = \omega l_2, \quad l_3' = \omega^2 l_3, \quad \omega^3 = 1.$$

If $\mathfrak{c} = l_1''\{a^2\}+l_2''\{b^2\}+l_3''\{c^2\}$, and $[\mathfrak{a}\mathfrak{b}] = \mathfrak{c}^2$, then

$$l_2l_3'+l_2'l_3 = 2l_2''l_3'', \quad l_3l_1'+l_3'l_1 = 2l_3''l_1'', \quad l_1l_2'+l_1'l_2 = 2l_1''l_2''.$$

Hence
$$\mathfrak{c}^2 \equiv l_2l_3[bc]^2+\omega l_3l_1[ca]^2+\omega^2 l_1l_2[ab]^2,$$
$$\mathfrak{c} \equiv l_1\{a^2\}+\omega^2 l_2\{b^2\}+\omega l_3\{c^2\}. \quad \text{(Cf. § 99·4.)}$$

11. *Condition that* $[\mathfrak{a}\mathfrak{b}] = 0$. This means that $[\mathfrak{a}\mathfrak{b}p^2] = 0$ for all points p of the plane.

Suppose, if possible, that $[\mathfrak{a}^3] \neq 0$, and take p on $\mathfrak{a}$, then $[\mathfrak{a}p^2]$ is the tangent at p repeated. But as $[\mathfrak{b}.\mathfrak{a}p^2] = 0$, therefore all tangents to $\mathfrak{a}$ touch $\mathfrak{b}$; hence $\mathfrak{a} \equiv \mathfrak{b}$, and so $[\mathfrak{a}^2] = 0$, contrary to the hypothesis that $[\mathfrak{a}^3] \neq 0$.

Hence $[\mathfrak{a}^3] = 0$, and similarly $[\mathfrak{b}^3] = 0$. Hence we can take $\mathfrak{a} = \{a, a'\}$, $\mathfrak{b} = \{b, b'\}$, and then $[\mathfrak{a}\mathfrak{b}.p^2] = 0$ means that $[ap]$, $[a'p]$ separate $[bp]$, $[b'p]$ harmonically for all p.

Hence a, a' are collinear with b, b' and separate them harmonically. (Ex. 16, p. 431, furnishes a special case.)

12. If $\mathfrak{a}$, $\mathfrak{b}$ be non-degenerate, then the equation

$$\{(\mathfrak{a}-k\mathfrak{b})^3\} = 0, \quad \text{or} \quad [\mathfrak{a}^3]-3k[\mathfrak{a}^2\mathfrak{b}]+3k^2[\mathfrak{a}\mathfrak{b}^2]-k^3[\mathfrak{b}^3] = 0,$$

is satisfied, in general, by three values, k_1, k_2, k_3, of k; that is, there are three degenerate conics in the pencil $\mathfrak{a}-k\mathfrak{b}$. These conics will be point-pairs; let $\{p, q\}$ be one of them, and let $\mathfrak{a}-k_1\mathfrak{b} = \{p, q\}$.

Then $[(\mathfrak{a}-k_1\mathfrak{b})[pq]] = [\{p, q\}[pq]] = 0$, by (11).

Hence $\mathfrak{a}[pq] \equiv \mathfrak{b}[pq]$, so that $[pq]$ has the same pole for all conics of the pencil $\mathfrak{a} - k\mathfrak{b}$, $(k \neq k_1)$, and in particular for $\mathfrak{a} - k_2\mathfrak{b}$, $\mathfrak{a} - k_3\mathfrak{b}$. Hence the pole of pq is the point where the joins of the last point-pairs meet.

Hence *the joins of the point-pairs in the pencil* $\mathfrak{a} - k\mathfrak{b}$ *form a triangle self-polar for all conics of the pencil.*

If a, b be points not on the conic $\mathfrak{a}$, there are points c, d such that

$$\mathfrak{a} \equiv \{a, b\} - k\{c, d\}, \quad (k \text{ scalar});$$

for we need only find the degenerate conics of the pencil

$$\mathfrak{a} - k'\{a, b\}.$$

13. *The poles of a line* L *for conics of the pencil* $\mathfrak{a} + k\mathfrak{b}$ *lie in general on a line.* For these poles are $[\mathfrak{a}L] + k[\mathfrak{b}L]$, and if $[\mathfrak{a}L] \neq [\mathfrak{b}L]$, they lie on the line $[\mathfrak{a}L . \mathfrak{b}L]$. If $[\mathfrak{a}L] \equiv [\mathfrak{b}L]$, they all coincide at this point; this is the case of 12.

Dually, *the polars of a point* p *for conics of the pencil* $\mathfrak{A} + k\mathfrak{B}$ *have in general one common point* $[\mathfrak{A}p . \mathfrak{B}p]$, but if $[\mathfrak{A}p] \equiv [\mathfrak{B}p]$ they all coincide with this line.

If L is the polar of p for $\mathfrak{A} + k\mathfrak{B}$, then $[L . \mathfrak{A}p . \mathfrak{B}p] = 0$. Hence this equation in p gives the locus of the poles of L for conics of the pencil. Since the equation contains p twice, the locus is a conic, the '*polar conic*' of L for the pencil.

The polar conic goes through the vertices of the common self-polar triangle of $\mathfrak{A}$ *and* $\mathfrak{B}$, for these vertices satisfy $[\mathfrak{A}p . \mathfrak{B}p] = 0$, or $\mathfrak{A}p \equiv \mathfrak{B}p$.

If s is on the polar conic of L, and $[\mathfrak{A}s . \mathfrak{B}s] = q$, then

$$[\mathfrak{A}s . q] = 0, \quad [\mathfrak{B}s . q] = 0, \quad [Lq] = 0.$$

Hence *the polar conic of* L *is the locus of points conjugate to points* q *of* L *for conics of the pencil* $\mathfrak{A} + k\mathfrak{B}$.

14. *Steiner correspondence.* To a point p make correspond the point $p' \equiv [\mathfrak{A}p . \mathfrak{B}p]$. Then

$$[\mathfrak{A}p . p'] = [\mathfrak{B}p . p'] = 0, \quad p \equiv [\mathfrak{A}p' . \mathfrak{B}p'].$$

The correspondence between p and p' is involutory and in general one-to-one. The points of the common self-polar triangle of $\mathfrak{A}$ and $\mathfrak{B}$ are the only exceptional points, a vertex corresponds to all points of the opposite side. The only self-corresponding points are the cuts of $\mathfrak{A}$ and $\mathfrak{B}$.

To points p of a line L, not a side of the common self-polar triangle, correspond points p' such that $[L.\mathfrak{A}p'.\mathfrak{B}p'] = 0$. Thus these lie on a conic circumscribed to the self-polar triangle. To the lines $L + kM$ of a pencil correspond the conics

$$[L.\mathfrak{A}p'.\mathfrak{B}p'] + k[M.\mathfrak{A}p'.\mathfrak{B}p'] = 0$$

of a pencil. The duals of 11 ... 14 are clear.

15. If a, b and c, d be conjugate point-pairs for $\mathfrak{A}$, then $\mathfrak{A}\{a, b\} = 0$, $\mathfrak{A}\{c, d\} = 0$. Hence the conic $\mathfrak{b} = k_1\{a, b\} + k_2\{c, d\}$ is apolar to $\mathfrak{A}$. But $\mathfrak{b}$ is in the pencil of conics touching the lines $[ac]$, $[bc]$ $[ad]$, $[bd]$, and by choice of k_1, k_2 it can be made $\equiv \{e, f\}$, where $e \equiv [ac.bd]$, $f \equiv [ad.bc]$. Hence $\mathfrak{A}\{e, f\} = 0$.

Hence, *if* (a, b), (c, d), (e, f) *be pairs of opposite vertices of a quadrilateral, and two pairs be conjugate pairs for a conic, so is the third pair.* (Cf. § 115, Ex. 10.)

16. By absorbing weights, any four independent points can be taken as $a, b, c, d = a + b + c$. Then

$$[bc.ad] \equiv b + c, \quad [ca.bd] \equiv c + a, \quad [ab.cd] \equiv a + b,$$

$$a^2 + b^2 + c^2 + (a + b + c)^2 = (b + c)^2 + (c + a)^2 + (a + b)^2$$

(algebraic squares).

Hence, *if abcd be a polar quadrangle, then* $[bc.ad]$, $[ca.bd]$, $[ab.cd]$ *is a self-polar triangle.*

The last identity can be written as a special case of (27):

$$[ab.cd]^2 + [bc.ad]^2 + [ca.bd]^2 = [bcd]^2 a^2 + [cad]^2 b^2 + [abd]^2 c^2 + [abc]^2 d^2.$$

§ 119. *Apolar spreads of conics.*

1. We may regard $\{e_1^2\}$, $\{e_2^2\}$, $\{e_3^2\}$, $\{e_2, e_3\}$, $\{e_3, e_1\}$, $\{e_1, e_2\}$ as unities in a spread of step six. Then any conic envelope $\mathfrak{a}$ is a linear combination of these unities.

Dually, any conic locus is a linear combination of $\{E_1^2\}$, $\{E_2^2\}$, $\{E_3^2\}$, $\{E_2, E_3\}$, $\{E_3, E_1\}$, $\{E_1, E_2\}$.

Now, if $\mathfrak{B}$, $\mathfrak{a}$ be expressed in this fashion, and $[\mathfrak{B}\mathfrak{a}] = 0$, there is a relation between the coefficients of $\mathfrak{B}$ and $\mathfrak{a}$, which is linear

in both sets. Hence, if $[\mathfrak{B}\mathfrak{a}_1] = 0, \ldots, [\mathfrak{B}\mathfrak{a}_5] = 0$, we have five linear equations for the coefficients of $\mathfrak{B}$. Hence, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_5$ are linearly independent, then $\mathfrak{B}$ is fixed apart from its weight.

Thus *there is just one conic apolar to five independent conics.*

2. If also $[\mathfrak{B}\mathfrak{a}_6] = 0$, then $\mathfrak{a}_1, \ldots, \mathfrak{a}_6$ are dependent.

3. If $\mathfrak{B}_1, \mathfrak{B}_2$ be independent, and apolar to $\mathfrak{a}_1, \ldots, \mathfrak{a}_4$, also independent, then all conics of pencil $k_1\mathfrak{B}_1 + k_2\mathfrak{B}_2$, and no others, are apolar to $\mathfrak{a}_1, \ldots, \mathfrak{a}_4$. This follows from the theory of linear equations.

If $\{a, b\}$ is in the system of $\mathfrak{a}_1, \ldots, \mathfrak{a}_4$, then $\mathfrak{B}_1\{a, b\} = 0$, $\mathfrak{B}_2\{a, b\} = 0$; hence a, b are conjugate for the pencil $k_1\mathfrak{B}_1 + k_2\mathfrak{B}_2$. Hence if a moves on a line, then b describes the polar conic of a for the pencil.

4. If $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ be independent and each apolar to $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$, also independent, then all conics of the 'net' $k_1\mathfrak{B}_1 + k_2\mathfrak{B}_2 + k_3\mathfrak{B}_3$, and no others, are apolar to $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$.

5. Since $\mathfrak{a} = \{a, b\}$ is apolar to $\mathfrak{B}$, if, and only if, a, b are conjugate points for $\mathfrak{B}$, therefore six pairs of points

$$\{a_i, b_i\}, \quad (i = 1, \ldots, 6),$$

if conjugate pairs for a conic, are connected by $\sum_1^6 k_i\{a_i, b_i\} = 0$ for some scalars k_i.

If $\{a_i, b_i\}$, $(i = 1, \ldots, 5)$, are five point pairs, and $\sum_1^5 k_i\{a_i, b_i\} = 0$ for some k_i, then a_1, b_1 are conjugate points for all conics for which the remaining four pairs are conjugate pairs. In particular, *the polars of a point p, for all conics with respect to which four given point-pairs are conjugate pairs, meet in a point.*

6. *If $a_1a_2a_3$ and $b_1b_2b_3$ be polar triangles for a conic, and $a_4a_5a_6$, $b_4b_5b_6$ be another pair of polar triangles for the conic, then $\{a_i, b_i\}$, $(i = 1, \ldots, 6)$, are dependent.*

*If $\{a_i, b_i\}$, $(i = 1, \ldots, 6)$, be dependent pairs, and be separated in any way into two sets, we obtain two pairs of polar triangles for the same conic, and the twelve sides of these two pairs of triangles form a dependent set of six line-pairs.**

* Rosanes, *Crelles Journ.* 90 (1881), p. 303.

*If $L_1, \ldots, L_4$ and $M_1, \ldots, M_4$ be two quadrilaterals, and we make the vertices correspond as follows, $[L_1L_2]$ to $[M_3M_4]$, $[L_2L_3]$ to $[M_1M_2]$, ..., $[L_3L_4]$ to $[M_1M_2]$, then we have a dependent system of six point-pairs. If five of the joins of corresponding vertices concur, the sixth goes through the common point.**

§ 120. *Nul triads of conics.*

1. These are triads of conics $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ such that $[\mathfrak{abc}] = 0$. If $\mathfrak{c}$ is apolar to the harmonic conic $[\mathfrak{ab}]$ of $\mathfrak{a}$, $\mathfrak{b}$, the relation between $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ is symmetrical, and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ is a nul triad.

By (30), if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ is a nul triad, and we take the poles p, q, r of any line L for $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, then the polars of p, q, r for $[\mathfrak{bc}]$, $[\mathfrak{ca}]$, $[\mathfrak{ab}]$ are concurrent.

If $[\mathfrak{a}\{a, b\}\{c, d\}] = 0$, then a, b separate c, d harmonically on a conic apolar to $\mathfrak{a}$; c, d are conjugate for the harmonic conic of $\mathfrak{a}$ and $\{a, b\}$; a, b are conjugate for the harmonic conic of $\mathfrak{a}$ and $\{c, d\}$.

2. If each triad of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ is a nul triad, then the harmonic conics of $[\mathfrak{ab}]$ and $[\mathfrak{cd}]$, of $[\mathfrak{bc}]$ and $[\mathfrak{ad}]$, of $[\mathfrak{ca}]$ and $[\mathfrak{bd}]$ are in a pencil. By (27).

3. By (39),

$$[\mathfrak{AB}.\mathfrak{cd}] = [\mathfrak{Ac}][\mathfrak{Bd}] + [\mathfrak{Bc}][\mathfrak{Ad}] - 2[(\mathfrak{AcB})\mathfrak{d}].$$

By (40), $$[(\mathfrak{AcB})\mathfrak{d}] = [(\mathfrak{AdB})\mathfrak{c}].$$

Substitute $[\mathfrak{ad}]$, $[\mathfrak{bd}]$ for $\mathfrak{A}$, $\mathfrak{B}$, then

$$2[([\mathfrak{ad}]\mathfrak{c}[\mathfrak{bd}])\mathfrak{d}] = [\mathfrak{ad}^2][\mathfrak{bcd}] + [\mathfrak{bd}^2][\mathfrak{acd}] - [\mathfrak{ad}.\mathfrak{bd}.\mathfrak{cd}].$$

By (28), $$[\mathfrak{cd}.\mathfrak{d}^2] = [\mathfrak{cd}^2]\mathfrak{d} + \tfrac{1}{3}[\mathfrak{d}^3]\mathfrak{c}.$$

Hence $$[\mathfrak{ab}.\mathfrak{cd}.\mathfrak{d}^2] = [\mathfrak{cd}^2][\mathfrak{abd}] + \tfrac{1}{3}[\mathfrak{d}^3][\mathfrak{abc}].$$

By (27),

$$2[\mathfrak{ad}.\mathfrak{bd}] + [\mathfrak{ab}.\mathfrak{d}^2] = 2[\mathfrak{abd}]\mathfrak{d} + [\mathfrak{ad}^2]\mathfrak{b} + [\mathfrak{bd}^2]\mathfrak{a}.$$

Hence
$$\begin{aligned} 2[\mathfrak{ad}.\mathfrak{bd}.\mathfrak{cd}] &= -[\mathfrak{ab}.\mathfrak{cd}.\mathfrak{d}^2] + 2[\mathfrak{abd}][\mathfrak{cd}^2] \\ &\quad + [\mathfrak{bcd}][\mathfrak{ad}^2] + [\mathfrak{acd}][\mathfrak{bd}^2] \\ &= [\mathfrak{bcd}][\mathfrak{ad}^2] + [\mathfrak{acd}][\mathfrak{bd}^2] \\ &\quad + [\mathfrak{abd}][\mathfrak{cd}^2] - \tfrac{1}{3}[\mathfrak{abc}][\mathfrak{d}^3], \qquad (48) \end{aligned}$$

$$\begin{aligned} 4[([\mathfrak{ad}]\mathfrak{c}[\mathfrak{bd}])\mathfrak{d}] &= [\mathfrak{bcd}][\mathfrak{ad}^2] + [\mathfrak{acd}][\mathfrak{bd}^2] \\ &\quad - [\mathfrak{abd}][\mathfrak{cd}^2] + \tfrac{1}{3}[\mathfrak{abc}][\mathfrak{d}^3]. \qquad (49) \end{aligned}$$

* Rosanes, *loc. cit.*

In particular, (48) gives: *if each triad of* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ *is a nul triad, so is* $\mathfrak{ad}, \mathfrak{bd}, \mathfrak{cd}$; *if* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *is a nul triad, and they are apolar to* $\mathfrak{d}^2$, *then* $\mathfrak{ad}, \mathfrak{bd}, \mathfrak{cd}$ *is a nul triad.*

If we put $\mathfrak{b} = \mathfrak{c}$ in (48), and $[\mathfrak{b}^2] = \mathfrak{B}$, $[\mathfrak{d}^2] = \mathfrak{D}$, then, *if* $\mathfrak{a}$ *is apolar to* $\mathfrak{B}$, *and if* $\mathfrak{a}, \mathfrak{b}$ *are apolar to* $\mathfrak{D}$, *then so is* $[\mathfrak{ad}]$ *to* $[\mathfrak{bd}]^2$.

4. By (37), $\qquad [\mathfrak{ac}.\mathfrak{bc}] = [\mathfrak{abc}]\,\mathfrak{c} + (\mathfrak{a}\mathfrak{C}\mathfrak{b})$.

Take $\mathfrak{b} = \mathfrak{a}$, and we have: *the harmonic conic of* $\mathfrak{a}, \mathfrak{c}$, *the conic* $\mathfrak{c}$, *and the reciprocal of* $\mathfrak{C}$ *for* $\mathfrak{a}$ *are in a pencil.*

Next, take $\mathfrak{c} = \{p, q\}$, then $\mathfrak{C} = -\frac{1}{2}[pq]^2$

$$(\mathfrak{a}\mathfrak{C}\mathfrak{b}) = [\mathfrak{abc}]\,\mathfrak{c} - \tfrac{1}{2}\{\mathfrak{a}[pq], \mathfrak{b}[pq]\}.$$

Hence *if* $[\mathfrak{abc}] = 0$ *and* $\mathfrak{c} = \{p, q\}$, *then* $[\mathfrak{ac}.\mathfrak{bc}]$ *is a point-pair* $\{a, b\}$, *where* $a = \mathfrak{a}[pq]$, $b = \mathfrak{b}[pq]$.

Conversely, if $\mathfrak{c} = \{p, q\}$, $(\mathfrak{a}\mathfrak{C}\mathfrak{b}) = -\frac{1}{2}\{a, b\}$, then $[\mathfrak{ac}.\mathfrak{bc}]$ is a point-pair, if $\{[\mathfrak{ac}.\mathfrak{bc}]^3\} = 0$, that is, if $([\mathfrak{abc}]\,\mathfrak{c} + (\mathfrak{a}\mathfrak{C}\mathfrak{b}))^3 = 0$.

This gives

$$[\mathfrak{abc}]\,[\mathfrak{c}.\mathfrak{c}.\{a, b\}] - \tfrac{1}{2}[\{a, b\}\{a, b\}\,\mathfrak{c}] = 0,$$

or, by (37), since $(\mathfrak{a}\mathfrak{C}\mathfrak{b}) = -\frac{1}{2}\{a, b\}$,

$$[[\mathfrak{ac}.\mathfrak{bc}]\{a, b\}\{p, q\}] = 0.$$

Hence the harmonic conic of $\mathfrak{L} = [\mathfrak{a}\{p, q\}]$ and $\mathfrak{M} = [\mathfrak{b}\{p, q\}]$ is a point-pair if, and only if, *either* p, q are conjugate for $[\mathfrak{LM}]$, *or* the harmonic conic of $\{a, b\}$ and $\{p, q\}$ is apolar to $[\mathfrak{LM}]$. In the first case, the point-pair is $\{a, b\}$ where $a = \mathfrak{a}[pq]$, $b = \mathfrak{b}[pq]$. In the second case, it is the third pair of vertices of the quadrangle *abpq*.

Examples. 23. If $[pq]$ does not touch $\mathfrak{a}$, it has the same pole for $\mathfrak{a}$ and for $[\mathfrak{a}\{p, q\}]$.

24. If $\mathfrak{b} = \{p, q\}$, $[\mathfrak{ad}] = \mathfrak{L}$, $[\mathfrak{bd}] = \mathfrak{M}$, and r be a common point of $\mathfrak{L}, \mathfrak{M}$, then $[\mathfrak{abd}] = 0$ is the necessary and sufficient condition that tangents to $\mathfrak{L}, \mathfrak{M}$ at r are harmonically separated by p, q, and then p, q are conjugate for $[\mathfrak{LM}]$.

§ 121. *The line at infinity and the circular points.*

1. Let e_0 be a fixed point, e_1, e_2 perpendicular unit vectors, $[e_0 e_1 e_2] = 1$. Write i for $\sqrt{-1}$.

Let $j_1 = e_1 + \mathrm{i}e_2$, $j_2 = e_1 - \mathrm{i}e_2$ be called the '*circular points at infinity*'.

Denote $[e_1 e_2]$ by U, $\{j_1, j_2\}$ by $\mathfrak{e}$, then

$$[j_1 j_2] = -2\mathrm{i}[e_1 e_2], \quad \{j_1, j_2\} = \{e_1^2\} + \{e_2^2\}, \quad \{[j_1 j_2]^2\} = -4\{U^2\},$$
$$[U\mathfrak{e}] = 0, \quad [\mathfrak{e}^3] = 0, \quad [\mathfrak{e}^2] = -\tfrac{1}{2}\{[j_1 j_2]^2\} = 2\{U^2\}.$$

If p be a point, then $[\mathfrak{e}p^2]$ is the pair of isotropic lines through p.

If $p = e_0 + \mathrm{p}_1 e_1 + \mathrm{p}_2 e_2$ is of weight one, then $[pU] = 1, [p^2\mathfrak{e}^2] = 2$.

If $q = e_0 + \mathrm{q}_1 e_1 + \mathrm{q}_2 e_2$ is also of weight one, then $[e_1 pq] = \mathrm{p}_2 - \mathrm{q}_2$; similarly, $[e_2 pq] = \mathrm{q}_1 - \mathrm{p}_1$.

But $\mathfrak{e} = \{e_1^2\} + \{e_2^2\}$. Hence $[\mathfrak{e}p^2q^2]$ represents the square of the distance between p and q.

If v is a vector, $[\mathfrak{e}v^2]$ is proportional to the square of its magnitude.

For, if $v = \mathrm{k}_1 e_1 + \mathrm{k}_2 e_2$, then

$$[(e_1^2 + e_2^2)(\mathrm{k}_1 e_1 + \mathrm{k}_2 e_2)^2] = \mathrm{k}_1^2[e_2 e_1]^2 + \mathrm{k}_2^2[e_1 e_2]^2 = (\mathrm{k}_1^2 + \mathrm{k}_2^2)\{U^2\}.$$

If L is a unit rotor, $[\mathfrak{e}L^2] = 1$; $[\mathfrak{e}L], = [e_1 L]e_1 + [e_2 L]e_2$, is the vector perpendicular to L, and $[\mathfrak{e}\{L, M\}]$ is the scalar product of L, M.

2. The circle $\mathfrak{c}$, centre p, radius r, as an envelope, is given by $\{p^2\} - \mathrm{r}^2\mathfrak{e}$.

For $[\mathfrak{c}L^2] = [pL]^2 - \mathrm{r}^2[\mathfrak{e}L^2]$, and this vanishes when the distance from p to L is r.

The same circle, as locus, is $[\mathfrak{c}^2]$, or $-2\mathrm{r}^2[p^2\mathfrak{e}] + \mathrm{r}^4[\mathfrak{e}^2]$; hence, if $\mathrm{r} \neq 0$, its equation, in q, is

$$[p^2\mathfrak{c}q^2] - \mathrm{r}^2[Uq]^2 = 0,$$

or, if q is of weight one, $[p^2\mathfrak{e}q^2] - \mathrm{r}^2 = 0$.

The point-circle, centre p, as locus, is $[p^2\mathfrak{e}]$.

Any circle, as locus, goes through j_1, j_2; for if

$$\mathfrak{K} = [p^2\mathfrak{c}] - \mathrm{r}^2\{U^2\}, \quad \text{then} \quad [\mathfrak{K}j_1^2] = [\mathfrak{K}j_2^2] = 0.$$

3. If $\mathfrak{c}$ is any conic, the pole $[\mathfrak{c}U]$ of U is its '*centre*'. From § 118·13 we have the locus of centres of conics touching four lines, or through four points.

The '*director circle*' of $\mathfrak{c}$ is $[\mathfrak{c}\mathfrak{e}]$; the expression represents a circle since $[\mathfrak{c}\mathfrak{e}j_1^2] = [\mathfrak{c}\mathfrak{e}j_2^2] = 0$.

4. By the last remark of § 118·12, any conic $\mathfrak{c}$ can be put in the form $\{f_1, f_2\} - \mathrm{k}^2\mathfrak{e}$, where k is a scalar, and f_1, f_2 points of unit weight, or vectors, called '*foci*'.

If L be a unit rotor, then $[\mathfrak{c}L^2] = [f_1 L][f_2 L] - \mathrm{k}^2$. Hence the product of the perpendiculars from f_1, f_2 on to a tangent to $\mathfrak{c}$ is constant.

If f_1 is a vector, take it for e_1, then

$$[\{f_1, f_2\}\mathfrak{e}] = \{[f_1 e_1], [f_2 e_1]\} + \{[f_1 e_2], [f_2 e_2]\} = \{[f_2 e_2], U\},$$
$$[\mathfrak{c}\mathfrak{c}] = \{[f_2 e_2], U\} - \mathrm{k}^2[\mathfrak{e}^2] = \{[f_2 e_2] - 2\mathrm{k}^2 U, U\}.$$

Now $[f_2 e_2] - 2\mathrm{k}^2 U$, or $[(f_2 - 2\mathrm{k}^2 e_1) e_2]$, is a line perpendicular to the direction of e_1, at a distance $-2\mathrm{k}^2$ from f_2. Hence the 'director circle' is this line (the directrix), together with the line at infinity.

5. The conic locus $\mathfrak{C}$ is a '*rectangular hyperbola*' if the point-pair $\mathfrak{e}$ is a conjugate pair for $\mathfrak{C}$, that is, if $[\mathfrak{C}\mathfrak{e}] = 0$.

Thence, if $\mathfrak{a} = \{f_1, f_2\} - \mathrm{k}^2\mathfrak{e}$, then $[\mathfrak{C}\mathfrak{a}] = [\mathfrak{C}\{f_1, f_2\}]$.

Hence *the conic envelopes apolar to a rectangular hyperbola have foci conjugate for the hyperbola.*

6. The circle loci $\mathfrak{K} = [p^2 . \mathfrak{e}] - \mathrm{r}^2\{U^2\}$ form a spread of step four; those apolar to a conic envelope $\mathfrak{a}$ form a spread of step three, the point-circles in this spread are of form $[p^2\mathfrak{e}]$, and they satisfy $[\mathfrak{a} . p^2\mathfrak{e}] = 0$.

Hence $[\mathfrak{a}\mathfrak{e} . p^2] = 0$, so that p is on the director circle of $\mathfrak{a}$. Now three such point-circles will serve as a basis for the spread of apolar circles, and these point-circles are on $[\mathfrak{a}\mathfrak{e}]$, that is, orthogonal to $[\mathfrak{a}\mathfrak{e}]$. Hence *the circle loci apolar to* $\mathfrak{a}$ *are orthogonal to its director circle.*

If $\mathfrak{a}$ is a rectangular hyperbola, $[\mathfrak{a} . \mathfrak{a}\mathfrak{e}] = 0$; hence its director circle is apolar to $\mathfrak{a}$, and hence is self-orthogonal, and thus is a point-circle. *The director circle of a rectangular hyperbola is a point-circle.*

7. If $\sum_1^{\mathrm{r}} \mathrm{k}_i \mathfrak{a}_i = 0$, then $\sum_1^{\mathrm{r}} \mathrm{k}_i [\mathfrak{a}_i \mathfrak{e}] = 0$. Hence, *in general, a spread of conic envelopes of step* r *has a spread of director circles of step* r.

We have an exceptional case when the $[\mathfrak{a}_i \mathfrak{e}]$ are connected by another linear relation; e.g. conics which touch the sides of a rectangle form a spread of step two, but have the same director circle.

Examples. 25. If $\mathfrak{k}_1 = \{p_1^2\} - \frac{1}{2}r_1^2\mathfrak{c}$, $\mathfrak{k}_2 = \{p_2^2\} - \frac{1}{2}r_2^2\mathfrak{e}$, where p_1, p_2 are proper points of unit weight, then

$$[\mathfrak{k}_1\mathfrak{k}_2\mathfrak{e}] = [p_1^2p_2^2\mathfrak{e}] - \tfrac{1}{2}r_1^2[p_2^2\mathfrak{e}^2] - \tfrac{1}{2}r_2^2[p_1^2\mathfrak{e}^2] = \overline{p_1p_2}^2 - r_1^2 - r_2^2.$$

Hence circle-envelopes $\mathfrak{k}$, $\mathfrak{k}'$ with centres p, p', radii r, r_1, are orthogonal if $[(\mathfrak{k} + \frac{1}{2}r^2\mathfrak{e})(\mathfrak{k}' + \frac{1}{2}r_1^2\mathfrak{e})\,\mathfrak{e}] = 0$.

26. The circle on diameter pq is $[\{p, q\}\,\mathfrak{e}]$. Hence if f_1, f_2 and f_1', f_2' separate one another harmonically on a circle, then the harmonic conic of any conics with foci f_1, f_2 and f_1', f_2' is a circle.

27. The harmonic conic of $\mathfrak{a}$, $\mathfrak{b}$ is a rectangular hyperbola or degenerate, if and only if $[\mathfrak{abe}] = 0$, that is, if and only if the director circles $[\mathfrak{ae}]$, $[\mathfrak{be}]$ are orthogonal. For if $[\mathfrak{ae}]$ is apolar to $\mathfrak{b}$, it is orthogonal to its director circle.

28. The line L conjugate to U for the pencil of conics $\{a, b\} + k\{c, d\}$ touching $[ac]$, $[ad]$, $[bc]$, $[bd]$ satisfies $[(a+b)\,L] + k[(c+d)\,L] = 0$, and hence goes through $\frac{1}{2}(a+b)$, $\frac{1}{2}(c+d)$. Thus we have the line of centres.

29. If
$$\sum_{i=1}^{4} k_i L_i^2 = \{L, U\},$$
then
$$\Sigma k_i[L_i p]^2 = [Lp]\,[Up]$$
for all points p. But $[Up] = 1$, hence $\Sigma k_i[L_i p]^2 = [Lp]$.

Then L is the line of centres of the conics $\mathfrak{c}$ touching L_1, L_2, L_3, L_4. For if $[\mathfrak{c}L_i^2] = 0$, $(i = 1, \ldots, 4)$, then $[\mathfrak{c}\{L, U\}] = 0$.

This may also be seen as follows:*

$$\sum_{i=1}^{4} k_i L_i^2 = \tfrac{1}{4}\{(L+U)^2 - (L-U)^2\}.$$

Hence a conic touches $L_1, \ldots, L_4$, $L+U$, $L-U$, and the centre is equidistant from $L+U$ and $L-U$, and hence is on L.

If $p = [L_1L_2]$, the polar of p is $k_3[L_3p]\,L_3 + k_4[L_4p]\,L_4$, which is a line through $[L_3L_4]$.

30. The conic loci through points a, b, c, d are of form
$$\mathfrak{C} = \{[bc], [ad]\} + k\{[ca], [bd]\}.$$
If bc, ca are perpendicular respectively to ad, bd, then
$$[\mathfrak{c}\{[bc], [ad]\}] = 0, \quad [\mathfrak{e}\{[ca], [bd]\}] = 0;$$
hence
$$[\mathfrak{C}\mathfrak{e}] = 0.$$
Hence $\mathfrak{C}$ is a rectangular hyperbola. *The conics through four orthocentric points are rectangular hyperbolas.*

* Serret, *Géométrie de direction* (1869), p. 140.

The loci of the centres of these conics is, by § 118·13,

$$[U.\mathfrak{A}p.\mathfrak{B}p] = 0,$$

where $\mathfrak{A} = \{[bc], [ad]\}$, $\mathfrak{B} = \{[ca], [bd]\}$.

Since $[\mathfrak{e}\mathfrak{A}] = 0$, $[\mathfrak{e}\mathfrak{B}] = 0$, $\mathfrak{e} = \{j_1, j_2\}$, we have $[\mathfrak{A}j_1.\mathfrak{B}j_1] \equiv j_2$. Hence $[U.\mathfrak{A}j_1.\mathfrak{B}j_1] = 0$. Thus *the locus of the centres is a circle.*

31. *The diameters of the five quadrilaterals formed by $L_1, \ldots, L_5$ are concurrent.*

For, if $k_{12}L_2^2 + k_{13}L_3^2 + \ldots + k_{15}L_5^2 = \{M_1, U\}$,

$k_{21}L_1^2 + k_{23}L_3^2 + \ldots + k_{25}L_5^2 = \{M_2, U\}$, and so on,

then $k_{23}\{M_1, U\} - k_{13}\{M_2, U\} = \{k_{23}M_1 - k_{13}M_2, U\}$

is a linear conbination of L_1^2, L_2^2, L_4^2, L_5^2, and hence is $\equiv \{M_3, U\}$. The point of concurrence is the centre of the conic touching the five lines.

32. *The centres of the six conics each of which touches five of the six given lines $L_1, \ldots, L_6$ lie on a conic.*

For take points such that $L_1 = ab'$, $L_2 = b'c$, $L_3 = ca'$, $L_4 = a'b$, $L_5 = bc'$, $L_6 = c'a$; and let o_i be the centre of the conic touching all the lines except L_i; let M_{ij} be the diameter of the quadrilateral formed by the lines L omitting L_i, L_j. Then $M_{ij} \equiv [o_i o_j]$. But, § 5, Ex. 33, $[M_{12}M_{45}.M_{23}M_{56}.M_{34}M_{61}] = 0$.

Hence the theorem follows from Pascal's Theorem, and the Pascal line of the hexagon of the centres in the order taken is

$$b'c + c'a + a'b - bc' - ca' - ab'.$$

33. The circles in the set of conics $k_1L_1^2 + \ldots + k_4L_4^2$, where $L_1, \ldots, L_4$ are lines, include (when $k_4 = 0$) the polar circle of $L_1L_2L_3$, and any circle which divides harmonically the diagonals of the quadrilateral, for example, the circumcircle of the diagonal triangle. The point-circles arise when the expression equals $L^2 + M^2$, where L, M are perpendicular lines. Then since

$$\sum_{i=1}^{4} k_i L_i^2 - L^2 - M^2 = 0,$$

it follows that the lines L_i and two perpendicular lines through the centre of a point-circle of the set, touch a conic. Thence the director circles of conics of the set are coaxal. (Serret, *l.c.* § 119.)

34. The locus of the centres of conics, for which L_1, L_1'; L_2, L_2'; L_3, L_3'; L_4, L_4' are conjugate pairs, is a line $\Sigma k_i\{L_i, L_i'\}$. Five pairs of lines define a conic for which they are conjugate pairs; its centre is the radical centre of the circles in the set $\sum_{i=1}^{5} k_i\{L_i, L_i'\}$. (Cf. § 119·5.)

35. *The Steiner parabola and the Apollonian hyperbola.* The conics $\mathfrak{a}$ and $\mathfrak{a}-k\mathfrak{e}$ are confocals, by 4. The polars L of p for $\mathfrak{a}-k\mathfrak{e}$ envelope the polar conic $[p.\mathfrak{a}L.\mathfrak{e}L] = 0$. *This is a parabola,* for $L = U$ satisfies the equation, since $[\mathfrak{e}U] = 0$. Since

$$[p.\mathfrak{a}L.(\mathfrak{a}-k\mathfrak{e})L] = 0,$$

the tangents at p to the confocals through p touch the polar conic. Since $\mathfrak{e}L$ is the vector perpendicular to L, therefore *the perpendicular $[p.\mathfrak{e}L]$ from p to L goes through the pole $\mathfrak{a}L$ of L for $\mathfrak{a}$. The parabola touches the normals to $\mathfrak{a}-k\mathfrak{e}$ at the points of contact of tangents from p.*

Let q be the pole of L for $\mathfrak{a}$, let $[\mathfrak{a}^2] = \mathfrak{A} \neq 0$, then $q = [\mathfrak{a}L]$, $L = [\mathfrak{A}q]$; hence $[p.q.\mathfrak{e}L] = 0$. The point q describes the reciprocal of the parabola with respect to $\mathfrak{a}$. When $L = U$ then $[\mathfrak{e}L] = 0$, $q = [\mathfrak{a}L]$ is the centre of $\mathfrak{a}$, thence the *reciprocal of the parabola goes through the centre of $\mathfrak{a}$; it also goes through p,* (since $q = p$ satisfies the equation) *and through points q whose polars L are perpendicular to pq, in particular through the feet of the normals from p.*

36. The poles of two arbitrary lines with respect to conics of a confocal system are corresponding points of two similar ranges.

37. The director circle of a conic, the circle on any chord as diameter, and the point circle whose centre is the pole of the chord are coaxal.

38. If tangents are drawn from a fixed point to confocals, the circumcircles of the triangles formed by pairs of tangents and the chord of contact are coaxal.

39. Erect a theory of point-pairs in involution on a conic, shewing that a, b separate c, d harmonically if $[ab\,|cd] = 0$, and that $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ are pairs in an involution on the conic, if the corresponding products are dependent. Extend this to generators on a regulus.

§ 122. *Algebraic products of three points and of three lines.**

1. The '*algebraic product*' of three points a, b, c of any weights is the point-triad a, b, c, in any order, and is denoted by $\{a, b, c\}$, the order of a, b, c being irrelevant. If one of the points be multiplied by a scalar, so is the product. We write $\{a^2, b\}$ for $\{a, a, b\}$ and $\{a^3\}$ for $\{a, a, a\}$. Algebraic products are added formally.

* H. Grassmann, d.J., *Projektive Geometrie*, Band II, 2 (1927).

The equation $\{a_1, b_1, c_1\}+\{a_2, b_2, c_2\}+\ldots = 0$

means that, for all lines L, we have

$$[a_1 L]\,[b_1 L]\,[c_1 L]+[a_2 L]\,[b_2 L]\,[c_2 L]+\ldots = 0.$$

Two sums $\mathfrak{a}$, $\mathfrak{b}$ of algebraic products are '*equal*' if $\mathfrak{a}-\mathfrak{b} = 0$.

Dually, we have the algebraic product of three lines, $\{L, M, N\}$.

2. The '*outer product*' of $\{a_1, a_2, a_3\}$ and $\{L, M, N\}$ is the scalar $\frac{1}{6}\Sigma[a_i L]\,[a_j M]\,[a_k N]$, the sum over all permutations i, j, k of 1, 2, 3. It is denoted by $\{[a_1, a_2, a_3]\{L, M, N\}]$.

For example, $[\{a, b, c\}\{L^3\}] = [aL]\,[bL]\,[cL]$,

and it would have been possible to start with this formula.

The distributive law being assumed for these products, if $\mathfrak{a}$ is a sum of algebraic products of point-triads, and L is a line, then $[\mathfrak{a}\{L^3\}]$ is defined. The set of lines L which satisfy $[\mathfrak{a}\{L^3\}] = 0$ envelope a cubic envelope, which we also denote by $\mathfrak{a}$.

Dually, if $\mathfrak{A}$ be a sum of algebraic products of line-triads, then $[\mathfrak{A}\{p, q, r\}]$ is a scalar; the locus of points p which satisfy $[\mathfrak{A}\{p^3\}] = 0$ is a cubic locus, which we also denote by $\mathfrak{A}$.

3. The '*outer product*' of $\{a_1, a_2, a_3\}$ and $\{L, M\}$ is

$$\tfrac{1}{6}\Sigma[a_i L]\,[a_j M]\,a_k,$$

where the sum is over all permutations i, j, k of 1, 2, 3. It is denoted by $[\{a_1, a_2, a_3\}\{L, M\}]$. Its outer product with N is clearly $[\{a_1, a_2, a_3\}\{L, M, N\}]$.

The '*outer product*' of $\{a_1, a_2, a_3\}$ and L is

$$\tfrac{1}{3}([a_1 L]\{a_2, a_3\}+[a_2 L]\{a_3, a_1\}+[a_3 L]\{a_1, a_2\}).$$

It is denoted by $[\{a_1, a_2, a_3\}L]$. Its outer product with M is $[\{a_1, a_2, a_3\}\{L, M\}]$.

Using the distributive law, we have now products $[\mathfrak{a}\{L, M\}]$, and $[\mathfrak{a}L]$; the first is a sum of points, and hence is a point; the second is a sum of products of point-pairs, and represents a conic envelope.

Dually, $[\mathfrak{A}\{p, q\}]$ is a sum of lines, and hence is a line, and $[\mathfrak{A}p]$ is a sum of products of line-pairs, and represents a conic locus.

The outer product of $[\mathfrak{A}p]$ and q is $[\mathfrak{A}\{p, q\}]$; the outer product of $[\mathfrak{A}\{p, q\}]$ and r is $[\mathfrak{A}\{p, q, r\}]$.

$[\mathfrak{A}\{p, q\}] = 0$ means $[\mathfrak{A}\{p, q, r\}] = 0$ for all points r.

$[\mathfrak{A}p] = 0$ means $[\mathfrak{A}\{p, q, r\}] = 0$ for all points q, r. This will be the case if, and only if, $[\mathfrak{A}\{p, q, q\}] = 0$ for all points q.

§ 123. *Cubic curves.*

1. We shall deal with the cubic locus denoted by $\mathfrak{A}$, and, for simplicity, we usually omit the square brackets from $[\mathfrak{A}\{p^3\}]$, $[\mathfrak{A}\{p, q, r\}]$, and so on.

If $p + kq$ (k scalar) is on $\mathfrak{A}$, then $[\mathfrak{A}\{p + kq\}^3] = 0$, that is,

$$\mathfrak{A}\{p^3\} + 3k\mathfrak{A}\{p^2, q\} + 3k^2\mathfrak{A}\{p, q^2\} + k^3\mathfrak{A}\{q^3\} = 0. \qquad \text{(i)}$$

2. If p is a fixed point, the conic $\mathfrak{A}p$, called the '*polar conic*' of p, is the locus of points q such that $\mathfrak{A}\{p, q^2\} = 0$; the line $\mathfrak{A}\{p^2\}$, called the '*polar line*' of p, is the locus of points q such that $\mathfrak{A}\{p^2, q\} = 0$.

If the polar conic of p goes through q, then the polar line of q goes through p.

For, if $\mathfrak{A}\{p, q^2\} = 0$, then $\mathfrak{A}\{q^2, p\} = 0$.

The polar line of p for the conic $\mathfrak{A}p$ is the polar line of p for the cubic $\mathfrak{A}$.

For the former line is the outer product of $\mathfrak{A}p$ and p, that is, $\mathfrak{A}\{p^2\}$.

3. If p is on $\mathfrak{A}$, then $\mathfrak{A}\{p^3\} = 0$, the line $[pq]$ cuts $\mathfrak{A}$ again in $p + kq$, where k is given by

$$3\mathfrak{A}\{p^2, q\} + 3k\mathfrak{A}\{p, q^2\} + k^2\mathfrak{A}\{q^3\} = 0.$$

If q is on the polar line $\mathfrak{A}\{p^2\}$ of p, then $\mathfrak{A}\{p^2, q\} = 0$, the equation (i) in k has an additional zero root, and hence $\mathfrak{A}p^2$ meets $\mathfrak{A}$ in two coincident points at p. Hence $\mathfrak{A}\{p^2\}$ *is the tangent at p.*

The polar conic of the point p on $\mathfrak{A}$ is $\mathfrak{A}p$, its tangent at p is $\mathfrak{A}\{p^2\}$, the tangent to $\mathfrak{A}$ at p. Hence *the polar conic of a point on $\mathfrak{A}$ touches $\mathfrak{A}$ at the point.*

4. In the last paragraph it was tacitly assumed that $\mathfrak{A}\{p^2\} \neq 0$. If $\mathfrak{A}\{p^2\} = 0$, that is, if $\mathfrak{A}\{p^2, q\} = 0$ for all points q, then *all* lines through p cut $\mathfrak{A}$ again in p. Hence $\mathfrak{A}\{p^2\} = 0$ *is the condition that p is a double point of $\mathfrak{A}$.*

If p is a double point of $\mathfrak{A}$, and $[pq]$ a tangent at p, then equation (i) has three roots $k = 0$, and hence $\mathfrak{A}\{p, q^2\} = 0$. Hence, if q is on a tangent at a double point p, it is on the conic $\mathfrak{A}p$, which must hence be a line-pair. *The polar conic of a double point p is the line-pair of tangents at p.*

Not all cubics have double points; for if $\mathfrak{A}\{p^2\} = 0$, then

$$\mathfrak{A}\{p^2, e_i\} = 0, \quad (i = 1, 2, 3).$$

Take $p = x_1 e_1 + x_2 e_2 + x_3 e_3$ and we obtain three homogeneous quadratic equations for x_1, x_2, x_3 which, in general, have no common root.

5. *If r be any point, the polar conic $\mathfrak{A}r$ goes through all double points p,* for since $\mathfrak{A}\{p^2\} = 0$, we have $\mathfrak{A}\{r, p^2\} = 0$. *The polar conic also goes through the points of contact of tangents from r.* For, if q be such a point of contact, then the tangent at q is $\mathfrak{A}\{q^2\}$, and since it goes through r, we have $\mathfrak{A}\{q^2, r\} = 0$, and hence $\mathfrak{A}\{r, q^2\} = 0$. Thus q is on $\mathfrak{A}r$.

6. If p is on $\mathfrak{A}$, but not a double point, and so $\mathfrak{A}\{p^3\} = 0$, $\mathfrak{A}\{p^2\} \neq 0$, then the line $[pq]$ will meet $\mathfrak{A}$ in three coincident points if $\mathfrak{A}\{p^2, q\} = 0$, $\mathfrak{A}\{p, q^2\} = 0$. Hence this is the condition that p should be a '*flex*', and $[pq]$ a tangent there. If these conditions hold, then, for all scalars l,

$$\mathfrak{A}\{p, (p + lq)^2\} = \mathfrak{A}\{p^3\} + 2l\mathfrak{A}\{p^2, q\} + l^2\mathfrak{A}\{p, q^2\} = 0.$$

Hence each point on $[pq]$ is on $\mathfrak{A}p$, the polar conic of p. Hence *the polar conic of a flex p is a line-pair, consisting of the tangent at p, and another line,* called the '*harmonic polar*' of p.

7. Thus the polar conic of a double point, or of a flex, is a line-pair. If q be any point whose polar conic $\mathfrak{A}q$ is a line-pair, with double point r, then the equation $\mathfrak{A}\{q, (r + ls)^2\} = 0$ in l has two zero roots for all points s. If this is so, then $\mathfrak{A}\{q, r^2\} = 0$, and hence r is on $\mathfrak{A}q$, and also $\mathfrak{A}\{q, r, s\} = 0$ for all points s, that is $\mathfrak{A}\{q, r\} = 0$.

Hence *$\mathfrak{A}q$ is a line pair if, and only if, there is a point r such that $\mathfrak{A}\{q, r\} = 0$, and then r is the double point of the line pair.*

Since $\mathfrak{A}\{q, r\} = \mathfrak{A}\{r, q\}$, therefore, *if r be the double point of the polar conic of q, then q is the double point of the polar conic of r.*

8. To find the locus of double points of polar conics of $\mathfrak{A}$. This is the locus of points q such that $\mathfrak{A}\{q, r\} = 0$, as r varies. Since $\mathfrak{A}\{q, r, e_i\} = 0$, $(i = 1, 2, 3)$, therefore the lines $\mathfrak{A}\{q, e_i\}$, $(i = 1, 2, 3)$, go through r, and hence the equation of the locus of q is

$$[\mathfrak{A}\{q, e_1\} . \mathfrak{A}\{q, e_2\} . \mathfrak{A}\{q, e_3\}] = 0.$$

As this is a cubic in q, the locus is a cubic curve, the '*Hessian*' $\mathfrak{H}$ of $\mathfrak{A}$.

If r is the double point of $\mathfrak{A}q$, then by the last remark of 7, both q and r are on $\mathfrak{H}$; they are called '*conjugate points*' of $\mathfrak{H}$.

$\mathfrak{H}$ goes through all double points and flexes of $\mathfrak{A}$, since their polar conics are line-pairs.

Conversely, if p is on $\mathfrak{A}$, and has a line-pair for polar conic, then $\mathfrak{A}\{p^3\} = 0$, $\mathfrak{A}\{p, r\} = 0$, for some point r. Hence

$$\mathfrak{A}\{p, r, p + lr\} = 0$$

for all scalars l, in particular $\mathfrak{A}\{p^2, r\} = 0$, $\mathfrak{A}\{p, r^2\} = 0$. Hence, *either* p is a flex, *or* $\mathfrak{A}\{p^2\} = 0$, and then p is a double point.

Thus $\mathfrak{H}$ *meets* $\mathfrak{A}$ *in all its flexes and double points and nowhere else.*

If the cubic has no double points, it has accordingly nine flexes.

§ 124. *Nets of conics.*

1. If $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ be three independent conic loci, they determine a net of conics

$$\mathfrak{C} = k_1\mathfrak{C}_1 + k_2\mathfrak{C}_2 + k_3\mathfrak{C}_3, \quad (k_1, k_2, k_3 \text{ scalars}).$$

The '*Jacobian*' $\mathfrak{H}$ of the net is the locus of points p whose polars $\mathfrak{C}_1 p$, $\mathfrak{C}_2 p$, $\mathfrak{C}_3 p$ for $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ concur. The Jacobian depends on the net only, for if these polars meet in q, then

$$[q . \mathfrak{C}_1 p] = [q . \mathfrak{C}_2 p] = [q . \mathfrak{C}_3 p] = 0,$$

and hence $[q . \mathfrak{C} p] = 0$. Thus the polars of p for all conics of the net go through q.

We call p, q '*conjugate points*' of the net. They satisfy $\mathfrak{C}\{p, q\} = 0$ for all conics $\mathfrak{C}$ of the net.

The equation of the Jacobian of the net is $[\mathfrak{C}_1 p . \mathfrak{C}_2 p . \mathfrak{C}_3 p] = 0$, and hence the Jacobian is a cubic locus.

2. If $\mathfrak{C}$, in the net, is a line-pair with double point r, then $\mathfrak{C}r = 0$; hence

$$k_1\mathfrak{C}_1r + k_2\mathfrak{C}_2r + k_3\mathfrak{C}_3r = 0, \quad [\mathfrak{C}_1r.\mathfrak{C}_2r.\mathfrak{C}_3r] = 0.$$

Hence *the double points of degenerate conics of the net lie on the Jacobian of the net; conversely, each point of the Jacobian is such a double point.*

3. Since the point-pairs consisting of opposite vertices of a complete quadrilateral are dependent, and as a, b are conjugate pairs of the Jacobian, if $\mathfrak{C}\{a, b\} = 0$ for all $\mathfrak{C}$ of the net, we have: *if two pairs of opposite vertices of a complete quadrilateral be conjugate pairs for the net, so is the third pair.*

4. Let c_1, c_2, c_3 be independent conic envelopes, each apolar to each of the independent conic loci $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$; then c_1, c_2, c_3 determine a net which we call '*apolar*' to the net determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$. The '*Jacobian*' $\mathfrak{h}$ of the net of envelopes $\mathfrak{c}$ is defined as the envelope of lines L whose poles for c_1, c_2, c_3 are collinear. It is a cubic envelope. The poles of L for all conics of the net of the $\mathfrak{c}$ lie on a line M tangent to $\mathfrak{h}$, and vice versa. These '*conjugate lines*' are the joins of the point-pairs which are degenerate conics of the net of the $\mathfrak{c}$. For dualise 1, 2.

5. If p, q be conjugate points on $\mathfrak{H}$, then $\mathfrak{C}\{p, q\} = 0$ for all conics $\mathfrak{C}$ of the net of the $\mathfrak{C}$; hence the point-pair $\{p, q\}$ is a degenerate conic of the net of the $\mathfrak{c}$, for the last equation means that $\mathfrak{C}$ and $\{p, q\}$ are apolar. Since a conic of the net of the $\mathfrak{C}$ is fixed by two points, there is one conic of that net through p, q, and it must be apolar to $\{p, q\}$. But a degenerate conic, one line of which is $[pq]$, is apolar to $\{p, q\}$. Hence the net of the $\mathfrak{C}$ contains a degenerate conic, one line of which is $[pq]$. And $[pq]$ touches $\mathfrak{h}$. Thence:

If p, q be conjugate points of $\mathfrak{H}$, then $[pq]$ touches $\mathfrak{h}$; the third cut of $[pq]$ and $\mathfrak{H}$ is the double point of the conic of which $[pq]$ is part.

Dually, *if L, M be conjugate lines touching $\mathfrak{h}$, then $[LM]$ is on $\mathfrak{H}$; the third tangent from $[LM]$ to $\mathfrak{h}$ is the join of the point-pair conic of the net of the $\mathfrak{c}$, one of whose points is $[LM]$.*

6. If p be a point such that the tangents from p to conics c_1, c_2, c_3 are pairs in involution, then these pairs c_1p^2, c_2p^2, c_3p^2 are

linearly dependent, that is, there are scalars k_1, k_2, k_3 such that $(k_1c_1+k_2c_2+k_3c_3)p^2=0$; hence p is a point of a point-pair conic of the net of the $\mathfrak{c}$, and hence p is on $\mathfrak{H}$. (Cf. Ex. 16, p. 431.)

Thence $\mathfrak{H}$ *is the locus of a point from which tangents to conics of the net of the* $\mathfrak{c}$ *are in involution.*

Dually, $\mathfrak{h}$ *is the envelope of lines cut in involution by the net of the* $\mathfrak{C}$.

7. If $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ be independent conic loci, the line $[pq]$ is cut at $p+kq$ by $\mathfrak{C}_1$, where

$$\mathfrak{C}_1\{p^2\}+2k\mathfrak{C}_1\{p,q\}+k^2\mathfrak{C}_1\{q^2\}=0.$$

Hence the line is cut in involution by the conics, if

$$\begin{vmatrix} \mathfrak{C}_1\{p^2\}, & \mathfrak{C}_1\{p,q\}, & \mathfrak{C}_1\{q^2\} \\ \mathfrak{C}_2\{p^2\}, & \mathfrak{C}_2\{p,q\}, & \mathfrak{C}_2\{q^2\} \\ \mathfrak{C}_3\{p^2\}, & \mathfrak{C}_3\{p,q\}, & \mathfrak{C}_3\{q^2\} \end{vmatrix}=0.$$

If $\mathfrak{C}_1=A^2+A_1^2+\ldots$, $\mathfrak{C}_2=B^2+B_1^2+\ldots$, $\mathfrak{C}_3=C^2+C_1^2+\ldots$,

then the determinant is the sum of determinants such as

$$\begin{vmatrix} [Ap]^2, & [Ap][Aq], & [Aq]^2 \\ [Bp]^2, & [Bp][Bq], & [Bq]^2 \\ [Cp]^2, & [Cp][Cq], & [Cq]^2 \end{vmatrix}$$

and this equals

$$\begin{aligned} &-([Bp][Cq]-[Cp][Bq]) \\ &\qquad\times([Cp][Aq]-[Ap][Cq])([Ap][Bq]-[Bp][Aq]) \\ &=-[BC.pq][CA.pq][AB.pq]=-[BCL][CAL][ABL], \end{aligned}$$

where $L=[pq]$.

Hence the equation of the envelope $\mathfrak{h}$ is given by the vanishing of a sum of terms like $[BCL][CAL][ABL]$. Dually the equation of the locus $\mathfrak{H}$ is given by the vanishing of a sum of terms like $[bcp][cap][abp]$, where

$$\mathfrak{c}_1=a^2+a_1^2+\ldots,\quad \mathfrak{c}_2=b^2+b_1^2+\ldots,\quad \mathfrak{c}_3=c^2+c_1^2+\ldots.$$

8. Now let $\mathfrak{A}$ be a cubic locus; the polar conics of all points of the plane with respect to $\mathfrak{A}$ form a net of conics, for the polar conic of e_i is $\mathfrak{A}e_i$, $(i=1,2,3)$, and these three conics are not in a pencil, since the e_i are independent. The Jacobian of the net of

polar conics has equation $[\mathfrak{A}e_1p.\mathfrak{A}e_2p.\mathfrak{A}e_3p]=0$, hence *the Hessian* $\mathfrak{H}$ *of the cubic locus is the Jacobian of the net of polar conics.* For $[\mathfrak{A}ep]=\mathfrak{A}\{e,p\}$.

The envelope $\mathfrak{h}$ of the join of conjugate points on $\mathfrak{H}$ is called the '*Cayleyan*' of the cubic. Its properties are given above.

9. The equation of $\mathfrak{H}$ can be written $[\mathfrak{A}pe_1.\mathfrak{A}pe_2.\mathfrak{A}pe_3]=0$, hence the polars of the non-collinear points e_1, e_2, e_3 for the conic $\mathfrak{A}p$ are concurrent, hence $\mathfrak{A}p$ is degenerate, and $[(\mathfrak{A}p)^3]=0$. If $\mathfrak{A}=A^3+B^3+\ldots$, this can be written

$$\Sigma[Ap]\{A^2\}.[Bp]\{B^2\}.[Cp]\{C^2\}=0$$

or

$$\Sigma[Ap][Bp][Cp][ABC]^2=0,$$

which connects it up with the equation in symbolic invariant theory.

10. By § 124·3, if p, p' and r, r' be conjugate pairs on $\mathfrak{H}$, then $q=[pr.p'r']$ and $q'=[pr'.p'r]$ are also conjugate points on $\mathfrak{H}$. Let p tend to r, and p' to r', then *tangents to* $\mathfrak{H}$ *at conjugate points* p, p' *of* $\mathfrak{H}$ *meet at* q *the point conjugate to the third cut* q' *of* $[pp']$ *and* $\mathfrak{H}$.

The pencil of conics $\mathfrak{A}(kp+k'p')$ or $k\mathfrak{A}p+k'\mathfrak{A}p'$ contains the line-pairs $\mathfrak{A}p$, $\mathfrak{A}p'$, $\mathfrak{A}q'$ with the respective double points p', p, q. Hence $pp'q$ is the common self-polar triangle of these conics. Hence, *if* p *is on* $\mathfrak{H}$, *the polar line* $\mathfrak{A}p^2$ *of* p *for* $\mathfrak{A}$ *is the tangent to* $\mathfrak{H}$ *at* p'; for $\mathfrak{A}p^2=\mathfrak{A}p.p\equiv[p'q]$.

We have $\mathfrak{A}\{p^2,q\}=0$, and similarly $\mathfrak{A}\{p'^2,q\}=0$, hence the conic $\mathfrak{A}q$ goes through p, p', q'; but as it is a line-pair, the line $[pp']$ is a part of the conic. Hence, *if* p, p' *be conjugate points of* $\mathfrak{H}$, *and* q' *the third cut of* $[pp']$ *and* $\mathfrak{H}$, *then* $[pp']$ *is part of the polar conic of* q (the conjugate point to q') *for* $\mathfrak{A}$; the other part is $[tq']$, where t is the point of contact not on $[pp']$ of a tangent from q to $\mathfrak{A}$.

§ 125. *Flexes of a cubic locus.*

1. If p, q be flexes of a cubic locus $\mathfrak{A}$, and $[pq]$ cuts $\mathfrak{A}$ again in $s=p+kq$, then s is a flex. For, if the flex tangents at p, q meet in r, then $\mathfrak{A}\{p^3\}$, $\mathfrak{A}\{q^3\}$, $\mathfrak{A}\{p^2,r\}$, $\mathfrak{A}\{p,r^2\}$, $\mathfrak{A}\{q^2,r\}$, $\mathfrak{A}\{q,r^2\}$ all vanish.

Hence $\mathfrak{A}\{s, r^2\} = 0$, and since $\mathfrak{A}\{s^3\} = 0$, we have

$$\mathfrak{A}\{p^2, q\} + k\mathfrak{A}\{p, q^2\} = 0. \qquad (i)$$

Now s is a flex, if, and only if, there is a point t such that

$$\mathfrak{A}\{s, t^2\} = \mathfrak{A}\{s^2, t\} = 0.$$

We shew that if the scalar l be suitably chosen, then $t = p + lr$ satisfies these conditions. For

$$\mathfrak{A}\{s, (p+lr)^2\} = \mathfrak{A}\{s, p^2\} + 2l\mathfrak{A}\{s, p, r\}, \quad \text{since} \quad \mathfrak{A}\{s, r^2\} = 0,$$

$$\begin{aligned}\mathfrak{A}\{s^2, p+lr\} &= \mathfrak{A}\{(p^2 + 2k\{p, q\} + k^2q^2),\ (p+lr)\} \\ &= 2k\mathfrak{A}\{p^2, q\} + k^2\mathfrak{A}\{p, q^2\} + 2kl\mathfrak{A}\{p, q, r\}.\end{aligned}$$

Both expressions vanish by (i), if

$$\mathfrak{A}\{p^2, q\} + 2l\mathfrak{A}\{p, q, r\} = 0,$$

and this gives l, unless both $\mathfrak{A}\{p^2, q\}$ and $\mathfrak{A}\{p, q, r\}$ vanish, which is impossible, since it would mean that q was on the tangent at p.

We shall assume that a cubic locus has two real flexes, it then has another, collinear with these two. We shall also assume that a cubic is at any rate fully determined when three of its flexes and the tangents there are given. Such questions are better treated by ordinary algebra than by the methods of this book.

2. If A, B, C be tangents at the collinear flexes p, q, s and if $D = [pq]$, then

$$\mathfrak{A} \equiv \{A, B, C\} - k\{D^3\}, \quad (k \text{ scalar}). \qquad (1)$$

If A, B meet in r, it will be sufficient to prove

$$\mathfrak{B}\{p, r^2\} = \mathfrak{B}\{p^2, r\} = 0, \quad \text{where} \quad \mathfrak{B} = \{A, B, C\} - k\{D^3\}.$$

Now

$$\begin{aligned}[\{A, B, C\}\{p, r^2\}] &= \tfrac{1}{3}([Ap][Br][Cr] + [Ar][Bp][Cr] \\ &\quad + [Ar][Br][Cp]) = 0,\end{aligned}$$

$$\begin{aligned}[\{A, B, C\}\{p^2, r\}] &= \tfrac{1}{3}([Ap][Bp][Cr] + [Ap][Br][Cp] \\ &\quad + [Ar][Bp][Cp]) = 0,\end{aligned}$$

$$[\{D^3\}\{p, r^2\}] = [Dp][Dr][Dr] = 0,$$

$$[\{D^3\}\{p^2, r\}] = [Dp][Dp][Dr] = 0.$$

Hence $$\mathfrak{B}\{p, r^2\} = \mathfrak{B}\{p^2, r\} = 0.$$

Thus any cubic locus can be written in form (1).

The harmonic polar of p goes through $[BC]$, as is easily shewn.

3. Consider the cubic locus

$$\mathfrak{C} = \{A^3\} + \{B^3\} + \{C^3\} - 3\mathrm{k}\{A, B, C\}, \qquad (2)$$

where $a = [BC]$, $b = [CA]$, $c = [AB]$, $[ABC] = 1$.

We have $\mathfrak{C}\{a, p\} = [Ap]\,A - \frac{1}{2}\mathrm{k}([Bp]\,C + [Cp]\,B)$,

$$[\mathfrak{C}\{a,p\}\,\mathfrak{C}\{b,p\}\,\mathfrak{C}\{c,p\}] = [Ap]\,[Bp]\,[Cp] - \tfrac{1}{4}\mathrm{k}^2([Ap]^3 + [Bp]^3 + [Cp]^3) - \tfrac{1}{4}\mathrm{k}^3[Ap]\,[Bp]\,[Cp].$$

The equation of the Hessian is accordingly

$$\{A, B, C\} - \tfrac{1}{4}\mathrm{k}^2\{A^3 + B^3 + C^3\} - \tfrac{1}{4}\mathrm{k}^3\{A, B, C\} = 0,$$

or $\{A^3 + B^3 + C^3\} - 3\mathrm{h}\{A, B, C\} = 0$, where $3\mathrm{h} = (4 - \mathrm{k}^3)\,\mathrm{k}^{-2}$.

Since the equation (2) may be written

$$\{(\mathrm{k}A + B + C), (\mathrm{k}A + \epsilon B + \epsilon^2 C), (\mathrm{k}A + \epsilon^2 B + \epsilon C)\} = (\mathrm{k}^3 - 1)\{A^3\}, \qquad (\epsilon^3 = 1,\ \epsilon \neq 1), \quad (3)$$

we see, by comparing the equation with (1), that the tangents at the flexes are $\mathrm{x}_1 A + \mathrm{x}_2 B + \mathrm{x}_3 C$, where x_1, x_2, x_3 have the following values:

$$\begin{array}{lll} \mathrm{k}, 1, 1; & \mathrm{k}, \epsilon, \epsilon^2; & \mathrm{k}, \epsilon^2, \epsilon; \\ 1, \mathrm{k}, 1; & \epsilon, \mathrm{k}, \epsilon^2; & \epsilon^2, \mathrm{k}, \epsilon; \\ 1, 1, \mathrm{k}; & \epsilon, \epsilon^2, \mathrm{k}; & \epsilon^2, \epsilon, \mathrm{k}. \end{array}$$

The flexes themselves are

$$[A(B + C)], \quad [A(B + \epsilon C)], \quad [A(\epsilon B + C)],$$

that is,

$$b - c, \quad b - \epsilon c, \quad b - \epsilon^2 c,$$

and similarly

$$c - a, \quad c - \epsilon a, \quad c - \epsilon^2 a, \qquad a - b, \quad a - \epsilon b, \quad a - \epsilon^2 b.$$

They are situated by threes on twelve lines. (Cf. § 13·4.)

4. Cubic loci of form (2) have the same flexes for all k; in particular, the Hessian of a cubic locus has the same flexes as the cubic locus.

5. The polar line for a cubic locus $\mathfrak{A}$ of a point p on its Hessian $\mathfrak{H}$ touches $\mathfrak{H}$ at the conjugate point p' (§ 124·10). Let w be a flex of $\mathfrak{A}$; the polar line $\mathfrak{A}w^2$ is the tangent at w; the polar conic $\mathfrak{A}w$ is this tangent together with the harmonic polar H of w. But w is also a flex of $\mathfrak{H}$ and its conjugate point w' is the double point of $\mathfrak{A}w$, which is the cut of H and $\mathfrak{A}w^2$. Hence *the tangent at w touches $\mathfrak{H}$ at the cut of H and the tangent.*

Examples. 40. *Three conics by their intersection in pairs give three complete quadrangles, the sides of these quadrangles touch a cubic envelope, which is the Cayleyan of the cubic of which the conics are polar conics.*

Hence *the three conics each round two of three triangles circumscribed to a conic meet in a point.*

If a triangle is inscribed in a fixed conic and circumscribed to a fixed parabola, its circumcentre describes a line.

41. The condition* that $\mathfrak{A}p$ touches $L = [qr]$ is $[(\mathfrak{A}p)^2 L^2] = 0$ or $[(\mathfrak{A}p)^2 [qr]^2] = 0$, or $\mathfrak{A}\{p, q^2\}\,\mathfrak{A}\{p, r^2\} - (\mathfrak{A}\{p, q, r\})^2 = 0$.

If L is fixed, p describes a conic (the '*polo-conic*' of L) which is the locus of points whose polar conics touch L; it is the envelope of the polar lines of points on L.

For the polar line of $q + \mathrm{k}r$ is

$$\mathfrak{A}\{q^2\} + 2\mathrm{k}\mathfrak{A}\{q, r\} + \mathrm{k}^2\mathfrak{A}\{r^2\},$$

and its envelope is $\{\mathfrak{A}\{q^2\}, \mathfrak{A}\{r^2\}\} - (\mathfrak{A}\{q, r\})^2$.

It is the locus of poles of L with respect to the polar conics of its points. It degenerates when L touches $\mathfrak{h}$.

42. The polar conic of a point p for a cubic and the polar conic of p for the Hessian of the cubic are apolar.

43. The polar conics of

$$\mathfrak{A} = \{A^3\} + \{B^3\} + \{C^3\} - 3\mathrm{k}\{A, B, C\} \qquad \text{(i)}$$

form the net given by

$$A^2 - \mathrm{k}\{B, C\}, \quad B^2 - \mathrm{k}\{C, A\}, \quad C^2 - \mathrm{k}\{A, B\}.$$

The conics apolar to these are in the net $a^2 - \mathrm{k}'\{b, c\}$, and so on, if $\mathrm{k}\mathrm{k}' = -2$, $[ABC] = 1$, and $[BC] = a$, and so on.

Hence if $\mathfrak{a} = \{a^3\} + \{b^3\} + \{c^3\} - 3\mathrm{k}'\{a, b, c\}$, (ii)

then $[\mathfrak{A}\mathfrak{a}] = 0$. Since the Hessian of $\mathfrak{A}$ is of form (i), with $\frac{1}{3}(4 - \mathrm{k}^3)\mathrm{k}^{-2}$ for k, its Cayleyan is of form (ii) with

$$\tfrac{1}{3}(4 - \mathrm{k}'^3)\mathrm{k}'^{-2} \quad \text{or} \quad \tfrac{1}{3}(2 + \mathrm{k}^3)\mathrm{k}^{-1} \quad \text{for} \quad \mathrm{k}'.$$

Hence $[\mathfrak{A}\mathfrak{h}] = 3 + \frac{1}{6}3\mathrm{k}(2 + \mathrm{k}^3)\mathrm{k}^{-1} = \frac{1}{2}(\mathrm{k}^3 + 8)$,

$[\mathfrak{H}\mathfrak{h}] = 3 + \frac{1}{6}(2 + \mathrm{k}^3)(4 - \mathrm{k}^3)\mathrm{k}^{-3} = \frac{1}{6}(8 + 20\mathrm{k}^3 - \mathrm{k}^6)\mathrm{k}^{-3}$.

The 'invariants' of $\mathfrak{A}$ are often given in the form

$$\mathrm{i} = \tfrac{3}{2}\mathrm{k}(\mathrm{k}^3 + 8), \quad \mathrm{j} = -\tfrac{3}{4}(\mathrm{k}^6 - 20\mathrm{k}^3 - 8),$$

and are thus connected with $[\mathfrak{A}\mathfrak{h}]$, $[\mathfrak{H}\mathfrak{h}]$.

* For 41, 42, 43, see Clifford, *Mathematical Papers* (1882), p. 532.

§ 126. *Algebraic products in spreads of step four.* Products of pairs of points or of pairs of planes.*

1. As the basis of our spread we take e_1, e_2, e_3, e_4 with $[e_1e_2e_3e_4] = 1$, and let

$$\epsilon_1 = |e_1 = [e_2e_3e_4], \quad \epsilon_2 = |e_2 = -[e_1e_3e_4],$$
$$\epsilon_3 = |e_3 = [e_1e_2e_4], \quad \epsilon_4 = |e_4 = -[e_1e_2e_3],$$

so that $[\epsilon_2\epsilon_3\epsilon_4] = -e_1$, and so on.

2. Algebraic products of two points can be introduced and treated as in step three; the dual construct here is the algebraic product of two planes. Products of the same type are added formally.

If π, ρ be planes, the outer product of $\{a, b\}$ and π, and of $\{a, b\}$ and $\{\pi, \rho\}$, may be defined as follows:

$$[\{a, b\}\pi] = \tfrac{1}{2}([a\pi]\,b + [b\pi]\,a),$$
$$[\{a, b\}\{\pi, \rho\}] = \tfrac{1}{2}([a\pi]\,[b\rho] + [b\pi]\,[a\rho]).$$

Thus $[\{a, b\}\{\pi, \rho\}]$ is the outer product of $[\{a, b\}\,\pi]$ and ρ.

The distributive law is assumed for these products. Hence if $\mathfrak{a}$ is a sum of products such as $\{a, b\}$, and $\mathfrak{B}$ a sum of products such as $\{\pi, \rho\}$, with any coefficients, we can speak of the outer products $[\mathfrak{a}\pi]$, $[\mathfrak{a}\{\pi, \rho\}]$, $[\mathfrak{a}\mathfrak{B}]$.

Dually, if p, q be points, we have products $[\mathfrak{B}p]$, $[\mathfrak{B}\{p, q\}]$, $[\mathfrak{B}\mathfrak{a}] = [\mathfrak{a}\mathfrak{B}]$.

The equation $\mathfrak{a} = \mathfrak{b}$ means that, for all planes π, we have

$$[\mathfrak{a}\{\pi^2\}] = [\mathfrak{b}\{\pi^2\}].$$

3. The expression $\mathfrak{a} = \Sigma k_{ij}\{a_i, a_j\}$ is zero, if $[\mathfrak{a}\{\pi^2\}] = 0$ for all planes π, that is, if $\Sigma k_{ij}[a_i\pi]\,[a_j\pi] = 0$ for all planes π.

If $\mathfrak{a} \neq 0$, we say that $\mathfrak{a}$ represents the envelope of planes π which satisfy $[\mathfrak{a}\{\pi^2\}] = 0$. This is a quadric envelope, and there is no restriction if we assume $k_{ij} = k_{ji}$. We can express $\mathfrak{a}$ as a sum of point-squares. (Cf. § 113·3.)

Dually, $\mathfrak{A} = \Sigma k_{ij}\{\alpha_i, \alpha_j\}$ is zero, if $\Sigma k_{ij}[\alpha_i p]\,[\alpha_j p] = 0$ for all points p. If $\mathfrak{A} \neq 0$, we say $\mathfrak{A}$ represents the locus of points p which

* E. Müller, *Wiener Ber.* (1924).

satisfy $[\mathfrak{A}\{p^2\}] = 0$. This is a quadric locus, and we can suppose $k_{ij} = k_{ji}$. We can express $\mathfrak{A}$ as a sum of plane-squares.

We shall denote the sums of products of pairs of planes by such letters as $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ..., reserving $\mathfrak{Q}$ for sums of products of pairs of lines introduced later.

4. Taking the harmonic property as the definition of conjugate points with respect to a quadric, we can shew, as in the plane, that $[\mathfrak{A}p]$ is the polar plane of p for the quadric locus $\mathfrak{A}$.

Dually, $[\mathfrak{a}\pi]$ is the pole of π for the quadric envelope $\mathfrak{a}$.

$[\mathfrak{A}\{a, b\}] = 0$, if, and only if, a, b are conjugate points for $\mathfrak{A}$.

$[\mathfrak{a}\{\pi, \rho\}] = 0$, if, and only if, π, ρ are conjugate planes for $\mathfrak{a}$.

5. Let $a_1, \ldots, a_4$ be independent points.

Let $\mathfrak{a} = \sum_{i=1}^{4} \{a_1, b_i\} \neq 0$, where $b_i = \sum_{j=1}^{4} k_{ij} a_j$, $k_{ij} = k_{ji}$.

Then b_1 is the pole of $\pi = [a_2 a_3 a_4]$ for $\mathfrak{a}$.

For
$$\begin{aligned} 2[\mathfrak{a}[a_2 a_3 a_4]] &= [a_1\pi]\, b_1 + [b_1\pi]\, a_1 + [b_2\pi]\, a_2 + [b_3\pi]\, a_3 \\ &\qquad + [b_4\pi]\, a_4 \\ &= [a_1\pi]\, b_1 + k_{11}[a_1\pi]\, a_1 + k_{21}[a_1\pi]\, a_2 \\ &\qquad + k_{31}[a_1\pi]\, a_3 + k_{41}[a_1\pi]\, a_4 \\ &= [a_1\pi]\,(b_1 + k_{11} a_1 + k_{21} a_2 + k_{31} a_3 + k_{41} a_4) \\ &= 2[a_1\pi]\, b_1 \\ &\equiv b_1, \quad \text{since } [a_1\pi] \neq 0. \end{aligned}$$

Thus $a_1 a_2 a_3 a_4$ *and* $b_1 b_2 b_3 b_4$ *are polar tetrahedra.*

Conversely, if $a_1, \ldots, a_4$ be independent points, and b_1, b_2, b_3, b_4 be poles of $[a_2 a_3 a_4]$, $[a_3 a_4 a_1]$, $[a_4 a_1 a_2]$, $[a_1 a_2 a_3]$ for a quadric $\mathfrak{a}$, then $\mathfrak{a}$ is represented by $\sum_{i=1}^{4} \{a_i, b_i\} = 0$, if the points b_i be suitably weighted.

If, then, $b_i = \Sigma k_{ij} a_j$, $(k_{ij} = k_{ji})$, we have, using outer products of points,

$$[a_1 b_1] + [a_2 b_2] + [a_3 b_3] + [a_4 b_4] = \Sigma k_{ij} [a_i a_j],$$

and since $[a_i a_j] = -[a_j a_i]$, this vanishes if $k_{ij} = k_{ji}$.

Hence the joins of corresponding vertices of polar tetrahedra lie on a regulus. (Cf. § 30, Ex. 20.)

If $a_i = \Sigma k_{ij} e_j$, then $\Sigma\{a_i, e_i\}$ is a '*normal form*', if $k_{ij} = k_{ji}$.

If $a_1 a_2 a_3 a_4$ be a self-polar tetrahedron for $\mathfrak{a}$, then $\mathfrak{a} = \sum_{i=1}^{4} k_i \{a^2\}$.

6. The expression $k_1\{a, b\}+k_2\{c, d\}$ represents a quadric $\mathfrak{a}$ through the lines $[ac]$, $[bc]$, $[ad]$, $[bd]$; since, for example, if $[\pi a]$, $[\pi c]$ vanish, then $[\mathfrak{a}\{\pi^2\}] = 0$; thus any plane through $[ac]$ is tangent to $\mathfrak{a}$.

The poles of a plane π for these quadrics lie on a line, for

$$2[\mathfrak{a}\pi] = k_1[a\pi]\,b+k_1[b\pi]\,a+k_2[c\pi]\,d+k_2[d\pi]\,c.$$

The polar planes of a point for these quadrics go through a line.

7. Let $\sum_{i=1}^{4} k_i\{a_i, b_i\} = 0$, where the a are independent, and the b are independent; then *the tetrahedra* $a_1a_2a_3a_4$ *and* $b_1b_2b_3b_4$ *are in Möbius relation* (§ 39).

For example, multiply by $\{[b_2b_3b_4]^2\}$, then we have

$$k_1[a_1b_2b_3b_4]\,[b_1b_2b_3b_4] = 0;$$

hence $[a_1b_2b_3b_4] = 0$; similarly $[b_1a_2a_3a_4] = 0$. Hence a_1 is on $[b_2b_3b_4]$, a_2 on $[b_1b_3b_4]$, ..., b_1 on $[a_2a_3a_4]$, ..., and so on. Thus the tetrahedra are in Möbius relation.

We can suppose $b_i = \sum_{j=1}^{4} k_{ij}a_j$, where we do not now assume $k_{ij} = k_{ji}$.

Then $[b_1a_2a_3a_4] = 0$ gives $k_{11} = 0$. Similarly $k_{ii} = 0$, $(i = 1, ..., 4)$.

And since $[a_1b_2b_3b_4] = 0$, we get, if we expand the relevant determinant,

$$k_{23}k_{34}k_{42} = -k_{24}k_{43}k_{32}.$$

Similarly, we have three other equations; any three of the four equations imply the fourth. In particular, if

$$\sum_{1}^{4}\{a_i, b_i\} = 0, \quad b_i = \sum_{j=1}^{4} k_{ij}b_j, \quad \text{then} \quad \sum_{i,j=1}^{4} k_{ij}\{a_i, b_j\} = 0,$$

and hence $k_{ij} = -k_{ji}$. The general case can be reduced to this by absorbing weights, taking $a_i' = k_i a_i$.

Conversely, if two tetrahedra be in Möbius relation, we can weight their vertices so that $\sum_{i=1}^{4}\{a_i, b_i\} = 0$.

Thus if $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$ be tetrahedra in Möbius relation, there is a nul system (§ 97) in space transforming the first into the second.

Further, if three of the pairs (a_i, b_i) be conjugate for a quadric $\mathfrak{A}$, so is the fourth pair. For $\Sigma\{a_i, b_i\} = 0$ implies $\Sigma\mathfrak{A}\{a_i, b_i\} = 0$.

8. If $\{a_1^2\}, \{a_2^2\}, \ldots, \{a_{10}^2\}$ be dependent, say $\sum_1^{10} k_i\{a_i^2\} = 0$, and no four of $a_1, \ldots, a_{10}$ are dependent, then $a_1, a_2, \ldots, a_{10}$ lie on a quadric. For through nine general points goes a quadric locus $\mathfrak{A}$, and $\sum_1^{10} k_i \mathfrak{A}\{a_i^2\} = 0$.

If $\{a_1^2\}, \{a_2^2\}, \ldots, \{a_9^2\}$ be dependent, and no four of $a_1, \ldots, a_9$ be dependent, then $a_1, \ldots, a_9$ lie on the curve of intersection of two quadrics, for any quadric through eight of them goes through the ninth point.

If $\{a_1^2\}, \{a_2^2\}, \ldots, \{a_8^2\}$ be dependent, then any quadric through seven of them goes through the eighth. In particular, if four are coplanar, so are the other four.

Conversely, it can be shewn that if eight points $a_1, \ldots, a_8$ be associated, (§ 32), then $\{a_1^2\}, \{a_2^2\}, \ldots, \{a_8^2\}$ are dependent. In particular, this is the case if the eight points are on a twisted cubic, for a quadric through seven points of a twisted cubic contains it.*

If $a, b', c, a', b, c', d, d_1$ be associate points, then, with appropriate weights, $a^2+b^2+c^2+a'^2+b'^2+c'^2$ and $d^2+d_1^2$ represent the same point-pair $\{p, q\}$. Hence p, q are separated harmonically by the cuts of $[pq]$ with $[abc]$ and $[a'b'c']$, with $[a'bc]$ and $[ab'c']$, and so on, and by d, d_1. Thus these cuts are pairs of an involution on $[pq]$.

In particular the theorem holds when d, d_1 are replaced by any points on the twisted cubic through a, b', c, a', b, c'. Only one line through d cuts the planes mentioned in involution; and only one bisecant of the cubic goes through d. Hence $[dd_1]$ is a bisecant.

9. If $\mathfrak{a} = \{a_1^2\} + \ldots + \{a_h^2\}$, and $a_1, \ldots, a_h$ are on the quadric $\mathfrak{B}$, then

$$[\mathfrak{B}\{a_i^2\}] = 0, \quad (i = 1, \ldots, h);$$

hence

$$[\mathfrak{a}\mathfrak{B}] = [\mathfrak{B}\{a_1^2\}] + \ldots + [\mathfrak{B}\{a_h^2\}] = 0.$$

If, also, $\mathfrak{a} = \{p_1^2\} + \ldots + \{p_h^2\}$, and $p_1, \ldots, p_{h-1}$ are on $\mathfrak{B}$, then p_h is on $\mathfrak{B}$.

Take, for instance, h = 4; we then have: *if there is one tetrahedron whose vertices are on* $\mathfrak{B}$ *which is self-polar to* $\mathfrak{a}$, *then there*

* We take this fact from the usual algebraical treatment.

are ∞^3 such tetrahedra. The vertices of two such tetrahedra are associate points. If eight associate points be divided in any way into two sets of four points, there is a quadric for which the two tetrahedra so determined are self-polar.

If we take h = 5, then the pole of $[a_1 a_2 a_3]$ is on $[a_4 a_5]$, the pole of $[a_2 a_3 a_4]$ is on $[a_5 a_1]$, and so on. For

$$[\{a_1^2\} + \ldots + \{a_5^2\}[a_1 a_2 a_3]] = [a_4 a_1 a_2 a_3]\, a_4 + [a_5 a_1 a_2 a_3]\, a_5.$$

If we take h = 6, then the pole of $[a_1 a_2 a_3]$ is on $[a_4 a_5 a_6]$, the pole of $[a_2 a_3 a_4]$ is on $[a_5 a_6 a_1]$, and so on.

10. If $\{a_1, b_1\}$, ..., $\{a_8, b_8\}$ be dependent, say, $\sum_{i=1}^{8} k_i\{a_i, b_i\} = 0$, and we separate these pairs in any way into two sets of four point-pairs, we get two pairs of polar tetrahedra for the same quadric. For, if π satisfies $k_1[a_1\pi][b_1\pi] + \ldots + k_4[a_4\pi][b_4\pi] = 0$, then it satisfies $k_5[a_5\pi][b_5\pi] + \ldots + k_8[a_8\pi][b_8\pi] = 0$.

The sixteen faces of the four tetrahedra form a dependent system of eight plane-pairs, for example $\{[a_1 a_2 a_3], [b_1 b_2 b_3]\}$.

11. If a line L cuts the faces *bcd*, *cda*, *dab*, *abc* of a tetrahedron in a_1, b_1, c_1, d_1 respectively, then $\{a, a_1\}$, $\{b, b_1\}$, $\{c, c_1\}$, $\{d, d_1\}$ are dependent pairs; for

$$\{[L.bcd], a\} - \{[L.cda], b\} + \{[L.dab], c\} - \{[L.abc], d\} = 0,$$

as is easily seen on expanding. (Cf. § 17, Ex. 19.)

Examples. 44. Among the ten cuts in which a line L meets the faces of a complete five-point are five sets of six points in involution; the five pairs of double points of the five sets are again in involution.

45. *Consider, in a spread of even step* 2r, *pairs of points* a_i, b_i $(i = 1, \ldots, r)$, *whose products* $\{a_i, b_i\}$ *are dependent, and shew that we have thus a generalisation of Möbius tetrahedra.* Connect this up with the work in § 110.

§ 127. *Algebraic products of pairs of lines in spreads of step four.*

1. *Def.* The '*outer product*' of $\{a^2\}$, $\{b^2\}$ is $\{[ab]^2\}$, a line repeated.

The '*outer product*' of $\{a^2\}$, $\{b^2\}$, $\{c^2\}$ is $\{[abc]^2\}$, a plane repeated.

The '*outer product*' of $\{a^2\}$, $\{b^2\}$, $\{c^2\}$, $\{d^2\}$ is $[abcd]^2$, a scalar.

By the distributive law, we can now use products $[\mathfrak{ab}]$, $[\mathfrak{abc}]$, $[\mathfrak{abcd}]$, where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ represent quadric envelopes. These products are respectively sums of line-pairs, of plane-pairs, and of scalars.

2. The algebraic product of two lines L, M is the pair of lines, in either order, and is written $\{L, M\}$. Such products are added formally.

The equation $\Sigma k_i\{L_i, M_i\} = 0$ means that

$$\Sigma k_i[L_i N][M_i N] = 0$$

for all lines N.

If $\mathfrak{Q} = \Sigma k_i\{L_i, M_i\} \neq 0$, then $\mathfrak{Q}$ represents the set of lines N such that $\Sigma k_i[L_i N][M_i N] = 0$. We can express $\mathfrak{Q}$ as a sum of squares of lines.

3. *Def.* The '*outer product*' of $\{L^2\}$ and $\{M^2\}$ is the scalar $[LM]^2$, and is written $[\{L^2\}\{M^2\}]$.

Assuming the distributive law, it follows that (cf. § 114·4) if

$$\mathfrak{Q} = \Sigma k_i\{L_i, M_i\}, \quad \text{then} \quad [\mathfrak{Q}\{N^2\}] = \Sigma k_i[L_i N][M_i N].$$

Thus $\mathfrak{Q}$ represents the set of lines N such that $[\mathfrak{Q}\{N^2\}] = 0$.

We can introduce the products $[\mathfrak{Q}\{L, M\}]$.

4. *Def.* If p is a point, L, N lines, π, ρ planes, the '*outer product*' of $\{p^2\}$ and $\{L^2\}$ is $\{[pL]^2\}$, that of $\{\pi^2\}$, $\{\rho^2\}$ is $\{[\pi\rho]^2\}$, that of $\{L^2\}$, $\{\pi^2\}$ is $\{[L\pi]^2\}$, that of $\{L^2\}$, N is $[L^2 N] = [LN]L$. The reader will easily verify that these definitions fit together.

With $\mathfrak{Q}$ as above, we have, as in § 114·7,

$$[\mathfrak{Q}N] = \tfrac{1}{2}(\Sigma k_i[L_i N] M_i + \Sigma k_i[M_i N] L_i),$$
$$[[\mathfrak{Q}N] M] = [\mathfrak{Q}\{N, M\}].$$

By the distributive law, we can now use products $[\mathfrak{a}\{L^2\}]$, $[\mathfrak{aQ}]$, where $\mathfrak{a}$ represents a quadric envelope and $\mathfrak{Q}$ is a sum of line-pairs, as well as $[\mathfrak{AB}]$, $[\mathfrak{ABC}]$, $[\mathfrak{ABCD}]$, $[\mathfrak{AQ}]$, where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ represent quadric loci.

We shall need $[\mathfrak{a}[bq]^2] = [\mathfrak{a}b^2 . q^2]$. It is easily shewn.

5. If we have normal forms

$$\mathfrak{a} = \sum_{i=1}^{4} \{a_i, e_i\}, \quad \mathfrak{b} = \sum_{i=1}^{4} \{b_i, e_i\}, \quad \mathfrak{c} = \sum_{i=1}^{4} \{c_i, e_i\}, \quad \mathfrak{d} = \sum_{i=1}^{4} \{d_i, e_i\},$$

then, as in the plane,

$$[\mathfrak{ab}] = \sum_{i,k=1}^{4} \{[a_i b_k] - [a_k b_i], [e_i e_k]\}, \quad \tfrac{1}{2}[\mathfrak{a}^2] = \sum_{i,k=1}^{4} \{[a_i a_k], [e_i e_k]\},$$
$$[\mathfrak{abc}] = \{\pi_1, \epsilon_1\} + \{\pi_2, \epsilon_2\} + \{\pi_3, \epsilon_3\} + \{\pi_4, \epsilon_4\}, \tag{1}$$

where, for example,

$$\pi_1 = \begin{vmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \end{vmatrix}$$

is a determinant whose elements are *points*. This determinant is to be expanded as usual, and the terms regarded as *outer products*. The expansion begins with $[a_2 b_3 c_4] - [a_2 b_4 c_3] + \ldots$. Writing (234) for π_1, then $\pi_2 = -(134)$, $\pi_3 = (124)$, $\pi_4 = -(123)$.

Similarly $[\mathfrak{abcd}]$ can be expressed as a four-rowed determinant of points. In particular

$$\tfrac{1}{6}[\mathfrak{a}^3] = \{\alpha_1, \epsilon_1\} + \{\alpha_2, \epsilon_2\} + \{\alpha_3, \epsilon_3\} + \{\alpha_4, \epsilon_4\}, \tag{2}$$

where
$$\alpha_1 = [a_2 a_3 a_4], \quad \alpha_2 = -[a_1 a_3 a_4],$$
$$\alpha_3 = [a_1 a_2 a_4], \quad \alpha_4 = -[a_1 a_2 a_3],$$
$$\tfrac{1}{24}[\mathfrak{a}^4] = [a_1 a_2 a_3 a_4].$$

If $[\mathfrak{a}^3] = \mathfrak{A}$, the dual of (1) is, by (2), since $[\epsilon_r \epsilon_s] = |[e_r e_s]$,

$$\tfrac{1}{72}[\mathfrak{A}^2] = \sum_{i,k=1}^{4} \{[\alpha_i \alpha_k], |[e_i e_k]\}$$
$$= [a_1 a_2 a_3 a_4] \sum_{i,k=1}^{4} \{[a_i a_k], [e_i e_k]\} = \tfrac{1}{24}[\mathfrak{a}^4] \cdot \tfrac{1}{2}[\mathfrak{a}^2].$$

Hence $[\mathfrak{A}^2] = \frac{3}{2}[\mathfrak{a}^4][\mathfrak{a}^2]$, similarly $[\mathfrak{A}^3] = \frac{9}{4}[\mathfrak{a}^4][\mathfrak{a}^4]\mathfrak{a}$,
$$[\mathfrak{A}^4] = \tfrac{9}{4}[\mathfrak{a}^4]^3, \quad [\mathfrak{a}^2\mathfrak{A}] = \tfrac{3}{2}[\mathfrak{a}^4]\mathfrak{a}. \tag{3}$$

In (2) we can take $\epsilon_i = \Sigma k_{ij} \alpha_j$, where k_{ij} are scalars with $k_{ij} = k_{ji}$.

The polar plane of p for $\mathfrak{A}$ is then

$$\tfrac{1}{6}[\mathfrak{A}p] = [a_2 a_3 a_4][\epsilon_1 p] - [a_1 a_3 a_4][\epsilon_2 p] + [a_1 a_2 a_4][\epsilon_3 p] - [a_1 a_2 a_3][\epsilon_4 p].$$

Thence

$$\tfrac{1}{6}[\mathfrak{a}[\mathfrak{A}p]] = -[a_1 a_2 a_3 a_4]([\epsilon_1 p] e_1 + [\epsilon_2 p] e_2 + [\epsilon_3 p] e_3 + [\epsilon_4 p] e_4)$$
$$= [a_1 a_2 a_3 a_4] p,$$
$$[\mathfrak{a}[\mathfrak{A}p]] = \tfrac{1}{4}[\mathfrak{a}^4] p \equiv p.$$

If $\mathfrak{Q} = \frac{1}{2}[\mathfrak{a}^2] = \Sigma\{A_i, E_i\}$, where $E_1, \ldots, E_6$ are independent lines, we can take $A_i = \sum_{j=1}^{6} k_{ij} E_j$, where $k_{ij} = k_{ji}$. (Cf. § 116·1.)

Then, if L be any line, and E_i be the edges of $e_1 e_2 e_3 e_4$,

$$[\mathfrak{Q}L] = \sum_i [A_i L] E_i, \quad [\mathfrak{Q}[\mathfrak{Q}L]] = [a_1 a_2 a_3 a_4] L.$$

Thus $$[\mathfrak{a}^2[\mathfrak{a}^2 L]] = \tfrac{1}{6}[\mathfrak{a}^4] L \equiv L.$$

§ 128. *Formulae.*

1. Since $[\mathfrak{a}L^2]$ and $[\mathfrak{b}L^2]$ are sums of plane-pairs, therefore $[[\mathfrak{a}L^2][\mathfrak{b}L^2]]$, which we write $[\mathfrak{a}L^2 . \mathfrak{b}L^2]$, is the sum of expressions like $[a^2L^2 . b^2L^2]$, when $\mathfrak{a}$, $\mathfrak{b}$ are each expressed as the sum of squares of points.

Now $[aL . bL] = [abL]\, L$, hence, by § 127,

$$\begin{aligned}[a^2L^2 . b^2L^2] &= [[aL]^2 [bL]^2] = \{[aL . bL]^2\} \\ &= [abL]^2 \{L^2\} = [a^2b^2L^2]\{L^2\}.\end{aligned}$$

Hence, by the addition of such equations,

$$[\mathfrak{a}L^2 . \mathfrak{b}L^2] = [\mathfrak{a}\mathfrak{b}L^2]\{L^2\}. \qquad (4)$$

Similarly $$[\mathfrak{a}[\mathfrak{b}L^2]] = [\mathfrak{a}\mathfrak{b}L^2].$$

2. If p, π be incident, $[\mathfrak{a}p^2\pi^2] = [\mathfrak{a}\pi^2]p^2$; if L, π be incident, $[\mathfrak{a}L^2\pi^2] = [\mathfrak{a}\pi^2]L^2$; if $\mathfrak{Q}$ lies in π, then $[\mathfrak{a}\mathfrak{Q}\pi^2] = [\mathfrak{a}\pi^2]\mathfrak{Q}$.

These follow from § 56·13.

3. Since $$[ab . \gamma] = [a\gamma]\, b - [b\gamma]\, a,$$

if we add the squares of such expressions, our usual process gives

$$\begin{aligned}[\mathfrak{a}\mathfrak{b} . \mathfrak{C}] &= [\mathfrak{a}\mathfrak{C}]\,\mathfrak{b} + [\mathfrak{b}\mathfrak{C}]\,\mathfrak{a} - 2\Sigma[a\gamma][b\gamma]\{\mathfrak{a}, \mathfrak{b}\} \\ &= [\mathfrak{a}\mathfrak{C}]\,\mathfrak{b} + [\mathfrak{b}\mathfrak{C}]\,\mathfrak{a} - 2(\mathfrak{a}\mathfrak{C}\mathfrak{b}), \qquad (5)\end{aligned}$$

where $(\mathfrak{a}\mathfrak{C}\mathfrak{b})$ is the symmetrical part of the matrix product of $\mathfrak{a}$, $\mathfrak{C}$, $\mathfrak{b}$, formed as in § 117·1.

$(\mathfrak{a}\mathfrak{C}\mathfrak{b})$ represents the quadric envelope of planes whose poles for $\mathfrak{a}$, $\mathfrak{b}$ are conjugate for $\mathfrak{C}$.

4. Since $[ab . \gamma\delta] = [ab\gamma . \delta]$, we get, from 3,

$$[\mathfrak{a}\mathfrak{b} . \mathfrak{C}\mathfrak{D}] = [\mathfrak{a}\mathfrak{C}][\mathfrak{b}\mathfrak{D}] + [\mathfrak{b}\mathfrak{C}][\mathfrak{a}\mathfrak{D}] - 2[(\mathfrak{a}\mathfrak{C}\mathfrak{b})\,\mathfrak{D}].$$

Dually $$[\mathfrak{C}\mathfrak{D} . \mathfrak{a}\mathfrak{b}] = [\mathfrak{a}\mathfrak{C}][\mathfrak{b}\mathfrak{D}] + [\mathfrak{b}\mathfrak{C}][\mathfrak{a}\mathfrak{D}] - 2[(\mathfrak{C}\mathfrak{a}\mathfrak{D})\,\mathfrak{b}].$$

Hence $$[(\mathfrak{a}\mathfrak{C}\mathfrak{b})\,\mathfrak{D}] = [(\mathfrak{C}\mathfrak{a}\mathfrak{D})\,\mathfrak{b}]. \qquad (6)$$

In particular, $$\tfrac{1}{2}[\mathfrak{a}^2 . \mathfrak{C}^2] = [\mathfrak{a}\mathfrak{C}]^2 - [(\mathfrak{a}\mathfrak{C}\mathfrak{a})\,\mathfrak{C}]. \qquad (7)$$

5. By the usual process, we obtain

$$[\mathfrak{abc}.\mathfrak{D}] = [\mathfrak{aD}][\mathfrak{bc}]+[\mathfrak{bD}][\mathfrak{ca}]+[\mathfrak{cD}][\mathfrak{ab}] - 2[(\mathfrak{aDb})\mathfrak{c}]-2[(\mathfrak{bDc})\mathfrak{a}]-2[(\mathfrak{cDa})\mathfrak{b}], \quad (8)$$

$$[\mathfrak{abcd}]\mathfrak{E} = [\mathfrak{aE}][\mathfrak{bcd}]+\ldots+[\mathfrak{dE}][\mathfrak{abc}] - 2[(\mathfrak{aEb})\mathfrak{cd}]-\ldots-2[(\mathfrak{cEd})\mathfrak{ab}]. \quad (9)$$

Replacing the last terms of (8) by means of (5), we have

$$[\mathfrak{abc}.\mathfrak{D}] = [\mathfrak{abDc}]+[\mathfrak{bcDa}]+[\mathfrak{caDb}] - [\mathfrak{aD}][\mathfrak{bc}]-[\mathfrak{bD}][\mathfrak{ca}]-[\mathfrak{cD}][\mathfrak{ab}]. \quad (10)$$

6. As particular cases of (8), (9), (10), we have

$$[\mathfrak{a}^3.\mathfrak{D}] = 3[\mathfrak{aD}][\mathfrak{a}^2]-6[(\mathfrak{aDa})\mathfrak{a}],$$
$$[\mathfrak{a}^4]\,\mathfrak{E} = 4[\mathfrak{aE}][\mathfrak{a}^3]-12[(\mathfrak{aEa})\mathfrak{a}^2],$$
$$[\mathfrak{a}^3.\mathfrak{D}] = 3[\mathfrak{a}^2\mathfrak{Da}]-3[\mathfrak{aD}][\mathfrak{a}^2].$$

Take $\mathfrak{E} = \{\pi^2\}$, where π is a plane, then $(\mathfrak{a}\{\pi^2\}\mathfrak{a}) = [\mathfrak{a}\pi]^2$,

$$[\mathfrak{a}^4]\{\pi^2\} = 4[\mathfrak{a}\pi^2][\mathfrak{a}^3]-12[\mathfrak{a}^2[\mathfrak{a}\pi]^2]. \quad (11)$$

Hence $$12[\mathfrak{a}^2\mathfrak{b}[\mathfrak{a}\pi]^2] = 4[\mathfrak{a}^3\mathfrak{b}][\mathfrak{a}\pi^2]-[\mathfrak{a}^4][\mathfrak{b}\pi^2]. \quad (12)$$

7. If p, q, r, s be points, then the usual process gives

$$[\mathfrak{AB}[pq]] = \begin{vmatrix} [\mathfrak{A}p], & [\mathfrak{A}q] \\ [\mathfrak{B}p], & [\mathfrak{B}q] \end{vmatrix}. \quad (13)$$

There are similar determinant expressions for $[\mathfrak{ABC}[pqr]]$ and $[\mathfrak{ABCD}[pqrs]]$.

The terms of the expanded determinants are to be regarded as outer products.

In particular,

$$[\mathfrak{A}^2[pq]] = 2[\mathfrak{A}p.\mathfrak{A}q], \quad [\mathfrak{A}^3[pqr]] = 6[\mathfrak{A}p.\mathfrak{A}q.\mathfrak{A}r]. \quad (14)$$

Hence $[\mathfrak{A}^2[pq]]$ is the polar line of $[pq]$ with respect to $[\mathfrak{A}^2]$ (or to $[\mathfrak{a}^2]$ if $[\mathfrak{a}^4] \neq 0$).

8. Since

$$[abcd]\pi = [a\pi][bcd]-[b\pi][acd]+[c\pi][abd]-[d\pi][abc],$$

and $$[a^2\pi.b^2c^2d^2] = [[a\pi]\,a\{[bcd]^2\}] = [a\pi][abcd][bcd],$$

we have, on multiplying the first equation by $[abcd]$, and employing the usual process,

$$\{[abcd]^2\}\pi = [a^2\pi . b^2c^2d^2] + [b^2\pi . a^2c^2d^2] + [c^2\pi . a^2b^2d^2] + [d^2\pi . a^2b^2c^2],$$

$$[abcd]\pi = [a\pi . bcd] + [b\pi . acd] + [c\pi . abd] + [d\pi . abc]. \quad (15)$$

Similarly, we find

$$[abcd]\pi = -[bcd . a\pi] - [acd . b\pi] - [abd . c\pi] - [abc . d\pi].$$

In particular,

$$[a^4]\pi = 4[a\pi . a^3], \quad (16)$$

$$[ab^3]\pi = [a\pi . b^3] + 3[b\pi . ab^2], \quad (17)$$

$$\tfrac{1}{2}[a^2b^2]\pi = [b\pi . a^2b] + [a\pi . ab^2]. \quad (18)$$

Similar to (15), we have

$$[abcd]L = [abL . cd] + [bcL . ad] + \ldots + [adL . bc] \text{ (six terms)}, \quad (19)$$

$$[abcd]p = [abcp . d] + [bcdp . a] + \ldots \text{ (four terms)}, \quad (20)$$

$$[a^4]L = 6[a^2L . a^2], \quad [a^4]p = 4[a^3p . a], \quad (21)$$

$$[a^2b^2]L = [a^2L . b^2] + [b^2L . a^2] + 4[abL . ab]. \quad (22)$$

§ 129. *Interpretations. The harmonic complex and surface.*

1. $[aL^2]$, if not zero, represents a plane-pair with equation $[aL^2p^2] = 0$, or $[a[Lp]^2] = 0$. Hence $[aL^2]$ represents the pair of tangent planes from L to a.

$[ap^2]$, if not zero, represents the set of lines L such that $[ap^2L^2] = 0$, that is, the tangent lines to the cone from p to a.

2. If $[aL^2] = 0$, then $[aL^2p^2] = 0$, $[a[Lp]^2] = 0$ for all points p; hence each plane through L touches a. Thus L is a generator of a. Conversely, if L is a generator of a, then $[aL^2] = 0$.

Dually, $[\mathfrak{A}\pi^2]$, if not zero, represents the lines through the cut of $\mathfrak{A}$ and π.

$[\mathfrak{A}L^2] = 0$, if, and only if, L is a generator of $\mathfrak{A}$.

3. If $L = [pq]$, then $[a^2L] \equiv [a^3p . a^3q]$ represents the polar line of L with respect to a^2. (§ 128·7.)

4. If $[\mathfrak{a}^4] = 0$, then $\mathfrak{a}$ represents a conic envelope; if $[\mathfrak{a}^3] = 0$, a point-pair. If $[\mathfrak{a}^3] \neq 0$, we often write $\mathfrak{A}$ for $[\mathfrak{a}^3]$, and so on.

If $[\mathfrak{A}^4] = 0$, then $\mathfrak{A}$ represents a cone locus: if $[\mathfrak{A}^3] = 0$, a plane-pair.

5. If $[\mathfrak{a}^4]$, $[\mathfrak{b}^4] \neq 0$, then $[\mathfrak{a}\mathfrak{b}^3] = 0$ implies $[\mathfrak{A}^3\mathfrak{B}] = 0$. Hence if there is a tetrahedron in $\mathfrak{b}$ self-polar to $\mathfrak{a}$, there will be one and hence ∞^3 round $\mathfrak{a}$ self-polar to $\mathfrak{b}$; and dually. (§ 126·9.) We say $\mathfrak{a}$ and $\mathfrak{B}$ are '*apolar*', $\mathfrak{b}$ and $\mathfrak{A}$ are '*apolar*'.

6. *The harmonic complex.* Let $\mathfrak{a} = a_1^2 + a_2^2 + \ldots$, $\mathfrak{b} = b_1^2 + b_2^2 + \ldots$.

If $[\mathfrak{a}\mathfrak{b}] = 0$, then $[a_i b_k] = [a_k b_i]$, and all point-pairs (a_i, b_i) are collinear; thence $\mathfrak{a}$, $\mathfrak{b}$ represent point-pairs on a line separating one another harmonically. (§ 118·11.)

Dually, if $[\mathfrak{A}\mathfrak{B}] = 0$, then $\mathfrak{A}$, $\mathfrak{B}$ are plane-pairs through a line, separating one another harmonically. Now $[\mathfrak{a}L^2]$, $[\mathfrak{b}L^2]$ are pairs of tangent planes to $\mathfrak{a}$, $\mathfrak{b}$ respectively through L. Hence if $[\mathfrak{a}L^2 . \mathfrak{b}L^2] = 0$ they separate one another harmonically. By (4), this condition can be written $[\mathfrak{a}\mathfrak{b}L^2]\{L^2\} = 0$ or $[\mathfrak{a}\mathfrak{b}L^2] = 0$. Thus $[\mathfrak{a}\mathfrak{b}]$, if not zero, represents the set of lines such that pairs of tangent planes through them to $\mathfrak{a}$, $\mathfrak{b}$ separate one another harmonically. This set of lines is the '*harmonic complex*' corresponding to $\mathfrak{a}$ and $\mathfrak{b}$.

7. If L be a line on a common tangent plane of $\mathfrak{a}$ and $\mathfrak{b}$, and through the point of contact of the plane with $\mathfrak{b}$, then $[\mathfrak{b}L^2]$ represents that tangent plane repeated. It touches $\mathfrak{a}$, hence $[\mathfrak{a}[\mathfrak{b}L^2]] = 0$, that is, $[\mathfrak{a}\mathfrak{b}L^2] = 0$. Hence *all such lines are in the harmonic complex.*

8. Let *pqrs* be a self-polar tetrahedron of $\mathfrak{b}$, and

$$\mathfrak{b} = p^2 + q^2 + r^2 + s^2,$$

then
$$[\mathfrak{a}\mathfrak{b}] = [\mathfrak{a}p^2] + [\mathfrak{a}q^2] + [\mathfrak{a}r^2] + [\mathfrak{a}s^2].$$

Let L be a common tangent line to the cones $[\mathfrak{a}p^2]$, ..., $[\mathfrak{a}s^2]$, then
$$[\mathfrak{a}p^2L^2] = \ldots = [\mathfrak{a}s^2L^2] = 0; \quad \text{hence} \quad [\mathfrak{a}\mathfrak{b}L^2] = 0.$$

Hence *all lines touching the four cones tangent to* $\mathfrak{a}$ *whose vertices are vertices of a self-polar tetrahedron of* $\mathfrak{b}$ *are in the harmonic complex.*

9. Let $\mathfrak{c}$, $\mathfrak{d}$ separate $\mathfrak{a}$, $\mathfrak{b}$ harmonically in the pencil $\mathfrak{a}+k\mathfrak{b}$, say,

$$\mathfrak{a}=\mathfrak{c}+k\mathfrak{d},\quad \mathfrak{b}=\mathfrak{c}-k\mathfrak{d},\quad \text{then}\quad [\mathfrak{a}\mathfrak{b}]=[\mathfrak{c}^2]-k^2[\mathfrak{d}^2].$$

Hence *the complex contains all common tangents of any two quadrics which separate* $\mathfrak{a}$, $\mathfrak{b}$ *harmonically.*

10. If $[\mathfrak{a}^3]=\mathfrak{A}$, then, by (3), $[\mathfrak{A}^2]=\frac{3}{4}[\mathfrak{a}^4][\mathfrak{a}^2]$.

Hence, by use of 9 we have: if $[\mathfrak{a}\mathfrak{b}L^2]=0$, then $[\mathfrak{A}\mathfrak{B}L^2]=0$.

The harmonic complex of $\mathfrak{a}$ *and* $\mathfrak{b}$ *is the complex of lines cut harmonically by* $\mathfrak{A}$ *and* $\mathfrak{B}$.

11. *Outer products in the subspreads.* If we have a section of a figure by a plane π, we often wish to form outer products in π; denote these by $[\ ]_\pi$.

For example, $[\mathfrak{A}\pi^2.\mathfrak{B}\pi^2]_\pi$ is the harmonic conic of $\mathfrak{A}\pi^2$ and $\mathfrak{B}\pi^2$ regarded as the sections of $\mathfrak{A}$, $\mathfrak{B}$ by π. (Cf. § 129·2.)

If $\pi=[p_1p_2p_3]$, then

$$[\alpha\pi]=[\alpha p_1][p_2p_3]+[\alpha p_2][p_3p_1]+[\alpha p_3][p_1p_2],$$

$$\begin{aligned}[\alpha\pi.\beta\pi]_\pi &= ([\alpha p_2][\beta p_3]-[\alpha p_3][\beta p_2])[p_3p_1\cdot p_1p_2]_\pi+\ldots+\ldots\\ &= ([\alpha\beta.p_2p_3]p_1+[\alpha\beta.p_3p_1]p_2+[\alpha\beta.p_1p_2]p_3)[p_1p_2p_3]\\ &= [\alpha\beta.\pi][p_1p_2p_3]=[\alpha\beta.\pi]\pi.\end{aligned}$$

$$\begin{aligned}[\alpha\pi.\beta\pi.\gamma\pi]_\pi &= [\alpha\beta\gamma.p_1p_2p_3][p_2p_3\cdot p_3p_1\cdot p_1p_2]_\pi\\ &= [\alpha\beta\gamma.p_1p_2p_3][p_1p_2p_3]_\pi^2\ldots=[\alpha\beta\gamma\pi]\{\pi^2\}.\end{aligned}$$

Hence
$$[\mathfrak{A}\pi^2.\mathfrak{B}\pi^2]_\pi=[\mathfrak{A}\mathfrak{B}\pi^2]\{\pi^2\},$$
$$[\mathfrak{A}\pi^2.\mathfrak{B}\pi^2.\mathfrak{C}\pi^2]_\pi=[\mathfrak{A}\mathfrak{B}\mathfrak{C}\pi^2]\{\pi^4\}.$$

This is the analogue of the Clebsch transference principle in invariant theory.

Dually, $\mathfrak{a}p^2$, $\mathfrak{b}p^2$, $\mathfrak{c}p^2$ are tangent cones from p; denoting outer products in the spread of elements through p by $[\ ]_p$, we have

$$[\mathfrak{a}p^2.\mathfrak{b}p^2]_p=[\mathfrak{a}\mathfrak{b}p^2]\{p^2\},\quad [\mathfrak{a}p^2.\mathfrak{b}p^2.\mathfrak{c}p^2]_p=[\mathfrak{a}\mathfrak{b}\mathfrak{c}p^2]\{p^4\}.$$

12. *The harmonic surface.* $[\mathfrak{a}\mathfrak{b}\mathfrak{c}]$ denotes the surface of points p such that $[\mathfrak{a}\mathfrak{b}\mathfrak{c}p^2]=0$, and hence $[\mathfrak{a}p^2.\mathfrak{b}p^2.\mathfrak{c}p^2]_p=0$. *It is* therefore *the locus of points* p *from which tangent cones to* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *form a nul triad* (cf. § 120), and is called the '*harmonic surface*' of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$. It is a quadric locus.

Thus $[\mathfrak{a}\mathfrak{b}^2]$ is the locus of points p from which the tangent-cone locus $[(\mathfrak{b}p^2)^2]$ to $[\mathfrak{b}^2]$ is apolar to the tangent-cone envelope $[\mathfrak{a}p^2]$

to $\mathfrak{a}$. The point of contact q of $\mathfrak{b}$ and any common tangent plane to $\mathfrak{a}$, $\mathfrak{b}$ is on this locus; for $\mathfrak{B}q$ is the tangent plane and it touches $\mathfrak{a}$; hence $[\mathfrak{a}[\mathfrak{B}q]^2] = 0, \quad [\mathfrak{a}[\mathfrak{b}q]^2] = 0, \quad [\mathfrak{a}\mathfrak{b}^2.q^2] = 0.$

13. Since if p is on $[\mathfrak{a}\mathfrak{b}^2]$, the cones from p to $[\mathfrak{b}^2]$ and to $\mathfrak{a}$ are apolar, there are triads of planes through p which touch the cone to $\mathfrak{a}$ and are self-polar to the cone to $\mathfrak{b}$, and hence self-polar to $\mathfrak{b}$, and one of these planes can be any arbitrary tangent plane to the cone to $\mathfrak{a}$. Any of these triads, together with the polar plane of p for $\mathfrak{b}$, form a self-polar tetrahedron of $\mathfrak{b}$.

14. If $[\mathfrak{a}\mathfrak{b}^3] = 0$, and $[\mathfrak{a}\mathfrak{b}^2[\mathfrak{b}\pi]^2] = 0$, then, by (12), $[\mathfrak{a}\pi^2] = 0$. Now $\mathfrak{b}\pi = p$, say, is the pole of π for $\mathfrak{b}$; $[\mathfrak{a}\mathfrak{b}^2p^2] = 0$ means p is on $[\mathfrak{a}\mathfrak{b}^2]$.

Hence *if* $\mathfrak{a}$, $\mathfrak{B}$ *be apolar, and* p *be on* $[\mathfrak{a}\mathfrak{b}^2]$, *then the polar plane of* p *for* $\mathfrak{b}$ *touches* $\mathfrak{a}$.

Conversely, *if* $\mathfrak{a}$, $\mathfrak{B}$ *be apolar, and the plane* π *touches* $\mathfrak{a}$, *the pole of* π *is on* $[\mathfrak{a}\mathfrak{b}^2]$.

From 13, we now have,

If $\mathfrak{a}$, $\mathfrak{B}$ *be apolar, and* π *any tangent plane of* $\mathfrak{a}$, *there are* ∞^1 *triads of planes touching* $\mathfrak{a}$, *which, with* π, *constitute a self-polar tetrahedron of* $\mathfrak{b}$, *and the vertices of these tetrahedra are on* $[\mathfrak{a}\mathfrak{b}^2]$.

15. By (18), *if* $[\mathfrak{a}^2\mathfrak{b}^2] \neq 0$, *and* p_1, p_2 *be the poles of the same plane* π *for* $\mathfrak{a}$, $\mathfrak{b}$, *then the polar planes of* p_1 *for* $\mathfrak{a}\mathfrak{b}^2$, *and of* p_2 *for* $\mathfrak{a}^2\mathfrak{b}$ *meet in a line lying on* π; *if* $[\mathfrak{a}^2\mathfrak{b}^2] = 0$, *these polar planes coincide.*

By (17), *if* $\mathfrak{a}$, $\mathfrak{B}$ *be apolar, and* p_1, p_2 *be the poles of the same plane* π *for* $\mathfrak{a}$, $\mathfrak{b}$, *then the polar planes of* p_1 *for* $\mathfrak{B}$ *and of* p_2 *for* $\mathfrak{a}\mathfrak{b}^2$ *coincide*; *if* $\mathfrak{a}$, $\mathfrak{B}$ *be not apolar, these polar planes meet in a line which lies on* π.

If $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *be dependent, then* $[\mathfrak{a}\mathfrak{b}\mathfrak{c}]$ *goes through the cut of* $[\mathfrak{a}^2\mathfrak{b}]$ *and* $[\mathfrak{a}\mathfrak{b}^2]$.

16. The dual to (5) gives us

$$[\mathfrak{A}^2\mathfrak{b}] = 2[\mathfrak{A}\mathfrak{b}]\,\mathfrak{A} - 2(\mathfrak{A}\mathfrak{b}\mathfrak{A}).$$

By (3), $$[\mathfrak{A}^2] = \tfrac{3}{2}[\mathfrak{a}^4][\mathfrak{a}^2].$$

Hence $$\tfrac{3}{4}[\mathfrak{a}^4][\mathfrak{a}^2\mathfrak{b}] = [\mathfrak{A}\mathfrak{b}]\,\mathfrak{A} - (\mathfrak{A}\mathfrak{b}\mathfrak{A}). \qquad (23)$$

Hence $\mathfrak{A}$, $[\mathfrak{a}^2\mathfrak{b}]$, $(\mathfrak{A}\mathfrak{b}\mathfrak{A})$ (the reciprocal of $\mathfrak{b}$ for $\mathfrak{A}$) are dependent.

Also, *if* $\mathfrak{A}$, $\mathfrak{b}$ *be apolar, the harmonic quadric* $[\mathfrak{a}^2\mathfrak{b}]$ *is the reciprocal of* $\mathfrak{b}$ *for* $\mathfrak{A}$.

17. *Nul quadruples* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ of quadrics are defined by $[\mathfrak{abcd}] = 0$. The harmonic quadric of any three quadrics of the quadruple is apolar to the fourth.

By (15), *the poles of any plane* π *for* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ *are such that their polar planes for* $[\mathfrak{bcd}]$, $[\mathfrak{cda}]$, $[\mathfrak{dab}]$, $[\mathfrak{abc}]$ *meet in a point.*

Thus we have four linearly dependent collineations in space.

§ 130. *Sums of products of pairs of rotors.*

1. We denote such sums by Ω, Ω_1, Ω_2, ... Each such sum can be reduced to the form

$$k_{11}\{E_1^2\} + \ldots + k_{66}\{E_6^2\} + 2k_{12}\{E_1, E_2\} + \ldots + 2k_{56}\{E_5, E_6\}, \qquad \text{(i)}$$

where $E_1, \ldots, E_6$ are the rotors

$$E_1 = [e_2e_3], \quad E_2 = [e_3e_1], \quad E_3 = [e_1e_2],$$
$$E_4 = [e_1e_4], \quad E_5 = [e_2e_4], \quad E_6 = [e_3e_4].$$

We can write Ω in the form $\{A_1, E_1\} + \ldots + \{A_6, E_6\}$, where $A_i = \sum_{j=1}^{6} k_{ij}E_j$, with suitable scalars k_{ij} satisfying $k_{ij} = k_{ji}$.

2. The equation $\Omega = 0$ means $[\Omega L] = 0$ for all lines L.

If $\Omega \neq 0$, then Ω represents the set of lines L such that $\Omega\{L^2\} = 0$ (§ 127·2). This set is a '*quadratic complex*'.

If $\Omega = \{A_1, E_1\} + \ldots + \{A_6, E_6\}$, then $\Omega\{p^2\} = \sum_{j=1}^{6} \{[A_j p]\,[E_j p]\}$. Thus $\Omega\{p^2\}$ is a sum $\mathfrak{P}$ of products of plane-pairs through p and represents the locus of points q, such that $\mathfrak{P}\{q^2\} = 0$; these form a quadric cone locus with vertex p. Dually, $\Omega\{\pi^2\}$ is a sum of products of point-pairs on π, and represents a conic envelope in plane π.

3. If $[\Omega\{L, M\}] = 0$, the lines L, M are '*conjugate*' for Ω.

If L is fixed, then $[\Omega L]$ is a screw (§ 127·4) whose nul lines are conjugate to L for Ω. Since (i) can in the general case be expressed as the sum of six squares (cf. § 91), therefore, in general, there are sets of six lines $L_1, \ldots, L_6$, such that $\Omega = \{L_1^2\} + \ldots + \{L_6^2\}$; each line L is conjugate to the lines of the linear complex defined by the other five lines L.

4. A special case of $\mathfrak{Q}$ occurs when $\mathfrak{Q} = \mathfrak{a}^2$, where $\mathfrak{a}$ represents a quadric envelope. $\mathfrak{Q}$ then represents the set of tangent lines to $\mathfrak{a}$. If $[\mathfrak{a}^2\{L, M\}] = 0$, the polar line of L for $\mathfrak{a}$ cuts M and vice versa; that is, L, M are conjugate lines for $\mathfrak{a}$.

5. By the usual methods employed here we can shew

$$[\mathfrak{abc}.\mathfrak{Q}] = [\mathfrak{bc}\mathfrak{Q}]\,\mathfrak{a} + [\mathfrak{ca}\mathfrak{Q}]\,\mathfrak{b} + [\mathfrak{ab}\mathfrak{Q}]\,\mathfrak{c} - 2(\mathfrak{a}[\mathfrak{c}\mathfrak{Q}]\,\mathfrak{b}) - \ldots, \quad (24)$$

$$[\mathfrak{ab}.\mathfrak{c}\mathfrak{Q}] = [\mathfrak{ac}\mathfrak{Q}]\,\mathfrak{b} + [\mathfrak{bc}\mathfrak{Q}]\,\mathfrak{a} - 2(\mathfrak{a}[\mathfrak{c}\mathfrak{Q}]\,\mathfrak{b}). \quad (25)$$

Together these give

$$[\mathfrak{ab}.\mathfrak{c}\mathfrak{Q}] + [\mathfrak{bc}.\mathfrak{a}\mathfrak{Q}] + [\mathfrak{ca}.\mathfrak{b}\mathfrak{Q}] = [\mathfrak{abc}.\mathfrak{Q}] + [\mathfrak{bc}\mathfrak{Q}]\,\mathfrak{a} + [\mathfrak{ca}\mathfrak{Q}]\,\mathfrak{b} + [\mathfrak{ab}\mathfrak{Q}]\,\mathfrak{c}. \quad (26)$$

6. The outer product $[\mathfrak{Q}\mathfrak{a}]$ is a sum of products of plane-pairs, and represents the quadric locus of points p such that $[\mathfrak{Q}\mathfrak{a}p^2] = 0$.

Now $$[\mathfrak{Q}p^2.\mathfrak{a}p^2]_p = [[\mathfrak{Q}\mathfrak{a}p^2]p^2].$$

Hence, if p is on the locus, then the cone locus $[\mathfrak{Q}p^2]$ is apolar to the cone envelope $[\mathfrak{a}p^2]$ of tangent lines from p to $\mathfrak{a}$.

We call $[\mathfrak{Q}\mathfrak{a}]$ the '*harmonic surface*' of $\mathfrak{Q}$ and $\mathfrak{a}$.

7. If $[\mathfrak{Q}_1\mathfrak{Q}_2] = 0$, the complexes $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ are '*apolar*'; then for any point p and plane π, $\mathfrak{Q}_1\{p^2\}$, $\mathfrak{Q}_2\{\pi^2\}$ are apolar; that is, the envelope $\mathfrak{Q}_2\{\pi^2\}$ in plane π and the conic locus which is the cut of π by $\mathfrak{Q}_1\{p^2\}$ are apolar.

If $[\mathfrak{Q}\mathfrak{a}^2] = 0$, the set of tangents to $\mathfrak{a}$ is apolar to $\mathfrak{Q}$; the quadrics $\mathfrak{a}$ and $[\mathfrak{Q}\mathfrak{a}]$ are apolar. Dually, we may consider $[\mathfrak{Q}\mathfrak{A}^2] = 0$ and derive various theorems from (24), (25), (26).

If $a_1, \ldots, a_4$ be a self-polar tetrahedron of $\mathfrak{a}$, and its edges be lines of $\mathfrak{Q}$, then, with suitable weights,

$$\mathfrak{a} = \{a_1^2\} + \ldots + \{a_4^2\}, \quad \mathfrak{a}^2 = \sum_{i,j}\{[a_i a_j]\}^2, \quad [\mathfrak{Q}\mathfrak{a}^2] = 0.$$

8. If $\mathfrak{Q}$ is the square of a screw S, $\mathfrak{Q} = \{S^2\}$, its lines are the nul lines of S repeated. $[\mathfrak{Q}p^2] = [S^2p^2] = [Sp]^2$, the nul plane of S repeated.

If S is a rotor, the corresponding complex is a special linear complex.

The equation $[\mathfrak{a}^2L.\mathfrak{b}^2L] = 0$ is satisfied by all lines L whose polars $[\mathfrak{a}^2L]$, $[\mathfrak{b}^2L]$ for $\mathfrak{a}$, $\mathfrak{b}$ meet; since L is the cut of the polar

planes for $\mathfrak{a}$, $\mathfrak{b}$ of the point of meeting of these polar lines (or is the join of the poles of the plane through the lines), the complex of lines L is tetrahedral (§ 51). If $[\mathfrak{a}^2L.\mathfrak{b}^2L] = 0$, it is clear geometrically that $[[\mathfrak{a}^2L.\mathfrak{b}^2]L] = 0$, $[[\mathfrak{b}^2L.\mathfrak{a}^2]L] = 0$. Hence, by (22), $[[\mathfrak{ab}L.\mathfrak{ab}]L] = 0$, $[\mathfrak{ab}L.\mathfrak{ab}L] = 0$. Hence the screw $[\mathfrak{ab}L]$ is a rotor, and the corresponding complex is special.

9. If $\mathfrak{Q} = \{S^2\}$, then

$$[\mathfrak{Q}\mathfrak{b}p^2] = [S^2\mathfrak{b}p^2] = [S^2p^2\mathfrak{b}] = [[Sp]^2\mathfrak{b}].$$

Hence $[\mathfrak{Q}\mathfrak{b}]$ in this case represents the locus of points p whose nul planes for S touch $\mathfrak{b}$; we call it the '*reciprocal*' of $\mathfrak{b}$ for S. Dually, the reciprocal of $\mathfrak{A}$ for S is $[S^2\mathfrak{A}]$.

Since S is involutory, the reciprocal of $[\mathfrak{b}S^2]$ should be $\mathfrak{b}$; that is, we should have $[\mathfrak{b}S^2.S^2] \equiv \mathfrak{b}$. This is so, for, if q be any point, $[qS.S] = \frac{1}{2}[SS]q$.

Hence $$[q^2S^2.S^2] = [qS.S]^2 = \tfrac{1}{4}[SS]^2\{q^2\},$$

$$[\mathfrak{b}S^2.S^2] = \tfrac{1}{4}[SS]^2\mathfrak{b} \equiv \mathfrak{b}, \quad \text{if} \quad [SS] \neq 0.$$

The equation $[\mathfrak{a}S^2.\mathfrak{b}] = [\mathfrak{a}.\mathfrak{b}S^2] = [[\mathfrak{ab}]S^2]$ is easily shewn and interpreted.

10. If $[\mathfrak{Q}_1\mathfrak{Q}_2] = 0$, there are ∞^{10} sets of six lines of $\mathfrak{Q}_1$ mutually conjugate for $\mathfrak{Q}_2$.

If there be one such set, then $[\mathfrak{Q}_1\mathfrak{Q}_2] = 0$.

If $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ be *general* sums of products of pairs of rotors, we can find $e_1, \ldots, e_4$ so that $\mathfrak{Q}_1 = \Sigma k_{ij}[e_ie_j]^2$, $\mathfrak{Q}_2 = \Sigma k'_{ij}[e_ie_j]^2$. If then $[\mathfrak{Q}_1\mathfrak{Q}_2] = 0$, we have $\sum_{ij} k_{ij}k'_{ij} = 0$.

In particular, if $[\mathfrak{a}^2\mathfrak{b}^2] = 0$, there is an infinite number of tetrahedra self-polar to one of $\mathfrak{a}$, $\mathfrak{b}$ and whose edges touch the other. The existence of one such tetrahedron is a sufficient condition for this situation.

If $[\mathfrak{a}^2\mathfrak{b}^2] = 0$, then the harmonic complex $[\mathfrak{ab}]$ is apolar to itself, the harmonic surface $[\mathfrak{a}^2\mathfrak{b}]$ is apolar to $\mathfrak{b}$, and $[\mathfrak{ab}^2]$ is apolar to $\mathfrak{a}$.

If $[\mathfrak{a}^2\mathfrak{b}^2] = 0$, and $[\mathfrak{ab}^3] = 0$, then, by (7),

$$[\mathfrak{a}\mathfrak{B}]^2 = [(\mathfrak{a}\mathfrak{B}\mathfrak{a})\mathfrak{B}] = 0, \quad [(\mathfrak{b}\mathfrak{A}\mathfrak{b})\mathfrak{A}] = 0;$$

hence each of $\mathfrak{a}$, $\mathfrak{b}$ is apolar to its reciprocal in the other.

11. If $\mathfrak{a} = \Sigma k_{ij}\{e_i, e_j\}$, $\mathfrak{b} = \Sigma l_{ij}\{e_i, e_j\}$,

then $[\mathfrak{a}^2] = \Sigma k_{ij}k_{jl}\{[e_i e_j], [e_j e_l]\}$
$+ \Sigma k_{ij}k_{lm}\{[e_i e_l], [e_j e_m] + [e_i e_m], [e_j e_l]\}$.

If $\mathfrak{a} = k_1\{e_1^2\} + \ldots + k_4\{e_4^2\}$, $\mathfrak{b} = l_1\{e_1^2\} + \ldots + l_4\{e_4^2\}$,
$L = l_{12}\{e_1, e_2\} + \ldots + l_{34}\{e_3, e_4\}$, $[e_1 e_2 e_3 e_4] = 1$,

then $\mathfrak{a}^2 = \sum_{ij} k_i k_j [e_i e_j]^2$,

$[\mathfrak{a}^2\mathfrak{b}^2] = k_1k_2l_3l_4 + k_1k_3l_2l_4 + \ldots + \ldots + k_3k_4l_1l_2$,
$[\mathfrak{a}^2 L] = k_1k_2l_{34}[e_3e_4] + k_1k_3l_{24}[e_2e_4] + \ldots$.

These connect up our symbols with the more usual treatment by coordinates.

§ 131. *A fundamental sum of products of pairs of rotors.*

1. If e_1, e_2, e_3, e_4 be four base points, let

$$\mathfrak{E} = \{[e_1e_2], [e_3e_4]\} + \{[e_2e_3], [e_1e_4]\} + \{[e_3e_1], [e_2e_4]\}.$$

Then $\mathfrak{E} \neq 0$, for if L be any line in the plane $[e_1e_2e_3]$, we have

$$2[\mathfrak{E}L] = [e_1e_2][e_3e_4L] + [e_2e_3][e_1e_4L] + [e_3e_1][e_2e_4L],$$

the other terms vanishing. If $[\mathfrak{E}L] = 0$ for all L in plane $[e_1e_2e_3]$, these lines must cut $[e_1e_4]$, $[e_2e_4]$, $[e_3e_4]$, which cannot happen since e_1, e_2, e_3, e_4 are independent.

2. If $p = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$, then

$$\begin{aligned}[\mathfrak{E}p^2] &= \{[e_1e_2p], [e_3e_4p]\} + \{[e_2e_3p], [e_1e_4p]\} + \{[e_3e_1p], [e_2e_4p]\} \\ &= \{(x_4\epsilon_3 - x_3\epsilon_4), (x_2\epsilon_1 - x_1\epsilon_2)\} + \ldots \\ &= (x_2x_4\{\epsilon_1, \epsilon_3\} + x_1x_3\{\epsilon_2, \epsilon_4\} - x_2x_3\{\epsilon_1, \epsilon_4\} - x_1x_4\{\epsilon_2, \epsilon_3\}) + \ldots \\ &= 0.\end{aligned}$$

Hence, if $\mathfrak{a}$, $\mathfrak{b}$ represent any quadrics, then $[\mathfrak{E}\mathfrak{a}] = 0$, $[\mathfrak{E}\mathfrak{a}\mathfrak{b}] = 0$, $[\mathfrak{E}\mathfrak{a}^2] = 0$; $\mathfrak{E}$ is apolar to all quadrics and all harmonic complexes.

3. All lines are in the set represented by $\mathfrak{E}$, for

$$[\mathfrak{E}[pq]^2] = [\mathfrak{E}p^2.q^2] = 0.$$

Hence, by § 130·10, *there are* ∞^{10} *sets of six lines mutually conjugate for a given quadric.*

We shall see that this fact is intimately connected with the double-six theorem; now the latter depends on a theorem peculiar to quadrics in step six, (§ 107). The corresponding peculiarity here is the existence of a product-sum $\mathfrak{E}$ apolar to all quadrics, and the structure of that sum is peculiar to step four.

4. Since $$[\mathfrak{E}[e_1 e_2]] = \tfrac{1}{2}[e_1 e_2],$$
therefore, if S is a screw, we have $[\mathfrak{E}S] = \tfrac{1}{2}S$,
$$[\mathfrak{E}\{S_1, S_2\}] = [[\mathfrak{E}S_1]\, S_2] = \tfrac{1}{2}[S_1 S_2].$$
Hence, if S_1, S_2 are in involution, they are conjugate for $\mathfrak{E}$.

§ 132. *F. Schur's Theorem.**

If three generators L, M, N of a regulus are mutually conjugate for a quadric $\mathfrak{b}$, *there are* ∞^1 *triads of such generators.*

We have $\mathfrak{b}^2\{M, N\} = \mathfrak{b}^2\{N, L\} = \mathfrak{b}^2\{L, M\} = 0$.

If $x_1 L + x_2 M + x_3 N$, $y_1 L + y_2 M + y_3 N$, $z_1 L + z_2 M + z_3 N$ be lines, then, forming their outer products with themselves,

$$\left.\begin{aligned} x_2 x_3[MN] + x_3 x_1[NL] + x_1 x_2[LM] &= 0, \\ y_2 y_3[MN] + y_3 y_1[NL] + y_1 y_2[LM] &= 0, \\ z_2 z_3[MN] + z_3 z_1[NL] + z_1 z_2[LM] &= 0, \end{aligned}\right\} \qquad \text{(i)}$$

and if they be conjugate in pairs for $\mathfrak{b}$, then

$$\left.\begin{aligned} \mathfrak{b}^2(y_1 z_1\{L, L\} + y_2 z_2\{M, M\} + y_3 z_3\{N, N\}) &= 0, \\ \mathfrak{b}^2(z_1 x_1\{L, L\} + z_2 x_2\{M, M\} + z_3 x_3\{N, N\}) &= 0, \\ \mathfrak{b}^2(x_1 y_1\{L, L\} + x_2 y_2\{M, M\} + x_3 y_3\{N, N\}) &= 0. \end{aligned}\right\} \qquad \text{(ii)}$$

The condition for equations (i) to be consistent is

$$\begin{vmatrix} x_2 x_3, & x_3 x_1, & x_1 x_2 \\ y_2 y_3, & y_3 y_1, & y_1 y_2 \\ z_2 z_3, & z_3 z_1, & z_1 z_2 \end{vmatrix} = 0.$$

The condition for equations (ii) to be consistent is

$$\begin{vmatrix} y_1 z_1, & y_2 z_2, & y_3 z_3 \\ z_1 x_1, & z_2 x_2, & z_3 x_3 \\ x_1 y_1, & x_2 y_2, & x_3 y_3 \end{vmatrix} = 0.$$

* Schur, *Math. Ann.* 18 (1881), p. 1. (The theorem is completed below.)

These conditions are each equivalent to

$$\begin{vmatrix} x_1^{-1}, & y_1^{-1}, & z_1^{-1} \\ x_2^{-1}, & y_2^{-1}, & z_2^{-1} \\ x_3^{-1}, & y_3^{-1}, & z_3^{-1} \end{vmatrix} = 0.$$

When x_1, x_2, x_3 are fixed, we have, from (i), (ii), five linear equations in y_1^{-1}, y_2^{-1}, y_3^{-1}, z_1^{-1}, z_2^{-1}, z_3^{-1}; these fix uniquely the ratios $y_1 : y_2 : y_3$ and $z_1 : z_2 : z_3$.

Cor. If L_1, L_2, L_3 be generators of the regulus through L, M, N and L_2, L_3 be conjugate for the quadric, and L_3, L_1 conjugate, then L_1, L_2 are conjugate.

§ 133. *Quadrics associated with a hexagon in space.**

1. Let $ab'ca'bc'$ be the hexagon; let $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ represent the quadrics through the triads of lines $[b'c]$, $[c'a]$, $[a'b]$ and $[aa']$, $[bb']$, $[cc']$ and $[bc']$, $[ca']$, $[ab']$ respectively. We may thus, by § 32·2, take $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ to be the expressions

$$\{[cb'c'], [aba']\} - \{[bc'a'], [cab']\},$$
$$\{[aa'b'], [bcc']\} - \{[cb'c'], [aba']\},$$
$$\{[bc'a'], [cab']\} - \{[aa'b'], [bcc']\}.$$

As the sum of these vanishes, the quadrics are dependent, as was shewn earlier, (§ 32).

2. Let $\quad d = [bca' . cab' . abc'], \quad [abcd] = 1,$

$$[bcd] = \alpha, \quad [cda] = -\beta, \quad [dab] = \gamma, \quad [abc] = -\delta.$$

Weight the points so that

$$a' = k_2 b + c + d, \quad b' = a + k_3 c + d, \quad c' = k_1 a + b + d,$$

and let $\quad e = a + b + c + d, \quad f = k_1 a + k_2 b + k_3 c + d,$

$$\epsilon = \alpha + \beta + \gamma - \delta, \quad \phi = k_1 \alpha + k_2 \beta + k_3 \gamma - \delta.$$

3. Substituting in the expressions for $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, simplifying, and omitting scalar factors, we have

$$\mathfrak{U} = \alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \epsilon^2,$$
$$\mathfrak{V} = \mathfrak{U} - k_1 k_2 k_3 \mathfrak{W},$$
$$\mathfrak{W} = k_1^{-2}\alpha^2 + k_2^{-2}\beta^2 + k_3^{-2}\gamma^2 - \delta^2 - \phi^2.$$

* Baker, *Journ. Lond. Math. Soc.* 1 (1926).

4. The quadric through the lines $[b'c]$, $[c'a]$, $[a'b]$, as an envelope, has equation

$$[\pi.b'c.c'a.a'b.\pi] = 0.$$

Hence it is represented by the sum of point-products

$$[abb'c']\{c,a'\}-[aa'b'c']\{b,c\}-[abcc']\{a',b'\}-[cac'a']\{b,b'\},$$

which reduces to $a^2+b^2+c^2-d^2-e^2$.

Thus and similarly we find

$$\begin{aligned}
\mathfrak{u} &= a^2+ b^2+ c^2-d^2-e^2,\\
\mathfrak{v} &= (k_1-1)^2a^2+(k_2-1)^2b^2+(k_3-1)^2c^2-e^2-f^2,\\
\mathfrak{w} &= k_1^2a^2+ k_2^2b^2+ k_3^2c^2-d^2-f^2.
\end{aligned}$$

5. If the quadric envelope $\mathfrak{p}$ reciprocates the opposite sides of the hexagon $aa'cc'bb'$ into one another, turning the lines aa', bb', cc' respectively into bc', ca', ab', and if $\mathfrak{q}$ does the same for the hexagon $ab'ca'bc'$, and $\mathfrak{r}$ the same for the hexagon $aa'bb'cc'$, then $\mathfrak{p}$ reciprocates $\mathfrak{V}$ into $\mathfrak{v}$, $\mathfrak{q}$ reciprocates $\mathfrak{U}$ into $\mathfrak{w}$, and $\mathfrak{r}$ reciprocates $\mathfrak{V}$ into $\mathfrak{u}$.

We find, as in § 99·3, constant factors being arbitrary,

$$\begin{aligned}
\mathfrak{p} &= k_1(k_1-1)a^2+k_2(k_2-1)b^2+k_3(k_3-1)c^2-f^2,\\
\mathfrak{q} &= k_1a^2+ k_2b^2+ k_3c^2-d^2,\\
\mathfrak{r} &= -(k_1-1)a^2- (k_2-1)b^2- (k_3-1)c^2-e^2,
\end{aligned}$$

6. *Comparison with matrices.* If p is any point, $[\mathfrak{U}p]$ is the polar plane of p for $\mathfrak{U}$; thus $\mathfrak{U}$ gives rise to a polarity which changes points into planes. Dually, if $\mathfrak{u}$ is the corresponding quadric envelope, so that $[\mathfrak{U}^3] \equiv \mathfrak{u}$, then $\mathfrak{u}$ gives rise to a polarity which changes planes into points. The two polarities are inverse to each other, when the weights of points are ignored, and in any fixed frame the matrices of $\mathfrak{u}$, $\mathfrak{U}$ are inverse matrices, apart from a constant factor. We can therefore connect the present work with matrices, or linear transformations, if we regard $\mathfrak{u}$ as a matrix, and write $\mathfrak{u}^{-1}$ for $\mathfrak{U}$. Similarly write $\mathfrak{v}^{-1}$, $\mathfrak{w}^{-1}$ for $\mathfrak{V}$, $\mathfrak{W}$.

7. The equations above give at once

$$(1)\ \mathfrak{q}+\mathfrak{r}=\mathfrak{u}, \qquad (2)\ \mathfrak{r}+\mathfrak{p}=\mathfrak{v}, \qquad (3)\ \mathfrak{p}+\mathfrak{q}=\mathfrak{w},$$

and since $\mathfrak{p}$ reciprocates $\mathfrak{W}$ into $\mathfrak{v}$, and so on, we have (§ 99, 128·3)

$$(4)\ \mathfrak{v} = \mathfrak{p}\mathfrak{w}^{-1}\mathfrak{p}, \qquad (5)\ \mathfrak{w} = \mathfrak{q}\mathfrak{u}^{-1}\mathfrak{q}, \qquad (6)\ \mathfrak{u} = \mathfrak{r}\mathfrak{v}^{-1}\mathfrak{r}.$$

Of these we need only assume (1), (2), (3), (4), and hence we can define all the quadrics in terms of $\mathfrak{p}$, $\mathfrak{w}$.

For (2), (3), (4) give

$$\mathfrak{r}+\mathfrak{p} = \mathfrak{v} = \mathfrak{p}\mathfrak{w}^{-1}\mathfrak{p} = \mathfrak{p}(\mathfrak{p}+\mathfrak{q})^{-1}\mathfrak{p}.$$

Hence $$(\mathfrak{r}+\mathfrak{p})\mathfrak{p}^{-1}(\mathfrak{p}+\mathfrak{q})\mathfrak{p}^{-1} = \mathfrak{J},$$

$$\mathfrak{J} = (\mathfrak{r}\mathfrak{p}^{-1}+\mathfrak{J})(\mathfrak{J}+\mathfrak{q}\mathfrak{p}^{-1}) = \mathfrak{J}+\mathfrak{r}\mathfrak{p}^{-1}+\mathfrak{q}\mathfrak{p}^{-1}+\mathfrak{r}\mathfrak{p}^{-1}\mathfrak{q}\mathfrak{p}^{-1},$$

$$\mathfrak{r}\mathfrak{p}^{-1}+\mathfrak{q}\mathfrak{p}^{-1}+\mathfrak{r}\mathfrak{p}^{-1}\mathfrak{q}\mathfrak{p}^{-1} = 0, \quad \mathfrak{r}+\mathfrak{q}+\mathfrak{r}\mathfrak{p}^{-1}\mathfrak{q} = 0,$$

$$\mathfrak{p}^{-1}+\mathfrak{q}^{-1}+\mathfrak{r}^{-1} = 0.$$

Hence the loci $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ are in a pencil.

The three quadric loci which reciprocate the sides of the hexagons aa'cc'bb', ab'ca'bc', aa'bb'cc' into the opposite sides are in a pencil.

To deduce (5), (6) from (1), (2), (3), (4): we have from the last equation written:

$$(\mathfrak{p}+\mathfrak{q})\mathfrak{q}^{-1}(\mathfrak{q}+\mathfrak{r})\mathfrak{q}^{-1} = \mathfrak{J}, \quad (\mathfrak{q}+\mathfrak{r})\mathfrak{r}^{-1}(\mathfrak{r}+\mathfrak{p})\mathfrak{r}^{-1} = \mathfrak{J}.$$

Hence from (1), (2), (3), $\mathfrak{q}^{-1}\mathfrak{u}\mathfrak{q}^{-1} = \mathfrak{w}^{-1}$, $\mathfrak{r}^{-1}\mathfrak{v}\mathfrak{r}^{-1} = \mathfrak{u}^{-1}$, equivalent to (5), (6).

8. Since $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ are dependent, so must $\mathfrak{u}^{-1}$, $\mathfrak{v}^{-1}$, $\mathfrak{w}^{-1}$ be.

Let $$\mathfrak{w}^{-1} = x\mathfrak{u}^{-1}+y\mathfrak{v}^{-1}.$$

By (1), (2), (3), (6),

$$\mathfrak{w} = \mathfrak{u}+\mathfrak{v}-2\mathfrak{r} = \mathfrak{u}-2\mathfrak{r}+\mathfrak{r}\mathfrak{u}^{-1}\mathfrak{r},$$

$$(\mathfrak{u}-2\mathfrak{r}+\mathfrak{r}\mathfrak{u}^{-1}\mathfrak{r})(x\mathfrak{u}^{-1}+y\mathfrak{v}^{-1}) = \mathfrak{J},$$

$$(\mathfrak{u}\mathfrak{r}^{-1}-2\mathfrak{J}+\mathfrak{r}\mathfrak{u}^{-1})(x\mathfrak{r}\mathfrak{u}^{-1}+y\mathfrak{r}\mathfrak{v}^{-1}) = \mathfrak{J}.$$

Put $\mathfrak{u}\mathfrak{r}^{-1} = \mathfrak{d}$, then, since $\mathfrak{r}\mathfrak{v}^{-1} = \mathfrak{u}\mathfrak{r}^{-1}$, we have

$$(\mathfrak{d}-2\mathfrak{J}+\mathfrak{d}^{-1})(x\mathfrak{d}^{-1}+y\mathfrak{d}) = \mathfrak{J}, \quad (\mathfrak{d}-\mathfrak{J})^2(x\mathfrak{J}+y\mathfrak{d}^2) = \mathfrak{d}^2.$$

The equation $\det(\mathfrak{u}-\lambda\mathfrak{r}) = 0$, or $\det(\mathfrak{d}-\lambda\mathfrak{J}) = 0$, is hence of form $$(\lambda-1)^2(x+y\lambda^2) = \lambda^2,$$

or $$y\lambda^4-2y\lambda^3+(x+y)\lambda^2-2x\lambda+x = \lambda^2.$$

Now $\det(\mathfrak{u}-\lambda\mathfrak{r}) = 0$ can be written, (p. 293, Ex. 7),

$$[\mathfrak{u}^4]-4\lambda[\mathfrak{u}^3\mathfrak{r}]+6\lambda^2[\mathfrak{u}^2\mathfrak{r}^2]-4\lambda^3[\mathfrak{u}\mathfrak{r}^3]+\lambda^4[\mathfrak{r}^4] = 0.$$

Comparing coefficients, we find

$$[\mathfrak{u}^4][\mathfrak{r}^4] = 4[\mathfrak{u}^3\mathfrak{r}][\mathfrak{u}\mathfrak{r}^3].$$

Similarly $$[\mathfrak{u}^4][\mathfrak{q}^4] = 4[\mathfrak{u}^3\mathfrak{q}][\mathfrak{u}\mathfrak{q}^3],$$

and the equations obtained by cycling $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ and $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ simultaneously.

9. If three generators of the same system of a quadric $\mathfrak{u}$ be mutually conjugate for a quadric $\mathfrak{q}$, we can take $a, \ldots, c'$ so that $b'c$, $c'a$, $a'b$ are these generators and bc', ca', ab' their polar lines for $\mathfrak{q}$. Then $\mathfrak{q}$ reciprocates the quadric through the first triad of lines into that through the second triad.

We can then take $\mathfrak{q}$, $\mathfrak{u}$ to have the equations above.

Then $$[\mathfrak{u}^4][\mathfrak{q}^4] = 4[\mathfrak{u}^3\mathfrak{q}][\mathfrak{u}\mathfrak{q}^3].$$

Hence this is the condition that one regulus of $\mathfrak{u}$ is in the Schur relation (§ 132) to $\mathfrak{q}$; as it only involves the quadrics, therefore the other regulus of $\mathfrak{u}$ is also in that relation; and further the relation between the quadrics is reciprocal.

The following pairs of quadrics are in Schur's relation:

$$(\mathfrak{w}, \mathfrak{q}), \quad (\mathfrak{u}, \mathfrak{r}), \quad (\mathfrak{v}, \mathfrak{p}), \quad (\mathfrak{u}, \mathfrak{q}), \quad (\mathfrak{v}, \mathfrak{r}), \quad (\mathfrak{w}, \mathfrak{p}).$$

10. *Relation to the double-six theorem.* Write $L_1 = b'c$, $L_2 = c'a$, $L_3 = a'b$, $M_1 = bc'$, $M_2 = ca'$, $M_3 = ab'$. Then L_1, L_2, L_3 are generators of a regulus on $\mathfrak{u}$; M_1, M_2, M_3 of one on $\mathfrak{w}$; L_1, L_2, L_3, and similarly M_1, M_2, M_3, is a triad of mutually conjugate lines for $\mathfrak{q}$.

Another line L_4, which is conjugate to L_1, L_2, L_3 for $\mathfrak{q}$, must meet the polars M_1, M_2, M_3 of L_1, L_2, L_3 for $\mathfrak{q}$; and any such line is conjugate to L_1, L_2, L_3.

When L_4 is fixed, if M_4 is its polar line for $\mathfrak{q}$, there are two lines L_5, L_6 which meet M_1, M_2, M_3, M_4. As these are conjugate to L_4, they are conjugate to one another, (§ 132, *Cor.*), as well as to L_1, L_2, L_3.

Thus we have six lines $L_1, \ldots, L_6$ mutually conjugate for $\mathfrak{q}$. Their polar lines $M_1, \ldots, M_6$ for $\mathfrak{q}$ are also mutually conjugate for $\mathfrak{q}$.

Hence L_i, M_i, $(i = 1, \ldots, 6)$ form a double-six derived from the hexagon $ab'ca'bc'$, and the line L_4 meeting bc', ca', ab'.

Examples. 46. Call two point-pairs on a sphere '*conjugate*' when the joins are lines conjugate for the sphere. Then all circles through either pair are cut orthogonally by just one circle through the other pair. If five point-pairs are mutually conjugate, there is a point-pair concyclic with each of the five pairs.

47. A line L_1 in space can be represented by two points p_1, q_1 in a fixed plane such that, if p_1q_1, p_2q_2 be parallel, then L_1, L_2 intersect, and conversely. Translate the double-six theorem into a theorem on the plane.

48. Taking e_1, e_2, e_3, e_4 as basic points, let

$$\begin{aligned} a &= e_1+e_2, & b &= e_1-e_2, & c &= e_3+e_4, & d &= e_3-e_4, \\ a_1 &= e_4-e_2, & b_1 &= e_4+e_2, & c_1 &= e_1-e_3, & d_1 &= e_1+e_3, \\ a_2 &= e_3-e_2, & b_2 &= e_3+e_2, & c_2 &= e_1-e_4, & d_2 &= e_1+e_4. \end{aligned}$$

The three tetrahedra, $abcd$, $a_1b_1c_1d_1$, $a_2b_2c_2d_2$ are '*desmic*', each edge of one cuts two opposite edges of the other. Thus bc, ad cut both b_1c_1 and a_1d_1. Any two of the tetrahedra are in four-fold perspective from the vertices of the third, for example,

$$\begin{aligned} b-c_1 &= c-b_1 = d+a_1 = d_1-a = a_2, \\ c-a_1 &= a-c_1 = d+b_1 = d_1-b = b_2, \\ a-b_1 &= b-a_1 = d+c_1 = d_1-c = c_2, \\ a+a_1 &= b+b_1 = c+c_1 = d_1-d = d_2. \end{aligned}$$

The vertices of any pair of tetrahedra are associated points, for using algebraic products,

$$a^2+b^2+c^2+d^2 = a_1^2+b_1^2+c_1^2+d_1^2 = a_2^2+b_2^2+c_2^2+d_2^2.$$

Deduce geometrical theorems from the following identities in algebraic products:

$$\begin{aligned} d^2+a^2-b^2-c^2 &= 2(b_1c_1-a_1d_1) = 2(b_2c_2-a_2d_2), \\ d^2-a^2+b^2-c^2 &= 2(c_1a_1-b_1d_1) = 2(c_2a_2-b_2d_2), \\ d^2-a^2-b^2+c^2 &= 2(a_1b_1-c_1d_1) = 2(a_2b_2-c_2d_2). \end{aligned}$$

$$\begin{aligned} 2(bc+ad) &= d_1^2+a_1^2-b_1^2-c_1^2 = 2(b_2c_2+a_2d_2), \\ 2(ca+bd) &= d_1^2-a_1^2+b_1^2-c_1^2 = 2(c_2a_2+b_2d_2), \\ 2(ab+cd) &= d_1^2-a_1^2-b_1^2+c_1^2 = 2(a_2b_2+c_2d_2). \end{aligned}$$

$$\begin{aligned} 2(bc-ad) &= 2(b_1c_1+a_1d_1) = d_2^2+a_2^2-b_2^2-c_2^2, \\ 2(ca-bd) &= 2(c_1a_1+b_1d_1) = d_2^2-a_2^2+b_2^2-c_2^2, \\ 2(ab-cd) &= 2(a_1b_1+c_1d_1) = d_2^2-a_2^2-b_2^2+c_2^2. \end{aligned}$$

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