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BELL'S MATHEMATICAL SERIES
FOR SCHOOLS AND COLLEGES

General Editor: WILLIAM P. MILNE, M.A., D.Sc.

PLANE GEOMETRY
PRACTICAL AND THEORETICAL
VOL. II.

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PLANE GEOMETRY

PRACTICAL AND THEORETICAL

PARI PASSU

BY

V. LE NEVE FOSTER, M.A.

SOME TIME ASSISTANT MASTER AT ETON

VOL. II.



LONDON

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PREFACE

IN this Book the aim is to develop practical and theoretical geometry *pari passu*. Appeal is made to the common sense of a beginner, before the *minutiae* are considered, and the opening is with Geometrical Drawing only, so that explanation of the use of instruments is at first given. Throughout the book are found applications to every-day affairs, to games, to Surveying, to Architecture, to Navigation, and to Engineering. A large number of diagrams has been provided; in a book the pictorial argument is the effective one, and memory of a picture is much more real and lasting than a vague recollection of something which was only read.

Reference is made continually to the historical side.

In the Examples the aim has been to give the figures for at least the earliest cases (even the simplest), and often later, too, if clearness was thereby gained. To many of the riders HINTS have been added. Geometry is quite a hard enough subject, even if difficulties are reduced to a minimum. The encouragement in success is considerable, and affords the best chance of future development. The "last page" has been strengthened, points on it can be "pricked off," and it thus affords an unlimited scope for possible questions, allowing the gain of much time in avoiding the tedious preparation of a figure (which too often turns out to be in a bad, and sometimes impossible, position on the page).

Throughout, bookwork and examples are either blank or are preceded by one asterisk (*), or by two asterisks (**), showing what is appropriate to first, second and third readings respectively.

Style is insisted on, and the "hints" (collected at the beginning of Volume I.) are of *constant* application.

Of the 20 propositions in this volume more than half are on Circles, and they include properties involving Chords, Angles and Tangents. The ready transference from Angles to Arcs (and *vice versa*), and its practical application to positions on the Earth, is convenient. The Distance of the Visible Horizon, and the value of continuous curves in Architecture and Railway Lines, are not lost sight of.

Very little in the way of words is added to the illustrations of Algebraical Identities, for no examiner will ever read the explanations. If the diagram is neat, then the only words needed are those which say what the diagram illustrates. The extensions of Pythagoras' Theorem are considered in two propositions. An important property of a Median of a Triangle is fully gone into.

On Proportion there are seven propositions. The use of Similar Triangles affords abundant opportunities of calculating Heights and Distances.

The practical examples were not concocted to fill a gap in the book. Very many have occurred in practice, and the rest are based on realities. It is hoped that many will gain ideas from these and, above all, learn to apply those ideas to problems of their own setting, very likely far from their mathematical school-rooms.

Over half of the 700 odd examples are of the formal rider type, for it is felt that, though one can learn to apply the ideas one has already got by tackling abundant examples, one will not get many new ideas if that method only is adopted.

The kind permission of the Controller of His Majesty's Stationery Office to reprint some questions from certain Civil Service Army Examination Papers has been of great value.

My warmest thanks are also due to Mr. C. H. Allcock for placing at my disposal a large collection of riders.

To many others I am grateful for advice, suggestions and help, but most especially to Mr. W. Hope-Jones, whose fertility of imagination and ungrudging assistance in checking the accuracy of every calculation have alone rendered possible the production of this book.

V. L. N. F.

August, 1921.

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Briefly the Propositions are :

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TO THE STUDENT.

Throughout, the bookwork and examples are either blank, or have one asterisk * at the side, or two asterisks * * at the side.

It is suggested that in the first reading you should confine your ideas to the unmarked parts. On a second reading you should also tackle the * parts. On a third reading you should be familiar with all.

Last Page.—This reference in several examples in succeeding chapters is to the pricking-off figure which appears at the end of the volume.

CHAPTER XVIII.

CIRCLES.

§ 1. The shape of a circle is familiar to all, and one frequently can judge fairly by eye whether something is round or not. In Geometry one must have something more exact than mere appearances, and either of the following defines a circle.

Definition 1. A **circle** is a plane figure contained by one line, called the **circumference**, and is such that all straight lines drawn from a certain point, within the figure, to the circumference are equal. That point is called the **centre**, and the equal straight lines are called **radii**.

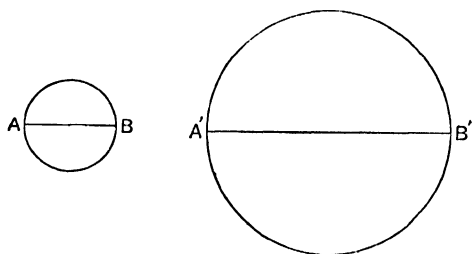
Definition 2. A **circle** is the locus of a point (in a plane) equidistant from a given point, called the **centre**.

A straight line, through the centre, terminated by the circumference, is called a **diameter**. A circle is symmetrical with respect to every diameter; and, since there are an infinite number of diameters, a circle is symmetrical in an infinite number of ways. Circles form the basis of many geometrical patterns.

§ 2. All circles are of the same shape; so, if we can find how big the circumference is compared to the diameter for one particular circle, we can infer that this same comparison holds in all cases.

If we take a figure and look at it through a magnifying glass so that *one* line is trebled, then *all* lines will be trebled. Hence the ratio (or comparative value) of two lines on the given figure will not be altered in the magnified figure. Now we can always magnify a circle so as to be the same size as another circle, hence

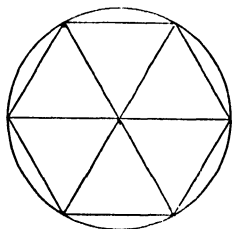
the ratio of any two lines in the first figure will be the same as the ratio of the corresponding lines in the second figure.



Hence

$$\frac{\text{Circumference of first circle}}{\text{Diameter AB}} = \frac{\text{Circumference of second circle}}{\text{Diameter A'B'}} = \text{etc.}$$

The ratio $\frac{\text{circumference}}{\text{diameter}}$ is represented always by the Greek letter π .



[By putting together 6 equilateral triangles, to form a regular hexagon, it is easy to see that the perimeter of the hexagon is 6 times the radius of the circle which just goes round all 6, or is 3 times the diameter. The circumference of the circle is clearly more than the perimeter of the hexagon, as an elastic band tightly round a pencil of hexagonal section would show; and hence the inference $\pi > 3$. Measuring the circumference of a round table, plate (or such like), with a tape measure, and also the diameter, gives data for some value of π more accurate.]

Now $\pi = 3\frac{1}{7}$ **approximately**. The correct value of π can be calculated to as many places of decimals as we like, but on principles involving higher mathematics. To know many places of

decimals is quite useless in G.D. In calculations it may be a different matter. In this book $3\frac{1}{7}$ or $\frac{22}{7}$ is used for the approximate value.

$\pi = 3\frac{1}{7}$ or $\frac{22}{7}$ or 3.142857 is right to about 1 in 2500. (Approximation too big.)

$\pi = 3.1416$ is right to about 1 in 400,000. (Approximation too big.)

[Say, circumference = π diameter,
and then circumference = $3\frac{1}{7}$ times diameter approximately.

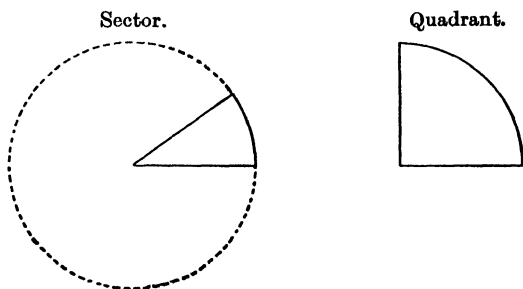
To leave out the word "approximately" is wrong.

In calculations in which you have used $\pi = 3\frac{1}{7}$ beware against giving any sevenths in the answer. 3 significant figures is right.]

If d stands for the diameter of a circle and r its radius, we have

$$\text{circumference} = \pi d = 2\pi r.$$

§ 3. A **sector** of a circle is the figure bounded by two radii and a part (called an **arc**) of the circumference. The most important sector is a **quadrant**.

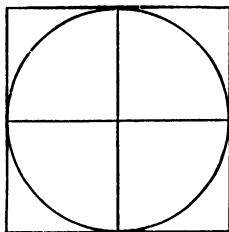


The size of the angle that the radii make (*i.e.* the angle at the centre of the circle) determines the fraction the **arc of the sector** is of the whole circle. In a quadrant the angle is 90° .

An **Example** follows: The diameter of a wheel is 4 ft. 3 in. The wheel has 8 spokes. What is the distance (to the nearest inch) between the spokes, measured along the rim?

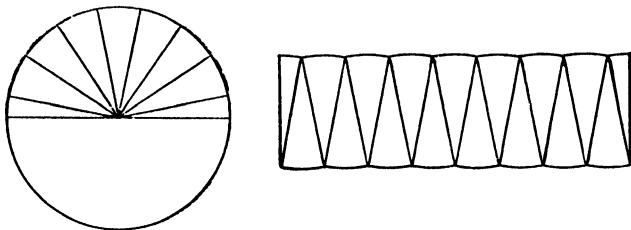
$$\begin{aligned}\text{Distance} &= \frac{1}{8} \times \pi d = \frac{1}{8} \times \frac{22}{7} \times 51'' = 20.04'' (-) \\ &= 1 \text{ ft. 8 in. about.}\end{aligned}$$

§ 4. The **area** of a circle may be judged fairly by eye. Enclosing the circle in a square shows that the area is less than 4 times the square on the radius.



But the following is better :

Imagine a circle cut in half along one of its diameters. Imagine it cut up into many equal sectors (here it is shown as only 7 equal sectors and 2 halves). Imagine the other half circle cut into 8 equal sectors. Then let all the sectors be fitted together like two combs. A rectangle will be formed if an infinite number of sectors is imagined (of course, the top and bottom are a bit corrugated when only 8 sectors are taken).

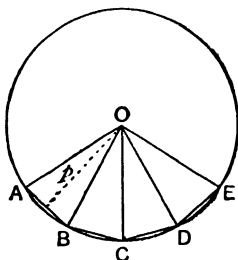


This rectangle is a half circumference long and a radius wide, *i.e.* πr by r , and so its area $= \pi r^2$.

Hence **area of circle** $= \pi r^2$.

[*N.B.*—The area of a circle might be given as $\frac{1}{4}\pi d^2$, but, as this involves a fraction, the use of πr^2 is universal. Hence in many *calculations* we use the *radius*, and in many practical examples we are apt to *measure* the *diameter*, *e.g.* diameter of a bicycle wheel, bore of a gun. The difference, over and over again, leads to mistakes.]

Another way of finding the area of a circle is to consider the area of an inscribed regular polygon of many sides.



The area of $\triangle AOB = \frac{1}{2}AB \times p$, etc.

$\therefore$ by addition,

area of inscribed regular polygon $= \frac{1}{2} (AB + BC + \dots) p$,

and, in limit, area of circle $= \frac{1}{2} \times \text{circumference} \times r$
 $= \frac{1}{2} \times 2\pi r \times r$
 $= \pi r^2$.

For the **area of a sector of a circle** we have only to see what the Angle of the Sector is, to settle how its area compares with the completed circle.

§ 5. If an arc subtends an angle of (say) 20° at the centre of a circle, it is **immaterial whether we talk about the arc of 20° or the angle of 20°** . Thus, if the positions of two places on the same meridian (*i.e.* one place exactly north of the other) is known, we can say how many degrees there are in the arc between them.

Venice and Rome are on the same meridian.

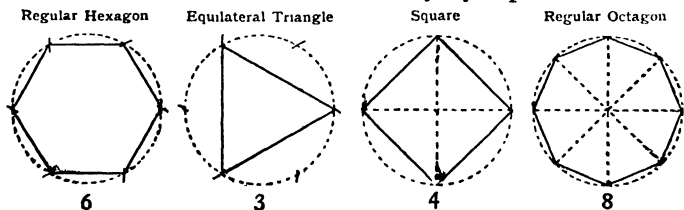
Venice 45° N. lat.,

Rome 42° N. lat.

There is an arc of 3° between them, or that **arc** subtends an **angle** of 3° at the centre of the globe.

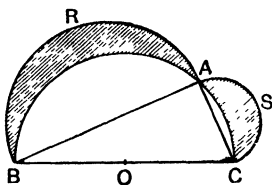
§ 6. **Regular Figures** (all the sides equal and all the angles equal) which have 3, 4, 6 or 8 sides can be constructed accurately, more especially if we first draw a circle on which to find the positions of the corners. Reference to the figure on page 230, which shows a regular hexagon (composed of 6 equilateral triangles), explains why the radius of a circle “steps” round the

circumference exactly 6 times, and it is this fact which makes a regular hexagon and equilateral triangle (join alternate of the six points) easy to draw. For 4 sides (a square) diameters at right angles are needed; and for 8 sides (a regular octagon) bisections of angles will give the necessary points. A mariner's compass (Vol. I.) can be constructed in this way by repeated bisections.



§ 7. Also there are rigid geometrical ways for inscribing regular figures of **5 sides** and **15 sides**, and, of course, for doubling the number of sides as often as we like, but **in practice** calculate the appropriate angle between successive radii and use a **protractor**.

* § 8. The following, proved by **Hippocrates** (about 430 B.C.) is interesting.



The two shaded lunes = area of $\triangle ABC$.

BAC is semicircle on BC as diameter.

BRA „ „ AB „

ASC „ „ CA ,

We know that

the two semicircles BRA and ASC

= the one semicircle BAC in area.

Take away the two white segments;

$\therefore$ the two shaded lunes = the $\triangle ABC$ in area.

[This was the first time that an area bounded by curved lines was proved equal to an area bounded by straight lines.]

EXAMPLES 18 a (GEOMETRICAL DRAWING).

1. In a circle of radius 4 cm. inscribe a regular hexagon.
2. Inscribe a square in a circle of radius 5.2 cm.
3. Inscribe an equilateral triangle in a circle, radius 1.3".
4. In a circle, whose radius is 1.9", inscribe a regular octagon.
5. In a circle (radius 3.7 cm.) inscribe two squares whose sides cut each other at 45° .
6. In a circle (radius 2") inscribe two equilateral triangles cutting each other symmetrically.
7. In a circle, whose radius is 1.75", inscribe a regular pentagon.
(Use a protractor for making the appropriate angle at the centre.)
8. Find A, B, C, D and E, consecutive points of a regular pentagon in a circle (radius 2"), and join them in the following order: AC, CE, EB, BD, DA, to make a star.
(Use the protractor for angles at the centre.)
9. Describe a circle of 1" radius and divide it into three equal sectors.
10. Construct a regular pentagon in circle 2" radius, and determine its area by two different methods.
(Use the protractor for angles at the centre.)

EXAMPLES 18 b (CALCULATIONS).

π is to be taken as $3\frac{1}{7}$ unless otherwise is indicated.

1. Draw a circle of radius 6.5 cm.; open out your dividers to measure 2 cm., and step them round the circumference of the circle. What is the length of the circumference found in this way?

From this result calculate (to 2 places of decimals) the ratio $\frac{\text{circumference}}{\text{diameter}}$. [Do not use a formula to find the circumference.]

2. Explain the meaning of π .

A roller, whose diameter is measured to be 21", is found to revolve very nearly fifty-five times in 100 yards. Obtain an approximate value of π from these data.

3. Draw a straight line 10 cm. long. On it P, Q, R, S and T are points 1, 3, 5, 7 and 9 cm. from one end. With centres P, Q, ... , etc., alternately above and below the line, draw five semicircles each with radius 1 cm. to form a continuous curve. Calculate the length of the curved line.

4. How would you show that for circles of different radii, $\frac{\text{circumference}}{\text{diameter}}$ was approximately the same?

A circular track is 220 yards long on the inside of the track, which is 10 yards wide. How much too far to the nearest yard would a man run in a 4 lap race, if he ran on the outside of the track the whole time?

5. A circular ring has outer and inner radii 6 cm. and 5 cm.; by how many centimetres does the outer circumference exceed the inner circumference? What would the result be if the radii were 6000 cm. and 5999 cm.?

6. Imagine the Equator of the Earth a perfect circle. If there were 2 strings, the one tightly round the Earth at its Equator, and the other making an enormous circle everywhere $3\frac{1}{2}$ feet off the surface of the Earth at the Equator, how much longer is the latter string than the former?

[HINT. Call the radius of the tight string r feet, and do not give r any numerical value.]

7. The diameters of the front and back wheels of a carriage are 4 ft. 4 in. and 5 ft. 3 in. respectively. How far has the carriage travelled when the front wheel has made 420 more revolutions than the hind wheel?

8. In your opinion is the following statement plausible? Give reasons for your answer.

To secure full credit your answer must fulfil three conditions:

- (1) It must be as simple as possible.
- (2) It must be easily intelligible.
- (3) It must be convincing.

A circle, whose area is one acre, would have a diameter of about 80 yards.

9. Calculate (i) the area of a circle whose radius is 2.5 cm., and (ii) the radius of a circle whose circumference is 17 cm. Be careful not to give the results to a greater degree of accuracy than your data warrant. [Show checks.]

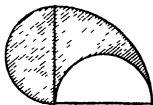
10. A square with sides 5 cm. is drawn, and on its sides, externally, are described semicircles. Calculate the area of the figure drawn.

11. A square lawn, measuring 50 yards each way, contains a circular pond of diameter 7 yards, and is surrounded by a gravel path 5 yards wide. Find the area of the pond and of the gravel path.

12. A rectangular flower-bed, 23' by 16', has in it five circular slug traps, each of diameter 2 feet 11 inches. What is the circumference of each trap, and what area of the bed is safe for a slug?

13. A quadrant of a circle has radius 6". Find the length of its arc and its area.

* 14. Semicircles are drawn on the radii of a quadrant as diameters. Prove that the shaded area is equal to the area of the quadrant.



* 15. Two circles of radii 5 cm. and 2.5 cm. respectively have the same centre. Find the area enclosed between the circles in square centimetres.

If two lines are drawn from the common centre, making an angle of 30° with one another, find the area enclosed between these two lines and the circumferences of the circles.

* 16. Through a point O at the centre of a sheet of cardboard, a straight line OABCD is drawn, OA, AB, BC, CD being equal in length. Circles are then drawn with O as centre and radii OA, OB, OC, OD respectively, and the spaces between these circles painted different colours to form a target.

If the area of the bull is 1 sq. inch, what are the areas of the three coloured strips?

* 17. The radius of one circle is greater than that of another by 1 inch. The difference in the areas is 22 square inches. Find their diameters.

[HINT. Take $r+1$ and r inches for the radii.]

* 18. Find the area of a plane section of a sphere distant 6 ft. from its centre, the radius of the sphere being 10 ft.

* 19. A figure bounded by a semicircle and its diameter is drawn. [The radius is 5 cm.] By "stepping" round its boundary (with dividers opened to $\frac{1}{2}$ cm.) $51\frac{1}{2}$ steps were taken. From this obtain roughly the value of π . [It is not necessary to draw a figure, or actually to take the steps.]

* 20. The angle of a sector of a circle is 70° . Round its boundary (including the radii) 32 half-cm. steps with dividers are taken. The radii are each 5 cm. long. From this obtain a value for π . The figure need not be drawn.

* 21. I possess a field in the shape of a rhombus ABCD, in which the diagonals AC, BD are 140 and 80 yards respectively. Calculate its area.

I have two donkeys, which I tether to the corners A and C by cords 70 yards long. Draw a plan of the field, and measure the angles at A and C, and hence find out what area they can graze over.

Hence calculate what area of grass will not be eaten.

* 22. PQR is an equilateral triangle of side 10 cm. The perpendicular PN from P to QR of course bisects the base [you need not prove this]. Using Pythagoras' Theorem calculate the length of PN. If semicircles are described on the 3 sides of PQR externally to it, calculate the area of the whole figure so formed.

* 23. A piece of strapping plaster 5 metres long is wrapped tightly round a reel 1.4 cm. in diameter, and the total diameter is then 6.3 cm. What is the thickness of the plaster?

* 24. A circle of radius 1 inch is rolled round the outside of an equilateral triangle whose side is 4 inches. Find the length of the locus of its centre.

* 25. ABCDE is a regular five-sided figure. Calculate the angles ABC, ABD, ABE. What is the ratio of the angle ACD to the angle CAD?

* 26. In a regular figure of n sides, where n is odd, the ends of any side are joined to the opposite vertex; show that each angle at the base equals $\frac{n-1}{2}$ times the angle at the vertex.

* 27. Two goal-posts, X, Y, are 8 yards apart. Taking each as centre, 2 equal circles are drawn to intersect exactly 4 yards behind and in front of the goal-line, at points A and B. Draw a plan, stating your scale clearly.

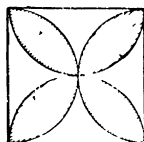
(a) Calculate the length of the radius of the circles.

(b) Prove that AXBY is a square.

(c) Calculate the area of the figure which looks an eight.

(d) Check the results in (a) by measurement, and in (c) by counting squares.

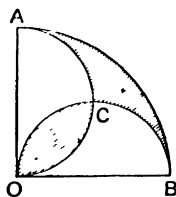
- * 28. Semicircles are described internally on the 4 sides of a square as diameters. If the side of the square is s , find the area of the shaded portion.



- * 29. Arcs of circles are described with equal radii, each half the side of the equilateral triangle, whose angles are the centres. Calculate the area shaded.



- * 30. Semicircles are described on the two radii of a quadrant as diameters. Prove that the two shaded areas are equal.



- * 31. ABCD is the shape of each of the two vertical sides of a coal box. The circular arcs have the same centre O and radii 20 in. and 8 in. The radii are inclined at an angle of 40° .

$abcd$ is the side corresponding to ABCD. Bb , the hinge of the door $BbcC$, is 10 in. long.

The coal box stands on 4 small legs to prevent its rocking.

Coal weighs 80 lb. per cu. ft.

(a) Make a sketch of the coal box.

(b) Calculate the areas of the sectors OAB and OCD, and hence of the side ABCD (to nearest sq. in.).

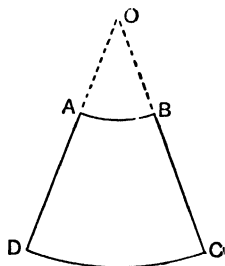
(c) Draw this side on squared paper, and check the preceding result by counting squares.

(d) Using the result of (b), determine the volume of the box in cubic inches.

(e) Calculate the lengths of the circular arcs AB and CD.

(f) Determine the whole area of thin metal needed to make the box, exclusive of legs, hinges, handle and ornaments (to the nearest sq. in.).

(g) If the thin metal weighs 1 lb. per sq. ft., and the

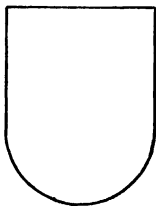


etceteras weigh 2 lb., give the weight of the empty box (to the nearest lb.).

(h) Lumps of coal being irregular, pack together badly. Suppose, when the box is "full," that coal only occupies three-quarters of its internal volume. Find the number of cubic inches of coal in a "full" box and the weight of that coal (to the nearest lb.).

(i) Give the weight of a box "full" of coal.

- * 32. A piece of metal composed of a square (with sides 10 cm.) and a semicircle weighs 560 grammes. When the semicircle is cut off, the square remaining weighs 400 grammes. If these weightings are considered accurate, what value of π is deduced?



- * * 33. Six equal shilling coins lie flat on a table, each touching a central shilling coin, and touching each other consecutively. A rubber band encloses all seven, tightly stretched round the outer six. The diameter of a shilling coin is 0.9"; how long is the stretched rubber band?

- * * 34. One of the simplest approximations for the *rectification of the circle* is "Inscribe, in the circle, a square, and to three times the diameter of the circle add a fifth of the side of the square." Prove that the result will differ from the circumference of the circle by less than one-eighteenth-thousandth part of it. Is the approximation too large or too small?

$$\pi = 3.141593 - \dots,$$

$$\sqrt{2} = 1.41421 + \dots$$

- * * 35. A right circular cylinder has a diameter 8 inches and height 10 inches. A string is attached to a point P in the circumference of the base and is wound tightly round the cylinder, ascending uniformly; and, after 3 turns, is attached to a point Q in the circumference of the top (Q being vertically above P). Find the length of the string.

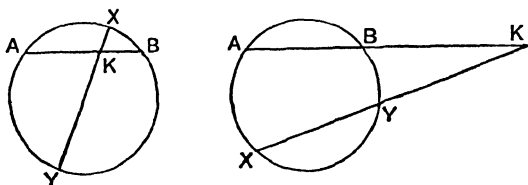
- * * 36. A dirty football bouncing on a stone floor left a circular mark 6 inches in diameter; assuming the circumference of the round football to be $27\frac{1}{3}$ ", calculate the depth to which it was squashed in by the impact.

- * * 37. Two spheres of radii 6 ft. and 8 ft. have their centres 10 ft. apart: find the area of their common section.
[The two spheres have a common section a plane, rather like two soap bubbles coalescing.]
- * * 38. Two straight lines AOD, BOC cut at right angles. $AO=BO=8$ cm., $CO=DO=6$ cm. With centre O and radius OA the circular arc AB is drawn, with centre C and radius CA the arc AK is drawn, and with centre D and radius DB the arc BK is drawn. Find the area of the curvilinear figure AKB, measuring the necessary angles.
[K is to be in the quadrant AOB.]
- * * 39. What is the distance travelled in space (due to the rotation of the earth about its axis) in 1 hour by a person living in 60° N. latitude? [The earth is a sphere of 4000 miles radius.]

CHAPTER XIX.

CHORDS.

§ 1. **Definition.** A **chord** of a circle is the straight line joining two points on the circumference of a circle. Thus a chord is a *terminated* straight line. It has ends, though it is quite possible to say that the chords **AB** and **XY** meet at **K**, understanding that the chords are produced, if necessary.



In the second figure the whole straight lines **ABK** and **XYK** are **secants**, and the parts **AB** and **XY** chords.

The most important chord of a circle is a **diameter**.

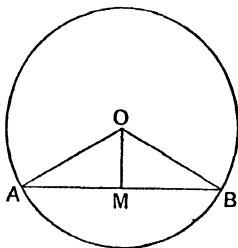
The perfect symmetry of a circle with regard to every diameter (which necessarily passes through the centre) might lead us to infer the truth of the following proposition and its converse. Formal proofs are, however, given.

PROPOSITION 18.

§ 2. General Enunciation. (i) *A straight line drawn through the centre of a circle bisecting a chord (which is not a diameter) must be at right angles to the chord; and (ii), conversely, a straight line drawn through the centre of a circle at right angles to a chord must bisect that chord.*

(i.)

Particular Enunciation. O is the centre of the circle, AB is the chord bisected at M. To prove OM is at right angles to AB.



Construction. Join OA, OB.

Proof. In the $\triangle$ s OMA, OMB,

$$\therefore \begin{cases} AM = BM \text{ (chord bisected),} \\ OM \text{ is common,} \\ OA = OB \text{ (radii),} \end{cases}$$

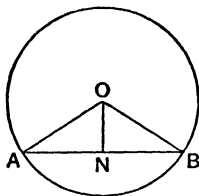
$\therefore \triangle$ s OMA, OMB are congruent,

and, in particular, $\angle OMA = \angle OMB$, but these are adjacent angles,
and therefore OM is at right angles to AB. **Q.E.D.**

For (ii.) see overleaf.

(ii.)

Particular Enunciation. O is the centre of the circle, AB is the chord, and ON is at right angles to AB. To prove that the point N bisects the chord AB.



Construction. Join OA, OB.

Proof. In the rt. $\angle$ d triangles ONA, ONB,

$$\therefore \begin{cases} OA = OB \text{ (radii),} \\ ON \text{ is common,} \\ \angle ONA \text{ and } \angle ONB \text{ are right angles,} \end{cases}$$

$\therefore \triangle$ s ONA, ONB are congruent,

and, in particular, $AN = BN$,

and therefore the point N bisects the chord AB.

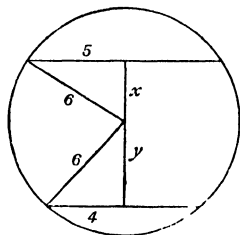
Q.E.D.

3. It is noticeable in the preceding figure that $\angle AOB$ is double $\angle AON$, and this is sometimes useful.

§ 4. Now calculation involving the lengths of chords very often makes use of the facts embodied in the proposition (half chords in right-angled triangles). For example :

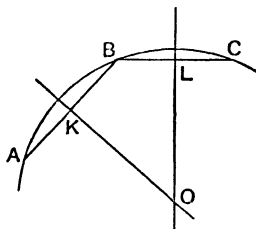
A circle has radius 6". Two parallel chords, on either side of its centre, are 10" and 8" long. How far, to the nearest tenth of an inch, are they apart ?

We get $x^2 + 5^2 = 6^2$ (Pyth.),
and $y^2 + 4^2 = 6^2$ (Pyth.).
Solving these, $x = 3.32...$,
 $y = 4.47...$;
 $\therefore x + y = 7.79...$



And the distance apart is about 7.8".

§ 5. This proposition gives a means of constructing a circle through any 3 given points (not in the same straight line). The thing to remember is that the straight line joining two of these points will eventually be a chord of the circle, and therefore bisect it at right angles. O will be the centre of the circle which passes through A, B and C.



[KO is the locus of points equidistant from A and B.

LO " " " " " " B and C.

Hence O is equidistant from A, B and C.

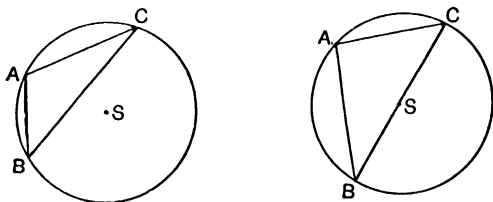
The loci cut at only one point, and this shows that there is only one circle.]

N.B.—If A, B and C were in the same straight line, the perpendicular bisectors would be parallel and would never meet.

§ 6. To **circumscribe** a triangle with a circle is only the same problem over again, *but is most important*. The perpendicular bisectors of *two* of the sides gives the centre (the sides will eventually be chords of the circle). [Only use the perpendicular bisector of the third side as a check.] The circle is sometimes called the **circumcircle** of the triangle.*

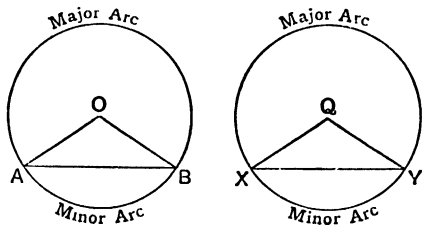
It should be noticed that, if the $\triangle ABC$ is obtuse-angled at A, the centre of its circumcircle is *outside* the $\triangle ABC$ (away from A).

If the $\triangle ABC$ is right-angled at A, the centre of its circumcircle is *on* BC.



* § 7. Since the perpendicular bisectors of the 3 sides of a triangle pass through the centre of the circumcircle they must be **concurrent**. *This is most important.**

§ 8. In **equal circles equal chords must cut off equal arcs** (the major arcs being equal and also the minor arcs). [For proof the $\triangle AOB$, $\triangle XQY$ can be proved congruent; and, in particular, $\angle AOB = \angle XQY$, and hence minor arc $AB =$ minor arc XY , etc.] The converse is also true. These facts should be used when necessary; *they are most important.*†



* The proposition (P) is given fully in the Appendix.

† The propositions (Q) and (R) are given fully in the Appendix

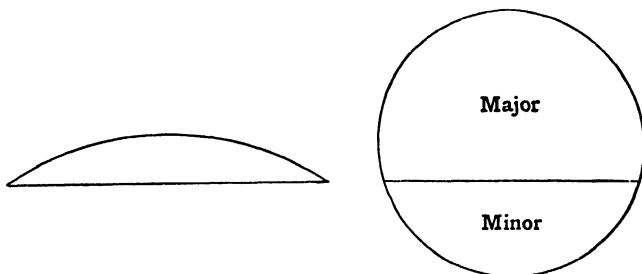
§ 9. Of course, in the figure of § 8, the minor arc AB is distinctly longer than the chord AB, and that is always the case if the angle (AOB) at the centre is considerable ; but if the angle is small the difference is not great, as the following table shows :

Angle	Arc greater than chord by about
1°	1 in 60,000
5°	1 in 3,000
10°	1 in 800
20°	1 in 200

Thus, **when the angle is small, the chord and arc are nearly equal.**

[Now the height of a man at a distance equal to the length of a football ground subtends an angle of about 1°, and it would do perfectly well, in G.D., to reckon the straight height of the man (it is actually a chord of an enormous circle) as the arc of that circle (whose centre is at the eye of the observer, and whose radius is the length of the football ground ; it would be a vertical circle if the man were standing upright). If the angle were large of course, the distinction must be carefully attended to.]

§ 10. A **segment** of a circle is bounded by an arc and by a chord. There are, of course, major and minor segments.



The **area of a segment** is best considered as the difference of areas of a sector and triangle. If the angle is not given, the centre should be found by bisecting two arcs (chords at right angles), and the angle measured, or calculated trigonometrically.



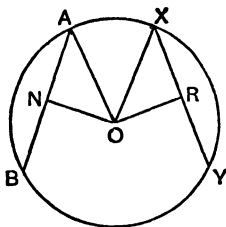
§ 11. The *distance* of a point from a line is the perpendicular distance, and this must be remembered in the proposition which follows.

PROPOSITION 19.

General Enunciation. (i) *Equal chords of a circle must be equidistant from the centre*; and (ii), *conversely, chords which are equidistant from the centre must be equal*.

(i.)

Particular Enunciation. AB and XY are the chords given equal. O is the centre. ON and OR are at right angles to AB and XY respectively. To prove $ON = OR$.



Construction. Join AO and XO.

Proof. ON and OR, at right angles to the chords AB and XY, necessarily bisect them.

In the rt. $\angle$ d $\triangle$ s ANO, XRO,

$$\therefore \begin{cases} AO = XO \text{ (radii),} \\ AN = XR \text{ (halves of equals),} \\ \angle ANO \text{ and } \angle XRO \text{ are rt. } \angle \text{s,} \end{cases}$$

$\therefore \triangle$ s ANO, XRO are congruent,

and, in particular, $ON = OR$.

Q.E.D.

(ii) (Converse.)

Particular Enunciation. (The same figure) AB and XY are the chords. O is the centre. ON and OR, at right angles to the chords AB and XY respectively. ON is given equal to OR. To prove $AB = XY$.

Construction. Join AO and XO.

Proof. ON and OR, at right angles to the chords AB and XY, necessarily bisect them.

In the rt. $\angle$ s ANO, XRO,

$$\therefore \begin{cases} AO = XO \text{ (radii),} \\ ON = OR \text{ (given),} \\ \angle ANO \text{ and } \angle XRO \text{ are rt. } \angle \text{s,} \end{cases}$$

$\therefore \triangle$ s ANO, XRO are congruent,

and, in particular, $AN = XR$, but these are halves of AB and XY ;

$\therefore$ chord $AB = XY$.

Q.E.D.

EXAMPLES 19 a (GEOMETRICAL DRAWING).

1. Prick off the points A, D, E (last page). Construct a circle to pass through those points. Measure its radius.

2. Prick off the points T, R, C (last page). Construct a circle to pass through those points. Measure its radius.

3. AB is a chord of a circle, centre O, radius 8 cm. The angle AOB is 110° . Find the length of AB.

4. Draw a triangle ABC in which $a = 2.4''$, $b = 2.8''$, $c = 3.3''$. Measure the largest angle. Construct a circle through A, B and C, and measure its radius. State your construction.

5. In a triangular field with sides 370 yards, 300 yards and 260 yards is planted a tree equidistant from the corners of the field. Find its distance from a corner by drawing a figure to scale.

- * 6. A triangular park has sides 300 yards, 400 yards and 600 yards. It has lodges at each corner. It is desired to build a house equidistant from the lodges. Show by a diagram drawn to scale that the house must be situated outside the park. What is the distance ?
- * 7. Prick off the points C, R, Q (last page). Construct a circle to pass through those points. [Considerable care will be needed, for the perpendicular bisectors are so nearly parallel.] What is the radius ?
- * 8. ABCD is a square, side 2"; on the diagonal BD (remote from A) is described an equilateral triangle BDE. Draw a circle passing through E, also through F and G (the middle points of AB and AD respectively). Measure its radius.
- * 9. Draw a straight line AB 4" long and find a point C 3" from A and $2\frac{1}{4}$ " perpendicularly from AB. Describe a circle to pass through the points A, B and C. Measure its radius.
- * 10. Draw a triangle with sides 10.4 cm., 9.2 cm., 7.9 cm. Circumscribe it with a circle. Also draw a circle to pass through the middle points of the sides of the triangle. Measure the radii of the circles.
[Don't think that the second circle has not got to cut the sides of the triangle. It does, and ought to do so.]
- * 11. Draw two straight lines meeting at an angle of 55° . Describe a circle, $1\frac{1}{2}$ " radius, cutting off from these lines chords of $1\frac{3}{4}$ " and $2\frac{1}{4}$ " respectively. How far apart are the ends of the chords furthest from the angle ?
- * * 12. Draw a triangle with sides 3.7", 2.5" and 2.93". Draw perpendiculars from the angular points to the opposite sides. Through the feet of perpendiculars describe a circle. Measure the radius.
- * * 13. Two parallel chords of a circle whose lengths are 10" and 16" are 2.9" apart; find, by drawing, the radius of the circle.

EXAMPLES 19 b (CALCULATIONS).

1. A chord of a circle is 30" long and 8" from the centre. Calculate the radius of the circle.
2. The diameter of a circle is 50 inches, a chord is 48 inches. Calculate the distance of the chord from the centre.

3. In a circle whose radius is 6.5 in. a chord is 12.6 in. Find the distance of the chord from the centre.

4. In a circle of radius 25 inches, two parallel chords of lengths 48 and 30 inches are drawn. Find the distance of the centre from each of them, and hence find the perpendicular distance between them.

5. Calculate the distance, from the centre, of a chord of 14 inches in a $\odot$ of radius 25 inches.

Calculate the distance between two parallel chords of lengths 14 inches and 48 inches in the same $\odot$.

6. Calculate the distance between two parallel chords of a circle, of radius 6.5 cm., whose lengths are 12 cm. and 10.4 cm.

7. In a circle radius 13 feet, two parallel chords are drawn respectively 24 ft. and 10 ft. Find the distance of each from the centre and from each other.

* 8. AB is a chord 6" long of a circle whose centre is O and whose radius is 5". Calculate the height and area of the triangle ABO.

* 9. Two parallel chords of a circle, whose radius is 5 inches, are 8 in. and 4 in. long respectively. Find their distances from the centre of the circle, and also the distance between them. (Lengths to be calculated correct to 2 decimal places of an inch.)

* 10. Two parallel chords of the same circle (whose radius is 65 cm.) are 77 cm. apart. One chord is 120 cm. long. Calculate the length of the other chord.

* 11. Calculate (correct to two decimal places) the distances of two parallel chords of length 16" and 14" from the centre of a circle of radius 17". Find also the distance between the chords.

* 12. What is the locus of the middle points of all chords, 8" long, of a circle of radius 5" ?

* 13. In a certain circle of unknown radius, a chord which is 3 in. longer than the radius is 5 in. from the centre. By obtaining and solving an equation, determine the radius of the circle.

Check your result by drawing an accurate figure, and show clearly how the figure does check your result.

* 14. Draw a square ABCD, side 2"; on AB describe an equilateral triangle ABE outside the square. Describe a circle to pass through the points B, D, E. What is its radius?

* 15. A chord of a segment of a circle is 10" long and the height of the arc is 1". What is the radius?

[HINT. First draw a rough figure. On it mark the given dimensions. Work with a half chord and a right-angled Δ . Call the radius r . Calculate r from an equation.]

Afterwards make a correct scale drawing.

* 16. A chord of a segment of a circle is 42" long and the height of the arc is 1". Calculate the radius. [You will not be able to draw an accurate figure.]

* 17. How high is the segment of a circle of radius 29" if the chord of the segment is 40"? [There are two cases to consider.]

* * 18. Two straight lines BAO and XOY meet at O at an angle of 45° . If BO = 17 in. and AO = 7 in., find the radius of a circle which passes through A and B and has its centre on XY.

* * 19. A tunnel is built in the form of a circular tube, and a level road is made by filling up a small segment of the circular section throughout the length of the tunnel. The road is 24 ft. broad. The height from the road to the top of the arch is 18 ft. Calculate the diameter of the tube.

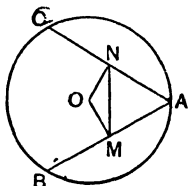
* * 20. If the Earth is a sphere of 3960 miles radius, and if two places, on its surface, 1000 miles apart were connected by a straight tunnel, how far below the surface of the Earth would the centre of the tunnel come?

* * 21. Two places on the surface of the Earth (a sphere of 3960 miles radius) are 200 mi. apart. They are connected by a straight, flat tunnel. Show that the depth of the centre of the tunnel below the surface of the Earth is not very different to the depth of the deepest mine on the Earth, say about $1\frac{1}{4}$ mi.

* * 22. If the Sun, distant about 92 million miles from the Earth, subtends an angle of about $\frac{1}{2}^\circ$, show that his diameter is roughly 100 times as great as the Earth's diameter, which is (say) 8000 miles. [The angle is so small that you can safely take chords and arcs equal. The data are rough too.]

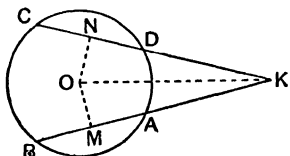
EXAMPLES 19 c (RIDERS).

1. ABC is a circle whose centre is O. The chords AB and AC are equal. OM and ON are perpendicular to the chords AB and AC. MN is joined. Prove that the triangle OMN is isosceles.



2. BA and CD, two equal chords of a circle, whose centre is O, are produced to meet outside the circle in K. Prove that $AK = DK$.

[HINT. Draw the perpendiculars OM and ON; join OK. Consider the $\triangle$ s OKM, OKN.]



3. There are two circles with the same centre; a straight line cuts the larger one in X and Y and the smaller one in A and B: prove $AX = BY$.

4. ABDC is a circle. Chords AB and CD produced towards B and D cut at K. If $KB = KD$, prove that $AB = CD$.

5. AB, AC are equal chords of a circle with centre O: prove that AO bisects the angle at A.

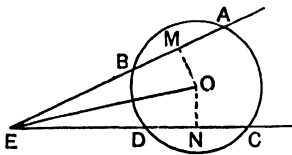
6. Two circles, having centres A and B, cut at a point K. Through K a line is drawn parallel to AB to meet the circles again at X and Y. Prove that $XY = 2AB$.

7. From a point A on a circle a diameter and two chords AB and AC are drawn. If the diameter bisect BC, prove that AB is equal to AC.

8. What is the locus in space of a point equidistant from three given points? When is there no such point?

9. Two chords of a circle AB, CD are produced to meet outside the circle at E. OE bisects the angle AEC, O being the centre of the circle. Prove that $AB = CD$.

[HINT. Construction as in figure.]



10. Two equal chords AB, DC of a circle ABCD intersect outside the circle at P. Prove $\angle PAD = \angle PDA$.

- ✓ * 11. AB and AC are two equal chords of a circle whose centre is O. Prove that AO is at right angles to BC.
- * 12. AB is a diameter of a circle, PQ a chord not at right angles to AB. AM and BN are perpendiculars drawn on PQ (produced if necessary). Prove that $PM = QN$.
- * 13. A and B are the centres of two non-intersecting equal circles. O is the middle point of AB. A straight line is drawn through O cutting the circles in P, Q and R, T respectively. Prove that $PQ = RT$.
- * 14. Prove that of two unequal chords in any circle the greater must be nearer to the centre of the circle than the less.
- ✓ * 15. Two circles intersect at A and B, and parallel straight lines CAD, EBF are drawn meeting the circles in C, D, E, F ; prove that CD is equal to EF.
[HINT. From the centres draw perpendiculars to the chords and obtain a rectangle.]
- * * 16. If two circles intersect at A and B, the straight line drawn through A parallel to the line joining their centres to meet the circles is greater than any other line through A.
- * * 17. Two circles cut in B and A, and PBQ is a line cutting them in P and Q ; find when the area PAQ is greatest.
- ✓ * * 18. AB and CD are two chords of a circle which cut at right angles at E within the circle ; if O is the centre, show that $AB^2 + CD^2 + 4OE^2 = 8OA^2$.
- * * 19. If the diagonals of a quadrilateral ABCD intersect at E, the centres of the circles described about the four triangles EAB, EBC, ECD, EDA are the angular points of a parallelogram.

CHAPTER XX.

ANGLES AND ARCS.

§ 1. In equal circles the arcs which subtend equal angles at the centres must be equal. The converse is true. We have already seen in Chapter XVIII. that it is immaterial whether we consider an arc of (say) 20° , or whether that arc subtends an angle of 20° at the centre. In Chapter XIX. (page 246) it was seen that equal chords cut off equal arcs. A ready transference from angles to arcs and chords is often convenient. Now the connexion between angles at the centre and angles at the circumference is considered in §§ 2 and 3. It is as well to read these two sections before attempting any exercises.

§ 2. We have already seen that arcs of circles and the angles they subtend at the centre of the circle are identical in size; but, in practice, we are far more likely to be concerned with **angles at a point on the circumference** of the circle; and consequently the relative sizes of an angle at the centre of a circle and an angle on the circumference subtending the same arc have to be considered. The fact that the former is twice the latter is of fundamental importance, and the proof thereof is embodied in the following proposition.

5. Two circles cut at A, B and parallel straight lines CAD, EBF are drawn meeting the circles in C, E and D, F respectively; show that the straight lines CE, DF are each equal to AB.

[~~HINT~~. Join EA. Prove arc CE = arc AB, etc.]

6. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn. Show that the chord PX is equal to the chord QY.

Angle at the centre double the angle at the circumference.

7. ABC is an arc of a circle with centre O, which is less than a semicircle, and B is the middle point of the arc: prove that angle $ACB = \frac{1}{2}AOC$.

8. C is the centre of a circle, and A, B, D are 3 points on the circumference; $\angle ACB = 50^\circ$; $\angle CAD = 25^\circ$. Prove that BD is parallel to AC. (D and B are on the same side of CA.)

9. It is required to inscribe in a given circle a triangle ABC with angles 50° , 60° , 70° . Sketch the figure. Let O be the centre of the circle. How big must the angle BOC be? Write a description of the construction. (An accurate figure is not required.)

10. XYZ is a given triangle. Show how to inscribe in a given circle a triangle equiangular to XYZ.

Miscellaneous.

* 11. On a base $BC = 5$ cm. construct a triangle BAC whose other sides are each 8 cm. : measure the vertical angle A, and on BC and on the same side of BC construct another isosceles triangle BDC, so that the angle BDC is double the angle BAC.

Show that a circle with D as centre and DA as radius will pass through B and C.

* 12. AB and CD are two chords of a circle, cutting at E at right angles; prove that the sum of the arcs AC and BD is half the circumference of the circle. (E is inside the circle.)

[HINT. Arc AC + arc BD is measured by twice angle E, etc.]

* 13. If AB be a fixed diameter and DE an arc of constant length in a fixed circle, and the straight lines AE, BD intersect at P, the angle APB is constant.

[HINT. $\angle APB$ is measured by $\frac{1}{2}$ what two arcs?]

* 14. Prove that the bisector of the vertical angle of any triangle and the perpendicular bisector of the base always meet on the side of the triangle. Is this theorem true in the case of an isosceles triangle?

- * 15. AB is a fixed chord of a circle ; and P is any point on the circumference. Prove that the bisector of the angle APB will always pass through a fixed point.
[HINT. Use the rider immediately preceding, and note that the point is on the circum-circle.]
- * 16. From any point A on a circle of centre O a straight line AD is drawn perpendicular to any chord BC ; prove that the angles CAD, OAB are equal.
[HINT. Draw ON perpendicular to BA. Prove the Δ s CAD, OAN equiangular.]
- * 17. Two circles, centres O and P, intersect at A and B. Through B a straight line HBK is drawn cutting the circles in H and K. Prove $\angle OAP = \angle HAK$.
[HINT. Prove $\angle O = \angle H$, $\angle P = \angle K$, etc.]
- * 18. A circle, centre O, circumscribes a triangle ABC ; AD is the perpendicular on BC ; show that the angle OAD is equal to the difference between the angles ABC and ACB.
[HINT. $\angle OAD = \angle DAB \pm \angle OAB$.]
- * 19. ABCDE is a regular pentagon, AC and AD meet BE in X and Y respectively ; show that BX is equal to EY.
- * 20. A circle is described through the angular points of a square ABCD. P is a point on the minor arc AB. Prove that PC and PD trisect the angle APB.
- * 21. AC, BD are two perpendicular chords of a circle ABCD ; prove that the sum of the arcs AB and CD is equal to the sum of the arcs BC and AD.
- * 22. ABC is a triangle inscribed in a circle. The bisector of the angle A meets the circumference in D. DM is the perpendicular on BC. Prove that BM = MC.
- * * 23. Two circles intersect in A and B. Two straight lines PAQ, PBR are drawn from any point P on one circle to cut the other circle again in Q and R. Prove that the length of the chord QR is constant.
[HINT. Join QB. $\angle QBR = \angle P \pm \angle Q$.]
- * * 24. Two chords, AC and BD, intersect, inside a circle (centre O), at E. Show that the size of the angle AED is the arithmetic mean between the sizes of the angles AOD and BOC.
[HINT. Recollect arc AD + arc BC is measured by twice angle E.]

- ** 25.** Two segments of circles are on the same base AB and on the same side of it, and a point C is taken on one of them; if AC, BC cut the other at R and S, prove that RS is an arc of constant length.

[HINT. $\angle RAS = \angle ACB \sim \angle ASB$.]

- ** 26.** A, B, C, D are four points on a circle; X, Y, Z and W are the middle points of the arcs AB, BC, CD and DA respectively of the circumscribing circle; show that XZ and YW are at right angles.

[HINT. If XZ and WY cut at K, then

2 angle WKX is measured by arc WX + arc ZY.]

- ** 27.** O is the centre of a circle BCDE; A is a point outside the circle; and ACB and ADE are straight lines such that BCE is an acute angle. Show that the angle COD and twice the angle CAD are together equal to the angle BOE.

[HINT. $\angle CAD$ is measured by $\frac{1}{2}$ (arc BE - arc CD), etc.]

- ** 28.** If two chords of a circle are perpendicular to each other, the sum of the squares on them, together with four times the square on the line joining their point of intersection to the centre, is equal to twice the square of the diameter.

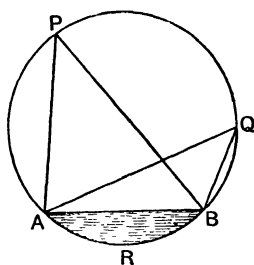
- ** 29.** PQRS is *any* quadrilateral inscribed in a circle. The angles P and R are bisected by PK and RL, which meet the circumference in K and L. Prove that KL must be a diameter of the circle.

CHAPTER XXI.

ANGLES IN THE SAME SEGMENT.

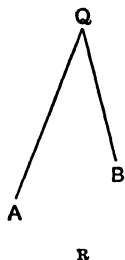
§ 1. We have already seen that a **segment** of a circle is bounded by a chord and arc, and we name it by at least 3 letters, necessarily starting from and finishing at the ends of the chord. If the chord is AB, the unshaded segment (the major segment) might be called by any of the following names :

The segment APB or AQB or APQB or BPA or BQA or BQPA, but by letters in no other order.

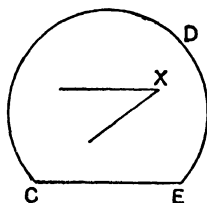


The angles P and Q are said to “stand” on the arc AB. In practice, if one stood at A and B one can measure the value of $\angle AQB$, even if Q is a distant and unapproachable object.

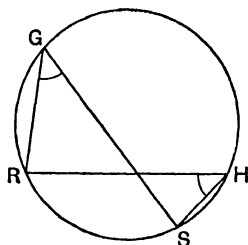
Angles are said to be “in” the segment if the arms of the angle go through the ends of the chord and the vertex of the angle is on the arc of the segment. Thus $\angle APB$ and $\angle AQB$ are both said to be angles “in” the segment APQB.



Doubtless the $\angle X$ is inside the segment CDE ; but $\angle X$ is not technically "in" that segment.



Frequently the chord of the segment is not drawn at all, but the angles RGS and RHS are both in the segment RGHS, imagining the chord RS to complete the (major) segment RGHS.

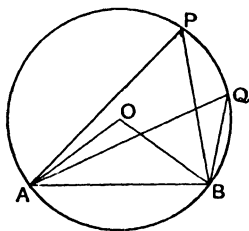


The fact that angles in the same segment of a circle are necessarily equal is proved in the next proposition.

PROPOSITION 21.

§ 2. General Enunciation. *Angles in the same segment of a circle must be equal.*

Particular Enunciation. The angles are $\angle APB$ and $\angle AQB$ in the segment $APQB$. To prove $\angle APB = \angle AQB$.



Construction. Let O be the centre of the circle. Join OA and OB .

Proof. $\angle AOB = 2\angle APB$, for angle at centre is double angle at circumference standing on the same arc,

and $\angle AOB = 2\angle AQB$, for the same reason ;

$\therefore \angle APB = \angle AQB$.

Q.E.D.

§ 3. The truth of the converse of this is of very great importance. It is comparatively easy to measure, with instruments, the angle that two (perhaps distant) objects subtend at the eye: and if they can be identified and if their positions are accurately charted, the fact that our own position must lie on a certain arc of a circle narrows down the possibilities of our whereabouts considerably; another condition may perhaps tie it down absolutely.

The enunciation of this converse is given in various forms.

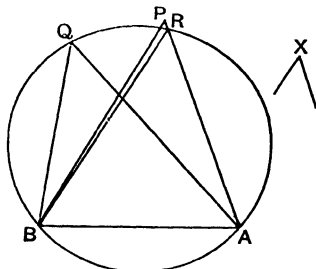
One form is, *If the straight line which joins two fixed points subtend equal angles at two other points on the same side of the line, a circle can be drawn to pass through the four points.*

In the following enunciation the locus idea is embodied.

PROPOSITION 22.

General Enunciation. *The locus of points, on the same side of a given straight line, at which that given straight line subtends a given angle, is a certain arc of a circle passing through the ends of that straight line.*

Particular Enunciation. AB is the given straight line and $\angle X$ is the given angle; the locus of a point, on the same side of AB, at which AB subtends an angle equal to $\angle X$ is a certain arc of a circle.



(Part i.)

Construction. Let $\angle AQB$ be equal to $\angle X$.

Circumscribe $\triangle AQB$ with a circle.

Let P be a point on the locus

(i.e. a point such that $\angle APB = \angle X$).

Suppose P does not lie on that circle.

Let AP (produced if necessary) meet the circle at R.

Join RB.

Proof.

$\angle X = \angle AQB$ (given),

and $\angle AQB = \angle ARB$ (in the same segment),

and, in $\triangle RPB$, ext. $\angle ARB >$ int. opp. $\angle APB$,

hence $\angle X > \angle APB$; but $\angle X = \angle APB$ (given),

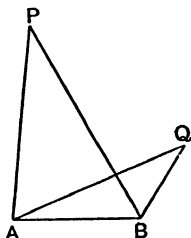
so that $\angle APB$ is equal and unequal to $\angle X$, which is impossible;
 $\therefore$ the point P must lie on a certain arc of a circle, passing through A and B.

(Part ii.)

To prove that *all* points on that arc are satisfactory is a repetition of the preceding proposition (21) and is generally taken for granted.

Q.E.D.

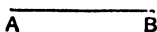
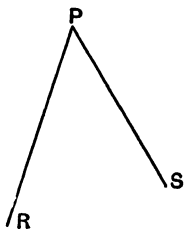
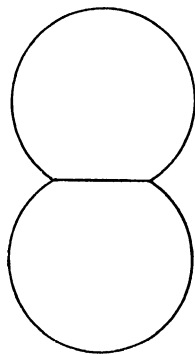
N.B.—If for some reason we know $\angle P = \angle Q$, then the four points A, B, Q and P lie on the same circle, but **the circle need not be drawn.**



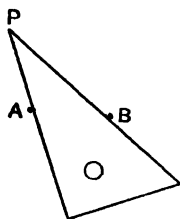
If the words “*on the same side of a given straight line*” were omitted in the enunciation there would be two arcs, which the figure sufficiently indicates.

[The idea of “locus” may be well shown in two ways :

(i) First, on tracing paper draw two lines PR and PS (longish for choice) meeting at the given angle (call it X). Secondly, quite separately and on a piece of plain paper, draw the given straight line AB. Now move the tracing paper about, compelling PR to pass through A, while PS passes through B. By making pin pricks through P in various positions the locus may be shown clearly.



(ii) Drawing pins are put in firmly at A and B. A set-square is moved about with two of its sides against the drawing pins. The locus of P (*i.e.* the path traced out by P) is again clear.

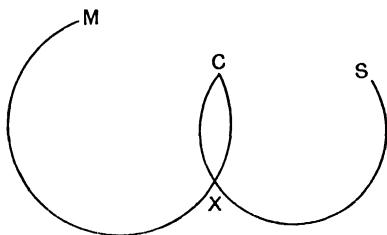


Of course (i) has the advantage that on the tracing paper we can draw two straight lines at any angle that we please, and is certainly to be preferred if we have a piece of tracing paper, (ii) ties us down to using the particular angle the set-square has.]

*** § 4.** Finding one's position on a map is often an easy task, if one is on familiar ground, or if one is at or near some well defined object (such as at a mountain top or a well-marked bend in a river). If one is near a road, which one can identify, it is frequently not a difficult job, but if one is out in the open the use of the locus in this proposition may help. If we can identify 3 landmarks in nature and their positions on the map, we can solve the problem.

[The method is shown clearly on the following page.]

M is a mountain top, C is a clump of trees, S is a church spire.



Suppose we are at X (at present its position is not known).

$\angle MXC$ is observed to be 30° ,

$\angle CXS$ is observed to be 40° .

M is on the left and S is on the right.

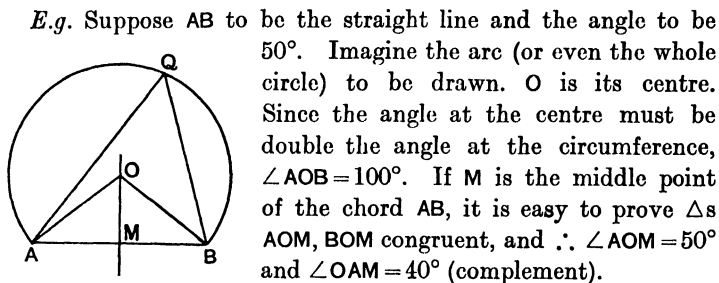
Now, as far as M and C are concerned, we are on a certain arc of a circle; and, as far as C and S are concerned, we are on a certain arc of another circle.

The intersection of the two loci (arcs of circles in this case) gives our position, X, accurately.

(M on the left, etc., makes us put the arcs on the proper side of the chords MC and CS.)

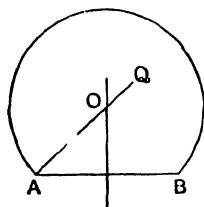
§ 5. That leads us to the problem :

On a given straight line to construct a segment of a circle containing a given angle.



E.g. Suppose AB to be the straight line and the angle to be 50° . Imagine the arc (or even the whole circle) to be drawn. O is its centre. Since the angle at the centre must be double the angle at the circumference, $\angle AOB = 100^\circ$. If M is the middle point of the chord AB, it is easy to prove $\triangle AOM$, $\triangle BOM$ congruent, and $\therefore \angle AOM = 50^\circ$ and $\angle OAM = 40^\circ$ (complement).

Hence we have the following construction. Make the angle BAQ 40° , and the perpendicular bisector of AB will give the centre O required.

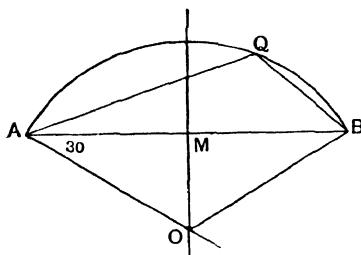


There is a slight variation if the angle is to be obtuse (say 120°). Again imagine the arc to be drawn and O to be its centre, so that the reflex angle AOB is 240° ;

$$\therefore \angle AOB = 120^\circ; (360 - 240 = 120)$$

$$\therefore \angle AOM = 60^\circ;$$

$$\therefore \angle OAM = 30^\circ \text{ (complement).}$$



The construction is now clear; it involves making angle 30° and perpendicularly bisecting AB .

Doubtless we can have rules for the size of the angle we have to make and on which side of AB to make it, but the rough preliminary figure will make us put the arc in the correct place, and that is what we need in practice.

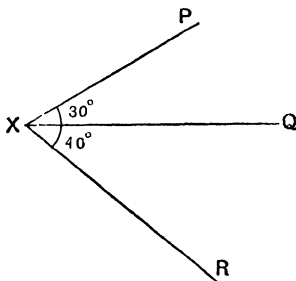
* § 6. So it comes to this, we can find our position on a map, by the intersection of loci, if three well-defined landmarks are visible. *They must be well placed.*

The words in italics are important. Our position must not be on the circle through the three landmarks (for then the two arcs would really be the same and would not cut at a definite point). In practice, our position must not nearly be on that circle (for otherwise we should get a bad intersection,

and the position, X, would be very unreliable). The most favourable cases (the best intersections) occur when we are distinctly inside the triangle formed by the 3 landmarks, or (if outside that triangle) the centre landmark should be nearest to us.

[Of course in a fog this fails, and in the midst of the ocean (where landmarks are not visible) this fails too. Then we have to fall back on dead-reckoning (knowing how fast and in what direction we have travelled since last we knew our position accurately); and, on the ocean, if it is not foggy, on star or sun observations.]

The observed angles (again say 30° and 40°) can be correctly drawn side by side on tracing paper; *e.g.* XP, XQ and XR (the lines had better be longish really). If the tracing paper is moved about till XP passes through M, XQ through C and XR through S simultaneously, this will give the position of X. With careful drawing this should give the position of X on the map well, provided we should get good intersections, but the results would be very unreliable otherwise.



EXAMPLES 21 a (GEOMETRICAL DRAWING).

1. On a base of 2" describe a segment of a circle containing an angle of 55° . Draw one angle in that segment and measure it. Is your figure satisfactory?

[HINT. First draw a rough figure. AB is the base and O the centre of the segment (*i.e.* of the circle of which the segment

is a part). Calculate the size of $\angle AOB$. Then draw $\triangle AOB$ correctly, etc.]

2. On a base 8 cm. describe a segment of a circle containing an angle of 130° . Draw one angle in that segment and measure it. Is your figure satisfactory? [Your angle should be within a degree or two.]

3. Draw a circle (centre O and radius 2"). Divide it into two segments, one of which contains an angle of 50° , by a chord AB. Measure AB.

[HINT. First roughly draw the figure to find out at what angle the radii OA and OB should be inclined.]

4. Draw a circle (with centre O and radius 6 cm.). Divide it into two segments, one of which contains an angle of 145° . Measure any angle in the other segment. [Do you know what it ought to be?]

5. Prick off the 5 points P, Q, R, S, T (last page). On PQ construct a segment of a circle containing an angle of 56° , and on ST construct a segment of a circle containing an angle of 70° . Let the segments cut at K and L. Measure KR and LR in centimetres.

* 6. Construct a segment of a circle, on a base of 7 cm., containing an angle of 49° .

Using same figure, on the same base of 7 cm. describe two triangles with altitude 5 cm. and vertical angle 49° . Measure the base angles of one of the triangles.

* 7. On a line QR 3" long draw any triangle PQR such that the vertical angle QPR is 72° . Draw a circle through PQR and write down the condition that all points lying on the arc QPR must satisfy.

With the help of the above construct a triangle having a base of 3", height of 1.7", and vertical angle 72° . Measure the shortest side of the triangle.

* 8. P, Q and R are three points such that $PQ=8$ cm., $\angle PQR=150^\circ$, and $QR=9$ cm. P represents a windmill, Q a church, and R a chimney, on a map. A person at an unknown point X sees the windmill on the left and the chimney on the right. The line PQ subtends an angle of 40° at X, and the line QR 66° . Determine the position of X by geometrical construction, stating briefly in words your processes. Measure QX on the map. [$\angle Q$ points towards X.]

- * 9. One is inside the triangle CMS. C is a clump of trees, M a mountain top and S a church spire. All these are visible from one's position X. On the map $MS = 10$ miles, $SC = 7$ miles, $CM = 11$ miles. One observes $\angle MXS$ to be 125° and $\angle CXM$ to be 115° . Draw an accurate diagram to show one's position X. How far is one from the spire?
- * 10. One is anxious to identify one's position K, at present unknown, and one observes a tower (T), a peculiar fan-shaped tree (F) and a windmill (W) in that order from left to right. All these are marked on the map. $TF = 4$ miles, $FW = 4.7$ miles, $\angle TFW = 145^\circ$. One observes $\angle TKF = 50^\circ$, $\angle FKW = 40^\circ$. Determine the position K. Measure KF. [K and F are on the same side of TW.]
- * 11. Draw a triangle ABC with base BC 3" and area 4.2 sq. in. (first calculating the height) and vertical angle A 50° . Measure the longest side.
- * 12. Draw a triangle with base 10 cm., area 17 cm.², and vertical angle 100° . Measure the shortest side of the triangle.
- * 13. Show, by a correct diagram, that it is impossible to draw a triangle of area 12 cm.², on a base 8 cm., if the vertical angle is to be 115° .
- * 14. A, B and C form the angular points of a triangle, of which the side $AB = 620$ yards, $BC = 513$ yards and $AC = 995$ yards. From a point D you observe that the angle between A and B is 46° , and the angle between B and C is 49° . Find the position of D, and its distance from B. D is on the opposite side of AC to B.
- * 15. Prick off the points P, Q, R (last page). Find, by construction, a point X at which PQ subtends an angle of 60° and QR an angle of 110° . Measure XQ in centimetres.
- * 16. Draw two lines OX, OY at right angles to each other, such that $OX = 10$ cm., $OY = 5$ cm. Find the position of a point P, such that $\angle OPX = 135^\circ$, $\angle OPY = 90^\circ$. Justify your method. Measure OP.
- * 17. XYZ is a triangle in which $YZ = 10$ cm., $ZX = 7$ cm. and $XY = 9$ cm. Find, by geometrical construction, a point P inside the triangle, such that each of the sides subtends an angle of 120° at it. Measure PX. State in words how the same problem can be solved by using tracing paper.

- * 18. XYZ is a triangle in which $YZ = 12$ cm, $ZX = 11$ cm. and $XY = 10$ cm. Find, by construction, the position of a point P in it, such that $\angle YPZ = 130^\circ$ and $\angle ZPX = 110^\circ$. Measure PZ.
- * * 19. Draw the quadrilateral PQRS, such that $SQ = 9$ cm., area of $\triangle PQS = 18$ sq. cm., $\angle P = 70^\circ$, $\angle R = 127^\circ$, $SR = 6$ cm. and $PS < PQ$. Measure PR.
- * * 20. Draw a figure ABCD. Given $AB = 2''$. $\angle ACB = \angle ADB = 30^\circ$, $CD = 3''$, and $\angle ABC = \angle DAB$. Find its area. [Larger figure.]
- * * 21. Draw a parallelogram with one angle 75° and with its diagonals 8 cm. and 10 cm. Measure the short sides of the parallelogram.

EXAMPLES 21 b (CALCULATIONS).

(The figures need not necessarily be drawn to scale; however, accurate scale drawings provide useful checks.)

1. ABC is a circle having its centre at O. The angle $BAC = 75^\circ$. How many degrees are there in the angle OCB ?

2. A man cycles round a circular track at the rate of 15 miles per hour. An observer at the edge of the track notices that in looking at the cyclist he has to turn his line of sight through 35° every 20 seconds, except when the cyclist passes him during that interval. Find the radius of the circular track.

3. ABCD is a quadrilateral in a circle. The diagonals AC and BD meet at K. $\angle CAD = 40^\circ$, $\angle AKD = 102^\circ$, $\angle CDB = 36^\circ$. Sketch the figure (not necessarily to scale). Give the sizes of $\angle ACB$ and $\angle ACD$ with reasons. Suppose O is the centre, join AO and BO. Give the size of $\angle OBA$ with reasons.

4. Through a point P outside a circle two straight lines PAB, PCD are drawn cutting the circle in A, B, C, D. BC and AD intersect in K. If the $\angle BAD = 60^\circ$, $\angle AKC = 70^\circ$, $\angle ACP = 55^\circ$, find the size of all the remaining angles, giving reasons.

* 5. ABCD is a quadrilateral inscribed in a circle of radius 4 cm., and DC and AB produced meet at K. If $AB = BC = 4$ cm., draw the figure roughly and calculate the size of the angle ADB : and if $\angle KCB = 70^\circ$, calculate the angle at which the diagonals AC and BD intersect.

* 6. Draw a circle of radius 5 cm. and a diameter AOC, O being the centre of the circle. Draw a chord AB above AC so that

$\angle BAC = 24^\circ$, and a chord AD below AC so that $\angle CAD = 52^\circ$. Join BC, CD, BD, BO, DO.

Find, by calculation, the number of degrees in every angle between all lines that meet and are next each other in the figure, and mark the degrees in the angles in the figure.

- * 7. AB and CD are any two chords of a circle cutting inside the circle at Z, while AC and DB cut (when produced) outside the circle at V. It is known that the angle $ACD = 53^\circ$ and the angle $V = 31^\circ$. Sketch the figure (which need not be accurate), and in each of the other nine angles write its size. In particular, separately, give reasons for your answers in the four following cases :

(a) $\angle DCV$, (b) $\angle D$, (c) $\angle ABD$, (d) $\angle DZB$.

- * * 8. O is the centre of a circle, A, B, C, D points on the circumference. $\angle DOC = 110^\circ$, $\angle BDC = 50^\circ$, $\angle ACD = 55^\circ$; AB, DC meet when produced in F. (*F will be too far away to appear on your paper.*) Calculate the number of degrees in the angles BCA, BFC.

EXAMPLES 21 c (RIDERS).

1. Two chords AB, CD of a circle cut at E. Show that the triangle ACE is equiangular to the triangle DBE.

2. BC is a chord of a circle; P a point on the circumference, Q a point outside the circle on the same side of BC as P. Prove that the angle BQC is less than the angle BPC.

[HINT. Let BQ or CQ cut the circle at X and join XC (or XB).]

3. B and C stand on a circular track watching A run round it. Prove that during any given time B and C must turn their heads through equal angles.

[HINT. Suppose A and Z are two positions of the runner.]

4. Show how to construct a triangle, given the base, vertical angle and altitude.

[HINT. Construct base, on it a segment containing given angle, then a parallel, etc.]

* 5. AB, CD are two parallel chords of a circle of centre O; if AD, BC cut at E show that A, C, O, E are concyclic.

[HINT. Prove $\angle AEC$ and $\angle AOC$ are each twice $\angle B$.]

* 6. AB and CD are parallel chords of a circle. If DA and CB meet at E, prove that EAB is an isosceles triangle.

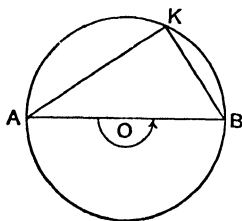
- *7. PQRS is a quadrilateral inscribed in a circle. RM is drawn perpendicular to QS meeting it at M, and RN is drawn perpendicular to QP meeting it at N. Prove that the angle PRS is equal to the angle NRM.
[HINT. Prove RMNQ cyclic.]
- *8. Construct a triangle on a given base with a given vertical angle and such that the point of intersection of the bisector of the vertical angle with the base may be at a given point.
[HINT. Get the circum-circle; consideration of Examples 20, 14 helps.]
- *9. ABC is a triangle inscribed in a circle. ADR perpendicular to BC cuts the circle again at R; CL perpendicular to AB cuts AR at O. Prove that $CR = CO$.
[HINT. Prove CDLA cyclic.]
- *10. AB is a fixed diameter of a circle and CD an arc of constant length; if AD, BC meet in P and AC, BD in Q, prove that the locus of P, and also of Q, is a circle.
- *11. AB is the diameter of a circle; another circle centre B is drawn cutting the first circle in C and D, and the diameter AB in E. If CE produced meets the first circle in F, then $\angle FAB = \angle FBA + \angle CAB$.
[HINT. Prove $\triangle CEB$ is isosceles.]
- * *12. AB, CD are two chords of a circle which cut at right angles at E; if EF be perpendicular to AC, and FE produced meets BD at K. Prove that K is the mid-point of BD.
[HINT. Prove $\triangle KED$ is isosceles.]
- * *13. Two circles cut at A and B, and a straight line PQRS crosses AB and cuts one of the circles in P, R and the other in Q, S; prove that the angles PAQ, RBS are equal.
[HINT. $\angle PAQ = \angle PAB - \angle QAB$.]
- * *14. OA, OB, OC are successive radii of a circle. A circle drawn round the triangle OAB cuts the straight line joining A to C in D. Prove that OD bisects the angle BOC.
- * *15. From a fixed point O on a given circle two chords OP, OQ are drawn equally inclined to a fixed chord. Prove that PQ is fixed in direction.
- * *16. Construct a triangle on a given base with a given vertical angle such that (1) the sum, (2) the difference of the other two sides may be equal to a given line.

CHAPTER XXII.

THE ANGLE IN A SEMICIRCLE.

PROPOSITION 23.

§ 1. **General Enunciation.** (i) *The angle in a semicircle must be a right angle ;* (ii) *the angle in a segment greater than a semicircle must be less than a right angle ;* and (iii) *the angle in a segment less than a semicircle must be greater than a right angle.*



(i.)

Particular Enunciation. O is the centre, AB is the diameter of the semicircle. $\angle AKB$ is an angle in the semicircle. To prove $\angle AKB$ must be a right angle.

Proof. $\angle AOB = 2\angle AKB$ (angle at centre twice angle at circumference standing on the same arc),

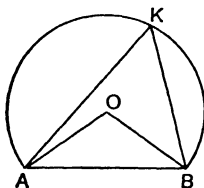
but $\angle AOB$, being a straight angle, is 2 right angles ;

$\therefore \angle AKB$ must be 1 right angle.

Q.E.D.

(ii.)

Particular Enunciation. O is the centre, AB is a chord. O is necessarily inside the segment AKB, which is given greater than a semicircle. $\angle AKB$ is an angle in the segment. To prove $\angle AKB$ must be less than a right angle.



Construction. Join OA and OB.

Proof. $\angle AOB = 2\angle AKB$ (angle at centre twice angle at circumference standing on the same arc).

Now OAB is a triangle so that $\angle AOB$ is necessarily less than 2 right angles,

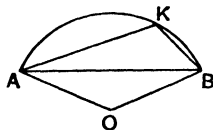
$\therefore \angle AKB$ is less than 1 right angle.

Q.E.D.

(iii.)

Particular Enunciation. O is the centre, AB is a chord. O is necessarily outside the segment AKB, which is given less than a semicircle. $\angle AKB$ is an angle in the segment. To prove $\angle AKB$ must be greater than a right angle.

Construction. Join OA and OB.



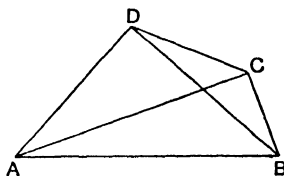
Proof. The reflex $\angle AOB = 2\angle AKB$ (angle at centre twice angle at circumference standing on the same arc), and reflex $\angle AOB$ is necessarily greater than two right angles ;

$\therefore \angle AKB$ is greater than 1 right angle.

Q.E.D.

§ 2. The **converse** of the proposition “the angle in a semicircle is a right angle” may be enunciated: *The locus of points at which a given straight line subtends a right angle is the circle on that straight line as diameter* (of course only a semicircle if the points are confined to only one side of the line).

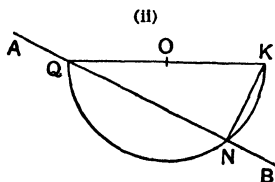
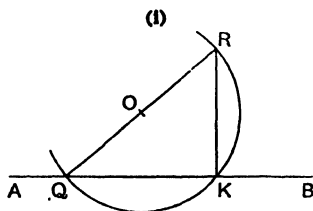
This is only a special case of Prop. 22 (though a very useful case), and a separate proof is not given, but in any case the truth of this **converse is usually assumed**. Notice that some points may be on the same circle, or be **cyclic** as they are called (the word *conconcyclic* is also used), and yet the circle need not be drawn; for instance, if the angles ADB and ACB are both right angles, the points A, B, C and D are concyclic (or $ABCD$ is said to be a cyclic quadrilateral); and $\angle BAC = \angle BDC$ (in the same segment, etc.).



§ 3. The original proposition affords a ready means of constructing a **perpendicular to a given line**, especially useful if the perpendicular will come at (or near) the end of the line.

(i) From a point K on the given line AB . Take any point O . Describe a circle, with centre O and radius OK . Let the circle cut AB again at Q . Join QO and produce it to cut the circle at R . Join RK .

(For proof. Of course QKR is a semicircle, etc.)

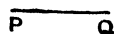


(ii) From a point K not on the line AB . Take any point Q on the line AB . Join QK . Bisect QK at O . With centre O and

radius OK describe a semicircle cutting AB (produced if necessary) at N. Join KN.

To get reliable results we must arrange to get good intersections, so in (i) take the position of O and in (ii) the position of Q judiciously.

One may talk about the perpendicular from a point to a line, recollecting that the line may have to be produced to meet the perpendicular.



EXAMPLES 22 (RIDERS).

1. On a base AB, and on opposite sides of it, two triangles ACB, ADB are described, with right angles at C and D. Prove that angle CAB = angle CDB.

[HINT. Prove ACBD cyclic.]

2. Prove that a diameter of a circle is larger than any chord of the circle which does not pass through the centre.

[HINT. Suppose AP is the chord, and AOB a diameter, consider the $\triangle APB$.]

3. In the triangle ABC, BP and CQ are drawn to meet the opposite sides at right angles. Prove $\angle PBC = \angle QCB$.

[HINT. Prove PQCB cyclic.]

4. ABC is a triangle. Perpendiculars AD and BE are let fall from A and B on to the opposite sides of the triangle and meet in K. Prove that circles will pass through ABDE and CDKE.

5. OA is any radius of a given circle whose centre is O. On OA as diameter another circle is described. If AXY be any straight line drawn through A to meet the two circles in X and Y respectively, show that Y is the middle point of AX.

[HINT. Prove OY perpendicular to the chord AX.]

6. O is the centre and BOC a fixed diameter of a circle whose radius is 5 cm. A is a point on BOC such that BA = 2 cm. and AC = 8 cm. Chords $B_1AC_1, B_2AC_2, \dots$ are drawn through A. [Draw 5 of these, making angles $30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ$ with BAC.] Carefully mark the middle points $P_1, P_2, \dots$ of these chords.

Does the locus of P appear to be a circle? If so, what is its diameter? [The question is continued overleaf.]

Give a formal geometrical proof that the locus of the middle points of all chords (of a given circle) which pass through a given fixed point must be a circle.

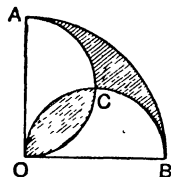
✓ 7. Prove that the circles described on two sides of a triangle as diameters intersect again on the base.

[HINT. Draw only one circle, and prove that the other circle will pass through the intersection of the first circle and the base.]

✓ 8. A circle described on one of the equal sides of an isosceles triangle as diameter must pass through the middle point of the base.

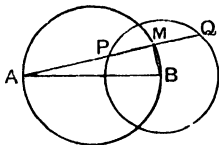
9. Semicircles are described on the radii OA, OB of a quadrant as diameters. They cut at C. Prove A, C, B are in a straight line bisected at C.

[You should note Ex. 18 b 30 too.]



✓ 10. A fixed point A is joined to a variable point P on a given circle. Find the locus of the middle point of AP.

* 11. A is a point outside a circle with centre B, and on AB as diameter another circle is described. A line APQ is drawn cutting the original circle at P, Q: prove that the second circle bisects PQ.



[HINT. How big is the angle AMB?]

✓ 12. PQR is a triangle, with Q an obtuse angle. Prove that the circles on PR and PQ as diameters intersect on the line RQ produced.

* 13. ABC is a triangle, and AD is the perpendicular from A on to BC; AE is the diameter of the circle ABC: prove that the triangles ABD and AEC are equiangular.

* 14. The ends A, B of a straight rod AB of given length move on two fixed straight lines OX, OY which are at right angles to one another. Find the locus of the middle point of the rod AB.

* 15. AB is a diameter and CD a chord of a circle; AM and BN are drawn perpendicular to CD produced and meet the circle in P and Q respectively; show that PM = BN and QN = AM.

[HINT. Prove MNPB a rectangle, etc.]

* 16. ABC is any triangle and on BC as diameter a circle is described; if DE be the diameter parallel to AC, show that DC, EC are the bisectors of the angle ACB and its exterior angle.

✓ 17. AOB is the diameter of a circle, centre O. BPC and APD

are chords of the circle intersecting in P, such that COD is a right angle. Prove that $AC = CP$.

[HINT. $\angle CAP = \frac{1}{2}\angle COD =$ how many degrees? etc.]

- *18. On AB, a diameter of a given circle, an equilateral triangle ABC is described. If AC meet the circle in D, and from D, DE be drawn perpendicular to AB, show that AE is a quarter of AB.

[HINT. Prove $\triangle DAE$ and $\triangle BAD$ both have angles 30° , 60° and 90° .]

- *19. ABCD is a parallelogram, of constant area, on a given base AB. Show that all circles described on the variable side AD as diameter pass through two fixed points.

[HINT. The altitude of the parallelograms (through A) is needed.]

- *20. FG is a given chord of the circle and K is a point within the circle. Draw, where possible, a chord through K which shall be bisected by FG. When is this impossible?

- *21. AD, BE are drawn from A and B perpendicular to the opposite sides of the triangle ABC and meet at O. AD is produced to meet the circum-circle of ABC at R. Prove that $OD = DR$.

[HINT. OEDC, AEDB, BACR are successively cyclic.]

- *22. A triangle XYZ has side XY equal to side XZ, and U, V are the feet of the perpendiculars from Y on XZ and from Z on XY respectively. Prove that UV is parallel to YZ.

- *23. Two straight lines OB and OAC make an acute angle at O: X and Z are the feet of the perpendiculars from A and C upon OB, and Y is the foot of the perpendicular from B on OAC. Prove that the angles XYZ and ABC are equal.

- *24. ABFD, ABGE are two circles intersecting in A and B. DAE is drawn at right angles to AB. DB and EB are produced to cut the circles again in G and F respectively. Prove that BA bisects the angle FAG.

[HINT. Prove $\angle FBD$ equal to the complement of $\angle FAB$, etc.]

- *25. The sides of a triangle ABC are bisected in D, E, F, and AL, BM, CN are drawn from the angular points perpendicular to the opposite sides; show that the circle which goes through D, E, F also passes through L, M, N.

[HINT. Prove FAED is a parallelogram and FAL an isosceles triangle, etc.]

- *26. ABC is a triangle, and the angles at B and C are bisected by BD and CE respectively; AD and AE are drawn at right angles to them. Prove that DE is parallel to BC.

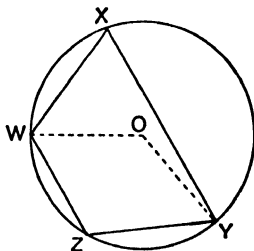
CHAPTER XXIII.

OPPOSITE ANGLES OF CYCLIC QUADRILATERALS.

PROPOSITION 24.

§ 1. General Enunciation. *The opposite angles of a quadrilateral inscribed in a circle must be supplementary.*

Particular Enunciation. WXYZ is the quadrilateral inscribed in a circle, whose centre is O, to prove $\angle X$ and $\angle Z$ are supplementary (and, similarly, $\angle XWZ$ and $\angle XYZ$).



Construction. Join WO and YO.

Proof. $\angle WOY = 2\angle X$ (angle at centre double angle at circumference standing on the same arc),

and reflex $\angle WOY = 2\angle Z$ (for the same reason) ;

by addition $\angle WOY$ and reflex $\angle WOY = 2\angle X + 2\angle Z$,

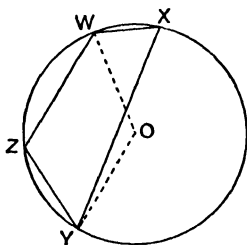
but $\angle WOY$ and reflex $\angle WOY = 4 \text{ rt. } \angle s$;

$\therefore \angle X + \angle Z = 2 \text{ rt. } \angle s$.

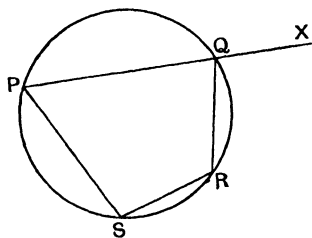
(Similarly, etc., joining XO and ZO.)

Q.E.D.

[*N.B.*—No one ever has any trouble in following this argument, but when O comes outside the quadrilateral $WXYZ$ there is apparently (*quite unnecessarily*) some trouble. Letter for letter the same proof applies.]



Since the sum of the 4 angles of *any* quadrilateral is 4 right angles, if one pair are supplementary, of course the other pair are.



This proposition proves

$$\angle PQR + \angle S = 2 \text{ rt. } \angle s ;$$

it is frequently just as useful to realize that

$$\angle XQR = \angle S$$

(a necessary **corollary** to the proposition).

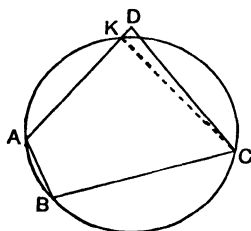
[The student is advised to mark *these* equal angles, with small bars, on the figure itself.]

§ 2. The following is the converse of the preceding proposition, proved by “*reductio ad absurdum*.”

PROPOSITION 25.

General Enunciation. *If the opposite angles of a quadrilateral are supplementary, a circle can be drawn through the 4 angular points of the quadrilateral.*

Particular Enunciation. ABCD is the quadrilateral, in which it is given that $\angle A$ and $\angle DCB$ are supplementary, to prove that a circle can be drawn through the 4 points A, B, C and D.



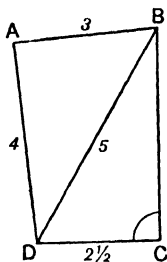
Construction. Draw a circle through the 3 points A, B, C, and (if it does not pass through the 4th point D) let it cut AD, produced if necessary, at K. Join KC.

Proof. $\angle A + \angle DCB = 2 \text{ rt. } \angle \text{s}$ (given),
 and $\angle A + \angle KCB = 2 \text{ rt. } \angle \text{s}$ (cyclic quadrilateral KABC);
 $\therefore \angle DCB = \angle KCB$,
 $\therefore$ KC and DC coincide.

$\therefore$ the circle through the 3 points A, B, C must also pass through the 4th point D. **Q.E.D.**

The **converse** of the corollary at the end of Proposition 24 is true; it is a necessary sequel to Proposition 25, and is often useful.

§ 3. We must repeat that a **figure may be cyclic** for some reason or another, and then the various facts we have proved about figures in a circle hold, **though quite possibly the circle is not drawn**.



For instance, ABCD is a quadrilateral in which the diagonal $DB = 5''$, $DA = 4''$, $AB = 3''$, $DC = 2\frac{1}{2}''$, $\angle DCB = \text{rt. } \angle$.

$$\text{Now } 5^2 = 4^2 + 3^2;$$

$$\therefore \angle DAB = \text{rt. } \angle \text{ (converse of Pyth. Theorem),}$$

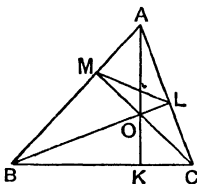
$$\text{but } \angle DCB = \text{rt. } \angle \text{ (given);}$$

$$\therefore \text{ABCD is cyclic quadrilateral (opp. } \angle \text{s supplementary);}$$

$$\therefore \angle DAC = \angle DBC \text{ (in same segment),}$$

etc.

* § 4. That the perpendiculars (to the opposite sides) from the angular points of any triangle are concurrent is very important.



ABC is *any* triangle. Only *two* perpendiculars, BL and CM, are drawn. They meet at O. AO, produced if necessary, meets BC at K. To prove that the angles at K must be right angles.

We want to prove an angle at K a right angle.

By construction there are 2 cyclic quadrilaterals LMBC and ALOM.

Hence $\angle LMC$ (or $\angle LMO$) is equal to both $\angle LBC$ (or $\angle OBK$) and $\angle OAL$.

$\therefore$ in the Δ s KBO, LAO

we have $\angle OBK = \angle OAL$ (proved)

$\angle BOK = \angle AOL$ (vertically opposite)

$\therefore$ 3rd $\angle OKB = \angle OLA$, which is a right angle; (etc.).

The point of concurrency is called the **orthocentre** of the triangle. [It is noticeable that the orthocentre is *outside* the triangle if it is obtuse-angled. You should draw an obtuse-angled triangle and see where its orthocentre comes.]

Especially, if we are concerned also with other "centres" of the ΔABC , the letter O is usually reserved for the orthocentre.

EXAMPLES 23 a (CALCULATIONS).

1. ABCD is a circle whose centre is O. The angle BCD is 100° ; calculate the size of the angle BOD, and give reasons.

[HINT. What size is the angle BAD?]

2. P, Q and R are points on the circumference of a circle, whose centre is O. The angle PQR is 140° ; calculate the size of the angle POR.

[HINT. Take any point X in the major arc PR. Join PX and RX. How many degrees are there in the angle PXR?]

3. O is the centre of a circle PQR. The angle POR is 112° and Q is in the minor arc PR. What is the size of the angle PQR?

4. A, B, C, D are four points on the circumference of a circle; if the angle ACB contains 70° , what is the size of the angle ADB? [Consider the two cases, when C and D are on the same side, and on opposite sides of the chord AB.]

* 5. ABCD is a quadrilateral in a circle. The centre of the circle, O, lies within the triangle ABD, and $AB = AD$. The angle $BCD = 138^\circ$. Calculate the number of degrees in the angle ODA.

* 6. A log, whose cross-section is a rectangle with one side three-quarters of the other, is cut from a circular tree trunk of diameter 25 inches; what will be the lengths of the sides of the log, and what percentage of the wood will be cut away, if a maximum log is made?

* 7. O is a point outside a circle and OAB, OCD are secants cutting the circle at A, B and C, D: given that the angles OBC, OCA, BXD are respectively 34° , 67° , 110° (X being the intersection of AD and BC), find by calculation the size of the angles AOC, ACB, CBD and ACD, showing your reasoning clearly and concisely.

* 8. Two unequal circles intersect at A and B: O, the centre of the smaller circle, is outside the larger circle. AO produced in both directions meets the smaller circle again at X, and the larger circle at Y: XB produced meets the larger circle again at Z. If AXB is an angle of x° , find the number of degrees (giving your reasons shortly) in the angles XAB, BAY, AYZ, BZY: and prove that AZ passes through the centre of the larger circle.

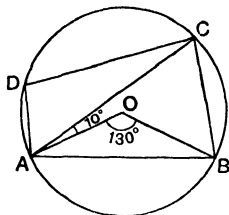
* 9. KLMN is a quadrilateral inscribed in a circle. KM passes through the centre O, and $\angle LKM = 33^\circ$, $\angle NKM = 40^\circ$. Find the size of each of the following angles: KNL, NMK, LOM, KOL, LMN.

At what angle do the diagonals LN and KM intersect?

* 10. Draw a circle of radius 5 cm., and a chord AB in it of length 5 cm. A line AD is drawn through A to meet the circle again at D so that $\angle BAD = 104^\circ$, and a line through B to meet the circle again at C so that $\angle ABC = 68^\circ$. Find the number of degrees in the angles ADC, BCD, DCA, BCA.

- * 11. ABCD is a quadrilateral inscribed in a circle whose centre is O. It is given that $\angle OAB = 40^\circ$, $\angle DAB = 56^\circ$, and that $CB = CD$. Calculate the angles BOD, ABD, CBD, giving full reasons in each case clearly.
- * * 12. ABCD is a circle; AB and DC produced meet at O outside the circle, and AC, BD, AD, BC are joined. If $\angle BOC = 20^\circ$, $\angle OBC = 60^\circ$, $\angle BDC = 15^\circ$, find the size of all the remaining angles of the figure.
- * * 13. ABCD is a quadrilateral inscribed in a circle whose centre is O and radius 5 cm. $BC = 5$ cm., the angle $ABC = 122^\circ$, and the angle $COD = 104^\circ$. Construct the quadrilateral and *calculate* the size of each of its angles and the angles between the diagonals.

- * * 14. Copy the accompanying figure roughly and write down the values of the angles ACB, CBA, CDA, OBC.



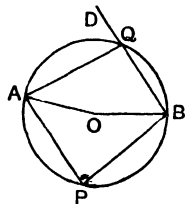
- * * 15. If an irregular polygon of $2n$ sides be inscribed in a circle, what would be the sum of its alternate (*i.e.* every other) interior angles?

EXAMPLES 23 b (RIDERS).

1. Q is any point on the arc AB of a circle with centre O. AQ and BQ are drawn and BQ is produced to D. Prove

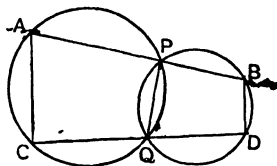
$$\angle AQD = \frac{1}{2} \angle AOB.$$

[HINT. Take any point P on the arc away from AQB. Join AP and BP.]



2. Two circles intersect at P and Q, and through P and Q chords APB, CQD are drawn cutting the circles in A, C and in B, D respectively. Show that AC is parallel to BD.

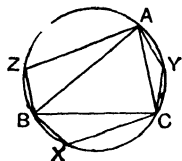
[HINT. Join PQ. Consider the cyclic quadrilaterals APQC and PBDQ.]



3. AB and CD are two chords of a circle, and cut each other, outside the circle, at K. Prove that the triangles KAC, KDB are equiangular.

4. A triangle is inscribed in a circle; show that the angles in the segments exterior to the triangle are together equal to four right angles.

[HINT. ABXC is a cyclic quadrilateral, etc.]



5. No parallelograms except rectangles can have all their angular points upon the circumference of a circle.

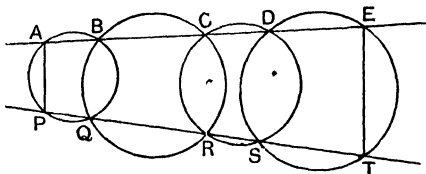
6. If the diagonals of a quadrilateral and also a pair of opposite sides be equal, the other pair of opposite sides will be parallel. Show also that the quadrilateral can be inscribed in a circle.

7. AED is a triangle; if C, B be points on DE, AE respectively, such that the angle EBC is equal to ADE, prove that a circle can be described about ABCD.

8. PQRSTV is an irregular hexagon inscribed in a circle; prove that the sum of the angles P, R and T is four right angles.

* 9. Four circles cut as indicated in the figure and ABCDE and PQRST are straight lines. Prove that AP must be parallel to ET.

[HINT. See Rider 2.]



* 10. ABC is an isosceles triangle; a circle passes through A, B and cuts the base BC, produced if necessary, in D; show that the circle described about ACD is equal to the first circle.

* 11. ABCD is a quadrilateral inscribed in a circle, and AB, DC are produced to meet in E; show that if EA = ED, then AD and BC are parallel.

[HINT. $\triangle EAD$ isosceles. ABCD cyclic.]

- *12. Two circles intersect in A and B, and through A and B are drawn two parallel straight lines PAQ, RBS cutting one circle in P, R, and the other in Q, S. Prove that PQSR is a parallelogram.

[HINT. Join AB. Prove $\angle P + \angle Q = 180^\circ$.]

- *13. Points P, Q, R are taken on the sides BC, CA, AB of a triangle; prove that the circles AQR, BRP, CPQ meet in a point.

[HINT. Take only 2 circles, and prove 3rd circle passes through their point of intersection.]

- *14. ABCD is a trapezium in which AB is parallel to DC, a circle is drawn through A and B cutting AD and BC produced, if necessary, at X and Y; prove that XYCD is cyclic.

- *15. ABCD is a parallelogram and BE (making with AB the angle ABE equal to BAD) meets DC produced, if necessary, in E. Prove that the circles round BCD and BED will be equal.

- *16. Show that the bisectors of one of the angles of a quadrilateral inscribed in a circle and of the opposite exterior angle meet on the circumference.

- *17. Squares are described, externally, upon the sides of a right-angled triangle; show that the straight line, which joins the right angle to the intersection of the diagonals of the square on the hypotenuse, is perpendicular to the line which joins the intersections of the diagonals of the other squares.

- *18. ABCD is a parallelogram and K is any point on the diagonal AC. Prove that the circles circumscribing the triangles AKB, DKC intersect again on the diagonal BD.

[HINT. Let one circle cut BD at L, then prove other circle passes through L.]

- *19. Two circles cut at A and B. C is a point external to both circles. The line CA meets the circles in D and E, the line CB in F and G. Prove that FD is parallel to GE.

- *20. P is any point in the side BC of triangle ABC; a circle BPK cuts AB again in R, and a circle CPK cuts AC again in Q; show that the points A, Q, R, K lie on a circle.

- *21. ABCD is a quadrilateral inscribed in a circle: the angle ABC is bisected by BE, which meets the circumference at E, and AD, BE produced meet at F. If ED be joined, show (giving

your reasons clearly) that the angles CDE and EBC are equal, and also the angles EDF and ABE : hence show that DE bisects the angle CDF.

- * 22. Two circles cut at A, B, and through A a straight line is drawn cutting the circles in C, D respectively, if the angle CBD be bisected by a straight line which cuts CD at E, prove that the locus of E is two arcs of circles.

[HINT. Prove $\angle AEB$ constant.]

- * 23. Two circles whose centres are A and B cut in C and D ; AC cuts the B circle in E, and BC cuts the A circle in F. Show that ABDEF is cyclic.

[HINT. BEC is an isosceles Δ . Prove

$$\angle AEB + \angle ADB = 180^\circ.$$

Similarly, etc.]

- * 24. In a triangle ABC, M is the foot of the perpendicular from B on AC, and X and Z are the feet of the perpendiculars from A and C on any line through B. Prove that the angle XMZ is equal to $\angle ABC$.

[HINT. $\angle AMX = \angle ABX$, why? $\angle CMZ = \angle ZBC$, why?]

- * 25. ABCD is a quadrilateral in a circle ; AB, DC produced meet at E, and AD, BC produced at F. If the circles about the triangles EBC, FCD cut at G, show that E, F, G are in the same straight line.

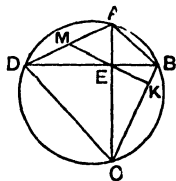
[HINT. Prove $\angle CGF + \angle CGE = 180^\circ$.]

- * 26. Three circles intersect in D, and their other points of intersection are A, B, C ; AD cuts the circle BDC at E, and EB, EC cut the circles ADB, ADC respectively at F and G ; show that the points F, A, G are collinear.

[HINT. Prove $\angle FAD + \angle DAG = 180^\circ$.]

- * * 27. If the diagonals of a quadrilateral in a circle be at right angles, prove that the line joining their intersection to the middle point of one side is perpendicular to the opposite side.

[HINT. M is the centre of the circle through D, E, A, why?]



- * * 28. ABC is an equilateral triangle, and P a point on the arc BC of the circumscribing circle. Prove that $PA = PB + PC$.

[HINT. From PA cut off PK equal to PB. Prove KA equal to PC. $\Delta s ABK, CBP$.]

- * * 29. ABCD is a quadrilateral inscribed in a circle, AB, DC produced meet in E and AD, BC produced in F. If the circle round the triangle ADE cut EF in G, show that CDFG, BCGE, ABGF are cyclic quadrilaterals.
- * * 30. The bisector of the vertical angle B of a triangle ABC meets the circumscribing circle in D. DE is drawn at right angles to BC. Prove that $AB + BC = 2BE$.
- * * 31. ABCD is an irregular quadrilateral. AB and DC produced meet at E; BC and AD produced meet at F. The circles circumscribing the triangles EBC and FCD meet again at S. Show that the circles circumscribing the triangles EAD and BAF also pass through S.
- * * 32. Prove that the quadrilateral formed by the bisectors of the angles of any quadrilateral is cyclic.
- * * 33. AB is a chord of a circle, ED is the diameter at right angles to AB, and P is any point on the arc ADB. Prove that PD bisects the angle between PB and AP produced.

Orthocentre.

(Assume that the perpendiculars from the angles of a triangle to the opposite sides are concurrent.)

- * * 34. ABC is an acute-angled triangle: D, E and F are feet of the perpendiculars from A, B and C upon the opposite sides respectively. O is the orthocentre. FODCE and OEAF are cyclic quadrilaterals, and hence prove that OE bisects the angle DEF.
- * * 35. ABC is a triangle. Perpendiculars to the opposite sides AD, BE, CF meet at O. Prove that the circle DEF will pass through P, the middle point of OA.
- * * 36. ABC is a triangle in which the perpendiculars AD, BE and CF from the angular points to the opposite sides meet in O, and OA, OB and OC are bisected in X, Y and Z respectively; prove that the circle round X, Y and Z will pass through D.

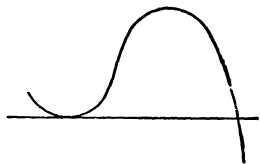
CHAPTER XXIV.

TANGENTS.

§ 1. **Definition.** A **tangent** to a circle is a straight line which meets a circle and, when produced, does not cut it.

[In the definition avoid the word "touch"; it may be only translating the Latin word *tangent* by the English word "touch," and mere translation is not a definition.]

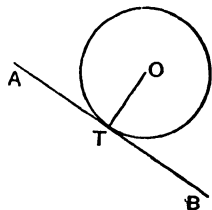
N.B.—For curves, other than circles, a definition similar to this may not necessarily serve. The straight line is a tangent to the curve all right, and yet, when produced, cuts the curve.



The point at which the tangent meets the circle is called the **point of contact**. In this figure AB is the tangent; it meets the circle at T , and T is the point of contact.

If O is the centre of the circle, then OT is called the radius at the point of contact.

The fact that the radius OT and the tangent AB must be at right angles is of fundamental importance and is next considered (but it is not part of the definition nor is it a substitute for the definition).



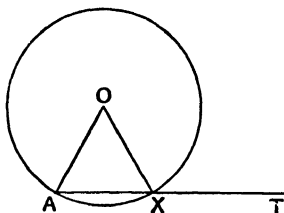
It is easier to prove that from a point on the circumference of a circle the line at right angles to the radius must be a tangent, and the proposition is given in this form.

There can be only one tangent to a circle at one point on the circumference.

PROPOSITION 26.

§ 2. General Enunciation. *If from a point on the circumference of a circle a straight line be drawn at right angles to the radius, the straight line must be a tangent.*

Particular Enunciation. O is the centre, OA is a radius, $\angle OAT$ is a right angle. To prove AT must be a tangent.



Construction. If AT is not a tangent, let it cut the circle again at X. Join OX.

Proof. OAX is an isosceles triangle (radii equal);

$$\therefore \angle OXA = \angle OAX.$$

Now $\angle OAX$ (or $\angle OAT$) was given a right angle;
 $\therefore$ two angles of the $\triangle OAX$ are right angles, and this is impossible, so that AT must be a tangent. **Q.E.D.**

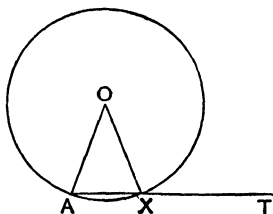
* § 3. The alternative proof given below is preferred by some.

[§ 9 of this chapter must be read first.]

PROPOSITION 26 (Aliter)

General Enunciation. *If from a point on the circumference of a circle a tangent be drawn, it must be at right angles to the radius at the point of contact.*

Particular Enunciation. O is the centre, OA and OX radii, AXT a secant. To prove, when X and A coincide, that AT must be at right angles to OA.



Proof. OAX is an isosceles triangle (radii equal) ;

$$\therefore \angle OAX = \angle OXA ;$$

also the 3 angles of $\triangle OAX = 2$ rt. $\angle$ s.

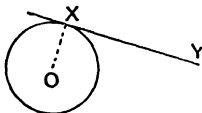
Revolve the secant AXT about A until X coincides with A.

AT will become a tangent and $\angle OAX$ will become one right angle (for $\angle AOX$ will become nothing), so that the tangent must be at right angles to the radius at the point of contact.

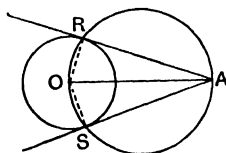
Q.E.D.

§ 4. The use of this fact is the key to the **construction of a tangent** to a given circle. The following are instances :

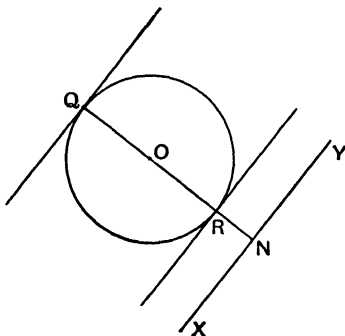
(i) To draw a tangent to the circle at X . O is the centre, so draw XY perpendicular to the radius OX .



(ii) To draw tangents to the circle from a point A outside it. The circle on the diameter OA will give the points R and S , and because the angle in a semicircle is a right angle we shall get AR at right angles to the radius OR , etc.

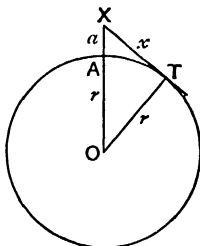


(iii) To draw tangents to a circle parallel to XY . The perpendicular ON gives the points Q and R (and the rest is easy).



§ 5. The **distance of the visible horizon** is of great practical importance. The question is also considered in Chap. XXXVII.

The figure is a section of the Earth, through its centre



Our eyes are at X and the visible horizon is at T, so that XT is a tangent to the circle (hence $\angle XTO = \text{rt. } \angle$).

$$x^2 + r^2 = (a + r)^2 \text{ (Pyth.) ;}$$

$$\therefore x^2 + r^2 = a^2 + 2ar + r^2 ;$$

$$\therefore x^2 = a^2 + 2ar.$$

Now, in practice, a is trivial compared to r ; the height up of our eyes is very small compared to the radius of the Earth [even the height of Mount Everest itself is nowhere near a thousandth part of the Earth's diameter], and so we can safely (in practical cases on the Earth) ignore a^2 compared to $2ar$. Of course the figure above is enormously exaggerated,

so that $x^2 = 2ar$ nearly,

$$\text{or } x = \sqrt{2ar}.$$

Now we shall have our height in *feet*; the diameter of the Earth is about 7920 *miles*, and we want the distance of the visible horizon in *miles*. To reduce the *feet* to *miles* divide by 5280 (*i.e.* 3×1760);

$$\begin{aligned} \therefore x \text{ (in miles)} &= \sqrt{\frac{7920}{5280} a \text{ (in feet)}} \\ &= \sqrt{1\frac{1}{2} \times a \text{ (in feet)}}. \end{aligned}$$

So, in practice, to get the distance in miles of the visible sea horizon, multiply the height of your eyes, in feet above sea-level, by $1\frac{1}{2}$ and take the square root.

Two examples worked out follow, but see also page 396.

Example 1. If one's eyes are 5 ft. above the ground and one is standing on a pier 25 ft. above sea-level, how far away is the horizon ?

$$\begin{aligned}\text{The distance is } & \sqrt{1\frac{1}{2} \times (25 + 5)} \text{ miles,} \\ & \text{or } \sqrt{45} \text{ miles,} \\ & \text{or } 6.708 \text{ miles,}\end{aligned}$$

i.e. somewhat under 7 miles.

Example 2. On the coast of Carnarvonshire one was watching, with a telescope, to identify certain liners making Liverpool. A vessel rounded Anglesey, and, as her peculiar funnels were fully visible (and no more), there was no question of her identity. Her deck was known to be 20 ft. above the water, so the fact that her funnels were wholly apparent led one to suppose 20 ft. was hidden. One's eyes were 100 ft. above the water. How far away was the vessel ?

$$\begin{aligned}\text{The distance is } & \sqrt{1\frac{1}{2} \times 20} + \sqrt{1\frac{1}{2} \times 100} \text{ miles,} \\ & \text{or } \sqrt{30} + \sqrt{150} \text{ miles,} \\ & \text{or } 5.48 + 12.25 \text{ miles,} \\ & \text{or } 17.73,\end{aligned}$$

i.e. about 17.7 miles.

[It is difficult to judge the complete visibility of each funnel at that distance, and in practice one seldom knows one's own height as accurately as that, and so perhaps the vaguer answer "about 20 miles" is really wiser if our data are not over-reliable.]

§ 6. To circumscribe regular figures about a circle one should get points of the *inscribed* regular polygon on the principles laid down in Chapter XVIII. (the chief are those with 3, 4, 6 or 8 sides), and at each of those points draw tangents. In practice, when 2 of those tangents (next to each other) have been drawn (very carefully), it is as well, and quicker, to draw the circle circumscribing this polygon and to aim at *inscribing* in this.

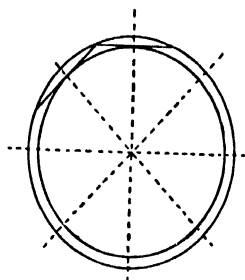
For example, to circumscribe a given circle with a regular octagon.

First get the 8 radii.

Secondly, two tangents very carefully.

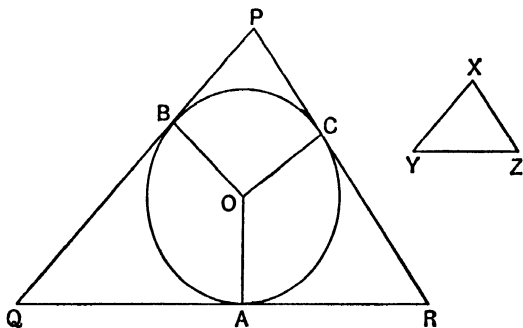
Thirdly, the outer circle.

"Stepping" round, or "bisecting angles," should give the rest.



§ 7. It is always understood that the **length of the tangent** from A means the length up to the point of contact (see § 4, fig. (ii) ; thus the lengths of the tangents are here AR and AS, though the lines may or may not be produced further. **It is to be noticed that the two tangents are of the same length.** [It is easy to prove triangles ORA, OSA congruent.] The fact that the angles at R and S are right angles makes the use of Pythagoras' Theorem in calculations possible. RS is called the **chord of contact**. The fact that $\angle ROS$ is the supplement of $\angle RAS$ helps in the following :

To circumscribe a given circle (centre O) with a triangle equiangular to $\triangle XYZ$.



Take OA any radius.

Make $\angle AOB$ the supplement of $\angle Y$ and $\angle AOC$ the supplement of $\angle Z$. Tangents at A, B and C complete the $\triangle PQR$.

§ 8. If a quadrilateral, PQRS, circumscribes a circle, the sums of its opposite sides must be equal,

$$\text{i.e. } PQ + RS = QR + SP,$$

for we have $a = w$ (tangents),

$$b = x \quad (\quad , \quad),$$

$$c = y \quad (\quad , \quad),$$

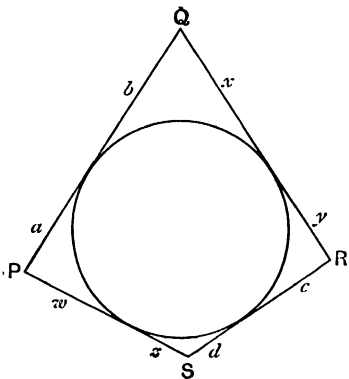
$$d = z \quad (\quad , \quad),$$

and, therefore, by addition

$$PQ + RS = QR + SP.$$

[Notice the order on the sides.]

Notice the distinction between a quadrilateral inscribed in a circle (Prop. 24) and a circle inscribed in a quadrilateral.



The **converse** is true for convex quadrilaterals.

* § 9. Another way of looking upon a **tangent** is considering it as a **secant cutting the circle at two coincident points**; in other words, the chord that it makes is absolutely *nil*.

Fig. i.

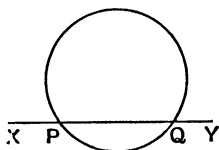


Fig. ii.

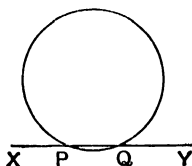
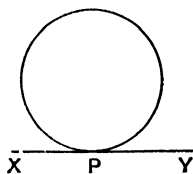


Fig. iii.



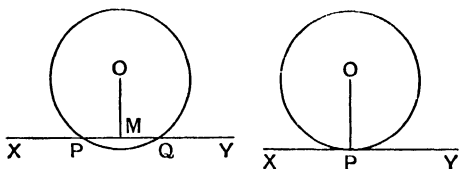
In Fig. i. the secant XY cuts the circle at two distinctly separate points P and Q. PQ is a chord.

In Fig. ii. the secant XY cuts the circle at two neighbouring points P and Q. PQ is a chord (but a small chord).

In Fig. iii. P and Q coalesce at P. The chord is *nil*. XY is a tangent, P the point of contact.

If we look upon a tangent in this way (as the limiting position of a secant when the chord is *nil*), then various facts we know about chords may (*mutatis mutandis*) have their counterparts with respect to tangents.

For instance, we might note in one figure, if M is the middle point of the chord PQ , then OM is perpendicular to XY ; in the other figure, the radius OP is perpendicular to the tangent XY .



EXAMPLES 24 a (GEOMETRICAL DRAWING).

1. Draw a pair of tangents to a circle of 1" radius from a point 1.7" from the centre of the circle. Measure the chord of contact.

2. To a circle, whose radius is 3 cm., draw a pair of tangents inclined at an angle of 70° to each other. Measure the chord of contact.

[HINT. Draw a rough figure first. At what angle would the corresponding radii be? In the accurate figure draw the radii first.]

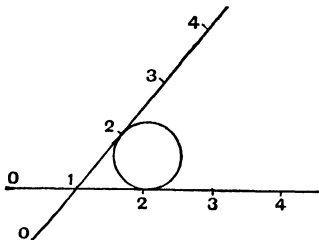
3. Draw a circle of radius 3 cm. and construct two tangents to it which enclose an angle of 78 degrees. Measure the line joining their points of contact.

4. Describe an equilateral triangle about a circle whose radius is 1". Measure the three sides of your triangle. Are they equal?

[HINT. "Step" the radius round the circumference and join alternate points to the centre to get 3 radii equally spaced.]

5. Describe a square about a circle whose radius is 3 cm. Measure both diagonals.

6. A halfpenny (diameter 1 inch) is touched, at their 2" points, by two rulers. These rulers have their 1" points coincident. How far apart are the 4" points of the rulers? Determine the answers from an accurate diagram drawn to scale.



7. A point lies $3\frac{1}{2}$ " distant from the centre of a given circle (radius 1"). Draw a tangent from the point, and also draw the parallel tangent.

8. Prick off the points C, P and E (last page). With centre C describe a circle of radius 3 cm. From P draw a tangent to that circle. Also construct the parallel tangent. How many centimetres from E is the point of contact of the second tangent?

9. Circumscribe a circle (radius 1.1") with a regular hexagon. Measure the longest diagonal of the hexagon. Do all 3 agree?

10. Given a circle of radius 1", draw three tangents to it, so that two of them intersect at an angle of 30° , and the third cuts the other two at equal angles. What are the lengths of the sides of the triangle formed by the 3 tangents?

11. Circumscribe a circle whose radius is 1.43" with a regular octagon. Measure the longest diagonal of the octagon. Do all 4 agree?

* 12. Draw a triangle PQR such that $PQ = 3"$, $QR = 2.6"$, $RP = 2.2"$. Construct a circle of radius 2" to touch PQ and to pass through R. How many solutions are there?

* 13. Draw an isosceles triangle with base equal to 3.5 cm. and sides AB, AC equal to 5 cm. Describe a circle to touch AB at A and to pass through C. Measure the radius.

* 14. Draw a triangle ABC having $AB = 8$ cm., $BC = 7$ cm., $CA = 6$ cm. Find by construction the centre of the circle which touches BC at C and passes through A, and draw the circle.

* 15. State the construction for drawing from a point (whose position is to be determined) on a given straight line a tangent of a given length to a given circle. The tangent from a point P to a circle of radius 1.6" is 1.2" long. Find the distance of P from the centre of the circle.

* 16. Draw a circle of 3 cm. radius and a line 7 cm. from its centre. Find by the intersection of loci the position of a point, which is 2 cm. from the given line, and such that the tangent from it to the given circle is 8 cm. long.

* 17. Draw a straight line AB, 3" long, and describe a circle with B as centre such that the length of the tangent drawn to it from A is 2". Explain your construction, and measure the length of the radius of the circle.

[HINT. On AB, as diameter, describe a semicircle.]

* 18. Draw a circle of radius 6.5 cm. Place in it 4 chords 5 cm. long, and draw a circle touching these chords. Now draw a chord of the bigger circle 10.4 cm. long, and construct a chord parallel to it 5 cm. long. Find the distance between these parallel chords by measurement and by calculation.

* 19. XABCDY is a straight line, $XA = 10$ units, $AB = BC = CD = 3$ units. OA is perpendicular to XY and equal to 8 units.

Construct 4 separate circles, all passing through the point O, and touching the straight line XY separately at A, B, C and D.

[The construction is to be given in words for the one case of the circle touching XY at B.] Mark the 4 centres a, b, c, d respectively.

Join the points a, b, c, d by a fair curve (thus showing the locus of centres of circles which pass through O and which touch XY). By the inspection of this locus determine the distance, from the extremity X, of the point-of-contact-with-XY of the circle whose radius is 5 units.

On a separate sheet of paper repeat the original line XY and the point O. On it construct a circle with radius 5 units to pass through O and to touch XY. [Give the construction in words.] Measure the distance of its-point-of-contact-with-XY from the extremity X of the line.

[As a check compare with the value obtained from the locus.]

* 20. A quadrilateral has two opposite sides equal (8" each) and the other two sides parallel (6" and 4"). Draw it to half scale. It is rounded off by describing circular arcs on the sides 6 and 4, not semicircles, but such that the sides 8, 8 melt into the curves without deflection. Draw this complete figure, showing construction lines.

[HINT. To have the non-parallel sides equal the figure is necessarily an isosceles trapezium and symmetrical. Draw the axis of symmetry first.]

* * 21. Show the construction for drawing a tangent to a circular arc of large radius, whose centre is out of reach, from a point on the circumference.

* * 22. Use a big plate to draw an arc of a circle. On the arc take a point. At that point construct a tangent to the circle.

* * 23. Draw two straight lines inclined at an angle of 65° , and draw four equal circles (radii $\frac{1}{2}$ ") to touch them.

- ** 24.** Draw two straight lines meeting at an angle of 105° , and draw four equal circles ($0.7''$ radii) to touch them.
- ** 25.** Draw a circle with radius $1''$. Circumscribe it with a kite-shaped quadrilateral. The top of the kite is to be $1.4''$ above and the bottom of the kite is to be $2.4''$ below the centre of the circle. Measure the shorter diagonal of the kite. Are the diagonals of the kite correctly at right angles?
- ** 26.** On a chord AB $4\frac{1}{4}''$ long draw the segment of a circle to contain an angle of 55° . Produce AB to C , making $BC = 2\frac{1}{2}''$. Through C draw a tangent to the segment.
- ** 27.** The railway from P to Q consists of a circular arc AB , and two tangents PA and BQ . The radius of the arc is two miles, and the radii to the ends of the arc make an angle of 28° . $PA = 1$ mile, $BQ = \frac{1}{2}$ mile. Show the railway on a scale of $2''$ to the mile, and measure the distance from P to Q as the crow flies.
- ** 28.** Draw two parallel straight lines, $2.4''$ apart, and take a point $0.72''$ from one of them. Draw a circle to pass through that point and to touch the two lines.
- ** 29.** P is a point $3\frac{1}{2}''$ from the centre of a circle of radius $2\frac{1}{2}''$. Draw a secant through P , so that the portion of it intercepted by the circle may be $3''$.
- ** 30.** Draw a circle, radius 4.2 cm., and take any point 7.8 cm. from its centre. Through this point draw a straight line cutting the circle, so that the chord is 5 cm. long.
- ** 31.** Draw two tangents to a circle, radius $1.75''$, inclined at an angle of 50° . From the point where these tangents intersect draw a secant to the circle, so that the chord is $3''$ long.

EXAMPLES 24 b (CALCULATIONS).

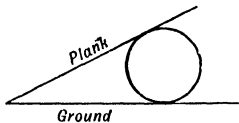
- Two tangents, AT and BT , to a circle (whose centre is O) meet at T at an angle of 50° . What are the sizes of the angles AOB and AOT ?
- With the same notation as in the preceding question, if $\angle AOB = 96^\circ 54'$, what is the size of $\angle ATB$?
- With the figure of Question 1, if $OA = 3''$ and $AT = 4''$, what is the length of OT ?

4. What is the radius of a circle if tangents to it from a point c feet from the centre are l feet long?

5. How far from the centre of a circle of radius $2\cdot5''$ must a point be taken so that the tangents from the point may be $5\cdot5''$ long?

6. Give the construction for drawing two tangents to a circle from an external point (no proof required). Calculate the lengths of the tangents to a circle of $2''$ radius from a point $2\cdot9''$ from the centre of the circle.

7. A plank, 6 ft. long, rests against a garden roller and has one of its ends on the ground. The points of contact of the plank and roller, both with the ground, are 4 ft. apart. What length of plank extends beyond its point of contact with the roller?



* 8. From a point $34''$ from the centre of a circle of radius $16''$ two tangents are drawn to the circle. Find the length of the chord of contact and its distance from the centre.

[HINT. First calculate the length of a tangent.

Secondly, find the area of triangle.

“Tangent—radius—distance from the centre”
in two ways.]

* 9. The angle between the tangents drawn to a circle at the ends of a chord AB is x degrees. Find the size of the angles in the segments cut off by the chord.

* 10. Two bicycle wheels with diameters $28''$ and $24''$ stand upright on a level road, and their points of contact with the ground are $31''$ apart. How far apart are (1) their centres, (2) their top points?

* 11. Draw accurately a $\odot$ radius 6 cm., and as many tangents to it as possible from a fixed point $7\cdot5$ cm. from the centre. Explain your construction, calculate the length of the tangent and check by measurement.

* 12. O is the centre of a circle whose radius is r , and XY is a straight line distant c from the centre. $c > r$. ON is perpendicular to XY . P is a point on XY and $PN = d$. What is the length of tangents to the circle from P ?

* 13. The inscribed circle of the triangle ABC touches BC at D ,

CA at E, and AB at F. If $\angle FDE = 40^\circ$ and $\angle DEF = 75^\circ$, find the number of degrees in the angles A, B and C.

- * 14. If the angles of a triangle ABC are α , β and γ , and D, E and F are the points of contact of the in-circle, what are the sizes of the angles of the triangle DEF?

- * 15. Make a table to show how far away is the visible horizon when one's height is 50 ft., 100 ft., 150 ft., ..., etc. Illustrate this table graphically.

[The scales you should adopt will depend upon the squared paper you use. Some half dozen points, well spaced, should be enough to control your graph.]

From the graph how far could you see when 333 ft. up? How high must you be up to see 20 miles?

- * 16. How high must an airman be up to see 100 miles all round?

- * 17. How far, at sea-level, can the light of a lighthouse 125 ft. high be seen? If a passenger on a steamer is 40 ft. above sea-level, what is the distance? Give this distance in sea miles too. [1 sea mile = 6080 feet.]

- * 18. I am on a cliff, overlooking the sea, with an island in front of me. The top of the island is just flush with the horizon beyond. From the map I see the height of the island, 136 feet, and that the island is 4 miles away from me. How high up am I?



- * * 19. The diameter of the moon (which is nearly round) is about 2000 miles. The elevations of two craters on her surface are 10,000 feet and 20,000 ft. The craters are 200 miles apart. Are the tops of the craters mutually visible?
- * * 20. A tangent of constant length is drawn to a circle; find the locus of its extremity.

If PT, P'T' are two positions of the tangent inclined to each other at angle θ , find the area bounded by the tangents and the two arcs PP', TT'. The lengths of the tangents are l .

- * * 21. A garden roller, placed up against a step 6" high, touches the ground at a distance of 12" from the foot of the step. Find the diameter of the roller.

[Hint. Call the radius r , and work with a right-angled triangle with sides r , $r - 6$ and 12.]

- * * 22. A piece of thread equal in length to the circumference of a reel has one end attached to the reel, and is partly wrapped round the reel so that $\frac{1}{3}$ of its length is in contact with the reel and the whole thread is taut. Find the distance of the other end of the thread from the centre of the reel.
- * * 23. Draw a circle of radius 3 cm. and construct two tangents to it which enclose an angle of 78 degrees. Measure the line joining their points of contact, and calculate the length of the arc between these points.
- * * 24. Supposing that the eyes of a midge are 1 mm. above the surface on which it walks, and that the circumference of an association football is 70 cm. Are the eyes of one midge visible to another, if the midges (5 cm. apart) are crawling on the surface of a football?

EXAMPLES 24 c (RIDERS).

1. Prove that tangents at the ends of any chord of a circle make equal angles with the chord.

2. The locus of the centres of all circles which touch a given straight line at a given point is a straight line.

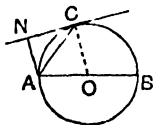
3. What is the locus of the centre of a circle which touches two given straight lines (1) which intersect, (2) which are parallel?

4. Show that all tangents drawn from points on the outer of two concentric circles to the inner are of the same length.

5. AB is the diameter of a circle and C any point on its circumference. AN is perpendicular to the tangent at C. Prove that $\angle NAB = \text{twice } \angle CAB$.

[HINT. AN and OC parallel.]

N.B.—This rider is also proposed in Examples II. 26. 7.



* 6. AB and CD are parallel tangents to a circle centre O. A and C are points of contact. A third tangent to the circle at R cuts the first two tangents at B and D. Prove that the angle BOD is a right angle.

* 7. P is a point external to a given circle whose centre is C. With C as centre describe a circle with radius twice that of

the first circle. With P as centre and PC as radius describe a third circle, cutting the second in Q. Join CQ, cutting the first circle in R. Prove that PR touches the first circle.

- * 8. T is any point outside a circle ABC whose centre is the point O. Through T two lines are drawn, TA touching the circle and TBC cutting it. If M is the mid-point of BC, show that $\angle AMT = \angle AOT$.

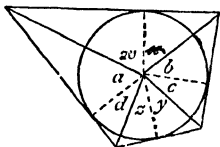
[HINT. Why is AMOT cyclic?]

- * 9. TA and TB are two tangents to a circle at right angles to one another, AC is a chord of the circle and BD is the perpendicular from B on AC. Prove that $BD = CD$.

[HINT. Prove $\angle ACB = 45^\circ$.]

- * 10. If a parallelogram circumscribe a circle, prove that it must be a rhombus.

- * 11. If a quadrilateral be described about a circle, prove that the angles subtended at the centre of the circle by a pair of opposite sides are supplementary.



[HINT. Prove $\angle a = \angle w$, etc. What do all 8 angles at the centre make up?]

- * 12. If two circles intersect one another, and if a chord be drawn through a point of intersection, then the inclination of the tangents at the ends of this chord will be equal to the angle at which the circles cut one another.
- * * 13. Given a square, show how to obtain from it a regular octagon by cutting off the corners. [No proof is required.]
- * * 14. Through a given point within a circle draw a chord of given length. When is this impossible?
- * * 15. A line cuts the circumference of a circle, centre O, in the points P, Q. What is the size of the angle OPQ if POQ is (i) 20° , (ii) 10° , (iii) 4° , (iv) 2° , (v) 1° ? If the line be made to revolve about P so that Q comes nearer and nearer to P, what will the angle OPQ eventually become? What property of a tangent does this give?
- * * 16. TA, TB are two fixed tangents to a circle whose centre is O, and any other tangent to the circle cuts TA, TB respectively in P and Q. Prove that PQ subtends a constant angle at the centre of the circle, and that the perimeter of the triangle TPQ is constant.

- * * 17.** O, P are two fixed points, OA, OB two variable straight lines which contain a constant angle ; if PM, PN be drawn perpendicular to OA, OB, show that MN touches a fixed circle.

[HINT. Prove PMNO cyclic, and MN a chord of constant length in that circle, etc.]

- * * 18.** CD and EF are two given parallel straight lines, and A is a given point in CD. B is a given point on the side of EF remote from A. It is required to determine the position of a point P in CD, such that, when the straight line PB is drawn crossing EF at Q, then PQ may be equal to AP. Prove that the perpendicular distance of the line PB from the point A is equal to the distance between the parallel lines. Hence solve the problem, and show that there are in general two possible positions for the point P.

Draw the figure making the perpendicular distance between the parallel lines equal to 3 cm., $AB = 4$ cm. and angle $DAB = 70^\circ$. Determine from your drawing the two possible lengths of PQ.

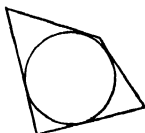
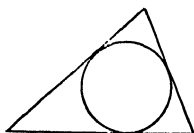
CHAPTER XXV.

IN- AND E-CIRCLES.

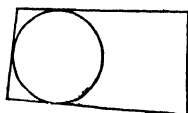
§ 1. To **inscribe** a circle in a figure is to construct a circle in that figure touching the sides. It is always possible to inscribe a circle in a triangle (it is the greatest circle, too, that can be got into the triangle), in a sector also, but it is the exception if a circle can be made to touch all the sides of a quadrilateral.

The inscribed circle is called the **in-circle** for short.

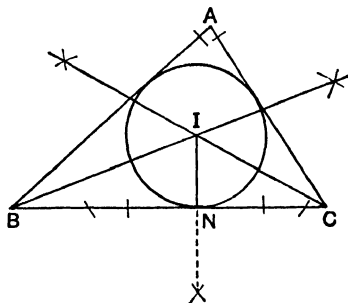
In-circles.



No in-circle possible.



To find the **centre of the in-circle of a triangle**, bisect two angles. Then draw a perpendicular. One can see at a glance if the circle



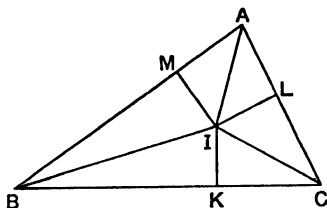
when drawn, does not fit well To get good results bisect the angles well, so take long radii for the first pair of arcs, and then

choose your radii to get good intersections. For two reasons let these intersections be outside the triangle, first because they are out of the way of where **I** will come, and, secondly, because (granted there is a good intersection) the further from **B** the better, for little errors then hardly affect the position of **I**. Get the accurate position of the point of contact with one side by constructing a perpendicular, and draw the circle.

Bisecting two angles, **B** and **C**, ensures that **I** is equidistant from the sides, and the right angles at **N** ensure the correct touching of the side **BC**, and likewise of the sides **CA** and **AB**.

Do not bisect the third angle of the triangle except as a check.

*** § 2.** It is important that the bisectors of the 3 angles of any triangle are concurrent.



An *outline* * of the proof is given.

The *two* angles **B** and **C** are bisected by **BI** and **CI**.

AI is joined.

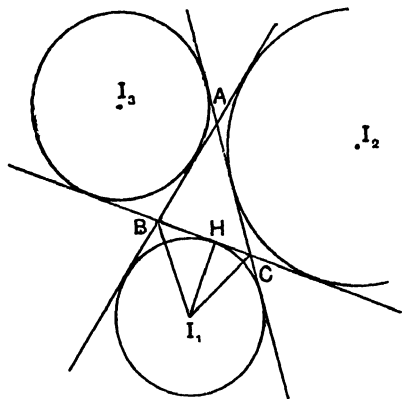
IK, **IL**, **IM** are perpendicular to the sides.

- (i) $\triangle$ s **BIK**, **BIM** congruent ; and, in particular, **IK** = **IM**.
- (ii) $\triangle$ s **CIK**, **CIL** congruent ; and, in particular, **IK** = **IL** ;
hence **IL** = **IM**.
- (iii) rt. $\angle$ d $\triangle$ s **AIM**, **AIL** congruent ; and, in particular, **AI** bisects $\angle$ BAC, etc.

If we are concerned with other centres of $\triangle$ ABC, usually the letter **I** is reserved for the **in-centre**.

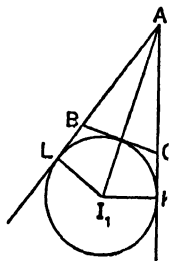
* The proof is given in full in the Appendix. Proposition (S).

§ 3. An **e-scribed** circle of a triangle touches one of the sides of the triangle and the other two sides produced. An escribed circle is called an **e-circle** for short. There are always 3 e-circles of a triangle. [Don't think that there are more than 3.] Bisecting two exterior angles and drawing one perpendicular will give

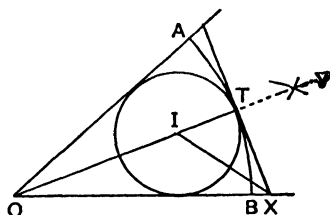


us the necessary position and size for one circle. Carrying out the same construction again twice will give us the others, but recollect I_3, B, I_1 are necessarily collinear, so that it is quite unnecessary to bisect separately two angles at B, merely producing I_1B backwards is enough (if BI_1 originally was a good bisector).

[As a matter of fact the bisector of the angle A will also pass through I_1 (this can be proved by joining AI_1 and considering the congruent triangles AI_1K, AI_1L).]



§ 4. For the **in-circle of a sector** there will be no difficulty in OT , the bisector of the angle at O . [*N.B.*— A and B are automatically equidistant from O , so take A and B as centres for the arcs at Y , aim at getting good intersections.] Then draw the tangent TX , necessarily at right angles to OT , to meet OB produced at X . The bisector of the angle X will give I . Then, with centre I and radius IT , describe a circle.

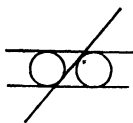


§ 5. To construct a **circle to touch 3 given straight lines** (every one of them) is the same problem as in-scribing or e-scribing a circle, and that means bisecting angles. If the lines will not meet on the paper, we have already (Vol. I.) got a way of bisecting the angle. If two lines happen to be parallel, the parallel line midway between them might be considered the bisector. The number of possibilities depends upon the relative positions of the 3 given straight lines.

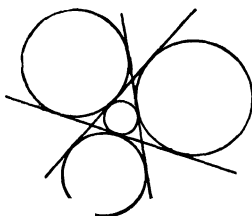
0 solutions.



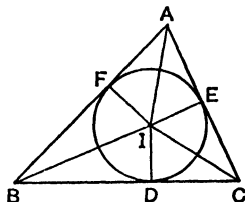
2 solutions.



4 solutions.



* § 6. In the $\triangle ABC$, a , b , c are the sides (cyclic order is preserved, a is the side opposite A , etc.). s is the semi-perimeter, so that $2s \equiv a + b + c$.



r is the radius of the in-circle.

(i) It is noticeable that

$$AF = AE \text{ (tangents),}$$

$$BD = BF \text{ ("),}$$

$$DC = CE \text{ (");}$$

and, therefore, by addition

$$AF + (BD + DC) = \text{semi-perimeter of } \triangle ABC ;$$

$$\therefore AF + a = s ; \quad \therefore AF = s - a.$$

So that $AE = AF = s - a$, $BF = BD = s - b$, $CD = CE = s - c$.

$$(ii) \quad \text{Area of } \triangle BIC = \frac{1}{2} ID \times BC = \frac{1}{2} ra.$$

$$\text{Similarly, } \triangle CIA = \frac{1}{2} rb$$

$$\text{and } \triangle AIB = \frac{1}{2} rc ;$$

and, therefore, by addition $\triangle ABC = rs$.

$$(iii) \quad AL = AK \text{ (tangents) ;}$$

$$\therefore AB + BL = AC + CK ;$$

$$\therefore AB + BN = AC + CN \text{ (tangents) ;}$$

$$\therefore AB + BN = \text{semi-perimeter of } \triangle ABC ;$$

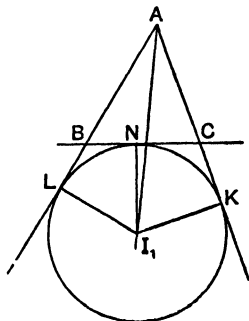
$$\therefore AB + BN = s ;$$

$$\therefore AL = s ;$$

$$\text{but } AB = c ;$$

$$\therefore BL = s - c,$$

and similarly for the other escribed circles.



* § 7. The following **summary** of the **concurrency** of certain lines in *any* triangle should be noted :

(1) **Medians** (Vol. I. Chap. XIII. § 6).

Point of concurrency, **Centre of Gravity** (or **Centroid**).

Medians **trisection** each other.

The centre of gravity is *always inside* the triangle.

The letter G is generally used.

(2) **Perpendicular Bisectors of the Sides** (Chap. XIX. § 7).

Point of concurrency, **Circum-centre**.

The circum-centre is *outside* the triangle, if the latter is *obtuse-angled*.

The letter S is generally used.

(3) **Perpendiculars from the Angles to the Sides** (Chap. XXIII. § 4).

Point of concurrency, **Orthocentre**.

The orthocentre is *outside* the triangle, if the latter is *obtuse-angled*.

The letter O is generally used.

(4) **Bisectors of Angles** (Chap. XXV. § 2 this chapter).

Point of concurrency, **In-centre**.

The In-centre is *always inside* the triangle.

The letter I is generally used.

(N.B.—In *general* these four points are separate; but, of course, they coincide in the *special* case of an Equilateral Triangle.)

These letters G, S, O, I should be used certainly whenever you are concerned with more than one. They are appropriate if the triangle is called ABC.

EXAMPLES 25 a (GEOMETRICAL DRAWING).

1. Prick off the points P, R, T (last page). Draw the $\triangle PRT$. In the triangle inscribe a circle.

[**HINT.** To get good results two angles have to be bisected very accurately, so every precaution should be taken to secure good bisectors.]

2. Draw a triangle with sides 6, 7 and 8 cm., and inscribe a circle. Does it touch the sides properly?

3. Draw a right-angled triangle with hypotenuse 3.3" and one side 1.8". Inscribe a circle in the triangle. Measure the radius.

4. Draw a triangle with angles of 106° and 34° , the side between them being 2.3" long. By bisecting the angles, find the centre of the biggest circle which can be drawn inside the triangle. Draw this circle, and measure its diameter.

5. Show how to cut from a triangular sheet of metal (sides $1\frac{3}{4}$ ", 2", $2\frac{1}{4}$ ") the greatest possible circle. Measure the radius of this circle and calculate (from it) the area of the circle.

6. Draw a sector, radius 3" and angle 37° . Inscribe a circle in it. Measure its radius.

7. Draw a quadrant, radius 7 cm., and inscribe a circle in it. What is its radius?

8. Draw a sector, radius 1.9" and included angle 30° . Produce the bounding radii and draw a circle to touch these lines (produced) and the arc of the sector. What is its radius?

9. Draw a circle C of radius 5 cm., and take a point P 9 cm. from its centre O. PO cuts the circle at R.

Construct the two tangents PA, PB from P to the circle C.

Next construct a second circle to touch the two tangents PA, PB, and to touch the circle C at R.

All construction lines should be very finely drawn and shown.

10. Draw a triangle with sides $3\frac{1}{2}$ ", 3" and 1", and draw a circle touching the two longer sides produced, and the shorter side. Measure its radius.

11. Draw a triangle with sides 4 in., 3.3 in., 1.4 in. Inscribe a circle and also escribe a circle to touch the shortest side. Measure the radius in each case.

12. Draw a triangle with sides 1.3", 1.4" and 0.9". Produce all the sides. Draw the three escribed circles. Do they touch the sides properly? [Recollect the necessity for accurate bisection.] How far apart are the centres of the two largest circles?

13. Draw a triangle with base 3.4 cm. and base angles 73° and 57° . Produce all the sides and draw the three escribed circles. Measure the three radii.

- * 14. Draw the kite-shaped figure ABCD, in which $AB = AD = 1\frac{1}{4}"$, $BD = 1\frac{1}{2}"$, $CB = CD = 2"$, and in it inscribe a circle. Measure the radius of the circle. Does it touch the sides correctly ?
- * 15. Draw the "kite" PQRS, in which $PQ = PS = 4$ cm., $QS = 5$ cm., $RQ = RS = 4.8$ cm. Inscribe in it a circle. Measure the radius.
- * 16. Draw a circle having a radius $1.25"$. Trisect this circle (by construction) by straight lines drawn from the centre to the circumference. In each of the three sectors thus formed draw a circle touching the two radii and the circumference of the first circle. What is the radius of one of these circles ?
- * 17. On opposite sides of a straight line $2\frac{1}{2}"$ long, as base, describe isosceles triangles with vertical angles 100° and 70° . Inscribe, in the quadrilateral so formed, a circle. What is the radius of the circle ?
- * 18. On a base of $2\frac{1}{2}"$ describe an isosceles triangle whose vertical angle is 40° , and on the opposite side of the base describe an equilateral triangle. In the figure so formed inscribe a circle. Measure the radius.
- * 19. In a square of side 10 cm. inscribe four equal circles each touching *one* side of the square and two other circles. Find, by measurement, the radii of the circles.
- * * 20. Draw a quadrilateral with sides respectively 9 cm., 6 cm., 4.5 cm. and 7.5 cm., and angle between the first two sides 80° . Why can a circle be inscribed in it ? Inscribe a circle in it and measure the radius.
- * * 21. Prick off the 4 points A, B, C, F (last page). Make the quadrilateral ABCF, producing all the sides both ways. Describe 4 circles, each to touch one side of the quadrilateral, and the two adjacent sides produced.
- * * 22. If the sides of a cyclic quadrilateral are 4, 4, 5, 5 inches, show that a circle can be inscribed in it, and find, by drawing, the radii of the inscribed and circumscribed circles.
- * * 23. The centres of the escribed circles of a triangle ABC form another triangle whose sides are 5, 6 and 7 cm. respectively. Construct the triangle ABC. Measure the sides of the $\triangle ABC$.

EXAMPLES 25 b (RIDERS).

1. Show how to describe a circle which shall touch two parallel lines and another straight line which cuts them.

[HINT. Think of the locus of the centre of circles which touch 2 intersecting straight lines.]

2. Find the centre of a circle which cuts off equal chords from the sides of a triangle.

[HINT. Equal chords are equidistant from the centre.]

3. The triangle formed by joining the centres of the escribed circles of a triangle ABC is acute-angled.

[HINT. Show that one angle is $\frac{B}{2} + \frac{C}{2}$, etc.]

4. Construct a right-angled triangle having given a side and the radius of the inscribed circle.

5. TA and TB are two tangents to a given circle. State how to construct another circle touching each of the lines and the given circle.

* 6. Given the inscribed circle of an isosceles triangle and the vertical angle, construct the triangle.

[HINT. Draw the circle first and think at what angle radii will have to be so that tangents will give the triangle.]

* 7. Show that a circle can be described through the centres of the four circles which can be drawn each touching one side and the two adjacent sides produced of any quadrilateral.

* 8. The inscribed circle of a triangle ABC touches the side AB in L, and the escribed circle (away from A) touches AB in M; prove that $LM = BC$.

[HINT. $LM = AM - AL$.]

* 9. The inscribed circle and an escribed circle of the triangle ABC touch the base (not produced) at R and S respectively. Prove that $BS = CR$ and $BR = CS$.

* 10. ABC is a triangle. The in-circle touches the sides at X, Y, Z. The angles of the triangle XYZ are 40° , 75° and 65° respectively. Calculate the angles of the triangle ABC.

* * 11. Prove that the diameter of the inscribed circle of a right-angled triangle and the hypotenuse are together equal to the other two sides of a triangle.

* * 12. Given the base BC of a triangle and the vertical angle, find the locus of the centre of the inscribed circle of the triangle.

* * 13. Given the vertical angle of a triangle and the lengths of the two parts into which the base is divided by the point of contact of the inscribed circle, construct the triangle.

[HINT. The preceding rider is useful.]

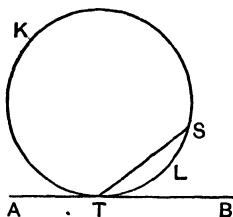
* * 14. I is the in-centre of a triangle ABC , E the centre of the escribed circle which touches AB , AC when produced ; if AI produced cut the circum-circle of ABC in D , show that DB , DE , DI are all equal.

[HINT. Prove $\triangle DBI$ isosceles, etc.]

CHAPTER XXVI.

ALTERNATE SEGMENT.

§ 1. ATB is a tangent of the circle. ST is a chord (from the point of contact). ST divides the circle into two segments TKS and TLS . The (major) segment TKS is said to be “**alternate**” to the angle STB , and the (minor) segment TLS is said to be “**alternate**” to the angle ATS .



§ 2. The following proposition is to prove that $\angle STB$ must be equal to any angle in the segment TKS ; now, since all angles in the same segment are equal, we can do the *apparently* unfair thing, and consider the angle that is particularly suited to us. That is the course adopted in the following proofs. The tangent makes 2 angles with the chord, and the chord divides the circle into 2 segments, so that there are 2 cases to consider.

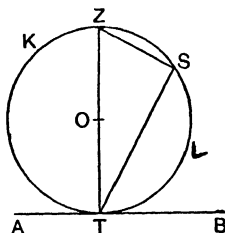
PROPOSITION 27.

General Enunciation. *If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent must be equal to the angles in the alternate segments.*

Particular Enunciation. ATB is the tangent and TS the chord, to prove (i) $\angle STB = \text{any angle in the segment TKS}$ and (ii) $\angle STA = \text{any angle in the segment TLS}$.

(i.)

Construction. O is the centre, so that TOZ is a diameter. Join ZS.



Proof. $\angle ZST = \text{rt. } \angle$ ($\angle$ in semicircle);

$\therefore \angle Z + \angle ZTS = \text{rt. } \angle$ (3 angles of $\triangle = 2 \text{ rt. } \angle$ s);

but $\angle STB + \angle ZTS = \text{rt. } \angle$ (tangent and diameter at rt. $\angle$ s);

$\therefore \angle STB = \angle Z$;

$\therefore \angle STB = \text{any angle in the segment TKS.} \quad \text{Q.E.D.}$

(ii.)

Construction. (In addition to the foregoing) L is any point in the segment TLS. Join TL and LS.

Proof.

$\angle STB + \angle STA = 2 \text{ rt. } \angle$ s (adjacent),

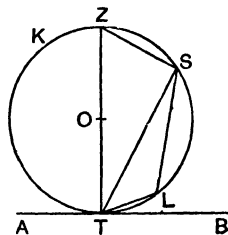
and $\angle Z + \angle L = 2 \text{ rt. } \angle$ s (TLSZ cyclic);

but $\angle STB = \angle Z$ (proved in i.);

$\therefore \angle STA = \angle L$;

$\therefore \angle STA = \text{any angle in the segment TLS.}$

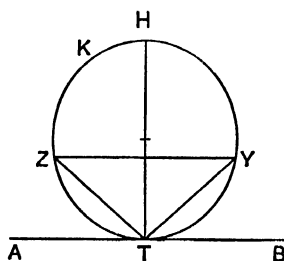
Q.E.D.



PROPOSITION 27. (Aliter.)

§ 3. General Enunciation. *If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent must be equal to the angles in the alternate segments.*

Particular Enunciation. ATB is the tangent, TY the chord, to prove (i) $\angle YTB =$ any angle in the segment TKY, and (ii) as on preceding page.



(i.)

Construction and Proof.

Draw YZ parallel to BA. Join TZ.

Let TH be the diameter through T and, therefore, perpendicular to both BA and YZ.

$$\begin{aligned}\angle YTB &= \angle Y \text{ (alternate)} \\ &= \angle Z \text{ (by symmetry with respect} \\ &\quad \text{to the diameter TH) ;}\end{aligned}$$

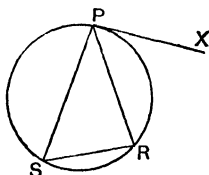
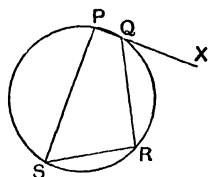
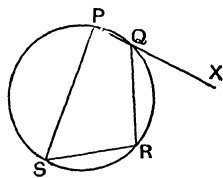
$\therefore \angle YTB =$ any angle in the segment TKY.

(ii) See preceding page.

Q.E.D.

§ 4. The **converse**, namely, if the angle between a straight line meeting a circle and a chord is equal to an angle in the alternate segment, then the straight line must be a tangent is true. Its truth is generally assumed, but could be proved by *reductio ad absurdum*. It is often useful.

* § 5. If the idea of a **tangent** is the **limiting position** of a **secant**, when the two ends of the chord ultimately coincide, we can argue out this proposition as an extension of the cyclic quadrilateral, Proposition 24. After that we saw $\angle XQR = \angle S$. Now suppose P and Q were very close, still $\angle XQR = \angle S$. If P and Q are made to coincide at P, then PX becomes the tangent and $\angle XPR = \angle S$; that is, the angle between the tangent and chord is equal to the angle in the alternate segment. [The student should mark the equal angles on the figures themselves.]



EXAMPLES 26 (RIDERS).

✓ 1. If the chord AB of a circle is parallel to the tangent at the point C of the circle, show that C bisects the arc AB.

[HINT. Prove $\angle CAB = \angle CBA$, and remember that equal angles subtend equal arcs.]

✓ 2. PQR is a triangle inscribed in a circle; if the tangent to the circle at the point P is parallel to QR, prove that the triangle PQR is isosceles. [HINT. Prove $\angle Q = \angle R$, etc.]

3. PQ is a chord of a circle, and R is its middle point; RS, drawn perpendicular to PQ, meets the circumference in S. Show that the perpendicular SN, from S on the tangent at P, is equal to RS.

✓ 4. If the tangent at a point C of a circle cut any chord AB produced in the point T, prove that the angles ACT and CBT are equal, and also that the angles BCT and CAT are equal.

[HINT. Besides the figure draw $\triangle ACT$ and $\triangle CBT$ separately; and on them mark equal angles.]

✓5. Draw a triangle ABC and produce CB to D ; make an angle ACE equal to ABD ; CD and CE being on the same side of AC . Show that CE will touch the circle which circumscribes the triangle ABC .

✓6. AB is a tangent to a circle at B , and BC is a chord of the circle. If the bisector of the angle ABC meets the circle again in D , prove that $CD = DB$.

✓7. AOB is the diameter of a circle, whose centre is O . C is any point on the circumference of the circle. AN is drawn perpendicular to the tangent at C . Prove that CA must bisect the angle NAB .

[A proof needing "Alternate Segment" is required. The same rider is proposed 24c 5, where a different proof is expected.]

8. AB is a chord and AC a tangent to a circle at A . CPQ is a straight line parallel to AB , meeting the circle in P and Q . Prove that the triangles PAC and BAQ are equiangular.

✓9. AB is a diameter and AP a chord of a circle. AQ is a chord bisecting the angle BAP . Prove that the tangent at Q is perpendicular to AP .

10. ABC is any circle, and XAY is the tangent at A : AK bisects the acute angle CAY , K being on the circumference of the circle: if $CAY = 70^\circ$, and the lines BK , AC cut at an angle of 80° , calculate the size of the angles CAB , CBK , AKB , showing your reasoning shortly and clearly. [There are two sets of answers.]

*11. Two circles cut at A , B , and a straight line PAQ cuts the circles at P and Q . The tangents at P and Q meet in T . Prove that the points P , B , Q , T lie on a circle.

[HINT. Join AB . $\angle TPQ = \angle PBA$. Why? etc.]

*12. A tangent TA touches a circle at A , and from A there are three successive chords AB , AC , AD , such that

$$\angle TAB = \angle BAC = \angle CAD.$$

Prove that $\angle ADC$ must be equal to twice the $\angle ACB$.

*13. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at E : a circle is drawn through A , B and E . Prove that the tangent to this circle at E is parallel to CD .

*14. BAC is the tangent at A to a circle, a chord DE when produced cuts BC at C , and the line drawn from D at right angles to DE cuts BC at B ; show that if $AB = AC$ then the arc AD is double the arc AE .

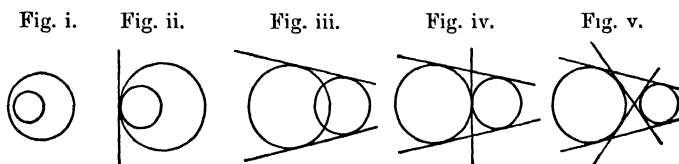
- *15. AB, AC are two chords of a circle ABC . DE , parallel to the tangent at A , cuts AB, AC in D and E respectively. Prove that a circle can be described about the quadrilateral $BDEC$.
- *16. QR is a chord of a circle, TR is the tangent at R ; a straight line through Q perpendicular to the tangent meets it in T and the circumference of the circle again in P ; PM is the perpendicular from P on QR . Prove that the angles QPM, TPR, TMR, TRM are all equal.
- *17. ABC is a triangle inscribed in a circle; tangents at A, B, C are drawn, and when produced form a triangle DEF . If the angles of the triangle ABC are $40^\circ, 60^\circ, 80^\circ$, calculate the angles of the triangle DEF .
- *18. ABC is a triangle in which A is an obtuse angle. Find in terms of the angles of the triangle ABC the angles of the triangle formed by the tangents at A, B and C to the circle circumscribed to the triangle ABC .
- *19. C is the middle point of an arc ACB of a circle; prove that C is the centre of a circle which touches the chord AB and the tangents at A and B to the given circle.
- *20. The angle A of a triangle ABC is a right angle, and the internal and external bisectors of it cut BC at D and E . Show that if BC is bisected at F , then FA is a tangent to the circle described around ADE .
21. A tangent is drawn to a circle from a point P , touching the circle at Q . PAB is the diameter of the circle drawn through P . If another circle be drawn through Q to touch AB at B , show that AQ is parallel to BR , where PQ cuts the latter circle at R .
- *22. Two circles cut at A, B , and P is taken on the circumference of one circle. PA, PB produced, if necessary, cut the other circle at C, D : prove that CD is parallel to the tangent at P .
- *23. The tangent at A to the circle circumscribing the triangle ABC , of which the side AC is less than AB , cuts BC produced in D , and AC produced cuts the circle circumscribing the triangle ABD in E . Prove that $AD = DE$.
- *24. The tangent at P to one circle is parallel to a chord AB of another circle; PA cuts the circles in N and R , and PB cuts the circles in L and M . Prove that NL is parallel to MR .

- * 25. The tangent at a point A to a circle of centre O meets a radius OB produced at C ; if AD be drawn perpendicular to OB, show that BA bisects the angle CAD.
- * * 26. The inscribed circle of a triangle ABC touches AB, AC at D, E respectively, and the line from A to the centre of the circle cuts the circle at F ; show that F is the centre of the inscribed circle of ADE.
- * * 27. ABC is a triangle inscribed in a circle, and TAS is the tangent at A. Prove that the bisectors of the angles TAB and BAS meet the internal and external bisectors of the angle C on the circumference of the circle and at opposite ends of a diameter.
- * * 28. Two tangents drawn to a circle from a point P outside it touch the circle at A and B. Prove that the angle between the tangents must be double that between the chord of contact AB and the diameter through either end of it.
- * * 29. AB is a chord of a circle whose centre is O ; on the tangent at A a length AC is taken equal to AB, and BC meets the perpendicular ON on AB, in D. Prove that the angle BDN is equal to (or complementary to) a quarter of the angle AOB.
- * * 30. PA, PB are tangents to a circle at A and B. Prove that $\angle APB$ is equal to the difference of the angles in the two segments into which AB divides the circle.
- * * 31. ABC is a triangle inscribed in a circle. Tangents at B and C meet at T. TK is drawn through T, parallel to BA, to meet AC produced (if necessary) at K. Prove that the triangle KAB is isosceles.

CHAPTER XXVII.

COMMON TANGENTS.

§ 1. **Definition.** If a straight line touch two circles, it is said to be a **common tangent** to the two circles.



In Fig. i. there can be no common tangents. One circle is entirely inside the other.

In Fig. ii. there is 1 common tangent. The circles touch internally.

In Fig. iii. there are 2 common tangents (both **direct**). The circles cut.

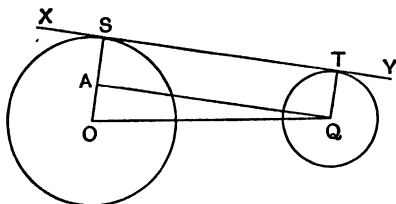
In Fig. iv. there are 3 common tangents. The circles touch externally.

In Fig. v. there are 4 common tangents (**two direct** and **two indirect or transverse**). The circles are entirely outside each other.

§ 2. There is no difficulty in constructing the common tangent at the point of contact of two circles touching internally or externally. It is only **the construction of direct and indirect common tangents** which needs to be considered, and, since these both occur in Fig. v. we shall confine our attention to the common tangents of two circles entirely outside each other.

Then comes the question : “ How are they to be drawn ? ”

For thinking out the correct method it is not a bad plan to imagine the construction complete, and to try to argue out the necessary consequences, then (by that method) to hit upon the correct procedure.



It might be something like this :

XY is to be a common tangent ;

$\therefore$ angles at S and T must be right angles.

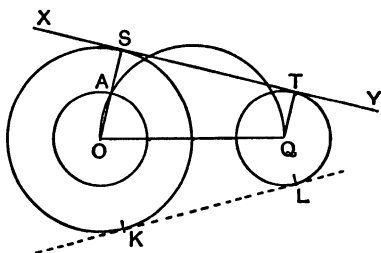
$STQA$ would be a rectangle if QA were parallel to TS ;

$\therefore$ angles at A must be right angles (the angle in a semicircle would do), and SA is equal to TQ ,

so that OA is equal to the difference between OS and QT .

This gives the clue to the following construction :

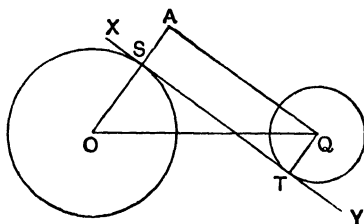
With centre O (necessarily the centre of the larger circle), and radius equal to the **difference** of the given radii, describe a circle (an arc, in the right position, is enough). On OQ as diameter



describe a semicircle (an arc, in the right position, is enough) to cut the former at A . Join OA and produce to find S ; through Q draw QT parallel to OS to find T . Then the join ST gives the **direct** common tangent XY . A repetition of the construction will

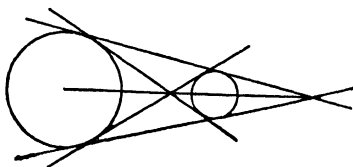
give the second (direct) common tangent. **Symmetry** affords a quicker construction ; thus, put one leg of the dividers at any point on OQ (it must not be at O or Q , and had better not be anywhere near them), and with the other leg stretch to S and so get K , again stretch to T and so get L . The join KL gives the second (direct) common tangent.*

The construction for the **indirect (or transverse)** common tangents may be thought out in the same way.



Here OA must be equal to the **sum** of the radii of the given circles. In other respects the construction of the indirect (or transverse) common tangents is similar to the construction given for the direct common tangents.

The calculation of the **lengths** of the common tangents, necessarily between the points of contact, S, T , is often an easy matter if we recollect $AQ = ST$, and (instead) calculate AQ from the right-angled triangle OAQ ; but it is a question of trigonometry to calculate at what angle the common tangents (produced if necessary) cut the line of centres.



It is to be noticed that the two direct common tangents meet the line of centres, produced, at the *same* point, and that the two indirect (or transverse) common tangents also meet the line

* The proposition is given fully in the Appendix (T).

of centres at the same point. This may be assumed from symmetrical considerations, or can be proved by consideration of similarity, so that this proof would be inappropriate until later.

EXAMPLES 27 a (CALCULATIONS).

1. Two circles touch each other externally; find the length of the common tangent in terms of the radii.

2. Construct (with explanation) a common tangent to two circles whose radii are 7 cm. and 4 cm. respectively and whose centres are 5 cm. apart. Calculate the length of this tangent between its points of contact with the circles.

3. Draw two circles with radii 3 cm. and 4 cm. and centres 10 cm. apart. Construct the common tangents. Calculate the length of one of the direct common tangents.

4. Give reasons for or against the following statement: Two circles have their centres 43 cm. apart and the lengths of their direct common tangents are 47 cm.

5. An endless tight band passes over two circular pulleys whose radii are 4" and 1", and whose centres are 6" apart. Find the length of the band correct to the nearest quarter of an inch. The band does not cross.

6. Draw two circles of radii 0.9" and 1.4" respectively, with their centres 3.6" apart. Construct one of their transverse common tangents, and calculate its length. Check by measurement.

7. A wheel of diameter 8' is driven by open belting from a smaller wheel of diameter 1', the centres of the wheels being 10' apart. Assuming the belt to be taut,

(a) Show that the length of belting not in contact with the wheels is about 18 ft. 9 in.

(b) Measure the angle subtended at the centre by the part in contact with the smaller wheel.

* 8. Two pulley wheels, 9 in. and 24 in. in diameter, are in the same vertical plane, their centres being 20 in. apart. Calculate the length of the belt which will go round them and cause them to revolve in opposite directions.

[Calculate the angles, trigonometrically, if you can; if not, measure them on an accurate diagram.]

- * 9. Take 2 points A and B 10 cm. apart. With centre A and radius 5 cm. describe a circle. With centre B and radius 3 cm. describe a circle.

- (a) *Construct* the two transverse common tangents, *also* stating in words the steps of the construction. By inking in, or otherwise, make a "figure of eight" clear.
- (b) *Calculate* the exact length of each of the common tangents.
- (c) *Measure* the reflex angles subtended by the circular arcs at the two centres.
- (d) *Calculate* the lengths of the circular arcs.
- (e) Give the total length of the "figure of eight" curve.
- (f) Check this length by stepping with dividers (make your check clear).
- (g) *Calculate* the areas of the two sectors (which subtend reflex angles at the centres).
- (h) *Calculate* the areas of the two quadrilateral portions bounded by tangents and radii.
- (i) Give the total area of the two loops.
- (j) Check this area by counting squares (make your check clear).

- * 10. What is the length of the common tangents of two circles, radii r_1 and r_2 , if the centres are c apart? Only the case when $c < r_1 + r_2$ is to be considered.

- * 11. If the distance between the centres of two wheels (whose radii are r_1 and r_2) is c , what is the length of their direct common tangents, and also of their transverse common tangents? [$c > r_1 + r_2$]

- * 12. A bath is designed in the following way:

A and B are two points 38 inches apart. A circle with centre A and radius 12 inches is described, and a circle with centre B and radius 8 inches is described. Common tangents are drawn to the two circles, and the figure represents the ground plan of the bath.

Draw the ground-plan to scale. Determine its area by considering it as the sum of two trapezia and two sectors.

Check this area by counting squares.

If the bath has vertical sides, calculate its depth (to the nearest inch), so that it may contain 30 gallons when half-full.

- * 13. The inner boundary of a running track is designed in the following way :

Two circles with radii 11 half-cm. and 3 half-cm. have their centres 17 half-cm. apart. Direct common tangents are drawn to the circles. [Preferably carry out the accurate geometrical construction, but no verbal explanation is asked.]

Calculate the length of each common tangent. [Check by measurement.]

Measure angles, and so calculate the lengths of the circular arcs of the track. [Check by stepping with dividers.]

What is the total length of the boundary of the track ?

If a half-cm. represents 10 yards, find the length of one lap.

Runners in a 3-mile race start along a tangent where it leaves the larger circular arc and keep the area enclosed by the track always on their right-hand side. How many laps must the winner run ? Mark with the letter P the position of the finish.

- * 14. The smaller cog-wheel (hub) of a bicycle has diameter 7 cm., and the larger cog-wheel (crank) has a diameter of $17\frac{1}{2}$ cm. The centres of these wheels are $55\frac{1}{2}$ cm. apart. Draw a diagram (omitting the cogs) to scale. Construct common tangents. *Measure* angles at centres. *Measure* the least possible length of a taut chain which could go round the wheels. There are 22 cogs on the small wheel (so that 22 links of chain would exactly surround it). Remembering that there must be a whole number and that the chain must not be absolutely taut, how many links should there be in the bicycle chain ?

- * * 15. The distance between the centres of two wheels is a and the sum of their radii is c ; find the length of the string which crosses between them and just wraps round them. [$a > c$.]

- * * 16. Draw 2 circles (on squared paper) of radii 3.5 and 2.5 cm., with centres 7.2 cm. apart. Construct their direct common tangents.

Find, by 2 separate methods, (a) the length of a belt passing round the circles, and (b) the area enclosed by this belt.

State which of your 2 methods in (b) is the more accurate, and calculate the error per cent. in the other answer.

- * * 17. The radius of the inscribed circle of a triangle ABC is 0.6", of the escribed circle which touches BC and the other two sides produced is 1.6". The centres of the two circles are 2.6" apart. Draw the triangle ABC, and calculate the length of BC.

EXAMPLES 27 b (RIDERS).

1. Show that the direct common tangents to two circles are equal. Prove the same for the two transverse common tangents.

2. Prove that a common tangent (whether direct or transverse) must be shorter than the line joining the centres.

[HINT. In the figure on pages 328 or 329 prove $ST < OQ$.]

* 3. Two circles cut at P and Q, and XY is one of their common tangents. Prove that XY subtends supplementary angles at P and Q.

* * 4. Draw a straight line so that the chords which are cut off from it by two fixed circles may be equal to two given straight lines.

[HINT. First draw the circles that those chords must touch.]

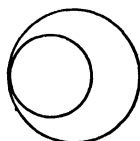
* * 5. If a straight line touch two circles the chords which join the points of contact to the points in which the line of centres meets the circles are parallel, two and two.

CHAPTER XXVIII.

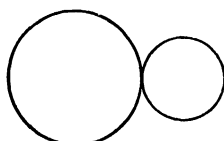
CIRCLES TOUCHING.

§ 1. Two circles may touch **internally** or **externally**. The contact is illustrated in the following figures :

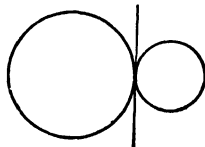
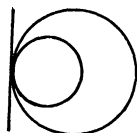
Internal Contact.



External Contact.



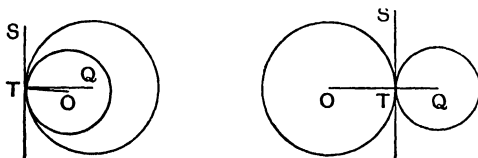
If two circles touch, they necessarily have the same tangent at the point of contact.



PROPOSITION 28.

§ 2. General Enunciation. *If two circles touch, the point of contact must lie on the straight line through the centres.*

Particular Enunciation. O and Q are the centres and T the point of contact. To prove O , Q and T are in the same straight line.



Construction. Draw the common tangent ST . Join OT and QT .

[We want to prove O , Q and T in the same straight line, so purposely we draw the figure slightly wrongly to make OT and QT separate.]

Proof. Of one circle the radius OT is at right angles to the tangent TS .

Of the other circle the radius QT is at right angles to the tangent TS .

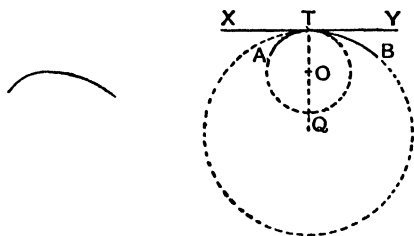
$\therefore OT$ and QT are both at right angles to TS ;

$\therefore O$, Q and T must (really) be in the same straight line.

Q.E.D.

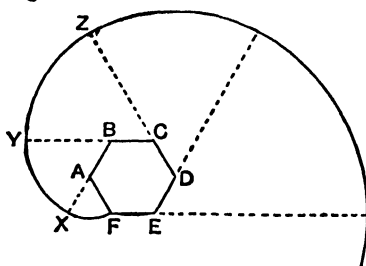
It follows, from this proposition, that the distance between the centres is exactly equal to the **difference** or **sum** of the radii according as the contact is **internal** or **external**.

* § 3. This proposition is of extreme importance in many practical problems. It very frequently happens that only arcs of the circles (and by no means the whole circles) come in, and it is quite possible not to detect an arc as a piece of a circle (of course it needn't be). When arcs join imperceptibly so that there is no sudden change of direction, the curves might do for railway curves without fear of derailment, or might illustrate an arch in architecture. The only curves we are concerned with now are circles, and in all probability only arcs of circles. How are we to make them **continuous** ?



Imagine the whole of the two circles to be drawn. The direction at the point where the curvature changes is to be unaltered. The tangent must be the same, and hence the radii must be in the same direction. So that Q, O and T must be in the same straight line. The essence of all continuous curves (we are only concerned with arcs of true circles now) is that the point of change of curvature and the two centres must be in the same straight line.

The locus of the end of a piece of cotton unwound from a reel of hexagonal section is an illustration.



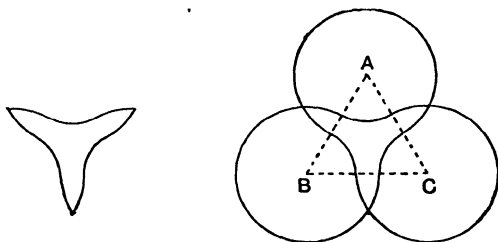
With centre A, and radius AF, describe the arc FX.

"	B,	"	BX,	"	"	XY.
"	C,	"	CY,	"	"	YZ.
			etc.			

B, A, X are necessarily in the same straight line.

C, B, Y	"	"	"	"	"	"
			etc.			

- * § 4. The appearance of many geometrical patterns may seem elaborate, but often they are easy to draw if they are made up of continuous circular arcs. It is extremely easy to draw a circle with a pair of compasses; it is extremely hard to draw at all accurately any other curve of the particular size and shape that we want.



The left-hand figure looks comparatively elaborate, but the **continuous curves are easy to draw** as arcs of circles with centres A, B and C. The straightness of the sides of the triangle ABC ensures the centres and the points of change of curvature being in the same straight line.

To illustrate further this collinearity (or being in the same straight line) **two examples** are given:

- (i) "The section of the egg-shaped sewer."

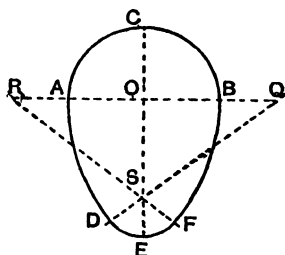
O is the centre of the semicircle ACB

S " " " arc DEF.

Q " " " " AD.

Q, O, A are necessarily collinear.

Q, S, D " " "



[The position of Q and the relative sizes of the radii are a matter of design.]

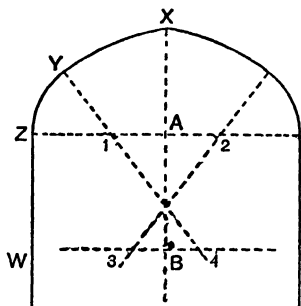
(ii) "The four-centred arch."
 (Four-centred arches are a common feature of "perpendicular" architecture.)

WZ is upright.

2, 1, Z are necessarily collinear.

4, 1, Y are necessarily collinear.

[The proportions are a matter for the taste of the architect.]



EXAMPLES 28 a (GEOMETRICAL DRAWING).

Circles with given radii.

1. Prick off the points P and R (last page). With centre R, and radius 3 cm., describe a circle. Construct two circles, each with radius 4 cm., to pass through P and to touch the other circle externally.

[HINTS. How far from R must the new centre be? How far from P must it be?]

2. Prick off the points Q and T (last page). With centre Q, and radius 4 cm., describe a circle. Construct two circles, each with radius 9 cm., to pass through T and to touch the other circle internally.

3. Draw two circles with radii 2 cm. and 3 cm. with their centres 7 cm. apart. Give in words the construction for drawing a circle with radius 8 cm. to touch both. Carry out this construction in your accurate drawing. Actually draw only one circle to fulfil these conditions, but mark with numbers 1, 2, 3, ... etc., the centres of all circles of radius 8 cm. which would touch the two given circles. In all, how many solutions are there beside the single case you draw?

4. Draw a circle with centre C and radius 4.5 cm.: take a point P 7 cm. from C. Construct two circles to touch PC and also the given circle, one with radius 3 cm., the other with radius 5 cm.: state the loci on which the centres of each of these circles must lie, and measure the distance XY, when X and Y are the points of contact between the new circles and the original circle.

- * 5. Prick off the points S, Q, R (last page). Produce SR both ways right across your paper, construct a circle to pass through Q and to touch SR at the point R. Give the words of your construction. How many solutions are possible? Now construct a circle of 2.5 cm. radius to touch externally the circle you have just drawn, and also SR. Give the words of your construction. How many solutions are possible?
- * 6. Draw two $\odot$'s of radii 3 cm. and 5 cm. with their centres 10 cm. apart.

Draw as many circles as possible of radius 4.5 cm. to touch the two given $\odot$'s.

Explain your construction very briefly.

How many $\odot$'s are there?

- * 7. POQ and ROS are two straight lines cutting one another at O at an angle of 56° . Construct 4 circles of radius 2 cm. to touch both of these lines. What is the area of the quadrilateral formed by joining their four centres? Do the circles touch each other? Give the *construction* in words for describing a circle of radius 8 cm. to touch and enclose the two circles which are diagonally nearest each other.
- * 8. Draw accurately a $\odot$ of radius 5 cm. and a straight line passing through its centre. Draw as many circles as you can of radius 8 cm. to touch the given circle and the straight line, explaining very briefly how you do it.

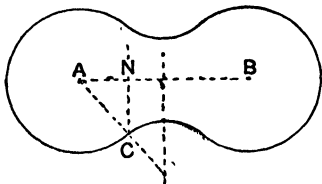
Measure in each case the distance between the centre of the 5 cm. $\odot$ and the point of contact of the constructed $\odot$ with the straight line.

[You had better draw this on half scale.]

‘Continuous’ Curves.

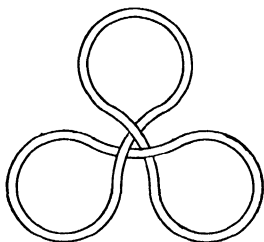
- * 9. XY is a straight line 1 cm. long. With centres X and Y alternately, describe semicircles above and below XY to form a continuous spiral. The first semicircle is to have a radius 1 cm.
- * 10. In a circle whose radius is 1" place a regular hexagon, and on alternate sides *inside* describe semicircles. Connect these by circular arcs to form a continuous curve.
- * 11. Draw the figure accurately. $AB = 7$ cm., $AC = 3$ cm., $AN = 2$ cm. The arcs are all circular.

[HINT. The centre of the lower small arc must be in AC produced.]



- * 18. Make a pattern something like the accompanying, though the particular size and radii you adopt are a matter for your taste.

[HINT. Take the corners of a very perfect equilateral triangle for centres.]



Circle touching line or circle at given point, etc.

- * 19. Construct a circle with centre O and radius an inch: draw a radius OA , and produce it to B so that $AB = 1.3''$: take a point C on the circumference so that $\angle AOC = 30^\circ$: construct a circle to pass through B and touch the original circle at C . Measure its radius.

Make your construction quite clear on the figure.

- * 20. O is the centre of a circle of radius 5 cm. OAP is a straight line, $OA = 5$ cm., $OP = 10$ cm. B, C, D are points on the circle on the same side of OAP , such that $\angle POB = 10^\circ$, $\angle POC = 20^\circ$, $\angle POD = 30^\circ$.

Construct four circles all passing through P , and touching the given circle *separately* at A, B, C, D . Draw a graph co-ordinating the angles POA, POB, POC, POD with the radii of the circles touching the given circle at A, B, C, D .

From the graph read off the angle corresponding to a radius of 4 cm.

On a separate piece of paper draw the original circle of radius 5 cm. and take the same point P . Construct a circle of radius 4 cm. to pass through P and to touch the given circle at E (say). Measure the angle POE , and compare this result with value of the angle obtained from the graph.

- * * 21. ABC is a triangle, B is a right angle. $AB = 5$ cm., $BC = 2.6$ cm. With centre A and radius 2 cm. a circle is drawn. Draw two other circles to touch this circle and to touch the straight line BC at the point C . What are the radii of the two circles?
- * * 22. ABC is an equilateral triangle. With centres B and C two equal circles are drawn touching externally. Show how to describe a circle to touch these two and to pass through the point A . Discuss two cases. Measure the radii of the circles.

**** 23.** ANB is a straight line. $AN = NB = 2.5$ cm.

With centres A and B and radii 2 cm. two equal circles are drawn.

Construct four circles with radii 1, 2, 3 and 4 cm. respectively to touch the two given circles externally all on the same side of ANB. Give the construction in words for the first circle.

With the letters P_1, P_2, P_3 and P_4 on these circles mark the points which are farthest from N.

Draw a graph to show how NP varies as the radius of the constructed circle varies.

From the graph determine what should be the radius of the constructed circle, if for it $NP = 6$ cm.

Re-draw the straight line ANB and the two circles whose centres are A and B. Take PN perpendicular to AB and equal to 6 cm. Give in words the geometrical construction for drawing a circle to pass through P and to touch each of the given circles externally. Construct the circle and measure its radius.

[As a check compare with the value obtained from the graph.]

EXAMPLES 28 b (CHIEFLY RIDERS).

1. A circle XYZ touches circle XLK externally in X. YXL and ZXK are any two lines through X cutting the circles in Y, L and Z, K respectively. Show that YZ is parallel to LK.

[HINT. Draw the common tangent TXS at X.]

2. Two circles KAB and KCD touch internally at K. AB is a chord of the outer circle. Prove that, if KA, KB cut the inner circle at C and D respectively, then AB and CD are parallel.

[HINT. Draw the common tangent TK (on the A and C side). $\angle TKC = \angle CDK$, why? Again, $\angle TKC = \angle ABK$, why? etc.]

3. PAQ is a straight line passing through the point of contact A of two circles, and cutting the circles again at P and Q. Prove that the tangents at P and Q are parallel. The contact can be either external or internal. Choose only one case.

[HINT. Draw the common tangent at A.]

4. AB, CD are two parallel chords of a circle; if AC, BD produced meet at F, show that the circles about the triangles FAB, FCD touch one another.

[HINT. Prove that the same line touches both circles.]

5. Two circles touch externally at A ; and are touched by a straight line at H and K. Prove $\angle HAK$ is a right angle (first prove that the tangent HK is bisected by the tangent at A).

* 6. Two circles which touch one another externally at A are touched by a straight line in B, C respectively. Prove that the circle whose diameter is BC passes through A.

[HINT. Use the rider, Question 5.]

* 7. Two circles whose centres are A and B touch at C. A line touches them at D and E respectively, and the tangent to both at C cuts DE in F. Prove that the angle AFB is a right angle.

* 8. Two circles touch externally, and on the part of their common tangent (intercepted between the points of contact) as diameter a circle is described : show that it will touch the line joining their centres.

* 9. Two circles touch externally at A, through A a line BAC is drawn terminated by the circles. Show that the diameters through B and C are parallel.

* 10. Two circles touch one another externally at A ; P, Q are two points one on each circle such that PQ passes through A : if PG, QG are any two straight lines cutting the circles at R and S, prove that G, R, A, S are concyclic.

* 11. Two circles touch internally at A : a chord is drawn cutting the larger in X, Y and the smaller in B, C : prove the angles XAB and YAC are equal.

* 12. Two circles touch internally at A, and a chord BC of the outer circle touches the inner circle at D. Prove $\angle BAD = \angle CAD$.

* 13. Two circles touch internally at A, and a chord ABC cuts them at B and C. Prove that the tangent at B (to the inner circle) cuts the outer circle at points which are equidistant from C.

* 14. Show how to describe a circle, of given radius, to touch a given circle and a given straight line. Discuss the limits of possibility and the number of solutions.

* 15. A quadrilateral ABCD has the sides AB, DC parallel, and AC, BD cut at E. Prove that the circles circumscribed about the triangles ABE, DEC touch one another.

* 16. ABC is a triangle ; BA, CA are produced through A to points D and E so that DE and BC are parallel. Prove that the circles ABC and ADE touch one another.

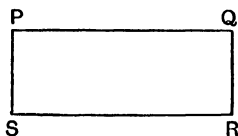
- * 17. The line AD bisects the angle BAC of a triangle and meets the side BC at D, and a circle is drawn through A touching BC at D. Prove that this circle will touch the circle circumscribing ABC at the point A.
- * * 18. Two circles touch externally at A, and a direct common tangent touches them at B, C respectively. Any line through A cuts them at D, E respectively. If DB, EC produced meet at F, show that the angle DFE is a right angle.
- * * 19. Two circles whose centres are R and S touch each other in the point A, and a direct common tangent touches them at B and C respectively; BRD, CSE are diameters of the two circles. Show that BE, CD intersect in A.
[HINT. Join BA, CA, AE. Prove BA and AE in the same straight line. Similarly, etc.]
- * * 20. The centres A and B of two circles which touch each other externally are fixed, but their radii vary; show that each of their direct common tangents touches a fixed circle.
- * * 21. Two circles touch externally at T. PQ is a chord of the one which when produced touches the other at R. Prove that RT must bisect the exterior angle PTQ.

Miscellaneous.

- * * 22. AOB is the diameter of a circle whose centre is O, and on AO and BO as diameters are also described circles. Construct geometrically a circle to touch the larger circle internally and the smaller circles externally, and also calculate its radius.
- * * 23. Each of two equal circles (radius R) passes through the centre of the other. Calculate the radius of a third circle which touches each of the two given circles and the line of centres. (Consider two cases.)
- * * 24. Two circles with given radii R and r have their centres c apart. Give the construction for drawing a circle, with radius equal to k , to touch the two given circles. ($R > r$.)
If $c > R + r$ and $k = \frac{1}{2}(c - R + r)$, how many solutions are there?

CHAPTER XXIX.

RECTANGLES.



§ 1. The rectangle PQRS is said to be **contained** by any two adjacent sides. [Since these two sides *completely* determine the rectangle, it is sufficient. Compare the brief expression “a carpet 15 by 10.”] It is written **rect. PQ . PS** ; and, when there can be no ambiguity, simply **PQ . PS** is enough. It is frequently necessary to discuss rectangles, which are not drawn at all, and the rectangle XY . XZ is considered.



The square on XY is usually written **sq. on XY** or, more shortly still, **XY²**.

§ 2. A line may be divided **internally** or **externally** ; it is necessary to produce the line in the latter case. In the figure above XZ is divided (internally) at Y, whereas XY is divided (externally) at Z. In the former case XY and YZ are called **segments** of XZ. In the latter case XZ and ZY (preferably ZY, and not YZ) are called **segments** of XY.

§ 3. **Algebra** should be used freely in many examples. It affords a shorter and more convenient notation. Write $(x+y)^2$ rather than AB^2 or the square on AB . Write xy rather than $AC \cdot CB$ or the rect. $AC \cdot CB$.



The solution of an **Equation** frequently solves a problem, and often a $\pm$ has to be **interpreted**.

The following example will make the point clear :

Where must a line, 5" long, be divided so that the square on one part is equal to the rectangle contained by the whole line and the other part ?



AB is the line. P is the (supposed) point. Mark x and $5-x$ for the segments of the line.

By hypothesis, $x^2 = 5(5-x)$.

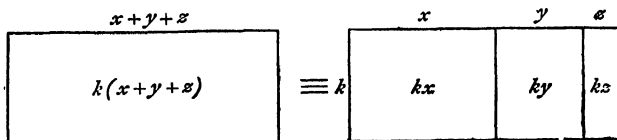
Solving this quadratic equation, we get $x = 3.09\dots$ or $-8.09\dots$.

It is easy to interpret $3.09\dots$ for the length of AP .

The $-8.09\dots$ means that Q , measured **backwards**, is another possible position and AQ is $-8.09\dots$.

§ 4. The **illustration of Algebraical Identities** is by diagrams (preferably two diagrams for each identity). The main thing is that the diagrams should be clear, so that neatness is essential.

§ 5. Illustration A.



The figures illustrate the identity $k(x+y+z) \equiv kx + ky + kz$.

[Similarly figures can be drawn to illustrate the identity

$$k(x+y+z+\dots) \equiv kx + ky + kz + \dots]$$

EXAMPLES 29 (RIDERS).

1. Draw a double diagram to illustrate geometrically the algebraical identity $a(a+b) \equiv a^2 + ab$.

2. Illustrate the identity $(x+y)^2 \equiv x(x+y) + y(x+y)$.

* 3. A, B, C and D are four points, in order, on a straight line. Prove $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

[HINT. Let AB, BC, CD be respectively x , y , z . Use Algebra.]

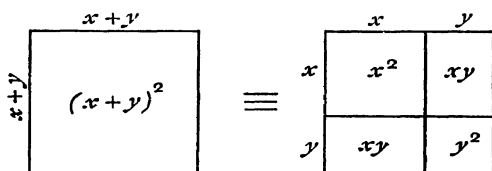
* 4. ACBD is a straight line such that $AC = CB$ and $BD^2 = 2AC^2$; show that $AD \cdot DB = CD \cdot AB$.

[HINT. $AC = CB = x$, $BD = y$, and $y^2 = 2x^2$. Use Algebra.]

CHAPTER XXX.

SQUARES ON SUM AND DIFFERENCE.

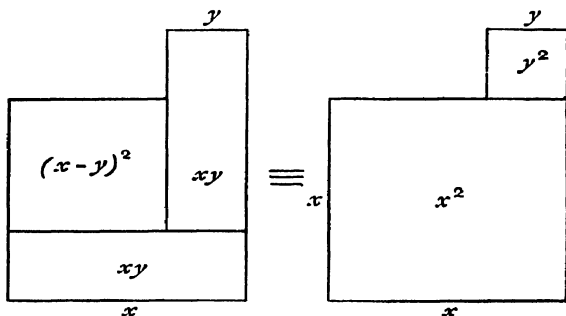
§ 1. Illustration B.



The figures illustrate the identity

$$(x + y)^2 \equiv x^2 + 2xy + y^2.$$

§ 2. Illustration C.



The figures illustrate the identity

$$(x - y)^2 + 2xy \equiv x^2 + y^2,$$

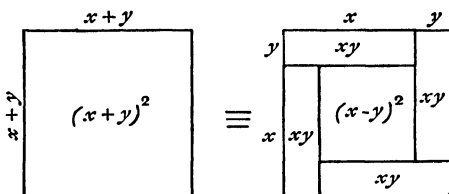
that is

$$(x - y)^2 \equiv x^2 - 2xy + y^2.$$

- § 3. Since $(x + y)^2 \equiv x^2 + 2xy + y^2$
 and $(x - y)^2 \equiv x^2 - 2xy + y^2$,
 (1) by addition, $(x + y)^2 + (x - y)^2 \equiv 2x^2 + 2y^2$; *
 (2) by subtraction, $(x + y)^2 - (x - y)^2 \equiv 4xy$.

The following figures illustrate the second identity, namely

$$(x + y)^2 \equiv (x - y)^2 + 4xy.$$



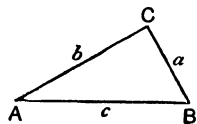
EXAMPLES 30 (RIDERS).

1. Prove that the sum of the squares on each of two parts of a line is less than the square on the whole line by twice the rectangle contained by those parts.

[HINT. Call the parts of the line x and y , so that the whole line is $x + y$. Use Algebra.]

2. Show that the square on the sum of the two sides of a right-angled triangle exceeds the square on the hypotenuse by four times the area of the triangle.

[HINT. If C is the rt. angle, we have to prove $(a + b)^2 = c^2 + 4 \text{ area } \triangle ABC$. What is area $\triangle ABC$ in terms of a and b ?]



* 3. ABC is an isosceles triangle having $AB = AC$; if BM be drawn perpendicular to AC , show that $BM^2 = CM^2 + 2AM \cdot MC$.

[HINT. $BM^2 = AB^2 - AM^2 = AC^2 - AM^2$, etc.]

* 4. If AB be divided at C so that $AC^2 = 2BC^2$, show that

$$AB^2 + BC^2 = 2AB \cdot AC.$$

[HINT. $AC = x$, $CB = y$ and $x^2 = 2y^2$. Use Algebra.]

* This conclusion is arrived at in another way in Chapter XXXII. § 6 on p. 357.

- * 5. C is a point on a given straight line AB ; if AC be bisected at M and BC at N, show that $AN^2 + 3BN^2 = 3AM^2 + BM^2$.

[HINT. Mark the pieces of the line x, x, y, y . Use Algebra.]

- * 6. Show that the area of a rectangle of given perimeter is greatest when the rectangle becomes a square.

[HINT. Call the sides of the rectangle x and y , and use the identity $4xy = (x+y)^2 - (x-y)^2$.]

- * 7. If ABC is an equilateral triangle, and BC is produced to D, making CD equal to BC, and AD is joined ; prove that the square on AD is three times the square on BC.

- * 8. A straight line AB is divided in C so that $AC^2 = AB \cdot CB$. In AC is taken a point D, so that $AD = BC$. Prove that $AD^2 = AC \cdot DC$.

- * 9. ABC is an isosceles triangle having $AB = AC$, and BD is drawn perpendicular to AC ; show that $BC^2 = 2AC \cdot CD$.

- * * 10. ABC is a triangle, D is the foot of the perpendicular from A on BC. If $AB^2 = AC^2 + 3BC^2$, show that C is the mid-point of BD.

- * * 11. A, B are two points on the same side of, but at different distances from, a straight line PQ. Find a point R in PQ such that the sum of the squares on AR, RB is a minimum.

[HINT. First draw perpendiculars to PQ from A and B.]

- * * 12. ABCD is a quadrilateral having AB parallel to DC ; prove that $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot DC$.

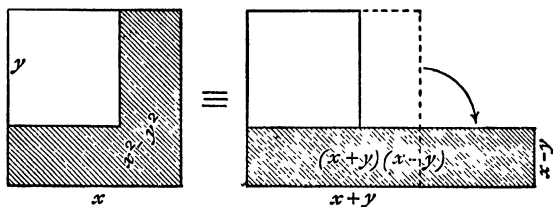
- * * 13. O is the centre of the circle ABDC. The chords AB and CD produced meet at right angles at T. Prove that

$$2AC^2 + 2BD^2 = AB^2 + CD^2 + 4OT^2.$$

CHAPTER XXXI.

DIFFERENCE OF SQUARES.

§ 1. Illustration D.

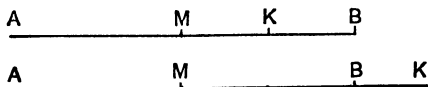


The figures illustrate the identity

$$x^2 - y^2 \equiv (x + y)(x - y).$$

This identity is especially useful in logarithmic calculation.

§ 2. The line AB is bisected at M and divided unequally (whether internally or externally) at K.



$$AK = x,$$

$$KB = y;$$

$$\therefore \text{ by addition, } AB = x + y,$$

$$\text{so that } AM = \frac{1}{2}(x + y).$$

$$\text{Now } MK = AK - AM = x - \frac{1}{2}(x + y) = \frac{1}{2}(x - y).$$

The facts

$$AM = \frac{1}{2}(AK + KB) \quad \text{and} \quad MK = \frac{1}{2}(AK - KB) \quad \text{are useful.}$$

§ 3. In riders to work with squares is easier than with rectangles, so be inclined to exchange the rectangle AK . KB for $AM^2 \sim KM^2$.

EXAMPLES 31 (RIDERS).

1. ABC is an isosceles triangle, and X is any point in the base BC. Prove that $AC^2 = AX^2 + BX \cdot XC$.

[HINT. Call the equal sides b , and the portions of the base x and y . You will need M the middle point of the base BC.]

2. P is a point on the diagonal AC of a rhombus ABCD ; show that the difference of the squares on AB, BP is equal to the rectangle AP . PC.

[HINT. Just like the preceding rider, for $\triangle ABC$ is necessarily isosceles.]

3. AB is a straight line bisected at C and produced to D, prove that

$$AD^2 = BD^2 + 2AB \cdot CD.$$

4. A variable straight line cuts one of two concentric circles at A, B and the other at C, D ; show that the rectangle AC . BC is constant.

* 5. If XY be drawn parallel to the base BC of an isosceles triangle ABC, then the difference of the squares on BY and CY is equal to the rectangle contained by BC and XY.

* 6. From a point M on a chord AB of a circle, whose centre is O, a line is drawn at right angles to OM, meeting the circle at N. Prove that $MN^2 = AM \cdot MB$.

* * 7. A triangle ABC is such that $AB^2 - AC^2$ is constant where B and C are fixed points ; prove that the locus of A is a straight line perpendicular to BC.

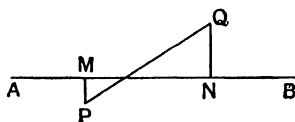
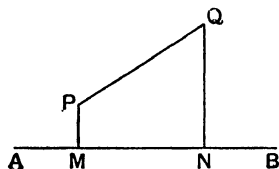
* * 8. The base BC of a triangle ABC is bisected at D and produced to E so that $CE = CD$. Prove that the difference of the squares on AB and AE is equal to three times the difference of the squares on AD and AC.

CHAPTER XXXII.

PROJECTIONS, ETC.

§ 1. If PM is drawn perpendicular to AB , the point M is called the **projection** of the point P on AB .

The line MN is the projection of the line PQ on AB .

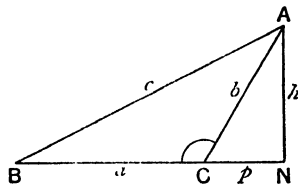


PROPOSITION 29.

§ 2. General Enunciation. *In an obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of those sides and the projection of the other on it.*

Particular Enunciation. ABC is the triangle. $\angle ACB$ is obtuse. AN is perpendicular to BC produced. To prove AB^2 is greater than BC^2 and AC^2 by twice the rectangle BC . CN.

$$i.e. \quad c^2 = a^2 + b^2 + 2ap.$$



Proof.

$$\begin{aligned} c^2 &= (a + p)^2 + h^2 \text{ (Pyth., } \triangle ABN) \\ &= a^2 + 2ap + p^2 + h^2 \\ &= a^2 + 2ap + b^2 \text{ (Pyth., } \triangle ACN) \end{aligned}$$

$$\text{or } c^2 = a^2 + b^2 + 2ap.$$

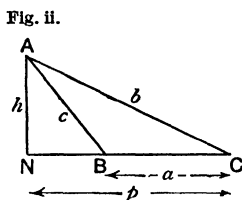
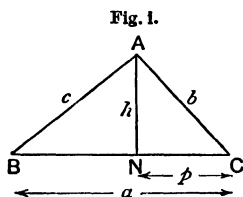
Q.E.D.

PROPOSITION 30.

§ 3. General Enunciation. *In any triangle, the square on the side opposite an acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by one of those sides and the projection of the other on it.*

Particular Enunciation. ABC is the triangle. $\angle C$ is acute. AN is perpendicular to BC (produced if necessary). To prove AB^2 is less than CB^2 and CA^2 by twice the rectangle BC . CN.

$$\text{i.e. } c^2 = a^2 + b^2 - 2ap.$$



Proof. In Fig. i. $BN = a - p$. In Fig. ii. $NB = p - a$.
 $\therefore$ in both cases $BN = a \sim p$.

[$a \sim p$ means the difference between a and p .]

$$\begin{aligned} c^2 &= (a \sim p)^2 + h^2 \text{ (Pyth., } \triangle ABN) \\ &= a^2 - 2ap + p^2 + h^2 \\ &= a^2 - 2ap + b^2 \text{ (Pyth., } \triangle ACN) \end{aligned}$$

$$\text{or } c^2 = a^2 + b^2 - 2ap.$$

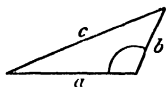
Q.E.D.

§ 4. Props. 29 and 30 are an **extension of Pythagoras' Theorem**.

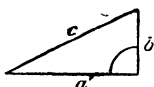
Square opp. obtuse $\angle$.

Square opp. rt. $\angle$.

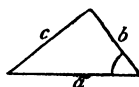
Square opp. acute $\angle$.



Case 1.



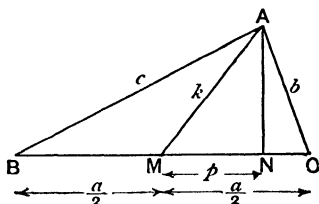
Case 2.



Case 3.

If the sides a and b are kept fixed in length, the square on c is greater than, equal to, or less than the sum of the squares on a and b , according as the angle included between the sides a and b is greater than, equal to, or less than a right angle.

* § 5. The combination of these two propositions leads to a most important property of a **median** of a triangle.



ABC is the triangle. The base BC is bisected at M, so that AM is a median. AN is perpendicular to BC.

$$\text{In } \triangle AMB, \quad c^2 = \left(\frac{a}{2}\right)^2 + k^2 + 2\left(\frac{a}{2}\right)p.$$

$$\text{In } \triangle AMC, \quad b^2 = \left(\frac{a}{2}\right)^2 + k^2 - 2\left(\frac{a}{2}\right)p.$$

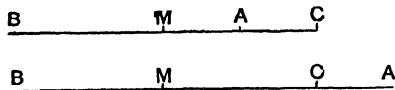
$$\text{By addition,} \quad c^2 + b^2 = 2\left(\frac{a}{2}\right)^2 + 2k^2,$$

$$\text{or} \quad BA^2 + AC^2 = 2BM^2 + 2AM^2.$$

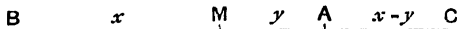
In words this becomes : *The sum of the squares on two sides of a triangle is equal to twice the square on half the base and twice the square on the median which bisects the base.* This is known as **Apollonius' Theorem**.

* § 6. It is to be noted that the vertex, A, of the triangle is *any* point. If that point, A, were actually in BC (produced if necessary), we get

$$BA^2 + AC^2 = 2BM^2 + 2MA^2.$$



In words this becomes : *If a straight line is divided equally and unequally (whether internally or externally), the sum of the squares on the unequal parts is equal to twice the square on half the line and twice the square on the line between the points of section.*



That is $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2.*$

EXAMPLES 32 a (RIDERS).

1. ABC is a triangle with the angle ACB obtuse ; AD is drawn perpendicular to BC to meet BC produced at D. If BE is also drawn perpendicular to AC to meet AC produced in E, prove that $BC \cdot CD = AC \cdot CE$.

2. ABC is *any* triangle ; BE, CF are drawn perpendicular to the opposite sides. Prove that rectangle $AF \cdot AB = AE \cdot AC$.

3. ABC is an isosceles triangle with an obtuse angle at C ; prove that if AD be drawn at right angles to BC produced, the square on AB is equal to twice the rectangle $BC \cdot BD$.

4. The base BC of an isosceles triangle ABC is produced to D so that $CD = BC$; prove that $AD^2 = AC^2 + 2BC^2$.

5. Each of two equal circles passes through the centre of the other ; if A, B be their centres and C, D their points of section, and if CA produced cut one circle again in P and AB produced cut the other in Q, show that $PQ^2 = 7AB^2$.

* This result was arrived at in a different way in Chapter XXX. on p. 349.

6. C being the obtuse angle of an obtuse-angled triangle ABC , D and E the feet of the perpendiculars from A and B on the opposite sides, prove that $AB^2 = BC \cdot BD + AC \cdot AE$.

7. If squares $ABDE$, $ACFG$ be described outwards on the sides AB , AC of *any* triangle ABC , show that the sum of the squares on EG , BC is double the sum of the squares on AB , AC .

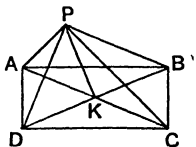
8. ABC is a right-angled triangle. BC is the hypotenuse. If $BEFC$, $CGHA$ and $ANMB$ are the squares described on the respective sides (externally to the triangle), prove that

$$AE^2 + AF^2 = 3BC^2 + 8(\text{area of triangle } ABC) \text{ and } EM^2 + FG^2 = 5BC^2.$$

Medians.

- * 9. $ABCD$ is a rectangle and P any point; show that $PA^2 + PC^2 = PB^2 + PD^2$.

[This rider has been proposed under Pythagoras' Theorem, but it is neater to consider PK a median of $\triangle PAC$, and again PK as a median of $\triangle PBD$.]



- * 10. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

[HINT. Recollect that the diagonals of a parallelogram bisect each other.]

- * 11. $ABCD$ is a square inscribed in a circle and P is any point on the circumference: prove that the sum of the squares on PA , PB , PC and PD is twice the square on the diameter of the circle.

[HINT. Regard PAC as one triangle and PBD as another.]

- * 12. If AP , BQ , CR are the medians of the triangle ABC , prove that $3(BC^2 + CA^2 + AB^2) = 4(AP^2 + BQ^2 + CR^2)$.

- * 13. Find the locus of a point P which moves in a plane such that $AP^2 + BP^2 = a$ constant, where A and B are fixed points in the plane in which P moves.

[HINT. If M is the middle point of AB , then PM is a median of $\triangle PAB$.]

- * 14. X , Y are two fixed points on the circumference of a given circle. Find the position of a point P on the circumference, if $PX^2 + PY^2$ is a maximum.

- * 15. BC is divided into three equal parts BD, DE, EC, and A is any point (not necessarily in BC) ; prove that

$$AB^2 + AC^2 = AD^2 + AE^2 + 4DE^2.$$

- * 16. ABCD is a parallelogram : R is any point. Prove that the difference between $AR^2 + CR^2$ and $BR^2 + DR^2$ is independent of the position of R.

- * 17. ABCD is a parallelogram and P any point ; show that

$$2(AP^2 + PC^2) + BD^2 = 2(BP^2 + PD^2) + AC^2.$$

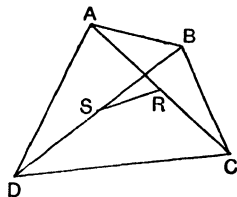
- * 18. Show that the difference of the squares on two sides of a triangle is three times the difference of the squares on the lines joining the vertex to the points of trisection of the base.

- * 19. Show that the sum of the squares on the sides of any quadrilateral exceeds the sum of the squares on the diagonals by four times the square on the line which joins the middle points of the diagonals.

What does this theorem become when the quadrilateral is a parallelogram ?

[HINT. AS is median of $\triangle ABD$,
CS " " $\triangle CBD$.

Add. Similarly, etc.]



- * 20. D is the point of trisection of the side BC of a triangle ABC which is nearer to B. Show that

$$6AB^2 + 3AC^2 = 9AD^2 + 2BC^2.$$

- * 21. The hypotenuse AB of a right-angled triangle ABC is trisected at D and E ; show that $CD^2 + CE^2 = 5DE^2$.

- * * 22. If two chords of a circle intersect at right angles within the circle, show that the sum of the squares on the four segments of the chords is equal to the square on a diameter of the circle.

- * * 23. If CD be any chord parallel to the diameter AB of a semicircle, and if P be any point in AB, prove that

$$CP^2 + PD^2 = AP^2 + PB^2.$$

- * * 24. If the medians of a triangle ABC meet at G, prove that

$$3(AG^2 + BG^2 + CG^2) = BC^2 + CA^2 + AB^2.$$

- * * 25.** ABC is a triangle and P any point in its plane ; prove that $AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3GP^2$, where G is the point of intersection of the medians of the triangle ABC.

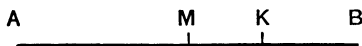
EXAMPLES 32 b (ON CHAPTERS XXIX-XXXII).

MISCELLANEOUS.

Square roots are to be taken correct to 3 significant figures, and calculations are to be made accordingly.

Use Algebra freely.

1. M is the middle point of AB, and the point K divides AB unequally, so that $AK = 19''$ and $KB = 7''$. How long are AM and MK ?



2. M is the middle point of PQ, and $PL = 65$ cm. and $QL = 15$ cm. How long are PM and ML ?



3. How must a line 100 cm. long be divided so that the rectangle contained by the two segments has an area of 2000 sq. cm. ? Interpret both answers.

4. How must a line 1 inch long be divided so that the square on one part is equal to the rectangle contained by the whole line and the other part ? Interpret the double answer.

[Answers to the hundredth of an inch.]

5. How far must a line 34 in. long be produced so that the rectangle contained by the whole line when produced and the part produced may have an area 131 sq. in. ? What is the meaning of the double answer ?

6. How should a line 32 inches long be divided so that the square on the whole line exceeds the sum of the squares on its two parts by 204 sq. in. ? Interpret the apparent two solutions.

7. State whether the triangles, whose sides are given below, are right-angled, obtuse-angled or acute-angled :

- (1) 33", 30", 14" ;
- (2) 50 cm., 28 cm., 41 cm. ;
- (3) 101 ft., 99 ft., 20 ft. ;
- (4) $a^2 + 2$, $2a$, $a^2 - 2$.

8. Show that a triangle, with sides $n^2 + 1$, $n^2 - 1$, $2n$, must be right-angled whatever value n may have. [$n > 1$.]

9. A triangle has sides 29 cm., 25 cm. and 6 cm. How long is the projection of the 6 cm. side on the 25 cm. side (produced) ?

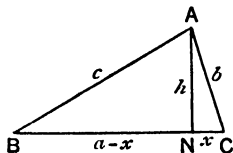
10. A triangle has sides 28 cm., 25 cm. and 17 cm. What shape is it ? Calculate the length of the projection of the 25 cm. side on to the 28 cm. side.

11. Find the length of the projection of the side AC on to the side BC of $\triangle ABC$.

[HINT. Solve two equations :

$$x^2 + h^2 = b^2$$

$$\text{and } (a-x)^2 + h^2 = c^2.]$$



12. Using the preceding results, find an expression (involving a , b and c) for the area of a triangle whose sides are known. Simplify this expression by putting $2s$ instead of $a + b + c$.

* 13. A triangle has sides 6, 8 and 12 inches. The longest of its medians corresponds to which side bisected ? Calculate the length of the projection of the longest median on each of the other two sides.

* 14. Calculate the lengths of the medians of a triangle whose sides are 8", 10" and 12".

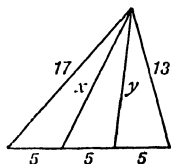
* 15. Calculate the lengths of the medians of a triangle whose sides are a , b and c .

* 16. If the lengths of the medians of a triangle are 10 cm., 11 cm., 12 cm., calculate the lengths of the sides of the triangle.

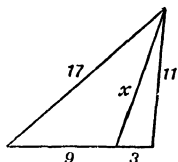
* 17. The medians of a triangle are l , m , n ; find the lengths of the sides.

- * 18. Calculate x and y .

[HINT. x is a median of one triangle and y of another.]



- * 19. Calculate x .



- * 20. In a triangle ABC , $a=42$, $b=26$, $c=40$. AD is a median. M is the mid-point of AD . Calculate CM .

- * * 21. Two circles intersect at A and B ; and a common tangent touches them in C and D . Prove that

$$AC^2 + AD^2 + BC^2 + BD^2 = AB^2 + CD^2 + 4EF^2,$$

where E and F are the middle points of AB and CD .

- * * 22. AB is divided at C so that $AC^2 = AB \cdot CB$, and AC is divided at D so that $AD^2 = AC \cdot DC$. Prove that $AD = CB$.

- * * 23. Produce a given straight line AB to C so that the rectangle $AC \cdot CB$ may be equal to the square on another given straight line.

CHAPTER XXXIII.

RATIO AND PROPORTION.

§ 1. We are accustomed to compare two things in respect of their size, and such expressions as “ I am twice as heavy as you ” or “ This article costs three times as much as that ” are of every day occurrence. We naturally compare things of the *same* kind, and the impossibility of comparing things *not* of the same kind is apparent when the absurdity of comparing 2 dogs and 4 chairs is pointed out.

Ratio is the mutual relation to one another of 2 quantities *of the same kind* with regard to magnitude.

The ratio of a to b is written $a : b$ or a/b or $\frac{a}{b}$, either of the two former ways is to be preferred to the fraction if clearer type is thereby used. [For the notation, compare $11\frac{3}{4}d.$ and $11\frac{3}{4}d.$ for elevenpence three farthings.]

a is called the **antecedent** and b the **consequent**. $a^2 : b^2$ is called the **duplicate ratio** of $a : b$.

§ 2. If two ratios are equal, the four quantities are said to be in **Proportion**. If a, b, c, d are in proportion, it is written $a : b = c : d$ or $a : b :: c : d$ (rather old-fashioned) or $a/b = c/d$ or $\frac{a}{b} = \frac{c}{d}$ (often smaller type, and, if so, perhaps not so clear as the ‘dots’). It is read “ a is to b as c is to d .” d is called the **fourth proportional** to a, b and c . [We can consider the proportion

$$10 \text{ pence} : 5 \text{ pence} = 6 \text{ houses} : 3 \text{ houses},$$

for each is merely the 2 to 1 ratio.]

If the ratios consist of more than 2 things, a similar notation is used; *e.g.* $x : y : z = a : b : c$. Proportions consist of an even number (at least 4) of things. **If three things are in proportion, it is understood that the second is repeated.** Thus 5, 10, 20 are in proportion, since $5 : 10 = 10 : 20$.

[The word "proportion" is often used when strictly the term "ratio" would be more correct; but in any special case no mistake is likely to be made, and so it does not matter. The expression "A mixture in the proportion of 2 to 1" is common enough.]

If a, b, c are in proportion, c is said to be a **third proportional** to a and b . And b is said to be a **mean proportional** between a and c .

The numbers 3, 6, 12, 24, 48, ... are said to be in **continued proportion** [$3 : 6 = 6 : 12 = 12 : 24 \dots$]; continued proportion is the same as geometrical progression.

§ 3. The following are tacitly assumed in elementary geometry. They are necessary facts with which we are familiar in Algebra :

$$\text{if } a : b = c : d,$$

then (1) $a : c = b : d$, **alternando or alternately,**

(2) $b : a = d : c$, **invertendo or inversely,**

(3) $a + b : b = c + d : d$,

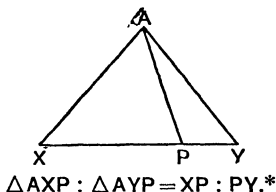
(4) $a - b : b = c - d : d$,

(5) $a + b : a - b = c + d : c - d$,

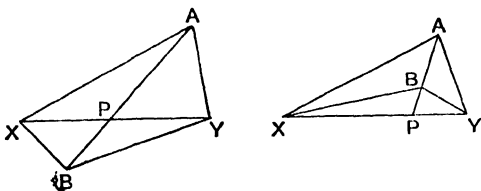
(6) $ad = bc$, **the product of extremes = product of means.**

[*N.B.*—Do not mix up the ratio sign (double dot) with the single dot used between the sides of a rectangle.]

§ 4. It is important that the **Areas of Triangles of the same height vary as their bases**. Much the most frequent case is "bases in the same straight line and the heights automatically equal, for the triangles have a common vertex."



A useful extension of the above is when the bases are the same and heights different (frequently on opposite sides of the base).



Here, for instance, $\Delta AXY : \Delta BXY = AP : PB$.

EXAMPLES 33 (CHIEFLY RIDERS).

1. A point P is taken in the diagonal BD of a parallelogram ABCD, such that $BP : PD = 2 : 5$; show that $\Delta ABP : \Delta ABC = 2 : 7$.

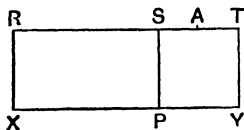
[HINT. Consider ΔABD instead of ΔABC . It has the same area.]

What is the numerical relation, $\Delta ABP : \Delta APD$?

Then what is the numerical relation, $\Delta ABP : \Delta ABD$?]

* It is intuitive that the rectangles RSPX and STYP are proportional to their bases XP and PY, and the ΔAXP and ΔAYP are equal in area to half these rectangles.

(You would naturally join AX, AP and AY to get these triangles.)



2. ABC is a triangle and M the middle point of BC ; find a point P in AB, such that the area of MBP is one-fifth of ABC.

[HINT. Imagine P correctly placed.

$$\begin{aligned}\triangle BPM &\text{ has got to be } \frac{1}{5} \triangle ABC ; \\ \therefore \triangle BPM &\text{ should be } \frac{2}{5} \triangle ABM.] \end{aligned}$$

3. ABC, ADC are right-angled triangles on opposite sides of their hypotenuse AC, which is 2.6" long. AB=BC and AD=1". Find by calculation the lengths of AB, BC, CD, and of the perpendiculars BM, DN dropped from B and D on to AC. What is the value of AN : NM : MC ? Check by drawing a figure.

4. X is a point in the side BC of a triangle ABC, and O is a point in AX. If BX : XC = 2 : 3, and XO : OA = 3 : 4, prove that $\triangle OBC : \triangle OAB = 15 : 8$. What ratio does $\triangle OAB$ bear to $\triangle ABC$?

5. Through a point O within a triangle straight lines are drawn from the angles A, B, C cutting the opposite sides in D, E, F respectively ; show that $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$.

[HINT. First prove $\frac{OD}{AD} = \frac{\triangle BOC}{\triangle ABC}$, etc.]

* 6. Prove that any triangle is divided by the three medians into six triangles of equal area.

* 7. O is a point inside the triangle ABC, such that the triangles AOB, AOC are equal in area. Prove that AO produced will bisect BC.

[HINT. If AO produced meets BC at K, first prove

$$\triangle BOK = \triangle COK \text{ in area.}]$$

* 8. AM, BN are medians of a triangle ABC, intersecting at G. In what ratio are the areas of the triangles AGB, NMG, ABC ? [Recollect that the medians of a triangle trisect each other.]

* 9. Find a point O within a triangle ABC, such that the area of BOC is double that of AOB and four times that of AOC.

[Imagine the problem solved ; if BO produced cuts CA at Y, then $\triangle BOC : \triangle AOB = CY : YA$, this gives the position of Y, etc.]

* * 10. The lines joining the angular points A and B of a triangle ABC to an internal point O cut the opposite sides at P and Q, such that BP : PC = 1 : 2 and CQ : QA = 2 : 3. If CO cut AB at R, prove that AB = 4RB.

- ** 11.** ABC is a triangle. AL, BM and CN are the perpendiculars from the angular points to the opposite sides, and they are produced to meet the circumcircle in L', M' and N' respectively.

Prove that $\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4$.

[HINT. If O is the orthocentre, recollect OL' is bisected at L, and similarly.]

CHAPTER XXXIV.

PROPORTIONAL LINES.

PROPOSITION 31.

§ 1. General Enunciation. (i) *If a straight line is drawn parallel to one side of a triangle, the other two sides must be divided proportionally ; and (ii) the converse.*

(i.)

Particular Enunciation. XY is drawn parallel to the side BC of the triangle ABC, cutting AB at X and AC at Y, to prove $AX : XB = AY : YC$.

Construction. Join XC and BY.

(See figures on opposite page.)

[*N.B.*—In Fig. i. XY cuts AB and AC ; in Fig. ii. XY cuts AB and AC produced ; in Fig. iii. XY cuts BA and CA produced. The same proof holds in all cases.]

Proof. $\triangle BXY = \triangle CXY$ in area (same base and between same ||s).

$$\therefore \triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY ;$$

but $\triangle AXY : \triangle BXY = AX : XB$ (bases in same st. line and common vertex),

and $\triangle AXY : \triangle CXY = AY : YC$ (for same reason) ;

$$\therefore AX : XB = AY : YC.$$

Q.E.D.

(ii) **Converse.**

Particular Enunciation. In the $\triangle ABC$, $AX : XB = AY : YC$, to prove XY is parallel to BC .

Construction. Join XC and BY .

Fig. i.

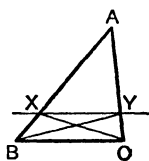


Fig. ii.

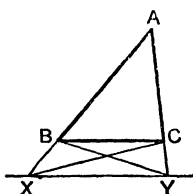
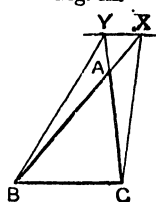


Fig. iii.



Proof. $AX : XB = \triangle AXY : \triangle BXY$ (bases in same st. line and common vertex) and $AY : YC = \triangle AXY : \triangle CXY$ (for same reason);

but $AX : XB = AY : YC$ (given);

$\therefore \triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY$;

$\therefore \triangle BXY = \triangle CXY$,

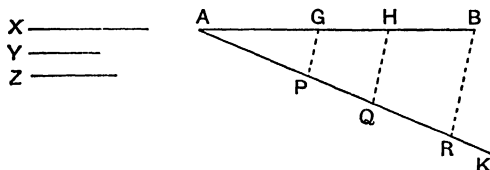
and the $\triangle BXY$ and $\triangle CXY$ are on the same base XY ;

$\therefore XY$ must be parallel to BC .

Q.E.D.

§ 2. It is the use of this proposition which affords a means of **dividing a line in a given ratio.**

AB is the line to be divided in the ratio $X : Y : Z$.



AK is any line. AP , PQ and QR are cut off equal to X , Y and Z respectively.

RB is joined.

QH and PG are parallel to RB .

The line AB is divided at G and H in the ratio $X : Y : Z$.

(Aim at good intersections.)

§ 3. The same sort of construction is used for a fourth proportional to three given lines.

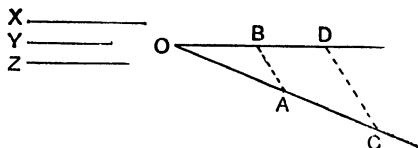
METHOD 1.

X, Y, Z are the lines. Draw any 2 lines meeting at O .

$$OA = X,$$

$$OB = Y,$$

$$AC = Z.$$



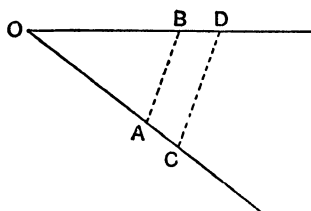
Join A and B . Draw a parallel through C to get D .
The line BD is the fourth proportional required.

METHOD 2.

If the lines are longish, the figure is apt to become unwieldy, measuring everything from O is convenient.

$$\begin{array}{l} X \text{ —————} \\ Y \text{ —————} \\ Z \text{ —————} \end{array}$$

$$\begin{array}{l} OA = X, \\ OB = Y, \\ OC = Z. \end{array}$$



The line CD is parallel to AB .
 OD is the fourth proportional required.

N.B.—To construct a **third proportional** to two given lines is the same. It only means that the second of the lines is repeated.

EXAMPLES 34 a (GEOMETRICAL DRAWING).

Dividing a line in a given ratio.

1. Draw a line 4" long. Divide it in the ratio $X : Y$. Measure the longer portion in inches. (Example 2.)

2. Draw a straight line 12 cm. long and divide it in the ratio $X : Y : Z$.
 Measure the middle-sized part in centimetres.

X _____
 Y _____
 Z _____

3. Construct a triangle with perimeter 12 cm., and with sides in the ratio $X : Y : Z$ (Example 2).

4. Divide a line 4" long into three parts in the ratio 1 : 2 : 3. Measure the shortest part in cm.

5. The letter **E** is to be made from a straight line, 16 cm. long, divided in the ratio 8 : 5 : 3 : 5 (the first piece for the upright). Construct the letter, and measure the length of the centre bar (not including the upright) in cm.

6. 4th and 3rd Proportionals.

- (1) Find a 4th proportional to A, B and C in inches.
- (2) " 4th " " C, D and E " "
- (3) " 4th " " D, E and F " centimetres.
- (4) " 3rd " " A and B " "
- (5) " 3rd " " B and A " "

A _____
 B _____
 C _____
 D _____
 E _____
 F _____

7. Determine the lengths of third proportionals (greater and less) to lengths of 3.5 cm. and 5.1 cm.

8. If $A = 7.8$ cm., $B = 9.3$ cm., $C = 8.5$ cm.; find X , Y and Z in the following cases :

- (1) $A : B = C : X$. (2) $A : Y = C : B$. (3) $B : Z = A : B$.

9. Construct a line whose length is $\frac{2.37 \times 5.27}{4.77}$ inches. Measure it in cm.

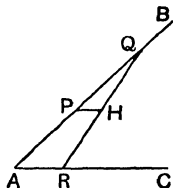
10. Construct a line whose length is $\frac{(3.75)^2}{4.21}$ inches. Measure it in cm.

11. Draw a rectangle whose sides are 2.73" and 3.14". On a straight line whose length is 2.35" construct a rectangle of the same area. Measure its diagonal.

EXAMPLES 34 b (RIDERS).

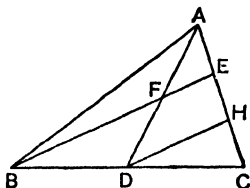
1. AB and AC are two given unlimited straight lines, and H a point between them. Draw a straight line having its ends on AB and AC, and its middle point at H. (*Note*.—Do not assume that H is equidistant from AB and AC: there is no reason why it should be.)

[HINT. Draw HP parallel to CA, and cut off PQ equal to AP. Join QH, etc.]



2. D is the middle point of the base BC of a triangle ABC. F is the middle point of AD. BF produced meets AC at E; show that E is a point of trisection of AC.

[HINT. Through D draw DH parallel to BE. Then prove CH=HE, also HE=EA.]



3. P is a point within an angle BAC, BPC a transversal meeting the sides of the angle in B and C. Show how to draw it so that BC may be *trisected* at P.

[HINT. The same sort of construction as in Question 1 is needed.]

4. AB, CD, EF are parallel straight lines; ACE, BDF cut them so that AC is $\frac{3}{4}$ CE. Prove $BD = \frac{3}{4}DF$.

5. ABC, DBC are two triangles on the same base BC and on the same side of it; through any point P in BC straight lines PE, PF are drawn parallel to BA, BD respectively and meeting CA, CD respectively in E, F; show that EF is parallel to AD.

[HINT. CE : EA = CP : PB, *why* ?

and CF : FD = CP : PB, *why* ?

$\therefore$ what ?]

6. ABC is a triangle and XY is drawn parallel to BC, cutting the sides in X and Y: YZ is drawn parallel to AB, meeting BC in Z: prove that $\frac{AX}{BX} = \frac{BZ}{CZ}$.

7. ABCD is a quadrilateral, through any point P in BD straight lines PQ, PR are drawn parallel to AB, BC respectively to cut AD, DC respectively in Q, R; prove that QR is parallel to AC.

8. Two given straight lines AB and AC meet at A, and Q is a given point on neither line. Through Q draw a straight line PQR meeting AB at P and AC at R, so that $PQ : QR = 2 : 3$.

[HINT. Same sort of construction as in Question 1.]

* 9. ABC is a triangle; D is the middle point of BC, and E divides CA into two parts, such that $CE = 2EA$; show that BE will bisect AD.

* 10. A straight line EF drawn parallel to the diagonal BD of a parallelogram ABCD cuts AB, AD, produced if necessary, in E, F respectively; show that the triangles BCE, CDF are equal in area.

* 11. If from the angle A of a parallelogram ABCD a straight line be drawn cutting the diagonal BD in E and the sides in P, Q, show that $AE^2 = PE \cdot EQ$.

* * 12. Through any point P in the side BC of a triangle ABC straight lines PD, PE are drawn parallel to AB, AC respectively and cutting the sides at D, E; show that the triangle PDE is a mean proportional between the triangles PBE, PCD.

* * 13. AB, CD, EF are three straight lines which meet (or which meet when produced) in a point; if AC, CE be parallel to BD, DF respectively, show that AE, BF are also parallel

* * 14. ABCD is a rhombus, N is the mid-point of CD and M the mid-point of AD. BN, BM meet AC in F and E respectively. Prove that $AE = EF = FC$.

* * 15. From any point in the base of a triangle straight lines are drawn parallel to the sides. Find the locus of the point of intersection of the diagonals of all parallelograms so formed.

CHAPTER XXXV.

SIMILAR FIGURES.

§ 1. From our early childhood we are familiar with similar figures. Any map of a country is intended to show that country of exactly the same shape, but, of course, reduced in size. It is to be noticed that

- (1) **Every pair of corresponding sides are proportional.**
- (2) **All corresponding angles are equal.**

Formal geometry is merely the means of finding out the minimum number of things which must be known to ensure these two facts being obtained.

A study of the formal definition which follows, and the accompanying figures, should convince you of the necessity for the **two** conditions.

Definition. Two figures are said to be similar if they have their corresponding angles equal **and** their corresponding sides in proportion.

Original.



Not similar to original.

Corresponding angles equal,
but corresponding sides not in
proportion.



Not similar to original.

Corresponding sides in
proportion, **but** corre-
sponding angles not equal.



Similar to original.

Corresponding angles equal and
corresponding sides in proportion.

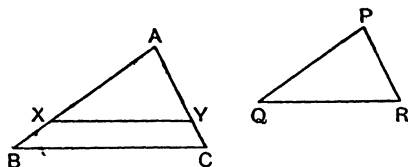


§ 2. That two **Polygons** should be **similar** these **TWO** conditions must obtain, but in the case of **Triangles** **ONE** condition is a necessary sequel to the other. The 3 separate cases, angles, sides, two sides and included angles (corresponding to the congruent triangles with which we are familiar in Vol. I.), are considered in the next 3 propositions, and the truths they embody are of great practical importance. All scale drawing assumes them, and scattered throughout the book are examples in which scale drawing is necessary.

PROPOSITION 32.

§ 3. General Enunciation. *If two triangles are equiangular their corresponding sides must be in proportion ($\therefore$ the triangles must be similar).*

Particular Enunciation. ABC and PQR are the equiangular triangles ($\angle A = \angle P$, etc.). To prove $AB : AC = PQ : PR$, etc.



Construction and Proof. Place $\triangle PQR$ in the position AXY (this is possible since $\angle A = \angle P$),

so that $\angle AXY = \angle ABC$,

and these are corresponding ;

$\therefore XY$ is parallel to BC ;

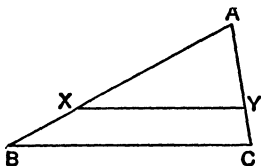
$\therefore AB : AX = AC : AY$;

alternately $AB : AC = AX : AY$;

$\therefore AB : AC = PQ : PR$.

By putting $\triangle PQR$ into other corners of $\triangle ABC$ other corresponding sides can be proved in proportion. **Q.E.D.**

§ 4. A necessary and valuable sequel to Proposition 31, which is obtained from **Proposition 32**, is: If XY is parallel to one side BC of the $\triangle ABC$, then $XY : BC = AX : AB$.



* § 5. **The Graphical Adjustment of Traverses** makes use of this.

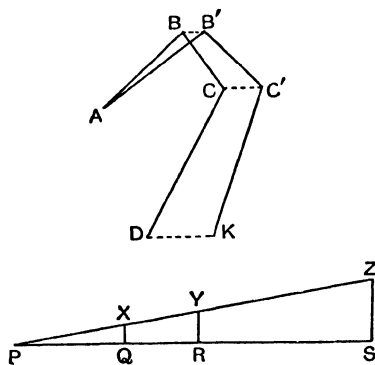
It frequently happens that travellers in entirely unmapped, or inadequately mapped, regions bring home valuable information from traverses carried out with light and non-bulky instruments, even estimating distances by the time taken in traversing them and obtaining angles with a prismatic compass. If the traverse begins and ends at known spots, the adjustment is as follows :

TRAVELLER'S NOTES.

Lap.	Bearing in Degrees.	Distance in Miles.
AB	50	15
BC	145	10
CD	200	23

The Traveller's Traverse ABCD is plotted. Suppose K is the correct position of the end. Join DK, and draw parallels through C and B to it.

Now separately draw a straight line PQRS, so that PQ=AB, QR=BC, RS=CD. At S erect a perpendicular SZ equal to DK. Join PZ. The perpendiculars* QX and RY give the corrections to B and C, and AB'C'K is the *adjusted* traverse. There is no suggestion



that the *adjusted* traverse is necessarily the *correct* traverse, it only spreads out the inevitable errors in estimating distances and a certain vagueness in possible differences in variation of the compass *en route*. However, the *adjusted* traverse is consistent, and is likely to be more nearly correct.

* Perpendiculars make good intersections, but theoretically QX, RY and SZ can be any parallels.

§ 6. The fact that equiangular triangles must be similar is of great practical importance. All **Maps** represent nature on a small scale, and practically all *great lengths are calculated after measuring angles*.

§ 7. If $a : x = x : b$, then x is called the **Mean Proportional** between a and b . Since the product of the means = the product of the extremes, $x^2 = ab$, and so $x = \sqrt{ab}$.

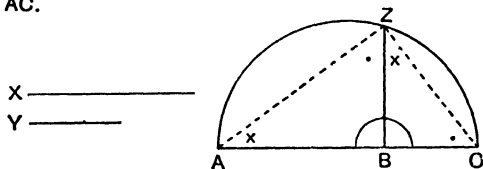
[A mean proportional between two quantities is the square root of their product—in Algebra it is called the geometric mean; while the mean, or average, of two quantities is different from this, it is half the sum of the two quantities—the arithmetic mean in Algebra.]

§ 8. To construct a **Mean Proportional** between 2 given straight lines will be found the essence of area problems that we shall consider later. The construction of a mean proportional depends upon equiangular triangles.

A mean proportional between X and Y is required.

METHOD 1.

$AB = X$, $BC = Y$. Semicircle on AC as diameter. BZ perpendicular to AC .



$\angle A = \text{complement of } \angle AZB$;

$\therefore \angle A = \angle BZC$ (using angle in semicircle);

also $\angle ABZ = \text{rt. } \angle = \angle ZBC$;

$\therefore \angle BZA = \angle C$

(third angles of $\triangle ABZ$ and $\triangle ZBC$),

so that $\triangle ABZ$ is equiangular to $\triangle ZBC$;

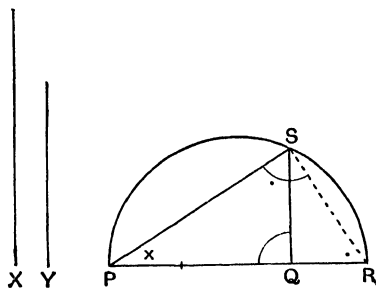
$\therefore AB : BZ = BZ : BC$,

so that BZ is a mean proportional between X and Y .

[Notice $\underline{BZ}$ is a mean proportional between $\underline{BA}$ and $\underline{BC}$.]

METHOD 2.

$PR = X$, $PQ = Y$. Semicircle on the longer as diameter. QS perpendicular from the end of the shorter.



$\angle PQS = \text{rt. } \angle$ (const.),

$\angle PSR = \text{rt. } \angle$ (angle in semicircle);

$\therefore \angle PSR = \angle PQS$,

and $\angle P$ common to Δ s PSR and PQS ;

$\therefore \angle R = \angle PSQ$ (third angles of same Δ s),

so that ΔPSR is equiangular to ΔPQS ;

$\therefore PR : PS = PS : PQ$,

so that PS is a mean proportional between X and Y .

[Notice PS is a mean proportional between PQ and PR .]

[As a matter of fact, Method 2 is the more useful in practice; and you should have no difficulty in remembering which lines, if you notice all three lines PQ , PR , PS , start from the point P . If the lines X and Y are at all long, this method is all the more convenient.]

EXAMPLES 35 a (GEOMETRICAL DRAWING).**Mean Proportionals.**

1. Construct a mean proportional to the lines 2" and 2.8". Measure it.

2. Construct a mean proportional between A and B. Measure it in centimetres.

A _____

B _____

3. Construct a mean proportional between X and Y. Measure it in inches.

X _____
Y _____

4. If $P = 4.73''$ and $Q = 3.71''$; find Y, when $Y^2 = PQ$.

5. If $A = 3.21''$, $B = 4.31''$; find X, when $A : X = X : B$.

6. Find Z when $Z : A = B : Z$, if $A = 8.3$ cm., $B = 7.2$ cm.

7. Find, by geometrical constructions,

(1) $\sqrt{6}$ (i.e. a mean proportional between 2 and 3);

(2) $\sqrt{3.6}$ (i.e. $\sqrt{4 \times 0.9}$ or $\sqrt{3 \times 1.2}$ or $\sqrt{2 \times 1.8}$, etc.).

8. Find the value of $\sqrt{323 \times 497}$ by a geometrical construction.

* 9. Construct a Mean Proportional between $2.4''$ and $4.5''$; also, separately, construct a Mean Proportional between $3''$ and $3.6''$. Measure each.

[Note $4.5 \times 2.4 = 10.8$ and $3.6 \times 3 = 10.8$; so that the two results should agree.]

Adjustment of Traverses.

* 10. The accompanying table gives the details of a "closed" traverse. E ought to coincide with A, but the distances were got by moderate pacing and the bearings by a not-very-good prismatic compass; and the discrepancy is considerable. Plot the traverse and adjust it graphically to $AB'C'D'A$. What is the direct distance and bearing of $B'C'$?

Lap.	True Bearing in Degrees.	Distance in Yards.
AB	35	160
BC	120	160
CD	190	210
DE	315	240

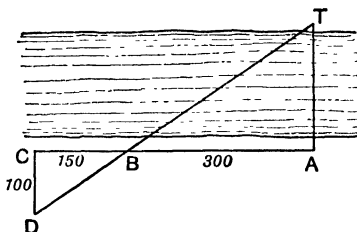
* 11. A traveller, in Africa, brings home details of a prismatic compass traverse. It began at A and ought to have finished at Z, so that AZ direct is 35.4 mi. on a magnetic bearing of 74° . Plot the traverse ABCDE, adjust it graphically to $AB'C'D'Z$. Measure the bearing and distance of the lap $B'C'$.

Lap.	Magnetic Bearing in Degrees.	Distance in Miles.
AB	55	11
BC	355	8
CD	65	11
DE	100	14

EXAMPLES 35 b (CALCULATIONS).

1. An upright 3 ft. stick casts a shadow 4 ft. long. At the same time the length of the shadow of a tree is 92 ft. How high is the tree ?

2. The width of the river is required. From A (opposite to the tree T) to B (where a stick is placed) is 300 yds. From B to C is 150 yds. A right-angle turn is made at C and CD is 100 yds., where at D both B and T appear in the same straight line. Calculate the width of the river.



3. To find the distance of an inaccessible windmill (W) from the point A, the following measurements are made :

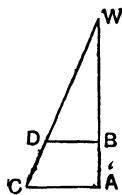
AC = 125 paces,

AB = 90 paces,

BD = 80 paces ;

also A, B, W are in a straight line and C, D, W are in a straight line, BD and AC are parallel.

Calculate AW in paces, and check by means of a figure drawn to scale.

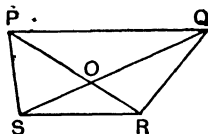


[N.B.—Angle A may or may not be a right angle ; it is immaterial.]

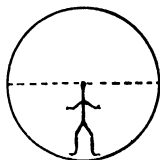
4. In the figure of Question 3, if AC = 300 yards, BD = 270 yards DC = 150 yards, find WD.

5. In the trapezium PQRS, in which PQ is parallel to SR, PQ = 105 mm., RS = 63 mm., PR = 80 mm., and SO = 42 mm. Calculate QS. (Notice that the length of PR need not be given for the calculation.)

[The accompanying figure is not meant to be accurately to scale. First draw a rough figure, like that, on which to mark dimensions. Then settle upon the procedure.]



6. The length of the barrel of a rifle is about 2 ft. 6 in. The calibre of the rifle is .303 in. The breech is opened and everything is removed, so that it is possible to look clearly down the barrel with the eye at one end. A distant man, standing upright, say 5 ft. 6 in. high, appears just to "fill up" half a diameter. How far away is the man? [Give the answer to the nearest 50 yards.]

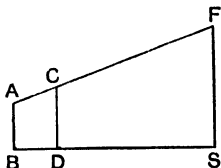


7. ABCD is a trapezium with AB parallel to DC and $\frac{5}{7}$ of it; if $BD = 18$ cm. and $CD = 35$ cm., find OB when O is the intersection of BD and AC.

8. Explain in words, with a diagram, how the height of a flagstaff may be obtained by the length of its shadow and the length of the shadow of a known stick.

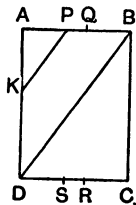
9. The height of the flagstaff FS may be obtained as in the figure, which is not drawn to scale. The heights of the sticks AB and CD are 4' 3" and 7' 9", and the sticks are 4' 6" apart. The distance from the flagstaff to the nearer, and taller, stick is 54'. BDS is straight. How high is the flagstaff?

[Hint. Through A draw APQ parallel to BDS, and work with similar triangles ACP, AFQ.]



10. To find the height of a tree, a stick is fixed upright in the ground 25 feet from the tree. A boy goes 4 feet further from the tree, and sees the top of the tree in a line with the top of the stick. The boy's eye is 5 feet from the ground, and the stick is 12 feet high. Find the height of the tree.

11. ABCD is a football ground, a yards long and b yards wide. P, Q, R, S are the goal posts. The goal is d feet wide. From P the line PK is drawn parallel to the diagonal BD. Calculate AK. [Don't mix up yards and feet.]



12. Two villages A and B are 5200 yards apart. C is a point in the straight line between them 3000 yards from A and 2200 yards from B.

The heights of A, B and C are respectively 729 feet, 403 feet, 543 feet.

There is no other high ground in the neighbourhood. Is A visible from B?

[It is such a near thing that little reliability can be put on results deduced from a scale diagram, so calculations on the lines of Question 9 are much better.]

* 13. ACB is a triangle, right-angled at C. CD is perpendicular on AB. Given $CD = 3$ in. and $AB = 10$ in., calculate BD and DA.

* 14. ABC is a triangle, B being a right angle and $AB = 3BC$. Circles are described on AB and BC as diameters. Find the length of the common chord of the circles in terms of BC.

* 15. The distance of the Marconi post (M) from A is required.

At A, an object E was in line with M.

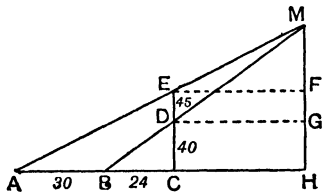
$AB = 30$ yds.

At B, an object D was in line with M, and the line ED produced is perpendicular to AB, meeting AB at C. $BC = 24$ yds., $CD = 40$ yds., $DE = 45$ yds.

[There has been no attempt to draw the figure to scale; if it were drawn to scale, the intersection at M would be so bad that results obtained by measurement on a diagram would be quite unreliable.]

Calculate AM.

[HINTS. Call MH y yards and AH x yards. Consider the similar triangles MHA and ECA, also the similar triangles MHB and DCB. Obtain two equations, and so determine x and y . Now use Pythagoras' Theorem for the triangle MHA.]



* * 16. The centres of two equal golf balls (A and B) with diameters d inches lie at distances a and b inches ($a > b$) from the centre of the hole whose diameter is D inches, and the centres of the balls and hole are collinear. Show that, if it is possible to hole the ball A without lofting it or without moving B, $a < \frac{Db}{D-2d}$. It is assumed that the ball A runs in a straight line when it is struck.

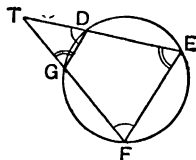
- ** 17.** A, D are goal posts at one end of a football ground, and B, C are goal posts (opposite to them) at the other end. At four collinear points W, X, Y, Z (all outside the ground) B and D, B and A, C and D, C and A appear respectively in the same straight line. If $WX=a$, $XY=b$, $YZ=c$, $BC=AD=d$, prove that the length of the ground is

$$\frac{(a+b+c)\sqrt{b^2-d^2}}{c \sim a}.$$

EXAMPLES 35 c (RIDERS).

1. D, E, F, G are points on a circle, and ED, FG meet at T. Prove that the triangle DGT is equiangular to the triangle FET, and hence similar.

[Do not generalize from a special case. In the figure GD and FE look suspiciously parallel, and they need not be so. You should draw a "general" figure.]



[HINT. $\angle TDG = \angle F$. Why?]

2. ABCD is a parallelogram; E is the middle point of AD; EB, AC are joined, intersecting in K. Prove that AK is one-third of AC.

[HINT. Consider the equiangular triangles AKE and CKB.]

3. ABCD is a parallelogram, P is a point in DC, such that $3DP=DC$. If PB and AC meet at G, find the ratio PG : GB.

[HINT. Consider the equiangular triangles PGC and BGA.]

4. The diagonal BD of a parallelogram ABCD is divided at E, so that BE is one-third of ED, and AE, DC produced meet in F. Prove that FC is double AB.

[HINT. Consider the triangles AEB and FED.]

5. If from D one of the angular points of a parallelogram ABCD, a straight line is drawn meeting AB at E, and CB at F, show that CF is a fourth proportional to EA, AD and AB.

6. Show that all isosceles triangles whose vertical angles are equal must be similar.

7. PQ is drawn parallel to the base BC of a triangle ABC, so that $2AP = PB$. Show that if BQ, CP intersect in R, $BR = 3RQ$.

Q is on AC.

[HINT. First equiangular $\triangle$ s APQ, ABC.

Then equiangular $\triangle$ s BRC, QRP.]

8. AB is the diameter of a circle whose centre is O ; the tangents at A and B are cut by the tangent at any point C in D and E ; show that $AO^2 = CD \cdot CE$.

[HINT. CO is a mean proportional between DC and CE if you can prove $\angle DOE$ a right angle.]

9. ABC is an isosceles triangle having $AB = AC$; the circle with B as centre, BC as radius, cuts AC at D. Prove that BC is a mean proportional to AC, CD.

[HINT. Prove the triangles BCD, ACB similar.]

10. From Y, the end of a straight line XY, YB is drawn, and produced half its own length to S. SX is joined, and produced half its own length to R. RB cuts XY at C. Prove that $XC = \frac{1}{4}$ of XY.

[HINT. Through S draw SL parallel to XY to cut RCB at L. Prove $SL = 3XC$; also prove $SL = \frac{1}{2}CY$, etc.]

* 11. If the bisector of the angle A of a triangle ABC meets the base in K and the circumcircle in D, prove that the square on DB is equal to the rectangle DK . DA.

[HINT. Prove $\triangle$ s BKD, ABD equiangular.]

* 12. C is a point on a circle whose diameter is AB, and CD is a perpendicular on the tangent at B. Show that the diameter of the circle is a third proportional to CD and CB.

* 13. ABCD is a trapezium with AB parallel to DC ; AC and BD intersect at O : a line through O parallel to DC cuts AD in X and BC in Y. Prove that $OX = OY$.

* 14. Two circles cut at A and B, and the tangents at A cut the circumferences at C, D. Prove that AB is a mean proportional between BC, BD.

* 15. XYZ is a triangle inscribed in a circle, XN is the altitude of the triangle and XD a diameter of the circle ; prove that $XY : XD = XN : XZ$.

[HINT. Prove $\triangle$ s XNZ, XYD equiangular.]

- * 16. The base BC of the triangle ABC is trisected at D (so that $BD = \frac{1}{3}$ of BC) and AB is bisected at E. CE and AD meet at O. Show that $DO = \frac{2}{3}OA$.

[HINT. Through D draw DF parallel to BA to meet EO at F.]

- * 17. Through M the middle point of the side BC of a triangle ABC a straight line is drawn which cuts AB, AC respectively at D, E and the line through A parallel to BC at F; show that $MD : ME = FD : FE$.
- * 18. Two triangles ABC, ABD are on the same base AB, and between the same parallels. A straight line parallel to AB cuts AC and BC in E and F, and AD and BD in G and H. Prove $EF = GH$.
- * 19. ABCD is a quadrilateral of which the sides AB, DC are parallel; prove that the line joining the middle points of AB and CD passes through the intersection of the diagonals AC, BD.
- * 20. ABC is a triangle, AD the perpendicular from A upon BC; if BM, CN be drawn perpendicular to the tangent at A to the circumscribing circle, show that $AD : BM = CN : AD$.
- * 21. Prove that the rectangle contained by the perpendiculars from any point K on the circumference of a circle, on two tangents AB, AC to the circle, is equal to the square on the perpendicular from K on their chord of contact BC.
- * 22. ABC is a triangle; find a point P in BC such that if PD be drawn parallel to AB to cut AC in D, the triangles PAB, PCD may be equal in area.
- * 23. ABC is a triangle and P is in BC, so that $2BP = PC$; and AC is bisected at E. If BE and AP meet at Q, find $AQ : QP$.
- * 24. PQ, PR are equal chords of a circle, and S is any point on the arc PR; PS and QR produced intersect in T; show that the triangles STR and SQP are equiangular.
- * * 25. Find the locus of a point which is such that its distances from two given straight lines are in a constant ratio.
- * * 26. If K be any point on the side BC of a triangle ABC, show that the radii of the circles which circumscribe AKC and AKB are proportional to AC and AB.

- * * 27. ABCD is any plane quadrilateral, and P, Q, R, S are respectively the centres of gravity of the triangles BCD, CDA, DAB and ABC; prove that the quadrilateral PQRS is similar to ABCD.

[N.B.—The C.G. of a $\triangle$ is at the point at which its medians meet.]

- * * 28. A quadrilateral has two of its sides parallel; show that the straight line which joins the middle points of its diagonals is half the difference of the parallel sides.
- * * 29. If a and b are the lengths of the parallel sides of a trapezium, find the length of a straight line (parallel to them) which bisects the trapezium.

[Hint. Call the length x and use Algebra.]

- * * 30. ABC is a triangle. D is the mid-point of BC and E is on AC, such that $AE = \frac{1}{3}AC$. If AD and BE intersect in F, prove that $BF = \frac{2}{3}BE$.

- * * 31. With any point X (in the circumference of a circle XYR) as centre, and with any radius, a circle YQZ is described cutting the former circle at Y and Z; from X, any line XPQR is drawn cutting the circumferences at Q and R and the chord YZ in P. Show that XQ is a mean proportional between XP and XR.

- * * 32. AD is the bisector of the angle A of a triangle ABC, and CF is the median: through P, the intersection of AD and CF, PH is drawn parallel to AB to cut BC in H: prove that $PH : BH = AC : BC$.

- * * 33. Two fixed straight lines XY and QR cut one another at T; and S is a fixed point on QR. Find the position of a point P (also on QR), such that $SP = \frac{1}{2}PM$. (PM is the perpendicular from P to XY.)

- * * 34. ABC is a straight line bisected at B. DE, FG are any straight lines parallel to AC (FG lying between DE and AC). DA, DB, EB, EC meet FG in F, H, K, G respectively: and the sum of FH and KG = HK. Prove that CK and AH meet on DE.

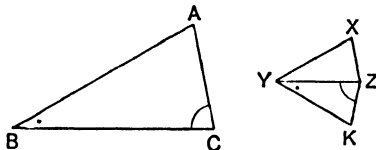
CHAPTER XXXVI.

2 $\triangle$ s WITH 3 CORRESPONDING SIDES PROPORTIONAL AND 2 $\triangle$ s WITH 2 SIDES PROPORTIONAL AND INCLUDED $\angle$ s EQUAL.

PROPOSITION 33.

§ 1. General Enunciation. *If two triangles have the sides of the one proportional to the corresponding sides of the other the triangles must be equiangular ($\therefore$ the triangles must be similar).*

Particular Enunciation. XYZ and ABC are the triangles in which $YZ : ZX : XY = BC : CA : AB$. To prove $\angle X = \angle A$, etc.



Construction. Make $\angle ZYK$ (on the side away from X) equal to $\angle B$.

Make $\angle YZK$ (on the side away from X) equal to $\angle C$.

Proof. Since two angles of $\triangle KYZ$ are equal to two angles of $\triangle ABC$, the third angles K and A must be equal ;

$\therefore \triangle KYZ$ and $\triangle ABC$ are equiangular ;

$\therefore YZ : KY = BC : AB$;

but $YZ : XY = BC : AB$ (given) ;

$\therefore KY = XY$.

Similarly $KZ = XZ$,

so that $\triangle KYZ$ and $\triangle XYZ$ are congruent,
but $\triangle KYZ$ is equiangular to $\triangle ABC$ (const.) ;

$\therefore \triangle XYZ$ is equiangular to $\triangle ABC$;

$\therefore \angle X = \angle A$, etc.

Q.E.D.

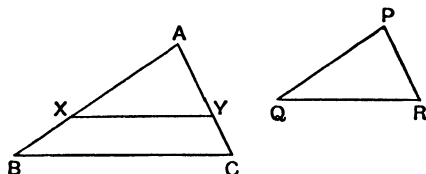
The above proposition is of extreme importance in copying triangles to scale.

§ 2. It is **easier to make lines correct than angles**, and so it is wiser to measure lines rather than angles on a small scale; thus plans of rooms and houses, diagrams of machinery, etc., are constructed from lengths and not angles. This proposition ensures that the angles of triangles are of the proper size.

PROPOSITION 34.

§ 3. **General Enunciation.** *If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles must be similar.*

Particular Enunciation. ABC and PQR are the triangles. $\angle A = \angle P$ and $BA : AC = QP : PR$. To prove $\triangle ABC$ is similar to $\triangle PQR$.



Construction and Proof. Place $\triangle PQR$ in the position AXY (this is possible for $\angle A = \angle P$), so that $AX = PQ$ and $AY = PR$.

Now $BA : AC = QP : PR$ (given);

$\therefore BA : AC = XA : AY$;

$\therefore$ alternately $BA : XA = AC : AY$,

so that XY is parallel to BC ;

$\therefore \angle AXY = \angle B$ (corresponding),

but $\angle AXY = \angle Q$ (same angle);

$\therefore \angle B = \angle Q$.

Similarly $\angle C = \angle R$,

so that $\triangle ABC$ and $\triangle PQR$ are equiangular;

$\therefore \triangle ABC$ is similar to $\triangle PQR$.

Q.E.D.

EXAMPLES 36 (RIDERS).

1. If AD, drawn from the angle A of a triangle perpendicular to the opposite side BC, be a mean proportional between BD and CD, show that BAC is a right angle.

[N.B.—This is the converse of one method of constructing a mean proportional. You need to prove $\triangle$ s BAD, ACD similar, and so get $\angle$ BAD equal to the complement of $\angle$ DAC.]

2. In the base AB of a triangle ABD a point C is taken, such that the rectangle AB.AC is equal to the square on AD. Show that the angle A is equal to the difference of DBC, DCB.

[HINT. Prove $\triangle$ s DAC, BAD similar.]

* 3. OAB, OA'B' are two similar triangles. Prove that

$$AA' : BB' = OA : OB,$$

and that the angle between AA' and BB' is equal to the angle AOB.

[HINT. Prove $\angle$ OAA' = $\angle$ OBB', and consider $\triangle$ s OAA', OBB'.]

* 4. ABC is a triangle, D a point within it, BEC a triangle on the other side of BC, such that the angles CBE, BCE are respectively equal to ABD, BAD; show that the triangles ABC, BDE are similar.

* 5. ABC is an isosceles triangle, right-angled at A. BDEC is a rectangle described on the side of BC remote from A, with $BC = 2BD$. If AB is bisected at F, prove that DFC is a right angle.

* 6. ABC is a triangle, and the tangent at C to the circle circumscribing it cuts AB produced in D. Prove

$$AD : BD = AC^2 : BC^2.$$

** 7. O is a fixed point. P is any point upon a given straight line XY; if Q be taken upon OP, so that the rectangle OP.OQ is constant, show that the locus of Q is a circle.

** 8. O is a fixed point, P a variable point on a given circle; if OP be divided at Q in a constant ratio, show that the locus of Q is a circle.

CHAPTER XXXVII.

INTERSECTING CHORDS OF A CIRCLE.

§ 1. The property of intersecting chords of a circle, embodied in the next proposition and its corollary, is of great practical importance. By it often we are able to determine the size of a circle whose centre is quite out of range.

PROPOSITION 35.

General Enunciation. *If two chords of a circle intersect (inside or outside the circle) the rectangle contained by the segments of the one must be equal to the rectangle contained by the segments of the other.*

Particular Enunciation. AB and XY are the chords, cutting at K, to prove rect. AK . KB = rect. XK . KY.

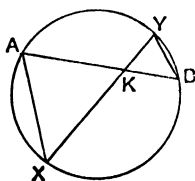


Fig. i.

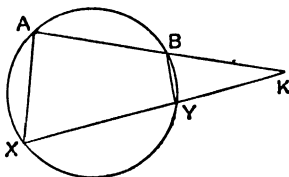


Fig. ii.

Construction. Join AX and BY.

Proof. In $\triangle$ s KAX, KYB,

$\angle AKX = \angle YKB$	Fig. i. (vert. opp.)	Fig. ii. (same angle)
$\angle KAX = \angle KYB$	(same segment)	(cyclic quadrilateral)

$\therefore \triangle$ s KAX and KYB are equiangular ;

$\therefore AK : XK = KY : KB ;$

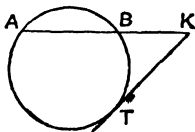
$\therefore \text{rect. } AK . KB = \text{rect. } XK . KY$

(for product of extremes = product of means).

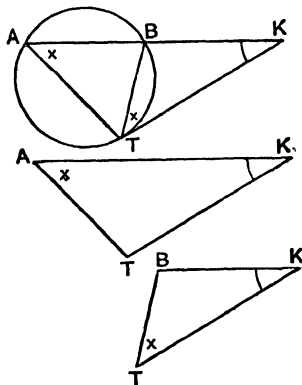
Q.E.D.

[See the next page.]

* **Corollary.** In Fig. ii., if X and Y coincide at T , the secant KYX becomes the tangent KT , and $KT^2 = \text{rect. } KA \cdot KB$; i.e. *the square on the tangent is equal to the rectangle contained by the whole secant and the part outside the circle.*



§ 2. The property embodied in the corollary can be proved, independently of the tangent as the limit of a secant, as follows :



We have to get $\triangle s$ AKT and TKB equiangular, and it is not a bad plan to draw the triangles separately.

$\angle AKT = \angle TKB$ (same angle),

$\angle TAK = \angle BTK$ (alternate segment) ;

$\therefore \triangle s$ AKT, TKB are equiangular ;

$\therefore AK : KT = KT : BK$;

$\therefore KT^2 = \text{rect. } KA \cdot KB$

(for product of extremes = product of means).

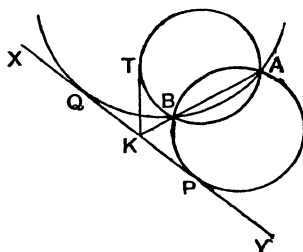
This fact is especially useful in determining how far lighthouses may be seen owing to the curvature of the Earth.

The **Converse** of the proposition and corollary is true, and is usually assumed. [It could be proved by *Reductio ad absurdum*.]

* § 3. The **Construction of a circle through 2 given points and touching a given line** depends upon this proposition.

METHOD 1.

A and B are the points. XY is the straight line.



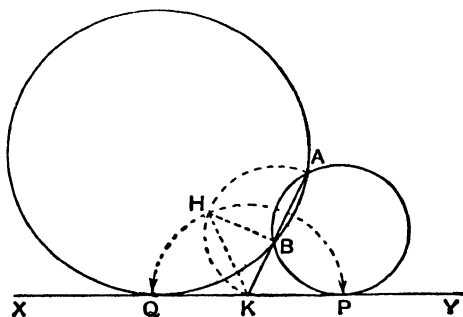
Draw *any* circle through A and B. AB produced cuts XY at K. From K draw KT a tangent to this circle.

Cut off KP and KQ equal to KT.

Circles through the 3 points A, B, P or A, B, Q are the circles required. [$KP^2 = KT^2 = KA \cdot KB$.]

METHOD 2.

A and B are the points. XY is the straight line.



AB produced cuts XY at K.

KH is the mean proportional between KA and KB.

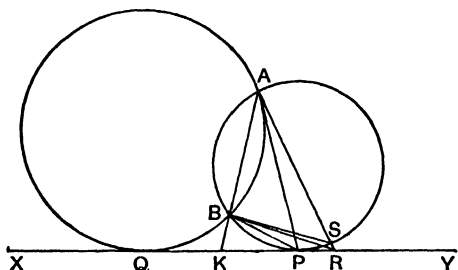
Cut off KP and KQ equal to KH.

Circles through the 3 points A, B, P or A, B, Q are the circles required.

[$KP^2 = KA \cdot KB$, so that KP will be a tangent.]

N.B.—There is little to choose between the two methods.

It is noticeable that in XY at P (and Q) the line AB subtends a **maximum angle**, for at any other point R the line AB subtends



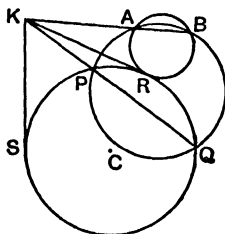
$\angle ARB$, which (being an interior opposite angle of $\triangle BSR$) is less than $\angle ASB$, but $\angle ASB = \angle APB$ in the same segment ;

$\therefore \angle ARB < \angle APB$, etc.

AB subtends, in XY , a maximum angle at P and at Q . Probably these two maxima are quite different, and one "maximum" may be greater than the other "maximum." The only stipulation for a maximum is that it should be greater than anything in the *immediate neighbourhood*.

*** § 4. The Construction of a circle through 2 given points and touching a given circle is as follows :**

A and B are the points, and C is the given circle.



Any circle through A and B is drawn cutting the C circle at points P and Q .

BA and QP produced give K. KR and KS are tangents. The circles ABR and ABS fulfil the conditions.

[Only one circle is drawn here. The contact is external. In *this case* the contact of the ABS circle would be internal. The number of solutions, at most 2, depends on the relative positions of the points A, B and the circle C.]

§ 5. The three following **examples** illustrate the use of this proposition :

(1) It is required to find the curvature of a certain part of wooden arch over a door.

A trident-shaped instrument is used. A and B, the outside legs, are fixed. The central leg X is movable in a slot in the cross-bar of the instrument. The leg X is graduated.

In a particular instrument $AB = 10.4$ cm. and the graduation showed (when all three legs were in contact with the arch, and when the instrument was held perpendicularly) that X was depressed 0.8 cm.

The figure (which is not drawn to scale), giving the complete circle, shows

$$XK \cdot KY = AK \cdot KB ;$$

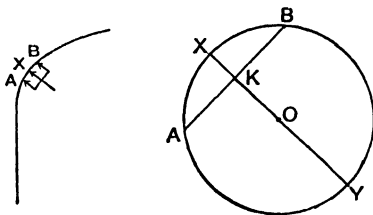
$$\therefore 0.8 \times KY = 5.2 \times 5.2 ;$$

$$\therefore KY = \frac{5.2 \times 5.2}{0.8} = 33.8,$$

$$\text{and } XY = XK + KY = 0.8 + 33.8 = 34.6 ;$$

$$\therefore XO = 17.3,$$

so that the **radius of curvature** is 17.3 cm.



[To get good results in practice the distance XK has to be very accurately measured. Oculists employ instruments, on this principle, to get the radius of curvature of spectacle lenses.]

(2) About how many miles away is the **visible horizon** when one is standing with one's eyes a feet above sea-level? The earth is a sphere of 7920 miles diameter.*

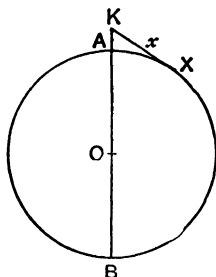
O is the centre of the earth. One's eyes are at K. KA (very enormously exaggerated) is a feet $= \frac{a}{5280}$ mi. KX is a tangent. The sea horizon is at X.

$$\begin{aligned} KB &= 7920 \text{ mi.} + \text{some trivial amount} \\ &= 7920 \text{ mi.} \end{aligned}$$

$$KX^2 = KA \cdot KB,$$

$$x^2 = \frac{a}{5280} \times 7920 = a \times 1\frac{1}{2};$$

$$\therefore x = \sqrt{a \times 1\frac{1}{2}},$$



so that the approximate distance in miles may be obtained by this very simple rule.

"Half as much again as the height in feet. The square root gives the distance in miles."

[At sea distances are reckoned in nautical miles, and you may have afterwards to transfer the land miles, obtained by this rule, to sea miles.

1 land mile = 5280 feet. 1 sea mile = 6080 feet.]

[You must recollect that the figure is very enormously exaggerated, and that really, in practice, AX and KX are *very very nearly equal*. If KA were as high as Mount Everest there is only about $\frac{1}{10}$ % difference. If something very exact is needed various things should be taken into consideration: the Earth is not an exact sphere; the diameter is not exactly 7920 miles; light, being refracted, does not travel in quite a straight line. In that case calculate trigonometrically, but this is good enough for rough purposes. Of course it does not apply on spheres not having this diameter; and on the Moon, for instance, the $1\frac{1}{2}$ is wrong.]

* 7920 miles is very nearly correct; it is more than the polar diameter and less than the equatorial diameter, and they only differ by about 27 miles.

(3) Using the approximate rule $x = \sqrt{a \times 1\frac{1}{2}}$, about how much above sea-level must my eyes be to see the top 50 feet of a 110-foot lighthouse 16 miles away ?

(The figure is grossly exaggerated of course.)

RA is the lighthouse (110 ft.).

I must see RK (50 ft.) ;

$\therefore KA = 60$.

My eyes are at H. I need to find HX. KTH just touches the Earth.

$$KT = \sqrt{1\frac{1}{2} \times 60} \text{ miles roughly}$$

$$= \sqrt{90}$$

$$= 9.5 \text{ about.}$$

$$\therefore HT = 16 - 9.5 = 6.5.$$

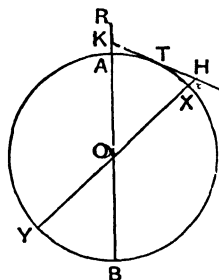
$$\text{Now } HT^2 = HX \cdot HY ;$$

$$\therefore (6.5)^2 = 1\frac{1}{2}x ;$$

$$\therefore 42.25 = \frac{3}{2}x ;$$

$$\therefore x = 28.2(-)$$

so that my eyes must be about 28 feet above sea-level.

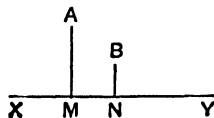


EXAMPLES 37 a (GEOMETRICAL DRAWING).

1. Draw a circle with radius 7 cm., and in it draw four chords AB, CD, EF and GH intersecting at any point X, where X is 3 cm. from the centre of the circle. Measure AX, XB, CX, XD, etc. Find the values of $AX \times XB$, $CX \times XD$, $EX \times XF$ and $GX \times XH$.

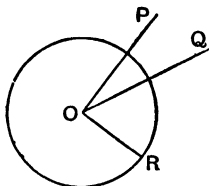
2. Draw a circle with radius 2" and take a point P 3" from its centre. Draw a tangent PQ and three secants PAB, PCD and PEF. Measure PQ, PA, PB, ..., etc. Find the values of PQ^2 , $PA \times PB$, $PC \times PD$ and $PE \times PF$.

3. XY is a straight line. M and N are two points on it 6 cm. apart. The perpendiculars AM and BN are respectively 9 cm. and 4 cm. Construct the smaller of the two circles, which pass through A and B and touch XY. Measure its radius.



4. Draw a diagram showing two points P and Q, each 4 cm., from a given straight line XY, and on the same side of it, and 3 cm. apart. Construct the *one* circle that passes through P and Q and touches XY. Measure its radius.

- * 5. $OP = 6.6$ cm., $OQ = 7.4$ cm., $\angle POQ = 23^\circ$, $OR = 4$ cm. Construct two circles to pass through P and Q and to touch the given circle. Measure their radii.



- * 6. A line ABC is 2500 feet long, BC representing a ship 500 feet long. A submarine is moving on a course AT, the angle CAT being 35° . Find by construction the point on AT at which the ship presents the widest target. [See Ex. 37 b, 18 too.]
- * 7. Draw a circle 3" diameter. On a diameter produced mark two points 1.75" and 3.5" respectively from the centre, on opposite sides of it. Draw a circle to touch the first circle and pass through the two points. Measure the radius of the circle.
- * 8. Show the construction for drawing a tangent to a circular arc of large radius (whose centre is out of reach) from a point outside the circumference.
- * 9. AP (2") touches a circle at P. ABC is a secant passing through the centre of the circle, AB being $1\frac{1}{2}$ ", B and C being on the circle. Give a construction for finding C. What is the radius of the circle? [This is to be "by construction" and not "by calculation."]
- * 10. Draw a circle (centre O) with radius 2 cm. and take two points A and B such that $OA = 3$ cm., $OB = 4$ cm., $AB = 3.7$ cm. Find the point on the circumference of the circle at which AB subtends a maximum angle.

EXAMPLES 37 b (CALCULATIONS).

1. A, B, C are 3 points in a straight line ; $AB = 5$ cm., $BC = 4$ cm. Through A and B any circle is drawn ; calculate the length of a tangent drawn from C to this circle.

2. A circle with radius 3 cm. is drawn and a point P 5 cm. from its centre is taken. One tangent from P to the circle is drawn. Calculate the length of the tangent.

3. Two lines AB, CD, of length 12 and 18 inches, bisect each other. If a circle is drawn through ABC, where will it cut CD ?

4. Smeaton's Eddystone Lighthouse (1756) was 72 ft. high (High Water to Focal Plane of Light) ; whereas Douglass' (erected by Trinity House in 1882) is 133 ft. high. By how many miles is the range of vision increased ?

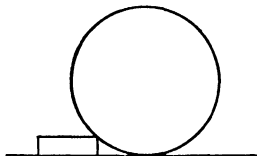
5. An airman, over the sea, is at a height of 2000 ft. How far away is the sea horizon ?

6. The Peak of Teneriffe has two tops, 12,200 ft. and 9880 ft. How far away (in each case ignore fractions of a mile) is the sea horizon from each of the two tops ?

7. AKB and XKZ are two straight lines cutting one another. $AK = 4$ yards, $KB = 5$ yards, $KX = 1.5$ yards and $KZ = 20$ yards. The circle through A, X and B cuts KZ at Y. How long is KY ?

8. A brick, 3" high, is used to prevent a carriage wheel slipping. The brick is 9" from the point of contact of the wheel with the road. What is the radius of the wheel ?

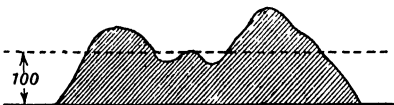
[The figure is not to scale.]



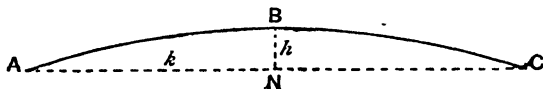
9. ABC is a straight line drawn from A, cutting a circle in B, C. $AB = 28$ mm., $AC = 63$ mm.; find the length of the tangent from A.

10. To find the radius of curvature of an arch a trident-like instrument was used (see page 395). The outside legs were 10.2 cm. apart. The reading of the centre leg was 0.7 cm. What is the radius of curvature ? If the centre leg reading was 0.68 cm. and 0.72 cm., as taken by separate observers afterwards, what radii would they deduce ?

11. In a ship, on approaching a coast, the centre hump (known to be 100 ft. high) of 3 familiar hill-tops just becomes visible; one's eyes are 20 ft. above sea level. How far is the ship from the coast ? Give this distance (1) in land miles and (2) in sea miles. [1 sea mile = 6080 feet.]



12. ABC is an arc of a circle. The chord AC is bisected at N and NB is perpendicular to AC . $AN=k$ and $BN=h$. Find a formula for the radius of the circle in terms of h and k only.



13. If KT is a tangent to a circle from a point K , and KAO joins K to the centre O and cuts the circle at A , state and prove a relation between x , y and r , where $KT=x$ in., $KA=y$ in. and r is the radius of the circle.

14. In liners there are many decks, so that the sea horizon is at different distances from the various decks. How much further can be seen from the "Boat Deck" of a certain vessel, 46 feet above sea-level, than from the "Main Deck," 6 feet above sea-level? Give this distance (1) in land miles and (2) in sea miles. [1 sea mile = 6080 feet.] Height of eye is 5 ft.

* 15. AOB is a chord of a circle whose centre is C . $AO=3$ in., $BO=7$ in., $CO=2$ in. Find the length of the radius of the circle. What does your result prove as to the position of C ?

* 16. If the decks (20 ft. and 30 ft. above the water) of two vessels are just visible from each other, how many miles are the vessels apart? Give this distance in sea miles too. [1 sea mile = 6080 ft.]

* 17. O is the centre of a circle whose radius is 5 cm. $AOKB$ is a diameter, $AK=7$ cm. and KC is drawn perpendicular to AB both ways to meet the circumference at C and D .

Hence *calculate* the length of KC to the nearest mm. Check by measurement.

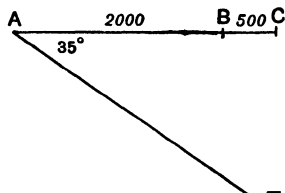
How does the figure afford a means of determining the square root of 21? [i.e. $\sqrt{7 \times 3}$.]

Explain how you could use the diagram to find two numbers whose sum is 10 and whose product is $(3.5)^2$.

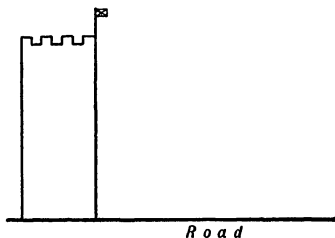
Check your answers by the solution of the equation

$$x^2 - 10x + (3.5)^2 = 0.$$

* 18. A ship, 500 feet long, is at anchor at BC . A submarine is on the course AT , AB being 2000 ft. How far must the submarine go so that the ship may present the widest target? Does the angle 35° matter?



- * 19. The tower is 138 ft. high, and the flagstaff is 22 ft. high. At what point on the road does the flagstaff subtend the greatest angle ?



- * 20. Snowdon, in Wales, is 3560 ft. high and 100 miles away the Wicklow Mountains, in Ireland, rise to 3039 ft. Are the top 2000 ft. of the Wicklow Mountains visible to one standing on the summit of Snowdon ? [It is to be assumed that no high ground interrupts the view.]
- * 21. Snaefell (highest mountain in the Isle of Man) is 2034 ft. high. It is distant 73 miles from Great Orme's Head (coast of North Wales). How high up on Great Orme's Head must you be to see at least the top 500 ft. of the Isle of Man ?
- * 22. Cape Gris-nez lighthouse (on the French side of the Straits of Dover) is 233 ft. above high water. If it is 24 miles from Dover, can it be seen from high-water level there ? If one is standing up at the water's edge with one's eyes 5 ft. above sea-level, is the light visible ? If one is at Dover Castle (375 ft. above sea-level), is the Cape Gris-nez light visible ?
- * * 23. A, B, C are three mountain tops in a straight line.
 $AB = 15 \text{ km.}$, $BC = 8 \text{ km.}$
 A is 700 metres high. B is 481 m. C is 378 m.
 First suppose the Earth is flat. Is A visible from C ?
 Secondly, suppose the Earth is round, and that a section through the 3 mountains and the centre of the Earth is a circle of 40,000 km. circumference, is A visible from C ?
 [Recollect that the circle is so big that the arc ABC and chord AC are practically equal.]
- * * 24. Taking the diameter of the moon as 2000 miles, how far apart must two lunar mountains (5000 ft. and 4000 ft. high respectively) be so that from the top of the former it is possible to see the top 1000 ft. of the latter, if no other high ground interrupts the view ?

EXAMPLES 37 c (RIDERS).

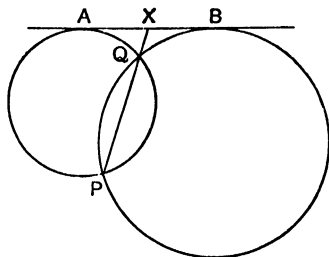
1. Two circles intersect at A and B, and any point P is taken on AB produced. The tangents from P to the two circles are PY and PZ. Prove $PY = PZ$.

[HINT. $PY^2 = PA \cdot PB = PZ^2$. *Why?*]

2. Two circles cut in P and Q; prove that PQ produced bisects their common tangents.

[HINT.

$AX^2 = XP \cdot XQ = BX^2$. *Why?*]



3. From T, a point outside a circle, a secant TYZ and a tangent TX are drawn; prove that TX is a mean proportional between TY and TZ.

4. ABC is a fixed straight line. Find the locus of the points of contact of tangents drawn from C to circles which pass through A and B.

[HINT. If CP is one of those tangents, prove CP^2 constant.]

5. Two circles intersect in A and B, and P is any point on AB. If a line through P intersects one circle in Q and R, and another line through P intersects the other circle in S and T, prove $PQ \cdot PR = PS \cdot PT$.

* 6. Two chords, HK and EF, cut one another within a circle at the point X; assume that $\text{rect. } XH \cdot XK = \text{rect. } XE \cdot XF$. In the special case in which HK is a diameter and EF any chord perpendicular to it, prove that XE is a mean proportional between XH and XK.

* 7. Three circles cut one another, their common chords being AB, CD, EF. Prove that AB, CD, EF meet in a point.

[HINT. This can be proved by *Reductio ad absurdum*. Let AB and CD cut at X. Join EX, and produce it to cut the two circles at G and H separately. Prove that G and H must coincide.]

- * 8. Two circles intersect in A and B. Through any point Q, in AB produced, straight lines QHK, QLM are drawn, of which QHK cuts one circle in H and K, and QLM cuts the other circle in L and M. Prove that a circle can be drawn through the 4 points H, K, L, M.
- * 9. AD, BE, CF are the perpendiculars from the angles of an acute-angled triangle upon the opposite sides and O their point of intersection; show that $\text{rects. } AO \cdot OD, BO \cdot OE \text{ and } CO \cdot OF$ are equal.
[HINT. Prove BFEC cyclic, etc.]
- * 10. If from a point P on the circumference of a circle a perpendicular be drawn to any chord, its square is equal to the rectangle contained by the perpendiculars from the extremities of the chord on the tangent at P.
- * 11. PA, PB are two tangents to a circle centre O. PO produced cuts AB at C, and the circle at D and E.
Prove that the rectangle $DP \cdot PE = \text{rectangle } OP \cdot PC$.
- * * 12. ABC is a triangle. If the tangent to the circumcircle at A meets the side BC produced at D, prove that
 $BD : DC = BA^2 : AC^2$.
- * * 13. Two circles cut at B and A. From the point P, which lies on BA produced, tangents PY and PZ are drawn to the circles. The tangents at A to the two circles meet PY and PZ respectively at D and E. Prove that the perimeters of the triangles DPA and EPA are equal.
- * * 14. The side BC of a triangle ABC is bisected at D, and AD is produced to cut the circumscribing circle at E; show that the sum of the squares on AB, AC is equal to twice the rectangle AD . AE.
- * * 15. If two chords intersect at a point inside the circle, the difference of their squares is equal to the difference of the squares on the difference of their segments.
- * * 16. C is the middle point of a chord AB of a circle. The tangents at A and B meet in D; show that the segments of any chord through C subtend equal angles at D.

CHAPTER XXXVIII.

BISECTOR OF THE VERTICAL ANGLE OF A TRIANGLE.

PROPOSITION 36.

§ 1. General Enunciation. (i) *The interior bisector of an angle of a triangle must divide the opposite side internally in the ratio of the sides containing the angle, and (ii) likewise the exterior bisector externally.*

Particular Enunciation. ABC is the triangle.

(i)
AX bisects $\angle BAC$ internally;
to prove $BX : XC = BA : AC$.

(ii)
AY bisects $\angle BAC$ externally;
to prove $BY : YC = BA : AC$.

Fig. i.

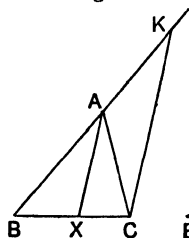
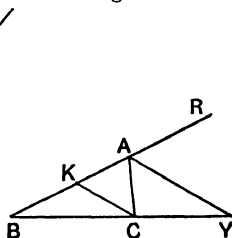


Fig. ii.



Construction.

Draw CK parallel to XA to meet BA, produced, at K.

Draw CK parallel to YA to meet BA at K.

Proof.

$\angle BAX = \angle AKC$ (corr.),
 and $\angle XAC = \angle ACK$ (alt.),
 but $\angle BAX = \angle XAC$ (given);
 $\therefore \angle AKC = \angle ACK$;
 $\therefore AK = AC$.

In the $\triangle BKC$,

AX is parallel to KC;
 $\therefore BX : XC = BA : AK$;
 $\therefore BX : XC = BA : AC$.

$\angle RAY = \angle AKC$ (corr.),
 and $\angle YAC = \angle ACK$ (alt.),
 but $\angle RAY = \angle YAC$ (given);
 $\therefore \angle AKC = \angle ACK$;
 $\therefore AK = AC$.

In the $\triangle BKC$,

AY is parallel to KC;
 $\therefore BY : YC = BA : AK$;
 $\therefore BY : YC = BA : AC$.

Q.E.D.

§ 2. The **converse** of this proposition is usually assumed. It could, however, be substantiated by writing this proof out practically backwards.*

* This Proposition is given in full in the Appendix U.

EXAMPLES 38 (RIDERS).

1. If the bisectors of the angles ABC, CDA of a quadrilateral ABCD meet at K on AC, prove that $AB : BC = AD : DC$.

[HINT. $\angle B$ of $\triangle ABC$ is bisected ;

$\therefore AB : BC = AK : KC$,

and $\angle D$ of $\triangle ADC$ is bisected ;

$\therefore$ what ? etc.]

2. The distance between the goal posts A and B is c feet. A football is situated b feet from one post and a feet from the other. At what point in AB should a kick be aimed so that the ball may most probably pass between the posts without touching them ? It is to be assumed that there is no curl on and that the ball travels straight.

3. Construct a triangle having given the vertical angle, the base and the ratio of the other two sides.

[HINT. Draw the circumcircle.]

* 4. D is the middle point of the base BC of a triangle ABC, and the angles ADB, ADC are bisected by DE, DF which meet AB, AC respectively in E, F ; show that EF is parallel to BC.

* 5. ABC is a triangle. O is the middle point of BC, and AO is produced to T. The lines bisecting internally the angles BOT and COT cut externally the sides AB, AC in D and E. Prove that DE is parallel to BC.

* 6. AB is a diameter of a circle, CD a chord perpendicular to it, and P any point in CD ; if AP, BP produced cut the circle in E, F, show that the sides of the quadrilateral CEDF are in proportion.

* * 7. AB is a straight line divided internally and externally at X and Y, so that $AX : XB = AY : YB = 3 : 1$. If P is any point such that $AP = 3PB$, prove that the locus of P is a circle through X and Y.

* * 8. The locus of a point in space whose distances from two fixed points are in a constant ratio is a sphere.

* * 9. If I be the centre of the inscribed circle of a given triangle ABC and AI produced cut BC at D, show that

$$AI : ID = AB + AC : BC.$$

* * 10. The diagonal AC of a quadrilateral ABCD bisects the angle BAD, and AC is a mean proportional between AB, AD ; the diagonals intersect in E ; prove that the ratio $BE : ED$ is the square of the ratio $BC : CD$.

CHAPTER XXXIX.

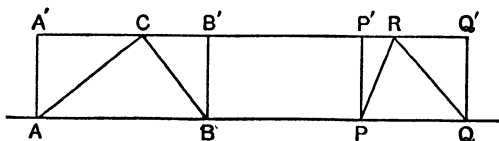
THE AREAS OF SIMILAR FIGURES VARY AS THE SQUARES ON CORRESPONDING SIDES.

§ 1. If two rectangles have the same altitude (between the same parallels is the most usual case), it is **intuitive** that their areas are proportional to their bases, so that

area of $ABB'A'$: area of $PQQ'P'$ = $AB : PQ$,

and $\triangle ABC = \frac{1}{2}$ rectangle $ABB'A'$ (etc.) ;

hence area of $\triangle ABC : \triangle PQR = AB : PQ$.

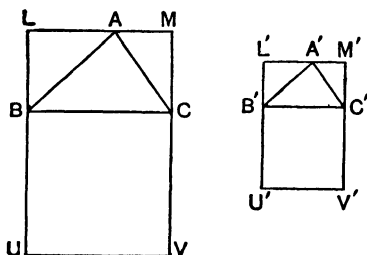


PROPOSITION 37.

§ 2. General Enunciation. *The ratio of the areas of similar triangles is equal to the ratio of squares on corresponding sides.*

Particular Enunciation. ABC , $A'B'C'$ are the triangles. $BCVU$, $B'C'V'U'$ are the squares. To prove

$$\triangle ABC : \triangle A'B'C' = \text{square } BCVU : \text{square } B'C'V'U'.$$



Construction. Complete the figures by drawing LAM parallel to BC , producing UB and VC to meet LAM . Similarly with dashed letters.

Proof.

Rectangle $LMCB$: square $BCVU$ = $LB : BU$ (by intuition*),
 likewise rectangle $L'M'C'B'$: square $B'C'V'U'$ = $L'B' : B'U'$,
 but $LB : BU = L'B' : B'U'$ (corresponding lines on similar figures);
 $\therefore$ rect. $LMCB$: square $BCVU$ = rect. $L'M'C'B'$: square $B'C'V'U'$.

Alternately,

$$\text{rect. } LMCB : \text{rect. } L'M'C'B' = \text{square } BCVU : \text{square } B'C'V'U',$$

$$\text{now } \triangle ABC = \frac{1}{2} \text{ rect. } LMCB,$$

$$\text{and } \triangle A'B'C' = \frac{1}{2} \text{ rect. } L'M'C'B';$$

$$\therefore \triangle ABC : \triangle A'B'C' = \text{square } BCVU : \text{square } B'C'V'U',$$

$$\text{i.e. } \triangle ABC : \triangle A'B'C' = BC^2 : B'C'^2.$$

Q.E.D.

* See page 407.

§ 3. Since any **polygon** can be divided up into triangles, we realize the areas of polygons vary as the squares on corresponding sides; and since any figures, even with curved boundaries, can be divided up into a very large number of triangles whose total area agrees very closely with the area of the figure, it follows that **areas of similar figures vary as the squares on corresponding sides**. In a map, if lengths are one-thousandth of their natural size, areas would be one-millionth

(one million = thousand $\times$ thousand).

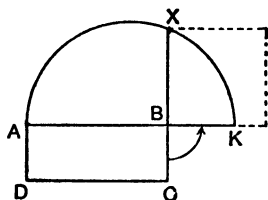
If you double the dimensions of a carpet you make it four times the size. If you treble the radius of a circle, you make its area nine times as big.

§ 4. To construct a square equal in area to a given polygon resolves itself into the problem of making a square equal to a rectangle, for the polygon is first reduced to a triangle, and a rectangle on the same base as the triangle and of half its height is equal to it in area.

Making a square equal in area to a given rectangle can be done in either of the following methods :

METHOD 1.

ABCD is the rectangle.



AB is produced. $BK = BC$.

The semicircle on AK as diameter is drawn.

CB produced cuts the semicircle at X.

The square on BX is the square required.

For BX is the mean proportional between AB and BK;

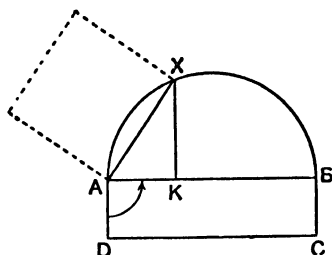
$$\therefore AB : BX = BX : BK ; \quad \therefore BX^2 = AB \cdot BK$$

(product of means = product of extremes) ;

$$\therefore BX^2 = AB \cdot BC ; \quad \therefore BX^2 = \text{rect. } ABCD.$$

METHOD 2.

ABCD is the rectangle.



$$AK = AD.$$

The semicircle on AB as diameter is drawn.

KX is perpendicular to AB.

The square on AX is the square required.

For AX is the mean proportional between AB and AK ;

$$\therefore AB : AX = AX : AK ;$$

$$\therefore AX^2 = AB \cdot AK$$

(product of means = product of extremes) ;

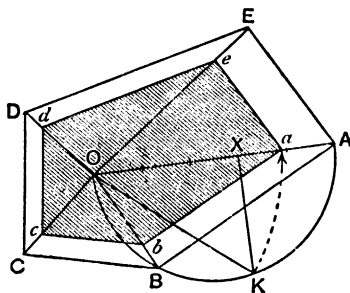
$$\therefore AX^2 = AB \cdot AD ;$$

$$\therefore AX^2 = \text{rect. } ABCD.$$

Method 2 is often the more convenient in practice ; and, if the sides of the rectangle are at all long, produces a less unwieldy figure.

§ 5. Two examples follow :

(1) To **decrease** a given figure $ABCDE$ to a similar figure of $\frac{3}{5}$ the size.



Take any point O (inside, outside, or on a side of $ABCDE$).

Join OA , OB , etc.

Divide one of them (a longish one for choice), say OA , into 5 equal parts.

$OX = 3$ parts.

XK perpendicular cuts the semicircle on OA as diameter at K .

$Oa = OK$.

Draw ab parallel to AB ,

then bc parallel to BC ,

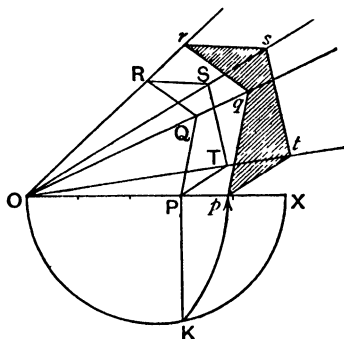
etc.

(the last pair of points e , a are joined, but of course ea should be parallel to EA).

$abcde$ is $\frac{3}{5}$ of $ABCDE$ and similar to it.

(2) To **increase** a given figure PQRST to a similar figure of $\frac{5}{3}$ the size.

Take any point O (inside or outside or on a side of PQRST).
Join OP, OQ, ... , etc., and produce.



Divide one of them (a longish one for choice), say OP, into 3 equal parts. OX, on OP produced, is 5 parts.

PK perpendicular to OX meets the semicircle on OX as diameter at K.

$$Op = OK.$$

Draw pq parallel to PQ,
then qr parallel to QR,
etc.

(the last pair of points t, p are joined, but of course tp should be parallel to TP).

$pqrst$ is $\frac{5}{3}$ of PQRST and similar to it.

EXAMPLES 39 a (GEOMETRICAL DRAWING).

Reduction to squares of equal area.

1. Construct a rectangle with sides of 2.4" and 1.8", and construct a square of equal area. Measure its sides.

2. Draw a triangle with sides 10.3 cm., 7.3 cm. and 5.8 cm., and construct a square of equal area. Measure its sides.

3. Draw a triangle with base 7.9 cm. and base angles 75° and 60° . Make a square of equal area. Measure its diagonals.

* 4. Construct the figure ABCD, having given $AB=3''$, $\angle ABC=108^\circ$, $\angle ABD=42^\circ$, $\angle BAC=30^\circ$ and $\angle BAD=82^\circ$. Measure and write down the lengths of AD, DC and CB. Find the length of a side of a square of equal area.

* 5. Construct two different triangles with data: $AB=4\frac{1}{4}''$, $AC=3''$, angle $ABC=35^\circ$; and make a square equal to the sum of their areas. Measure its sides.

* 6. Construct the figure ABCDE, having given $AB=2.5''$, $AD=3.6''$, $AE=2.46''$, $DE=2.58''$, $\angle BAC=27^\circ$, $\angle DAC=25^\circ$, $\angle ABC=45^\circ$. Reduce it to an equivalent triangle and make a square equal to this triangle. Measure the side of the square.

Decreasing and Increasing figures to similar figures in given ratios.

7. Draw a triangle with sides 10, 9, 8 cm., and construct a similar triangle $\frac{2}{3}$ the area. Measure its shortest side.

8. Draw a triangle with two sides $3.6''$ and $3''$ and included angle 60° . Construct a similar triangle $\frac{5}{4}$ the area. Measure its longest side.

9. Construct an isosceles triangle whose base is $2.4''$ and area 5 sq. inches. Divide it into a triangle and trapezium in the ratio of 2:3 by a line parallel to the base. Measure that line.

10. Draw a circle whose diameter is 7 cm., and divide it into four equal concentric annuli. What is the radius of the central circle and the width of the narrowest annulus?

* 11. Construct a quadrant of a circle, radius $2.5''$, and divide it into 3 parts of equal area by arcs having the same centre as the arc of the original quadrant. What is the width of the middle area?

* 12. Construct a parallelogram, given two sides $2''$ and $2.75''$ and included angle 60° . Divide it into three equal parts by lines parallel to the shorter diagonal. What is the length of one of those lines?

* 13. Construct a quadrilateral ABCD, given $AB=3.2''$, $BC=2.15''$, $CD=3.7''$, $AC=4.3''$, the angle $CAD=42^\circ$, and the angle D is obtuse. Make a similar quadrilateral $\frac{5}{3}$ its area. Measure its shortest side.

- * 14. Construct the irregular figure $ABDC$, having given $AB = 2.75''$, $\angle BAC = 120^\circ$, $\angle BAD = 45^\circ$, $\angle ABC = 30^\circ$, $\angle ABD = 90^\circ$ (all angles to be made without the use of the protractor), and make a figure similar to it, and half as large again. Measure the side corresponding to AB .

EXAMPLES 39 b (CHIEFLY RIDERS).

1. D, E are the middle points of the sides AB, AC of a triangle; show that the area of $BCED$ is three times the area of ADE .

[HINT. $\triangle ADE : \triangle ABC = 1 : 4$. Why?]

2. The base BC of a triangle ABC is bisected at D , and BC is also divided at F , so that CF is a mean proportional between CD and CB . Prove that FK drawn parallel to AB will bisect the triangle ABC .

[HINT. $\triangle KFC : \triangle ABC = FC^2 : BC^2$. Why?]

3. Through the vertex A of a triangle ABC a straight line is drawn cutting the base BC at D , so that $\angle DAC = \angle ABC$: prove that $BC : CD = AB^2 : AD^2$.

[HINT. Prove $\triangle s ADC$ and BAC similar.]

4. $ABCD$ is a square whose diagonals AC and BD intersect in O . OB and OC are produced to P and Q respectively, so that $BP = OB$ and $CQ = OC$.

Prove that the area of the quadrilateral $BPQC$ is three-quarters that of the square.

5. A ditch is 30 yds. long and its cross-section is an equilateral triangle with 10' sides. How many cubic feet of earth must have been taken out in making it; and what is the weight of water in it, if it comes halfway up the sides? [1 cu. ft. of water weighs about 62 lb.]

6. Through a point O at the centre of a sheet of cardboard, a straight line is drawn. Four circles are then drawn with O as centre, and the spaces between these circles painted different colours to form a target.

If the area of the bull is 20 sq. inches and all the spaces are of the same area, what is the diameter of the bull and the widths of the three coloured strips?

7. ABC is a triangle. The point Z divides the line AB, internally, so that $AZ:ZB=3:1$. ZX is drawn parallel to AC to meet BC at X. ZY is drawn parallel to BC to meet AC at Y. What is the ratio of the area of the parallelogram ZYCX to the triangle ABC?

8. A diverging beam of light issues from a point 8 inches behind the circular front glass of a lamp. If the area of the glass is 50 square cm., what is the area lighted up on a screen placed square to the light 10 feet from the *point* of light? The screen is moved towards the lamp, and placed 16 inches from the *glass*. Show that the area lighted up on the screen is 450 square cm.

* 9. ABC is a triangle and D, E, F the feet of the perpendiculars from A, B, C upon the opposite sides; show that the area of BCEF is to that of AEF as BE^2 is to AE^2 .

* 10. ABCD is a square, and Q is a point in AD such that $AQ=\frac{1}{4}AD$, and P is in DC such that $DP=\frac{1}{4}DC$. BQ and AP meet in R. Find what fraction the area of the triangle AQR is of the square ABCD.

* 11. D, E, F are the feet of the perpendiculars from the angles A, B, C of a triangle ABC to the opposite sides; prove that the triangle ABC is to the triangle DBF in the duplicate ratio of AB to BD. Also that the figure AFDC is to the triangle BFD in the duplicate ratio of AD to BD.

* 12. D is a point in the side AB of a triangle ABC, such that $DB=\frac{2}{3}AD$. DE parallel to BC meets AC in E, and parallels through D and E to AC and AB respectively meet BC in F and G. Find what fraction the area DFGE is of the whole triangle ABC.

* * 13. ABC is a triangle and squares ABDE, CAFG are described externally on the sides AB, CA; show that the triangles ADF, AEG are equal in area.

* * 14. ABC and PQR are triangles such that the medians of the former are equal to the sides of the latter. Prove that
area of ABC : area of PQR = 4 : 3.

* * 15. Through the vertices A, B, C of an equilateral triangle straight lines are drawn perpendicular to the sides AB, BC, CA respectively, so as to form another equilateral triangle. Compare the areas of the two triangles.

APPENDIX.

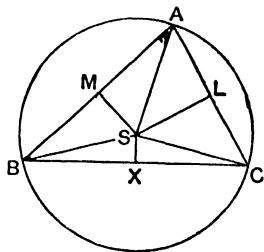
For convenience certain propositions (proofs of which were only outlined in the text) are given in full here.

PROPOSITION P.

General Enunciation. (i) *To circumscribe a circle about a given triangle, and (ii) to prove that the perpendicular bisectors of the three sides of a triangle must be concurrent.*

Particular Enunciation. ABC is the triangle. (i) It is required to circumscribe it with a circle, and (ii) to prove that the perpendicular bisectors of BC, CA and AB meet at a point.

Construction. LS and MS are the perpendicular bisectors of the two sides CA and AB.



They meet at S. SX is perpendicular to BC.

AS, BS, CS are joined.

Proof. (i) In the triangles BSM, ASM,

$$\therefore \begin{cases} BM = AM \text{ (construction),} \\ MS \text{ common,} \\ \angle BMS = \angle AMS \text{ (rt. } \angle \text{s),} \end{cases}$$

$\therefore \triangle \text{s BSM, ASM are congruent,}$
and, in particular, $BS = AS$.

Similarly $CS = AS$;

$\therefore AS = BS = CS$, and S is the centre of the circle circumscribing the $\triangle ABC$ (i). **Q.E.F.**

(ii) Now SX is, by construction, perpendicular to BC ; and BC is a chord of the circle ;

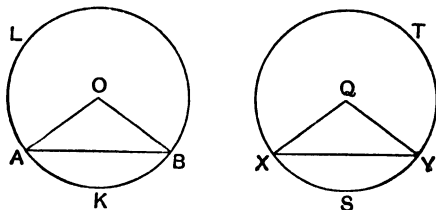
$\therefore$ SX bisects BC,

so that the perpendicular bisectors of the sides BC, CA and AB are concurrent (ii). **Q.E.D.**

PROPOSITION Q.

General Enunciation. *In equal circles equal chords must cut off equal arcs (the minor arcs being equal and the major arcs being equal).*

Particular Enunciation. AB and XY are the equal chords of the equal circles, whose centres are O and Q; to prove that the (minor) arc AB = the (minor) arc XY, and likewise the major arcs.



Construction. Join AO, BO, XQ and YQ.

Proof. In the Δ s AOB, XQY,

$$\therefore \begin{cases} AO = XQ \text{ (given),} \\ OB = QY \text{ (given),} \\ AB = XY \text{ (given);} \end{cases}$$

$\therefore$ the Δ s AOB, XQY are congruent,

and, in particular, $\angle AOB = \angle XQY$.

Now equal angles, in equal circles, subtend equal arcs;

$\therefore$ (minor) arc AKB = (minor) arc XSY,

and the whole circumferences of the circles are equal; and

$\therefore$ by subtraction, the (major) arc ALB = (major) arc XTY.

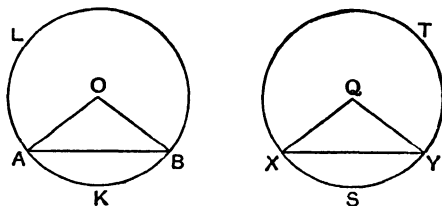
Q.E.D.

The next proposition (**R**) is the converse of the preceding (**Q**).

PROPOSITION R.

General Enunciation. *In equal circles equal arcs must be subtended by equal chords.*

Particular Enunciation. O and Q are the centres of the equal circles. It is given that the arc AKB = arc XSY, to prove chord AB = chord XY.



Construction. Join AO, BO, XQ and YQ.

Proof. Because arc AKB = arc XSY,

$$\therefore \angle O = \angle Q.$$

Now, in the Δ s AOB, XQY,

$$\therefore \begin{cases} AO = XQ \text{ (given),} \\ OB = QY \text{ (given),} \\ \angle O = \angle Q \text{ (proved) ;} \end{cases}$$

$\therefore$ the Δ s AOB, XQY are congruent,

and, in particular, $AB = XY$,

so that chord AB = chord XY.

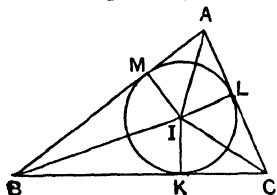
Q.E.D.

PROPOSITION 8.

General Enunciation. (i) *To inscribe a circle in a triangle, and (ii) to prove that the bisectors of three angles of a triangle must be concurrent.*

Particular Enunciation. ABC is the triangle. (i) It is required to inscribe a circle in it, and (ii) to prove that the bisectors of its 3 angles A, B and C are concurrent.

Construction. Bisect the two angles B and C by straight lines meeting at I. Join AI.



Draw IK, IL and IM perpendicular to BC, CA and AB respectively.

Proof. In the $\triangle$ s BIK, BIM,

$$\therefore \begin{cases} \angle KBI = \angle MBI \text{ (construction),} \\ \angle IKB = \angle IMB \text{ (construction),} \\ BI \text{ common;} \end{cases}$$

$\therefore$ the $\triangle$ s BIK, BIM are congruent,
and, in particular, $IK = IM$.

Similarly $IK = IL$;

$\therefore IL = IM$,

so that a circle with centre I, and radius IK, will also pass through K, L and M, and because the angles at K, L and M are right angles (construction) the circle must touch the sides (i).

Now, in the rt. $\angle$ d. $\triangle$ s AIL, AIM

Q.E.F.

$$\therefore \begin{cases} AI \text{ common,} \\ IL = IM \text{ (proved),} \\ \angle ALI \text{ and } \angle AMI \text{ are right angles by construction,} \end{cases}$$

$\therefore$ the $\triangle$ s AIL, AIM are congruent,
and, in particular, $\angle IAL = \angle IAM$,
so that AI bisects $\angle BAC$.

Hence the bisectors of the 3 angles of $\triangle ABC$ are concurrent (ii).

Q.E.D.

PROPOSITION T.

General Enunciation. *To construct a common tangent to two given unequal circles.*

Particular Enunciation. O and Q are the centres of the two given circles. [O is the centre of the larger circle.]

Fig. i.

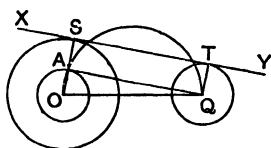
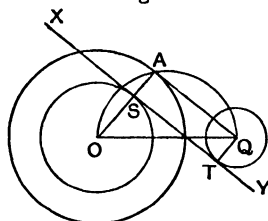


Fig. ii.



Construction. Join OQ. On it, as diameter, describe a semicircle. With centre O and radius equal to the *difference* of the 2 given radii, for the direct common tangent (Fig. i.), or to the *sum* of the 2 given radii, for the indirect common tangent (Fig. ii.), describe a circle to cut the semicircle at A. Join OA, and let it (produced if necessary) cut the O circle at S.

Draw QT parallel to AS.

Join ST to get XY.

Proof. OS is the radius of the larger circle.

OA is the difference (Fig. i.) or sum (Fig. ii.) of the radii ;

$\therefore$ by subtraction $AS = \text{radius of smaller circle}$;

$\therefore$ AS and QT are equal and parallel ;

$\therefore$ ASTQ is a parallelogram,

and angles at A are right angles (angle in semicircle) ;

$\therefore$ ASTQ is a rectangle ;

$\therefore \angle S$ and $\angle T$ are right angles ;

$\therefore$ XY touches both circles.

Q.E.F.

The following proposition is the converse of 36.

PROPOSITION U.

General Enunciation. *If the base of a triangle is divided (i) internally or (ii) externally in the ratio of the sides, then the join of the point of the division to the vertex of the triangle must bisect the vertical angle (internally or externally).*

Particular Enunciation. ABC is the triangle.

(i)

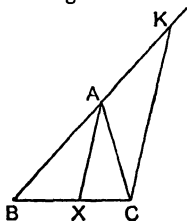
X divides BC internally,
so that

$$BX : XC = BA : AC ;$$

to prove AX bisects

$\angle BAC$ internally.

Fig. i.



(ii)

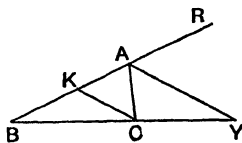
Y divides BC externally,
so that

$$BY : YC = BA : AC ;$$

to prove AY bisects

$\angle BAC$ externally.

Fig. ii.



Construction and Proof.

Draw CK parallel to XA
to meet BA, produced, at K ;

Now XA and CK are parallel,
so that $BX : XC = BA : AK$,

but $BX : XC = BA : AC$ (given),

hence $AK = AC$;

$\therefore \angle AKC = \angle ACK$,

but $\angle AKC = \angle BAX$ (corr.),

and $\angle ACK = \angle CAX$ (alt.) ;

$\therefore \angle BAX = \angle CAX$;

$\therefore AX$ bisects $\angle BAC$ int.

Draw CK parallel to YA
to meet BA at K ;

Now YA and CK are parallel,
so that $BY : YC = BA : AK$,

but $BY : YC = BA : AC$ (given),

hence $AK = AC$;

$\therefore \angle AKC = \angle ACK$,

but $\angle AKC = \angle RAY$ (corr.),

and $\angle ACK = \angle CAY$ (alt.) ;

$\therefore \angle RAY = \angle CAY$;

$\therefore AY$ bisects $\angle BAC$ ext.

Q.E.D

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ANSWERS

ANSWERS.

NOTE AS TO ANSWERS.

ALL these answers have been calculated (using 4-figure tables and Trigonometry, if necessary), and results are given to a degree of accuracy at which you might aim.

In Geometrical Drawing you will get results varying in reliability according to the intersections involved and to your skill; but too great reliance cannot be put on answers obtained *solely* by Geometrical Drawing.

Frequently the (+) or (-) will give you some clue. [Answer 9.87(+) means more than 9.87, but less than 9.875.] Between ... and ... has been given for results somewhere near midway. Notice that "(very nearly)" only implies that the approximation is very accurate; it may be too much or too little. In square brackets are given some numbers which may be useful as checks.

In Calculations π has been taken as $3\frac{1}{2}$ (unless the contrary is implied). After using $3\frac{1}{2}$ it is *definitely wrong* to give an answer like 19.643 sq. cm., not merely unnecessarily accurate. 3 significant figures is the proper way. (As a matter of fact, when $3\frac{1}{2}$ has been used, the third significant figure is generally right, but not invariably correct.)

EXAMPLES 18 a. (Page 235.)

10. 9.5 sq. in.(+).

EXAMPLES 18 b. (Pages 235 to 241.)

- | | | |
|---|---|-----------------|
| 1. About $40\frac{1}{2}$ cm., $3.13(-)$. | 2. $3.12(-)$. | 3. 15.7 cm.(+). |
| 4. 251 yds.(+). | 5. 6.3 cm.(-); the same. | 6. About 22 ft. |
| 7. 10,900 yds.(+)=6 mi. 340 yds.(+). | | 8. About right. |
| 9. (i) 20 sq. cm.(-); (ii) 2.7 cm. (very nearly). | 10. 64.3 sq. cm.(-). | |
| 11. About $38\frac{1}{2}$ sq. yds., 1100 sq. yds. | 12. About $9' 2''$; about $334\frac{1}{2}$ sq. ft. | |
| 13. $9.43''(-)$; 28.3 sq. in.(-). | 15. 59 sq. cm.(-); about 4.9 cm. | |
| 16. 3 sq. in., 5 sq. in., 7 sq. in. | 17. About $6''$ and $8''$ | |
| 18. 201 sq. ft.(+). | 19. 3.15 | 20. $3.1(-)$. |

21. 5600 sq. yds., very nearly $59\frac{1}{2}^\circ$; 5100 sq. yds.(-); 500 sq. yds.(+).
 22. 8.66... cm. [$5\sqrt{3}$]; about 161 sq. cm. 23. 0.6 mm.(-).
 24. $18.3''$ (-). 25. 108° , 72° , 36° ; 2:1.
 27. Between 5.6 yds. and 5.7 yds. [$4\sqrt{2}$]; 183 sq. yds.(-).
 28. $s^2\left(\frac{\pi}{2} - 1\right)$ or about $0.57s^2$.
 29. $\frac{1}{4}\left(\sqrt{3} - \frac{\pi}{2}\right)$ times the square of a side of the given triangle, or about 0.04 times etc.
 31. (b) 117 sq. in. (nearly) $\left[\frac{400\pi}{9} - \frac{64\pi}{9} = \frac{112\pi}{3}\right]$; (d) 1170 cu. in. (nearly);
 (e) about 5.6" and 14"; (f) about 670 sq. in.; (g) 7 lb.; (h) about 878 cu. in.; (i) about 48 lb.
 32. 3.2 33. $8.23''$ (-). 34. Approximation too small.
 35. $76''$ (+). 36. $1.2''$ (-). 37. About 72.4 sq. ft.
 38. 19.8 sq. cm.(+). 39. 524 mi.(-).

EXAMPLES 19 a. (Pages 249 and 250.)

1. About 3.8 cm. or $1.5''$ 2. About 5.5 cm. or $2.2''$
 3. 13.1 cm. (very nearly). 4. $78\frac{1}{2}^\circ$ (-); between $1.68''$ and $1.69''$
 5. Between 186 yds. and 187 yds. 6. 338 yds.(-). 7. About $2.8''$
 8. About $1.66''$ 9. $2.02''$ (-). 10. About 5.4 cm., 2.7 cm.
 11. $2.90''$ (+). 12. $0.93''$ (-). 13. $9.6''$ (-).

EXAMPLES 19 b. (Pages 250 to 252.)

1. $17''$ 2. $7''$ 3. $1.6''$ 4. $7''$, $20''$, $13''$ or $27''$
 5. $24''$; $17''$ or $31''$ 6. 6.4 cm. or 1.4 cm.
 7. 5 ft., 12 ft., 17 ft. or 7 ft. 8. $4''$, 12 sq. in.
 9. $3''$, $4.58''$; $7.58''$ or $1.58''$ 10. 78 cm.
 11. $15''$, $15.49''$; $30.49''$ or $0.49''$ 12. Concentric circle with radius $3''$
 13. About $7.11''$ (and a negative solution). 14. $2''$ 15. $13''$
 16. $221''$ 17. $50''$ or $8''$ 18. $13''$ 19. 26 ft.
 20. About 32 mi.

EXAMPLES 20. (Pages 257 to 260.)

14. They coincide; they may still be said to "meet" at the point (among others) outside the triangle.

EXAMPLES 21 a. (Pages 270 to 273.)

3. Between $3.06''$ and $3.07''$ 4. 35° .
 5. About 2.9 cm. and about 5 cm. 6. 33° and 98° (very nearly).
 7. About $1.77''$ 8. About 4.1 cm. 9. Very little over 3 mi.

10. Very little over 2.8 mi. 11. Between 3.89" and 3.30"
 12. 4.1 cm. (-). 14. About 680 yds. 15. About 2.7 cm. or 1.1" (-).
 16. 2.8 cm. (-). 17. 3.4 cm. (-). 18. 7.5 cm. (+).
 19. 8.1 cm. (-). 20. 7.6 sq. in. (+). 21. 4.4 cm. (very nearly).

EXAMPLES 21 b. (Pages 273 and 274.)

1. 15° . 2. 120 yds. 3. 38° , 66° , 52° .
 4. $\angle B = \angle D = 10^\circ$, $\angle BCD = 60^\circ$, $\angle ACB = 65^\circ$, $\angle KAC = 45^\circ$, $\angle CAP = 75^\circ$
 $\angle P = 50^\circ$, $\angle AKB = 110^\circ$. 5. 30° , 70° (or 110°).
 6. Beside the given angles the following occur: 14° (twice), 24° (twice),
 38° , 48° , 52° (twice), 62° (twice), 66° , 76° , 104° , 118° (twice), 132° .
 7. The nine angles are: 22° , 75° , 75° , 105° , 127° ; (a) 127° , (b) 22° , (c) 53° ,
 (d) 105° , of which the last four are those for which reasons are required.
 8. 20° , 5° .

EXAMPLES 23 a. (Pages 286 to 288.)

1. 160° . 2. 80° . 3. 124° . 4. 70° or 110° . 5. 21° .
 6. 15° , 20° ; about 39%. 7. 42° , 37° , 33° , 113° .
 8. $(90 - x)^\circ$, $(90 + x)^\circ$, 90° , $(90 - x)^\circ$.
 9. 57° , 50° , 66° , 114° , 107° ; 83° (or 97°). 10. 112° , 76° , 46° , 30° .
 11. 112° , 74° , 28° .
 12. 15° , 35° (twice), 45° (twice), 50° (twice), 85° (twice), 100° , 130° (twice).
 13. $\angle A = 82^\circ$, $\angle B = 122^\circ$ (given), $\angle C = 98^\circ$, $\angle D = 58^\circ$; 80° (or 100°).
 14. 65° , 80° , 100° , 55° . 15. $(2n - 2)$ rt. $\angle$ s.

EXAMPLES 24 a. (Pages 301 to 304.)

1. $1.62''$ (-). 2. 4.9 cm. (+). 3. Between 4.6 cm. and 4.7 cm.
 4. Between 3.46" and 3.47" 5. 8.5 cm. (-).
 6. $2.68''$ (+). 8. About 1.7 cm. or 6.8 cm. 9. $2.54''$ (very nearly).
 10. Between $5.03''$ and $5.04''$ } or { between $2.96''$ and $2.97''$
 Between $2.60''$ and $2.61''$ } { between $1.53''$ and $1.54''$
 11. Between $3.09''$ and $3.10''$ 12. Two solutions. 13. 3.8 cm. (+).
 15. $2''$ 17. Between $2.23''$ and $2.24''$ 18. 2.1 cm. or 9.9 cm.
 19. 14 units or 6 units. 25. Between $2.39''$ and $2.40''$ 27. 2.43 mi. (-).

EXAMPLES 24 b. (Pages 304 to 307.)

1. 130° , 65° . 2. $83^\circ 06'$. 3. $5''$. 4. $\sqrt{c^2 - l^2}$ ft.
 5. $6.04''$ (+). 6. $2.1''$. 7. 2 ft. 8. $28.2''$ (+).
 9. $\left(90 - \frac{x}{2}\right)^\circ$, $\left(90 + \frac{x}{2}\right)^\circ$. 10. (1) About $31.06''$; (2) about $31.26''$
 11. 4.5 cm. 12. $\sqrt{c^2 + d^2 - r^2}$. 13. 100° , 30° , 50° .
 14. $\frac{1}{2}(\beta + \gamma)$, etc., or $\frac{1}{2}(\pi - \alpha)$, etc. 15. 22 mi. (+); 270 ft. (-)
 16. Under 7000 ft. [6666 $\frac{1}{2}$ is too great a pretence to accuracy.]

17. 14 mi. (—), [13·69], between 21 mi. and 22 mi. = $18\frac{1}{2}$ sea miles (about).
 18. About 220 ft. [It is better to give only an approximate answer here.]
 19. Nowhere near visible. 20. $\frac{1}{2}\theta$. [Recollect that θ is in radians.]
 21. $30''$.
 22. $r\sqrt{1 + \frac{1}{9}\pi^2}$, where r is the radius of the red or about $4\cdot3r$ etc.
 23. Between 4·6 cm. and 4·7 cm. ; between 5·3 cm. and 5·4 cm.
 24. Nowhere near visible.

EXAMPLES 25 a. (Pages 315 to 317.)

3. Between $0\cdot63''$ and $0\cdot64''$ 4. $1\cdot14''(+)$.
 5. $0\cdot56''(-)$, about $0\cdot98$ sq. in. 6. $0\cdot72''(+)$.
 7. 2·9 cm. (very nearly). 8. $0\cdot66''(+)$. 10. Between $0\cdot50''$ and $0\cdot51''$
 11. $0\cdot50''(-)$, between $0\cdot73''$ and $0\cdot74''$ 12. Between $2\cdot71''$ and $2\cdot72''$
 13. Between 2·6 cm. and 2·7 cm., 4·2 cm. (+), 3·1 cm. (—). 14. $0\cdot66''(-)$.
 15. Between 2·0 cm. and 2·1 cm. 16. $0\cdot58''$ (very nearly).
 17. $0\cdot93''$ (very nearly). 18. $1\cdot14''(-)$. 19. 2·1 cm. (—).
 20. Between 3·1 cm. and 3·2 cm. 22. $2\cdot22''(+)$, about $3\cdot20''$
 23. 3·6 cm. (—), between 3·2 cm. and 3·3 cm., 1·4 cm.

EXAMPLES 25 b. (Pages 318 and 319.)

10. 100° , 30° , 50° .

EXAMPLES 26. (Pages 323 to 326.)

10. 65° , 35° , 45° , or 45° , 35° , 65° (two solutions).
 17. 100° , 60° , 20° . 18. A — B — C, 2B, 2C.

EXAMPLES 27 a. (Pages 330 to 332.)

1. $2\sqrt{r_1 r_2}$. 2. 4 cm. 3. About 9·95 cm. [$\sqrt{99}$].
 4. Impossible; (the length of a common tangent is necessarily not greater than the distance between the centres).
 5. About $29\frac{1}{4}''$ 6. $2\cdot77''(-)$. 7. (b) About 139° .
 8. Between $106''$ and $107''$
 9. (b) 6 cm. ; (c) $286^\circ(+)$, same ; (d) 25 cm. (—), 15 cm. (—) ; (e) 52 cm. (—).
 (g) About $62\cdot4$ cm², and about $22\cdot5$ cm². ; (h) $18\frac{1}{2}$ cm², $6\frac{1}{2}$ cm². ;
 (i) about $110\cdot4$ cm².
 10. $\sqrt{c^2 - (r_1 - r_2)^2}$ 11. $\sqrt{c^2 - (r_1 - r_2)^2}$; $\sqrt{c^2 - (r_1 + r_2)^2}$.
 12. 1090 sq. in. (+) ; about $15''$
 13. 15 half cm. ; $236^\circ(+)$; $124^\circ(-)$; two arcs = 52 half cm. (—) ; boundary = 82 half cm. (—) ; lap = 820 yds. (—) ; 6·4 laps (+).
 14. $169^\circ(+)$, $191^\circ(-)$, about 149·5 cm., at least 150.
 15. $2\sqrt{a^2 - c^2} + c\left(\pi + 2\sin^{-1}\frac{c}{a}\right)$.
 16. (a) About 33·4 cm. ; (b) about 72·7 sq. cm. 17. 2·4 in.

ANSWERS

v

EXAMPLES 28 a. (Pages 338 to 342.)

3. 7 more [8 in all]. 4. 4.2 cm.(-), or between 0.6 cm. and 0.7 cm.
5. 1, 3 (including contact at R). 6. 4.
7. About 19.3 sq. cm; no. 8. Between 10.2 cm. and 10.3 cm.
13. About 17.5 chains. 14. 60 sq. cm.(-). 19. $1.16''$ (+).
20. $23\frac{1}{2}^\circ$ (very nearly). 21. 2.0 cm.(-); 4.6 cm.(+).
22. Between 0.27 and 0.28 or between 1.02 and 1.03 times a side of the $\triangle$.
23. 2.4 cm.(-).

EXAMPLES 28 b. (Pages 342 to 344.)

14. Maximum number of solutions is 8.
22. $\frac{1}{2}R$ (where R is the radius of the larger circle).
23. $\frac{1}{3}R, \frac{\sqrt{3}}{4}R$ (that is $0.375R$, about $0.433R$). 24. 3 solutions.

EXAMPLES 32 b. (Pages 360 to 362.)

1. $13'', 6''$ 2. 25 cm., 40 cm.
3. The two parts of the line are 72.4 cm.(-) and 27.6 cm.(+).
4. $+0.62''$... from an end or $-1.62''$... from an end.
5. Forwards $3.49''$..., or backwards $37.49''$...
6. The two parts of the line are $28.41''$... and $3.59''$...
7. (1) Acute-angled; (2) obtuse-angled; (3) right-angled; (4) obtuse-angled.
9. 3.6 cm. 10. Acute-angled; 20 cm. 11. $\frac{a^2 + b^2 - c^2}{2a}$.
12. $\frac{1}{2}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} = \sqrt{s(s-a)(s-b)(s-c)}$.
13. Longest median corresponds to shortest side; $9\frac{1}{2}$ in. and $9\frac{3}{4}$ in.
14. About $10.30''$, $8.89''$ (-), $6.78''$ (+).
15. $\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ (and two similar expressions preserving cyclic order).
18. 13.8 cm.(+), 12.8 cm.(-), 11.5 cm.(+).
17. $\frac{1}{2}\sqrt{2m^2 + 2n^2 - p^2}$ (and two similar expressions preserving cyclic order).
18. [$\sqrt{199}$, $\sqrt{159}$], about 14.1 units(+), about 12.6 units(+).
19. [$\sqrt{136}$], between 11.6 units and 11.7 units.
20. [$\frac{1}{2}\sqrt{1537}$], about 19.6.

EXAMPLES 33. (Pages 365 to 367.)

3. $AB = BC = 1.84''$ (-), $CD = 2.4''$, $BM = 1.3''$, $DN = 0.92''$ (+);
AN : NM : MC = 50 : 119 : 169.
4. 8 : 35. 8. 4 : 1 : 12.

ANSWERS

EXAMPLES 34 a. (Pages 371 and 372.)

1. Between 2.2 cm. and 2.3 cm.
2. 4.0 cm.(+).
4. 1.7 cm.(-).
5. 2.3 cm.(-).
6. (1) Between 1.3" and 1.4"; (2) about 0.4"; (3) about 2.7 cm.; (4) about 2.6 cm.; (5) about 5.2 cm.
7. 7.4 cm.(+), 2.4 cm. (very nearly).
8. (1) Between 10.1 cm. and 10.2 cm.; (2) 8.5 cm.(+); (3) 11.1 cm.(-).
9. Between 6.6 cm. and 6.7 cm.
10. 8.5 cm.(-).
11. 4.34"(-).

EXAMPLES 35 a. (Pages 379 and 380.)

1. Between 2.36" and 2.37"
2. 2.8 cm. to 2.9 cm.
3. About 3.0"
4. 4.19"(-).
5. 3.72" (very nearly).
6. 7.7 cm.(+).
7. (1) 2.45(-); (2) 1.90(-).
8. Between 400 and 401.
9. Between 3.28" and 3.29"
10. About 157 yds. and 122°.
11. 6½ mi. and 357° (both very nearly).

EXAMPLES 35 b. (Pages 381 to 384.)

1. 69 ft.
2. 200 yds.
3. 250 paces.
4. 1350 yds.
5. 112 mm.
6. 350 yds. [The question says "nearest 50 yards." Though 363 yds.(+) may be obtained by working, some of the data must only be approximate.]
7. 7.5 cm.
9. About 50 ft. [49½].
10. About 56 ft. [55½].
11. $\left(\frac{b}{2} - \frac{d}{6}\right)_b^a$ yds.
12. Not visible.
13. 9" and 1" (or *vice versa*).
14. $\frac{8}{10}\sqrt{10}$ times BC, i.e. 0.95(-) times BC.
15. Very little over 1007 yds. [$\sqrt{540^2 + 850^2}$]; but with necessarily approximate data the answer "About 1000 yds." is wiser.

EXAMPLES 35 c. (Pages 384 to 387.)

3. 2 : 3.
23. 3 : 1.
29. $\sqrt{\frac{1}{2}(a^2 + b^2)}$.

EXAMPLES 37 a. (Pages 397 and 398.)

1. 40 sq. cm.
2. 5 sq. in.
3. Just over 4.6 cm.
4. 2.3 cm.(-).
5. 1.8 cm.(-) or 5.8 cm.(-).
6. About 2240 ft.
7. 2.79" (very nearly).
9. 0.58"(+).

EXAMPLES 37 b. (Pages 398 to 401.)

1. 6 cm.
2. 4 cm.
3. 5" from D.
4. About 3½ mi. [$\sqrt{199.5} - \sqrt{108}$].
5. Under 55 mi. [$\sqrt{3000} = 54.77$].
6. 135 mi.(+); 122 mi.(-).
7. 13½ yds.
8. 15"
9. 42 mm.

10. 19 cm.(-) [18.9(+)] ; 19.5 cm.(-) ; about 18.4 cm.
 11. (1) About $17\frac{3}{4}$ mi. ; (2) about $15\frac{1}{2}$ sea miles.
 12. $\frac{1}{2h}(h^2 + k^2)$. 13. $y^2 + 2yr = x^2$.
 14. (1) About $4\frac{3}{4}$ mi. [$\sqrt{76\frac{1}{2}} - \sqrt{16\frac{1}{2}}$] ; (2) about 4 sea miles.
 15. 5" ; C is on AB. 16. 12 mi.(+), about $10\frac{1}{2}$ sea miles.
 17. $\sqrt{21}$ cm. = 4.6 cm.(-) ; numbers are 8.6(-) and 1.4(+).
 18. About 2240 ft. [$\sqrt{2000 \times 2500}$] ; angle 35° does not matter.
 19. Between 148 ft. and 149 ft. from the base of the tower.
 20. Easily visible. 21. 418 ft.(-). 22. No ; no ; yes (easily).
 23. Yes ; no. 24. Not more than 77.3 mi.

EXAMPLES 38. (Page 406.)

2. At a point in AB at distances $\frac{bc}{a+b}$ ft. from A and $\frac{ac}{a+b}$ ft. from B.

EXAMPLES 39 a. (Pages 412 to 414.)

1. 2.08"(-). 2. Between 4.5 cm. and 4.6 cm. 3. 8.6 cm.(-).
 4. About 2.42", between 3.36" and 3.37", 2.24"(+) ; between 2.69" and 2.70"
 5. Between 2.91" and 2.92" 6. Between 2.37" and 2.38"
 7. 6.5 cm.(+). 8. Between 4.02" and 4.03"
 9. 1.52"(-) [it is taken that $\triangle$ is $\frac{2}{3}$ area and trapezium $\frac{1}{3}$, and not *vice versa*].
 10. $1\frac{1}{2}$ cm., about 0.47 cm. 11. About 0.60" 12. About 2.01"
 13. 1.12"(-). 14. 3.37"(-).

EXAMPLES 39 b. (Pages 414 and 415.)

5. About 3900 cu. ft., about 27 tons. [Do not pretend to greater accuracy. The data must be only approximate.]
 6. Bull diameter 5"(+) ; widths of spaces 1"(+), about 0.8", 0.7"(-).
 7. 3 : 8. 8. 1.125 sq. m. 10. $\frac{1}{16}$. 12. $\frac{9}{16}$. 15. 1 : 3.

PRICKING-OFF DIAGRAM

Pricking off.

Constant pricking off wears out the exact points. It is not a bad plan to prick off the points very occasionally, on to a loose piece of paper, to keep that piece of paper in the book, and to use that piece of paper. It has the additional advantage of allowing you to prick off your figure in the position on your paper most convenient to you.

It is not the exactness of your answers which is aimed at. It is that you adopt the correct principle and *clearly*. In the book answers are given to two significant figures, or even less, and that is quite enough. (This paper is sure to expand and contract a bit in different weathers. Areas can vary certainly 1%, so that an answer to many significant figures is only a pretence to accuracy.)

