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MECHANICS AND PROPERTIES OF MATTER

A TEXTBOOK OF PHYSICS

By R. C. BROWN, B.Sc., Ph.D.

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Volume 2 HEAT—contains the text of pp. 277–547

Volume 3 SOUND—contains the text of pp. 549–690

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MECHANICS AND PROPERTIES OF MATTER

by

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With Diagrams

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PREFACE

This book is intended to be a straightforward presentation of the principles of Physics up to approximately the standard required by the G.C.E. (Advanced Level) and Intermediate examinations.

There is a noticeable tendency to include in the "A" Level curriculum (but not so much in the examination itself) more and more topics which were originally regarded as being appropriate to a General Degree course. It seems to me that this practice carries with it the danger of leaving the less brilliant students with an inadequate understanding of the fundamental principles of elementary classical Physics. I have therefore deliberately restricted the scope of this book, but have included what I hope will be helpful discussions of principles. In writing it I have assumed that the student has reached the stage in his education and mental development at which he is prepared to do some reasonably hard thinking. By using the calculus very sparingly I hope I have emphasized that mathematical analysis is not necessarily a substitute for physical reasoning. The significance of the last line of a mathematical treatment of a physical phenomenon depends as much upon the physical principles incorporated in the first line as upon the elegance of the mathematics itself.

I have retained the customary methods of exposition where these have seemed to me to be adequate and sufficiently illuminating, but I have attempted more original modes of presentation of some of those topics which, as an examiner, I have found to be but dimly understood by the average student. It is to be hoped that these efforts will be recognizable to teachers and helpful to students.

My indebtedness to several friends for assistance is gratefully acknowledged. The late Dr. A. H. Ferguson gave valuable advice throughout the writing of the book. Mr. A. C. Stevenson has read a number of the Mechanics chapters and has made helpful suggestions. Dr. J. W. Fox gave similar help with some of the other topics, and Dr. R. E. Jennings has been good enough to provide many of the answers to the numerical questions. The publishers' artist has been very patient in interpreting my own pencil drawings. Figs. 159 and 160 were made with the help of photographs kindly supplied by Messrs Short & Mason, London, although the line drawings are not intended to represent the original instruments in every detail.

I wish to thank the following examining bodies for having given permission for the inclusion of questions from past papers:—

The Delegates of Local Examinations, Oxford (O.).
The University of Cambridge Local Examinations Syndicate (C.).
The University of London (L.).
The Joint Matriculation Board (J.M.B.).

The various examinations are indicated in the text by the above initials in conjunction with letters according to the following scheme:—

Matriculation—M.
School Certificate—S.C.
Higher School Certificate—H.S.
Intermediate—I.
First Medical—Med.

In transcribing examination questions I have usually refrained from modifying the various modes of writing the names of physical units. The reader is thereby enabled to become acquainted with other common methods of expressing units in addition to the "index" convention which is used in the text.

R. C. Brown.

PREFACE TO SECOND EDITION

In revising this volume for a second printing I have included an account of the experimental determination of the constant of gravitation. My thanks are due to Mr. G. Ullyott for having read and constructively criticized the manuscript of this new section. A number of minor alterations have also been made which, it is hoped, are improvements on the original text.

I am indebted to reviewers and correspondents who have made helpful criticisms of the first edition.

R. C. Brown.

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VOLUME I

MECHANICS AND PROPERTIES OF MATTER

Chapter I

THE QUANTITIES USED IN DESCRIBING MOTION

1. INTRODUCTION

The scope of the science of mechanics can be outlined very briefly as follows. It first deals with the motions of bodies and establishes mathematical equations which describe those motions in terms of distance and time. Next a set of principles known as **Newton's laws of motion** are introduced. These correlate all types of motion by making use of the idea of force as the agent which causes bodies to change their direction or speed of motion. This branch of mechanics, which is called **dynamics,** has an offshoot—**statics**—which concerns itself with the conditions under which bodies are in equilibrium. When dynamics and statics are applied to liquids they become **hydrodynamics** and **hydrostatics** respectively.

Length and Time.—We shall begin our study of motion by assuming a familiarity with the ideas of length and time. Length or distance is measured fundamentally by means of a divided scale graduated by comparison with a standard of length which is adopted by international agreement and carefully preserved by the governments of various countries. The passage of time is determined by means of an oscillating mechanical system such as a pendulum. The unit in which time is measured in the great majority of scientific observations and calculations is the **second** and is defined as 1/86,164·100 of a **sidereal day,** which is the interval between two successive transits of a fixed star across any selected meridian.

There are two systems of length units with which we must become familiar. In most scientific work and for all purposes on the continent of Europe the **metre** is used as the standard of length. Its subdivision, the **centimetre,** is the commonest unit of length in physics. In everyday life and also in engineering in Britain the British system of length measure is used. The **yard** is the standard, and when this system is used in mechanics the commonest unit is the **foot.**

2. DISPLACEMENT

Specification of the Change of Position of a Body.—The fundamental quantity appertaining to the motion of a body is its change of position or **displacement.** Suppose that a body changes its position from a point A to a point B (Fig. 1), the points being plotted in the usual way with respect to axes OX and OY. Then the displacement of the body with respect to the axes is described in magnitude and direction by the straight line AB, the sense (whether from A to B or B to A) being indicated by the arrow. The magnitude and direction of the displacement is governed only by the positions of A and B. In specifying the displacement it is immaterial whether the body actually follows the straight path or goes by another quite irregular route.

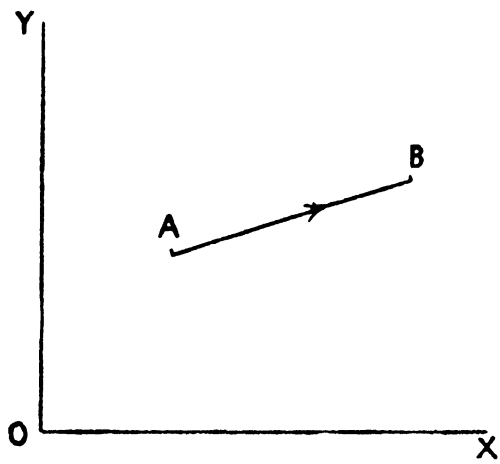


FIG. 1

Evidently the complete specification of the displacement of the body from A involves a statement of both the length and direction of the line AB.

It must be clearly understood that *all* displacements are relative. The displacement from A to B in Fig. 1 is relative to the axes OX and OY because the positions of the points A and B are plotted with respect to these axes regardless of the fact that we do not know how OX and OY have moved relatively to other objects. We are, of course, accustomed to regarding the surface of the earth as a fixed object with respect to which displacements of bodies occur and are specified, but we know that the earth is moving relatively to the sun and that the sun moves relatively to the other stars. Consequently, motion of bodies with respect to the earth is in no sense absolute motion, although we frequently think of it as such, forgetting the existence of the earth's motion.

Vector Quantities.—A quantity such as displacement which requires a statement of direction as well as magnitude is called a **vector quantity.** Other similar quantities in mechanics, such as velocity, acceleration, force and momentum, derive their vector character from the fact that their definition involves displacement either directly or indirectly.

A physical quantity which does not possess a directive quality, and which is therefore completely specified when its magnitude is given, is called **a scalar quantity.** Length, mass, time, temperature and energy are examples.

We represent a vector such as AB by writing $\vec{AB}$, the arrow indicating that the displacement is from A to B and not *vice versa.*

The methods of adding and subtracting vector quantities are more complicated than those which apply to numbers or scalar quantities, and we shall examine them by taking displacement vectors as examples.

The Addition of Displacements.—The sum of two displacements is the single displacement to which they are equivalent when taken together. Thus it has the same kind of meaning as the sum of two numbers, with the distinction that in the case of the vectors the direction of each separate vector influences both the magnitude and direction of the sum. Thus in Fig. 2, if a body suffers a displacement from A to B and then from B to C,

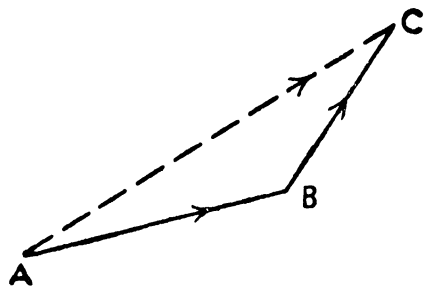


FIG. 2

its total or **resultant** displacement is the vector $\vec{AC}$, and we can write

$$\vec{AB} + \vec{BC} = \vec{AC}$$

Therefore, in order to add two vectors we draw an arrow to represent the first in magnitude and direction, and at its head we place the tail of the arrow which represents the second vector. Their sum is then represented in magnitude and direction by the arrow which joins the tail of the first vector to the head of the second. The sense of $\vec{AC}$, *i.e.* from A to C, and not the reverse, should be carefully noted.

The order in which the vectors are placed on the diagram is immaterial, just as the sequence in which a column of figures is added up does not affect the result. Evidently if we start from the same point A and draw AD equal to BC (Fig. 3) in magnitude and direction (*i.e.* parallel to it) and then draw DC equal to AB we arrive at the same result for the sum. In fact, the sum of two vectors may be regarded as the diagonal of a parallelogram of which the vectors are the adjacent sides, care being taken to choose the correct diagonal. (See Fig. 3.)

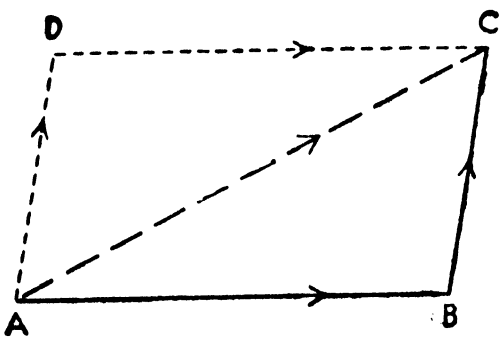


FIG. 3

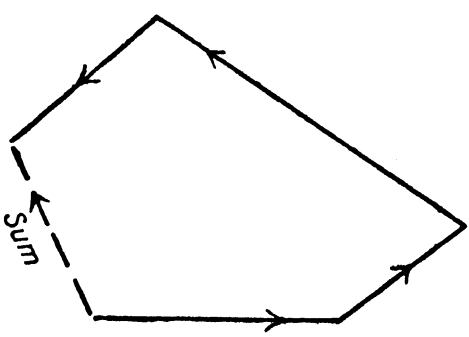


FIG. 4

Vector addition can be extended to more than two vectors by continuing to place arrows, tail to head, and joining the first tail to the last

head to give the resultant (Fig. 4). The sum of a number of vectors is zero if, when the last of the series is drawn, its head coincides with the tail of the first, so that no resultant can be drawn. In this case the separate vectors when placed tail to head form a closed polygon. This is obvious in the case of displacements, because if a body undergoes successive changes of position which finally bring it to its original position its total displacement is zero.

Two equal and opposite displacements, *e.g.* from A to B and then back to A, evidently cancel each other completely, so that

$$\vec{AB} + \vec{BA} = 0$$

or

$$\vec{AB} = -\vec{BA}$$

The sum of any number of given vectors can always be found graphically by making a scale drawing. Where only two vectors are to be added, *e.g.* $\vec{AB}$ and $\vec{BC}$ in Figs. 2 and 3, the sum may be found by the formula which gives the length of AC in terms of AB, BC and the angle between them. Thus

$$AC^2 = AB^2 + BC^2 - 2AB.BC.\cos A\hat{B}C$$

For more than two vectors, successive applications of this equation may be made.

The Subtraction of Displacements.—We have shown on page 3 (Fig. 2) that

$$\vec{AB} + \vec{BC} = \vec{AC}$$

It therefore follows that

$$\vec{AC} - \vec{AB} = \vec{BC}$$

and that

$$\vec{AC} - \vec{BC} = \vec{AB}$$

The method of subtracting two vectors is, therefore, to draw two arrows to represent the vectors, their tails being placed together. The arrow joining their heads then represents their difference. There are two possibilities as to the sense of the difference according to which way round the subtraction is to be made. Thus in Fig. 5 (i) we have

$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$\therefore \vec{AC} - \vec{AB} = \vec{BC}$$

while in Fig. 5 (ii)

$$\vec{AB} = \vec{AC} + \vec{CB}$$

$$\therefore \vec{AB} - \vec{AC} = \vec{CB}$$

The correctness of any particular vector subtraction can be verified in the same way as can a simple numerical subtraction, *i.e.* by remembering

that the result of subtracting, say, x from y is the quantity which when added to x gives y.

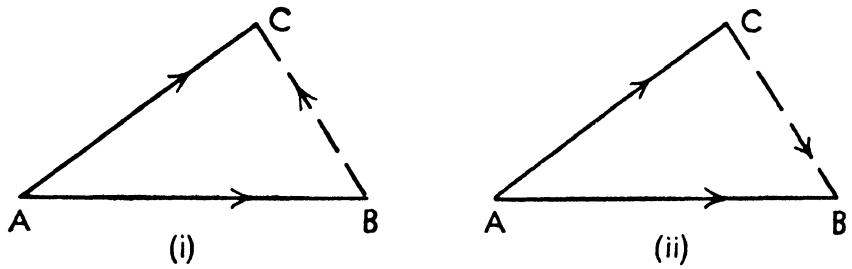


FIG. 5

Example.—*Subtract a displacement of* 5 *miles in a northerly direction from a displacement of* 3 *miles in a north-easterly direction.*

The two vectors are drawn (Fig. 6) in the correct directions with their lengths equal to 5 and 3 convenient units (*e.g.* inches) and their tails together. The dotted vector is then the required difference.

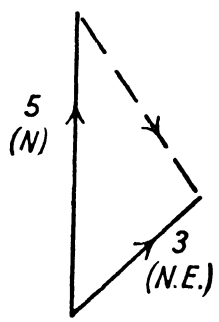


FIG. 6

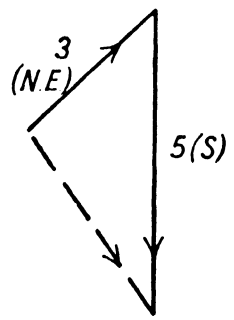


FIG. 7

This can be checked by noting that the sum of 5 miles N. and the dotted vector is equal to 3 miles N.E. The exercise is, of course, the exact equivalent of finding the vector sum of 3 miles N.E. and 5 miles in a *southerly* direction (Fig. 7). Vector subtraction is merely vector addition with a reversal of the sense of the vector which is to be subtracted. This is just the same as in arithmetic and algebra. To subtract a from b we add $-a$ to b.

Relative Displacement.—Examples of the addition and subtraction of vectors occur in cases where a body undergoes two or more simultaneous displacements. For instance, when a man walks on the deck of

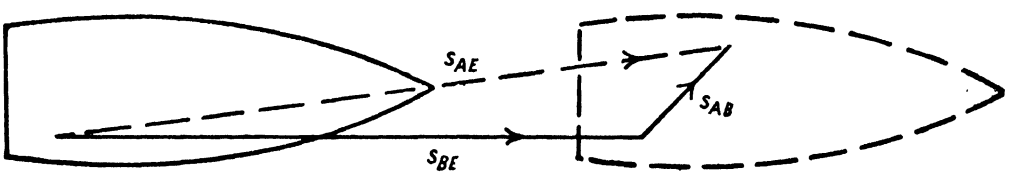


FIG. 8

a moving ship, his total displacement with respect to the earth's surface is equal to the sum of his displacement relative to the ship and the ship's displacement relative to the earth. This is illustrated in Fig. 8.

If we denote the man by the letter A, the ship by B and the earth by E, and represent the displacements of A and B relative to the earth by s_{AE} and s_{BE} respectively, and the displacement of A relative to B by s_{AB}, then we have

$$s_{AE} = s_{AB} + s_{BE}$$

or

$$s_{AB} = s_{AE} - s_{BE}$$

The second method of writing the equation has an important application in connection with relative velocity. It can be expressed in words by the statement that the displacement of A relative to B is equal to the displacement of A relative to the earth *minus* the displacement of B relative to the earth (or *plus* the displacement of B reversed).

Components.—Any two vectors are said to be the **components** of the third vector which is their sum. Thus in Fig. 9 (i) the vectors s_1 and s_2 are the components of the vector s in the directions OX and OY to which they are respectively parallel.

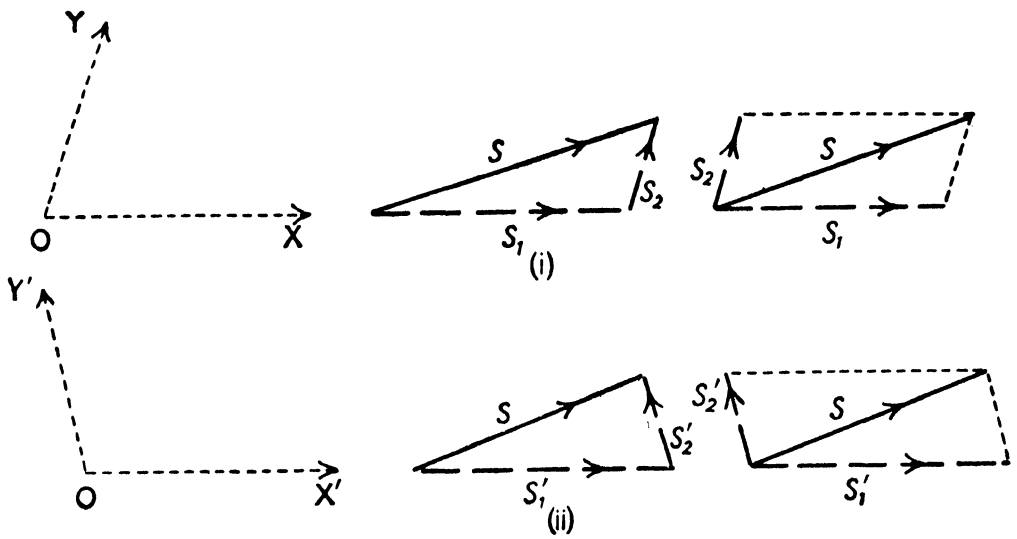


FIG. 9

In Fig. 9 (ii) the components of the same vector s are shown in directions OX′ and OY′. It will be noticed that the components are different from those in Fig. 9 (i). In each case the triangle and parallelogram constructions are both shown. It cannot be too clearly emphasized that the combined effect, or sum, of the two components of a displacement is the exact equivalent of the displacement itself. This principle, which is true for all vector quantities, has its origin in the fact that displacement is defined as the straight line joining two positions of a body regardless of the path actually taken. All paths which join the two positions (*e.g.* s_1 and s_2 in Fig. 9 (i) and s_1' and s_2' in (ii)) are therefore equivalent from the point of view of determining the displacement.

Rectangular Components.—It is frequently very useful to replace a single vector by two components which are respectively parallel to two mutually perpendicular directions. These are called the **rectangular components,** or **resolved parts,** of the vector in the directions chosen.

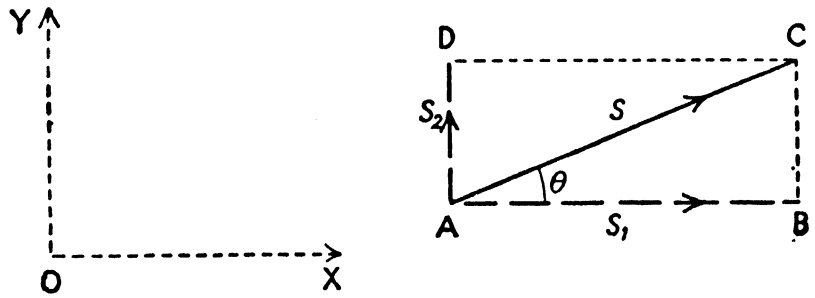


FIG. 10

In Fig. 10 the components of s along the two mutually perpendicular directions OX and OY are found by completing the parallelogram (actually a rectangle) of which s is the diagonal, the adjacent sides being parallel to OX and OY. This is done by dropping perpendiculars CB and CD from C on to lines parallel to OX and OY drawn through A.

By the theorem of Pythagoras,

$$AB^2 + BC^2 = AC^2$$

or
$$AB^2 + AD^2 = AC^2 \qquad \text{since } BC = AD$$

Therefore, since the lengths of the arrows are proportional to the magnitudes of the vectors, the last equation means that

$$s_1^2 + s_2^2 = s^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We also have

$$\frac{s_1}{s} = \frac{AB}{AC} = \cos\theta$$

$$\therefore s_1 = s\cos\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and
$$\frac{s_2}{s} = \frac{AD}{AC} = \frac{BC}{AC} = \sin\theta$$

$$\therefore s_2 = s\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where θ is the angle which OX makes with the direction of s. It is evident, also, that

$$\frac{s_2}{s_1} = \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Noticing that equation (3) can be written

$$s_2 = s\cos(90° - \theta)$$
$$= s\cos D\hat{A}C$$

we can formulate a rule which embodies both equations (2) and (3) as follows: When the rectangular components of a displacement or other vector are to be determined, the magnitude of each is obtained by multiplying the magnitude of the original vector by the cosine of the angle between this vector and the direction of the component required.

Suppose that, of the two rectangular directions chosen for the components of a given vector s, one direction is actually parallel to s and the other is, therefore, perpendicular to s. Then the component of s in its own direction is equal to $s \cos 0°$, which is equal to s, while in the perpendicular direction the component is $s \cos 90°$, which is zero. Thus a vector has no component in a direction perpendicular to itself. This principle is illustrated when we realize that a body which is travelling, say, due north (Fig. 11) is, at the instant when it is at A, neither approaching nor receding from a point B which is due east of A and so lies in a direction perpendicular to the path of the body. When it is at A the body can, however, be said to be approaching C and receding from D—its motion has a positive component in the direction AC and a negative one in the direction AD.

FIG. 11

We have spent some time in discussing the properties of displacement vectors because the other vector quantities with which we shall have to deal later on are derived from them. The important principles to grasp are:

1. Vector addition and subtraction.
2. The meaning of relative displacement.
3. The properties of rectangular components.

3. SPEED

Definition of Speed.—Everyone is familiar with the idea of speed and its measurement in units such as miles per hour, feet per second, cm. per second, and so on. To determine a speed it is clearly necessary to measure a distance and a time.

The **average speed** of a body during a given interval of time is defined as the distance travelled, *measured along the actual path taken*, divided by the time taken to cover that distance. The word "average" is used because the definition does not exclude the possibility of variation in the rate of travel during the time interval. The **actual** or **instantaneous** speed of a body at any instant is defined as the distance which it covers in a *very small* time interval divided by that interval. The smallness of the time is necessary in order to avoid the possibility of variation of speed during the time occupied by the observation.

If Δl is the distance travelled in a time interval Δt, then the average speed during the interval is $\Delta l/\Delta t$, and the instantaneous speed is the limiting value of this quotient when Δt, and therefore Δl, tends to zero; thus:

$$\text{instantaneous speed} = \operatorname{Lim}_{\Delta t \to 0} \frac{\Delta l}{\Delta t}$$
$$= \frac{dl}{dt} \text{ in calculus notation.}$$

It may be noted that the instantaneous speed of a body is the distance which the body would cover in unit time if its speed remained constant at the value which it has at the given instant.

Calculation of the Instantaneous Speed of a Body.—From the practical point of view it would be extremely inaccurate if we attempted to calculate the speed of a body at any instant by measuring how far it travelled in, say, $\frac{1}{100}$th of a second. We therefore use a method which is applicable to all physical determinations of this kind, *i.e.* to all determinations of the instantaneous value of the quotient of the increments of two observed quantities. A starting-point is chosen somewhere near the beginning of the path of the body and the times are noted at which it reaches certain measured distances from this point. We can conveniently, but not necessarily, call the time zero when the body passes through the chosen starting-point. A graph of distance (l) against time (t) is then drawn, which is known as a **distance-time graph.** Suppose it has a shape like that shown in Fig. 12. Let distance and time have values l and t at P, on the graph, and l' and t' at P′ (Fig. 12 (i)). Then the average speed during the interval $t' - t$ is given by

$$\text{average speed} = \frac{l' - l}{t' - t} = \frac{\text{P'Q}}{\text{PQ}}$$

where P′Q and PQ are lines parallel to the axes and are measured according to the scales marked on the axes of distance and time respectively. If P and P′ are joined by a straight line which cuts the axis of time at M, and if PN is perpendicular to the time axis, then triangles PMN and P′PQ are similar, so that

$$\frac{\text{P'Q}}{\text{PQ}} = \frac{\text{PN}}{\text{MN}}$$

and we can therefore write

$$\text{average speed} = \frac{\text{PN}}{\text{MN}}$$

Now suppose that the time interval over which the speed is calculated is made smaller by bringing P′ nearer to P. The quantities P′Q and PQ become smaller, but their ratio is always equal to the ratio PN/MN, the value of which changes, of course, with the change of position of the

point M. Finally let the points P and P′ become coincident. This means that the line P′PM has become a tangent to the curve at P (or P′), and the speed given by the quotient PN/MN has become the **instantaneous** speed corresponding to the point P on the graph, because the time interval has become indefinitely small. In order, therefore, to find the instantaneous speed at any point, a tangent to the distance-time curve is drawn at that

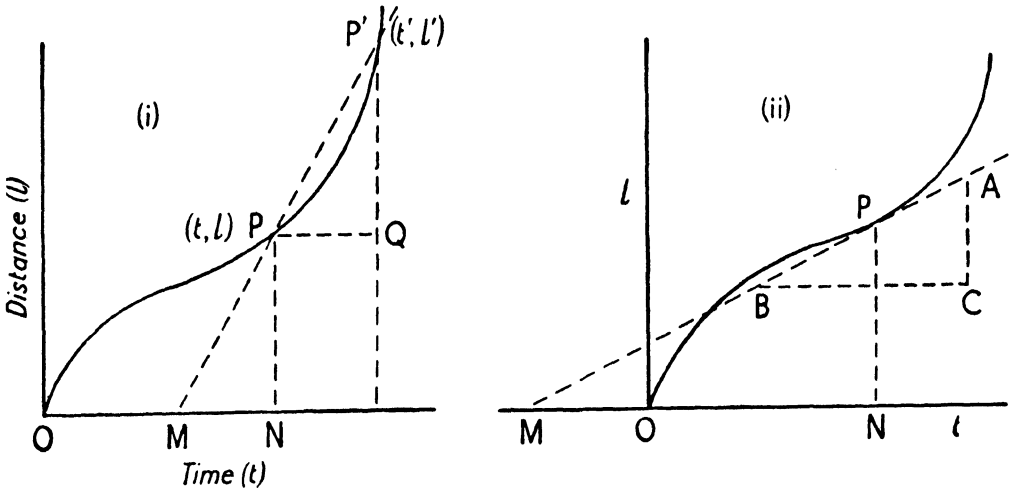


FIG. 12

point and the expression PN/MN evaluated (Fig. 12 (ii)). This may be done directly, not forgetting that the lengths PN and MN must be measured in terms of the units stepped off on the axes to which they are respectively parallel. Alternatively, the ratio of any pair of vertical and horizontal lines such as AC and BC may be evaluated.

The process of finding instantaneous speed is a particular case of calculating what is called the **slope** or gradient of a graph at a given point. It is the graphical equivalent of the process of differentiation in calculus.

Speed-time Graph.—A graph showing how the speed of a body varies with time can be constructed if a series of values of the instantaneous speed is obtained from the distance-time graph by drawing tangents. Let the speed-time graph for a particular case be as shown in Fig. 13. At times t_1 and t_2 the speed is represented by the ordinates AB and CD respectively. Now the area of the figure ACDB is equal to the sum of all the strips like *acdb* into which it can be divided. This is true whether the number of strips taken is large or small. Suppose it is very large so that each individual strip is very narrow. The area of a strip then becomes indistinguishable from that of a rectangle such as *aedb*, because as the strip narrows, the area *ace* diminishes at a greater rate than that of the strip itself on account of the diminution of both *ce and ae.* Therefore, if we make the number of strips very large, the total area ACDB can be taken as being equal to the sum of the areas of all the *rectangles* such as *aedb.* But for a very narrow strip the time interval is so small that the speed does not change appreciably and can be looked upon as being

represented by *ab* throughout the interval. The distance travelled during the small interval is therefore given by the product $ab \times bd$, which is equal

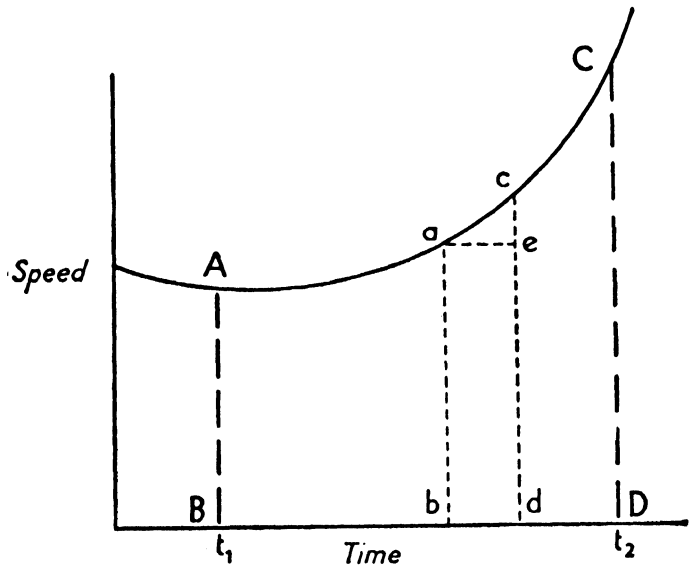


FIG. 13

to the area of the rectangle *aedb*. For an infinite number of strips of zero width (which we are at liberty to imagine) we can write:

Area ACDB = sum of areas of rectangles like *aedb*
= sum of distances travelled during all the successive indefinitely small time intervals between t_1 and t_2
= total distance travelled between times t_1 and t_2

Therefore, in order to calculate the distance travelled by a body from its speed-time graph, we measure the area under the curve between the two ordinates drawn at the beginning and end of the specified time interval. The word "area" here does not, in general, mean the actual area in sq. cm. or sq. in. as measured on the graph paper. Account must be taken of the scales used on the speed and time axes.

The above process of finding distance travelled by means of a speed-time graph is the graphical equivalent of integration in calculus. We write, using calculus notation,

$$\text{distance} = \int_{t_1}^{t_2} \text{speed} \times dt$$

where *dt* represents an indefinitely small time interval. The expression for the distance travelled is therefore the limit of the sum of the products (speed × interval) as the interval tends to zero.

4. VELOCITY

Definition of Velocity.—Suppose a body changes its position from A to B (Fig. 14) so that its displacement is represented in magnitude and

direction by the vector $\vec{AB}$. If the change of position occurs in a time t, then the **average velocity** of the body during the time interval t is defined as AB/t in the direction AB.

The distinction between speed and velocity must be carefully noted. In the first place, AB is the *displacement* of the body and not necessarily the path actually traversed, which might be of the nature indicated by the dotted line. It follows therefore that the magnitude of the average velocity between A and B is different from the average speed which is given by the length of the *path* divided by t. Secondly—and because of the use of displacement in its definition—velocity is a vector quantity requiring the specification of both magnitude and direction, while speed is a scalar quantity.

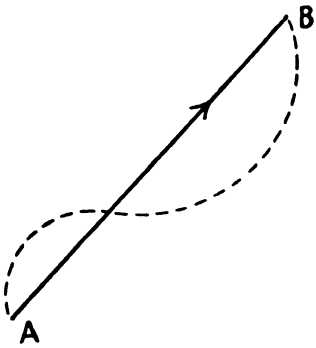


FIG. 14

The actual velocity of a body at any instant—its **instantaneous velocity**—is defined as the displacement which the body undergoes in a very short time interval divided by that time interval. The time is made small (zero in the limit) so as not to allow a change of velocity to occur. If the displacement of a body changes by Δs in a time Δt, then

$$\text{average velocity} = \frac{\Delta s}{\Delta t}$$

and

$$\text{instantaneous velocity} = \operatorname{Lim}_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Evidently the direction of the instantaneous velocity is the same as that in which the body is actually moving at the instant considered, *i.e.* it is tangential to the path of the body at the point in question (Fig. 15).

It may be noted that the instantaneous velocity of a body is the displacement which it would undergo in unit time if it continued to move without a change of speed or direction.

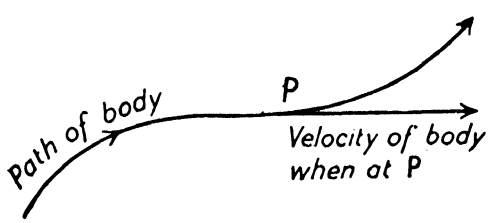


FIG. 15

The magnitude of the instantaneous velocity is necessarily equal to the instantaneous speed, because the very short path described by the body in a short time interval is straight and therefore identical with the displacement.

Since velocity includes both speed and direction, it follows that the velocity of a body is not **uniform,** *i.e.* constant with time, unless the body is moving at a constant speed in a *constant direction*, that is to say, in a straight line. In these circumstances, the velocity at any instant is

the same as the average velocity during any finite interval of time, and the magnitude of the velocity is the same as the speed.

Units of Velocity.—Since the magnitude of a velocity is the quotient of a displacement by a time, its units will be the same as those of speed, *i.e.* ft. per sec., cm. per sec., etc. In this book we shall almost invariably use the index notation, writing cm. sec.$^{-1}$ for cm. per sec. It is useful to remember that 60 m.p.h. is the same as 88 ft. sec.$^{-1}$.

Addition of Velocities.—It is frequently necessary to add and subtract velocities, and these operations are carried out according to the rules for vectors already established for the case of displacements. Velocity is simply displacement occurring in unit time.

It should be realized that when we assign a magnitude and direction to the velocity of a body, we are really expressing its velocity *relative* to something (usually the earth's surface) which is, for this particular purpose, imagined to be stationary. Unless it is stated otherwise, we shall always mean "velocity relative to the earth" when we specify the velocity of a body.

A case of addition of velocities occurs in an example already considered on page 5, namely when a man walks on the deck of a moving ship. Let the velocities of the man and ship with respect to the earth be u_A and u_B respectively, and let the velocity of the man relative to the ship be u_{AB}. Then the velocity of the man relative to the earth is equal to the sum of the velocities which he possesses by virtue of his movement relative to the ship and the movement of the ship relative to the earth. Thus u_A is the vector sum of u_{AB} and u_B, that is,

$$u_A = u_{AB} + u_B$$

and the addition is performed as in Fig. 16, or simply by a triangle. The principle is identical with that used for adding displacements on page 2.

Clearly when u_{AB} and u_B are parallel to each other, the vector summation becomes simple arithmetical addition of magnitudes. When a man walks at 2 m.p.h. (u_{AB}) along the corridor of a train travelling at 60 m.p.h. (u_B), his velocity relative to the earth (u_A) is either 62 or 58 m.p.h., according to whether he is walking towards or away from the engine. A simple example of this kind enables the more general principle of vector addition to be recalled when the velocities are in different directions.

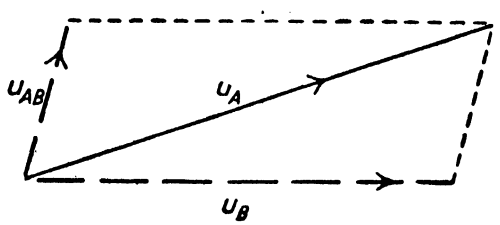


FIG. 16

Relative Velocity.—The following discussion will be considerably clarified if the example of the man on the ship's deck is borne in mind. If two bodies A and B are moving with respect to the same object (*e.g.* the earth) with velocities u_A and u_B respectively, then the velocity of A

relative to B really means the velocity which A would appear to have to an observer who is moving with B. We have already discovered how to determine this since it is the u_{AB} of the previous example. Thus

$$u_A = u_{AB} + u_B$$

so that

$$u_{AB} = u_A - u_B$$

The relative velocity of A to B is found, therefore, by subtracting (vectorially, of course) the velocity of B from that of A, according to the rule laid down on page 4. This is shown in Fig. 17. The result, and particularly the sense of u_{AB}, should always be checked by verifying that the actual velocity of A is equal to its velocity relative to B *plus* the velocity of B.

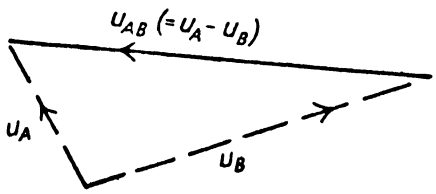


FIG. 17

Evidently the velocity of B relative to A (denoted by u_{BA}) is, by analogy with the last equation, given by

$$u_{BA} = u_B - u_A$$

so that

$$u_{AB} = -u_{BA}$$

A particular and very simple case occurs when u_A and u_B are parallel. Suppose two trains A and B are travelling in parallel directions with speeds of 60 m.p.h. (u_A) and 40 m.p.h. (u_B) respectively. Then, taking the direction of motion as the positive direction, we have

$u_{AB} = u_A - u_B = 60 - 40 = 20$ m.p.h., in the direction of motion of the trains.

$u_{BA} = u_B - u_A = 40 - 60 = -20$ m.p.h., in the direction of motion of the trains.

$= 20$ m.p.h., in the direction opposite to that of the trains.

This is illustrated in Fig. 18.

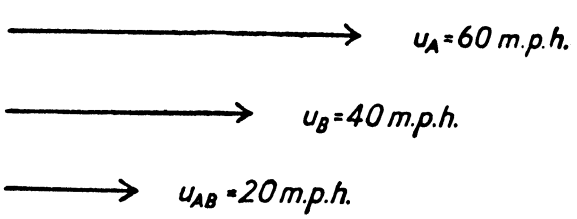


Fig. 19 illustrates the case when B is travelling in the opposite direction to A and the direction of motion of A is taken as positive.

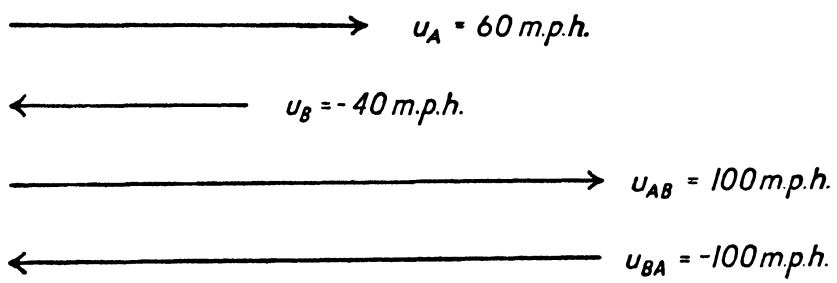


FIG. 19

There are various ways of looking at the question of relative velocity. One common method which may be helpful if the foregoing idea of vector subtraction is found to be a little abstract can be outlined briefly as follows. To find the velocity of A relative to B, imagine a velocity equal and *opposite* to u_B to be superimposed on both A and B. This does not affect their relative velocity, but B is now at rest, so that the actual velocity of A, which is now the resultant or vector sum of u_A and $-u_B$ (note the reversal of direction), is the required velocity of A relative to B. The student should make his own diagram of this method of tackling the problem, and should verify that it is identical with the above method.

Example on Relative Velocity.—*A steamer is moving with a velocity of* 8 *m.p.h. in an easterly direction, and a motor-boat puts out from a quay when the steamer is two miles north-west of it. If the motor-boat can travel with a speed of* 10 *m.p.h., in what direction must it be steered in order to reach the steamer in the minimum of time? Calculate this time.*

If the steamer is to be reached in the smallest possible time, the velocity of the motor-boat *relative to the steamer* must always be directed from the position of the boat to that of the steamer. This relative velocity must therefore have a north-westerly direction throughout the journey since it obviously has this direction initially. Therefore, the actual velocity of the motor-boat must be such that the vector subtraction of the velocity of the steamer from that of the boat shall give a north-westerly velocity. It may be helpful to recall that finding the velocity of the boat relative to the steamer is the equivalent of superimposing on both a velocity equal and opposite to that of the steamer so that this is now effectively at rest.

In Fig. 20 (i) a vector $\vec{AB}$, 8 units long, represents the easterly velocity of the steamer. From its head B a line is drawn running N.W. and S.E. Since in vector subtraction the tails of the two vectors to be subtracted lie together, the tail of the 10-unit vector, representing the velocity of the motor-boat, will lie at A and its head will lie on the N.W.–S.E. line. With centre A and radius 10 units describe two arcs to cut this line at C and C′. Of these two points C is the one required, because the relative velocity must (i) be directed towards the north-west, and (ii) be the result of subtracting $\vec{AB}$ from $\vec{AC}$ and not the other way round. The course of the motor-boat can therefore be found by making the construction and measuring the angle which $\vec{AC}$ makes with the northerly direction, *i.e.* the angle CÂN. Alternatively we can use trigonometry and proceed as follows.

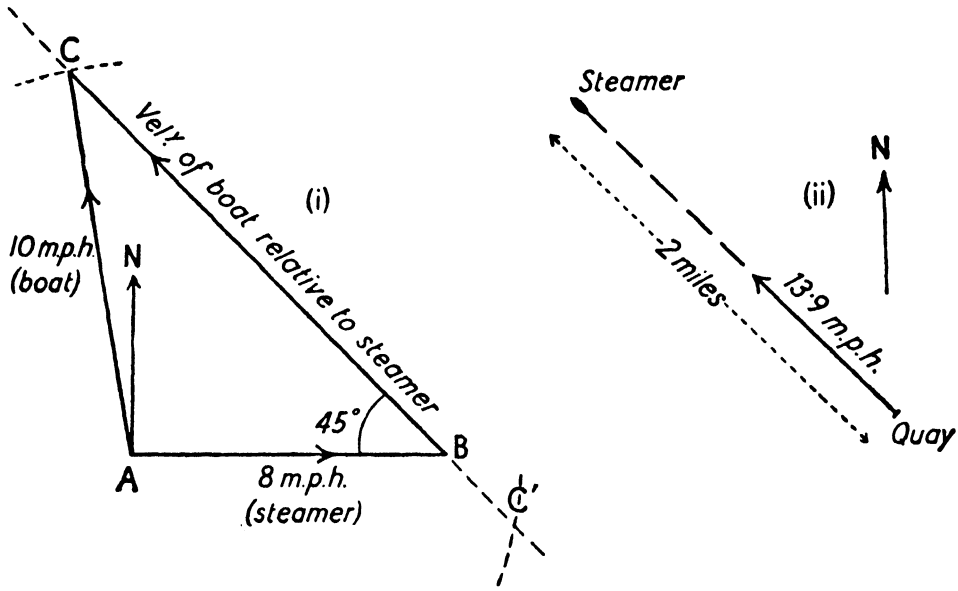


FIG. 20

Since
$$\frac{AC}{\sin A\hat{B}C} = \frac{AB}{\sin A\hat{C}B}$$

therefore
$$\sin A\hat{C}B = \frac{AB}{AC} \times \sin A\hat{B}C$$
$$= \frac{8}{10} \times \sin 45°$$
$$= \frac{8}{10} \times \frac{1}{\sqrt{2}}$$
$$= 0{\cdot}4 \times \sqrt{2}$$
$$= 0{\cdot}5656$$
$$\therefore A\hat{C}B = 34° 26'$$

and
$$C\hat{A}B = 180° - 45° - 34° 26'$$
$$= 100° 34'$$
$$\therefore C\hat{A}N = 100° 34' - 90°$$
$$= 10° 34'$$

The motor-boat must therefore be steered in a direction 10° 34′ west of north.

The magnitude of the relative velocity $\vec{BC}$ is given by
$$BC = \frac{AC}{\sin A\hat{B}C} \times \sin C\hat{A}B$$
$$= 10 \times \sqrt{2} \times \sin (100° 34')$$
$$= 14{\cdot}14 \times 0{\cdot}9830$$
$$= 13{\cdot}9 \text{ m.p.h.}$$

Therefore, if the motor-boat is approaching the steamer with a velocity of

13·9 m.p.h., and the distance it has to cover is 2 miles (Fig. 20 (ii)), the time taken is

$$\frac{2}{13{\cdot}9} \text{ hours}$$

$$= \frac{120}{13{\cdot}9} \text{ minutes}$$

$$= 8 \text{ min. } 38 \text{ sec.}$$

Components of Velocity.—It is often convenient to replace a single velocity by its components in two mutually perpendicular directions. They are calculated by the method explained on page 7. To find the

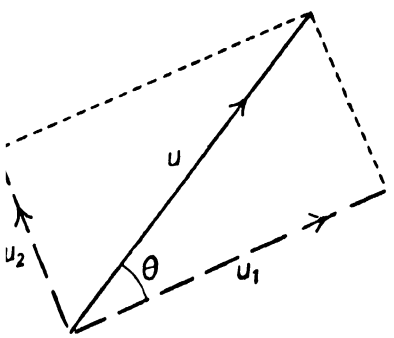
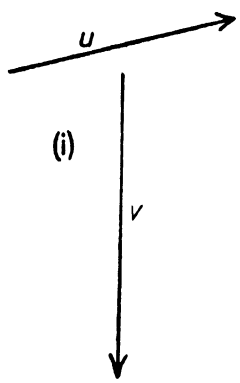


FIG. 21



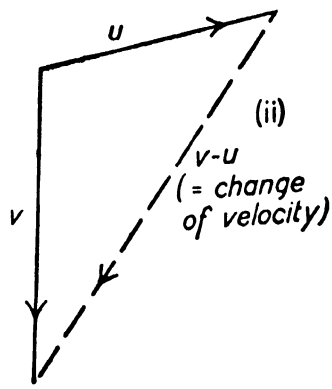


FIG. 22

component of a velocity in any direction, the velocity must be multiplied by the cosine of the angle between the direction of the velocity and the direction of the required component. Thus in Fig. 21

$$u_1 = u \cos \theta$$

$$u_2 = u \cos (90° - \theta)$$

$$= u \sin \theta$$

Change of Velocity.—The change in the velocity of a body is found in exactly the same way as the change of any other quantity, that is to say, by subtracting the initial value from the final value, the subtraction in this case, of course, being performed vectorially. In Fig. 22 (i), the vector u represents the initial velocity and v represents the final velocity. The change of velocity is obtained by subtracting u from v vectorially. This is done in Fig. 22 (ii). The construction can be checked by remembering that the change of any quantity is that which must be added to its initial value in order to give the final value. In the figure, u and $v - u$ when added together give v.

5. ACCELERATION

Definition of Acceleration.—Suppose that in a time t the velocity of a body undergoes a change of magnitude R in a given direction. Then

the magnitude of the **average acceleration** of the body during the interval t is equal to the magnitude R divided by t, and the direction of the acceleration is that of the change of velocity. Acceleration is, therefore, rate of change of velocity and is a vector quantity. The magnitude of the **instantaneous acceleration** of a body is the magnitude of the change of velocity which occurs in a very small time interval divided by that interval, and its direction is that of the change of velocity. The relation of acceleration to velocity is similar to that of velocity to displacement. Thus

$$\text{instantaneous acceleration} = \operatorname{Lim}_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

The instantaneous acceleration of a body is the same thing as the change of velocity which would occur in unit time if the velocity continued to change at the rate prevailing at the instant considered. (Compare the definitions of average and instantaneous velocity.)

Uniform Acceleration occurs when the velocity of a body continues to change at a constant rate in the same direction. In this case, the instantaneous value of the acceleration is always the same and is equal to its average value over any time interval.

In everyday language the word "acceleration" implies an *increase* of speed, and for this reason the word "retardation" is sometimes used in mechanics to denote a rate of decrease of speed. The term is superfluous, however. All vector quantities can be either positive or negative according to whether their sense is with or against the direction which we have arbitarily chosen as positive in any particular example.

It should be clearly understood that acceleration, in the sense of our definition, occurs when a body changes its *direction* even if its speed remains constant. This is of great importance, for instance, in connection with uniform motion in a circle.

Units of Acceleration.—The units in which the magnitude of an acceleration is expressed can be derived as follows. By definition,

$$\begin{aligned} \text{acceleration} &= \text{velocity} \div \text{time} \\ &= (\text{displacement} \div \text{time}) \div \text{time} \\ \therefore \text{Unit of acceleration} &= \text{unit of displacement} \div (\text{unit of time})^2 \\ &= \text{unit of length} \div (\text{unit of time})^2 \end{aligned}$$

Acceleration can therefore be expressed in such units as feet per sec.^{2}, otherwise written as feet per sec. per sec. or feet sec.$^{-2}$. The other units commonly used in physics are cm. per sec.^{2}, which we shall write as cm. sec $^{-2}$

EXAMPLES I

1. Explain how the sum and difference of two vector quantities may be obtained.
If an aircraft flies a distance of 5 miles in a north-easterly direction, what will be its apparent displacement to an observer in another aircraft which simultaneously flies 8 miles due north?

2. A vessel travels 12 miles in a north-easterly direction in 3 hours. The current is flowing due north with a speed of 2 m.p.h. What will the velocity of a floating object drifting with the current appear to be to an observer on the vessel?

3. An aircraft can cruise at 200 m.p.h. in still air. What course must it keep in order to travel south-west when the wind is blowing at 40 m.p.h. from the east? What will be the aircraft's actual speed relative to the ground?

4. Describe how to find (*a*) the sum, (*b*) the difference of two velocities not in the same straight line. Explain the procedure adopted.
A stream is flowing due north at 3 m.p.h. In what direction must a boat be steered through the water in order that its resultant velocity may be 4 m.p.h. to the south-east? What is then the velocity of the boat relative to the water? (L.M.)

5. A vessel A is steaming due north at a speed of 18 m.p.h., and a vessel B is steaming south-east at a speed of 15 m.p.h. Find, graphically or otherwise, the magnitude and direction of the velocity of A relative to B.
If B is initially 30 miles due north of A, find the shortest distance between the two vessels during the subsequent motion. (L.I.)

6. Distinguish between *resultant* and *relative* velocity.
A passenger in a train at rest notices that a wind is blowing from 30° south of west, but when the train is moving due east with a uniform velocity of 30 m.p.h. the wind appears to blow from 30° south of east. Explain how this is possible and find the true velocity of the wind. (L.M.)

7. A ship sails north-west by compass through a tide running at 5 knots, and the captain finds that after 2 hours it has made 4 nautical miles south-west. Determine the direction of the current and the speed of the ship relative to the water. (O.H.S.)

8. Two ships are steaming directly towards a point O on courses inclined at 30°. Their speeds are 16 and 8 knots, and their distances from O at a certain time are 20 and 12 nautical miles respectively. Find their shortest distance apart. (O.H.S.)

9. ABCD is a square, and the lines AB, AC, AD represent vectors in magnitude and direction, the order of the letters determining the sense of the vectors. Show on a clear diagram the line which represents AB + AC − AD. (C.H.S.)

10. A motor-boat with a maximum speed of 8 m.p.h. in still water crosses a river, 1 mile wide and flowing at 2 m.p.h., from a point A on one bank to a point B on the opposite bank and back again to A. Find by geometrical construction the least time the boat can take if B is 3 miles from, and on the upstream side of, A. (C.H.S.)

11. Explain, with the help of a diagram, what is meant by *relative velocity*.
A car, A, is travelling southward at 30 m.p.h., and is one mile north of a cross-road at 12 noon. A second car, B, is travelling eastward at 40 m.p.h., and is two miles west of the cross-road at 12 noon. Calculate (*a*) the velocity of B relative to A, and (*b*) the time at which the cars are nearest together. (L.I.)

12. Define *velocity*, and show how two velocities may be added together when their directions are different.
A car is moving forward on a horizontal road with a velocity v. A, B and C are points on the extreme circumference of one of the tyres. A is vertically below the centre of the wheel, B is vertically above the centre, and C is on the front of the wheel at the same horizontal level as the centre. Write down the magnitudes and directions of the instantaneous velocities of A, B and C with respect to the road. State also the horizontal components of these velocities. (L.I.)

Chapter II

MOTION WITH UNIFORM ACCELERATION

1. FUNDAMENTAL EQUATIONS

Definition of Quantities.—In this chapter we deal with uniformly accelerated motion. It is important not only on account of the principles involved, but also because, apart from the effects of air resistance, it is the type of motion described by bodies under the action of their own weight, *i.e.* the earth's gravitational force upon them.

We deal first with a body which is moving in a straight line so that the magnitude of its average velocity is always identical with its average speed, because the distance travelled along its path is the same as its displacement in that direction. Suppose that when the body is at A (Fig. 23) it has a velocity u and that we begin reckoning time from this

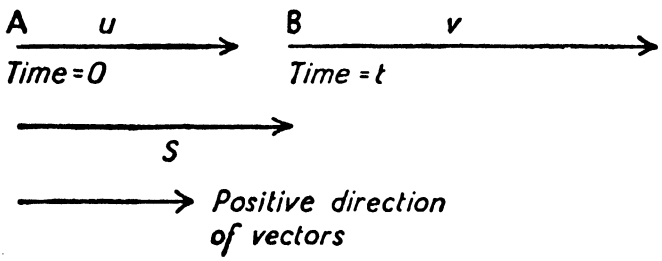


FIG. 23

instant. After an interval of time t, the body is at B and its velocity is now v. The displacement is $\vec{AB}$. Let $\vec{AB} = s$. In the figure, u, v and $\vec{AB}$ have been shown in the same direction, which we shall call the positive direction. Similarly the acceleration is made positive (let it be f) by supposing that v is greater than u. We shall return to the important question of algebraic signs after the equations have been established.

Relation between u, v, f and t.—By definition of acceleration we can write

$$f = \frac{v-u}{t}$$

or $$ft = v - u$$

or $$v = u + ft \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation shows that, since f is constant, the velocity of the body

increases at a constant rate, *viz.* f units of velocity for each unit of time which elapses.

Relation between s, u, f and t.—Since the speed of the body is increasing by equal amounts for equal increments of time, the speed-time

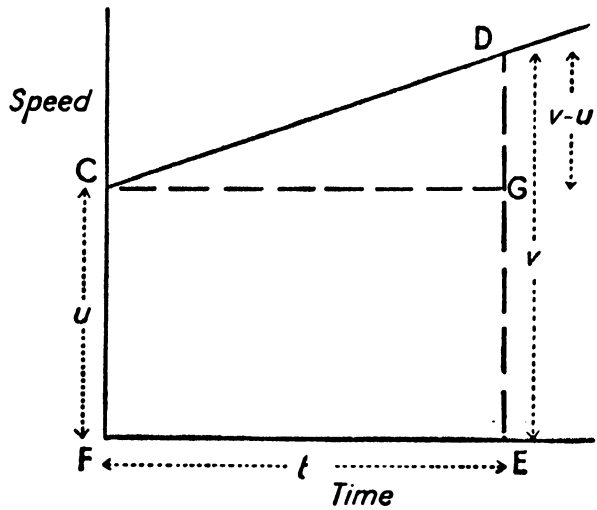


FIG. 24

graph is a straight line such as is shown in Fig. 24. The distance travelled by the body in the time t is equal to the area CDEF (page 11). Thus

$$\begin{aligned} s &= \text{area CDEF} \\ &= \text{rectangle CGEF} + \text{triangle CDG} \\ &= (\text{CF} \times \text{FE}) + \tfrac{1}{2}(\text{DG} \times \text{CG}) \\ &= ut + \tfrac{1}{2}(v-u)t \\ &= ut + \tfrac{1}{2}(ft \times t) \quad \text{since } f = \frac{v-u}{t} \end{aligned}$$

$$\therefore s = ut + \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Average Velocity.—By definition, the average velocity is equal to s/t, which by equation (2) is equal to

$$\begin{aligned} &(ut + \tfrac{1}{2}ft^2) \div t \\ &= u + \tfrac{1}{2}ft \\ &= u + \tfrac{1}{2}\frac{v-u}{t}t \\ &= u + \tfrac{1}{2}(v-u) \\ &= \frac{u+v}{2} \end{aligned}$$

$$\therefore \text{Average velocity} = \frac{u+v}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At first sight it might appear that this result, *viz.* that the average velocity is the arithmetic mean of the initial and final velocities, could have been written down immediately. Equation (3), however, depends on the fact that the acceleration is uniform, and it does not apply to cases of varying acceleration unless the time interval is very small. Equation (3) is sometimes helpful in problems on uniformly accelerated motion.

Relation between u, v, f and s.—We have shown that

$$v = u + ft$$

Squaring, we obtain

$$v^2 = u^2 + 2uft + f^2t^2$$
$$= u^2 + 2f(ut + \tfrac{1}{2}ft^2)$$

Therefore by equation (2)

$$v^2 = u^2 + 2fs \quad . \quad . \quad . \quad . \quad (4)$$

Summary.—For uniformly accelerated motion in a straight line, where f is the acceleration, u the initial velocity, v the final velocity t units of time later, and s is the distance travelled, we have

$$v = u + ft \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = ut + \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Average velocity} = \frac{u+v}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$v^2 = u^2 + 2fs \quad . \quad . \quad . \quad . \quad (4)$$

These equations should be memorized.

Note as to Algebraic Signs.—All the vector quantities have the same direction in the above equations. If any of them is *given* as being in the reverse direction to that which is taken as positive in any particular example, a *minus* sign must be placed in front of its magnitude when it is inserted in equations (1) to (4). On the other hand, the sign of an unknown quantity which is to be found by means of the equations should be left as positive even if it can be readily foreseen that its direction is really opposite to the positive direction. The negative sign will appear in front of the numerical value of this quantity when the equations are solved. For example, if u and v are both in the same direction but v is smaller than u, then equation (1) will automatically give a negative value for f. The time t is, of course, always positive.

When the initial velocity is zero ($u = 0$), the body is said to "start from rest," and we are left with only the second term in each of the equations (1), (2) and (4). Similarly if f is zero only the first terms remain. Thus in the general case when neither u nor f is zero the values of v, s and v^2 are seen to be made up of two parts. The first part is the contribution which the initial velocity makes to these quantities, and the second part is due to the effect of the acceleration. The two effects are superimposed.

Example.—*A body has an initial velocity of* 30 *ft. sec.*$^{-1}$ *and suffers a uniform acceleration in its direction of motion of* 5 *ft. sec.*$^{-2}$. *Find* (a) *the velocity after* 4 *sec.*, (b) *the distance travelled in* 4 *sec.*, (c) *the velocity after the body has travelled* 270 *ft.*, (d) *the distance travelled in the* 9*th second.*

(*a*) $v = u + ft$

$\therefore$ after 4 sec. $v = 30 + (5 \times 4)$

$= 50$ ft. sec.$^{-1}$

(*b*) $s = ut + \frac{1}{2}ft^2$

$\therefore$ after 4 sec. $s = (30 \times 4) + (\frac{1}{2} \times 5 \times 4^2)$

$= 160$ ft.

(*c*) $v^2 = u^2 + 2fs$

$\therefore$ when the body has travelled 270 ft.

$v^2 = 30^2 + (2 \times 5 \times 270)$

$= 3600$

$\therefore$ $v = \sqrt{3600}$

$= 60$ ft. sec.$^{-1}$

(*d*) After 8 sec. the velocity of the body is equal to

$30 + (5 \times 8)$

$= 70$ ft. sec.$^{-1}$

Therefore at the beginning of the 9th second the velocity is 70 ft. sec.$^{-1}$, and the distance travelled in one second from this instant is given by

$$s = ut + \tfrac{1}{2}ft^2$$

where $u = 70$ ft. sec.$^{-1}$ and $t = 1$ sec.

$\therefore$ distance in 9th second $= 70 + (\frac{1}{2} \times 5 \times 1)$

$= 72{\cdot}5$ ft.

2. MOTION DUE TO GRAVITY

The Acceleration due to Gravity.—When a body is allowed to fall freely, its motion is influenced by two factors. Its weight, *i.e.* the gravitational attraction which the earth exerts upon it, gives the body its tendency to fall, while the resistance due to the air, including the buoyancy, opposes this tendency and diminishes the rate of fall. The effect of air resistance and buoyancy is more noticeable in the case of a body like a feather or a piece of tissue paper, because the ratio of the magnitude of the effect to that of the weight of the body is greater than it is for bodies of high density such as stones or pieces of metal.

If bodies are allowed to fall in a vacuum the effect of the air is eliminated, and the classical **guinea and feather experiment** is intended to demonstrate the effect of gravity alone. A coin and a feather are released simultaneously from the top of a long vertical glass tube from which the air has been pumped, and they are observed to fall side by side and to reach the bottom together. When air is present, the coin reaches the bottom first. The principle may also be illustrated by a very simple

experiment as follows. An ordinary tin lid is held horizontally with its rim upwards and a small piece of paper is placed in it. When the lid is dropped, the paper remains in it so that both reach the floor simultaneously. The lid drags down the air above it as it falls so that the paper experiences no buoyancy or air resistance.

This and many other experimental observations lead us to believe that under the action of gravity alone all bodies, whatever their weight, take the same time to reach the ground when released from the same height. This is strictly true only when comparisons are made at the same spot. The time taken does vary slightly from place to place on the earth's surface. The direction of fall of a body which is released can be regarded as being the definition of the vertical direction at the particular place where the fall occurs.

Most people who were previously ignorant of the matter would probably forecast that a heavier body would fall more rapidly than a light one, presumably because of our experience with bodies falling *in air.* The belief that this was true had existed for many centuries (it was part of the philosophy of the Greek thinker Aristotle who lived in the fourth century B.C.) before it was challenged by Galileo at the end of the sixteenth century. The story goes that Galileo, who was professor of mathematics at Pisa, demonstrated the equality of the times of fall by dropping objects of unequal weight from the top of the Leaning Tower. Present-day historians of science, however, can find no evidence of this having taken place. Galileo derived his results from experiments on bodies rolling down an inclined plane. Such motion has all the characteristics of free motion under gravity, but is slowed down so that observations of distance and time can be made more easily.

One of Galileo's results is contained in the statement that *the square of the time of fall is proportional to the height at which the body is released from rest.* Thus if s is the height and t is the time of fall,

$$s \propto t^2$$

or

$$s = kt^2$$

where k is a constant which is independent of s and of the weight of the body.

Now the distance travelled by a body under uniform acceleration is given by

$$s = ut + \tfrac{1}{2}ft^2$$

which, when the body is released from rest ($u = 0$), reduces to

$$s = \tfrac{1}{2}ft^2$$

The last equation shows that the proportionality of s and t^2 is a characteristic of *uniformly* accelerated motion from rest, and we therefore deduce that, since it obeys this relationship, a body falling freely under gravity

suffers a constant acceleration which is found to be the same for all bodies and is directed vertically downwards. This is known as the **acceleration due to gravity,** and it has a value of about 981 cm. sec.$^{-2}$ or 32·2 ft. sec.$^{-2}$. It is given the symbol g.

The acceleration g varies slightly from place to place over the earth's surface, and at any particular place it diminishes with altitude because the attraction due to the earth becomes weaker as the distance from its centre increases. This effect is small, however. An increase of altitude of one mile causes a decrease of g of about one part in 2000.

Motion under Gravity.—Since g is a constant acceleration it can be substituted directly for f in equations (1) to (4) on page 22, and these are then applicable to vertical motion under the action of gravity. In the special case where the body starts from rest ($u=0$) we have

$$v=gt \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$s=\tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\text{average velocity}=\frac{v}{2}$$

and $$v^2=2gs \quad . \quad . \quad . \quad . \quad . \quad (7)$$

all vectors being positive when directed vertically downwards. These equations show that:

(i) the velocity at any instant is proportional to the time which has elapsed since release (equation (5));
(ii) the distance fallen from rest is proportional to the square of the time (equation (6));
(iii) the square of the velocity at any point is proportional to the distance fallen (equation (7)).

If the body is **projected vertically downwards,** u has a positive value and the appropriate equations are simply equations (1) to (4) with g substituted for f.

Body Projected Vertically Upwards.—In this case we should substitute $-u$ for u in the equations if vectors are regarded as positive when they are directed downwards.

Alternatively we can choose the *upward* direction as positive so that u is positive, and f is equal to $-g$. Adopting this alternative, we find that equation (1) becomes

$$v=u-gt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which we can calculate v at any time t after the instant of projection. When the velocity v is zero, we have

$$0=u-gt$$

or $$t=\frac{u}{g}$$

When t is less than u/g in equation (8), v is positive, *i.e.* the body is moving upwards, while if $t > u/g$, v is negative and the body is falling. It has therefore reached its highest point when t is equal to u/g. At this point and time it is momentarily at rest.

The value of s at the highest point is obtained by putting $v = 0$ in the equation

$$v^2 = u^2 - 2gs \quad . \quad . \quad . \quad . \quad (9)$$

which gives $$s = \frac{u^2}{2g}$$

The maximum height reached is therefore $\frac{u^2}{2g}$

We next put $s = 0$ in the equation

$$s = ut - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and obtain $$ut = \tfrac{1}{2}gt^2$$

Thus either $$t = 0$$

or $$t = \frac{2u}{g}$$

The solution $t = 0$ refers to the instant of projection, while the other solution is the time taken for the body to rise and fall to the point from which it was projected. This point is also characterized by $s = 0$, because s refers to *displacement* and not to distance travelled, and this is zero when the body has returned to its original level.

Clearly, the time taken for the body to fall from its maximum height to its original level is equal to the total time, $2u/g$, *minus* the time taken to rise, which we have calculated as u/g. Thus the time of fall is also equal to u/g. The times for rise and fall are equal, therefore.

If we put $s = 0$ in the equation

$$v^2 = u^2 - 2gs$$

we obtain $$v = \pm u$$

The solution $v = +u$ refers to the first occasion on which s is zero, *i.e.* at the moment of projection, while v is equal to $-u$ when the body has returned to its original level. It then has a downward velocity equal to its initial upward velocity. In fact the velocity of the falling body at any point above its starting level is equal and opposite to the velocity which it had at the same height while it was rising.

If the body is projected upwards from a point above the ground so that it continues to fall after having returned to its original level, its subsequent motion is still described by the same equations provided it is remembered that s is the *displacement* from the point of projection (not the actual distance travelled) and t is measured from the instant of

projection. The displacement s will be negative after the body has fallen below its original level.

Clearly, after the maximum height has been reached, the motion of the body is identical with that of a body released from this same height, and it can be worked out on this basis. The above methods are preferable, however, not only on account of their neatness, but also because they illustrate that the fundamental equations apply to all cases of uniformly accelerated motion, no matter whether the body is increasing its speed continuously or whether it reverses its direction during the motion.

The results just obtained are shown in Fig. 25.

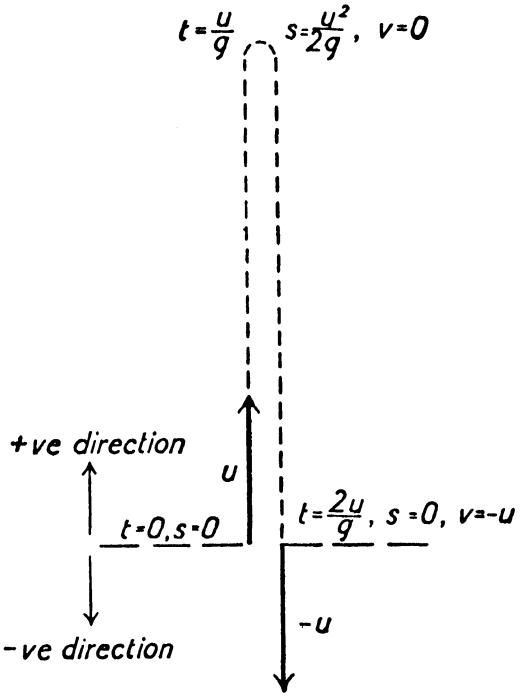


FIG. 25

3. PROJECTILES

Discussion of Principles.—In this section we analyse the motion of a body which is projected with a velocity u which makes an angle θ with the horizontal. Examples are a ball which is thrown, hit or kicked, a shell from a gun and water from the nozzle of a hose.

The foregoing discussion on the *vertical* motion of a body has emphasized the fact that when equations (1) to (4) on page 22 are applied to the motion, s is the vertical *displacement* and not necessarily the actual distance travelled. Now the displacement of a body as it moves from one point to another in its trajectory can be resolved into its vertical and horizontal components, and the rates of change of these are respectively the vertical and horizontal components of the velocity of the body. These two components can be treated quite independently of each other. We know that the acceleration due to gravity governs the variations of the vertical displacement and velocity, and we can apply the equations which have already been established for purely vertical motion regardless of the fact that in the case we are now considering the body also possesses horizontal motion. This horizontal motion is not affected by gravity—in fact the direction of the horizontal at any place is really defined as the direction in which gravity has no accelerative effect.

The fundamental principle of the independence of the horizontal and

vertical motions of a projectile may perhaps be more clearly understood by considering what occurs in a railway train which is moving horizontally with constant velocity. If a passenger throws a ball vertically upwards it returns to his hand just as it would if he were stationary. Its motion *as observed in the train* is exactly the same as when the train is standing still. The equations for vertical motion under gravity are therefore applicable. To an observer outside the train, however, the ball would be seen to undergo a horizontal movement which is unnoticed by the passenger. With respect to the stationary observer, *i.e.* to the earth, the path of the ball is exactly the same as if it had been projected at a certain angle to the vertical by a stationary thrower. The total horizontal distance which the ball covers during the interval between throwing it up and catching it is equal to the distance travelled by the train in this time, since the thrower's hand is not moved with respect to the train. Before, after *and during* its flight, the ball has the same horizontal velocity as the train relative to the earth.

Vertical Motion of the Projectile.—Fig. 26 shows the splitting up of the initial velocity u, making an angle θ with the horizontal, into its vertical and horizontal components. These are u_1 and u_2 respectively, and we know that

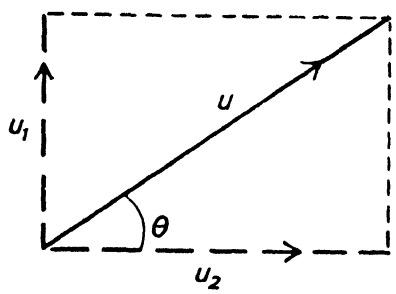


FIG. 26

$$u_1 = u \sin \theta$$

$$u_2 = u \cos \theta$$

and $$u_1/u_2 = \tan \theta$$

Treating the vertical motion first, we can quote the following results from page 26:—

Time for projectile to reach its maximum height $= \dfrac{u_1}{g} = \dfrac{u \sin \theta}{g}$

Time for projectile to fall to its original level from its highest point $= \dfrac{u_1}{g} = \dfrac{u \sin \theta}{g}$

Total time of flight $= \dfrac{2u_1}{g} = \dfrac{2u \sin \theta}{g}$

Maximum height reached = {height at which vertical component of velocity is zero} $= \dfrac{u_1^2}{2g} = \dfrac{u^2 \sin^2 \theta}{2g}$

Horizontal Motion.—At the same time as the projectile undergoes a vertical displacement, its horizontal displacement changes with constant velocity (u_2) because the acceleration due to gravity has no horizontal component. Therefore, at any time t after projection, the horizontal displacement of the body from its starting point is equal to $u_2 t$. The

horizontal range is therefore

$$\begin{aligned} & u_2 \times (\text{total time of flight}) \\ &= u_2 \times \frac{2u_1}{g} \\ &= \frac{2u^2 \sin\theta \cos\theta}{g} \\ &= \frac{u^2 \sin 2\theta}{g} \end{aligned}$$

Thus for a given *speed* of projection the range is proportional to sin 2θ. It is therefore greatest when sin 2θ has its maximum value, which is unity.

If $\sin 2\theta = 1$

then $2\theta = 90°$

and $\theta = 45°$

This is the condition for *maximum horizontal range* for a given speed of projection.

The *instantaneous velocity* of the projectile in the direction of motion at any time t is found by adding vectorially the instantaneous value of its vertical component of velocity which is given by $u_1 - gt$ (equation (8)) to its horizontal velocity which is always u_2. We already know that at the end of its flight its vertical velocity is $-u_1$, so that the magnitude of its actual velocity when it strikes the ground is

$$\begin{aligned} & \sqrt{(-u_1)^2 + u_2^2} \\ &= \sqrt{u_1^2 + u_2^2} \\ &= u \end{aligned}$$

The angle which this velocity makes with the positive horizontal direction is

$$\begin{aligned} & \tan^{-1} \frac{-u_1}{u_2} \\ &= -\left(\tan^{-1} \frac{u_1}{u_2}\right) \\ &= -\theta \end{aligned}$$

Shape of the Trajectory.—Fig. 27 (i) shows many of the results we have established. The trajectory is exactly symmetrical about a vertical line drawn through its highest point. It can be shown that the path is actually a *parabola* as follows.

Choose the highest point as an origin of co-ordinates, O, with axes

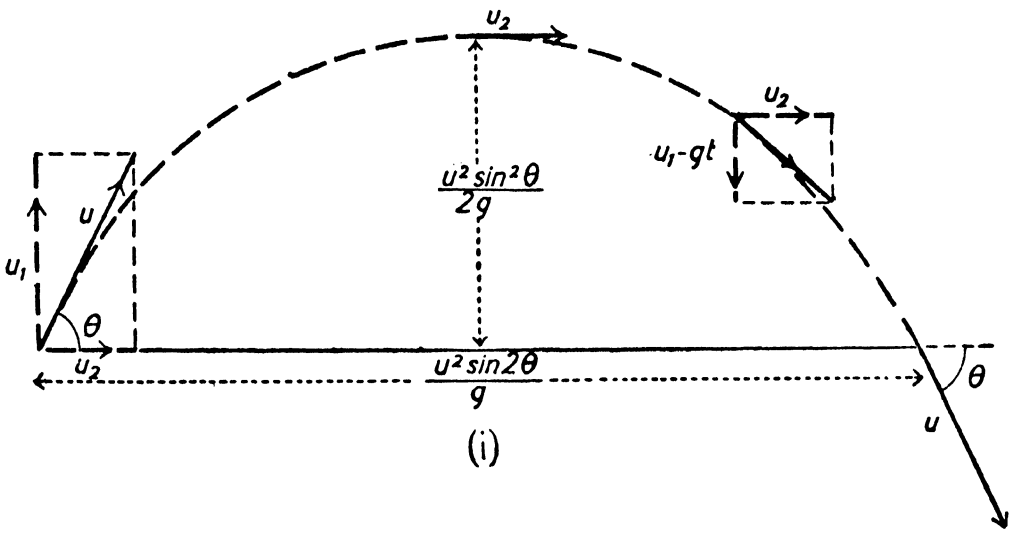


FIG. 27

OX and OY as shown in Fig. 27 (ii). Let the body reach P in t seconds after passing through O. If the co-ordinates of P are (x, y) the body has travelled a horizontal distance x with a constant velocity u_2, so that

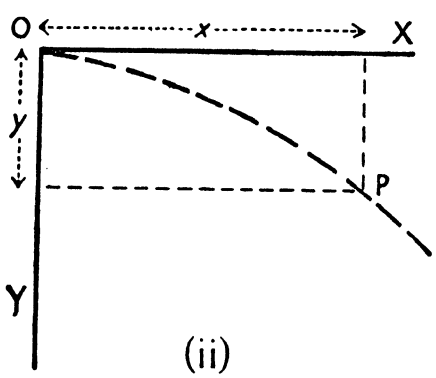


FIG. 27

$$x = u_2 t$$
$$= ut \cos\theta$$

and $$t = \frac{x}{u \cos\theta}$$

In the same time the body has also fallen through a vertical distance y with an acceleration g and zero initial velocity. Therefore

$$y = \tfrac{1}{2} g t^2$$

Substituting for t from the previous equation we get

$$y = \frac{g}{2u^2 \cos^2\theta} x^2$$

or for a given initial velocity

$$y = \text{constant} \times x^2$$

which is the equation of a parabola.

There are many examples of the projection of bodies which involve no other principles than those contained in the foregoing analysis. The point at which the body is brought to rest may be above or below the horizontal level of its point of projection. If a body is projected hori-

zontally from a point above ground level, its trajectory is identical with the second half of Fig. 27 (i). The following example illustrates this.

Example.—*An aeroplane flying horizontally with a speed of* 200 *m.p.h. at a height of* 3600 *ft. over level ground releases a bomb. Calculate* (a) *the time taken for the bomb to fall,* (b) *the distance of the point at which the bomb strikes the ground from the point on the ground vertically below the aeroplane at the instant of release,* (c) *the velocity of the bomb when it strikes the ground.*

In Fig. 28 (i) A is the point from which the bomb is released, *i.e.* projected horizontally with a velocity equal to that of the aeroplane. The point B is 3600 ft. vertically below A, and C is the point at which the bomb strikes the ground.

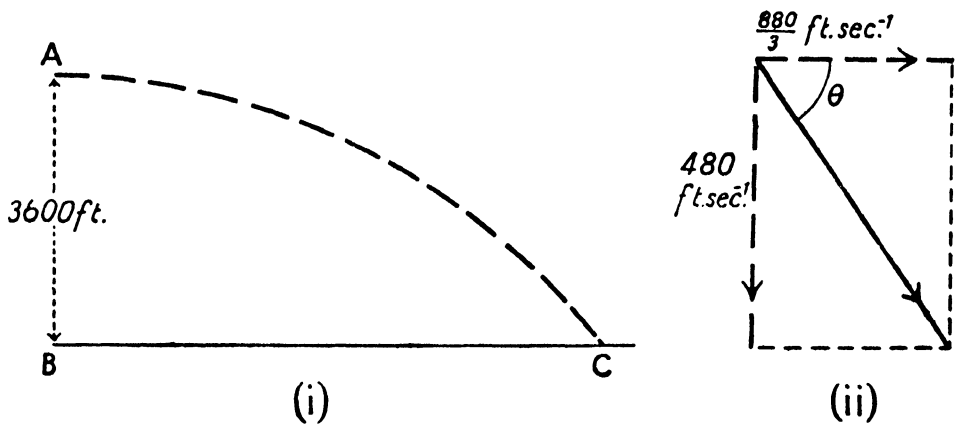


FIG. 28

(*a*) Dealing first with vertical motion we have

$$s = \tfrac{1}{2}gt^2$$

where s is vertical displacement. Therefore

$$t = \sqrt{\frac{2s}{g}}$$

$$= \sqrt{\frac{2 \times 3600}{32}}$$

$$= 15 \text{ sec.}$$

(*b*) The horizontal motion continues with a constant velocity of 200 m.p.h. during these 15 sec. (neglecting air resistance).

$$\text{But } 200 \text{ m.p.h.} = \frac{200 \times 88}{60} \text{ ft. sec.}^{-1}$$

$$\therefore \qquad \text{BC} = \frac{15 \times 200 \times 88}{60} = 4400 \text{ ft.}$$

(*c*) The vertical component of the velocity of the bomb when it strikes the ground is given by

$$v = u + gt$$

where u, the initial vertical component, is zero and t is 15 sec., and the downward direction is taken as positive. Therefore

$$v = 32 \times 15$$

$$= 480 \text{ ft. sec.}^{-1}$$

The horizontal component is 200 m.p.h. at every instant. This, expressed in ft. sec.$^{-1}$, is

$$200 \times \frac{88}{60} \quad \text{or} \quad \frac{880}{3} \text{ ft. sec.}^{-1}$$

Therefore at the time of impact the magnitude of the velocity is equal to

$$\sqrt{480^2 + \left(\frac{880}{3}\right)^2}$$
$$= 564 \text{ ft. sec.}^{-1}$$

The direction of the velocity is given by

$$\tan \theta = 480 \div \frac{880}{3}$$

where θ is the angle which the velocity makes with the horizontal. Thus

$$\tan \theta = 1{\cdot}6364$$
$$\therefore \quad \theta = 58° \, 34'$$

EXAMPLES II

1. Explain the construction and use of a displacement-time graph.

A stone falls vertically from rest. Two seconds later another stone is projected vertically downwards with a velocity of 96 ft. sec.$^{-1}$ from the same point. When and where will the second stone overtake the first? Draw on squared paper a displacement-time graph to represent the motion of each stone. (L.M.)

2. Distinguish between *average* and *instantaneous* velocity.

A train accelerates from rest at a rate of 1 ft. sec.$^{-2}$ for 50 sec.; it continues at a uniform speed for 5 miles and is then brought uniformly to rest in a distance of 250 yards. Find (*a*) the instantaneous velocity of the train when it has travelled 200 yards from its starting-point, (*b*) its maximum velocity, (*c*) the time taken to travel the 250 yards while coming to rest, (*d*) its average velocity for the whole journey. Give the results in ft. sec. units. (L.M.)

3. A lift, starting from rest, moves with uniform acceleration for 6 sec. and with constant velocity for the next 6 sec. It is then uniformly retarded and comes to rest 3 sec. later at a height of 504 ft. above its starting-point. Find the acceleration during the first stage of the motion. (O.H.S., abridged.)

4. Establish the equation $s = ut - \frac{1}{2}gt^2$ for the motion of a body projected upwards: the symbols have their usual meanings.

If $u = 30$ ft. sec.$^{-1}$, $t = 4$ sec., what is s? Interpret the meaning of the negative sign. (L.M.)

5. Define *displacement*, *velocity* and *acceleration.*

If the distance s ft. travelled by a body in t sec. is expressed by the formula $s = 3t + 2t^2$, plot a graph showing how distance varies with time in the first 5 sec. of the motion. From the graph deduce a value for the velocity of the body at the end of 3 sec. (L.M.)

6. A particle is projected vertically upwards with a velocity of 92 ft. sec.$^{-1}$, and 2 sec. later another particle is projected vertically upwards from the same point with a velocity of 68 ft. sec.$^{-1}$. Find the height above the point of projection at which they meet, and find also the time which has elapsed since the projection of the first particle. (L.I.)

7. A particle starting from rest travels distances of 80 ft., 200 ft. and 120 ft. in successive equal intervals of 8 sec. in each of which it is uniformly accelerated. Calculate these accelerations and make a sketch of the velocity-time graph of the whole motion. (C.H.S.)

8. A ball is projected vertically upwards from ground-level with an initial velocity of 40 ft. sec.$^{-1}$. It passes the lower edge of a window-frame with a velocity of 10 ft. sec.$^{-1}$. With what velocity must the ball be projected so as just to reach the window-frame? (L.M.)

9. A cricket-ball was thrown vertically upwards from ground-level and was observed to remain in the air for 4·8 sec. How high did it rise and with what velocity was it thrown? If, instead of rising freely, the ball hit a telegraph-wire 25 ft. from the ground, calculate the time when it struck the wire and the velocity at the time of impact. (L.M.)

10. A body is projected from a point 80 ft. above level ground with a velocity of 128 ft. sec.$^{-1}$ in a direction making an angle of 30° with the horizontal. Calculate (*a*) the maximum height reached, (*b*) the total time taken to reach the ground, (*c*) the horizontal distance travelled. (L.I.)

11. Define *velocity* and *acceleration.*

A body is projected horizontally from the top of a tower with a velocity of 30 ft. sec.$^{-1}$. If it takes 3 seconds to reach the ground, find (*a*) the height of the tower, and (*b*) the distance from the foot of the tower of the point at which the body strikes the ground. (L.I.)

12. Define *uniform acceleration.*

A body is projected with a velocity of 72 ft. sec.$^{-1}$ vertically upwards at the edge of a cliff. Calculate the maximum height which it reaches above the point of projection. If it takes 6 sec. from the instant of projection to reach the beach below, calculate the height of the cliff. (L.I.)

13. A body is projected horizontally with a speed of 30 ft. sec.$^{-1}$ from a tower 64 ft. high. Calculate (*a*) the time taken to reach the ground, (*b*) the distance from the foot of the tower of the point at which the body strikes the ground.

Find the direction in which the body should be projected from the tower in order to reach the ground in 1·5 sec., the initial speed being the same as in the first case. (L.I.)

14. Define *velocity* and *acceleration.*

A body is projected from a point above level ground with a velocity of 64 ft. sec.$^{-1}$ at an angle of 60° above the horizontal and takes 4 sec. to reach the ground. Calculate (*a*) the height of the point of projection above the ground, (*b*) the horizontal distance between the point of projection and the point at which the body strikes the ground, (*c*) the maximum height reached by the body. (L.I.)

15. A particle is projected from the top of a cliff 128 ft. high with a velocity of 140 ft. sec.$^{-1}$ at an elevation of $\tan^{-1}(\frac{4}{3})$. Find the distance from the foot of the cliff at which it strikes the sea, and show that its direction of motion then makes an angle of approximately 59° 45′ with the horizontal. (L.I.)

16. Two vertical posts of height h are distant d apart on a horizontal plane. A particle is projected under gravity from a point in the horizontal plane and just clears the tops of the two posts. If u and v are the horizontal and vertical components of the velocity of projection, show that $4u^2(v^2 - 2gh) = g^2d^2$.

If when the particle is passing the first post its direction of motion is inclined at an angle of $\tan^{-1}\frac{1}{2}$ to the horizontal, show that $u^2 = gd$ and $4v^2 = g(d + 8h)$. (J.M.B.H.S.)

Chapter III

UNIFORM MOTION IN A CIRCLE. SIMPLE HARMONIC MOTION

1. UNIFORM MOTION IN A CIRCLE

Suppose that a small body—or **particle** as it is called if its size is quite negligible—is moving in a circular path with constant *speed* v, so that in each unit of time the particle moves round an arc of length v. Its velocity is changing continuously because its direction of motion is changing. It therefore suffers an acceleration although its speed is constant.

Expression for the Acceleration.—In Fig. 29 (i), the point A marks the position of the particle at a given instant, and B is its position after a

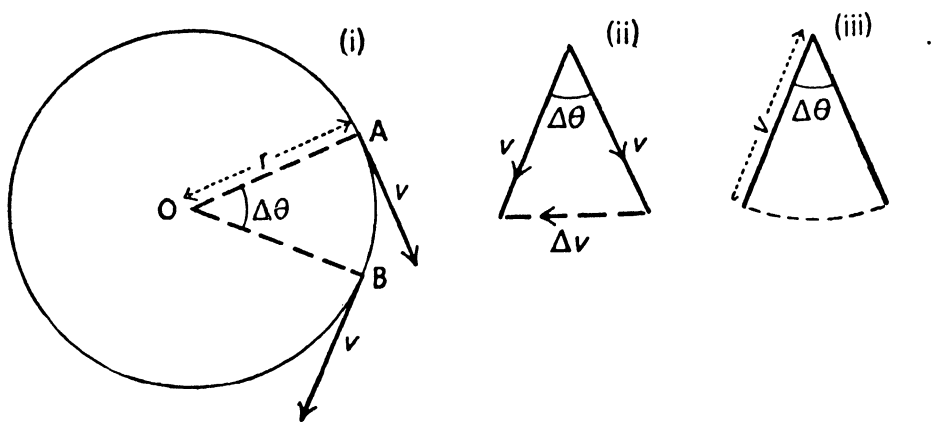


FIG. 29

short interval of time which we denote by the symbol Δt. We have therefore

$$\text{arc AB} = v\Delta t$$

The two vectors of length v drawn tangentially to the circular path at A and B represent the instantaneous velocities at these points.

The change of velocity which occurs between A and B is obtained by subtracting the velocity at A from the velocity at B. The result of this is the velocity Δv shown in Fig. 29 (ii). Since one vector v is perpendicular to OA and the other is perpendicular to OB, the angle between them ($\Delta\theta$ in Fig. 29 (ii)) is equal to the angle $A\hat{O}B$. Thus

$$\Delta\theta = \hat{AOB} = \frac{\text{arc AB}}{\text{radius}} = \frac{v\Delta t}{r}$$

where r is the radius of the circle.

Furthermore, if the time interval Δt is extremely small, $\Delta\theta$ and Δv are very small and the length of the vector Δv is indistinguishable from the length of the circular arc (Fig. 29 (iii)) subtending an angle $\Delta\theta$ at the centre of a circle of radius v. This arc has a length $v\Delta\theta$ so that when Δt is very small, we have

$$\Delta v = v\Delta\theta = \frac{v \cdot v\Delta t}{r}$$
$$= \frac{v^2\Delta t}{r}$$

Therefore, the magnitude of the instantaneous acceleration (f) is equal to the change of velocity divided by the time and is given by

$$f = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2\Delta t}{r\Delta t} = \frac{v^2}{r}$$

Again, when Δt is made very small, Δv in Fig. 29 (ii) also becomes very small, and its direction can then be specified by saying that it is at right angles to either of the vectors v which are themselves tangents to the circle at A and B. Thus when Δt is so small that A and B coincide, the direction of the change of velocity, and therefore the direction of the acceleration, is along AO or BO (the two are coincident) and the acceleration is stated to be always directed *towards the centre* of the circular path.

Angular Velocity.—The rate of motion of the particle round the circle can be specified either by a statement of its linear speed or by giving the **angular velocity** of the radius joining the point to the centre. This is the number of radians through which the radius rotates in unit time. If the angular velocity is ω radians per unit time, then in unit time the particle moves round an arc of length $r\omega$ which must be equal to v (Fig. 30). Therefore

$$\omega = \frac{v}{r}$$

Substituting for v in the expression for f we have

$$f = \frac{v^2}{r} = r\omega^2$$

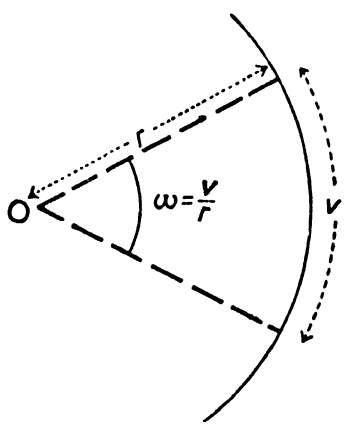


FIG. 30

2. SIMPLE HARMONIC MOTION

Definition.—In Fig. 31 a point Q is moving round the circumference of a circle with constant linear speed v. The straight line AOB is any

diameter of the circle, and QP is perpendicular to AOB. As Q moves round the circle, the foot of the perpendicular oscillates to and fro between A and B and passes through every point between A and B twice for each complete rotation of the line OQ. The motion of P defined in this way is said to be **simple harmonic motion** (S.H.M.).

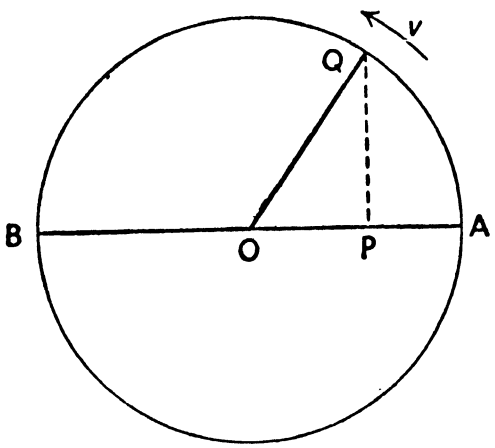


FIG. 31

The **time period** T of the motion is the time interval between two consecutive passages of P through any given point in AOB in the same direction. It is therefore the time in which OQ makes one complete revolution (2π radians), and if ω is the (constant) angular velocity of OQ, we have

$$\omega = \frac{2\pi}{T}$$

The **phase** of the oscillation at any instant is the angle through which OQ has turned since it was last coincident with any arbitrarily chosen radius. For instance, if the radius OA is chosen, the angle $A\hat{O}Q$ in Fig. 31 represents the phase. Clearly this can have any value between 0 and 2π. When it reaches 2π it becomes zero again. The phase can also be represented by the fraction of the time period which has elapsed since OQ was last in the arbitrarily chosen direction.

The **amplitude** of the oscillation is the maximum distance from O which the point P reaches during its motion. In Fig. 31 the length OA (or OB) is the amplitude.

Characteristics of S.H.M.—The displacement of P from O is constantly changing. We shall call the displacement positive when it is in the direction OA, *i.e.* when P lies between O and A, and negative in the opposite direction.

If a time interval t has elapsed since Q passed through A, then

$$Q\hat{O}P = \omega t$$

and therefore

$$OP = OQ \cos Q\hat{O}P$$
$$= OQ \cos \omega t$$

or

$$x = r \cos \omega t$$

where x represents the displacement of P from O and r is the radius of the circle and also the amplitude of the S.H.M. A displacement-time graph of P is therefore identical in shape with the graph of the cosine of an angle against the angle itself. This is shown in Fig. 32, in which times are expressed as fractions of the time period. The displacement

is zero when $t = T/4$, $3T/4$, $5T/4$, etc., because at these times the angle ωt, *i.e.* $2\pi t/T$, is an odd multiple of $\pi/2$ and its cosine is zero.

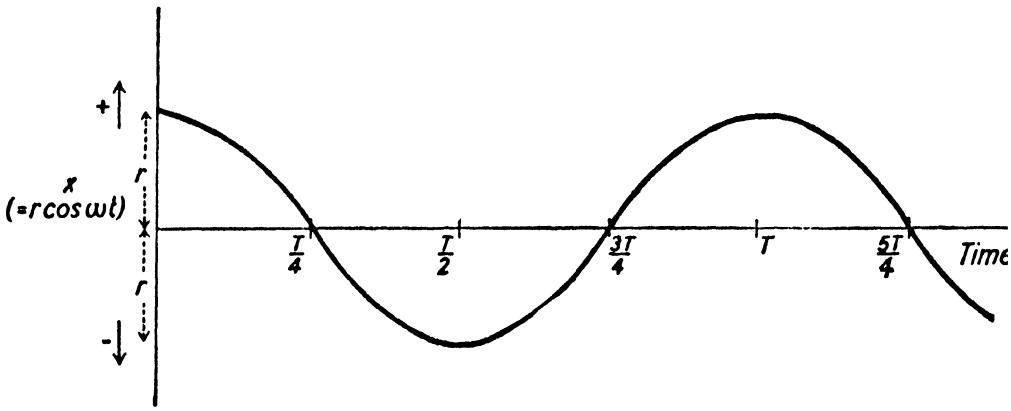


FIG. 32

A little thought will show that the **velocity** of P is changing continuously, being instantaneously zero at A and B and a maximum at O. We can investigate this variation quantitatively by supposing that at any instant the velocity v of the point Q is split up into two components, *viz.* v_1 parallel to the direction of AOB and v_2 at right angles (Fig. 33). The angle between v_2 and v is equal to $A\hat{O}Q$, *i.e.* to ωt, because v_2 and v are

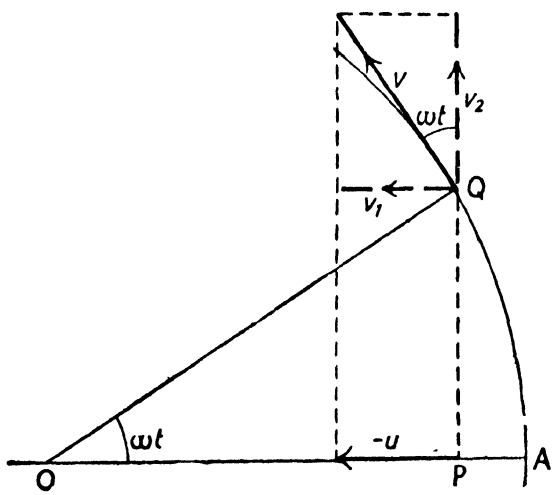


FIG. 33

respectively perpendicular to OA and OQ. Now since the direction of QP is always perpendicular to the fixed direction OA, v_1 is the velocity of P as well as of Q. Therefore if we represent the velocity of P in the direction OA by u, and since at the instant considered v_1 is opposite to the direction OA, we can write

$$u = -v_1 = -v \sin \omega t$$
$$= -r\omega \sin \omega t$$

The velocity-time graph of P is therefore a sine curve, and clearly

$$u = 0 \text{ when } \sin \omega t = 0$$

i.e. when $\omega t = 0, \pi, 2\pi, 3\pi$, etc.

or $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \frac{3\pi}{\omega}$, etc.

or $t = 0, \frac{T}{2}, T, \frac{3T}{2}$, etc.

These times occur when P is at A or B (Fig. 31), *i.e.* at the instants when P is reversing its direction.

On the other hand u has its maximum value (which is equal to v) when $\sin \omega t = 1$, *i.e.* when $t = T/4, 3T/4, 5T/4$, etc., these being instants when P is passing through O.

The **acceleration** of P is investigated in exactly the same way. Since Q is moving round a circle it has an acceleration of v^2/r or $r\omega^2$ directed towards O. In Fig. 34 this acceleration is split up into f_1 parallel to AO

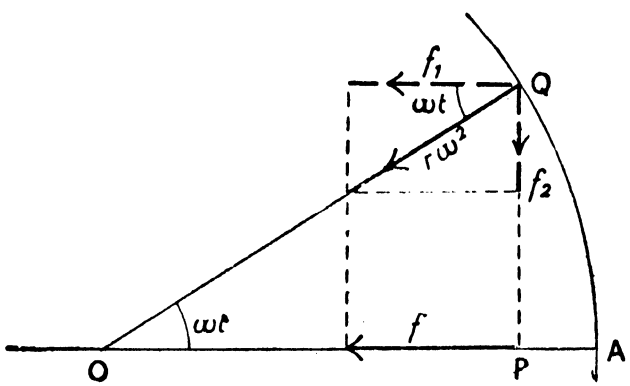


FIG. 34

and f_2 in a perpendicular direction. By the same argument as we used in connection with velocity, the instantaneous acceleration of P is f_1 directed towards O, or if f represents the acceleration of P in the positive direction, *i.e* in the direction OA, we have

$$f = -f_1 = -r\omega^2 \cos \omega t,$$

since the angle between f_1 and f is equal to $A\hat{O}Q$ which is ωt.

But $$x = r \cos \omega t$$

$$\therefore f = -\omega^2 x$$

This last result is very important. It must be noticed that the sign, *i.e.* the sense, of f is always opposite to that of the displacement x, since ω^2 is necessarily a positive quantity. When P lies to the right of O (x positive), the acceleration is negative and therefore directed towards O.

When P is to the left of O (x negative), f is positive and is again directed towards O. Therefore we can state that when a particle is performing S.H.M. it constantly suffers an acceleration towards the centre of its oscillation, and the magnitude of the acceleration is equal to a positive constant multiplied by the distance of the particle from the centre. We use the word "distance" instead of displacement here because in this form of the statement OP must always be considered positive regardless of the position of P. The reversal of sign of f is allowed for by stating that it is always directed "towards O".

The above relation between the acceleration of the particle and its distance from O is known as the **criterion for S.H.M.** Its twofold nature should always be borne in mind—both the direction and the magnitude of the acceleration are involved.

We shall deal with several examples of S.H.M. later. The most important is the motion of a pendulum when its oscillations are small. Other instances are:—a weight hanging from a helical spring, a ball-bearing rolling from side to side in a concave cylindrical or spherical surface, a body hanging from a wire performing torsional oscillations, *i.e.* twisting and untwisting the wire.

EXAMPLES III

1. A particle describes a circular path of radius r at constant speed v. Find the magnitude and direction of the particle's average acceleration during (*a*) one-quarter, (*b*) one-half of a complete revolution.

2. The driving wheel of a locomotive has a radius of 40 in. Calculate the acceleration of a point at the top of the wheel 4 sec. after the locomotive has started from rest with a uniform acceleration of $\frac{1}{2}$ ft. sec.$^{-2}$.

3. A fly-wheel of radius 40 cm. rotates about its centre O at the rate of 120 r.p.m. A point A is situated on the rim while another point B is 30 cm. from O, OB being perpendicular to OA. Calculate (*a*) the velocity, (*b*) the acceleration of B relative to A.

4. What are the maximum values of the velocity and acceleration of the end of the prong of a tuning-fork performing S.H.M. with a frequency of 256 sec.$^{-1}$ if its amplitude is 2 mm.?

5. A particle performs S.H.M. with an amplitude of 10 cm. and a time period of $\frac{1}{2}$ sec. Calculate its displacement, velocity and acceleration $\frac{1}{12}$ sec. after passing through its mean position.

6. Find the smallest fraction of the time period of a simple harmonic oscillation which elapses between the instants when the body is at its central position and when it has a displacement equal to half its maximum displacement.

7. A particle is subjected simultaneously to two mutually perpendicular simple harmonic motions of the same amplitude and time period, their phases differing by 90°. Show that the particle describes a circular path with a uniform speed.

8. A particle is moving in a straight line with simple harmonic motion. When it is at a distance of 3 ft. from O, the centre of oscillation, and is moving away from O, its speed is 8 ft. sec.$^{-1}$ and its acceleration is 12 ft. sec.$^{-2}$. Find the time of a complete oscillation and the amplitude. Show that the particle will next be at O in approximately $1\frac{1}{4}$ sec. (O.H.S., abridged).

Chapter IV

NEWTON'S LAWS OF MOTION

1. THE IDEA OF FORCE

Bodies do not *of themselves* begin to move. We feel convinced of the truth of this statement by our own observations, and we instinctively seek an external cause when we see a body beginning to move. Our minds lean strongly towards the belief that every "effect" has a "cause." Moreover we know that in order to set a body in motion with our own hands it is necessary to make a muscular effort, the intensity of which depends on the amount of motion communicated to the body and also upon some property of the body (to which we shall shortly give the name "mass") by virtue of which it appears to resist our efforts to move it. Here the "cause" of the body's motion seems very real. We are also conscious of the necessity of using our muscles when we change the direction of a moving body as well as when we speed it up or slow it down. When we catch a cricket ball we are perhaps primarily aware of the reaction of the ball on our hands rather than our action upon it, and this is an important aspect of the matter which, together with the other points just mentioned, are dealt with in Newton's three laws of motion.

All the effects mentioned above—changes both of speed and direction—are described by the term "acceleration." When we accelerate a body with our own hands we are conscious of exerting upon it an influence or **force,** to use the mechanical term. However, there are many accelerations which occur without the direct action of our muscles, *e.g.* when bodies collide, when shots are fired by the action of a charge of explosive, when bodies are released or projected and suffer an acceleration of g vertically downwards, when a magnet is brought near to a piece of iron, and so on.

Our system of mechanics is based on the idea that **every acceleration is due to the action of a force,** whether we exert that force with our own muscles or not. So firmly do we believe in this principle that we instinctively search for the origin of the force whenever we observe an accelerated body. Accelerated motion and motion under the action of a force are synonymous expressions, and the qualitative definition of force is contained in the somewhat clumsy phrase "that which produces an acceleration." This statement of the meaning of the word "force" is embodied in Newton's first law, which constitutes the creation of the idea that the nature of the causes of all accelerations is the same.

Newton's First Law of Motion can be written thus: **Every body continues in its state of rest or of uniform motion in a straight line unless it is acted upon by a force.** In other words, if a body is accelerated we say that it is acted upon by a force *by definition* of the term "force." There can be no question of verifying the law experimentally or otherwise, because it is in reality a definition.

It should be noted that the law recognizes the equivalence of a state of rest and a state of motion with uniform velocity. Both are characterized by the absence of acceleration.

The perpetual rectilinear motion of a body which has been given an initial velocity is not met with in practice. Every moving body slows down and eventually comes to rest. Its retardation can always be ascribed to agencies like air resistance and friction between the body itself and the surroundings through which it is moving. Retarding forces of this nature can never be entirely eliminated but can, of course, be reduced in magnitude.

2. THE QUANTITATIVE DEFINITIONS OF MASS AND FORCE

Newton's Second Law can be stated as follows: **When a body is accelerated, the magnitude of the force causing the acceleration is equal to the product of the mass of the body and the magnitude of the acceleration. The direction of the force is the same as that of the acceleration.**

Since force has both magnitude and direction it is a vector quantity.

Thus the second law lays down the method by which magnitudes can be ascribed to the forces which we suppose to be the causes of accelerations. It does so, however, by introducing a hitherto undefined physical property of the body concerned, *viz.* **mass.** It is clearly impossible to define quantitatively both mass and force by means of the single equation which represents the second law, namely

$$\text{force} = \text{mass} \times \text{acceleration}$$

Attempts to do so merely amount to arguing in a circle. For this reason some people are inclined to suggest that we must assume that we have an instinctive knowledge of the definition of mass. But inasmuch as we can determine the mass of a body quantitatively there must be a quantitative definition of this property which, like all other quantitative definitions, contains within it the statement of how the mass is to be measured. Those who define mass as "the quantity of matter in a body" must obviously be using some more exact definition when they actually determine a mass. We conclude, therefore, that there must be some other quantitative statement in which mass or force is concerned and which, together with the second law, will provide us with two simultaneous equations for the

determination of these two hitherto unknown quantities. This further statement is contained in the third law to which we proceed immediately.

Newton's Third Law states that **to every action there is an equal and opposite reaction.** Suppose that a body A acts upon another body B (*e.g.* by colliding with it) with the result that B is given an acceleration. According to the second law, B experiences a force equal to its mass multiplied by its acceleration. The third law states that, while this is occurring, A is also experiencing a force of the same magnitude but opposite in direction.

Once again this is a matter of definition and not of experimental observation (except in so far as we can observe that the accelerations of A and B are opposite in direction, which means that the action and reaction are also opposite). Since the third law is necessary in order to establish a system of measurement for both mass and force, it follows that any experimental proof of it would involve begging the question just as it would in the case of the other two laws.

The Comparison of Masses.—The second law suggests that we could find the ratio of the masses of two bodies by acting on them with the same force. The product (mass × acceleration) would then be the same in each case, so that the ratio of the masses would be determined by taking the inverse ratio of the observed accelerations. But this presupposes that we have some means of knowing when we are acting on the two bodies with the *same* force. The only criterion for the equality of two forces which the second law provides is that they should produce equal accelerations in the *same* mass, and we are concerned here with two different masses. The third law furnishes the necessary criterion in this case by its statement as to the equality of the magnitudes of action and reaction when the two bodies interact.

Suppose that we have the means of projecting two bodies in such a way that they collide with each other, with the result that each undergoes a change of velocity. Suppose also that we can plot their paths and deduce instantaneous values of their velocities and accelerations. Nowadays this can be achieved by high-speed cinematography, but it is not necessary to discuss the experimental method used. All that matters in an argument of this kind is that the experiment should involve the observation of only those physical quantities which have been previously defined independently of the matter in hand.

We shall call the bodies A and B and suppose that f_A and f_B represent two simultaneous values of their respective accelerations. Then the application of Newton's laws to the collision can be set out as follows.

First Law.—Both A and B suffer an acceleration. Therefore each is acted upon by a force.

Second Law.—The force acting on A (P_A) is given by

$$P_A = m_A f_A$$

and that on B is given by

$$P_B = m_B f_B$$

where m_A and m_B are the respective masses of A and B. The physical meaning of mass will be deduced from the discussion which follows. For the time being we suppose only that it is a scalar quantity capable of measurement.

Third Law.—At every instant the force with which A acts on B (P_B) is equal and opposite to the force with which B reacts on A (P_A). Therefore

$$P_A = -P_B$$

so that

$$m_A f_A = -m_B f_B$$

and

$$\frac{m_A}{m_B} = -\frac{f_B}{f_A} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus by combining the three laws (or definitions, as they really are) we have arrived at a method of comparing the masses of two bodies.

In practice, the directions of f_A and f_B are always observed to be opposite so that the accelerations are always opposite in sign and the ratio of m_A to m_B is always positive.

Standards, Units and Definition of Mass.—In order to assign a numerical value to the mass of any particular body it is now only necessary to choose any convenient body to represent a standard of mass in just the same way as we choose a standard of length.

In the British or Foot-Pound-Second (F.P.S.) system the arbitrary standard of mass is the **pound** and the unit is also the pound.

In the Centimetre-Gram-Second (C.G.S.) system the standard is the **kilogram,** which is the mass of a cylinder of 90 per cent. platinum and 10 per cent. iridium (the International Prototype Kilogram) kept in Paris. The unit of mass in the C.G.S. system is the **gram,** which is one-thousandth of the mass of a kilogram.

The *fundamental* method of determining the mass of a body in terms of a chosen standard is, therefore, to perform a collision experiment, such as that indicated above, between the standard and the unknown mass. Then, since the mass of the standard is unity by definition, equation (1) becomes

$$\text{Mass of body} = \frac{\text{instantaneous acceleration of standard during collision}}{\text{simultaneous opposite acceleration of body}}$$

A more general definition of the mass of a body derived directly from the second law is

$$\text{Mass of body} = \frac{\text{acceleration produced when a given force acts on the standard}}{\text{acceleration produced when an equal force acts on the body}}$$

the method of ensuring the equality of the forces not being specified.

In practice the commonest method of determining mass is by weighing, but this is, as we have seen, not the fundamental method. Mass and weight are different properties of matter—bodies would possess mass even in the absence of the earth's gravitation—and some discussion is necessary before the determination of mass by weighing can be justified.

Any confusion in one's mind as to the distinction between mass and weight is often difficult to resolve, but the above discussion of the determination of mass should assist clear thinking. The question of the weights of the two bodies does not enter into the discussion; indeed, the experiment would be easier to perform if, for the time being, the effect of gravity on the bodies (*i.e.* their weights) could be eliminated. Another distinction between mass and weight which emphasizes their dissimilarity is that mass is a scalar quantity having no directional property, while weight, being a force, is a vector quantity.

The Nature of Mass.—Mass is a property which is possessed by all matter, and it is the value of its mass which determines the result of an attempt to accelerate a body. A large mass shows a reluctance to move when acted upon by a force, and when once it is moving it is more difficult to bring to rest than a small mass. The significance of the term **inertia,** which is synonymous with mass, will be appreciated in this connection. Newton's first law is sometimes called the **law of inertia.**

In the Newtonian system of mechanics mass is supposed to be indestructible. For instance the sum of the masses of the products of a chemical reaction is equal to the total mass of the original ingredients. This suggests that the mass of a body is associated with its individual atoms and is unaffected by the order in which these are arranged. In atomic physics, however, there are instances of particles uniting with apparent loss of mass and a release of energy. This is in agreement with one of the forecasts of the theory of relativity. Experiments on nuclear transformation verify that energy is generated at a rate of 9×10^{20} ergs for every gram of mass which disappears.

The mass of a body is found to be independent of its velocity for all practical purposes in mechanics. According to the theory of relativity, however, mass should increase with velocity and would become infinite if the body travelled with the speed of light (3×10^{10} cm. sec.$^{-1}$). In other words it would require an infinite force to bring this about, and the speed of light is therefore regarded as being unattainable by material bodies. This increase of mass at high speeds has actually been detected. For instance, when electrons are given velocities of the order of 10^{10} cm. sec.$^{-1}$ their masses are found to increase quite appreciably.

Absolute Units of Force.—The magnitude of the force acting on an accelerated body is simply the mass of the body multiplied by its acceleration according to the equation

$$P = mf$$

Therefore, when once the mass of a body has been determined by means of the principles set out above, the force acting upon it on any subsequent occasion is easily calculated.

In the F.P.S. system where mass is expressed in pounds and acceleration in ft. sec.$^{-2}$, the units in which the force is expressed are called **poundals.** The poundal may therefore be defined as that force which produces an acceleration of one ft. sec.$^{-2}$ when it cts on a mass of one pound.

In the C.G.S. system (mass in gm. and acceleration in cm. sec.$^{-2}$) the unit of force is the **dyne.** A dyne is that force which produces an acceleration of one cm. sec.$^{-2}$ when it acts on a mass of one gram.

These force units are called *absolute units* to distinguish them from units of force of a different type which will be mentioned later.

3. THE VECTOR PROPERTIES OF FORCES

The Addition of Forces.—It is worth while reminding ourselves of the properties of vectors at this stage, because forces are among the most important vectors encountered in mechanics.

If two or more forces act simultaneously on a body, the magnitude and direction of the single force (*i.e.* the **resultant**) to which they are equivalent is given by the methods of vector addition already explained on page 3. Thus the resultant of P_1 and P_2 in Fig. 35 (i) is given by P in Fig. 35 (ii), which is the diagonal of a parallelogram whose sides represent P_1 and P_2 in magnitude and direction. In Fig. 35 (iii) the resultant is found by the triangle construction.

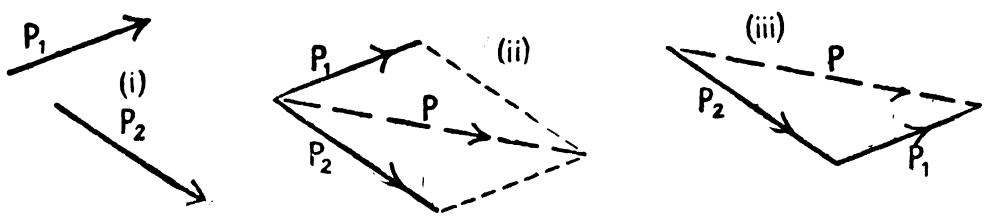


FIG. 35

The resultant of a system of more than two forces can be found by successive applications of the parallelogram of forces, *i.e.* by adding the third force P_3 vectorially to the resultant of P_1 and P_2 and so on, or else

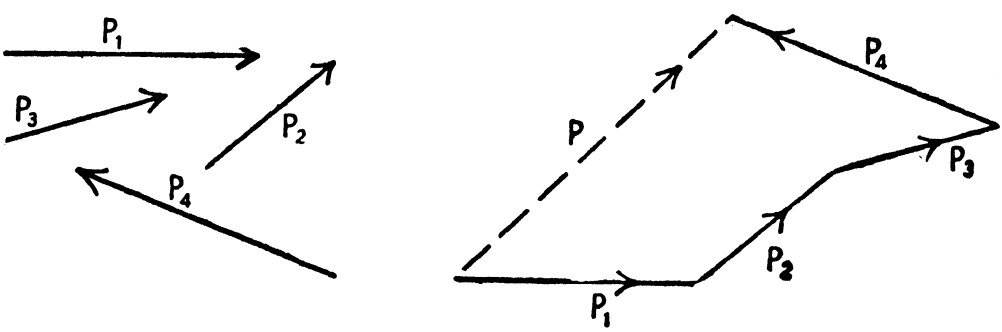


FIG. 36

by a polygon of forces. In Fig. 36 the resultant P of four forces is given by the fifth side of a pentagon. The order in which the vectors are placed on the diagram is immaterial so long as they are always placed tail to head.

The Components of a Force are the separate forces whose vector sum is equal to the original force in magnitude and direction. In particular the two rectangular components of the force P in Fig. 37 are P_1 and P_2

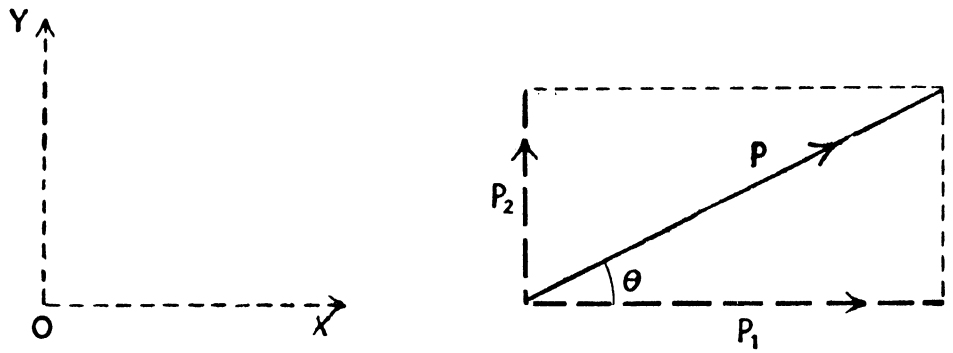


FIG. 37

in the directions OX and OY. As already shown on page 7, if the angle between P and the direction OX is θ, we have

$$P_1 = P \cos \theta$$

$$P_2 = P \sin \theta$$

$$\frac{P_2}{P_1} = \tan \theta$$

and

$$P_1^2 + P_2^2 = P^2$$

It follows from this that a force has no component, *i.e.* it is unable to produce an acceleration, in a direction at right angles to its own direction. The component P_1 is zero if θ is equal to a right angle.

Finding the Resultant of a System of Forces by Components.— Suppose that the rectangular components of each member of a system of forces are found by using equations like $P_1 = P \cos \theta$ and $P_2 = P \sin \theta$, and that we find the sum of all the components in the direction OX and the sum of all the components in the direction OY, remembering to *subtract* any components whose sense is opposite to that which is chosen as positive. Call these two totals $\Sigma P \cos \theta$ and $\Sigma P \sin \theta$. We then have two forces (namely the two totals of the components) at right angles to each other, and these forces are together equivalent to the original system. The resultant of the

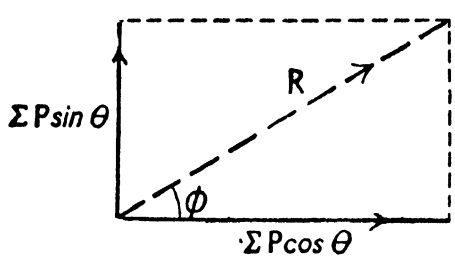


FIG. 38

system (R) must therefore be the resultant of the two totals $\Sigma P \cos\theta$ and $\Sigma P \sin\theta$ and is given by

$$R^2 = (\Sigma P \cos\theta)^2 + (\Sigma P \sin\theta)^2$$

Furthermore if the resultant R makes an angle ϕ with the OX direction we have (Fig. 38)

$$\tan\phi = \frac{\Sigma P \sin\theta}{\Sigma P \cos\theta}$$

4. MOMENTUM

Definition of Momentum.—The momentum of a body is defined as its mass multiplied by its velocity. It is a vector quantity on account of the vector character of velocity, and the direction of the momentum of a body is the direction of its velocity. Thus momentum can be resolved into components in different directions in the same way as all other vectors, such as displacement and force, and when two or more momenta are added together the addition must be performed vectorially.

If a body has a mass m and a velocity u its momentum is mu, and according to Newton's first law this quantity remains constant so long as no force acts on the body. When a force is applied to the body, the momentum will change because of the acceleration, *i.e.* the change in u. It can be taken as an experimental fact that m is constant except in cases where the velocity is so large as to be attainable only by small particles such as atoms and electrons.

Thus we can write

$$\begin{aligned}\text{rate of change of momentum} &= \text{rate of change of } (mu)\\ &= m \times \text{rate of change of } u\\ &= mf\end{aligned}$$

Therefore Newton's second law can be expressed in terms of momentum as follows:—

$$\text{force} = \text{rate of change of momentum}$$

Conservation of Momentum.—When two bodies A and B, which are not acted upon by any other force, interact so as to modify each other's motion, Newton's third law requires that at every instant the rate of change of momentum of A shall be equal and opposite to the rate of change of momentum of B. Therefore the momentum in a given direction which A loses during any given interval of time must be equal to the momentum gained by B in the same direction and during the same time. In other words the total momentum is constant, provided that neither of the bodies is acted upon by an external force other than that due to their mutual interaction. This principle is known as the **conservation of momentum,** and is true for any number of interacting bodies.

In the case of two bodies we can apply the conservation of momentum very simply as follows:—

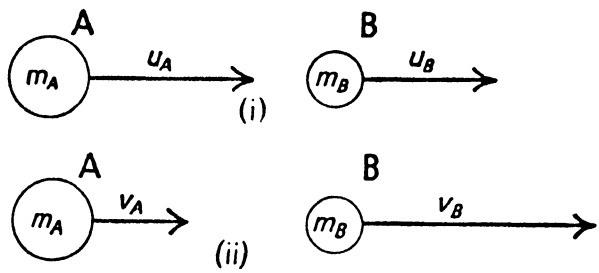


FIG. 39

Fig. 39 (i) shows the conditions at some instant before collision, and (ii) shows the velocities after collision. The principle of the conservation of momentum requires that the total momentum in the common direction of motion shall be unchanged by the collision, so that we can write

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

from which any one of the quantities can be calculated if all the others are given. It should be noticed that in this example all the velocities are in the same direction. If, in any particular case, a velocity is given as being in the opposite direction to that which is arbitrarily chosen as the positive direction, then a *minus* sign must be used when its magnitude is inserted in the equation.

Applications of the principle of the conservation of momentum are discussed in Chapter VIII.

5. GRAVITATIONAL UNITS OF FORCE

Newton's Universal Law of Gravitation states that every particle of matter in the universe attracts every other particle with a force which is proportional to the product of the masses of the two particles and inversely proportional to the square of their distances apart.

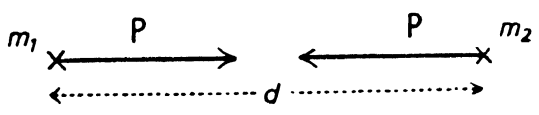


FIG. 40

In Fig. 40, where the force of attraction between two particles is denoted by P, we have

$$P \propto \frac{m_1 m_2}{d^2}$$

or

$$P = G\frac{m_1 m_2}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where G is the **universal constant of gravitation,** whose value is very small indeed (about $6{\cdot}7 \times 10^{-8}$ when the masses are expressed in gm., the distance in cm., and the force in dynes). Clearly G is the force of

attraction in dynes between two particles of mass 1 gm. situated 1 cm. apart.

Equation (2) applies not only to particles of negligible size but also to two hollow or solid spheres of uniform density whose masses are m_1 and m_2. In this case d is the distance between their centres, and they are said to behave as though their masses were concentrated at their centres.

Evidently the determination of the value of G is a matter of considerable difficulty owing to its extremely small value, but it has been measured by using fairly large masses and determining the force of attraction between them by means of sensitive chemical or torsion balances. There are also other less direct means. A fuller account is given in Section 8 of Chapter IX (page 151A).

Gravitation due to the earth.—Suppose that the earth has a mass M and can be considered as a sphere of radius R made up of concentric spherical shells each of uniform density. Then the force of attraction which the earth exerts on a body of mass m situated on its surface (*i.e.* at a distance R from the centre) is equal to

$$G\frac{Mm}{R^2}$$

This force is what we refer to as the **weight** of the body. If g is the acceleration with which the body falls to the earth, then, by Newton's second law, the force of attraction causing the acceleration is equal to mg absolute units. We therefore have

$$mg = G\frac{Mm}{R^2}$$

or

$$g = \frac{GM}{R^2}$$

Thus g is independent of the mass of the body, which fact we have already stated as an experimental observation. The acceleration due to gravity will, however, vary with height above the earth's surface. At a height h it will be equal to

$$\frac{GM}{(R+h)^2}$$

which indicates that in order to reduce g from say 981 to 980 cm. sec.$^{-2}$ (that is to say by about one part in 1000) it is necessary that h should be about $\frac{1}{2000}$th of the radius of the earth, or about 2 miles. Also g varies from place to place over the earth's surface because the earth is not exactly spherical. The distance from the centre of the earth to points on its surface diminishes from the equator to the poles, and this fact, combined with the effect of the earth's rotation which is greatest at the equator, causes the value of g to vary from about 978 cm. sec.$^{-2}$ at the equator to about 983 cm. sec.$^{-2}$ at the north pole.

Although the mass of a body is constant, its weight mg varies when the

body is taken from one place to another on the earth's surface on account of the variation of g. This variation of weight would be detected if the weighing were done by a sensitive spring- or torsion-balance. In fact, this principle has been used to investigate the variation of g over the earth's surface. The variation of weight cannot be detected, however, if the body is "weighed" by a common (or chemical) balance, because the force with which the earth attracts the known weights alters in exactly the same way as the attraction upon the body which is being weighed. Exactly the same weights are required to counterpoise a given body wherever the weighing is performed. In principle, the condition for the equilibrium of the balance can be reduced to the equality of the *weights* of the contents of the pans. If the masses of these contents are m_1 and m_2 respectively, and if g has the same value on both sides of the balance, we have

$$m_1 g = m_2 g$$

or

$$m_1 = m_2$$

Thus weighing by means of a chemical balance really determines the *mass* and not the weight of the unknown body, because the masses of the known weights are constant and independent of the value of g.

Gravitational Units of Force.—Instead of expressing the gravitational force acting on a mass of m gm. as mg dynes, it is frequently convenient to express it as m **gm. weight.** The gm. wt. is called a **gravitational unit of force,** and is defined as the force with which the earth attracts a mass of 1 gm., or, simply, the weight of a mass of 1 gm. Clearly, in accordance with Newton's second law,

1 gm. wt. is equivalent to g dynes

which emphasizes the important fact that the gm. wt. is *not* an absolute unit, since its magnitude varies from place to place according to the variation of g. For this reason the forces involved in the magnitudes of such physical properties of materials as elasticity, surface tension and viscosity are always expressed in absolute units even although they may be actually measured in gravitational units in the first place. The conversion is made by multiplying the value of the force measured in gm. wt. by the value of g in cm. sec.$^{-2}$ *at the place where the determination is made.* Very often in the C.G.S. system it is sufficiently accurate to use 981 cm. sec.$^{-2}$ for the value of g in this conversion, because the accuracy of the determination does not justify the use of the exact value of g at the particular place.

In the F.P.S. system the gravitational unit of force is the **pound weight,** and is the force with which the earth attracts a mass of 1 lb. Thus

1 lb. wt. is equivalent to g poundals

where g is the acceleration due to gravity in ft. sec.$^{-2}$ at the particular

spot in question. It is frequently sufficiently accurate to use 32·2 or even 32 ft. sec.$^{-2}$ in this connection.

EXAMPLES IV

1. Explain how Newton's laws of motion lead to quantitative definitions of mass and force. Distinguish between absolute and gravitational units of force.

2. Discuss whether (*a*) a common balance, (*b*) a spring-balance, determines mass or weight.

3. The coaches of a train weigh 300 tons. The train starts from rest and reaches a speed of 30 m.p.h. in 55 sec. on a level track. Calculate the force in tons weight exerted by the locomotive. Assume that the force is uniform and that the motion of the train is opposed by a frictional force of 10 lb. wt. per ton wt. of the train.

4. A bullet of mass 30 gm. travelling with a horizontal velocity of 200 metres sec.$^{-1}$ penetrates 5 cm. into a quantity of sand. Supposing the acceleration of the bullet to be uniform, calculate its value and that of the retarding force due to the sand.

5. Define *momentum.*

How are a force and the time for which it must act to destroy the momentum of a body related to the momentum?

A projectile is fired into sand and is brought to rest in 0·1 sec. When a similar projectile moving with twice the velocity is fired into loam it is brought to rest in $\frac{1}{8}$ sec. Compare (*a*) the average resistances offered by sand and loam, and (*b*) the penetrations of the projectile in sand and loam. (L.M.)

6. A body of mass 2 lb. slides with an acceleration of 12 ft. sec.$^{-2}$ down a plane inclined at 30° to the horizontal. Calculate the vertical and horizontal components of the force acting on the body.

Chapter V

THE ACTION OF FORCES ON RIGID BODIES

1. ANALYSIS OF THE MOTION OF A BODY

Conception of a Rigid Body.—It is helpful in mechanics to regard the solid bodies with which we have to deal as being composed of a large number of idealized particles each having mass but no size. The body is said to be **rigid** if the relative positions of its particles remain constant however the body changes its position. That is to say, a rigid body can be regarded as a collection of particles joined together by inextensible connecting-rods of no mass.

Linear and Angular Motion.—In this chapter the discussion is limited to motion in one plane only. Suppose that a body of any shape changes its position in one plane from (*a*) to (*c*) in Fig. 41 (i). It is evident that this movement can be regarded as a combination of a movement from (*a*) to (*b*), in which every particle of the body undergoes a linear displacement equal to OO′, together with a rotation of the body through an angle θ from (*b*) to (*c*). The rotation is specified by the angle through which any straight line drawn in the body rotates with respect to a fixed direction. In Fig. 41 (ii) the same change of position is analysed into a *different* linear displacement OO′ from (*a*) to (*b*), but, it should be noted, the *same* rotation or angular displacement θ. In each case the change of position of the body from (*a*) to (*c*) has been specified by stating the angle through which any line in it has rotated together

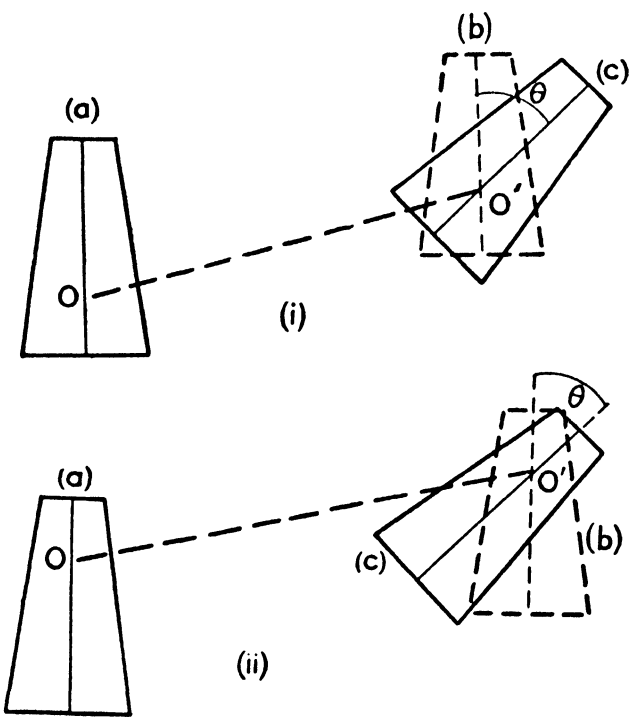


FIG. 41

with the linear displacement of any particular point in the body. Effectively we are treating the body as if it were pivoted at O and representing its change of position from (*a*) to (*c*) as being made up of a movement of the pivot from O to O′ without a rotation of the body, together with a rotation through an angle θ about the pivot. Clearly there is an infinite number of ways of analysing a given change of position in this manner since there is an unlimited choice of points such as O. The total displacement of any point in the body when the latter moves from (*a*) to (*c*) is the vector sum of OO′ and the displacement which the point has by virtue of the rotation from (*b*) to (*c*) about O′.

Angular Velocity and Acceleration.—Suppose that at any instant a body represented by the full outline in Fig. 42 is rotating about the point O; that is to say, the point O is stationary and the body can be regarded as being pivoted at that point. Let OA be any line drawn on the body through O. After a short interval of time Δt its position will become OA′ as a result of the rotation of the body into the dotted position. The **angular displacement** of the body is the angle $A\hat{O}A'$. Actually this angular displacement may be measured by the angle between the initial and final directions of *any* straight line drawn on the body. The instantaneous **angular velocity** of the body (ω) is given by

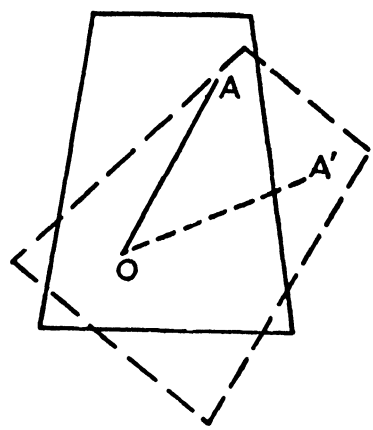


FIG. 42

$$\omega = \operatorname*{Lim}_{\Delta t \to 0} \frac{A\hat{O}A'}{\Delta t} \text{ radians sec.}^{-1}$$

The **angular acceleration** of the body (α) is defined as the rate of change of angular velocity. Thus

$$\alpha = \text{rate of change of } \omega$$

Now if v is the instantaneous linear velocity of A in its direction of motion (*i.e.* at right angles to OA), and if the time is very short,

$$\begin{aligned} v &= \operatorname*{Lim}_{\Delta t \to 0} \frac{AA'}{\Delta t} \\ &= \operatorname*{Lim}_{\Delta t \to 0} \frac{OA \times A\hat{O}A'}{\Delta t} \\ &= OA \times \operatorname*{Lim}_{\Delta t \to 0} \frac{A\hat{O}A'}{\Delta t} \\ &= OA \,.\, \omega \\ &= r\omega \end{aligned}$$

where $r = OA = OA'$

Furthermore if f is the instantaneous linear acceleration of the point A in its direction of motion, then

$$\begin{aligned} f &= \text{rate of change of } v \\ &= \text{rate of change of } r\omega \\ &= r \times \text{rate of change of } \omega, \text{ since } r \text{ is constant} \\ &= r\alpha \end{aligned}$$

Combination of Linear and Angular Acceleration.—The state of accelerated motion of a body can be represented in a manner exactly analogous to the way in which its change of position was represented in Fig. 41. The same diagram will serve if we substitute a linear acceleration f for the linear displacement OO′ and an angular acceleration α for the angular displacement θ. In order to find the actual acceleration of any point in the body it is necessary to add f vectorially to the acceleration which the point has by virtue of α and its position with respect to the point whose actual acceleration is f. This is exactly the same as the method of finding the displacement of any point on the body in Fig. 41, and is best understood by thinking in terms of displacement. We are at liberty to choose any point for the purpose of analysing the acceleration in this way.

2. ANALYSIS OF A SYSTEM OF FORCES

Line of Action of a Force.—The effect which a force has on the motion of a body cannot be completely determined unless in addition to its magnitude and direction, the position of its line of action is stated. Its *point* of application is immaterial when the force is acting on a rigid body, because if a force P (Fig. 43) has a point of application A, we can imagine that equal and opposite forces P_1 and P_2 each equal in magnitude to P are applied at any other point B in the line of action of P. The combined effect of P and P_2 is merely to introduce a tension between the points A and B which is sustained by the body because it is rigid. This has no effect on the motion, which can therefore be regarded as being due to P_1 acting at B.

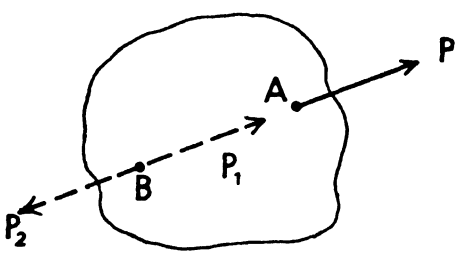


FIG. 43

Rotational Effects of Forces. Moments.—In Fig. 44 a single force P is shown acting on a free body at various points A, B and C in turn. We can guess that when applied at A the force would cause a clockwise angular acceleration, while at C it would cause an anticlockwise angular acceleration, and at some intermediate position such as B it would give the body a purely linear acceleration with no rotation.

In order to deal with the effect of the position of the line of application of a force with respect to the body acted upon, we find it necessary to

introduce a quantity called the **moment of a force** which is defined as follows.

The moment of the force P about a point such as O (Fig. 45) is equal

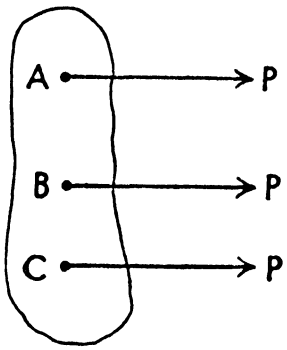


FIG. 44

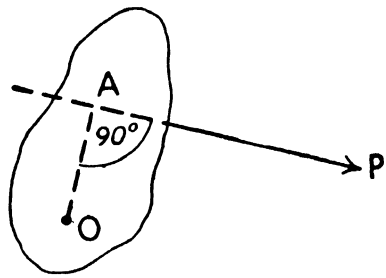


FIG. 45

to the product of the magnitude of P and the perpendicular distance between O and the line of action of the force. Thus

$$\text{moment of } P \text{ about O} = P \times \text{OA}$$

the units being dyne cm., poundal ft., or similar units.

The Resultant of a System of Forces.—Hitherto we have spoken of the resultant of a number of forces as being simply their vector sum.

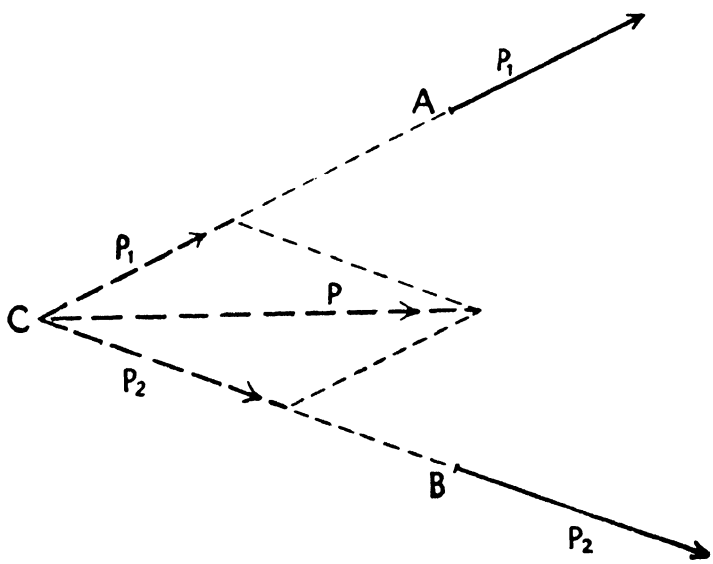


FIG. 46

There is an additional consideration to be taken into account, however. In order that a single force shall be completely equivalent to a system of forces it is necessary that, in addition to its magnitude and direction being obtained by vector summation, its line of action shall be such that its turning effect shall be the same as the combined turning effects of the

separate forces. In the case of a system of only two forces such as P_1 at A and P_2 at B in Fig. 46 it is evidently necessary that their resultant P shall not only be their vector sum as given by the parallelogram construction, but it must also have a line of action which passes through the point of intersection (C) of the lines of action of P_1 and P_2. Otherwise P would have a moment about C while the separate forces P_1 and P_2 do not. By working the matter out fully it can be shown that when the line of action of P passes through C the moment of P about *any* point is equal to the sum of the moments of P_1 and P_2 about that same point. The single force P is therefore exactly equivalent to P_1 and P_2 in every way. This principle can be applied to a system of any number of forces.

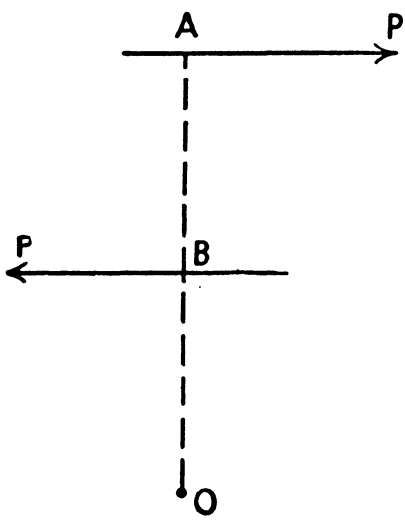


FIG. 47

Couples.—It is possible for a system of forces to have no force resultant while at the same time the sum of the moments of the individual forces about any point is *not* zero. In this case the system cannot be replaced by a *single* force. It occurs when the last stage in the vector addition of the forces consists in adding two equal and opposite forces so as to give zero, but their lines of action are not coincident. Such a pair of forces (Fig. 47) is called a **couple.** The sum of the moments of the two forces P about a point such as O is equal to

$$\begin{aligned}(P \times \mathrm{OA}) - (P \times \mathrm{OB}) &\text{ in a clockwise direction} \\ &= P\,(\mathrm{OA} - \mathrm{OB}) \\ &= P \times \mathrm{AB}\end{aligned}$$

the line ABO being perpendicular to both forces. This moment is the same whatever the position of the point O and it is called the **moment of the couple.** The perpendicular distance (AB) between the lines of action is called the **arm of the couple.**

Reduction of a System of Forces.—To recapitulate, we can say that by the process of vector addition it is possible to find the magnitude and direction of the resultant of a system of forces. Its line of action can be obtained by using the principle that the moment of the resultant about any point must be equal to the sum of the moments of the separate forces about that point. Thus any system of forces can be reduced to a single resultant having a definite magnitude, direction and line of action. We have just seen that in the special case where the magnitude of the resultant is zero, the system reduces, in general, to a couple.

Suppose that a system of forces acting on the body in Fig. 48 has a resultant P with a line of action passing through A as shown. At any other point in the body it is possible to apply a pair of equal and opposite forces without affecting the motion. Let P_1 and P_2, each of them equal in magnitude to P and parallel to the direction of P, be applied at B. The motion which the body would perform under the action of the single force P at A is identical with that which would be caused by the simultaneous action of the three forces. These can be regarded as a single force P_1 (equal to P) acting at B, and a couple consisting of the equal and opposite forces P and P_2. Thus the original system of forces of which P was the resultant has been replaced by a single force (P_1) and a couple. This is an important general principle, the essence of which is as follows. The magnitude, direction and line of action of the resultant of a given system of forces are fixed by the principles of vector addition and moments. We can, however, transfer the line of action of the resultant to any other parallel line provided that we allow for the fact that in its modified position the resultant no longer exerts the same turning effect as the original force system. This is done by introducing a couple. Any system of forces may be replaced by the vector sum of its constituents having any line of action together with a couple. In the particular case where the vector sum of the forces of the original system is zero the system reduces, in general, to a couple only.

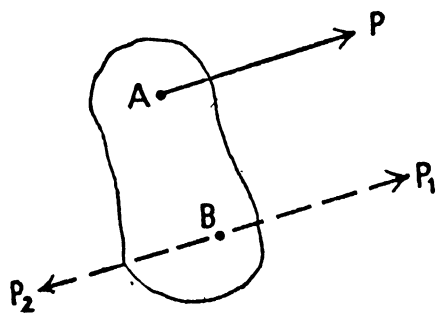


FIG. 48

3. THE ACTION OF FORCES ON BODIES

Fundamental Principles.—Suppose that a system of forces acting on the body in Fig. 49 can be reduced to a single force P whose magnitude, direction and line of action are as shown in the diagram. At any instant let any particle of the body have an acceleration whose component in the direction of P is f. This is equivalent to saying that the particle is being acted upon by a force having a component p in the direction of P, where, by Newton's second law,

$$p = mf$$

if m is the mass of the particle. It is necessary to realize that the force p is, in general, composed of two parts, namely (i) the force which the other particles of the body exert upon the particular particle in question by virtue of the rigidity of the body (the so-called **internal** force), and (ii) any

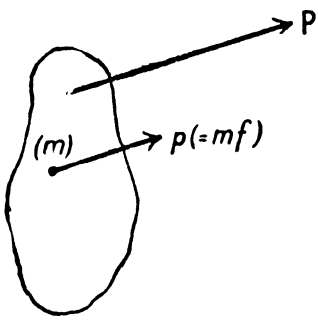


FIG. 49

external force which may be applied to the particle by some outside agent. Now the internal force which any particle A exerts on any other particle B is, by Newton's third law, equal and opposite to the force with which B reacts on A. Therefore if we add together all the force components like p for all the particles of the body, the components of the internal forces cancel each other in pairs, and consequently the result of the summation is equal to the sum of the components of the external forces only. This is P, and we have

$$P = \Sigma p = \Sigma mf \quad . \quad . \quad . \quad . \quad (1)$$

This equation is valid no matter where the line of action of the force P may be.

Newton's Second Law Applied to Rotation about a Fixed Axis. Moment of Inertia.—In order to discover how to deal with the rotational effect of a force we shall discuss the case of a body which is supposed to be pivoted at any point O (Fig. 50). Any particle of the body such as A is instantaneously moving in a direction at right angles to the line joining it to O. If the angular acceleration of the body is α, then the linear acceleration of the particle in the direction of its motion is $r\alpha$, where r denotes the length of OA. If p is the force acting on the particle in a direction at right angles to OA, then

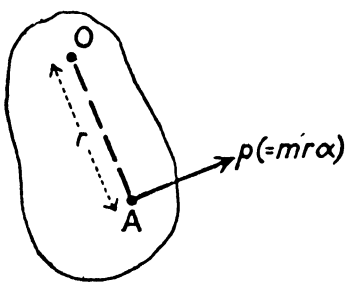


FIG. 50

$$p = mr\alpha$$

where m is the mass of the particle.

The moment of p about O is pr and is given by

$$pr = mr^2\alpha$$

This is the moment of the *actual* force acting on the particle, because the force can always be resolved into two components which are respectively parallel and perpendicular to the direction of OA, and only the latter has any moment about O. The moment pr is made up of the moments due to both internal and external forces, but if we add together the products pr for all the particles, the moments due to the internal forces cancel each other in pairs just as the internal forces themselves do. We are therefore left with the total moment about O of the external forces acting on the body. If we represent this by N_O we have

$$N_O = \Sigma pr = \Sigma mr^2\alpha$$

Since α is the same for all particles, we can write

$$N_O = \alpha\Sigma mr^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The quantity Σmr^2 is a constant property of the body and is known as the **moment of inertia** of the body about O (more precisely about an axis through O perpendicular to the plane of motion). For regularly shaped bodies the moment of inertia about certain axes can be readily calculated. For example, for a cylinder of mass M and radius a the moment of inertia about its geometrical axis is equal to $\frac{Ma^2}{2}$, while for a sphere it is $\frac{2Ma^2}{5}$ about any axis passing through the centre.

If I_O is the appropriate value of Σmr^2 about O for the body in Fig. 50 equation (2) becomes

$$N_O = I_O\alpha \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The similarity between this equation and Newton's law of motion for linear acceleration (from which it is derived) is obvious. In equation (3) the moment of the forces replaces the resultant force, while moment of inertia and angular acceleration replace mass and linear acceleration respectively. It should be noted that the moment of the external forces and the moment of inertia are both taken about the *same* axis, which has been supposed to be *fixed* so that the body is rotating about it.

Centre of Mass.—It can be assumed that associated with every body there is one point which has the property that if the line of action of an external force passes through this point no angular acceleration is produced by that force alone; that is to say, every particle of the body suffers the same linear acceleration. This point is called the **centre of mass** of the body. Later on (page 128) we shall define and discuss the **centre of gravity** of a body which is the point at which the weight of the body can be regarded as acting. Now, it is an experimental fact that when a body is allowed to fall freely it does not, of itself, acquire a rotational motion. Each particle suffers the same vertical acceleration g under the action of a force mg. Therefore the force which we regard as responsible for the motion of the body, namely its weight, must be acting at the point which we have just defined as the centre of mass because the weight causes no rotation. Thus it follows that the centre of mass and the centre of gravity of a body are one and the same point.

If a body is acted upon by a force P whose line of action passes through the centre of mass G of the body (Fig. 51), we can apply equation (1), *viz.*

$$P = \Sigma mf$$

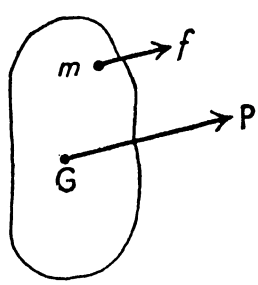


FIG. 51

In this particular case f is the common component of the acceleration of *every* particle in the direction of P, so that it can be placed before the

summation sign, and we have

$$P = f\Sigma m$$

or

$$P = Mf \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where M is the mass of the body, *i.e.* the sum of the masses of its individual particles.

The force P has no component in a direction at right angles to itself. Therefore, when equation (4) is applied to this direction, we find that the component of the acceleration of the body in this direction is zero. This means that f, which was originally defined as the component of the acceleration of every particle in the direction of P, is in fact the *actual* acceleration of every particle and therefore of the body itself.

Thus for a force acting at the centre of mass, the calculation of the acceleration is the same as if we regarded the body as a particle of mass M.

General Solution of the Motion of a Rigid Body.—We are already equipped with the means of calculating the angular acceleration of a body which is pivoted (equation (3)), and also of finding the linear acceleration of a body which is entirely free and has a force applied at its centre of mass (equation (4)). The problem now facing us is to devise a method of finding the state of acceleration of a rigid body when it is acted upon by *any* given system of forces. We consider first how equation (3) can be applied to the general case of a body which is not pivoted.

In general, if any one point on a moving body were to be brought to rest by introducing a pivot at that point, the motion of the body would be modified, because the point in question which was previously being accelerated would now have no acceleration. Or, to look at it in another way, the pivot would exert on the body an additional force which previously did not exist. However, if the pivot were introduced *at the centre of mass*, the force due to the pivot would not modify the *angular* acceleration of the body, because a force applied at this point produces no angular acceleration of itself. Therefore, in dealing with angular acceleration, it is always possible to regard the centre of mass of a body as a pivoted point, even if in fact it is not. This principle enables us to calculate by means of equation (3) the angular acceleration produced by any system of forces acting on a body. Thus if N_G is the moment of the external forces about G and I_G is the moment of inertia of the body about G, we have

$$N_G = I_G\alpha \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and α can be calculated when we know N_G and I_G. The fact that an axis through G can be treated as a fixed axis for the purpose of calculating α is sometimes referred to as the principle of the **independence of translation and rotation.**

Suppose that the state of motion of a body under the action of a system

of forces is such that it can be represented by a linear acceleration f of the centre of mass together with an angular acceleration α. We proceed now to express this motion in terms of forces as follows.

The linear acceleration can be regarded as being due to the action of a force of magnitude Mf applied at G in the direction of f. Furthermore, as we have just seen, the angular acceleration α must be caused by a force or forces whose moment about G is equal to $I_G\alpha$, where I_G is the moment of inertia of the body about G.

The actual forces acting on the body must, in common with every system of forces, be reducible to a single force having a definite magnitude, direction and line of action. Let this force be P acting at A (Fig 52). We have to find the magnitude, direction and line of action (*i.e.* the distance r) of the force P in terms of M, I_G, f and α. Let two opposite forces, P_1 and P_2, be applied at G, each being equal in magnitude to P and parallel to its direction. This does not affect the motion of the body. The original system of forces of which P is the resultant is now replaced by the single force P_1 at G and the couple consisting of P and P_2. If we suppose that P_1 is equal to Mf, *i.e.*

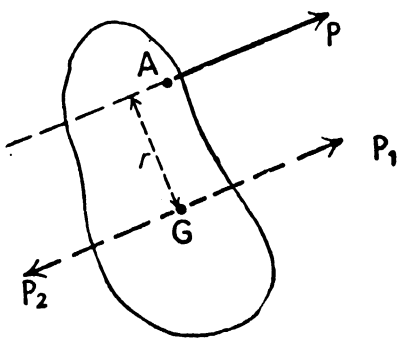


FIG. 52

$$P_1 = P = Mf \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then P_1 at G has the correct magnitude, direction and point of application for producing the linear acceleration f, but it will cause no angular acceleration. It follows therefore that α, which is the remaining element of the body's state of acceleration, must be attributable to the couple consisting of P and P_2, since this is the remaining constituent of the applied force system. Thus, using the principle that angular acceleration can always be calculated by taking moments of the applied forces about G, we have

$$Pr = I_G\alpha \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore P_1, P_2 and P, taken together, are responsible for the linear and angular acceleration of the body. Since the application of P_1 and P_2 had no resultant effect, it follows that the single force P, which is given by equations (6) and (7), is actually responsible for the motion. This last force may, of course, be the resultant of a system of forces.

Reversing the above process of reasoning, we can say that when we are given the magnitudes, directions and lines of action of a number of forces acting on a body we can calculate their effect as follows:—

(i) The direction of the linear acceleration of the centre of mass is given by the direction of the resultant of the forces.

(ii) The magnitude of the linear acceleration of the centre of mass is given by equation (6), where P is the magnitude of the resultant of the forces, *i.e.* their vector sum.

(iii) The magnitude and sense (clockwise or anticlockwise) of the angular acceleration is given by equation (7), where Pr represents the moment of the resultant about G, or, what is equivalent, the algebraic sum of the moments of the separate forces about G.

Calculations (i) and (ii) may be performed somewhat differently by finding the algebraic sum of the components of the forces in two convenient perpendicular directions and so, by dividing by M, finding the components of the acceleration of G in these directions. By adding these two components vectorially the magnitude and direction of the linear acceleration of G can be found.

It follows from the above that if a body is acted upon by a couple only, its centre of mass acquires ***no*** linear acceleration because the resultant of the forces is zero. A couple, therefore, produces angular acceleration and no linear acceleration of G.

Summary.—It is useful at this stage to summarize the argument by which the fundamental equations of dynamics are arrived at. We can do so as follows:—

1. The state of acceleration of a body can be completely specified by stating the angular acceleration of the body (*i.e.* of a line drawn on it) together with the actual linear acceleration of a chosen point.

2. Every system of forces can be replaced by a single force which represents their vector sum in magnitude and direction and has a line of action which is such that its moment about any point is equal to the sum of the moments of the separate forces about that point. It is possible to assign to the resultant any other line of action which is parallel to the actual one. If this is done, the original system of forces is equivalent to this single force in its new position together with a couple (to allow for the altered turning effect of the force) whose moment is equal to the product of the force by the distance between the actual and assigned lines of action. This transposition is effected when P at A in Fig. 52 is replaced by P_1 at G and the couple of moment Pr.

3. The centre of mass G of a body is the point at which a force can be applied without causing angular acceleration. When a force P is applied at G every point of the body acquires a linear acceleration equal to P/M, where M is the mass of the body.

4. When a body has an angular acceleration α, the sum of the moments about G of the forces applied to the body is equal to $I_G\alpha$. This relationship treats G as a fixed axis or pivot, and is justified because if a pivot were introduced at G the force exerted by it would not modify the angular acceleration.

5. Given that the linear acceleration of the centre of mass of a body is f and that its angular acceleration is α, we can proceed as follows in order to find the force which would produce this state of motion. The linear acceleration of G would be caused by a force of magnitude Mf acting in the direction of f at G. But this force would cause every particle of the body to have the same acceleration f, and there would be no angular acceleration. We have "localized" the force at G, *i.e.* we have assigned to it a line of action which is not necessarily the same as that of the actual resultant force acting on the body. We must therefore expect that it will be necessary to represent the actual forces acting on the body by a couple as well as by the force at G. This couple must be responsible for the other element in the motion of the body which is not accounted for by the force Mf acting at G, namely the angular acceleration, and it must therefore have a moment equal to $I_G\alpha$.

In the final analysis we must be able to represent the system of forces by a single force having a certain magnitude, direction and line of action. If this force is P and has a line of action distant r from G it is equivalent to the force Mf at G together with the couple $I_G\alpha$ provided that

(i) its direction is that of f,

(ii) its magnitude is given by

$$P = Mf,$$

(iii) its line of action is given by

$$Pr = I_G\alpha.$$

6. By reversing the argument we are enabled by these three relations to find f and α when a body for which M and I_G are known is acted upon by a system of forces whose resultant is P.

It will be noticed that the principles have been established in what is logically and fundamentally the correct order. Forces and couples are really abstract quantities which we postulate as being the causes of accelerations. In the above argument we have begun with a given state of motion and have deduced the (simplest) system of forces, namely the single force P, which can be regarded as being responsible for that motion.

Theorem of Parallel Axes.—Let the body in Fig. 53 be acted upon by a system of forces which reduces to the single force P acting in the position shown. The motion of the body will consist of a linear acceleration of G in the direction of P given by

$$P = Mf_G \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and an angular acceleration α given by

$$Pr = I_G\alpha \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where r is the perpendicular distance between G and the line of action of P.

Consider any point on the dotted line AG which passes through G and is perpendicular to P and f_G in Fig. 53. The linear acceleration of this point in the direction of P is equal to $f_G + f'$, where f' is the acceleration of the point in the direction of P caused by the angular acceleration α. At G, f' is zero because f_G is the actual acceleration of G. On the opposite side of G from P, f' is in the opposite direction to f_G, and there will in general be one point on the dotted line at which their magnitudes are equal, so that the resultant acceleration of this point is zero. If this point is O, we have

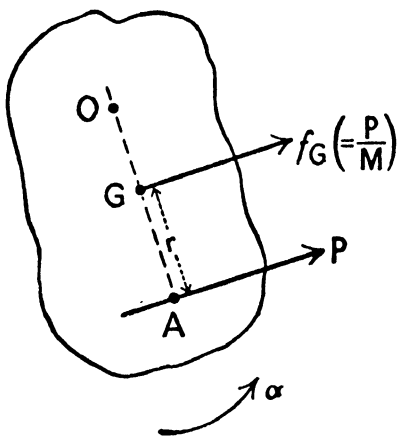


FIG. 53

$$f' \text{ at O} = -f_G$$

Since the point O is unaccelerated we can regard the body as being pivoted there, so that $f_G = \text{OG} \times \alpha$, and taking moments about O gives

$$P(r + \text{OG}) = I_O\alpha \quad . \quad . \quad . \quad . \quad (10)$$

where I_O is the moment of inertia of the body about an axis through O perpendicular to the plane of motion. Thus, by equations (9) and (10) we have

$$I_G\alpha + (P \times \text{OG}) = I_O\alpha \quad . \quad . \quad . \quad . \quad (11)$$

and by equations (8) and (11)

$$I_G\alpha + (Mf_G \times \text{OG}) = I_O\alpha$$

or

$$I_G + (M \times \text{OG}^2) = I_O \quad . \quad . \quad . \quad . \quad (12)$$

since f_G is equal to $\text{OG} \times \alpha$.

The quantities P, r, f_G and α do not appear in this last equation, which, therefore, merely expresses a geometrical relation relevant to the body itself and independent of its motion. It is known as the **theorem of parallel axes,** and states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass *plus* the product of its mass by the square of the perpendicular distance between the axes.

Centre of Percussion.—The discussion in the foregoing section shows that, when a force is applied to the body at a point A in a direction

at right angles to AG, there is a certain point O which acquires no motion and that there exists a relationship between the relative positions of G, A and O which can be derived from the equations just established. This means that, when the body is pivoted at O, the point A has the property that if a force is applied to the body at A in a direction at right angles to OA the pivot experiences no force, since even if the body were not pivoted at O it would still rotate about that point under the action of the force. The point A which has this property is called the **centre of percussion,** corresponding to the particular position of the pivot. Everyone who has hit a ball with, say, a tennis racket has discovered that for a given position of the hand (*i.e.* pivot) there is one point at which the ball may strike the racquet and yet cause no jarring of the hand. This point is the centre of percussion.

The relationship between the positions of the pivot, centre of mass and centre of percussion can be derived by combining equations (10) and (12), from which we obtain

$$P(r+\mathrm{OG}) = I_G\alpha + (M\alpha \times \mathrm{OG}^2) \quad . \quad . \quad . \quad (13)$$

Substituting for α from equation (9) gives

$$Pr + (P \times \mathrm{OG}) = Pr + \frac{MPr \times \mathrm{OG}^2}{I_G}$$

or

$$I_G = Mr \times \mathrm{OG} \quad . \quad . \quad . \quad . \quad (14)$$

The quantity I_G is often written as Mk^2, where k is known as the **radius of gyration** of the body about an axis through G perpendicular to the plane of motion. Making this substitution in equation (14) we have

$$k^2 = r \times \mathrm{OG} \quad . \quad . \quad . \quad . \quad (15)$$

which is the required relationship between the positions of O, A and G since r is equal to AG.

EXAMPLES V

1. The points A, B, C, D are the vertices of a square of side a. Forces of magnitude 5, 2, 3, 4 lb. wt. act along the lines AB, BC, CD, DA in the directions indicated by the order of the letters, and a force of $9\sqrt{5}$ lb. wt. acts through the point B in the direction from B to the middle point of CD. Find the single force at A and the couple to which this system of forces is equivalent. Show also that the resultant force of the system passes through C. (J.M.B.H.S.)

2. Assuming that the moment of inertia of a uniform circular cylinder of mass M and radius a is $Ma^2/2$ about its axis, derive an expression for the moment of inertia about an axis through its centre perpendicular to its plane of a ring of rectangular cross-section, having a mass M and internal and external radii a_1 and a_2.

3. Calculate the angular acceleration of a fly-wheel of mass 10 kg. and radius 12 cm. when a force of 0·6 kg. wt. is applied tangentially to its rim. It may be assumed that the mass of the fly-wheel is concentrated at its rim.

4. Define the *moment of inertia* of a body about a given axis.

Describe how the moment of inertia of a fly-wheel can be determined experimentally.

A horizontal disc rotating freely about a vertical axis makes 100 r.p.m. A small piece of wax of mass 10 gm. falls vertically on to the disc and adheres to it at a distance of 9 cm. from the axis. If the number of revolutions per minute is thereby reduced to 90, calculate the moment of inertia of the disc. (J.M.B.H.S.)

5. A body of mass 1 kg. is free to rotate about an axis about which its moment of inertia is 5×10^5 gm. cm.^{2}. If the moment of inertia about a parallel axes through its centre of mass is 4×10^5 gm. cm.^{2}, calculate the distance of the axis of rotation from (*a*) the centre of mass, (*b*) the centre of percussion.

Chapter VI

WORK, POWER AND ENERGY

1. WORK

Definition of Work.—In the mechanical sense the term **work** has a quantitative meaning and is defined as follows.

In Fig. 54 (i) the upper arrow represents a force P having its point of application at A. The force may be regarded, for instance, as due to

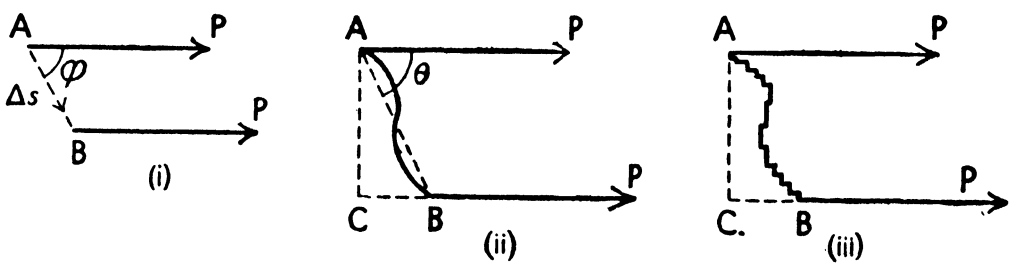


FIG. 54

the action of a stretched string or spring which is attached to a body at A. Suppose that during the course of a subsequent movement of the body the point of application moves through a short distance Δs to B, the force P having acted all the time in the same direction. The *displacement* of the point of application is represented by the vector $\vec{AB}$ and has a magnitude Δs.

The work done by the force P is defined as the **magnitude of P multiplied by the component of $\vec{AB}$ in the direction of P.** The magnitude of this component of the displacement is equal to $\Delta s \cos \phi$, where ϕ is the angle between the direction of P and the direction of the displacement, so that the work done by P is equal to

$$P\Delta s \cos \phi$$

Evidently an identical expression for the work would be obtained if we regarded the work done as the product of the magnitude of the displacement and the component of P in the direction of the displacement. This definition is equally admissible.

We next consider the movement of the point of application of a force through a longer distance which is not necessarily straight, *e.g.* along the

path AB in Fig. 54 (ii). Any path, no matter what its shape, can be regarded as made up of a series of alternate small steps parallel to or perpendicular to the direction of the force. The path AB is analysed in this way in Fig. 54 (iii). No work is done while the point of application of the force is moving along a step which is perpendicular to P, because in the definition $\cos \phi$ is zero in such cases. Work is done only in the other steps, for all of which $\cos \phi$ is unity. Thus

$$\text{Total work } (W) \text{ done by } P = P \times \text{sum of steps parallel to } P$$
$$= P \times \text{CB}$$

if AC is perpendicular to P and CB is parallel to P. The *straight* line AB is the displacement of the point of application, and if it has a length s and makes an angle θ with the direction of P, then

$$\text{CB} = s \cos \theta$$

and

$$W = Ps \cos \theta$$
$$= P \times \text{the component of the displacement in the direction of } P$$

Thus, regardless of the shape of the path described by the point of application, the work done by a force is equal to the product of the force and the component of the displacement of the point of application in the direction of the force.

Particular cases of the performance of work are represented in Fig. 55 (i) to (iv).

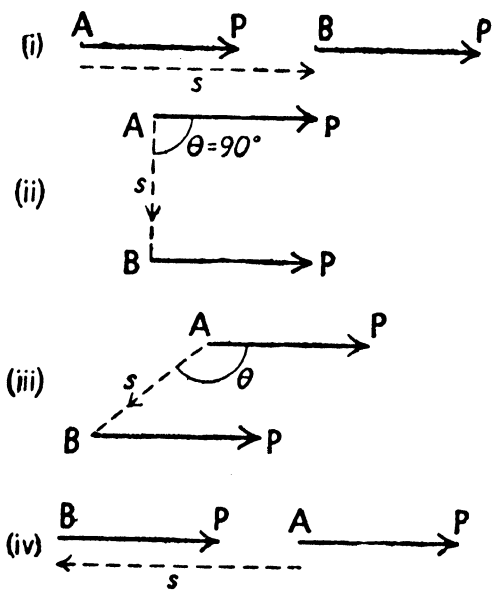


FIG. 55

(i) The displacement is in the same direction as P. The work done by P is Ps because $\theta = 0$ and $\cos \theta = 1$.

(ii) The displacement is perpendicular to the direction of P. The work done by P is zero. ($\cos \theta = 0$.)

(iii) The displacement makes an obtuse angle with the direction of the force. The work done by P is $Ps \cos \theta$, which is negative since $\cos \theta$ is negative. When the work done *by* a force is negative, an equal positive amount of work is said to have been done *against* the force.

(iv) The displacement is in the exactly opposite direction to the force. Thus θ is 180° and the expression for W becomes $-Ps$. Thus a quantity of work equal to Ps is done *against* the force.

While each of the examples is important, particular notice should be taken of case (ii). No work is done by or against a force when the point of application suffers a displacement in a direction at right angles to the force. This is true whatever may be the actual path of the point of application. Thus the total work done by or against gravity (*i.e.* the weight of a body) is zero not only when a body is moved horizontally, but also when it moves along *any* path whose beginning and end are on the same horizontal level. In Fig. 56 the total work done by or against *mg* is zero when a body of mass *m* moves by either the straight or curved path from A to B on the same horizontal level. It follows that the work done by *mg* over any section of the curved path is equal to the work done against *mg* in the remaining parts of the path.

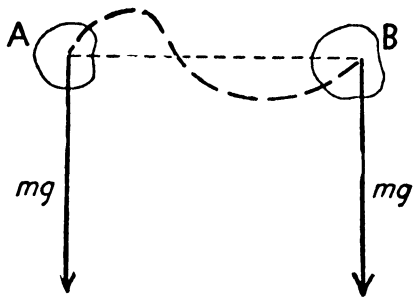


FIG. 56

Absolute Units of Work.—If a force is expressed in dynes and its displacement in cm., the corresponding work unit is given the name **erg.** The erg is, therefore, the absolute unit of work in the C.G.S. system and is the amount of work done when the point of application of a force of one dyne moves through one cm. in the direction of the force.

Since the erg is a very small unit of work, it is convenient to have a larger absolute unit in the C.G.S. system. This is the **joule,** which is defined as being equal to 10^7 ergs.

The absolute unit of work in the F.P.S. system is the **foot-poundal,** which is the amount of work done by a force of one poundal when its point of application moves through one foot in the direction of the force.

Gravitational Work Units are units of work corresponding to gravitational force units. The commoner unit is the **foot-pound** (strictly speaking, ft. lb. wt.), which is the work done by a force of one lb. wt. when its point of application moves one foot in the direction of the force. The earth attracts a body of mass of *M* lb. with a force of *M* lb. wt., so that in order to raise the body through a vertical distance of *h* ft. it is necessary to perform *Mh* ft. lb. of work against the force of gravity.

Evidently one ft. lb. is equal to *g* ft. poundals, where *g* is approximately 32·2.

The corresponding C.G.S. unit of work is the **cm. gm. wt.,** but although it is sometimes convenient to use this in calculations, results are seldom expressed in terms of this unit.

There are several other units which are used in particular cases, *e.g.* ft. tons, metre kilograms, etc.

Work done by a Couple.—Suppose that two equal and opposite forces P are applied tangentially to the rim of a fly-wheel which is pivoted centrally (Fig. 57). As the wheel rotates, let the forces change their directions so as to remain always tangential to the rim. The movement of the points of application of the forces is always in the direction of the force, *i.e.* tangential, and for a rotation of θ radians the total displacement of each point of application in the direction of the force is equal to $r\theta$. Therefore the work done by the couple is

$$2r\theta P,$$

or

$$N\theta$$

where N is the moment of the couple ($=2Pr$).

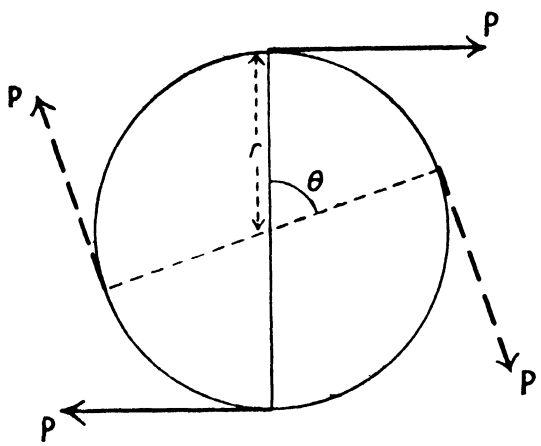


FIG. 57

Alternatively the expression may be derived by supposing that the motion of the wheel is caused by our pulling with a force P in a constant direction on both of two strings wrapped round it.

2. POWER

Definition.—In mechanics the word "power" has a definite quantitative meaning and should never be used in any other sense.

The power of a system at any instant is the rate at which it is doing work. If the power is varying with time, its instantaneous value must be determined by dividing the work done in a small interval of time by the time.

Units.—Evidently absolute units of power would be ergs sec.$^{-1}$ and ft. poundals sec.$^{-1}$ in the C.G.S. and F.P.S. systems respectively. In the former system, since the erg is a very small work unit, power is often expressed in **watts,** one watt being defined as a rate of working of **one joule sec.$^{-1}$.** The watt is very frequently used as a measure of electrical power, and is clearly equal to a rate of working of 10^7 ergs sec.$^{-1}$. A kilowatt is a rate of working of 1000 watts.

In the F.P.S. system we have the important unit of power known as the **horse-power,** which is defined as a rate of working of 550 ft. lb. sec.$^{-1}$. It is therefore a gravitational and not an absolute unit, and is mostly used in engineering. It is a useful exercise in units for the student to calculate the number of watts equivalent to one H.P. The result is 746. Thus 1 H.P. is approximately equivalent to $\frac{3}{4}$ kilowatt.

Example.—*Calculate the rate of working when* 15 *cub. ft. of water are raised per minute through a vertical distance of* 20 *feet.* (1 *cub. ft. of water weighs* $62\frac{1}{2}$ *lb.*)

$$\text{Weight of water raised per minute} = 15 \times 62\tfrac{1}{2} = 937{\cdot}5 \text{ lb. wt.}$$

$$\therefore \text{ Rate of working} = 20 \times 937{\cdot}5 \text{ ft. lb. min.}^{-1} = \frac{20 \times 937{\cdot}5}{60} \text{ ft. lb. sec.}^{-1}$$

$$\therefore \text{ H.P.} = \frac{\text{no. of ft. lb. per sec.}}{550} = \frac{20 \times 937{\cdot}5}{60 \times 550} = 0{\cdot}57 \text{ H.P.}$$

Determination of Horse-power.—Various methods are used for the determination of the horse-power of an engine. The following is the fundamental principle common to several.

Suppose that the engine is driving a shaft or drum (Fig. 58) round which a rope or band (a "brake") is passed and held stationary. This means that the vertical portions of the rope must be in a state of tension. The weight W and the spring-balance provide suitable means for creating the tensions and allowing them to be measured. Thus T_1 is equal to W and T_2 is given by the reading of the spring-balance.

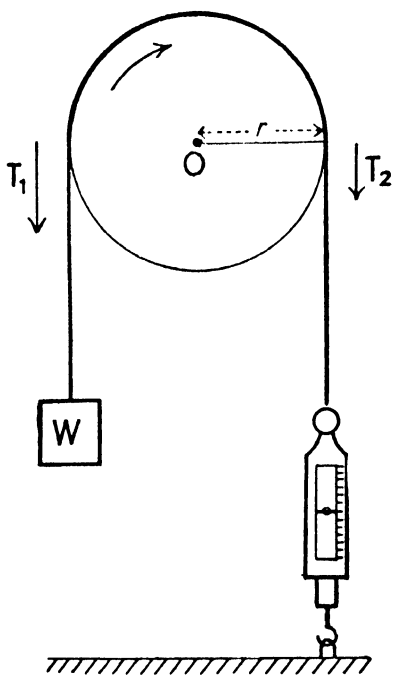


FIG. 58

The moment of T_1 and T_2 about O is equal to

$$T_1 r - T_2 r$$

i.e.

$$(T_1 - T_2)\, r$$

There is therefore a couple of moment $(T_1 - T_2)r$ acting on the shaft, and the shaft must act with an equal and opposite couple on the rope. If the shaft makes n revolutions per minute, the number of radians through which it turns in one second is

$$\frac{2\pi n}{60}$$

so that the work done per sec. by the engine against the couple exerted on it by the friction of the band is

$$\frac{2\pi n}{60}(T_1 - T_2)r$$

which will be in ft. lb. sec.$^{-1}$ if T_1 and T_2 are in lb. wt. The horse-power is therefore

$$\frac{2\pi n}{60 \times 550}(T_1 - T_2)r \quad . \quad . \quad . \quad . \quad (1)$$

This quantity is known as the **brake horse-power,** and (owing to friction in the engine) it is always less than the **indicated horse-power,** which is the rate of working of the force which the steam or other gases in the cylinder of the engine exerts on the piston. This latter quantity can be estimated by a device which, during successive cycles, records simultaneously the pressure in the cylinder and the movement of the piston.

An exactly similar formula to (1) will serve for the calculation of the horse-power transmitted by an endless belt which is driving a shaft or pulley. The forces T_1 and T_2 are then the tensions in the two sides of the belt, and their difference $T_1 - T_2$ is called the **effective tension** because it is the effective force which is acting on the circumference. The larger tension will occur on the side towards which the shaft is being rotated by the belt and, of course, the application of the formula assumes that there is no slipping.

Power and Velocity.—If the point of application of a force P has a velocity v in the direction of the force, it moves a distance v in unit time and the work done by P in unit time, *i.e.* the power, is equal to Pv. In order to obtain the horse-power we must use the formula $\frac{Pv}{550}$ where P is in lb. wt. and v in ft. sec.$^{-1}$.

If the velocity is not constant, the horse-power is varying with time, and the formula gives its instantaneous value if v is the instantaneous velocity.

***Example.**—Calculate the horse-power when a train of total mass* 300 *tons travelling at* 30 *m.p.h. has an acceleration of* $\frac{1}{2}$ *ft. sec.*$^{-2}$ *on a level track, and the frictional resistance to motion is* 10 *lb. wt. per ton.*

The total work done can be separated into two parts:

(i) The work done against friction per sec.
$$= \text{frictional force} \times \text{velocity in ft. sec.}^{-1}$$
$$= 10 \times 300 \times 44 \text{ ft. lb. sec.}^{-1}$$
$$= 132{,}000 \text{ ft. lb. sec.}^{-1}$$

(ii) The work done by the force which is causing acceleration.
The magnitude of this force $= \text{mass} \times \text{acceleration}$
$$= 300 \times 2240 \times \tfrac{1}{2} \text{ poundals}$$
$$= \frac{300 \times 2240 \times \frac{1}{2}}{32} \text{ lb. wt.}$$
$$= 10{,}500 \text{ lb. wt.}$$

$\therefore$ Rate of working of this force $= 10{,}500 \times 44$ ft. lb. sec.$^{-1}$
$= 462{,}000$ ft. lb. sec.$^{-1}$

$\therefore$ Total rate of working $= 132{,}000 + 462{,}000$ ft. lb. sec.$^{-1}$
$= 594{,}000$ ft. lb. sec.$^{-1}$
$$= \frac{594{,}000}{550} \text{H.P.}$$
$$= 1080 \text{ H.P.}$$

There are many variations of this example. For instance, given the total H.P., mass, velocity and acceleration we can find the frictional resistance; or given the total H.P., mass and frictional resistance we can find the acceleration at any given velocity.

If the train is travelling up an incline, additional work must be done against the component of the weight acting down the incline. Thus if the track were not level but had an inclination of 1 in 60, this may be taken to mean that the sine of the angle which the track makes with the horizontal is $\frac{1}{60}$. Thus the component of the weight of the train acting down the plane is $\frac{1}{60}$th of the weight of the train, *i.e.* 5 tons wt. or 5×2240 lb. wt. When the train is going up the incline at 30 m.p.h. the rate of doing work is $5 \times 2240 \times 44$ ft. lb. sec.$^{-1}$ and the horse-power expended is

$$\frac{5 \times 2240 \times 44}{550}$$

i.e. 896 H.P.

3. ENERGY

Definition of Energy.—When a body or system of bodies performs work by reason of a force which it exerts externally it is said to lose an amount of **energy,** equal to the amount of work which it performs. This statement can be regarded as a quantitative definition of energy. Qualitatively, the energy of a body is often stated to be its "ability to perform work." Clearly energy is expressed in work units.

Energy due to Various Causes.—There are several ways in which a body can perform work. A moving body, for instance, can do work (*i.e.* it possesses energy) by virtue of its speed, and when its speed is reduced it is said to lose **kinetic energy.** An example of this occurs when a moving railway truck is brought to rest as it compresses the springs of the buffers by exerting a force on them which is equal and opposite to the force which they exert on it. Similarly a moving hammer drives a nail into wood, doing work against the resistance of the wood; a jet of water turns a paddle wheel which can be used to do work; air in motion drives a windmill which raises water by means of a pump. We shall return to a fuller discussion of kinetic energy later on.

The form of energy which bodies possess by virtue of their position in a field of force such as the earth's gravitational field or the magnetic field surrounding a magnet is called **potential energy.** It is due to the force which the body experiences when it is situated in the field. The most impressive example of such energy is to be found in elevated lakes and rivers, the water from which can be made to do enormous amounts of work when it is allowed to fall to lower levels.

The term "potential energy" is also applied to the **energy of strain** possessed by a stretched or coiled spring or a compressed gas.

Potential Energy of Position.—As a general definition we can state that the increase of the potential energy of a body with respect to a field of force is the work which must be done against the action of the field when the body undergoes a displacement, *e.g.* when a body is raised against the downward pull of gravity. In order to cause the displacement it is necessary to exert a force on the body which is equal and opposite to that exerted by the field at all stages during the movement. Thus it is the work done by this external force *against* the force due to the field which constitutes the increase of potential energy. If the body moves in such a way that the force due to the field does work, there is a negative increase, *i.e.* a loss, of potential energy. This occurs when a body falls to the earth. For the same path between two points A and B the gain of potential energy when the body moves from A to B is equal to the loss of energy for a reverse movement from B to A because the quantities of work done respectively against and by the field are numerically equal. If the change of potential energy between any two points is independent of the path taken by the body the field of force is said to be **conservative.**

For the time being we shall confine ourselves to a consideration of the effect of the earth's gravitational field (*i.e.* the potential energy which bodies possess by virtue of their weight and position relative to the earth), and we shall suppose that g is constant over any particular region with which we are concerned.

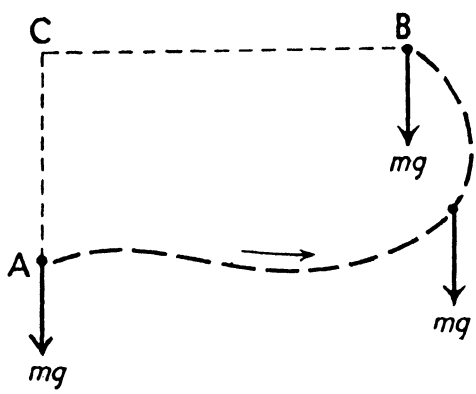


FIG. 59

Gravitational Potential Energy.—Let a body of mass m be moved from a point A (Fig. 59) to a point B which is at a higher level, and suppose that it follows *any* path such as that indicated by the curved line. It constantly experiences a downward vertical force of mg absolute units, and its displacement in the vertical direction is equal to AC, if AC and CB are respectively vertical and horizontal lines. We have, therefore,

$$\text{Gain of P.E. of body} = \text{work done against gravity} = mg \times \text{AC absolute work units}$$

It should be noticed that the gain of potential energy depends only on the vertical distance between the ends of the path and is independent of the shape or length of the actual path traversed. The body and the uniform gravitational field constitute a conservative system. The fact that the nature of the path does not affect the change of potential energy is due, in the first place, to our definition of work. Its truth can be

appreciated when we realize that any curved path can be regarded as a very large number of small consecutive vertical and horizontal steps as explained on page 68. No work is done by or against *mg* when the body takes a horizontal step.

When a body moves to a lower horizontal level, *e.g.* from B to A, it loses an amount of potential energy equal to the product of *mg* and the vertical displacement in a downward direction. We can imagine this loss of energy being converted into work by supposing that during its fall the body is acted upon by a force *mg* vertically upwards so that it falls without acceleration. It then does an amount of work against this upward force equal to the potential energy which it loses. A case like this would occur if two bodies of equal weight were attached to the ends of a string which passed over a frictionless pulley free to rotate about a horizontal axis. In such a system the work which one of the bodies is capable of performing by virtue of its loss of potential energy when it falls is used in imparting an equal amount of potential energy to the other body which is raised. The potential energy of the *system*, by which we mean the sum of the energies of the separate interacting bodies, is unaltered by the movements of the bodies. If friction is present (as it always is in practice) the total potential energy is diminished by motion of the above system, because the potential energy lost by the descending body is partly used in doing work against friction, and consequently it would not be able to raise so large a weight on the other side. If the weights were made equal it would be necessary to provide some external force which would perform the necessary work against friction in order to make the system just move without acceleration.

Absolute Value of the Potential Energy of a Body.—If a body were allowed the necessary freedom of movement, it would continue to fall and lose potential energy until it reached the centre of the earth. The total gravitational potential energy of a body is, therefore, the work which would be necessary in order to move it from the centre of the earth to its actual position. This is not a very fruitful conception, however, because in the normal way we are only concerned with the *differences* of potential energy which occur when bodies rise or fall through comparatively short distances near the earth's surface. Thus in any particular problem it is usual to choose a convenient arbitrary level at which potential energy is reckoned to be zero.

Kinetic Energy has already been referred to as the energy possessed by a moving body by virtue of its speed. In order to derive an expression for the magnitude of the kinetic energy, we simply calculate the amount of work which the body can perform while it is being brought to rest.

Let the initial velocity of the body be u, and let a constant force P act upon it in a direction opposite to its motion (Fig. 60). The body therefore suffers a constant acceleration in a direction opposite to u. If

the distance in which the body is brought to rest is s, we have, since the final velocity is zero,

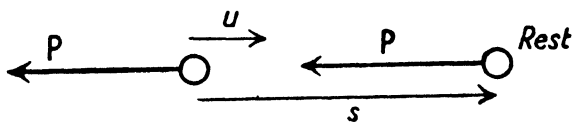


FIG. 60

$$0 = u^2 + 2fs$$

or $$f = -\frac{u^2}{2s}$$

the negative sign showing that the acceleration f is opposite to u and s as we should expect. The acceleration in the direction of P is therefore $+\frac{u^2}{2s}$, and if m is the mass of the body,

$$P = \frac{mu^2}{2s}$$

by Newton's second law.

The original kinetic energy of the body is equal to the work which it does against P, *i.e.*

$$\text{K.E.} = Ps = \frac{mu^2}{2s} \cdot s$$
$$= \tfrac{1}{2}mu^2$$

Therefore, when moving with a speed u, a body possesses kinetic energy of $\frac{1}{2}mu^2$ *absolute work units.*

The above calculation applies equally well to the reverse process of performing work on a body which is originally at rest. The gain of kinetic energy of a body is equal to the amount of work necessary to give it its velocity, excluding, of course, any work which is done incidentally against other agencies, such as friction or gravity.

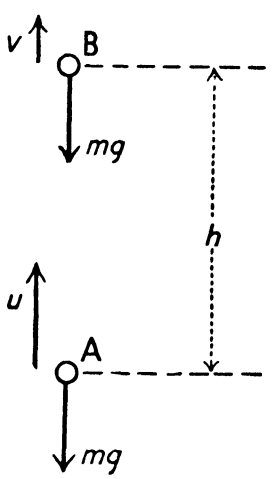


FIG. 61

The Conservation of Mechanical Energy.—Consider what happens when a body is projected with a velocity u vertically upwards against gravity so that it eventually comes to rest (Fig. 61). This is a particular case of a moving body performing work against a constant force such as we have already dealt with in deriving the expression for the kinetic energy of a body. When the body has risen from A to B its velocity has been reduced from u to v. We can write

Loss of K.E. in going from A to B = work done by body against mg in going from A to B, by definition of K.E.
= gain of P.E. of body in going from A to B, by definition of P.E.

Thus between any two points in its path the change of kinetic energy is exactly balanced by an equal and opposite change of potential energy. This is true whether the body is rising or falling.

This principle is known as the **conservation of mechanical energy,** and may be expressed by the statement that the total amount of mechanical energy of both kinds which the body possesses is always constant. It applies only to the motion of a body *without friction, i.e.* to a *conservative system.* For if the motion of the rising body is opposed by a frictional force as well as by gravity, we have, by definition of kinetic energy,

Loss of K.E. = work done by the body against gravity (*i.e.* gain of P.E.) + work done by the body against friction.

So that for a given loss of kinetic energy the body gains less potential energy than when friction is absent.

Similarly when the body is falling,

Gain of K.E. = work done on the body by gravity (*i.e.* loss of P.E.) − work done against friction.

Thus the body gains an amount of kinetic energy which is less than the potential energy which it loses.

When a body is falling in a fluid medium, even if this is as slightly viscous as air, it experiences a resistance to motion which increases with its speed. Eventually the resistance becomes so large as to prevent further acceleration, *i.e.* it becomes equal to the weight of the body *minus* the buoyancy of the medium. When this so-called *terminal velocity* is attained, further loss of potential energy of the falling body is completely used up in doing work against the resistance, and the kinetic energy of the body ceases to increase.

The constancy of the sum of the two kinds of energy when there is no friction is expressed by the equation

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 = mgh$$

where u and v are velocities at points separated by a vertical distance h (Fig. 61). When this equation is divided throughout by m it becomes identical with

$$v^2 = u^2 - 2gh$$

which is, of course, the form of the equation

$$v^2 = u^2 + 2fs$$

which is applicable to this case.

It is not surprising that the energy principle should lead to this equation, because this actual equation was used in the derivation of the expression $\tfrac{1}{2}mu^2$.

It has already been mentioned (page 44) that when transmutations of atomic nuclei occur, some mass seems to disappear and energy appears as kinetic energy of the products of the transformation. Some of the particles originally present in the nuclei are shot off at high speeds. The energy produced bears a constant ratio to the mass destroyed (9×10^{20} ergs for every gm. of mass), so that the phenomenon is not regarded as contrary to the principle of conservation but rather as an indication that mass and energy have similar physical natures.

The Conservation of Energy Demonstrated for a General Case.— In Fig. 62 a body is moving up a *smooth* surface. At A its speed is u and at B it is v. There are two forces acting on the body, namely its weight mg vertically downwards and the force due to the surface. This latter always acts *normally* to the surface at the point of contact because

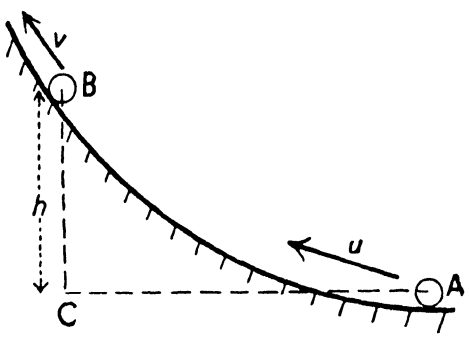


FIG. 62

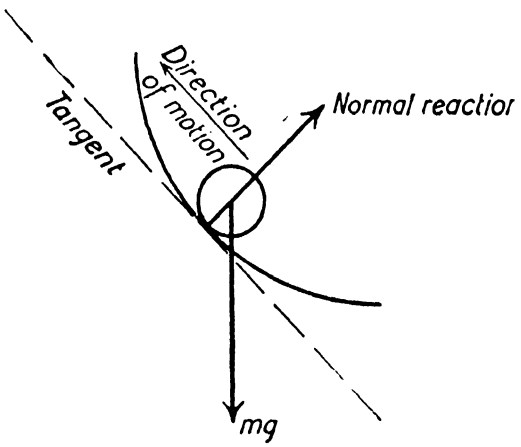


FIG. 63

the surface is stated to be smooth, and in the absence of friction a surface can only exert a force perpendicular to itself. It is only by virtue of friction that a surface can exert a tangential force. The normal force exerted by the surface is called the **normal reaction,** and since the motion of the body never has a component in the direction of the normal, if it is always in contact with the surface, it follows that the normal reaction does no work on the body. This is illustrated in Fig. 63. Thus the sole effect of the normal reaction is to change the direction of the body without altering its speed. Therefore the kinetic energy which the body loses in going from A to B is all used up in doing work against gravity, *i.e.* in giving potential energy to the body. Once again, therefore, the sum of the kinetic and potential energies of the body is constant, and we can write

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 = mgh$$

or

$$v^2 = u^2 - 2gh$$

where h is the vertical height between A and B, *i.e.* the height CB in Fig. 62.

Whenever this relationship is used for cases other than vertical motion, it should be realized that it is based on the definitions of kinetic and potential energies and is not simply the equation for *rectilinear* motion under gravity which is only a particular case. The equation is equally applicable to a body sliding *down* a smooth surface. It is always necessary, however, that the surface should be smooth. When there is friction, the total energy diminishes by an amount equal to the work done against friction during the motion.

In the figures, the body has been represented as having a circular section. This is not necessary to the argument, and it must be remembered that the body will not roll unless there is friction. We are concerned only with sliding; when a body is rolling it possesses kinetic energy of rotation as well as of translation, and the matter is more complicated.

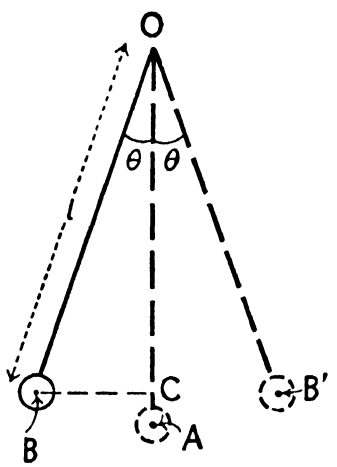


FIG. 64

Example. — Let a weightless inextensible string of length l be attached at one end to a fixed point O (Fig. 64) and at the other end to a small body or particle of mass m and negligible size. This system is called a "simple pendulum," and we shall discuss its properties more fully later on. The point A, vertically below O, represents the lowest possible position of the particle or "bob," and we can regard its potential energy as being zero at this point. At any other point, say B, where the string makes an angle θ with the vertical, we have

$$\begin{aligned}\text{At B the P.E. of the bob} &= mg \times \text{AC, if BC is horizontal}\\ &= mg(\text{OA} - \text{OC})\\ &= mg(\text{OA} - \text{OB}\cos\theta)\\ &= mgl(1 - \cos\theta)\end{aligned}$$

Suppose the pendulum is held in the position OB and then released. At every stage of the subsequent motion the total mechanical energy is the same, and is equal to the potential energy which the bob had at B. In particular, when it reaches its lowest point A, the bob has no potential energy, and if the speed of the bob at this point is u, then its kinetic energy is equal to the potential energy which it has lost. Therefore

$$\tfrac{1}{2}mu^2 = mgl(1 - \cos\theta)$$

or

$$u^2 = 2gl(1 - \cos\theta)$$

After passing through A the bob will swing to B′, which is at the same horizontal level as B, and then back again through A to B, and so on. In actual fact air resistance is continually diminishing the energy of the pendulum and the swings gradually die down.

Kinetic Energy of Rotation.—Let the body in Fig. 65 be rotating with an angular velocity ω about an axis through O perpendicular to the plane of the paper. Any particle A, of mass m, situated at a distance r from O has a velocity $r\omega$ perpendicular to OA. Its kinetic energy is therefore $\frac{1}{2}m(r\omega)^2$. We now add up all the separate kinetic energies of the particles composing the body. This is, of course, a purely scalar addition because energy has no directive property—it depends on speed and not on velocity. Thus

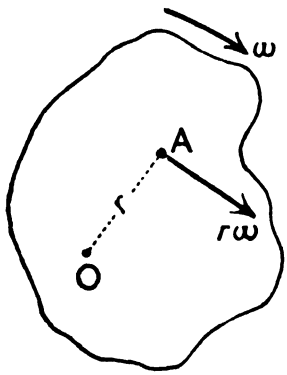


FIG. 65

K.E. of whole body
$= \Sigma\frac{1}{2}m(r\omega)^2$
$= \frac{1}{2}\omega^2\Sigma mr^2$, since ω is the same for all particles
$= \frac{1}{2}I_0\omega^2$

where I_0 is the moment of inertia of the body about the axis of rotation.

In particular, if the axis of rotation passes through the centre of mass, the body has an amount of kinetic energy equal to $\frac{1}{2}I_G\omega^2$. The kinetic energy of rotation of a body, such as a flywheel pivoted at its centre, represents the work which the body can do against a couple opposing its motion, or alternatively the work which was done by the couple which established the motion.

In the general case of a moving body we can regard the motion of each particle as being made up of the linear velocity which it possesses in common with the centre of mass, together with the speed which it has by virtue of the angular velocity of the body supposing that the centre of mass were pivoted. The total energy of the particles due to the linear velocity (u_G) of the centre of mass is

$$\Sigma\tfrac{1}{2}mu_G^2$$
$$= \tfrac{1}{2}Mu_G^2, \text{ since } u_G \text{ is a constant, and } \Sigma m = M$$

while the total energy due to rotation is

$$\tfrac{1}{2}I_G\omega^2$$

Thus the total kinetic energy of the body is

$$\tfrac{1}{2}Mu_G^2 + \tfrac{1}{2}I_G\omega^2$$

This means that if the motion of the body were destroyed by the action of forces upon it, it could do $\frac{1}{2}Mu_G^2$ absolute units of work against a retarding force applied at G, and $\frac{1}{2}I_G\omega^2$ absolute units of work against a couple opposing its rotation.

The Importance of Energy in Physics.—It should be remembered that energy is a quantity whose definition can be traced back to Newton's laws of motion, whence are derived the ideas of mass and force. In mechanics it is often more convenient to discuss a problem in terms of

energies of the bodies concerned than it is to apply Newton's laws directly in their original form. For instance, the velocity of the bob of the simple pendulum on page 79 was worked out very easily by the energy principle.

The conservation of energy tells us at once that no real system of bodies will continue to move indefinitely unless we supply it from outside with sufficient energy to compensate for the energy which it constantly loses on account of the inevitable friction. All devices for achieving perpetual motion can be immediately discarded by the application of this principle.

However, the importance of energy extends beyond the bounds of what we call "mechanics." We now know, for instance, that what was originally called "heat" is of exactly the same nature as mechanical energy—a discovery which a hundred years ago welded heat and mechanics into what is essentially one subject.

EXAMPLES VI

1. Define *work* and *power*. State the units in which they are measured.

A train of mass 180 tons moves along a horizontal track at a maximum speed of 40 m.p.h. If the resistance due to friction is 10 lb. wt. per ton, what is the horse-power of the engine? (1 H.P. = 550 ft. lb. wt. per sec.). (L.M.)

2. A pump raises 10 cub. ft. of water per min. through a height of 20 ft. and delivers it through a nozzle of diameter 1·2 in. Calculate the work per sec. converted into (*a*) potential energy, (*b*) kinetic energy. Calculate also the horse-power expended.

3. A train of total mass 200 tons is ascending an incline of 1 in 112 with an acceleration of 1 ft. sec.$^{-2}$. When the speed of the train is 15 m.p.h. the horse-power developed by the engine is 784. Prove that the resistance to motion is 8 lb. wt. per ton of the train. (O.H.S.).

4. A 5-ton motor lorry is running on a level road 45 miles per hour. It comes to an uphill stretch of the road, which has the effect of reducing the speed to a steady 15 m.p.h. Assuming that the engine is working throughout at a constant rate and that the resistance, apart from gravity, is 40 lb. wt. per ton, calculate the horse-power exerted and show that the gradient of the hill is 1 in 28. (O.H.S.)

5. A railway train, the coaches of which are of mass 400 tons, is running on a level track, and at the instant at which its speed is 37½ m.p.h. its acceleration is $\frac{1}{10}$ ft. sec.$^{-2}$, and the tension in the coupling joining the engine to the coaches is 5¼ tons weight. Calculate (i) the power transmitted by the coupling (give your answer in H.P.), (ii) the resistance to the motion of the coaches in tons weight. (C.H.S.)

6. Derive an expression for the kinetic energy of a moving body.

A vehicle of mass 2 tons travelling at 30 m.p.h. on a horizontal surface is brought to rest in a distance of 40 ft. by the action of its brakes. Calculate the average retarding force. What horse-power must the engine develop in order to take the vehicle up an incline of 1 in 10 at a constant speed of 30 m.p.h. if the frictional resistance is equal to 50 lb. wt.? (L.I.)

7. Establish the expression for the kinetic energy of a particle of mass m moving with a speed u.

A stone of mass ½ lb. is released from a point 20 ft. above the surface of some mud and penetrates to a depth of one foot below the surface before coming to rest. Calculate the average resisting force which the mud exerts on the stone. (L.I.)

8. What is meant by the *conservation of energy*? Give examples.

A body of mass 40 gm. starts from rest and slides down a plane inclined at 30° to the horizontal. After it has travelled 100 cm. down the plane its velocity is 80 cm. sec.$^{-1}$. Calculate the work done against friction. (L.I.)

9. State the principle of the *conservation of energy*, and discuss its application to the motion of a pendulum (*a*) if there is no resistance opposing the motion, (*b*) when resistance is present.

The maximum angular displacement of an oscillating simple pendulum is observed to diminish from 10° to 8°. Calculate the percentage loss of energy of the pendulum which has occurred. (L.I.)

10. Explain why the oscillations of a simple pendulum of given length die down more rapidly when its bob consists of a celluloid ball than when the bob is a lead sphere of the same size.

11. Explain what is meant by the *moment of inertia* of a body about a given axis

Find from first principles an expression for the kinetic energy of a rigid body rotating with angular velocity ω radians sec.$^{-1}$ about a fixed axis.

A fly-wheel of moment of inertia 6×10^5 gm. cm.^{2} is mounted on an axle 4 cm. in diameter. A string wound round the axle supports a mass of 1000 gm. Calculate the angular velocity of the wheel after the mass has fallen a vertical distance of 50 cm. from rest, and find the moment of the couple that must be applied to the wheel to bring the system to rest after a further fifteen revolutions. (O.H.S.)

12. A cricket-ball of mass 0·3 lb. and diameter 0·25 ft. is bowled at 30 ft. sec.$^{-1}$, spinning about an axis through its centre with an angular velocity of 80 radians sec.$^{-1}$. Calculate (*a*) its rotational energy and (*b*) its total kinetic energy. (The moment of inertia of a sphere about an axis through its centre is $0{\cdot}4mr^2$, where m is the mass and r the radius.) (O.H.S.)

13. Define *work*, *watt* and *horse-power*.

A belt transmits power from a wheel $3\frac{1}{2}$ ft. in diameter which revolves 120 times per minute. If the difference between the tensions on the slack and tight sides of the belt is 220 lb. wt., what horse-power does the belt transmit? (L.Med.)

14. Define *force* and *power*, and give two units in which each is measured.

A gun 60 cm. long discharges 50 gm. of shot with a muzzle velocity 300 metres per sec. Calculate the average force acting on the shot, and the power in watts developed. (L.Med.)

15. Water flows in a horizontal channel, 2 ft. square in section, at a rate of $3\frac{3}{4}$ m.p.h., and then falls through a height of 120 ft. The energy of the escaping water is used to drive a turbine. If 80 per cent. of the available energy is thus utilized, calculate the horse-power developed. (Assume that the mass of 1 cub. ft. of water is $62\frac{1}{2}$ lb.) (L.M.)

Chapter VII

FRICTION

Whenever an attempt is made to move one solid surface over another with which it is in contact, a force which we term **friction** is brought into play. Friction may be sufficient to prevent the motion entirely, or alternatively it may make the relative acceleration of the surfaces less than it otherwise would be. As an example we consider the simple case of a solid block resting on a horizontal surface (Fig. 66), and acted upon by a horizontal force P of adjustable magnitude. This may be applied by means of a spring-balance, or by a string which passes over a frictionless pulley and has a scale pan for weights or lead shot attached at its other end.

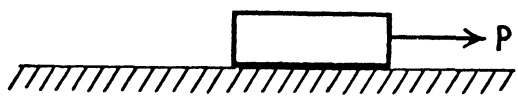


FIG. 66

Static Friction.—The first observation which can be made is that, in the absence of jolting or tapping, no motion occurs until P reaches a certain value which, for successive trials, is not particularly constant, although its average value over a large number of observations is fairly consistent. When P is less than this value, the body is in equilibrium under the action of P and an equal and opposite force due to friction. Thus we learn that the frictional force can have any value between zero (when P is zero) and a certain maximum which is called the limiting value of the **static friction,** and is, of course, equal and opposite to the value of P necessary to cause motion. The fact that no very consistent value can be obtained for the limiting friction is due to the impossibility of replacing the block in exactly the same position for each determination. The frictional force is governed by the relative positions of protuberances and pits which are inevitable even in the smoothest of real surfaces.

Kinetic or Sliding Friction.—It is also possible to determine the force necessary to maintain the body in motion *at constant velocity* when once its motion has been initiated by, for instance, tapping the surface on which it is standing. When this occurs, since the acceleration is zero, the force P must be equal and opposite to the frictional force which is operative under these conditions. It is found by experiment that this is less than the limiting static friction. This is made obvious by the observation that when the horizontal force P is applied to the stationary undisturbed body and gradually increased until motion occurs, the body

moves with acceleration unless P is reduced immediately after it has caused the motion to begin. The frictional force present during motion is called the **sliding or kinetic friction,** and in elementary mechanics can be regarded as being independent of the velocity of the body.

The Laws of Friction as found by simple experiments are usually stated as follows:—

(i) The frictional force always opposes attempted relative motion.

(ii) The maximum frictional force between two surfaces of a given nature is independent of the area of contact between them, provided that the force pressing them together is the same.

(iii) The maximum frictional force between two given surfaces is proportional to the normal force or "reaction" with which the surfaces are pressed together. Thus if F is the maximum frictional force and R is the normal reaction,

$$\frac{F}{R}=\mu, \text{ which is constant}$$

The number μ is called the **coefficient of friction,** and refers to static or kinetic friction according as F is the limiting static frictional force or the kinetic frictional force.

It should be understood that in actual fact μ is only approximately constant. The inconsistency of successive values of the static friction between two given surfaces for a given value of R has already been referred to, and it is only possible to give a rough value to μ in these circumstances. Furthermore, although the value of the force required to maintain unaccelerated motion (*i.e.* the kinetic friction) can be determined with fair consistency, it is found that μ for kinetic friction varies to some extent with R. The common laboratory experiment with a block of wood sliding on a horizontal wooden surface reveals, if carefully performed, that μ diminishes as R increases.

Determination of Coefficient of Friction.—Evidently one of the simplest methods of determining μ is to adopt the arrangement depicted

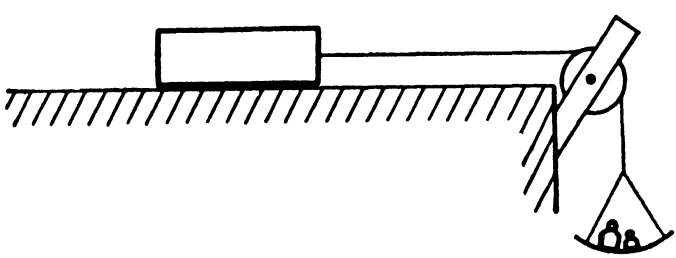


FIG. 67

in Fig. 67, which enables either the static or kinetic frictional force to be measured. In determining the static friction it is necessary that the system should not be jolted or disturbed in any way while the weights in the

pan are being gradually increased. With kinetic friction, however, the weights are adjusted until, when the body is given a slight start by gently tapping the horizontal surface, it moves very slowly and without acceleration. The surface upon which it is sliding may not be absolutely uniform, so that it may be necessary to adjust the weights to a kind of average value which, over the whole length of the motion, produces no acceleration even though the speed of the motion varies slightly from place to place. The value of the frictional force is given by the weights in the pan *plus* the weight of the pan itself, while the normal reaction R is equal to the weight of the body *plus* any weights which may be placed on it for the purpose of varying R.

A second method of determining μ is to place the body on the board or other plane surface and, for static friction, gradually to tilt the plane without jerking it until the body suddenly slides down. For kinetic friction the inclination is adjusted until the body slides down without acceleration when it has been started by tapping. In both cases the inclination of the plane to the horizontal is measured. The forces acting on the body are indicated in Fig. 68. They are shown as having a common

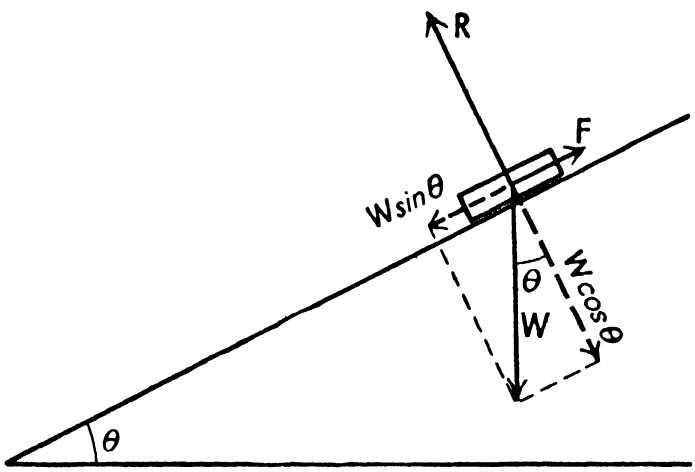


FIG. 68

point of application, which is not actually the case but is a justifiable simplification when we are not concerned with the rotation of the body. The weight of the body W can be resolved into two rectangular components thus,

$W \sin \theta$ down the plane
$W \cos \theta$ perpendicular to the plane

If the body is on the point of sliding when the inclination is θ, it has no acceleration so that there is no resultant force acting on it in any direction. Consequently

$$R = W \cos \theta$$

and

$$F = W \sin \theta$$

where R is the normal reaction and F is the limiting static friction. But the coefficient of static friction is given by

$$\mu = \frac{F}{R}$$

$$\therefore \mu = \frac{W \sin \theta}{W \cos \theta} = \tan \theta \quad . \quad . \quad . \quad . \quad (1)$$

The calculation is exactly the same for the coefficient of kinetic friction if θ in this case represents the inclination of the plane which just allows the body to slide down without acceleration. Since kinetic friction is less than the limiting static friction the corresponding inclination is also less.

The fact that the critical inclination for both static and kinetic friction determinations is independent of the weight of the body sometimes puzzles the student, until it is pointed out that, if weights are added to the body, both the frictional force ($\mu W \cos \theta$) and the driving force ($W \sin \theta$) are increased in the same proportion. When the body is sliding without acceleration, weights may be put on it without any noticeable change in the motion unless they are sufficient to cause a change in the value of the coefficient of kinetic friction.

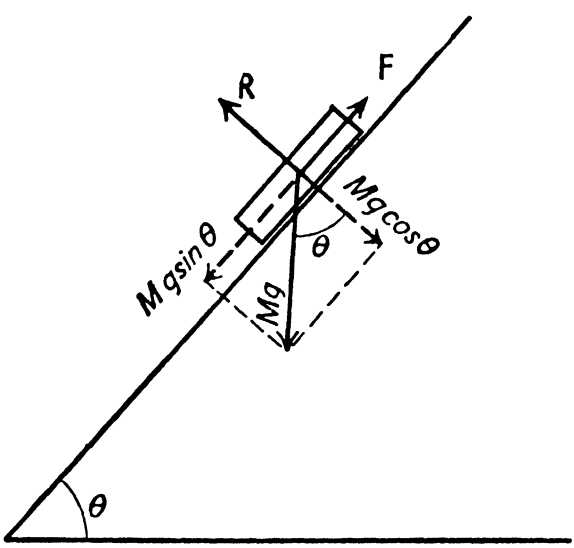


FIG. 69

Accelerated Motion on a Rough Inclined Plane.—Suppose that the inclination (θ) of a rough plane exceeds that which is necessary for the unaccelerated sliding of a body of mass M placed upon it (Fig. 69). The force due to gravity down the plane is $Mg \sin \theta$, and the force due to friction (F) acting up the plane is given by

$$F = \mu R = \mu Mg \cos \theta$$

where R is the normal reaction and μ is the coefficient of sliding friction.

Equating the resultant force in the direction of motion to the product of the mass and acceleration, we have

$$Mg \sin \theta - \mu Mg \cos \theta = Mf$$

or

$$f = g (\sin \theta - \mu \cos \theta) \quad . \quad . \quad . \quad . \quad (2)$$

It will be noticed that if f is zero so that the body is sliding with constant velocity, the last equation reduces to equation (1).

When the body slides a distance s down the plane, the work done against friction is Fs, which is equal to

$$\mu Mg \cos \theta . s \text{ absolute work units}$$

This quantity of work represents the excess of the loss of potential energy over the gain of kinetic energy which occurs as a result of the displacement s. It is actually converted into heat energy.

If the body in Fig. 69 is acted upon by a force P absolute units directed up the plane and, as a result, acquires an acceleration f up the plane, we have, by Newton's law

$$P - Mg \sin \theta - \mu Mg \cos \theta = Mf \quad . \quad . \quad . \quad (3)$$

The frictional force in this case is directed *down* the plane because the motion is upwards. Any one of the quantities in equation (3) may be calculated if all the others are given, and there are many ways in which exercises involving the use of this equation can be set.

EXAMPLES VII

1. A plane, 4 metres long, is inclined at 30° to the horizontal. Find the time taken for a body to slide from the top to the bottom of the plane, starting from rest, (*a*) when there is no friction between the body and the plane, (*b*) when the coefficient of kinetic friction is 0·2. (L.I.)

2. A body slides without acceleration down a plane when the latter is inclined at 20° to the horizontal. With what acceleration will the body move down the plane when the inclination is 30°?

3. A body slides down a plane inclined at 30° to the horizontal with an acceleration of $0{\cdot}2g$. For what inclination will it just move without acceleration, and what will be the acceleration when the inclination is 45°?

4. A body of mass 500 gm. is moved up an inclined plane by means of a force parallel to the plane. If the coefficient of friction is 0·3, the inclination of the plane is 30° and the acceleration up the plane is 100 cm. sec.$^{-2}$, calculate the magnitude of the force.

Chapter VIII

THE DYNAMICS OF SOME SIMPLE SYSTEMS

1. UNIFORM ACCELERATION

We consider first an arrangement such as that shown in Fig. 70, where a body of mass M is caused to move along a horizontal surface by the action of the weight of another mass m attached to it by a light string passing over a pulley. Let the tension in the string be T absolute force units. If the pulley is small (so as to have a negligible moment of inertia) and frictionless, T is the same in both parts of the string. Let the force of kinetic friction between the body and the surface when motion is occurring be F absolute units, and let f be the acceleration of the body. Then, applying Newton's law to the body of mass M, we have

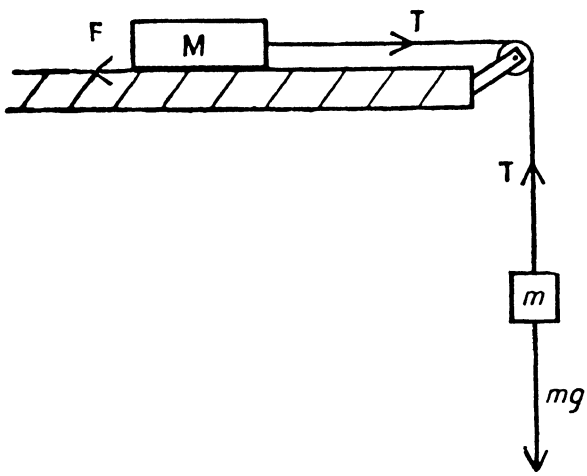


FIG. 70

$$T - F = Mf \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The string exerts an upward force of T on m so that the equation of motion of m is

$$mg - T = mf \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the acceleration of m being of the same magnitude as that of M if the string is inextensible. By adding equations (1) and (2) we obtain

$$mg - F = (M + m)f$$

or

$$f = \frac{mg - F}{M + m} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The tension in the string is obtained by substituting for f in equation (2). Thus

$$mg - T = \frac{m}{M+m}(mg - F)$$

or
$$T = mg\left(1 - \frac{m}{M+m}\right) + \frac{m}{M+m} \cdot F$$
$$= mg \cdot \frac{M}{M+m} + \frac{m}{M+m} \cdot F$$
$$= \frac{m}{M+m}(Mg + F) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In solving a numerical question of this kind, the student would be expected to begin with first principles and not to quote the above expressions for f and T.

A system of this kind is sometimes used in a teaching laboratory in order to provide a demonstration of the use of Newton's law of motion. By using a trolley with light, freely-running wheels it is possible to make F very small. The trolley passes under a small inked brush attached to the end of a horizontal vibrating metal reed of known time period (*e.g.* one-tenth of a second), and the acceleration f can be determined from measurements made on the wave trace which the brush produces on a sheet of paper pinned on the top of the trolley. This arrangement is known as **Fletcher's Trolley.**

Force Necessary to Produce a Given Vertical Acceleration.—Let a body of mass M be acted upon by a vertical upward force P (Fig. 71). Then the resultant upward force on the body is $P - Mg$, where P is in absolute units. Thus, if f is the upward acceleration, we have

$$P - Mg = Mf$$

or
$$P = M(g + f)$$

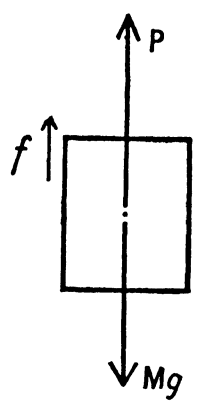


FIG. 71

If f is in the downward direction and is therefore negative, P is still directed upwards unless the downward acceleration is equal to or greater than g.

Examples of the use of this equation occur when it is required to calculate the force which the floor of an accelerated lift exerts upon a body resting on it, or to find the reading of a spring-balance hanging from the ceiling of the lift and supporting the body at its lower end. The reading of the balance in this case is P absolute force units or P/g gravitational units.

Atwood's Machine.—This consists essentially of the system shown diagrammatically in Fig. 72. Two equal masses M are attached to the ends of a light string passing over a light pulley whose bearings are made as frictionless as possible. A rider of small mass m can be placed on one of the masses so that when the system is released the loaded mass descends

with an acceleration f while the other rises with an upward acceleration of the same magnitude. If the friction on the bearings of the pulley is negligible, and if the pulley is so light that it requires only a negligible couple to give it its angular acceleration, then the tension in the string is the same on both sides of the pulley. Let this be T absolute units. The equation of motion of the left-hand mass is, then,

$$T - Mg = Mf$$

and for the other side we have

$$(M+m)g - T = (M+m)f$$

By adding these two equations we eliminate T and obtain

$$mg = (2M+m)f$$

or

$$f = \frac{mg}{2M+m}$$

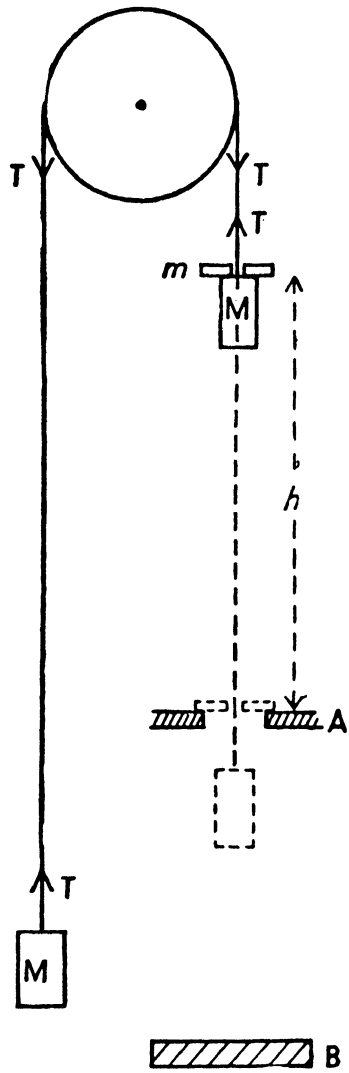


FIG. 72

It may be noted that this equation shows that the acceleration is equal to the effective force on the system (mg) divided by the total mass. The equation may be used to calculate g if the masses are known and f is determined experimentally.

There is a variety of ways in which the acceleration of the above system can be determined. One of the simplest is to arrange that the loaded mass falls through a hole in platform A (Fig. 72) and leaves the rider resting upon it. The system subsequently moves with a constant velocity equal to that which it possessed at the instant of removal of the rider. This velocity (v, say) can be determined by noting with a stop watch the time taken for the right-hand mass to fall from the first platform to a second one, B, fixed at a known distance (*e.g.* 2 metres) below. If the loaded mass falls through a height h (Fig. 72) from its point of release to the point A where the rider was removed, f can be calculated from the equation

$$f = \frac{v^2}{2h}$$

The Atwood machine is sometimes used as a laboratory exercise for the determination of g, but the results are often disappointing for several

reasons. Firstly, it is difficult to determine the times of fall under constant velocity with much accuracy because these are of the order of only a few seconds. Secondly, friction is always present and makes the observed acceleration smaller than it otherwise would be. This effect can be allowed for by adding sufficient small weights (*e.g.* pieces of wire) to the right-hand mass in order to cause the system to move with constant velocity (as tested by timing) when this mass is descending and there is no rider. Lastly, the pulley cannot be massless, and therefore requires a couple to give it angular acceleration. The effect of this is to add a constant term (actually the moment of inertia of the pulley divided by the square of its radius) to the total mass, $2M+m$. This term can be eliminated by replacing the masses M by another pair and repeating the observations.

Rotation of a Flywheel.—Suppose that a flywheel is pivoted with its axis of rotation horizontal. Let one end of a length of light string be attached to a point on the circumference of the wheel, while the other end is tied to a small mass m. When the wheel has been rotated several times, a certain amount of string will be wrapped round the rim, and the remainder will hang vertically from the rim under the action of the weight of m. The tension in the string will then cause the flywheel to rotate while m descends.

The downward acceleration f of the weight can be evaluated by determining the time t in which the weight falls through a measured height h after the system has been released from rest. We then have

$$f=\frac{2h}{t^2}$$

Denoting the tension in the vertical part of the string by T absolute force units, and applying Newton's second law to the mass m, we can write

$$mg-T=mf$$

or

$$T=m(g-f)$$

so that T can be evaluated from the observations mentioned above.

If a is the radius of the flywheel and I is its moment of inertia about its axis of rotation, its angular acceleration $\alpha(=f/a)$ can be related to T by taking moments about the axis and writing

$$Ta-\Phi=I\alpha$$

where Φ is a retarding couple due to friction. This equation shows that, if for a series of different values of m, we calculate corresponding values of Ta and α from observations of the motion, the graph relating these two quantities will be a straight line, provided that Φ is constant. The slope of the graph gives the value of I, while the intercept gives Φ.

There are several variants of this experiment, in some of which the string is wound round the axle instead of round the rim of the flywheel. In this case a is the radius of the axle.

2. PROBLEMS INVOLVING THE USE OF MOMENTUM

Force due to a Jet of Liquid.—Suppose a jet of liquid whose mass per unit volume (*i.e.* density) is ρ is delivered horizontally from a nozzle and strikes a vertical wall (Fig. 73). Let the linear velocity of the liquid be v as it strikes the wall and let the area of cross-section of the stream at this point be a. In unit time a volume of liquid equal to av has impinged on the wall. The mass of this volume is ρav, and its momentum before striking was ρav^2 in a horizontal direction. Therefore, in unit time this quantity of horizontal momentum is destroyed if we suppose that the liquid does not rebound at all but merely falls vertically under its own weight. The liquid must therefore have been acted upon by the wall with a force of ρav^2 absolute units, since this is its rate of change of momentum. The force which the jet exerts upon the wall is therefore also ρav^2 absolute units by Newton's third law.

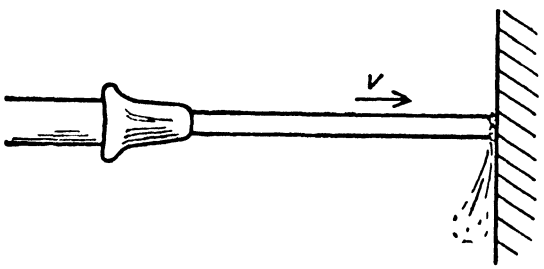


FIG. 73

It is, of course, fallacious to suppose that the liquid's horizontal momentum is completely destroyed. What actually happens depends upon the shape of the surface upon which the jet impinges. If this is curved so as to send the liquid back, the force will be increased because the rate of change of momentum will then be greater than ρav^2. The maximum possible force which the jet could exert would be $2\rho av^2$, but this would require that the liquid should be sent back with a velocity equal and opposite to v and therefore with unchanged kinetic energy. This can never be achieved in practice because some of the original energy of the liquid is always converted into heat.

Impact between Two Bodies.—We now return to a question which has already been partially dealt with on page 48. Suppose that the two spheres A and B in Fig. 74 collide so that their velocities are changed from u_A to v_A and from u_B to v_B respectively. Then by the conservation of momentum we have

$$M_A u_A + M_B u_B = M_A v_A + M_B v_B \quad . \quad . \quad . \quad (5)$$

where velocities are positive if they are directed from left to right. Equation (5) enables any one of the velocities to be determined provided that the other three velocities and M_A and M_B are given, but it does not

enable us to calculate the final velocity of each body when only the two initial velocities and the masses are given. In order to accomplish this a second relationship is necessary, and this is provided by an *experimental* law due to Newton which states that for bodies composed of a given *pair of materials* there is a constant ratio between their velocity of

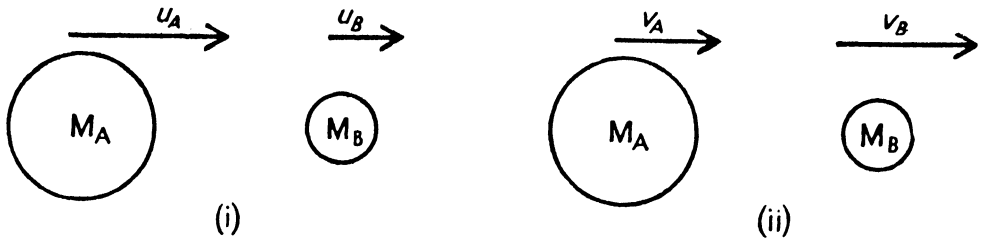


FIG. 74

separation after impact and their initial velocity of approach. If this ratio, which is called the **coefficient of restitution,** is denoted by e, we can write

$$\frac{v_B - v_A}{u_A - u_B} = \text{a constant} = e \quad . \quad . \quad . \quad . \quad (6)$$

The order in which the velocities are subtracted in equation (6) should be carefully noted. When all the velocities are positive, $v_B - v_A$ is the final velocity of *separation* (*i.e.* it is positive when the distance between the bodies is actually increasing) and $u_A - u_B$ is the initial velocity of *approach.*

The value of the coefficient of restitution necessarily lies between zero and unity. The former value occurs when the bodies adhere together upon collision. This is called **inelastic collision** and involves a loss of kinetic energy. Two lumps of clay or plasticine behave in this way. A value of unity for the coefficient (**perfectly elastic collision**) would imply that collision and rebound involved no loss of mechanical energy; for instance, a ball falling vertically on to a horizontal surface with which it had a coefficient of restitution of unity would bounce upward with an initial speed equal to that with which it struck the surface, and it would therefore rise to the level from which it was released. The mechanical energy of the system would not diminish and perpetual motion would ensue. Impact, however, involves the deformation of the colliding bodies. This is strikingly demonstrated by photographs of tennis and golf balls at the instant of their being struck. The subsequent rebound is due to the elasticity of the bodies, *i.e.* their tendency to regain their shape. During this deformation some heat is developed at the expense of the mechanical energy of the bodies, and the relative velocity of separation is always less than the initial velocity of approach. The coefficient of restitution, therefore, is always less than unity in practice. A few values of e are given in the following table:—

Substances	Coefficient of Restitution
Glass/glass . . .	0·94
Ivory/ivory . . .	0·81
Iron/iron . . .	0·66
Lead/lead . . .	0·20
Iron/lead . . .	0·13

Ball Bouncing on a Horizontal Surface.—Let the ball in Fig. 75 be released from a height h_1 above a horizontal surface with which its coefficient of restitution is e. Immediately before striking the surface its velocity, v_1, is given by

$$v_1 = \sqrt{2gh_1}$$

and, supposing that the horizontal surface is so rigidly fixed as to acquire no velocity, the upward velocity (v_2) of the ball immediately after impact is given by

$$v_2 = ev_1$$

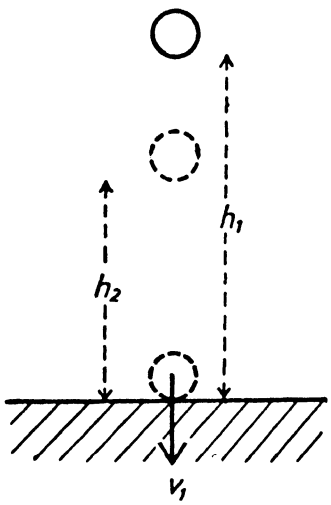


FIG. 75

If the ball rises to a height h_2, we have

$$v_2 = \sqrt{2gh_2}$$

so that $$\sqrt{2gh_2} = e\sqrt{2gh_1}$$

and $$e = \sqrt{\frac{h_2}{h_1}}$$

It will be noticed that this particular problem can be fully worked out without making use of the principle of the conservation of momentum. Indeed if we do apply this principle we find that, supposing the horizontal surface to be immobile, we obtain the result that $v_1 = -v_2$. This is because the surface *cannot* be immobile in fact. The nearest approach to this condition is achieved when the surface is rigidly fixed to the earth, in which case the momentum which the earth receives, *viz.* an equal and opposite amount to that which is lost by the ball, must be inserted in the equation which expresses the conservation of momentum. The velocity acquired by the earth as a result of the bouncing of the ball is, of course, extremely small on account of its comparatively enormous mass.

Hicks's Ballistic Balance.—In this apparatus two platforms are suspended at their corners by vertical strings arranged in such a way that the platforms can swing in one vertical plane only and remain horizontal while doing so (Fig. 76). When the platforms are in their lowest positions their edges are just in contact with each other, and when they collide they adhere together on account of the penetration of needle-points attached to one of the platforms into soft material, such as cork, which is stuck on to the other. It is first necessary to show how the velocity with which a platform reaches its lowest point after it is released from

a given position can be expressed in terms of the horizontal distance travelled (x). It is this distance which is measured during the experiment, the vertical displacement (y) being comparatively small. The dotted line

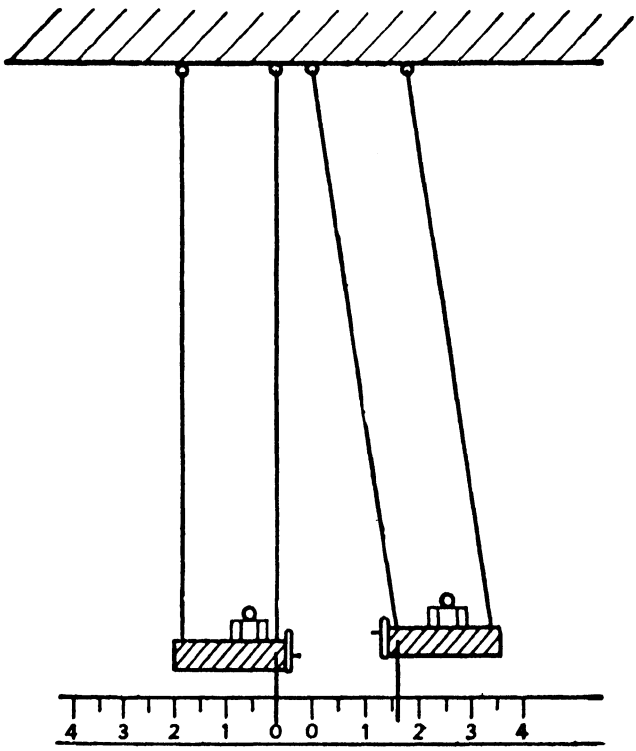


FIG. 76

AB (Fig. 77) is the path described by the lower end of one of the strings as the platform to which it is attached falls from its position of release to its lowest position. The line AB is, therefore, a circular arc, and if l is the length of each of the strings, the well-known relation between the segments of intersecting chords of a circle gives

$$(2l-y)y=x^2$$

If y is small in comparison with l, we can neglect it and write

$$2ly=x^2$$

or

$$y=\frac{x^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By the conservation of energy (page 78) the velocity v, with which a platform of mass M reaches its lowest point after being released, is given by

$$Mgy=\tfrac{1}{2}Mv^2$$

or

$$v^2=2gy \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus, combining equations (7) and (8), we have

$$v^2 = \frac{gx^2}{l}$$

or

$$v = x\sqrt{\frac{g}{l}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus it is possible to take v as being proportional to x since g and l are constant. The distance x is measured on a horizontal scale by means

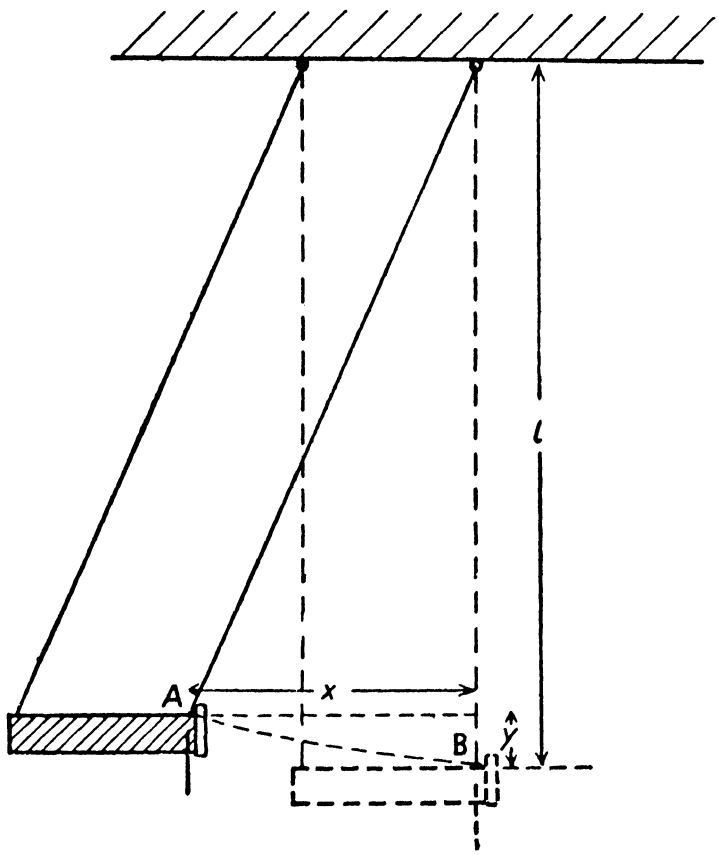


FIG. 77

of pointers attached to the platforms. Equation (9) also allows us to calculate the velocity with which a platform leaves its lowest position on being hit, provided that we observe the horizontal distance (x) through which it moves from that position before coming to rest.

There are several ways of experimenting with the balance, and in each case the calculation is based on the conservation of momentum.

As a general case, suppose that each platform is loaded so that the total masses of each are M_1 and M_2 respectively, and that they are released simultaneously from positions where the readings of their pointers on the horizontal scale are x_1 and x_2 on opposite sides of the zero. After

colliding and adhering together they swing to a horizontal distance x_3 from their position of rest. Then if v_1, v_2 and v_3 are the velocities at the lowest position corresponding to the horizontal displacements x_1, x_2 and x_3, we can express the conservation of momentum by the equation

$$M_1v_1 - M_2v_2 = (M_1 + M_2)v_3$$

Since v is proportional to x, we can write this as

$$M_1x_1 - M_2x_2 = (M_1 + M_2)x_3$$

Evidentally x_3 will have a positive sign if $M_1x_1 > M_2x_3$, which is the case when, after impact, the composite mass swings towards the side from which M_2 came. This equation can be verified if the masses M_1 and M_2 are previously known by weighing.

The student should consider for himself the possibilities of (*a*) adjusting the relative values x_1 and x_2 so that the bodies are completely stationary after collision, *i.e.* $x_3 = 0$; (*b*) making either x_1 or x_2 zero by drawing aside only one platform and leaving the other in its position of rest.

It might be questioned whether the collision between the platforms actually occurs at their lowest position when they are released simultaneously from different distances. Provided the strings are long compared with the horizontal displacement this is so, however, for the same reason that the oscillations of a simple pendulum take the same time whatever their amplitude, provided this is small.

A similar arrangement to the ballistic balance may be used for studying the elastic collision between two bodies such as ivory balls, and the coefficient of restitution between them can be determined experimentally. The student should be able to verify that the coefficient is equal to the ratio of the horizontal distance to which the balls separate after the collision to the horizontal distance between their initial points of release.

Ballistic Pendulum.—This consists of a box containing sand and suspended by strings of length l as shown in Fig. 78, so that as it swings it does not rotate. While the pendulum is hanging at rest, a bullet of mass m is fired into the side of the box with a velocity u. The pendulum swings to one side through a horizontal distance x, the bullet remaining embedded in the sand. We have already shown in connection with the ballistic balance that the velocity with which the pendulum leaves its position of rest is equal to $x\sqrt{\frac{g}{l}}$ (equation (9)), so that if M is the mass of the box and sand, the conservation of momentum gives

$$mu = (M + m)x\sqrt{\frac{g}{l}} \quad . \quad . \quad . \quad . \quad (10)$$

from which the velocity of the bullet u can be calculated if x is observed.

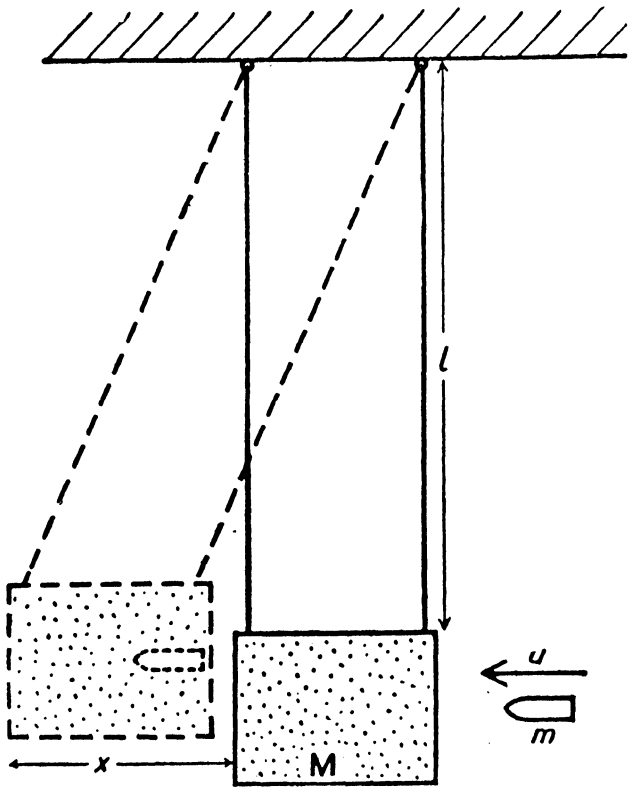


FIG. 78

3. MOTION IN A CIRCLE

Centripetal and Centrifugal Forces.—On page 35 it was shown that when a body is travelling with a speed v in a circular path of radius r its constantly changing direction implies that it is suffering an acceleration of v^2/r directed towards the centre of the circle. According to Newton's second law, therefore, the body is under the influence of a force which is also directed towards the centre and is equal to mv^2/r, where m is the mass of the body. This is called the **centripetal** force. It causes the deviation from the rectilinear motion which, by Newton's first law, would persist if the force were absent. When a stone is swung round in a sling, the centripetal force which keeps it in its circular path is the tension in the sling. (The movements of the hand with which the sling is set in motion are necessary in order to speed it up and keep it moving against the resistance of the air. The *circular* character of the motion of the stone is due simply to the tension.) By Newton's third law the stone exerts on the sling a **centrifugal** force which is equal and opposite to the centripetal force exerted upon it by the sling. Other examples of this kind of motion are to be found, for instance, when a vehicle turns a corner, and in the rotation of the planets in orbits about the sun which attracts them with a gravitational force.

The Banking of a Track.—A vehicle such as a car or a railway engine cannot *of itself* change the direction of its motion. For this to occur an *external* force must act upon it through its contact with the road or railway lines. In the case of a car on a horizontal road surface the centripetal force is provided by the friction which is brought into play between the front wheels and the road when the steering gear is actuated. The force necessary for the successful negotiation of a corner (*i.e.* mv^2/r) must not exceed the maximum value of the frictional force. Thus skidding occurs when v is too large and r too small.

Safety in cornering can be achieved by "banking" the track, *i.e.* by constructing it so that it slopes at right angles to the direction of motion and downwards towards the centre of the curve. In this way it is possible for a vehicle to turn a corner without any reliance being placed on friction.

Suppose that the conditions under which the car in Fig. 79 is turning

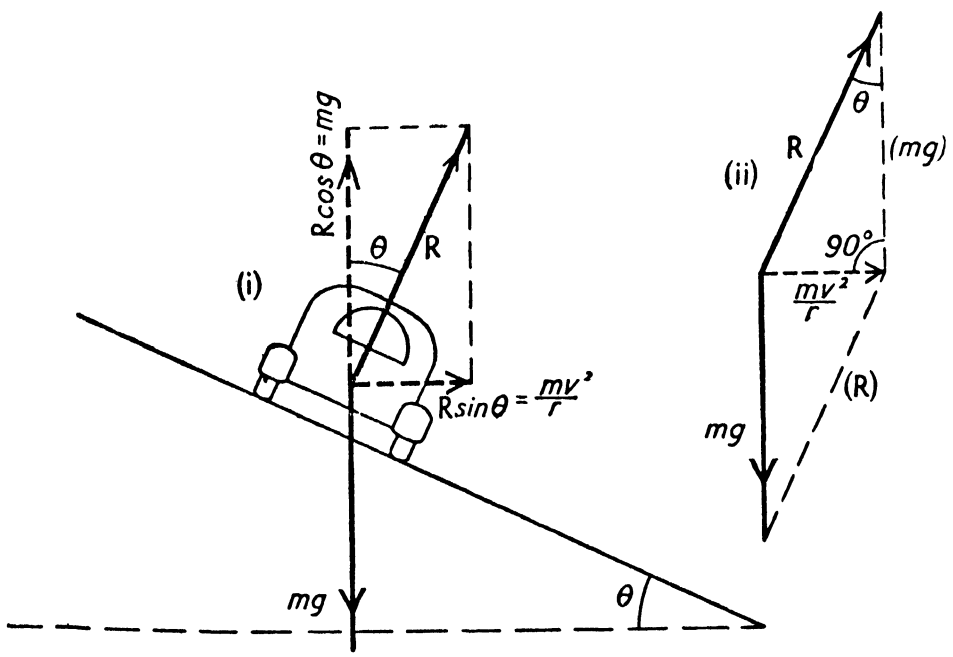


FIG. 79

to the right are such that friction contributes nothing to the centripetal force. This force is therefore derived entirely from the normal reaction of the track, R.

The vertical component of R is $R \cos \theta$ and must be equal and opposite to the weight of the car mg, since the car is supposed to remain in the same horizontal plane and therefore has no vertical acceleration. Thus

$$R = \frac{mg}{\cos \theta}$$

The horizontal component of R is $R \sin \theta$ and must be equal to the

centripetal force (in fact it *is* the centripetal force), since mg has no horizontal component. Therefore

$$\frac{mv^2}{r} = R \sin \theta$$
$$= \frac{mg \sin \theta}{\cos \theta}$$
$$= mg \tan \theta$$

Thus

$$\tan \theta = \frac{v^2}{gr}$$

Another way of looking at the matter is shown in Fig. 79 (ii). The two forces R and mg combine to give a horizontal force of mv^2/r. The above expression for tan θ can be written down by inspection of this figure.

For a given speed and radius of curve, therefore, a definite angle of banking is required in order to turn the corner without relying on friction. In a car-racing track the banking is graded, becoming steeper towards the outer rim, and the driver chooses a position appropriate to his speed.

The Centrifuge.—When small particles of matter are in suspension in a liquid whose density (ρ) is lower than theirs (σ), they gradually settle to the bottom under the action of gravity. The downward force on a particle of volume V is given on page 192 as

$$Vg(\sigma - \rho)$$

Particles whose density is lower than that of the liquid move upwards.

There are many examples of the use of this principle to achieve a separation, *e.g.* the less dense fat particles which constitute the cream can be skimmed from the surface of milk after it has been allowed to stand. Such processes are often slow, however, and a machine called the **centrifuge** has been devised which produces results more rapidly. The principle is illustrated in Fig. 80 (i) and (ii). A number of tubes similar to test-tubes are filled with the suspension and hung in the machine, which is then set in motion, with the result that they are made to rotate about an axis in the same way as the horses on a roundabout. The tubes are hinged near their tops so that their lower ends rise like the governors of a steam-engine. Thus the tubes are eventually horizontal and whirling round in a horizontal plane about a vertical axis O.

If the particle at A in Fig. 80 (ii) were replaced by the liquid which it displaces, this portion of liquid would be moving round O in a circle of constant radius r and would not drift along the tube. Therefore the centripetal force exerted upon it by the surrounding liquid must be just sufficient to maintain this circular motion, *i.e.* it must be equal to $V\rho v^2/r$ where V is the volume of the particle and ρ is the density of

the liquid. This is, therefore, the force exerted on the particle. But in order that the actual particle of density σ should describe a circular path of radius r with a speed v, a centripetal force of $V\sigma v^2/r$ would be necessary. If $\sigma > \rho$, therefore, the actual centripetal force falls short of what is required by

$$V\frac{v^2}{r}(\sigma - \rho)$$

Thus the particle drifts towards the closed end of the tube for the same reason that a car skids when the friction of the road fails to provide a

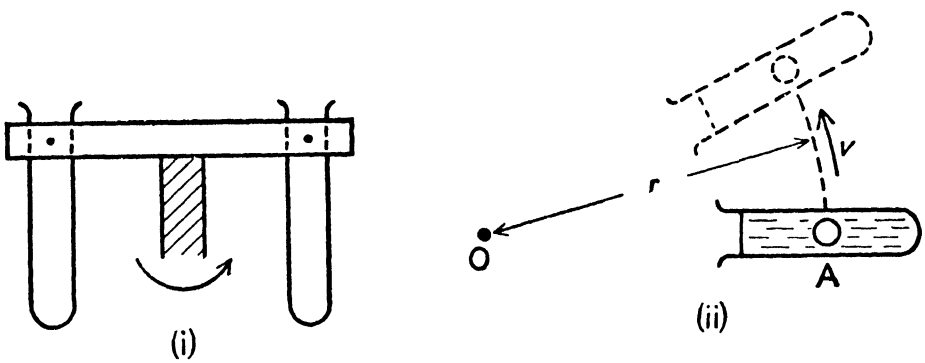


FIG. 80

sufficiently large centripetal force for the turning of a particular corner at a given speed.

If the above expression is compared with the previous one for the downward force due to gravity on a particle in suspension, it is seen that, in the case of the centrifuge, g is replaced by v^2/r. By making v^2/r large, therefore, it is possible to establish conditions in the centrifuge which correspond to very large values of g. In this way sedimentation can be greatly speeded up, and it is even possible to cause drifts of actual molecules which would not occur under the action of gravity. With modern technique it has been possible to obtain values of v^2/r approaching a million times that of gravity. In such centrifuges the dimensions are comparatively small, and their construction has necessitated the solving of many unusual mechanical problems on account of the very high speeds involved.

Planetary Motion.—Let a planet of mass m be moving round a sun of mass M in a circular orbit of radius R (Fig. 81). The force of attraction with which M acts upon m is, by Newton's law of gravitation (page 49),

$$G\frac{mM}{R^2}$$

It is this force which holds the planet in its circular path and must be equal to mv^2/R where v is the linear speed with which the planet is

describing its orbit. Thus

$$G\frac{mM}{R^2}=\frac{mv^2}{R}$$

$$=\frac{m}{R}\left(\frac{2\pi R}{T}\right)^2$$

where T is the time taken by the planet to complete its orbit, *i.e.* the length of the planet's year. Rearrangement of the last equation gives

$$T^2=\frac{4\pi^2R^3}{GM}$$

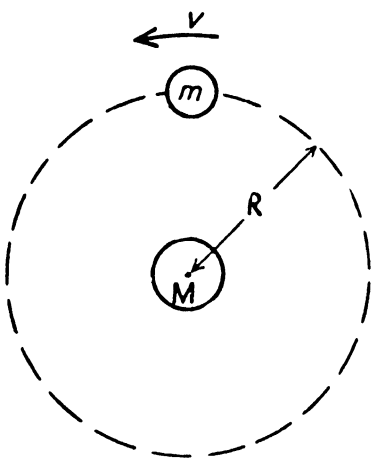


FIG. 81

This shows that for any particular "sun" having a number of planets the square of the duration of each planet's year is proportional to the cube of the radius of its orbit. In the early part of the seventeenth century Kepler enunciated three laws concerning the motion of the planets of the solar system. He based his statements on the astronomical observations which were then available, and his third law stated the above relationship between the length of the planets' years and the radii of their orbits. We have just shown that Newton's law of gravitation leads to this result—a fact which provides very strong evidence in favour of Newton's theory. It can be shown that the ellipse as well as the circle is a possible orbit when the force of attraction obeys an inverse square law and, in fact, the planets of the solar system do describe ellipses.

It should be realized that the gravitational force merely holds the planet in its orbit. It performs no work, and the energy of the system remains constant, being equal to that which the planet originally possessed when the solar system came into existence.

Exactly the same principles, of course, apply to the moon's motion round the earth, and the last equation expresses the relation between the radius of the moon's orbit (R) and the time required for the moon to make one complete rotation (T). In this case the mass M is, of course, that of the earth. Let the radius of the earth be r. Then if g is the value of the acceleration due to gravity at the earth's surface, *i.e.* the weight of a mass of 1 gm., we have

$$g=\frac{GM}{r^2}$$

and substituting for GM in the previous equation, we obtain

$$T^2=\frac{4\pi^2}{g}\cdot\frac{R^3}{r^2}$$

Substituting the known values

$$R = 3{\cdot}84 \times 10^{10} \text{ cm.}$$

and

$$r = 6{\cdot}37 \times 10^{8} \text{ cm.}$$

we have

$$T = \frac{2\pi}{6{\cdot}37 \times 10^{8}} \sqrt{\frac{3{\cdot}84^3 \times 10^{30}}{980}}$$

$$= 2{\cdot}37 \times 10^{6} \text{ seconds, approximately}$$

$$= \frac{2{\cdot}37 \times 10^{6}}{86400} \text{ days}$$

$$= 27{\cdot}5 \text{ days, approximately}$$

This result is in excellent agreement with the actual observed time.

4. SYSTEMS PERFORMING SIMPLE HARMONIC MOTION

The Criterion for Simple Harmonic Motion.—Simple harmonic motion has been defined on page 36 and its characteristics worked out. It is a type of motion which can be recognized by several of its features, but the criterion usually adopted is

$$\frac{\text{Acceleration of the body towards its position of rest}}{\text{Distance of the body from its position of rest}} = \text{a positive constant}$$

By the position of rest of the body is meant the position from which it can be released without acquiring any motion; that is to say, the position in which it experiences no resultant force.

It has also been shown on page 36 that if the positive constant referred to in the criterion is denoted by ω^2, then the time period of the oscillation T is given by

$$T = \frac{2\pi}{\omega}$$

Thus the general method of testing theoretically whether the motion of an oscillating body is S.H.M. is to evaluate the ratio of the acceleration to the distance by an application of Newton's laws or their equivalent (such as the conservation of mechanical energy). If the ratio is found to be a positive constant the motion can be said to be S.H.M., and its time period can be calculated by the last equation.

The criterion applies equally well for both linear and angular acceleration and displacement.

Body Suspended by a Helical Spring.—Fig. 82 depicts the system. It is assumed that the spring obeys **Hooke's law,** which states that its extension is proportional to the extending force. Experiment shows that this is frequently true provided the extension is not too large. The

upper end of the spring is fixed, and if the mass M is raised or lowered and is then released it will perform oscillations up and down a vertical line. At any instant during the oscillation let the distance of the mass from its position of equilibrium be x. The length of the spring therefore differs from its equilibrium length by an amount x, so that by Hooke's law we can say that at the instant considered it is being extended or compressed by a force proportional to x. Let this be μx, where μ is a constant independent of x but depending on the dimensions and material of the spring. The constant μ us the force (in absolute units) necessary to extend or compress the spring by one unit of length, and in the C.G.S. system it would be expressed in dynes cm.$^{-1}$.

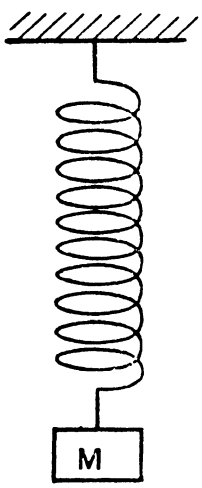


FIG. 82

From Newton's third law it follows that the mass M is acted upon by a force μx in the opposite direction to the force acting on the spring. Thus if M is at a distance x below its equilibrium position, the spring is extended and its lower end is acted on by a *downward* force μx, so that M is acted upon by an equal *upward* force. Similarly when M is a distance x above its position of rest it experiences a *downward* force μx due to the compression of the spring. The force on M is always directed towards its position of rest, and for this reason it is called the **restoring force** on M.

If at the instant when it is at a distance x from its equilibrium position the mass M has a vertical acceleration f in the direction of the restoring force, we have by Newton's second law

$$Mf = \mu x$$

and f is always directed towards the position of rest because the force μx is so directed. Thus

$$\frac{f}{x} = \frac{\mu}{M} = \text{a positive constant} = \omega^2$$

and the motion is S.H.M. Furthermore, the time period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{\mu}} \quad . \quad . \quad . \quad . \quad (11)$$

In establishing this expression for T we have taken it for granted that the only moving mass is that of the body hanging on the spring, and this is not perfectly true. The coils of the spring itself have mass and vibrate up and down with amplitudes diminishing from the lowest coil upwards. It is possible, by making certain assumptions, which are frequently not

justifiable in practice, to calculate the correction necessary for the mass of the spring, and the following equation results:—

$$T = 2\pi\sqrt{\frac{M + \frac{1}{3}m}{\mu}} \quad . \quad . \quad . \quad . \quad (12)$$

where m is the mass of the spring.

Without assuming the factor 1/3, suppose that the correction to be added to M is km, where k is assumed to be independent of T and M. The quantity km is called the **effective mass** of the spring, and we write

$$T = 2\pi\sqrt{\frac{M + km}{\mu}}$$

or, squaring and rearranging,

$$M = \frac{\mu}{4\pi^2} \cdot T^2 - km \quad . \quad . \quad . \quad . \quad (13)$$

Thus if we alter the value of M by attaching different known masses to the spring and observe the time period for each one, a straight-line graph should be obtained when M is plotted against T^2 (Fig. 83). When the best straight line passing through the experimental points is produced backwards so as to meet the axis of M at A, the intercept OA, *i.e.* the value of M when T^2 is zero, is seen by equation (13) to be equal to $-km$. Thus km can be read off on the scale used on the M axis and the value of k found if the mass of the spring is known by weighing.

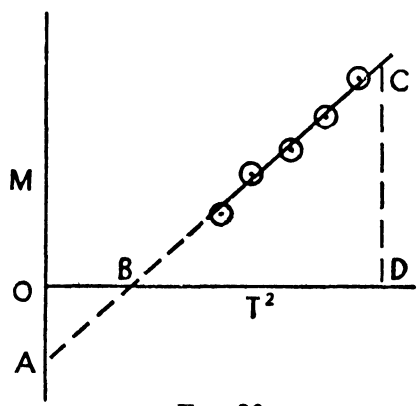


FIG. 83

Again, the slope of the graph is CD/BD, the lengths being measured on the appropriate scales, and this is equal to the coefficient of T^2 in equation (13), so that

$$\mu = 4\pi^2 \cdot \frac{\text{CD}}{\text{BD}}$$

Thus μ can be determined.

Now let a "static experiment" (as opposed to the foregoing "dynamic experiment") be performed in which the steady extension of the spring is observed for different masses attached to its end. This can be done by fixing a horizontal pointer to the lower end of the spring, erecting a vertical millimetre scale and using a series of hooked weights (Fig. 84). A graph of the extension against the mass attached will be a straight line if Hooke's law is obeyed, and it will pass through the origin if extensions

are measured from the position of the pointer when no load is attached. The slope of this graph will give the mass required to produce unit extension. Let this be M_1. Then the *force* required to produce unit extension is $M_1 g$, so that

$$\mu = M_1 g$$

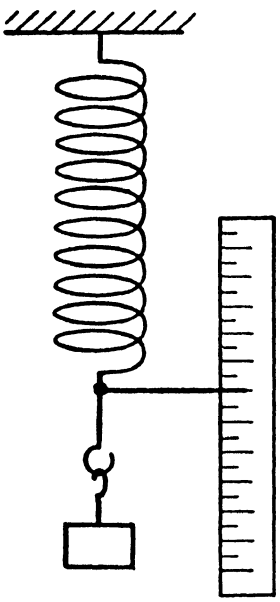


FIG. 84

Since μ has already been found from the slope of the graph in the dynamic experiment, a value for g, the acceleration due to gravity, can be obtained.

The combination of the dynamic and static experiments to find the value of g and the effective mass of the spring is a common laboratory exercise. Actually it is not necessary to have a set of known weights in order to determine g. It is obvious that this must be so, since the units of acceleration do not include mass. Suppose we merely have a set of different unknown masses and hang each in turn on the spring, observing both the time period (T) and the static extension (x) from the unloaded position in each case. Then for any particular one of the masses M we have

$$T = 2\pi\sqrt{\frac{M+km}{\mu}}$$

and

$$Mg = \mu x$$

Eliminating M from these two equations, we obtain

$$x = \frac{g}{4\pi^2} \cdot T^2 - \frac{gkm}{\mu}$$

Therefore a straight-line graph should be obtained by plotting the extension (x) produced by each unknown mass against the square of time period when the mass is attached. The slope of the graph is equal to $g/4\pi^2$, so that g can be determined.

The Simple Pendulum.—This consists of a small bob (theoretically a **particle** of no size) hanging by a light inextensible thread from a fixed point of suspension. The bob is drawn aside and released, and the pendulum swings in a vertical plane. Suppose that at any instant during the motion the string (which, of course, always remains straight and taut) makes an angle θ with the vertical (Fig. 85). The weight of the bob, mg, can be resolved into two components, *viz.* $mg \cos\theta$ parallel to the direction of the string and $mg \sin\theta$ in a direction tangential to the path of the bob. The instantaneous acceleration f of the bob in the direction

of this force is due entirely to the latter component (the tension in the string being perpendicular to f), so that

$$mf = mg \sin \theta \quad . \quad . \quad . \quad (14)$$

or

$$f = g \sin \theta$$

The angular acceleration of the pendulum α (page 53) is given by

$$\alpha = \frac{f}{l} = \frac{g}{l} \cdot \sin \theta$$

while the angle it makes with its position of rest is θ. To test for S.H.M. we examine the ratio of α to θ. Thus

$$\frac{\alpha}{\theta} = \frac{g}{l} \cdot \frac{\sin \theta}{\theta}$$

and the motion is *not* S.H.M., because the right-hand side of this equation is not constant. However, if we suppose that the oscillations are sufficiently small to enable us to write

$$\frac{\sin \theta}{\theta} = 1$$

the right-hand side becomes simply g/l, which is constant. Under these conditions, therefore, the oscillations of the pendulum are simple harmonic, and

$$\frac{g}{l} = \omega^2$$

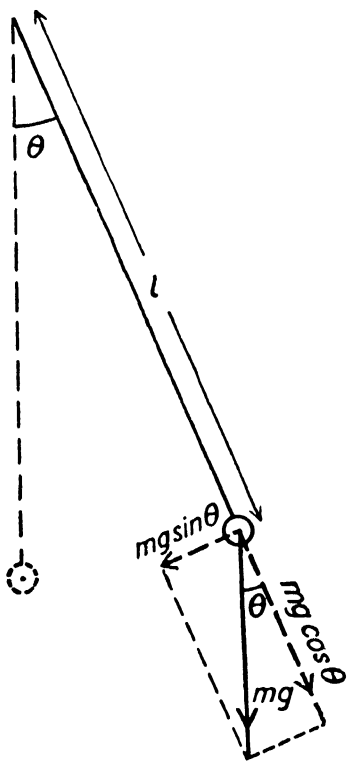


FIG. 85

so that the time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad (15)$$

It must be emphasized that this formula applies only to small oscillations. When θ is 5° its circular measure is 0·0873 radian and its sine is 0·0872, so that the condition that $\frac{\sin \theta}{\theta}$ is unity is fulfilled to one part in 870, but when θ is 10° it is only true to one part in 170. Within the limits in which the approximation is valid the time period is independent of the amplitude of the oscillation. This is a property of all systems which vibrate with simple harmonic motion.

The simple pendulum provides a fairly accurate method of determining g. Frequently the bob is a small sphere of lead or iron, and the length l must be taken from the point of suspension to the centre of the bob.

The equation for the time period of small oscillations may be verified experimentally by finding T for various lengths of the pendulum (*e.g.* by timing 100 complete oscillations and dividing the total time by 100) and then plotting a graph of T^2 against l. Since

$$T^2 = \frac{4\pi^2}{g} \cdot l$$

the graph should be a straight line passing through the origin (Fig. 86). The slope $\left(\frac{AB}{OB}\right)$ will be equal to $\frac{4\pi^2}{g}$, so that

$$g = 4\pi^2 \cdot \frac{OB}{AB}$$

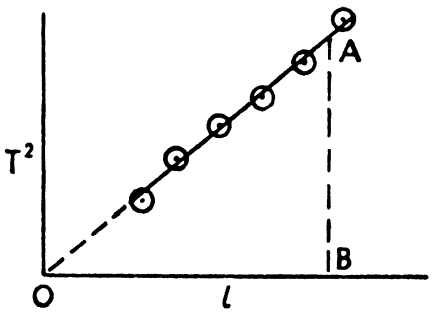


FIG. 86

where OB is measured in the units used on the l axis and AB in those on the T^2 axis.

The length of a simple pendulum which has a period of two seconds is equal to $\frac{g}{\pi^2}$, and if g is taken as 981 cm. sec.$^{-2}$ this length is very nearly one metre. Such a pendulum is said to "beat seconds"—it passes through its vertical position at intervals of one second, first in one direction and then in the other, and is called the **seconds pendulum.** Its length, of course, varies slightly from place to place because of the variation of g. The length of a simple pendulum whose *period* is one second is one-quarter of that of the seconds pendulum, *i.e.* about 25 cm.

It will be noticed that the mass of the bob does not appear in the expression for the time period of a simple pendulum. This is because m was cancelled from both sides of equation (14). On the left-hand side m represents the mass of the bob and on the right-hand side mg is its weight. The fact that the time period for a given length of pendulum is independent of the mass of the bob really depends, therefore, on the proportionality of mass and weight. Newton showed experimentally that, for a given length, the time period is always the same no matter what the mass of the bob, and he regarded this as a proof that mass and weight are proportional.

The Compound Pendulum.—Any rigid body pivoted so as to be able to swing in a vertical plane is described as a "compound pendulum". A pendulum of irregular shape is shown in Fig. 87. It is pivoted at the point O, which is called the **centre of suspension.** In its equilibrium position the pendulum hangs with its centre of mass, G, vertically below O, and if it is drawn aside and released it will oscillate under the action of its weight Mg, the point of application of which is G. If OA is a horizontal line meeting the vertical line through G in A, when the pendulum is in the position shown in Fig. 87, then by taking moments about a

horizontal axis through O perpendicular to the plane of motion, we have

$$Mg \times \mathrm{OA} = I_O \alpha$$

or
$$Mgh \sin\theta = I_O \alpha \quad . \quad . \quad (16)$$

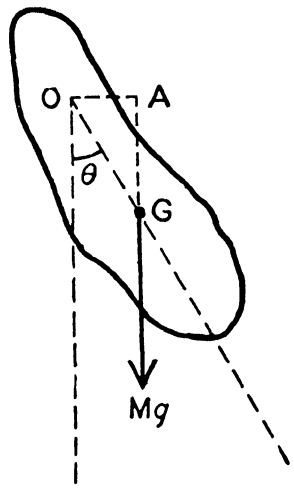


FIG. 87

where h is the distance OG, I_O is the moment of inertia of the body about the axis of rotation through O, and α is the instantaneous angular acceleration towards the position of rest.

We test whether the motion is simple harmonic by evaluating α/θ, and obtain

$$\frac{\alpha}{\theta} = \frac{Mgh}{I_O} \cdot \frac{\sin\theta}{\theta}$$

Thus, as with the simple pendulum, the oscillations are not simple harmonic unless the angle θ is always small enough to enable us to use the approximation

$$\frac{\sin\theta}{\theta} = 1$$

With this condition,

$$\frac{\alpha}{\theta} = \frac{Mgh}{I_O} = \text{a positive constant} = \omega^2$$

so that the motion is S.H.M. and the time period is given by

$$T = 2\pi\sqrt{\frac{I_O}{Mgh}} \quad . \quad . \quad . \quad . \quad (17)$$

By the theorem of parallel axes (page 63) we can write

$$I_O = I_G + Mh^2$$

where I_G is the moment of inertia of the body about an axis through G perpendicular to the plane of rotation. Writing Mk^2 for I_G, where k denotes the radius of gyration of the body about the axis through G, we obtain

$$I_O = M(k^2 + h^2)$$

Thus

$$T = 2\pi\sqrt{\frac{k^2 + h^2}{hg}} \quad . \quad . \quad . \quad . \quad (18)$$

The time period of an oscillating system is sometimes specified by stating what is called the length of the "equivalent simple pendulum," *i.e.* the simple pendulum which has the same time period. If its length is l in

the present case, then we have

$$T = 2\pi\sqrt{\frac{l}{g}}$$

and comparing this with equation (18), we obtain

$$l = \frac{k^2 + h^2}{h}$$

or

$$h^2 - lh + k^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Now, for a given compound pendulum the value of k is fixed, and since equation (19) is a quadratic in h, it follows that for any given time period as specified by a given value of l there are, in general, two values of h, *i.e.* two possible distances of the centre of suspension from G. These distances are the two roots of equation (19). Let them be denoted by h_1 and h_2. Then by the theory of quadratic equations we have

Sum of the roots = – (coefficient of h)
Product of the roots = constant term

So that

$$h_1 + h_2 = l \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$h_1 h_2 = k^2 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It is only the *distance* of the centre of suspension from G which governs the time period, so that the pendulum in Fig. 88 would have the same time period for any point of suspension on either of the two circles described with radii h_1 and h_2 and centre G. In particular, consider a pair of such points which fulfil the following conditions:—

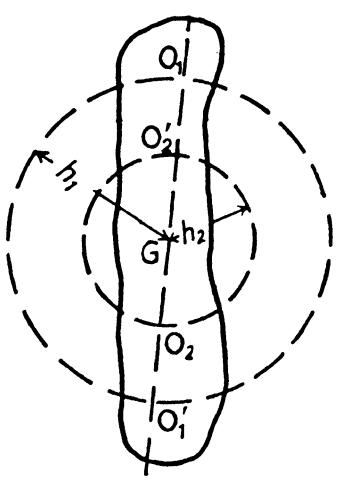


FIG. 88

(*a*) They lie on a straight line passing through G.
(*b*) They lie on opposite sides of G.
(*c*) They are *not* equidistant from G.

In Fig. 88 O_1 and O_2 are a pair of such points, and evidently

$$O_1O_2 = h_1 + h_2 = l$$

Therefore, in general, for any centre of suspension such as O_1 we can find another centre of suspension, O_2, giving the same time period and fulfilling the above conditions. The distance between the two points $(h_1 + h_2)$ is equal to the length of the simple pendulum having the given time period, so that

$$T = 2\pi\sqrt{\frac{\overline{O_1O_2}}{g}}$$

From this equation g can be calculated if the points O_1 and O_2 are located, their distance apart measured, and T determined. It will be realized, of course, that when oscillating about O_2 the pendulum is inverted with respect to its position when it is suspended from O_1. The other pair of points O_1' and O_2' also fulfil the same conditions and give the same time period, so that in general there are four points lying on a given straight line through G about which the pendulum would have the same given time period.

The point O_2 is called the **centre of oscillation** corresponding to the centre of suspension O_1. *Vice versa*, O_1 is the centre of oscillation for oscillation about O_2. The position of the centre of oscillation can be discovered by suspending a simple pendulum from the same point in front of the compound pendulum. The length of the former is then adjusted until its time period is equal to that of the compound pendulum, and when both are at rest the bob of the simple pendulum hangs in front of the centre of oscillation (Fig. 89). If the centre of mass (which is identical with the centre of gravity) is also located by finding the point about which the compound pendulum will balance, or by other suitable means, the lengths h_1 and h_2 can be measured and the radius of gyration of the pendulum k can be calculated from equation (21), *i.e.*

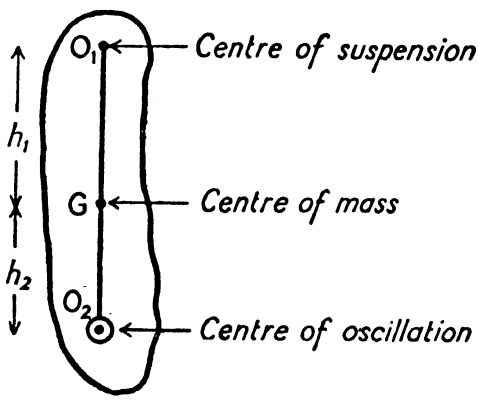


FIG. 89

$$k = \sqrt{h_1 h_2}$$

A comparison of this equation with equation (15) on page 65 shows that centre of oscillation and centre of percussion are identical.

Determination of g by a Compound Pendulum.—The principle of the interchangeability of the centres of suspension and oscillation has been used in accurate determinations of g. The disadvantage of the simple-pendulum method lies in the impossibility of setting up a simple pendulum in practice. For instance, the bob can never be made indefinitely small, and the string can never be weightless and at the same time inextensible. Captain Kater in 1817 was the first to use the principle of what is called the **convertible pendulum,** although the theory of the pendulum had been known for many years previously. Fig. 90 shows Kater's pendulum diagrammatically. It had two fixed knife edges, k_1 and k_2, on opposite sides of G and at different distances from it. The "great weight" W was fixed, but the position of G could be altered by adjusting the positions of the smaller weights w_1 and w_2. Adjustments were made until the time periods about both knife edges were judged to be equal over periods of trial extending

up to 24 hours. The value of g was then given by the simple-pendulum equation, l being the distance between the knife edges and T the common time period.

Since Kater's time it has been found necessary to make allowances for

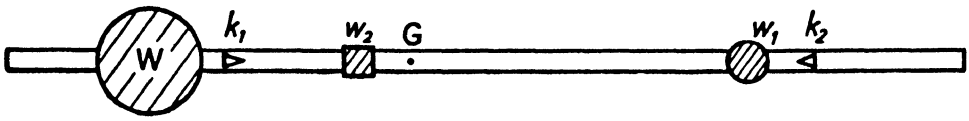


FIG. 90

certain small errors due to the impossibility of obtaining perfectly sharp knife edges and a perfectly rigid support for the agate plane on which the knife edges rest in turn. The effect of the surrounding air has also to be taken into account.

An instructive experiment on the properties of the compound pendulum can be carried out with a uniform metal rod having holes drilled along its length at regular intervals. A horizontal knife edge clamped at the edge of the bench can be inserted in each hole in turn so that the bar hangs freely from it. The time period is measured for each hole on both sides of the centre of mass. A graph of the time period, T, against the distance of the knife edge from one end of the bar is shown in Fig. 91.

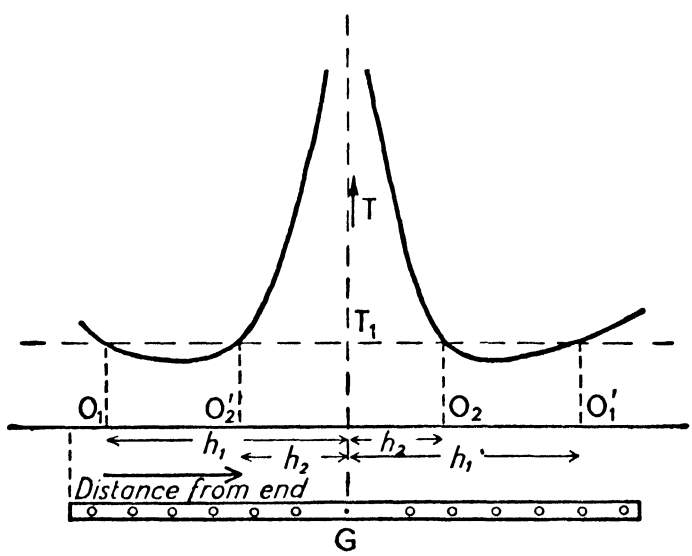


FIG. 91

The two halves of the curve are exactly symmetrical. A horizontal line corresponding to a time period T_1 cuts the graph, in general, at four points, which are O_1, O_2, O_1' and O'_2. The mean of O_1O_2 and $O_1'O_2'$ as measured on the graph is taken. Let it be l. Then

$$T_1 = 2\pi\sqrt{\frac{l}{g}}$$

from which g can be calculated. Further results can be obtained by taking another horizontal line corresponding to another time period.

When the roots of the quadratic equation (19) are equal, each is equal to k since their product is always k^2. The distances h_1 and h_2 are equal at the minima, so that the horizontal distance between the two minima is equal to $2k$. It will be noticed that it is not necessary to locate G in order to calculate either g or k.

If the pendulum is made unsymmetrical, *e.g.* by loading one end, the curves obtained are still symmetrical as regards their shape, but since G is now nearer the loaded end, the curve at that end is cut off sooner than in Fig. 91, while the other portion extends over a correspondingly larger range.

Torsional Oscillations.—In Fig. 92 a rigid body is suspended by a wire attached to a fixed support. If the body is twisted about a vertical axis coinciding with that of the wire and then released, it performs rotational oscillations. At any instant during the oscillation let the bottom of the wire be twisted through θ radians with respect to the top. In order to keep it stationary in this position it would be necessary to act upon the body with a couple equal and opposite to the restoring couple due to the twist in the wire. It is an experimental fact that the couple required is proportional to θ, and we can call it $N\theta$, where N is the couple required to maintain a twist of 1 radian, and is called the **torsional constant** of the wire. When it is oscillating freely the body is therefore acted upon by a restoring couple $N\theta$ at the instant at which the twist in the wire is θ, and its equation of motion is

FIG. 92

$$N\theta = Ia \quad . \quad . \quad . \quad . \quad (20)$$

where a is its instantaneous angular acceleration towards its position of rest and I is its moment of inertia about its axis of rotation. From equation (20) we have

$$\frac{a}{\theta} = \frac{N}{I} = \text{a positive constant} = \omega^2$$

Thus the motion is simple harmonic, without the limitation to small oscillations which we have had in the case of pendulums, and

$$T = 2\pi\sqrt{\frac{I}{N}} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If I is known, *e.g.* if the body is a cylinder rotating about its geometrical axis, N can be calculated when T has been observed. This having been done, the wire may be used for the determination of moments of inertia

of bodies of irregular shape which are not susceptible to calculation by simple formulae.

EXAMPLES VIII

1. A rectangular block resting on a smooth table is connected, by means of a light thread which passes over a light frictionless pulley, to an equal mass hanging freely. Find the acceleration of the system. (L.Med.).

2. A block of mass 300 gm. is placed on a plane inclined at 30° to the horizontal, the coefficient of sliding friction being 0·2. A light string attached to the block passes over a pulley at the top of the plane so that it is parallel to the line of greatest slope, and a mass of 250 gm. is suspended at the other end of the string. Calculate the acceleration of the block and the tension in the string.

3. Define *poundal* and *pound weight.*

A man of mass 150 lb. stands in a lift. Calculate in lb. wt. the force which the floor of the lift exerts on his feet when the lift has an acceleration of 2 ft. sec.$^{-2}$ (*a*) upwards, (*b*) downwards. In what circumstances is the force equal to 150 lb. wt.? (L.I.)

4. A light string, which passes over a frictionless pulley and hangs vertically on each side of it, carries at each end a mass of 100 gm. What is the tension in the string? A mass of 1 gm. is added at one end of the string. Calculate the acceleration of the masses and the tension in the string. (L.I.)

5. Define the terms *force* and *momentum* and state the relation between them.

Two masses A and B of $3\frac{3}{4}$ lb. and $4\frac{1}{4}$ lb. respectively hang freely from the ends of a light string passing over a smooth peg. The system is released and 1 second later a mass C of 1 lb. is removed from B. Calculate the tension in the string and the acceleration of the masses before and after the removal of C, explaining the method used. Through what further distance will A move before coming to rest? (L.M.)

6. Two blocks, each of 4 lb. mass, are attached to each other by a light string P and are pulled over the surface of a smooth horizontal table by a 1 lb. weight attached by another string Q to one of the blocks and hanging over the edge of the table. Calculate the acceleration of the 1 lb. weight and the tension in each of the strings when the system is moving freely with P taut and at right angles to the edge of the table.

Find the acceleration of the 1 lb. weight and the tension in Q if the motion of the blocks over the table is opposed by a constant force of 0·5 lb. wt. (L.M.)

7. State Newton's second law of motion and use it to calculate:

(*a*) The force exerted on a vertical wall by a horizontal jet of water of diameter 2 cm., the rate of flow being 1 litre per sec. (Assume that the water strikes the wall normally and does not rebound.)

(*b*) The reaction exerted by the floor of a lift on a mass of 150 lb. resting on it when the lift has an upward acceleration of 12 ft. sec.$^{-2}$. (L.I.)

8. How is force related (*a*) to momentum, (*b*) to energy?

A jet of water, 1 sq. in. in section, is projected horizontally, with a velocity of 48 ft. sec.$^{-1}$, against a vertical wall. Find (i) the force, in lb. wt., exerted on the wall, supposing that the water does not rebound; (ii) the horse-power required to drive the jet. Assume that 1 cub. ft. of water weighs 62·5 lb. and that 1 horse-power is 550 ft. lb. wt. per sec. (L.M.)

9. A gun weighing 70 lb. fires a bullet weighing 0·1 lb. with an initial horizontal velocity of 1750 ft. sec.$^{-1}$. If the gun is free to recoil on a smooth horizontal plane, find its momentum and kinetic energy when the bullet leaves it.

If the gun fires a stream of bullets at the rate of 240 a minute, find the average force in lb. wt. required to keep the gun at rest. (O.H.S.)

10. Distinguish between the principles of *conservation of mechanical energy* and *momentum.*

A bullet of mass 15 gm. travelling horizontally with a velocity of 200 metres

sec.$^{-1}$ embeds itself in a block of wood of mass 5 kg. resting on a horizontal surface. If the coefficient of kinetic friction between the block and the surface is 0·1, calculate the distance travelled by the block. (L.I.)

11. Define *kinetic energy.*

A 28-lb. shot is fired with an initial velocity of 1200 ft. sec.$^{-1}$ from a gun weighing 30 cwt. Find the initial kinetic energy of the shot and of the gun in ft. tons wt. If the gun recoils 18 in., find, in tons wt., the average value of the retarding force. (L.M.)

12. A ball is dropped from a height of 1 metre above a horizontal surface. If the coefficient of restitution between the ball and the surface is 0·8, calculate from first principles the height to which the ball rises on the first bounce, and express its loss of mechanical energy as a percentage of the kinetic energy just before impact. (L.I.)

13. A ball, allowed to fall freely so as to bounce on a level floor, rises to one-quarter of the height from which it was dropped. Show that the coefficient of restitution is $\frac{1}{2}$.

Such a ball, moving on a smooth horizontal table with a velocity U, impinges directly on an exactly similar ball which is at rest but free to move. Assuming the same coefficient of restitution, find the velocity of each ball after the impact. (C.H.S.)

14. State Newton's experimental law of impact.

A particle of mass m moving with velocity u collides with a stationary particle of mass M. The coefficient of restitution is $\frac{1}{2}$. Find the velocities of the particles after collision. Prove that the loss of kinetic energy due to the collision is $3Mmu^2/\{8(M+m)\}$. (J.M.B.H.S., abridged.)

15. Explain the terms *potential energy, kinetic energy.*

A mass of 1 kg. is hung from one end of a light string 1 metre long fixed at the other end. The string is held taut horizontally and the mass is then released. Calculate its velocity when the string (*a*) makes an angle of 30° with the vertical, (*b*) is vertical. At the bottom of its swing the kilogram mass strikes another equal mass and both move on together. Through what height will they rise? (L.M.)

16. Explain the action of a centrifuge when used to hasten the deposition of sediment from a liquid.

A pendulum bob of mass 1 kg. is attached to a string 1 metre long and made to revolve in a horizontal circle of radius 60 cm. Find the period of the motion and the tension in the string. (C.H.S.)

17. Define *acceleration.*

Find from first principles the acceleration in c.g.s. units of (*a*) a body sliding down a plane inclined at 30° to the horizontal, the coefficient of kinetic friction being 0·3; (*b*) a point on the rim of a wheel of 20 cm. diameter rotating at a rate of 420 r.p.m. (L.Med.)

18. Describe an accurate method of finding the value of the acceleration due to gravity. Give the theory of the method, and discuss carefully the steps taken to attain accuracy.

Assuming the earth to be a uniform sphere of radius $6{\cdot}4 \times 10^8$ cm., and considering only the effect of the earth's rotation about its axis, find the value of g at the equator if its value at the poles is 983 cm. sec.$^{-2}$. (O.H.S.)

19. Define *simple harmonic motion.*

Show that for small amplitudes the motion of a simple pendulum is simple harmonic and that its time period is equal to $2\pi\sqrt{l/g}$, where l is its length.

Calculate the original time period of a simple pendulum if an increase of length of 50 cm. changes the time period by $\frac{1}{2}$ sec. (L.I.)

20. Explain what is meant by *simple harmonic motion.*

Deduce an expression for the periodic time of the vertical oscillations of a mass suspended by a helical spring, the extension of which is proportional to the load. The mass of the spring may be neglected.

Describe in some detail how such an arrangement might be used to obtain an estimate of the acceleration due to gravity. (J.M.B.H.S.)

21. A rigid body is free to rotate in a vertical plane about a pivot which is attached at a point 32 cm. from its centre of gravity. For small oscillations under gravity the body has the same time period as a simple pendulum of length 50 cm. For what other distance between the pivot and the centre of gravity will the time period be the same, and where must the pivot be placed to give the minimum time period?

22. A flat helical spring hangs vertically and when loaded with a mass of 14 gm. shows an extension of 1·75 cm. The time period of vertical oscillations of the suspended mass is 0·3 sec. Calculate (*a*) the effective mass of the spring, (*b*) the maximum velocity of the load if the amplitude of its oscillation is 0·8 cm. (L.I.)

23. The mass of a fly-wheel is 20 lb., and a mass of 1 lb. hangs by a string wrapped round the axle which is horizontal. This mass is observed to descend from rest through 5 ft. in 8 seconds. Find the radius of gyration of the fly-wheel, being given that the radius of the axle is 2 in. Neglect the mass of the axle and friction. (L.I.)

24. A wheel of mass M lb. and radius a ft. can turn without friction about its axis which is horizontal. One end of a light string is attached to a point on the rim. The string is coiled round the rim and carries at its free end a mass m lb. Prove that when motion is allowed to take place the angular acceleration of the wheel is $\frac{mga}{Mk^2+ma^2}$ radians sec.$^{-2}$, where k (ft.) is the radius of gyration of the wheel about its axis.

Initially the string passes exactly twice round the rim, and the wheel is stationary. Given that $M=6m$, $a=2$ and $k=\sqrt{2}$, prove that the string becomes completely uncoiled in $\sqrt{(2\pi)}$ sec. (O.H.S.)

Chapter IX

THE EQUILIBRIUM OF FORCES

1. GENERAL CONDITIONS FOR EQUILIBRIUM

In Chapter V we have dealt with the action of a system of forces on a body and derived the following relationships:—

$$\left.\begin{matrix}\text{Vector sum of forces}\\ \text{acting on a body}\end{matrix}\right\} = \text{mass of body} \times \left\{\begin{matrix}\text{linear acceleration of}\\ \text{centre of mass}\end{matrix}\right.$$

$$\left.\begin{matrix}\text{Sum of moments of forces}\\ \text{about a fixed axis or axis}\\ \text{through centre of mass}\end{matrix}\right\} = \left\{\begin{matrix}\text{moment of inertia of body}\\ \text{about fixed axis or axis}\\ \text{through centre of mass}\end{matrix}\right\} \times \left\{\begin{matrix}\text{angular}\\ \text{accelera-}\\ \text{tion}\end{matrix}\right.$$

A body is said to be in equilibrium when it has neither linear nor angular acceleration. It may, therefore, be completely at rest, or it may possess *uniform* linear or angular velocity or both. A body moving with uniform velocity is in equilibrium as well as one which is stationary.

The above relations enable us to write down immediately the conditions which must be satisfied by the forces acting on a body which is in equilibrium. In the first place, the linear acceleration of the centre of mass is zero, so that for equilibrium:

(i) The vector sum, or resultant, of the forces acting on the body must be zero.

This condition can be stated in another way which is sometimes more convenient. Evidently if the forces have no resultant, the sum of their components in any direction is zero. Every system of forces, however, whether it is in equilibrium or not, fulfils this condition in **one** direction, namely the direction at right angles to its resultant. Therefore it cannot be stated with certainty that a given system of forces is in equilibrium, *i.e.* has no resultant, until it has been shown that the sum of the components of the forces is zero in **two different** directions. Only two directions are necessary, provided they are different. They need not be at right angles to each other, although they are often chosen to be so for simplicity of calculation. We state the condition for no linear acceleration in its alternative form, therefore, as follows:—

(i*a*) The sum of the components of the forces in any chosen direction is zero, **and**

(i*b*) the sum of the components of the forces in **any other** (usually perpendicular) direction is also zero.

We next consider the rotation of the body. The angular acceleration of a body is not *necessarily* zero when there is no resultant force acting on it, because the forces may constitute a couple, which, as we know, causes angular acceleration without acceleration of G (page 62). For complete equilibrium, *i.e.* absence of angular as well as of linear acceleration, therefore, it is necessary to add a further condition. Since the moment of a couple is the same about all points, it follows that if its moment about *any* point is zero it is also zero about all points, that is to say, the couple is non-existent. We therefore frame the remaining condition as follows:—

(ii) The sum of the moments of the forces about any point is zero.

It will be noticed that while conditions (i), (i*a*), (i*b*) and (ii) are all necessarily satisfied when the body is in equilibrium, yet each one is not *individually* sufficient to ensure equilibrium. We have already seen that condition (i) (or its equivalent (i*a*) and (i*b*)) does not rule out the possibility of angular acceleration. Similarly the fulfilment of condition (ii) for any one point about which moments are taken does not, of itself, ensure equilibrium, because the point chosen might happen to lie on the line of action of the resultant force, unless this is shown to be non-existent by the fulfilment of condition (i).

Taken together, (i) (or (i*a*), (i*b*)) and (ii) constitute what are called **necessary and sufficient** conditions for complete equilibrium. Each is individually necessary and is satisfied when there is equilibrium, but the satisfaction of any one of them does not necessarily mean that equilibrium exists.

It may be added that it is possible to express the conditions entirely in terms of moments. Thus, forces are in equilibrium when the sum of their moments is zero about any three points not in the same straight line. The *three* non-linear points are necessary, because although two of them might happen to lie in the line of action of the resultant force (in which case the total moment about each of them would be zero without the resultant's necessarily being zero), yet the third one cannot lie on this line, and if the total moment about it is also zero, it must mean that there is no resultant and also that the force system is not a couple.

Example.—*The lower end of a uniform ladder of weight* 100 *lb. and length* 30 *ft. rests on a horizontal floor while the upper end rests against a vertical wall. If the inclination of the ladder is* 45° *and the coefficient of friction at each end is* 0·5, *calculate how far up the ladder a man weighing* 150 *lb. can climb before slipping begins.*

In Fig. 93, AB represents the ladder and the arrows show all the forces acting on it. Its weight acts from its midpoint, G, since the ladder is uniform. Suppose

that slipping begins when the man has reached the point C. This means that the frictional forces F_1 and F_2, which prevent slipping, have reached their maximum possible values, which are given by

$$F_1 = 0{\cdot}5R_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$F_2 = 0{\cdot}5R_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since R_1 and R_2 are the normal reactions.

Since there is equilibrium, all the conditions necessary for equilibrium must be

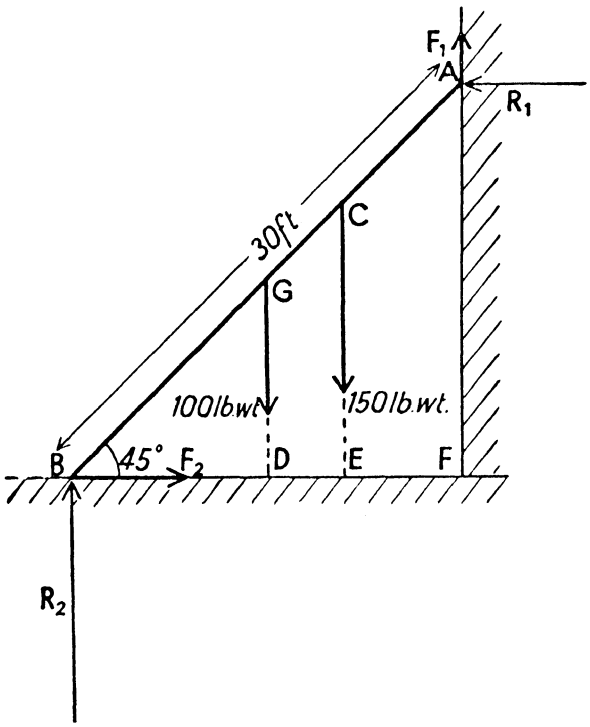


FIG. 93

satisfied. We apply these conditions in turn and obtain three independent relationships between the forces and distances.

(*ia*) We first equate the horizontal components of the forces to zero and obtain

$$F_2 - R_1 = 0$$

$$\therefore\ R_1 = F_2 = 0{\cdot}5R_2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*ib*) Next, the sum of the vertical components must be zero, so that

$$R_2 + F_1 - 100 - 150 = 0$$

or

$$R_2 = 250 - F_1$$

Substituting for R_2 and F_1 from equations (1) and (3) in the last equation, we obtain

$$2R_1 = 250 - 0{\cdot}5R_1$$

or

$$R_1 = 100 \text{ lb. wt.}$$

so that

$$F_1 = 0{\cdot}5R_1 = 50 \text{ lb. wt.}$$

(ii) The magnitudes of all the forces having been found, we are now in a position to take moments in order to find the line of application of the man's weight, *i.e.* the position of C. Any point may be chosen for taking moments. Let it be B. We then have

$$(100 \times \text{BD}) + (150 \times \text{BE}) - (F_1 \times \text{BF}) - (R_1 \times \text{AF}) = 0$$

But
$$BD = BG \cos 45° = \frac{15}{\sqrt{2}}$$
$$BE = BC \cos 45° = \frac{BC}{\sqrt{2}}$$
$$BF = BA \cos 45° = \frac{30}{\sqrt{2}}$$
$$AF = BA \sin 45° = \frac{30}{\sqrt{2}}$$

When these values are substituted in the last equation, we obtain

$$\left(100 \times \frac{15}{\sqrt{2}}\right) + \left(150 \times \frac{BC}{\sqrt{2}}\right) - \left(F_1 \times \frac{30}{\sqrt{2}}\right) - \left(R_1 \times \frac{30}{\sqrt{2}}\right) = 0$$

or
$$15 \times BC = 3(F_1 + R_1) - 150$$
$$= 3(50 + 100) - 150$$
$$\therefore BC = 30 - 10$$
$$= 20 \text{ ft.}$$

2. THE EQUILIBRIUM OF THREE FORCES

The conditions for equilibrium which were laid down in the previous section apply to all systems of forces; but for certain particular systems they can be expressed in other forms.

Two Forces.—When only two forces act on a body it is evident that for equilibrium they must (i) be equal in magnitude, (ii) have opposite directions, and (iii) have the same line of action.

Three Forces. Triangle of Forces.—The conditions for no linear acceleration of the centre of mass of a body which is acted upon by three non-parallel forces is a particular case of the general condition which states that their vector sum must be zero. Applying the principle of vector summation we can state the **theorem of the triangle of forces** as follows:—

When three forces are in equilibrium they can be represented in magnitude and direction by the three sides of a triangle taken in order.

In Fig. 94 the three forces, P, Q and R, are in equilibrium and can be represented by the sides of either of the triangles ABC and A′B′C′, which are, of course, identical. In each case the vector sum of the forces is zero.

The phrase "taken in order" merely means that the arrows on the sides of the triangle must all point either in a clockwise or an anticlockwise direction, or in other words, as explained on page 45, the vectors must be placed tail to head. The fact that the sides of the triangle represent the forces in magnitude means that the lengths of the sides are pro-

portional to the magnitudes of the forces, so that

$$\frac{P}{AB}=\frac{Q}{BC}=\frac{R}{CA}$$

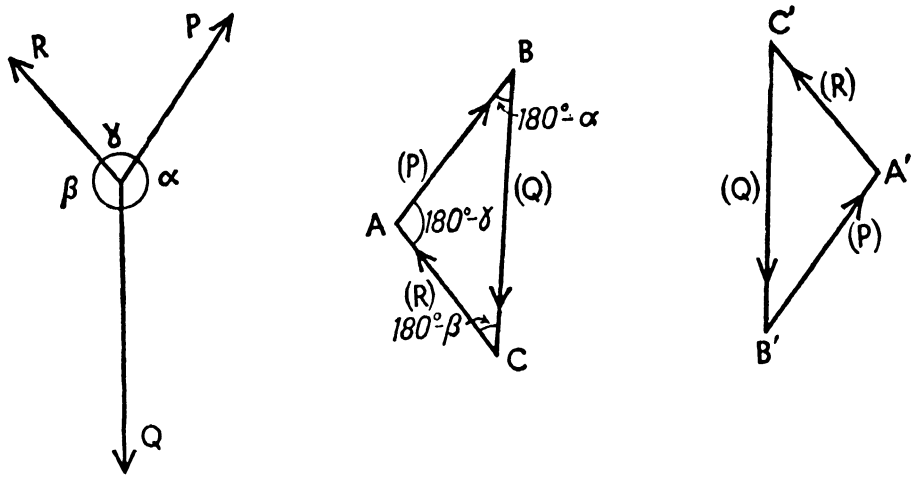


FIG. 94

It is a **necessary** condition for the equilibrium of three forces that it should be possible to construct a triangle whose sides represent the forces in magnitude and direction. It is not a **sufficient** condition, however, because there still remains the possibility that the three forces are equivalent to a couple. This is illustrated in Fig. 95, where the resultant of P and Q (*viz.* R', found by a parallelogram) is equal and opposite to R but does not act along the same straight line. The forces R and R' constitute a couple, and therefore P, Q and R are not in equilibrium, although they can be represented by the sides of either of the triangles ABC or ADC. The further necessary condition is, therefore, that the sum of the moments of the forces must be zero, or *in the particular case of three forces*, that their **lines of action must be concurrent,** *i.e.* must meet in a point. This does not mean that all three forces must necessarily have the same point of application, it is only necessary for the line of action of R to pass through the point of intersection of the lines of action of P and Q in order that the forces shall have no moment.

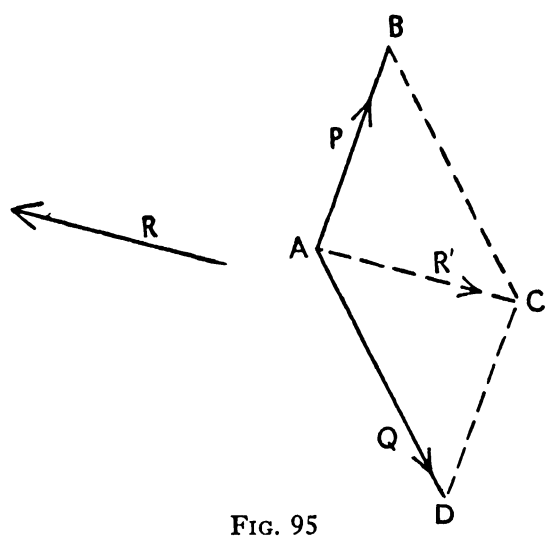


FIG. 95

Lami's Theorem.—If the angles between the directions of the forces shown in Fig. 94 are α, β and γ, then in the triangle ABC we have

$$\hat{A} = 180° - \gamma$$

$$\hat{B} = 180° - \alpha$$

$$\hat{C} = 180° - \beta$$

Furthermore, by the well-known property of a triangle,

$$\frac{AB}{\sin \hat{C}} = \frac{BC}{\sin \hat{A}} = \frac{CA}{\sin \hat{B}}$$

i.e.

$$\frac{AB}{\sin (180° - \beta)} = \frac{BC}{\sin (180° - \gamma)} = \frac{CA}{\sin (180° - \alpha)}$$

But since the lengths of the sides are proportional to the magnitudes of the forces, and the sine of an angle is equal to the sine of its supplement, these last relationships can be written

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha}$$

This is Lami's theorem, and it can be stated as follows:—

When three forces are in equilibrium, the magnitude of each of the forces is proportional to the sine of the angle between the directions of the other two.

Experimental Verification of the Theorem of the Triangle of Forces.—This may be done in a variety of ways. In Fig. 96 two freely running pulleys are fixed near the two upper corners of a vertical drawing-board covered with paper, and light strings with known weights P, Q and R attached are arranged as shown. The strings will attain an equilibrium position depending upon the relative values of the weights. With the help of a set-square a dot is made on the paper immediately behind the junction of

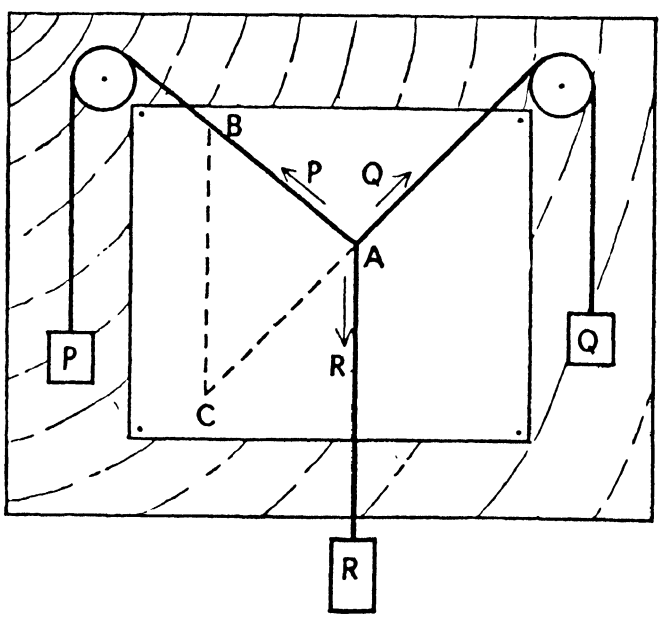


FIG. 96

the strings (which is the body whose equilibrium is being considered) and other dots are made along the three strings so that their directions can be drawn on the paper. Next, any triangle is constructed with its sides parallel to the three directions. It is most convenient to form the triangle by producing one of the lines backwards and then drawing the third side of the triangle parallel to one of the other two directions and in *any* convenient position. Thus

$$\begin{aligned} &\text{AB is parallel to } P \\ &\text{BC} \quad \text{,,} \quad \text{,,} \quad R \\ &\text{CA} \quad \text{,,} \quad \text{,,} \quad Q \end{aligned}$$

The three forces are, by construction, represented in direction by the three sides of the triangle, and it remains to be shown that they are also represented in magnitude. To do this, the lengths of the sides of the triangle are measured, and it can be verified that

$$\frac{P}{\text{AB}} = \frac{R}{\text{BC}} = \frac{Q}{\text{CA}}$$

Evidently if we assume the truth of the triangle of forces theorem it is possible to use these last relationships to calculate two of the weights P, Q and R, provided that the remaining one is known.

Example.— *A uniform rod of mass* 12 *lb. and length* 4 *ft. is pivoted at one end and held in a horizontal position by a string* 6 *ft. long. One end of the string is attached to the rod at a point* 3 *ft. from the pivot, while the other is tied to a fixed support vertically above the pivot. Calculate the tension in the string and the magnitude and direction of the reaction of the pivot.*

Fig. 97 illustrates the arrangement. The rod is represented by AB, the pivot being at A, and the string is stretched between C and D. The weight of the rod acts vertically downwards at its centre of gravity G, which is at its midpoint. since the rod is uniform. Thus

$$\text{AG} = \text{GB} = 2 \text{ ft.}$$

and we are given that

$$\text{AC} = 3 \text{ ft.}$$

$$\therefore \text{CB} = 1 \text{ ft.}$$

Let T be the tension in the string and R be the reaction of the pivot. Then since the rod is in equilibrium, the lines of action of the three forces acting on it, *viz.* T, R and its weight, must be concurrent and must therefore meet at E.

Any triangle whose sides, taken in order, are parallel to the directions of the three forces will be a triangle of forces, and, in particular, DAE is such a triangle. In order to find the magnitudes of T and R it is therefore necessary to calculate the lengths of the sides of this triangle. We have

$$\text{DA} = \sqrt{\text{DC}^2 - \text{AC}^2} = \sqrt{6^2 - 3^2} = 3\sqrt{3} \text{ ft.}$$

Also, since EG is parallel to DA, it divides the two sides AC and DC of the triangle DAC in the same proportion. Therefore

$$\frac{\text{ED}}{\text{CD}} = \frac{\text{AG}}{\text{AC}}$$

or

$$ED = CD \times \frac{AG}{AC} = 6 \times \tfrac{2}{3} = 4 \text{ ft.}$$

and

$$CE = 6 - 4 = 2 \text{ ft.}$$

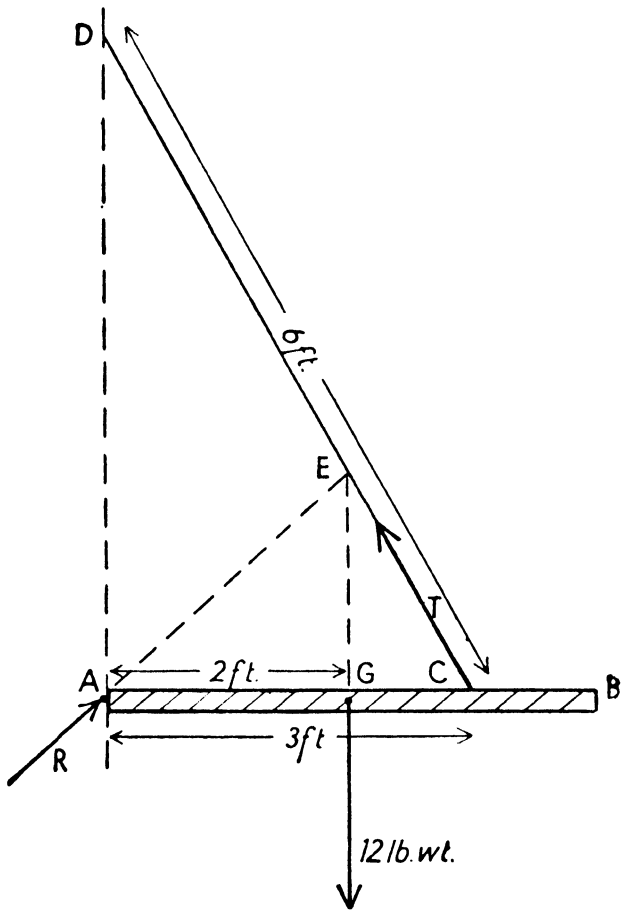


FIG. 97

Also, by Pythagoras' theorem,

$$EG = \sqrt{CE^2 - GC^2}$$
$$= \sqrt{4 - 1}$$
$$= \sqrt{3} \text{ ft.}$$

Therefore the third side of the triangle, AE, is given by

$$AE = \sqrt{EG^2 + AG^2} = \sqrt{3 + 4} = \sqrt{7} \text{ ft.}$$

We have, therefore, in the triangle of forces DAE,

$$\frac{T}{ED} = \frac{12}{DA}$$

or

$$T = 12 \times \frac{ED}{DA} = 12 \times \frac{4}{3\sqrt{3}} = \frac{16}{\sqrt{3}} = 9{\cdot}2 \text{ lb. wt.}$$

Similarly,
$$\frac{R}{AE}=\frac{12}{DA}$$

or
$$R=12\times\frac{AE}{DA}=12\times\frac{\sqrt{7}}{3\sqrt{3}}=4\sqrt{\frac{7}{3}}=6{\cdot}1 \text{ lb. wt.}$$

Finally, the angle $E\hat{A}G$ is given by
$$\tan E\hat{A}G=\frac{EG}{AG}=\frac{\sqrt{3}}{2}$$
$$\therefore\ E\hat{A}G=41^\circ \text{ approx.}$$

Thus the required answers are:

(i) The tension in the string is 9·2 lb. wt.
(ii) The reaction of the pivot is 6·1 lb. wt. and makes an angle of 41° with the horizontal.

It will be realized that these results can be obtained by graphical construction as well as by calculation. It is also possible, of course, to use the *general* conditions for equilibrium by resolving the forces vertically and horizontally and by taking moments about any convenient point.

3. THE EQUILIBRIUM OF PARALLEL FORCES

General Case. Any Number of Forces.—When a body is acted upon by parallel forces, the first condition for equilibrium, *viz.* that the vector sum of the forces is zero, can be expressed very simply. Thus if the forces are P, Q, R, S, etc., we have
$$P+Q+R+S+\ \ldots\ =0$$

It is necessary to choose a direction which shall be regarded as positive and to affix a negative sign in front of the terms in this equation which represent forces whose direction is opposite to this. That is to say, the vector sum of the forces becomes their **algebraic** sum.

If the forces acting on the body in Fig. 98 are in equilibrium, then
$$P-Q+R-S+T=0$$

where P, Q, R, etc. stand for the magnitudes of the forces.

As in all the previous cases of equilibrium, so in the case of parallel forces, this single condition is not a sufficient condition for complete equilibrium because it would be satisfied by a system of forces which reduces to a couple and therefore causes rotation. The other necessary condition is that the sum of the moments of the forces about any point is zero. Thus if O (Fig. 98) is the chosen point and a straight line is drawn through O perpendicular to the lines of action of the forces to cut them at A, B, C, D and E, we have, for equilibrium,
$$(P\times OA)-(Q\times OB)+(R\times OC)-(S\times OD)+(T\times OE)=0$$

or, if the point O′ is chosen,
$$(P\times O'A)-(Q\times O'B)+(R\times O'C)+(S\times O'D)-(T\times O'E)=0$$

A relation of this kind fixes the positions of the lines of action of the forces. The points of application may, of course, lie anywhere along these lines.

An alternative way of specifying the necessary conditions of equilibrium states that the sum of the moments of the forces about any **two** points, the line joining which is not parallel to the forces, is zero. We have already seen that the moments about any one point must be zero in order that the forces shall have no turning effect, and the zero moment about another point replaces the former condition of zero algebraic sum of the forces themselves. The condition of zero moment about any one point is not sufficient in itself, because the chosen point might happen to lie in the line of action of the resultant, but zero moment about any other point eliminates the possibility of there being a resultant provided the second point does not also lie on the line of action of the resultant. This last possibility is eliminated if the line joining the two points is not parallel to the direction of the forces.

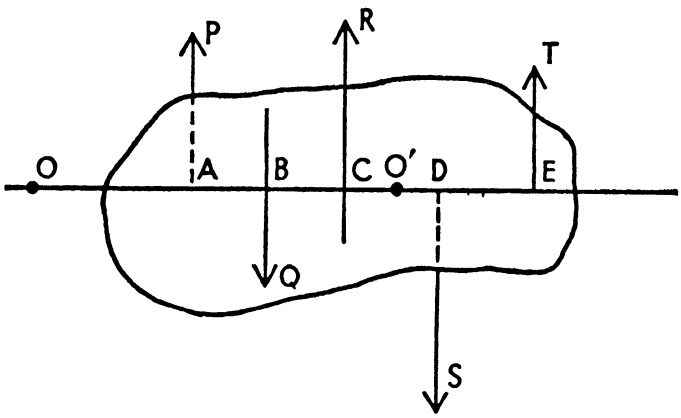


FIG. 98

Example.—*A small horizontal bridge consists of a single uniform footway,* 20 *ft. long and weighing* 400 *lb., supported at its two ends. Calculate the force on each support when a man weighing* 150 *lb. is standing* 6 *ft. from one end of the bridge.*

Since the footway is uniform its weight acts at its centre, and the other three forces acting on it are shown in Fig. 99, P and Q being unknown. The simplest

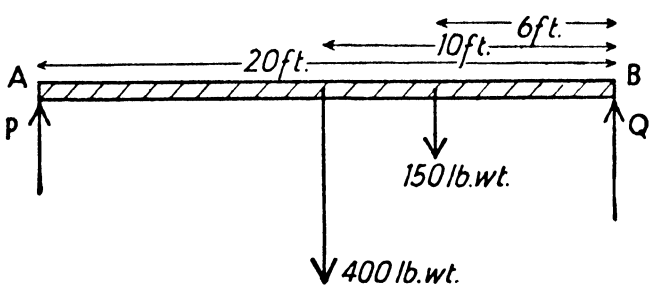


FIG. 99

method of calculating P and Q is to take moments about, say, B and so obtain an equation for P, and then to find Q from the fact that the sum of the forces is zero.

Thus, for moments about B we have, for equilibrium,

$$20P - (10 \times 400) - (6 \times 150) = 0$$

$$\therefore P = \frac{4000 + 900}{20}$$

$$= 245 \text{ lb. wt.}$$

Also

$$P + Q - 400 - 150 = 0$$

$$\therefore Q = 400 + 150 - P$$

$$= 400 + 150 - 245$$

$$= 305 \text{ lb. wt.}$$

Alternatively Q could have been found by taking moments about A.

Resultant of a Number of Parallel Forces.—It should be noticed that if any one of the forces in Fig. 98 is reversed it becomes the resultant of the other forces. Thus R *reversed* is the resultant of P, Q, S and T, since R itself is in equilibrium with them and therefore counterbalances their resultant. The two conditions for equilibrium can therefore be used to give the magnitude and line of action of the resultant of a number of parallel forces provided that the reversal is not forgotten.

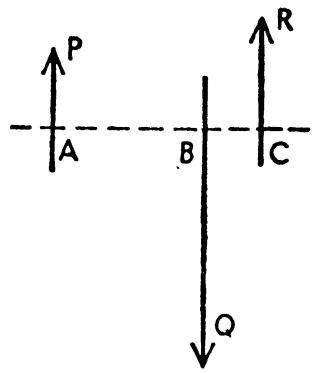


FIG. 100

Three Parallel Forces.—When only three parallel forces are in equilibrium, such as P, Q and R (Fig. 100), we have

$$P + R - Q = 0$$

or

$$P + R = Q$$

for the first condition.

Secondly, if a straight line perpendicular to their lines of action intersects these lines at A, B and C, we have, by taking moments about B,

$$P \times AB = R \times BC$$

or

$$\frac{P}{BC} = \frac{R}{AB}$$

$$= \frac{P + R}{BC + AB} \text{ by a theorem in proportion}$$

$$= \frac{Q}{AC}, \text{ since } P + R = Q$$

Thus when three parallel forces are in equilibrium, the magnitude of each force divided by the perpendicular distance between the lines of action of the other two forces is a constant. As in the general case just dealt with, any one of the three forces *reversed* is the resultant of the other two.

The relation between three forces in equilibrium can be tested experimentally with the arrangement shown in Fig. 101. A wooden rod which is as light as possible is supported horizontally by two spring-balances whose readings give the magnitudes of P and R, while Q is provided by hanging a weight on the rod. The relationships between the magnitudes of P, Q and R and between the distances between A, B and C can be verified for different cases by altering the positions of A, B and C. This assumes that the weight of the rod is negligible. If the rod itself is sufficiently heavy to cause P and R to show measurable readings when Q is removed, the experiment may be performed using the weight of the rod (which acts from its centre if the rod is uniform) as the third force and dispensing with Q altogether.

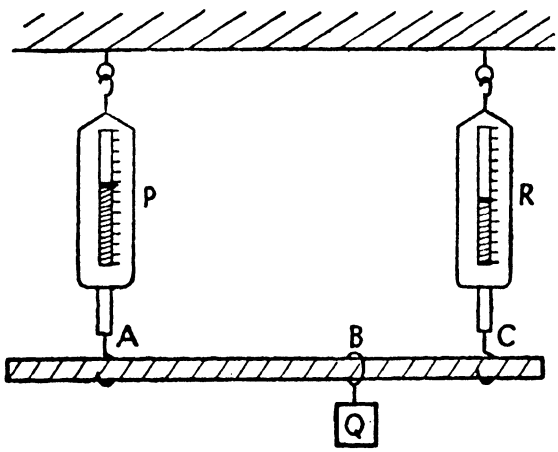


FIG. 101

Another example of the equilibrium of three parallel forces occurs in the system illustrated in Fig. 102, where a rigid rod is pivoted at its centre (so that its own weight has no turning effect) and has a weight hanging from it on either side of the pivot. These weights are altered and their positions adjusted each time so that the rod is in equilibrium as tested by the absence of any tendency to rotate one way or the other. Within the limits of experimental error it will be found that

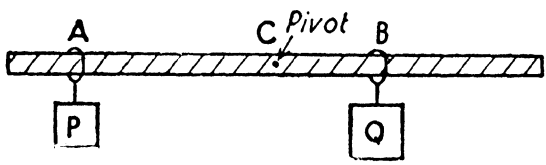


FIG. 102

$$P \times \mathrm{AC} = Q \times \mathrm{BC}$$

Evidently the reaction of the pivot on the rod must be equal to

$$P + Q + \text{weight of rod}$$

The truth of this last relationship can be tested if the pivot is supported from a spring-balance.

4. CENTRE OF GRAVITY

The Weight of a Body due to the Weight of its Particles.—We can regard the weight of a body as being made up of the separate attractive forces which the earth exerts upon the ultimate particles into which the

body can theoretically be divided. In Fig. 103 (i) the weights of some of the individual particles of a flat body or lamina are represented by w_1, w_2, w_3, w_4, all directed vertically downwards. The total weight of the body (W) is the resultant of all the small forces, and, since they are

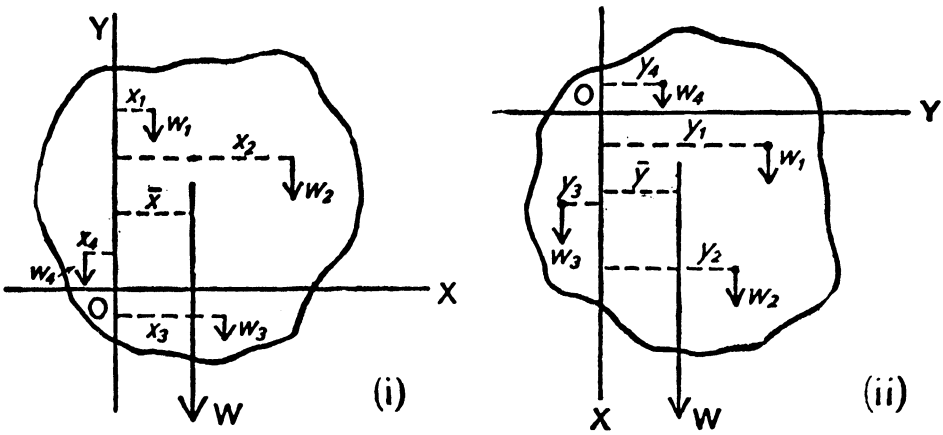


FIG. 103

all parallel and in the same direction, this is simply their arithmetic sum. Thus

$$W = w_1 + w_2 + w_3 + w_4$$
$$= \Sigma w$$

Location and Definition of Centre of Gravity.—Since the force W is the resultant of the weights of the particles, its line of action must be such that the moment of W about any point is equal to the moment of the separate weights about the same point. We can choose any point for the purpose of taking moments. Suppose that the point has been chosen and that OY in Fig. 103 (i) is a vertical line passing through it. The moment of any given vertical force is the same about every point in OY so that no particular point need be marked. Let the horizontal distances of the lines of action of W, w_1, w_2, etc. from OY be $\bar{x}$, x_1, x_2, etc., as shown in diagram. Then by taking moments we find that the position of the line of action of W relative to OY must be such that

$$W\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 - w_4x_4 + \ldots$$
$$= \Sigma wx$$

or
$$\bar{x} = \frac{\Sigma wx}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The term Σwx represents the *algebraic* sum of the moments of the weights of the particles, clockwise moments being positive and anticlockwise negative.

Now suppose that the body is rotated in its own plane, say through

90°, so that the line OY is now horizontal and another line OX which was previously horizontal is now vertical (Fig. 103 (ii)). Let the perpendicular distances of the new lines of action of W, w_1, w_2, etc. from OX be $\bar{y}, y_1, y_2$, etc. Then by taking moments about any point in OX we obtain in just the same way as for $\bar{x}$

$$\bar{y} = \frac{\Sigma wy}{W} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the line of action of the total weight of the body is, in the first case, a line parallel to OY at a distance $\bar{x}$ from it, and in the second case, a line parallel to OX at a distance $\bar{y}$ from it, the values of $\bar{x}$ and $\bar{y}$ depending upon the shape of the body and the way in which its weight is distributed. The point of application of the weight of the body must, therefore, lie on both these lines and consequently must be situated at their point of intersection G, Fig. 104. This point is called the **centre of gravity** (C.G.) of the body, which may be defined as the **point of application of the weight of the body.** For a three-dimensional body the C.G. is situated at the point of intersection of three *planes* (corresponding to two *lines* in the two-dimensional case) each of which is fixed by an equation such as (1) or (2).

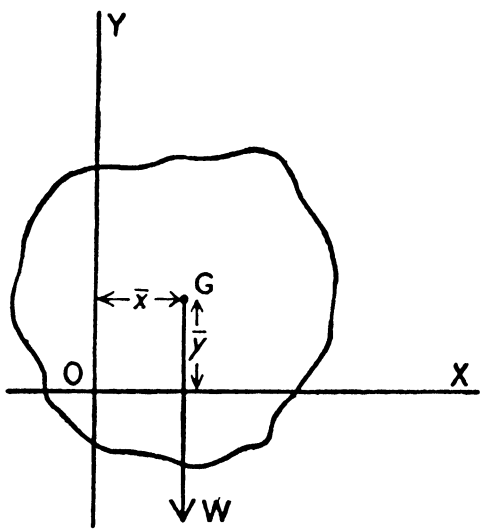


FIG. 104

The positions of the axes OX and OY with respect to which the position of G has been found in the foregoing can be fixed quite arbitrarily with reference to the body. Suppose we decide that the origin O shall *coincide with G* so that both $\bar{x}$ and $\bar{y}$ are zero. Equations (1) and (2) then reduce to

$$\frac{\Sigma wx}{W} = 0$$

and

$$\frac{\Sigma wy}{W} = 0$$

or

$$\Sigma wx = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\Sigma wy = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The directions of the axes with respect to the lamina have not been specified; the only condition for the validity of these last equations is that both axes should pass through G. It therefore follows that equations

of the same type hold good for *any* line passing through G provided that the distances like x and y are perpendicular to the particular line in question, and provided also that the distance is positive for particles lying on one side of the line and negative for those on the other side. This means that the algebraic sum of the moments of the weights of the separate particles about G is always zero for all positions of the body. This result may also be deduced by remembering that, since the resultant weight of the body acts through G, the body would not rotate under the action of its weight if it were pivoted at G.

It has already been shown (page 59) that since g has the same value for all the particles of a body, no rotation occurs when the body is released and allowed to fall from rest, and therefore the force acting on it (*viz.* its weight) must have its point of application at the body's centre of mass. Thus centre of gravity and centre of mass are one and the same point provided the body is not sufficiently large to occupy a region over which g varies perceptibly. All equations referring to the position of the one point, therefore, apply to the other. If we replace w, the weight of a particle, by mg, where m is its mass, the two equations (3) and (4) which specify the position of the centre of gravity become

$$\Sigma mgx = 0$$

and

$$\Sigma mgy = 0$$

or, if g is the same for all particles,

$$\Sigma mx = 0$$

and

$$\Sigma my = 0$$

These last two equations apply to the centre of mass and are, in the more mathematical treatment of mechanics, used to define it and to deduce its properties. In the present more descriptive method of approach we have used one of those properties as the initial definition, *viz.* that a single force applied at the centre of mass causes no rotation of the body (page 59).

Centre of Gravity of Two Particles.—Let the two particles of weights w_1 and w_2 at A and B in Fig. 105 be rigidly connected. The distance of each particle from the line AB is zero, so that for this line the relationship $\Sigma wx = 0$ is true and therefore it must pass through the C.G. of the two particles. Let G be the C.G. and imagine a line drawn through G at right angles to AB. For this line also, since it passes through G, the sum of the products of the weights and their perpendicular distances from the line must be zero. Therefore

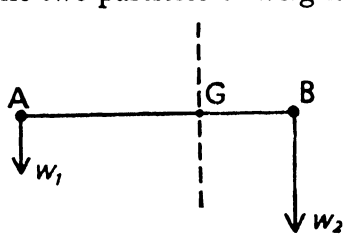


FIG. 105

$$w_1 \times AG = w_2 \times GB$$

Thus the C.G. of two rigidly connected particles lies on the straight line joining them and divides the distance between them into two parts whose lengths are in the *inverse* ratio of the weights at their ends.

Centre of Gravity of Three Particles.—Let the three particles at A, B and C in Fig. 106 have weights w_1, w_2 and w_3 respectively and be rigidly connected. Let their C.G. be at G. Choose a line which passes through G and, for the sake of simplicity, through one of the particles, *e.g.* the line AGD. If x_2 and x_3 are the perpendicular distances of B and C from AGD (produced if necessary), we have, since AGD is known to pass through the C.G.,

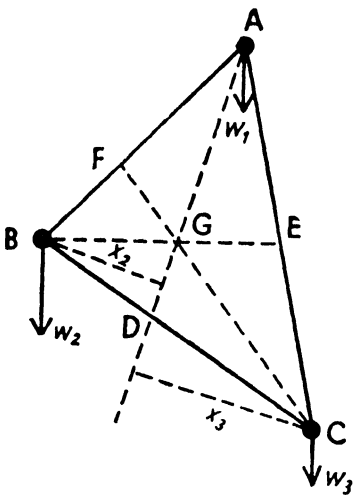


FIG. 106

$$(w_1 \times 0) - w_2x_2 + w_3x_3 = 0 \text{ by equation (3)}$$

or
$$\frac{x_2}{x_3} = \frac{w_3}{w_2}$$

By similar triangles

$$\frac{x_2}{x_3} = \frac{\text{BD}}{\text{DC}}$$

$$\therefore \frac{\text{BD}}{\text{DC}} = \frac{w_3}{w_2}$$

so that we can state that G lies on a line joining A to a point in BC which divides BC in the inverse ratio of w_2 and w_3 and is, in fact, the C.G. of these two weights. By a similar argument we conclude that G also lies on a line joining B to a point E which divides AC in the inverse ratio of w_1 and w_3. The C.G. of the system (G) is therefore the point of intersection of all three lines of this type. Since D is the C.G. of w_2 and w_3, these two weights can be replaced by a single weight $w_2 + w_3$ at D, and G is then the C.G. of this weight and w_1. Thus, from what has just been proved for the C.G. of two particles, we have

$$w_1 \times \text{AG} = (w_2 + w_3) \times \text{GD}$$

or
$$\frac{\text{AG}}{\text{GD}} = \frac{w_2 + w_3}{w_1}$$

Experimental Determination of the Position of the C.G. of a Lamina.—It is evident that if a body is pivoted freely at any point (*e.g.* O in Fig. 107) it comes to rest with its C.G. vertically below the pivot. The body is then hanging in equilibrium under the action of its own weight, W, acting through G and the reaction of the pivot at O, which must be equal and opposite to W and must have the same line of action.

In order to find the position of the C.G. of a lamina (*e.g.* a piece of wooden board) it is suspended as freely as possible from a point near its edge and, with the help of a plumbline, a vertical line is drawn on the body

passing through the point of suspension. When this is repeated, using another point of suspension, the intersection of the two lines is the C.G. If only two lines are drawn, the second point of suspension should, for maximum accuracy, be so chosen as to make the lines intersect at approximately a right angle. It is preferable to draw three or more lines and to take the C.G. as the centre of the small figure which they enclose at their intersections.

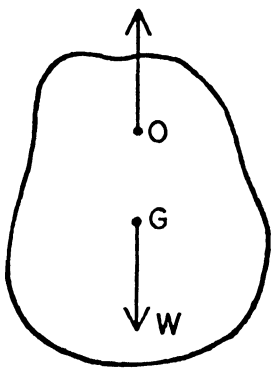


FIG. 107

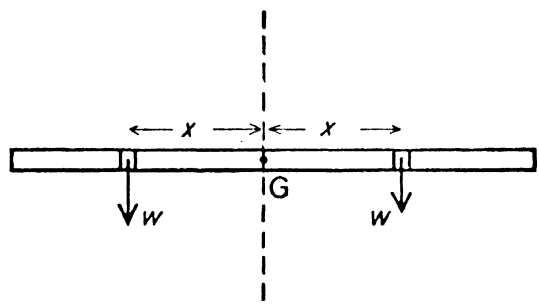


FIG. 108

The Centres of Gravity of Certain Symmetrical Bodies.—We can take a thin straight uniform rod (Fig. 108) as the simplest of the examples we are about to consider. The word "uniform" in this connection means that the weight per unit length of the rod is constant all along the rod. The centre of gravity, G, of the whole rod must lie on the line joining all the separate centres of gravity of the short lengths into which it can be divided and must, therefore, be somewhere on the axis of the rod. It is easy to see that G must lie midway between the ends, because if we imagine the rod to be divided into very short lengths of equal weight, these segments will be symmetrically distributed about the centre since the rod is uniform. Thus for each product wx on the left-hand side of a line passing through the geometrical centre and perpendicular to the rod there is an identical product for another short length of rod symmetrically situated on the other side. The products therefore cancel each other in pairs, and the condition $\Sigma wx = 0$ is true for a line passing through the geometrical centre of the rod, which must therefore be the C.G.

A similar argument extended to two or three dimensions (lamina or solid) leads to the conclusion that in cases where there is a centre of symmetry, *e.g.* a circular or square lamina, sphere, cube, spheroid, circular cylinder, etc., the C.G. of the body lies at this point if the weight per unit area or per unit volume is uniform throughout the body.

Other cases in which such symmetry does not exist can be worked out by simple methods. As an example we can take the triangular lamina ABC in Fig. 109. Imagine the triangle to be divided into a number of narrow

strips parallel to one of the sides, *e.g.* BC. The C.G. of each strip is its geometrical centre, and we can imagine that each one is replaced by a particle of the same weight as itself situated at its centre. All the particles therefore lie on the median joining A to D, the midpoint of AC. The C.G. of these particles (which is the same as the C.G. of the original triangle) must therefore lie somewhere on AD. A similar argument shows that the C.G. must also lie on the other two medians BE and CF so that it must be situated at their point of intersection. It is a geometrical property of this point that it divides each median in the ratio of 2 : 1, making

$$AG = \tfrac{2}{3} AD, \text{ etc.}$$

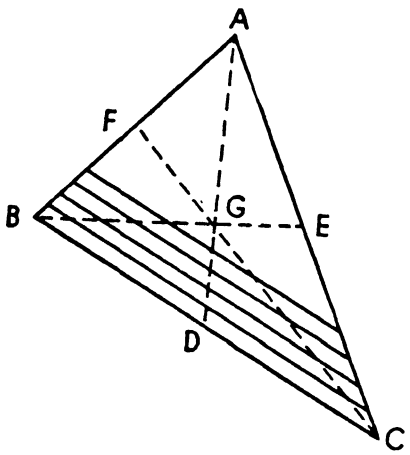


FIG. 109

In a similar manner the C.G. of a parallelogram can be shown to be the point of intersection of its diagonals by imagining the figure divided into strips parallel to the directions of its sides.

5. TYPES OF EQUILIBRIUM

Stable Equilibrium.—There are three types of equilibrium, *viz.* stable, unstable and neutral. The first of these is the commonest, and is typified by such a system as a cube resting on a horizontal table as in Fig. 110 (i).

The defining characteristic of stable equilibrium is that if the body is given a small displacement, the forces which then act upon it are such as

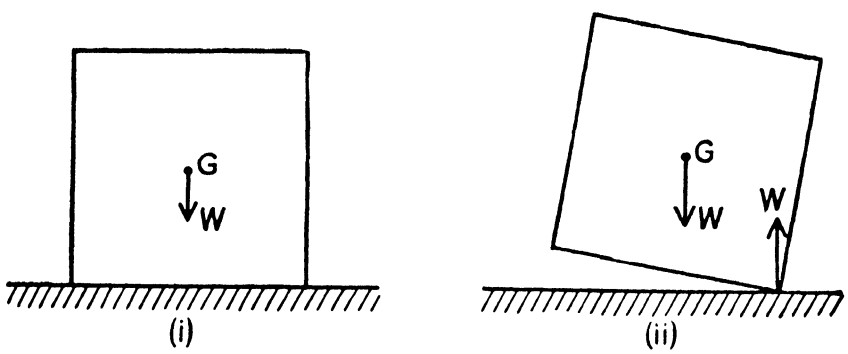


FIG. 110

to restore it to its original position. In Fig. 110 (ii) the cube has been tilted slightly on one edge and will evidently fall back to its original position under the action of the couple consisting of its own weight W acting through G and the equal and opposite reaction of the table acting at the

edge. In the case of a system such as this, in which the weight of the body plays a part in the equilibrium, the equilibrium is stable if the small displacement causes a *rise* of the C.G., as it obviously does in Fig. 110. Evidently the displacement necessitates the performance of work by an external force and a corresponding increase of the potential energy of the body. It follows that the potential energy of a body is a minimum when it is in a position of stable equilibrium.

Unstable Equilibrium.—An example of this type of equilibrium is to be found in a cube standing on one of its edges as in Fig. 111 (i). At first sight it appears irrational to describe this condition of the body as one of equilibrium when common experience teaches us that the cube would not remain in such a position for more than a second or two.

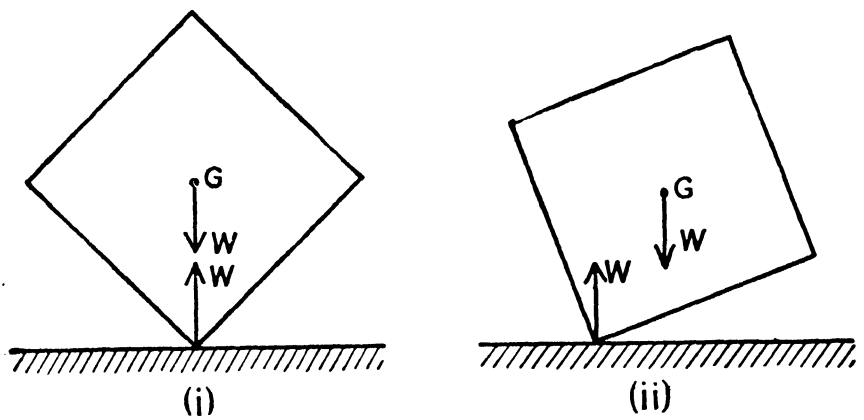


FIG. 111

Nevertheless the forces acting on the body fulfil the conditions for equilibrium. The reaction of the table is equal and opposite to the weight of the cube and has the same line of action if G is vertically above the edge on which the cube is resting, so that there is no turning effect. The respect in which the condition of the body differs from one of stable equilibrium lies in the fact that when the body is given a small displacement (Fig. 111 (ii)) the forces which then act upon it are such as to *increase* the displacement. This is the exact opposite of the condition for stable equilibrium and accounts for the impossibility of balancing a cube on one edge or corner. The slightest disturbance causes an ever-increasing displacement from the equilibrium position. The small displacement involves a *lowering* of the C.G., *i.e.* a decrease of P.E., so that a position of unstable equilibrium is a condition of maximum P.E.

Neutral Equilibrium.—Neutral equilibrium represents the transitional condition between stable and unstable equilibrium. It occurs when a body is in such a position that a small displacement has no effect on the forces acting upon it. An example is a uniform sphere resting on a horizontal surface (Fig. 112). As the sphere is rolled, the reaction at the point of contact is always equal and opposite to the weight and has

the same line of action. Also, since the C.G. does not rise or fall, there is no work done during the displacement and therefore no change of P.E.

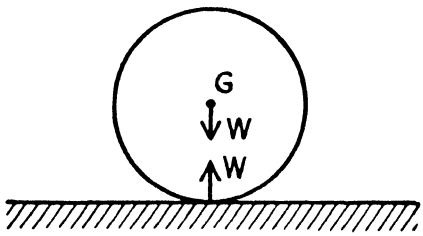


FIG. 112

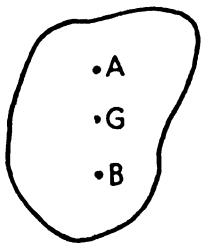


FIG. 113

A body is in a state of neutral equilibrium when it is pivoted at its centre of gravity, G (Fig. 113), since its condition is unaltered by a rotation. If it is pivoted at A vertically above G it is in stable equilibrium, while a pivot at B, vertically below G, would give unstable equilibrium.

Although in the above discussion we have used examples in which the force of gravity plays a part, the distinction between the three types of equilibrium is not dependent on the action of gravity. It is purely a matter of the way in which the forces acting on the body in question are modified by a small displacement.

6. MACHINES

Definitions.—A machine, in the sense in which the word is used in mechanics, is a device whereby a force applied at one place can cause another force to be exerted elsewhere. Very frequently the machine is designed for the purpose of making the second force larger than the one which causes it, but this is not necessarily always the case. The *movement* of the first force and the corresponding movement of the second is an essential feature. Examples of machines are levers, pulley systems, the screw-jack, etc.

The principle of the machine is shown diagrammatically in Fig. 114. A force, E, called the **effort,** is applied to the machine and moves through a distance s_E in its own direction with the result that another force, L, (the **load**) of possibly different magnitude and direction is caused by the

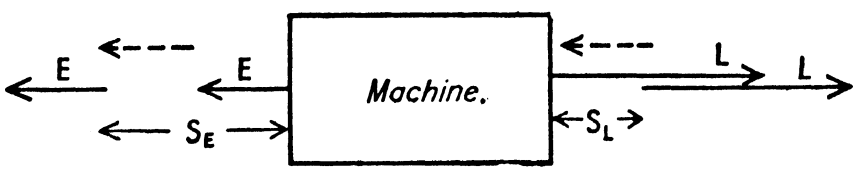


FIG. 114

machine to move through a distance s_L in a direction opposite to its own. An example will make the matter clearer. In a system of pulleys and chains used for hoisting heavy articles the load is the weight of the article, while the effort is the force with which the hands haul on the chain.

The Equilibrium of Forces

There are three quantities which are used in order to specify performance of a machine. They are:

$$\text{Mechanical Advantage (M.A.)} = \frac{\text{Load}}{\text{Effort}} = \frac{L}{E}$$

Velocity Ratio (V.R.)

$$= \frac{\text{Distance moved by effort in its own direction}}{\text{Distance moved by load contrary to its own direction}} = \frac{s_E}{s_L}$$

$$\text{Efficiency (Eff.)} = \frac{\text{Work done by machine against load}}{\text{Work done on machine by effort}} = \frac{Ls_L}{Es_E}$$

The efficiency is sometimes multiplied by 100 so as to express it as a percentage. Evidently the above three quantities are related by the equation

$$\text{Eff.} = \frac{\text{M.A.}}{\text{V.R.}}$$

where efficiency is expressed as a fraction and *not* as a percentage.

It is impossible to make a completely frictionless machine, so that in practice some of the work done on the machine is used against friction and is not available for performing work against the load. By the conservation of energy the work done against friction and against the load must add up to the work done on the machine by the effort. Therefore unless friction is entirely absent, the work done on the load is less than that done by the effort, and the efficiency is always less than 100 per cent.

In practice the efficiency varies very much from one type of machine to another. A lever, for example, has a high efficiency while that of a screw-jack is low. However, the effectiveness of a machine is rather a matter of its mechanical advantage than of its efficiency. Since efficiency is always less than unity, it follows that in every machine

$$\text{M.A.} < \text{V.R.}$$

In other words, the V.R. represents the maximum possible M.A. which a particular machine would give if it were frictionless. The V.R. is a function of the design and dimensions of the machine, and can always be calculated or determined simply by the measurement of distances. On the other hand, the M.A. cannot be calculated from the construction and dimensions of the machine because the magnitude of the frictional resistance is unknown. It must be determined by experiment with actual forces. If we calculate from the dimensions of the machine the ratio of L to E, supposing that the machine is in equilibrium under the action of these forces *alone*, we arrive at the value which the M.A. would have if there were no friction (efficiency unity), *i.e.* the same value as the V.R. Examples of this principle are given in connection with the different types of machine which are dealt with later.

Experimental Determination of M.A.—The general method of doing this is to place a known load on the machine and, by means of weights or a spring-balance, to apply just sufficient effort to cause the system to move without acceleration after being given a slight start. The effort is then overcoming the kinetic friction in the machine and balancing the load. The M.A. is then calculated as the ratio of the load to the effort.

Levers.—Fig. 115 shows a simple lever, the **fulcrum,** or point about which the lever turns, being marked F. As the effort E which is applied at A, moves downwards, the load L applied at B is caused to move upwards. If the lever rotates slightly through an angle θ so as to take up the position A′FB′, the velocity ratio is AA′/BB′. For a small rotation AA′ and BB′ can be regarded as arcs of circles with centres at F, so that

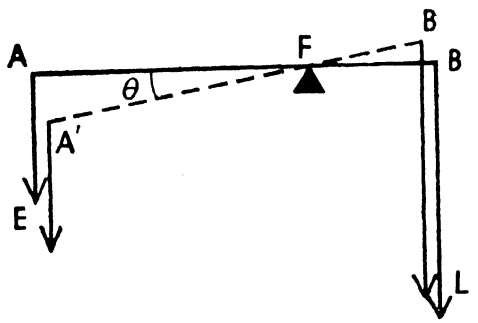


FIG. 115

$$AA' = AF \times \theta$$

and

$$BB' = BF \times \theta$$

$$\therefore \text{V.R.} = \frac{AA'}{BB'} = \frac{AF}{BF}$$

This ratio would also be equal to the M.A. if there were no friction at the fulcrum and if the weight of the lever itself had no moment about the fulcrum. This latter condition is satisfied if the lever is rotating in a horizontal plane or if its C.G. is at F. It should be noted that, as already explained, the M.A. is found to be AF/BF if we suppose that the lever is in equilibrium under the action of L and E and we equate their moments about F, *i.e.* if we ignore friction. Examples of this type of lever are the crow-bar and the "village"-pump handle. Scissors, pliers, pincers, etc. also work on the same principle, the two levers in each case having a common fulcrum.

In another type of lever the fulcrum is not between the load and the effort (Fig. 116). The velocity ratio is equal to BF/AF, and the M.A. is also equal to this if friction and the weight of the lever itself can be ignored. A device such as this has many applications, examples being the tin-opener and a pair of nut-crackers.

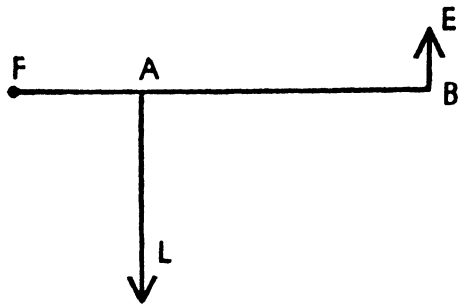


FIG. 116

Many levers in practical use are not straight but "bent," and the load and effort may have any relative directions convenient for the particular purpose for which the lever is

used. Examples are the hammer, when used for removing nails, and the bent lever used for connecting the rods which operate railway points (Fig. 117).

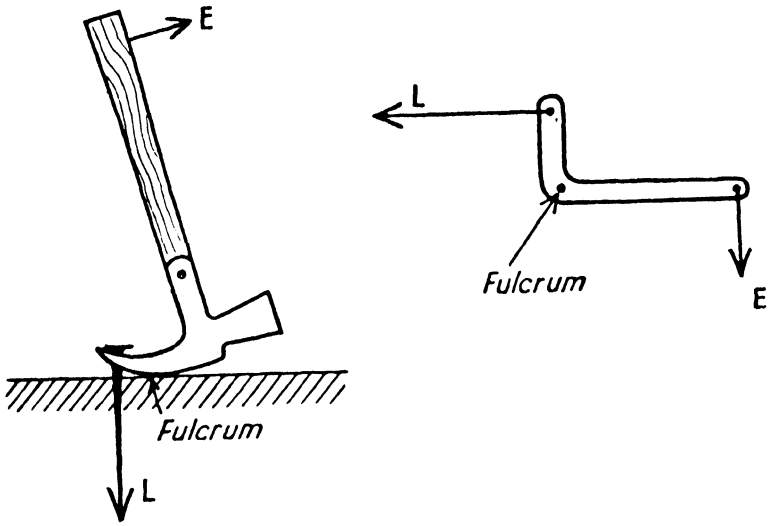


FIG. 117

The Inclined Plane is one of the oldest and commonest devices for enabling man to raise heavy loads. If the body on the plane in Fig. 118 is moved up the plane through a distance s_E under the action of the effort E, then the vertical distance travelled is s_L and the weight of the body is the load L. The V.R. is given by

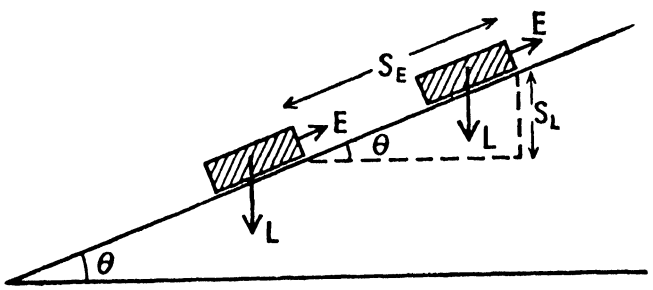


FIG. 118

$$\text{V.R.} = \frac{s_E}{s_L} = \frac{1}{\sin\theta}$$

Thus, if there were no friction (*i.e.* an efficiency of unity) the M.A. would also be equal to $\frac{1}{\sin\theta}$. This is also easily proved from a consideration of the statics of the inclined plane. Thus

L = wt. of body = mg

E = force necessary to balance component of weight down the plane

= $mg \sin\theta$

$$\therefore \text{M.A.} = \frac{L}{E} = \frac{1}{\sin\theta}$$

In practice, friction causes the M.A. to be less than this, but by minimizing

friction, particularly by the use of rollers or wheels, the efficiency is increased, and the M.A. corresponding to a given value of θ is brought nearer to the above value. The student should be able to show that if the coefficient of sliding friction is μ, then

$$\text{M.A.} = \frac{1}{\sin\theta + \mu\cos\theta}$$

and

$$\text{Eff.} = \frac{1}{1 + \mu\cot\theta}$$

Machines like the Windlass and Capstan.—In machines of this type the M.A. is derived from the action of a long handle which rotates an axle or drum, on to which a rope is wound. An example of the type used for raising water from a well is shown in Fig. 119. If the radius of the drum is r and the length of the crank of the handle is R, then one complete turn

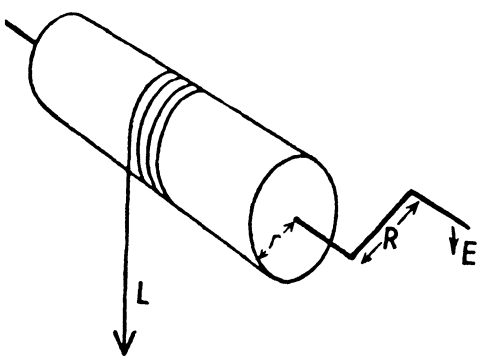


FIG. 119

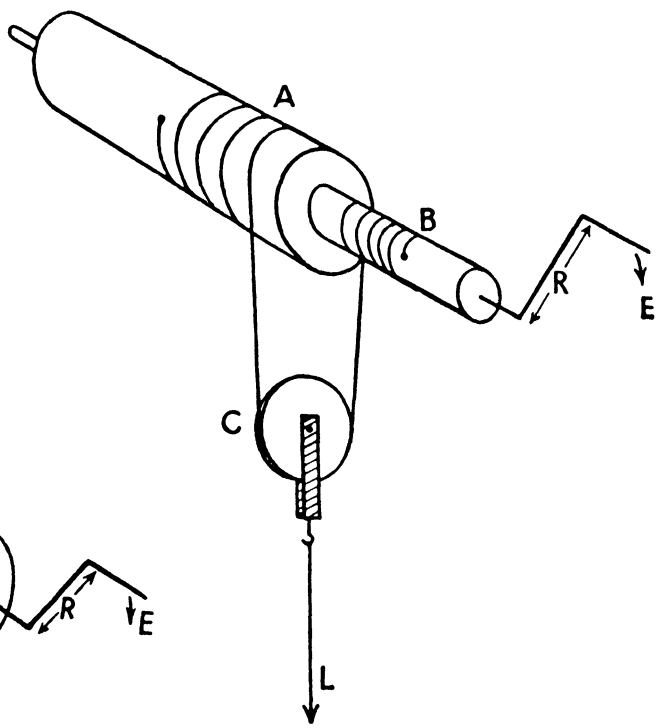


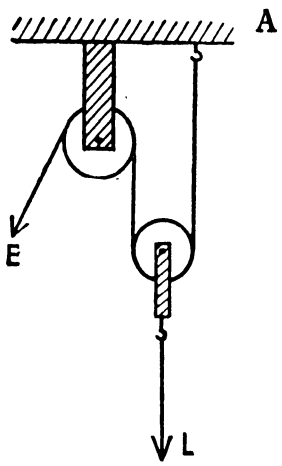
FIG. 120

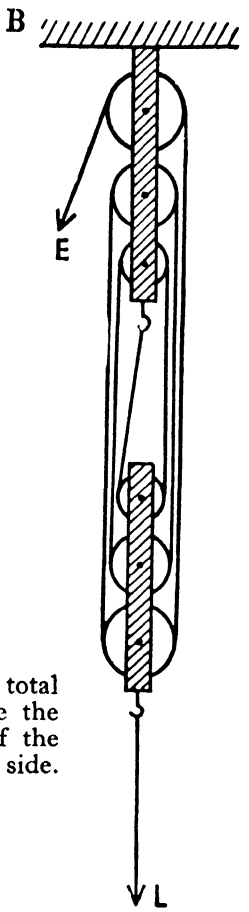
causes a movement of the load by $2\pi r$ and of the effort (which is applied at right angles to the crank) by $2\pi R$. Therefore

$$\text{V.R.} = \frac{R}{r}$$

and the mechanical advantage approaches this value when friction is small.

Differential Wheel and Axle.—The arrangement is shown in Fig. 120. A string is wound in the way illustrated round two drums A and B of different radii, r_A and r_B, and a pulley, C, hangs in the loop and supports the load. One turn of the handle causes a length of string $2\pi r_A$ to wind

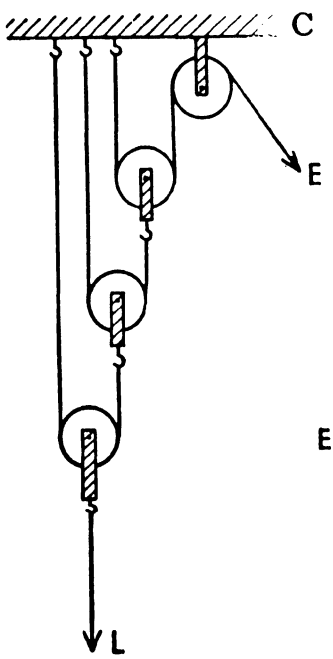




A V.R. = 2

B V.R. = 6

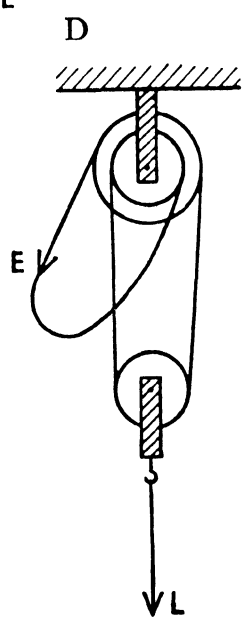
In general V.R. = n, the total number of pulleys. In practice the pulleys in each block are all of the same size and are placed side by side.



C Archimedes' System.

V.R. = 8

In general V.R. = 2^n, where n = number of pulleys, excluding the top fixed one.



D Weston Differential Pulley Block.

The two upper pulleys of radii R (larger) and r are rigidly fixed to each other. There must be no slipping at the upper pulleys, therefore a chain whose links fit into sockets in the pulleys is used.

$$\text{V.R.} = \frac{2R}{R - r}$$

(see differential wheel and axle).

FIG. 121

on to A and a length $2\pi r_B$ to leave B. The hanging loop has therefore diminished its length by $2\pi(r_A - r_B)$, and since this decrease is shared equally between the two sides, the pulley C is raised by $\pi(r_A - r_B)$. Meanwhile the point of application of the effort moves through $2\pi R$ in the direction of the effort if R is the length of the crank. Therefore

$$\text{V.R.} = \frac{2\pi R}{\pi(r_A - r_B)} = \frac{2R}{r_A - r_B}$$

Thus a large V.R. and correspondingly large M.A. can be obtained by making the difference between the radii of the drums small.

It is possible to make a machine of a similar type with only one drum by attaching to a fixed point the end of the string which in the above case is wrapped round B. The V.R. is then given by the above expression with r_B equated to zero.

Pulley Systems.—There is a great variety of ways in which pulleys and ropes or chains can be used to achieve mechanical advantage for lifting and hauling. Fig. 121 shows four different types of pulley systems. The student should be able to verify the values given for the V.R.s by considering the construction and action of the machines. In all cases the M.A. falls short of the V.R. (efficiency < 1) because of (*a*) friction, (*b*) the fact that all those pulleys which are not rigidly fixed to the top support are raised during the hoisting of the load. With increasing load the effects of both (*a*) and (*b*) become relatively smaller, and M.A. consequently becomes more nearly equal to V.R. but never reaches that value. The weight of the pulleys has no effect when the machine is used for hauling horizontally, so that M.A. is greater for hauling than for hoisting.

The Screw-Jack.—A simple representation of a screw-jack is shown in Fig. 122. The **pitch** of the screw (p) is the distance between two consecutive threads measured along the axis of the screw. One revolution causes the screw to rise or fall through a distance p. If the load acts along the axis of the screw, *e.g.* the weight L shown in Fig. 122, and the effort E is applied tangentially to an arm at a distance R from the axis of the screw, then the V.R. is equal to $2\pi R/p$. The M.A. usually falls very short of this value because friction is considerable, but when p is small and R is large, a high M.A. can nevertheless be achieved. An example of this device is the jack used for raising the axle of a car. In this the screw does not rotate, but the

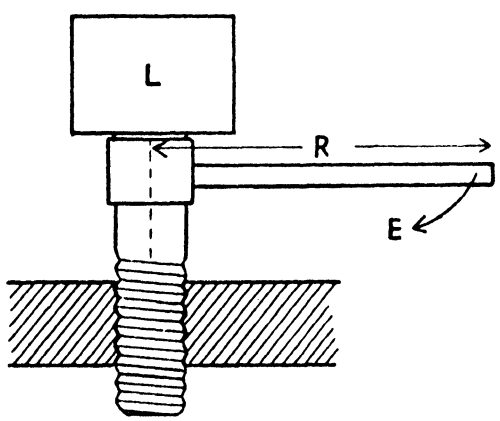


FIG. 122

thread surrounding it is rotated by two bevelled cogs operated from a long shaft which is turned by a handle at its far end. Further M.A. may be introduced by the ratio of the number of cogs on the wheels and the radius of the handle.

Variation of M.A. with Load. The Law of a Machine.—It has already been explained in connection with pulley systems that friction and the weights of parts of the machine which are raised during its operation cause the M.A. to fall short of the V.R., but that this deficiency becomes less as the load increases. For any given machine, say a pulley system, a series of corresponding values of L and E can be determined experimentally and the M.A. calculated for each pair. A graph of M.A. against load is of the type shown in Fig. 123. The machine requires some effort to make it work without acceleration even when L is zero, so that the M.A. (L/E) is initially zero. With increasing load the M.A. rises rapidly at first and then at a decreasing rate. It approaches the value of the V.R. but never reaches it.

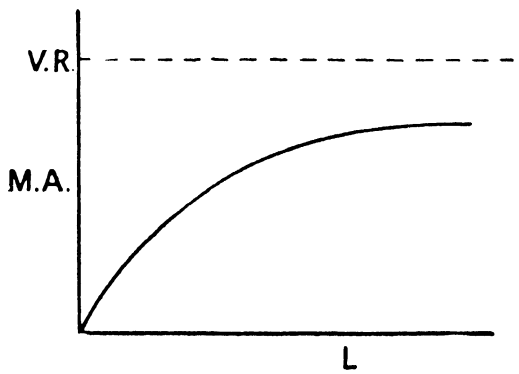


FIG. 123

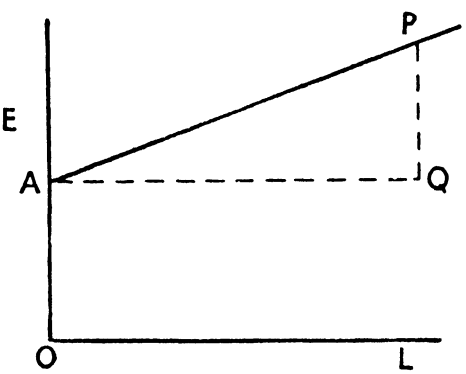


FIG. 124

If the same set of readings is used to construct a graph of E against L, a straight line such as that in Fig. 124 is frequently obtained. The effort OA represents the force necessary to work the machine when no load is applied. It is the force required to overcome the weights of the pulleys, etc., and also the kinetic friction. Since the graph is a straight line, it is evident that the *additional* effort over and above OA which is required to overcome any given load is proportional to the load. Nevertheless the load, divided by the additional effort which is actually responsible for raising it, is *not* equal to the V.R. That is to say, the efficiency of the machine is *not* unity even when allowance is made for the force required to operate it at no load. This is because the frictional resistance is not constant but increases with the load—friction is proportional to normal reaction.

The **law of the machine** is the equation connecting E and L, which can be derived from the graph in Fig. 124. When this is a straight line, the equation is of the form

$$E = mL + c$$

where m and c are constants.

If P is any point on the best straight line which can be drawn through the experimental points, and PQ and AQ are perpendicular to each other, then

$$m = \frac{PQ}{AQ}$$

provided the lengths are measured in the units used on the axes to which they are respectively parallel. Also

$$c = OA$$

measured in the force units used on the axis of E.

Overhauling.—It is common experience that a motor-car jack will support the car even when the effort which was used to raise it is completely removed. On the other hand in some machines, notably levers and pulley systems with small velocity ratios, the load causes the system to operate backwards when the effort is removed. This is known as **overhauling,** and a simple example is the falling bucket of a well windlass whose handle is allowed to go free. In general, when the effort is removed there is only the friction within the machine to prevent the load from "taking charge". Therefore if the machine does not overhaul, the frictional resistance must be at least equal to the load. When the resistance is equal to the load and the machine is being operated in the usual way, the effort has to overcome a force which is effectively double the actual load, provided we can assume that the resistance has the same magnitude (though, of course, the opposite direction) whichever way the machine is operating. In these circumstances half of the work done by the effort is used in doing work against frictions, and the other half is effective in moving the load. The efficiency is therefore 50 per cent. A machine will overhaul, therefore, unless its efficiency is less than 50 per cent., that is to say, unless the resistance exceeds the load. This is only an approximate criterion, since we have not considered the weights of those parts of the machine which rise or fall during its operation. The directions of these forces are not reversed when the machine operates backwards.

7. THE COMMON BALANCE

Construction of a Balance.—The student should take the earliest opportunity of studying the construction of an actual balance. One of the commonest mistakes in students' drawings of a balance is the failure to realize that both the central knife edge *and* the knife edges at the ends of the beam are rigidly fixed to the beam itself. In practice the knife edges are frequently made of a hard material called agate.

The central knife edge O (Fig. 125) points downwards and acts as the pivot about which the beam turns. When the balance is in use this knife edge is supported either on an agate plane or in a groove, of wider angle than its own, cut out of a piece of agate (Q in the figure). The terminal

knife edges, A and B, act as supports for the pans and their contents. There are various ways of hanging the pans. One method, by means of a stirrup, is shown in the small drawing in Fig. 125. A grooved piece of agate fixed to the stirrup rests on the knife edge. When the arrestment of the balance is lowered by turning back a handle or knob in front of the base, the support S is lowered so that the beam rests on fixed supports

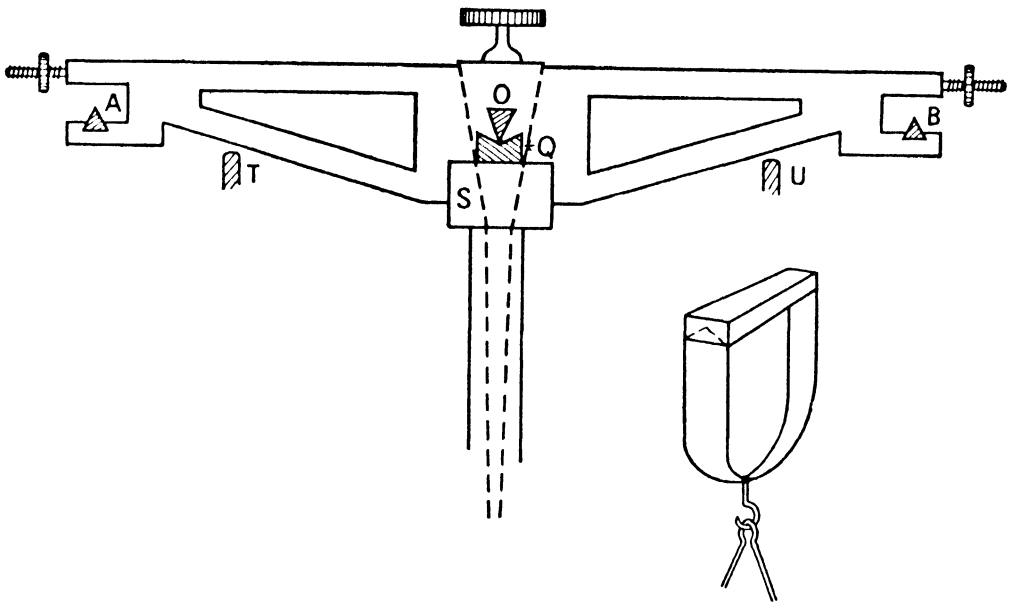


FIG. 125

T and U, and its weight is taken off the central knife edge. At the same time the end knife edges are left free because the lowering of the beam causes the stirrups which were previously resting on them to drop into fixed supports while the pans themselves now rest on the base of the balance. A vertical pointer (shown dotted in the drawing) is rigidly fixed to the beam and moves over a small scale fixed at the bottom of the pillar.

The correct use of a balance must be learnt in the laboratory. Weighing is one of the most accurate physical measurements provided that care is taken, and especially if the balance and weights are not allowed to deteriorate by rough usage.

Theory of the Balance.—The theory of the balance is more complicated than might at first sight be supposed. In Fig. 126, which is a simplified diagram of a balance beam, A and B are the knife edges which support the pans, and O is the central knife edge about which the beam is free to turn. The weight, w, of the beam and all objects (*e.g.* the pointer) *rigidly* connected to it acts at the centre of gravity G, which, it should be noted, lies below C. In the diagram, the distances OC ($=b$) and OG ($=h$) are made to appear very much larger than they are in practice. It is assumed for simplicity that OG is perpendicular to AB.

If AB is inclined at an angle θ to the horizontal, the other angles marked in the diagram are also equal to θ. Various horizontal, and vertical construction lines are drawn in order to assist in taking moments about O. The lengths of the **arms** of the balance AC and BC are shown as a_1 and

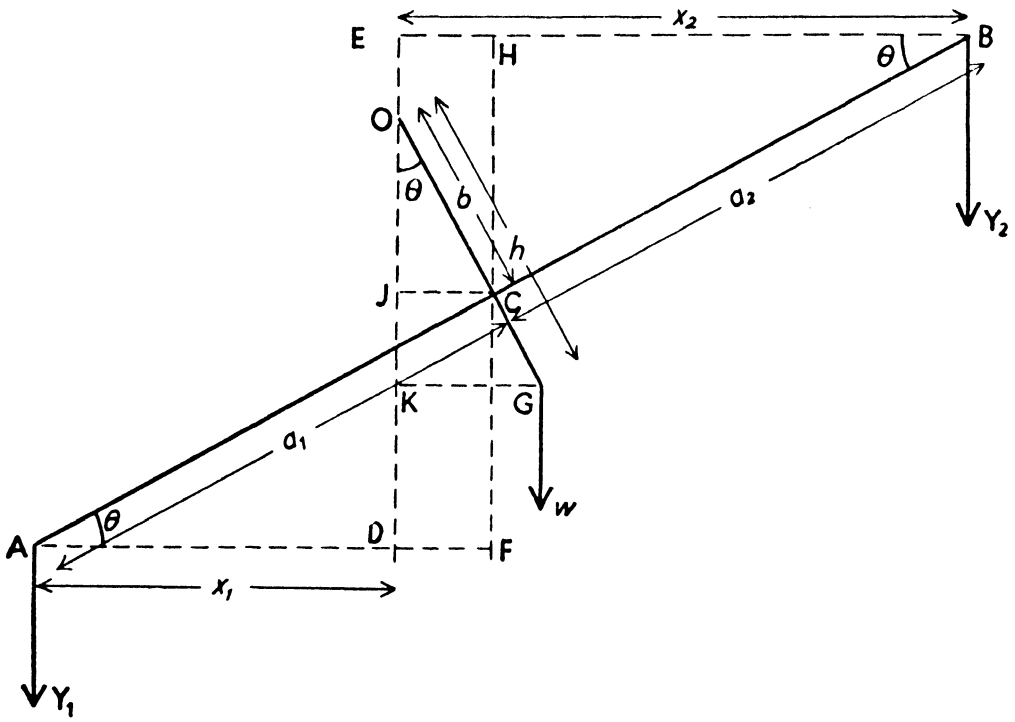


FIG. 126

a_2 and are not assumed to be equal in the first place. When the beam is in the deflected position shown, the effective arms are the horizontal distances x_1 and x_2, and we have

$$\begin{aligned} x_1 = \text{AD} &= \text{AF} - \text{DF} \\ &= \text{AF} - \text{JC} \\ &= a_1 \cos\theta - b \sin\theta \end{aligned}$$

Also

$$\begin{aligned} x_2 &= \text{BE} \\ &= \text{BH} + \text{EH} \\ &= \text{BH} + \text{JC} \\ &= a_2 \cos\theta + b \sin\theta \end{aligned}$$

If the beam is in equilibrium, taking moments about O gives

$$\begin{aligned} Y_1 x_1 &= Y_2 x_2 + (w \times \text{GK}) \\ &= Y_2 x_2 + wh \sin\theta \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

where Y_1 and Y_2 are the vertical forces on A and B.

This equation represents the equilibrium between the deflecting couple $Y_1x_1 - Y_2x_2$ and the restoring couple $wh \sin \theta$ due to the weight of the beam.

Substituting for x_1 and x_2 in equation (5), we have

$$Y_1(a_1 \cos \theta - b \sin \theta) = Y_2(a_2 \cos \theta + b \sin \theta) + wh \sin \theta$$

Dividing by $\cos \theta$, we obtain

$$Y_1a_1 - Y_1b \tan \theta = Y_2a_2 + Y_2b \tan \theta + wh \tan \theta$$

or
$$\tan \theta = \frac{Y_1a_1 - Y_2a_2}{(Y_1 + Y_2)\, b + wh} \quad . \quad . \quad . \quad (6)$$

The procedure to be followed when a body is weighed against a set of weights may be devised by noting equation (5). The lengths x_1 and x_2 vary with θ but are constant *for a given value of* θ, and so is the expression $wh \sin \theta$.

Suppose that the balance pans have weights P_1 and P_2 respectively, and that when they are empty the equilibrium angle of the beam is θ. We then have, by equation (5),

$$P_1x_1 - P_2x_2 = wh \sin \theta \quad . \quad . \quad . \quad . \quad (7)$$

Now let a body of actual weight W_1 be placed in the left-hand pan, and suppose that weights from a box to the value of W_2 must be placed in the right-hand pan in order to bring the deflection to the previous value as indicated by the pointer. Then

$$(P_1 + W_1)x_1 - (P_2 + W_2)x_2 = wh \sin \theta \quad . \quad . \quad . \quad (8)$$

Subtracting equation (7) from equation (8), we obtain

$$W_1x_1 = W_2x_2 \quad . \quad . \quad . \quad . \quad (9)$$

Thus W_1 and W_2 are *not* equal. But if W_1 is now transferred to the right-hand pan and weights (W_2') are put in the left-hand pan in order to bring the pointer once again to the same reading, we have

$$W_2'x_1 = W_1x_2 \quad . \quad . \quad . \quad . \quad (10)$$

Dividing equation (9) by equation (10) gives

$$\frac{W_1}{W_2'} = \frac{W_2}{W_1}$$

or
$$W_1 = \sqrt{W_2W_2'}$$

Thus, by adopting the above procedure it is possible to deduce the true weight of the body, W_1. The ratio of the effective arms of the balance x_1/x_2 for the particular deflection θ can be found by eliminating W_1 from

equations (9) and (10), and we have

$$\frac{x_1}{x_2} = \sqrt{\frac{W_2}{W_2'}}$$

The makers of the balance usually aim at making the three knife edges

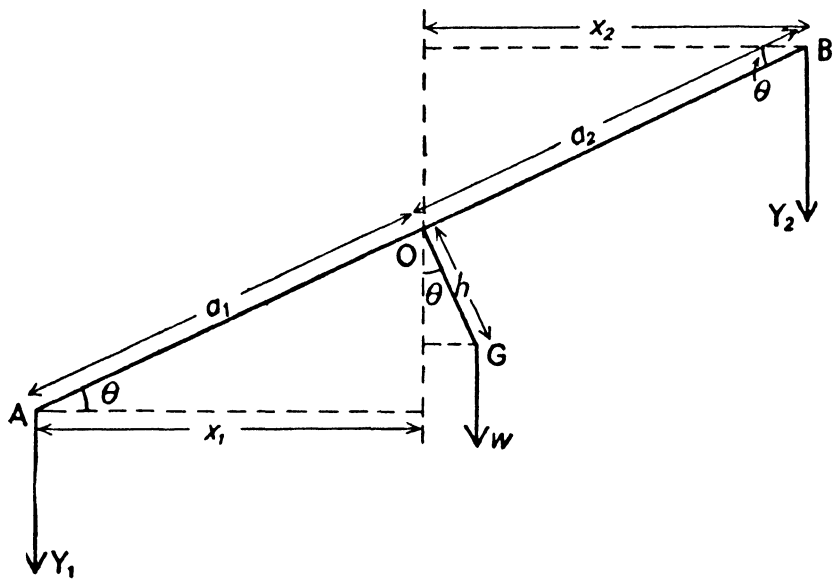


FIG. 127

coplanar so that O coincides with C, and b is equal to zero. The system is then as shown in Fig. 127. In this case

$$x_1 = a_1 \cos \theta$$

and

$$x_2 = a_2 \cos \theta$$

and, either from first principles or by putting $b = 0$ in equation (6), we find the equation of equilibrium to be

$$\tan \theta = \frac{Y_1 a_1 - Y_2 a_2}{wh}$$

The difference between this and the previous case, in which O was above the line AB, lies in the fact that *stable* equilibrium is not possible in the present case unless the centre of gravity of the beam is below the pivot O. If G were coincident with O it would be possible to achieve *neutral* equilibrium by a correct choice of Y_1 and Y_2, but a slight additional load on one side would result in a deflection of 90°, because rotation of the beam would not bring a restoring moment into play. In the first case we considered (Fig. 126), it is possible for stable equilibrium to occur even if G coincides with O.

Strictly speaking, it is still necessary, even when A, C and B are

coplanar, to follow the interchange procedure when weighing. Thus, using the same symbols as before, we have for the empty pans

$$P_1a_1 \cos \theta_1 - P_2a_2 \cos \theta_2 = wh \sin \theta$$

or

$$P_1a_1 - P_2a_2 = wh \tan \theta$$

When the unknown weight is on the left-hand side and the deflection is again θ,

$$(P_1 + W_1)a_1 - (P_2 + W_2)a_2 = wh \tan \theta$$

so that by subtraction

$$W_1a_1 = W_2a_2 \quad . \quad . \quad . \quad . \quad (11)$$

and when the weights are interchanged and W'_2 is now required to balance W_1, the deflection again being θ, we have

$$W_2'a_1 = W_1a_2 \quad . \quad . \quad . \quad . \quad (12)$$

Thus by dividing equation (11) by equation (12) we obtain

$$W_1 = \sqrt{W_2W_2'}$$

and

$$\frac{a_1}{a_2} = \sqrt{\frac{W_2}{W_2'}}$$

Thus, in contrast with the former case, it is now possible to determine the ratio of the actual arms of the balance.

In order to be able to determine the weight of a body without interchanging, it is clearly necessary that a_1 and a_2 should be exactly equal, because then equation (11) gives

$$W_1 = W_2$$

It should be noted, however, that the condition that the three knife edges are coplanar is also necessary unless the balance beam is always exactly horizontal.

Factors Affecting the Sensitivity of a Balance.—The sensitivity of a balance can be defined as the deflection of the pointer due to a given difference of weight on the two sides, *e.g.* one centigram for a coarse balance or one milligram for a sensitive one. Alternatively, sensitivity may be expressed as the difference of weight necessary to produce a deflection of one division, in which case the more highly sensitive the balance the smaller is the figure representing its sensitivity.

For the small angles through which a balance beam is deflected we can assume that θ in radians is equal to $\tan \theta$, and in discussing sensitivity we can assume that the arms of the balance are equal. Let each be of length a. Then equation (6) becomes

$$\theta = \frac{(Y_1 - Y_2)a}{(Y_1 + Y_2)b + wh}$$

and, taking the first definition of sensitivity, we have

$$\text{sensitivity is proportional to } \frac{\theta}{Y_1 - Y_2}$$

$$\textit{i.e.}\ \text{proportional to } \frac{a}{(Y_1 + Y_2)\, b + wh}$$

From this we can deduce the following:—

(i) Sensitivity is increased by increasing a, the length of the balance arms. This, however, increases the weight of the beam which tends to reduce the sensitivity.

(ii) Unless the knife edges are coplanar ($b = 0$), sensitivity decreases with increasing load on account of the factor $Y_1 + Y_2$ in the first term of the denominator.

(iii) Sensitivity is large when the dimensions b and h are small. As already mentioned, b is usually made to be zero, but if h were also made zero there would be no restoring moment, and the smallest value of $Y_1 - Y_2$ would cause a very large deflection (theoretically 90°). It is often possible to vary the position of G by raising or lowering a small weight attached to the beam above its centre. The values of b and h are, of course, affected by the bending of the balance beam under the action of increasing loads. These will tend to depress the terminal knife edges A and B, and so increase b and possibly h, thus making the balance less sensitive.

(iv) For high sensitivity the weight of the beam, w, must be as small as possible consistent with rigidity.

When an unloaded balance is set gently swinging it may be compared to a compound pendulum (page 109) in which the restoring moment is $wh \sin \theta$. Thus the time period of a balance is large when wh is small, *i.e.* when the sensitivity is high.

The Steelyard.—A steelyard uses a modification of the common-balance principle for weighing heavy objects by means of smaller weights. The **Roman steelyard** is shown diagrammatically in Fig. 128. It consists of a steel bar pivoted at O with a widened portion at one end. The C.G. of the beam is at G. The object to be weighed is hung at A, and a movable weight w can be placed at various points along the graduated bar. Suppose that when there is no load at A the beam is balanced by

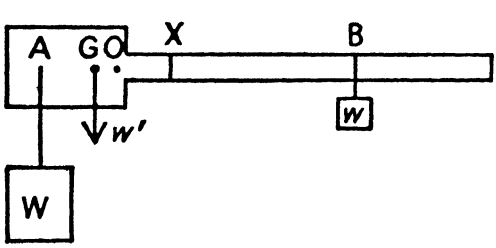


FIG. 128

adjusting the position of w to some point X. Then by taking moments about O we have

$$w' \times OG = w \times OX \quad . \quad . \quad . \quad . \quad (13)$$

where w' is the weight of the beam.

Now let an unknown weight W be attached at A and suppose that the beam is counterpoised when w is moved to B. Evidently the additional moment due to W is balanced by the increased moment due to the shift of w, so that

$$W \times OA = w \times XB \quad . \quad . \quad . \quad . \quad (14)$$

a result which can be obtained by writing down the equation of the moments when W is attached and subtracting equation (13) from it.

Equation (14) shows that W is proportional to XB since OA and w are constant. The point X can therefore be marked as zero on the scale. Suppose that w is 1 lb., and that the distance between the zero mark X and the point where w must be hung when W is 1 stone is divided into 14 equal divisions. Then each division will represent 1 lb., *e.g.* if the beam balances when w is at the 9th division then $W=9$ lb. If W is heavier than 1 stone it can be weighed by placing additional weights at the 14th division of the scale and making a fine adjustment with the 1-lb. weight at the appropriate distance along the scale. Thus if a weight of 3 lb. must be hung at the 14th mark and a 1-lb. weight at the 6th mark in order to balance a given weight W, then W is 3 stone 6 lb.

In the **Danish steelyard** there is no movable weight. The position of the pivot itself can be moved along the bar, and a constant weight is permanently fixed at the far end of the bar on the opposite side to the object (W) which is being weighed, which is always attached at A (Fig. 129).

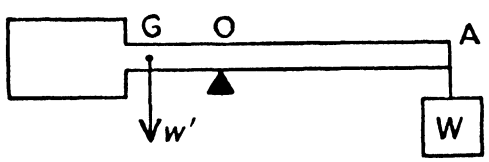


FIG. 129

If the total weight of the bar is w', taking moments about O gives

$$w' \times OG = W \times OA$$

The graduations of the scale along GA (with the zero at G), on which the position of the pivot O gives directly the value of W, are not equally spaced but are crowded together near the end A. The student should verify that if $W=nw'$, OG, the scale reading, is equal to $\frac{n}{n+1}$ of the fixed distance GA.

8. THE DETERMINATION OF THE CONSTANT OF GRAVITATION

On page 48 it was stated that Newton's law of gravitation is expressed by the equation

$$P = G\frac{m_1 m_2}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where P is the force of attraction between two particles of masses m_1 and m_2 situated at a distance d apart. The constant of gravitation G is a quantity which depends on the units in which P, m_1, m_2 and d are expressed, but when these units are fixed G is independent of the magnitudes of the masses and of their distances apart.

The above statement of the law of gravitation refers to the force between *particles*, *i.e.* masses of negligible size, so that in order to find an expression for the force between any pair of bodies upon which actual experiments can be performed it is necessary to make calculations with this law as the starting-point. One of the shapes most commonly used in experiments of this sort is the sphere, and when the law of gravitation between particles is used to calculate the gravitational effect of a *spherical shell of uniform density*, it is found that, as regards points outside it, the shell behaves as though all its mass were concentrated in a single particle situated at its centre. Thus if we have two such separate shells of masses m_1 and m_2 with their centres a distance d apart, the force of attraction between them is given by equation (15). By regarding a solid sphere as being made up of concentric shells, we conclude that equation (15) would also be true for two solid spheres composed of shells each of uniform density (*e.g.* the earth) or, what is more important in laboratory experiments, two solid spheres each of uniform density.

The gravitational effect of a uniform spherical shell at points *within* its hollow space is found by calculation to be zero. This is because the attraction of each small portion of the shell on a particle placed inside it is exactly equal and opposite to the attraction due to another portion on the opposite side, and when all the portions and their counterparts are taken together the whole shell has been accounted for.

Assuming that the earth is a solid sphere of radius R and is composed of concentric shells, each of which is of uniform density, then since its attraction on bodies outside it is the same as if its mass M were concentrated at its centre, the force it exerts on a small body of mass m situated at its surface is $G\frac{Mm}{R^2}$. This force is the weight of the body, which is mg if g is the acceleration due to gravity at the place where m is situated. Equating the two expressions, therefore, we obtain (as on page 49) the equation

$$g = \frac{GM}{R^2}$$

Thus when the value of G has been found experimentally we can calculate M by this equation, provided that R is known. This is why the process of determining G is sometimes referred to as "weighing the earth." The student should be able to decide for himself why this phrase is misleading, if not meaningless.

When M has been found, the mean density of the earth Δ can be found from the equation

$$\Delta = \frac{M}{\text{Volume of earth}} = \frac{M}{\frac{4}{3}\pi R^3}$$

Experiments using Natural Masses.—During the eighteenth and nineteenth centuries a number of experiments were done, principally in Scotland and the Andes, in which the attraction exerted on a body by a known natural mass, *e.g.* a mountain, was compared with the force exerted on it by the remainder of the earth. The direction which we call "vertical" at any place is the direction of a freely hanging plumb-line, and, in general, this does not pass exactly through the centre of the earth. Local protuberances, such as mountains, hills and even buildings, exert very small sideways attractions on a plumb-bob. In using this effect to determine the mass of the earth, a telescope T (Fig. 129*a* (i)) was used to observe

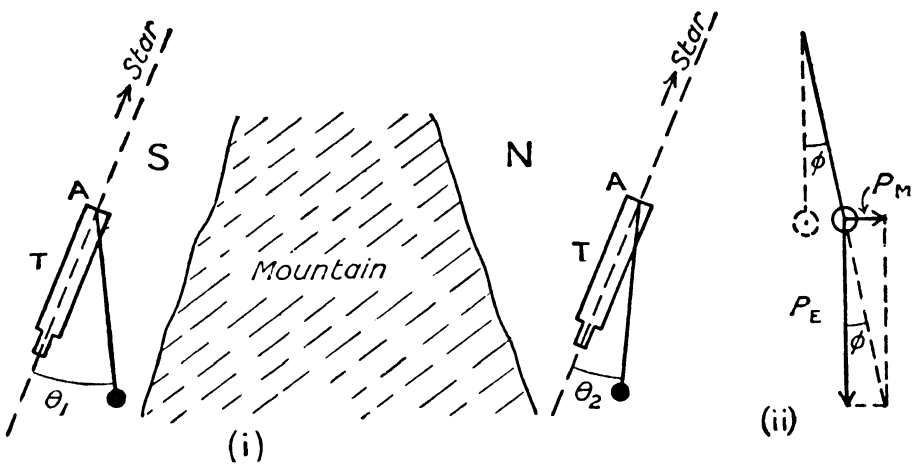


FIG. 129*a*

a fixed star. The axis of the instrument could be rotated about a horizontal axis perpendicular to the plane of the drawing at A, and the angle between the telescope axis and the direction of a plumb-line hanging from A could be measured. The star was sighted on the cross-wire of the telescope at a given hour of the day for two positions of the arrangement on opposite sides N and S of a mountain of suitably simple shape, and the angles θ_1 and θ_2 between the axis of the telescope and the direction of the plumb-line were measured in each case. The two broken lines representing the axis of the telescope in the drawing were therefore parallel, and the difference between θ_1 and θ_2 was due to the sideways pull of the mountain

on the plumb-bob as well as to the fact that even in the absence of the mountain the directions of the plumb-line would not have been quite parallel because each would then have passed through the centre of the earth. After the latter effect had been allowed for, the remaining difference between the two directions of the plumb-line was due to the pull of the mountain, and half this remainder was the angle ϕ (Fig. 129*a* (ii)) by which the mountain pulled the plumb-line out of the direction in which it would have hung in either position N and S if the mountain had not been present. A knowledge of this angle allowed the pull of the mountain P_M to be compared with that of the earth P_E, so that the mass of the earth could be calculated when the mass of the mountain and its effective distance from the plumb-bob had been estimated by surveying and density determinations on samples of rocks. The uncertainties associated with these processes and the smallness of the angle ϕ (a few seconds of arc) prevent such experiments as these from giving accurate results for the mean density of the earth, although they are of historical interest.

Another way of comparing the attraction of a mountain with that of the earth is to use a pendulum to determine g (*i.e.* the force of gravity on unit mass) at sea-level and then at the top of the mountain. The amount by which the elevation should reduce g by reason of the greater distance from the centre of the earth can be calculated from the inverse square law. The observed reduction is less than this, because g at the summit is made greater than it otherwise would be by the force of the attraction of the mountain on the pendulum. Thus the effect of the mountain on the value of g can be determined; let it be Δg. Then $\Delta g/g$ is equal to the ratio of the attraction exerted on the pendulum by the mountain to that exerted by the earth. If the distance of the pendulum from the centre of attraction of the mountain is r and from the centre of the earth R (which is, with sufficient approximation, the radius of the earth), we have, calling the mass of the mountain m,

$$\frac{\Delta g}{g} = \frac{m}{r^2} \div \frac{M}{R^2}$$

from which M, the mass of the earth, can be calculated. As in the plumb-line experiments, the uncertainty in the values of m and r limit the accuracy of the result.

A similar type of experiment has been done in which the value of g at the bottom of a mine is compared with that at sea-level. At a depth h below sea-level the complete spherical shell of thickness h between the point considered and the earth's surface has no gravitational effect, so that the value of g is due only to the inner sphere of radius $(R-h)$. This causes g to diminish as h increases in spite of the increasing proximity to the centre of the earth.

Cavendish's Determination of *G*.—In the late eighteenth century Rev. John Michell devised a torsion-balance method of measuring the force of gravitational attraction between two lead spheres. He did not

live to do more than construct the apparatus which subsequently came into the hands of Cavendish, who largely rebuilt it and made determinations of G in 1797–98.

The apparatus used by Cavendish is shown diagrammatically in Fig. 129*b* (i). It was situated in an enclosure A built in an out-house in the garden of Cavendish's house at Clapham Common, near London.

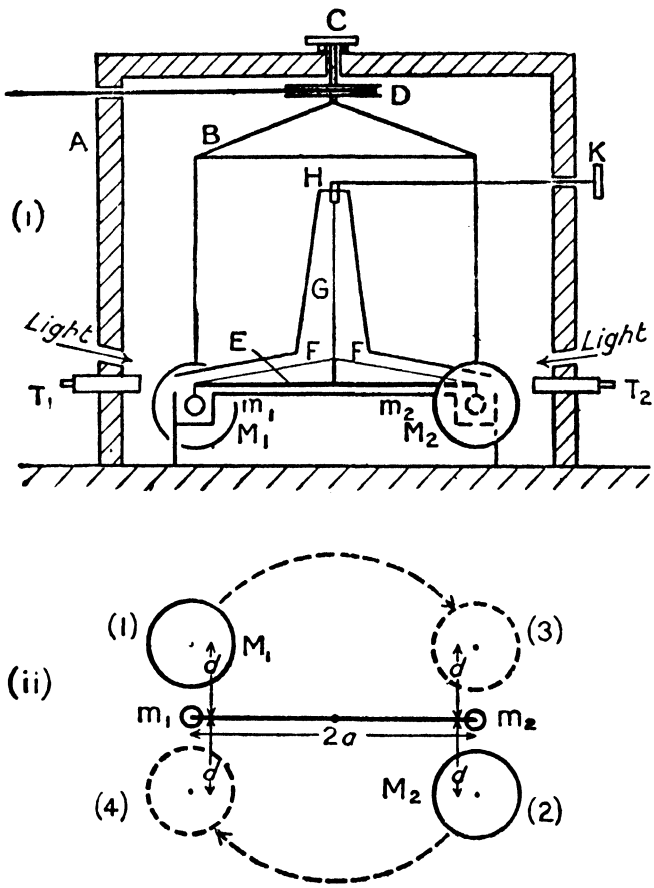


FIG. 129*b*

The purpose of the enclosure was to prevent draughts and convection currents of air which would have disturbed the delicate torsion balance. From the top of the enclosure was suspended a framework B which supported two lead spheres M_1 and M_2 each 12 in. in diameter and having a mass M of over 300 lb. The framework could be made to swivel at C about a vertical axis by pulling on a rope passing round the pulley D. In this way M_1 was moved from position (1) to position (3) in Fig. 129*b* (ii) and M_2 from (2) to (4). The masses on which M_1 and M_2 exerted attractive forces were two lead spheres m_1 and m_2 each 2 in. in diameter and having a mass m of about $1\frac{1}{2}$ lb. These hung by short wires from the ends of a wooden beam E about 6 ft. long strengthened by stays FF. The torsion fibre G supporting the beam was made of silvered

copper wire about 40 in. long. The suspended system used by Cavendish in the actual determination of the gravitational constant had a time period of about 7 min. and was housed in an inner case at the top of which was the torsion head H which could be rotated by turning the knob K so as to alter the equilibrium position of the torsion beam.

A horizontal vernier scale was fixed at each end of the torsion beam, perpendicular to its length, and as the beam rotated, the verniers moved over corresponding fixed scales. Thus the position of the beam could be read by means of the scales and verniers, which were illuminated by light from lamps outside the enclosure, as indicated in the figure, and observed through the telescopes T_1 and T_2.

Suppose that the large masses are in positions (1) and (2) and are sufficiently near to m_1 and m_2 respectively to cause a measurable deflection θ (radians) of the torsion beam from the direction in which it would set if the large masses were absent. If the distance between the centre of each large mass and that of the nearer small suspended mass is d when the beam is in equilibrium, then there is a deflecting force of $G\frac{Mm}{d^2}$ on each suspended mass. The relative positions of the masses were such that these deflecting forces were as nearly as possible perpendicular to the torsion beam, in which case they form a couple of moment $\frac{GMm}{d^2} \cdot 2a$, where a is *half* the distance between the centres of the small masses. For equilibrium in this position we have

$$N\theta = \frac{GMm}{d^2} \cdot 2a \quad . \quad . \quad . \quad . \quad (16)$$

where N is the torsional constant of the fibre, *i.e.* the couple required to give the fibre a twist of one radian.

The experiment consisted in reading the verniers when the large masses were in positions (1) and (2) and then changing their positions to (3) and (4) and taking the vernier readings again. The change of reading of each vernier then corresponded to a rotation of the beam through 2θ between the two deflected positions of the beam. Thus θ was determined. It is necessary to know N in order to calculate G from equation (16). This was done by finding the time period T of torsional oscillations of the suspended system and calculating its moment of inertia I about its axis of rotation from a knowledge of the distribution of mass on the torsion beam and its fittings, including m_1 and m_2. N could then be found from the equation (page 113)

$$T = 2\pi\sqrt{\frac{I}{N}}$$

Cavendish's experiment was remarkably accurate. He made corrections

for the attraction of each large mass on the remote suspended mass and for several other small effects. The suspended system did not actually come to rest to allow the vernier readings to be taken, so successive swings to right and left of the equilibrium position were observed and the reading for the position of rest was deduced from these. Cavendish obtained the value 5·45 gm. cm.$^{-3}$ for the mean density of the earth, which corresponds to a value for G of $6{\cdot}75 \times 10^{-8}$ dyne cm.^{2} gm.$^{-2}$.

Boys's Experiment.—Several determinations of G were made during the nineteenth century by methods similar to that of Cavendish, although the experiments were more elaborate. The size of Cavendish's apparatus made it difficult to avoid temperature gradients between its various parts, and the resulting convection currents are liable to disturb the suspended system. If a torsion apparatus of the Cavendish type is made smaller, its sensitivity is reduced because the masses are diminished and so is the arm ($2a$) of the deflecting gravitational couple. The time period of torsional oscillation is also reduced because I is diminished by the reduction in size and mass of the suspended system. In fact, the sensitivity of the apparatus is proportional to the square of the time period. Thus if, while diminishing the size of the apparatus, we modify the torsion fibre so as to reduce N in the same proportion as I, then the time period is unaltered and the sensitivity remains the same as it was in the larger apparatus. If the same material is used for the fibre in both cases, the condition of unaltered time period can be fulfilled only by making the fibre inconveniently long or so thin that it would not be strong enough to support the suspended system.

The use of very fine torsion fibres made by drawing out fused quartz was discovered by Boys. These can be made fine enough to retain adequate sensitivity in a small apparatus while their strength is sufficient to support the suspended system. Accordingly, Boys was able to construct a sensitive apparatus with a torsion beam less than an inch long. This

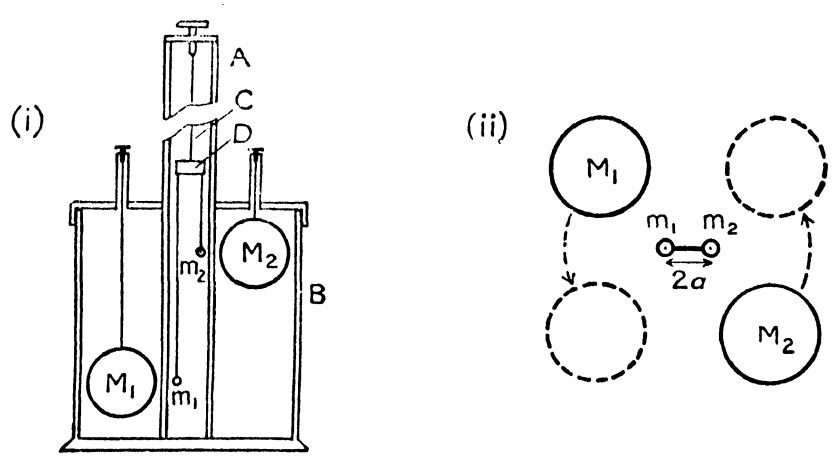


FIG. 129c

being so, the masses m_1 and m_2, which were gold spheres $\frac{1}{4}$ in. diameter and weighing $2\frac{1}{4}$ gm., had to be suspended at different levels from the ends of the beam by fine wires, otherwise each large mass would attract each of them almost equally. In Boys's apparatus the suspended system was contained in a vertical tube A (Fig. 129*c* (i)), and the large masses M_1 and M_2 (lead spheres $4\frac{1}{4}$ in. diameter), which were, of course, suspended at the same levels as the corresponding small ones, were situated in an outer container B. The lid of B could be rotated so as to alter the positions of the masses M_1 and M_2 which were suspended from it. The quartz fibre C was 17 in. long; and the beam D, only 0·9 in. long, was made of glass and was silvered so as to act as a reflector for a beam of light by which its deflection was observed. The masses were first disposed as shown in Fig. 129*c* (ii), and the direction of the beam for this arrangement was determined. Then M_1 and M_2 were moved to the dotted positions and the direction was again found. The equation of equilibrium was more complicated than for the Cavendish apparatus because the deflecting forces were not perpendicular to the beam.

The result obtained by Boys, which was published in 1895, was $G = 6{\cdot}658 \times 10^{-8}$ dyne cm.^{2} gm.$^{-2}$ and $\Delta = 5{\cdot}53$ gm. cm.$^{-3}$, and his experiment is regarded as being one of the most reliable determinations of these constants.

Common-Balance Experiments.—Several important determinations of G have been made by means of a common balance. In principle the experiment consists in suspending two equal spherical masses, one from each side of a balance beam. A large heavy sphere is placed first under one and then under the other of the suspended masses. Thus the gravitational attraction between the large mass and the suspended mass above it is transferred from one side of the balance to the other. The corresponding change in the reading of the balance pointer is noted. When this is halved it represents the effect of placing the large mass beneath one of the suspended masses, and if we know by previous calibration the increase of weight on one side of the balance which gives a change of pointer reading of one division, we can calculate the gravitational pull between the large mass M and one suspended mass m. Equating this to $G \cdot \frac{Mm}{d^2}$, where d is the distance apart of the masses, allows G to be calculated.

One of the best known experiments of this type was performed by Poynting (1891). The gravitational pull between the two spheres in Poynting's apparatus was only about 0·2 mg. wt.

Heyl's Experiments.—More recently (since about 1930) important experiments have been done by Heyl at the Bureau of Standards in Washington. In these, a torsion balance has been used, but not in the static way of Cavendish and Boys. Fig. 129*d* is a diagrammatic plan of the arrangement. The torsion beam carried two small spheres, m_1 and m_2,

and large steel cylinders, M_1 and M_2, could be placed either on the axis of the beam (positions (1) and (2), known as the "near position") or on the perpendicular bisector of the beam ((3) and (4), the "far position"). When the torsion beam was set swinging about a vertical axis through O, the gravitational pull of M_1 and M_2 on m_1 and m_2 increased the restoring couple on the suspended system when M_1 and M_2 were in the near position and decreased it (to a smaller extent) in the far position. Therefore the time period in the near position is less than that in the far position by an amount which can be calculated in terms of G. This calculation is very complicated when M_1 and M_2 are cylinders. Actual observation of the two time periods allows G to be found. Heyl's mean result was $G = 6{\cdot}669 \times 10^{-8}$ dyne cm.^{2} gm.$^{-2}$.

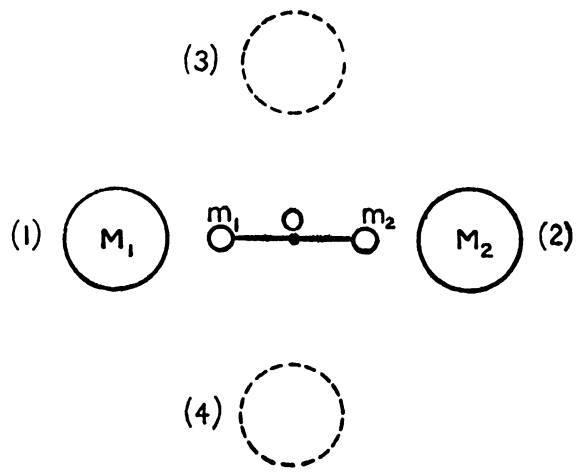


FIG. 129*d*

The accurate determination of G is a difficult matter, and it is not surprising that experiments by different workers, and even different experiments by the same worker, should yield slightly different values of G. The discrepancies are not large enough, however, to indicate that G is anything but a universal constant independent of the properties of any particular material. In this, gravitational force differs from magnetic and electrical forces which depend very much on the nature of the material in which the magnetic poles or electric charges are situated. A critical assessment of all the most reliable results for G suggests the value $6{\cdot}670 \times 10^{-8}$ dyne cm.^{2} gm.$^{-2}$ with an uncertainty of $\pm 0{\cdot}005 \times 10^{-8}$.

EXAMPLES IX

1. Two forces make an angle of 60° with one another. The magnitude of one of the forces is 4 units and of the resultant is 6 units. Find the magnitude of the other force and the direction of the resultant.

Describe an experiment you would carry out to check the result. (L.M.)

2. What conditions are satisfied by a number of coplanar forces which are in equilibrium?

The lower end of a straight uniform rod rests on a horizontal surface, while its upper end rests against a smooth vertical wall. If the coefficient of static friction between the rod and the horizontal surface is 0·2, calculate the inclination of the rod when it is about to slip. (L.I.)

3. State a set of conditions sufficient to ensure that a system of coplanar forces shall be in equilibrium.

ABCD is a plane non-re-entrant quadrilateral in which the sides AD and BC are parallel. Show that forces represented in magnitude and direction by $\overrightarrow{AB}$ acting in the side AB, $\overrightarrow{DA}$ acting in the side BC, $\overrightarrow{CD}$ acting in the side CD, and $\overrightarrow{BC}$ acting in the side DA are in equilibrium. (O.H.S.)

4. State the conditions of equilibrium for a body under the action of a number of coplanar forces.

If the length of a simple pendulum is l cm. and the mass of its bob m gm., calculate the work done in displacing the bob through a *horizontal* distance of $l/2$ cm. from its equilibrium position. Find also the horizontal force which will keep the bob in its new position. (L.Med.)

5. What is meant by (*a*) the moment of a force about an axis, (*b*) the resolved part of a force in a given direction?

A forearm is held horizontally, with the upper part of the arm vertical, and a 2-kg. weight is placed in the hand so that its centre of gravity is 30 cm. from the elbow-joint. Assuming that the upper and lower ends of the biceps muscle are attached at points 12 cm. and 5 cm. respectively from the elbow-joint, find (*a*) the tension in the muscle, (*b*) the horizontal and vertical components of the reaction at the joint. (Neglect the weight of the forearm itself.) (L.Med.)

6. A string ABCD has its ends A and D attached to two fixed points at the same level and at a distance 3 ft. apart. To the points B and C particles of mass 4 lb. and 1 lb. respectively are attached. When the system hangs in equilibrium, B and C are vertically below the points of trisection B′ and C′ of the horizontal line AD, and B′B = 2 ft. Prove that the vertical and horizontal components of the reaction at A are 3 and $1\frac{1}{2}$ lb. wt. respectively. Show also that C′C = 1 ft. 4 in., and find the tensions in the parts AB, BC, CD of the string. (J.M.B.H.S.)

7. What is the resultant of three forces which can be represented in magnitude and direction by the sides of a triangle taken in order (*a*) if the forces act at a point, (*b*) if they act along the sides of the triangle, (*c*) if, in the latter case, one of the forces is reversed? Describe an experiment to demonstrate the truth of your statement in case (*a*).

8. What is the principle of the triangle of forces?

An elastic cord, 6 ft. long, is attached to two points A and B, 6 ft. apart, A being vertically above B. A light string CD, attached to a point C on the elastic cord 2 ft. from B, is pulled sideways until the lengths AC and CB are 5 ft. and 3 ft. respectively. Assuming that the tension in the elastic cord is proportional to its extension, and that the cord stretches by 1 ft. when a weight of 2 lb. is hung from it, find, by graphical construction, the magnitude and direction of the tension in C.D. (L.M.)

9. (*a*) Three coplanar forces act at a point as follows: (i) 3 units in a direction 30° N. of E., (ii) 4 units 45° W. of N., (iii) 2 units 60° S. of W. Find graphically the magnitude and direction of the force which balances these forces.

(*b*) Three coplanar cords are attached to a smooth ring, which is in equilibrium when the tensions in the cords are 3, 5 and 6 units respectively. Find the angles between the cords and explain your method. (L.M.)

10. Show that, if three forces maintain a rigid body in equilibrium, they must either (*a*) all be parallel to one another, or (*b*) all be inclined to one another.

A bar CD, 4 ft. long, weighing 20 lb., whose centre of gravity is not at its middle point, is supported by two light chains AC and BD fixed at two points A and B in the same horizontal line 6 ft. apart. AC is 2 ft. long and is inclined at 30° to the vertical, while BD is inclined at 45° to the vertical. Draw a diagram of the arrangement on squared paper and determine the tensions in AC and BD. (L.M.)

11. Under what conditions is a rigid body in equilibrium under the action of three non-parallel forces?

The ends of a strong cord 63 cm. long are fixed at two points P and Q in a horizontal line and 45 cm. apart. A mass of 1 kg. is suspended from a point O in the cord 27 cm. from P. Calculate the tensions in the segments PO and OQ of the cord, and state the units in which they are expressed. (L.Med.)

12. State one set of conditions under which a system of three non-parallel coplanar forces is in equilibrium.

A uniform rod of weight 1 lb. per ft. length rests on a smooth horizontal peg and against a smooth vertical wall to which it is inclined at an angle of 55°. If the peg is 6 in. from the wall, find, graphically or otherwise, the length of the rod. Find also the force on the peg in magnitude and direction. (C.H.S.)

13. State the conditions of equilibrium of a rigid body under coplanar forces. How would you illustrate one of the conditions stated?

A trap-door is hinged at one edge and held in a horizontal position by a cord of length 4 ft. attached at the midpoint of the opposite edge, and fixed to a point 3 ft. vertically above the midpoint of the hinge. Find the tension in the cord and the reaction at the hinge if the weight of the trap-door is 20 lb. (C.H.S.)

14. Under what conditions is a system of coplanar forces in equilibrium?

A picture weighing 2 lb. is supported by a string passing over a nail. The string is 26 in. long and its two ends are attached to the picture at points 10 in. apart. Calculate the tension in the string. (L.I.)

15. State the conditions for equilibrium of three non-parallel coplanar forces.

A uniform trap-door, 3 ft. square and weighing 20 lb., is held open by a rope, 4 ft. long, joining the midpoint of the edge of the door farthest from the hinge to a hook 5 ft. vertically above the hinge. Calculate the tension in the rope. (L.I.)

16. Explain how the position of the line of action of the resultant of two like parallel forces may be calculated.

A uniform metre stick, mass 100 gm., is suspended horizontally by two vertical strings A and B attached at the zero and 80 cm. marks respectively. The string A breaks when the tension in it exceeds 50 gm. wt., and B when the tension in it exceeds 100 gm. wt. Determine the positions on the stick within which an additional mass of 40 gm., resting on it, may be moved without either string breaking. (L.M.)

17. Under what conditions are a number of parallel forces acting on a body in equilibrium?

A uniform metre rod weighing 400 gm. is smoothly hinged at one end so that it can move in a vertical plane. As its other end a weight of 1 kg. is attached. The rod is held in a horizontal position by a string 80 cm. long joining a point 64 cm. from the hinged end to a fixed point vertically above the hinge. Find the tension in the string. (L.I.)

18. What conditions must be satisfied when parallel coplanar forces are in equilibrium?

A uniform metre stick, weighing 205 gm., is supported horizontally by two vertical threads attached at the 40 cm. and 70 cm. marks respectively. A weight of 100 gm. is fixed to the stick at the zero end and 300 gm. at the 100 cm. end. Calculate the tension in each thread. If the loaded metre stick is supported by a single thread, where must this be attached in order that the stick shall be horizontal? (L.I.)

19. State the conditions for the equilibrium of a rigid body acted upon by a number of parallel forces.

A light straight rod 95 cm. long, from which are suspended a mass 120 gm. at one end A and a mass *m* gm. at the other end B, is balanced horizontally about a point C in the rod. The two masses are interchanged and the whole is now balanced horizontally about a point D 5 cm. from C towards B. Calculate the mass *m*. (L.Med.)

20. Show how to find the magnitude and line of action of the resultant of two like parallel forces whose magnitudes and lines of action are given.

A uniform beam, of length 8 ft. and mass 50 lb., is supported in a horizontal position by two vertical strings attached to its end points. The breaking tension of each string is 40 lb. wt. Find the greatest mass of a particle that can be placed anywhere on the beam without causing a string to break.

On what portion of the beam can a particle of 20 lb. be placed without causing a string to break? (O.H.S.)

21. Define *centre of gravity*, and explain how the centre of gravity of a system of massive particles can be calculated.

Three particles of masses 3, 4 and 5 units respectively are placed at the corners A, B and C of an equilateral triangle of 6 cm. side. Find the distance of their centre of gravity (*a*) from AB, (*b*) from a line through A perpendicular to AB. (L.M.)

22. A pendulum is made by fixing a brass cylinder 4 cm. in diameter at the end of a circular brass rod 1 cm. in diameter and 24 cm. long, the cylinder being coaxial with the rod. What must be the length of the cylinder so that the centre of gravity of the whole shall be 25 cm. from the upper end of the rod? (L.M.)

23. Define *centre of gravity*.

Prove that the centre of gravity of a uniform triangular lamina coincides with that of three equal masses placed at the corners of the triangle.

The sides AB, BC, CD, DA of a uniform square lamina are 4 in. long. AC and BD intersect at O. If the portion BOC is removed, find the distances of the centre of gravity of the remainder from AB and AD.

24. What determines whether the equilibrium of a body is stable, unstable or neutral?

A solid wood cube of 3 in. side having a metal plate 0·25 in. thick covering its upper face is placed on an inclined plane with an edge perpendicular to the slope of the plane. The plane is tilted till the cube topples over. Find the inclination of the plane when this happens, supposing that there is no slip and that the specific gravity of the metal is 12 times that of the wood. (L.M.)

25. Explain with illustrations how the stability of a body depends on the position of its centre of gravity.

A uniform bar ACDB is 80 cm. long and has a mass of 120 gm. The bar rests on two supports C and D, 30 cm. apart, and a load of 40 gm. is hung from A. The bar is pushed across the supports in the direction D to C. Find the position on the supports when it is on the point of toppling over.

Where must the bar be placed in order that the supports may carry equal loads? (L.M.)

26. Explain and illustrate the statement that "in a machine, what is gained in speed is lost in power."

In a system of pulleys the efficiency is 40 per cent., the weight of the lower block is 10 lb., and the effort moves 6 ft. for every foot moved by the load. Draw a diagram of the arrangement, and find the *useful* load which can be lifted by an effort of 20 lb. wt. (L.M.)

27. Explain the terms *machine, mechanical advantage, efficiency*.

A load of 2 cwt. has to be lifted by a force of 56 lb. wt. Draw a diagram of a suitable system of pulleys if its efficiency is (*a*) 100 per cent., (*b*) 50 per cent. Calculate in each case the work done by the effort if the load is raised 6 ft. (L.M.)

28. Define a *machine*. What are meant by the terms *mechanical advantage, velocity ratio, efficiency*, and how are these quantities related?

A screw jack with 4 threads per inch is used to raise one wheel of a car weighing 2400 lb. If the efficiency of the jack is 60 per cent. and the car weight is distributed equally on the wheels, what minimum force must be applied at the end of the arm of the jack, one yard in length, in order to raise the wheel, and how much work is done in lifting the wheel one inch off the ground? (L.Med.)

29. Describe the common balance, paying particular attention to the question of sensitivity.

When a 20-mg. weight is placed on one pan of a balance the pointer of length 25 cm. moves through 1 cm. If the mass of the beam and pointer is 100 gm. and the knife edges are in line, with the two outer ones 12 cm. from the centre edge, find the position of the centre of gravity of the beam.

What modifications are required to make this balance twice as sensitive? (L.Med.)

30. Give a diagram showing the essentials of the construction and action of a common (chemical) balance.

State what is meant by the *sensitivity* of a balance and explain how this is affected by its construction. (L.Med.)

31. Describe a balance, pointing out the factors which determine its accuracy, sensitiveness and period of oscillation.

Explain how the true weight of a body may be found when the lengths of the arms of the balance are not exactly equal. (L.Med.)

32. Explain the action of the common balance, and give a clear account of the factors which influence its sensitivity.

An object is placed on the left-hand pan of a balance and it is found that 23·640 gm. must be put on the right-hand pan in order to counterpoise it. When the object is placed on the right-hand pan 24·113 gm. are needed in the left-hand pan. Calculate (*a*) the ratio of the lengths of the arms of the balance, and (*b*) the true mass of the object. (L.I.)

33. Describe an accurate method of measuring the value of the universal gravitational constant, and give the theory of the experiment.

The planet Neptune travels round the sun with a period of 165 years. Show that the radius of its orbit is approximately thirty times that of the earth's orbit, both being considered as circular. (O.H.S.)

Chapter X

THE EQUILIBRIUM OF FLUIDS

1. DEFINITIONS AND FUNDAMENTALS

Definition of a Fluid.—The fundamental difference between solids and fluids lies in the fact that fluids can be made to flow while solids cannot, unless extremely large forces are applied to them.

Flowing is fundamentally a matter of the change of shape of a quantity of substance. A liquid poured from a beaker, for instance, becomes a vertical stream and then a flat pool on the floor. The simplest example of a change of shape is shown in Fig. 130. The cross-section of a piece of material ABCD which was originally square has become a rhombus A′B′C′D′. This deformation is called a **shear,** and can be produced by the application of four equal forces P tangentially to the faces of the material which are perpendicular to the drawing. The diagonals of the deformed figure, A′C′ and B′D′, remain perpendicular to each other after deformation and their directions are unaltered. If the material is solid, a given set of shearing forces—provided that their magnitude does not exceed a certain limit—will produce a certain degree of deformation and no more. When the forces are removed the solid goes back to its original undeformed shape. The solid is said to possess **rigidity,** which is the particular type of elasticity called into play when a substance is sheared. When the material subjected to shearing forces is fluid, however, the deformation goes on increasing indefinitely no matter how small the deforming forces may be. In other words the substance flows. When the forces are removed the flow subsides, but the fluid does not return to its original shape. The fluid has no rigidity. It is true, of course, that the *rate* of flowing or yielding produced by given forces varies very much from one fluid to another. For instance, it is much slower in liquids like glycerine, treacle and heavy oil than it is in water or in gases, but if there is any flow at all, no matter how slow, the substance is considered to be fluid in this respect. The rate at which a fluid yields to shearing forces is governed by its **viscosity,** a property which is discussed in a later chapter.

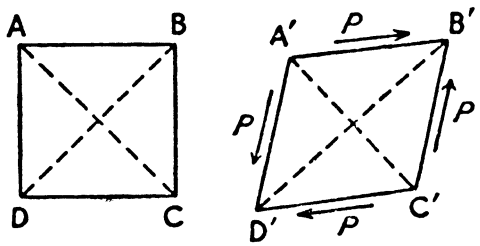


FIG. 130

Suppose for the moment that the sheared material in Fig. 130 is solid

and that the deformation is such that a state of equilibrium has been reached. Consider the portion of the solid between the top surface A'B' and a plane FE parallel to A'B'. This portion is shown isolated in Fig. 131. On the ends A'F and B'E equal and opposite tangential forces act which are the appropriate fractions of the larger forces P acting uniformly over the faces A'D' and B'C' of the complete solid. The original tangential force P acts over A'B'. It is evident that for equilibrium the material below FE must act upon the portion above with a force P' which is equal and opposite to the force P acting on A'B'. This force P' is brought into existence by the deformation and acts tangentially over the plane FE.

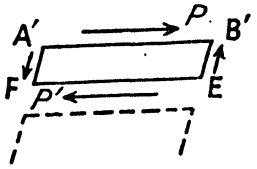


FIG. 131

We conclude therefore that, while it is possible for such a force as P' to exist in a solid, the fact that a fluid is unable to remain at rest under the action of shearing forces indicates that it cannot exist in a fluid. This means that any two adjacent portions of a fluid which are relatively at rest are unable to exert forces upon each other which are tangential to the surface separating them. A force such as P' cannot exist in a fluid unless there is relative motion. This is taken to be the defining property of a fluid. The absence of tangential force means that only normal force can exist, for any force other than one which is perpendicular to the separating surface would have a tangential component. Thus the definition can be stated as follows: In the fluid state of matter two adjacent portions which are relatively at rest can only exert upon each other forces which are purely normal to the surface separating the two portions.

Liquids and gases in general conform to the definition of a fluid stated above; but there are some border-line cases of which pitch is an example. This substance will flow if it is subjected to small forces for long periods, but it can also be hammered, chipped or sawn like a solid.

Definition of Fluid Pressure.—Fluids are usually in a state of compression. This is obviously true of a gas contained in a closed vessel, and of the water below the surface of a lake or sea. In the water example the compression is produced by the weight of the liquid above the point in question and becomes very large at great depths. The atmosphere around us is in a state of compression for exactly the same reason.

Imagine a small plane of area ΔA to be described in a fluid which is at rest. We have already shown that the force (F, say) which the fluid on one side of the area exerts upon that lying on the other side must be normal to ΔA. The average pressure p over the area ΔA is defined by

$$p = \frac{F}{\Delta A}$$

If ΔA is sufficiently small to allow the area to be regarded as no more than

a single point, p is the pressure *at that point.* Otherwise p is the mean pressure over the area.

The units of pressure can be immediately deduced from the definition. In the C.G.S. system the absolute units of pressure are dynes cm.$^{-2}$. Pressure units in common use in English engineering are lb. wt. in.$^{-2}$, usually written as lb. per sq. in.

The pressure in a fluid is considered to be positive if the two portions of fluid separated by ΔA are being pressed together as in the examples mentioned at the beginning of this section. It is possible, but not so common, to have a negative pressure in a liquid.

Pressure at a Point is Independent of the Direction in which ΔA is Taken.—Consider the equilibrium of a portion of fluid contained in a very small triangular prism. The prism must be so small that there is no variation of pressure from point to point over any one of its surfaces. Let the lengths of the sides of the triangular section be a, b and c, as shown in Fig. 132 (i), and let the length of the prism be x. If the pressures

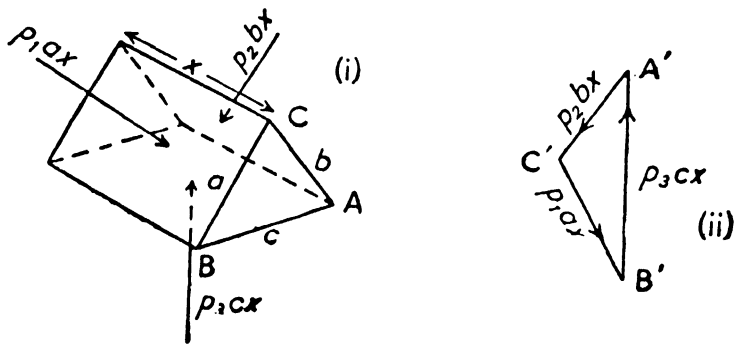


FIG. 132

over the three rectangular faces are respectively p_1, p_2 and p_3, the normal forces on the faces due to the fluid outside the prism are p_1ax, p_2bx and p_3cx respectively. These are shown by arrows in Fig. 132 (i). If the prism of fluid is in equilibrium with the fluid surrounding it, the forces on its rectangular faces can be represented by a triangle of forces A′B′C′ (Fig. 132 (ii)) whose sides are parallel to the forces and therefore proportional to their magnitudes. Since the side of a triangle is proportional to the sine of the opposite angle, we can write

$$\frac{p_1ax}{\sin \hat{A}'} = \frac{p_2bx}{\sin \hat{B}'} = \frac{p_3cx}{\sin \hat{C}'}$$

A little thought shows that $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$ and $\hat{C} = \hat{C}'$, since the forces are respectively perpendicular to the rectangular faces of the prism, so that the above relation can be written

$$\frac{p_1ax}{\sin \hat{A}} = \frac{p_2bx}{\sin \hat{B}} = \frac{p_3cx}{\sin \hat{C}}$$

But in the triangle ABC

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

and the last two sets of equations can only be simultaneously true if

$$p_1 = p_2 = p_3$$

Thus the pressure in the neighbourhood of the small prism is the same whatever the orientation of the area over which it is calculated. This is not true if the prism is large enough to allow a variation of pressure over its faces, but we initially precluded this possibility.

Thrust on a Surface of any Shape in a Fluid in which Pressure is Uniform.—In Fig. 133 ABCD is the periphery of an irregular surface—not necessarily plane—described in a fluid in which the pressure is the same at every point. It is required to find the force which the fluid on one side of ABCD exerts on that on the other side. Let any plane EFGH be drawn, and let perpendiculars to this plane be dropped from every point in the periphery of ABCD. The feet of the perpendiculars lie on the dotted line KLMN. The area enclosed by KLMN is said to be the *orthogonal projection* of ABCD on to the plane EFGH.

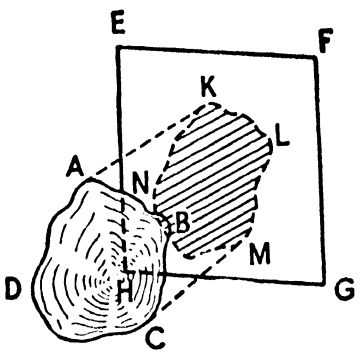


Fig. 133

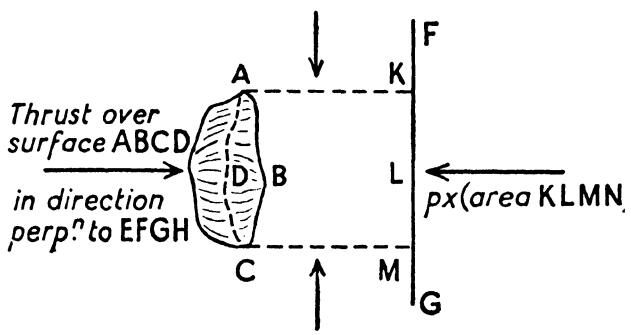


Fig. 134

If the uniform pressure in the fluid is p, then the fluid inside the figure ABCDKLMN is acted upon by the fluid on the far side of KLMN with a force equal to $p \times$ (area KLMN) in a direction perpendicular to this area. The forces due to the fluid round the sides of the enclosed portion are all at right angles to the sides and therefore have no component in the direction of the force $p \times$ (area KLMN) (Fig. 134). Consequently the force, or thrust, exerted by the fluid on the outside of the irregular surface ABCD must balance the force $p \times$ (area KLMN), and so must be equal and opposite to it.

Therefore in order to evaluate the component of the thrust due to a uniform fluid pressure over any irregular surface such as ABCD, it is necessary to project the surface on to a plane whose direction is perpendicular to that in which the component of the thrust is required. The

thrust component in this direction is then equal to the product of the pressure and the area of the projection. The *total* thrust is perpendicular to the direction of EFGH for which the projected area is a maximum.

It may be noticed that if the irregular surface is a *closed* surface it has no periphery, and therefore its projection on to any plane has no area. Consequently the total thrust on any closed surface exposed to uniform pressure is zero. We could have anticipated this result by remembering that if the fluid inside the closed surface is to be in equilibrium under the action of the forces due to the surrounding fluid, the sum of the components of these forces must be zero in every direction.

As an application of the above formula for the thrust on a surface of any shape, we can consider the case of a cylindrical vessel containing gas at a uniform pressure p and having an area of cross-section A. The thrust on the ends in the direction of the axis of the cylinder is equal to pA whatever may be the shape of the ends (Fig. 135).

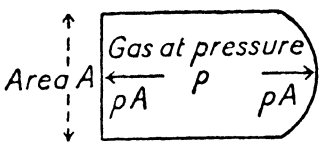


FIG. 135

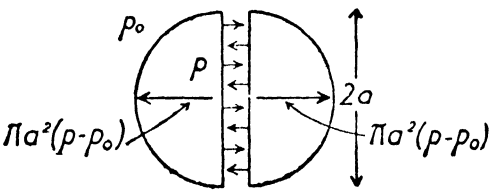


FIG. 136

Similarly, the force acting on each of the two halves of a spherical surface such as that of a balloon or soap-bubble is, by the same principle, equal to

$$\pi a^2(p - p_0)$$

where p and p_0 are the gas pressures inside and outside the surface respectively and a is its radius (Fig. 136). This force is balanced by the tension in the spherical surface acting round the circumference of the great circle separating the two hemispherical surfaces. The forces are indicated in Fig. 136, where the two hemispheres are shown slightly separated for clarity. Each half of the surface is in equilibrium under the action of the force $\pi a^2(p - p_0)$ and the tension acting at the junction of the two halves. The sphere as a whole is in equilibrium under the action of the two equal and opposite forces $\pi a^2(p - p_0)$.

2. THE HYDRAULIC PRINCIPLE

Fundamental Principle.—Consider a vessel of any shape with two cylinders let into its sides as shown in Fig. 137. Let each cylinder contain a perfectly fitting piston, and let their areas of cross-section be A_1 and A_2 respectively. Suppose that when forces F_1 and F_2 as shown are applied to the pistons, equilibrium is maintained and neither piston moves. The

resultant force on each piston is therefore zero, and if the pressure throughout the fluid is p, we can write

$$F_1 = pA_1$$

and

$$F_2 = pA_2$$

so that

$$\frac{F_1}{F_2} = \frac{A_1}{A_2}$$

If the connection between the two cylinders is made *via* a pipe instead of the vessel shown in Fig. 137, the conditions are unaltered, and it is clear that we have a means of "transmitting" a force over considerable distances. When an external force F_1 is applied to one of the pistons, the other piston will exert a force F_2 outwards. The magnitude of F_2 can be any multiple or fraction of F_1 according to the ratio of the cross-sectional areas of the pistons. The system is a machine whose mechanical advantage (F_2/F_1) is governed by the relative sizes of the pistons. We cannot, of course, derive more work from the system than is put into it, as the following analysis shows.

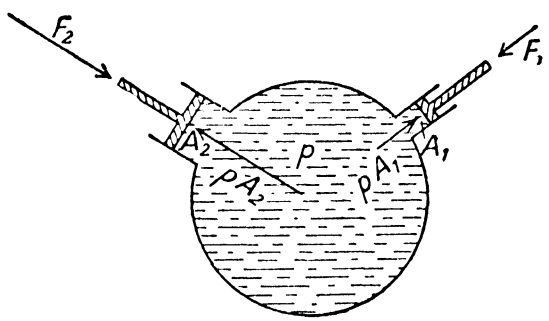


FIG. 137

Let the smaller piston be pushed in through a distance x_1 so that a volume of liquid x_1A_1 is expelled from the smaller cylinder into the vessel. An equal volume must enter the large cylinder, and if the displacement of its piston is x_2 we have

$$x_1A_1 = x_2A_2$$

The quantities of work done in each cylinder are F_1x_1 and F_2x_2 respectively. But

$$F_1x_1 = pA_1x_1 = pA_2x_2 = F_2x_2$$

The quantities of work are therefore equal (efficiency 100 per cent.) under ideal conditions. In reality, of course, there is friction between the pistons and the cylinder walls, and also resistance to flow (viscosity) within the liquid itself.

The idea of using fluid pressure to cause a large force to move through a (necessarily) small distance by the application of a small force through a large distance is known as the **hydraulic principle.** This has many applications, one of which is described in the next paragraph.

The Bramah Hydraulic Press.—This machine, illustrated in Fig. 138, is used for exerting large forces for pressing metals, etc. The narrow plunger of cross-section A_1 is forced into fluid contained in the left-hand cylinder by applying a downward force to the handle of the lever B. (Note that the lever itself provides some mechanical advantage.) During this stroke the valve a_1 is closed by the pressure of liquid above it while a_2 is forced open. Thus the large piston of cross-section A_2 is raised by the fluid entering the right-hand cylinder, and exerts a large force by means of which it can compress an object C placed on the top of it, as shown. When the plunger is raised, the pressure in the small cylinder is reduced so that a_2 closes while a_1 opens and allows more liquid to flow past it from the reservoir D. The machine can therefore be worked continuously, the net effect being the transference of liquid from the reservoir to the right-hand cylinder. After use, the press can be released by opening a cock in a tube (not shown) which connects the large cylinder directly to the reservoir.

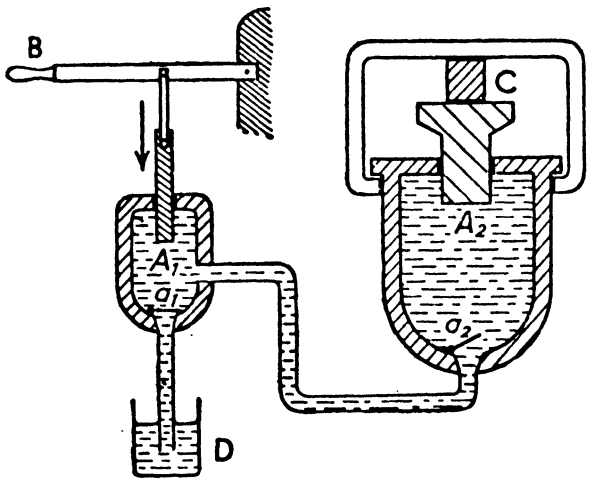


FIG. 138

A machine similar to this, known as a **hydraulic jack,** is used for raising heavy vehicles. The same principle may also be used for working lifts. Car brakes sometimes operate on the hydraulic principle, the force being transmitted to each brake-shoe through tubes containing liquid which is compressed when the brake pedal is pressed down.

3. PRESSURE IN A LIQUID DUE TO GRAVITY

Dependence of Pressure on Depth in a Liquid.—In Fig. 139 XY represents the plane surface of a liquid with the atmosphere above it. Let atmospheric pressure be B absolute units of pressure. Consider the equilibrium of a vertical column of liquid of uniform cross-section ΔA and height h, its upper end being coincident with the surface. There is a downward force $B\Delta A$ on the top of the column due to the atmosphere, while an upward force $p\Delta A$ acts on the bottom of the column if p is the

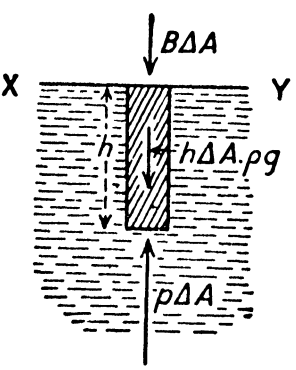


FIG. 139

pressure in the liquid at the depth h. The fluid round the sides of the column exerts only horizontal forces on the column since the sides are supposed vertical. Therefore for equilibrium of the vertical forces we have

$$\begin{aligned} p\Delta A - B\Delta A &= \text{weight of liquid in column} \\ &= \text{volume of liquid in column} \times \text{density} \times g \\ &= h\Delta A . \rho g \end{aligned}$$

Dividing through by ΔA we obtain

$$p - B = hg\rho$$

or

$$p = B + hg\rho$$

Thus at the depth h below the liquid surface the total pressure is made up of that due to the atmosphere (or whatever gaseous pressure exists immediately above the surface) together with the additional pressure $hg\rho$, where ρ is the density of the liquid and g is the acceleration due to gravity. In calculating the total pressure it must be remembered that B and $hg\rho$ must be expressed in the same units.

Experimental Verification.—There are many ways of verifying that the pressure due to a liquid at a depth h below the surface is proportional to h. In the experiment to be described, a uniform glass tube is suspended vertically from a spring-balance (Fig. 140). A paper scale is placed inside the tube with its zero at the lower end of the tube, which is closed. The tube can be immersed in water or any other liquid to any required depth by adjusting the position of the beaker containing liquid. It may be necessary to place some lead shot in the tube in order to make it remain upright when immersed. This is done before any readings are taken. The reading of the spring-balance is first noted when the tube is hanging in air (w_0), and then a series of spring-balance readings is taken for different depths of immersion of the tube as measured on the scale inside it. Let the area of cross-section of the tube be A. Its weight is w_0 gm. or $w_0 g$ dynes. When it is immersed to a depth h, and the spring-balance reading is w, the tube is in equilibrium under the action of the following vertical forces:—

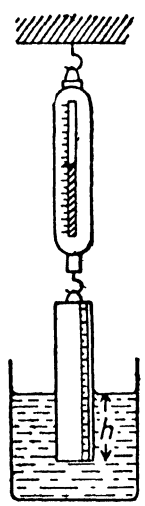


FIG. 140

The force due to atmospheric pressure $(B) = BA$ downwards
The weight of the tube $= w_0 g$ downwards
The force due to the balance $= wg$ upwards
The force due to the pressure in the liquid $= (B + p)A$ upwards

where p is the pressure at the bottom of the tube due to the liquid itself.

Therefore for equilibrium we have

$$BA + w_0 g = wg + (B + p)A$$

or

$$p = (w_0 - w)\frac{g}{A}$$

Now g and A are constant, so that p is proportional to $w_0 - w$, and to verify that p is proportional to h it is necessary to show that $w_0 - w$ is proportional to h. This is best done by drawing a graph of $w_0 - w$ against h. The relation is verified by the fact that the graph is a straight line passing through the origin (Fig. 141). Furthermore, if we substitute $hg\rho$ for p in the last equation, we obtain

$$h\rho = \frac{w_0 - w}{A}$$

from which it can be seen that the slope of the graph, *i.e.* the value of $(w_0 - w)/h$ for any point on it, is equal to ρA. The slope can be evaluated (PM/OM) and ρ found if A is known by measurement of the tube.

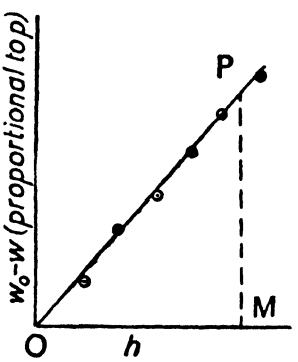


FIG. 141

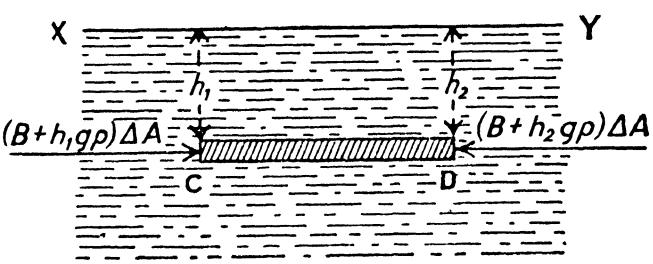


FIG. 142

The Surface of a Liquid at Rest is Horizontal.—In the previous section the term "vertical" has signified the direction in which gravity acts, and "horizontal" is, in the first place, defined as being perpendicular to this direction. Consider now the equilibrium of a horizontal column of liquid CD (Fig. 142), of small uniform cross-section ΔA. We are about to prove that the surface XY of the liquid is horizontal, so that we must not initially suppose that it is so. Let the ends of the column lie at vertical depths h_1 and h_2 below the surface. The pressure in the liquid at C is equal to $B + h_1 g\rho$, so that there is a horizontal force on the vertical end C equal to $(B + h_1 g\rho)\Delta A$. Similarly there is an opposite horizontal force on the end D equal to $(B + h_2 g\rho)\Delta A$. There are no other forces parallel to the axis of the column, since the forces exerted by the surrounding liquid on the horizontal sides are all at right angles to this direction, and so also is the weight of the column. Therefore for equilibrium of the horizontal forces parallel to the axis of the column

we have

$$(B+h_1g\rho)\Delta A=(B+h_2g\rho)\Delta A$$

whence

$$h_1=h_2$$

Therefore the surface XY must be horizontal, since we have just shown that the ends of the column CD, which was supposed horizontal, lie at equal depths below it.

The foregoing consideration of the equilibrium of a horizontal liquid column shows that the pressure is the same at all points on a given horizontal plane in a liquid of uniform density provided that the liquid is at rest. Furthermore, the difference of pressure between any two points in a liquid at rest is equal to $hg\rho$, where h is the *vertical* height of one point above the other irrespective of whether the two points do actually lie on the same vertical line.

These principles can be demonstrated with a piece of apparatus known as **Pascal's vases,** shown in Fig. 143. The lower ends of several glass vessels of different shapes and sizes are joined to a horizontal tube. When water or any other liquid is poured into the apparatus, the level eventually becomes the same in each vessel. The explanation may be stated as follows. Points which are on the same horizontal level have the same pressure, as has already been shown. Consequently, by virtue of the expression $hg\rho$, they must lie at equal depths below the surfaces, which must therefore all lie in the same horizontal plane.

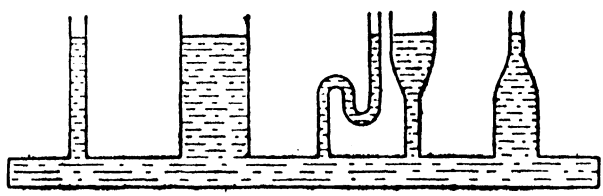


FIG. 143

An even more striking experiment may be done with two vessels like C and D in Fig. 144. The lower end of each vessel is open, has the same size, and can be screwed into a support at EE. The bottom of the vessel is then closed by the platform F attached to a lever which carries a weight on the other side of the fulcrum. With the weight in a given position, the height of water necessary to cause the platform to be forced downwards is the same whichever vessel is used, although the actual weights of water in the two vessels are very different. The net downward force on the platform is equal to $hg\rho$ multiplied by the area which is in contact with the water, and is therefore the same when h, the height of water, is the same regardless of the shape of the vessel. Apart from the foregoing

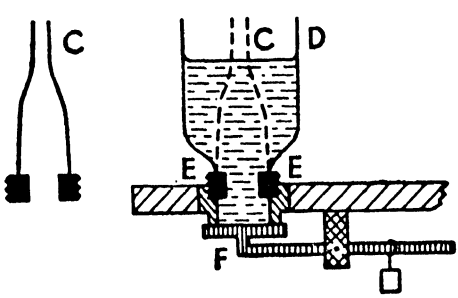


FIG. 144

explanation, this apparent paradox can be explained by realizing that in the case of the wide-topped vessel D the weight of the water which is not vertically above the bottom is supported by the sloping sides, while in the other vessel C the sides provide a downward force equal and opposite to that with which the water presses upwards on them, and this force is eventually supported by the bottom.

The U-Tube.—In a liquid contained in a U-tube of any shape (Fig. 145) the pressure is the same at any two points such as C and D on the same horizontal level. Each pressure is equal to $B+hg\rho$, where B is atmospheric pressure and h is the depth of the point below the surface. Consequently the depths are equal and the two surfaces are therefore at the same level. (Note that the principle is the same as in Pascal's vases.)

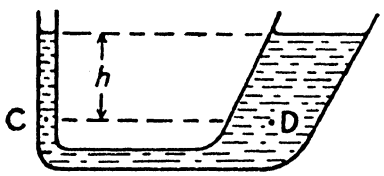


FIG. 145

It will be noticed that the equilibrium does not depend on the *weights* of liquid in the two tubes, which may have any values according to the cross-sectional areas of the tubes. Neither is it necessary that the tubes should be vertical.

In Fig. 146 two liquids, 1 and 2, which do not mix are placed in a U-tube and have come to rest in the positions shown. The junction of the liquids is at D, and the point C in the other limb is on the same horizontal level. Both C and D are in liquid 1, and CD is horizontal so that the pressures at the two points are equal. Let atmospheric pressure be B and the densities of the two liquids be ρ_1 and ρ_2 respectively. Then

$$B+h_1g\rho_1=B+h_2g\rho_2$$

or

$$h_1\rho_1=h_2\rho_2$$

This forms the basis of a method of determining the ratio of the densities of two immiscible liquids (page 198). It should be noted that the two limbs of the tube need not have the same diameter.

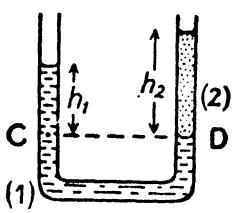


FIG. 146

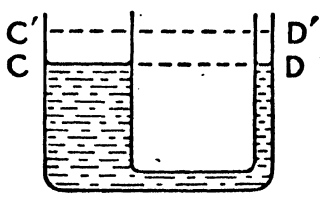


FIG. 147

The Hydrostatic Paradox.—It is worth while considering numerically the equilibrium of a liquid in a U-tube such as that shown in Fig. 147. With a certain quantity of liquid in the apparatus the surfaces stand at C and D on the same horizontal level. Suppose more of the same liquid is added and the surfaces reach equilibrium at C′ and D′, again on the same

level. Then

$$CC' = DD'$$

We can regard the weight of the additional liquid in the column CC′ as being "balanced" or "supported" by the smaller weight of the narrow column DD′. The weights are evidently not equal to each other—in fact they are in the ratio of cross-sections of the tubes—and the "balancing" is of the type met with in a machine such as a lever.

Suppose, for example, that the wide tube is a vessel of 50 cm. diameter while the diameter of the narrow tube is 5 mm. The diameters are in the ratio of 100 to 1, and the ratio of their cross-sections is therefore the square of this, which is 10,000 to 1. Thus in order to support, say, the weight of a man on the left-hand side, *e.g.* 80 kg., only 8 gm. of liquid is required in the narrow tube. If the liquid is water, this 8 gm. in the tube of 5 mm. diameter would occupy a height of $8 \div (\pi \times 0{\cdot}25^2)$ or about 41 cm. This apparent balancing of unequal weights (the so-called "hydrostatic paradox") is, of course, an example of the hydraulic principle.

Representation of Pressure in Terms of Head of Liquid.—It is often convenient to express pressures in such units as cm. of mercury, feet of water and so on rather than in absolute units. By "a pressure of 15 cm. of mercury" is meant the pressure which would support (or would be exerted by) a vertical column of mercury 15 cm. high. In order to find its value in absolute units, *e.g.* dynes cm.$^{-2}$, it is necessary to multiply 15 cm. by the acceleration due to gravity and the density of mercury. Neither of these is a definite constant quantity: g depends on position with respect to the earth and ρ on the temperature of the mercury column. Thus the height of a liquid column, or **head** of liquid as it is called, is not an absolute measure of pressure, and can be converted into absolute units only when the appropriate values of g and ρ are known. When great accuracy is not required it is sufficient to take g as 980 cm. sec.$^{-2}$ and ρ for mercury as 13·6 gm. cm.$^{-3}$.

Example.—*If atmospheric pressure is* 76 *cm. of mercury, calculate the pressure at a point* 30 *ft. below the surface of the sea in* (a) *dynes cm.*$^{-2}$, (b) *cm. of mercury*, (c) *feet of sea-water. The densities of mercury and sea-water may be taken as* 13·6 *and* 1·03 *gm. cm.*$^{-3}$ *respectively. One inch is equal to* 2·54 *cm.*

(*a*) In absolute units the required pressure is equal to

$$(76 \times 980 \times 13{\cdot}6) + (30 \times 12 \times 2{\cdot}54 \times 980 \times 1{\cdot}03) \text{ dynes cm.}^{-2}$$
$$= 1{,}013{,}000 + 923{,}000 \text{ dynes cm.}^{-2}, \text{ (approx.)}$$
$$= 1{,}936{,}000 \text{ dynes cm.}^{-2}$$

(*b*) If a head of 30 ft. of sea-water is equivalent to, say, h cm. of mercury, we have

$$30 \times 12 \times 2{\cdot}54 \times g \times 1{\cdot}03 = h \times g \times 13{\cdot}6$$

$$\therefore \quad h = \frac{30 \times 12 \times 2{\cdot}54 \times 1{\cdot}03}{13{\cdot}6} \text{ cm.}$$

$$= 69 \text{ cm.}$$

$\therefore$ Total pressure $= 76 + 69 = 145$ cm. of mercury.

(*c*) If a head of 76 cm. of mercury is equivalent to, say, H feet of sea-water, we have

$$76 \times g \times 13{\cdot}6 = H \times 12 \times 2{\cdot}54 \times g \times 1{\cdot}03$$

$$\therefore \quad H = \frac{76 \times 13{\cdot}6}{12 \times 2{\cdot}54 \times 1{\cdot}03}$$

$$= 32{\cdot}9 \text{ ft.}$$

$$\therefore \text{ Total pressure} = 32{\cdot}9 + 30$$

$$= 62{\cdot}9 \text{ ft. of sea-water}$$

Thrust on a Plane Surface Immersed in a Liquid.—In Fig. 148 (i) ABCD is a plane surface of irregular outline situated in a liquid, part of the horizontal surface of which is represented by KLMN. The plane ABCD is inclined at an angle θ to the horizontal. This is made clear in the sectional diagram in Fig. 148 (ii). The force exerted on ABCD by the fluid above it is purely normal, and over any small area ΔA situated at a depth h below the surface the pressure due to the liquid is $hg\rho$ and the normal force on the small area is $hg\rho\Delta A$ absolute units.

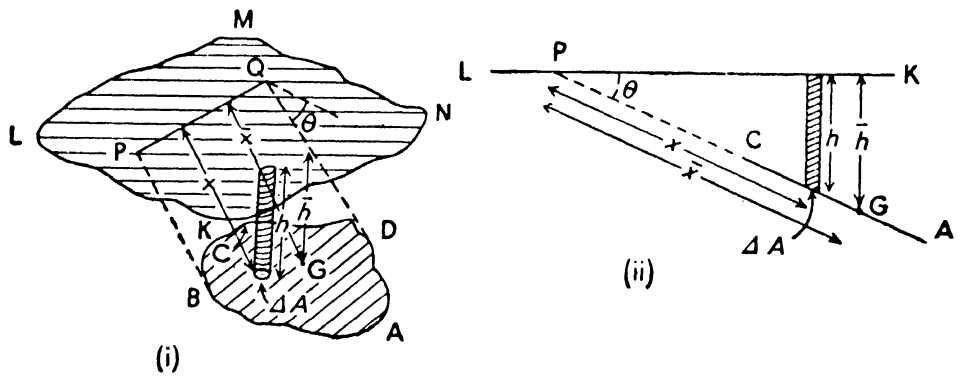


FIG. 148

Suppose that the plane ABCD is produced to meet the liquid surface in the straight line PQ, and let the distance of the small area ΔA from PQ be x. Then

$$h = x \sin \theta$$

and the total normal thrust on ABCD is equal to the sum of all the small forces, *i.e.*

$$\text{Total thrust} = \Sigma hg\rho\Delta A$$

$$= \Sigma x \sin \theta . g\rho\Delta A$$

$$= g\rho \sin \theta . \Sigma x\Delta A$$

since g, ρ and θ are constant for all points in ABCD. The term $\Sigma x\Delta A$ reminds us of the theory of centres of gravity (page 129). Suppose that ABCD were a uniform lamina of weight, say, w per unit area. The weight of the small area ΔA would be $w\Delta A$, and the total weight of ABCD would be wA, where A, its area, is given by

$$A = \Sigma\Delta A$$

Suppose that the distance of G, the centre of gravity of the lamina, from he line PQ is $\bar{x}$. Then by the property of the centre of gravity

$$wA\bar{x} = \Sigma xw\Delta A$$
$$= w\Sigma x\Delta A \text{ since } w \text{ is constant}$$

Thus

$$A\bar{x} = \Sigma x\Delta A$$

The point G which obeys this last equation is called the **centroid** of the area ABCD.

The expression for the thrust now becomes

$$g\rho \sin\theta . \bar{x}A$$

But if $\bar{h}$ is the depth of G below the surface of the liquid, we have

$$\bar{x} \sin\theta = \bar{h}$$

and the thrust can now be written

$$\bar{h}g\rho A \text{ absolute units of force}$$

Therefore the thrust is equal to the area of the submerged surface multiplied by the pressure ($\bar{h}g\rho$) at the centroid of the area. In order to take atmospheric pressure into account it is only necessary to add B to $\bar{h}g\rho$, so that the total pressure at G becomes $B + \bar{h}g\rho$, and the total normal thrust is $(B + \bar{h}g\rho)A$.

Centre of Pressure.—The point of application of the total thrust on ABCD is *not* the point G as one might at first suppose, but another point called the **centre of pressure** whose position depends on the shape of the area ABCD.

Example.—*A flat plate in the shape of an equilateral triangle of edge 2 ft. is immersed in sea-water of density* 64·4 lb. ft.$^{-3}$. *One of the edges is coincident with the water surface and the plate is inclined at an angle of* 30° *to the horizontal. Calculate the thrust due to the sea-water on one side of the plate.*

The length of a median of the triangle (Fig. 149 (i)) is 2 sin 60° or $\sqrt{3}$ ft. Therefore the centroid G lies at a distance of $\sqrt{3}/3$ ft. from the base of the median (see page 134). In Fig. 149 (ii) C marks the position of the edge of the plate lying in the surface, and CD is the plane of the plate. The depth ($\bar{h}$) of G below the surface is given by

$$\bar{h} = \text{CG} \sin 30° = \frac{\sqrt{3}}{3} \times \frac{1}{2} = \frac{\sqrt{3}}{6} \text{ ft.}$$

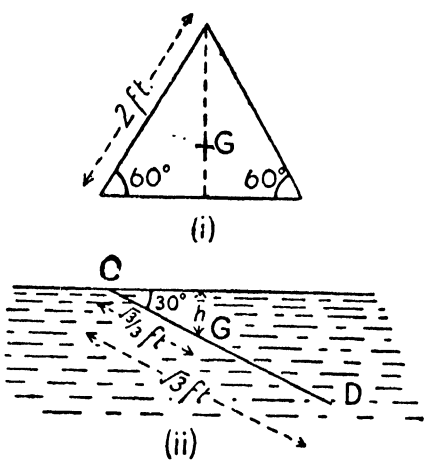


FIG. 149

while the area of the plate is given by

$$A = \tfrac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3} \text{ ft.}^2$$

Therefore, by the formula just established, the thrust is equal to

$$\begin{aligned}&\bar{h}g\rho A\\&=\frac{\sqrt{3}}{6}\times 32\times 64{\cdot}4\times\sqrt{3}\text{ poundals}\\&=16\times 64{\cdot}4\text{ poundals}\\&=\frac{16\times 64{\cdot}4}{32}\text{ lb. wt.}\\&=32{\cdot}2\text{ lb. wt.}\end{aligned}$$

The Siphon.—This device is illustrated in Fig. 150. Before the siphon will operate, the bent tube must be filled with liquid. This may be done by filling the tube before it is placed in position and holding the thumbs over the two ends while the tube is being arranged as shown in the drawing. Alternatively the tube may be placed in position while it is empty and the liquid drawn into it by applying suction at its lower end B.

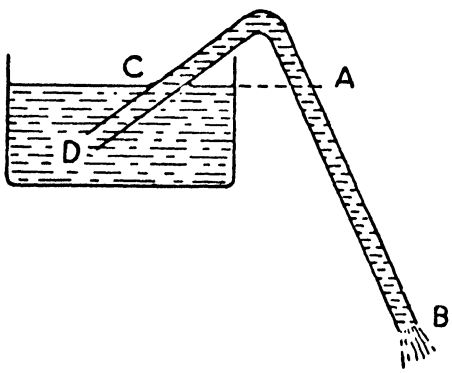


FIG. 150

Suppose that the tube is full of liquid which is at rest. Then atmospheric pressure exists at A because this point is on the same horizontal level as the plane surface in the beaker. But the pressure is also atmospheric at the end B. The column AB therefore experiences equal forces at its two ends due to atmospheric pressure, but it is not in equilibrium because of the component of its weight acting down the tube. It therefore begins to flow down the tube. As soon as it begins to move, the pressure at A is diminished below that at C and liquid flows from C to A. Thus there is a continuous flow of liquid into the tube at D and out at B. This goes on until the level of the surface in the beaker falls below the level of *either* of the ends of the tube. In the case illustrated the surface will reach the level of D first and the flow will then stop for an obvious reason. On the other hand, if D happens to be lower than B, the flow stops when the surface in the beaker falls to the level of B, because when this occurs A and B coincide and the weight of the column AB, which is the motive force of the system, becomes zero.

Constant Pressure Head.—It is sometimes important in physical experiments to maintain a flow of water or other liquid at a steady rate. This necessitates the maintenance of a constant pressure in the liquid as it enters the apparatus. As a rule the pressure of the main water supply is not sufficiently constant for this purpose.

One form of constant-pressure device is shown in Fig. 151. It consists of a funnel fitted with a wide central tube A, from the lower end of which a rubber tube leads to a sink. Water from the main flows in at B and

past a baffle C, which spreads out the stream. The tube D is joined to the apparatus which is to be supplied by the constant flow. The rate of inflow at B is made sufficiently large to maintain an outflow through A but not to fill this tube. Thus the level in the funnel is maintained constant at the top of A, or, more exactly, slightly above this. The pressure available in the apparatus to which D is connected is adjusted by raising and lowering the whole funnel.

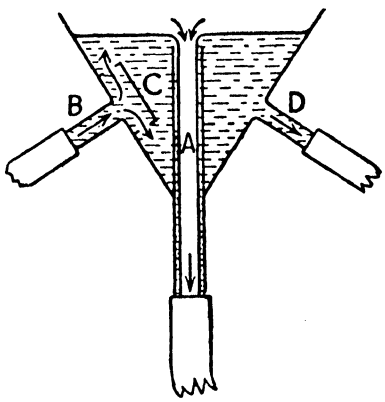


FIG. 151

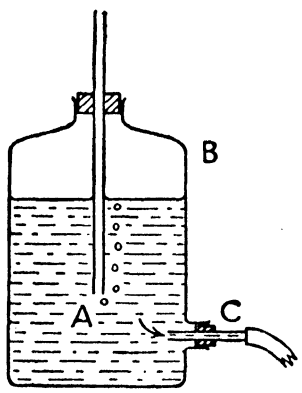


FIG. 152

Another very useful constant-head appliance is the **Marriotte bottle** illustrated in Fig. 152. It is more suitable than the previous device in cases where a constant flow of liquid other than water is required, because it is not wasteful of liquid. It is also preferable when a steady flow of water at constant temperature is required, because the temperature of main water sometimes fluctuates. The bottle B has an outlet C, while its neck is closed by a well-fitting bung through which passes a vertical tube whose lower end A is well below the level of the liquid in the bottle. When liquid flows out of C into the rest of the apparatus the pressure in the bottle is reduced, so that air from the atmosphere passes down the vertical tube and leaves A in a stream of bubbles, eventually reaching the space above the liquid. The pressure at A is always atmospheric pressure because this point is in direct communication with the atmosphere through the tube. Consequently the pressure in the liquid at the outlet C is always constant so long as the level of the liquid in the bottle does not fall below A. The pressure head may be adjusted by raising and lowering the vertical tube or by altering the height of the whole bottle.

EXAMPLES X

1. Describe the hydraulic press and explain its mode of action.

The diameters of the pistons in a hydraulic press are 1 in. and 12 in. respectively. The effort available is 14 lb. wt. and is applied by a lever system of mechanical advantage 4. Calculate the total thrust developed in the press. (L.M.)

2. A square ABCD, of side a, is immersed vertically in liquid, with the side AB in the free surface. Show that if P is a point in BC such that $BP = \frac{2}{3}a$, the liquid thrust on the triangle PCD is twice that on the triangle ABP. (L.I.)

3. A closed cubical box of side 2 ft. is half full of oil, specific gravity 1·3, and half full of water. It is tilted about one edge which is horizontal until the faces about this edge are inclined at 45° to the horizontal. Find the liquid thrust on one of the vertical faces of the box (i) if the oil and water are not mixed, (ii) if the oil and water are thoroughly mixed. Take the density of water to be 62·5 lb. ft.^{3} (L.I.)

4. A lock gate is 6 ft. wide and the water on the one side is 14 ft. deep; the level on the other side is 6 ft. lower, the water thus being 8 ft. deep. Determine the resultant thrust on the gate and explain your working. (The density of water may be taken as 62·5 lb. per cub. ft.) (L.M.)

5. Describe the hydraulic press and explain its action.

The plunger and ram of a hydraulic press have diameters of 1 in. and 4 in. respectively. The plunger is lowered by means of a lever giving a mechanical advantage of 10. Assuming that 60 per cent. of the work done on the plunger is usefully employed, what force is exerted on the ram when a force of 15 lb. wt. is applied to the end of the lever? (L.M.)

Chapter XI

ATMOSPHERIC PRESSURE

1. THE EXISTENCE AND EFFECTS OF ATMOSPHERIC PRESSURE

It is a simple matter to demonstrate that air has weight by weighing a glass bulb of capacity, say, one litre or more, first when it is full of air and then after the air has been pumped out. The density of air under ordinary conditions is about 1·2 gm. per litre, so that the weight of air contained in a vessel of this size is quite detectable.

The earth's atmosphere exerts a pressure on account of its weight in the same way as liquids do; but the effects of atmospheric pressure are rarely directly obvious in our everyday activities. Our bodies are adapted to its action, but we feel physical effects when any considerable change of pressure takes place, *e.g.* during aviation. Even the comparatively small changes of atmospheric pressure which occur when we go up or down in a lift are sometimes detectable in the ears when the equalization of the pressure on the two sides of the ear drum is prevented by a stoppage in the passage which connects the inner side of the drum with the atmosphere *via* the nose.

There are many simple experiments which demonstrate the existence of atmospheric pressure. The student is probably already familiar with several of these, and we need do no more than recall, for example, the airtight cannister which caves in as it cools after having been stoppered while water was boiling in it, the rubber "sucker" which adheres to a smooth surface (especially when wetted), the partially inflated toy balloon which swells when placed in a vessel from which the air is pumped. A striking demonstration of the magnitude of the force which arises from the action of atmospheric pressure over a large area was given in the middle of the seventeenth century by Otto von Guericke at Magdeburg, using what have come to be known as the **Magdeburg hemispheres.** These were two metal hemispheres of about 15 in. diameter whose circular rims were placed together with a leather washer between them (Fig. 153). When the air was pumped out of the closed vessel thus formed, it required eight horses a side to pull the hemispheres apart. The success of the experiment is entirely dependent on the accuracy with which the metal can be worked. It should be noticed

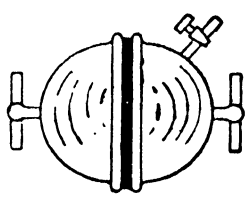


FIG. 153

that the force holding the two halves together is equal to atmospheric pressure multiplied by the area of the diametral plane (page 160).

2. THE MEASUREMENT OF ATMOSPHERIC PRESSURE

The Barometer.—In the seventeenth century Torricelli, a former pupil of Galileo, made the first measurement of atmospheric pressure with a barometer. His experiment can be repeated by completely filling with mercury a long glass tube closed at one end, and then closing the open end with the thumb while inverting the tube and placing the open end below the surface of some mercury in a trough. When the thumb is removed, some mercury flows out of the tube leaving a column as shown in Fig. 154. Since the tube was originally completely full of mercury and nothing has been allowed to enter it, it follows that the space above the column must be a vacuum except for the practically negligible pressure of the mercury vapour. A vacuum created in this way by the withdrawal of liquid from a vessel is referred to as a **Torricellian vacuum.**

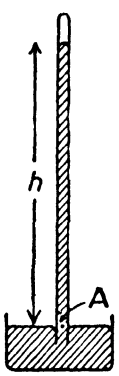


FIG. 154

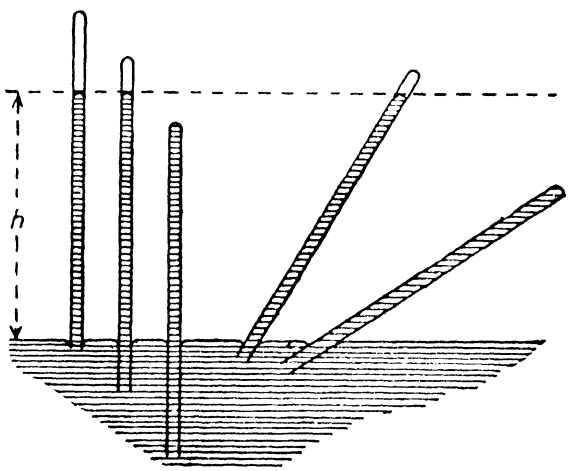


FIG. 155

The pressure at the point A in the tube is equal to that of the atmosphere, since it is on the same level as the mercury surface in the trough. But the pressure at A is due entirely to the column of height h and so is equal to $hg\rho$, where ρ is the density of the mercury. Thus atmospheric pressure as measured by the barometer is equal to $hg\rho$ absolute units or simply h cm. of mercury.

If the tube is gradually lowered into the mercury trough, the height of the column above the surface in the trough remains constant until the top of the column comes into contact with the closed top of the tube. Also if the tube is tilted, the *vertical* height remains h until the mercury completely fills the tube. Both these points are illustrated in Fig. 155.

Standard Atmospheric Pressure.—At any one spot the pressure of the atmosphere varies slightly from time to time. At sea-level it rarely

goes outside the range 74 – 78 cm. of mercury. For several purposes it is convenient to specify some arbitrary pressure as a standard. Standard atmospheric pressure (sometimes called "normal," but "standard" is better) is defined as the pressure due to a column of 76 cm. of mercury at 0° C., situated at sea-level in latitude 45°. Note that g and ρ as well as h must be specified when a pressure is being defined. The temperature of 0° C. fixes the value of the density of mercury at 13·595 gm. cm.$^{-3}$, while sea-level and latitude 45° fix the value of g at 980·66 cm. sec.$^{-2}$, so that in absolute units

$$\text{Standard atmospheric pressure} = 76 \times 980{\cdot}66 \times 13{\cdot}595$$
$$= 1{\cdot}013 \times 10^6 \text{ dynes cm.}^{-2} \text{ (approx.)}$$

At sea-level the average value of atmospheric pressure is about the same as standard pressure.

Other Units for Atmospheric Pressure.—Standard atmospheric pressure is sometimes adopted as a unit, called the **atmosphere,** especially for the measurement of high pressures.

It will be noticed that standard pressure is almost 10^6 dynes cm.$^{-2}$—a fact which has caused the introduction of the unit known as the **bar,** which is defined as 10^6 dynes cm.$^{-2}$. For meteorological purposes atmospheric pressure is expressed in **millibars,** *i.e.* thousandths of a bar. A barometric height of 76 cm. corresponds to 29·9 inches of mercury, and to about 14·7 lb. wt. per sq. in. The height of a column of water corresponding to 76 cm. of mercury is 76 × 13·6 cm., which is equal to 1034 cm. or 33·9 ft. The height of a water barometer would fall slightly short of this value because of the pressure due to the water vapour in the top of the barometer tube.

The Variation of Atmospheric Pressure with Altitude.—We have shown that the pressure due to a liquid is $hg\rho$ absolute units at a point whose depth below the surface is h. This formula is derived on the assumption that the liquid above the point has a uniform density of ρ right up to the surface in spite of the fact that the pressure to which it is subjected varies with depth. In other words, the liquid is treated as being incompressible, and this is usually a justifiable assumption, because very large pressures are necessary in order to change the density of a liquid appreciably. With a large quantity of gas such as the earth's atmosphere, however, the diminution of pressure with increasing altitude is accompanied by an appreciable fall of density. This being so, the fall of pressure due to a given increase of height is smaller at higher altitudes because of the smaller density, so that the graph of pressure against altitude is roughly as shown in Fig. 156. The pressure of the air decreases to about 1 cm. of mercury at a height of about 35 km. Actually the relation between pressure and altitude is complicated by the diminution of temperature with altitude, and the influence of this on equilibrium is discussed on page 552 (Vol. 2).

If we assume that the density of the atmosphere changes very little from its sea-level value (approximately 0·0012 gm. cm.$^{-3}$) during the

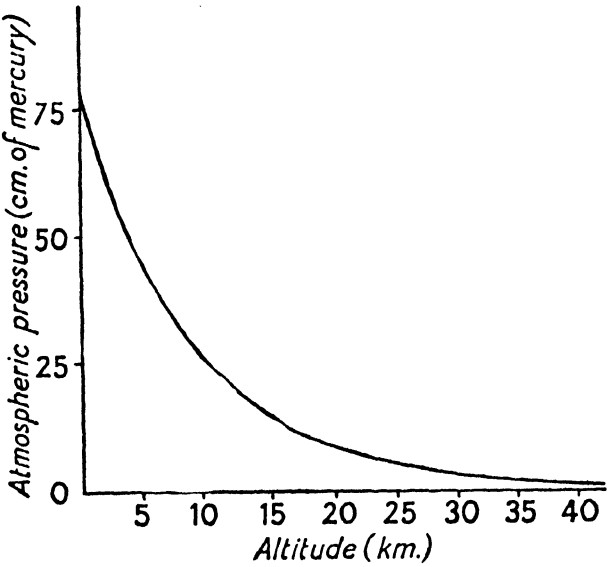


FIG. 156

comparatively small rise of, say, 100 metres, the change of pressure corresponding to this increase of elevation would be

$$100 \times 100 \times 980 \times 0{\cdot}0012 \text{ dynes cm.}^{-2}$$

or

$$\frac{100 \times 100 \times 980 \times 0{\cdot}0012}{980 \times 13{\cdot}6} \text{ cm. of mercury}$$

$$= 0{\cdot}9 \text{ cm. of mercury approx.}$$

Since this change of pressure is comparatively small, the assumption of constant density is fairly well justified and, for low altitudes, the diminution of pressure for a rise of 100 metres can be taken to be as calculated above, *i.e.* approximately 9 mm. of mercury per 100 metres. Conversion to F.P.S. units gives the rate of diminution as 1 inch of mercury in roughly 900 ft.

It is of course possible, but not very realistic, to suppose that the density of the atmosphere remains 0·0012 gm. cm.$^{-3}$ throughout and then to calculate the height to which the atmosphere would have to extend in order to exert the pressure which it actually does at sea-level. If H is this height, we have

$$H \times g \times 0{\cdot}0012 = 76 \times g \times 13{\cdot}6$$

or

$$H = \frac{76 \times 13{\cdot}6}{0{\cdot}0012} \text{ cm.}$$

$$= 8{\cdot}6 \text{ km. (approx.)}$$

$$= 5{\cdot}3 \text{ miles (approx.)}$$

In actual fact the pressure has fallen only to about 30 cm. of mercury at this height which, roughly, marks the beginning of what is called the **stratosphere.**

Fortin's Barometer.—This is a Torricellian type of barometer adapted for easy and accurate observation. A vertical linear scale A (Fig. 157) is engraved on the metal case of the instrument in the neighbourhood of the top of the mercury column. This scale is of sufficient length to compass the extreme variations of atmospheric pressure, and an ivory point B is accurately fixed at what would be the zero of the scale if it were continued downwards. The level of the mercury in the reservoir can be adjusted by means of the screw D which, when rotated, raises or lowers the bottom of a bag of chamois leather E with which the lower part of the reservoir F is lined. The top part of the reservoir is made of glass so that the ivory point can be viewed from outside. It should be realized that the interior of the reservoir must be in communication with the atmosphere, although the aperture G through which this is effected is made very small so as to prevent the entry of dust and other contamination.

Assuming that the instrument has been set up correctly in the first place, the first adjustment to be made before taking an observation is to rotate the screw D slowly until the surface of the mercury in the reservoir is made just to touch the ivory point. This is necessary each time the instrument is used, because a change of atmospheric pressure causes some mercury to leave or enter the vertical tube, thus altering the level in the reservoir. Having adjusted this level to the ivory point, which is at the zero of the scale, we can be sure that the reading of the top of the column against the scale A gives the true height of the column above the level in the reservoir.

In order to achieve greater accuracy of observation a small metal plate H carrying a vernier scale can be made to travel vertically by turning the knob K. This is operated until the lower horizontal edge of H (which coincides with the zero of the vernier) appears just to touch the highest point of the mercury surface, which, owing to capillarity,

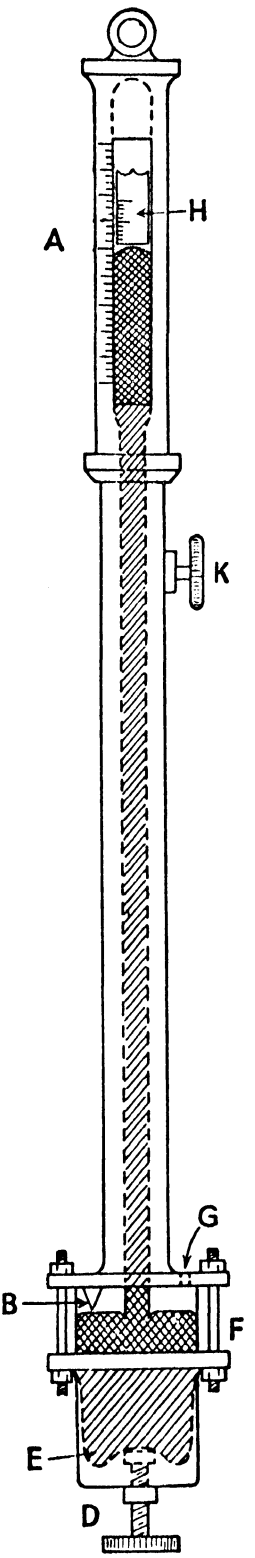


FIG. 157

is convex upwards (Fig. 158). The reading of the bottom edge of H on the scale A is then taken to the next lower millimetre, and the vernier gives the 10ths or 20ths of a millimetre. The reading shown in Fig. 158 is 76·26 cm.

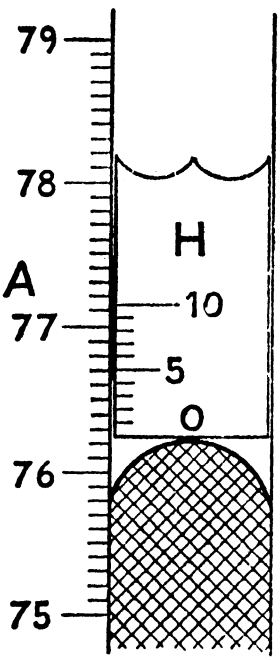


FIG. 158

Where possible, the student should become familiar at first hand with the technique of taking an observation with an actual Fortin barometer.

Corrections to the Barometer Reading.—Certain corrections must be taken into account during the determination of barometric pressure if an accuracy of the order of 0·1 mm. is desired.

Firstly, allowance must be made for the variations in the length of the scale as its temperature changes. If the temperature at which the instrument is used is θ deg. C. higher than that at which the scale is correct, it is necessary to *add* to the observed reading the product of the reading itself × the coefficient of linear expansion of the scale × θ. This is explained in the chapter on the expansion of solids in Volume II, where it is also shown that the correction is usually very small.

Next, if we are going to express atmospheric pressure in cm. of mercury it is necessary, for comparison purposes, to correct the observed barometric height to what it would be if g were 980·66 cm. sec.$^{-2}$ (its value at sea-level and latitude 45°) and if ρ were 13·595 gm. cm.$^{-3}$ (the density of mercury at 0° C.). In order to do this we remember that both the corrected and the observed barometric heights represent columns of mercury supported by the same *absolute* pressure. Therefore we can write

(Corrected height) × 980·66 × 13·595 = (observed height) × (value of g at the instrument) × (density of mercury at the temperature of the instrument)

The calculation of the correction due to the variation of the density of mercury with temperature is explained in the chapter on the expansion of liquids in Part II. The effect of the variation of g is usually negligible.

There is also the effect of the surface tension of mercury to be considered (see Chapter XVII). On account of the fact that a mercury meniscus in a glass tube is convex upwards, the column is actually slightly depressed below its true height. It is hardly possible to calculate a correction for this effect, which is fortunately quite small in a good barometer with a fairly wide tube of, say, 1 cm. diameter. The makers of barometers usually provide a table from which the total correction due to all effects can be found when the temperature of the instrument is known.

The Aneroid Barometer.—This instrument works on quite a different principle from that of the Torricellian barometer. It consists of a flat circular box A (Fig. 159) made of thin metal and with its top and bottom corrugated for flexibility. The box is permanently exhausted of air and is prevented from collapsing by the stout spring B, against which it pulls

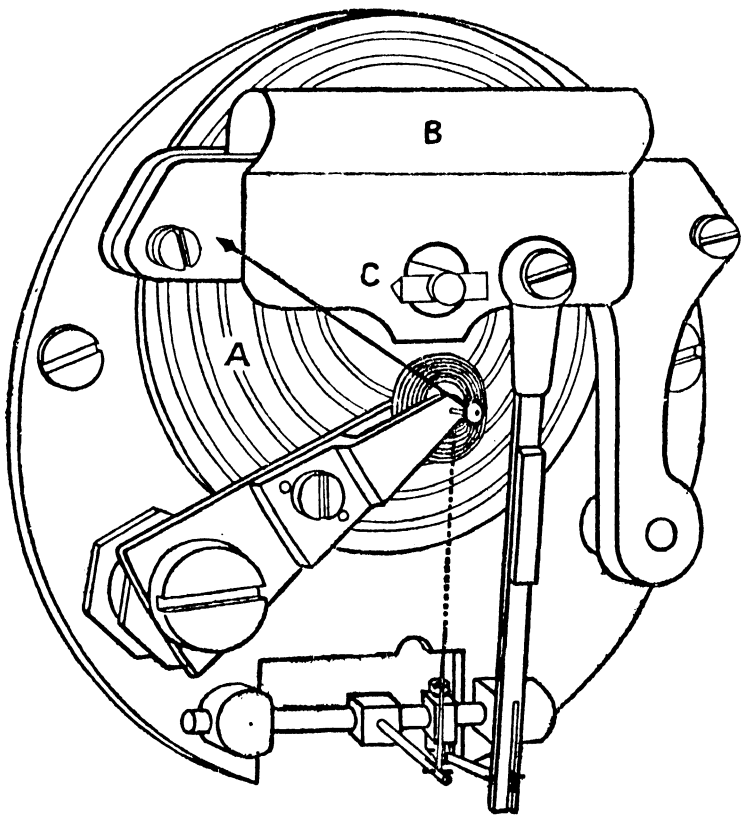


FIG. 159

by means of the knife-edge C. Changes of atmospheric pressure cause the top of the box to be pushed in or out against the controlling action of the spring. Although these movements are very small they are magnified by the series of levers shown in the figure and are eventually imparted to the needle by a light chain which is kept taut by a hair-spring. The needle moves over a circular scale marked in inches of mercury. It is necessary to set or graduate every aneroid barometer with the help of some standard, because they are not absolute instruments.

It frequently happens that an ordinary household barometer of the aneroid type is intended for use as a weather forecaster, relying on the somewhat rough-and-ready assumption that the lower the atmospheric pressure falls the more unpleasant will be the weather. In this case, and if the various types of weather are already inscribed on the face of the instrument, it is necessary to adjust its pointer at the place where the barometer is to be used so that it records the barometric pressure which

would prevail *at sea-level* at that place at the same time. Otherwise, if the barometer is adjusted so as to indicate correctly the pressure of the atmosphere to which it is exposed, the variation of this pressure with altitude as well as with weather conditions would cause the instrument to give false indications of the latter unless it is used at sea-level.

An aneroid barometer can also be used to estimate height above sea-level. It is evident that in this case the zero of the altitude scale is the value of atmospheric pressure at the same time at sea-level vertically below its actual position. For moderate altitudes atmospheric pressure falls by one inch of mercury for an increase of altitude of approximately 900 ft. (see page 176). When used for this purpose the instrument is known as an **altimeter.**

A **barograph** (Fig. 160) is an aneroid barometer which gives a continuous record of the value of atmospheric pressure. The instrument

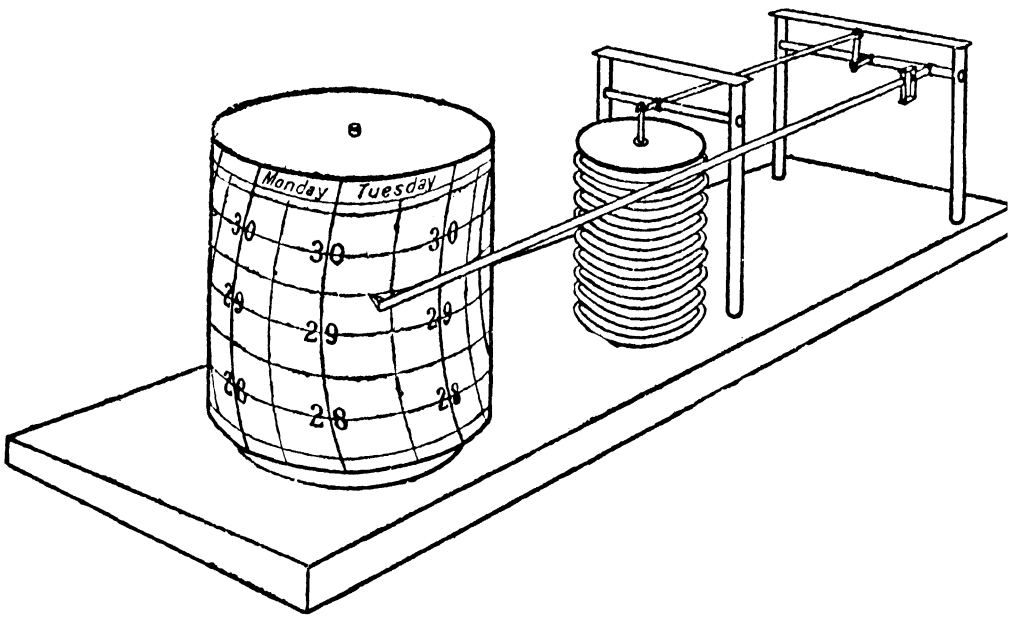


FIG. 160

actuates a long lever at the end of which is a pen which traces a line on a sheet of paper wrapped round a revolving drum. Several vacuum boxes are placed one on top of the other in the type of barograph shown. This is done in order to achieve the necessary sensitivity.

EXAMPLES XI

Certain examples appropriate to this chapter but involving the use of Boyle's law are included in Examples XV.

1. Describe two simple experiments to demonstrate that the atmosphere exerts a pressure.

If the height of the barometer falls from 30 in. to 29 in. of mercury, calculate in lb. wt. the change in the total thrust on one side of a surface 4 sq. yd. in area. (Assume that the specific gravity of mercury is 13·6 and that the weight of 1 cub. ft. of water is 62·5 lb.) (L.M.)

2. Define *pressure* and describe how you would determine the pressure of your gas supply in dynes per cm.^{2}.

The pressure of the atmosphere is 14·5 lb. wt. per in.^{2}. If the density of the air remained 1·25 gm. per litre irrespective of height, what would be the thickness of the layer of air covering the earth's surface? State the units (*e.g.* feet or metres) in which your result is expressed. (Assume that 1 lb. wt. is equal to 454 gm. wt. and that 1 ft. is equal to 30 cm.) (L.M.)

3. Derive an expression for the pressure due to a liquid at a depth h below its surface, and describe how the variation of this pressure with h can be investigated experimentally.

Taking the density of air as 1·2 gm. litre^{-1} and of mercury 13·6 gm. cm.$^{-3}$, calculate the change in the reading of a mercury barometer when it is taken from the bottom to the top of a building 40 metres high. Explain qualitatively how and why the rate of change of barometric reading with altitude varies as the altitude increases. (L.I.)

Chapter XII

THE MEASUREMENT OF PRESSURE. PUMPS

1. METHODS OF MEASURING PRESSURE

Manometers.—The principle of the simple U-tube manometer by which the pressure of a gas may be measured has already been explained (page 166). The pressure of the gas contained in the vessel shown in Fig. 161 is equal to atmospheric pressure *plus* $hg\rho$, where ρ is the density of the liquid in the manometer. If the level of the liquid in the open limb of the U-tube is below that in the closed limb, the pressure difference $hg\rho$ must be subtracted from atmospheric pressure instead of being added to it. Atmospheric pressure must, of course, be expressed in absolute units when $hg\rho$ is added or subtracted. It is frequently desirable, but not always necessary, that the manometric liquid should have a low vapour pressure so as to avoid the diffusion of its vapour into the rest of the apparatus.

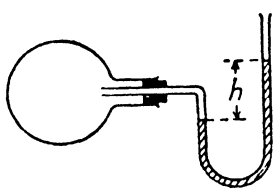


FIG. 161

Where the pressure to be measured is always less than atmospheric, an arrangement such as that shown in Fig. 162 is frequently used. The pressure in the vessel is atmospheric pressure *minus* $hg\rho$.

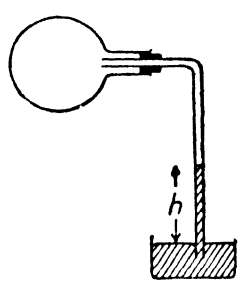


FIG. 162

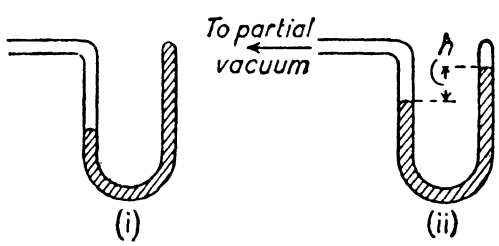


FIG. 163

For the approximate estimation of the pressure inside a piece of apparatus which is being evacuated an indicator of the type shown in Fig. 163 is sometimes used. The liquid is mercury, and before the apparatus is exhausted the mercury extends right up to the end of the closed limb (Fig. 163 (i)). As evacuation proceeds, the mercury comes away from the closed end of the tube leaving a Torricellian vacuum (Fig. 163 (ii)). The

pressure in the apparatus is simply the head of mercury h represented by the difference of level in the two limbs.

A **differential manometer** may be used for the accurate measurement of small pressure differences. One type of instrument is shown in Fig. 164. Its essential features are the wide reservoirs C and D at the top of each limb of the U-tube, and the use in the apparatus of two immiscible liquids of slightly different densities ρ_1 and ρ_2. Let the depth of the junction E of the two liquids be h_1 and h_2 below the free surfaces in the reservoirs. If the air (or gas) pressure is the same above each surface, we have

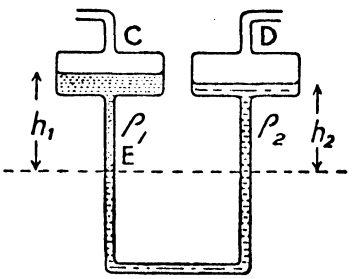


FIG. 164

$$h_1 g\rho_1 = h_2 g\rho_2 \quad . \quad . \quad . \quad . \quad (1)$$

Let the gas pressure above C be now increased by a small amount Δp. This will cause h_1 and h_2 to change to, say, h_1' and h_2' respectively, and we have

$$\Delta p + h_1' g\rho_1 = h_2' g\rho_2 \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting equation (1) from equation (2) gives

$$\Delta p + (h_1' - h_1)g\rho_1 = (h_2' - h_2)g\rho_2 \quad . \quad . \quad . \quad (3)$$

Suppose now that the area of cross-section of the reservoirs is so much greater than that of the connecting tube that the change of level in the reservoirs due to the application of Δp is negligible compared with the accompanying movement of the common surface E. Thus $h_1' - h_1$ and $h_2' - h_2$ are now both equal to the depression of E caused by Δp. This movement can be measured with a travelling microscope. Let it be d. We then have from equation (3)

$$\begin{aligned}\Delta p &= (h_2' - h_2)g\rho_2 - (h_1' - h_1)g\rho_1 \\ &= dg(\rho_2 - \rho_1)\end{aligned}$$

Thus, by making the difference of the densities of the manometric liquids small, it is possible to cause a measurable movement d by a very small pressure change. For example, if $\rho_2 - \rho_1$ is, say, 0·01 gm. cm.$^{-3}$, a value of d of 1 mm. would correspond to a value of Δp as small as 0·001 cm. of water.

An instrument known as the **McLeod gauge** enables very low gas pressures to be measured. A known volume of the rarefied gas is compressed into a narrow graduated tube until its pressure is measurable, and its new volume is then noted. An application of Boyle's law allows the original pressure to be calculated.

High Pressures.—In researches on the compressibility of gases during

the nineteenth century, experimenters such as Amagat and Cailletet used columns of mercury to measure pressures up to several hundreds of atmospheres. More modern experiments, notably by Bridgman in America, have taken pressures up to 50,000 atmospheres and even above. It is, of course, quite impracticable to measure such pressures by means of a mercury column. A direct method of measuring high pressures is to note the weight which must be placed on a piston of known cross-section in order to prevent it from being forced out of the apparatus. In order to avoid the buckling of the piston under the enormous compression involved, a differential piston is sometimes used. The principle of this is shown in Fig. 165. The liquid under high pressure exerts an upward force on the small area of the shoulder C, which is equal to the difference between the cross-sectional areas of the upper and lower parts of the piston D. If a single piston of the same small area were used it would be unable to withstand the force of compression.

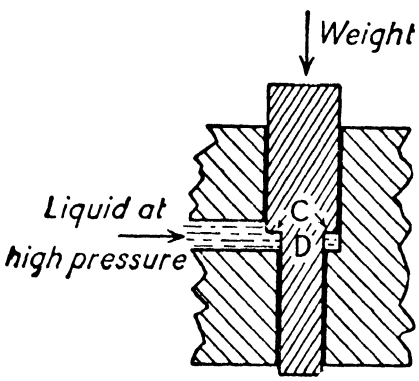


FIG. 165

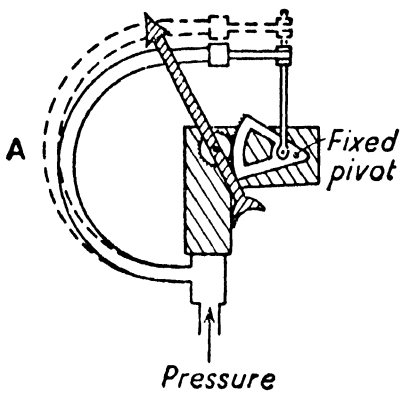


FIG. 166

Secondary Gauges.—A very common pressure gauge which is frequently used in connection with boilers and gas cylinders etc. is the **Bourdon gauge.** The action of this depends on the fact that a tube of oval cross-section which is bent into the arc of a circle tends to become straighter when the pressure inside it is increased. The instrument is shown diagrammatically in Fig. 166. A flattened tube A, made of phosphor-bronze or beryllium-copper, is closed at one end, and the other end, into which the gas at high pressure is admitted, is fixed. As the tube uncoils it moves to the dotted position, and so causes a rotation of the pointer over a scale graduated in lb. per sq. in. or other pressure units.

For the measurement of very high pressures Bridgman has used with great success the change of electrical resistance of a manganin wire with the pressure of the fluid in which it is placed. It is necessary to calibrate a pressure gauge of this kind against an absolute gauge such as the loaded piston, but once standardized it is extremely convenient.

2. PUMPS

The Lift Pump.—We deal first with pumps which are designed for the pumping of water and other liquids, and the simplest of these is the lift pump, such as is used for raising water from a well. Fig. 167 illustrates the pump diagrammatically, and shows the state of affairs when it has been operated for a sufficient length of time for the delivery of the water to have begun. At the instant considered in the figure, the plunger, or piston, is being raised so that it creates a low pressure in the water beneath it in the cylinder. This low pressure has the effect of closing the valve *a* in the piston and opening *b* at the bottom of the cylinder. Atmospheric pressure acting on the water in the well forces it towards the low pressure, *i.e.* up the pipe and past *b* into the cylinder, provided that the total height does not exceed the height of a water barometer. Thus the cylinder is filled on the upstroke while water is flowing out of the spout. As soon as the downstroke begins, the pressure below the piston is increased so that *a* opens and *b* closes. In this way the piston is returned to the bottom of the cylinder without allowing water to run back down the feed pipe. Standard atmospheric pressure can support a column of about 34 ft. of water, so that it is not possible for a lift pump to raise water through a greater height than this. Actually the working height is a few feet less than this on account of mechanical imperfections.

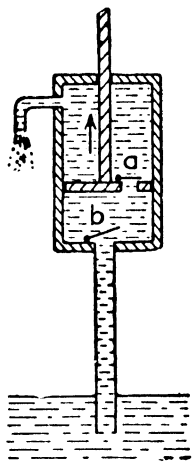


FIG. 167

When the pump is first operated, it merely pumps air until by doing so it has reduced the air pressure in its cylinder sufficiently to cause the water to rise into it. This may never occur if the valves and piston fit badly. Hence the necessity, in some cases, of pouring water into the cylinder. This starts the action by sealing the gaps between badly fitting parts.

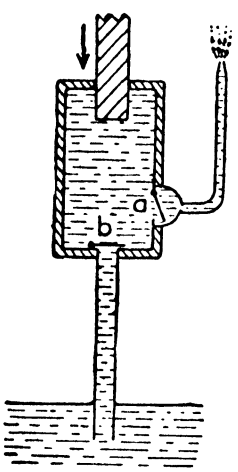


FIG. 168

The Force Pump.—In this appliance (Fig. 168) the raising of the water into the cylinder on the upstroke corresponds exactly with the action of the lift pump. On the downstroke of the plunger (shown in the figure), however, *b* closes and *a* opens so that water is forced into the outlet tube and sent to a higher level or discharged with high velocity through a hose and nozzle. The maximum height of the pump itself above the level of the reservoir is again about 30 ft., but there is no theoretical limit to the height to which the water can be forced above the level of the pump.

The Air Pump.—As regards the disposition of the valves, the air pump

(Fig. 169) for producing a partial vacuum is similar to the lift pump. Its action can be described as follows. Suppose that the air in the vessel to be evacuated (the "receiver") is initially at atmospheric pressure (p_0), and that the volume of this vessel and the connecting tube is V. Let the piston be initially at the bottom of the cylinder, which has a volume v. As the piston is drawn up to the top of the cylinder it produces a low pressure beneath it, while the pressure above it is atmospheric on account of the vent c. Thus b opens and a closes. When the piston has reached the top it has swept out the air above it, and the air which was in the receiver now fills the cylinder and receiver. Its volume has changed from V to $V+v$ and its pressure (p_1) is now given by

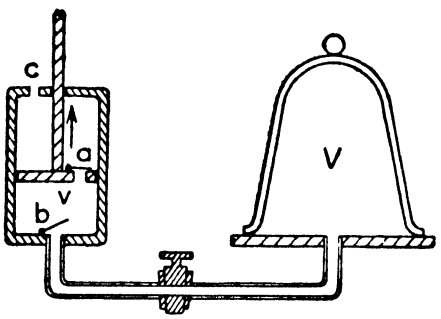


FIG. 169

$$p_1 = p_0 \frac{V}{V+v}$$ according to Boyle's law (page 225)

The commencement of the downstroke immediately closes b and opens a, so that the pressure in the receiver remains p_1 until the piston reaches the bottom. The next upstroke repeats the action of the first. The air in the receiver is again expanded to a volume of $V+v$, and its pressure (p_2) after two strokes is given by

$$p_2 = p_1 \frac{V}{V+v} = p_0 \left(\frac{V}{V+v} \right)^2$$

After n upward strokes the pressure in the receiver is given by

$$p_n = p_0 \left(\frac{V}{V+v} \right)^n$$

Naturally in practice the reduction of pressure by any given number of strokes is not so great as that given by this formula. The simple theory has assumed perfect fitting of the valves and piston, and also that only an infinitesimal difference of pressure causes the operation of the valves.

It should be noticed that, even theoretically, the pressure can never be reduced to zero however large n may be. The reduction of pressure caused by each stroke diminishes as the evacuation proceeds, because it is always equal to the constant fraction $v/(V+v)$ of the pressure at the beginning of the stroke.

If both the valves a and b operate in the reverse directions to those shown in Fig. 169, the action of the pump is to send air into the vessel to which it is connected instead of withdrawing it. It is then called a

compressor or **condensing air pump**. An example of this is the cycle pump, in which the valve *a* is provided by the way in which the washer (piston) fits against the walls of the barrel, and *b* is the valve on the tyre.

The " Hyvac " Pump.—This is a type of air pump which is in general use for the evacuation of apparatus down to a pressure of about one-thousandth of a millimetre of mercury. The principle of its working is shown in Fig. 170.

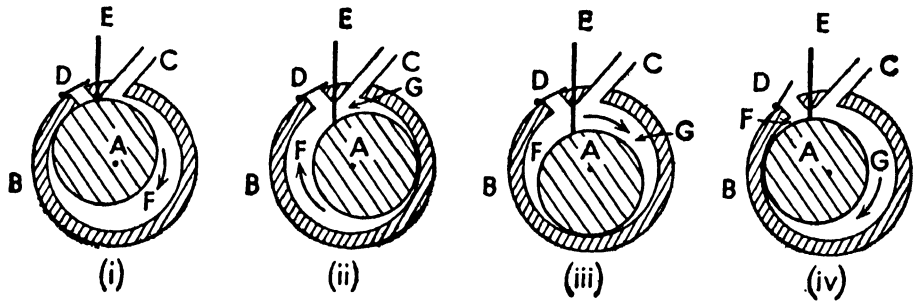


FIG. 170

A cylindrical metal rotor A is mounted eccentrically on an axle and always touches the inside surface of the hollow cylinder B as the axle rotates. The pipe C is the inlet and is joined to the vessel to be exhausted. The other opening in the cylinder is the outlet through which air is forced after it has opened the valve D. A metal shutter E passes through the cylinder wall and is kept pressed against the rotor A by a spring (not shown) acting on its upper edge. In position (i) air from the vessel to be exhausted fills the space F, which extends round the rotor from C almost to the outlet. In (ii) this air is imprisoned and is being compressed while more air is entering the space G whose volume is increasing. This continues through stage (iii) until in (iv) the increasing pressure in F has caused the valve D to open, and the air which was originally in the pump in (i) is ejected. The pump is immersed in a tank of oil so as to keep it sealed.

EXAMPLES XII

1. The piston of an air pump for reducing pressure has a diameter of 7 cm. and the length of its stroke is 20 cm. If the volume of the vessel to be evacuated is 5 litres and that of the connecting tube is negligible, calculate from first principles the percentage pressure reduction caused by three strokes of the piston.

Draw diagrams to show the action of the pump, and explain why a perfect vacuum cannot be obtained. (L.I.)

2. A quantity of gas, the pressure of which is being determined by a McLeod gauge, is transferred from a bulb of one litre volume to a tube of cross-sectional area 1 sq. mm., where it occupies a length of 5 cm. and exerts a pressure of 2 cm. of mercury. Calculate the original pressure of the gas.

3. Explain how very high liquid pressures can be measured. A differential piston, the diameters of which are 3·000 and 2·980 cm. respectively, is held in equilibrium by a force of 200 kg. wt. Calculate the pressure (in atmospheres) in the liquid to which the piston is exposed.

Chapter XIII

DENSITY AND SPECIFIC GRAVITY ARCHIMEDES' PRINCIPLE

1. DENSITY AND SPECIFIC GRAVITY

Density.—The density of a material is defined as

$$\frac{\text{Mass of a given quantity of material}}{\text{Volume of this same quantity of material}}$$

that is, the mass per unit volume of the material. We have already used this quantity (usually denoted by the symbol ρ) when occasion necessitated its introduction into formulæ, and it is now necessary to discuss it further.

The units in which densities are expressed are implied in the definition. In the C.G.S. system the commonest units are gm. cm.$^{-3}$, while lb. ft.$^{-3}$, usually written lb. per cub. ft., are often used in the F.P.S. system.

The densities of pure metals cover a wide range of values. For instance, at ordinary temperatures sodium has a density of 0·97 gm. cm.$^{-3}$, aluminium 2·7, iron 7·9, copper 8·9, lead 11·4, mercury 13·6, gold 19·3, and platinum 21·5. The densities of many common liquids do not differ very much from that of water. At a temperature of 0° C. and a pressure of one atmosphere hydrogen has a density of 0·09 and chlorine 3·2 gm. litre^{-1}. The densities of many common gases fall within this range.

When the temperature of a given piece or quantity of material is altered, its mass remains constant but its volume changes—almost always increasing with rise of temperature—so that the density of a given substance usually diminishes when it is heated. In solids this is a very small change for each deg. C. rise of temperature (say about one part in 20,000 and often less than this), in liquids it is greater, and in gases it is as high as one part in less than 300.

Water has its maximum density at 4° C., and its value in gm. cm.$^{-3}$ is very nearly unity. This happens because the kilogram, which is the standard of mass in the C.G.S. system, was originally chosen to be as nearly as possible the mass of 1000 cm.^{3} of pure water at its maximum-density temperature.

Since the time when the original standard kilogram and metre were chosen, more precise experiments have shown that the density of pure water at 4° C. is not quite exactly 1 gm. cm.$^{-3}$, although the difference is only about 3 parts in 100,000. The **litre** is now defined, *not* as 1000 cm.^{3},

but as the exact volume of a kilogram of pure air-free water at its maximum-density temperature and standard pressure. Thus the maximum density of pure water at standard pressure is exactly 1 gm. per millilitre by definition.

The Specific Gravity of a material is defined as

$$\frac{\text{Weight of a given volume of material}}{\text{Weight of the same volume of water}}$$

Specific gravity is thus a ratio of two similar physical quantities, namely weights, and therefore, unlike density, it has *no units*. Provided that both the weight of substance and the weight of water are expressed in the same units, the value of the specific gravity of a given material is always the same no matter what these units may be.

When g is a constant, *i.e.* at any particular spot, weights are proportional to masses. Therefore we can substitute mass for weight in the definition of specific gravity. If we do this and then suppose that the given volume is unity—it can, of course, have any value—both numerator and denominator become densities, so that

$$\text{Specific gravity of a substance } (S) = \frac{\text{Density of the substance } (\rho)}{\text{Density of water } (\rho_{\text{water}})}$$

Specific gravity is sometimes called **relative density,** since it is the density of the substance relative to that of water.

When great accuracy is required and for purposes of comparison it is clear that, in stating the numerical value of a specific gravity, the temperature both of the substance and of the water must be specified. These are always stated in accurate tables, and the temperature chosen for the water is frequently, though not necessarily, 4° C. To the degree of accuracy to which the density of the water used for comparison is equal to unity in the C.G.S. system (gm. cm.$^{-3}$), we can state that the specific gravity of a substance is numerically equal to the density of the substance expressed in gm. cm.$^{-3}$. For this reason the terms density and specific gravity are often used indiscriminately, but the fundamental distinction between them should always be borne in mind. So also should the fact that their almost exact numerical equality holds good only when density is expressed in gm. cm.$^{-3}$.

If the specific gravities of two substances are S_1 and S_2 and their densities are ρ_1 and ρ_2 respectively, we have

$$S_1 = \frac{\rho_1}{\rho_{\text{water}}}$$

$$S_2 = \frac{\rho_2}{\rho_{\text{water}}}$$

$$\therefore \quad \frac{S_1}{S_2} = \frac{\rho_1}{\rho_2}$$

Thus the ratio of the specific gravities of two substances with respect to the same specimen of water is equal to the ratio of their densities.

With this introduction we pass on to a discussion of Archimedes' principle, and then to methods of determining densities and specific gravities.

2. ARCHIMEDES' PRINCIPLE

Statement of Principle.—It is common experience that fluids exert an upthrust or buoyancy on bodies which are wholly or partially immersed in them. This may result in the apparent cancellation of the weight of the immersed body, as when a cork or a boat floats on water or a balloon ascends in the air. In every case, even if the immersed body does not actually float, its apparent weight is found to be less in the fluid than it is in a vacuum. The origin of this upthrust is the action of gravity on the fluid surrounding the immersed body. This causes the pressure in the fluid immediately underneath the body to be greater than that on the top. The magnitude of the upthrust can be calculated in any particular case by using this idea of pressure difference, but it is difficult to do so when the immersed body has an irregular shape. The following simple argument is sufficient to establish the general principle by means of which the upthrust may be calculated in all cases.

Fig. 171 represents a quantity of fluid at rest, and we consider the equilibrium of a portion A of any shape and size. The forces acting on A can be analysed into (i) its weight acting vertically downwards, and (ii) the total force due to the pressure of the surrounding fluid acting on the surface of A. Since A is in equilibrium, it follows that the force (ii),

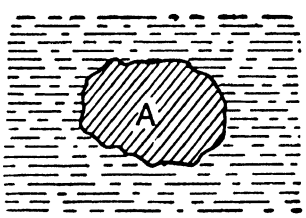


FIG. 171

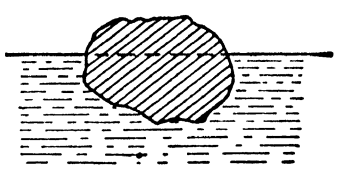


FIG. 172

which is the upthrust on A, is equal and opposite to the weight of A. Now suppose that A is replaced by a quantity of some other material of the same shape and size. The weight of this material will, in general, be different from the weight of the fluid A which it has replaced, but since the upthrust is due to the action of the surrounding fluid, this latter force remains unaltered, and we conclude that the upthrust on any body immersed in a fluid is equal to the weight which the body would have if it were composed of the same fluid in which it is immersed. If the body is only partially immersed (Fig. 172), the upthrust is equal to the weight which the *immersed portion* (below the dotted line) would have if it were

composed of fluid, because the upthrust would support this amount of fluid if the body were not present.

In all cases, therefore, the upthrust is equal to the weight of the fluid which the body displaces, *i.e.* the fluid which would fill the space created by the removal of the body from its actual position. This law is summed up as follows in what is known as **Archimedes' principle:—**

When a body is wholly or partially immersed in a fluid, it experiences an upthrust equal to the weight of the fluid which it displaces.

The point of application of the upthrust is the same as the point of application of the weight of the displaced fluid, *i.e.* the centre of gravity of the displaced fluid. This is *not necessarily* the centre of gravity of the immersed body unless this happens to be totally immersed and of uniform density.

Experimental Illustration of Archimedes' Principle.—In the case of a solid body which sinks in a liquid, the principle can very easily be verified.

Fig. 173 shows an arrangement called the **hydrostatic balance** by means of which the apparent weight of a body can be found when it is suspended in a liquid. The body is first weighed in air by hanging it from the hook on the left-hand side of the balance by means of a fine thread or wire and counterpoising with weights in the right-hand pan as usual. Let this weight be W_0. Next a wooden bridge, as shown in the figure, is placed over the left-hand pan so as to support a beaker of liquid of known density in which the solid is made to hang totally immersed. Weights are again put on the right-hand pan to counterpoise after it has been made certain that the balance is swinging freely and that the solid is not touching the sides or bottom of the beaker or the surface of the liquid. The second weight (W_1) will be less than W_0, and the upthrust can be calculated simply by evaluating $W_0 - W_1$, which is sometimes called the "loss of weight" of the solid when it is transferred from air to liquid. In order to find the weight of liquid displaced by (*i.e.* equal in volume to) the solid it is necessary to multiply the previously determined density of the liquid (ρ) by the volume of the solid (V). This latter quantity may be found by measurement of its dimensions if the solid is regular, *e.g.* cube or sphere, or by placing the solid in a graduated measuring cylinder partly filled with water and noting the increase of volume.

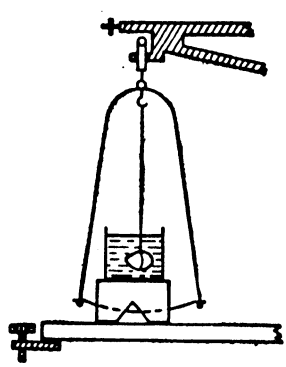


FIG. 173

If Archimedes' principle is correct, we should find that

$$W_0 - W_1 = V\rho$$

Naturally there are many other direct and indirect ways of verifying

the principle. One point worth mentioning in connection with experiments on buoyancy is that the upthrust on a body can also be determined by noting the apparent increase of weight of a beaker of liquid standing on the scale pan when the body is introduced into it and suspended from some fixed support (Fig. 174).

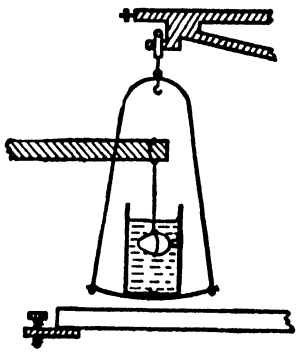


FIG. 174

The Equilibrium of Wholly or Partially Immersed Bodies.—Let a body of volume V and density σ be wholly immersed in a fluid of density ρ (Fig. 175). The mass of the body is $V\sigma$ and its weight is $V\sigma g$ absolute force units. The displaced fluid has a volume V, a mass $V\rho$ and a weight $V\rho g$. Thus the body is acted upon by $V\sigma g$ downwards and $V\rho g$ upwards, and the resultant downward force on it is

$$Vg(\sigma - \rho)$$

If $\sigma < \rho$ (*e.g.* cork or wood in water, most solids in mercury), the above expression is negative, which means that the body experiences a net *upward* force and will be accelerated upwards. If $\sigma > \rho$ (*e.g.* metals in water), the resultant force acts downwards.

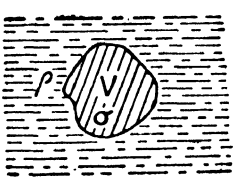


FIG. 175

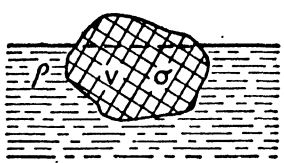


FIG. 176

In each case the magnitude of the force required to maintain the body in equilibrium is equal to $Vg(\sigma - \rho)$. This, for example, would be the tension in a vertical string by which the body is suspended when $\sigma > \rho$, or by which it is prevented from rising when tethered to a point below it in the case of $\sigma < \rho$. The same expression also gives the apparent weight of the body when it is immersed in the fluid.

Fig. 176 shows a body of total volume V and density σ with only part of its volume (v) immersed in a liquid of density ρ. Its weight is $V\sigma g$ and the upthrust is $v\rho g$, so that the resultant downward force is

$$(V\sigma - v\rho)g$$

This may be either positive or negative, *i.e.* directed downwards or upwards, according to whether $V\sigma$ is greater or less than $v\rho$. In either case it is the magnitude of the force necessary to hold the body in equilibrium.

Suppose that the body in Fig. 176 is in equilibrium without the action of any other force than its weight and the upthrust, that is to say, the body is *floating* with a volume v below the liquid surface. The net force on it is zero, so that

$$(V\sigma - v\rho)g = 0$$

or

$$V\sigma = v\rho$$

and the fraction of its total volume which is immersed (v/V) is equal to σ/ρ.

An often-quoted example is the iceberg. The specific gravity of the ice may be taken as 0·92 and that of sea-water as 1·03. We have, therefore,

$$\begin{aligned}\frac{v}{V} &= \frac{\text{density of ice}}{\text{density of sea-water}} \\ &= \frac{\text{specific gravity of ice}}{\text{specific gravity of sea-water}} \\ &= \frac{0{\cdot}92}{1{\cdot}03} \\ &= \frac{8}{9}\ \text{(approx.)}\end{aligned}$$

Thus only about one-ninth of the volume of the iceberg is above the water surface.

When a body floats freely in pure water, the fraction of its volume which is immersed is given by

$$\frac{v}{V} = \frac{\sigma}{\rho_{\text{water}}} = \text{specific gravity of the solid}$$

Example.—*A block of wood of volume* 1000 *cm.*^{3} *floats half immersed in a liquid of specific gravity* 1·2. *Calculate the volume of brass of specific gravity* 8·7 *which must be attached to the wood in order that the combination shall just float in a liquid of specific gravity* 1·1.

Let the densities of the two liquids be denoted by $\rho_{1\cdot2}$ and $\rho_{1\cdot1}$ respectively, and the densities of the wood, brass and pure water by ρ_{wood}, ρ_{brass} and ρ_{water}. Let specific gravities be denoted by S with similar subscripts.

When the block is floating half immersed we have

$$\frac{1}{2} = \frac{v}{V} = \frac{\rho_{\text{wood}}}{\rho_{1\cdot2}} = \frac{S_{\text{wood}}}{1{\cdot}2}$$

$$\therefore\ S_{\text{wood}} = 1{\cdot}2 \times \tfrac{1}{2} = 0{\cdot}6$$

Suppose that x cm.^{3} of brass must be attached in order that the brass and wood together shall just float (completely immersed) in liquid of density $\rho_{1\cdot1}$. The downward force is the sum of the weights of the wood and brass, which is equal to

$$(1000\rho_{\text{wood}} + x\rho_{\text{brass}})g$$

while the upthrust is the weight of displaced liquid, *i.e.*

$$(1000 + x)\rho_{1\cdot1}g$$

For equilibrium these are equal, so that

$$1000\rho_{\text{wood}} + x\rho_{\text{brass}} = (1000 + x)\rho_{1\cdot1}$$

Dividing each term by ρ_{water} in order to convert densities to specific gravities, we obtain

$$1000S_{\text{wood}} + xS_{\text{brass}} = (1000 + x)1\cdot1$$

$$\therefore \quad (1000 \times 0\cdot6) + 8\cdot7x = (1000 + x)1\cdot1$$

$$\therefore \quad 7\cdot6x = 500$$

so that

$$x = 65\cdot8 \text{ cm.}^3$$

Thus 65·8 cm.^{3} of brass must be attached.

Correction of Weighings for Air Buoyancy.—Whenever a common balance is used, both the object weighed and the weights themselves displace some air, and consequently they exert smaller downward forces on the pans of the balance than they would if they were in a vacuum. A correction can be made by adding to the observed weight of the object the weight of air which it displaces (*i.e.* its volume multiplied by the density of air), and subtracting the weight of air displaced by the weights themselves, which is similarly found. When a large light vessel such as a flask containing vapour is weighed during the determination of vapour density, the correction to be added is appreciable and must be taken into account. The correction for the buoyancy of the air on the weights themselves is often small enough to be neglected because of the small volume of the weights. When the object weighed occupies the same volume as tne weights themselves, no correction is necessary because both sides experience the same buoyancy.

Stability of Floating Objects.—We shall consider an example to which the present discussion is particularly relevant, namely the stability of a ship. Fig. 177 (i) shows the section of a ship which is in equilibrium under the action of its weight acting at its centre of gravity G and the upthrust acting through H, which is the centre of gravity of the displaced liquid and is called the **centre of buoyancy.**

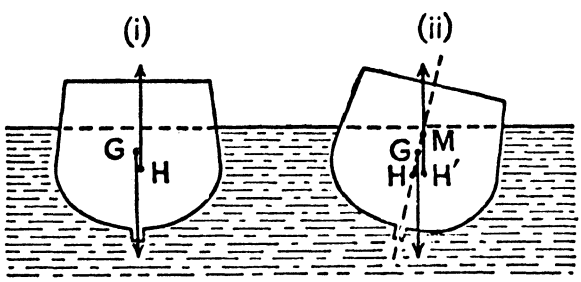


FIG. 177

In Fig. 177 (ii) the ship has been tilted sideways, and although G still occupies the same position relative to the ship, the centre of buoyancy has moved to H′. The ship is stable, however, because the two forces acting on it constitute a couple whose direction is such as to restore it to its upright position. The stability or otherwise of a given ship in a given position can be judged by noting the position of the **metacentre** M, which is the point at which the line of action of the buoyancy (*i.e.* the vertical line through H′) meets the line which passes through G and is

vertical when the ship is in its equilibrium position. The ship is stable when M is above G.

Fig. 178 shows a ship which is unstable and will list further under the action of the couple. The metacentre is below G. The instability is brought about by the high position of G with respect to the ship. It is realized, of course, that G is the position of the resultant centre of gravity of the ship and its cargo and passengers.

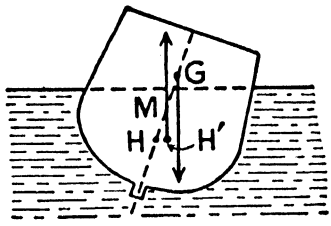


FIG. 178

3. EXPERIMENTAL DETERMINATION OF DENSITY AND SPECIFIC GRAVITY

In this section, which deals mainly with experiments, we shall give the underlying theory of various methods of determining density. Details of experimental procedure must be learnt in the laboratory.

Density.—The direct determination of density involves the separate determination of the mass and volume of a given quantity of substance. For solids the mass is found by weighing, and the volume can be found by measurement of dimensions and computation in the case of regular solids. The volume of an irregular solid may be found by placing in a measuring cylinder some liquid in which the solid sinks but is insoluble and then noting the increase of volume which occurs when the solid is immersed. The density of a liquid can be determined by measuring a convenient volume of it from a pipette or burette into a weighed vessel and determining its mass.

Most determinations carried out in a laboratory lead in the first place to the specific gravity of the substance concerned rather than to its density. This is because the methods involve the use of water in some way or another. In a case of this kind the density of the substance may be calculated in any desired units by multiplying its specific gravity by the density of the water which was used expressed in the units required. For example, if the specific gravity of a solid is found to be 7·2, its density in lb. per cub. ft. is $7{\cdot}2 \times 62{\cdot}5$, *i.e.* 450 lb. per cub. ft., if we take the density of water as 62·5 lb. per cub. ft.

Determinations of specific gravities of both solids and liquids are dealt with in the following sections, the methods being grouped together according to the principles involved.

FIG. 179

Specific Gravity Bottle.—One of these bottles is shown in Fig. 179. The ground glass stopper through which a small hole passes fits well into the neck of the bottle. The simplest determination that can be made with a specific gravity (or "relative density") bottle is the specific gravity of a liquid. The bottle, together with the stopper, is first weighed when it is

empty and dry (W_0), and it is then completely filled with the liquid. When the stopper is firmly inserted, liquid passes out through the hole, and the excess of it on the top of the stopper and the outside of the bottle is removed with blotting-paper. The liquid then completely fills the bottle and the hole in the stopper. The bottle and its contents are weighed (W_1). It is then emptied, completely filled with pure water in the same way and again weighed (W_2). The change from liquid to water may involve the careful washing and drying of the bottle unless the original liquid happens to mix well with water, in which case only washing with water is necessary. The quantities $W_1 - W_0$ and $W_2 - W_0$ are weights of equal volumes of liquid and water—the design of the bottle is meant to ensure the equality of volumes—so that the specific gravity of the liquid is equal to

$$\frac{W_1 - W_0}{W_2 - W_0}$$

The density of the liquid at the temperature of the experiment can be calculated by multiplying the specific gravity by the density of the water used. The experiment is capable of considerable accuracy.

This type of method may also be used to determine the specific gravity of a finely divided solid which does not dissolve in water, *e.g.* sand, lead shot, ball bearings, etc. The procedure is as follows, and is shown diagrammatically in Fig. 180.

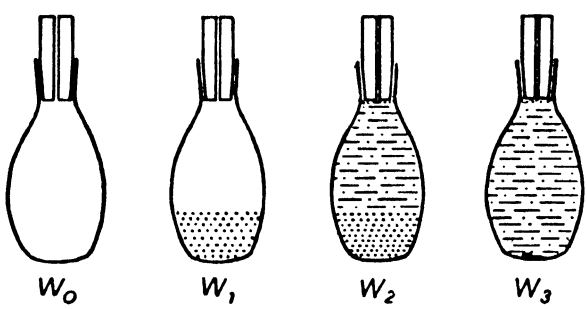


FIG. 180

The bottle is first weighed dry and empty (W_0) and then with a reasonable amount of the solid in it (W_1). With the solid still in the bottle, water is added to fill the bottle in the usual way and the total weight found (W_2). The contents are then emptied out and the bottle is completely filled with water and weighed (W_3). The weight of the solid is $W_1 - W_0$, and the weight of water in the bottle during the third weighing is $W_2 - W_1$. The weight of water completely filling the bottle in the last weighing is $W_3 - W_0$. The weight of water which occupies the same volume as the solid is equal to the weight of water in the last weighing *minus* the weight of water in the third weighing, *i.e.* $W_3 - W_0 - (W_2 - W_1)$. Therefore the specific gravity of the solid is given by

$$\text{Specific gravity of solid} = \frac{\text{Weight of solid}}{\text{Weight of an equal volume of water}}$$

$$= \frac{W_1 - W_0}{W_3 - W_0 - W_2 + W_1}$$

If the solid is soluble in water, *e.g.* a crystalline salt, a liquid in which it is insoluble, say paraffin, may be substituted for the water. The above expression would then give the specific gravity of the solid relative to the liquid used, and to obtain the specific gravity relative to water it is necessary to multiply by the specific gravity of the liquid, the determination of which merely involves one more weighing, *viz.* the bottle completely full of water.

The Hydrostatic Balance.—The apparatus is shown and described on page 191.

Insoluble solid which sinks in water. This is the simplest specific-gravity determination by the hydrostatic-balance method. The solid is suspended from the left-hand arm of the balance by a fine thread or wire and weighed in air (W_1) and then in water contained in a beaker standing on a bridge as described on page 191 (W_2). The upthrust is $W_1 - W_2$, and by Archimedes' principle this is equal to the weight of water which is displaced by (and therefore has an equal volume with) the solid. Thus the specific gravity of the solid is equal to

$$\frac{W_1}{W_1 - W_2} \quad \text{or} \quad \frac{\text{Weight in air}}{\text{"Loss of weight"}}$$

Specific gravity of a liquid. If the solid referred to in the last experiment is now weighed in a liquid in which it sinks but does not dissolve (W_3), the weight of liquid having a volume equal to that of the solid is the upthrust $W_1 - W_3$. Therefore the specific gravity of the liquid is equal to

$$\frac{W_1 - W_3}{W_1 - W_2} \quad \text{or} \quad \frac{\text{Loss of weight in liquid}}{\text{Loss of weight in water}}$$

The hydrostatic balance, like the specific gravity bottle, is also capable of considerable accuracy. It can be used to determine the densities of liquids such as molten metals at high temperatures if the solid (or "sinker") which is weighed when suspended in the hot liquid has a sufficiently high melting-point.

Insoluble solid which floats in water. The solid is first weighed in air (W_1). Then a sinker, such as a piece of lead or a brass weight, of sufficient weight and density is attached to the solid so that the combination sinks in water. The solid and sinker are together weighed in water (W_2), and finally the weight of the sinker alone *in water* is found (W_3). The weight of the solid alone in water is equal to the weight of the combination *minus* that of the sinker, *i.e.* $W_2 - W_3$. This quantity is actually negative in practice as we should expect, because the solid alone in water experiences a resultant *upward* force, so that its apparent weight in water is negative. Nevertheless the upthrust, $W_1 - (W_2 - W_3)$, is equal to the weight of water which the body displaces. (Note that this is actually greater than W_1, the weight of the solid in air.) The specific gravity of

the solid is then

$$\frac{W_1}{W_1-(W_2-W_3)} \quad \text{or} \quad \frac{W_1}{W_1-W_2+W_3}$$

A Practical Exercise.—*Determine separately the specific gravities of two solids such as wax and brass as described above, and then determine the composition of a lump of wax in which some brass is embedded.*

Let the separate specific gravities be S_b and S_w. The composite body of wax and brass is weighed in air (W_1) and then in water (W_2). The weight W_2 can be either positive or negative according to whether the wax contains sufficient brass to make it sink or not. In the latter case W_2 must be determined with the help of an auxiliary sinker.

Let the weights of wax and brass in the composite body be W_w and W_b respectively. Then we have

$$W_w + W_b = W_1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Further, the weight of water equal in volume to the wax is $\frac{W_w}{S_w}$ and to the brass is $\frac{W_b}{S_b}$, and the total weight of water which would be displaced by the composite solid is the sum of these two expressions. But this weight of displaced water is the upthrust on the composite solid when it is weighed in water, *i.e.* $W_1 - W_2$. We therefore have the second equation

$$\frac{W_w}{S_w}+\frac{W_b}{S_b}=W_1-W_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, from equation (1), we substitute $W_1 - W_w$ for W_b in equation (2), we obtain

$$\frac{W_w}{S_w}+\frac{W_1-W_w}{S_b}=W_1-W_2$$

or

$$W_w=\frac{W_1-W_2-\frac{W_1}{S_b}}{\frac{1}{S_w}-\frac{1}{S_b}}$$

Similarly

$$W_b=\frac{\frac{W_1}{S_w}-(W_1-W_2)}{\frac{1}{S_w}-\frac{1}{S_b}}$$

Methods Involving Balancing Columns.—U-*tube.* Two immiscible liquids, (1) and (2), are placed in a U-tube. Suppose that they come to rest as shown in Fig. 181, their junction being at D. It has already been shown (page 166) that if C is on the same horizontal level as D,

$$h_1\rho_1 = h_2\rho_2$$

where ρ_1 and ρ_2 are the respective densities of the liquids.

Suppose liquid (1) is water. Then the specific gravity of liquid (2) (S_2) is given by

$$S_2=\frac{\rho_2}{\rho_1}=\frac{h_1}{h_2}$$

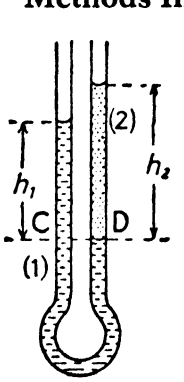


FIG. 181

Thus in the case shown in the figure the specific gravity of (2) is less than unity.

Hare's apparatus enables the principle of balancing columns to be used for the determination of the specific gravity of a liquid that mixes with water. It consists of an inverted U-tube with a short tube A attached as shown in Fig. 182. A piece of rubber tube which can be closed with a clip is fixed to the small tube. The open ends of the U-tube are placed under the liquids (1) and (2), and the liquids are drawn up the vertical tubes by sucking with the mouth at B. The clip is then closed so that the liquid columns remain stationary, and the heights h_1 and h_2 are read on vertical scales. The reduced pressure in the air above each column is the same on both sides. The pressure is also the same (atmospheric) in each column at the level of the liquid in its beaker. These levels are, of course, not necessarily the same on each side. The two columns of height h_1 and h_2 therefore both represent the same change of pressure, namely from atmospheric pressure to the pressure in the air at the top of the U-tube. We have therefore

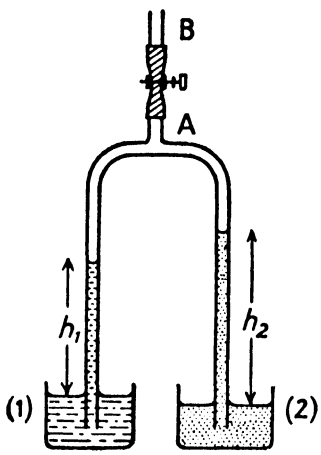


FIG. 182

$$h_1\rho_1 = h_2\rho_2$$

or

$$\frac{\rho_2}{\rho_1} = \frac{h_1}{h_2}$$

If liquid (1) is water, the above ratio is equal to the specific gravity of liquid (2).

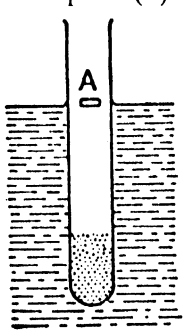


FIG. 183

Hydrometers are instruments for determining specific gravities, and we shall explain the principles of two types.

Constant-immersion hydrometer. A very simple form of constant-immersion hydrometer can be made as follows. Sufficient sand is put into a fairly long test-tube to make it float upright in water (Fig. 183). A small slip of gummed paper A is stuck inside the test-tube at about the level of the water surface, and then the amount of sand in the test-tube is adjusted until the paper is as accurately as possible on a level with the surface when the hydrometer is floating freely. The tube is then removed from the water, dried and weighed with the sand still in it (W_1). Next the test-tube is floated in a liquid whose specific gravity is to be determined, and the quantity of sand in it is adjusted until the paper slip is level with the surface. The test-tube is again dried and weighed (W_2). The same volume of liquid is displaced by the hydrometer in each case,

and for equilibrium the weight of this volume of liquid is equal to the total weight of the hydrometer. Therefore W_1 and W_2 are weights of equal volumes of water and liquid respectively, and the specific gravity of the liquid is equal to W_2/W_1.

Constant-weight or common hydrometer. This consists of a bulb and stem (usually of glass) of the shape shown in Fig. 184. Very frequently, as shown in the drawing, an additional bulb containing some mercury or lead shot is attached below the main bulb in order to lower the centre of gravity and ensure that the hydrometer remains upright when it floats. When the instrument floats freely in a liquid, its depth of immersion depends upon the specific gravity of the liquid. When once the scale (frequently marked on paper inside the hollow stem) is properly fixed, the reading of the liquid surface against it gives the specific gravity either directly or by a simple calculation. It will be realized at once that the higher readings of specific gravity are at the bottom of the stem. The reading in water is marked unity, that is 1·00 or 1·000 according to the degree of accuracy of the instrument.

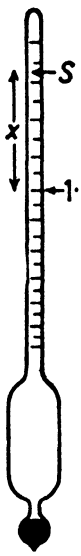


FIG. 184

Suppose that in a liquid of density ρ and specific gravity S the hydrometer floats with the liquid surface at a distance x above the unity mark. Let the volume of the hydrometer up to the unity mark be V and the (uniform) cross-section of the stem be A. The volume of the bulb and stem up to the division marked S will therefore be $V+Ax$.

Now since the hydrometer always has the same weight, it always displaces the same weight of any liquid in which it is floating freely (Archimedes' principle). In the case of water this weight is $V\rho_{\text{water}}$, and in the case of a liquid of density ρ it is $(V+Ax)\rho$. Therefore

$$V\rho_{\text{water}} = (V+Ax)\rho$$

or

$$\frac{\rho_{\text{water}}}{\rho} = \frac{V+Ax}{V}$$

so that

$$\frac{1}{S} = 1 + \frac{A}{V}\cdot x \quad . \quad . \quad (3)$$

since ρ/ρ_{water} is equal to S.

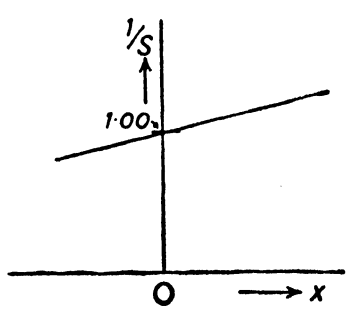


FIG. 185

Thus equal increments of x, *i.e.* equal distances along the stem, correspond to equal changes of $1/S$ (*not* of S). A graph of $1/S$ against x is a straight line as shown in Fig. 185. The value of x may be positive or negative according to whether the surface of the liquid is above or below the unity mark. The intercept on the $1/S$ axis is unity, while the slope of the line is A/V.

The student should be able to deduce that the change of x caused by a given change of S is small when S is large. This means that the graduations (which indicate equal changes of S) are more crowded near the bottom of the stem. The departure from uniformity of scale is fairly small, however, unless the stem is wide so that the range of S is large.

***Example* (*i*).**—*A common hydrometer reads* 1·00 *when placed in water contained in a measuring cylinder, and the rise of the water surface shows that it has displaced* 20·0 *cm.*^{3}. *If the distance between the* 1·00 *and* 0·99 *marks on the hydrometer is* 0·5 *cm., calculate the area of cross-section of the stem and the distance between the* 0·90 *and* 0·89 *marks.*

If the density of the liquid of specific gravity 0·99 is ρ, and the area of cross-section of the stem is A, we have, since the hydrometer always displaces the same weight of liquid,

$$20{\cdot}0 \times \rho_{\text{water}} = (20 + 0{\cdot}5A)\rho$$

i.e.
$$\frac{\rho_{\text{water}}}{\rho} = \frac{20 + 0{\cdot}5A}{20}$$

or
$$\frac{1}{0{\cdot}99} = \frac{20 + 0{\cdot}5A}{20}$$

whence
$$0{\cdot}5A = \frac{20(1 - 0{\cdot}99)}{0{\cdot}99}$$

and
$$A = 0{\cdot}404 \text{ cm.}^2 \text{ (approx.)}$$

If the distance of the 0·90 and 0·89 marks from the 1·00 mark for water are x_1 and x_2 respectively, we have

$$\frac{1}{0{\cdot}90} = \frac{20 + 0{\cdot}404x_1}{20}$$

and
$$\frac{1}{0{\cdot}89} = \frac{20 + 0{\cdot}404x_2}{20}$$

whence
$$x_1 = \frac{20(1 - 0{\cdot}90)}{0{\cdot}404 \times 0{\cdot}9}$$
$$= 5{\cdot}50 \text{ cm.}$$

and
$$x_2 = \frac{20(1 - 0{\cdot}89)}{0{\cdot}404 \times 0{\cdot}89}$$
$$= 6{\cdot}13 \text{ cm. (approx.)}$$

Thus the distance between the 0·90 and 0·89 marks is 6·13 − 5·50 or 0·63 cm. It will be noticed that graduations representing the same difference of specific gravity (*viz.* 0·01) are further apart when the specific gravity is small.

***Example* (*ii*).**—*Calculate the weight of aluminium (specific gravity* 2·7*) which must be attached to the bottom of a hydrometer weighing* 24 *gm. in order to make it read* 0·8 *when floating in a liquid of specific gravity* 0·9.

Let the weight of aluminium required be W gm., so that the total weight of the loaded instrument is $24 + W$ gm. This downward force must be balanced by the weight of the displaced liquid of specific gravity 0·9 when the loaded hydrometer is floating freely in it. The weight of liquid of specific gravity 0·8 which is displaced by the hydrometer alone when it is floating in such a liquid is, of course, equal to the weight of the hydrometer, *i.e.* 24 gm. But when it is loaded the hydrometer

sinks to the 0·8 mark in a liquid of specific gravity 0·9, so that the weight of *this* liquid which is displaced by the immersed portion of the hydrometer itself is *not* 24 gm. but the weight of a quantity of the denser liquid which has the same volume as 24 gm. of liquid of specific gravity 0·8. This weight is $\frac{24 \times 0{\cdot}9}{0{\cdot}8}$, since, for equal volumes, substances have weights proportional to their specific gravities. Again, the weight of liquid of specific gravity 0·9 which is displaced by the aluminium is the weight of this liquid which has the same volume as W gm. of aluminium, *i.e.* $W \times \frac{0{\cdot}9}{2{\cdot}7}$.

For equilibrium the weight of liquid displaced is equal to the weight of the hydrometer and aluminium. Therefore

$$\frac{24 \times 0{\cdot}9}{0{\cdot}8} + \left(W \times \frac{0{\cdot}9}{2{\cdot}7}\right) = 24 + W$$

or

$$24\left(\frac{0{\cdot}9}{0{\cdot}8} - 1\right) = W\left(1 - \frac{0{\cdot}9}{2{\cdot}7}\right)$$

Therefore

$$W = \frac{24 \times \frac{0{\cdot}1}{0{\cdot}8}}{\frac{2{\cdot}7 - 0{\cdot}9}{2{\cdot}7}}$$

$$= \frac{3 \times 3}{2}$$

$$= 4{\cdot}5 \text{ gm.}$$

EXAMPLES XIII

Certain examples appropriate to this chapter but involving the use of Boyle's law are included in Examples XV.

1. Describe a common type of hydrometer and explain its action. The stem of a hydrometer has a diameter of 4 mm., and the marks on it corresponding to water and a liquid of specific gravity 1·25 are 5 cm. apart. Find (*a*) the weight of the hydrometer, (*b*) the position it assumes in a liquid of specific gravity 1·10. (L.M.)

2. State Archimedes' principle, and describe an experiment to verify it.
The specific gravities of cork, glass and paraffin oil are 0·15, 2·5 and 0·80 respectively. What mass of glass must be attached to a piece of cork of mass 4 gm. so that both just sink in paraffin oil? (L.M.)

3. The fabric and cage of a balloon of 50,000 cub. ft. capacity weigh 1 ton. What extra load can be lifted if the balloon is filled with (*a*) hydrogen, (*b*) helium? (Assume that the densities of air, helium and hydrogen are 0·080, 0·011 and 0·0055 lb. per cub. ft. respectively.) (L.M.)

4. How would you compare the densities of two liquids using a test-tube and some sand?
A uniform tube 20 cm. long with a bulb containing mercury at its lower end floats vertically in water with 10 cm. of its length above the surface. On adding 2·1 gm. of mercury an additional 5 cm. of the tube sinks below the surface, and the total weight is then 10·3 gm. Find the external volume of the bulb. (L.M.)

5. State Archimedes' principle and describe an experiment to illustrate it.
When a hole drilled in a wood sphere 4·2 cm. in diameter is filled with lead, the sphere floats just submerged in a liquid of specific gravity 1·03. If the specific gravity of the wood is 0·63 and of the lead 11·4, find the volume of the hole. (L.M.)

6. Describe the constant-weight hydrometer and explain the principle upon which its use depends.

If the distance between the 0·70 and 1·00 graduations on the stem of a constant-weight hydrometer is 5 cm., what is the ratio of the volume below the 0·70 graduation to the volume of 1 cm. of the stem? (L.M.)

7. How would you show experimentally that the pressure exerted at a point in a liquid is proportional to the depth below the free surface?

A U-tube with vertical limbs has a cross-section of 1 sq. cm. and is partly filled with mercury. 20 c.c. of water are poured into one limb and 10 c.c. of oil of specific gravity 0·8 into the other. Find the difference in the heights of the mercury in the two limbs. (Assume that the specific gravity of mercury is 13·6.) (L.M.)

8. Show how to establish the truth of the principle of Archimedes, either theoretically or experimentally.

A ship of displacement 25,000 tons, having a sectional area of 42,000 sq. ft. at the water line, sinks to a certain depth in fresh water and rides 7 in. higher in sea-water Find the density of the latter. (1 cub. ft. of water weighs 62½ lb.) (L.M.)

9. State Archimedes' principle and describe an experiment to verify it.

A beaker of water rests on one pan of a balance and is counterpoised by weights in the other pan. When a piece of metal is suspended from a stand to hang in the water without touching the beaker, the counterpoising weights must be increased by 5·4 gm., and when the metal is allowed to rest on the bottom of the beaker a further increase of 32·94 gm. is necessary to restore the balance. Explain these results and deduce the specific gravity of the metal. (L.M.)

10. What do you understand by the *pressure* at a point in a liquid?

A cylinder is sufficiently long to ensure that the upper end is always above the surface of salt water (sp. gr. 1·02) in which it is immersed vertically. The lower end of the cylinder is closed by a flat brass plate which has the same diameter as the cylinder. If the plate is 1 cm. thick and the sp. gr. of brass is 8·6, how far below the surface of the salt water must the plate be immersed in order that the fluid pressure will just hold the plate against the cylinder? (L.M.)

11. State Archimedes' principle, and describe an experiment to demonstrate it.

A body weighs 100 gm. in air and 20 gm. in water. It floats on a certain liquid with one-eighth of its volume above the surface. Find the specific gravity of the liquid. (L.I.)

12. State the principle of Archimedes and use it to derive an expression for the resultant force experienced by a body of weight W and density σ when it is totally immersed in a fluid of density ρ.

A solid weighs 237·5 gm. in air and 12·5 gm. when totally immersed in a liquid of specific gravity 0·9. Calculate (*a*) the specific gravity of the solid, (*b*) the specific gravity of a liquid in which the solid would float with one-fifth of its volume exposed above the liquid surface. (L.I.)

13. State the principle of Archimedes and describe an experiment which illustrates it.

A solid is composed of v c.c. of metal of specific gravity 8·0 surrounded by V c.c. of wax of specific gravity 0·8. The solid floats in a liquid of specific gravity 1·2 with one-fifth of its total volume exposed above the liquid surface. Calculate the ratio of V to v. (L.I.)

14. A closed circular cylinder is completely immersed in water with its axis inclined at an angle θ to the vertical. Calculate the resultant liquid thrust on the curved surface of the cylinder in terms of the weight W of water displaced by the cylinder. (L.I.)

15. A solid, lighter than water, is held completely immersed in a vessel containing 100 c.c. of water by means of a string fastened to a point in the base of the vessel, and the tension in the string is 1·6 gm. wt. When 60 c.c. of a second liquid of specific gravity 1·2 are added and thoroughly mixed with the water (with no change of volume) the tension in the string is 2·2 gm. wt. Find the specific gravity and weight of the solid. (L.I.)

16. A piece of material A weighs 30·4 gm. in air and 26·4 gm. in water; a piece of material B weighs 67·2 gm. in air and 61·4 gm. in water. A piece of an alloy formed from A and B weighs 54·6 gm. in air and 48·6 gm. in water. Find the volume of each material in the piece of alloy, assuming there to have been no diminution of volume when the alloy was made. (L.I.)

17. State the principle of Archimedes. How would you verify it for a body which (*a*) sinks, (*b*) floats in a given liquid?

A lead cylinder 20 cm. long floats upright in mercury. Water is poured on the mercury so that the cylinder is entirely submerged. Find the change in the position of the cylinder relative to the mercury surface due to the addition of the water. (Sp. gr. of mercury = 13·6; sp. gr. of lead = 11·3.) (L.M.)

Chapter XIV

FLUIDS IN MOTION

1. GENERAL PRINCIPLES

Stream-lines.—Whenever a fluid is flowing it is possible, in imagination at any rate, to select any one particle and to trace its path. If each successive particle passing through any chosen fixed point in the fluid subsequently follows the same path, the flow is said to be **orderly** or **stream-line flow** and the path is a **stream-line.** For example, when a fluid flows steadily past a fixed solid sphere or cylinder the stream-lines are as shown in Fig. 186. They can be demonstrated by introducing a fine jet of coloured liquid (or smoke in the case of a flowing gas) at points like A and C. The liquid or smoke is then drawn out into a filament which marks the stream-line.

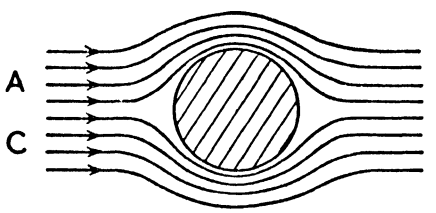


FIG. 186

There is another possible type of flow, known as **disorderly** or **turbulent** flow, in which the fluid is churned up as it moves, so that the velocity of the particles passing through any given point varies with time both in magnitude and direction in an irregular way.

Tubes of Flow.—In a case of orderly flow as described above, it is possible to regard the fluid as streaming through imaginary tubes. The fluid between any two stream-lines remains between them as it flows along in just the same way as it would if it were contained in a tube whose walls coincided with these lines. Such an imaginary tube is called a **tube of flow,** and the usefulness of this conception lies in the fact that we can apply the laws of motion to the fluid passing through any tube without having to take into account the motion of the rest of the fluid. We can say immediately that where stream-lines are close together, *i.e.* where a tube of flow becomes narrow, the velocity of the fluid must be high, just as water passing through a pipe flows more rapidly when it reaches a constriction.

Bernoulli's Equation.—Suppose that a fluid is passing through a non-uniform tube of flow such as that illustrated in Fig. 187. Let its linear speed change from v_1 to v_2 as the fluid travels from X to Y, at which places the areas of cross-section are A_1 and A_2 respectively and the pressures are p_1 and p_2. We must suppose that pressure and velocity are

uniform over any particular cross-section, and in general this means that the tube must always be supposed narrow. For this reason the equation we are about to establish applies in reality to a tube of flow which is so narrow as to be, in fact, a single stream-line.

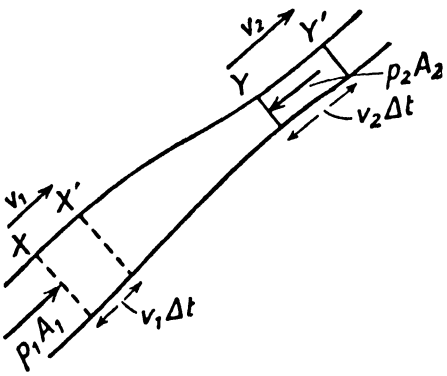


FIG. 187

The flow of the fluid which lies between X and Y is governed by the action upon it of the fluid in contact with its ends. At X this action is a force p_1A_1 in the direction of flow, and at Y it is a force p_2A_2 in the direction opposite to the flow at this place.

After a short time interval Δt, one end of the column of fluid XY has moved from X to X′ through a distance $v_1\,\Delta t$, so that at X the work done *on* the fluid XY by the force p_1A_1 is equal to

$$p_1A_1v_1\Delta t$$

Similarly, at the other end, the work done against p_2A_2 *by* the fluid XY in the same time Δt is

$$p_2A_2v_2\Delta t$$

We shall suppose that the fluid is incompressible, which means that the volume of the fluid which was originally between X and Y remains constant as it flows along the tube. Thus the volume swept out in a given time by each end of the thread or column of fluid must be the same. Therefore we have

$$\text{Volume between X and X}' = \text{Volume between Y and Y}'$$

or

$$A_1v_1\Delta t = A_2v_2\Delta t$$

The work done by the fluid at the end Y can therefore be written as

$$p_2A_1v_1\Delta t$$

and the total work done *on* XY is

$$(p_1 - p_2)A_1v_1\Delta t$$

We now consider the change of mechanical energy which occurs when the fluid moves from XY to X′Y′. In the new position the potential energy of the thread of fluid XY has increased by the difference between the P.E. of the fluid between Y and Y′ and the P.E. of the fluid which was originally between X and X′. The masses of these two portions of fluid are equal, since the mass crossing every cross-section of the tube in a given time must be the same everywhere. Each has a mass of $\rho A_1v_1\Delta t$, and if h_1 and h_2 are the heights of X and Y above some arbitrary horizontal

level, the increase of P.E. in time Δt is

$$(h_2 - h_1)g\rho A_1 v_1 \Delta t$$

Strictly speaking, h_1 and h_2 should refer to the heights of the centres of gravity of the two quantities of liquids considered, but XX′ and YY′ are supposed to be small, since Δt is small, so that the heights of the centres of gravity can be regarded as the heights of X and Y.

In a similar manner we deduce that the increase of kinetic energy of the thread as it moves from XY to X′Y′ is equal to the K.E. of the fluid which appears in YY′ *minus* the K.E. of that which disappears from XX′. Thus the gain of K.E. is

$$\tfrac{1}{2}\rho A_1 v_1 \Delta t.(v_2^2 - v_1^2)$$

We now assume that the whole of the work done on the fluid is converted into potential and kinetic energy. In actual fact this is never quite true, because all fluids possess viscosity, which means that in order to maintain a state of flow it is necessary to do work against the viscous forces which come into play when the various parts of the fluid are in relative motion. The viscosity of liquids like treacle and glycerine is very large. The work done against the viscous forces appears as heat, and, in reality, the gain of mechanical energy of a given portion of fluid is equal to the work done on it *minus* the heat developed. Ignoring this dissipation of energy, we equate the work done to the gain of mechanical energy and write

$$(p_1 - p_2)A_1 v_1 \Delta t = (h_2 - h_1)g\rho A_1 v_1 \Delta t + \tfrac{1}{2}\rho A_1 v_1 \Delta t.(v_2^2 - v_1^2)$$

or

$$p_1 + h_1 g\rho + \tfrac{1}{2}\rho v_1^2 = p_2 + h_2 g\rho + \tfrac{1}{2}\rho v_2^2 \quad . \quad . \quad . \quad (1)$$

This is known as Bernoulli's equation, and it is usually stated by saying that for all the points on a given stream-line the quantity $p + hg\rho + \frac{1}{2}\rho v^2$ is a constant. Each of the three terms of this expression represents a quantity of energy possessed by unit volume of the fluid. Thus p, the so-called "pressure energy," is the work which unit volume could perform if it were allowed to move to a place where the surrounding pressure is zero, $hg\rho$ is the P.E. of unit volume (mass ρ) and is the work it could do if it fell to the level above which h is measured, while $\frac{1}{2}\rho v^2$ is the K.E. of unit volume, *i.e.* the work it could do if it were brought to rest. Bernoulli's equation expresses the constancy of the sum of these three types of energy.

If the fluid is actually at rest, so that

$$v_1 = v_2 = 0$$

the equation reduces to the familiar relationship between pressure and vertical height in a fluid, *viz.*

$$p_1 - p_2 = (h_2 - h_1)g\rho$$

2. APPLICATION OF BERNOULLI'S EQUATION

Efflux of Liquid from a Wide Vessel.—Fig. 188 shows a wide vessel with a hole in its side just above the bottom. Liquid contained in the vessel will flow out of the aperture, and AB and CD are typical streamlines. They are the paths which particles initially in the surface would traverse if sufficient liquid were added to maintain the level of the surface constant.

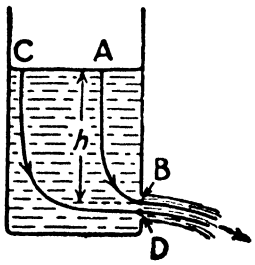


FIG. 188

We apply Bernoulli's equation to the points A and B.

At A: p_1 = atmospheric pressure = P.

$h_1 = h$, if the arbitrary horizontal level from which heights are measured passes through B.

$v_1 = 0$, if we suppose that the rate of fall of the surface is negligible compared with the speed at B on account of the large area of the vessel compared with that of the aperture.

At B: $p_2 = P$, since p_2 in equation (1) signifies the pressure of the fluid into which the contents of the tube of flow are discharging, and this is atmospheric.

$h_2 = 0$.

$v_2 = v$, say, the velocity of the liquid at B.

Substituting in Bernoulli's equation, we have

$$P + hg\rho + 0 = P + 0 + \tfrac{1}{2}\rho v^2$$

so that $$hg\rho = \tfrac{1}{2}\rho v^2$$

or $$v^2 = 2gh \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the potential energy which unit volume of liquid (mass ρ) loses by falling from the surface to a depth h is converted into kinetic energy, and the velocity can be calculated according to this principle. It is the same velocity as would be acquired by free fall. Actually the velocity acquired is always smaller than $\sqrt{2gh}$ owing to the viscosity of the liquid which brings about the conversion of potential energy into heat. This effect is most noticeable, of course, when the liquid is very viscous—treacle or heavy oil, for example. Equation (2) is known as Torricelli's theorem.

When the aperture is small the velocity at D differs only slightly from that at B, because the depth of the two points below the surface is practically the same. In fact, v can be regarded as the linear velocity of the water in the jet. The conservation of energy and equation (2) suggest that if a nozzle is attached to the orifice so as to send a jet upwards, it would reach the same level as the surface inside the vessel. Viscosity prevents the liquid from reaching this level in practice.

The jet of outflowing liquid does not become a cylinder having the

diameter of the orifice immediately it enters the air but has a shape depicted in Fig. 189. Some distance beyond the orifice the jet reaches its minimum cross-section (about 0·62 of the cross-section of the orifice). At this place the stream-lines become parallel to each other and the velocity becomes uniform. This region is called the **vena contracta** or "contracted vein."

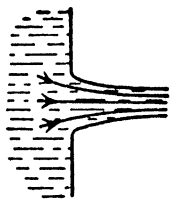


FIG. 189

The Pitot Tube.—It should be noted that the quantity p in Bernoulli's equation—the so-called "static pressure"—is the pressure which actually exists in the moving liquid. This means that in order to determine p it is necessary to measure the force per unit area on a surface, the presence of which in the liquid does not interfere with the prevailing conditions of motion. A rigid surface placed in the liquid would modify the direction and speed of motion of the liquid in its neighbourhood unless it is either (*a*) moving with the liquid or (*b*) placed parallel to the direction of the stream-lines. The walls of the tube through which the liquid is flowing are subjected to the static pressure p, since they fulfil condition (*b*), and the simplest way of measuring p at any place is by means of a manometer such as C (Fig. 190 (i)) which is connected to a hole in the tube wall.

If a stationary surface, however small, is placed perpendicular to the direction of flow, the liquid impingeing on it is brought to rest, and a stagnant region is created in front of the surface. Consequently, if the pressure in the liquid is p at a place where its speed v is not influenced by the presence of the surface, the pressure in the region in front of the surface is $p + \frac{1}{2}\rho v^2$, since the speed here is zero. This expression therefore represents the force per unit area on the front of the surface. The conditions necessary for the determination of this total pressure are realized by an arrangement such as the tube D in Fig. 190 (i). The orifice of D is perpendicular to the liquid flow, and the liquid in it is stationary when steady conditions have been attained. It follows that the difference between the two pressures corresponding to the heights of the columns in C and D is the "dynamic pressure" $\frac{1}{2}\rho v^2$. Thus v, the linear speed of flow, can be calculated from observations of the two heights. This is the principle of the Pitot tube, which, in one of its practical forms shown in Fig. 190 (ii), consists of a tube placed parallel to the direction of flow with its orifice facing the oncoming liquid. The pressure at this surface ($p + \frac{1}{2}\rho v^2$) is registered by a manometer connected at D. An opening in the side of the Pitot tube is independently joined to another manometer at C, which therefore measures the static pressure p.

Flow of Liquid through a Constricted Tube.—Fig. 190 (iii) represents a horizontal tube which is constricted at one place. The vertical side tubes allow the pressure (above atmospheric) in the liquid flowing through the main tube to be registered by the height to which the liquid itself rises

in them. Comparing conditions at X and Y by Bernoulli's equation, we have

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

where p_1 and p_2 are the pressures, and v_1 and v_2 the liquid velocities at the

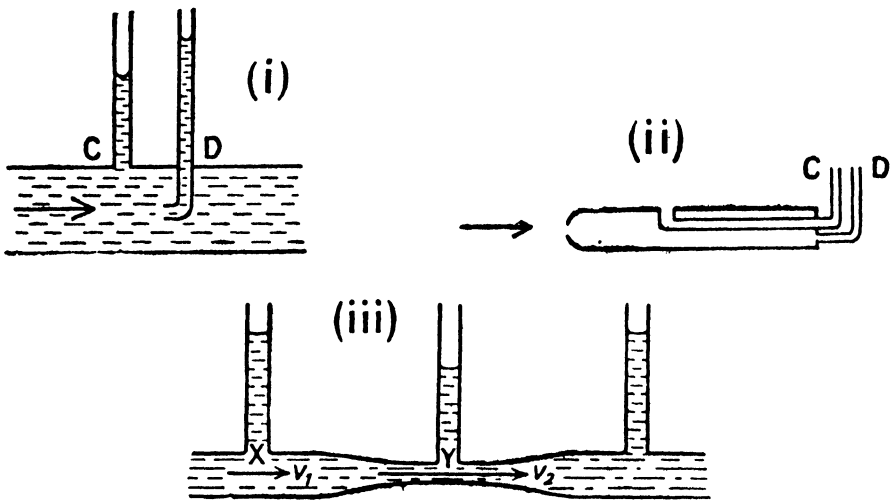


FIG. 190

two places. The horizontal levels of X and Y are supposed to be the same, so that h_1 and h_2 in the general equation are equal.

If A_1 and A_2 are the cross-sectional areas at X and Y respectively, then

$$A_1 v_1 = A_2 v_2$$

since the same volume of liquid must pass every section of the tube per second. Therefore

$$\begin{aligned} p_1 - p_2 &= \tfrac{1}{2}\rho(v_2^2 - v_1^2) \\ &= \tfrac{1}{2}\rho v_1^2\left(\frac{v_2^2}{v_1^2} - 1\right) \\ &= \tfrac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2} - 1\right) \\ &= \tfrac{1}{2}\rho v_1^2 . \frac{A_1^2 - A_2^2}{A_2^2} \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

Thus

$$p_1 > p_2$$

when

$$A_1 > A_2$$

which means that when the liquid flows through the constriction, the pressure is reduced on account of its increased velocity. We can say, therefore, that apart from variations of pressure from point to point due to

differences of level, the pressure in a moving fluid is small at places where the velocity is large. This is true of both liquids and gases, although the last equation applies only to an incompressible fluid and therefore not to gases.

The **Venturi meter** for measuring the rate of flow of water through a pipe uses the principle just described. The difference of pressure between two points, one in a normal part of the pipe itself and the other in a constriction, is measured, and if A_1 and A_2 are known, the velocity of flow (v_1) can be calculated by equation (3).

In Fig. 191 A is a tube fitted with a flange B. When air is blown through A, and B is placed close up against a flat surface C (say the top of a match-box), C and B *adhere* to each other instead of being separated by the current of air as might at first be supposed. At the perimeter of B the air current enters the atmosphere and its pressure is atmospheric. The velocity of the air near the centre is larger than it is near the edge, so that, on the whole, the pressure of the air between B and C is less than atmospheric. Consequently the two surfaces are forced together by the action of the atmosphere on their outer faces.

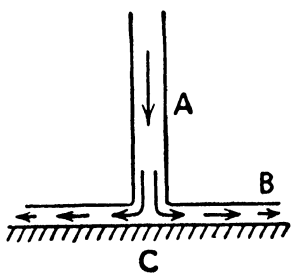


FIG. 191

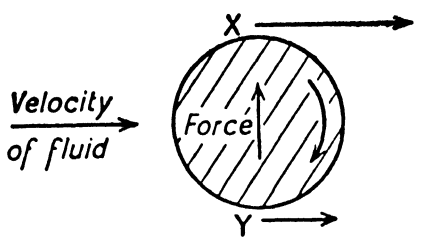


FIG. 192

Force on a Rotating Body in a Stream of Fluid.—Suppose a cylinder (Fig. 192) is being rotated in a current of fluid. The cylinder drags a shell of fluid round with it, and the combination of the velocity which is due to this effect with the velocity of the stream itself gives a high fluid velocity at X and a low one at Y. Therefore the pressure in the fluid is greater at Y than at X, and the cylinder experiences an upward force in the direction shown. It is quite possible to produce this effect by rotating a cylinder in a current of air, and a **rotor ship** was once constructed in which a vertical rotating cylinder took the place of a sail. A similar thing occurs when a cricket, tennis or golf ball is spinning as it moves through the air. Here the air is stationary and the ball moves bodily. If the circle in Fig. 192 represents the spinning ball and we suppose it to be moving to the left through still air, it suffers an upward force. This can therefore be brought about by "back-spinning" or "chopping" the ball when it is struck. Giving the ball a spin in tne opposite direction ("top-spin") produces a downward force, which brings the ball to the ground sooner than normally.

Water-Pump.—The low pressure in and around a fast jet of water is used in the water- or filter-pump for producing a partial vacuum (Fig. 193).

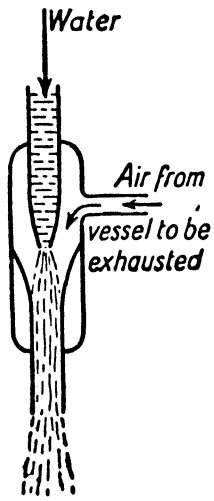


FIG. 193

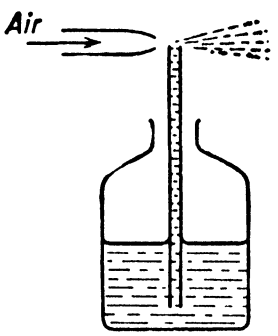


FIG. 194

The "Atomizer."—A strong jet of air blown across the top of a tube standing in liquid will, on account of reduced pressure, cause the liquid to rise and flow out at the top of the tube. Here it is broken up into a cloud of droplets. This is the principle of the **scent-spray** and **"atomizer"** (Fig. 194).

The Rate of Working of a Pump.—When a pump is used to keep a liquid in motion, it does so by creating the pressure difference which is necessary to drive the liquid through the pipes against the resistance due to viscosity and also against the force of gravity if the level of the liquid is being raised. Suppose that the liquid pressures on the two sides of the piston differ by an average amount p when the pump is working. The piston therefore exerts a net force pA on the liquid, where A is the area of the piston which is in contact with the liquid; and when the piston moves with a speed v the work done in unit time is pAv. But Av is the volume swept out by the piston in unit time and is therefore equal to V, the volume of liquid pumped in unit time. Thus the average rate of working of the pump is pV.

During part of the forward stroke the piston is giving an acceleration to the liquid, so that the pressure difference against which it is then working is greater than when it is moving with constant speed, because at constant speed the piston has to overcome only the resistance due to viscosity and pressure head. Thus if p' is the pressure difference at constant speed, the expression $p'V$ represents the rate at which the pump is working against viscous and gravitational forces only, while the rate of working pV includes also the work done in giving kinetic energy to the liquid, since p is the average value of the pressure taken over the whole

stroke. The rate of working necessary to give the liquid in the cylinder a linear speed v is $\frac{1}{2}\rho v^2 V$, ρV being the mass of liquid flowing in unit time. This expression added to $p'V$ is equal to pV. It should be realized that if the static pressure difference in the pump is deduced from pressure observations made at points in the system elsewhere than in the cylinder, it is necessary to calculate the corresponding pressures in the cylinder by using Bernoulli's equation in order to allow for the dependence of pressure on the cross-section of the stream (equation (3)).

EXAMPLES XIV

1. Derive an expression for the work done in moving a volume V of a liquid from rest through a uniform tube against an average resisting pressure P when the final velocity of the liquid is v.

Explain how the pressure and velocity of a fluid vary as it flows through a constriction in a tube, and show how your explanation accounts for the mode of action of the Venturi meter. (L.Med.)

2. Calculate the mean rate of working in watts of a heart which discharges 75 c.c. of blood per beat against an average pressure of 12 cm. of mercury, the pulse frequency being 72 per minute. (L.Med.)

3. Explain the meanings of the terms *static* and *dynamic pressures* in a flowing liquid. Explain how these pressures can be measured and how a knowledge of their values enables the speed of flow to be determined.

4. Describe and explain two ways in which the rate of flow of liquid through a pipe can be determined by means of pressure measurements.

Calculate the volume of water flowing per sec. through a horizontal pipe given that there is a pressure difference of 1 cm. of water between two points at which the radii are 0·5 cm. and 0·3 cm. respectively.

5. What volume of water must flow per sec. through a pipe of diameter 1·2 cm. in order that the dynamic pressure shall be equivalent to 0·25 cm. of water? How may this pressure be measured?

6. The diameter of a pipe is 2 cm. at a certain place and 1 cm. at a place which is 40 cm. vertically lower. Calculate the difference of pressure between the two points when water is flowing through the pipe at a rate of 30 c.c. per sec.

7. A man using a stirrup pump works at $\frac{1}{10}$ H.P. and thereby forces water at 2 gal. a minute through the nozzle, whose cross-sectional area is 0·01 sq. in., and which is held at a height of 12 ft. above the level from which the water is being pumped. Find the efficiency of the pump to two significant figures. (Assume that 1 cub. ft. of water contains $6\frac{1}{4}$ gal. and weighs 62·5 lb.) (O.H.S.)

Chapter XV

ELASTICITY

1. INTRODUCTION

When several forces are applied to a body they produce, in general, a change in the shape or size of the body, unless they happen to have a common point of application. We shall deal only with forces which are in equilibrium with each other so that they produce no acceleration. Two simple examples are shown in Fig. 195. In (i) a rod is stretched by two equal and opposite forces, and in (ii) it is bent by two equal and opposite couples. The extent of the deformations produced in a body by given forces is dependent upon the dimensions and material of the body. In order to correlate the changes of shape and size with the forces which cause them we use quantities known as **strain** and **stress.**

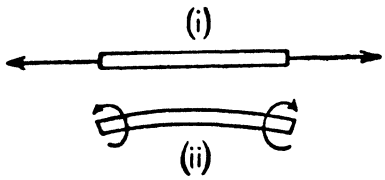


FIG. 195

Strain expresses the magnitude of the change of size or shape of the body and is always expressed as a *fractional change*, *e.g.* change of length per unit length or change of volume per unit volume. Strain has no units since it is a ratio.

Stress is a quantity which expresses the magnitude of the forces which exist in a strained material. Consider a plane surface of area A described in a strained body, and suppose that the material on one side of A exerts a force F upon the material on the other side (Fig. 196). The force F can, in general, be resolved into two rectangular components, one (F_1) perpendicular to A and the other (F_2) in the plane of A. Then

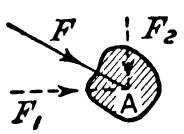


FIG. 196

$$\text{Average normal stress} = \frac{F_1}{A}$$

$$\text{Average tangential stress} = \frac{F_2}{A}$$

The word "average" must be used because the stresses might vary from point to point over the area unless this is made indefinitely small.

It will be realized that the pressure in a fluid is a particular type of stress. There is no tangential stress in a fluid at rest, by definition of the nature of a fluid, so that pressure is a purely normal stress (page 157). The units of stress are dynes cm.$^{-2}$, lb. wt. in.$^{-2}$, etc.

The stress across any surface can always be related to the external forces acting on the body by considering the equilibrium of a portion of the body. As a very simple example, consider a rod BC of uniform cross-section A (Fig. 197) stretched by two equal and opposite forces F. Imagine a transverse plane to be described at D, and consider the equilibrium of DC. The forces acting on it are the external force F at C and the force which the material on the left of D exerts upon that on the right. The latter force (shown dotted in the figure) must be equal and opposite to the force F at C, so that the stress at the section D is purely normal and is equal to F/A. It is the same at every section of the rod.

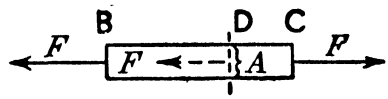


FIG. 197

The Stretching of a Wire.—As a particular example of the way in which the deformation of a body varies with the stress, we describe the results of experiments on the stretching of a wire. We can imagine that the wire is suspended vertically and that its change of length (extension) is measured for increasing weights attached to its lower end. A graph is then drawn between the extension and the load, and Fig. 198 shows the trend of the graph for steel.

The first part of the curve, OP, is accurately straight, thus showing that extension is proportional to load and that strain is proportional to stress if the stretching has only a negligible effect on the cross-sectional area of the wire. This relationship was first discovered by Hooke in 1679, and the statement that **strain is proportional to stress** is known as **Hooke's law.** It is commonly obeyed by metals and alloys when the strain is small.

The point at which Hooke's law begins to fail (P in Fig. 198) is known as the **limit of proportionality.** This point coincides approximately (but not necessarily) with the **elastic limit** of the material, which is the point at which an appreciable part of the extension of the wire becomes permanent. Up to this point the wire will return to its original length (zero extension) when the load is removed, but beyond the elastic limit the recovery is not complete.

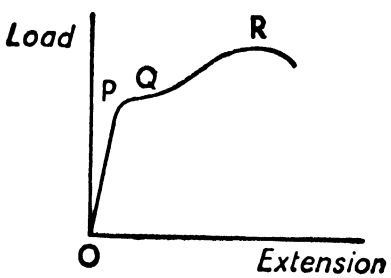


FIG. 198

Soon after the elastic limit is passed, the extension increases much more rapidly with increasing load than previously. The elongation of the wire is now more in the nature of a flow—rather like glass which is drawn out after being softened by heating in a flame. The point at which this region begins is called the **yield-point,** Q. Eventually a stage is reached at which a small additional load causes the extension to go on increasing with time, and in order to bring it to a halt it is necessary to

remove weights. This explains the fall of the curve after the point R is reached. Finally the wire breaks, and the value of the force at the point R, divided by the original area of cross-section, is called the **breaking stress.**

It is interesting to note that beyond the initial stage of small extension the shape of the graph of stress against strain is not the same as that of the load-extension graph on account of the considerable diminution of the area of cross-section as stretching proceeds. A feature of the stress-strain graph is the absence of a maximum such as R in Fig. 198. This indicates that the *material* becomes continuously more resistant to stretching as the extension increases even though the *wire* itself weakens.

2. TYPES OF STRESS AND STRAIN. ELASTIC MODULI

Modulus of Elasticity.—We shall henceforth confine ourselves to cases of change of size and shape for which Hooke's law is applicable. This implies that the strains are always considered to be small. There are various types of strain with which we have to deal. In each case the ratio of the change of stress to the accompanying strain is a constant independent of the strain, provided Hooke's law is obeyed. The constant is known as the **elastic modulus** appropriate to the particular type of stress and strain in question. For a given type, the value of the modulus is a characteristic physical property of the material which is being strained. Since strain has no units and

$$\text{Elastic modulus} = \frac{\text{change of stress}}{\text{accompanying strain}}$$

it follows that the units of a modulus are the same as those of stress, *e.g.* dynes cm.$^{-2}$ or lb. wt. in.$^{-2}$.

In general, a thermal change occurs in a material which is strained. A wire cools when it is stretched and a gas becomes warm when it is compressed. The opposite effects occur when the processes are reversed. In order that the material shall remain at a constant temperature during the establishment of the strain, therefore, it is necessary to allow heat to pass into or out of it. When conditions are such as to allow this, the process is said to be " isothermal " and the modulus is the **isothermal modulus.** We shall usually assume in future that the changes discussed are isothermal. If no heat is allowed to enter or leave the material which is being strained, the change is said to be **adiabatic** and is accompanied by changes of temperature. The values of the adiabatic and isothermal moduli differ from each other, the former being the larger. This is because the change of temperature in the adiabatic case is always such as to oppose the establishment of the strain. Thus when a gas is compressed adiabatically it gets warmer and so tends to expand against the force which is compressing it.

Bulk Modulus.—This refers to the change of volume of a solid, liquid or gas when it is submitted to a change of pressure. In the case of a liquid or a gas we can imagine the substance contained in a cylinder fitted with a piston which can be moved in or out, while for a solid the change of pressure can be pictured as being brought about by immersing the material in a liquid and then changing the pressure in the liquid by means of a piston.

If the volume of the substance is V at a given pressure and it changes by an amount ΔV when the pressure changes by Δp, we have

$$\text{Bulk modulus } k = \frac{\text{change of stress}}{\text{accompanying strain}} = \Delta p \div \frac{\Delta V}{V}$$

$$= V.\frac{\Delta p}{\Delta V}^{*}$$

The experimental determination of the bulk modulus for solids and liquids is not easy, because very large pressures must be applied and measured in order to produce measurable changes of volume. Typical values of bulk moduli are: steel $1{\cdot}7 \times 10^{12}$, copper $1{\cdot}4 \times 10^{12}$, water $0{\cdot}2 \times 10^{12}$ dynes cm.$^{-2}$. The bulk moduli of gases are dealt with in a later section.

Young's Modulus.—This modulus applies to solids only, and concerns the deformation of the solid by the application of equal and opposite forces along one dimension. When the uniform rod or wire in Fig. 199 is stretched by the equal and opposite forces F, its length increases from, say, l to $l+\Delta l$. The strain produced by the forces is therefore $\Delta l/l$, and we have already seen that the stress is F/A where A is the area of cross-section. Therefore

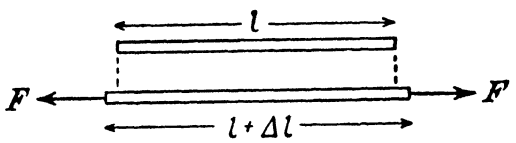


FIG. 199

$$\text{Young's modulus } Y = \frac{\text{change of stress}}{\text{accompanying strain}} = \frac{F}{A} \div \frac{\Delta l}{l}$$

$$= \frac{Fl}{A\Delta l}$$

A solid body is said to be elastically **isotropic** if its value of Young's modulus is the same in all directions. This is frequently not the case with metallic wires on account of the structure which is imparted to them during the process of drawing.

* If ΔV and Δp are regarded as *increases* of V and p respectively, one or other of them will always be negative, so that, on this convention, a *minus* sign is inserted in front of the expression for k in order that the modulus shall have a positive value.

Example.—*A steel wire 2 metres long hangs vertically from a fixed hook and is required to support a load of 15 kg. wt. Calculate the minimum diameter allowable if its extension under this load must not exceed 3 mm. Young's modulus for steel is* $2{\cdot}0 \times 10^{12}$ *dynes cm.*$^{-2}$.

Let the minimum allowable diameter be d cm., so that the cross-section A is equal to $\pi d^2/4$. Then we have

$$\begin{aligned} \text{Stretching force } F &= 15 \text{ kg. wt.} \\ &= 15{,}000 \times 980 \text{ dynes} \\ \text{Length of wire} &= 2 \text{ metres} \\ &= 200 \text{ cm.} \\ \text{Extension } \Delta l &= 3 \text{ mm.} \\ &= 0{\cdot}3 \text{ cm.} \end{aligned}$$

Then

$$Y = \frac{Fl}{A\Delta l}$$

$$\therefore\ A = \frac{\pi d^2}{4} = \frac{Fl}{Y\Delta l}$$

so that

$$\begin{aligned} d &= \sqrt{\frac{4Fl}{\pi Y \Delta l}} \\ &= \sqrt{\frac{4 \times 15{,}000 \times 980 \times 200}{\pi \times 2 \times 10^{12} \times 0{\cdot}3}} \\ &= \sqrt{\frac{0{\cdot}0196}{\pi}} \\ &= 0{\cdot}079 \text{ cm.} \end{aligned}$$

Poisson's Ratio.—When an isotropic solid is stretched by the application of a pair of equal and opposite forces, it is always found that its dimensions diminish in directions perpendicular to the line of action of the forces. Thus, when the cube shown in Fig. 200 is stretched by the forces F, it contracts laterally and its new shape is represented by the dotted lines. The ratio

$$\frac{\text{lateral contraction per unit width}}{\text{longitudinal extension per unit length}}$$

is called **Poisson's ratio.** In the figure, the ratio is equal to the diminution of the dimensions at right angles to the forces divided by the increase of length in the direction of the forces, since width and length were initially equal. It can be shown theoretically that Poisson's ratio for an isotropic solid cannot exceed 0·5. In practice it is usually found to be about 0·3.

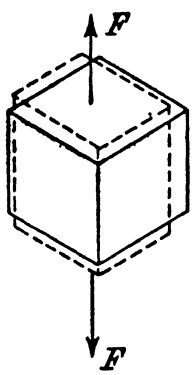


FIG. 200

Shear. Rigidity Modulus.—The type of strain known as a "shear" consists in the deformation depicted in Fig. 201 (i) and (ii), in which the square section of a cube of material is deformed into the shape of a rhombus.

In Fig. 201 (i) the strained and unstrained solid are shown in the relative positions which they actually occupy when deforming forces F are applied tangentially to the four faces of the cube as indicated in the diagram. No forces are applied at right angles to the plane of the paper. The square section is elongated along one diagonal and equally shortened along the other, the directions of the diagonals being unaltered. It can be shown that the linear dimensions of the cube are not appreciably

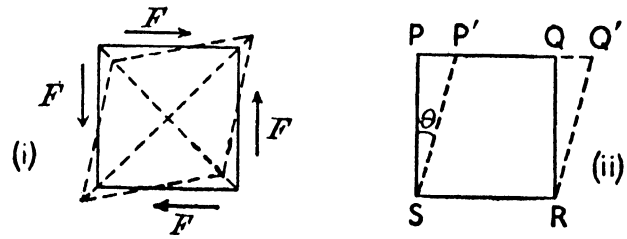


FIG. 201

changed provided that the strain is small. We can therefore imagine the deformed shape to be rotated as in Fig. 201 (ii) so that its base coincides with the base of the original cube. When it is represented in this way the shear can be regarded as the sliding of layers parallel to the base through distances proportional to their distance from the base. A pile of sheets of notepaper or a pack of cards can be used to illustrate this type of deformation, which is a change of shape but not of size. It should be remembered, however, that the actual strain due to the tangential forces takes place in the way shown in Fig. 201 (i).

The shearing strain is defined as the ratio PP'/PS, *i.e.* the relative parallel displacement of two planes (PQ and SR) divided by their distance apart. Provided that the deformation is small, this ratio may be taken to be equal to the circular measure of the angle θ, which is known as the **angle of shear.** In Fig. 201 (i) the angle between the displaced and the original directions of the side of the square is $\theta/2$ in each case.

The shearing stress is defined as F/A, where F is the tangential force applied uniformly over each of the four faces of the cube, and A is the area of each face.

The modulus connecting shearing strain and stress is the **rigidity modulus** n, which is defined by

$$n = \frac{\text{change of shearing stress}}{\text{accompanying shearing strain}}$$

$$= \frac{F}{A\theta}$$

It is interesting to note that the stresses set up in the material by the tangential forces F may be related directly to the aspect of the shear

represented in Fig. 201 (i), namely the elongation and contraction along the diagonals. Suppose that a unit cube whose section PQRS is shown in Fig. 202 (i) is acted on by tangential shearing stresses p, and consider the equilibrium of the portion PQR. Since the area of each face is unity, the faces PQ and QR are acted on by tangential forces p as shown in the figure. Vector addition of these forces gives their resultant as $p\sqrt{2}$ acting at right angles to the plane PR. This is represented by a dotted vector in the diagram. For the equilibrium of PQR it is necessary that the portion PSR should act on PQR across PR with a force $p\sqrt{2}$ at right

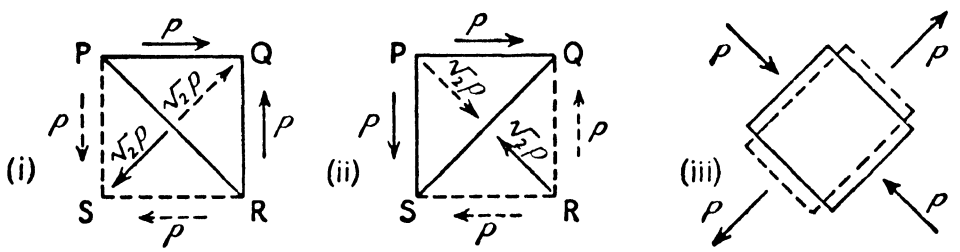


FIG. 202

angles to PR in a direction away from PR as shown by the full arrow. It therefore follows that the stress across PR due to this force is purely normal and has a value p, since the area of the plane PR is $\sqrt{2}$. It is a tensile stress stretching the material along the diagonal QS. In Fig. 202 (ii) an exactly similar analysis is made of the equilibrium of PQS, and this leads to the result that the stress across QS is purely normal, equal to p and directed so as to compress the material along the diagonal PR.

It follows from the above treatment that the condition of the stressed and strained material of the cube PQRS is exactly the same as that of a cube whose faces are acted upon by normal stresses p in the manner shown in Fig. 202 (iii). The consequent change of shape is indicated by the dotted outline.

We have seen that when a cube of isotropic material is submitted to a change of pressure its size changes but its shape remains cubical. On the other hand a shear causes change of shape but not of size. It follows that every strain can be analysed into a uniform compression or dilatation combined with a shear.

Torsion.—Consider a straight rod or wire of circular section which is clamped at one end while the other end is twisted about the axis of the rod or wire (Fig. 203 (i)). The rod may be regarded as composed of a series of thin discs, and the deformation may be represented as the rotation of each disc about the axis of the rod through an angle proportional to its distance from one end. This is illustrated in Fig. 203 (ii) and (iii). In (ii) the rod is unstrained and a straight line is drawn on its surface parallel to its axis. When the strain has occurred (Fig. 203 (iii)), the

line is split up into a series of marks on the edges of the discs. Thus the material of a twisted rod or wire is in a state of shear like the cube in

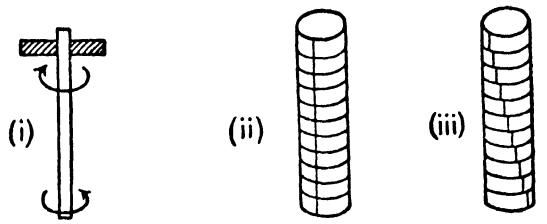


FIG. 203

Fig. 201. The rod is, of course, acted upon by two equal and opposite couples.

3. EXPERIMENTAL DETERMINATION OF YOUNG'S MODULUS

The Stretching of a Wire.—The extension of a wire under various loads can be investigated by the simple apparatus shown in Fig. 204. A long thin wire is looped near its centre and fixed to a rigid support by clamping it between two metal plates as shown at A. A piece of brass B, on which a millimetre scale is engraved, is attached near the lower end of one segment of the wire C, while a vernier E corresponding to this scale is fixed to the other segment D at the same level. The distance from A to the scale and vernier should be as large as possible—*e.g.* two metres or more—otherwise the extensions may be too small to measure accurately. The segment C is the comparison wire and is kept taut by a weight permanently attached to its lower end. A large pan or "hook" for the reception of kilogram and half-kilogram weights is hung from D.

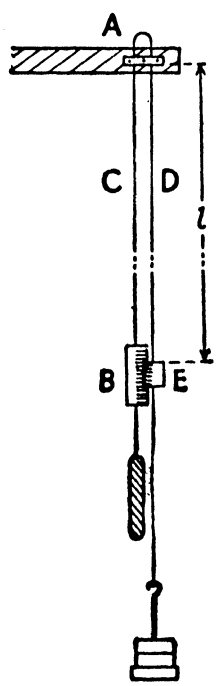


FIG. 204

The purpose of the two wires hanging side by side is to ensure that the change of vernier reading when D is loaded shall be the true extension of the length l (see Fig. 204), and shall not be vitiated by changes of the temperature of the wire or the yielding of the support. These effects are eliminated because they raise or lower both the scale and the vernier equally.

Sufficient load (W_0) is initially put on D to ensure that it is taut. The diameter of D is then measured by a micrometer screw gauge at several positions along its length, two readings being taken at right angles to each other at each place. The mean diameter can then be deduced after allowance has been made for the zero error of the screw gauge.

Next, the reading of the vernier against the mm. scale is taken. Let it be r_0. The load on D is then increased to W_1, which should be sufficient to change the reading by between 0·3 and 0·5 mm. Let the new reading be r_1. The load is increased to W_2, W_3, . . ., etc., and the corresponding scale readings taken (r_2, r_3, . . ., etc.). If it is only required to determine Young's modulus, the wire should not be extended beyond the limit of proportionality and elastic limit. Therefore the load should be reduced to W_0 from time to time and the zero reading (r_0) checked. This will show a substantial increase as soon as the elastic limit has been exceeded. If desired, the experiment may then be continued as an investigation of the behaviour of the wire right up to the breaking point.

The results may be tabulated as follows:—

Load	Load in excess of W_0	Reading	Extension produced by load in excess of W_0
W_0	0	r_0	0
W_1	$W_1 - W_0$	r_1	$r_1 - r_0$
W_2	$W_2 - W_0$	r_2	$r_2 - r_0$
.	. .	.	. .
.	. .	.	. .
.	. .	.	. .
W_n	$W_n - W_0$	r_n	$r_n - r_0$

It is now necessary to use the results in order to determine the mean increase of extension per unit increase of load. The easiest way to do this is *not* to find the average value of the ratio of corresponding pairs of figures in the second and fourth columns, but to add up the figures in the second column, then to add up those in the fourth column and finally to find the ratio of the two totals. Apart from the fact that it involves less arithmetical labour, this procedure gives a more accurate mean, because it automatically gives more weight to the larger (and therefore more accurate) results.

When the mean extension per kg. is divided by 1000×980 to bring it to cm. dyne^{-1}, it is equal to $\Delta l/F$ in the expression for Young's modulus (page 217). The area of cross-section (A) is $\pi d^2/4$, where d is the mean diameter of the wire, and l is the length of the wire D between the support A and the top of the vernier E. Hence Young's modulus for the material of the wire can be calculated. A graph of extension against

load should be plotted and the mean extension per kg. may be derived from the slope.

The Potential Energy of a Stretched Wire.—In order to derive an expression for this quantity we consider how to calculate the mechanical work which is done when a wire is stretched. Suppose that Fig. 205 represents the graph of stretching force F against extension l, and that it is required to find the work done when the extension increases by AB. Erect ordinates at A and B to cut the graph at D and C respectively, and suppose that a series of narrow rectangles is described in the figure ABCD in the manner shown. Let the width of each rectangle represent an increase of extension equal to Δl cm. If the height of a given rectangle represents a stretching force of F dynes its area is $F\Delta l$. But this product is the work done in ergs when a force of F dynes moves through Δl cm. It is not quite equal to the work done during the extension of the wire by Δl cm. because F is the force at the *beginning* of the step; but the smaller Δl is considered to be, the more nearly does the area of the rectangle represent the actual work done. We can imagine the area ABCD divided into an infinite number of rectangles of infinitesimal width, in which case the area of each is accurately equal to the work done during the small step which it represents, and the total area of the rectangles accurately represents the total work done in extending the wire from A to B. But the total area of the rectangles is the area ABCD when each is indefinitely narrow, so that the work done when the wire is extended from A to B is equal to the area ABCD. This is true no matter what may be the shape of the graph, and is, in fact, a particular case of the process of integration already discussed in connection with speed-time graphs on page 11.

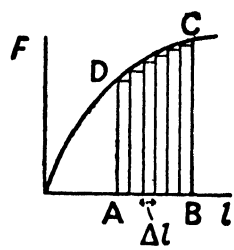


FIG. 205

We now consider the stretching of a wire over the region for which Hooke's law is obeyed. Fig. 206 shows the rectilinear graph. Provided the elastic limit is not exceeded, the wire returns to its original length when the stretching force is removed. Thus if this force were due, say, to a load of lead shot, the shot could be removed one by one and, as each was removed, those remaining would be raised until the last shot was taken away. The wire would then have done work in raising shot (only the last one is raised all the way, however), and it therefore possessed potential energy of strain in its stretched condition. The amount of this energy is the work which the wire can do in returning to its original length and is the same as the work which was done in stretching it, since the graph is the same for loading and unloading. Therefore when the stretching force is F dynes and the extension is x cm., the potential energy of the wire is equal to the area of the right-angle

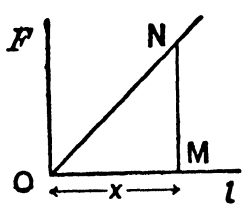


FIG. 206

triangle OMN, MN being equal to F. Thus

$$\text{Potential energy} = \tfrac{1}{2} \times \text{MN} \times \text{OM}$$
$$= \tfrac{1}{2}Fx \text{ ergs}$$

This formula also applies to a stretched helical spring and to all other systems in which the extension or deflection is proportional to the applied force.

If l is the original length of the wire and A is its area of cross-section, its volume is equal to lA, so that

$$\text{Potential energy per unit volume} = \frac{\tfrac{1}{2}Fx}{lA}$$
$$= \tfrac{1}{2} \times \text{stress} \times \text{strain}$$

since F/A is stress and x/l is strain.

But Young's modulus Y is given by

$$Y = \frac{\text{stress}}{\text{strain}}$$

$$\therefore \text{ Potential energy per unit volume} = \tfrac{1}{2}Y(\text{strain})^2$$

Similar expressions apply to potential energy of compression (bulk modulus) and of shear (rigidity modulus).

The Bending of a Beam.—When a " beam " such as a metre rule is bent by the action of two equal and opposite couples (Fig. 207), it is found that its strained condition can be represented entirely in terms of stretching and contraction. This means that the modulus of elasticity involved in the deformation is Young's modulus. The beam is regarded as a collection of thin filaments lying parallel to its axis. Those on the convex side of the beam are extended and those on the concave side are contracted. Somewhere in the interior there is a layer of filaments

FIG. 207

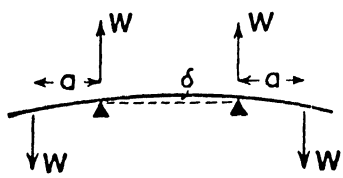


FIG. 208

whose length is the same as it was when the beam was straight. This layer is known as the **neutral surface,** and its intersection with the plane in which the couples act is shown as a dotted line in the figure and is called the **neutral axis.**

It is possible to achieve such bending quite simply by supporting the beam horizontally and symmetrically on two knife edges (Fig. 208) and attaching two equal weights W symmetrically. If the weight of the beam is negligible, the reaction of each knife edge is an upward force W, and each couple has a moment of Wa. It can be shown that between the knife

edges the shape of the beam is that of a circular arc. The radius of this arc can be calculated by measuring the elevation (δ) of the midpoint of the beam and the distance between the knife edges. It is connected with Young's modulus and the dimensions of the beam by an equation which we shall not attempt to establish. This and other ways of bending a beam, therefore, provide methods for determining Young's modulus which are sometimes preferable to the stretching of a wire, because, as already mentioned, the process of drawing a wire may modify the elastic properties of the material.

4. THE BULK MODULUS OF GASES

At the beginning of this section it is worth recalling that the only modulus of elasticity which a gas possesses is its bulk modulus, and also that there is a distinction between the isothermal and adiabatic moduli which is particularly important in the case of gases.

Isothermal Relation between Pressure and Volume of a Gas. Boyle's Law.—The compressibility of air occupied the attention of Robert Boyle, Earl of Cork, in the middle of the seventeenth century, and from the results which he obtained over a limited range of pressure and volume Boyle deduced the following law. **At constant temperature the pressure of a given mass of gas is inversely proportional to its volume.** If pressure is denoted by p and volume by V, Boyle's law is represented by

$$p \propto \frac{1}{V}$$

or

$$p = \frac{K}{V}$$

or

$$pV = K$$

where K is a constant for a given mass of a particular gas at a given temperature.

The piece of apparatus shown in Fig. 209 resembles that of Boyle, and is frequently used in teaching laboratories at the present time. The bent glass tube of about 1 cm. diameter contains a quantity of clean dry mercury which traps some dry air in the closed limb. The volume of the air can be taken to be proportional to the length of the air space provided that the cross-section of the tube is uniform in this region and the effect of the curvature of the closed end is not large. Let the volume be expressed as V in any convenient units. If atmospheric pressure is B cm. of mercury, the pressure of the enclosed air is given by

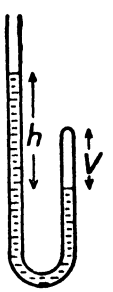


FIG. 209

or

$$\left.\begin{aligned} p &= B + h \\ p &= B - h \end{aligned}\right\} \text{cm. of mercury}$$

according as the level in the open limb is above or below that in the closed limb.

The experiment consists in varying the quantity of mercury in the apparatus so that h and V change step by step and observing simultaneous values of h and V. When the product pV is calculated for each pair of values it is found to be sensibly constant, thus verifying the law.

Alternatively the readings may be used to test the law graphically as follows. If the law is true, we have

$$pV = K$$
$$\therefore (B+h)V = K$$

or

$$h = \frac{K}{V} - B$$

where h is the height of the level in the open limb above that in the closed limb and consequently may have a negative value.

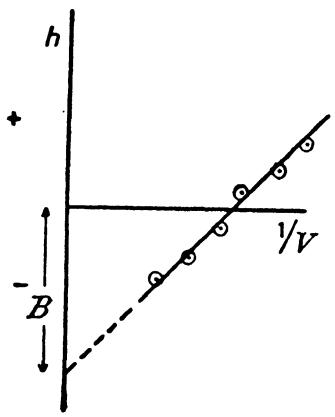


FIG. 210

If this equation is true, the graph of h against $\frac{1}{V}$ is a straight line which has an intercept of $-B$ on the h axis. This is shown in Fig. 210, where the observed portion is represented by the full line and includes negative values of h. The two separate facts that the graph is rectilinear when the experimental values are plotted and that the intercept is found to be numerically equal to the height of the barometer (found independently) constitute a verification of the original equation, *i.e.* of Boyle's law. It should be noted that the height of the barometer cannot be deduced from the experiment unless the validity of the law is assumed.

Example.—*A faulty barometer, which contains a little air above the mercury, reads* 745 *mm. when the atmospheric pressure is* 765 *mm. of mercury. What is the true atmospheric pressure on a day when the barometer reads* 735 *mm., the temperature being the same as before? The closed end of the tube is* 80 *cm. above the surface of the mercury in the reservoir.* (L. Med.)

Fig. 211 (i) and (ii) illustrate the example. In (i) atmospheric pressure equal to 765 mm. of mercury exists at the level of the mercury surface in the reservoir. Therefore since the actual height of the column is only 745 mm., the air at the top of the barometer tube must be exerting a pressure of (765 − 745) or 20 mm. of mercury. It occupies a length of tube equal to (800 − 745) or 55 mm., so that its volume is equal to $55A$ mm.^{3}, where A is the cross-section of the tube in mm.^{2} Similarly in (ii), where true atmospheric pressure is B mm. of mercury, say, the pressure of the air is $(B - 735)$ mm. of mercury and its volume is $(800 - 735)A$ or $65A$ mm. Applying Boyle's law to the two conditions of the gas, we have

(pressure × volume) in (i) = (pressure × volume) in (ii)

or

$$20 \times 55A = (B - 735)\, 65A$$

or

$$B = \frac{20 \times 55}{65} + 735$$

$$= 16{\cdot}9 + 735$$

$$= 752 \text{ mm. of mercury (very nearly)}$$

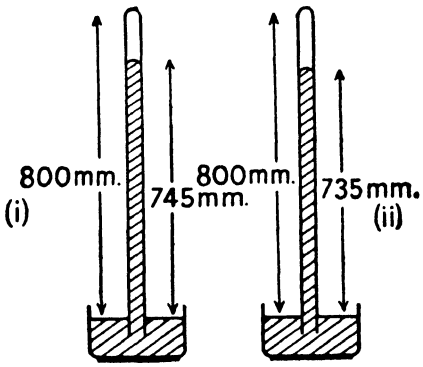


FIG. 211

Deviations from Boyle's Law.—The above experiment, and others of a similar nature performed with more convenient apparatus, will be found to verify Boyle's law within the possible limits of accuracy. The range of pressure is, however, comparatively small—say from 0·5 to 1·5 atmospheres.

The isothermal relation between p and V for gases has been investigated up to very high pressures (15,000 atmospheres and more), particularly by Bridgman in the United States. The experiments involve a highly specialized design of apparatus and technique of pressure measurement. Results show that as the pressure is increased the volume occupied by the gas at a given pressure is in general different from that predicted by Boyle's law. This failure of Boyle's law at high pressures is due partly to the finite size of the gas molecules. At low pressures the mean distance between the molecules is large compared with their size, and Boyle's law is obeyed under these conditions in which the compression of a gas is merely a matter of diminishing the separation of its molecules. As the pressure is increased, the nature of the process of compression gradually changes until it eventually involves the actual compression of the molecules themselves, because these are now effectively in contact with each other. The matter is more fully discussed on page 436 (Vol. 2).

Isothermal Bulk Modulus of a Gas which obeys Boyle's Law.—Let a given mass of gas initially have a pressure p and a volume V, and suppose that its pressure is increased at constant temperature by a *small* amount Δp with the result that its volume is diminished by a small amount ΔV. The initial and final states are related by Boyle's law, so that

$$pV = (p + \Delta p)(V - \Delta V)$$
$$= pV + V\Delta p - p\Delta V - \Delta p \Delta V$$

Since Δp and ΔV are very small, we can neglect their product in comparison with the other terms in the equation, which then becomes

$$p\Delta V = V\Delta p \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The change of stress is Δp, and the corresponding strain is $\Delta V/V$,

so that the isothermal bulk modulus (k_θ) is given by

$$k = \frac{\text{change of stress}}{\text{accompanying strain}}$$
$$= V\frac{\Delta p}{\Delta V}$$
$$= p, \quad \text{by equation (1)}$$

Thus the isothermal bulk modulus of a gas which obeys Boyle's law is equal to its pressure expressed in dynes cm.$^{-2}$. The modulus is not a constant of the gas itself—in fact, for a given pressure, it is independent of the nature of the gas. As the pressure—and therefore the bulk modulus—increases it becomes more and more difficult to produce a given fractional decrease of volume. The difficulty becomes even greater when the pressure becomes so high that the gas molecules are crowded together and are themselves compressed. At the very high pressures reached in Bridgman's experiments the gases were no more compressible than liquids.

Adiabatic Bulk Modulus of a Gas which obeys Boyle's Law.—It can be shown that if a gas obeys Boyle's law and its pressure and volume are altered adiabatically (*i.e.* by allowing no heat to enter or leave the gas), the relation between them is

$$pV^\gamma = \text{a constant}$$
$$= C, \text{ say}$$

where γ is the ratio of the specific heat of the gas at constant pressure to its specific heat at constant volume. For air and other gases or mixtures of gases whose molecules consist of two atoms, the value of γ is about 1·4.

Let the pressure of the gas be increased adiabatically by a small amount Δp, so that its volume diminishes by, say, ΔV. Its temperature will rise slightly. The original and final pressures and volumes are related by the equation

$$pV^\gamma = (p+\Delta p)(V-\Delta V)^\gamma$$
$$= V^\gamma(p+\Delta p)\left(1-\frac{\Delta V}{V}\right)^\gamma$$
$$= V^\gamma(p+\Delta p)\left(1-\gamma\frac{\Delta V}{V}\right)$$

by the binomial theorem if ΔV is small enough to allow us to neglect $\left(\frac{\Delta V}{V}\right)^2$ and higher powers of this fraction in comparison with unity. Thus

$$p = (p+\Delta p)\left(1-\gamma\frac{\Delta V}{V}\right)$$

or

$$\Delta p = \gamma p\frac{\Delta V}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

provided we can neglect the quantity $\gamma \Delta p \frac{\Delta V}{V}$, which will be very small since it contains the product of two small quantities. Therefore, the adiabatic bulk modulus (usually denoted by k_s) is given by

$$k_s = \frac{\text{change of stress}}{\text{accompanying strain}}$$
$$= V\frac{\Delta p}{\Delta V}$$
$$= \gamma p, \quad \text{by equation (2)}$$

Therefore, the adiabatic value of the bulk modulus of a gas is γ times greater than its isothermal value. This is of importance in connection with the speed with which sound waves are propagated in a gas.

EXAMPLES XV

1. Describe how you would measure Young's modulus for a metal, provided in the form of a wire. State, giving reasons, the precautions you would take to secure an accurate result.

A steel wire 2 metres long, of area of cross-section 2 sq. mm., is loaded until it extends by 5 mm. Young's modulus for the material is 2×10^{12} dynes cm.$^{-2}$. Calculate (*a*) the force required to produce the extension, (*b*) the work done on the wire. (O.H.S.)

2. What is meant by saying that a substance is "elastic"?

A vertical brass rod of circular section is loaded by placing a 5-kilogram weight on top of it. If its length is 50 cm., its radius of cross-section 1 cm., and Young's modulus of the material $3{\cdot}5 \times 10^{11}$ dynes per sq. cm., find (*a*) the contraction of the rod, (*b*) the energy stored in it. (C.H.S.)

3. Define *elastic limit* and *Young's modulus*, and describe how you would find their values for a copper wire.

What stress would cause a wire to increase in length by one-tenth of 1 per cent. if Young's modulus for the wire is 12×10^{11} dynes per sq. cm.? What load in kg. wt. would produce the stress if the diameter of the wire is 0·56 mm.? (L.I.)

4. Define *Young's modulus*, and point out the conditions to which it applies.

When the load on a wire is increased slowly from 3 to 5 kg. wt. the elongation increases from 0·61 to 1·02 mm. How much work is done during this extension of the wire?

Find a value for Young's modulus of the material of the wire if it is 1 metre long and has a cross-sectional area of 0·40 sq. mm. (L.I.)

5. Define *Young's modulus*, and describe how its value may be determined for a material in the form of a wire.

Two copper wires, A and B, each of length 1 metre and diameter 1 mm., hang vertically 12 cm. apart. Their lower ends are joined by a uniform horizontal rod. Calculate the extension of each wire when a 6-kilogram weight is attached to the rod 8 cm. from A and 4 cm. from B. (Young's modulus for copper $= 12{\cdot}5 \times 10^{11}$ dynes cm.$^{-2}$.) (L.I.)

6. A certain wire has a length L_1 and a circular cross-section of radius a, while another wire of the same material has a length L_2 and a rectangular cross-section measuring $2a$ by $a/2$. If the extension produced in each wire by a given force is the same, find the ratio of L_1 to L_2. (L.I.)

7. A kilogram of air is contained in a closed vessel whose capacity is 1000 cm.^{3}. Find the ratio of its pressure to the atmospheric pressure assuming the atmospheric density to be 0·00129 gm. cm.$^{-3}$. (L.I.)

8. A thin hollow sphere, of radius 2 ft., containing air, has a small hole pierced at its lowest point A. It is lowered into water with the point A underneath, until the water inside has risen to a height of 3 ft. above A. Find the depth of the centre of the sphere if the water barometer stands at 34 ft. (The volume of a segment of height h of a sphere of radius a is $\pi h^2(a - \frac{1}{3}h)$.) (L.I.)

9. The envelope and load of a balloon together weigh 190 kg. When filled with hydrogen the balloon exerts a pull on the mooring rope of 50 kg. wt. What is its volume?

If the balloon were let go, assuming that the hydrogen escapes so as to keep the pressure inside and out always the same, find the density of the air at the maximum height to which the balloon would rise.

(Density of air = 1·29, of hydrogen = 0·09, gm. per litre under the conditions prevailing at ground-level. Temperature variations may be neglected.) (L.Med.)

10. State Boyle's law.

The inside of a glass tube 61 cm. long, open at one end, is covered with a soluble pigment. After a sea sounding, in which the tube is lowered open end down, the pigment is found to be dissolved up to a point 51·5 cm. from the open end. Neglecting temperature changes and taking the densities of sea-water and mercury to be 1·03 and 13·6 gm. per c.c. respectively, and the barometric pressure to be 76 cm. of mercury, calculate the sea-depth to which the open end has been lowered. (L.Med.)

11. A small balloon of volume 30 litres contains hydrogen at a pressure which is $1\frac{1}{4}$ times the prevailing atmospheric pressure. It is tethered by a light string which, on account of the horizontal force exerted by the wind, makes an angle of 30° with the vertical. If the envelope of the balloon weighs 16 gm., calculate the force due to the wind, taking the densities of air and hydrogen at the prevailing atmospheric temperature and pressure to be 1·2 and 0·08 gm. litre^{-1} respectively.

Enunciate any two of the principles or laws involved in the solution of the problem. (L.Med.)

12. Describe an experiment to show how the density of a gas varies with its pressure at constant temperature.

An oxygen cylinder contains 2 cub. ft. of gas at 200 atmospheres pressure. Some of the gas is used and the pressure falls to 140 atmospheres. What fraction of the mass of gas initially present was let out? What volume would the used gas occupy at 1·2 atmospheres pressure? (Assume that the temperature does not alter.) (L.M.)

13. State Boyle's law, and sketch the apparatus you would use to demonstrate its truth, and state how the pressure of the gas is determined.

A diving bell is sunk to the bottom of a lake 40 ft. deep and air is pumped in to drive out the water forced into the bell. By what fraction of its initial value must the mass of air inside be increased? (Assume that the pressure of the atmosphere is 15 lb. wt. per in.^{2} and that 1 cub. ft. of water weighs 62·4 lb.) (L.M.)

14. Describe the construction of the mercury barometer. How would you detect the presence of air in the space above the mercury?

A faulty barometer reads 74 cm. when the true height is 76 cm. On raising the tube through 20 cm. the mercury column increases to 75 cm. Find the length of the space above the mercury in the first instance. (L.M.)

15. A piece of glass tubing, of cross-section 1 sq. cm., has a bulb at one end and its other end is joined by rubber tubing to another glass tube. The apparatus is set up for a Boyle's law experiment, the bulb being uppermost and the mercury showing in each glass tube. The levels are the same in both tubes when the mercury is 10 cm. below the bulb. On altering the levels by raising the open tube the mercury rises 5 cm. in the closed and 13 cm. in the open tube. Calculate the volume of the bulb. (Assume that the barometric pressure is 76 cm. of mercury.) (L.M.)

16. One limb A of a U-tube containing mercury is closed, and the other limb B is open, there being a vacuum at the top of A, and the difference of level of the mercury in the two limbs being 30 in. The top of B is now sealed, thus enclosing a volume of air which occupies a length of 20 in. of the tube at atmospheric pressure. If the top of A is then opened, find how far the mercury rises in B. (L.I.)

Chapter XVI

VISCOSITY

1. THE FLOW OF FLUIDS

At the beginning of Chapter X it was explained that a fluid *at rest* cannot permanently resist an attempt to make it flow, *i.e.* to change its shape. This is, in fact, the definition of the fluid state of matter. The phrase "at rest" is important. Although all fluids begin to yield to a shearing force as soon as it is applied to them, they evidently offer a resistance when once the motion has begun. This resistance is ascribed to the **viscosity** of the fluid and varies greatly from one fluid to another. For instance, when treacle and water are poured through the same funnel in turn, their rates of flow are very different.

Viscous resistance comes into play as soon as flow begins, that is to say, as soon as there is relative motion between the various parts of the fluid. If the external forces causing the flow are kept constant, the rate of flow eventually becomes constant, and a steady state is reached in which the resisting force has become as large as the applied force and no further acceleration occurs. When the applied force is removed, the flow subsides because the viscous forces remain and oppose the relative motion between the various parts of the fluid. Thus a liquid which is being stirred comes to rest after the stirrer has been removed. The kinetic energy which it possessed when in motion is used up in doing work against the viscous forces and is eventually converted into heat.

Since shearing involves *tangential* stress, it follows that the viscous forces with which a fluid resists shearing must also be tangential. The statement (page 157) that two adjacent portions of a fluid can act upon each other only by forces which are normal to the plane separating them, is true only when there is no relative motion between them. Relative motion calls into play a tangential force which opposes it. This is considered in more detail in the following paragraphs.

Effects and Definition of Viscosity.—Consider a fluid which is flowing in parallel straight streamlines with a velocity which varies from one streamline to the next in the manner indicated by the arrows in Fig. 212. A given mass of fluid which was originally cubical in shape (PQRS) becomes increasingly deformed as if by a shear (P′Q′R′S′) as it moves down the stream. Thus the flow evidently involves the sliding of

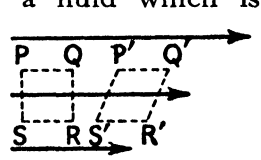


FIG. 212

parallel layers of fluid over each other. In Fig. 213 two such layers are shown slightly separated for the sake of clarity. Their velocities are indicated by the arrows on the right-hand side. The lower layer, being the slower, exerts a tangential retarding force F on the upper layer, and itself experiences an equal and opposite force tending to speed it up (Newton's third law). The tangential stress is F/A, where A is the area of contact between the layers.

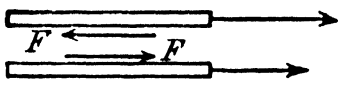


FIG. 213

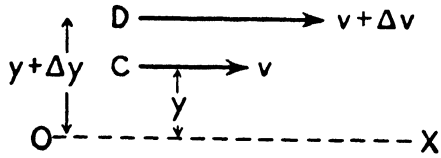


FIG. 214

The **velocity gradient** existing at any place in a fluid is defined as follows. In Fig. 214 the velocities of the fluid at two points C and D are v and $v+\Delta v$ respectively. The distances of these points from a fixed plane OX drawn parallel to the direction of the velocities are y and $y+\Delta y$ respectively. The velocity gradient in a direction perpendicular to the velocities is then $\Delta v/\Delta y$. Suppose that the points C and D approach each other so as to become points on *adjacent* layers of fluid. The separation Δy tends to zero, and the velocity gradient is

$$\operatorname*{Lim}_{\Delta y \to 0} \frac{\Delta v}{\Delta y} = \frac{dv}{dy}$$

Newton stated that the tangential stress (F/A) between two adjacent layers in a given fluid is proportional to the velocity gradient in a direction perpendicular to the layers. Thus

$$\frac{F}{A} \propto \frac{dv}{dy}$$

or

$$F = \eta A \frac{dv}{dy}$$

where η is a quantity depending upon the nature and temperature of the fluid. It is known as the **coefficient of viscosity,** or simply the **viscosity** of the fluid, and may be taken to be defined by the last equation. There are many fluids—elements, compounds and mixtures—for which η is independent of velocity gradient and therefore a constant for a given fluid at a given temperature.

The expression for η is

$$\eta = \frac{F}{A} \div \left(\frac{dv}{dy}\right)$$

and in the C.G.S. system the units of the right-hand side will be

$$\frac{\text{dynes}}{\text{cm.}^2} \div \left(\frac{\text{cm. sec.}^{-1}}{\text{cm.}}\right)$$

or

$$\frac{\text{gm. cm. sec.}^{-2}}{\text{cm.}^2} \div \left(\frac{\text{cm. sec.}^{-1}}{\text{cm.}}\right)$$

or

$$\text{gm. cm.}^{-1}\ \text{sec.}^{-1}$$

These are, therefore, the units of coefficient of viscosity. They are frequently given the name **poise** after Poiseuille, who discovered experimentally the laws governing the rate of flow of liquids through narrow tubes. Values of some viscosities in poises are shown in the following table. The liquids are at 20° C. and the gases are at 0° C.

Water . .	0·0101	Air . . .	$1{\cdot}71 \times 10^{-4}$
Benzene . .	0·0065	Hydrogen .	0·86 „ „
Mercury . .	0·0157	Oxygen . .	1·87 „ „
Glycerine. .	8·3		

Flow of Liquid in a Canal.—We can investigate further the part played by viscosity in the flow of a liquid by considering a liquid flowing in a wide canal or channel. Suppose that a steady flow of liquid in a canal is maintained from left to right (Fig. 215). The layer of liquid which is in contact with the bed of the canal is at rest, and the velocities of layers above this increase as the surface is approached. In order that the flow shall be maintained it is necessary that there shall be a diminution of pressure from left to right at any given level in the liquid. Consider a rectangular block of liquid whose section in the plane of the drawing is PQRS, and let the area of each of its vertical faces perpendicular to the drawing (*i.e.* PS and QR) be a. If the difference of pressure between these two faces is p, the liquid is acted upon by a horizontal force pa in the direction of flow. When a steady state has been reached the resultant of the viscous forces exerted on PQRS by the liquid above and below it is equal and opposite to the force pa. Since velocity increases from the bed upwards, PQRS is moving faster than the liquid below it, which is therefore retarding it with a force of, say, F_1. Similarly the liquid above PQRS tends to accelerate it with a force F_2. When the steady state is reached we must have

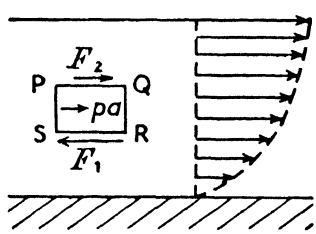


FIG. 215

$$pa - F_1 + F_2 = 0$$

Thus, since pa is positive, F_1 must be greater than F_2, and the velocity gradient, to which each force is proportional, must be less at the top face

than at the bottom face. Therefore the diminution of velocity per unit vertical distance decreases as we go from the bed to the surface, and the distribution of velocity must be of the kind shown on the right-hand side of the figure.

Turbulence.—In the example just discussed, each element of the liquid remains at a constant depth below the surface as it flows down the stream. This state of affairs does not *necessarily* occur in all circumstances. It is confined to rates of flow below a certain critical value. At higher speeds the liquid is, so to speak, churned up as it flows along the canal, and the streamlines are not parallel straight lines. The flow is then said to be **turbulent** in contrast with the **steady, laminar, orderly,** or **non-turbulent** flow (each of the terms is used) which has been assumed to exist in the above discussion. (See page 205.)

Flow of Liquid through a Tube.—When a liquid passes through a tube the flow may be turbulent or non-turbulent, according as the velocity is high or low. The transition from one type of flow to the other was investigated by Osborne Reynolds during the last century. His experimental method is illustrated in Fig. 216. A jet which delivered coloured liquid was placed near the entrance to the tube. Turbulence was indicated by a uniform coloration of the liquid in the tube (Fig. 216 (ii)), whereas orderly flow gave a single thread of colour (i).

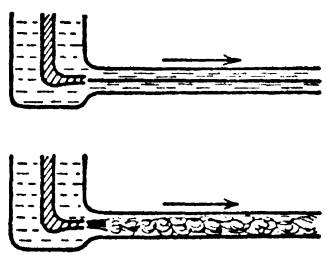


FIG. 216

Reynolds defined what has come to be known as **Reynolds' number** (R) by

$$R = \frac{va\rho}{\eta}$$

where a is the radius of the tube, ρ is the density of the liquid, and v is the mean linear speed, which is obtained by dividing the volume of liquid passing per second by the area of cross-section. He found that the flow was turbulent if R exceeded a value in the neighbourhood of 1000. Actually there is no absolutely definite transition point. As the speed of flow of a given liquid through a particular tube is gradually increased, the onset of turbulence may be encouraged by fortuitous disturbances, while if these are absent orderly flow may persist for higher velocities. In other words, there is a condition of instability when Reynolds' number is in the region of 1000.

When orderly flow occurs in a tube of circular section, the distribution of velocity across each section can be shown, both theoretically and experimentally, to be of the type depicted in Fig. 217 (i). The liquid immediately in contact with the walls has no velocity and, as we should

expect, the maximum velocity occurs at the greatest distance from the walls, *i.e.* on the axis of the tube. The shape of the curve joining the heads of the arrows is parabolic. The velocity distribution in turbulent flow is shown in Fig. 217 (ii).

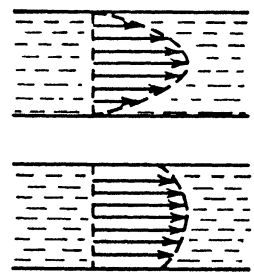


FIG. 217

From the definition of viscosity it is possible to derive an expression for the volume (Q) of liquid flowing through the tube per second when the flow is orderly. It is

$$Q = \frac{\pi p a^4}{8\eta l}$$

where p is the difference of pressure in dynes cm.$^{-2}$ between the ends of the tube, a is the radius of the tube and l is its length. Thus p/l is the fall of pressure per unit length of tube, or the " pressure gradient," and the equation shows that Q is proportional to the pressure gradient when a is constant. The fourth power of the radius appears in the expression because the linear velocity on the axis of the tube is found to be proportional to a^2, while the volume of liquid delivered in unit time is proportional to both this velocity and to the area of cross-section of the tube πa^2.

The way in which Q depends on the pressure gradient and the radius of the tube when the flow is orderly was discovered experimentally by Poiseuille. Previous experimenters had failed to obtain a consistent law because the flow which they measured was sometimes turbulent and sometimes orderly, and, as has already been mentioned, the laws for the two cases are different. Unlike his predecessors, Poiseuille, who was a surgeon and was interested in the flow of blood through veins, confined his attention to narrow tubes and small rates of flow and so discovered the law for non-turbulent flow.

2. DETERMINATION OF THE VISCOSITY OF LIQUIDS BY FLOW METHODS

Viscosity of Water by Flow through a Tube.—When a large amount of liquid is available (*e.g.* water), the apparatus shown in Fig. 218 may be used for the determination of viscosity by the direct application of the formula. The water flows into the apparatus from some kind of constant-pressure head. The type shown in the figure is a Marriotte's bottle (page 171). The pressure at the lower end C of the vertical tube is always atmospheric, provided that C is below the surface of the water. The pressure difference between the two ends of the capillary tube AB is indicated by the difference of level h of the water in the two vertical tubes D and E. The level in D will be only slightly below that of C, because the resistance to flow in the tubes joining the bottle to A is very

small on account of their comparatively large width. The water leaves the apparatus through the bent tube F (sometimes constricted at the

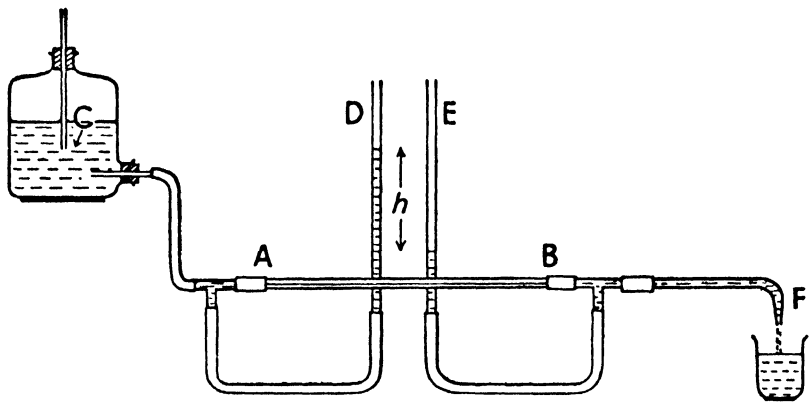


FIG. 218

outlet) and is collected in a measuring cylinder or weighed beaker. The rate of flow can be controlled by adjusting the height of the bottle above the level of the capillary tube.

The radius of the capillary tube is determined before the apparatus is set up. This is usually done by filling the tube with mercury to within a centimetre or so of its ends, measuring the length of the column and then emptying the mercury into a weighed watch-glass so as to obtain its weight. Assuming that the density of mercury is 13·59 gm. cm.$^{-3}$, the volume of the mercury thread can be calculated, and when this is divided by its length the result is the mean cross-section of the tube, πa^2. Dividing by π and squaring gives the value of a^4.

When the apparatus has been set up, measurements are first taken with a low rate of flow so as to be sure that the flow is non-turbulent. The difference of level h and the volume of water collected at the exit in a given time are determined. From the latter the volume discharged per second (Q) can be calculated. Further determinations are made at increasing values of h until the graph of Q against h (Fig. 219) departs from the original straight line and so indicates that turbulence has set in and the formula no longer applies. The value of η at the particular temperature of the experiment can be calculated, using the mean value of Q/h derived from the slope of the first straight portion of the graph. It must be remembered that the p in the formula is $hg\rho$ where ρ is the appropriate density of water.

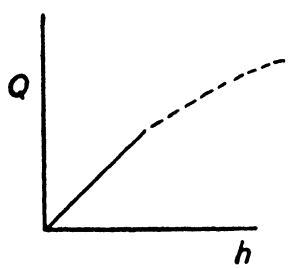


FIG. 219

The Ostwald Viscometer.—There are many instruments, known as " viscometers," for investigating the viscosity of liquids by timing their flow through narrow tubes or apertures. As a rule these do not give

absolute values of viscosity as the foregoing experiment does, but they are suitable for comparison purposes, either with different liquids or with the same liquid at different temperatures.

One of the most satisfactory types for precision work is the Ostwald viscometer. It consists of a U-tube in which two bulbs and a length of capillary tube are incorporated (Fig. 220). The instrument is always filled with the *same volume* of liquid. By applying suction to A or pressure to B the liquid surface is brought above the mark etched at C, its other surface then being in the region of E. The liquid is then released and the time for its upper surface to fall between the marks C and D is accurately determined. Evidently this is the time during which a volume of liquid equal to the volume (V) of the instrument between C and D has flowed through the capillary tube. The pressure difference causing the flow diminishes as the flow proceeds; but, since the viscometer is always charged with the same volume, the average head (*i.e.* difference of level) during the flow is the same in every case. If this is $\bar{h}$, the average value of the pressure difference p is $\bar{h}g\rho_1$ for a liquid of density ρ_1, and $\bar{h}g\rho_2$ for another of density ρ_2. If the corresponding viscosities are η_1 and η_2 and the times of flow are t_1 and t_2, we have

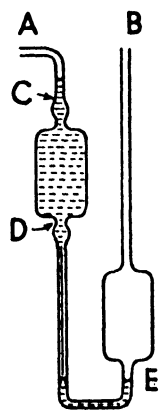


FIG. 220

$$Q_1 = \frac{V}{t_1}$$

and

$$Q_2 = \frac{V}{t_2}$$

Thus for the first liquid the equation for orderly flow is

$$\frac{V}{t_1} = \frac{\pi \bar{h} g \rho_1 a^4}{8\eta_1 l}$$

and for the second

$$\frac{V}{t_2} = \frac{\pi \bar{h} g \rho_2 a^4}{8\eta_2 l}$$

where a and l are the radius and length of the capillary tube respectively. By division we obtain the ratio of the two viscosities as

$$\frac{\eta_1}{\eta_2} = \frac{t_1 \rho_1}{t_2 \rho_2}$$

Thus viscosities may be compared, or their actual values may be found if the viscometer is once calibrated with a pure liquid such as benzene, the viscosity of which is accurately known from an absolute determination. Temperature control is simple, because an Ostwald viscometer is usually

only a few inches tall and can be immersed in a bath whose temperature is thermostatically controlled. This is a very suitable method of investigating the variation of viscosity with temperature.

3. VISCOUS RESISTANCE TO THE MOTION OF A SPHERE

Motion of a Sphere through a Fluid.—When a sphere moves through a fluid (*e.g.* a ball-bearing sinking in a liquid, or a small drop of liquid falling through air), the disturbance which it creates in the fluid may be either of two kinds according to the velocity of the sphere. As the velocity of the sphere relative to the fluid increases, the motion of the fluid changes from orderly to turbulent at a certain critical velocity.

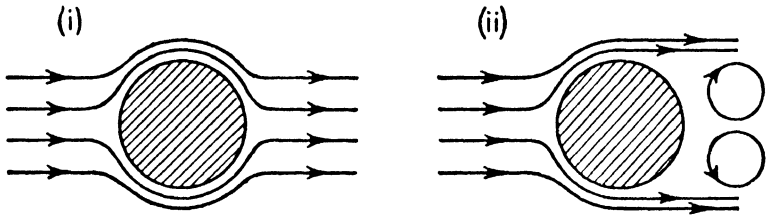


FIG. 221

Fig. 221 (i) indicates the streamlines which are followed by a slow stream of liquid moving past a stationary solid sphere. Conditions are similar when the sphere moves and the liquid is at rest, but in this case the lines indicate how the particles of liquid would appear to be moving to an observer who is himself moving with the sphere.

When the critical velocity is reached, eddies are formed in the wake of the sphere and the flow is turbulent. Fig. 221 (ii) indicates an initial stage in the development of turbulence. The eddies subsequently move downstream and others are formed behind the sphere. Such eddies can be observed when a rod is held vertically in still water and drawn along steadily in a horizontal direction.

Resistance to Motion. Stokes' Law.—When the motion of the fluid is not turbulent, the resistance F which the sphere experiences is found to be proportional to the viscosity of the fluid (η) and to the velocity v of the sphere through the fluid. The expression for F in absolute units was given by Stokes as

$$F = 6\pi\eta a v$$

where a is the radius of the sphere.

It was assumed by Stokes (and is borne out in practice) that there is no relative velocity between the sphere and the layer of liquid which is in contact with its surface. But since there is a relative velocity between the sphere and other parts of the fluid, it follows that there must be a velocity gradient in the fluid, and viscous forces between adjacent layers have to be overcome in order to maintain the motion. These constitute the resisting force. It should be noted that the streamlines are symmetrical

about a diametral plane perpendicular to the direction of the stream. This means that, although the liquid is accelerated as it flows past the sphere (this is indicated by the crowding of the streamlines at the sides), it is slowed down after passing, and the liquid has the same velocity at corresponding points in front of and behind the sphere. It has, therefore, received no permanent increase of kinetic energy, and the only work done by the force which maintains the motion is done against viscosity. In other words, when the liquid which has passed round the sphere gives up its temporarily acquired kinetic energy, it creates a pressure behind the sphere which is exactly equal to that which was required at the front of the sphere in order to give it the extra kinetic energy. This is an application of Bernoulli's equation (page 207). But for the existence of viscosity, therefore, it would be possible to maintain the relative motion of the sphere and the liquid without exerting a force at all and without doing any work.

When turbulent flow occurs, however, the liquid behind the sphere retains the kinetic energy which it acquired as it flowed past the sides. A circulation is set up which did not exist in front of the sphere, so that the pressure behind the sphere is less than in front on account of the greater fluid velocity. If the liquid were non-viscous, this distribution of velocity (if it could be established) would nevertheless cause a resisting force, and the work done in overcoming this would represent the kinetic energy associated with the eddies, which would, of course, persist indefinitely. Above the critical velocity the resistance due to turbulence is approximately proportional to the square of the velocity. This is because the kinetic energy of an element of fluid is proportional to the square of its velocity. The resistance of the air to the motion of fast vehicles and especially of aeroplanes increases rapidly when the speed exceeds that at which turbulence begins. It is possible, by modifying the shape of the body, to raise this critical velocity and thus to reduce the resistance at a given speed by ensuring orderly instead of turbulent flow. This is called **streamlining** the body. In the case of a sphere it is easy to see that turbulence is due to the failure of the streamlines to come together again behind the sphere. If a pointed " tail " (Fig. 222) is added, the bending of the lines behind the sphere is made less sharp and orderly flow persists at higher velocities, thus involving a smaller expenditure of power. The principle involved can be regarded as the filling up of the turbulent region by a portion of the solid body itself. With turbulent motion, air resistance diminishes when the pressure is reduced, on account of the smaller energy associated with the eddies of air of lower density. Thus for long-distance flights it is speedier and more economical to fly at great altitudes.

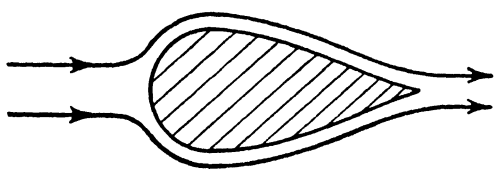


FIG. 222

If the air is suddenly compressed at some point, *e.g.* by an explosion which produces a large volume of gas, a wave of compression spreads out in all directions with a speed of about 750 m.p.h., and the disturbance is thereby dissipated. (Actually the speed is greater than this very near a large explosion.) This is the speed with which sound travels in air, because sound consists of a series of compressions and rarefactions generated in the air by a vibrating body. If an object is projected or propelled through the air at a greater speed than that of sound, the air compression which its motion produces in front of it is, so to speak, being built up faster than it can disperse. Consequently a very large pressure is generated in front of the moving object. The enormous resistance which therefore comes into play when the speed of a body approaches that of sound constitutes a problem in connection with high-speed air travel.

Determination of Viscosity by Falling Spheres.—Suppose that a sphere of radius a and density ρ is free to fall under the action of gravity in a liquid of density σ. The net downward force on it due to gravity is equal to the difference between its weight and that of the liquid which it displaces, *i.e.* $\frac{4}{3}\pi a^3(\rho-\sigma)g$ absolute units, and is a constant. If the downward velocity of the sphere is v, it experiences an upward force due to the viscosity of the liquid equal to $6\pi\eta av$, provided that the motion of the liquid is non-turbulent.

The resultant downward force is therefore

$$\tfrac{4}{3}\pi a^3(\rho-\sigma)g-6\pi\eta av$$

and the acceleration of the sphere is equal to the quotient of this force by its mass. But as v increases, the above expression diminishes and eventually becomes zero. From this stage onwards the sphere falls with a constant speed (known as its **terminal velocity**) of, say, v_0. If the liquid is still not turbulent when the speed v_0 is reached, we have

$$\tfrac{4}{3}\pi a^3(\rho-\sigma)g-6\pi\eta av_0=0$$

or

$$\eta=\frac{2a^2(\rho-\sigma)g}{9v_0}$$

This equation provides a simple absolute method of determining the viscosities of highly viscous liquids, such as heavy oils, for which the method of flow through a tube would be unsuitable. Unless corrections are made, it is necessary to satisfy as nearly as possible the conditions which Stokes assumed in calculating his expression for the resistance. One of these is that the liquid is infinite in extent, or, in other words, the effects of the walls and bottom of the containing vessel are ignored. It is therefore advisable to use a large quantity of liquid contained in a wide deep vessel such as a cylindrical glass jar (Fig. 223), and to determine the terminal velocity of the falling sphere when it is in the central region.

To do this, two horizontal levels, A and B, are marked by tying threads round the outside of the jar in such positions as to divide the total depth of the liquid into three approximately equal parts. Ball bearings are then dropped singly through the vertical glass tube, and their times of fall (t) between A and B are determined by a good stop-watch. It is necessary, of course, that the spheres should have reached their terminal velocity before passing A. This velocity can then be determined by the equation

$$v_0 = \frac{AB}{t}$$

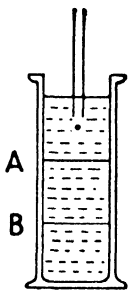


FIG. 223

The diameter of each ball should be measured with a micrometer screw gauge, and, if several of them are weighed, their density ρ can be calculated. It is important to keep the temperature of the liquid constant during the experiment and to record its value, because the viscosity of highly viscous liquids diminishes quite rapidly with rising temperature. It will be realized that this fact constitutes a difficulty in connection with lubrication—particularly with car engines. An oil which has a suitable viscosity at the normal running temperature may, unless it is correctly selected, be so viscous at air temperature as to make starting difficult.

4. MISCELLANEOUS

Non-Newtonian Liquids.—Newton's statement that the viscosity of a fluid is independent of velocity gradient is strictly true for a large number of liquids and gases; but there are also many cases in which it is not. Solutions (especially colloidal) frequently exhibit an effectively smaller viscosity when the velocity gradient is large, *i.e.* at high rates of flow. Such fluids are said to be **thixotropic.** Ordinary oil paint is a good example of this. Were it not for its thixotropy a paint which was sufficiently viscous not to drain down a vertical surface as soon as it was applied would require a great effort when it was being brushed on, because there is a very high velocity gradient in the thin layer during brushing. In actual fact the large velocity gradient effectively lowers the viscosity during the brushing. The viscosity under low rates of shear must not be too great, however, or the effect of surface tension will be insufficient to eliminate brush marks. Gelatine, liquid cement, clay, milk and blood are only a few of the large number of natural and artificial thixotropic substances.

There is a smaller class of substance whose effective viscosity *increases* with rate of shear. Such materials are said to show **dilatancy.** A suspension of rice starch in water shows this property at a certain concentration. If a ball is pulled through the liquid by means of an attached

wire its velocity is independent of the force over a wide range. If the force is applied by the hand it is very curious to feel the resistance automatically increasing as the effort is increased.

The Origin of Viscous Forces.—It seems probable that viscosity in liquids has its seat in the cohesive forces between molecules. Evidently relative motion between molecules involves the successive linking and parting of any one particular molecule with those with which the motion brings it in contact. A theory of liquid viscosity has been developed along these lines by Andrade. Since there are no cohesive forces between the molecules of a gas (or at any rate only very feeble forces), it might at first sight be thought that gases would have no viscosity. It must be remembered however that gas molecules move about with comparative freedom so that there is a constant exchange of molecules between any pair of adjacent layers. Thus when the layers are in relative motion, those molecules moving from the slower to the faster layer take with them their lower velocity and so slow down the faster layer. Similarly those going in the reverse direction speed up the slower layer. Therefore the relative motion will subside unless an external force is applied.

An expression for the viscosity of a gas can be worked out on the basis of the kinetic theory of gases, using the above ideas, and molecular diameters have been calculated by means of it. The formula obtained also predicts that the viscosity of a gas is independent of its pressure. This is found to be true over quite a wide range, although, as might be expected, viscosities diminish as very low pressures are reached. Again, the kinetic theory of gases predicts that the viscosity of a gas should increase with rising temperature in contrast with the behaviour of liquids. This is also found to be true.

EXAMPLES XVI

1. State the meaning of the term *viscosity*, and describe how the coefficients of viscosity of water and salt solution may be compared. What do you know about the variation of viscosity with temperature? (L.Med.)

2. What is the distinction between turbulent and non-turbulent flow of liquid through a tube?

Liquid flows through a tube at a steady rate of 12 cm.^{3} per min. under the action of a pump which creates a pressure difference of 20 cm. of water between the ends of the tube. Calculate the power expended, and explain why there is a slight rise of temperature as the liquid moves through the tube. (L.Med.)

3. Discuss fully the meaning of the statement that a liquid cannot sustain a shearing stress. Describe an experiment for the determination of the viscosity of water.

4. Describe some form of viscometer, and explain how it may be used to investigate the variation of the viscosity of a liquid with temperature.

5. Explain the distinction between turbulent and steady flow when a liquid is flowing (*a*) through a tube, (*b*) past a stationary solid sphere. How may the two types of flow be demonstrated in each case, and how does the nature of the flow determine the major factor governing the resistance?

6. A powder, containing particles of various sizes which are insoluble in water, is shaken up with water and allowed to stand. If the depth of water is 10 cm., find the diameter of the largest particles that remain in suspension one hour later, assuming that the viscous force experienced by a spherical particle of radius a cm. moving with velocity v cm. per second through a medium of coefficient of viscosity η c.g.s. units is given by $6\pi\eta av$. (The viscosity of water may be taken as 0·01 c.g.s. units and the density of the particles 2·8 gm. per c.c.) (L.Med., abridged.)

Chapter XVII

DIFFUSION AND OSMOSIS

1. DIFFUSION

When some copper sulphate or other coloured soluble substance is placed in the bottom of a cylindrical jar containing pure water it gradually dissolves and the coloration slowly extends to higher and higher levels. This occurs without the help of any artificial aid to mixing, such as stirring. The dissolved substance is said to **diffuse** and its concentration is eventually uniform throughout the liquid, although the attainment of this condition requires a very long time (theoretically an infinite time).

Diffusion occurs between any pair of substances which are miscible. It is comparatively rapid when two or more gases diffuse into each other.

Experiments on the diffusion of dissolved substances in liquids were begun by Graham in the middle of the nineteenth century. They reveal that, for a given temperature, the rate of diffusion depends on the nature of the dissolved substance and on the gradient of concentration in the direction of the diffusion. This latter quantity is defined in a way which is analogous to temperature gradient in connection with the conduction of heat. We imagine a straight line OX (Fig. 224) drawn in the mixture. Let two points A and B be at distances x and $x+\Delta x$ from O, and let the concentration of dissolved substance at these two points be C and $C+\Delta C$ respectively. The average concentration gradient between A and B in the direction OX is then $\Delta C/\Delta x$. To define the gradient at a *point* we suppose that Δx (and therefore ΔC) becomes indefinitely small, and the concentration gradient then becomes

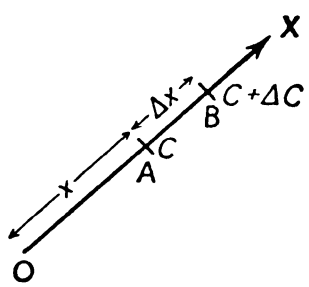


FIG. 224

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx} \text{ in the direction OX}$$

Experimental results on diffusion in liquids led to **Fick's law,** which states that, for a given dissolved substance, the mass diffusing across unit area in unit time is equal to

$$k\frac{dC}{dx}$$

where the x direction (*i.e.* the direction in which the concentration gradient is taken) is perpendicular to the given unit area. The quantity k is known as the **coefficient of diffusion** of the dissolved substance, and its value depends on the nature of the substance and the temperature of the solution. In actual fact the coefficient also depends upon concentration. Fick's law is analogous to the equation which relates the rate of conduction of heat through a substance with its thermal conductivity and the temperature gradient.

Explanation of Diffusion on the Kinetic Theory of Matter.—According to the kinetic theory of matter all molecules are constantly in a state of motion. It is supposed that in gases, and to a smaller extent in liquids, each individual molecule drifts about in a haphazard manner as a result of unhampered movements during the intervals between collisions with other molecules. In solids each molecule is, so to speak, anchored to one spot by the very close proximity of its neighbours, and its motion is almost entirely confined to vibration about a fixed position.

When very small particles are suspended in a liquid (*e.g.* gamboge in water) and examined under a high-power microscope, each particle is observed to be in a state of perpetual and random motion and to describe an irregular path. This so-called **Brownian motion** (discovered by a botanist, Robert Brown, early in the nineteenth century) is ascribed to the bombardment of the particles by the molecules of the liquid surrounding them.

We may presume that, as well as the suspended particles, each molecule of the liquid itself drifts from place to place in a haphazard manner as a result of collisions with others, and so also do the molecules of any dissolved substance. It is this motion which constitutes diffusion. The "randomness" of the motion, together with the enormous number of molecules of dissolved substance present in even dilute solutions, results, after a sufficiently long time, in a distribution which is found to be uniform when examined by the ordinary chemical or physical properties of samples of solution taken from various places. Nevertheless, if we were actually able to count the molecules which are present at any instant in extremely minute quantities of solution at various places we should find variations from one sample to another. But these differences become proportionately smaller as the size of the sample is increased. Similarly, if we could repeatedly count the number of solute molecules present in a very small volume at a number of given places we should find that the average of a large number of counts was the same for all places although individual counts would differ among themselves. The underlying principle is that random variations of any quantity tend to disappear with increasing size of the groups examined.

When the concentration of a solution is uniform, the average number of solute molecules crossing any plane is the same in both directions. This must be so because the concentration on each side of the plane remains

constant and the net transfer across the plane is therefore zero. Thus in Fig. 225 (i) there are equal numbers of molecules crossing the plane EF from left to right and from right to left, because the concentration (C_1) is the same on each side and remains so as time goes on. Similarly in Fig. 225 (ii). In this latter case the concentrations (C_2) are lower than C_1,

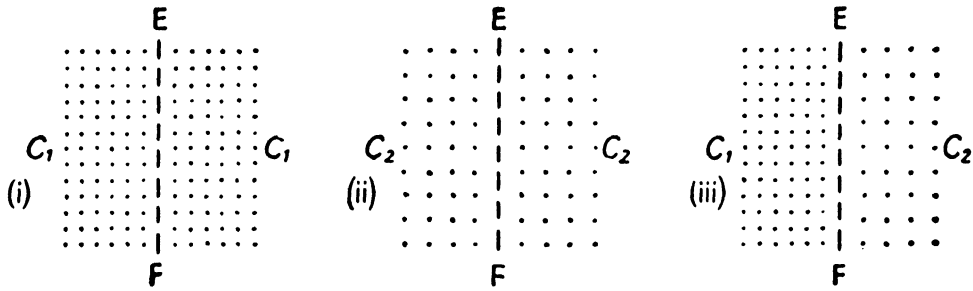


FIG. 225

so that the rate of transfer from each side is lower than in (i), because when molecules are moving at random, the number which cross any plane in a given time is evidently proportional to the actual number which participate in the random motion, *i.e.* to the concentration on the side of the plane from which they cross. In Fig. 225 (iii) the plane EF separates two regions of different concentration C_1 and C_2. The number of molecules crossing EF from left to right will be the same as in (i), and the number travelling in the opposite direction will be the same as in (ii). Consequently there is an excess moving from C_1 to C_2, which means that diffusion occurs towards places of low concentration and eventually brings about a uniform distribution.

According to the kinetic theory of gases the average kinetic energy of all gaseous molecules is the same at the same temperature. Thus for two different gases at the same temperature,

$$\tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m_2v_2^2$$

where m_1, m_2, v_1 and v_2 represent molecular masses and average velocities. At the same temperature and pressure equal volumes V of the two different gases contain the same number of molecules (Avogodro's hypothesis), say n. Let the masses of these equal volumes be M_1 and M_2 and their densities ρ_1 and ρ_2. Then

$$M_1 = nm_1$$
$$M_2 = nm_2$$

and

$$\rho_1 = \frac{M_1}{V}$$

$$\rho_2 = \frac{M_2}{V}$$

These equations give

$$\frac{\rho_1}{\rho_2}=\frac{M_1}{M_2}=\frac{m_1}{m_2}=\frac{v_2^2}{v_1^2}$$

or

$$\frac{v_1}{v_2}=\sqrt{\frac{\rho_2}{\rho_1}}$$

We should expect the rate of diffusion of a gas to be proportional to the average velocity of its molecules, and the last equation therefore suggests that the rates of diffusion of two gases should be in the inverse ratio of the square roots of their densities, provided that we make the comparison at the same temperature and pressure. This was confirmed by Graham when he allowed different gases at the same temperature and pressure to diffuse into a vacuum through a separating partition made of material with very fine pores, *e.g.* fine plaster or the clay of a tobacco pipe. Hydrogen diffuses four times as fast as oxygen since its density is one-sixteenth that of oxygen. The process is known as **effusion.**

The above effect may be demonstrated qualitatively by means of the apparatus shown in Fig. 226, in which A is a porous pot such as is used in electric batteries. A long glass tube B is connected to A by a gastight joint, and its lower end dips below some water C. The porous pot is surrounded by a large inverted beaker D, into which is passed a stream of hydrogen or coal-gas. This gas, being less dense than air, diffuses into the pot at a greater rate than the air contained in the pot diffuses out. Thus there is an increase of pressure in the pot which is indicated by the escape of bubbles from the lower end of B through C. When the jacket of light gas is removed, the water rises in B showing that the light gas which A now contains is passing out faster than air from the atmosphere is entering. The effects are reversed when A is jacketed by a heavy gas such as carbon dioxide.

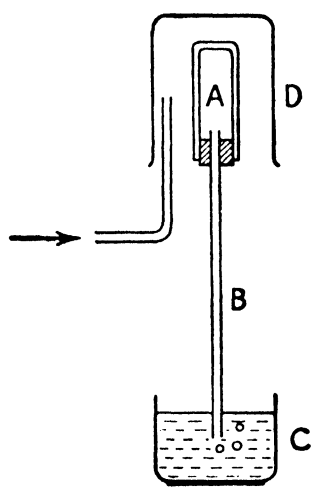


FIG. 226

When a mixture of gases is placed on one side of a porous plate its constituents diffuse through the plate at different rates, and the concentration of the denser components increases in the mixture which remains. This principle, applied successively, has been used to isolate the isotopes of gases which cannot be separated by chemical means.

2. OSMOTIC PRESSURE

Nature and Definition of Osmotic Pressure.—When a dissolved substance diffuses through a solvent and eventually reaches a uniform

distribution, its behaviour is analogous to that of a gas inasmuch as it distributes itself uniformly throughout the space which is available to it. The solute molecules are separated from each other by the act of dissolving them, so that they are no longer strongly bound together by the cohesive forces which kept them in the solid or liquid state before they were dissolved.

A further resemblance between a dissolved substance and a gas is revealed when a barrier or partition is placed in the solvent so as to prevent diffusion and thus to confine the dissolved substance to one region of the solvent. It is necessary that the partition should be of such a nature as to confine only the dissolved substance and to allow the solvent to pass freely through it. Such **semi-permeable membranes,** as they are called, do exist (*e.g.* parchment is impervious to dissolved sugar but not to water). They can also be made artificially. The action of a semi-permeable membrane may be likened to that of a net which encloses a catch of fish while allowing water to pass freely through its meshes.

Suppose that a solution is contained in a cylinder (Fig. 227) and that the semi-permeable membrane is made in the form of a piston which fits so perfectly against the walls of the cylinder as to prevent diffusion past its edges. This is, of course, impossible in practice, but it is a useful theoretical idea. The solute (indicated by dots in the drawing) is confined in the lower part of the cylinder and exerts an upward force on the piston in the same way as an ordinary fluid does when it is confined in a cylinder by a piston which is impervious to it. We can regard the solvent on both sides of the piston as having no effect on it whatever, and, by so doing, we ascribe the force on the piston to the action of the dissolved substance. If the downward force which must be applied to the piston in order to keep it in equilibrium is equal to F, and the area of the piston is A, then the **osmotic pressure** P of the solute (frequently referred to as the osmotic pressure of the *solution*) is defined by

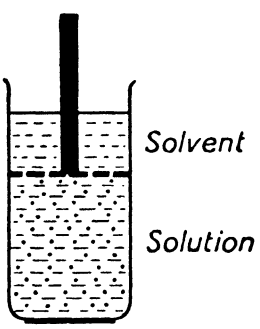


FIG. 227

$$P = \frac{F}{A}$$

Osmotic pressure increases with increasing concentration, which means that it is necessary to exert a greater downward force in order to keep the piston in equilibrium when it is near the bottom of the cylinder, or, conversely, if a greater force is applied the piston will move down to a lower equilibrium position.

Osmosis.—This process is the passage of *solvent* through a semi-permeable membrane under the action of osmotic pressure. A simple demonstration is shown in Fig. 228. A piece of parchment A is stretched

over the lower end of the vertical glass tube B, and a small amount of sugar solution is poured into the tube which is then partially immersed in a trough of water as shown. The level of the liquid in the tube begins to rise, and reaches a steady height after a considerable time. Water has entered the tube through the parchment membrane by osmosis, and the mechanism may be explained in terms of osmotic pressure as follows. The solute in the tube tends to increase its volume by the exertion of

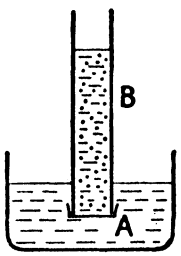


FIG. 228

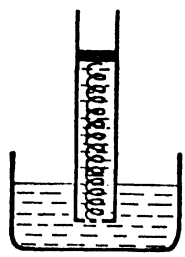


FIG. 229

its osmotic pressure, and the ability of the solvent to pass through the pores of the membrane makes the expansion of the solution possible. The process is the same as would occur if the trough and tube both contained pure water (Fig. 229), and if the driving force, *i.e.* the osmotic pressure of the solute, were replaced by a more tangible and obvious agent in the form of a compressed spring with its lower end pressing on the membrane and its upper end pushing on a piston with which the water is in direct contact. This arrangement is hydrostatically identical with a lift-pump except that the upward force applied to the piston is here exerted from below instead of above. It is clear that the long time which is necessary for the completion of osmosis is due to the large viscous forces opposing the flow of the water through the fine pores of the membrane. Osmosis would continue indefinitely while a supply of pure solvent is available were it not for the action of gravity on the raised column of solution. This creates an increasing force against which the osmotic pressure has to act.

It is interesting to note that osmosis would occur even if the membrane were replaced by, say, a metal plate which is impervious to both solute and solvent, provided that the system in Fig. 228 is totally enclosed in a vessel. If the solute is of a non-volatile character, the vapour in the vessel contains no solute molecules and is therefore pure solvent vapour and is in equilibrium with the pure solvent in the trough. The solution itself exerts a lower vapour pressure than this because of the presence of the non-volatile solute molecules in its surface. Therefore vapour condenses on the surface of the solution which, consequently, rises in the tube. The vapour pressure in the enclosure diminishes with height, however, so that the level of the solution ceases to rise when it has reached the height at which the vapour pressure in the enclosure has fallen to that of

the solution. The surface of the solution has acted as the semi-permeable membrane, allowing solvent but not solute molecules to pass through it.

In the model depicted in Fig. 229 the weight of the liquid in the tube above the outside level is held up by the spring which is itself supported by the membrane. The resultant downward force on the membrane is therefore equal to the weight of the raised column of liquid *plus* the weight of the spring itself (supposing that the piston is weightless). The principle is similar in the case of the solution. Suppose that the density of the solution (Fig. 230) when it has reached an equilibrium height h_1 above the membrane is ρ. Let the density of the pure solvent be ρ_0 and the depth of the membrane below its surface be h_2. According to the ordinary laws of hydrostatics, a liquid with a density ρ but no osmotic pressure would be in equilibrium with the solvent if its height (h_3) above the membrane (Fig. 230) were such that

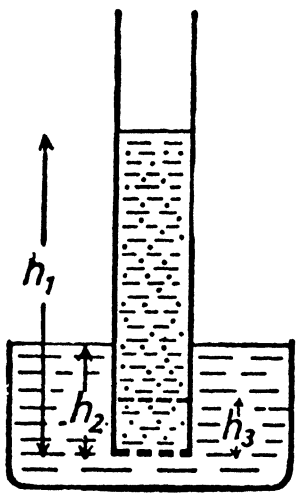


FIG. 230

$$h_2 g\rho_0 = h_3 g\rho$$

Therefore the additional head of solution above h_3 represents the osmotic pressure, and we have

$$\begin{aligned} P &= g\rho(h_1 - h_3) \\ &= h_1 g\rho - h_2 g\rho_0 \end{aligned}$$

which can be determined from observations. It should be noticed that the last equation shows that when the process of osmosis is complete, the osmotic pressure is equal to the difference of fluid pressure on the two sides of the membrane.

It must be realized that this experiment (and all others in which osmosis proceeds until a back pressure is generated which halts it) measures the osmotic pressure of the solution at its *final* concentration, which must therefore be determined at the end of the experiment. Another type of experiment has sometimes been used in which an artificial pressure is applied to the solution in order to prevent osmosis from taking place. The pressure necessary to do this is equal to the osmotic pressure of the solution which was originally put into the apparatus. Fig. 231 is a diagrammatic representation of the apparatus. An artificial membrane is deposited in the wall of the porous cylinder A, which contains pure solvent, the solution being placed in the metal jacket B

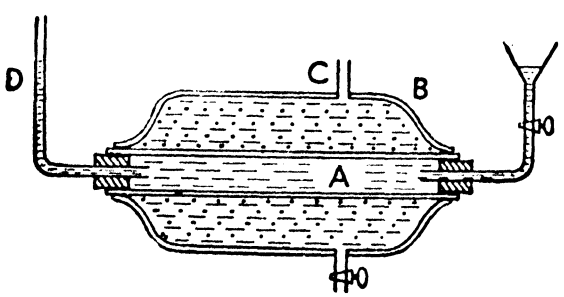


FIG. 231

surrounding A. Sufficient pressure is applied to the solution through the tube C in order to maintain the level of the solvent constant in the tube D. This pressure is the osmotic pressure of the solution.

The analogy of the spring which has been explained above makes it clear that the membrane itself is subjected to the osmotic pressure (the force of the spring) when equilibrium is attained, so that a parchment membrane supported round its perimeter is much too weak for high-pressure measurements. In experiments in which osmotic pressures of 100 atmospheres or more have been reached, the membrane consists of a layer of copper ferrocyanide deposited by electrolysis in the walls of a porous pot made from clay which has been carefully selected for uniformity. The pressure is not, of course, measured by allowing the osmosis to raise a column of solution, but by attaching a calibrated manometer (*e.g.* nitrogen-filled) to the porous pot containing the solution.

The Osmotic Pressure Equation for Dilute Solutions of Non-Electrolytes.—In addition to the previously mentioned qualitative similarity between the behaviour of a solute and a gas there is, in certain circumstances, a remarkable quantitative resemblance. Provided that the molecules of dissolved substance do not dissociate (*i.e.* split up into ions when in solution) as do the so-called electrolytes like NaCl and $CuSO_4$, and provided also that the solutions are dilute, it is found experimentally that the osmotic pressure of any substance is proportional to its concentration and to the absolute temperature of the solution. (Absolute temperature is approximately Centigrade temperature with 273 deg. C. added on.) Furthermore, the constant of proportionality is the same for all substances when concentration is expressed in terms of the number of gram-molecules of dissolved substance present in unit volume of solution. Thus, if C is the concentration in (gm.-mol.) cm.$^{-3}$ of a solution which exerts an osmotic pressure of P dynes cm.$^{-2}$ at an absolute temperature T, we have

$$P = KCT$$

where K is the same constant for all solutes. Suppose that V is the number of cm.^{3} of solution which contain one gram-molecule of dissolved substance, then

$$C = \frac{1}{V}$$

and the last equation can be written

$$PV = KT$$

Turning now to the behaviour of a gas, we find that when one gram-molecule of any ideal gas at an absolute temperature T is confined in a vessel of volume V it exerts a pressure p which is given by the equation

$$pV = MRT$$

where M is the molecular weight of the gas and R is a constant which is governed by the nature of the gas, although the product MR is the same for all gases. From experimental observations it is found that the coefficients of T are identical in the two cases, *i.e.*

$$K = MR$$

and the similarity between a dissolved substance and a gas is therefore complete. It is sometimes expressed by saying that for dilute solutions of non-electrolytes the osmotic pressure of the dissolved substance contained in a given solution is equal to the pressure which it would exert if it were in the gaseous state at the same temperature and if it occupied a volume equal to that of the solution. If this is true, an osmotic pressure of 76 cm. of mercury at 0° C. would be exerted by any dissolved substance when the concentration of the solution is such that 22·4 litres of solution contain a gram-molecule of solute. This corresponds to the fact that a gram-molecule of any ideal gas occupies 22·4 litres at standard temperature and pressure.

Example.—*Calculate the osmotic pressure of a solution which contains* 0·6 *gm. of sugar* ($C_{12}H_{22}O_{11}$) *in* 100 *cm.*^{3} *of solution at a temperature of* 20° *C.*

According to its chemical formula the molecular weight of the sugar is 342 to the nearest whole number. Therefore its osmotic pressure would be 76 cm. of mercury at 0° C. for a concentration of $\frac{342}{22{\cdot}4}$ gm. litre^{-1}. Its actual concentration is 6 gm. litre^{-1}. Therefore, since osmotic pressure is proportional to concentration and to absolute temperature, the required osmotic pressure is

$$76 \times 6 \times \frac{22{\cdot}4}{342} \times \frac{293}{273} = 32 \text{ cm. of mercury, approximately}$$

The fact that MR is the same for all ideal gases means that the pressure which an ideal gas exerts at any given temperature is proportional to the number of molecules contained in a given volume regardless of the nature of the molecules. This tallies with the fact that the constant MR is the same for mixtures as well as for pure gases. The identity of the equations for gaseous and osmotic pressures implies that the osmotic pressure also depends on the number of molecules present and not on their nature. We should therefore expect a dissociated electrolyte to exert a higher osmotic pressure than a non-electrolyte of the same molecular concentration because some of the molecules of the electrolyte are split into two ions, each of which contributes as much to the osmotic pressure as a single molecule. This is found to be the case.

It should be realized that the similarity between gaseous and osmotic pressures as regards their dependence upon volume and temperature does not necessarily imply that the mechanism by which the pressure is exerted is the same in both cases. Indeed, some people deny that there can be any fundamental similarity. It must be admitted, however, that a col-

lection of solute molecules suspended in a solvent and performing random motion is very similar to the kinetic-theory picture of a gas, and that the bombardment of the membrane by the solute molecules provides a fairly convincing explanation of osmotic pressure. Moreover, it is possible to form a picture of the process of osmosis as distinct from osmotic pressure in terms of molecular motion. Thus a membrane separating a solution from pure solvent suffers equal molecular bombardment on both sides if the ordinary hydrostatic pressure in the two liquids is the same. But whereas the bombardment from the solvent side is entirely due to molecules which pass through the membrane, a proportion of the molecules (*viz.* those of the solute) which strike the membrane on the other side are prevented by the nature of the membrane from passing through it. Therefore more solvent molecules enter the solution than leave it, so that the latter tends to expand and to become more dilute.

It should be understood that everything which has been said by way of explanation and description of osmosis in terms of either molecular motion or osmotic pressure deals only with the mechanical aspect of the phenomenon. The reason for the behaviour of the membrane as a kind of filter for certain—but not all—types of molecule is a complicated matter which we shall not attempt to discuss.

Some Consequences of Osmotic Pressure.—The linings of the walls of cells of living matter are often semi-permeable. Under the influence of the osmotic pressure of the dissolved substance which they contain, such cells will swell up when surrounded by pure solvents or solutions of lower osmotic pressure than the cell contents. When they are immersed in a solution of higher osmotic pressure, solvent diffuses out through the cell walls and they shrink. It is important that when liquid, *e.g.* glucose solution, is introduced into the blood-stream it should have the same osmotic pressure as that of the contents of the corpuscles, otherwise these will suffer damage. Solutions having equal osmotic pressures are said to be **isotonic,** while a solution of higher osmotic pressure than another is said to be **hypertonic.** Hypertonic solutions are used in order to extract fluid from regions of skin infection or wounds.

EXAMPLES XVII

1. Describe an experiment which demonstrates osmosis. Show how this phenomenon can be explained by the conception of osmotic pressure.

A solution containing 5 gm. of sugar per litre has an osmotic pressure of 25·2 cm. of mercury at 7° C. What is the concentration of a sugar solution which has an osmotic pressure of 33·6 cm. of mercury at 27° C.? (L.Med.)

2. Explain what is meant by *osmosis,* and describe how you would demonstrate and measure an osmotic pressure.

Calculate the osmotic pressure at 0° C. of a solution of cane sugar (molecular weight 342) containing 15 gm. of sugar per litre. The gram molecular volume of hydrogen at S.T.P. is 22·3 litres. (L.Med.)

Chapter XVIII

SURFACE TENSION

1. INTRODUCTION

A Liquid Surface is in a State of Tension.—In Fig. 232 are shown the outlines of a number of different-sized drops of mercury resting on a solid. If the only forces acting on each drop were the weight of the liquid and the reaction of the solid, the liquid would be spread out into a thin sheet. In other words, the centre of gravity of each drop would reach its lowest possible position. A glance at the actual shapes suggests, therefore, that there is some other force which opposes the action of the weight even when this is large. As the size, and therefore the weight, of the drop diminishes, the shape becomes more nearly spherical. This suggests that if the weight of the drop were entirely eliminated the force which opposes the weight would make the drop perfectly spherical. Such a state of affairs can actually be realized by choosing two liquids of equal density which do not mix with each other (*e.g.* aniline and a solution of common salt in water) and forming drops of one liquid within the other by means of a pipette. When this is done the drops remain in suspension and are perfectly spherical. If one of the drops is deliberately deformed it returns to its original shape. We conclude that there is a force which moulds the drop into a spherical shape when other forces are absent.

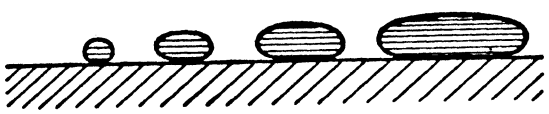


FIG. 232

Now the important characteristic of a sphere in this connection is that it, of all possible geometrical shapes, has the smallest area of surface for a given volume. The force which we are discussing, therefore, acts in such a way as to reduce the surface area of the liquid drop to its smallest possible value. In other words, we can regard it as a *tension* which acts parallel to the surface, although it must be said at once that the tension in a liquid surface is, in general, not increased by stretching as it is in the case of a solid membrane. When a surface is extended, its physical state and its tension remain constant provided that there is sufficient liquid in the interior to move into the surface as the stretching proceeds. This is always the case unless the liquid has been stretched into an exceedingly thin film.

It is also possible to describe the behaviour of a liquid drop by supposing

that each sq. cm. of its surface has a certain amount of potential energy. This would cause the drop to adjust its shape until its surface area is a minimum in accordance with the principle that every system moves towards a condition of minimum potential energy and is in stable equilibrium when the minimum is reached. The conceptions of the tension and of the potential energy are exactly equivalent to each other, as we shall see later on.

Many other observations point to the existence of surface tension. It is well known that a small sewing needle (particularly if it is smeared with vaseline) can be made to rest on the surface of water provided that it is laid on gently. Once it is immersed it sinks, of course, owing to its greater density. Certain insects rely on the action of surface tension to support them on the surface of a pond.

Perhaps the most striking demonstration of surface forces, however, is provided by the behaviour of thin liquid films such as are readily made with a soap solution. The envelope of an ordinary soap bubble provides evidence of surface tension by its spherical shape, and also by the fact that its tension compresses the air inside it and will force it out if an exit is provided. If a soap film is formed in the area enclosed by the bent wire ABC and the separate straight wire DE in Fig. 233, then it is necessary to apply a small force to DE in the direction shown in order to keep it in position, *i.e.* in order to prevent the film from contracting.

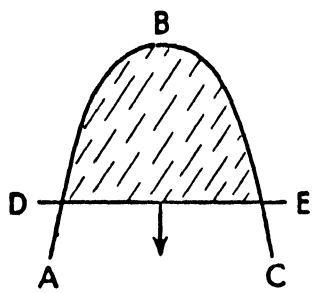


FIG. 233

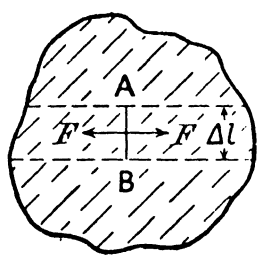


FIG. 234

Definitions of Surface Tension and Surface Energy.—The definition of surface tension is analogous to the definition of fluid pressure. Let a short straight line AB of length Δl (Fig. 234) be drawn in a liquid surface, and suppose that the state of tension of the surface is such that the surface on one side of AB exerts a pull of F dynes on the surface lying on the other side. In a *liquid* surface this force is at right angles to AB and is in the tangent plane of the surface at AB. Each portion of the surface exerts a pull on the other portion from which it is separated by AB. The **surface tension** (γ) is defined as the force exerted across AB per unit length of AB. Thus

$$\gamma = \frac{F}{\Delta l}$$

The units in which surface tension is expressed are dynes cm.$^{-1}$. Water has a surface tension of about 73 dynes cm.$^{-1}$ at 20° C., most organic liquids have values between 20 and 30 dynes cm.$^{-1}$, while the surface tension of mercury is about 480 dynes cm.$^{-1}$.

To calculate the force acting across a line drawn in a *curved* surface, we can imagine the line to be made up of a series of small elements each of which is so short as to be sensibly straight. For example, the total force which one half of the surface of a spherical drop exerts upon the other half (Fig. 235) is the sum of a large number of small forces like $\gamma\Delta l$. Each force is at right angles to the great circle separating the two halves (only one of which is shown in the figure) and they are all in the same direction. The total force in this direction is therefore $\Sigma\gamma\Delta l$, which is equal to $\gamma.\Sigma\Delta l$ or γl, if l is the total circumference of the circle.

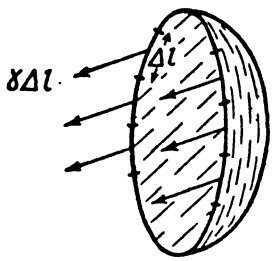


FIG. 235

We now consider the process of increasing the area of surface associated with a given volume of liquid. The simplest way of picturing this is to imagine that the liquid is in the form of a film ABCD contained in the plane wire framework shown in Fig. 236 (i). The side CD is supposed to be free to move, and it is found necessary to act upon it with a force P, as shown, in order to maintain it in equilibrium. Imagine that the film is divided into two portions by the line EF of length y parallel to CD. The portion on the left of EF acts upon that to the right with a force which is perpendicular to EF, directed towards the left, and has a magnitude of γ per unit length of EF. This force is therefore equal to $2y\gamma$ because the film has two surfaces each of width y. Therefore the equilibrium of the portion of the film CDFE (Fig. 236 (ii)) requires that

$$P = 2y\gamma$$

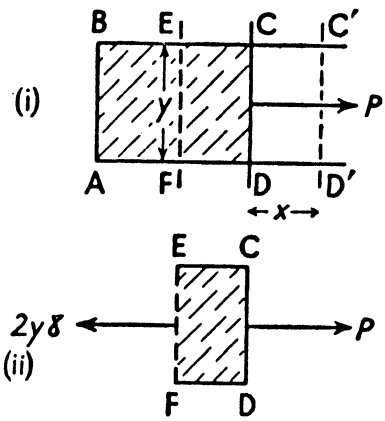


FIG. 236

Now suppose that CD moves to the right through a distance x to the position C′D′. It must be assumed that the movement occurs very slowly and without acceleration, and under these conditions the magnitude of the force P necessary to keep the film stretched would remain constant. The work done by P is Px, and as a result of this expenditure of work an area of surface equal to $2xy$ has been created. Therefore work is necessary in order to create a new surface, and, furthermore, when the surface shrinks it does work, *e.g.* if the movable rod returns to its original position CD it does an amount of work equal to Px. Thus each sq. cm. of surface has a certain amount

of energy associated with it, and we define the **free surface energy** per sq. cm. as the mechanical work required to produce one sq. cm. of new surface. Thus

$$\text{Free surface energy per sq. cm.} = \frac{Px}{2xy}$$
$$= \frac{2y\gamma x}{2xy}$$
$$= \gamma$$

The free surface energy per unit area is therefore numerically equal to the surface tension per unit length. The units ergs cm.$^{-2}$ and dynes cm.$^{-1}$ are, of course, identical.

The term "*free* surface energy" should be used rather than just "surface energy" in order to denote the *mechanical* energy stored in one sq. cm. of surface. A surface cools when it is extended, so that to keep its temperature constant it is necessary that it should absorb some heat. Heat is a form of energy, so that in actual fact the energy necessary to increase the area of a surface isothermally (*i.e.* at constant temperature) is greater than the mechanical work which must be done. The *total* surface energy is the sum of the mechanical work and the heat. When the surface is allowed to contract, both the mechanical (or free) energy and the heat are given back.

The extension of a liquid surface (*e.g.* the stretching of a film just discussed) consists essentially in the transference of molecules from the interior to the new surface, and experimental observations of what occurs when it is carried out show that this process requires the expenditure of mechanical work and heat.

2. SURFACE TENSION ASCRIBED TO COHESIVE FORCES

Cohesion.—We know that the molecules of a liquid are bound to each other by forces of cohesion. If this were not so, a liquid would behave like a gas and would fill any space into which it is placed. The action of cohesion can be observed when a liquid is put into a cylinder with a tightly fitting piston (Fig. 237), and an outward force is applied externally to the piston. Provided that the liquid is free from dissolved gases, the force required to cause a cavity to form in it may be so large as to create a negative pressure of many atmospheres in the liquid. Thus the cohesion between adjacent molecules of the liquid is able to oppose very large forces tending to pull them apart. Incidentally such an experiment also provides evidence for the existence of **adhesion** between the liquid and the walls of the cylinder and piston.

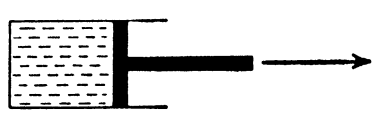


FIG. 237

Conditions in the Surface.—The magnitude of the force between

two molecules must be supposed to diminish very rapidly with their distance apart and to become zero at some finite but very small distance, probably of the order of two or three times the normal distance between adjacent molecules. Thus it is possible to describe an imaginary sphere round any chosen molecule such that if the centre of another molecule comes within this sphere it suffers an appreciable force of attraction, while if it is outside the sphere it does not. Such a sphere is called the " sphere of influence " of the molecule about whose centre it is described, and the radius of the sphere obviously represents the maximum distance between the centres of two molecules between which a force of attraction exists.

Consider a molecule at A (Fig. 238) which is well inside the liquid. Although the number and distribution of the other molecules within its sphere of influence vary from time to time owing to the thermal agitation of all molecules (including A), yet on the whole, *i.e.* over a considerable time, there is no excess in any part of the sphere, and the molecule at A is acted upon by attractive forces equally in all directions. On the other hand, for a molecule B, situated near the surface, there are more attracting molecules below it than there are above it, the excess being equal to the number of molecules which would have been contained in the top portion of the sphere had it not been cut off by the surface. Therefore any molecule such as B, whose sphere of influence extends beyond the surface of the liquid, experiences a force due to cohesion directed towards the interior and perpendicular to the surface. When a molecule is taken from the interior to the surface it begins to experience this force as soon as its sphere of influence touches the surface (C), and the force reaches its maximum value when the molecule is actually in the surface (D).

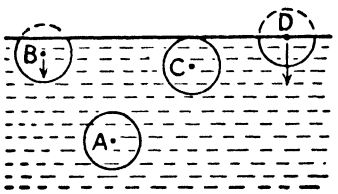


FIG. 238

When the area of a liquid surface is increased, each molecule which enters the surface has to be pulled or pushed there against the action of the inward pull. This explains why it is necessary to perform mechanical work in order to increase the surface area of a liquid. The external force such as P (Fig. 236) which performs the work does not, of course, act *directly* on the molecules which enter the surface. It may be regarded as initially producing a small separation, parallel to the surface, of the molecules already present in the surface, with the result that a molecule from the interior enters the surface to restore its original structure. Another slight separation then follows and another molecule moves to the surface, and so on. Thus, as the extension proceeds, the tension in the surface is kept constant by the entry of molecules from the interior.

The molecules in a liquid surface are all under the influence of the inward attraction, and to prevent their entering the interior it is necessary

to apply an external force in order to prevent the contraction of the surface which would result from their leaving it. This explains why, in the case of a plane film (Fig. 236), the force P is necessary to maintain equilibrium. In a spherical liquid drop the contraction of the surface is opposed by an excess pressure in the interior which is set up by the contractile tendency of the surface.

The foregoing arguments may be stated in a rather shorter form as follows. Since work must be performed on each molecule which moves from the interior to the surface, any given number of surface molecules has more potential energy (due to the inward attracting force) than the same number in the interior. The total amount of this excess mechanical energy for each sq. cm. of surface is the free surface energy, *i.e.* the work necessary to create one sq. cm. of surface. Surface energy necessarily implies the existence of surface tension, because the statement that a surface loses mechanical energy when it contracts is, according to the definition of energy, only another way of saying that it performs mechanical work. In order that it shall perform mechanical work it must exert a force on its surroundings, *e.g.* the sides of the frame in Fig. 236. This means that it is in a state of tension.

Note on the Reality of Surface Tension.—It is surprising that many people contend that surface tension is a purely fictitious property of a liquid and that it has no real existence. This fallacy seems to have its origin in the fact that the existence of surface *energy* can be readily explained by the cohesive forces between molecules. It has already been pointed out above, however, that the existence of surface energy implies a state of tension in the surface, and, in any case, the simplest laws of statics applied to the system in Fig. 236 require that the force P shall be balanced by an equal and opposite force which we call surface tension. The existence of surface tension does not depend on the nature of our theories about liquids, but on simple experimental observations such as the necessity of applying P in order to prevent the surface from shrinking. It can be shown, of course, that cohesive forces explain the existence of surface tension without introducing the idea of free surface energy, but it is not necessary to give the argument here.

3. SOME EFFECTS OF SURFACE TENSION

The Pressure Difference due to a Curved Surface.—In Fig. 239 (i) ABB′A′ represents a portion of a cylindrical liquid surface of radius R. Let $AA' = BB' = 1$ cm. Then each of these edges is acted upon by a force γ as shown in both the figures. The surface-tension forces on the curved edges AB and A′B′ are equal and opposite to each other, but each of the two forces γ acting on AA′ and BB′ has a component equal to $\gamma \sin \theta$ parallel to the direction of CO which bisects the angle $A\hat{O}B$ (Fig. 239 (ii)) into two equal angles θ. (It is easily shown that each force

makes an angle of $(90-\theta)°$ with the direction CO.) The components in a perpendicular direction are equal and opposite and cancel each other. There is, therefore, a net force of $2\gamma \sin\theta$ acting on the surface in the direction CO, and for equilibrium it is necessary that the pressure in the

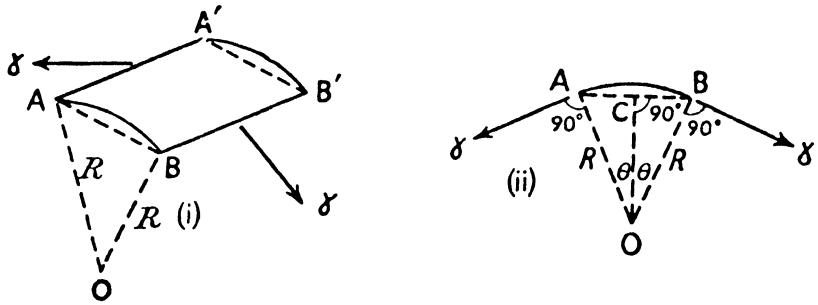


FIG. 239

fluid on the concave side of the surface should be greater than that on the convex side. If this excess pressure is p, then the force which it exerts in the direction OC on the curved surface ABB'A' is equal to p multiplied by the *plane* area ABB'A' which is perpendicular to OC (page 239). Thus

$$\begin{aligned} 2\gamma \sin\theta &= p \times (\text{plane area ABB'A'}) \\ &= p \times (\text{straight line AB}) \times 1 \\ &= p \times 2\text{AC} \times 1 \\ &= p \times 2R \sin\theta \end{aligned}$$

whence $$p = \frac{\gamma}{R}$$

This pressure excess will be expressed in dynes cm.$^{-2}$ if γ is in dynes cm.$^{-1}$ and R is in cm.

If a liquid surface is curved in two directions instead of in only one, there is a further excess pressure due to this additional curvature. A spherical surface is the equivalent of the superposition of two equal cylindrical surfaces with axes at right angles. Therefore inside a spherical surface of radius R there is an excess pressure of $2\gamma/R$. This is the case with a spherical bubble of air in the interior of a liquid, or with a spherical drop of liquid surrounded by air or another liquid. With an ordinary soap bubble it must be remembered that the film has two surfaces of equal radii (the thickness of the film being negligible), so that the pressure of the gas inside exceeds that outside by $4\gamma/R$ dynes cm.$^{-2}$. This is a comparatively small pressure since γ is about 30 dynes cm.$^{-1}$ for a soap solution, and if R is, say, 2 cm., the pressure is 60 dynes cm.$^{-2}$ or approximately 0·06 cm. of water. It is possible to measure a pressure such as this with a sensitive manometer and thus to determine the surface tension of a soap solution, but the method is not very accurate.

An effective demonstration of the fact that the pressure inside a small bubble is *greater* than that in a large one is made by means of the apparatus shown in Fig. 240. The slightly flared lower ends of the two branching tubes are dipped in soap solution and withdrawn, and bubbles of different sizes (D and E) are formed by blowing gently into the apparatus with the tap A open and B and C alternately closed and open. The tap A is then closed and both B and C are opened, with the result that the smaller bubble shrinks while the large one grows until their radii are equal as indicated by the dotted lines. The experiment is more successful when both bubbles are fairly small.

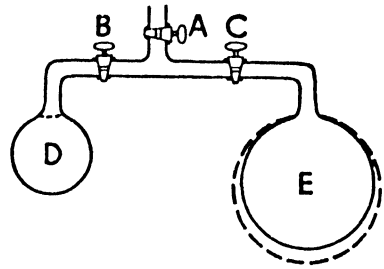


FIG. 240

Angle of Contact.—It is a well-known fact that an otherwise plane liquid surface is curved near the walls of the containing vessel. In Fig. 241 (i) and (ii) are depicted two of the many possible ways in which liquids and solids can be in equilibrium. The angle marked θ is, in each case, the inclination of the liquid surface to the common surface between the solid and liquid at the point of meeting. This is called the **angle of contact** and is characteristic of the pair of substances concerned.

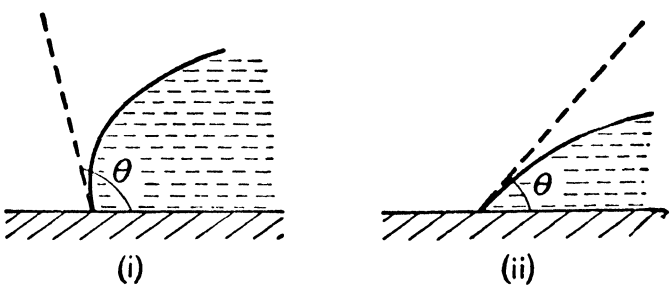


FIG. 241

Since the liquid surface is usually curved, its direction at the junction must be determined by drawing a tangent. The angle of contact is always measured in the liquid. Mercury and glass give a contact angle of about 140° (Fig. 241 (i)), while water and wax give an acute angle. Zero contact angle is very common, *e.g.* with water, alcohol, benzene, and many other liquids in contact with glass, provided that the glass is thoroughly clean and free from grease.

In actual fact the equilibrium value of the contact angle often depends on the direction of the motion by which the liquid has reached its final position. If the liquid has been moving towards the uncovered portion of the solid surface, θ is known as the **advancing** contact angle (θ_A), and is often greater than in the opposite case—the **receding** angle (θ_R). An example of this is provided by a raindrop which has come to rest after

sliding down a window-pane (Fig. 242). Thus a liquid and solid which give a receding angle of 0° often show finite advancing angles. This complication will, to some extent, be ignored in what follows.

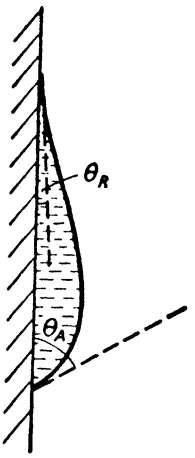


FIG. 242

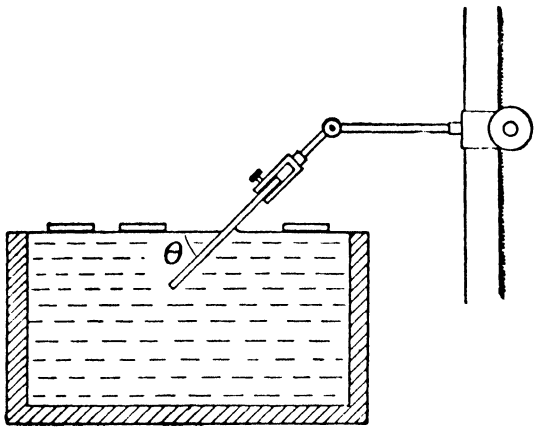


FIG. 243

Determination of Angle of Contact.—One of the simplest methods of determining contact angle is known as the **tilting-plate** method. The apparatus is shown in Fig. 243. The liquid is made to fill completely a rectangular tank which can be levelled. The surface of the liquid may be cleaned by moving across it strips of glass coated with paraffin wax which rest on the top of its sides. The ends of three of these are shown in the figure. The surface contamination is swept to one end and imprisoned on the surface by one of these movable barriers. The solid with which it is desired to measure the contact angle is in the form of a flat plate supported by the arrangement shown, which enables the experimenter to vary the inclination of the plate and also to move it vertically by a rack and pinion. To find the advancing contact angle the plate is set at various inclinations, and after each setting it is lowered into the liquid. When, on doing this, the surface of the liquid on one side of the plate remains plane right up to the plate (as shown on the left-hand side of the plate in the figure), the angle θ is measured with a protractor and taken to be the advancing angle. The receding contact angle is similarly found by pre-setting the inclination of the plate while it is partially immersed and then raising it.

Spreading and Wetting.—It is well known that a quantity of mercury placed on a horizontal glass surface shows little tendency to spread whereas a liquid like benzene does. Naturally the weight of the liquid assists spreading, but it is evidently not the only cause, because spreading can occur in a vertically upward direction. It is also clear that the surface tension of the liquid opposes spreading because it opposes the extension of the liquid/air surface. But we do not give a complete account of the

phenomenon until we take into consideration the condition of the liquid surface which is in contact with the solid.

We have seen how surface energy and surface tension can be attributed to the inward attraction experienced by the surface molecules of the liquid. Suppose that the air in contact with a liquid is replaced by a solid (Fig. 244). Liquid molecules near the surface of the solid will now experience a force of attraction F_S towards the solid. Let F_L be the force of attraction towards the interior of the liquid which these molecules experienced before the air was replaced by the solid. The net force on a molecule when the solid is present is $F_L - F_S$, and the surface tension in the liquid surface which was due to the force F_L is now due to $F_L - F_S$ and is therefore reduced. Let its value be represented by $\gamma_{(S)}$. It is quite possible for the nature of the solid and liquid to be such that F_S exceeds F_L with the result that $\gamma_{(S)}$ becomes a negative tension, *i.e.* a "surface pressure." When this is the case, molecules of the liquid show a tendency to crowd to the solid surface and the area of contact between the liquid and solid tends to increase. We can suppose that this is the cause of spreading.

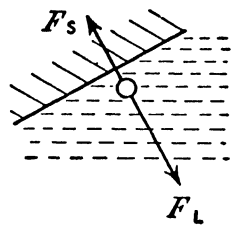


FIG. 244

Fig. 245 shows the state of affairs at the edge A of a liquid which has spread over a solid until a condition of equilibrium is reached. Suppose that the nature of the solid surface is not modified by the presence of the liquid, that is to say it has exactly the same properties on both sides of A. The force directly exerted by the solid on a liquid molecule at A would then be perpendicular to the solid surface and would have no tangential component. The condition of the liquid surface in contact with the solid is specified by a tension of $\gamma_{(S)}$ per unit length which, as we have seen, may be negative, in which case the surface is in a state of pressure. This must be the state of affairs in Fig. 245 if equilibrium exists, because, resolving parallel to the solid surface, the equilibrium of a line of molecules of unit length perpendicular to the paper at A is represented by

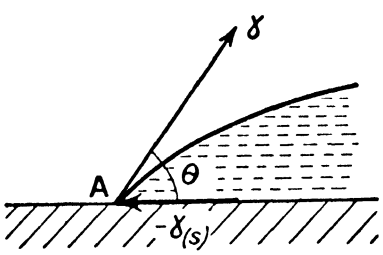


FIG. 245

$$\gamma \cos \theta + \gamma_{(S)} = 0$$

or

$$\gamma \cos \theta = -\gamma_{(S)}$$

where γ is the surface tension of the liquid against air. Thus when θ is less than 90° and its cosine is therefore positive, $\gamma_{(S)}$ must be negative and the surface is in a state of pressure. When θ is greater than 90°, $\gamma_{(S)}$ is positive and is a tension.

Evidently the angle of contact will be zero when γ and $-\gamma_{(S)}$ are numerically equal. It is quite possible for $-\gamma_{(S)}$ to exceed γ, in which case equilibrium appears to be impossible according to the last equation, because $\cos\theta$ cannot exceed unity, which is its value when θ is zero. In such a case the large value of $-\gamma_{(S)}$ would cause spreading to continue until the liquid is drawn out into a thin film, in which both the liquid/air and liquid/solid surfaces acquire a different character and tension from those bounding the liquid in bulk. Thus zero contact angle occurs in all cases in which $-\gamma_{(S)}$ is equal to or exceeds γ, and this explains why its occurrence is so frequent.

If the solid in Fig. 245 is replaced by another liquid with which the first does not mix, the spreading is governed by the equilibrium at A of *three* tensions, *viz.* the two surface tensions of the liquid/air surfaces and the tension in the composite surface separating the two liquids (the so-called **interfacial** tension). We shall not pursue this further, however.

Capillary Elevation.—This phenomenon is said to have been discovered by Leonardo da Vinci in the fifteenth century. If a clean glass capillary tube is fixed vertically in a beaker containing a liquid which wets glass (*e.g.* water, alcohol, benzene), the liquid rises up the tube and reaches a definite height h (Fig. 246 (i)), which depends on the nature of the liquid and the diameter of the tube. With a given liquid, the smaller the diameter the higher is the rise. The thickness of the walls of the tube has no effect on the phenomenon. The surface of the liquid at the top of the elevated column (the "meniscus" as it is called) is concave towards the air. This signifies an acute contact angle. With a liquid such as mercury which does not wet glass (obtuse contact angle), the meniscus is convex towards the air and there is a capillary depression (Fig. 246 (ii)). The shape of the surface in the beaker should be noted where it meets the outer walls of the tube. Examples of capillary elevation (or capillarity) are to be found in the action of blotting-paper and other absorbent materials.

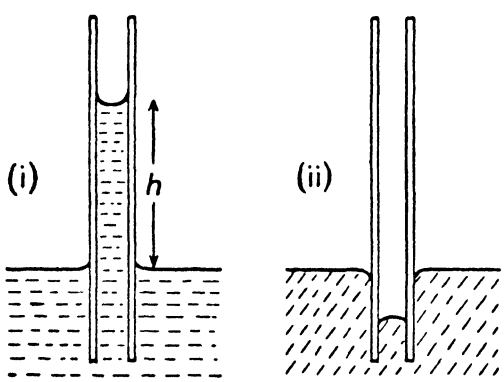


FIG. 246

Capillary rise can be regarded as a particular case of spreading. The "surface pressure" of the liquid in contact with the inner and outer walls of the tube acts so as to increase the area of these surfaces and thus causes spreading up the walls of the tube. The tendency to spread up the outer walls is shown by the curving upward of the surface in the beaker which would be plane if the tube were not present. Further spreading

up the outside of the tube is prevented by the weight of the raised liquid. Inside the tube the opposition offered to spreading by the weight of the raised liquid is smaller on account of the narrowness of the bore. Consequently a considerable elevation occurs, and the rise ceases when the weight of the raised column is equal to the force causing spreading.

If the inner radius of the tube is a, the volume of the column raised above the general level in the beaker is that of a cylinder of radius a and height h, *i.e.* $\pi a^2 h$. (This neglects the small effect due to the curvature of the meniscus, which will be mentioned later.) Thus the weight of the raised column in absolute units is

$$\pi a^2 h g \rho$$

The " surface pressure " causing the spreading is $-\gamma_{(S)}$ per unit length of the boundary of the surface. It acts upwards on the column round the line of contact of the meniscus with the tube. The length of this line is $2\pi a$, so that the force acting vertically upwards on the column is $-2\pi a \gamma_{(S)}$. Thus the liquid ceases to rise when the height of the column is such as to satisfy the equation

$$-2\pi a \gamma_{(S)} = \pi a^2 h g \rho$$

But if θ is the angle of contact at the edge of the meniscus, the equilibrium of a molecule at this point is given, as on page 263, by

$$-\gamma_{(S)} = \gamma \cos \theta$$

Substituting for $\gamma_{(S)}$ in the previous equation gives

$$2\gamma \cos \theta = a h g \rho$$

or

$$h = \frac{2\gamma \cos \theta}{a g \rho}$$

It will be realized that it is the surface layer of the meniscus which is being held up all round its perimeter by the upward force $-\gamma_{(S)}$ per unit length of this perimeter, while the surface layer itself supports the weight of the column by cohesion (Fig. 247). The system can be roughly likened to a clothes-line. The surface of the meniscus is the line and the column of liquid corresponds to the clothes, while the surface pressure in the liquid surface in contact with the walls of the tube is the support corresponding to the posts.

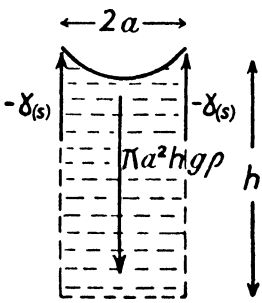


FIG. 247

We can, in fact, arrive at the last equation by treating the equilibrium of the system as we would that of a line carrying clothes. This is made clear in Fig. 248. Equating the vertical component of the tension in the surface of the meniscus to the downward force due to the weight

which the tension supports, we have

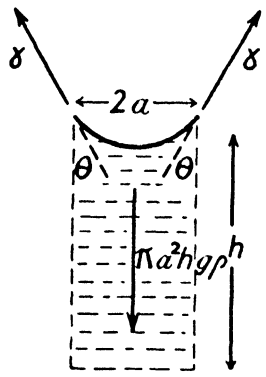


FIG. 248

$$2\pi a\gamma \cos\theta = \pi a^2 hg\rho$$

or

$$h = \frac{2\gamma \cos\theta}{ag\rho}$$

as before.

If the angle of contact is zero, the expression for the capillary elevation h becomes simply

$$h = \frac{2\gamma}{ag\rho}$$

and this is the most important case, because unless θ is zero its value in any particular tube is difficult to measure. Surface tension is rarely determined by the capillary-rise method unless θ is known to be zero. When θ is greater than 90° its cosine is negative so that h is negative, and there is capillary depression as already mentioned in the case of mercury in a glass tube ($\theta = 140°$ and $\cos\theta = -0{\cdot}766$).

So far we have treated the sides of the tube as being vertical and cylindrical, so that the thrust exerted by them on the raised column has no vertical component. This is usually the case in practice, but it is possible to consider what would happen if the tube had an irregular shape like that in Fig. 249. The liquid will not necessarily rise of its own accord to the position shown, but if it is drawn up by suction into the uniform part of the tube (radius a) it will remain in equilibrium at the same height as if the tube had a uniform radius of a all the way up. This can be proved by the following method, which uses the principles of hydrostatics and is also applicable to the uniform tube. Suppose that the angle of contact is zero and that the meniscus has a spherical shape. Its radius is therefore equal to the radius of the tube a. If p_A, p_B and p_O denote the pressures at A, B and C respectively, we have

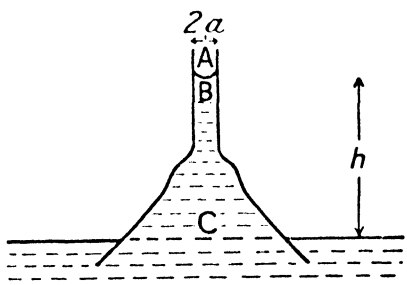


FIG. 249

$p_A - p_B$ = excess pressure on the concave side of the meniscus

$$= \frac{2\gamma}{a}$$

Also $p_O - p_B$ = pressure due to a depth of liquid h

$$= hg\rho$$

But $p_A = p_0 =$ atmospheric pressure, and therefore

$$\frac{2\gamma}{a} = hg\rho$$

or

$$h = \frac{2\gamma}{ag\rho}$$

Although this method of deriving the expression for h is equally suitable when the tube is uniform, it provides rather less explanation of the mechanism by which the liquid is drawn up the tube than does the original treatment given here.

Example.—*Two vertical uniform glass tubes, each of internal diameter* 0·2 *mm., are joined at their upper ends to a chamber in which the air pressure is less than atmospheric. The lower end of one tube dips into water contained in a wide vessel and the other dips into oil in a similar vessel. If the height of the water meniscus above the level of the water surface in the vessel is* 20 *cm., calculate the corresponding height for the oil. (The density of oil is* 0·9 *gm. cm.*$^{-3}$*, the surface tension of water is* 75 *dynes cm.*$^{-1}$ *and that of oil is* 25 *dynes cm.*$^{-1}$.) (L. Med.)

The arrangement is shown in Fig. 250. As the problem is evidently concerned with pressure relationships, it is best to use the fact that the pressure on the concave side of a meniscus in a tube of radius a is $2\gamma/a$. If p is the air pressure in the upper vessel and B is atmospheric pressure (both in absolute units), we have, in the case of the water (density 1).

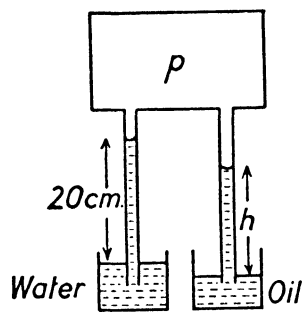


FIG. 250

pressure just below meniscus $= B - 20g$

$\therefore$ pressure above meniscus $= B - 20g + \frac{2 \times 75}{0{\cdot}01}$

Similarly on the oil side,

pressure above meniscus $= B - 0{\cdot}9hg + \frac{2 \times 25}{0{\cdot}01}$

But each of these pressures is equal to p, so that they are equal to each other, and

$$-20g + \frac{2 \times 75}{0{\cdot}01} = -0{\cdot}9hg + \frac{2 \times 25}{0{\cdot}01}$$

$$\therefore\ 0{\cdot}9hg = 20g - \frac{2 \times 50}{0{\cdot}01}$$

or

$$h = \frac{(20 \times 980) - 10{,}000}{980 \times 0{\cdot}9}$$

$$= 10{\cdot}9 \text{ cm. approx.}$$

Thus the height of the oil column is 10·9 cm.

Since no mention is made of contact angle in the question, it is evidently intended that this should be taken as zero for both liquids.

Correction for the Weight of Liquid in the Meniscus.—If h is the height of the *lowest point* of the meniscus above the level of the liquid outside the tube, then the total weight of the raised liquid is really greater than $\pi a^2 hg\rho$ by the weight of the liquid in the shaded portion in Fig. 251. Assuming that the meniscus is spherical and that the angle of contact is zero, the volume of this portion is equal to the difference between the volumes of a cylinder whose radius and height are both a and a hemisphere of radius a. Thus

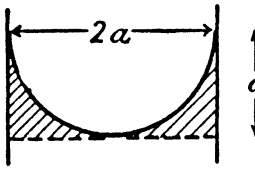


FIG. 251

$$\text{volume of shaded portion} = \pi a^3 - \tfrac{2}{3}\pi a^3$$
$$= \frac{\pi}{3}a^3$$

$$\text{Therefore weight of shaded portion} = \frac{\pi}{3}a^3 g\rho \text{ absolute units}$$

Allowing for this additional weight of raised liquid, the equilibrium of the column is given by

$$2\pi a\gamma = \pi a^2 hg\rho + \frac{\pi}{3}a^3 g\rho$$

or

$$\gamma = \frac{ag\rho}{2}\left(h + \frac{a}{3}\right)$$

Thus the correction known as the " weight of the meniscus " is allowed for by adding one-third of the radius to h. With water ($\gamma = 73$ dynes cm.$^{-1}$) in a tube of radius, say, 0·3 mm., h is of the order of 5 cm. and the correction is 0·01 cm., so that it amounts to about one part in 500 and is hardly worth taking into account in any but the most accurate experiments.

It should be realized that in actual fact the shape of the meniscus *cannot* be spherical for the following reason. The pressure above the meniscus is uniformly atmospheric all over its surface, while the pressure immediately below the surface varies from point to point because of the varying height in the liquid. Therefore the pressure difference on the two sides of the meniscus is not the same at all points and, since this difference depends on the radius of the liquid surface, the radius must vary from point to point over the meniscus which cannot, therefore, be spherical. It is a very complicated matter to work out the shape of the curve mathematically, and, as a matter of fact, it cannot be done exactly. As we should expect, the meniscus becomes more nearly spherical as the radius of the tube diminishes. It is obviously not spherical in a tube of, say, 1 cm. diameter.

Capillary Rise between Two Parallel Plates.—Capillary elevation

or depression occurs between two vertical plates supported side by side in a liquid. The vertical lines in Fig. 252 serve to represent the sections of the plates. Let the distance apart of the inner walls of the plates be d, so that, supposing the meniscus to be *cylindrical* in shape and the angle of contact to be zero, the radius of the surface of the meniscus is $d/2$. Therefore the pressure at a point just below the meniscus is less than that just above (*i.e.* atmospheric pressure) by $2\gamma/d$, and also, if the height of

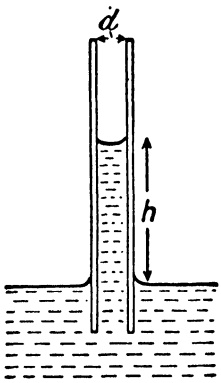


FIG. 252

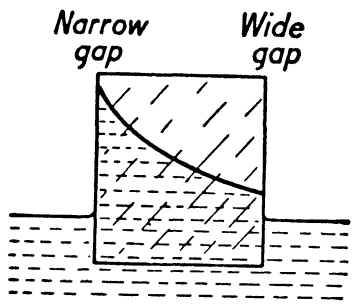


FIG. 253

the meniscus above the general level of the liquid outside the plates is h, the pressure just below the meniscus is less than atmospheric pressure by $hg\rho$. Thus the capillary elevation h is given by

$$hg\rho = \frac{2\gamma}{d}$$

or

$$h = \frac{2\gamma}{dg\rho}$$

This formula is subject to a correction for the " weight of the mensicus " similar to that applied to the case of the tube, although its actual value is different.

If the plates, while remaining vertical, are no longer parallel to each other but are inclined at a small angle so that d changes from one vertical edge to the other, the elevation varies continuously, diminishing towards the end where d is large. It can be proved that the surface of the elevated liquid has a hyperbolic shape and the appearance is as shown in Fig. 253. This has been made the basis of a method of determining γ.

4. EXPERIMENTAL DETERMINATION OF SURFACE TENSION

Plane Soap Film.—A frame like that shown in Fig. 254 is constructed of clean wire or thin glass rod or tube. (Platinum wire can be cleaned by heating it in a flame.) The frame is suspended from the left-hand arm of a balance and counterpoised when the lower ends of its vertical

members are immersed in a soap solution. The balance is then arrested. Next the dish of solution is raised so as to cover the frame completely and is then lowered to its original position, thus leaving a plane film in the frame above the surface of the solution. Weights are added to the right-hand side of the balance so as to counterpoise it again with its pointer at the same scale reading as before. In this way the amount of frame immersed is the same before and after the formation of the film, so that the buoyancy due to the liquid is the same. The additional weights (m gm.) balance the downward force exerted on the top of the frame by the film, and we have

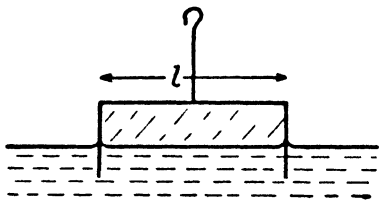


FIG. 254

$$mg = 2\gamma l$$

where l is the length of the top. This method is successful with any liquid which forms a stable film. It will be noticed that the above simple formula takes no account of the small effects due to (*a*) the weight of the liquid in the film, (*b*) the alteration in the pull exerted on the vertical wires when the film is formed. These effects are very small, however. As regards (*a*), it may be noted that the equilibrium of a vertical film requires that the tension in it shall be greater at the top than at the bottom because of the weight of the film itself. When no allowance is made for the weight of the film in the above experiment it would appear that it is the tension at the top which is being measured.

Capillary Rise.—This method should be used only in cases where it is known that the angle of contact is zero. If the tube is of glass, which is nearly always the case, suitable liquids are water, alcohol, benzene, and many other pure liquids and solutions. The tube must be cleaned before it is used, unless it has been freshly drawn after heating in a blowpipe flame, in which case the grease, etc. has probably been burnt off. The cleaning solution usually used for all glassware in surface-tension experiments is chromic acid, which is made by adding concentrated sulphuric acid to an aqueous solution of potassium bichromate. The tube should be soaked in this solution for some hours, making sure that the liquid completely fills the bore of the tube. After this the tube is washed with plenty of tap water and then with distilled water, and dried with clean warm air. A piece of thin glass tube or rod or a clean wire is bent as shown in Fig. 255 and attached to the tube with a rubber band. The tube is placed vertically in the liquid contained in a wide beaker or crystallizing dish, immersed nearly up to the rubber band and then withdrawn and clamped with the point A just touching the surface. The immersion and withdrawal ensure that the meniscus has *fallen* to the position which it finally occupies in the tube, so that it exhibits the receding

contact angle, which is more likely to be zero than is the advancing angle. When the apparatus is in adjustment, a cathetometer with a vertical scale is focused on the lowest point of the meniscus and the reading is taken. The dish of liquid is then removed and the cathetometer is lowered so that A is now on the cross-wires. The difference between the two cathetometer readings is the capillary rise h. The diameter of the tube is next measured with a travelling microscope, and, strictly speaking, this should be done at the place where the meniscus formerly stood. The tube should, therefore, be broken at this place and measurements made of the major and minor axes of both the ends so formed (the section is, in general, slightly elliptical). The mean diameter is thus obtained and halved to give the mean radius.

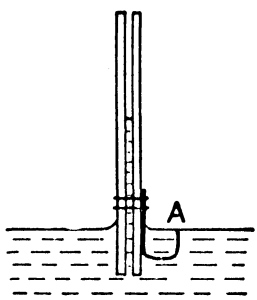


FIG. 255

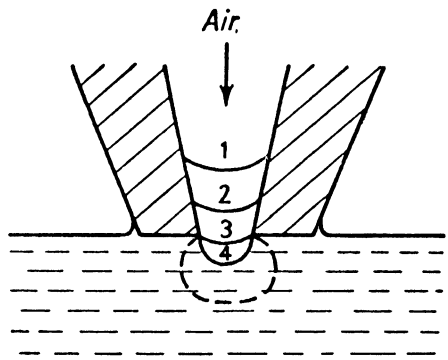


FIG. 256

Jaeger's Maximum Bubble-Pressure Method.—The essential part of this method is the determination of the maximum pressure which can be supported by an air bubble blown on the end of a jet which is immersed in liquid. Fig. 256 is a magnified drawing of a narrow tube or jet whose orifice has been set in the plane of a liquid surface. Capillarity will cause the meniscus to rise up the inside of the jet. Suppose that the pressure of the air in the tube above the meniscus is gradually increased. The meniscus will be forced down through positions such as 1 and 2 and will eventually reach 3. To a first approximation the meniscus may be assumed to be spherical in shape. Provided that the adhesion between the liquid and the material of the jet is not too weak (*i.e.* contact angle is zero or small), the perimeter of the meniscus (or bubble as it now is) remains on the edge of the orifice while the bubble grows under the action of the increasing air pressure in the jet. Its radius is diminishing at first during this growth until it reaches a minimum value equal to the radius of the orifice a in the position 4. Since the pressure excess which the bubble can support is inversely proportional to its radius, it follows that position 4 corresponds to a maximum pressure excess inside the bubble. A further slight increase of air pressure causes an expansion of the bubble to a radius greater than a. It cannot support the increased pressure and so swells rapidly and bursts. Thus the bubble becomes unstable when

its radius is equal to that of the orifice, and at the same time the pressure of the air in the jet reaches a maximum. The pressure in the liquid outside the bubble is practically atmospheric pressure, since the orifice is in the plane surface and the height of the bubble is very small. Thus γ can be determined by equating the maximum pressure as registered by a manometer to $2\gamma/a$.

One form of apparatus is shown in Fig. 257 (i). The jet may be made by drawing down a wider tube after it has been heated in a blowpipe and its walls have been allowed to thicken while soft. The appearance

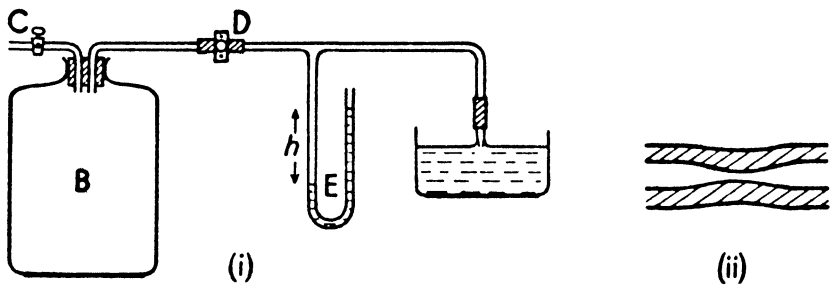


FIG. 257

of the drawn tube is then as in Fig. 257 (ii), and it may be cut anywhere in the constricted part according to the diameter of orifice required. If the walls of the constriction are thin a square fracture cannot be obtained. Alternatively the jet may be made by cutting squarely a ready-made thick-walled capillary tube. The rest of the apparatus consists of a large bottle B into which air can be pumped (*e.g.* by a bicycle pump) and from which, when the tap C is closed, air passes to the jet at a slow rate controlled by the screw clip D. After the orifice of the jet has been set in the plane of the surface of the liquid in the dish, the clip D is adjusted until bubbles are released at not less than, say, ten-second intervals. Between successive releases the manometer shows periodically a slow increase of difference of level, and the bursting of a bubble is followed by a quick fall of a few mm. as the bubble bursts. It is a simple matter to read the maximum difference of level h. We can then calculate γ from the equation

$$\frac{2\gamma}{a} = hg\rho$$

where a is the internal radius of the orifice and ρ is the density of the liquid in the manometer.

The above equation is accurate only for small orifices however. A correction is needed of the same nature as that applied to the capillary-rise equation, and a more exact equation is

$$\frac{2\gamma}{a} = hg\rho - \tfrac{2}{3}ag\sigma$$

where σ is the density of the liquid whose surface tension is being determined.

The original form of the experiment in which the jet orifice is immersed some distance (say h') below the liquid surface is still often used. In this case $h'g\sigma$ must be subtracted from $hg\rho$ in the above equation.

Jaeger's method is very satisfactory, and can readily be used for investigating the variation of surface tension with temperature. This is not the case with the capillary-rise method unless the apparatus is specially designed for immersion in a bath of liquid. When the angle of contact between the jet and the liquid is quite large (*e.g.* with mercury and glass), Jaeger's method becomes unreliable or, at any rate, less straightforward than otherwise.

5. SOME MISCELLANEOUS SURFACE-TENSION EFFECTS

Dependence of Surface Tension on Temperature.—The surface tension of all pure liquids and of most mixtures diminishes with temperature. The effect is not very large, *e.g.* between 0° and 100° C. the surface tension of water falls from 75·6 to 58·8 dynes cm.$^{-1}$. Over comparatively narrow temperature ranges (say 10° C. or so) the change may be regarded as being linear, but the relationship is really more complicated than this.

If the correct quantity of a liquid is placed in a closed vessel containing no air or other gas and the whole is heated, the liquid becomes progressively less dense and its vapour more dense until a stage is reached at which their densities are equal and the two phases become indistinguishable from each other in all respects. This is the **critical temperature,** and inasmuch as the surface separating the liquid and vapour disappears at this temperature, the surface tension and surface energy must become zero.

The Surface Tension of Solutions.—The addition of soluble organic chemicals frequently diminishes the surface tension of water, and it often happens that a small amount of dissolved substance produces almost as great a reduction of surface tension as a much larger quantity. It can be shown both theoretically and experimentally that when the addition of a soluble substance to a liquid causes a decrease of surface tension, the concentration of the solute is greater in the surface than in the interior. This tendency of the dissolved substance to collect in the surface is known as **adsorption,** and is evidently in accordance with the principle that a system moves to a condition of minimum potential energy. The more solute there is in the surface the lower is the surface energy.

Inorganic salts cause an increase in the surface tension of water, and are said to be negatively adsorbed because there is a paucity of solute in the surface as compared with the interior.

When shavings of camphor and certain other substances are scattered on a *clean* water surface they perform rapid movements. Owing to the irregular shape of each fragment the rate of solution differs from place to

place round its perimeter. The surface tension is higher where the concentration is less and the particle is pulled in this direction. The evaporation of the dissolved camphor from the surface keeps the process going. A "boat" cut from a sheet of mica in the shape shown in Fig. 258 will travel over the surface of water when a piece of camphor is attached at A owing to the reduction of surface tension at the stern.

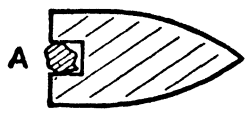


FIG. 258

Surface Films.—Oleic acid has a lower surface tension than water, and when a very small drop of it is placed on a large water surface it spreads out into a very thin film of large area. The spreading can be observed if the clean water surface is previously dusted with lycopodium powder. The powder is pushed back as the patch of oleic acid expands, and the clear part of the surface shows the extent of the spreading. Evidently when the boundaries of the film are eventually in equilibrium its surface tension must have become as high as that of the surrounding water. The thickness of the film at this stage may be calculated by dividing the volume of the original drop of oleic acid by the final area of the film. The result is about 10^{-7} cm., which is of the order of molecular dimensions and suggests that the film is only one molecule thick. Thus the film has spread until every molecule is as near the water surface as possible.

Fig. 259 is a plan of a shallow trough used in surface-film experiments. The trough is coated with paraffin wax and completely filled with water. A light barrier AB floats in the surface and its ends are joined to the sides of the trough by short pieces of platinum foil. The force which AB experiences in a direction parallel to the length of the trough can be measured by a delicate torsion balance attached to it. Another barrier CD (a waxed strip of glass) rests across the tank. Thus when a small quantity of insoluble substance is placed on the water surface between AB and CD, the film which it forms is confined within these limits. The force experienced by AB in a direction away from the film is equal to the length of AB multiplied by the difference between the surface tension of the clean water surface on the one side of AB and that of the film on the other. This difference can, therefore, be determined by measuring the force. It has become customary to regard the force on unit length of AB as being the so-called " surface pressure " of the film, which is thus numerically equal to the difference of the surface tensions. It is, so to speak, a two-dimensional pressure. By sliding CD to various positions and reading the torsion balance attached to AB, the variation of surface pressure with the area of the film can be investigated. Suppose that CD is initially at the far end of the tank and is gradually brought nearer to AB.

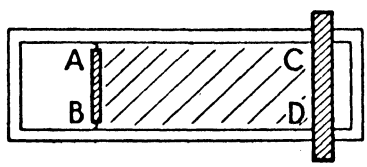


FIG. 259

With a film like oleic acid, no surface pressure is registered at first when the area ABDC is large, thus showing that the film is not occupying the whole of the area available to it or, in other words, that its surface tension is equal to that of water. The film has spread to its maximum extent, and the area which it covers is less than ABDC. As the area between the barriers is reduced, a stage is reached at which AB begins to experience a force which increases sharply as the area is further diminished. It is supposed that the molecules of the film are now being pressed together and offer considerable resistance to compression. Eventually some are squeezed off the water surface and rest on those which remain. The film is then more than one molecule thick. When a film which behaves in this way is one molecule thick it is said to be " coherent " or " solid " because its molecules are bound together by lateral attracting forces which prevent it from spreading indefinitely. Thus the film exerts no force on AB when the area available to it is greater than that to which its cohesion confines it. On the other hand, some substances form films which exert pressures no matter how large the area between AB and CD is made. In such cases the cohesion between molecules is not large enough to prevent indefinite spreading, and the film is called " gaseous " by analogy with the way in which a three-dimensional gas occupies the whole of the vessel in which it is confined.

EXAMPLES XVIII

1. Explain what is meant by the surface tension of a liquid, and describe, with full experimental details, how you would measure, as accurately as possible, the surface tension of a liquid which wets glass. Give the theory of the method.

A piece of thin wire is made into a plane ring of 3·5 cm. diameter, and a film of soap solution is formed on it. A bubble is made by blowing gently at right angles to the film. Given that the surface tension of soap solution is 25 dynes $cm.^{-1}$, find the maximum force exerted by the film on the wire during this process. (O.H.S.)

2. What is meant by the statement that the surface tension of water at a given temperature is 70 dynes per cm.?

Why, and to what height, does water at that temperature rise in a perfectly clean glass tube dipped vertically into it, the tube being of circular cross-section and 0·7 mm. internal diameter? (L.Med.)

3. Define (*a*) surface tension, (*b*) surface energy. Explain in general terms why small raindrops and small air bubbles in water are almost perfectly spherical.

A hemispherical soap bubble is blown on the end of a tube of diameter 1 mm., and it is found that the excess pressure inside is equivalent to 2·20 cm. of water. Calculate the surface tension of the soap solution. (L.Med.)

4. Define *surface tension*, and describe in detail a method, involving the use of a capillary tube, of determining the surface tension of alcohol.

The vertical limbs A and B of a Hare's apparatus consist of capillary tubing 0·5 mm. in diameter. Their lower ends dip into alcohol and water respectively, and the liquids are sucked up the limbs so that the height of water in B is 20 cm. above the free surface. Find the height of the alcohol in A above the free surface. The surface tensions of water and alcohol are 73 dynes per cm. and 22 dynes per cm. respectively, and the specific gravity of alcohol is 0·80. Assume that the angle of contact of alcohol and glass is zero. (L.I.)

5. What do you understand by the *surface tension* of a liquid? Explain why it is so much easier to separate two glass plates, between which is a thin film of water, by sliding one over the other rather than by a direct pull.

The lower end of a vertical capillary tube, bore 0·2 mm., is placed beneath the surface of water of surface tension 70 dynes cm.$^{-1}$. Calculate the excess pressure which must be exerted to prevent the water rising in the tube. (L.I.)

6. Mention some common phenomena which suggest that liquid surfaces are in a state of tension, and explain how surface tension can be accounted for in terms of molecular cohesion. Explain why water rises in a glass capillary tube.

7. Define *surface tension* of a liquid and describe a method of finding this quantity for alcohol.

If water rises in a capillary tube 5·8 cm. above the free surface of the outer liquid, what will happen to the mercury level in the same tube when it is placed in a dish of mercury? Illustrate this by the aid of a diagram. Calculate the difference in level between the mercury surfaces inside the tube and outside.

S.T. of water = 75 dynes/cm.
S.T. of mercury = 547 dynes/cm.
Angle of contact of mercury with clean glass = 130°.
Density of mercury = 13·6 gm./c.c.

(L.H.S.)

ANSWERS TO EXAMPLES

Examples I. Page 19

1. 5·69 miles, 51° 37′ S. of E.
2. 2·95 m.p.h., 16° 20′ S. of W.
3. 53° 8′ S. of W., 226·2 m.p.h.
4. 64° 6′ S. of E., 6·48 m.p.h.
5. 30·5 m.p.h., 20° 21′ W. of N., 10·43 miles.
6. 17·32 m.p.h.
7. 21° 25′ E. of S., 4·583 knots.
8. 1·6 miles.
9. 2AB.
10. 47·8 minutes.
11. 50 m.p.h., 36° 52′ N. of E., 2·64 minutes past 12.
12. 0, $2v$ parallel to road; $v\sqrt{2}$, 45° below horizontal; 0, $2v$, v.

Examples II. Page 32

1. After falling for 2 sec., 256 ft.
2. (*a*) 34·64 ft. sec.$^{-1}$, (*b*) 50 ft. sec.$^{-1}$, (*c*) 30 sec., (*d*) 46·7 ft. sec.$^{-1}$.
3. 8 ft. sec.$^{-2}$.
4. -136 ft.
5. 15 ft. sec.$^{-1}$.
6. 60 ft., 5 sec.
7. $2\frac{1}{2}$, $1\frac{1}{4}$, $-3\frac{3}{4}$ ft. sec.$^{-2}$.
8. 38·7 ft. sec.$^{-1}$.
9. 92·16 ft., 76·8 ft. sec.$^{-1}$, 0·35 sec., 65·5 ft. sec.$^{-1}$.
10. (*a*) 144 ft., (*b*) 5 sec., (*c*) 554·3 ft.
11. (*a*) 144 ft., (*b*) 90 ft.
12. 81 ft., 144 ft.
13. (*a*) 2 sec., (*b*) 60 ft., $38\frac{1}{2}$° below horizontal.
14. (*a*) 34 ft., (*b*) 128 ft., (*c*) 82 ft.
15. 672 ft.

Examples III. Page 39

1. (*a*) $\frac{2\sqrt{2}v^2}{\pi r}$, (*b*) $\frac{2v^2}{\pi r}$ in the direction of the instantaneous acceleration at the middle of the interval.
2. 1·3 ft. sec.$^{-2}$, at an angle of 67° 18′ below the forward horizontal direction.
3. (*a*) 200π cm. sec.$^{-1}$, making an angle of 143° 8′ with the velocity of A, (*b*) 800π cm. sec.$^{-2}$, making an angle of 143° 8′ with the acceleration of A.
4. 322 cm. sec.$^{-1}$, $5{\cdot}16 \times 10^5$ cm. sec.$^{-2}$.
5. $5\sqrt{3}$ cm., 20π cm. sec.$^{-1}$, $80\sqrt{3}\pi^2$ cm. sec.$^{-2}$.
6. $\frac{1}{\sqrt{3}}$.
8. π sec., 5 ft.

Examples IV. Page 51

3. 8·84 tons wt.
4. 4×10^7 cm. sec.$^{-2}$, 6×10^8 dynes.
5. $\frac{10}{18}$, 0·4.
6. 12, $12\sqrt{3}$ poundals.

Examples V. Page 65

1. $\sqrt{305}$ lb. wt., making an angle of $\tan^{-1}\frac{7}{16}$ with AD, $23a$ lb. wt. ft.
2. $\frac{M(a_1^2+a_2^2)}{2}$.
3. 4·9 radians sec.$^{-2}$.
4. 7290 gm. cm.^{2}.
5. 10 cm., 40 cm.

Examples VI. Page 81

1. 48 H.P.
2. 208·3, 7·9 ft. lb. sec.$^{-1}$, 0·39 H.P.
4. 24 H.P.
5. 1176 H.P., 4 tons wt.
6. 3388 lb. wt., 39·8 H.P.
7. 10·5 lb. wt.
8. $1{\cdot}83\times10^6$ ergs.
9. 36·2 per cent.
11. 12·75 radians sec.$^{-1}$, $2{\cdot}48\times10^6$ dyne cm.
12. (*a*) 6 ft. poundals, (*b*) 141 ft. poundals.
13. 17·6 H.P.
14. 382 kg. wt., $5{\cdot}625\times10^5$ watts.
15. 241 H.P.

Examples VII. Page 87

1. 1·23 sec., 1·58 sec.
2. 181·5 cm. sec.$^{-2}$.
3. 19° 6′, 0·46*g*.
4. 431 gm. wt.

Examples VIII. Page 114

1. $\frac{g}{2}$.
2. 85·7 cm. sec.$^{-2}$, 228 gm. wt.
3. (*a*) 159, (*b*) 141 lb. wt. Rest or uniform velocity.
4. 100 gm. wt., 4·9 cm. sec.$^{-2}$, 100·5 gm. wt.
5. 127½ poundals, 2 ft. sec.$^{-2}$; 111$\frac{3}{7}$ poundals, 2$\frac{2}{7}$ ft. sec.$^{-2}$; $\frac{7}{8}$ ft.
6. 3$\frac{5}{9}$ ft. sec.$^{-2}$, 14$\frac{2}{9}$ poundals (P), 28$\frac{4}{9}$ poundals (Q); 1$\frac{7}{9}$ ft. sec.$^{-2}$, 30$\frac{5}{9}$ poundals.
7. (*a*) $\frac{10^6}{\pi}$ dynes or 325 gm. wt., (*b*) 6600 poundals or 206¼ lb. wt.
8. (i) 31½ lb. wt., (ii) $1\frac{4}{7}$ H.P.
9. 175 poundal sec., 218¾ ft. poundals, 21$\frac{7}{8}$ lb. wt.
10. 182 cm.
11. 281·25 ft. tons wt., 2·34 ft. tons wt., 1·56 tons wt.
12. 64 cm., 36 per cent.
13. $\frac{1}{4}U$, $\frac{3}{4}U$.
14. $\frac{(2m-M)u}{2(m+M)}$, $\frac{3mu}{2(m+M)}$
15. (*a*) 412 cm. sec.$^{-1}$, (*b*) 443 cm. sec.$^{-1}$, 25 cm.
16. 1·794 sec., 1·25 kg. wt.
17. (*a*) 236 cm. sec.$^{-2}$, (*b*) 193×10^2 cm. sec.$^{-2}$.
18. 979·6 cm. sec.$^{-2}$.
19. 1·76 sec.
21. 18 cm., 24 cm.
22. 3·9 gm., 16·8 cm. sec.$^{-1}$.
23. 6·4 in.

Examples IX. Page 152

1. 2·90 units, 24° 44′ with the force of 4 units.
2. 68° 12′.
4. $\frac{(2-\sqrt{3})mgl}{3}$ ergs, $\frac{mg}{\sqrt{3}}$ dynes.
5. (*a*) 13 kg. wt., (*b*) 5 kg. wt., 10 kg. wt.
6. $\frac{3\sqrt{5}}{2}$ lb. wt., $\frac{\sqrt{13}}{2}$ lb. wt., 2·5 lb. wt.
7. (*a*) zero, (*b*) couple equal to double the area of the triangle, (*c*) double the reversed force acting parallel to the reversed force midway between it and the opposite corner of the triangle.
8. 6·53 lb. wt., $6\frac{1}{2}°$ above horizontal.
9. (*a*) 2·87 units, $64\frac{1}{2}°$ S. of E., (*b*) 86° 11′, 123° 45′, 150° 4′.
10. 14·6 lb. wt., 10·4 lb. wt.
11. 800 gm. wt., 600 gm. wt.
12. 21·83 in., 2·22 lb. wt.
13. $\frac{4}{3}Q$ lb. wt., $\frac{4}{3}Q$ lb. wt. in direction $\sin^{-1}\frac{3}{4}$ with horizontal.
14. $1\frac{1}{12}$ lb. wt.
15. 8 lb. wt.
16. 55 to 75 cm.
17. 3125 gm. wt.
18. 535 gm. wt., 70 gm. wt., 66·53 cm.
19. 108 gm.
20. 15 lb., within 2 ft. of the centre.
21. (*a*) 2·17 cm., (*b*) 3·25 cm.
22. 7·33 cm.
23. 2 in., $1\frac{5}{8}$ in.
24. 32° 58′.
25. DB = 20 cm., DB = 35 cm.
26. 38 lb. wt.
27. (*a*) 1344 ft. lb. wt., (*b*) 2688 ft. lb. wt.
28. 1·11 lb. wt., 83·3 ft. lb. wt.
29. 0·060 cm. below the central knife edge.
32. (*a*) 1·01, (*b*) 23·875 gm.

Examples X. Page 172

1. 8064 lb. wt.
3. (i) 371 lb. wt., (ii) 407 lb. wt.
4. 24750 lb. wt.
5. 1440 lb. wt.

Examples XI. Page 180

1. 2550 lb. wt.
2. 8430 metres or 28100 ft.
3. 0·35 cm.

Examples XII. Page 187

1. 35 per cent.
2. 1×10^{-4} cm. of mercury.
3. 2060 atmospheres.

Examples XIII. Page 202

1. (*a*) π gm., (*b*) 2·27 cm. higher than in water.
2. 25·5 gm.
3. (*a*) 1485 lb., (*b*) 1210 lb.
4. 4 c.c.
5. 1·44 c.c.
6. $\frac{5}{3}\frac{0}{0}$.
7. 0·88 cm.
8. 64·3 lb. ft.$^{-3}$.
9. 7·1.
10. 8·43 cm. (to the bottom of the plate).
11. 1·43.
12. (*a*) 0·95, (*b*) 1·19.
13. $\frac{V}{v} = 44$.
14. $W \sin \theta$.
15. 0·8, 6·4 gm.
16. 3·74 c.c. of A, 2·26 c.c. of B.
17. Moves 0·27 cm. out of the mercury.

Examples XIV. Page 213

2. 1·44 watts.
4. 13·4 c.c. per sec.
5. 25 c.c. per sec.
6. 39·3 cm. of water.
7. 0·63.

Examples XV. Page 229

1. (*a*) 10^8 dynes, (*b*) $2{\cdot}5 \times 10^7$ ergs.
2. (*a*) $2{\cdot}33 \times 10^{-4}$ cm., (*b*) 547 ergs.
3. 12×10^8 dynes cm.$^{-2}$, 3·01 kg. wt.
4. $1{\cdot}61 \times 10^5$ ergs, $12{\cdot}0 \times 10^{11}$ dynes cm.$^{-2}$.
5. 0·020 cm., 0·040 cm.
6. $\frac{L_1}{L_2} = \pi$.
7. 775.
8. 185 ft.
9. 2×10^5 litres, 1·02 gm. litre^{-1}.
10. 54·9 metres.
11. 9·82 gm. wt.
12. $\frac{3}{1}\frac{}{0}$, 100 cub. ft.
13. $\frac{5}{4}\frac{2}{8}$.
14. 19 cm.
15. 42·5 c.c.
16. 7 in.

Examples XVI. Page 242

2. $3{\cdot}9 \times 10^{-4}$ watts.
6. $5{\cdot}3 \times 10^{-4}$ cm.

Examples XVII. Page 253

1. 6·22 gm. litre^{-1}.
2. 74·3 cm. of mercury.

Examples XVIII. Page 275

1. 550 dynes.
2. 4·1 cm.
3. 27·0 dynes cm.$^{-1}$.
4. 19·8 cm.
5. 14,000 dynes cm.$^{-2}$.
7. 2·0 cm.

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