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A SCHOOL GEOMETRY

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SCHOOL GEOMETRY

by

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Specimen for consideration

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PREFACE

IN the beginning of this century Euclid's *Elements of Geometry* was expelled from English schools as an unsuitable book for the young, and one generally accepted consequence of this revolution has been that the fundamental theorems on the congruence of triangles and on parallels are, at first, assumed on experimental or intuitional grounds. The proofs are now deferred to a much later stage, when the pupil has already had a training, lasting for three or four years, in deductive methods, and is of an age to appreciate, more or less, what constitutes a sound proof. Now it is found that the proofs usually given of these fundamental theorems do not satisfy him; they are, indeed, often faulty and sometimes he perceives this clearly; at other times he merely feels vaguely uneasy, and the faint stirrings of his critical conscience may perhaps be misinterpreted as stupidity. This is a most unsatisfactory state of affairs, for Geometry is the only subject taken by all pupils in schools, wherein the power of deduction is made manifest; that is its main educational value—as distinct from the facts imparted. The proofs of these fundamental theorems, and of all others, should be made sufficiently rigorous to convince the pupil at the stage he has reached: they need not conform to any absolute standard of rigour, if indeed such a standard exists.

In this book the standard adopted for these theorems is approximately that of Euclid, modified in accordance with modern criticism, and it is the author's aim to raise the rigour to this standard *gradually*, at first to be content with a low standard and continuously to improve it, and to keep always in mind the age of the pupil and the sort of proof that will, or ought to, satisfy *him*.

I have adhered to the ordinary terminology except in one instance. I distinguish between 'line' and 'interval'; the first

is the infinite straight line; the second is a finite piece of a straight line. The common use of the word 'line' in two senses does, in fact, cause confusion, and some change is long overdue. In other cases where a word is commonly used in two senses, but no confusion results, I have conformed to custom, and contented myself with pointing out the double use. In choice of terms, in degree of exactness of expression, in order of treatment as in type of demonstration, I have been guided by psychological rather than by logical necessities, and a pupil who has mastered the latter part of this book will be able to view the earlier part from a stricter standpoint than I have myself adopted.

A large number of exercises is given, such as are not 'original' have been taken from the immense stores which exist in the old editions of Euclid. Many of these are of such an obvious nature that they can be done by all pupils; these are followed by harder ones for the better pupils and for revision purposes, and they are grouped round central ideas, so that the clue to one is often given by the preceding questions. Attention is also called to the large number of worked Examples, which usually embody important geometric truths.

I have interspersed sections on such subjects as irrational numbers, rational triangles, dissections, and so on, which never fail to evoke interest, and frequent notes calling attention to points which might otherwise escape notice. The facts of solid geometry have been introduced gradually, without proofs, and questions involving three dimensions have been included.

The book presupposes a course in experimental Geometry, the length of which depends on the ability of the pupil, but it contains all that is necessary for revising such a course. In the first part will be found a collection of drawing exercises, many of an unusual type. Experience shows that if these are taken at the beginning of the deductive course, very great interest is created, and in working through them, a pupil not only acquires proficiency in the use of instruments, but learns unconsciously a large amount of empirical geometric data, which are invaluable in his later theoretical reasoning.

Full consideration has been given to the demands of Examinations and the extra stress, that is laid on logic in this book is of value even from that standpoint, for it is the common experience of Examiners, that pupils brought up on stricter diet do, as a rule, much better than those nourished on some of the more careless and more loosely constructed 'modern' Geometries.

Acknowledgment is due to Mr H. E. Dudeney for permission to use some questions in his *Amusements in Mathematics*.

I am very greatly indebted to my colleague, Mr J. W. Maddock, and to my old friends, Mr H. Coates, Senior Mathematical Master at Mill Hill School, and Mr J. H. Doughty, Senior Mathematical Master at Leigh Grammar School, for their invaluable help, which began when the work was still in manuscript, and did not cease till it was in print.

And it is a pleasure once again to express my gratitude to the Syndics and officials of the University Press for the kindness and consideration they have shown me during the production of this book.

August, 1930

PART I

§ 1. Revision of the Fundamental Ideas.

Angles If a boy faces North and then turns completely round, he will again face North. We say he has made a complete turn. If he makes one-quarter of a complete turn, we say he has turned through one *right angle*. A right angle is divided into ninety equal parts, each part is called a *degree*.

EXERCISE 1 (Oral)

1 A boy faces North and then turns South. Through how many right angles has he turned and through how many degrees?

2 A boy faces North and then turns until he faces South-east. Through how many right angles has he turned and through how many degrees? (Two answers.)

3 Through how many degrees does the hour hand of a watch turn in 12 hours, 1 hour, 4 hours?

4 What angle do the hands of a clock make with each other at 3 p.m., 6 p.m., 2 p.m.? Express the answers in right angles and degrees.

5 A wheel has twenty spokes, what is the angle between two spokes that are next to each other?

6 Hold your pen and ruler so that the angle between them is 30° , 60° , 90° , 45° , 120° , 180° , as near as you can guess.

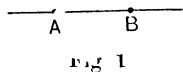
7 A flywheel makes one revolution each second, how long does it take to turn through 1° , 30° , 45° ?

8 If the earth made a complete turn round its axis in 24 hours, through how many degrees would it turn in (i) one hour, (ii) one minute?

9 Two wheels are geared so that one turns round four times while the other turns round once. Through how many degrees does the second wheel turn, when the first wheel turns half-way round, one-third way round?

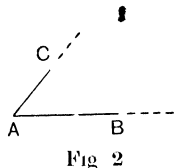
[It is advisable in work on the board not to use letters, at first, to denote points, angles, and so on, but to mark and point out the objects mentioned, even in places where letters have necessarily to be used in this book.]

Straight lines and intervals. Everyone has an idea of what is meant by a straight line. If you took a very long ruler and tried to draw a straight line on a sheet of paper, you would have to stop when you came to the edge of the paper. The straight lines which we shall talk about are supposed to go on in both directions without coming to an end. A piece of a straight line of definite length, with two ends, is called an 'interval'. We shall usually omit the word 'straight' and say 'line' instead of 'straight line'. When we say 'draw a line' or 'draw a straight line,' we mean you should draw as much of it as you can, or as much as is necessary. The 'line AB ' means the (infinite) line which goes through the points A and B (Fig. 1). The 'interval AB ' is the part of the line between A and B . If ABC be a triangle, the 'side AB ' is an interval, the 'line AB ' is that interval produced in both directions without end.



EXERCISE 2

- 1 Draw an interval two inches long.
- 2 Draw two intervals which have one end-point the same, as in Fig. 2, where AB and AC are the intervals. If the interval AB is produced beyond B , and the interval AC is produced beyond C , is the angle between the intervals altered?



- 3 Can you draw a straight line on the surface of the earth?
- 4 If a straight line were drawn through a point in England and a point in Australia, where would it go?
- 5 Draw as much of a straight line as you can, and mark off on it three intervals AB , BC , CD , each one inch long.

Two lines, or two intervals, or a line and an interval are said to 'cut' or 'meet' in a point which is on both. Never use the word 'touch' instead of 'cut' or 'meet'.

The angles a , b in Fig. 3 are called 'adjacent' angles, the angles x , y in Fig. 4 are called 'vertically opposite'. An angle is usually

† In later work we shall often say ' AB ' instead of 'the line AB '.

name, by three letters. Thus the angle a in Fig. 3 would usually be called the angle AOB or $\hat{AOB}$. OA , OB are the 'arms' of the angle, O is the 'vertex'. Notice that in the name of the angle, the middle letter denotes the vertex.

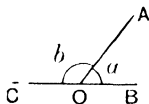


Fig. 3

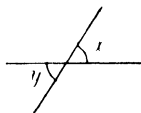


Fig. 4

§ 2 Perpendiculars.

EXERCISE 3

- 1 Draw a figure like Fig. 3, and measure the adjacent angles.
- 2 Draw a figure like Fig. 4, and measure the vertically opposite angles.
- 3 Draw a figure like Fig. 3, but so that the angles a and b are as nearly equal as you can make them. Notice that when these adjacent angles are equal, they are right angles.
- 4 Draw a figure like Fig. 3, and draw a line as nearly as you can at right angles to OA . Two lines at right angles to each other are also said to be 'perpendicular' to each other.

It is usual in Geometry to state quite clearly all the important things to be remembered. The following should be learnt.

Definition If two (straight) lines cut one another so as to make a pair of adjacent angles equal, the angles are called 'right angles,' and the lines are said to be 'perpendicular' to one another.

An 'acute' angle is an angle less than a right angle.

An 'obtuse' angle is an angle greater than a right angle.

Any two angles whose sum is two right angles are called 'supplementary' angles.

We now make some statements which can be seen to be true, and which will be found useful to us as we go on. We call them Assumptions.

Assumption 1 When two (straight) lines cut each other, the sum of two adjacent angles equals two right angles (or 180°)

Assumption 2. When two lines cut each other, any two vertically opposite angles are equal

Assumption 3. If the points B, C are on opposite sides of the line OA , and the sum of the angles AOB, AOC equals two right angles, then B, O, C are in one line

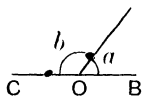


Fig. 3a

Assumption 4 All right angles are equal to one another.

We have distinguished between a line and an interval. It is now important to note another distinction. If AB is a line, the part of the line which starts at A , goes through B and goes on

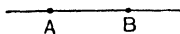


Fig. 5

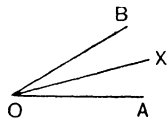


Fig. 6

then for ever, is called the 'ray' AB . Thus the arms of an angle AOB are the ray OA and the ray OB . If OX is a ray inside the angle AOB , so that the angles AOX, BOX are equal, then we say that the ray OX 'bisects' the angle AOB (Fig. 6)

EXERCISE 4

1 Draw, with your protractor, angles of the following size: 45° , 60° , 30° . What is the size of their supplementary angles?

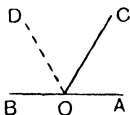


Fig. 7

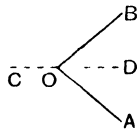


Fig. 8

2 (Fig. 7) The ray OD bisects $\angle COB$, $\angle COA = 60^\circ$. How big are $\angle BOD, \angle DOA$?

3 (Fig. 8) The ray OD bisects $\angle AOB$ $\angle AOB = 80^\circ$. What is the size of $\angle BOC$?

4 (Fig 9) What is the sum of the angles $\hat{AOB}$, $\hat{BOC}$, $\hat{COA}$? If $\hat{AOB}$ and $\hat{BOC}$ are each twice the size of $\hat{AOC}$, find the sizes of all these angles

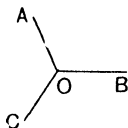


Fig 9

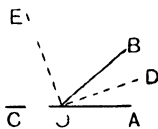


Fig 10

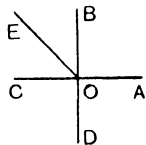


Fig 11

5 (Fig 10) The rays OD , OE bisect angles AOB , COB resp, $\hat{AOB} = 40^\circ$ Find the size of $\hat{COD}$ and of $\hat{EOD}$

6 In the same figure, if $\hat{AOB} = x^\circ$, find the size of $\hat{COD}$ in terms of x

7 (Fig 11) OA , OB are perpendicular lines, the ray OE bisects $\hat{COB}$ Find the size of $\hat{AOE}$

8 If in Fig 11 we have $\hat{BOE} = x^\circ$, find the sizes of angles AOE , COE , DOE in terms of x

[The following constructions should now be introduced, *without proofs*, it will probably be best not to use letters for points at first

(1) To bisect an interval, p 37 (2) To bisect an angle, p 36 (3) To draw a perpendicular to a given line from a given point on the line, p 35 (4) To draw a perpendicular to a given line from a given point not on the line, p 38 (5) To construct, by compasses, angles of 60° , 30° , 45° , 120° , p 49 (6) To draw a triangle, given the lengths of its sides (7) Construction of a regular hexagon, p 49]

§ 3. Kinds of triangles.

If we have three points not on the same line, the intervals joining them form a 'triangle'. The intervals are the 'sides' of the triangle, the points its vertices. If one angle of a triangle is a right angle, the triangle is called a 'right-angled triangle'. Only one angle of a triangle can be an obtuse angle or a right angle. If one angle of a triangle is an obtuse angle, the triangle is called an 'obtuse-angled triangle'. If all the angles of a triangle are acute angles, the triangle is called an 'acute-angled triangle'.

EXERCISE

1 Draw a triangle whose sides are 3, 4, 5 inches long. Measure its angles, and bisect them.

2 Draw a triangle whose sides are 3, $3\frac{1}{2}$, 4 inches long, from each vertex draw a perpendicular to the opposite side. These perpendicular intervals are called the '*altitudes*' of the triangle. Measure their lengths.

3 Draw a triangle whose sides are 2, 3, 4 inches long, bisect each side and join the middle point of each side to the opposite vertex. The intervals thus obtained are called the '*medians*' of the triangle. Measure their lengths.

4 To find the distance from a point to a line, we draw the perpendicular interval from the point to the line, the length of this interval is the distance required. Thus, if OP is perpendicular to the line AB , then the distance from O to the line is the length of the interval OP . P is called the *foot* of the perpendicular from O to AB . Draw an angle of 60° and bisect it. Take a point on the bisector 2 inches from the vertex, and measure its distance from either arm of the angle.

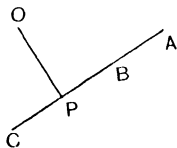


Fig. 12

5 AOB is a straight line, C any point not on it, the rays OX , OY bisect the adjacent angles AOC , BOC . Prove that XOY is a right angle.

6 AOB , COD are lines cutting in O , the ray OX bisects $\hat{DOB}$, XO is produced beyond O to Y . Prove that the ray OY bisects $\hat{COA}$.

7 In Fig. 13, the rays OX , OY bisect angles AOB , AOC resp. If $\hat{AOB} = x^\circ$, $\hat{AOC} = y^\circ$, find $\hat{XOY}$ in terms of x and y .

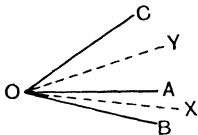


Fig. 13

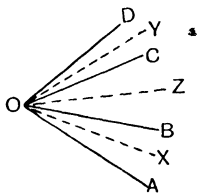


Fig. 14

8 In Fig. 14, the rays OX , OY bisect angles AOB , COD resp., $\hat{AOB} = a^\circ$, $\hat{BOC} = b^\circ$, $\hat{COD} = c^\circ$. Find $\hat{XOY}$ in terms of a , b , c .

9 If also the ray OZ bisects the angle BOC , and $\hat{A}OB = \hat{C}OD$, prove that the ray OZ bisects $\hat{XOY}$

*10 OA, OB, OC, OD are rays from O in that order, $\hat{A}OB = \hat{C}OD$, $\hat{BOC} = \hat{AOD}$. Prove that A, O, C are in one line, and B, O, D in one line

§ 4. Parallel lines.

Parallel lines are lines which run in the same direction, or are the same distance apart all the way. But for purposes of Geometry we use the following definition, which should be learnt by heart

Def Two lines are said to be 'parallel,' when they lie in the same plane and do not cut each other

(Remember that 'line' means 'a straight line,' infinite in both directions. A plane is a flat surface, like the top of a smooth table.)

EXERCISE 6

1 Draw a line, and a series of lines parallel to it, by means of the set-square

2 Can a line on the floor be parallel to a line on a wall of a room? Can two lines on opposite walls, or on neighbouring walls of a room, be parallel?

3 Copy the following figures, drawing parallel lines with your set-square. You need not trouble to draw them the same size as the figures here. In the figures some angles are marked, mark all angles, equal to these, by the same letter. Thus the answer for the first

figure is

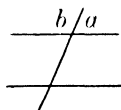


Fig 16

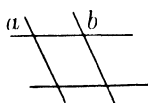


Fig 17

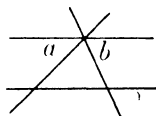


Fig 18

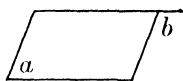


Fig. 19

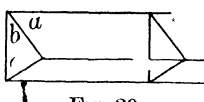


Fig. 20

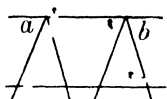


Fig. 21

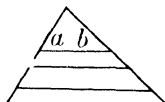


Fig. 22

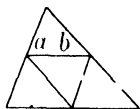


Fig. 23

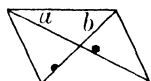


Fig. 24

In the following figures, the angles a are called a pair of 'alternate' angles, so are the angles b , the angles x are called a pair of 'corresponding' angles, so are the angles y .

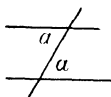


Fig. 25

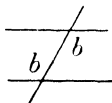


Fig. 26

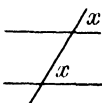


Fig. 27

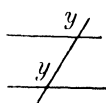


Fig. 28

The preceding examples should have made clear the following facts which we state as assumptions.

Assumption 5 If a line cuts two parallel lines, the alternate angles so formed are equal

Assumption 6 If a line cuts two parallel lines, the corresponding angles so formed are equal

Assumption 7. If a line cuts two other lines, and the alternate angles so formed are equal, then the lines are parallel, and if these angles are not equal, the lines are not parallel

Assumption 8 If a line cuts two other lines, and the corresponding angles so formed are equal, then the lines are parallel, and if these angles are not equal, the lines are not parallel

EXERCISE 7

ABC is any triangle, CX is a line parallel to BA . Mark the angles at C which are equal to the angles marked at A and B . What is the sum of the three angles at C ? What is the sum of the angles of the triangle ABC ?

Consideration of this exercise leads us to state

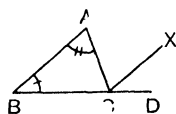


Fig. 29

Assumption 9. The sum of the angles of any triangle equals two right angles

[The following is a formal proof of this assumption. It will often be better to postpone the actual learning of it to a slightly later stage. It is inserted here for convenience.]

THEOREM 1.

The sum of the angles of any triangle equals two right angles.

Given. ABC is a triangle

To prove that the sum of the angles of triangle ABC equals two right angles

Proof. Through C draw the line CX parallel to AB . Produce the side BC of the triangle, beyond C , to D . Then since $CX \parallel BA$,

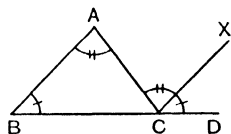


Fig 30

$$\hat{ABC} = \hat{XCD}, \quad (\text{corr. } \angle s),$$

and since $CX \parallel BA$,

$$\hat{BCA} = \hat{ACX}, \quad (\text{alt } \angle s)$$

$$\hat{ABC} + \hat{BCA} = \hat{XCD} + \hat{ACX} = \hat{ACD}.$$

Add to both sides the angle ACB ,

$$\therefore \hat{ABC} + \hat{BCA} + \hat{ACB} = \hat{ACD} + \hat{ACB}$$

But BC and CD are in one straight line,

$$\hat{ACD} + \hat{ACB} = 2 \text{ rt } \angle s, \quad (\text{adj. } \angle s).$$

$$\therefore \hat{ABC} + \hat{BCA} + \hat{ACB} = 2 \text{ rt } \angle s.$$

Quod erat demonstrandum.

Corollary. An exterior angle of a triangle equals the sum of the interior opposite angles. For we have proved that the exterior angle ACD equals the sum of $\hat{ABC}$ and $\hat{BCA}$.

Note. The symbol $\parallel$ may be used for 'is parallel to,' and $\perp$ for 'is perpendicular to.' The reasons for the statements made should be put in brackets, as above, after the statements.

Warning. Avoid writing a letter over a line or round a point. Letters on a figure should be made so carefully that it is impossible to misread them.

§ 5. Isosceles triangles.

An 'equilateral' triangle is one with all its sides equal.

An 'isosceles' triangle is one with (at least) two equal sides.

A 'scalene' triangle is one with no two sides equal

A 'quadrilateral' is a figure with four sides. $ABCD$ is a

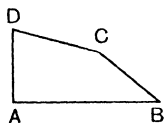


Fig 31

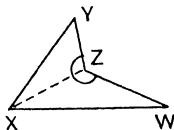


Fig 32

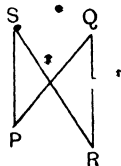


Fig 33

'convex' quadrilateral, $XYZW$ is a 're-entrant' or 'non-convex' quadrilateral. $PQRS$ is a 'crossed' quadrilateral, this will seldom be considered. In naming a quadrilateral we go round the vertices in order. Thus the quadrilateral in Fig 31 can be called $ABCD$ or $ADCB$ or $BCDA$, and so on, but not $ACDB$ or $ACBD$. The intervals AC and BD are the 'diagonals' of the quadrilateral $ABCD$.

A figure with any number of sides is a 'polygon'. In particular, a figure with five sides is a 'pentagon', with six, a 'hexagon', with seven, a 'heptagon', with eight, an 'octagon'.

We shall assume, at present, that our figures are in one plane.

EXAMPLE 1 The sum of the angles of a (plane) quadrilateral equals four right angles. (The quadrilateral need not be convex.)

Given $XWZY$ is a quad (Fig 32)

To prove that the sum of its angles equals four rt $\angle$ s

Proof. Join XZ

$$\text{In } \triangle XZW, \hat{X}ZW + \hat{Z}WX + \hat{W}XZ = 2 \text{ rt } \angle s$$

$$\text{In } \triangle XZY, \hat{X}ZY + \hat{Z}YX + \hat{Y}XZ = 2 \text{ rt } \angle s$$

But the sum of the angles of $\triangle$ s XZW and XZY equals the sum of the angles of quad $XWZY$

the sum of the angles of the quad $XWZY$ equals 4 rt $\angle$ s

Q.E.D.

An angle greater than two right angles, like the angle at Z of the quadrilateral $XWZY$ is called a '*reflex*' angle. It is marked in Fig. 32.

EXERCISE 8

1. Can (i) a triangle, (ii) a quadrilateral, be not in one plane?
2. Draw an interval 2 inches long, and, using your compasses, draw an equilateral triangle, which has this interval for one side.
3. Draw an isosceles triangle, whose equal sides are 3 inches long, and whose other side is 2 inches long. Measure its angles.

If ABC is an isosceles triangle and AB, AC are the equal sides, the point A is often called the '*apex*' or '*the vertex*' of the triangle, while the side BC is called the '*base*', the angle at A is called the '*vertical angle*', the angles at B, C the '*base angles*' of the isosceles triangle.

4. Draw an isosceles triangle whose equal sides are 2 inches long and whose base is 3 inches long. Measure its base angles.

5. Draw a triangle ABC given that $BC = 3$ inches, and the angles at B, C are each 40° . Measure the sides AB, AC .

The results of the last two questions lead us to make the following statements.

Assumption 10. The base angles of an isosceles triangle are equal.

Assumption 11. If two angles of a triangle are equal, the sides opposite the equal angles are equal.

From Assumption 10, it follows that *the angles of an equilateral triangle are all equal to each other*. By Assumption 9, their sum equals 180° . Hence *each angle of an equilateral triangle = 60°* .

EXAMPLE 2 *To draw, without a protractor, an angle of 60° with a given arm.*

Given a ray OA .

To find a point B such that $\angle AOB = 60^\circ$.

Construction. With centre O , and radius OA , draw a circle. With centre A , and radius AO , draw another circle to cut the first in B .

Then $\angle AOB = 60^\circ$.

Proof. The angles of $\triangle OAB$ are all equal, (the sides are equal).

The sum of the angles is 180° .

$$\angle AOB = 60^\circ.$$

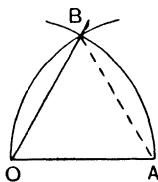


Fig. 34

EXERCISE 9

In answering these questions you may use any of the assumptions given above. You should draw figures freehand, they need not be accurate, but they must be neat. Give reasons for every statement you make.

- 1 What is the size of the angles of an equilateral triangle?
- 2 A right-angled triangle has one angle of 20° . How big are the other angles?
- 3 A right-angled triangle has two angles equal. How big are they?
- 4 Construct a right-angled triangle two of whose angles are equal.
- 5 A right-angled triangle has one acute angle twice the other. How big are these angles?
6. The triangle ABC has its side BC produced beyond C to D , $\hat{ACD} = 110^\circ$, $\hat{ABC} = 70^\circ$. How big is $\hat{BAC}$?
- 7 If, in the same figure, $\hat{ACD} = 2\hat{ABC}$, prove that $\hat{ABC} = \hat{CAB}$.
- 8 Prove that at least two of the angles of any triangle must be acute.

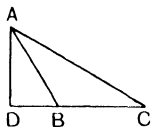


Fig 35



Fig 36

- 9 ABC is a triangle, $\hat{ABC} = 120^\circ$, $\hat{ACB} = 30^\circ$, AD is an altitude†. Prove that $\hat{BAC} = \hat{BCA}$, and find the size of $\hat{CAD}$ (Fig. 35).
- 10 The angles of a triangle are of sizes x° , $2x^\circ$, $3x^\circ$. Find x° .
- 11 In Fig. 36 the sizes of angles a , b , c are known. Find the size of d in terms of them. (Hint: Draw a diagonal.)
- 12 If, in Fig. 36, a , b , c are all equal, prove that $d = 3a$.
- 13 A , B , C are on a line, B is between A and C , O is a point not on the line, $\hat{OAB} = 60^\circ$, $\hat{OBC} = 80^\circ$, $\hat{OCB} = 80^\circ$. Find the sizes of $\hat{AOB}$, $\hat{BOC}$.
- 14 ABC is a triangle, AD an altitude, $\hat{BAC}$ is a right angle. Prove $\hat{ABC} = \hat{DAC}$.

† An altitude of a triangle is a perpendicular interval from a vertex on to the opposite side or *that side produced*.

15. ABC is a triangle, AD an altitude, $\hat{ABC} = \hat{DAC}$. Prove $\hat{BAC}$ is a right angle

16. AE is a bisector of BAC , AD is an altitude of triangle ABC . If $\hat{ABC} = 40^\circ$, $\hat{ACB} = 60^\circ$, find the size of $\hat{EAD}$

[The work now consists of theorems, riders, and drawing exercises. A collection of the latter will be found on pp. 39-45. It is suggested that the bookwork *should be kept well ahead of the riders*, and that the drawing exercises should be taken in alternate lessons.]

EXERCISE 10

In Nos. 1 to 18, ABC is an isosceles triangle with $AB = AC$

1. If $\hat{BAC}$ is a right angle, find the size of $\hat{ABC}$

2. If $\hat{BAC} = 120^\circ$, find the size of $\hat{ABC}$

3. If the side BC be produced beyond C to D , prove that

$$\hat{ACD} + \hat{ABD} = 180^\circ$$

4. If $\hat{BAC} = x^\circ$, find the size of $\hat{ABC}$ in terms of x

5. If $\hat{ABC} = y^\circ$, find the size of $\hat{BAC}$ in terms of y

6. If the side BA is produced beyond A to E , prove $\hat{EAC} = 2\hat{ABC}$

7. If the angles at B, C are each double the angle at A , find their size

8. If the angles at B, C are each thrice the angle at A , find their size

9. If $\hat{BAC} = 100^\circ$ and D is on the side BC , and $BD = DA$, find the sizes of $\hat{ADB}$ and $\hat{DAC}$

10. If $\hat{BAC} = 40^\circ$ and D is on the side BC produced beyond C , and $CD = AC$, find the sizes of $\hat{BDA}$ and $\hat{BAD}$

11. If $\hat{BAC} = 40^\circ$ and the bisectors of $\hat{ABC}$, $\hat{ACB}$ cut in I , find the size of $\hat{BIC}$

12. If the bisectors of $\hat{ABC}$, $\hat{ACB}$ cut in I , prove that

$$\hat{BIC} = 180^\circ - \hat{ABC},$$

whatever be the size of $\hat{BAC}$

13. If the bisector of $\hat{ABC}$ cuts AC in X , prove that $\hat{AXB} = 3\hat{XBC}$.

14. If the side AB is produced beyond B to E , prove that

$$2\hat{ABC} = \hat{AEC} + \hat{ACE}$$

15 If the side BC is produced beyond C to D , and $AC = CD$, prove that $\angle ABC = 2\angle CAD$

16 If the side BC is produced beyond C to D , and $AC = CD$, and DA is produced beyond A to E , prove that $\angle BAE = 3\angle ADB$

17 If the side BA is produced beyond A to F , and $AC = AF$, prove that $\angle BCF$ is a right angle

18 If the side BA is produced beyond A to F , prove that the bisector of $\angle CAF$ is parallel to BC

***19** B, C are points on the same side of the line OA , A, C are on opposite sides of the line OB , $OA = OB = OC$, $\angle AOB = 100^\circ$, $\angle OCB = 60^\circ$. Find the size of $\angle ABC$ and of $\angle AOC$.

***20** ABC is a triangle, the line AD is parallel to BC , D, B are on opposite sides of AC , $AD = AB$. Prove that the ray BD bisects $\angle ABC$

***21** In triangle ABC , $AB = AC$, $\angle A$ is an obtuse angle, D is the foot of the perpendicular from B on to the line AC , $ABD = 2\angle ABC$. Find the size of $\angle ABC$

***22** ABC is an isosceles triangle, $AB = AC$, CR is an altitude. Prove $\angle BCR = \frac{1}{2}\angle BAC$

***23** $\angle ACB$ is a right angle, D, E are on the interval AB , $AD = AC$, $BE = BC$. Prove $\angle DCE = 45^\circ$

[Riders on the converse isosceles theorem (Assumption 11) are postponed to Exercise 18, p. 32.]

§ 6. Congruent triangles

Exercise Draw a triangle ABC with the following measurements $\angle A = 60^\circ$, $AB = 3''$, $AC = 4''$. Measure BC . The construction is as follows. Make an angle YAZ of 60° , and cut off along the arms AY , AZ , intervals AB , AC of lengths $3''$, $4''$ respectively. Join BC . All triangles with the given measurements are obviously of the same size and shape. Triangles of the same size and shape are called 'congruent'. In Geometry we use the following definition of congruent triangles.

Def. Two triangles are called 'congruent' when and only when the three sides of one triangle have the same length as the three sides of the other, and the angles of the triangles which are opposite to the equal sides are also equal.

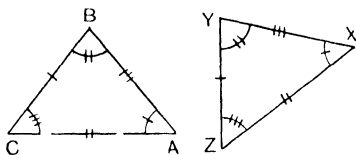


Fig 37

Thus the triangles ABC and XYZ are congruent, if

$$BC = YZ, CA = ZX, AB = XY,$$

and $\hat{A}BC = \hat{X}YZ, \hat{BC}A = \hat{Y}ZX, \hat{C}AB = \hat{Z}XY$.

But it is clear that we do not have to know *all* these six equations to be sure that the triangles are congruent. It will be enough, for instance, if we know that $AB = XY, BC = YZ, \hat{A}BC = \hat{X}YZ$.

$\hat{A}BC$ is called the angle 'included' between the sides AB and BC , so we may say:

Assumption 12 If two sides and their included angle in one triangle be equal to two sides and their included angle in another triangle, then the two triangles are congruent.

In Geometry we have to be careful about every statement we make, if we can consider it as quite obvious, we put it among our assumptions; if it is not quite obvious, or is difficult to see, or is not one of our assumptions, we have to prove it from our assumptions. The following examples will show what we mean by a proof.

EXAMPLE 3 If C is the mid-point of the interval AB , and D is any point on the perpendicular to AB at C , then $DA = DB$.

Given. $AC = BC, CD \perp AB$.

To prove that $AD = BD$.

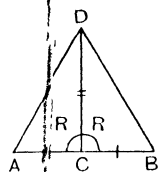
Proof In $\triangle ACD, BCD$,

$$\left\{ \begin{array}{l} AC = BC, \quad (\text{given}) \\ CD = CD \end{array} \right.$$

$$\left\{ \begin{array}{l} \hat{A}CD = \hat{BCD}, \quad (\text{rt } \angle) \end{array} \right.$$

$\triangle ACD, BCD$ are congruent, (two sides, incl. $\angle$)

$$DA = DB.$$



(Fig 38

Q.E.D.

EXAMPLE 4 Given ABC is a triangle, ABX , CBY are equilateral triangles, outside the triangle ABC .

To prove that $CX = AY$.

Proof. Since ABX is an equilateral triangle, (given),

$$\hat{ABX} = \hat{BXA} = \hat{XAB}.$$

The sum of these angles equals two right angles $\therefore \hat{AXC} = 60^\circ$.

Similarly $\hat{CAY} = 60^\circ$

Add to these the angle $\hat{ABC}$,

$$\hat{CBX} = \hat{YBA}$$

In Δ s CBX , YBA ,

$$\begin{cases} CB = YB, & (\text{sides of an equil } \Delta) \\ BX = BA, & (\text{sides of an equil } \Delta) \\ \hat{CBX} = \hat{YBA}, & (\text{proved}) \end{cases}$$

Δ s CBX , YBA are congruent, (two sides, incl $\angle$)

$$CX = YA$$

Q. E. D.

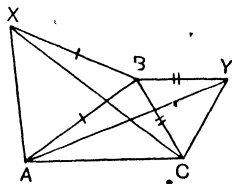


Fig. 39

EXERCISE 11 †.

Mark on the figure the facts you are given, and the facts which you discover in the course of your proof. Do not draw triangles which look like isosceles triangles, unless you are told they are isosceles.

1 (a) ABC is an isosceles triangle, $AB = AC$, the bisector AX of angle BAC meets BC in X . Prove that $BX = CX$ and $AX \perp BC$.

(b) ABC is an isosceles triangle, $AB = AC$, X is any point on the bisector of angle BAC . Prove that $BX = CX$.

(c) The line which bisects the base of a triangle at right angles goes through a vertex of the triangle. Prove that the triangle is isosceles.

(d) L is the foot of the perpendicular from A on to a given line, X is any point on the line, the interval AB is produced beyond B to C , $BC = AB$. Prove that AX , CX make equal angles with the given line.

2 (a) Points D , E are taken resp. on the equal sides AB , AC of an isosceles triangle ABC , $AD = AE$. Prove that $CD = BE$.

(b) The equal sides AB , AC of an isosceles triangle ABC are bisected resp. at X , Y . Prove that $CX = BY$.

† Some replies should be written out fully. But more will be done if a rough sketch and an oral proof are accepted for some of them.

3 (i) The intervals AB and CD bisect each other. Prove that $CB = AD$, $AC = BD$

(b) The intervals AB and CD bisect each other at right angles. Prove that the quadrilateral $ACBD$ has all its sides equal

(c) D, E are the mid-points of the sides AB, AC of any triangle ABC , DE is produced beyond E to F , $DE = EF$. Prove that $CF = AD$

4 ABC is an isosceles triangle, $AB = AC$, points D, E are taken on the line BC in the order D, B, C, E , so that $BD = CE$. Prove that $AD = AE$

Def The 'right bisector' of an interval is the line which is at right angles to the interval and goes through its mid-point

***5** The right bisectors of the sides AB, AC of any triangle ABC meet in O . Prove that $OA = OB = OC$

***6** (a) M, N are the mid-points of the sides AB, AC of $\triangle ABC$, the intervals CM, BN are produced beyond M, N resp. to X, Y ; $CM = MX, BN = NY$. Prove that $XA = YA$

(b) Prove also that X, A, Y are collinear. (Use alternate angles. Points which lie on the same line are called *collinear*.)

***7** $\hat{BAC}$ is an obtuse angle of $\triangle ABC$. The right bisectors of sides AB, AC cut BC in X, Y . Prove $\hat{XAY} + 2\hat{BAC} + 2\hat{CAY} = 180^\circ$. State a corresponding result when $\hat{BAC}$ is acute

Exercise Draw a triangle whose sides are of lengths $3'', 4'', 4\frac{1}{2}''$. Measure its angles. It is clear that all triangles whose sides have these lengths are of the same size and shape. We therefore state:

Assumption 13 If two triangles be such that the three sides of the one are respectively equal to the three sides of the other, then the triangles are congruent

EXAMPLE 5 Given ABC, ABD are isosceles triangles with $AC = BC, AD = BD$

To prove that CD is the right bisector of AB

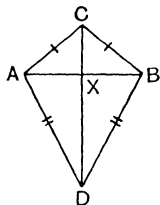


Fig. 40

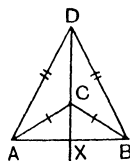


Fig. 41

Proof. Let the lines AB , CD cut in X .

In $\triangle s ADC, BDC$,

$$\begin{cases} AC = BC, & (\text{given}) \\ AD = BD, & (\text{given}) \\ CD = CD \end{cases}$$

$\triangle s ADC, BDC$ are congruent, (three sides)

$$\hat{ACD} = \hat{BCD}.$$

In $\triangle s ACX, BCX$,

$$\begin{cases} AC = BC, & (\text{given}) \\ CX = CX \\ \hat{ACX} = \hat{BCX}, & (\text{proved}). \end{cases}$$

$\triangle s ACX, BCX$ are congruent, (two sides, incl $\angle$)

$$AX = XB \text{ and } \hat{CXA} = \hat{CXB}$$

But A, X, B are in one st line $CX \perp AB$ (def)

Thus $AX = XB$, $CX \perp AB$ CD is the right bisector of AB . Q.E.D.

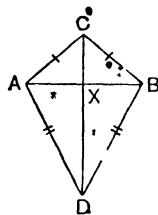


Fig 40

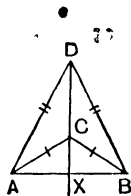


Fig 41

Warning Beginners are often tempted to prove the result as follows

In $\triangle s ACX, BCX$,

$$\begin{cases} AC = BC & (1) \\ CX = CX & (2) \\ \hat{CAX} = \hat{CBX}, & (\text{isos } \triangle) & (3) \end{cases}$$

$\triangle s ACX, BCX$ are congruent, (two sides, incl $\angle$)

But facts (1), (2), (3) concern two sides and a *not* included angle in the two triangles and therefore are *not* enough to prove that the triangles are congruent. The statements (1), (2), (3) are all true *whatever position the point X has on the interval AB*, and of course the triangles ACX, BCX cannot be congruent unless X is the mid point of the interval AB . The following exercises depend on the 'three sides congruence'.

EXERCISE 12

1. (a) $\triangle ABC$ and $\triangle ABD$ are isosceles triangles, $AC = BC$, $AD = BD$, $\hat{ACB} = 30^\circ$, $\hat{ADB} = 40^\circ$. Find the size of $\hat{CAD}$ when C, D are on opposite sides of AB .

(b) Find the size of $\hat{CAD}$ when C, D are on the same side of AB .

(c) In both cases, prove $\hat{CAD} = \hat{CBD}$.

2 (a) ABC is an isosceles triangle, $AB = AC$, X is the mid-point of BC . Prove $AX \perp BC$

(b) AB is a chord of a circle†, centre O , X is the mid-point of AB . Prove $OX \perp AB$

(c) The line joining the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base.

3. Two circles with centres A, B cut in C and D . Prove $\hat{ACB} = \hat{ADB}$.

4 AB is a chord of a circle, centre O ; XY is an equal chord of an equal circle, centre P . Prove $\hat{AOB} = \hat{XPY}$.

5 A, B are points on a circle, centre O , X, Y are points on a concentric circle, $AX = BY$. Prove $\hat{AOX} = \hat{BOY}$.

6 P is a point on a circle, centre O , a circle with centre P cuts the first circle in A and B . Prove that OP bisects angle $\hat{AOB}$.

7 The quadrilateral $ABCD$ has its opposite sides equal. Prove that its opposite angles are equal.

8 A quadrilateral has all its sides equal. Prove that its diagonals bisect each other at right angles.

*9. ABC and ABD are two triangles, $AC = BD$, $BC = AD$, C, D are on the same side of AB . How many pairs of congruent triangles are there in the figure? Give reasons.

Exercise Draw a triangle ABC such that $BC = 3''$, $\hat{ABC} = 50^\circ$; $\hat{BCA} = 40^\circ$. Measure its sides. It is clear that all triangles with these measurements are congruent.

Draw a triangle ABC such that $BC = 3''$, $\hat{ABC} = 50^\circ$, $\hat{CAB} = 30^\circ$. Calculate first the size of the angle $\hat{BCA}$. Measure the sides of the triangle. It is clear that all triangles with the given measurements are congruent. We may hence say,

Assumption 11 If two triangles are such that a side and any two angles of the one are equal respectively to a side and two angles of the other in a like situation, then the triangles are congruent.

There are two cases, which are illustrated by the following figures, which will also explain what we mean by a 'like situation'. In the first pair of triangles, the equal sides are between

† Use the fact that the radii of a circle are all equal to each other. A chord of a circle is an interval which joins two points on a circle. Concentric circles are circles with the same centre.

angles concerned, in the second pair, the equal sides are side to equal angles

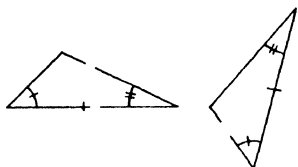


Fig. 42

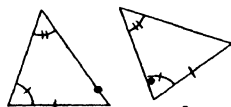


Fig. 43

77

we know that the sum of the three angles of any triangle equals two right angles, it follows that, if two angles of one triangle equal those of another, the third angle in the one equals the third angle in the other. Hence the second case reduces to the first.

EXERCISE 13

The following examples depend on the 'two angles and side' congruence.

In triangle ABC , the bisector of angle BAC is perpendicular to BC . Prove that $AB = AC$.

$ABCD$ is a quadrilateral, AC bisects $\hat{BAD}$ and $\hat{BCD}$. Prove that $AD = BC$, $CB = CD$.

D is any point on the bisector of $\hat{BAC}$, X, Y are the feet of perpendiculars from D on to AB, AC resp. Prove $DX = DY$.

X is the mid-point of AB , parallel lines are drawn through X , a line through X cuts them in Y, Z resp. Prove $XY = XZ$. (alternate $\angle$ s)

(a) ABC is an isosceles triangle, $AB = AC$, the bisectors of $\hat{B}$ and $\hat{C}$ cut sides AC, AB resp. in X, Y . Prove $BX = CY$.

(b) In the same triangle, BQ, CR are altitudes. Prove that $BQ = CR$.

A man finds the breadth of a river as follows: standing at A , he notes a post B on the other bank opposite to him. He walks to C , and puts his instrument there. He then walks to D , so that $AC = CD$, then at right angles to the bank to E , where B, C, E are in a line. He measures DE . Prove that this is the breadth of the river.

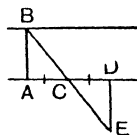


Fig. 44

N, M are the mid-points of sides AB, AC of any triangle ABC , a line CF parallel to AB meets the line NM in F . Prove $NM = MF$.

8 Two circles have the same centre O , a line cuts one in A, B , the other in C, D . Prove that $\hat{AOC} = \hat{BOD}$ and $AC = BD$

***9** (a) The bisectors of angles ABC, BCA of triangle ABC cut in I . Prove that I is equidistant from the sides of the triangle (that is, the distance from I to the side AB equals the distance from I to the side BC and equals the distance from I to the side CA , see p. 6)

(b) The sides AB, AC of triangle ABC are produced beyond B, C to X, Y resp., the bisectors of angles CBX, BCY cut in E . Prove that E is equidistant from the sides AB, BC

(c) Hence prove that E is equidistant from the sides AB, AC

***10** L is the mid-point of the side BC of triangle ABC . Prove that B, C are equidistant from AL (that is, the distance from B to AL equals the distance from C to AL)

***11.** AX bisects the angle BAC , perpendiculars from B, C to AX cut AB, AC in E, F resp. Prove $BE = CF$

***12** AOB is a right angle, $OA = OB$, X, Y are the feet of perpendiculars from A, B resp. on any line through O . Prove $OX = BY$, $OY = AX$

***13.** M, N are the mid-points of the sides AC, AB of triangle ABC , P is any point on the parallel through A to BC , PN, PM cut the line BC in D, E resp. Prove that $DE = BC$

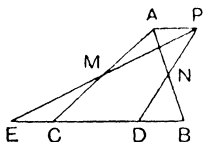


Fig 45

Exercise. Draw a triangle ABC , given that

$\angle B = 3^\circ$, $AC = 3\frac{1}{2}''$, $\hat{ABC}$ is a right angle.

Measure the other angles of the triangle. A

little reflection will probably convince the reader that all triangles with the given measurements are congruent, but as it is harder to see this than the other facts on congruent triangles we shall prove this is true, instead of assuming it. Also it is best not to assume more things than we can help.

THEOREM 2.

If two right-angled triangles be such that their hypotenuses are equal and another side of one equals a side of the other, then the triangles are congruent.

(The hypotenuse of a right-angled triangle is the side opposite the right angle.)

Given. ABC , XYZ are triangles, the angles at B , Y being right angles,

$$AC = XZ, AB = XY.$$

To prove that $\triangle s ABC, XYZ$ are congruent.

Proof Produce the side CB beyond B to D , making $BD = YZ$. Then in $\triangle s ABD, XYZ$,

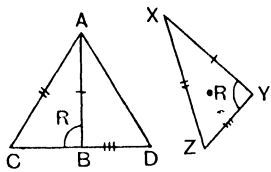


Fig 46

$$\begin{cases} AB = XY, & (\text{given}). \\ BD = YZ, & (\text{const}) \\ \hat{A}BD = \hat{X}YZ, & (\text{rt. } \angle s) \end{cases}$$

$\triangle s ABD, XYZ$ are congruent, (two sides, incl. $\angle$)

$$XZ = AD.$$

But

$$XZ = AC, \quad (\text{given})$$

$$AC = AD$$

$$\hat{ACB} = \hat{ADB}, \quad (\text{isos } \triangle).$$

In $\triangle s ABC, ABD$,

$$\begin{cases} AB = AB \\ \hat{ABC} = \hat{ABD}, & (\text{rt } \angle s) \\ \hat{ACB} = \hat{ADB}, & (\text{proved}) \end{cases}$$

$\therefore \triangle s ABC, ABD$ are congruent, (one side, two $\angle s$).

But $\triangle s ABD, XYZ$ are congruent, (proved)

$\triangle s ABC$ and XYZ are congruent

Q E D

EXERCISE 14

The following depend on the 'right angle congruence'.

1. ABC is an isosceles triangle, $AB = AC$, X is the foot of the perpendicular from A to BC . Prove $BX = CX$

($AB = AC$, $AX = AX$, $\hat{AXB}$ and $\hat{AXC}$ are right angles. $\triangle s AXB, AXC$ are congruent. You must mention that $\hat{AXB}$ and $\hat{AXC}$ are right angles, it is not enough to say merely that they are equal, because they are not included angles)

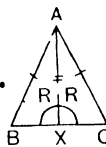


Fig 47

2 The perpendicular to a chord of a circle from the centre bisects the chord

(Use the fact that the radii of a circle are all equal. A 'chord' of a circle is an interval joining two points on the circle. If the chord goes through the centre, it is called a 'diameter'.)

3 O is a point inside an angle XAY , B, C are the feet of the perpendiculars from O to AX, AY resp., $OB = OC$. Prove $AB = AC$.

4 AB is a diameter of a circle, O its centre, P, Q are points on the circle, X, Y are feet of perpendiculars from P, Q resp. to AB , $PX = QY$. Prove $OX = OY$.

5 M is the mid-point of the side AB of triangle ABC , the distances of M from the lines CA, CB are equal. Prove $CA = CB$.

Table of conditions sufficient for congruence of triangles

Two triangles are congruent, when the following parts of one equal the corresponding parts of the other:

- (a) Two sides and *included* angle
- (b) Three sides
- (c) One side, two angles
- (d) Two sides and a right angle

The following are *not sufficient*:

- (i) Three angles
- (ii) Two sides and an angle which is *not* included

EXERCISE 15

General examples on congruence

1 Two circles cut in A, B . Prove that the line joining their centres bisects the interval AB at right angles.

2 $ABCD$ is a quadrilateral, AC bisects the angles DAB, DCB . Prove that AC bisects the interval BD at right angles.

3 AB bisects the intervals PQ and RS at right angles. Prove $PR = QS$.

4 (a) OAB is an isosceles triangle, $OA = OB$, C, D are points on the sides OA, OB resp. such that $OC = OD$, BC, AD cut in P . Prove that OP bisects the angle AOB .

(b) Using the last example, give a construction for bisecting an angle by means of a graduated ruler (e.g. a ruler marked in inches), without using compasses.

5 AB, CD are chords of a circle, centre O ; X, Y are the feet of perpendiculars from O to AB, CD . If $OX = OY$, prove $AB = CD$.

6 The bisectors of the angles BAC, ABC of triangle ABC cut in I . Prove that IC bisects the angle ACB .

7 The sides CA, CB of triangle ABC are produced to X, Y resp.; AE, BE bisect angles BAX, ABY resp. Prove that CE bisects the angle ACB .

8 ABC and ABD are congruent triangles, C, D are on opposite sides of AB , X is on the line AB . Prove $CX = DX$.

9 ABC is any triangle. $ABPQ$ and $BCXW$ are squares falling outside the triangle. Prove $CP = AW$.

10 On the base BC of $\triangle ABC$ a point D is taken. Draw a line so that if the triangle be folded along this line, then A falls on D .

***Def** Two quadrilaterals, convex or not, $ABCD$ and $PQRS$ are called '*congruent*' when the six intervals joining the points A, B, C, D in pairs equal the six corresponding intervals joining the points P, Q, R, S in pairs. The corresponding angles are then equal.

In ordinary language, the quadrilaterals are of the same size and shape.

If $A, B, C, D,$ and $A', B', C', D',$ are two sets of points and $AB = A'B', AC = A'C', AD = A'D', BC = B'C', BD = B'D', CD = C'D',$ and so on, the set of points $A, B, C, D,$ is said to be '*congruent*' to the set $A', B', C', D',$

If the set of points $A', B', C', D',$ were marked on a piece of paper and we could move the paper about, we should be able to fit this set on to $A, B, C, D,$, so that A' fell on A, B' on $B,$ and so on. We say A '*corresponds*' to A', B to $B',$ and so on. If the set A, B, C is congruent to the set $A', B', C',$ then $\triangle ABC, \triangle A'B'C'$ are congruent.

***11** If $ABCD$ and $A'B'C'D'$ be two convex quadrilaterals and the five intervals AB, BC, CD, DA, BD be equal resp. to the intervals $A'B', B'C', C'D', D'A', B'D',$ then the sixth intervals AC and $A'C'$ are equal, and the quadrilaterals are congruent. The set of points A, B, C, D is congruent to the set $A', B', C', D'.$

***12** ABC and XYZ are congruent triangles, the points A, B, C corresponding resp. to $X, Y, Z,$ AP, XW are altitudes. Prove $AP = XW$.

*13. ABC and XYZ are congruent triangles, the points A, B, C corresponding resp. to X, Y, Z , M, N are the mid-points of AB, XY resp. Prove $CM = ZN$.

*14. ABC and DEF are congruent triangles, O is a point inside $\triangle ABC$. Prove that there is a point P inside $\triangle DEF$ so that the set A, B, C, O is congruent to the set D, E, F, P .

Further studies on congruence and the isosceles triangle theorems will be found on p. 32.

Similar triangles are triangles of the same shape. They will be dealt with in detail later. In Geometry we adopt the following definition.

Def. Two triangles are similar when the angles of one are respectively equal to the angles of the other.

Thus if triangles ABC, XYZ are such that $\hat{BAC} = \hat{YXZ}$ and $\hat{ABC} = \hat{XYZ}$, we know that (since the three angles of any triangle add up to two right angles) the third angle $\hat{ACB} =$ the third angle $\hat{XZY}$. Thus the triangles are similar. Similar triangles are sometimes called 'equiangular' triangles. If triangles ABC and XYZ are similar, and we also know that a side of one equals the *corresponding* side of the other, then the triangles are congruent.

1. X, Y are resp. on sides AB, AC of $\triangle ABC$ and $XY \parallel BC$. Prove that triangles ABC, AXY are similar.

2. ABC and XYZ are similar triangles, O is inside $\triangle ABC$. Find a point P inside $\triangle XYZ$ so that triangles OAB, OBC, OCA are resp. similar to triangles PXY, PYZ, PZX .

§ 7. Parallelograms

Def. A parallelogram is a quadrilateral with both pairs of opposite sides PARALLEL.

We shall soon *prove* that the opposite sides of a parallelogram are *equal*, therefore this should not be stated in the definition. A definition should tell us just enough about the figure for us to know what the figure is like, and it should tell us no more.

THEOREM 3

The opposite sides of a parallelogram are equal

Given. $ABCD$ is a parallelogram

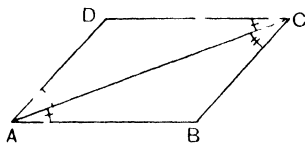


Fig 48

To prove that $AB = CD$, $BC = AD$

Proof Join AC In $\triangle ABC$, CDA ,

$$\begin{cases} \angle BCA = \angle DAC, & (\text{alt } \angle s, AB \parallel CD) \\ \angle BAC = \angle DCA, & (\text{alt } \angle s, BC \parallel AD) \\ AC = AC. \end{cases}$$

$\triangle ABC$, CDA are congruent, (two $\angle$ s, one side)

$$AB = CD, BC = AD \quad \text{Q.E.D.}$$

Cor. A diagonal of a parallelogram divides it into two congruent triangles

THEOREM 4

If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram

Given. $ABCD$ is a quadrilateral, $AB = CD$, $BC = AD$

To prove that $ABCD$ is a parallelogram

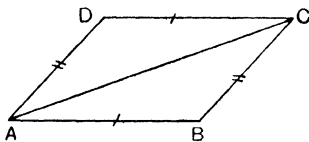


Fig 49

Proof Join AC . In $\triangle ABC$, CDA

$$\begin{cases} AB = CD, & (\text{given}) \\ BC = DA, & (\text{given}) \\ AC = AC. \end{cases}$$

$\therefore \triangle ABC, CDA$ are congruent, (three sides).

$$\therefore \hat{C} = \hat{A}$$

$\therefore AB \parallel CD$, (alt. $\angle$ s)

Also

$$\hat{ACB} = \hat{CAD}$$

$\therefore BC \parallel AD$, (alt. $\angle$ s)

$ABCD$ is a parallelogram

Q. E. D.

Note In Theorem 3, we were *given* that the opposite sides were parallel, and had to prove they were equal, in Theorem 4 it was the other way round, we were *given* that the opposite sides were equal and had to prove they were parallel. Theorems so related are called the '*converses*' of one another.

State the converses of the following theorems, and say which of the converses are true

- 1 A Prime Minister is a wise man
- 2 A sparrow is a biped
- 3 If two numbers are equal their doubles are equal
- 4 If two numbers are equal their squares are equal
- 5 The angles at the base of an isosceles triangle are equal

Def A parallelogram with two adjacent sides equal is a '*rhombus*'

Def A parallelogram which has one angle a right angle is a '*rectangle*'

Def A rectangle which has two adjacent sides equal is a '*square*'

Cor. A rhombus has *all* its sides equal a rectangle has *all* its angles right angles

Note A square is, by definition, both a rectangle and a rhombus. If, therefore, we prove anything about either a rectangle or a rhombus, the same theorem will be true for a square

EXERCISE 16.

1 Draw a parallelogram whose sides are 3" and $2\frac{1}{2}$ " long, and one of whose angles is 40° . Measure its diagonals. Bisect its angles.

2 Draw a rectangle whose sides are 3" and $2\frac{1}{2}$ " long. Measure its diagonals.

3 Draw a rhombus whose sides are 3" long, and one of whose angles is 60° . Measure its diagonals.

4. Draw a square whose sides are 3" long. Measure its diagonals.

Use the above definitions in the next questions.

5. Prove that all the sides of a rhombus are equal
6. Prove that all the angles of a rectangle are right angles
7. Prove that all the sides of a square are equal

THEOREM 5

- (i) *The diagonals of a parallelogram bisect each other*
- (ii) *Conversely, if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram*

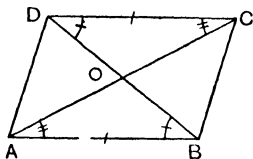


Fig 50

- (i) *Given.* $ABCD$ is a parallelogram. Its diagonals cut at O .
To prove that $AO = OC$, $BO = OD$

Proof In $\triangle AOB$, COB ,

$$\begin{cases} AB = CD, & (\text{opp. sides of parm.}) \\ \angle OAB = \angle OCB, & (\text{alt } \angle s, AB \parallel CD) \\ \angle OBA = \angle OCD, & (\text{alt } \angle s, AB \parallel CD) \end{cases}$$

$\therefore \triangle AOB, COB$ are congruent, (one side, two $\angle s$)

$$AO = OC, BO = OD$$

- (ii) *Given.* $ABCD$ is a quad. Its diagonals cut in O , $AO = OC$,
 $BO = OD$.

To prove that $ABCD$ is a parallelogram

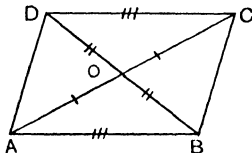


Fig 51

Proof In $\triangle s AOB, COD$,

$$\begin{cases} AO = OC, & (\text{given}) \\ BO = OD, & (\text{given}) \\ \hat{AOB} = \hat{COD}, & (\text{vert opp } \angle s) \end{cases}$$

$\therefore \triangle s AOB, COD$ are congruent, (two sides, incl. $\angle$)

$$\therefore \hat{OAB} = \hat{OCD}$$

$$\therefore AB \parallel CD \quad (\text{alt } \angle s)$$

Similarly, considering $\triangle s BOC, DOA$, we find $BC \parallel AD$.

$ABCD$ is a parallelogram.

Q E D

Exercise 17, Nos 1-9 can now be taken, if desired

THEOREM 6

If one pair of opposite sides of a quadrilateral are equal and parallel, the other pair of opposite sides are equal and parallel.

Given. $ABCD$ is a quadrilateral, $AB = CD$, $AB \parallel CD$

To prove that $AD = BC$, $AD \parallel BC$

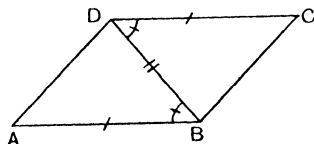


Fig 52

Proof. Join BD . In $\triangle s ABD, CDB$,

$$\begin{cases} AB = CD, & (\text{given}) \\ BD = BD \\ \hat{ABD} = \hat{CDB}, & (\text{alt } \angle s, AB \parallel CD). \end{cases}$$

$\therefore \triangle s ABD, CDB$ are congruent, (two sides, incl. $\angle$).

$$\therefore AD = BC,$$

and

$$\hat{ADB} = \hat{CBD}.$$

$$AD \parallel BC, \quad (\text{alt. } \angle s).$$

$\therefore$ the sides AD and BC are equal and parallel. Q E.D.

Cor. *If one pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram*

The following simple test for parallel lines is often useful.

The lines AB , CD of the figure are parallel, if and only if the angles a and b are supplementary. These angles are sometimes called 'allied angles'.

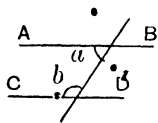


Fig 53

In the next theorem, $\angle$ denotes an angle by the letter at its vertex. This can only be done when it causes no confusion, viz. when there is only one (non-reflex) angle with the vertex in question.

THEOREM 7

The opposite angles of a parallelogram are equal. Conversely, if both pairs of opposite angles of a quadrilateral be equal, the quadrilateral is a parallelogram.

(i) *Given.* $ABCD$ is a parallelogram

To prove that $\hat{A} = \hat{C}$, $\hat{B} = \hat{D}$

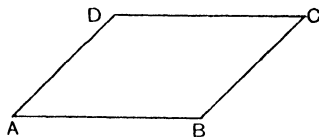


Fig 54

Proof $\hat{A} + \hat{B} = 2 \text{ rt. } \angle\text{s}$, (allied $\angle\text{s}$, $AD \parallel BC$)

$\hat{C} + \hat{D} = 2 \text{ rt. } \angle\text{s}$, (allied $\angle\text{s}$, $AB \parallel CD$)

$$\hat{A} + \hat{B} = \hat{C} + \hat{D}$$

$$\hat{A} = \hat{C} \quad \text{Similarly } \hat{B} = \hat{D}$$

Q.E.D.

(ii) *Given.* $ABCD$ is a quad. with $\hat{A} = \hat{C}$, $\hat{B} = \hat{D}$

To prove that $ABCD$ is a parallelogram

Proof. Since the sum of the angles of a quad. equals $4 \text{ rt. } \angle\text{s}$,

$$\therefore \hat{A} + \hat{B} + \hat{C} + \hat{D} = 4 \text{ rt. } \angle\text{s}$$

But

$$\hat{A} = \hat{C}, \hat{B} = \hat{D}, \quad (\text{given})$$

$$2(\hat{A} + \hat{B}) = 4 \text{ rt. } \angle\text{s}$$

$$\therefore \hat{A} + \hat{B} = 2 \text{ rt. } \angle\text{s}$$

$$\therefore AB \parallel CD, \quad (\text{allied } \angle\text{s}).$$

Similarly $AD \parallel BC$ $\therefore ABCD$ is a parallelogram.

Q.E.D.

EXAMPLE 6 If a diagonal of a parallelogram bisects an angle of the parallelogram, the parallelogram has all its sides equal, i.e. is a rhombus

Given. $ABCD$ is a parm, AC bisects $\hat{BAD}$

To prove that $AB = BC = CD = DA$

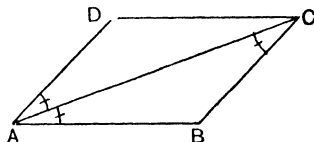


Fig 55

Proof $\hat{DAC} = \hat{ACB}$, (alt $\angle$ s, $AD \parallel BC$)

$\hat{DAC} = \hat{BAC}$, (given).

$\hat{ACB} = \hat{BAC}$.

$AB = BC$

But $AB = CD$, $DA = BC$, (opp sides of parm)

$AB = BC = CD = DA$

Q.E.D.

Exercise Prove the first part of Theorem 7 by the method of Theorem 3

EXERCISE 17

1. The diagonals of a rectangle are equal.
2. The diagonals of a rhombus cut at right angles
- 3, 4 State and prove the converses of 1 and 2.
- 5 The diagonals of a rhombus bisect its angles
- 6 The diagonals of a parallelogram cut in O , any line through O cuts two sides of the parallelogram in X, Y resp. Prove $XO = OY$.
7. $ABCD$ is a parallelogram, AC bisects $\hat{BAD}$. Prove AC bisects $\hat{BCD}$
- 8 It is often said that parallel lines are the same distance apart all the way. State this theorem more clearly, and prove it
9. The bisector of the angle BAC of triangle BAC cuts the side BC in D , E, F are points on AB, AC resp. such that $AEDF$ is a parallelogram. Prove $DE = DF$
- 10 A hexagon has its opposite sides equal and parallel. Prove that the intervals which join opposite vertices bisect each other
11. $ABCD$ is a parallelogram. The bisectors of the angles at A, B cut in P . PQ is parallel to the line BC and it cuts AB in Q . Prove $AQ = QB$

12 $ABCD$ is a parallelogram, X, Y are on the diagonal AC , $AX = CY$. Prove that $BXDY$ is a parallelogram

13 (a) $ABCD$ is a parallelogram, but not a rhombus. Prove that the bisectors of the angles at A and C are parallel

(b) $ABCD$ is any parallelogram. Prove that the bisectors of the angles at A and B are at right angles

(c) $ABCD$ is a parallelogram, but not a rhombus. Its angles are all bisected. Prove that these bisectors cut in four points which are the vertices of a rectangle

*(d) Prove that the diagonals of that rectangle, produced if necessary, bisect the sides of the parallelogram

14 The parallelograms $ABCD$ and $A'B'C'D'$ are congruent, if $AB = A'B', BC = B'C'$ and $\hat{ABC} = \hat{A'B'C'}$

EXERCISE 18

Riders on the isosceles triangle, including the converse theorem

1 (a) ABC is an isosceles triangle, $AB = AC$, the bisectors of angles ABC, ACB cut in I . Prove that IBC is an isosceles triangle

(b) The bisectors of the angles ABC, ACB of triangle ABC cut in I , $IB = IC$. Prove that ABC is an isosceles triangle

2 (a) ABC is an isosceles triangle, $AB = AC$, X, Y are on sides AB, AC resp., XY is parallel to BC . Prove $AX = AY$

(b) In the same triangle, if Y, Z be on sides AC, BC resp. and YZ is parallel to AB , prove $YZ = YC$

3 ABC is an isosceles triangle, $AB = AC$, perpendiculars from B, C to AC, AB resp. cut in X . Prove $BX = CX$

4 X, Y are on the side BC of triangle ABC . $AX = AY$, $\hat{BAX} = \hat{CAY}$. Prove $AB = AC$.

5 In the quadrilateral $ABCD$, $AB = AD$, $\hat{ABC} = \hat{ADC}$. Prove $BC = CD$

6 In the quadrilateral $ABCD$, $\hat{ABC} + \hat{ADC} = \hat{BCD}$ and $AC = AB$. Prove $AC = AD$

*7 In triangle ABC the angles at A, B are each twice the angle at C , the bisector of the angle at B cuts the side AC in D . Prove $AB = BD = CD$

*8 The side BA of triangle ABC is produced beyond A to X , the bisectors of angles CAB, CAX cut a parallel to AB , through C , in D, E resp. Prove $CE = CD$

*9 D, E are points on the side BC of triangle ABC such that $\hat{BAD} = \hat{ACB}$, $\hat{CAE} = \hat{AB}$. Prove that the bisector of $\hat{DAE}$ is perpendicular to BC .

*10 ABC is an isosceles triangle, $AB = AC$, a perpendicular to the side BC cuts the side AB in D and the side CA produced in E . Prove $AD = AE$.

*11 $ABCD$ is a square, ABE an equilateral triangle, E is inside the square, the line DE cuts the line BC in F . G is on AB , $DG = DE$. Prove $\triangle LGF$ is equilateral.

*12 ABC is an isosceles triangle. Find points D, E on the sides AB, AC resp. such that BD, DE, EC are all equal.

Def. A 'trapezium' is a quadrilateral with two sides parallel, and the other two not parallel, if the non-parallel sides are equal, the 'trapezium' is an 'isosceles trapezium' (Fig. 56)

*EXERCISE 19

$ABCD$ is an isosceles trapezium, $AB \parallel CD$, $AD = BC$. Prove the Theorems 1-7.

1 A, B are equidistant from the line CD .

2 $\hat{C} = \hat{D}$, $\hat{A} = \hat{B}$.

3 $AC = BD$.

4 If M is the mid-point of side CD , then $AM = BM$.

5 If also N is the mid-point of side AB , then MN is perpendicular to AB and CD .

6 If AC and BD cut in X , then X is on MN .

7 If the lines AD and BC cut in Y , then Y is on MN .

8 $ABCD$ is a quadrilateral with $\hat{A} = \hat{B}$, $\hat{C} = \hat{D}$. Prove that it is either an isosceles trapezium, or a rectangle.

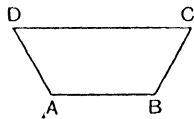


Fig. 56

EXERCISE 20

Parallelograms Harder questions

1 $ABCD$ is a parallelogram, X is the mid-point of side AB , DX, BC cut in Y . Prove $CB = BY$.

2 (a) $ABCD$ is a parallelogram, a parallel to BD through A cuts the lines BC, DC in X, Y resp. Prove that A, B, D are mid-points of the sides of triangle XYC .

(b) $ABCD$ is a parallelogram, the sides CB, CD are produced beyond P, D resp. to X, Y so that $BX = CB, DY = CD$. Prove that X, A, Y are collinear.

3 A, B, C, D are points in order on a straight line, $AB = CD$, $BCEF$ is a parallelogram, the lines AF, DE cut in K , $AB \perp BF$. Prove that $\hat{AKD}$ is a right angle.

4 $ABCD, ABPQ$ are parallelograms. Prove that either CD, PQ are in one line, or $CDQP$ is a parallelogram.

5 ABC, XYZ are triangles such that AX, BY, CZ are parallel, $AB \parallel XY, BC \parallel YZ$. Prove $AC \parallel XZ$.

6 ABC is any triangle, P any point on side AC , PQ and TS are parallel to BC , TW and QR are parallel to AC , $RS \parallel AB$. Prove $PW \parallel AB$ (Fig. 57)

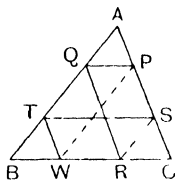


Fig. 57

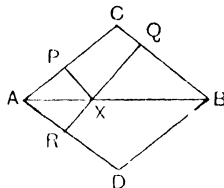


Fig. 58

***7.** (a) ABC is an isosceles triangle, $CA = CB$, X moves along the base AB , P, Q are feet of perpendiculars from X to sides AC, BC . Prove that $XP + XQ$ is constant (i.e. does not depend on the position of X on side AB). (Complete part: $ACBD$, let QX, AD cut in R . Prove $XP = XR, QR \perp AD$ from C on AD .)

***(b)** The point X moves so that the sum of its distances from two fixed lines is constant. What is its locus?

***8** ABC is an isosceles triangle, $CA = CB$, X is any point on AB , XP and XQ parallel to BC, AC resp. cut AC and BC resp. in P, Q . Prove that $XP + XQ$ is constant. Express this as a locus theorem.

***9** Through a given point draw a given line such that the part intercepted between two given parallel lines has a given length.

***10** C is a point inside angle AOB , draw a line through C which cuts OA, OB in X, Y resp. such that $CX = CY$.

§ 8. Formal Constructions and Drawing Exercises.

CONSTRUCTION 1

Given an angle, to draw an angle equal to it with given vertex and arm

Given The angle HAK , the point O and ray OX .

To draw an angle, with vertex O and arm OX , equal to $\hat{HAK}$.

Construction With centre A , draw a circle cutting the arms AB , AC of the given angle in B , C resp

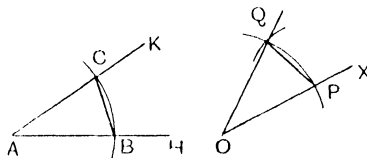


Fig 59

With centre O , and the same radius, draw a circle cutting the ray OQ in P .

With centre P , and radius equal to BC , draw a circle, and let Q be one of its cuts with the circle, centre O

Then $\hat{POQ} = \hat{BAC}$

Proof In $\triangle ABC$, OPQ ,

$$\begin{cases} AB = OP, & (\text{const}) \\ AC = OQ, & (\text{const}) \\ BC = PQ, & (\text{const}) \end{cases}$$

$\therefore \triangle ABC$, OPQ are congruent, (three sides)

$\therefore \hat{POQ} = \hat{BAC}$, i.e. $\hat{POQ} = \hat{BAC}$.

Quod erat faciendum

CONSTRUCTION 2.

To draw a perpendicular to a given line from a given point on the line

Given Line XY and a point O on it

To draw through O a perpendicular to XY .

Construction Let A be any point on XY .

With centre O , radius OA , draw a circle cutting XY again in B .

With centre A , radius greater than AO , draw a circle.

With centre B , and the same radius, draw a circle cutting the circle last drawn in C .

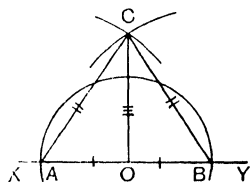


Fig 60

Then $CO \perp AB$

Proof. In $\triangle s AOC, BOC$,

$$\begin{cases} AO = BO, & (\text{const.}) \\ AC = BC, & (\text{const.}) \\ OC = OC \end{cases}$$

$\triangle s AOC, BOC$ are congruent, (three sides)

$$\therefore \hat{AOC} = \hat{BOC}.$$

But these are adjacent $\angle s$. $\therefore CO \perp AB$, i.e. $CO \perp XY$.

Q.E.D.

CONSTRUCTION 3

To bisect a given angle

Given the angle ABC .

To draw a ray which bisects the angle

Construction With centre B , draw a circle to cut the rays BA and BC in X, Y resp

With centres X, Y and radii equal to the interval XY draw circles, let them cut in P .

Then the ray BP bisects $\angle ABC$

Proof In $\triangle s XBP, YBP$,

$$\begin{cases} BX = BY, & (\text{const.}) \\ XP = YP, & (\text{const.}) \\ BP = BP \end{cases}$$

$\triangle s XBP, YBP$ are congruent,
(three sides)

$$\therefore \hat{PBA} = \hat{PBY}.$$

$\therefore$ the ray BP bisects the angle ABC

Q.E.D.

Note It is not necessary that the radii of the circles, with centre X, Y , should equal the interval XY , so long as the circles cut and the radii are equal.

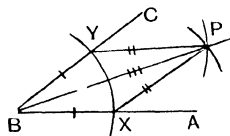


Fig 61

CONSTRUCTION 4.

To bisect a given interval at right angles.

Given an interval AB

To draw a line bisecting the interval AB at right angles

Construction Draw two circles with centres A, B resp and radii equal to the interval AB :

Let them cut in C and D . Let the line CD cut AB in X .

Then the line CD bisects the interval AB at right angles at X .

Proof In $\triangle ACD, BCD$,

$$\begin{cases} AC = BC, & (\text{const.}) \\ AD = BD, & (\text{const.}) \\ CD = CD \end{cases}$$

$\triangle ACD, BCD$ are congruent, (three sides)

$$\hat{ACD} = \hat{BCD} \quad \hat{ACX} = \hat{BCX}$$

In $\triangle ACX, BCX$,

$$\begin{cases} AC = BC, & (\text{const.}) \\ CX = CX. \\ \hat{ACX} = \hat{BCX}, & (\text{proved}). \end{cases}$$

$\therefore \triangle ACX, BCX$ are congruent, (two sides, incl $\angle$).

$$AX = XB,$$

$$\text{and } \hat{AXC} = \hat{BXC}$$

But $\hat{AXC}, \hat{BXC}$ are adj $\angle$ s,

$$\therefore CX \perp AB$$

$\therefore CD$ is the right bisector of the interval AB Q.E.F.

Note It is not necessary that the radii of the circles, with centres A, B , should equal the interval AB , so long as the circles cut and the radii are equal

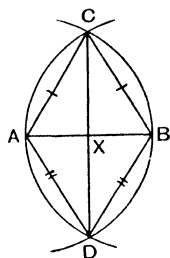


Fig 62

CONSTRUCTION 5

To draw a perpendicular to a given line from a point not on the line.

Given the line XY and a point C not on it.

To draw a line through C perpendicular to the line XY

Construction. Take any point A on XY .

With centre C and radius CA , draw a circle

Let it cut the line XY again in B

Draw circles with centres A, B resp. and radii equal to the interval AB .

Let these circles cut in D

Let CD cut XY in Z .

Then CZ is perpendicular to XY .

Proof In $\triangle CAD, CBD$,

$$\begin{cases} CA = CB, & (\text{const}) \\ DA = DB, & (\text{const}) \\ CD = CD \end{cases}$$

$\therefore \triangle CAD, CBD$ are congruent,
(three sides).

$$\therefore \hat{ACD} = \hat{BCD}. \therefore \hat{ACZ} = \hat{BCZ}$$

In $\triangle ACZ, BCZ$,

$$\begin{cases} CA = CB, & (\text{const}) \\ CZ = CZ. \\ \hat{ACZ} = \hat{BCZ}, & (\text{proved}). \end{cases}$$

$\triangle ACZ, BCZ$ are congruent, (two sides, incl. $\angle$).

$$\therefore \hat{AZC} = \hat{BZC}.$$

But $\hat{AZC}, \hat{BZC}$ are adj. $\angle$ s,

$$CZ \perp AB, \text{ i.e. } CZ \perp XY.$$

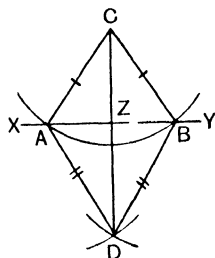


Fig. 6d

Q.E.F.

Note. The proofs of Constructions 4 and 5 should be compared with Example 5, pp 17-18.

CONSTRUCTION 6.

To draw through a given point a parallel to a given line

Given A line XY , and a point P not on it.

To draw through P a parallel to XY .

Construction Let Q, R be any points on XY

With centre R , draw a circle of radius equal to the interval PQ

With centre P , draw a circle of radius equal to the interval QR

These circles cut in a point S on the opposite side of PR from Q .

Then $PS \parallel XY$

Proof In the quad. $PQRS$,

$$PQ = RS, \quad (\text{const})$$

$$QR = PS, \quad (\text{const}).$$

$\therefore PQRS$ is a parm, (opp sides equal)

$\therefore PS \parallel QR$, i.e. $PS \parallel XY$.

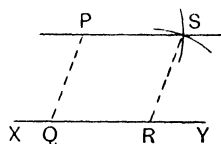


Fig 64

Q R P

Note The purpose of these constructions will be explained later. For the present we need only note that the only instruments which have been used are the ruler, for drawing lines, and the compasses, for drawing circles. The protractor and the set-square have not been used.

EXERCISE 21

Drawing Exercises

(1-14 drawings to scale, 15-34 loci and envelopes, 35-38 interesting constructions, 39-41 models of solids)

1. A picture is hung by a cord 5 ft. long, its ends are tied to two points on the top of the picture frame 4 ft. apart, the cord is slung over a nail. Draw a diagram to scale of the cord, and measure the angle the two straight pieces of the cord make at the nail.

2. A man is walking along a straight road which runs due North. He notices a monolith in front of him, whose direction is 20° East of North, after walking a mile, he finds its bearing is 40° East of North. How far from the road is it?

3. A goat is tethered by a chain 5 ft. long, which is attached by a stake 3 ft. away from a straight hedge. How much of the hedge can the goat nibble?

4. A man facing East notices an aeroplane whose elevation is 30° , another man, one mile away, facing West, notices that the elevation is 40° . How far is the aeroplane from the ground?

(The '*angle of elevation*' is the angle between the horizontal and a line from the man to the aeroplane)

5. A man on a cliff 500 ft high notices two objects out at sea, the angle of depression of one is 30° , of the other 60° . How far apart are they?

(The '*angle of depression*' is the angle between the horizontal and a line from the man to the object)

6. Three men, A , B , C , live on the prairie, A is 40 miles from B and 50 miles from C , B and C are 60 miles apart. They travel to meet one another, A and B each go 30 miles. How far does C go?

7. $ABCD$ is a rectangular field, $AB = 520$ yds, $BC = 430$ yds. A tree is situated inside the field, 300 yds from A , and 350 yds from D . How far is it from B and C ?

8. What is the length of the shadow of a pole 6 ft high, when the sun is 40° , 60° , 70° high? How high is the sun when the shadow is 6 ft long?

9. The angle of elevation of a tower from a certain point on the ground is 60° , 100 ft further off it is 50° . How high is the tower?

10. A road slopes at 1 in 3. At what angle is it inclined to the horizontal?

11. A tower is surmounted by a pinnacle. At a distance of 100 ft, the angle of elevation of the top of the tower, from a point on the ground, is 60° , the pinnacle subtends an angle of 10° at the same point on the ground. How tall is the pinnacle?

12. Two men start at the same point, one walks along a road at 3 miles per hour, the other walks along a road at right angles to the first at 4 miles per hour. How far apart are they after 3 hours?

13. A circle of radius 10 ft is divided into 8 equal pieces by suitable radii. What is the distance apart of the ends of two neighbouring radii?

14. What would be the distance if the circle were divided into 16 equal pieces?

15. Draw a line and take a point O at distance 2'' from it. Suppose a point P to move along the line. Join OP and produce to Q so that PQ is 1'' long. Sketch the locus of Q as P moves along the line. (The path of a moving point is called a '*locus*'. We can think

of OPQ as a rod which always goes through a ring at O , and moves so that a fixed point P on the rod always lies in a slot (Fig. 65)

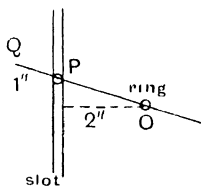


Fig. 65

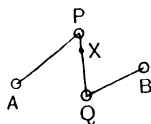


Fig. 66

16 AP , PQ , QB are three rods, jointed together at P , Q , while the ends A , B are fixed. By means of a Meccano set, or by using pieces of cardboard for the rods, sketch the locus of a point X of the rod PQ .

17 Draw a line, and take a point P at a distance from it of $1''$. Draw a large number of circles, through P , with their centres on the line.

18 Draw a circle of radius $1''$. Take a point P at a distance of $\frac{3}{4}''$ from the centre of the circle. Draw a large number of circles which go through P , and have their centres on the circle first drawn.

19 Do the same construction with P (i) on the circle, (ii) at a distance of $1\frac{1}{2}''$ from the centre. (Two separate figures.)

20 Draw a circle of radius $1\frac{1}{2}''$. Take a point O inside the circle at a distance of $1''$ from the centre. Suppose a point P moves round the circle. Join OP and produce to Q so that PQ is $1''$ long. Sketch the locus of Q as P moves round the circle.

21 Draw a line, and take a point O at a distance of $2''$ from it. P moves along the line. Sketch the locus of the mid-point of the interval OP .

22 Draw a circle of radius $1\frac{1}{2}''$, and take a point O , $\frac{1}{2}''$ from the centre. P moves round the circle. Sketch the locus of the mid-point of the interval OP .

23 Take the circle and point O as in the last question. Draw a lot of lines through O to cut the circle. Find the locus of the mid-points of the chords of the circle so obtained.

24 A fly walks very slowly along the minute hand of a clock. Sketch the locus described by the fly.

25 A , B are points $2''$ apart. Draw lines through A , and drop perpendiculars on them from B . Sketch the locus of the feet of the perpendiculars.

26 A ladder slides down a wall. Draw two lines at right angles to each other to represent the wall and floor. Draw an interval 3'' long, with an end on each line, to represent the ladder. Draw various positions of the ladder as it slides down. You will find they trace out a curve, called the 'envelope' of the moving interval.

27 Sketch the locus of the mid-point of the ladder in the last question.

28 Sketch the locus of the point of the ladder represented by a point 1'' from the top point of the moving interval.

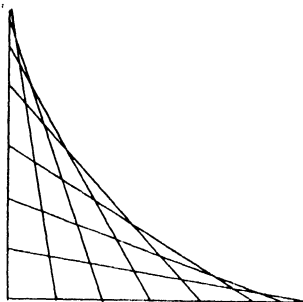


Fig 67

29 Two roads, OP , OQ , are at right angles to each other. One man is walking along OP towards O , another man is walking at the same rate along OQ away from O . Sketch the envelope of the line joining the men.

(Draw two perpendicular intervals, OP , OQ , 3'' long, and suppose that when one man is at P , the other is at O .)

30 Sketch the locus of the mid-point of the interval joining the men in the last question.

31 Two men start, from the same point, to walk round a circular track in the same direction, one walks twice as fast as the other. Sketch the envelope of the line joining them. (Draw a circle of 2'' radius, cut off equal chords along it, and put numbers against the points on the circle.)

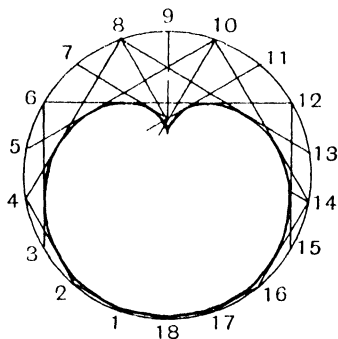


Fig 68

32 Cut out a triangle in cardboard. Take two fixed points X, Y on a piece of paper. Place the triangle in various positions, but so that two of its sides always go through the points X, Y . Sketch the loci of the vertices of the triangle, and by making holes in the cardboard sketch the loci of points like P . Sketch the envelope of the base BC .

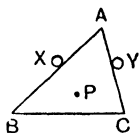


Fig 69

***33** A man walks at a steady pace along a straight road. A dog tries to catch him up, starting at a point not on the road and always heading towards him, and running at a steady rate. Sketch the path of the dog.

****34** Three boys chase each other. A chases B , B chases C , C chases A . Sketch their paths.

35 Two lines, AC and BD , meet off the paper. O is a given point. To draw a line through O which goes through the point where the given lines cut. Take any point C on the line and complete the figure, where XC, CD, DX are resp. parallel to OA, AB, BO . Then OX is the line required.

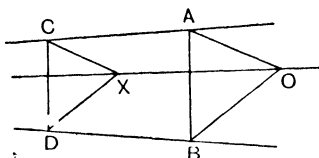


Fig 70

36 Copy the following figures on any convenient scale.

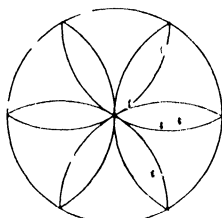


Fig 71

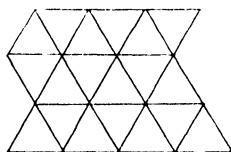


Fig 72

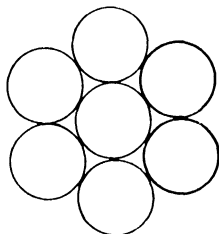


Fig 73

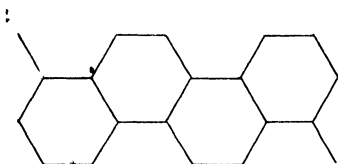


Fig 74

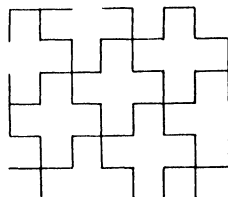


Fig 75

***37.** A boy draws a circle with a penny, and wishes to find its centre with his compasses, not using a ruler. The following is a construction due to Mascheroni (1750–1800). Take any point A on the circle, and with A as centre, and any radius r less than the diameter of the given circle, but greater than one-quarter of it, draw a circle cutting the given circle in M, B . With the same radius r , step off points C, D, E on the circle so that

$$AB = BC = CD = DE$$

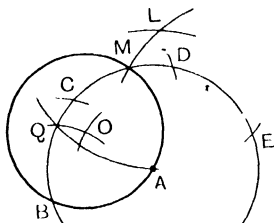


Fig. 76

With centres E, A , and radius equal to the interval EM , draw two circles cutting in L , with centre L , and radius LA , draw a circle cutting the circle BME at Q , with centres B and A , and radius equal to the interval BQ , draw two circles to cut in O . Then O is the required centre.

Def. A 'regular polygon' is one whose sides are all equal and whose angles are all equal. Eg. an equilateral triangle and a square are regular polygons.

***38** (a) Draw a pentagon (five-sided figure) whose sides are all equal, but whose angles are not all equal. Draw a pentagon whose sides are of any length, and four of whose angles are of size 108° . Measure the fifth angle.

(b) To draw a regular pentagon with a given interval AB for a side.

Construct Fig. 77, where $AC = CB$, $CD \perp AB$, $CD = AL$, $DE = AC$, $AF = AE$, $FH = FK = AH = BK = AB$.

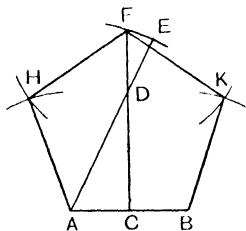


Fig. 77

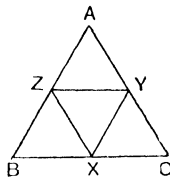


Fig. 78

39 Draw Fig. 78 on stiff paper, X, Y, Z are the mid-points of the sides. Fold the $\triangle ABC$ along the lines XY, YZ, ZX until the three points A, B, C meet each other. The solid figure so obtained is a 'tetrahedron.' If ABC is an equilateral triangle, the solid is a 'regular' tetrahedron.

40 Make a model of a *cube* out of stiff paper

***41** Make models out of stiff paper of the '*regular solids*,' whose front views are shown below. The back half is like the front half.

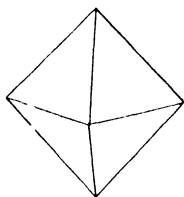


Fig 79

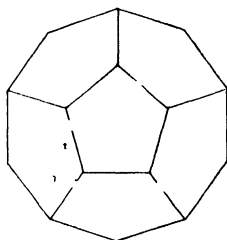


Fig 80

Fig 79 represents an '*octahedron*', its eight faces are equilateral triangles

Fig 80 represents a '*dodecahedron*', its twelve faces are regular pentagons

Fig 81 represents an '*icosahedron*', its twenty faces are equilateral triangles

Fig 82 is a '*development*' of an icosahedron, if we cut it out in paper we can fold it along the lines so as to make an icosahedron

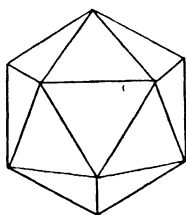


Fig 81

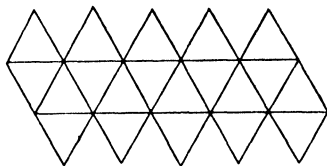


Fig 82

EXERCISE 22

Constructions

Construct the following figures, and *measure the intervals in brackets*

1. (a) An isosceles triangle, whose base is 3" long, and whose height is 2" Measure its sides

(v) An equilateral triangle of height 5 cm Measure its sides

2. A quadrilateral $ABCD$, where $AB = 2''$, $AD = BC = 1\frac{1}{2}''$, $\hat{BAD} = 120^\circ$, $\hat{ABC} = 140^\circ$. (CD)

† It is best to give these as an optional alternative to written home work.

3 A quadrilateral $ABCD$, where $AB = 2''$, $AC = BD = 1\frac{1}{2}''$, $\hat{A}BD = 40^\circ$, $\hat{B}AC = 30^\circ$. (CD)

4 An isosceles trapezium $ABCD$, where $AB = 3''$, $CD = 2''$, distance between parallel sides AB , CD is $2''$. (AD)

5 A rhombus $ABCD$, where $AC = 5$ cm, $\hat{BAD} = 40^\circ$. (BD)

6 A quadrilateral $ABCD$, where $AB = 1\frac{1}{2}''$, $BC = 2''$, $CD = 1\frac{1}{2}''$, $AD = 1''$, $BD = 2''$. ($\hat{A}C$)

7 A parallelogram $ABCD$, where $AB = 1\frac{1}{2}''$, $AC = 2\frac{1}{2}''$, $ED = 1''$ (AD)

8 A parallelogram $ABCD$, where $AB = 4.5$ cm, $AD = 6$ cm., $\hat{BAD} = 45^\circ$ (BD)

9 A parallelogram $ABCD$, where $AC = 1\frac{1}{2}''$, $BD = 2''$, angle between diagonals $= 40^\circ$. (AB)

EXAMPLE 7 To construct a triangle given two angles and the perimeter (i.e. the sum of its sides).

Given XY — perimeter of triangle, and two angles of triangle.
To construct the triangle.

Construction Draw rays XP , YQ , on the same side of XY , so that $\hat{Y}XP$, $\hat{X}YQ$ are half the given angles.

Let rays XP , YQ cut in C .

Draw lines CA , CB to cut the interval XY in A , B , and such that $\hat{X}CA = \hat{A}XC$ and $\hat{Y}CB = \hat{B}YC$.

Then ABC is the required triangle

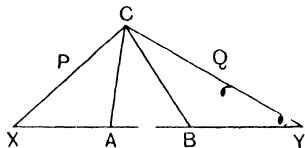


Fig 83

Proof Since ext $\angle$ of $\triangle XCA =$ sum of int opp $\angle$ s,

$$\begin{aligned} \therefore \hat{C}AB &= \hat{X}CA + \hat{A}XC \\ &= 2\hat{A}XC, \quad (\hat{X}CA = \hat{A}XC, \text{ const}) \\ &= \text{one of given angles,} \quad (\hat{A}XC = \frac{1}{2} \text{ given angle}) \end{aligned}$$

Similarly, $\hat{C}BA =$ the other given angle

Also $AX = AC$, ($\hat{X}CA = \hat{A}XC$).

Similarly $BY = BC$

$$AC + BC + AB = AX + BY + AB = XY.$$

the triangle ABC has the given perimeter and the given angles

10 Draw a triangle ABC such that $\hat{BAC} = 60^\circ$, $\hat{ABC} = 90^\circ$, perimeter = 3". Measure its side

11 Construct an isosceles triangle of perimeter 4", whose vertical angle is 30° . Measure its sides

*12. Construct a right-angled triangle, given the hypotenuse and one side

*13. Construct a right-angled triangle ABC , given the hypotenuse BC and the altitude from A

*14. Construct an isosceles triangle, given the lengths of its equal sides and its height

*15. Construct a triangle ABC , given its perimeter, the angle BAC , and the altitude from C

EXAMPLE 8 To construct a triangle ABC , where $AB = 1\frac{1}{2}"$, $\hat{ACB} = 60^\circ$, $BC = 1\frac{1}{4}"$

Construction Make an angle XAY of 60° , on the ray AY cut off an interval $AB = 1\frac{1}{2}"$. With centre B and radius $1\frac{1}{4}"$ draw a circle. This cuts AX in two points C_1 and C_2 . Hence we see that there are two non-congruent triangles ABC_1 and ABC_2 with the given measurements. Of course this is because we were given two sides and an angle *not* included. If the angle XAY had been obtuse, we should however have found only one triangle. The whole question is cleared up by the next theorem

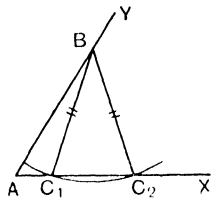


Fig 84

*THEOREM 8

If two triangles be such that two sides and a not included angle of one be equal to the corresponding parts of the other, then the third angles are equal or supplementary, and if they are equal the triangles are congruent

(By the third angles we mean those neither given as equal, nor included between the sides mentioned)

Given. In $\triangle s ABC, XYZ$,

$$AB = XY, AC = XZ, \hat{ABC} = \hat{XYZ}.$$

To prove that $\hat{ACB}$ and $\hat{XZY}$ are equal or supplementary

Proof. Either BC is equal to YZ , or it is not.

First, suppose $BC = YZ$, then in $\triangle s ABC, XYZ$,

$$\begin{cases} AB = XY, & \text{(given).} \\ BC = YZ, & \text{(given).} \\ \hat{A}BC = \hat{X}YZ, & \text{(given)} \end{cases}$$

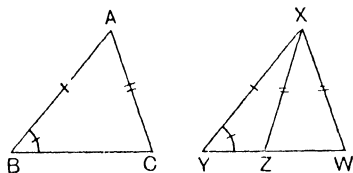


Fig 85

$\therefore \triangle s ABC, XYZ$ are congruent, (two sides, incl $\angle$).

$$\hat{A}CB = \hat{X}ZY.$$

Secondly, suppose BC is greater than YZ . Take a point W on ray YZ , so that $BC = YW$. Then, as in the first part, we have. $\triangle s ABC, XYW$ are congruent

$$\hat{A}CB = \hat{X}WY, \text{ i.e. } \hat{A}CB = \hat{X}WZ, \quad (1)$$

$$\text{and } AC = XW.$$

$$\text{But } AC = XZ, \text{ (given)}$$

$$XW = XZ \quad \hat{X}ZW = \hat{X}WZ, \text{ (isos } \triangle)$$

$$\text{by (1), } \hat{A}CB = \hat{X}ZW. \quad (2)$$

But $\hat{X}ZW$ and $\hat{X}ZY$ are supplementary (adj $\angle$ s),

$\therefore$ by (2), $\hat{A}CB$ and $\hat{X}ZY$ are supplementary.

Thus the third angles $\hat{A}CB$ and $\hat{X}ZY$ are either equal or supplementary.

Lastly, if BC is less than YZ , the proof is similar. Q.E.D.

Cor If two obtuse-angled triangles, or two right-angled triangles, or two acute-angled triangles be such that two sides and a not included angle of one equal the corresponding parts of the other, then the triangles are congruent, but this does not follow if one triangle be acute-angled, the other obtuse-angled.

***EXERCISE 23.**

1. Deduce the right angle congruence in the above way.
2. The bisector of the vertical angle of a triangle bisects the base. Prove that the triangle is isosceles.
3. Two sides of a quadrilateral are equal, the other two are parallel. Prove that the quadrilateral is either a parallelogram or a trapezium.
4. X is any point on the bisector of angle BAC , a circle, centre X , cuts rays AB, AC in two points each, the points on these rays nearest to A are E, F . Prove $AE = AF$.
5. M is the mid-point of the base AB of triangle ABC , the perpendicular intervals from M on to CA, CB are equal. Prove $CA = CB$.
6. $ABCD$ is a convex quadrilateral, the diagonals AC, BD cut in O , $AO = CO$, $AB \parallel CD$. Prove that lines AD, BC are either parallel or cut in a point P such that $PB = PD$.
7. $ABCD$ is a quadrilateral, $AB = BC = CD$, the angles at C, D are right angles. Prove that the quadrilateral is a square.
8. Draw a quadrilateral $ABCD$, where $AB = CD$, $\hat{ABC} = \hat{ADC}$, and which is not a parallelogram.

The reader should know already what a regular hexagon is, and how to construct a regular hexagon inscribed in a given circle, the following is a formal proof that the construction is correct

***CONSTRUCTION 7.**

To draw a regular hexagon inscribed in a given circle

Given. Circle, centre O

To construct a regular hexagon inscribed in the circle

Construction. Take any point A on the circle.

With centre A draw a circle through O . Let it cut given circle in B .

With centre B draw a circle through A . Let it cut given circle again in C .

With centre C draw a circle through B . Let it cut given circle again in D .

Then AB is a side of a regular hexagon inscribed in the circle.

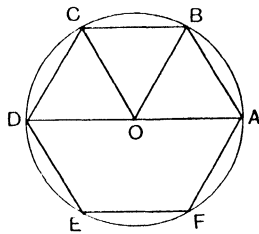


Fig. 86

Proof. We first prove that A, O, D are in one straight line.

$OA = AB = OB$, (in two equal circles).

$\therefore \triangle OAB$ has all its angles equal

But their sum equals two rt. $\angle$ s.

$\therefore \angle AOB$ equals $\frac{2}{3}$ of one rt. $\angle$.

Similarly, $\angle BOC$ and $\angle COD$ each equal $\frac{2}{3}$ of one rt. $\angle$

$\therefore \angle AOB + \angle BOC + \angle COD$ equals 2 rt. $\angle$ s

A, O, D are in one straight line

Similarly, if AF, FE, ED' are equal chords, then A, O, D' are in one straight line. Therefore D and D' coincide

Thus the hexagon $ABCDEF$ has all its sides equal

Also all its angles are equal, since each is the sum of two angles of an equilateral triangle

Hence the hexagon is regular

Q.E.F.

1. Describe the construction (Fig. 87) for drawing an angle of 90° , $OA, OB, OC, AB, BC, CD, BD$ are all equal

2. O, A are given points. Find with compasses only, a point D such that $OA = OD$, and A, O, D are in one straight line

§ 9. Angles of polygons.

THEOREM 9

The sum of the angles of a polygon of n sides equals $2n - 4$ right angles

Given. $ABC\cdots$ is a polygon of n sides (Fig. 88.)

To prove that the sum of its angles equals $2n - 4$ rt. $\angle$ s

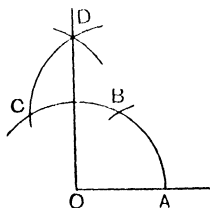


Fig. 87

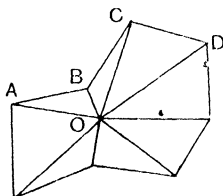


Fig. 88

Proof. Assume there is a point O inside the polygon, such that all the intervals $OA, OB, \dots$ from O to the vertices lie entirely inside the polygon.

The polygon is then dissected into n triangles by these intervals.

The sum of the angles of each triangle equals two rt. $\angle$ s.

the sum of the angles of all the triangles equals $2n$ rt. $\angle$ s.

The sum of the angles of the triangles at O equals four rt. $\angle$ s.

The sum of the other angles of the triangles equals the sum of the angles of the polygon.

the sum of the angles of the polygon equals $2n - 4$ rt. $\angle$ s.

Q. E. D.

I. Fig. 88 the angle of the polygon at B is 'reflex,' the polygon is 'non-convex' or 're-entrant.' Theorem 9 is true for such polygons as well as for convex polygons. There may be no point O inside the polygon having the assumed properties. The proof then fails, but the theorem, though unproved in that case, is still true.

The following (Theorem 10) holds *only* for convex polygons.

THEOREM 10.

If the sides of a convex polygon, taken in order, be produced, the sum of the exterior angles so formed equals four right angles.

Given a polygon $ABC\cdots$ of n sides, with the side AB produced beyond B to X , the side BC produced beyond C to Y , and so on.

[This is what is meant by producing the sides, taken in order.]

To prove that the sum of the ext. $\angle$ s $XBC, YCD, \dots$, and so on, equals four rt. $\angle$ s.

Proof. The sum of the int. $\angle$ s $ABC, BCD, \dots$ equals $2n - 4$ rt. $\angle$ s.

The sum of the int. $\angle$ ABC and the ext. $\angle$ XBC equals two rt. $\angle$ s, (adjacent $\angle$ s).

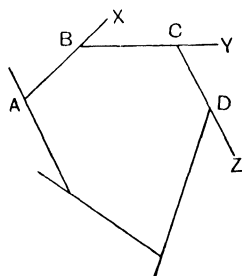


Fig 89

Similarly for the other adjacent int. $\angle$ s and ext. $\angle$ s. •

$\therefore$ the sum of all the int. and all the ext. $\angle$ s equals $2n$ rt. $\angle$ s.

$\therefore$ the sum of the ext. $\angle$ s alone equals

$$2n - (2n - 4) \text{ rt. } \angle\text{s} = 4 \text{ rt. } \angle\text{s}$$

Q.E.D.

EXERCISE 24.

1. (a) Find the size of the angles of a regular polygon of 5, 6, and 100 sides,

(b) and on n sides.

2. (a) A quadrilateral has two angles right angles. Prove the other two are supplementary.

(b) A quadrilateral has all its angles equal. What sort of a quadrilateral is it?

3. CR, BQ are altitudes of $\triangle ABC$. They cut in H . Prove that $\hat{R}AQ, \hat{R}HQ$ are supplementary.

4. The sides AB, AC of $\triangle ABC$ are produced beyond B, C resp. to X, Y , the bisectors of angles ABC, ACB cut in I , those of angles CBX, BCY in K . Prove that angles BIC, BKC are supplementary.

5. The opposite angles at A, C of a convex quadrilateral $ABCD$ are bisected. Show that one of the angles between the bisectors is $\frac{1}{2}(B \sim D)$ [$\sim$ means *difference between*]

6. How many sides has the regular polygon whose angles are (i) 135° , (ii) 156° , (iii) 162° ?

7. The angles of a regular polygon are A° . Prove that the number of its sides is $360/(180 - A)$.

8. $ABCDEF$ is a regular hexagon. Prove that $\hat{EAB}$ is a right angle.

9. $ABCDE$ is a regular pentagon. Find the angles of $\triangle ABD$.

10. $ABCDEFGH$ is a regular octagon. Find the acute angle between AE and FB and that between AE and CH .

11. The exterior angles of a pentagon are in the ratio $1 : 2 : 3 : 4 : 5$. Find the largest interior angle of the figure.

12. The interior angles of a quadrilateral are in the ratio $1 : 2 : 3 : 8$. Prove that it is not convex. What would happen if the angles were in the ratio $1 : 2 : 3 : 6$?

13. $ABCDEFG$ is a regular heptagon. Calculate the angles between the intervals from A to B, C, D, E, F, G respectively.

14. $ABCDEFG$ is a regular heptagon. Calculate the acute angle between AC and BD .

15 The angles of a regular polygon of n sides are double those of a regular polygon of m sides. Prove $\frac{4}{m} - \frac{2}{n} = 1$. Hence prove that the polygons can only be a regular hexagon and an equilateral triangle.

***16** Prove that six equilateral triangles, or four squares, or three regular hexagons can be fitted together at a point, so as to fill the plane round the point without overlapping, but that if four regular pentagons be fitted together at a point they must overlap.

Revision Papers I—X can now be taken, if desired

§ 10. Intercepts

THEOREM 11.

If a set of parallel lines intercept equal intervals on one transversal, they will intercept equal intervals on any transversal.

(A transversal of the parallel lines is a line cutting them.)

Given. AX, BY, CZ are parallel lines, lines ABC, XYZ are transversals $AB = BC$

To prove that $XY = YZ$

Proof Through X draw $XP \parallel AB$, let XP cut BY in P .

Through Y draw $YQ \parallel AB$, let YQ cut CZ in Q .

Then since XP, YQ are both parallel to AB , $XP \parallel YQ$

Also $XPBA$ is a parallelogram, ($XP \parallel AB, AX \parallel BP$)

$\therefore AP = AB$, (opp sides of parm.).

Similarly, $YQ = BC$.

But $AB = BC$, (given)

$\therefore XP = YQ$

In $\triangle$ s XPY, YQZ ,

$$\begin{cases} XP = YQ, & (\text{proved}). \\ \hat{PXY} = \hat{QYZ}, & (\text{corr. } \angle\text{s}; XP \parallel YQ). \\ \hat{PYX} = \hat{QZY}, & (\text{corr. } \angle\text{s}, PY \parallel QZ). \end{cases}$$

$\triangle$ s XPY, YQZ are congruent, (one side, two $\angle$ s)

$\therefore XY = YZ$.

Q E D

Notes. 1. We have said that since $XP \parallel AB$ and $YQ \parallel AB$, $\therefore XP \parallel YQ$, i.e. we have used an assumption which we now state.

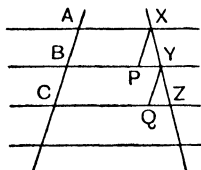


Fig 90

Assumption 15 *Lines parallel to the same line are parallel to one another.*

2 There is a temptation to reason as follows

Since $XP \parallel YQ$ and $PY \parallel QZ$,

$$\hat{XPY} = \hat{YQZ}$$

That this argument is faulty is plain from the following figure, where the angles $\hat{XPY}$ and $\hat{YQZ}$ are not equal, although $XP \parallel YQ$ and $PY \parallel QZ$

To state such an argument correctly we should have to regard the parallel lines as having directions, so that in Fig 90 the parallel lines PY and QZ would have the same direction, while in Fig 91 the parallel lines PY and QZ would have opposite directions. All this raises difficulties, and there is no need to face them at this stage because they can always be avoided

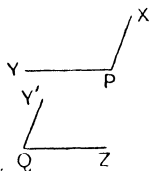


Fig 91

THEOREM 12

A line through the mid-point of one side of a triangle parallel to a second side bisects the third side

(The second side in this theorem is often called the base)

Given. ABC is a triangle, X the mid-point of AB , XY is parallel to BC , and cuts AC in Y .

To prove that $AY = YC$.

Proof Draw $CP \parallel AB$, and let CP cut line XY in P .

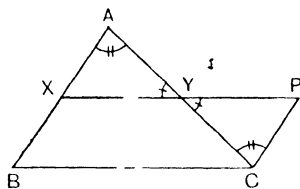


Fig 92

$PCBX$ is a parm ($\because BC \parallel XP$, $CP \parallel XB$).

$$\therefore PC = XB, \text{ (opp sides of parm)}$$

But $AX = XB$, (given)

$$\therefore AX = PC.$$

In $\triangle s AXY, CPY$,

$$\begin{cases} AX = CP, & \text{(proved).} \\ \hat{AXY} = \hat{CPY}, & \text{(vert opp)} \\ \hat{XAY} = \hat{YCP}, & \text{(alt } \angle s; CP \parallel AX) \end{cases}$$

$\therefore \triangle s AXY, CPY$ are congruent, (one side, two $\angle s$).

$$\therefore AY = YC.$$

Q.E.D.

THEOREM 13

The interval joining the mid-points of two sides of a triangle is parallel to the third side, and has a length equal to half the length of the third side

Given. $\triangle ABC$ is a triangle, X, Y the mid-points of AB, AC resp.

To prove that $XY \parallel BC$, $XY = \frac{1}{2}BC$

Proof Draw $CP \parallel AB$, and let CP cut line XY in P

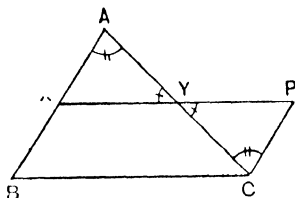


Fig 92

In $\triangle s \triangle XY, CPY$,

$$\begin{cases} AY = YC, & (\text{given}) \\ \hat{A}YX = \hat{C}YP, & (\text{vert opp}) \\ \hat{X}AY = \hat{Y}CP, & (\text{alt } \angle s, CP \parallel AX) \end{cases}$$

$\therefore \triangle s \triangle XY, CPY$ are congruent, (one side, two $\angle s$)

$$AX = CP \quad (i)$$

$$\text{and } XY = YP \quad (ii)$$

But $AX = XB$, (given)

by (i) $XB = CP$, and $XB \parallel CP$, (const)

$XBPC$ is a parm.

$XP \parallel BC$, i.e. $XY \parallel BC$,

and $XP = BC$ by (ii), $XY = \frac{1}{2}BC$ Q.E.D.

Note It is best (for examination reasons) to prove Theorem 13 without using Theorem 12. If we can use Theorem 12 we can prove the first part of Theorem 13 as follows: if XY be not parallel to BC , draw the parallel through X to BC and suppose it cuts AC in W . Then by Theorem 12, W is the mid-point of AC , thus the interval AC has two mid-points, Y and W , which is absurd. The method of proof in which we pretend that what we want to prove is *false*, and then deduce an 'absurdity,' is called the method of '*reductio ad absurdum*'.

Exercise. (i) If Z is any point on XY , and the line AZ cuts BC in D , then $AZ = ZD$

(ii) The altitude from A to BC is bisected at right angles by XY

CONSTRUCTION 8.

To divide a given interval into n equal parts

Given an interval AB

To divide it into n equal parts

Construction From A draw any ray AX , and mark off on it n equal intervals, AP_1 , P_1P_2 , P_2P_3 , ..., $P_{n-1}P_n$.

Join BP_n , and through P_1, P_2 , draw parallels to P_nB . Let these cut AB in Q_1, Q_2 , ...

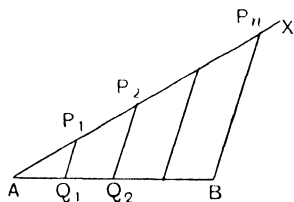


Fig 93

Then AB is divided into n equal parts by Q_1, Q_2 , ...

Proof The parallel lines P_1Q_1, P_2Q_2 , ..., P_nB cut off n equal intervals on the transversal AX , and therefore they cut off n equal intervals on the transversal AB .

∴ the interval AB is divided into n equal parts.

Q E D

J

EXERCISE 25

1. (a) L, M, N are the mid-points of the sides BC, CA, AB of triangle ABC . Prove that $ANLM$ is a parallelogram.

(b) If the points L, M, N be joined, then $\triangle ABC$ is dissected into four congruent triangles.

(c) Given a triangle, show how to divide it into two pieces which will fit together to make a parallelogram.

2. L, M, N are the mid-points of the sides of a triangle. Construct the triangle.

3. AP is an altitude of $\triangle ABC$, N, M are mid-points of sides AB, AC . Prove that triangles ANM, PNM are congruent.

4. $ABCD$ is a parallelogram, X, Y are mid-points of sides AB, CD . Prove that DX, BY trisect the interval AC .

5. The parallels AX, BY, CP, DQ are cut by one line in $\angle, B, C, D$, and by another in X, Y, P, Q . If $AB = CD$, prove $XY = PQ$.

6 (a) If, in Fig 92, Q is any point on base BC , prove that XY bisects interval AQ

(b) A is a given point, C moves on a line BC . Prove that the locus of the mid-point of AQ is a straight line

(c) Suppose in (b) that the interval AQ is produced beyond Q to R , so that $AQ = QR$. Prove that the locus of R is a straight line

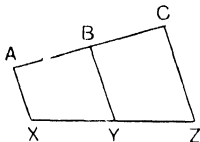


Fig. 94

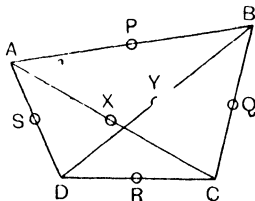


Fig 95

7 In Fig 94, AX , BY , CZ are parallel, $XY = YZ$. Prove $2BY = AX + CZ$

8 $ABCD$ is any quadrilateral, P , Q , R , S are the mid-points of its sides AB , BC , CD , DA , X , Y are the mid-points of its diagonals AC , BD . Prove the following (Fig 95)

(a) $PQRS$ is a parallelogram

(b) The intervals PR , QS bisect each other.

(c) If $AC = BD$, prove $PR \perp QS$

(d) If $AC \perp BD$, prove $PR = QS$

(e) If $BC \parallel AD$, then P , R , X , Y are collinear and $2PR = AD + BC$

(f) If BC and AD are not parallel, then intervals PR and XY bisect each other

(g) If $ABCD$ be neither a parallelogram nor a trapezium, then intervals PR , QS , XY bisect each other

9 ABC is any triangle, O any point inside it, L , M , N are mid-points of BC , CA , AB , X , Y , Z are mid-points of OA , OB , OC . Prove that XL , YM , ZN are concurrent.

EXAMPLE 9 The three medians of a triangle are concurrent (A median is a line joining a vertex to the mid-point of the opposite side)

Given. L , M are the mid-points of sides BC , AC of $\triangle ABC$, BM , AL cut in G (Fig 96)

To prove that CG cuts the side AB in its mid-point

Proof Let CG cut the side AB in N ; let P be the point on line CG such that $CG = GP$.

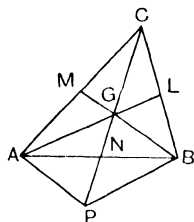


Fig 96

In $\triangle CAP$, $CM = MA$, $CG = GP$. $MG \parallel AP$, i.e. $BG \parallel AP$.

In $\triangle CBP$, $CL = LB$, $CG = GP$, $LG \parallel BP$, i.e. $AG \parallel BP$

$AGBP$ is a parallelogram, (def)

$AN = NB$, (diags bisect each other) Q.E.D.

Cor Also $GN = NP$ And $CG = GP$, $2NG = GC$ $3NG = NC$

G trisects CN and, similarly, it trisects BM and AL

Def The point where the medians cut is called the 'centroid' of the triangle. If the triangle ABC be cut out in paper, it will balance on a pin at the point G

10 The altitudes AP , BQ of $\triangle ABC$ cut in H , L , X , N are the mid-points of intervals BC , AH , AB . Prove that angle LNX is a right angle

***11** L , M , N are the mid-points of sides BC , CA , AB of $\triangle ABC$, BM cuts LN in X , CN cuts LM in Y . Prove that $4XY = BC$, $XY \parallel BC$

***12** O is a fixed point. P moves on a fixed circle. Prove that the locus of the mid-point of interval OP is a circle, whose radius is half that of the given circle.

***13** (a) X , Y are the feet of the perpendiculars from B , C on the bisector of the angle BAC of $\triangle ABC$. L is the mid-point of BC . Prove $LX = LY$ (Produce the intervals BX , CY to cut lines AC , AB resp. Fig 97.)

***(b)** $LX = \frac{1}{2}(AB + AC)$

***14** X is the foot of the perpendicular from B on the bisector of the angle BAC of $\triangle ABC$. Prove that the parallel through X to AC bisects AB .

***15** The intervals AB and CD bisect each other, P is any point, A' , B' , C' , D' are points on the lines PA , PB , PC , PD resp, such that $PA = AA'$, $PB = BB'$, $PC = CC'$, $PD = DD'$. Prove that intervals $A'B'$ and $C'D'$ bisect each other.

***16** ABC is a triangle, L is the mid-point of side BC , M of interval AL , the lines BM and AC cut in N . Prove that $3AN = AC$.

***17** (a) O is any point inside $\triangle ABC$, L , M are mid-points of BC , CA resp, the intervals OL , OM are produced to X , Y such that $OL = LX$, $OM = MY$. Prove that $XYAB$ is a parallelogram.

(b) Hence show that if N is the mid-point of AB , and Z is on line ON such that $ON = NZ$, then the intervals AX , BY , CZ bisect each other.

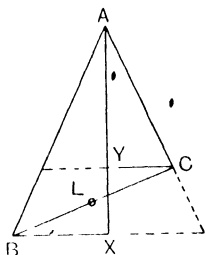


Fig 97

***18** P, Q are the feet of the perpendiculars from A to the bisectors of the angles ABC, ACB of $\triangle ABC$. Prove that $PQ \parallel BC$ (Produce the perps to cut BC)

***19** M is the mid-point of the base BC of triangle ABC , a line through M cuts the lines AB, AC in D, E resp., $AD = AE$. Prove $BD = CE$ (Through B draw a parallel to AC Fig 98)

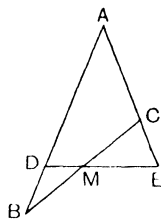


Fig 98

***20** A, B, C are three points in order on a line, $m AB = n BC$, m, n being integers, parallel lines through A, B, C cut another line in points X, Y, Z on the same side of AB . Prove that

$$m XY = n YZ$$

and $m AX + n CZ = (m + n) BY$

***21** Through the vertices A, B, C, D of a parallelogram parallel lines are drawn, a line, not meeting the parallelogram, cuts them in X, Y, Z, W . Prove that

$$AX + CZ = BY + DW$$

***22** G is the centroid of triangle ABC , parallel lines through A, B, C, G are cut by a line, not meeting the triangle, in X, Y, Z, P . Prove that

$$3GP = AX + BY + CZ$$

(Use No 20)

***23** The points B, C move on fixed lines $AX, AY, AB + AC$ is a constant length. Prove that the locus of the mid-point of the interval BC is a line. (Complete the parallelogram $ABPC$ and prove that the locus of P is a line Cf Exercise 20, No 8)

****24** (a) $ABCD$ is a convex quadrilateral, $AB = CD$, M, N are the mid-points of sides BC, AD . Prove that the line MN is equally inclined to AB and CD . (Make $NP = BN$, $DP = BA$, $\triangle DCP$ is isos $\triangle$, $MN \parallel CP$ Fig 99)

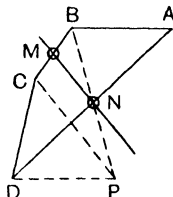


Fig 99

(b) BA, CD are fixed lines, A moves along BA and D moves at the same rate along CD . Prove that the locus of the mid-point of the interval AD is a line equally inclined to the fixed lines. State an exceptional case.

Construct a *diagonal scale* showing inches, tenths and hundredths of an inch. Fig 100 shows part of the scale $AB, BK, \dots$ are inch divisions, the sides of $ABCD$ are divided into tenths, division 4 on AB is joined to 5 on CD , and so on. Draw all these slant lines and all the horizontal lines. As we go upwards along

a slant line we advance $1/100$ of an inch to the left between each horizontal line and the next. Thus the distance of X from the line CB is $.46''$; the tenths are given by the numbers along AB , the hundredths by the numbers along BC . The distance XY is $1.46''$

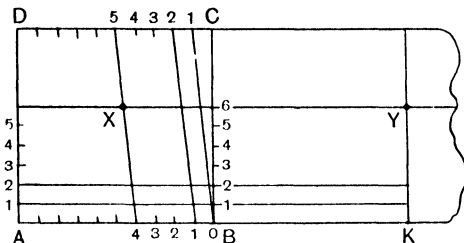


Fig 100

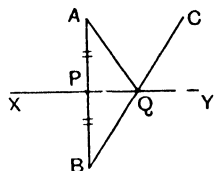


Fig 101

§ 11. Images.

Def Let XY be a fixed line, A a point, P the foot of the perpendicular from A to XY . If the interval AP be produced beyond P to B , so that $AP = PB$, then B is the 'image' or 'reflection' of A in XY . (Fig 101.)

EXERCISE 26

1 A, C are on the same side of the line XY , and B is the image of A in XY , CB cuts XY in Q . Prove that AQ and CQ are equally inclined to XY . (If XY were a sheet of glass, a ray of light in direction AQ would be reflected in direction QC .)

2 B, F are the images of A, E resp. in XY . Prove $AE = BF$.

3. Rays of light start from a point A and strike a sheet of glass, they leave the glass, on reflection, at the same angle at which they strike it. Prove that the reflected rays, if produced backwards, would pass through the image of A in XY .

4 How long a mirror does a six-foot man need, if he wishes to see himself completely? Does his distance from the mirror matter?

5. Two men, A and B , are looking at a mirror, A can see B 's eyes. Show by a figure that B could see A 's eyes.

6 (a) OA, OB are two lines, Q is the image of P in OA , R is the image of Q in OB . Prove that $\angle POR = 2\angle AOB$ or $360^\circ - 2\angle AOB$ (Fig. 102)

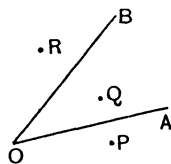


Fig 102

**(b)* The effect of reflecting any point, first in OA and then in OB , is the same as that obtained by rotating the point round O through an angle $2\hat{AOB}$

7. *(a)* AB, CD are parallel lines, Q is the image of P in AB , R is the image of Q in CD . Prove that the distance between P and R is twice that between AB and CD .

**(b)* The effect of reflecting any point, first in AB and then in CD , is the same as that obtained by moving the point by a translation perpendicular to AB , and of amount twice the distance between AB and CD .

8. X, Y, Z are the images of A, B, C resp. in a line. If A, B, C are the vertices of a triangle, then $\triangle XYZ$ is a congruent triangle

If two triangles in the same plane are congruent, and one can be moved *in the plane* until it covers the other, we say they are '*directly congruent*' if this cannot be done, they are '*indirectly congruent*'. The triangles ABC and XYZ of the last example are *indirectly congruent*. A reflection in a line turns E into E'

§ 12. Parallel lines and planes.

It is clear that two planes which have one point common, have a line common

Def. Two planes which do not cut are called '*parallel*' planes. If a line and a plane do not cut, the line and plane are called '*parallel*'

Note that two parallel *lines* lie, by definition, in the same plane

EXERCISE 27 (Oral)

1. Are all vertical lines parallel? Are all horizontal lines parallel?
2. Are all vertical planes parallel? Are all horizontal planes parallel?
3. Can you draw a vertical plane through a horizontal line, or a horizontal plane through a vertical line?

4. What is the (i) largest number, (ii) smallest number of the faces of a brick that can be horizontal at the same time?

What is the (i) largest number, (ii) smallest number that can be vertical at the same time?

Answer the same questions for edges of the brick.

5. How many lines can be drawn, through a point, parallel to (i) a line, (ii) a plane?

6. How many planes can be drawn, through a point, parallel to (i) a line, (ii) a plane?

7 How many planes can be drawn, through a given line, so as to be parallel to (i) a line, (ii) a plane? Consider both cases when the given line is, and when it is not, parallel to the other given line or plane

8 Give an example of two lines parallel to the same plane and *not* parallel to one another

In Fig. 103, 1 and 2 represent parallel planes, while 3 represents a plane cutting both. It should now be easy to see the truth of the following statements

If a plane cuts two parallel planes, it cuts them in parallel lines. If two planes are parallel to a third plane, they are parallel to one another

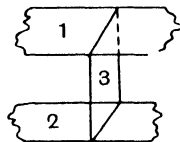


Fig 103

If two lines are parallel to a third line, (in the same plane, or not), they are parallel to one another

If two planes are parallel, any line drawn in one plane is parallel to the other plane (For if the line cut the other plane, the planes themselves would cut)

If OA and PC be different planes, and the lines OA , OB be parallel resp to the lines PC , PD , then the planes OAB , PCD are parallel (Fig 104.)

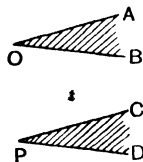


Fig 104

A 'parallelepiped' is a solid with six faces, all parallelograms, the opposite faces lying in parallel planes (see Fig. 105). A is a 'vertex,' $ABXY$ is a 'face', the interval AB is an 'edge'. Do not use the word 'side' where solids are concerned. The interval AC is a diagonal of the solid, the interval AX is a diagonal of a face of the solid

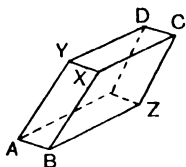


FIG 105

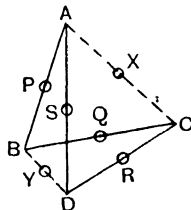


FIG 106

In Fig 106 AB, BC, CD, DA represent four intervals not in the same plane. $ABCD$ is called a 'skew' quadrilateral. If we join BD and AC we have the six edges of a 'tetrahedron.' The triangles ABC , etc. are the four *faces* of the tetrahedron.

EXERCISE 28

1. In Fig 105 prove that $ABCD$ is a parallelogram.
2. In the same figure, prove that the diagonals AC, BD , and YZ bisect each other.
3. Make a model of a skew quadrilateral $ABCD$, let P, Q, R, S be the mid-points of the sides AB, BC, CD, DA . Prove that $PQ \parallel AC$, $SR \parallel AC$, $PQRS$ is a parallelogram, the intervals PR and QS bisect each other (Fig 106).
 - (a) In the same quadrilateral, let X, Y be the mid-points of the 'diagonals' AC and BD . Prove that the intervals XY, PR and QS bisect each other.
 - (b) The intervals which join the mid-points of opposite edges of a tetrahedron bisect each other.
5. $ABCD$ and $ABPQ$ are parallelograms with a common side AB , but lying in different planes. Prove that $CDQP$ is a parallelogram.
6. AB, PQ, XY are three intervals, not in the same plane, but equal and parallel. B, Q, Y lie on the same side of plane APX . Prove that the triangles APX and BQY are congruent.

REVISION PAPERS

I 1. AE is a bisector of angle BAC , E is on side BC , AD is an altitude of $\triangle ABC$, $\hat{ABC} = 50^\circ$, $\hat{ACB} = 20^\circ$. Find the size of angles EAD, AED .

2. Construct a triangle whose sides are $3''$, $2\frac{1}{2}''$, $4''$ long. Measure the altitude which is perpendicular to the longest side.

3. $\angle ABC$ is an isosceles triangle, $AB = AC$, the side BC is produced beyond C to D , the interval DA is produced beyond A to E . Prove $\hat{BAE} = 2\hat{BDA} + \hat{CAD}$.

II 1. AE is a bisector of angle BAC , AD is an altitude of $\triangle ABC$. Prove $\hat{EAD} = 90^\circ - (B + \frac{1}{2}C) = \frac{1}{2}(C - B)$ ($\sim$ means 'difference between').

2. Construct a triangle ABC where $AB = 3''$, $BC = 3\frac{1}{2}''$, $\hat{ABC} = 40^\circ$. Measure the side AC .

3. M is the mid-point of the side AB of $\triangle ABC$, $ME = MC$. Prove $\angle CMB = 180^\circ - \angle C$.

III 1. AE is a bisector of $\angle BAC$; $\angle AEC$ is a right angle, AD is an altitude of $\triangle ABC$, $\angle B = x^\circ$. Find $\angle EAD$ in terms of x° .

2. Construct a $\triangle ABC$ where $\angle B = 90^\circ$, $\angle C = 40^\circ$, $AC = 2$ cm. Measure AB .

3. The angles at B, C of an isosceles $\triangle ABC$ are each double that at A , on sides BC, BA points D, E are taken so that $BD = BE$. Find the size of $\angle BED$.

IV. 1. $\angle BAC$ is a right angle, BD is an altitude of $\triangle ABC$; the bisector of $\angle DBC$ cuts AC in E . Prove $\angle BEA = \angle ABE$.

2. Construct the quadrilateral $ABCD$ where $AB = 2''$, $BC = 3''$, $CD = 2''$, $\angle B = 130^\circ$, $\angle C = 50^\circ$. Measure its diagonals.

3. The angles at B, C of an isosceles $\triangle ABC$ are each thrice those at A , on sides BC, BA points D, E are taken so that $BD = BE$. Find the size of $\angle BED$.

V 1. $ABCD$ is a quadrilateral, AC bisects angles DAB and ECB . Prove $\angle BAC = \angle CAD$.

2. $ABCD$ is a quadrilateral, $AB = 3''$, $BC = 2''$, $CD = 4''$, $DA = 2\frac{1}{2}''$, $AC = 2\frac{1}{2}''$. Construct it, and measure the diagonal BD .

3. $ABCDE$ is a regular pentagon. Prove $\angle ACD = \angle CAD$.

VI. 1. The bisectors of angles ABC, ACB of $\triangle ABC$ cut in E . Prove that $\angle BEC = 90^\circ + A/2$.

2. $ABCD$ is a quadrilateral, $AB = 3''$, $BC = 3''$, $AD = 4''$, $AC = 4''$, $BD = 3\frac{1}{2}''$. Construct it, and measure CD .

3. $ABCDE$ is a regular pentagon, AC, BE cut in F . Prove $\angle CBF = \angle FEB$.

VII 1. $ABCD$ is a parallelogram whose sides are 4 cm and 5 cm long. One of its angles is 60° . Construct it, and measure its diagonals.

2. ABC is an isosceles triangle, $AB = AC$. The sides AB, AC are produced beyond B, C resp. to X, Y so that $BX = BC = CY$. Prove that BY, CX cut at an angle equal to $\angle ABC$.

3. The sides AB, AC of any triangle are produced beyond B, C resp. to X, Y , the bisectors of $\angle CBX, \angle BCY$ cut in K . Prove that $\angle BKC = \frac{1}{2}(B + C)$.

VIII' 1 Construct a rhombus whose diagonals are 4 cm and 6 cm long. Measure its sides.

2 $\triangle ABC$ is an isosceles triangle, $AB = AC$, the bisector of the angle at B cuts the exterior bisector of the angle A at D . Calculate the angles of $\triangle ABD$ in terms of the angle A .

3 The side BC of any $\triangle ABC$ is produced beyond C to D . The bisectors of angles ABC , ACD cut in L . Find $\angle BLC$ in terms of the angles of $\triangle ABC$.

IX 1. $ABCD$ is an isosceles trapezium whose parallel sides are $3\frac{1}{2}"$ and $3\frac{1}{2}"$ long, their distance apart being $2"$. Construct it, and measure its sides.

2 From the ends of the base of an isosceles triangle, perpendiculars are drawn to the sides. Prove that they make with the base an angle equal to half the vertical angle.

3 AP , BQ are altitudes of $\triangle ABC$. Prove that they make an angle with each other equal to $\angle C$.

X 1 Construct a square whose sides are 5 cm long. Measure its diagonals.

2 $\triangle ABC$ is an isosceles triangle, $AB = AC$, the line EDF is perpendicular to BC , it cuts AB in D and CA in F . Prove that the parallel to BC through A bisects the interval FD .

3 X is inside $\triangle YAB$, OA , OB bisect angles $\angle AY$, $\angle BY$ resp. Prove $\angle AOB = \frac{1}{2}(\angle AXB + \angle AYB)$.

XI 1 Calculate the size of the angles of a regular figure of forty sides.

2 $\triangle ABC$ is any triangle. Right-angled isosceles triangles ABX , ACY are described outside the triangle on sides AB , AC , the right angles being at X , Y . Prove $CX = BY$.

3 If the right bisectors of the intervals AB , CD cut in O , and $\angle AOC = \angle BOD$, prove that $AC = BD$.

XII 1 A quadrilateral has two of its angles each double the size of the other two. Find the sizes of its angles.

2 A' , B' , C' are reflections (or images) of A , B , C in the line XY . Prove that $\angle ABC = \angle A'B'C'$. [See § 11, p. 60.]

3 $ABCD$ and $A'B'C'D'$ are convex quadrilaterals, AB , BC , CA , BD , CD are resp. equal to $A'B'$, $B'C'$, $C'A'$, $B'D'$, $C'D'$. Prove $AD = A'D'$.

XIII 1 A regular pentagon is drawn on base AB , AC is one of its sides, on AC another regular pentagon is drawn, AD is one of its sides, on AD a third regular pentagon is drawn, AE is one of its sides. Calculate the size of $\hat{BAE}$

2 A, B are two points, XY is any line, P, Q are images of A, B resp. in XY . If AB and XY cut in Z , prove that Z is on line PQ

3 ABC and XYZ are two congruent triangles, if $AB \parallel XY$ and $BC \parallel YZ$, prove that $CA \parallel ZX$

XIV 1 The angles on a pentagon are in the ratio 6 : 2 : 3 : 4 : 5. Find their sizes

2 A, B, C are three non-collinear points. Through C draw a line to pass between A and B such that the distances of A and B from the line are equal

3 $ABCD$ is a quadrilateral, the angles at A, B being right angles, M is the mid-point of side AB , DM bisects angle ADC . Prove that CM bisects angle BCD and that $CM \perp AB$

XV 1 The angles of a regular polygon are 144° . How many sides has it?

2 ABC is a right angle, $AB \perp BC$, the bisector of angle CAB cuts BC in E . Prove $AC^2 = BE \cdot AB$

3 If in the quadrilateral $ABCD$ we have $AD \perp BC$, $\hat{ABC} = \hat{BAD}$, prove that the intervals AB and CD have the same right bisector

XVI 1 ABC is a triangle, BI, CI bisect the angles at B, C . Through I is drawn a parallel to BC cutting AB in D and AC in E . Prove $DE = BD + CE$

2 $ABCD$ is a trapezium, $AB \parallel CD$, X, Y are the mid-points of AD, BC resp. Prove $2XY = (AB + CD)$

3 If the diagonals of a quadrilateral are perpendicular to each other, the intervals joining the mid-points of opposite sides are equal in length

XVII 1 $\hat{ABC}$ is a right angle, O is on interval AC , $OA = OB$. Prove $OB = OC$

2 $ABCD$ and $ABEF$ are parallelograms not in the same plane. Prove that $CDFE$ is a parallelogram

3. ABC is a triangle, the right bisectors of sides BC, AC cut in O , X, Y are the images of O in BC, CA resp. Prove that AX and BY bisect each other

XVIII 1 In $\triangle ABC$, $\hat{ABC} = \hat{ACB} + \hat{BAC}$. Divide it into two isosceles triangles.

2 A, B, C lie on a line, X, Y, Z lie on a parallel line, $AX \parallel CZ$, $AY \parallel BZ$. Prove $BX \parallel CY$.

3 Draw a triangle ABC , given the lengths of AB, AC and the length of the median from A . (Use diagonals of parallelogram bisect each other.)

XIX 1 ABC is an isosceles triangle such that the angles at B, C are each thrice those at A . D is any point on BC , E is on BA , $BE = BD$, $ED \perp AC$ cut in F . Prove $EF = EA$.

2 Show how to divide a triangle into four congruent triangles.

3 A quadrilateral has one pair of opposite sides equal, the other pair parallel. Prove that its diagonals are equal.

XX 1 A ladder slides down a wall. Prove that the locus of its mid-point is an arc of a circle.

2 ABC is a triangle. Through each vertex is drawn a parallel to the opposite side. Prove that we thus get a triangle whose size is four times the size of the given triangle.

(The following examples are difficult. Solutions are given in Part III.)

***3** Given a ruler with two marks on it, to construct, without compasses, a parallel through a given point to a given line.

***4** ABC is a triangle, the bisectors of the angles at B, C cut AC, AB resp. in D, E . $BD = CE$. Prove $\hat{ABC} = \hat{ACB}$.

13. Equivalent figures and areas.

Everyone has an idea of what is meant by such statements as 'this piece of cloth is of the same size as that,' or 'this field is larger than that field.' In Geometry we give rules for calculating and comparing the areas of figures. Two figures with the same area will be called *equivalent*†.

THEOREM 14

Two parallelograms on the same base and between the same parallels are equivalent.

Given. $ABCD$ and $ABEF$ are parallelograms on the same base AB and between the same parallels AB and CD .

To prove that the pairs are equivalent.

† This is not a formal definition, and should not be stressed. In Part IV we shall consider in more detail 'equivalence' and 'equality of area'.

Proof In $\triangle s AFD, BEC$,

$$\begin{cases} AD = BC, & (\text{opp sides of parm}). \\ \hat{ADF} = \hat{BCE}, & (\text{corr } \angle s \quad AD \parallel BC). \\ \hat{AFD} = \hat{BEC}, & (\text{corr. } \angle s. \quad AF \parallel BE) \end{cases}$$

$\triangle s AFD, BEC$ are congruent (one side, two $\angle s$).

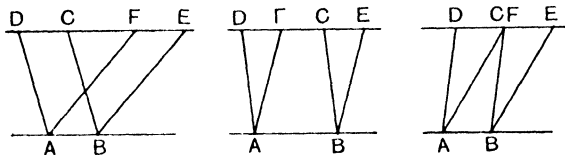


Fig 107

Subtracting each triangle in turn from the whole figure $DABE$, the parts left, viz the parms $ABCD$ and $ABEF$, are equivalent

Q E D

Cor. 1 Two parallelograms on equal bases and of equal heights are equivalent. For we can draw, on the same base and between the same parallels, two parallelograms congruent to the given two.

Cor. 2 Two parallelograms on equal bases and between the same parallels are equivalent.

Note The essential part of the proof of Theorem 14 is the phrase in italics. If we attempt to make up our parallelograms by adding pieces together, we shall have to give different proofs for the different figures.

Note carefully the meaning of the phrase 'Between the same parallels', the bases of the parallelograms are on one line, their opposite sides are on the same line parallel to the bases.

THEOREM 15

Two triangles on equal bases (or on the same base), and between the same parallels are equivalent.

Given. ABC, XYZ are triangles on equal bases AB, XY , and between the same parallels AB and CZ .

To prove $\triangle s ABC, XYZ$ are equivalent.

Proof Complete the parms $ABDC$ and $XYWZ$

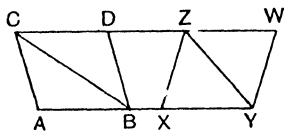


Fig 108

These pairs are on equal bases and between the same parallels, therefore they are equivalent.

But $\triangle ABC$ is half $\triangle BDC$, ($\triangle s ABC, DCB$ are congruent)

Similarly $\triangle XYZ$ is half $\triangle YWZ$.

$\therefore \triangle s ABC, XYZ$ are equivalent.

Q.E.D

Cor. 1 Two triangles on equal bases and of equal heights are equivalent.

For we can draw, on the same base and between the same parallels, two triangles congruent to the given triangles

Cor. 2. If a triangle and a parallelogram be on equal bases and between the same parallels, the triangle is equivalent to half the parallelogram

Note the following important case of Theorem 14. *If P is any point not on the line AB , and C is the mid-point of AB , then $\triangle s PAC, PBC$ are equivalent*

Exercise Write out the proof of Cor. 2 in full without assuming Theorem 15

EXERCISE 29

1 Draw a parallelogram $ABCD$ with sides $AB = 6$ cm, $BC = 4$ cm and $\hat{ABC} = 40^\circ$. Construct an equivalent parallelogram on the same base AB , but with an angle of 60° at B . Measure the new side

2 Transform the parallelogram $ABCD$ of No. 1 into an equivalent parallelogram with sides 6 cm and 6 cm. Measure the angles of the parallelogram

3 Draw a triangle ABC with sides $AB = 3''$, $BC = 3\frac{1}{2}''$, $CA = 3\frac{1}{2}''$. Draw an equivalent triangle ABD with $\hat{BAD} = 60^\circ$. Measure AD .

4 Draw a triangle ABC with $AB = 6$ cm, $BC = 4$ cm, $CA = 5$ cm. Draw an equivalent isosceles triangle ABD , ($AD = BD$). Measure AD

5 Draw a parallelogram $ABCD$ with sides $AB = 6$ cm, $BC = 8$ cm and $\hat{ABC} = 30^\circ$. By means of two applications of Theorem 14 reduce it to an equivalent rhombus whose sides are 7 cm. Measure the angles of the rhombus

THEOREM 16

If two equivalent triangles be on the same base and on the same side of it, they lie between the same parallels

Given. $\triangle ABC$ and $\triangle ABD$ are equivalent triangles on the same side of AB

To prove that $AB \parallel CD$

Proof. Suppose, if possible, that AB and CD are not parallel. Through C draw a line parallel to AB , it will cut AD in a point E , distinct from D

Then $\triangle s$ ABE , ABC are equiv, (same base, equal hts)

But $\triangle s$ ABC , ABD are equiv, (given)

$\therefore \triangle s$ ABD , ABE are equiv

But this is impossible because one is a part of the other,

$\therefore CD \parallel AB$

Q.E.D.

Cor. Equivalent triangles on equal bases have equal heights.

$\triangle ABC \sim \triangle DEF$ will be used to mean that the triangles ABC and DEF are equivalent

The following example illustrates the use of our theorems

EXAMPLE 10 *To show part of Theorem 13 by means of the equivalence of triangles*

Given D , E are mid-points of sides AB , AC of $\triangle ABC$

To prove that $DE \parallel BC$

Proof. Join DC , BE . Then

$$\triangle ADE \sim \triangle BDE, \quad (AD = DB)$$

$$\triangle AED \sim \triangle CED, \quad (AE = EC)$$

$$\triangle BDE \sim \triangle CED$$

But these $\triangle s$ are on the same base DE

$DE \parallel BC$

Q.E.D.

Note. Instead of drawing two separate figures for Theorem 16, we have put two points E in Fig. 109, corresponding to the fact that the parallel through C to AB may cut the interval AD or that interval produced beyond D .

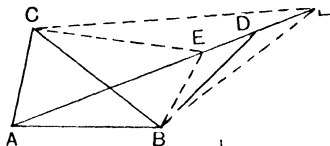


Fig. 109

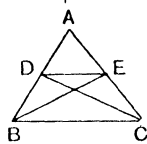


Fig. 110

EXERCISE 29 (*continued*)

6. ABC , ABD are congruent triangles on the same side of AB . Prove that A , B , C , D are the vertices of a trapezium

7 (a) C is the mid-point of the interval AB , O is any point not on line AB . Prove $\triangle OAC = \triangle OBC$

(b) C is the mid-point of the interval AB , O , P , C are in one line. Prove $\triangle OPA = \triangle OPB$

8 The intervals AB , CD bisect each other. Prove $\triangle ABC = \triangle ADC$ (Do not use congruence)

9 $ABCD$ is a trapezium, $AB \parallel CD$, through M , the mid-point of AD , a parallel to BC is drawn, it cuts the lines AB , CD in P , Q resp. Prove that $ABCD$ and $PBCQ$ are equivalent

10 $ABCD$ is a quadrilateral, $\hat{ABC} = \hat{BAD}$, $BC = AD$. Prove $AB \parallel CD$

11 $ABCD$ is a trapezium, $AB \parallel CD$, M is the mid-point of AB . Prove that the quadrilaterals $AMCD$ and $MBCD$ are equivalent

12 The diagonals of a parallelogram $ABCD$ cut in E , F is a point on side AD . Prove that the quadrilateral $BFCE$ is equivalent to $\triangle BCE$

13 (a) $ABCD$ is a trapezium, $AB \parallel CD$, a line parallel to AB cuts sides BC , AD in P , Q resp. Prove $\triangle BQC = \triangle APD$

*(b) If also R , S be any points on sides AB , CD , prove that $PRQS$ and $\triangle BQC$ are equivalent

*14 M , N are the mid-points of sides AB , AC of $\triangle ABC$, $MNPQ$ is a parallelogram, P , Q being on side BC . Prove that $MNPQ$ is equivalent to $\triangle MBN$

*15 $ABCD$ is a parallelogram, F is on side AB , the lines DF , BC cut in E . Prove that $\triangle ADE = \triangle DEC$

*16 ABC is a triangle, a parallel to the base BC cuts AC , AB in X , Y resp., CY , BX cut in O . Prove $\triangle AXB = \triangle AYC$, $\triangle COX = \triangle BOY$

*17 X is any point on the diagonal AC of parallelogram $ABCD$. Prove $\triangle ABX = \triangle ADX$

§ 14. The measurement of areas.

Def. The 'area' of a rectangle is the product of the lengths of two of its perpendicular sides

E.g. If these sides are of lengths 3" and 4", the area is 12 sq. ins., if the sides are of lengths 3 units and 4 units, the area is 12 square

units, the unit may be any length, and then the square unit is the area of the square on the unit length.

We can see that this is a reasonable definition of the area of a rectangle by the following argument. Let $ABCD$ be a rectangle, whose side AB is p units long, and whose side AD is q units long. Divide the sides into unit lengths, and draw parallels as in the figure. The rectangle is divided into pq square units, and thus has area pq .

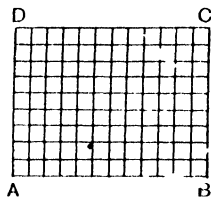


Fig. 111

By Theorem 14, a parallelogram is equivalent to a rectangle on the same base and between the same parallels. Hence *the area of a parallelogram is the product of the length of a side and the corresponding height*.

By Theorem 15, Cor. 2, a triangle is equivalent to half a rectangle on the same base and between the same parallels. Hence *the area of a triangle is half the product of the length of a side and the corresponding height*.

Since the lengths of sides and the heights are numbers, we can use Algebra.

EXAMPLE 11 *The area of a trapezium is the product of half the sum of the lengths of the parallel sides, and the distance between them.*

Given $ABCD$ is a trapezium, $AB \parallel CD$,
 h = distance between AB and CD .

To prove that area $ABCD = \frac{1}{2}(AB + CD)h$

Proof

$$\begin{aligned}\text{Area } ABCD &= \triangle ABD + \triangle BCD \\ &= \frac{1}{2} AB \cdot h + \frac{1}{2} CD \cdot h \\ &= \frac{1}{2} (AB + CD) h\end{aligned}$$

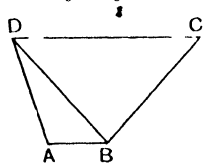


Fig. 112

Q.E.D.

EXERCISE 30.

1 Area $ABC = 15$ sq. in., $BC = 5$ in. Find the height AP of the triangle.

2 Area $ABC = 12$ sq. cm., height $AP = 15$ cm. Find the length BC .

3 In $\triangle ABC$, $AB = 4''$, $BC = 3''$, height $AP = 2''$. Find the height CR .

4 In the parallelogram $ABCD$, $AB = 4$ cm., $BC = 3$ cm., distance between AB and $CD = 2$ cm. Find the distance between BC and AD .

Find the area of a trapezium $ABCD$ when $AB \parallel CD$, and

5 $AB = 7''$, $CD = 3''$, distance between AB and CD is $4''$

6 $\hat{BAD} = 90^\circ$, $AB = 7''$, $CD = 3''$, $AD = 2''$

7. $AB = 10$ miles, $CD = 3$ miles, $\hat{BAD} = \hat{ABC} = 45^\circ$

8 $AB = 3$ cm., $BC = 2$ cm., $\hat{ABC} = 90^\circ$, $\hat{BID} = 135^\circ$

9 In trapezium $ABCD$, $AB \parallel CD$, $AB = 6''$, $CD = 8''$, $\triangle ABD = 15$ sq. in. Find the area of $\triangle BCD$

10 The diagonals of a quadrilateral $ABCD$ are perpendicular to each other and of lengths $7''$ and $9''$. Find the area of the quadrilateral

11 $ABCDE$ represents a field (Fig. 113). The angles marked are right angles, $AX = 100$ yds., $XY = 20$ yds., $YZ = 80$ yds., $ED = 120$ yds., $BX = 110$ yds., $CZ = 120$ yds., $YE = 140$ yds. Find the area of the field.

12 In Fig. 114 the angles marked are right angles, and lengths a, b, c, d are given. Find the area of shaded triangle in terms of a, b, c, d . What is the area if $a = 2''$, $b = 2.5''$, $c = 5''$, $d = 4''$?

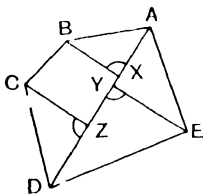


Fig. 113

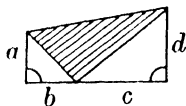


Fig. 114

13 (Fig. 115) $PQRS$ is a square. Find the area of the inner square in terms of a, b . What is the area, if the side $PQ = 6''$ and $a = 4''$?

14 (Fig. 116) $XYZW$ is a parallelogram, the angles marked are right angles. Find the area of $\triangle XWZ$ and hence of $XYZW$ in terms of a, b, c, d . Find the area when $a = 5''$, $b = 1''$, $c = 1''$, $d = 3''$.

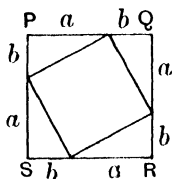


Fig. 115

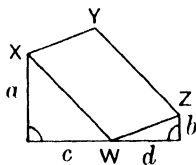


Fig. 116

15 (Fig 117) AD, XY, BC are parallel, the angles marked are right angles, $AX = XB$. Find the area of $ABCD$ in terms of a, b, c . Find the area when $a = 6''$, $b = 8''$, $c = 1''$

16 (Fig 118) The angles marked are right angles, $AB = 4''$, $AC = 7''$, $EF = 5''$. Find the length of DE

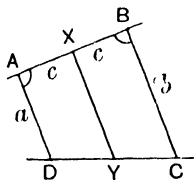


Fig 117

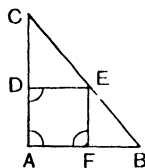


Fig 118

17 (Fig 119) $AE \parallel EB$, $AD \parallel DC$, $BC = a$, h = height of $\triangle ABC$. Find the area of $BCDE$

18 (Fig 120) The interval AB is divided into equal parts. Lines like EX, FY are called ordinates. Prove that the area of the figure is obtained by multiplying the length of AB by the sum of $\frac{1}{2}AC$, $\frac{1}{2}BD$ and all the other ordinates

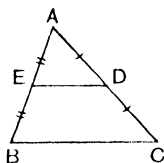


Fig 119

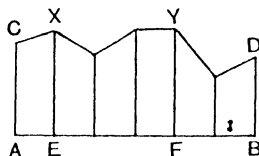


Fig 120

EXAMPLE 12 A second proof of Theorem 16, Cor
Given $\triangle ABC, DEF$ are equivalent triangles, $AB = DE$
To prove that $\triangle ABC, DEF$ have equal heights

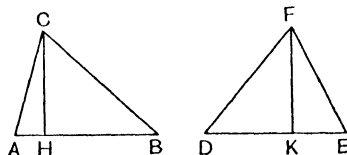


Fig 121

Proof Let CH, FK be the heights of the triangles

$$\triangle ABC = \frac{1}{2}AB \cdot CH$$

$$\triangle DEF = \frac{1}{2}DE \cdot FK$$

$$\frac{1}{2}AB \cdot CH = \frac{1}{2}DE \cdot FK, \quad (\text{given})$$

But

$$AB = DE, \quad (\text{given})$$

$$CH = FK.$$

Q.E.D.

The reasons for giving two proofs of this will be explained in Part IV. It is enough to learn one, and either will do.

EXAMPLE 13 (Cl. Exercise 20, No. 7, p. 34)

Given $\triangle ABC$ is an isosceles triangle, $AB = AC$, X moves on base BC ; P , Q are feet of perpendiculars from X to AC , AB .

To prove that $XP + XQ$ is constant

Proof

$$\begin{aligned}\triangle ABC &= \triangle ABX + \triangle ACX \\ &= \frac{1}{2}AB \cdot QX + \frac{1}{2}AC \cdot PX \\ &= \frac{1}{2}AB (PX + QX), \quad AB = AC\end{aligned}$$

But $\triangle ABC$ and AB are fixed,

$$PX + QX \text{ is constant}$$

Q.E.D.

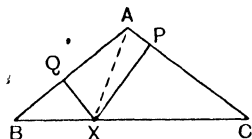


Fig. 122

Note that $AB \cdot QX$ means the product of the lengths of AB and QX .

Exercise. Prove in the same way, that if P be inside an equilateral triangle ABC , and X , Y , Z be the feet of the perpendiculars from P on to the sides of the triangle, then $PX + PY + PZ$ is constant as P moves about the inside of the triangle.

EXERCISE 31

1. If C is the mid-point of the interval AB , the area of the square on AC is one-quarter of the area of the square on AB .

2. If M , N are the mid-points of the sides AC , AB of $\triangle ABC$, then the area of $\triangle AMN$ is one-quarter of the area of $\triangle ABC$.

3. $ABCD$ is a quadrilateral, the diagonal BD is produced beyond D to E , so that $DE = BD$. Prove that the area of $ABCE$ is twice the area of $ABCD$.

4. (a) Any line through the point of intersection of the diagonals of a parallelogram divides the latter into two parts of equal area.

(b) Divide a parallelogram into two parts of equal area by a line through a given point.

5. If M , N are the mid-points of the sides AC , AB of $\triangle ABC$, and AP an altitude, then area $AMPN = \frac{1}{2} \triangle ABC$.

6. (a) AD is an altitude of $\triangle ABC$, the angle at A is a right angle. Prove $AD \cdot BC = AB \cdot AC$.

(b) ABC is any triangle, AP , BQ its altitudes. Prove

$$AP \cdot BC = BQ \cdot AC$$

(c) ABC is an isosceles triangle, $AB = AC$, the altitudes BP , CR from B , C cut in D . Prove $BC \cdot AD = 2AB \cdot DR$.

7. (a) A, B, C are collinear points; O is a point not on the line AB , $\triangle OAB = \triangle OBC$. Prove that B is the mid-point of the interval AC .

(b) If a quadrilateral be divided into four equivalent triangles by its diagonals, it is a parallelogram

8. ABC, ABD are equivalent triangles on opposite sides of AB . Prove that AB bisects the interval CD . (First show the heights are equal)

9. The intervals AE, CD cut in O , $\triangle OAC = \triangle OBD$. Prove $AD \parallel BC$.

10. M is the mid-point of side BC of $\triangle ABC$, N is the mid-point of the interval AM , AM, BN, CN are joined. Prove that the triangle is divided into four equivalent triangles.

11. G is the centroid of $\triangle ABC$. Prove that

$$\triangle GAB = \triangle GBC = \triangle GCA$$

12. (a) $ABCD$ is a quadrilateral, P, Q, R, S are the mid-points of its sides. Prove area $PQRS = \frac{1}{2}$ area $ABCD$.

(b) Through the vertices of a quadrilateral parallels are drawn to the diagonals. Prove that the area of the parallelogram so obtained is twice that of the quadrilateral.

(c) All quadrilaterals whose sides have the same mid-points are equivalent.

13. Through a point P of the diagonal BD of a parallelogram $ABCD$, parallels are drawn to the sides. Prove that the parallelograms AP and CP so formed are equivalent (Fig. 123).

14. The diagonals of a quadrilateral $ABCD$ cut at right angles. Prove

$$\text{area } ABCD = \frac{1}{2} BD \cdot AC$$

15. $ABCD$ is a parallelogram, P is a point inside it. Prove

$$\triangle PAB + \triangle PCD = \frac{1}{2} \text{ area } ABCD$$

16. $ABCD$ and $ABEF$ are parallelograms on opposite sides of AB . Prove area $CDFE = \text{area } ABCD + \text{area } ABEF$.

17. If two triangles have two sides of the one equal to two corresponding sides of the other, and the included angles supplementary, they are equivalent.

*18. (a) $ABCD$ is a trapezium, $AB \parallel CD$, its diagonals AC, BD cut in E . Through E is drawn a line parallel to AB , cutting AD, BC in Z, Y resp. Prove $EZ = EY$.

($\triangle ABD = \triangle ABC$, $\triangle AED = \triangle BEC$, assume $EZ \neq EY$ and use reductio ad absurdum.)

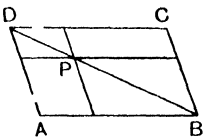


Fig. 123

***(b)** With the same figure, a line parallel to AB cuts AD , BD , AC , BC in P , Q , R , S resp. Prove $PQ = RS$.

***(c)** Prove No. 8 by reductio ad absurdum.

***(d)** M is the mid-point of the side CD of $\triangle XCD$, A , B are points on sides XD , XC resp., $AB \parallel CD$. Prove that XM bisects the interval AB (Fig. 124).

($\triangle XDM = \triangle XCM$, $\triangle D1M = \triangle BCM$, $\triangle X1M = \triangle XBM$, then use No. 8.)

***(e)** Let $\triangle XDC$ and points A , B , M be as in (c), AC , BD cut in E . Using (a), (d) prove that X , E , M are on one line.

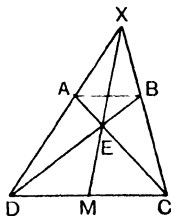


Fig. 124

(Fig. 124 is most important and should be carefully studied.)

***19** M , N are the mid-points of sides of AC , AB of $\triangle ABC$, CN , BM cut in G . Prove area $1NGM = \triangle BGC$.

***20** In Fig. 118, if $AFED$ is a square, prove that

$$AF(AB + AC) = AB \cdot AC$$

***21** ABC is an isosceles triangle, $AB = AC$, X , Y are points on side AB , Z , W are points on side AC , $XY = ZW$. As P moves along the side BC , prove that $\triangle PXY + \triangle PZW$ is constant.

***22** (a) $ABCD$ is a parallelogram, AB is produced beyond B to X , P is a point inside angle CBA . Prove that $\triangle P1B + \triangle PBC = \triangle PBD$.

(Using theorems on intercepts, prove that the sum of the perpendicular intervals on PB from A and C equals that from D .)

***(b)** State the corresponding theorem when P is inside the parallelogram.

***23** Through the vertices A , B , C of $\triangle ABC$ are drawn three parallel lines which cut the sides, or the sides produced, of the triangle in D , E , F resp. Prove $\triangle DEF = 2 \triangle ABC$ (Fig. 125).

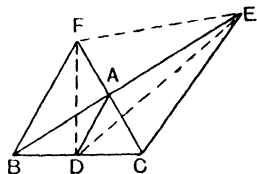


Fig. 125

***24** X , Y are the feet of the perpendiculars from B , C on to the bisector of angle BAC of $\triangle ABC$. Prove area $ABC = BX \cdot AY$ (cf. Fig. 97, p. 58).

***25** L is the mid-point of side AB of $\triangle ABC$. A line through L cuts the rays CA , CB in D , E resp. Prove $\triangle CDE > \triangle ABC$.

***26** $ABCD$ is a rectangle, E is on side BC , F on side CD . Prove $2 \triangle AEF + BE \cdot DF = \text{area } ABCD$.

***27** K is a point inside rectangle $ABCD$. Through K are drawn parallels to the sides, dividing the rectangle into four rectangles. Prove rect $KA \sim \text{rect. } KC = 2 \triangle BKD$.

§ 15. The Theorem of Pythagoras.

Consider the following problem: *A square is given, it is required to draw a square of twice the size.* It is no use putting two squares side by side, like $\square$, because the result is not a square, and it is no use making a square whose side is twice the side of the given square, because then we should have a square whose area was four times, and not twice, the area of the given square. Fig. 126 gives us the clue. If $ABCD$ is the given square, then the square $WXYZ$ is twice the given square, because the big square is made up of eight congruent triangles, and the given square of four of them. We can see this is so, and it is easy to write out a 'formal proof' like the others in this book. Now the big square has its sides equal to AC , i.e. equal to a diagonal of the given square. We can therefore say: *To draw a square whose area is twice that of a given square, we draw a square whose side equals a diagonal of the given square.*

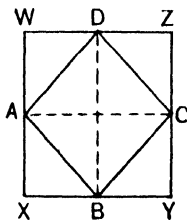


Fig. 126

Another problem: *Two squares are given side by side, like those shaded in the figure below. Cut two such squares out in paper. The problem is to cut the pair of squares into pieces that will fit together to make one square.* To do this, mark off $AX = CZ$ and cut along the lines XY , XZ , remove the triangles cut off, and place them in the positions of the unshaded triangles. It is possible to prove that the figure so formed is a square. The fact, however, which we wish to bring out now, is that the square we have made has one side XZ , while the squares we started from had sides of lengths CZ and CX resp. (because we made $AX = BC$, $AB = CX$). Now XCZ is a right-angled triangle. We can therefore say, *the area of the square on XZ equals the sum of the areas of the squares on CX and CZ .* This is called the Theorem of Pythagoras. The following proof of this theorem has been taught for over 2000 years, but many other proofs are known. An important one will be given later.

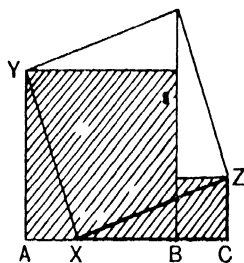


Fig. 127

THEOREM 17 (Pythagoras)

The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the two sides

Given $\triangle ABC$ is a triangle, right angled at B

To prove that sq on AC = sq on AB + sq on BC

Proof Let $ABED$ and $ACHF$ be the squares on AB and AC

Join CD and BF Through B draw a parallel to AF cutting FH in L

Since $\angle ABE = 90^\circ$ ($\angle$ of square), and $\angle ABC = 90^\circ$ (given),

E, B, C are in one line

Now $\angle DAB = \angle CAF$ (90°)

Add $\angle BAC$ to each $\angle DAC = \angle BAF$

In $\triangle DAC, BAF$,

$$\begin{cases} DA = BA, & (\text{sides of square}) \\ AC = AF, & (\text{sides of square}) \\ \angle DAC = \angle BAF, & (\text{proved}) \end{cases}$$

$\triangle DAC, BAF$ are congruent, (two sides, incl $\angle$)

Now $\triangle DAC = \frac{1}{2}$ area of sq on AB , (on same base AD , and between same parallels AD, CE)

$\triangle BAF = \frac{1}{2}$ area of rect AL , (on same base AF , and between same parallels AF, BL).

sq on AB = rect AL

Similarly, sq on BC = rect CL

Adding, sq on AB + sq on BC = rect AL + rect CL

sq on AC Q.E.D.

Note A rectangle or parallelogram may be denoted by a pair of opposite vertices

Warning The area of the square on AB is often written AB^2 . Carefully note that in Algebra, ab^2 means a multiplied by the square

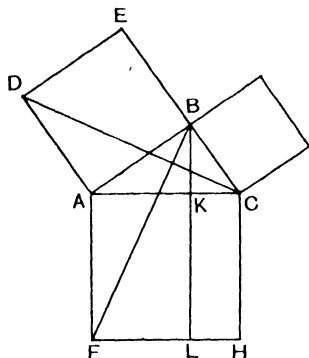


Fig 128

of b , while $(ab)^2$ means the square of the product of a and b . In Geometry AB^2 means the square of the length of AB , and perhaps it would be better to write it $(AB)^2$, but as we cannot square a point B , there is no danger in writing it as AB^2 , provided that we remember what ab^2 means in Algebra. The Theorem of Pythagoras can now be stated thus: *If $\hat{ABC}$ is a right angle, then $AP^2 + BC^2 = AC^2$.*

EXAMPLE 14 In the proof of the Theorem of Pythagoras, we proved that sq on AB = rect AL . Since the area of the rect AL in Fig. 128 is

$$AL \cdot AK = AC \cdot AK,$$

therefore, we can write $AB^2 = AK \cdot AC$.

If ABC be a triangle, right angled at B , and K be the foot of the perpendicular from B on to the hypotenuse, then $AB^2 = AK \cdot AC$. Similarly $CB^2 = CK \cdot CA$.

Pythagoras was a Greek philosopher who lived in South Italy. He founded a secret brotherhood whose members made researches in mathematics and philosophy. Apparently they eventually seized the Government of the district in which they lived, and in the ensuing disturbances, Pythagoras was killed and his brotherhood dispersed (500 B.C.). It is probable that the earliest proof of Theorem 17 is due to him.

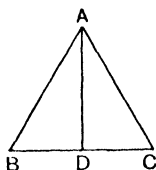


Fig. 129

EXAMPLE 15 ABC is an equilateral triangle whose side is a inches long. Find its area.

Let D be the mid-point of BC . Then $AD \perp BC$, and $BD = a/2$.

$$AD^2 = AB^2 - BD^2 = a^2 - (a/2)^2$$

$$= a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$AD = \frac{\sqrt{3}a}{2}$$

$$\Delta ABC = \frac{1}{2}BC \cdot AD = \frac{1}{2}a \cdot \frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4}$$

EXERCISE 32

1. A square has sides 1" long. Prove that its diagonal is $\sqrt{2}$ inches long.
2. An equilateral triangle has sides 2" long. Prove that its height is $\sqrt{3}$ inches. Calculate this to 3 decimal places.

3. Find the lengths of the hypotenuses of the right-angled triangles whose sides in inches are:

3, 4; 5, 12; 20, 21; 33, 56.

1, 2; 1, 3; 12, 3; 4, 6; $\sqrt{3}$, 2; $\sqrt{19}$, $\sqrt{6}$

4. A courtyard is 80 yds. wide and 150 yds long. How much distance is saved by cutting across from corner to corner instead of going along two sides?

5. The hypotenuse of a right-angled triangle is 89 ft., one side is 60 ft. How long is the other side? [Use $a^2 - b^2 = (a - b)(a + b)$]

6. Two towers stand on the same plane at a distance of 40 ft. apart; one is 100 ft., the other 120 ft. high. Find the length of a wire joining their top points.

7. A man walks 12 ft. North, then 8 ft. West, then 3 ft. South. How far is he then from his starting-point?

8. $ABCD$ is an isosceles trapezium, $AB = 32$ cm, $CD = 8$ cm; the distance apart of the parallel sides AB , CD is 5 cm. Find the length of AD .

9. $ABCD$ is an isosceles trapezium; $AB \parallel CD$, $AB = 198$ cm, $CD = 120$ cm, $AD = 89$ cm. Find the distance apart of the parallel sides.

10. Find the area of a trapezium $ABCD$ where $\hat{ABC}$ is a right angle; $BC \parallel AD$, $AB = 5'$, $BC = 8'$, $CD = 13'$.

11. A goat is tethered by a rope 9 ft. long at a distance 7 ft. from a straight hedge. What length of hedge can he nibble? If the length of rope is doubled, how much can he nibble then?

12. In $\triangle ABC$, $AB = 2''$, $AC = 1'$, $\hat{BAC} = 45^\circ$. Calculate the length of BC .

13. In Fig. 130, XA is a ladder, leaning against a wall, $AB = 8'$, $BX = 1'$. The ladder is turned into position XD , $XC = 7'$. Find the lengths of CD and AD .

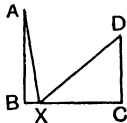


Fig. 130

*14. In triangle ABC , $BC = 2''$, $\hat{ABC} = 60^\circ$, $AB = 3''$. Find the length of AC .

*15. Find the area of the trapezium $ABCD$, $AB \parallel CD$, given $AB = 2$ cm, $AD = 3$ cm, $\hat{BAD} = 90^\circ$, $\hat{ABC} = 120^\circ$.

EXAMPLE 16. The sides of a triangle are 25, 29, 36 inches long. Find its area. (This is not a right-angled triangle.)

Fig 131)

Drop a perpendicular CD from C to AB , let

$AD = x$, $BD = y$, $CD = p$. Then,

from $\triangle ADC$, $x^2 + p^2 = 29^2$,

from $\triangle BDC$, $y^2 + p^2 = 25^2$.

Subtracting,

$$x^2 - y^2 = 29^2 - 25^2 = (29 + 25)(29 - 25) = 54 \cdot 4$$

But

$$x + y = 36$$

Dividing, $\therefore x - y = \frac{54 \cdot 4}{36} = 6$, $\left(\frac{x^2 - y^2}{x + y} = x - y \right)$

Solving the last two equations, we have

$$x = 21, \quad y = 15$$

$$p^2 = 29^2 - x^2 = 29^2 - 21^2 = (29 + 21)(29 - 21) = 50 \cdot 8 = 400.$$

$$p = 20$$

$$\therefore \triangle ABC = \frac{1}{2} \cdot 36 \cdot 20 = 360 \text{ sq. ins.}$$

16 Find the areas of the triangles whose sides have the following lengths (take the last side as base)

13, 20, 21, 13, 15, 14, 65, 87, 88, 125, 136, 99

17 Find, correct to two decimal places, the areas of the triangles whose sides have the following lengths

4, 5, 6, 4, 5, 7, 5, 6, 7

18 ABC is a triangle, right-angled at B , $AB = 20'$, $BC = 21'$; BD is an altitude. Calculate the length of AC and the area of the triangle. Hence find the length of BD .

***19** Find the area of a regular hexagon whose side is 1"

***20.** Two circles of radii 40' and 51' have centres 77' apart. Find the length of their common chord

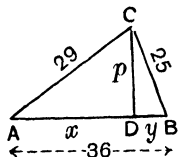


Fig 131

§ 16 Perpendicular lines and planes

Def A line is said to be 'perpendicular' or 'at right angles' to a plane, when it is perpendicular to every line, in the plane, which cuts it

Thus in Fig 132, OP is perpendicular to the plane XYZ , if it is perpendicular to every line which goes through O and lies in the plane.

To be sure that OP is perpendicular to

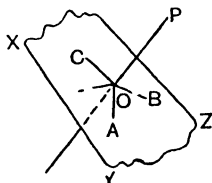


Fig. 132

plane ABC , it is obviously enough to know that it is perpendicular to *two* of these lines through O , e.g. OA and OB , it will then be perpendicular to *all* lines of the plane through O

Two planes which cut in a line are said to form a '*dihedral angle*' at the line. If two planes cut in a line AB , and the planes XAY and PBQ are both perpendicular to AB , then clearly $\angle XAY = \angle PBQ$ (Fig 133). These angles measure the dihedral angle†

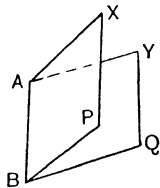


Fig 133

Exercise Open a book so that the dihedral angle is 60° , 90° , 120°

If the dihedral angle between two planes is a right angle, the planes are said to be '*perpendicular*' or '*at right angles*' to each other. It is easy to see that the following are true: (1) *If the line OP is perpendicular to plane ABC , all planes through OP are perpendicular to ABC* (see Fig 132). (2) *If two planes which cut in OP are both perpendicular to plane ABC , then the line OP is perpendicular to ABC* . (3) *Two lines perpendicular to the same plane are parallel*. (4) *Two planes perpendicular to the same line are parallel*.

Exercise Draw figures and make models illustrating these four facts.

Fig 134 represents three planes through O , *mutually perpendicular*, i.e. each of the planes OAB , OBC , OCA is perpendicular to each. Then the lines OA , OB , OC are mutually perpendicular, also OA is perpendicular to plane OBC , OB to OCA , OC to OAB .

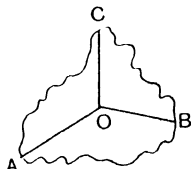


Fig 134

Def A parallelepiped (p. 62) in which all the dihedral angles are right angles is generally called a '*rectangular parallelepiped*'.

We shall call it a '*box*'. A box whose edges are all equal is a '*cube*'.

† An intuitive notion of a dihedral angle, and a knowledge of how it is measured, are all that are here required. A more exact treatment will be given in Part IV.

Fig 135 represents a box. Take a model of a box, and observe that $\hat{ABC}$ and $\hat{ACF}$ are both right angles. Let the edges of the box be of lengths $AB = a$, $BC = b$, $CF = c$. Then,

since $\hat{ABC} = 1 \text{ rt } \angle$, $AC^2 = a^2 + b^2$.

Since $\hat{ACF} = 1 \text{ rt } \angle$, $AF^2 = AC^2 + CF^2$
 $AF^2 = a^2 + b^2 + c^2$.

Therefore we can calculate the length of the diagonal AF of the box, when we know the lengths of its edges.

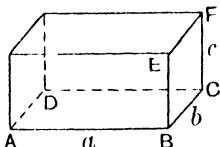


Fig 135

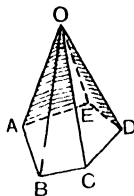


Fig 136

If we join the vertices of a plane polygon to a point outside the plane, we have a 'pyramid'. In Fig 136, $ABCDE$ represents a plane polygon, O is a point not on its plane. A pyramid on a triangular base is a 'tetrahedron'. If we join all the points of a circle to a point O not in the plane of the circle, we have a 'cone'. If X is the centre of the circle, OX is the 'axis' of the cone, the circle is called the 'base' of the cone. If the axis is perpendicular to the base, we have a 'right (circular) cone'. The lines from O to points of the circle are 'generators'. In a right circular cone the angle $\angle XO1$ is called the 'semi-vertical angle' (Fig 137).

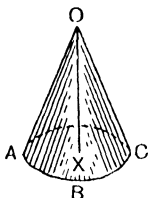


Fig. 137

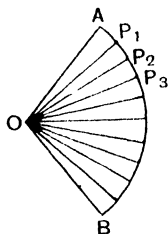


Fig 138

Exercise Cut out a figure like Fig 138, where AB is an arc of a circle, of which OA , OB are radii. Bend it round, till OA and OB coincide, so as to form a right circular cone. Notice that OA becomes a generator, its length is the 'slant height' of the cone, the arc AB

becomes the circumference of the base. In the same way a cone made of paper could be unwrapped to form a 'sector of a circle,' i.e. a figure like Fig. 138.

If we take any plane polygon, and through its points draw equal and parallel intervals on the same side of its plane, then extend them on a congruent plane polygon (Fig. 139). The figure obtained is called a 'prism'.

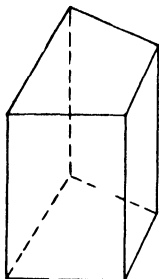


Fig. 139

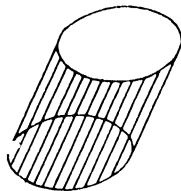


Fig. 140

If we take any plane curve, and through its points draw equal and parallel intervals on the same side of its plane, the figure obtained is a 'cylinder' (Fig. 140). If the curve is a circle, the figure is a 'circular' cylinder, if all the intervals are perpendicular to the plane of the circle, it is a 'right circular' cylinder. These are what are usually called cylinders in daily life. The (straight) lines which lie along the cylinder are called 'generators'.

EXERCISE 33

1. What is the length, to three decimal places, of a diagonal of a cube whose edge is 1" long?
2. A box has edges 2, 3, 6 inches long. What is the length of its diagonals?
3. A right circular cone has a base of radius 7", its height is 15", find its slant height.
4. A pyramid is on a square base, the sides of the square are 4 ft long. Its axis is perpendicular to the base, and is 6 ft high. Find the length of its slant edges.
5. Suppose Fig. 135 is a cube of edge 1'. Find the length $AB + BF$. If M is the mid-point of BC , find the length of $AM + MF$.

6 On the same cube, a spider is at A and a fly at F , the spider walks along the floor ABC and up the wall BCF to catch the fly. What is the length of shortest path he can take? *Prove what you say*

7 The great pyramid of Khufu is 451 ft high, the length of each side of the base is 755 ft, there is a flat space on the top about 30 ft square, neglecting this, find the area of a face of the pyramid, to two significant figures

§ 17. Further applications of the Theorem of Pythagoras.

In working out riders which depend on the Theorem of Pythagoras, it is usually best, *not to draw* the squares in question, but to proceed as follows

EXAMPLE 17 Given C and D are on a perpendicular to the line AB
To prove that $AC^2 + BC^2 = AD^2 + BD^2$

Proof Let CD cut AB in X . Then

$$\hat{CXA} = 1 \text{ rt } \angle \quad AX^2 + XC^2 = AC^2$$

$$\hat{CXB} = 1 \text{ rt } \angle \quad BX^2 + XC^2 = BC^2$$

$$\text{Subtracting,} \quad AX^2 - BX^2 = AC^2 - BC^2$$

$$\text{Similarly,} \quad AX^2 - BX^2 = AD^2 - BD^2.$$

$$AC^2 - BC^2 = AD^2 - BD^2.$$

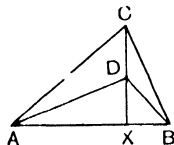


Fig 141

Q.E.D.

In the following exercise Algebra may freely be used. Notice that $2AB^2$ means twice the area of the square on AB , while $(2AB)^2$ means the area of the square on an interval whose length is twice that of AB .

EXERCISE 34

1 AP is an altitude of $\triangle ABC$, $AB = AC$. Prove $BP = CP$

2 Draw the following figure. $OA = AB = BC = CD = \dots = 1''$,
 $\hat{OAB} = \hat{OBC} = \hat{OCD} = \dots = 1 \text{ rt } \angle$. Prove that $OB = \sqrt{2}$, $OC = \sqrt{3}$,
 $OD = \sqrt{4}$, and so on

3 $\hat{ABC}$ is a right angle, D, E are on rays BA, BC . Prove
 $AE^2 + CD^2 = AC^2 + DE^2$

4 P is inside the rectangle $ABCD$. Prove $PA^2 + PC^2 = PB^2 + PD^2$
(Use the method of Example 17)

5 (a) $ABCD$ is a quadrilateral whose diagonals are at right angles. Prove $AB^2 + CD^2 = BC^2 + AD^2$

(b) The sum of the squares on the sides of a rhombus equals the sum of the squares on its diagonals

6 P is a point inside $\triangle ABC$, D, E, F are the feet of perpendiculars from P on to the sides BC, CA, AB . Prove

$$BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2$$

7 D, C are on the same side of AB , $\angle ACB$ and $\angle ADB$ are right angles, E, F are the feet of perpendiculars from A, B to the line CD . Prove $CE^2 + CF^2 = DE^2 + DF^2$

8 $\angle CAB$ is a right angle, E, F are the mid-points of the intervals AC, AB resp. Prove $BE^2 + CF^2 = 5EF^2$

9 $ABCD$ is a square, X is on the diagonal AC produced beyond C . Prove $XA^2 + XC^2 = XB^2 + XD^2$ (Method of Example 17)

10 X, Y, Z, W are points on the sides AB, BC, CD, DA of the rectangle $ABCD$, $XY = ZW$. Prove $AZ^2 + CW^2 = AY^2 + BX^2$

***11** ABC is a triangle, right-angled at B , from M , the mid-point of AB , a perpendicular is drawn to AC . If P is the foot of this perpendicular, prove $CP^2 = AP^2 + BC^2$

***12** AD is an altitude of $\triangle ABC$, L is the mid-point of side BC . Prove $AB^2 + AC^2 = 2BC \cdot DL$

***13** ABC is an isosceles triangle, $AD = AC$, P is on BC produced beyond C . Prove $PA^2 = AB^2 + PB \cdot PC$ (Method of Example 17)

***14** $ABCD$ is a trapezium, $AB \parallel CD$. Prove $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$

THEOREM 18. (Converse of the Theorem of Pythagoras)

If the square on one side of a triangle equals the sum of the squares on the other two sides, these two sides are at right angles to each other.

Given ABC is a triangle with $AB^2 + AC^2 = BC^2$

To prove that $\angle BAC$ is a right angle

Proof Construct a triangle XYZ with $XY = AB$, $XZ = AC$, and the angle YXZ a right angle

Then $YZ^2 = XY^2 + XZ^2$, (Pythagoras)

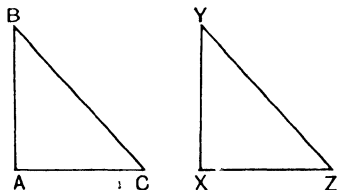


Fig. 142

But $XY = AB, XZ = AC, (\text{const.})$
 $YZ^2 = AB^2 + AC^2 = BC^2, (\text{given})$
 $\therefore YZ = BC$

In $\triangle ABC, XYZ,$

$$\begin{cases} AB = XY, & (\text{const.}), \\ AC = XZ, & (\text{const.}) \\ BC = YZ, & (\text{proved}) \end{cases}$$

$\triangle ABC, XYZ$ are congruent, (three sides)

But $\angle YXZ = \angle BAC = 90^\circ$ Q.E.D.

§ 18. Construction of a rectangle equivalent to a polygon.

Our object here is to prepare the way for proving the following theorem *If a polygon be given, a square can be drawn equivalent to it, by means of ruler and compasses*. We shall prove here that, by means of ruler and compasses, a rectangle can be drawn equivalent to a given polygon. It will then only remain to show that *If a rectangle be given, a square can be drawn equivalent to it, by means of ruler and compasses*.

CONSTRUCTION 9

To draw, on a given interval as base, a rectangle equivalent to a given rectangle

Given A rectangle $ABCD$, and an interval XY

To draw on XY as base a rectangle equivalent to $ABCD$.

Construction Produce the side BC to E , making $CE = XY$. Through E draw a parallel to AB , this line will cut the line AD in F , say

Join FC , this line will cut the line AB in K , say.

Through K draw a parallel to AD , this will cut the line CD in H , say, and the line FE in G , say.

Then $ECHG$ is a rectangle, equivalent to $ABCD$, and $EC = XY$

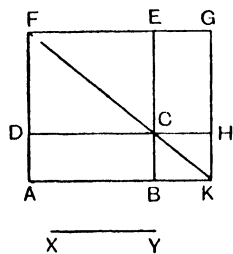


Fig. 143

On XY draw a rectangle congruent to $ECHG$. This is the rectangle required.

Proof Since $ECHG$ is clearly a rectangle, being a parallelogram with one angle a right angle, we have only to prove it is equivalent to $ABCD$.

Now $\triangle FAK, FGK$ are congruent, (FK is diagonal of parm.)

$\triangle FDC, FEC$ are congruent

$\triangle CBK, CHK$ are congruent

Taking $\triangle FDC$ and $\triangle CBK$ from $\triangle FAK$, we have rect $ABCD$ left

Taking $\triangle FEC$ and $\triangle CHK$ from $\triangle FGK$, we have rect $ECHG$ left

$\therefore$ rect $ABCD$ and $ECHG$ are equivalent

Q. E. D.

Note Do not draw the diagonal through the given rectangle

Exercise To draw, on a given interval as base, a parallelogram, with one angle given, equivalent to a given parallelogram

CONSTRUCTION 10

To draw, on a given interval as base, a rectangle equivalent to a given polygon

Given Any polygon $ABCD$ and an interval XY .

Required To draw, on XY as base, a rectangle equivalent to $ABCD$

Construction Divide the polygon into triangles in any way, and number the triangles 1, 2,

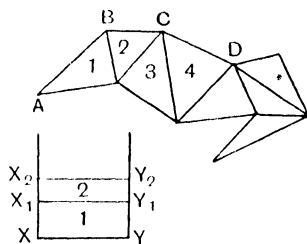


Fig. 144

On any side of triangle 1 as base, construct a rectangle whose height is half that of the triangle.

This rectangle is equivalent to triangle 1.

On the interval XY as base draw a rectangle $XY Y_1 X_1$ equivalent to this rectangle.

Then rectangle $XY Y_1 X_1$ is equivalent to triangle 1.

Similarly on $X_1 Y_1$ as base we can draw a rectangle $X_1 Y_1 Y_2 X_2$

equivalent to triangle 2, and on X_2Y_2 as base a rectangle $X_2Y_2Y_3X_3$ equivalent to triangle 3, and so on

Finally we have a rectangle equivalent to the given polygon
Q.E.D.

In simple cases we can give simpler constructions

EXAMPLE 18. *To draw a rectangle equivalent to a quadrilateral ABCD*

Construction Join AC. Bisect AB, BC, CD, DA in P, Q, R, S respectively

From A, C draw perpendiculars to the lines PQ and RS. These perpendiculars, and the line PQ, RS, determine the rectangle by their intersections. Prove this construction.

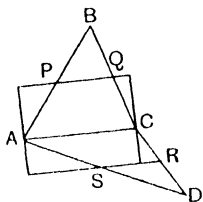


Fig 145

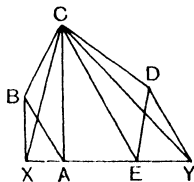


Fig 146

EXAMPLE 19 *To draw a rectangle equivalent to a convex pentagon ABCDE*

Construction Join CE. Through B draw a line parallel to CA, to cut the line AE in X

Through D draw a line parallel to CE, to cut the line AE in Y

Then $\triangle XCY$ is equivalent to the pentagon, and a rectangle on base XY and half the height of $\triangle XCY$ is the rectangle required

Proof To show that $\triangle XCY$ is equivalent to the pentagon ABCDE

$\triangle s ACB, ACX$ are on the same base AC and between the same parallels AC, XB they are equivalent

$\triangle s ECD, ECY$ are on the same base EC and between the same parallels EC, YD they are equivalent

Hence ABCDE, being the sum of $\triangle s ACB, ECD, ACE$, is equivalent to $\triangle XCY$, which is the sum of $\triangle s ACX, ECY, ACE$. Q.E.D.

Notes on theoretical constructions (1) Suppose we were set the following question To draw a rectangle of area 7 sq in on a base of 2". We should naturally calculate the height of the rectangle - $7/2 = 3\frac{1}{2}$ ", and proceed to draw a rectangle whose base was 2" and height $3\frac{1}{2}$ ".

This is a construction with numerical data (viz 7 sq in and 2"), and we have used calculation.

Thus we may suggest the following construction for a rectangle equivalent to a given polygon. Divide the polygon into triangles, as in Construction 10, measure the base and height of triangle 1, find half their product. This is the area of triangle 1. Do the same for each triangle and add the results. If the sum is s draw a rectangle on unit base and of height s . This is the rectangle required. But *this construction should never be given*, because (i) our data are now, not numerical, (ii) it is assumed, when the data are not numerical, that no measurements should be made, but that all constructions should be performed by ruler and compasses.

(2) The whole point of Constructions 9 and 10 is that they are performed by ruler and compasses. We will now explain what this means in detail. When we say a construction can be performed by ruler and compasses, we mean it can be reduced to the following fundamental constructions.

(a) Given two points, we can draw the line through them.

(b) Given two lines which meet, or a line and circle which meet, or two circles which meet, we can find the point or points of meeting.

(c) Given a point O , and an interval AB , we can draw a circle with centre O , and radius equal to AB .

Note carefully that we only need use the ruler to draw lines, we need not use it to cut off given lengths. The latter process can be carried out by Construction (c). The reader should go through the constructions already given, and satisfy himself that they can be reduced to the fundamental constructions. Thus our constructions are primarily of theoretical interest. Whether they are the best from the point of view of a practical draftsman is another question. Usually they are, but not always. All constructions in this book are by ruler and compasses, or by ruler alone, or by compasses alone.

(3) The constructions by ruler and compasses that we give are *ideally exact*. We will explain what this means. When we make a construction on paper, the lines and circles *that we actually draw* have a finite thickness, and thus two circles cut in a small area. We choose a point in that area, as near the middle as we can guess. Thus our drawings are always imperfect. When we say our constructions are ideally exact, we mean that the more perfect our instruments are, the more correct our final figure will be, and there is no limit to the degree of perfection we might reach. For instance, we have proved that, if we start at a point A on the circumference of a circle and place in the circle chords equal to the radius, end to end, we reach A again after drawing six chords. The better our instruments are, and the more carefully we draw, the more exactly this will be verified.

But if the construction were not ideally exact, the better our instruments were, and the more carefully we drew, the clearer it would be that our construction was incorrect

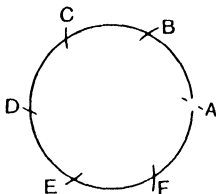


Fig 117

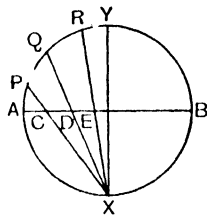


Fig 118

The following construction is sometimes given for drawing a regular heptagon (seven-sided figure) in a given circle. Let AB and XY be two diameters at right angles to each other. Divide AB into seven equal parts, by C, D, E , and so on. Draw the lines XC, XE and let them cut the circle in P, R . Then the chord PR is a side of the regular heptagon required. If you perform this construction very carefully you will find it is incorrect, and the more carefully you draw your figure the more evident will be the incorrectness of the construction. In fact it can be *proved* that there is no exact construction by ruler and compasses for a regular heptagon in a given circle. We obtain a better construction than the above, if we take, for the side of the heptagon, a length equal to the height of the equilateral triangle whose base is the radius, but this construction is again only approximate.

EXERCISE 35

In set A, calculation may be used, in set B, only ruler and compass constructions may be used. Constructions 9 and 10 can be assumed, if no short construction can be found.

A

- 1 Draw a triangle ABC , given $AB = 3''$, $\hat{ABC} = 15^\circ$, area = 6 sq. in. Measure AC .
- 2 Draw an isosceles triangle whose base is $3''$ and area $7\frac{1}{2}$ sq. in. Measure its sides.
- 3 A triangle has base $5''$ and height $1\frac{1}{2}''$. On a base of $3''$, draw an equivalent rectangle. Measure its diagonal.
- 4 Construct a parallelogram of area 5 sq. in. on a base of $3''$, with an angle of 60° . Measure its longest diagonal.

5 Construct a parallelogram of area 5 sq. in. on a base of 3", with a diagonal of 3". Measure its other diagonal.

6. Construct a parallelogram whose sides are 2" and 3" long and whose area is 4 sq. in. Measure its longest diagonal.

7. Construct a rhombus of area 3 sq. in. and sides of 2". Measure its longest diagonal.

B

1. Given a rectangle, construct an equivalent parallelogram with sides of given lengths. Is this always possible?

2. Given a rectangle, construct an equivalent parallelogram with a side and a diagonal of given length. Is this always possible?

3. Construct an isosceles triangle on a given base, equivalent to a given rectangle.

4. Construct a rhombus with sides of given length, equivalent to a given rectangle. Is this always possible?

5. Construct a rectangle on a given base equivalent to a given trapezium.

6. On a side of a regular hexagon, construct a rectangle equivalent to the hexagon.

7. On a given base, draw a triangle whose area is the sum of the areas of two triangles.

EXAMPLE 20 Given a triangle ABC , and a point P on the side AB . To draw a line through P which divides the triangle into two parts of equal area.

Construction. Find the mid-point X of side AB . Join CP .

Draw a parallel through X to CP and let it cut the side BC in Y . Then PY is the line required.

Proof. Join CX .

Then $\triangle YXC = \triangle YXP$, (on same base XY , between same parallels XY, CP)

Add to each the $\triangle BXY$

$$\triangle BXC = \triangle BPY$$

But $\triangle BXC = \frac{1}{2} \triangle BAC$, ($AX = XB$, altitudes from C the same).

$$\triangle BPY = \frac{1}{2} \triangle ABC.$$

PY is the line required

Q.E.F.

Exercise We have assumed that P is between A and X . Write out the proof when P is between B and X .

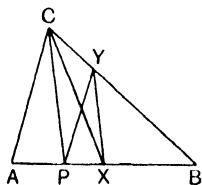


Fig. 119

*EXERCISE 36

1 ABC is a given triangle, X a point on the side BC . Find a point Y on the line BA such that $\triangle XBY \sim \triangle ABC$.

2 P is a point on the side AB of triangle ABC . Draw lines through P which will divide the triangle into three parts of equal area.

3 Given a quadrilateral, to draw a line through a given vertex which divides the quadrilateral into two parts of equal area. (Reduce quad. to equiv. triangle.)

4 Find a point P inside $\triangle ABC$ such that $\triangle s\ ABP, BCP, CAP$ are equivalent.

5 P is a point on the side AB of triangle ABC . Draw lines through P which will divide the triangles into (i) four, (ii) five parts of equal area.

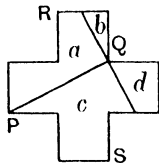


Fig 150

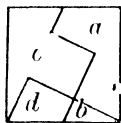


Fig 151

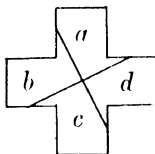


Fig 152

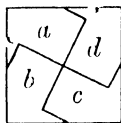


Fig 153

Some examples in dissection† (1) Fig 150 is a 'Greek cross' composed of five equal squares. To cut the cross into four pieces which will fit together to make one square. If a side of the cross is 1 unit long, its area is 5 square units. Now by the Theorem of Pythagoras, the square on the interval PQ is 5 square units. Whence a side of the required square must equal PQ in length. Two dissections are given by Figs 150, 152, the resulting squares are pieced together in Figs 151, 153. The problem can also be solved by folding. Cut the cross out in paper. Fold along the line RS , then fold along the perpendicular from P to RS . The resulting figure has a line of symmetry. Cut along this line. The pieces fit together as in Fig 153.

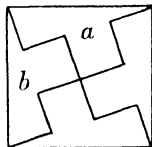


Fig 154

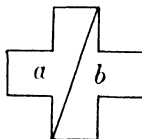


Fig 155

† These are taken from Dudeney's *Amusements in Mathematics*, a book which is full of interesting questions.

(2) To cut up a square so as to form two equal Greek crosses The required dissection is shown in Fig. 154, one of the resulting crosses in Fig. 155. The intervals inside the square of Fig. 154 are along lines connecting points of trisection of the sides.

(Revision Papers XXI to XXV can now be taken, if desired.)

§ 19. Rational right-angled triangles

These are right-angled triangles whose three sides are all a whole number of units in length. (The unit itself may be of any length.) For instance, the triangle whose sides are 3, 4, 5 inches long, is a rational triangle, and it is right-angled because $5^2 = 3^2 + 4^2$. Now if z be the length of the hypotenuse of a right-angled triangle and x, y be the lengths of its sides, then

$$z^2 = x^2 + y^2 \quad (i)$$

The problem of finding rational right-angled triangles is therefore the same as the problem of finding integers x, y, z to satisfy (i), i.e. we have to find two square numbers whose sum is a square number.

The general solution of the problem is as follows: take any two integers m and n , such that $m > n$, then calculate

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2 \quad (ii)$$

Then x, y, z satisfy (i), and so do any equimultiples of x, y, z .

$$\text{For } x^2 + y^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = z^2$$

Exercise. Fill up the gaps in the following table and verify that $x^2 + y^2 = z^2$. (This equation is most easily verified as follows: take line 4, $73^2 + 48^2 = (73 + 48)(73 - 48) = 25 \cdot 121 = 5^2 \cdot 11^2 = 55^2$.)

m	n	x	y	z
2	3			
3	4			
3	5			
3	8	55	48	73
5	6			
5	12			

When we say that (ii) is the *general* solution, we mean that the lengths of the sides of *all* rational right-angled triangles can be obtained from formulae (ii). This fact is too difficult to prove here.

It will be found that the product of x, y, z is always divisible by 60. Perhaps you will be able to discover the reason for this. The following facts are of interest. The proof of the theorems mentioned would involve work more advanced than any in this book.

(1) We have seen that it is possible to find two square numbers whose sum is a square number. But it is impossible to find two cube numbers whose sum is a cube, or two fourth powers whose sum is a fourth power. It was asserted by Fermat (1608-65) that, when $n > 2$, the sum of two n th powers can never be an n th power. Though this has been proved for a large number of values of n , it has never been shown *in general*.

(2) Every whole number, greater than 1, can be expressed as the sum of four squares, or three squares, or two squares. (The number $1^2 = 1$ is of course to be counted as a square.)

(3) Every odd number, not of the form $8n + 7$, can be expressed as the sum of two or three squares. ($8 \cdot 0 + 7 = 7$, $8 \cdot 1 + 7 = 15$, $8 \cdot 2 + 7 = 23$, and so on, are numbers of the form $8n + 7$.)

(4) Every prime number of the form $4n + 1$ can be expressed as the sum of two squares in just one way.

Express the following numbers as the sums of two, three, or four squares: 21, 30, 67, 79, 160.

§ 20. Incommensurable magnitudes and irrational numbers

We have seen that if a side of a square is 1 inch long, the length of a diagonal is then $\sqrt{2}$ inches. This naturally leads us to ask, 'What is the value of the square root of 2?' The important fact, which we now wish to make clear is that *there is no fraction* (proper or improper) *whatever which equals $\sqrt{2}$* . It is usual to call numbers, that can be put in the form of fractions, 'rational' numbers. Thus 2, $3\frac{1}{2}$, $\frac{126}{1877}$, and, in Algebra, -1 , $-\frac{2}{3}$, $-1\frac{1}{4}$ are rational numbers. Our statement is, therefore, *$\sqrt{2}$ is not a rational number*. Numbers, not rational, are called 'irrational'. We can, however, find rational numbers as near as we please to $\sqrt{2}$, e.g. $1\frac{1}{2}$, $1\frac{1}{100}$, $1\frac{207}{1000}$, $1\frac{2071}{5000}$ are nearer and nearer to $\sqrt{2}$, the last differing from $\sqrt{2}$ by less than $1/50,000$. Again the following fractions are alternately less than and greater than $\sqrt{2}$, but each is nearer to it than the preceding

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots$$

It is obvious that no terminating decimal (i.e. one which comes to an end) can represent $\sqrt{2}$. For, if there were such a decimal, and it ended, say, in the figure 7, its square, which ought to be just 2, would have a decimal part ending in 9. Similarly, it could not end in any given figure. The decimal which represents $\sqrt{2}$ can therefore never come to an end.

Theorem. We will now prove that *$\sqrt{2}$ cannot equal a fraction p/q , where p, q are whole numbers*.

Proof. If possible, suppose $\sqrt{2} = p/q$, where, of course $q \neq 1$. Square both sides

$$2 = p^2/q^2$$

Now we can suppose p/q is in its lowest terms, i.e. that p and q have no common factor (for if they had one, we could cancel it out at the beginning, and still have $q \neq 1$).

Now all factors of p^2 are factors of p , for we get p^2 by taking the

factors of p twice over (Thus if $p = 3^2 \cdot 7 \cdot 11^3$ then $p^2 = 3^4 \cdot 7^2 \cdot 11^6$) Similarly all factors of q^2 are factors of q . But p, q have no common factors, therefore p^2 and q^2 have no factors in common.

But $p^2/q^2 = 2$ gives $p^2 = 2q^2$, which clearly says *all factors of q^2 are factors of p^2* .

The two statements in italics contradict each other.

Therefore, $\sqrt{2}$ cannot be expressed in the form p/q , where p, q are integers. Q. E. D.

Cor. The decimal which represents $\sqrt{2}$ can never end (as we have already proved) and can never recur.

It is proved in Arithmetic, that a decimal which ends, or recurs, can be reduced to a vulgar fraction, i. e. to the form p/q , where p, q are integers. But $\sqrt{2}$ cannot be expressed in that form.

Other examples of irrational numbers are the square root of any number which is not a perfect square (e. g. $\sqrt{5}, \sqrt{7}$) and the number π .

When we transfer our results to Geometry, we reach a surprising conclusion.

Theorem. It is impossible to find an interval which will go a whole number of times into a side of a square, and also a whole number of times into a diagonal.

Proof. Suppose there is some interval, which goes q times into a side, and p times into a diagonal of a square. Then

$$\frac{\text{diagonal}}{\text{side}} = \frac{p}{q} \quad \text{But} \quad \frac{\text{diagonal}}{\text{side}} = \sqrt{2} \quad \sqrt{2} = \frac{p}{q},$$

where p, q are integers. But we have shown that this is impossible. Hence the theorem follows. Q. E. D.

Two intervals which are such that an interval can be found, which will go a whole number of times into each, are said to be '*commensurable*' with one another. Two intervals which are such that no such interval can be found, are said to be '*incommensurable*' with one another.

When on p. 72 we showed that it was reasonable to define the area of a rectangle as the product of the lengths of two perpendicular sides, we assumed that there was a unit interval, which went a whole number of times into both sides. Our discussion, therefore, only applied to rectangles whose sides were commensurable with one another. Nevertheless, the definition can be shown to be reasonable in all cases, but this is a difficult thing to do.

On the other hand, no such assumption was made in proving Theorems 14-17, and thus those proofs hold for all cases.

Though the existence of incommensurable intervals is of very great importance, it is of no consequence when measurements are made. *For all measurement is approximate only.* For instance, no man can

measure the length of a given interval absolutely accurately, if he is sure he has got the length correct to one thousandth of an inch, and its length is, say, 5.238 ins., he cannot be sure that the next decimal place is zero, it may be any figure. Obviously he cannot go on measuring more and more exactly, so as to get more and more figures of the decimal, for ever. When you measure an interval with a ruler, you try to make one end of the interval coincide with a mark on the ruler, this mark has a finite thickness, and you cannot be sure that its centre is just opposite the end of the interval, and is not a hair's breadth, or a thousandth part of a hair's breadth to one side. Hence, whether the length is a rational number or an irrational number, when expressed in inches, we shall only be able to get a few decimal places, and so shall be quite unable to say if the length is a rational, or an irrational number.

Summing up, if we knew a figure was an *exact* square, and its side was *exactly* 1 inch long, then we should know that the length of its diagonal was *exactly* $\sqrt{2}$ inches, and $\sqrt{2}$ is an irrational number. But no one can draw an *exact* square, or measure a length *exactly*.

Nevertheless the discovery of the existence of incommensurable magnitudes was one of the greatest advances made by the ancient Greeks, and its influence stretches over the whole field of exact science.

Historical Note. The Science of Geometry began in Ancient Egypt as a set of rules, many of them quite rough, for finding the area of pieces of land, and the word itself is the English equivalent of a Greek word, which originally meant 'earth-measurement'. This practical knowledge of the Egyptians was introduced to the Greeks about 600 B.C. by Thales who lived at Miletus, a Greek colony in Asia Minor, but the first important mathematical discoveries were made by Pythagoras (p. 80). Special cases of Pythagoras' Theorem (e.g. that the triangle whose sides are 3, 4, 5 units long is right-angled) were known to the Egyptians, and used for making right-angles in their pyramids, but tradition affirms that Pythagoras first discovered and proved the general proposition. His disciples discovered the regular solids (p. 45), and the fact that $\sqrt{2}$ is an irrational number (p. 96).

The most famous text-book of Elementary Geometry was written by Euclid who lived at Alexandria about 300 B.C. His work was divided into thirteen 'books,' the first of which corresponds roughly to Part I of the present book. The rest deal with circles, the properties of numbers, incommensurable magnitudes, solid geometry, and the regular solids. He also wrote a work on Conic Sections.

§ 21. Mensuration.

1. Area of parallelogram = base $\times$ height

2. Area of triangle = $\frac{1}{2}$ base $\times$ height.

3. Area of trapezium = $\frac{1}{2}$ sum of parallel sides $\times$ distance between them

4. Circumference of circle = $2\pi r$ (r = radius)

5. Area of circle = πr^2

6. Volume of prism or cylinder = area of base $\times$ height

7. Volume of parallelepiped = area of base $\times$ height

8. Volume of pyramid = $\frac{1}{3}$ area of base $\times$ height

9. Volume of cone = $\frac{1}{3}$ area of base $\times$ height

10. Area of curved surface of right circular cylinder = height $\times$ circumference of base

(‘Height’ in Nos. 6–10 means height perpendicular to base)

11. Area of curved surface of right circular cone = $\frac{1}{2}$ slant height $\times$ circumference of base

12. Surface of sphere = $4\pi r^2$ (r = radius)

13. Volume of sphere = $\frac{4}{3}\pi r^3$

14. Area of zone of sphere cut off by two parallel planes = $2\pi r \times$ distance between the planes

15. If a , b , c be the lengths of the sides of a triangle, and $s = \frac{1}{2}(a + b + c)$, then the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

$\pi = \frac{22}{7}$ approx., more nearly $\frac{355}{113}$, still more nearly

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288$

Exactly $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots)$ for ever

Some of the above results will be discussed later

PART II

§ 22. Symmetry properties of a circle.

[This section can be taken while the theorems on congruence are being revised.]

The property of circles which is taken as fundamental in the following work is this *all the radii of a circle are equal*. Thus if we are given a circle, there is a point, called the 'centre,' such that all points on the circle are at equal distances from it. We can therefore give the following formal definition of a circle.

Def. A 'circle' is the set of all points in a plane equidistant from a fixed point of the plane: the fixed point is called the 'centre.'

Note. According to this definition, a circle is a curved line, sometimes, however, the word 'circle' is used to denote the part of the plane inside the curved line, as e.g. when we speak of the area of a circle. Then, for definiteness, the curved line is called the 'circumference' of the circle. It is, of course, wrong to use the word 'circle' in two senses, and it is really unnecessary to use the word 'circumference' at all, but in this book we shall follow the usual customs, and the reader will find he can always guess in which sense the word 'circle' is being used in any particular passage.

We can give the following definition of a sphere:

Def. A 'sphere' is the set of all points equidistant from a fixed point: the fixed point is called the 'centre.'

The word 'radius' for a circle (or sphere) is also used in two senses. It may mean an interval from the centre O to a point A on the circle (or sphere), or, r may mean the *length* of that interval. Here again, the meaning can easily be guessed in any particular passage.

* A 'diameter' of a circle (or sphere) may mean an interval joining two points on the circle (or sphere), which goes through the centre, or it may mean the length of that interval.

A 'chord' of a circle (or sphere) is an interval joining two points of the circle (or sphere).

THEOREM 19

The right bisector of a chord of a circle goes through the centre of the circle.

Given. AB is a chord of a circle, centre O .

To prove that the right bisector of chord AB goes through O .

Proof. Let X be the mid-point of chord AB . Join O to A, X, B .

In $\triangle OXA, OXB$,

$$\begin{cases} AX = XB, & (\text{const.}) \\ OA = OB, & (\text{radii}). \\ OX = OX. \end{cases}$$

$\therefore \triangle OXA, OXB$ are congruent, (three sides).

$$\therefore \angle OXA = \angle OXB.$$

$\therefore \angle OXA$ is a rt. $\angle$, (def.)

$\therefore OX$ is the right bisector of chord AB .

$\therefore$ the right bisector of chord AB goes through the centre.

Q.E.D.

(We have assumed that the chord AB itself does not go through the centre. If it does, the theorem is 'trivial'.)

Cor. The perpendicular to a chord of a circle, from the centre, bisects the chord.

THEOREM 20

If two circles cut, the line joining their centres is the right bisector of their common chord.

Given. The circle, centre O , and the circle, centre P , cut in A and B .

To prove that OP is the right bisector of chord AB .

Proof. Draw the right bisector of chord AB .

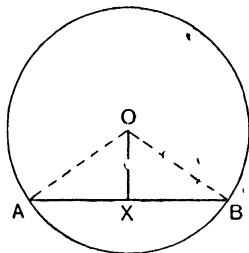


Fig. 156

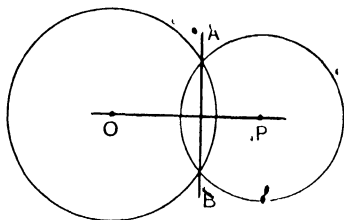


Fig. 157

Since AB is a chord of the circle, centre O ,

$\therefore$ this right bisector goes through O

Since AB is a chord of the circle, centre P ,

$\therefore$ this right bisector goes through P .

$\therefore$ this right bisector goes through O and P .

the line OP is the right bisector of the chord AB .

Q E D.

Exercise Prove Theorem 20, without using Theorem 19, from the properties of congruent triangles.

EXERCISE 37.

1. A circle of radius $3' 1''$ has a chord of length $5' 10''$. How far is it from the centre?

2. A circle of radius $2' 5''$ has a chord $1' 8''$ from the centre. What is its length?

3. Find the lengths of chords of a circle of radius 1 ft, which are distant $2''$, $3''$, $4''$, $6''$, $10''$ from the centre.

4. A circle is given of diameter 12 cm. Find the distances from the centre of chords whose lengths are 10 cm, 8 cm, 6 cm, 3 cm.

5. Two circles of radii $4''$, $5''$ have their centres $8''$ apart. Find the length of their common chord.

6. Two circles of radii $6''$, $7''$ have a common chord of length $4''$. Find the distance between their centres.

7. Two circles have their centres 12 cm. apart, their common chord is of length 8 cm, the radius of one circle is 10 cm. What is the radius of the other?

8. What is the radius of a circle if it has a chord of length a inches at a distance of b inches from the centre?

9. The line joining the mid-points of two parallel chords of a circle goes through the centre.

10. If the line which joins the mid-points of two chords of a circle, which are not diameters, is perpendicular to one chord, it is perpendicular to the other.

11. If the line which joins the mid-points of two chords of a circle, which are not diameters, goes through the centre, then the chords are parallel.

12. A is a point, not the centre, inside a given circle, through A draw a chord such that A is its mid-point. How many such chords are there?

13 A line cuts one circle in A, B and a concentric circle in C, D . Prove $AC = BD$.

14 AB is a chord of a circle, P, Q are points on the perpendicular diameter. Prove $\hat{PAQ} = \hat{PBQ}$.

15. It is known that a circle goes through a given point P , and has its centre on a given line AB . Find another point through which it must go.

***16** P, Q, P', Q' are points on a circle, P, Q' separate, P', Q There is a point A on the circle such that $AP = AP', AQ = AQ'$. Prove $PP' \parallel QQ'$.

***17** A, B, C, D are points on a line, B, D separate, $A, C, AB = CD$, circles with centres B, C and equal radii cut in E . Prove $EA = ED$.

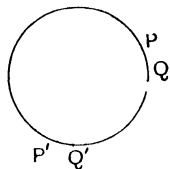


Fig 158

***18** Two circles of radii a, b have their centres at a distance a apart. Find the length of their common chord. What happens to the formula when the circles do not cut one another?

THEOREM 21

The locus of a point, which moves so as to be equidistant from two fixed points, is the right bisector of the interval joining the points.

(1) *Given.* A, B are fixed points, P is a point such that $PA = PB$.

To prove that P lies on the right bisector of the interval AB .

Proof. Let X be the mid-point of the interval AB . Join PA, PX, PB .

In $\triangle s PAX, PBX$,

$$\begin{cases} AX = BX, & (\text{const}) \\ PA = PB, & (\text{given}). \\ PX = PX. \end{cases}$$

$\therefore \triangle s PAX, PBX$ are congruent, (three sides)

$$\therefore \hat{PXA} = \hat{PXB}.$$

$\therefore PX$ is perpendicular to AB , (def.).

$\therefore P$ lies on the right bisector of interval AB .

$\therefore$ all points on the locus are on that right bisector.

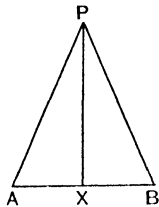


Fig 159

(2) *Given.* A, B are fixed points, X is the mid-point of interval AB , P is on the right bisector of interval AB .

To prove that $PA = PB$.

Proof. In $\triangle PAX, PBX$,

$$\begin{cases} AX = BX, & (\text{given}). \\ PX = PX. \\ \hat{PXA} = \hat{PXB}, & (\text{rt. } \angle\text{s}). \end{cases}$$

$\triangle\text{s } PAX, PBX$ are congruent, (two sides, incl. $\angle$).

$$PA = PB.$$

$\therefore P$ is on the locus.

all points on the right bisector of the interval AB are on the locus.

Hence the locus is that right bisector.

Q.E.D.

CONSTRUCTION 11.

To find the centre of a circle which passes through three given points, not on a straight line.

Given. A, B, C , three points, not on a straight line.

To find the centre of the circle through A, B, C .

Construction Draw the right bisectors XY and PQ of AB, BC resp. These right bisectors cut in the centre of the circle through A, B, C .

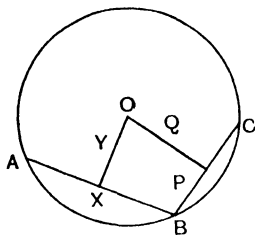


Fig 160

Proof. The right bisectors must cut, for if $XY \parallel PQ$, then, since $XY \perp AB$, and $PQ \perp BC$, we should have A, B, C on one straight line, contrary to the hypothesis. Let the right bisectors cut in O

Now every point on XY is equidistant from A and B , and every point on PQ is equidistant from B and C .

$\therefore$ the point O is equidistant from A, B and C .

$\therefore O$ is the centre of a circle which goes through A, B and C .

Q.E.F.

Cor. One and only one circle can be drawn through three non-collinear points.

EXERCISE 38

1. Draw a quadrilateral $ABCD$ with $AB = 2''$, $CD = 3''$, $BC = 2\frac{1}{2}''$, $\hat{A}BC = \hat{BCD} = 1 \text{ rt } \angle$. Draw a circle through A, D with its centre on BC . Measure its radius.

2. Draw a circle of radius $1\frac{1}{2}''$, and take radii OA, OB , such that $\hat{AOB} = 60^\circ$. On OA, OB resp. take points C, D , such that $OC = 1''$, $OD = 2''$. Draw a circle through C, D with its centre on the given circle. How many such circles are there?

3. Draw a circle of given radius to go through two given points. Under what circumstances are there (i) no such circles, (ii) more than one such circle?

Def. If ABC is a triangle, the circle through A, B, C is called the 'circumcircle' of the triangle, its centre is the 'circumcentre' of the triangle, its radius the 'circumradius.'

THEOREM 22

The three right bisectors of the sides of a triangle are concurrent.

Given ABC is a triangle

To prove that the right bisectors of its sides are concurrent

Proof. Every point on the right bisector of AB is equidistant from A and B .

Every point on the right bisector of BC is equidistant from B and C .

These right bisectors cut, for if they were parallel, the lines AB, BC perpendicular to them, would be in one straight line.

Let the right bisectors cut in O .

Then O is equidistant from A, B and C . $\therefore OA = OC$.

$\therefore O$ is on the right bisector of AC .

$\therefore$ the right bisectors of AB, BC and CA all cut in O .

Q E D.

Cor. The three right bisectors of the sides of a triangle meet in the circumcentre

Exercise. When is the circumcentre of a triangle outside it?

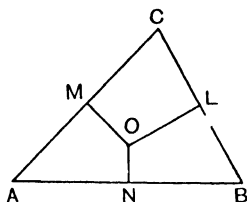


Fig 161

THEOREM 23.

(i) If two chords of a circle be equal, they are equidistant from the centre.

(ii) If two chords of a circle be equidistant from the centre, they are equal.

(iii) If two chords of a circle be unequal, the greater is nearer the centre.

(iv) If two chords of a circle be at different distances from the centre, the one nearer is the greater.

Given. AB, CD are chords of a circle, centre O , X, Y are the feet of the perpendiculars from O on to them

To prove that

(i) If $AB = CD$, then $OX = OY$.

(ii) If $OX = OY$, then $AB = CD$.

(iii) If $AB > CD$, then $OX < OY$.

(iv) If $OX < OY$, then $AB > CD$.

Proof. Since $OX \perp AB$, $AX = XB$.

Similarly $CY = YD$

Now

$$\begin{aligned} OX^2 + AX^2 &= OA^2 && (OX \perp XA, \text{Pythagoras}). \\ &= OC^2, && (OA = OC, \text{radii}). \\ &= OY^2 + CY^2, && (OY \perp YC, \text{Pythagoras}). \\ \therefore OX^2 + AX^2 &= OY^2 + CY^2. \end{aligned}$$

Using this result we find

(i) If $AB = CD$, then $AX = CY$.

$$\therefore AX^2 = CY^2. \quad OX^2 = OY^2. \quad \therefore OX = OY.$$

(ii) If $OX = OY$, then $OX^2 = OY^2$.

$$\therefore AX^2 = CY^2 \quad \therefore AX = CY \quad \therefore AB = CD.$$

(iii) If $AB > CD$, then $AX > CY$.

$$\therefore AX^2 > CY^2 \quad \therefore OX^2 < OY^2. \quad \therefore OX < OY.$$

(iv) If $OX < OY$, then $OX^2 < OY^2$.

$$\therefore AX^2 > CY^2. \quad \therefore AX > CY. \quad \therefore AB > CD.$$

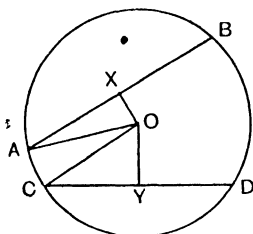


Fig 162

EXERCISE 39

1. $ABCD$ is a quadrilateral such that the right bisectors of sides AB, BC, CD are concurrent. Prove that the right bisector of AD goes through the same point ;

2. The locus of the mid-points of a set of parallel chords of a circle is a diameter.

3 The locus of the mid-points of a set of equal chords of a circle is a concentric circle, if the chords are not diameters.

4 Two circles cut in A, B Through A is drawn a line parallel to the line joining the centres, cutting the circles again in X, P . Prove that the length of XP is twice the distance between the centres

5 Two circles cut in A, B Through A, B are drawn parallel lines. Prove that the intervals XP, YQ 'intercepted on these lines' by the circles are equal (Fig 163)

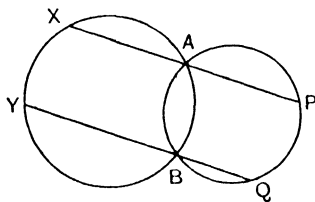


Fig 163

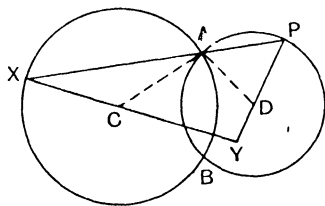


Fig 164

6 Two circles, centres C, D , cut in A, B . Through A is drawn a line cutting the circles again in X, P , the lines CX, PD cut in Y . Prove $C\hat{A}D = X\hat{Y}P$. (Fig 164)

7 If two chords of a circle bisect each other they are diameters.

8 (a) Two circles cut in A, B , C is the mid-point of the interval joining their centres, the line through A perpendicular to CA cuts the circles again in X, P . Prove that $AX = AP$

(b) Through a common point A of two circles draw a line so that the interval intercepted between the circles is bisected at A

9 A circle has two parallel chords of lengths $8''$ and $10''$, whose distance apart is $6''$. Find the radius of the circle (Use Algebra)

10 $ABCD$ is an isosceles trapezium, $AB \parallel CD$, $AB = 8''$, $CD = 6''$, distance between parallel sides = $1''$. Find the radius of the circum-circle of $ABCD$

11 Circles are drawn having the sides of a quadrilateral $ABCD$ for diameters. Prove that the common chord of the circles on diameters AB and BC is parallel to the common chord of the other two circles.

***12** The diagonals of the quadrilateral $ABCD$ cut in O . Prove that the circumcentres of OAB , OBC , OCD , ODA are the vertices of a parallelogram

***13** Describe a circle, with given centre to cut another circle at the ends of a diameter

***14** $PQRS$ is a quadrilateral inscribed in a circle, centre O , $\hat{PQR}$, $\hat{QRS}$ are right angles, a diameter of the circle cuts lines PQ , RS in A , B resp. Prove $OA = OB$

***15** Find the radius of the circumcircle of an isosceles triangle whose base is 8" and whose height is 2"

***16** Find the height of an isosceles triangle whose base is 12" and whose circumradius is $7\frac{1}{2}$ ".

***17** Prove that the radius of the circumcircle of an equilateral triangle, whose side is of length a , is $a/\sqrt{3}$.

***18** A wooden sphere of radius 10" has a piece sawn off along a plane, the radius of the circle which forms the base of the piece is 8" Find the height of the piece, and the area of the surface of the piece

***19** The radius of the earth is 4000 miles. A parallel of latitude is at a distance, measured along the polar axis, of 2000 miles from the equator. Find the radius of this parallel of latitude, and the area contained between it and the equator.

***20** (a) In the quadrilateral $ABCD$, $AB = AD$, $CB = CD$. A double cone is formed by rotating the quadrilateral round AC . If $AB = 3$ ", $BC = 5$ ", $CA = 7$ ", find the area of the circle formed by the interval BD in this rotation.

(b) Find also the area of the surface of the double cone

***21** Two circles cut in A , B . Through A draw a line cutting them in P , X resp. so that $PA = AX$.

(Revision Papers XXVI-XXX can now be taken, if desired.)

*§ 23. Some important examples

EXAMPLE 21. A , B are fixed points, P moves in a plane through A , B so that $PA^2 - PB^2$ is constant. To prove that the locus of P is a line perpendicular to AB .

Proof Assume $PA > PB$. Let C be the foot of the perpendicular from P on to AB . Let O be the mid-point of interval AB .

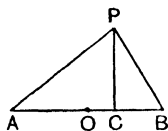


Fig 165

Then $PA^2 = PC^2 + CA^2$, $PB^2 = PC^2 + CB^2$, (Pythagoras).

$$\begin{aligned}
 \text{Subtracting, } PA^2 - PB^2 &= CA^2 - CB^2 \\
 &= (CA + CB)(CA - CB) \\
 &= AB \cdot \{CO + AO - (BO - OC)\} \\
 &= AB \{CO + AO - BO + OC\} \\
 &= 2AB \cdot CO, \quad (\because AO = BO).
 \end{aligned}$$

But $PA^2 - PB^2$ is const. (given).

$2AB \cdot CO$ is const CO is const C is a fixed point.

if P is on the locus, the foot of the perpendicular from P to AB is a fixed point. every point on the locus is on this perpendicular the locus is a line perpendicular to AB . Q.E.D.

Note To get the last step, we ought also to show that every point on the perpendicular is on the locus. It is usual to omit this second part in locus proofs, because it can usually be proved like the first part. In Theorem 21 we proved both parts.

Cor. If A, B, C, D be coplanar points such that

$$CA^2 - CB^2 = DA^2 - DB^2, \text{ then } CD \perp AB.$$

Conversely if $CD \perp AB$, then $CA^2 - CB^2 = DA^2 - DB^2$

Exercise The proof assumes that C is between A and B . Write out the proof when C is on AB produced beyond B

EXAMPLE 22 The three altitudes of a triangle are concurrent

Given AP, BQ, CR the altitudes of triangle ABC .

To prove they are concurrent.

Proof Let BQ, CR cut in H

$$\text{Since } BH \perp AC, \quad AB^2 - BC^2 = AH^2 - CH^2$$

$$\text{Since } CH \perp AB, \quad BC^2 - CA^2 = BH^2 - AH^2$$

$$\text{Adding,} \quad AB^2 - CA^2 = BH^2 - CH^2.$$

$$AH \perp BC.$$

the third altitude also goes through H

Q.E.D.

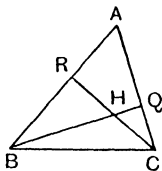


Fig 166

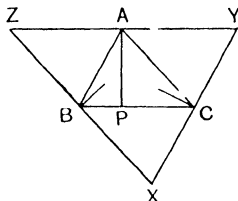


Fig 167

Second proof. Through A, B, C draw parallels to the opposite sides; these parallels cut in points X, Y, Z , say, opposite to A, B, C resp. Then $ACBZ$ and $ABCY$ are parallelograms.

$$AZ = BC, \quad AY = BC, \quad (\text{opp. sides of parms})$$

$$AZ = AY.$$

The altitude AP is perpendicular to BC .

$$AP \perp YZ, \quad (YZ \parallel BC)$$

AP is the right bisector of YZ .

Similarly, all the altitudes of $\triangle ABC$ are right bisectors of sides of $\triangle XYZ$.

But the right bisectors of the sides of any triangle are concurrent.

$\therefore$ the altitudes of $\triangle ABC$ are concurrent $Q.E.D.$

Def. The point where the altitudes of a triangle cut is called the 'orthocentre' of the triangle

*EXERCISE 40

1. Show how to draw a square equal to the difference of two given squares.

2. Find a point X on the line AB so that $XA^2 - XB^2$ is a given square.

3. A quadrilateral of rods, jointed at the corners, has its diagonals at right angles, show that they remain at right angles as the quadrilateral changes its shape, the rods of course maintaining always the same length.

*4. Given a quadrilateral of jointed rods, prove that if, in one position, the lines joining the mid-points of opposite sides are equal, this is the case in all positions.

*5. A, B are fixed points in space, P moves in space so that $PA^2 - PB^2$ is constant. Prove that P always lies in a fixed plane perpendicular to AB .

*6. $ABCD$ is a rectangle, P is any point, not necessarily in the plane ABC . Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

§ 24. Angles in segments, cyclic quadrilaterals.

THEOREM 24.

The angle, subtended at the centre of a circle by an arc of the circle, is double the angle subtended at the circumference by the same arc.

Given. AB is an arc of a circle, centre O , C is a point on the circle and not on the arc AB .

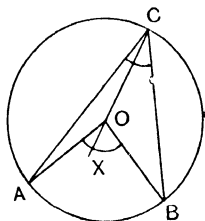
To prove that angle AOB is twice angle ACB .

Proof. Join CO and produce the interval CO beyond O to X .

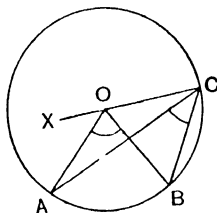
If the ray CO lies in $\hat{ACB}$, then the ray OX lies in $\hat{AOB}$.

Case 1.)

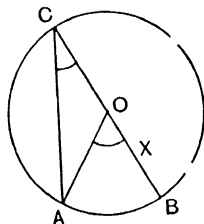
If the ray CO lies outside $\hat{ACB}$, then the ray OX lies outside $\hat{AOB}$. (Case ii)



Case (i)



Case (ii)



Case (iii)

Fig. 168

In these two cases we have, since $OA = OC$,

$$\hat{OAC} = \hat{OCA}, \quad (\text{isos. } \triangle).$$

$$\hat{AOX} = \hat{OAC} + \hat{OCA}, \quad (\text{ext. } \angle \text{ of } \triangle = \text{sum of int. opp. } \angle \text{s})$$

$$\therefore \hat{AOX} = 2\hat{OCA}.$$

$$\text{Similarly} \quad \hat{BOX} = 2\hat{OCB}.$$

Adding in case (i), subtracting in case (ii),

$$\therefore \hat{AOB} = 2\hat{ACB}$$

In case (iii), the ray CO goes through A or B . Suppose it goes through B . Then, as before

$$\hat{AOX} = 2\hat{OCA}.$$

$$\therefore \hat{AOB} = 2\hat{ACB}.$$

Q.E.D.

Note The angle subtended at the centre by the arc AB may be reflex. The proof still holds then

Def. A 'segment' of a circle is the part of the plane between an arc of a circle and the chord joining the ends of the arc. It is a 'major,' or 'minor,' segment according as the arc is greater than, or less than, a semicircle. An 'angle in a segment' is an angle whose vertex is on the arc of the segment and whose arms pass through the ends of the arc. E.g. in Fig. 168 the angle ACB is in the segment ACB .

THEOREM 25.

Angles in the same segment, of a circle are equal.

Given. $\hat{AXB}$, $\hat{AYB}$ are angles in the same segment of a circle, centre O .

To prove that $\hat{AXB} = \hat{AYB}$.

Proof $\hat{AOB} = 2\hat{AXB}$, ($\angle$ at centre, double $\angle$ at circumf.)

Similarly $\hat{AOB} = 2\hat{AYB}$.

$$\hat{AXB} = \hat{AYB}.$$

Q E D.

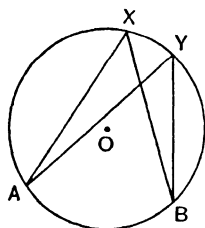


Fig 169

THEOREM 26.

An angle in a semicircle is a right angle, an angle in a major segment is acute, an angle in a minor segment is obtuse.

Given. $\hat{ACB}$ is an angle in a segment of a circle, centre O

To prove that $\hat{ACB}$ is a right angle, acute angle, or obtuse angle, according as the segment ACB is a semicircle, a major, or a minor segment

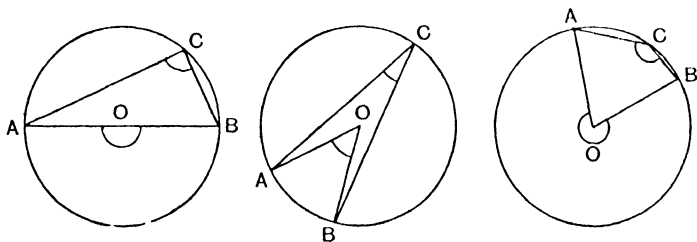


Fig 170

Proof. If ACB is a semicircle, then $\hat{AOB} = 2 \text{ rt. } \angle s$

If ACB is a major segment, then $\hat{AOB} < 2 \text{ rt. } \angle s$.

If ACB is a minor segment, then reflex $\hat{AOB} > 2 \text{ rt. } \angle s$.

But $\hat{AOB} = 2\hat{ACB}$.

$\therefore$ In the three cases we have resp $\hat{ACB}$ is a right angle, acute

THEOREM 27.

The circumcircle of a right-angled triangle is the circle on the hypotenuse as diameter.

Given. ABC is a triangle with the angle ABC a right angle, O is the mid-point of AC .

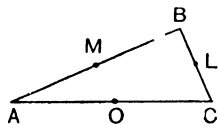


Fig 171

To prove that $OA = OB = OC$.

Proof Let L be the mid-point of side BC .

Since O, L are the mid-points of sides CA, CB resp.,

$\therefore OL$ is parallel to AB

$\therefore OL$ is perpendicular to BC , ($\angle ABC$ is a rt. $\angle$)

$\therefore OL$ is the right bisector of side BC . $\therefore OB = OC$.

But

$$OA = OC.$$

$$\therefore OA = OB = OC$$

Q.E.D.

Cor. If ABC and ADC are right angles, then A, B, C, D are on a circle on diameter AC . (B, D may be on the same or on opposite sides of AC .)

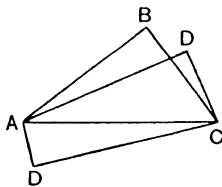


Fig 172

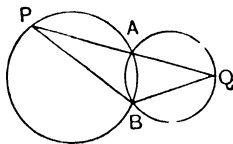


Fig 173

Note This is a partial converse of Theorem 25, the full converse will be proved later

EXAMPLE 23. Two circles cut in A, B ; a line through A cuts them again in P, Q . To prove that, as this line varies, while still passing through A , the angle PBQ remains constant. (Fig. 173)

Proof As the line PQ varies, $\angle APB$ remains constant, ($\angle$ s in same segment).

Similarly, $\angle AQB$ remains constant.

But $\angle APB + \angle AQB + \angle PBQ = 2 \text{ rt. } \angle$ s, (angles of $\triangle PRQ$).

$\therefore \angle PBQ$ remains constant.

Q.E.D.

EXAMPLE 24 AB, CD are two chords of a circle cutting in P . To prove that half the sum of the angles subtended at the centre O by the arcs BC, AD equals one of the angles between AB and CD .

Proof $\frac{1}{2}\angle COB = \angle CAB$, ($\angle$ at circumf. = $\frac{1}{2}\angle$ at centre).

Similarly, $\frac{1}{2}\angle AOD = \angle CDP$

$$\frac{1}{2}(\angle COB + \angle AOD) = \angle CAB + \angle CDP = \angle APD,$$

(ext $\angle$ of $\triangle APC$ = sum of int opp $\angle$ s)

Q.E.D.

Note The arcs BC and AD must be taken so as not to overlap.

Exercise State the theorem corresponding to Example 24 when the chords AB, CD produced, cut in P .

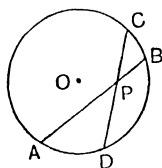


Fig 174

EXERCISE 41

1 Chords AB, CD of a circle cut in O . Prove that triangles AOC, BOD are similar, and that triangles AOD, BOC are similar.

2 The radii OA, OB, OC of a circle ABC cut a concentric circle XYZ in X, Y, Z . Prove $\angle ACB = \angle XYZ$.

3 The circumcentre O of the triangle ABC lies outside the triangle. If $\angle BAC = x^\circ$, find $\angle BOC$ in terms of x (Two cases)

4 O is the centre of circle ABC , $\angle OAC = 20^\circ$, $\angle OBC = 30^\circ$. Find the size of $\angle AOB$ (Two cases)

5 O is the centre of circle ABC , $\angle OAB = 40^\circ$, O, C are on the same side of AB . Find the size of $\angle ACB$.

6. (a) OA, OB, OC are radii of a circle, $\angle AOB = 140^\circ$, $\angle BOC = 120^\circ$. Find the sizes of the angles of triangle ABC . (Two cases.)

(b) Find also the sizes of these angles when $\angle AOB = 40^\circ$, $\angle BOC = 20^\circ$. (Two cases)

7 Circles are drawn on the sides AB, AC of triangle ABC as diameters. Prove that they cut on the line BC .

8 Two circles cut in A, B , AX, AY are their diameters through A . Prove that X, B, Y are collinear. (Join XB, BY . Consider angles XBA, YBA)

9 ABC is a circle, centre O ; $\angle OAB = 40^\circ$; $\angle ABC = 60^\circ$. Find the size of $\angle OCB$.

10 ABC is a circle, centre O ; B, O are on opposite sides of AC ; $\angle OAC = 35^\circ$. Find the size of $\angle ABC$.

11. AB, CD are perpendicular chords of a circle, centre O ; $\hat{AOD} = 125^\circ$. Find the size of $\hat{COB}$.

12. Prove that if the circumcentre of a triangle is inside the triangle, then the triangle is acute-angled.

13. AB and CD are parallel chords of a circle; AD, BC cut at P inside the circle. Prove $PA = PB$.

14. AB and CD are chords of a circle cutting in P , $AP = PC$. Prove $BP = DP$.

15. (a) A is any point on a circle, centre O , on OA as diameter another circle is drawn. Any line through A cuts the first circle in P , the second in Q . Prove $AQ = QP$.

(b) A is any point on a given circle. Find the locus of the mid-points of chords through A .

16. A circle through the vertex A of a parallelogram $ABCD$ cuts the lines AB, AC, AD in P, Q, R . Prove that the triangles ABQ and PQR are similar.

17. The chord BC of the circle ABC is produced beyond C to D . The bisectors of angles ACB, ACD cut the circle again in X, Y . Prove that XY is a diameter.

18. ABC is a triangle inscribed in a circle, centre O , CD is an altitude of the triangle. Prove $\hat{ACD} = \hat{BCO}$.

19. (a) ABC is a circle, centre O , X is on the arc AOB of circle AOB ; AX cuts the circle ABC again in Y . Prove $XY = XB$.

(b) The lines OX, BY are at right angles (Fig. 175.)

*20. APB and $ACDB$ are two arcs on the same side of their chord AB , A, P, D are in one line, B, P, C are in one line, AC, BD cut in Q . Prove that, as P moves over the arc APB , $\hat{AQB}$ is constant.

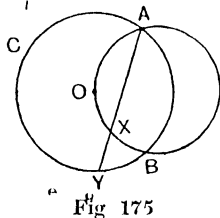


Fig. 175

*21. $AXYB$ is an arc on chord AB , $X\hat{A}Y, X\hat{B}Y$ are bisected by rays AR, BR . Prove that, as X, Y move on the arc, $A\hat{R}B$ is constant.

*22. AB and CD are perpendicular chords of a circle cutting in a point X , Y is the mid-point of BC . Prove that XY is perpendicular to AD .

*23. CD is a chord of a circle perpendicular to a diameter AB , E is any point on AB ; DE cuts the circle again in F . Prove that CA and CB bisect the angle ECF internally and externally.

*24. AX, BY are any two chords of a circle, perpendiculars to AB, BY at A, B , resp cut the circle again in C, D . Prove that CD is equal and parallel to XY .

Defn. A quadrilateral is 'cyclic' when a circle can be drawn through its four vertices. A set of points is 'cyclic' when a circle can be drawn through them all. Such a set of points is also called a 'concyclic' set.

THEOREM 28.

Opposite angles of a cyclic quadrilateral are supplementary.

Given. $ABCD$ is a quadrilateral whose vertices lie on a circle, centre O

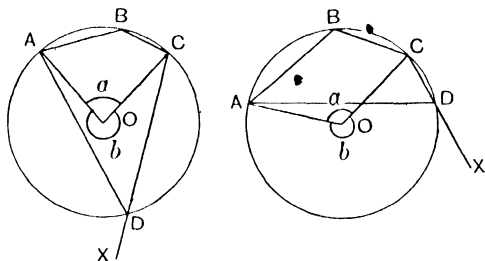


Fig 176

To prove that $\hat{ABC} + \hat{ADC} = 2 \text{ rt } \angle\text{s}$.

Proof Draw the radii OA, OC .

Then, of the two angles at O formed by the radii OA, OC , the angle marked a , subtended by the arc ABC , is twice $\hat{ADC}$, the angle marked b , subtended by the arc ADC , is twice $\hat{ABC}$. But the sum of the angles a and b is four right angles.

$$\therefore \hat{ABC} + \hat{ADC} = 2 \text{ rt } \angle\text{s}. \quad \text{Q.E.D.}$$

Cor. An exterior angle of a cyclic quadrilateral equals the interior opposite angle.

For, produce CD beyond D to X , then $\hat{ADX}$ and $\hat{ADC}$ are supplementary, but so are $\hat{ADC}$ and $\hat{ABC}$. $\therefore \hat{ADX} = \hat{ABC}$.

16. $ABCDEF$ is a hexagon inscribed in a circle. Prove that the sum of the angles at A, C, E equals four right angles.

17. The altitudes AP, BQ of a triangle ABC cut in H . Prove
 $\hat{AHQ} = \hat{BQ}, \hat{CPQ} = \hat{AQ}.$

18. Show how to construct a triangle ABC , given sides AC, BC and the altitude from A , all in length only.

19. Two triangles with bases, areas, and vertical angles equal, are congruent.

20. AP, BQ, CR are altitudes of $\triangle ABC$, L is the mid-point of EC . Prove $\hat{RQL} = \hat{QRL} = \hat{BAC}.$

21. CD is a chord of a circle perpendicular to the diameter AB , a line through A cuts the circle in X , and CD , produced beyond D , in Y . Prove $\hat{AXC} = \hat{DXY}.$

22. AB, CD are perpendicular chords of a circle, cutting in E ; the perpendicular from C to AD cuts AB in X . Prove $\hat{XE} = \hat{EB}.$

*23. AB is a common chord of two circles, centres C and D . The line PAX cuts the circles in P, X resp. Prove $\hat{CPB} = \hat{DXB}.$ (Draw a careful figure.)

*24. Two circles cut in A, B , X is on one circle, centre O , XA and XB cut the other circle in C, D resp. Prove $OX \perp CD.$ (Fig. 179)

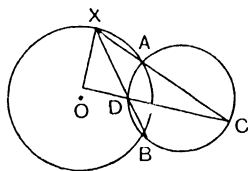


Fig. 179

*25. P is any point inside $\triangle ABC$, D, E, F are the feet of the perpendiculars from P to BC, CA, AB resp. Prove

$$\hat{BPC} - \hat{BAC} = \hat{EDF}.$$

*26. Two circles cut in A, B . Through A draw a line cutting the circles in P, X , so that $\hat{XBA} = \hat{PBA}.$

§ 25. Inequalities†.

We now return to the study of the triangle for a short time, proving some theorems that will be useful when we again take up the circle. It will probably be thought that some of the theorems which follow are so simple that they need no proof, and it is here that we may give some explanation as to why a proof is given at all. If a theorem

† This section may be postponed, in whole or in part.

which is *not* obvious is proved, it is in this way reduced to other theorems that are either obvious or have been previously proved; and anyone who assents to these latter theorems, and understands the proof, is thereby compelled to assent to the theorem proved. Thus at the very base of our work, we have a collection of theorems, like those we labelled 'assumptions,' which are regarded as obvious. If now we consider another theorem which seems obviously true, we may either add it to our assumptions, or we may try to 'prove' it from the assumptions already made, but when we pretend we are proving obvious theorems, what we are really doing, is not making them more certain (for they are supposed to be obvious already), but arranging them: if one theorem follows from three others, we can put this theorem in the same class as the other three, and the subject is being treated from the logical rather than from the geometric standpoint.

Thus a proof may serve two purposes: it may convince us of some new truth; or it may show us how old truths are connected together. So far, in this book, we have given proofs with the first aim in view; henceforth, sometimes one aim, sometimes the other, will predominate.

The theorems of this section can all be shown from the theorems on the *congruence* of triangles, without the use of any theorem which depends on *parallel lines*. We shall see later that this fact is very important from the point of view of the arrangement, or logical connection, of theorems.

THEOREM 29.

If a line cuts two other lines so as to make the alternate angles equal, then these two lines cannot cut each other.

Given. $\angle ABC$ and $\angle BCD$ are equal alternate angles.

To prove that the lines AB and CD cannot cut each other.

Proof. If possible, let the lines AB and CD cut in X .

First, suppose X and A are on the same side of BC .

On the ray CD there is an interval CZ of the same length as BX .

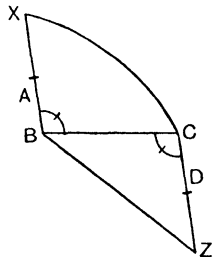


Fig 180

Then in $\triangle s XBC, ZCB$,

$$\begin{cases} BX = CZ, & (\text{const.}) \\ BC = BC. \\ \hat{XBC} = \hat{BCZ}, & (\text{given}). \end{cases}$$

$\therefore \triangle s XBC, ZCB$ are congruent, (two sides, incl. $\angle$).

$\therefore \hat{XCB} = \hat{ZCB}$. But $\hat{ZCB} = \hat{XBC}$, (given).

$\therefore \hat{XCB} + \hat{ZCB} = \hat{XBC} + \hat{ZBC}$.

But $\hat{XCB} + \hat{ZCB} = 2 \text{ rt. } \angle s$, ($\because X, C, Z$ are collinear);

$\therefore \hat{XBC} + \hat{ZBC} = 2 \text{ rt. } \angle s$

$\therefore X, B, Z$ are collinear.

$\therefore$ the lines AB and CD cut in two distinct points, viz. X and Z .

But this is impossible, since two lines cannot cut in two points. Hence the lines AB and CD cannot cut in a point on that side of BC on which A lies. Similarly they cannot cut in a point on the other side of BC . Q.E.D.

Cor. If a line cuts two other lines so as to make the corresponding angles equal, these two lines cannot cut each other. (Fig. 181.)

For, if $\hat{ABC}$ and $\hat{FCE}$ are equal corresponding angles, and FC be produced beyond C to D , then $\hat{FCE} = \hat{BCD}$.

$\therefore \hat{ABC}$ and $\hat{BCD}$ are equal alternate angles.

$\therefore$ the lines AB and CF cannot cut each other. Q.E.D.

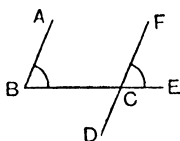


Fig 181

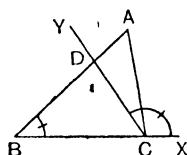


Fig 182

THEOREM 30 (The Fundamental Inequality).

An exterior angle of a triangle is greater than either of the interior opposite angles. (Fig. 182.)

Given. ABC is a triangle, ACX an exterior angle.

To prove that $\hat{ACX} > \hat{ABC}$.

Proof. Since AB and AC cut in A , $\therefore \hat{ACX}$ and $\hat{ABC}$ cannot be equal angles for if they were, then by the previous corollary, the lines BD and CA would not cut each other.

If possible, suppose $\hat{ACX} < \hat{ABC}$.

Then there is a ray CY on the same side of BC as A , such that $\hat{YCX} = \hat{ABC}$, and this ray will lie *inside* the angle $\hat{ACB}$, since $\hat{YCX} > \hat{ACX}$.

$\therefore$ the ray CY will cut the interval AB .

But this is impossible, since $\hat{YCX} = \hat{ABC}$.

$\therefore \hat{ACX}$ can neither be equal to, nor less than $\hat{ABC}$.

$\therefore \hat{ACX} > \hat{ABC}$

Similarly it can be shown that $\hat{ACX} > \hat{BAC}$.

Q E D.

Note We proved in Theorem 1, *Cor.* that an exterior angle of a triangle was equal to the sum of the interior opposite angles; it is hence greater than either of them. *But this proof depended on the assumptions we made about parallel lines.* The proofs of Theorems 29 and 30 do *not* depend on these assumptions, but they depend on the assumptions about congruent triangles. In fact, in Theorem 29 we have proved one of the assumptions about parallel lines (Assumption 7, first part) from the assumptions about congruent triangles, and in Theorem 29, *Cor* we have proved the first part of Assumption 8. We shall see later how important is the fact that this can be done.

The other theorems of this section are deduced from Theorem 30.

Exercise. Prove from Theorem 30 that a triangle cannot have two angles right angles.

THEOREM 31.

If one side of a triangle be longer than another, the angle opposite the longer side is greater than the angle opposite the shorter side.

Given. In triangle ABC , $AB > AC$

To prove that $\hat{ACB} > \hat{ABC}$.

Proof. Since $AB > AC$, there is a point D on the side AB such that $AD = AC$.

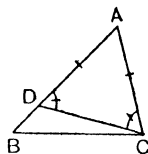


Fig 183

Then $\hat{ACD} = \hat{ADC}$, (isos. Δ).

But $\hat{ADC} > \hat{ABC}$, (ext. $\angle$ and int. opp. $\angle$),

and $\hat{ACB} > \hat{ACD}$, (D is on interval AF).

$$\therefore \hat{ACB} > \hat{ABC}.$$

Q.E.D.

THEOREM 32.

If one angle of a triangle be greater than another, the side opposite the greater angle is longer than the side opposite the smaller angle.

Given. In triangle ABC , $\hat{ACB} > \hat{ABC}$.

To prove that $AB > AC$

Proof. Either $AB > AC$, or $AB = AC$, or $AC > AB$.

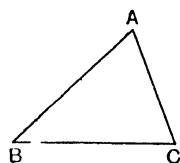


Fig 184

If $AB = AC$, then would $\hat{ABC} = \hat{ACB}$, (isos. Δ), contrary to the hypothesis.

If $AC > AB$, then would $\hat{ABC} > \hat{ACB}$, contrary to the hypothesis.

Thus $AB = AC$, and $AC > AB$ are false.

Hence

$$AB > AC.$$

Q.E.D.

THEOREM 33.

The shortest interval which can be drawn from a given point to a given line is the interval perpendicular to the line

Given. P is foot of the perpendicular from O on line AB , C any other point on AB .

To prove that $OP < OC$

Proof $\hat{OPC} = \hat{OPB}$, (rt. $\angle$ s).

$\hat{OPB} > \hat{OCP}$, (ext. $\angle$, int. opp. $\angle$).

$$\therefore \hat{OPC} > \hat{OCP}.$$

$\therefore OC > OP$, (greater side opp. greater $\angle$) Q.E.D.

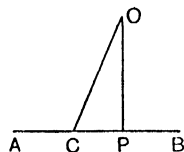


Fig 185

THEOREM 34

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given ABC is a triangle.

To prove that $AB + BC > AC$.

Proof Produce the side AB beyond B to D , making $BD = BC$.

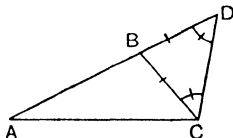


Fig 186

$$\therefore \hat{BCD} = \hat{BDC}, \quad (\text{isos. } \triangle).$$

$$\text{But } \hat{ACD} > \hat{BCD}.$$

$$\hat{ACD} > \hat{ADC}.$$

$$\therefore AD > AC, \quad (\text{greater side opp. greater } \angle).$$

$$\text{But } AD = AB + BC.$$

$$\therefore AB + BC > AC$$

Q.E.D.

Note This theorem may justly be considered as self-evident, the proof does not make it more certain, but it shows its relation to the other assumptions we have made. When we have made our other assumptions, there is no need to assume this theorem *also*.

The following has often been given as the definition of an interval: 'An interval is the shortest distance between two points.' There are many objections to this as a definition: (i) An interval is a set of points, and so it cannot be a *distance*, for that is not a set of points. (ii) Suppose we amend the definition, and say, 'An interval between two points is that "curve" joining the points whose length is less than that of any other curve joining them.' We then have a true *theorem*, but we cannot use it as a *definition*, for it supposes we already know what is meant by the *length of a curve*. Now the latter is a very difficult thing to define, and in any case, the definition of the length of a curve will assume that we already know what is meant by an *interval*, so that the suggested definition of an interval involves a vicious circle.

THEOREM 35†.

If two sides of one triangle be equal to two sides of another, but the angle between the two sides in one triangle be greater than the corresponding angle in the other triangle, then the base of the first triangle is greater than the base of the second triangle.

Given In triangles ABC, XYZ ,

$$AB = XY, \quad BC = YZ, \quad \hat{ABC} > \hat{XYZ}$$

To prove that $AC > XZ$

Proof Since $\hat{ABC} > \hat{XYZ}$, there is a ray BD inside ABC , such that $\hat{ABD} = \hat{XYZ}$, and there is a point D on the ray, such that $BD = YZ$

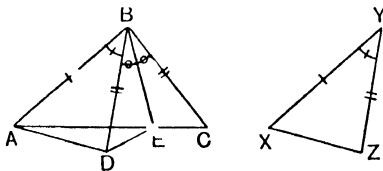


Fig 187

Then in $\triangle s ABD, XYZ$,

$$\begin{cases} AB = XY, & (\text{given}) \\ BD = YZ, & (\text{const}) \\ \hat{ABD} = \hat{XYZ}, & (\text{const}) \end{cases}$$

$\triangle s ABD, XYZ$ are congruent, (two sides, incl. $\angle$)

$$AD = XZ$$

If D lies on the line AC , then, clearly $AD < AC$ $XZ < AC$

If D is not on the line AC , bisect angle DBC by the ray BE , cutting AC in E . Join DE .

Then in $\triangle s DBE, CBE$,

$$\begin{cases} BD = BC, & (\text{const}). \\ BE = BE, \\ \hat{DBE} = \hat{CBE}, & (\text{const}). \end{cases}$$

$\triangle s DBE, CBE$ are congruent, (two sides, incl. $\angle$).

$$DE = EC$$

But $AE + ED > AD$. $AE + EC > AD$.

$$AC > AD. \quad AC > XZ.$$

Q.E.D.

† The only justification for giving this theorem and the next is that they are still set in Examinations

THEOREM 36.

If two sides of one triangle be equal to two sides of another triangle, but the third side of the one triangle is greater than the third side of the other, then the angle opposite the third side in the first triangle is greater than the angle opposite the third side in the other triangle.

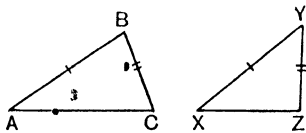


Fig 188

Given In triangles ABC , XYZ ,
 $AB = XY$, $BC = YZ$, $AC > XZ$.

To prove that $\hat{ABC} > \hat{XYZ}$.

Proof. Either $\hat{ABC} = \hat{XYZ}$, or $\hat{ABC} < \hat{XYZ}$, or $\hat{ABC} > \hat{XYZ}$

If $\hat{ABC} = \hat{XYZ}$, then $\triangle ABC$, XYZ are congruent, (two sides, incl $\angle$)

$AC = XZ$, contrary to the hypothesis

If $\hat{ABC} < \hat{XYZ}$, then $AC < XZ$ (previous theorem), contrary to the hypothesis.

Hence $\hat{ABC} = \hat{XYZ}$, and $\hat{ABC} < \hat{XYZ}$ are both false

$$\hat{ABC} > \hat{XYZ}.$$

Q.E.D.

EXAMPLE 27. Given D is a point inside $\triangle ABC$

To prove that

$$AD + DC < AB + BC, \hat{ADC} > \hat{ABC}.$$

Proof The interval AD , produced beyond D , will cut the side BC , in E , say.

Then, $DC < DE + EC$.

$$AD + DC < AE + EC$$

But

$$AE < AB + BE$$

$$AD + DC < AB + BE + EC.$$

$$AD + DC < AB + BC.$$

Also $\hat{ADC} > \hat{AEC}$, (ext $\angle$ of $\triangle DEC$, int opp).

$\hat{AEC} > \hat{ABC}$, (ext $\angle$ of $\triangle ABE$, int. opp.).

$$\hat{ADC} > \hat{ABC}.$$

Q.E.D.

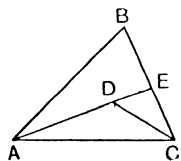


Fig 189

Note the following facts about inequalities, where a, b, c, d may mean the lengths of lines or the sizes of angles

(i) If $a > b$, then $b < a$

(ii) If $a > b$, then $a + c > b + c$

(iii) If $a > b + c$, then $a - c > b$.

(iv) If $a > b$ and $c > d$, then $a + c > b + d$

EXAMPLE 28 *P is any point inside triangle ABC.*

To prove that $PA + PB + PC$ is less than the perimeter and greater than the semiperimeter of the triangle (Fig. 190)

Proof.

$$PB + PC < AB + AC,$$

$$PC + PA < BC + BA,$$

$$PA + PB < CA + CB.$$

Adding, $2(PA + PB + PC) < 2(AB + BC + CA).$

$$\therefore PA + PB + PC < \text{perimeter}.$$

Also

$$PB + PC > BC,$$

$$PC + PA > CA,$$

$$PA + PB > AB.$$

Adding, $\therefore 2(PA + PB + PC) > AB + BC + CA.$

$$PA + PB + PC > \text{semiperimeter}.$$

Q.E.D.

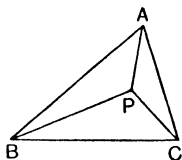


Fig. 190

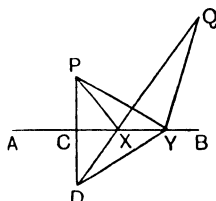


Fig. 191

EXAMPLE 29 *Two pipes are to be laid from given points to the same point of a straight road To find the point on the road so that the total length of the pipes is as small as possible*

Given. Two points P, Q on the same side of a line AB (Fig. 191)

To find the point X on AB so that $PX + XQ$ is as small as possible.

Construction Let C be the foot of the perpendicular from P on to AB , and let D be on the interval PC produced beyond C , so that $CD = PC$.

Let the lines QD, AB cut in X Let Y be any other point on AB .

Then $PX + XQ < PY + YQ$.

Proof. In $\triangle s XCP, XCD$,

$$\begin{cases} CX = CX, \\ CP = CD, & (\text{const}). \\ \hat{XCP} = \hat{XCD}, & (\text{rt. } \angle s). \end{cases}$$

$\triangle s XCP, XCD$ are congruent, (two sides, incl $\angle$).

$$PX = DX \quad \text{Similarly, } PY = DY.$$

$$PX + XQ = DX + XQ = DQ, \quad PY + YQ = DY + YQ.$$

But

$$DQ < DY + YQ.$$

$$PX + XQ < PY + YQ.$$

Q.E.D.

EXERCISE 43†

1. In an obtuse-angled triangle, the greatest side is that opposite the obtuse angle.

2. The greatest side of any triangle makes acute angles with the other sides.

3. ABC is a triangle with $AB > AC$, the bisectors of the angles B, C cut in I . Then $IB > IC$.

4. (a) In the quadrilateral $ABCD$, $AB > BC$, $AD > DC$. Prove $\hat{DCB} > \hat{DAB}$.

(b) If AD is the longest, BC the shortest side of the quadrilateral $ABCD$, prove $\hat{ABC} > \hat{ADC}$, $\hat{BCD} > \hat{BAD}$.

5. C, D are on a circle on opposite sides of the diameter AB , $AC > AD$. Prove $BC < BD$.

6. The altitudes of triangle ABC cut in H , if $AB > AC$, prove $BH > CH$.

7. ABC is a triangle, right-angled at B , D is on the side BC . Prove $AD < AC$.

8. A is on the interval BC ; OA is perpendicular to BC , $AC > AB$. Prove $OC > OB$.

9. (a) Of two chords of a circle, the greater subtends the greater angle at the centre.

(b) State and prove the converse of 9 (a).

10. (a) L is the mid-point of the side BC of triangle ABC ; $AB > AC$. Prove $\hat{ALB}$ is obtuse.

(b) Prove also $\hat{BAL} < \hat{CAL}$ [Complete the parallelogram $BACD$].

11. (a) The bisector of the angle BAC of triangle ABC cuts BC in D , prove $BA > BD$, $CA > CD$.

(b) If $AB > AC$, prove also $\hat{ADC}$ is acute, and $BD > DC$.

12. The difference between any two sides of a triangle is less than the third side.

13. AB is a diameter of a circle, centre O , C is on AB produced beyond B ; a line through C cuts the circle in X, Y . If $CX > CY$, prove $CA > CX$, $CB < CY$.

† It is better in this exercise to avoid where possible the use of parallel lines, and of the theorem that the sum of the angles of a triangle equals two right angles, and of such theorems as Pythagoras. We give a hint where this is not convenient.

14. L is a mid-point of side BC of triangle ABC . Prove

$$AB + AC > 2AL$$

[Complete the parallelogram $ABDC$.]

- 15 (a) The perimeter of a quadrilateral is greater than twice the longest diagonal

(b) It is greater than the sum of the diagonals.

- 16 (a) D is any point on the side BC of triangle ABC Prove

$$2AD + BC > AB + AC.$$

(b) Also $2AD < AB + AC + BC$

17. C is any point on the interval AB , P is any point. Prove

$$PA + PB + 2PC > 2AB$$

- 18 ABC is a triangle, P any point in space Prove that $PA + PB + PC$ is greater than the semiperimeter of the triangle

- 19 The sum of the diagonals of a convex quadrilateral is greater than the semiperimeter

20. (a) If P is any point in the plane of the quadrilateral $ABCD$, then $PA + PB + PC + PD >$ semiperimeter of the quadrilateral,

(b) and also $>$ sum of diagonals, unless the quadrilateral is convex, and P is the point where the diagonals cut

(c) If the quadrilateral is convex, then $PA + PB + PC + PD$ is a minimum, when P is at the cut of the diagonals

[Note. The problem of finding a point P inside a triangle ABC , such that $PA + PB + PC$ is a minimum, is much more difficult Fermat's problem See Part IV]

- 21 In triangle ABC , $AB > AC$. Prove that the right bisector of BC meets the side AB .

- 22 (a) D is any point on the side BC of the triangle ABC Prove that either $AD < AB$, or $AD < AC$

(b) If also E is any point on the side CA , then ED is less than the longest side of the triangle

(c) Any interval which has both its ends either on the sides of a triangle, or inside the triangle, is shorter than the longest side of the triangle

- 23 X, Y are inside the triangle ABC , $AXYC$ is a convex quadrilateral Prove that $AX + XY + YC < AB + BC$.

- 24 The internal and external bisectors of the angle BAC of triangle ABC cut BC in the points D, E resp., and they lie in order B, D, C, E . Prove that $AB > AC$, $CD < AC$, $CE > AC$, $DE > AE$, $DE > AD$.

25. (a) Of all parallelograms on the same base and with the same area, the rectangle has the least perimeter.

(b) Of all triangles on the same base and with the same area, the isosceles triangle has the least perimeter,

(c) and it also has the greatest vertical angle

*26. If P is a point inside an equilateral triangle, a triangle can be drawn whose sides are of lengths PA, PB, PC .

*27. If G is the centroid of triangle ABC , a triangle can be drawn whose sides are of lengths GA, GB, GC .

*28. If D, E, F be any points on the sides BC, CA, AB of triangle ABC then $AD + BE + CF > \frac{1}{2}s$ and $< \frac{3}{2}s$, where s is the semiperimeter of the triangle.

*29. In triangle ABC , $AB = AC$, D is on the side AC , E is on AB produced beyond B , $DC = EB$. Prove $ED > BC$.

§ 26. Tangents to circles.

Def. A 'tangent' to a circle is a line which meets the circle in just one point. The point is called the 'point of contact', the tangent is said to 'touch' the circle at the point.

Note on the words 'touch,' 'cut,' 'meet,' 'intersect' We have said that a tangent touches the circle, this word 'touch' should **never** be used except with regard to tangents. A line which meets the circle in two distinct points is said to 'cut,' or 'intersect,' the circle in these points, and the line itself is sometimes called a 'secant'. 'Meet' is a neutral word, covering both 'cut' and 'touch'.

THEOREM 37

A line drawn through a point on a circle perpendicular to the radius to the point is a tangent.

Given. A circle with centre O , A a point on a circle, $AT \perp OA$.

To prove that AT is a tangent at A .

Proof. By the definition of tangent, we must show that AT meets the circle in A only.

Suppose, if possible, that AT meets the circle again in B . Then $OA = OB$.

$$\therefore \angle OAB = \angle OBA.$$

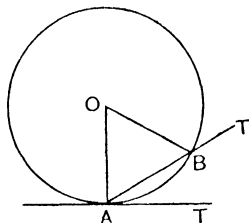


Fig. 192

But $\hat{OAB}$ is a rt. $\angle$, (given). $\therefore \hat{OBA}$ is a rt. $\angle$.

$\therefore$ two angles of $\triangle OAB$ are right angles, which is impossible.

$\therefore AT$ cannot meet the circle in any point except A .

$\therefore AT$ is a tangent to the circle at A , (def.) Q.E.D.

THEOREM 38

A tangent to a circle is perpendicular to the radius to the point of contact.

Given. AT is a tangent at A to a circle, centre O .

To prove that angle OAT is a right angle.

Proof. If possible, suppose angle OAT is not a right angle.

Draw a perpendicular from O to AT , the foot B of this perpendicular will be distinct from A .

Produce AB beyond B to C , making $BC = AB$.

Then in $\triangle s OBA, OBC$,

$$\begin{cases} OB = OB. \\ AB = CB, \quad (\text{const.}). \\ \hat{OBA} = \hat{OBC}, \quad (\text{rt. } \angle s). \end{cases}$$

$\therefore \triangle s OBA, OBC$ are congruent, (two sides, incl. $\angle$).

$\therefore OA = OC$.

$\therefore C$ is on the circle.

$\therefore$ the line AT meets the circle in another point besides A , which is impossible, since AT is a tangent.

Hence $\hat{OAT}$ is a right angle.

Q.E.D.

Cor. 1. Through a given point on a circle there is one and only one tangent to the circle.

Cor. 2. Every point on a tangent to a circle, except the point of contact, is outside the circle.

This follows at once from Theorem 33.

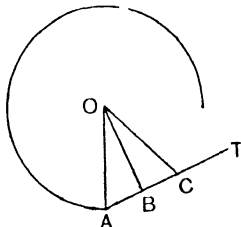


Fig 133

CONSTRUCTION 12

From a given point outside a given circle to draw the tangents to the circle.

Given A circle, centre O , and a point P outside it.

To construct the tangents to the circle which go through P .

Construction On OP as diameter describe a circle, let it cut the given circle in A and B .

Then PA , PB are tangents to the given circle

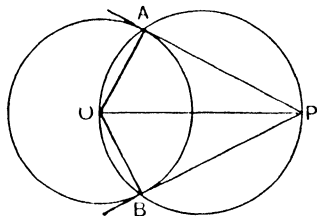


Fig. 194

Proof $\angle OAP$ is an angle in a semicircle, (const.)

$$OA \perp AP$$

$\therefore AP$ is a tangent to the circle at A , (tan. $\perp$ radius).

Similarly BP is a tangent to the circle at B

Q.E.D.

Cor. Two tangents can be drawn to a circle from an external point

THEOREM 39

The intervals along the two tangents from an external point to a circle have equal lengths.

Given The lines PA , PB touch a circle, centre O , at A , B resp

To prove that $PA = PB$.

Proof In $\triangle OAP$, OBP ,

$$\begin{cases} OA = OB, & (\text{radii}) \\ OP = OP, \\ \angle OAP \text{ and } \angle OBP \text{ are rt } \angle\text{s}, & (\text{tan. } \perp \text{ radius}). \end{cases}$$

$\triangle OAP$, OBP are congruent, (two sides and rt $\angle$)

$$\therefore PA = PB.$$

Q.E.D.

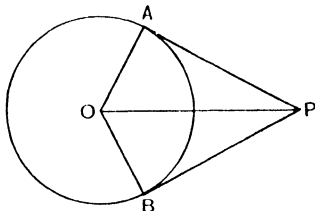


Fig. 195

Note We defined a tangent to a circle to be a *line* with a certain property—it is usual also to call an interval a tangent to a circle, when the line, of which it is a part, is a tangent, and one of its

end-points is on the circle. Thus Theorem 39 is usually stated as follows: *the tangents to a circle from an external point are equal*.

If the line PQ touches one circle in P and another in Q , then the line PQ is a 'common tangent' to the circles. The interval PQ is also called a common tangent.

EXERCISE 44.

1. P is at a distance of $3''$ from the centre of a circle of radius $1''$. How long is a tangent (interval) from P to the circle?

2. In Fig. 195, the radius $OA = 1''$, $OP = 2''$. What is the size of $\angle AOP$?

3. P is a point such that the tangent (interval) from P to a circle, of radius $1''$, is one inch long. What is the distance from P to the centre of the circle?

4. $\triangle ABC$ is an equilateral triangle inscribed in a circle. Prove that the tangents at A , B , C form an equilateral triangle.

5. (a) The lines PA , PB touch a circle, centre O , at A , B . Prove that $\angle POA = \angle POB$, $\angle OPA = \angle OPB$.

(b) Prove also AB is perpendicular to OP .

(c) Prove that $OAPB$ is a cyclic quadrilateral.

(d) Prove that $\angle OPA = \angle OAB$.

6. Tangents are drawn to concentric circles from a given point. Find the locus of their points of contact.

7. (a) Show how to draw a tangent to a circle parallel to a given line.

(b) Show how to draw a tangent to a circle perpendicular to a given line.

(c) Prove that the line joining the points of contact of two parallel tangents to a circle goes through the centre.

8. (a) A chord of given length moves so that its ends lie on a fixed circle. Prove that it always touches a fixed circle. Re-state this property, using the word 'envelope'. (Assume the chord is shorter than a diameter.)

(b) Through a given point draw a line such that its part within a given circle has a given length.

(c) Draw a chord of a given circle parallel to a given line and of a given length.

9. Tangents to a circle, centre O , at A , B , cut in P , a circle is drawn with centre P to pass through A , prove that its tangent at A goes through O .

If A, B, P are in one line, the circles through P with centres A, B resp. have the same tangent line at P , viz. the perpendicular to AB through P (see Figs. 196, 197)

Def. Two circles are said to *touch* at a point when they have the same tangent at that point, the point is called the '*point of contact*' of the circles.

Note The word 'touch,' in regard to circles, must **never** be used in any other sense. Two circles which 'meet' in a point, but do not touch there are said to '*cut*' or '*intersect*' at that point.

THEOREM 40.

If two circles touch, the line joining their centres goes through the point of contact

Given. Two circles, centres A, B , touching at P

To prove that A, B, P are collinear

Proof. Let PT be the common tangent to the circles at P , (def.).

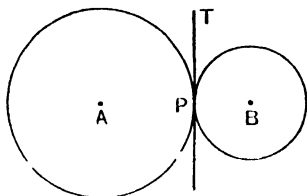


Fig. 196

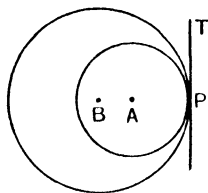


Fig. 197

Draw the perpendicular at P to PT

Since PT touches the circle, centre A , at P ,

$\therefore$ the perpendicular to PT at P goes through A , (tan $\perp$ radius).

Since PT touches the circle, centre B , at P ,

$\therefore$ the same perpendicular goes through B

$\therefore$ the same perpendicular goes through A, B, P .

$\therefore$ the points A, B, P lie on one line Q.E.D.

EXAMPLE 30. *If two circles touch at a point, they have no other point common.*

Given. Two circles, centres A, B , touching at P .

To prove that P is their only common point

Proof First, suppose that, if possible, they met also at a point C not on AB .

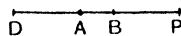


Fig. 198

Now A, B, P are collinear

Also, since P, C are on both circles, the centres A and B are on the right bisector of PC , which is impossible

Next, if the circles met also in a point D on AB , then both A and B would be the mid-points of the interval DP , which is impossible

Q.E.D.

Def Two circles are said to touch '*externally*,' if they touch, and lie on opposite sides of the tangent at the point of contact (Fig. 196).

Def Two circles are said to touch '*internally*,' if they touch, and lie on the same side of the tangent at the point of contact (Fig. 197).

EXERCISE 44 (continued)

10 (a) P moves on a given circle, PT is a tangent, the interval PT has a fixed length. Prove that T describes a concentric circle.

(b) Two circles are given. Find a point from which tangents to them have given lengths. How many such points are there?

(c) Find a point on a given line such that a tangent from it to a given circle has a given length. How many such points are there?

11 P, Q lie on concentric circles, centre C . Tangents at P, Q to the circles cut in T . Prove that P, Q, C, T lie on a circle.

12 A circle, centre O , goes through the centre P of a second circle, and cuts it in A, B . Prove that the tangents at A, B to the second circle cut on the first circle.

13 Two circles of radii $4''$, $5''$, touch (i) internally, (ii) externally. What is the distance between their centres?

14 Three circles whose centres are at A, B, C resp., touch each other externally. If $AB = 3''$, $BC = 5''$, $CA = 4''$, find their radii.

15 Two circles, centres A, B , touch externally, a circle, centre C , touches both internally. $AB = 3''$, $BC = 3''$, $CA = 4''$. Find the radii of the circles.

16 Two circles, of radii $4''$, $5''$, have their centres $12''$ apart. What is the radius of the smallest circle touching both?

17 (a) If two circles touch externally, the distance between their centres equals the sum of their radii, and if they touch internally, this distance equals the difference of their radii.

(b) Given a circle, show how to draw a circle with a given centre to touch the given circle (i) internally, (ii) externally.

(c) A moving wheel rolls round the outside of a fixed wheel. Prove that its centre describes a circle.

THEOREM 41

The locus of a point which moves so as to be equidistant from two given lines is the pair of lines bisecting the angles between the given lines.

(i) *Given.* OA, OB are two lines, P any point equidistant from them.

To prove that OP bisects one of the angles between the lines OA and OB

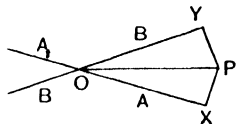


Fig 199

Proof. Let X, Y be the feet of the perpendiculars from P on to OA, OB resp

Then in $\triangle OPX, OPY$,

$$\begin{cases} OP = OP, \\ PX = PY, \text{ (given).} \\ \angle OXP \text{ and } \angle OYP \text{ are rt. } \angle s \end{cases}$$

$\triangle OPX, OPY$ are congruent, (two sides and rt $\angle$)

$$\angle XOP = \angle YOP.$$

OP bisects the angle XOY , which is one of the angles between the lines OA and OB .

Any point equidistant from OA and OB is on the bisector of an angle between OA and OB .

(ii) *Given.* P is any point on the bisector of one of the angles between OA and OB .

To prove that P is equidistant from OA and OB

Proof. Let X, Y be the feet of the perpendiculars from P on to OA, OB resp

Then in $\triangle OPX, OPY$,

$$\begin{cases} OP = OP, \\ \angle XOP = \angle YOP, \text{ (given)} \\ \angle OXP = \angle OYP, \text{ (rt. } \angle s). \end{cases}$$

$\triangle OPX, OPY$ are congruent, (one side, two angles).

$$PX = PY$$

Any point on the bisector of an angle between OA and OB is equidistant from OA and OB

Hence the locus is the pair of bisectors of the angles between OA and OB .

Q.E.D.

THEOREM 42

The three bisectors of the angles of a triangle are concurrent.

Given. ABC is a triangle.

To prove that the bisectors of its angles are concurrent

Proof. Bisect the angles at B and C , and let the bisectors cut in I .

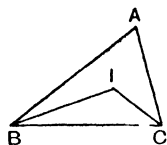


Fig 200

Then since I is on the bisector of $\hat{ABC}$,

$\therefore I$ is equidistant from AB and BC .

Since I is on the bisector of $\hat{ACB}$,

I is equidistant from AC and BC .

$\therefore I$ is equidistant from AB and AC .

I is on the bisector of $\hat{BAC}$.

$\therefore$ the three bisectors of the angles of the triangle cut in I .

Q.E.D.

CONSTRUCTION 13.

To draw the circle touching the three sides of a given triangle

Given triangle ABC

To construct the circle touching the sides AB , BC , CA .

Construction. Bisect the angles at B and C , and let the bisectors cut in I .

Let X , Y , Z be the feet of the perpendiculars from I on to the sides BC , CA , AB resp

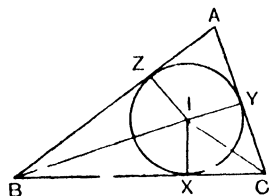


Fig 201

Then the circle, centre I , radius IX , is the circle required.

Proof. Since I is on the bisector of $\hat{ABC}$,

$\therefore IX = IZ$.

Since I is on the bisector of $\hat{ACB}$,

$\therefore IX = IY$.

$\therefore$ the circle, centre I , radius IX , goes through Y and Z .

And since the lines BC , CA , AB are resp. perpendicular to the radii IX , IY , IZ ,

$\therefore$ the circle touches these lines at X , Y , Z resp. Q.E.F.

CONSTRUCTION 14.

To draw a circle touching one side of a triangle and two sides produced.

Given triangle ABC .

To draw a circle touching the side BC and the sides AB and AC produced.

Construction. Produce AB beyond B to H , AC beyond C to K .

Bisect $\hat{CBH}$ and $\hat{BCK}$ and let the bisectors cut† in E .

Let X , Y , Z be the feet of the perpendiculars from E on the lines BC , CA , AB resp

Then the circle, centre E , radius EX , is the circle required.

Proof. Since E is on the bisector of an angle between BC and BA ,

$$\therefore EX = EZ.$$

Since E is on the bisector of an angle between BC and CA ,

$$\therefore EX = EY.$$

$\therefore$ the circle, centre E , radius EX , goes through Y and Z .

And since the lines BC , CA , AB are resp perpendicular to the radii EX , EY , EZ ,

$\therefore$ the circle touches these lines at X , Y , Z resp. Q.E.F.

Def. The circle which touches the sides of a triangle is called the ‘*incircle*,’ its centre the ‘*incentre*,’ its radius the ‘*inradius*’

Def. A circle which touches a side of a triangle and the other two sides produced, is called an ‘*excircle*,’ its centre an ‘*excentre*,’ its radius an ‘*exradius*.’ (There are three excircles.)

Note. If a triangle is not equilateral, its circumcentre, its incentre, its orthocentre (p. 111), and its centroid (p. 58) are all distinct points. A triangle has no ‘centre.’

† The bisectors cannot be parallel, for, if they were, then would

$$\hat{CBE} + \hat{BCE} = 2 \text{ rt } \angle\text{s} \quad \hat{CBH} + \hat{BCK} = 4 \text{ rt } \angle\text{s},$$

which is impossible, since $\hat{CBH}$, $\hat{BCK}$ are exterior angles of a triangle

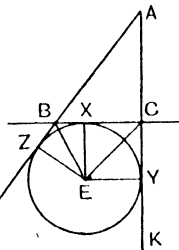


Fig. 202

Exercise Prove that these points coincide when the triangle is equilateral, and hence prove that the Π radius of such a triangle is one-half of the circumradius.

EXAMPLE 31 *The figure of the incentre and the centres of a triangle.*

Let I be the incentre, E_1, E_2, E_3 the centres of the circles touching resp. the sides BC, CA, AB . Produce BA beyond A to B_1 . (Fig. 203.)

Then AI bisects $\angle BAC$, AE_2 bisects $\angle CAB_1$

But $\angle BAC + \angle CAB_1 = 2 \text{ rt } \angle$ s,

$$\therefore \angle IAC + \angle E_2AC = 1 \text{ rt } \angle \quad \therefore IA \perp AE_2$$

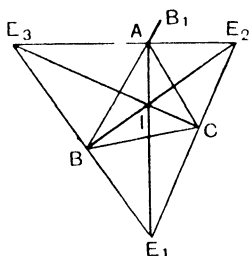


Fig. 203

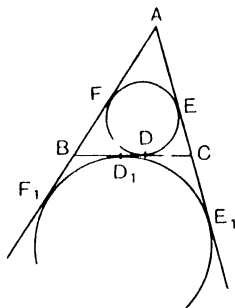


Fig. 204

Similarly,

$$IA \perp AE_3$$

E_2, A, E_3 are in one line

Similarly, the sides of triangle $E_1E_2E_3$ go through A, B, C

We have proved $IA \perp E_2E_3$. Similarly, $IB \perp E_1E_3, IC \perp E_1E_2$

I is the orthocentre of $\triangle E_1E_2E_3$

Exercise Start with $\triangle E_1E_2E_3$ and construct the rest of the figure

EXAMPLE 32 *Given* The incircle of $\triangle ABC$ touches BC, CA, AB in D, E, F resp. The circle which touches side BC , touches that side in D_1 , and sides AB, AC produced in F_1, E_1 resp. (Fig. 204)

To find the lengths of the intervals in the figure

Let the lengths of BC, CA, AB be a, b, c (Note that the side of length a is opposite the angle A , and so on)

Let $s = \frac{1}{2}(a + b + c)$, (s = semiperimeter)

Since the tangents from a point to a circle are equal,

$$AE = AF, \quad BF = BD, \quad CD = CE$$

and $AE + AF + BF + BD + CD + CE = \text{perimeter}$

$$s = AE + BD + CD$$

$$= AF + a$$

$$AF = s - a$$

(1)

Similarly, $BF = BD = s - b$, $CE = CD = s - c$. (ii)

Next $AF_1 = AE_1$, $BF_1 = BD_1$, $CE_1 = CD_1$,
 $2AF_1 = AF_1 + AE_1$
 $= AB + BF_1 + AC + CE_1$
 $= AB + BD_1 + AC + CD_1$
 $= a + b + c$
 $\therefore AF_1 = AE_1 = s$. (iii)

Lastly $BD_1 = BF_1$, $AF_1 = AB$, $s = c$,
 $BD_1 = s - c = CD$ (iv)

Similarly $CD_1 = s - b = BD$ (v)

Exercise Calculate all these lengths for the triangle whose sides are 3", 4", 5" long

EXERCISE 44 (continued)

18 Two circles touch externally at A , a common tangent touches them in B , C resp. Prove that the tangent at A bisects the interval BC , and hence that $\hat{BAC}$ is a right angle

Note When you are merely told that two circles touch, you must consider the case of internal *and* the case of external touching

19 (a) Two circles, centres O , P , touch at A , a line through A cuts them again in B , C . Prove that radii OB , PC are parallel (Fig. 205),

(b) and that AB , AC subtend equal angles at the respective centres

(c) Two circles touch at A , a tangent to one of them touches it at P , a parallel tangent to the other touches it at Q . Prove that either P , A , Q are collinear, or PA is perpendicular to QA .

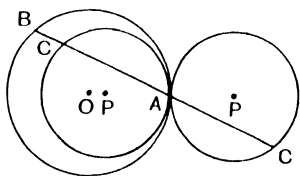


Fig. 205

20 Two circles, centres A , B , touch externally at T , a circle touching AB at T cuts the circles in P , Q resp. Prove that AP , BQ are tangents to the last circle

21 Two circles, centres O , P , touch at A , OX , PY are parallel radii. Prove that either $XA \perp YA$, or X , A , Y are collinear

22 (a) Two equal circles touch externally. Prove that all circles touching both have their centres on a straight line

(b) Two circles, centres A , B , touch externally, a circle, centre C , touches both externally. Prove that $AC \sim BC$ equals the difference between the radii of the circles

23 A circle, centre E , touches the side AB and the sides CA, CB produced of $\triangle ABC$. Prove $\hat{AEB} = 90^\circ = \frac{1}{2}\hat{ACB}$

24. O is the circumcentre, I the incentre of $\triangle ABC$. Prove $O\hat{A}I = 90^\circ - \frac{1}{2}B$ and hence $O\hat{A}I = (A/2 + B) - 90^\circ = \frac{1}{2}(B - C)$

25 I is the incentre of $\triangle ABC$. Prove that a circle, centre I , which cuts the sides of $\triangle ABC$, cuts off equal intervals from them

26 (a) $ABCD$ is a quadrilateral whose sides touch a circle. Prove $AB + CD = BC + AD$

(b) If the sides AB, CD touch the circle in E, F resp. prove $AE + DF = s - BC$, where s is the semiperimeter of the quadrilateral.

27 $ABCDFF$ is a hexagon whose sides touch a circle. Prove $AB + CD + EF = BC + DE + FA$

28 AB is a diameter of a circle, a tangent cuts the tangents at A, B in P, Q resp. Prove that area $ABQP = \frac{1}{2}AB \cdot PQ$

29 P is a point on the outer of two concentric circles, tangents PX, PY to the inner circle touch it at points X, Y , and cut the outer circle again at A, B . Prove $AB = 2XY$

30 (a) The incircle of $\triangle ABC$ touches the sides in D, E, F . Prove that three circles can be drawn, with centres A, B, C resp., to touch in pairs at D, E, F

(b) Given three points, show how to draw circles with these centres to touch each other in pairs

(c) Three circles with radii 1, 2, 3 inches touch each other externally. Prove that their centres are at the vertices of a right-angled triangle.

31 A circle is inscribed in $\triangle ABC$, show how to draw a circle to touch this circle and two sides of the triangle

32 AB, AC are radii of a circle, show how to draw a circle to touch this circle and AB, AC

33 $ABCD$ is a quadrilateral circumscribed to a circle, centre O . Prove $\hat{AOB} + \hat{COD} = 180^\circ$

34 A variable tangent to a circle, centre O , cuts two fixed tangents in A and B . Is the angle AOB constant?

35 PQ is a diameter of a circle, centre O , a tangent cuts the tangents at P, Q in A, B . Prove $PO^2 = PA \cdot QB$

36 r is the inradius of a right-angled triangle whose sides are a, b, c , c being the hypotenuse. Prove that $2r = a + b - c$

37 Two coins of radius 1" touch each other and touch the same line. Find the radius of the circle touching the line and the two coins. Use the Theorem of Pythagoras (Fig 206)

38. (a) Three equal coins of radius r touch each other. Find, in terms of r , the radius of the smallest disc which will just cover them.

(b) Find also, in terms of r , the radius of the largest disc which can be placed in the space between them.

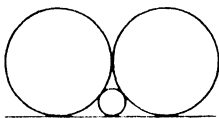


Fig 206

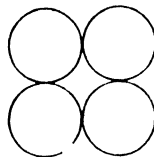


Fig 207

39 Four equal coins of radius r , with their centres at the vertices of a square, touch each other (Fig 207). Find the radius of the smallest disc which will just cover them.

40 A semicircle, centre O , radius r , is on diameter AB , other semicircles are described on diameters OA , OB . Find the radius of the small circle touching these semicircles (Fig 208)

***41** Two given circles touch internally at A , a variable line through A cuts them in P , Q . Prove that the locus of the mid-point of the interval PQ is an arc of a circle.

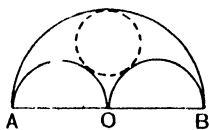


Fig 208

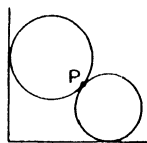


Fig 209

***42** Two (unequal) discs are placed in contact one above the other, the top one leaning against a wall. If the bottom one slips, find the locus of the mid-point of the interval between their centres (Fig 209)

***43** (a) Three circles touch externally in pairs at A , B , C , the lines AB , AC cut the circle through B , C in D , E . Prove that DE is a diameter of the latter circle.

(b) If X , Y , Z are the centres of the circles, then the triangle of XYZ is the circumcircle of ABC .

***44** ABC is a triangle inscribed in a circle, centre O , OX , OY are parallels to AB , AC resp. cutting the tangents at B , C in X , Y resp. Prove that XY touches the circle.

***45** D is the point of contact on the base BC of the incircle of $\triangle ABC$. Prove that the incircles of triangles ABD and ACD touch each other.

***46** A regular tetrahedron has an edge of length a , find its height. Three equal spheres of radius $1''$ lie on a table touching each other, an equal sphere is resting on them. Find the height of its centre above the table.

***47** Four equal spheres of radius $1''$ lie on a table touching each other in pairs, their centres being at the vertices of a square, an equal sphere rests on them. Find the height of its centre above the table.

****48** Six cylinders of radius 1 ft. and height 1 ft. are tightly packed in the smallest box which will hold them. How much of the box is unfilled?

****49** Equal spheres are packed together as tightly as possible. How many spheres does a sphere in the interior touch?

****50** How many spheres can be drawn to touch the four faces, produced if necessary, of a tetrahedron? (It is said that the first Archbishop Temple, when at school, answered this question at once.)

§ 27 Equal arcs and equal chords

THEOREM 13.

Equal chords of equal circles (or of the same circle) subtend equal angles at the centre, conversely, in equal circles (or the same circle), equal angles at the centre are subtended by equal chords.

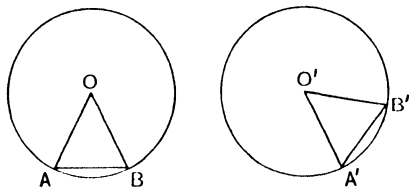


Fig. 210

(1) *Given.* AB and $A'B'$ are equal chords of the circles with centres O , O' , and equal radii.

To prove that $\angle AOB = \angle A'O'B'$.

Proof. In $\triangle AOB, A'O'B'$,

$$\begin{cases} OA = O'A', & (\text{equal radii}) \\ OB = O'B', & (\text{equal radii}) \\ AB = A'B', & (\text{given}) \end{cases}$$

$\therefore \triangle AOB, A'O'B'$ are congruent, (three sides)

$$\therefore \angle AOB = \angle A'O'B'.$$

(ii) *Given.* AB and $A'B'$ are chords of the circle with centres O, O' and equal radii, $\angle AOB = \angle A'O'B'$

To prove that $AB = A'B'$

Proof. In $\triangle AOB, A'O'B'$,

$$\begin{cases} OA = O'A', & (\text{equal radii}) \\ OB = O'B', & (\text{equal radii}) \\ \angle AOB = \angle A'O'B', & (\text{given}). \end{cases}$$

$\therefore \triangle AOB, A'O'B'$ are congruent, (two sides, incl. $\angle$)

$$AB = A'B'.$$

Q.E.D.

Cor. 1. Equal chords of equal circles (or of the same circle), subtend equal angles at the circumference, conversely, in equal circles (or the same circle), equal angles at the circumference are subtended by equal chords.

For the angle at the centre is double the angle at the circumference subtended by the same chord.

Cor. 2 In equal circles (or the same circle), supplementary angles at the circumference are subtended by equal chords.

At present we shall assume the following theorem as obvious. A 'proof' will be given in Part III.

THEOREM 44.

Equal arcs of equal circles (or of the same circle), subtend equal angles at the centre, conversely, in equal circles (or the same circle), equal angles at the centre are subtended by equal arcs.

Cor. 1 Equal arcs of equal circles (or of the same circle), subtend equal angles at the circumference, and conversely.

Cor. 2 In equal circles (or the same circle), equal chords cut off equal arcs, and conversely.

Proof of Cor. 2 (see Fig. 210). If $AB, A'B'$ be equal chords in equal circles with centres O, O' resp., then

$$\hat{AOB} = \hat{A'O'B'}.$$

are AB are $A'B'$ (Th 44).

Warning If AB, BC be equal arcs of the circle, centre O , then $\hat{AOC} = 2\hat{AOB}$, but the length of the chord AC is not twice that of the chord AB (Fig. 211)

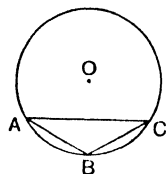


Fig. 211

EXERCISE 45

1 AB, BC, CD are equal chords of a circle, $\hat{ACB} = 20^\circ$. Find the size of $\hat{ADC}$

2 $A_1, A_2, \dots, A_8$ are eight points equally spaced round a circle. Find the sizes of the angles $A_1A_4A_5, A_1A_4A_6, A_1A_3A_7$.

3. AB and CD are perpendicular chords of a circle, the minor arc $BD = 2$ arc AC . Find the size of $\hat{ADC}$

4 AB and CD are chords of a circle inclined at an angle of 60° , the minor arc $BD = 3$ arc AC . Find the size of $\hat{ADC}$

5 PA, PB are tangents to a circle at A, B , $\hat{APB} = 70^\circ$, C is a point on the minor arc AB such that arc $AC = 2$ arc BC . Find the size of angle ABC .

EXAMPLE 33 Given Two circles cut in A, B , P moves on one circle, the lines PA, PB cut the other circle again in X, Y resp.

To prove that XY has a fixed length.

Proof. As P moves on the circle,

$\hat{APB}$ is constant, (angles in same segment on AB),

and $\hat{AYB}$ is constant, (angles in same segment on AB)

But $\hat{XAY} = \hat{APB} + \hat{AYB}$, (ext $\angle$ of $\triangle APY =$ sum of int opp $\angle$ s)

$\hat{XAY}$ is constant.

XY is a fixed length

Q.E.D.

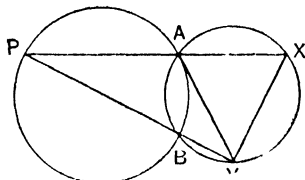


Fig. 212

Exercise With the hypotheses of the example, prove that XY envelopes a circle concentric with circle ABX . What happens when XY , in one position, is a diameter of circle ABX ?

EXAMPLE 34 Given I is the incentre of triangle ABC , AI cuts the circle ABC again in P

To prove that $PB = PC = PI$

Proof Since I is the incentre of triangle ABC ,

$$\angle BAI = \angle CAI$$

$$PB = PC, \quad (\text{equal chords subtend equal } \angle \text{ s})$$

Since I is the incentre of triangle ABC ,

$$\angle IBA = \angle IBC, \quad \angle IAB = \angle IAC$$

$$\angle BIP = \angle IBA + \angle IAB, \quad (\text{ext } \angle \text{ of } \triangle IAB = \text{sum of int. opp. } \angle \text{ s})$$

$$= \angle IBA + \angle IAC$$

$$= \angle IBC + \angle IAC, \quad (\angle IBC = \angle IAC, \text{ angles in same segment})$$

$$= \angle IBP$$

$$PB = PI, \quad (\text{isos. } \triangle)$$

Q.E.D.

EXERCISE 45 (continued)

6 A, B, C, D, E, F are on a circle, $\angle ABC = \angle DEF$, prove that arc $AC =$ arc DF

7 (a) A is a fixed point on a circle, X, Y move round the circle, $\angle XAY$ remains constant, prove that interval XY maintains a constant length

(b) and XY touches a fixed concentric circle

(c) A, B are fixed points on a circle, AX, BY are parallel chords. Prove that XY touches a fixed concentric circle

8 (a) $ABCD$ is a cyclic quadrilateral with $AB = CD$, prove that AD, BC are parallel

(b) State and prove the converse of (a)

(c) $ABCD$ is a trapezium inscribed in a circle, AD and BC are parallel. Prove that AB and CD subtend equal or supplementary angles at a point P on the circle

(d) With the hypothesis of (c), prove that the triangles ABC and DBC are congruent

(e) If AB, CD are equal arcs of a circle, prove that the triangles ABC and DBC are congruent

(f) AB, CD are equal chords of a circle which intersect inside the circle, prove that A, B, C, D are the vertices of a trapezium

(g) $\angle BCD$ is a quadrilateral inscribed in a circle, $AB = CD$, prove $AC = BD$

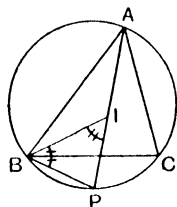


Fig. 213

9 (a) If ABC be a triangle inscribed in a circle, the bisector of the angle ABC goes through the mid-point of the arc AC opposite to B

(b) A, B are points on and C a point inside a circle, find a point P on the circle such that $\hat{APC} = \hat{BPC}$

10 (a) The right bisector of a chord of a circle goes through the mid-points of the arcs cut off by the chord

(b) The right bisector of the base of a triangle and the bisector of the vertical angle meet in a point on the circumcircle of the triangle (Fig. 214)

11 (a) A, B, C, D, E, F are on the same circle, $\hat{ABC} = \hat{DEF}$. Prove that A, C, D, F are the vertices of a trapezium

(b) AB, DE is a hexagon inscribed in a circle, AB is parallel to DE , and BC is parallel to EF . Prove that CA is parallel to FD

(c) AB, BC, CD, DE are equal chords of a circle. Prove that $AC = BD = CE$

12 A triangle ABC cut out in cardboard is moved about in a plane, so that its sides AB, AC always go each through a fixed point. Prove that any line through A , traced on the cardboard, also goes through a fixed point of the plane

13 (a) $ABCDE$ is a regular pentagon inscribed in a circle. Prove

$$\hat{EBC} = 2\hat{ABE} \text{ and } AB \parallel EC$$

(b) ABC is a triangle inscribed in a circle,

$$\hat{ABC} = \hat{ACB} = 2\hat{BAC},$$

the angles at B, C are bisected, and the bisectors are produced to cut the circle. Prove that the circumference is thus divided into five equal parts

14 (a) The bisector of an exterior angle of a quadrilateral inscribed in a circle, and the bisector of the interior opposite angle cut on the circle

(b) The bisectors of the opposite angles of a quadrilateral inscribed in a circle cut the circle at the ends of a diameter

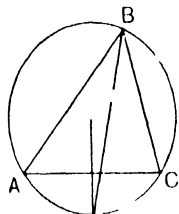


Fig. 214

EXAMPLE 35 Given L, M are the mid-points of arcs AB, AC of a circle, these arcs do not overlap, LM cuts the lines AB, AC in E, F

To prove that $AE = AF$

Proof Since ext. $\angle$ of $\triangle EBM =$ sum of int opp. $\angle$ s,

$$\therefore \hat{AEM} = \hat{EBM} + \hat{EMB},$$

$\therefore \angle$ subtended at circumf by arc AM

$+ \angle$ subtended at circumf by arc LB

" $\angle$ subtended at circumf by arc MC

$+ \angle$ subtended at circumf by arc LA ,

(equal $\angle$ s subtended by equal arcs)

$$= \hat{MLC} + \hat{ACL}$$

$$= \hat{AFL}, (\text{ext } \angle \text{ of } \triangle FCL = \text{sum of int opp } \angle \text{s})$$

$$\therefore \hat{AEF} = \hat{AFE} \quad AE = AF, (\text{isos } \triangle) \quad \text{Q.E.D.}$$

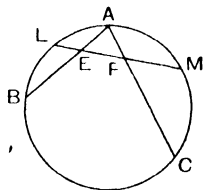


Fig 215

Note This illustrates the method of dealing with angles not at the centre or circumference we relate them to angles at the centre or circumference

15. (a) AB, LM are chords of a circle, cutting in E Prove $\hat{AEM} = \frac{1}{2}$ sum of $\angle$ s subtended at centre by certain arcs AM and LB

(b) If AB, LM are chords of a circle, perpendicular to each other, then are $AM \perp$ are LB half circumference Which arcs must be taken

(c) Let N be the mid-point of arc BC in Fig 215 Prove that AN, BM, CL are bisectors of the angles of $\triangle ABC$ and are perpendicular to the sides of $\triangle LMN$

(d) The incentre of $\triangle ABC$ is the orthocentre of $\triangle LMN$

*(e) If a.c. $AL \perp$ are MC are $BL \perp$ are AM , prove $AE = AF$

16 $\angle ABC$ is a right angle Show that the line from B to the centre of the square described on AC , outside the triangle, bisects the angle ABC

17 Two equal circles cut in A, B , through B is drawn a line cutting them again in P, X resp. Prove $AP = AX$

18. The altitudes of $\triangle ABC$ from the vertices A, B cut the circumcircle in X, Y resp. Prove that C is the mid-point of the arc XY

19 $ABCD$ is a trapezium inscribed in a circle, $AD \parallel BC$, the tangents at B, C cut in T , the diagonals cut in X Prove $TX \perp AD$ and $\hat{BTX} = \hat{CTX}$.

20 (a) P is on the circumcircle of $\triangle ABC$, lines from P perpendicular to AB , AC cut the circle again in Z , Y . Prove $ZY = BC$ (Fig. 216)

* (b) Hence show that, if the perpendicular from P to BC cuts the circle again in X , then triangles ABC , XYZ are congruent, and AX , BY , CZ are parallel.

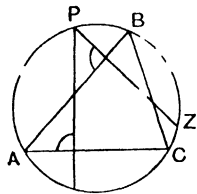


Fig. 216

21 P moves on a circle, lines are drawn from P perpendicular to two fixed diameters of the circle, they cut the circle again in Q , R . Prove that QR touches a fixed circle. State an exceptional case.

22 P moves on a circle, centre O , PA is a line perpendicular to a fixed diameter. Prove that the bisector of $\angle OPA$ goes through one of two fixed points.

EXAMPLE 36 Given The bisector of angle BAC of triangle ABC cuts the circle ABC again in X , E , F are the feet of perpendiculars from X on to AB , AC .

To prove that $BE = CF$

Proof Since $\angle BAX = \angle CAX$, (given)

$BX = CX$, (equal $\angle$ s subtended by equal chords)

Since X is on the bisector of $\angle BAC$,

$$EX = FX$$

In $\triangle BEX$, CFX ,

$$\begin{cases} BX = CX, & (\text{proved}) \\ EX = FX, & (\text{proved}) \\ \angle BEX \text{ and } \angle CFX \text{ are rt } \angle\text{s}, & (\text{given}) \end{cases}$$

$\triangle BEX$, CFX are congruent, (two sides and rt $\angle$)

$$BE = CF$$

Cor. $AE + AF = AB + AC$

Proof If E falls between A and B , then F does not fall between A and C †. For suppose E fell between A and B , and also F between A and C .

Then $\angle EXA > \angle BXA$, (ext $\angle$ of $\triangle EBX$ = int opp $\angle$)

$\angle FXA < \angle CAX$, (ext $\angle$ of $\triangle FCX$ < int opp $\angle$)

$$\angle EXA + \angle FXA > \angle BXA + \angle CAX$$

$$= 2 \text{ rt. } \angle\text{s}, \text{ (opp } \angle\text{s of cyclic quad)}$$

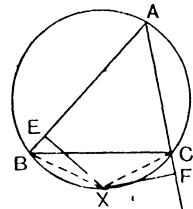


Fig. 217

† This vital step should be carefully noted

But as $\hat{A}E\hat{X}$ and $\hat{A}F\hat{X}$ are right angles, this is impossible

If E is between A and P , F cannot be between A and C , and F must fall on AC produced beyond C , for we shall assume as obvious that F cannot fall on AC produced beyond A .

Then $AE + AB = BE$, $AF = AC + CF$. But $BE = CF$.

$$AE + AF = AB + BE + AC + CF = AB + AF.$$

Similarly, if E is on AB produced beyond P , then F cannot be on AC produced beyond C , but must fall between A and C . Then the same result follows.

EXAMPLE 37 Consider the following fallacious proof that *all triangles are isosceles*

Proof Let ABC be any triangle, let the bisector of $\hat{BAC}$ and the right bisector of BC cut in X . Let Y, Z be the feet of the perpendiculars from X to AB, AC resp.

(i) Suppose, if possible, that X lies inside the triangle. Prove $BX = CX$, then that $\triangle XAY, \triangle XAZ$ are congruent. Hence $AY = AZ$, and $XY = XZ$. Now prove $\triangle BYX$ and CZX are congruent, and hence $BY = CZ$. Then since $AY = AZ$ and $BY = CZ$ we have $AB = AC$.

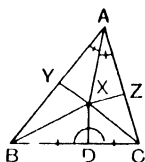


Fig. 218

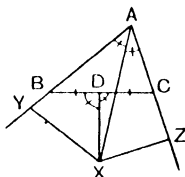


Fig. 219

(ii) Suppose X lies outside the triangle. With the figure shown (Fig. 219), prove as in part (i) that $AB = AC$.

Discover the fallacy in part (i) and in part (ii) of this proof.

(iii) Prove that, if the bisector of angle BAC and the right bisector of BC are assumed to have the mid-point of BC common, or to be parallel, then these lines coincide, and the triangle is isosceles. This part of the proof should not be fallacious.

(iv) Are any other cases thinkable?

(v) Where *must* the right bisector of the base and the bisector of the vertical angle cut each other?

Exercise AX bisects the angle BAC , circles are drawn through A, X , they cut AB in E and AC in F . Prove $AE + AF$ is constant.

***23** P, Q are on a circle, centre O , through A, B , BP and AQ cut on the circle AOB . Prove

$BP = AQ, AB \perp PQ$ (Fig. 220)

***24** ABC is a triangle inscribed in a circle, AX, BY are chords bisecting the angles at A, B , if $AX = BY$, prove either $CA = CB$ or $\hat{ACB} = 60^\circ$.

***25.** $ABCD$ is a cyclic quadrilateral. Prove that the bisectors of its angles cut the circumcircle in the vertices of a rectangle.

***26** ABC is a triangle inscribed in a circle, centre O , D is the mid-point of minor arc BC . Prove $\hat{ADO} = \frac{1}{2}(\hat{B} + \hat{C})$.

***27.** Two equal circles cut in A, B , a circle, centre A , cuts them in P, Q resp. on the same side of AB . Prove that B, P, Q are collinear.

***28** C is a point on circle ADE , the circle, centre C , through A cuts the circle ADE again in E , DA cuts the circle, centre C , in B . Prove $BE \perp CA$.

***29** A wheel rolls within another twice its size. Prove that each point of the small wheel describes a diameter of the large wheel.

***30** (a) If a quadrilateral be inscribed in a circle, and three of its sides be parallel to three given lines, the fourth is parallel to a given line.

(b) State and prove the corresponding theorem for a hexagon, octagon, etc.

***31.** If a pentagon be inscribed in a circle, and four sides be parallel to given lines, the fifth side has a fixed length.

EXAMPLE 38 Given $ABCD$ is a convex quadrilateral. To draw a square each of whose sides goes through one of the points A, B, C, D .

Consider the two sides of the square, one of which has to go through A and the other through B . If X is the common end-point of these sides, it lies on a semicircle on AB as diameter (for the $\angle$ in a semicircle is a rt $\angle$).

The bisector of angle AXB goes through the mid-point P of the other half of the circle AXB . Hence a diagonal of the square must go through P .

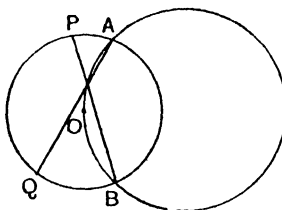


Fig. 220

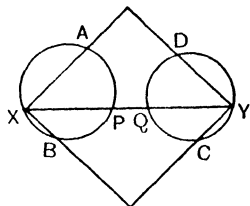


Fig. 221

Similarly, if Y is the common end-point of the sides of the square which pass through C, D resp., then the bisector of the angle CYD must go through the mid-point Q of the semicircle on CD opposite to Y .

But the bisectors of opposite angles of a square lie along the same diagonal.

Hence, if P, Q are distinct points, then PQ gives a diagonal of the square required, and the other points where the line PQ cuts the circles are opposite vertices of the square required.

When we have two opposite vertices of a square, it is a simple matter to complete the construction of the square. Q.E.D.

Note. If P and Q coincide, or the figure differs much from Fig. 221, it may not be possible to draw any square to satisfy the conditions, or it may even be possible to draw an infinite number.

§ 28. The alternate segment.

Explanation. If ABC be a segment of a circle on the chord AC , and ACD be an angle, such that B, D are on opposite sides of AC , then the segment ABC is said to be 'alternate' to the angle ACD .

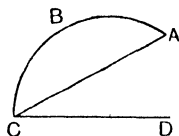


Fig. 222

THEOREM 15

If the angle between a chord of a circle and a ray through an end-point of the chord equals an angle in the alternate segment, then the ray is along a tangent to the circle at the end of the chord.

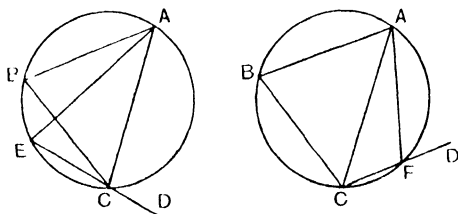


Fig. 223

Given. AC is a chord of a circle, ACD an angle equal to an angle ABC in the alternate segment of the circle.

To prove that the line CD touches the circle.

Proof If CD be not a tangent, it will cut the circle again, either in a point E on the same side of AC as B , or in a point F on the opposite side of AC from B

In the first case,

$$\angle ABC = \angle AEC, \quad (\angle s \text{ in same segment})$$

$$\text{But} \quad \angle ABC = \angle ACD, \quad (\text{given})$$

$$\therefore \angle AEC = \angle ACD,$$

which is impossible, (ext. $\angle$ of $\triangle AEC >$ int. opp. $\angle$).

In the second case,

$$\angle AFD = \angle ABC, \quad (\text{ext. } \angle \text{ of cyclic quad.} = \text{int. opp. } \angle)$$

$$\text{But} \quad \angle ABC = \angle ACD, \quad (\text{given})$$

$$\angle AFD = \angle ACD,$$

which is impossible, (ext. $\angle$ of $\triangle ACF >$ int. opp. $\angle$)

Hence CD cannot meet the circle again

CD is a tangent

Q.E.D.

THEOREM 46

An angle between a chord and a tangent to a circle at an end of the chord equals the angles in the alternate segment.

Given AB is a chord of a circle, BC a tangent at B .

To prove that $\angle ABC$ equals the angles in the alternate segment on AB

Proof Let D be a point on the circle on the side of the line AB opposite to C

We have to prove that $\angle ABC = \angle ADB$

If $\angle ABC$ is not equal to $\angle ADB$, draw a ray BE so that E, D are on opposite sides of AB , and $\angle ABE = \angle ADB$

Then BE touches the circle ABD at B , (previous theorem)

But the line BC , which is distinct from BE , also touches the circle ABD at B , (given),

which is impossible, (only one tan. at a given pt.). Q.E.D.

We give an alternative proof which does not assume Theorem 45

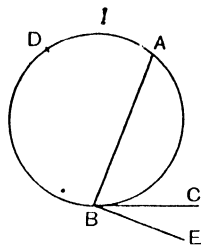


Fig. 224

THEOREM 46 (Second proof)

Given. AB is a chord of a circle, BC' a tangent at B

To prove that $\hat{ABC}$ equals the angles in the alternate segment on AB

First, suppose $\hat{ABC}$ is acute.

Draw the diameter BD

Then $\hat{ADB}$ is an angle in the segment alternate to $\hat{ABC}$.

Now

$$\hat{DAB} = 1 \text{ rt } \angle, \quad (\angle \text{ in semicircle})$$

$$\hat{ABD} + \hat{ADB} = 1 \text{ rt } \angle, \quad (\text{sum of } \angle\text{s of } \triangle ABD = 2 \text{ rt } \angle\text{s}).$$

$$\text{But } \hat{ABD} + \hat{ABC} = 1 \text{ rt } \angle, \quad (\text{tan } \perp \text{ radius})$$

$$\hat{ABD} + \hat{ADB} = \hat{ABD} + \hat{ABC},$$

$$\hat{ADB} = \hat{ABC}.$$

$\hat{ABC}$ = the angles in segment ADB .

Next, suppose $\hat{ABC}$ is obtuse, and let $\hat{AFB}$ be any angle in the alternate segment

Draw the diameter BD

Take E on line CB so that B is between E and C .

Then $BDFA$ is a cyclic quad

$$\therefore \hat{BDA} + \hat{AFB} = 2 \text{ rt } \angle\text{s}.$$

Now

$$\hat{ABE} + \hat{ABC} = 2 \text{ rt } \angle\text{s}, \quad (\text{adj } \angle\text{s}).$$

$$\therefore \hat{ABE} + \hat{ABC} = \hat{BDA} + \hat{AFB}.$$

But since $\hat{ABE}$ is acute,

$$\therefore \hat{ABE} = \hat{BDA}, \quad (\text{already proved})$$

$$\therefore \hat{ABC} = \hat{AFB}.$$

Lastly, if $\hat{ABC}$ is a right angle, the alternate segment is a semicircle, and the angles in a semicircle are right angles.

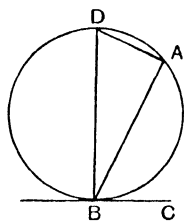


Fig 225

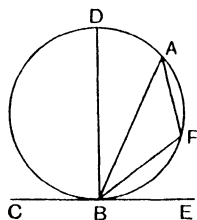


Fig 226

EXAMPLE 39 Two circles cut in A, B . Through P , a point on one are drawn lines PA, PB cutting the other in X, Y resp. Prove that XY is parallel to the tangent at P .

Proof Let PT be the tangent at P , where T is on the opposite side of the line AP from B .

Then

$$\hat{TPA} = \hat{PBA}, \quad (\text{alt. segment})$$

But $\hat{PBA} = \hat{AXY}$, (ext $\angle$ of cyclic quad - int opp $\angle$).

$$\hat{TPA} = \hat{AXY}.$$

$$TP \parallel XY, \quad (\text{alt } \angle s)$$

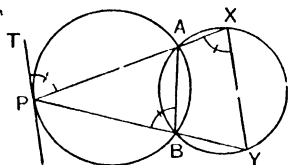


Fig 227

EXERCISE 46.

1 The incircle of $\triangle ABC$ touches its sides in X, Y, Z . The angles at X, Y of $\triangle XYZ$ are $30^\circ, 80^\circ$. What are the angles of $\triangle ABC$?

2 If in question 1, the angles at A and B of $\triangle ABC$ were $30^\circ, 40^\circ$, what are those of $\triangle XYZ$?

3 $ABCD$ is a quadrilateral circumscribed to a circle. Its sides touch the circle in X, Y, Z, W . The angles at A, B, C are $40^\circ, 90^\circ, 60^\circ$. Find those of the quadrilateral $XYZW$.

4 Find the angles at which the diagonals of the quadrilateral $XYZW$ cut in question 3.

5 (a) AB, AC are chords of a circle, AT is a tangent, AB bisects the angle TAC . Prove $AB = BC$.

(b) In triangle ABC , $AB = BC$, AB bisects angle CAD . Prove that AD touches circle ABC .

6 (a) If the chord AB of a circle is parallel to the tangent at C , then $AC = BC$.

(b) AB, AC are equal chords of a circle, a parallel to the tangent at A cuts them in D, E . Prove $AD = AE$.

7 (a) Tangents at A, B to a circle cut in T . Prove that the bisectors of angle BAT cut the arcs AB in their mid-points.

(b) Prove that the incentre of $\triangle ABT$ is on the minor arc AB .

8 ABC is a triangle inscribed in a circle, the bisector of angle ABC cuts the circle again in D , the tangent at D cuts the lines BA, BC in E, F . Prove $EDA = FDC$.

9 AT touches the circle ABC at A , through B is drawn a parallel to AT cutting AC in D . Prove that AB touches the circle BDC at B .

10 (a) Two circles touch at A . Through A are drawn two lines cutting one circle in P, Q , the other in X, Y . Prove $PQ \parallel XY$.

(b) Prove that triangles APQ, AXY are similar.

11 AB, CD are parallel chords of a circle, AD, BC cut in O . Prove that circles OAB, OCD touch at O .

12 Two circles touch at P , through P is drawn a line cutting them in B, C resp. Prove that the tangents at B, C are parallel.

13 APB and AQB are segments of two circles on the same side of the chord AB , B, P, Q are in one line. Prove $\angle PAQ$ equals an angle between the tangents at A .

14 OA, OB are perpendicular radii of a circle, C is on the radius OA , BC cuts the circle again in D . Prove that circle ACD touches BA at A .

15 AB is a diameter of a circle, centre O , the chords BC, AD cut in E . Prove that OC touches the circle CDE .

16 The tangent at P to a circle, centre O , cuts the radius OA produced beyond A at B , C is the foot of the perpendicular from P to OA . Prove that PA bisects angle BPC .

17 AT is a tangent to a circle at A , AB is a chord, a parallel through T to AB cuts the circle in C, D . Prove that triangles ACT and ABD are similar. Take C between T and D .

18 Two circles cut in A, B , the tangents at A cut them again in C, D . Prove that AB bisects the angle CBD . (Fig. 228)

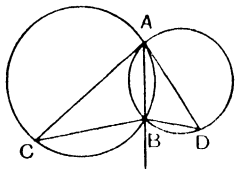


Fig. 228

19 Two circles $AHIB, AGB$ cut in A, B , the tangent at B to circle AGB cuts circle $AHIB$ in C, G , A, H are in one line. Prove that BG and CH are parallel.

20 The tangent at C to a circle cuts the chord AB , produced, in D , the circle, centre D , through C cuts chord AB in E . Prove $\angle ECA = \angle ECB$.

21 $\angle ABC$ is a right angle, the bisector of angle ACB cuts AB in E , D is the foot of the perpendicular from A on to CE . Prove that AD touches circle AEC .

22 (a) Tangents to a circle at B, C cut in T , A is on the major arc BC . Find $\angle BTC$ in terms of $\angle BAC$.

(b) The incircle of $\triangle ABC$ touches its sides in D, E, F . Find the angles of $\triangle DEF$ in terms of those of $\triangle ABC$.

23. The tangent at A to a circle cuts the chord BC , produced, in T . Prove that triangles TAC, TBA are similar.

24 (a) Two circles cut in A, B , the tangents at A to each circle cut the other in X, Y resp., X, B, Y are collinear. Prove $\angle XAY$ is a right angle.

(b) Prove that $\triangle ABX$ and $\triangle ABY$ are similar, even if X, B, Y are not collinear.

25 (a) Two circles touch internally at T , a line cuts one in A, D , the other in B, C . Prove $\angle ATB = \angle CTD$.

(b) If a line touches the inner at P and cuts the outer in Q, R , prove that $\angle QTP = \angle RTP$.

26 Two circles, centres A, B , cut in P . Prove that angle $\angle APB$, equals one of the angles between the tangents at P to the circles.

27 ABC is a triangle. Show how to draw a circle with its centre on AB to touch AC and go through B .

EXAMPLE 40 $ABCD$ is a quadrilateral whose sides touch a circle in X, Y, Z, W , XZ, WY cut in P . Prove

$$\angle XPW = \frac{1}{2} (\angle ADC + \angle ABC)$$

Proof $\angle XPW = \angle P\hat{W}Z + \angle P\hat{Z}W$

$$= \angle Y\hat{Z}C + \angle A\hat{X}W, \quad (i)$$

(since $\angle P\hat{W}Z = \angle Y\hat{Z}C, \angle P\hat{Z}W = \angle A\hat{X}W$, alt. seg.)

But $\angle CZ = \angle CY$, (tang. from point equal)

$$\angle Y\hat{Z}C = \angle Z\hat{Y}C$$

And $\angle Y\hat{Z}C + \angle Z\hat{Y}C + \angle Z\hat{C}Y = 2 \text{ rt } \angle s$

$$\angle Y\hat{Z}C = \frac{1}{2} (2 \text{ rt } \angle s - \angle Z\hat{C}Y)$$

Similarly, $\angle A\hat{X}W = \frac{1}{2} (2 \text{ rt } \angle s - \angle X\hat{C}W)$

$$\angle Y\hat{Z}C + \angle A\hat{X}W = \frac{1}{2} (2 \text{ rt } \angle s - \angle Z\hat{C}Y - \angle X\hat{C}W)$$

$$= \frac{1}{2} (\angle ADC + \angle ABC), \quad (\text{sum of } \angle s \text{ of quad})$$

$$= \frac{1}{2} (2 \text{ rt } \angle s)$$

$$\text{by (i), } \angle XPW = \frac{1}{2} (\angle ADC + \angle ABC)$$

28 $ABCD$ is a quadrilateral inscribed in a circle, its sides touch another circle in X, Y, Z, W resp. Prove that XZ, YW are at right angles.

***29** P, P' are the points of contact of tangents from a point T to a circle, Q, Q' are the points of contact of tangents from T to a concentric circle. Prove that $PQ, P'Q'$ make equal angles with the tangent at P . (Prove that P, Q, P', Q' and the common centre lie on a circle.)

***30** Tangents at C, D to a circle $AC'DB$ on diameter AB , cut in T , the chords BC, AD cut in E . Prove that T is the centre of circle ECD .

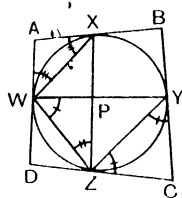


Fig. 229

***31** Two equal circles touch at T , TA , TB are perpendicular chords, one of each circle. Prove that interval AB is equal and parallel to the interval between the centres.

***32** The line PAQ touches a circle at A , AB is a diameter, the lines PB , QB cut the circle again in C , D . Prove that BCD and BQP are similar triangles. (Fig. 230)

***33** Two circles cut in A , the tangents at A to the circles cut them again in B and C , the line BC' cuts the circles again in X and Y . Prove $AX = AY$.

***34** L , M , N are the mid-points of the sides BC , CA , AB of $\triangle ABC$. Prove that the tangent at L to the circle LMN is parallel to the tangent at A to the circle ABC .

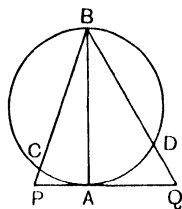


Fig. 230

***35** Two circles cut in A , B , through A are drawn two lines cutting one circle in P , Q , the other in X , Y . Prove that the angles between the lines PQ and XY equal the angles between the tangents to the circles at A .

***36** A circle through B , C cuts the sides AB , AC of $\triangle ABC$ in D and E , O is the circumcentre of $\triangle ABC$. Prove $OA \perp DE$.

***37** $ABCD$ is a cyclic quadrilateral whose diagonals cut in E , O is the circumcentre of $\triangle CED$. Prove $OE \perp AB$.

***38** $ABCDE$ is a pentagon inscribed in a circle, AB is parallel to DE , BC is parallel to AE . Prove CD is parallel to the tangent at A .

***39** C is the centre, CA a radius of a circle, B is a point on a perpendicular radius, the line AB cuts the circle again in D , the tangent at D cuts CB in E . Prove $BE = DE$. (Fig. 231)

***40** AB , CD are perpendicular diameters of a circle, centre O , the tangent at P to the circle cuts the line CD in E , while AP and BP cut this line in X , Y resp. Prove $EX = EY$. (Fig. 232)

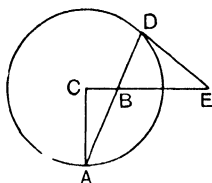


Fig. 231

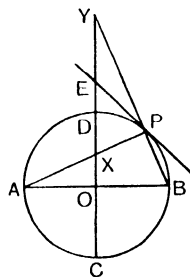


Fig. 232

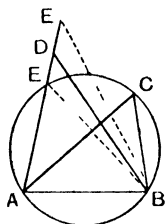
*41 The diameter AB of a circle is produced beyond B to C so that $BC = \text{radius}$, D is the point of contact of a tangent from C to the circle, the tangent at B cuts CD in E and AD in F . Prove DEF is an equilateral triangle

*42 Tangents at B, C to the circle ABC cut in T . Prove that the parallel through T to AC cuts AB on the diameter perpendicular to CT .

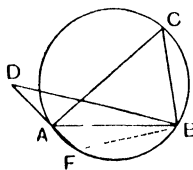
§ 29. Tests for concyclic points.

THEOREM 17.

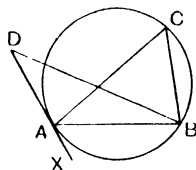
If an interval joining two points subtends equal angles at two points on the same side of it, then the four points are cyclic.



Case (i)



Case (ii)



Case (iii)

Fig. 233

Given. $\angle ACB, \angle ADB$ are equal angles, C, D lying on the same side of AB .

To prove that A, B, C, D are cyclic.

Proof. Draw the circle ABC , if possible, suppose the circle does not go through D . Then *either* it cuts the line AD on the interval AD , *or* on that interval produced beyond D , *or* on that interval produced beyond A , *or* the circle ABC touches the line AD at A .

Case (i). If the circle ABC cuts the line AD at a point E on the interval AD , or on that interval produced beyond D , then

$$\angle AEB = \angle ACB, \quad (\angle s \text{ in same segment}).$$

$$\text{But} \quad \angle ACB = \angle ADB, \quad (\text{given}).$$

$$\therefore \angle AEB = \angle ADB,$$

which is impossible, (ext. $\angle$ of $\triangle$ greater than int. opp. $\angle$).

Case (i). If the circle ABC cuts the interval AD produced beyond A in F , then $ACRF$ is a cyclic quadrilateral.

$\therefore \hat{ACB}$ and $\hat{AFB}$ are supplementary, (opp. $\angle$ s).

But $\hat{ACB} = \hat{ADB}$, (given).

$\therefore \hat{ADB}$ and $\hat{AFB}$ are supplementary,

which is impossible, (two $\angle$ s of Δ cannot be supp.).

Case (ii). If the circle ABC touches AD at A , and DA is produced beyond A to X , then

$\hat{XAB} = \hat{ACB}$, (alt. seg.).

But $\hat{ACB} = \hat{ADB}$, (given)

$\therefore \hat{XAB} = \hat{ADB}$,

which is impossible, (ext $\angle$ of Δ greater than int. opp. $\angle$)

Hence the circle ABC goes through D . Q E D.

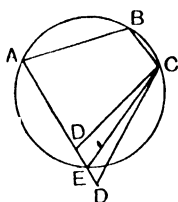
THEOREM 48

If the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

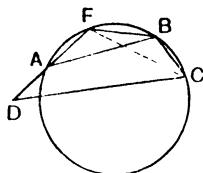
Given. $ABCD$ is a quadrilateral, the angles at B, D are supplementary.

To prove that $ABCD$ *is cyclic*

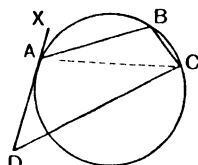
Proof. Draw the circle ABC , and suppose, if possible, that it does not go through D . Then either it cuts the interval AD , or that interval produced beyond D , or that interval produced beyond A , or the circle ABC touches the line AD in A .



Case (i)



Case (ii)



Case (iii)

Case (i). If the circle ABC cuts the interval AD , or that interval produced beyond D , in E , then $ABCE$ is cyclic

$\therefore \hat{ABC}$ and $\hat{AEC}$ are supplementary, (opp $\angle$ s)

But $\hat{ABC}$ and $\hat{ADC}$ are supplementary, (given)

$$\therefore \hat{AEC} = \hat{ADC},$$

which is impossible, (ext. $\angle$ of Δ greater than int. opp. $\angle$).

Case (ii). If the circle ABC cuts the interval AD , produced beyond A , in F , then F, B are on the same side of AC

$\therefore \hat{AFC} = \hat{ABC}$, (same segment)

But $\hat{ADC}$ and $\hat{ABC}$ are supplementary, (given),

$\hat{ADC}$ and $\hat{AFC}$ are supplementary,

which is impossible, (two $\angle$ s of Δ cannot be supp.).

Case (iii). If the circle ABC touches the line AD at A , then

$$\hat{DAC} = \hat{ABC}, \text{ (alt. segment)}$$

But $\hat{ABC}$ and $\hat{ADC}$ are supplementary, (given),

$\therefore \hat{DAC}$ and $\hat{ADC}$ are supplementary,

which is impossible, (two $\angle$ s of Δ cannot be supp.)

$\therefore$ the circle ABC must go through D Q.E.D.

Note Theorem 48 can be deduced at once from Theorem 44

EXERCISE 47

1 If an exterior angle of a quadrilateral equals the interior opposite angle, the quadrilateral is cyclic

2 A, B are fixed points, P moves so that the angle APB is constant. Find the locus of P , and prove that it is a circle only when APB is a right angle

3 In the quadrilateral $ABCD$, the sum of the angles at A and C equals the sum of the angles at B and D . Prove that the quadrilateral is cyclic

4 The bisectors of the angles A, B of ΔABC cut in I , the bisectors of the corresponding exterior angles cut in E . Prove that $AIBE$ is cyclic

5 $ABCD$ is a parallelogram, a circle, centre A and radius AD , cuts the line CD again in E . Prove that $ABCE$ is cyclic

6 Two intervals AB , CD cut in O , $OA = OC$, $OB = OD$. Prove that A , B , C , D are cyclic.

7 Four rods are hinged together to form a rhombus $ABCD$. The rod AB is fixed. Find the locus of the cut of the lines AC and BD .

8 $ABCD$ is a trapezium with AB parallel to CD , and $AD = BC$. Prove it is cyclic.

9 The line XY parallel to AD cuts the sides AB , CD of the cyclic quadrilateral $ABCD$ in X , Y resp. Prove that $BXYC$ is cyclic.

10 (i) The triangle ABC is inscribed in a circle, a parallel to the tangent at A cuts AB , AC in D , E resp. Prove that B , C , D , E are cyclic.

(b) If, also, a parallel to AC cuts the tangent at A in T and BC in D , then A , D , B , T are cyclic.

11 ABC is a triangle, right-angled at A , AP is an altitude. The circle on diameter AP cuts AB , AC in D , E resp. Prove $CDBE$ is cyclic.

12 $ABCD$ is a trapezium, $AB \parallel CD$, a circle through A , B cuts BC in E , AD in F . Prove that $CDFE$ is cyclic.

13 P moves on an arc AB of a circle, AP is produced to Q so that $PQ = PB$. Find the locus of Q .

14 P , Q are points on the base BC of triangle ABC , the lines AP , AQ cut the circle ABC again in X , Y . Prove that, if $AB = AC$, then P , Q , X , Y are cyclic.

15 P , Q are points on the sides AB , AC resp. of triangle ABC , the line CP , BQ cut the circle ABC again in X , Y . Prove that, if $PQ \parallel BC$, then P , Q , X , Y are cyclic.

16 (a) Two triangles have one side of one equal to a side of the other, and the angles opposite the equal sides are supplementary. Prove that the triangles can be fitted together along the equal sides, so as to form a cyclic quadrilateral.

(b) If two triangles have a side of one equal to a side of the other, and the angles opposite the equal sides be equal or supplementary, then the circumcircles of the triangles are equal.

(i) ABC is an isosceles triangle, $AB = AC$, D is a point on the line BC . Prove that the circumcircles of triangles ABD and ACD are equal.

EXAMPLE 41 The altitudes of a triangle are concurrent (See also p 110)

Given AP, BQ are altitudes of $\triangle ABC$, cutting in H

To prove that CH is perpendicular to AB

Proof Let CH, AB cut in S .

Then $\angle CPH, \angle CQH$ are rt $\angle$ s, (given)

C, P, H, Q are cyclic, (rt $\angle$ is $\angle$ in semicircle)

$\angle PQC = \angle PHC$, ($\angle$ s in same seg of circle $PCQH$)

$= \angle SHA$, (vert. opp)

Also $\angle APB, \angle AQB$ are rt $\angle$ s, (given).

B, P, Q, A are cyclic, (rt $\angle$ is $\angle$ in semicircle)

$\angle BQP = \angle BAH$, ($\angle$ s in same seg of circle $APBQ$)

But $\angle PQC = \angle SHA$, (proved)

and $\angle BQP = \angle PQC = \angle HQC$ rt $\angle$, (given)

$\angle BAH = \angle SHA$ rt $\angle$

$\angle HSA =$ rt $\angle$, (sum of $\angle$ s in $\triangle ASH = 2$ rt $\angle$ s)

$CH \perp AB$

Q.E.D.

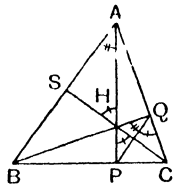


Fig 235

EXAMPLE 42 Given. H is the orthocentre of triangle ABC , and the line AH cuts the circumcircle again in K

To prove that $HP = PK$

Proof. B, P, Q, A are cyclic, (above)

$\angle BAH = \angle PQC$, ($\angle$ s in same seg of circle $ABPQ$)

But $\angle PAQ = \angle PBK$,

($\angle$ s in same seg of circle ABC)

$\angle PBH = \angle PBK$

In $\triangle$ s PBH, PBK ,

$$\begin{cases} \angle PBH = \angle PBK, & \text{(proved)} \\ \angle BPH = \angle BPK, & \text{(rt } \angle\text{s)} \\ BP = BP \end{cases}$$

$\triangle$ s PBH, PBK are congruent, (one side, two $\angle$ s)

$HP = PK$

Def If P, Q, S be the feet of the altitudes of triangle ABC , then the triangle PQS is called the 'pedal triangle' of triangle ABC

17 (a) In Fig 235, prove that $\angle QPH = \angle SPH$

(b) Prove that the orthocentre of a triangle is the incentre of the pedal triangle

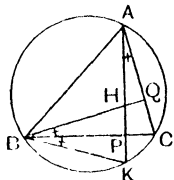


Fig 236

18 (a) An interval AB of fixed length moves with its ends on two lines which intersect in O . Find the locus of the circumcentre of $\triangle OAB$ (cf No 16 (b))

(b) The altitudes of $\triangle ABC$ meet in H . Prove that the circumradii of triangles ABC, BCH, CAH, ABH are all equal (cf No 16 (b))

19 AP, BQ are altitudes of triangle ABC , M, N are the mid-points of sides AC, BC resp. Prove that M, N, P, Q are cyclic

20 $ABCD$ is a cyclic quadrilateral. Prove that any quadrilateral whose four sides are perpendicular to the four sides of $ABCD$ is also cyclic

21 (a) The exterior angles of a convex quadrilateral are bisected. Prove that the quadrilateral so formed is cyclic

(b) Show the same for the bisectors of the interior angles

22 D is a point on the side BC of $\triangle ABC$. Prove that the circle through E touching AB in B , and the circle through D touching AC in C , meet on the circle ABC

23 Equilateral triangles are described on the sides of any triangle so as to lie outside it. Prove that their circumcircles meet in a point

24 (a) A cyclic quadrilateral has perpendicular diagonals which cut in P . Prove that the feet of the perpendiculars from P to the sides lie on a circle

(b) Prove that the mid-points of the sides also lie on that same circle

***25** The altitudes of $\triangle ABC$ from B, C , when produced, cut the circle ABC in D, E resp. P is any point on the circle ABC , PD, PE cut AC, AB resp. in X, Y . Prove that XY goes through the orthocentre. (Use Example 42. Join XH, YH . Prove $X\hat{H}D = P\hat{A}B$)

EXAMPLE 43 *The pivot theorem*

Given ABC is a triangle, D, E, F any points on its sides BC, CA, AB resp.

To prove that the circles AEF, BFD, CDE have one point common

Proof Let the circles AEF, CDE cut in O

Then $A\hat{F}O = O\hat{E}C$,

(ext. $\angle$ of cyclic quad $AEOF$ — int. opp. $\angle$)

$O\hat{E}C = O\hat{D}B$,

(ext. $\angle$ of cyclic quad $C'EOD$ — int. opp. $\angle$)

$A\hat{F}O = O\hat{D}B$

$BFOD$ is cyclic, (ext. $\angle$ — int. opp. $\angle$)

circle BFD goes through O

Q.E.D.

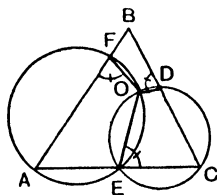


Fig 237

26. Prove the pivot theorem when D, E, F are any points on the lines BC, CA, AB . Take, e.g. F on BA produced beyond A .

27. $ABCD$ is a cyclic quadrilateral, the lines BA, CD cut in E , the lines BC, AD cut in F . Prove that the circles EAD, FDC cut on EF .

28. (Fig 237) $EODC$ is a cyclic quadrilateral, B, A are on the lines CD, CE resp. Prove that the circles AEO, BDO cut on AB .

29. (a) Three circles cut in O , their other cuts, in pairs, are D, E, F, A is on circle EOF , AE cuts circle EOD again in C , CD cuts circle DOF again in B . Prove that B, F, A are collinear.

(b) Prove that, for all positions of A on circle EOF , the triangles ABC so obtained are similar to one another.

*EXAMPLE 44 Given any four lines in a plane, such that by omitting each in turn we obtain four triangles. To prove that the circumcircles of these triangles have a common point.

Given Four lines forming four triangles ABC, CDE, BFD, EFA .

To prove that the circumcircles of these four triangles meet in a point.

Proof Let the circles ABC and CDE cut in S . Produce the intervals CS, DS beyond S to X, Y resp.

Then $\hat{BSX} = \hat{BAC}$, (ext. $\angle$ of cyclic quad $BACS =$ int. opp. $\angle$)

$$X\hat{S}Y = C\hat{S}D, \text{ (opp. } \angle\text{s)}$$

$$= C\hat{E}D, \text{ (}\angle\text{s in same seg. of circle } CDE\text{)}$$

$$B\hat{S}X - X\hat{S}Y = B\hat{A}C - C\hat{E}D$$

$$= A\hat{F}E, \text{ (ext. } \angle \text{ of } \triangle AFE = \text{sum of int. opp. } \angle\text{s)}$$

$$B\hat{S}Y = A\hat{F}E \quad B, F, D, S \text{ are cyclic, (ext. } \angle = \text{int. opp. } \angle)$$

Hence the circle BFD goes through S .

Similarly, the circle EFA goes through S .

Q.E.D.

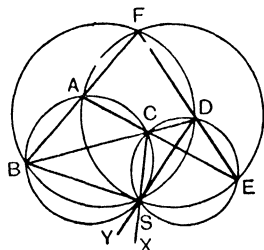


Fig 238

Exercise Prove this, assuming the pivot theorem, considering the triangle BFD , and the points A, C, E on its sides or sides produced. We find that the circles ABC, CDE, FEA have a common point. Similarly the circles ABC, CDE, BFD have a common point; similarly each three of the four circles have a common point. This however is not enough† to prove that the four circles have a common

† This fact is usually overlooked

point, for in Fig. 239 we have four circles, not all having a common point, though each three have a common point. Complete the proof by showing that we do not have here the case of Fig. 239

Note. The following very remarkable infinite set of theorems is much more difficult to prove.

[5] If we have five lines, by omitting each in turn, we have five sets of four lines, giving five points like S above. These lie on a circle C_5 .

[6] If we have six lines, by omitting each in turn, we have six sets of five lines, giving six circles like C_5 . These all have a common point P_6 .

[7] If we have seven lines, by omitting each in turn, we have seven sets of six lines, giving seven points like P_6 . These lie on a circle C_7 .

And so on. (It is assumed that none of the lines are parallel and that no three meet in a point.)

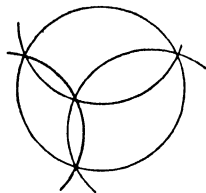


Fig. 239

***EXAMPLE 45 (Simson's line)** *The feet of the perpendiculars to the sides of a triangle from a point on the circumcircle are collinear.*

Given P is on the circumcircle of $\triangle ABC$,
 D, E, F are the feet of the perpendiculars from P to BC, CA, AB resp.

To prove that D, E, F are on a line

Proof. Join DF and FE

$\angle PEA, \angle FEA$ are rt. $\angle$ s, (given)

$\angle EAF$ is cyclic, (rt. $\angle$ is $\angle$ in semicircle)

$$\angle PFE = \angle PAE,$$

($\angle$ s in same seg. of circle PEA)

Also $\angle PAE = \angle PBC$, (ext. $\angle$ of cyclic quad $PACB$ - int. opp.)

$$\angle PFE = \angle PBC.$$

But $\angle PDB, \angle PFB$ are rt. $\angle$ s.

$\angle PDB$ is cyclic, (rt. $\angle$ is $\angle$ in semicircle).

$\angle PBC, \angle PFD$ are supp., (opp. $\angle$ s of cyclic quad.)

$\angle PFE, \angle PFD$ are supp., ($\angle PFE = \angle PBC$, proved).

D, F, E are collinear.

Q.E.D.

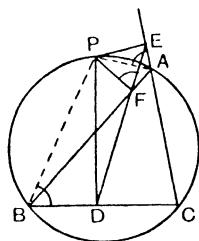


Fig. 240

Exercise (i) State and prove the converse of this example.

(ii) Prove from this example the 'vital step' of Example 36, p. 150.

***EXAMPLE 46** Given The following quadrilaterals are cyclic, $ABCD$, $ABYX$, $BCZY$, $CDWZ$, $DAXW$

To prove that $XYZW$ is cyclic.

Proof. Produce AX beyond X to A' , and CZ beyond Z to C'

Then the cyclic quads give

$$\widehat{YXA'} = \widehat{ABY}, \quad \widehat{WXA'} = \widehat{ADW}$$

$$(\text{adding}), \quad \widehat{WXY} = \widehat{ABY} + \widehat{ADW},$$

$$\text{and} \quad \widehat{YZC'} = \widehat{CBY}, \quad \widehat{WZC'} = \widehat{CDW}$$

$$(\text{adding}), \quad \widehat{YZW} = \widehat{CBY} + \widehat{CDW}$$

$$\widehat{WXY} + \widehat{YZW} = (\widehat{ABY} + \widehat{CBY}) + (\widehat{ADW} + \widehat{CDW})$$

$$= \widehat{ABC} + \widehat{ADC} = 2 \text{ rt } \angle s, \quad (\angle ABCD \text{ is cyclic}).$$

$$XYZW \text{ is a cyclic quad, (opp } \angle s \text{ supp)} \quad \text{Q.E.D.}$$

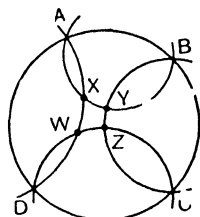


Fig 241

***EXAMPLE 47** The perpendiculars from the mid-points of the sides of a cyclic quadrilateral on to the opposite sides meet in a point

Given L, M, N, P are mid-points of the sides AB, BC, CD, DA of the cyclic quadrilateral $ABCD$, Q, R are the feet of the perpendiculars from L, N on to CD, AB resp, LQ, NR cut in S

To prove that the perpendiculars from P, M to BC, AD resp go through S

Sketch of proof The right bisectors of AB and CD cut in the centre O of circle $ABCD$

Prove that $LSNO$ is a parallelogram LN, OS bisect each other

Next, if the perpendiculars from P, M to BC, AD resp cut in S' , then, similarly, OS', PM will bisect each other

But LN and PM bisect each other (Exercise 25, 8 b), p 57)

Hence S and S' coincide

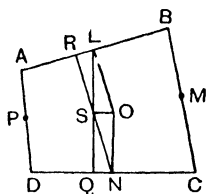


Fig 242

***EXAMPLE 48** Given $ABCD$ is a cyclic quadrilateral, the sides AB, CD , produced, cut in F , the sides AD, BC , produced, cut in E

To prove that the bisectors of angles AFC, AEB cut at right angles

Proof Let the bisector of $\widehat{AFC}$ cut BC, AD resp in X, Y , and let the bisector of $\widehat{AEB}$ cut CD, AB resp in W, Z

Then

$$\widehat{FZW} = \widehat{ZAE} + \widehat{ZEA}, \quad (\text{ext } \angle \text{ of } \triangle ZAE = \text{sum of int opp } \angle s).$$

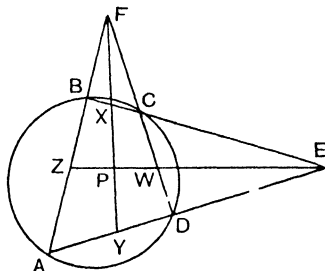


Fig 243

§ 30. Constructions.

CONSTRUCTION 15.

On a given interval to draw a segment of a circle such that the angles in the segment equal a given angle.

Given an interval AB , and an angle PQR .

To draw on AB a segment of a circle, such that the angles in the segment equal angle PQR .

Construction. Draw a ray AC , so that

$$\hat{BAC} = \hat{PQR}.$$

Through A draw a perpendicular to AC , and let it cut the right bisector of AB in O .

Draw the circle, centre O , radius OA .

This circle goes through B , and its segment on the opposite side of AB from C is such that angles in it equal $\hat{PQR}$.

Proof. Since O is on the right bisector of AB , (const.).

$\therefore$ the circle goes through B

Since $OA \perp AC$.

$\therefore AC$ touches the circle constructed, (tan. $\perp$ radius)

$\therefore \hat{BAC} = \text{angle in segment constructed, (alt. seg.)}$

But $\hat{BAC} = \hat{PQR}$, (const.).

$\therefore$ angles in segment = $\hat{PQR}$.

Q.E.F.

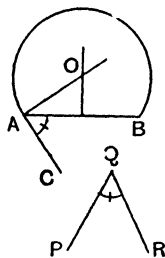


Fig 246

CONSTRUCTION 16.

To inscribe in a given circle a triangle similar to a given triangle.

Given. A triangle ABC and a circle.

To inscribe in the circle a triangle similar to ABC .

Construction. Let P be any point on the given circle. Draw a tangent at P , and on it take points X, Y on opposite sides of P .

Draw chords PQ, PR of the circle such that

$$\hat{XPQ} = \hat{BAC} \text{ and } \hat{YPR} = \hat{BCA}.$$

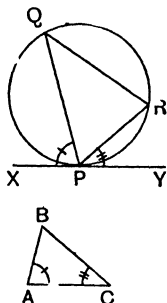


Fig 247

Join QR ; then $\triangle PQR$ is similar to $\triangle ABC$.

Proof. $\angle XPQ = \angle RQ$, (alt. seg.).

But $\angle XPQ = \angle BAC$, (const.).

$$\therefore \angle RQ = \angle BAC.$$

Similarly, $\angle QPR = \angle RPY = \angle BCA$;

$$\therefore \angle QPR = \angle ABC, \text{ (sum of } \angle\text{s of any } \triangle = 2 \text{ rt. } \angle\text{s).}$$

$$\therefore \triangle PQR \text{ is similar to } \triangle ABC.$$

Q.E.F.

CONSTRUCTION 17.

To circumscribe to a given circle a triangle similar to a given triangle.

Given. A circle, centre O , and a triangle ABC .

To circumscribe to the circle a triangle similar to ABC .

Construction. Take any point P on the circle, and draw radii OQ, OR so that

$$\angle POQ = 180^\circ - \angle BAC,$$

$$\angle POR = 180^\circ - \angle ABC.$$

At P, Q, R draw tangents to the circle; these cut in points X, Y, Z such that P, R, Q are on XY, YZ, ZX resp.

Then $\triangle XYZ$ is similar to $\triangle ABC$.

Proof. $\angle OPX$ and $\angle OQX$ are right angles, (tan. $\perp$ radius).

$$\therefore OPXQ \text{ is cyclic.}$$

$$\therefore \angle POQ = 180^\circ - \angle PXQ, \text{ (opp. } \angle\text{s of cyclic quad supp).}$$

$$\text{But } \angle POQ = 180^\circ - \angle BAC, \text{ (const.).}$$

$$\therefore \angle PXQ = \angle BAC, \text{ i.e. } \angle YXZ = \angle BAC.$$

$$\text{Similarly, } \angle XYZ = \angle ABC.$$

$$\therefore \angle XZY = \angle ACB, \text{ (sum of } \angle\text{s of any } \triangle = 2 \text{ rt. } \angle\text{s).}$$

$$\therefore \triangle XYZ \text{ is similar to } \triangle ABC.$$

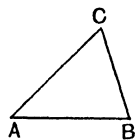
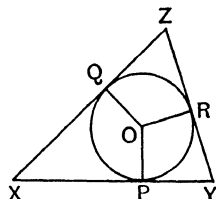


Fig 248

EXERCISE 48

If there are several solutions of the following constructions, give them all.

1 On the sides of any triangle draw segments of circles containing angles of 60°

2 Construct a triangle ABC where $AB = 2''$, $\hat{ACB} = 60^\circ$, altitude from C to $AB = 1.5''$ Measure AC

3 Construct a triangle ABC where $AB = 2''$, $\hat{ACB} = 45^\circ$, area $\triangle ABC = 2$ sq in Measure AC

4 Construct a triangle ABC where $AB = 3''$, $\hat{ACB} = 30^\circ$, $AC = 4''$ Measure BC

5. Draw a circle of radius $3''$ and draw a circumscribed triangle two of whose angles are 90° , 60° Measure its sides

EXAMPLE 50 To draw a triangle, given its base, vertical angle, and the sum of its other two sides

Construction Draw the base AB , and on AB describe a segment AXB of a circle containing an angle half the vertical angle

With centre A , radius = sum of sides, draw a circle cutting the segment in Y

Draw a line BC , cutting AY in C , such that $\hat{YBC} = \hat{CYB}$

Then ACB is the required triangle.

Proof Since $\hat{YBC} = \hat{CYB}$, $CB = CY$, (isos $\triangle$)

$AC + CB = AC + CY = AY =$ given sum of sides, (const)

and $\hat{ACB} = \hat{CYB} + \hat{YBC} = 2\hat{CYB}$

But $\hat{CYB} = \frac{1}{2}$ given vertical angle

$\hat{ACB} =$ given vertical angle

ACB is the triangle required

Q.E.D.

6 Draw a triangle ABC where $AB = 1\frac{1}{2}''$, $\hat{ACB} = 60^\circ$, perimeter = $4''$ Measure AC

7 Prove that a triangle of given base and given vertical angle has maximum area when it is isosceles

8 Draw a triangle, given its base, vertical angle, and the foot of the altitude on the base

9 Draw a triangle, given its base, vertical angle, and length of median to base.

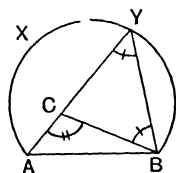


Fig. 249

§ 31. The common tangents to two circles.

Two circles which are external to one another (Fig. 250), obviously have four common tangents two of these, AB and CD , do not cut the interval joining the centres, these are 'direct' common tangents. Two other common tangents, EF and HK , do cut the interval joining the centres; these are the 'transverse' common tangents.

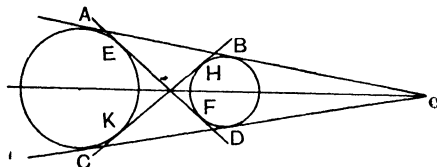


Fig 250

How many direct and transverse common tangents do two circles have (i) if they cut each other, (ii) touch each other, (iii) if one lies inside the other?

CONSTRUCTION 18.

To draw the direct common tangents to two circles, neither of which lies inside the other.

Given Two circles, centres O and P , neither circle inside the other

To draw then direct common tangents

Construction. Suppose the circle, centre O , has the larger radius.

With centre O , and radius equal to the difference of the radii of the given circles, draw a circle.

Let T be the point of contact of a tangent from P to this circle.

Join OT , and let the interval OT , produced beyond T , cut the circle, centre O , in X .

Through P draw a ray parallel to OX , and in the same sense as the ray OX . Let it cut the circle, centre P , in Y

Join XY . Then XY is a direct common tangent.

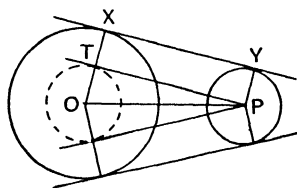


Fig 251

Proof. $OT = OX - PY$, (const.).

$$\therefore PY = OX - OT = TX.$$

Also $YP \parallel TX$, (const.).

$\therefore TPYX$ is a parallelogram.

But $\hat{OTP}$ is a right angle, (tan. $\perp$ radius).

$\therefore \hat{OXT}$ and $\hat{PYX}$ are right angles.

$\therefore XY$ touches both circles, (tan. $\perp$ radius).

Similarly, by drawing the other tangent from P to the constructed circle, the second direct common tangent can be drawn.

Q.E.F.

CONSTRUCTION 19.

To draw the transverse common tangent of two mutually external circles (i.e. circles, each of which is wholly outside the other).

Given. Two mutually external circles, centres O and P .

To draw their transverse common tangents.

Construction. With centre O , and radius equal to the sum of the radii of the given circles, draw a circle.

Let T be the point of contact of a tangent from P to this circle.

Join OT , and let the interval OT cut the circle, centre O , in X .

Through P draw a ray parallel to OT , in the opposite sense to the ray OX . Let it cut the circle, centre P , in Y .

Join XY . Then XY is a transverse common tangent.

Proof. $OT = OX + YP$, (const.).

$$\therefore YP = OT - OX = XT.$$

Also $YP \parallel XT$, (const.).

$\therefore TPYX$ is a parallelogram.

But $\hat{OTP}$ is a right angle, (tan. $\perp$ radius).

$\therefore \hat{OXY}$ and $\hat{PYX}$ are right angles

$\therefore XY$ touches both circles, (tan. $\perp$ radius).

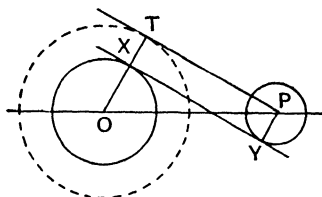


Fig 252

Similarly, by drawing the other tangent from P to the constructed circle, the second transverse common tangent can be drawn. Q.E.F.

Note. The construction for the direct common tangents breaks down when the circles have equal radii. Show how to draw these tangents in that case.

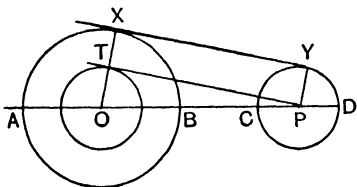


Fig 253

EXAMPLE 51. To find the length of the direct common tangent in terms of the radii and the distance between the centres of the circles. (Fig. 253)

$$XY^2 = TP^2 = OP^2 - OT^2 = OP^2 - (r_1 - r_2)^2,$$

where r_1, r_2 are the radii of the circles AXB, CYD resp.

EXAMPLE 52. To prove $XY^2 = AC \cdot BD$. (Fig 253)

$$\begin{aligned} XY^2 &= OP^2 - (r_1 - r_2)^2 = (OP + r_1 - r_2)(OP - r_1 + r_2) \\ &= (OP + OA - PC)(OP - OB + PD) = AC \cdot BD. \end{aligned}$$

Exercise. Square of length of transverse common tangent = $AD \cdot BC$.

EXERCISE 49.

1. Two wheels of radii 3', 4' have their centres 10' apart. Find the length of the straight portion of a belt which goes round them.

2. Three wheels of radii 4', 5', 6' have their centres at the vertices of an equilateral triangle of side 12'. Find the length of the straight portion of a belt that goes round them all.

3. Two circles touch. Prove that the square of the length of their direct common tangent equals four times the product of their radii.

4. Two circles of radii 3 cm. and 3 cm. touch a line, the distance between the points of contact is 12 cm. Find the distance between their centres (Two answers)

5. Two circles cut the line joining their centres, one in A, B , the other in C, D ; $AB = 4''$, $BC = 3''$, $CD = 2''$. Find the lengths of their direct and transverse common tangents

*6. Two circles are mutually external, prove that the difference of the squares on a transverse and a direct common tangent equals the product of the diameters

*7. Two circles are mutually external; a transverse common tangent has length 2'', a direct common tangent has length 5''. One circle has radius 1 5''. Find the radius of the other.

EXAMPLE 53 To draw a circle touching a given line at a given point and a given circle

Given A line AB and a point P on it, a circle, centre O

To draw a circle touching AB at P and touching the circle

Construction Let Q be the foot of the perpendicular from O to AB

Produce the interval QO , beyond O , to cut the circle in R

Join PR , and let it cut the circle again in S

Let the line OS , and the perpendicular to AB at P , cut in X

The circle, centre X , radius XS , is the circle required

Proof $PX \parallel QR$, (both are $\perp AB$)

$$\hat{XPS} = \hat{ORS}, \text{ (alt } \angle\text{s)}$$

$$= \hat{OSR}, \text{ (OR = OS, radii)}$$

$$= \hat{PSX}, \text{ (vert opp } \angle\text{s)}$$

$$XS = XP$$

the circle, centre X , radius XS , goes through P

It touches the line AB at P , ($XP \perp AB$)

It touches the circle, centre O , at S , (O, S, X are in one line)

the circle constructed is the circle required Q.E.D.

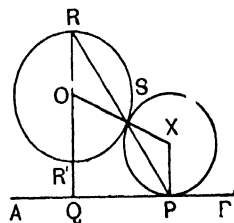


Fig. 254

Exercise Prove that there is a second circle satisfying the conditions obtained from the point R' in which the interval OQ cuts the circle, centre O .

EXAMPLE 54 To draw a circle to touch a given circle at a given point and a given line (Fig. 254)

Given A circle, centre O , and a point S on it, a given line AB

To draw a circle touching the given circle at S , and touching AB .

Construction Let Q be the foot of the perpendicular from O to AB

Produce the interval QO , beyond O , to cut the circle in R

Join RS , and let it cut the line in P

Let the line OS , and the perpendicular to AB at P , cut in X

The circle, centre X , radius XS , is the circle required

Proof As before. There are two circles satisfying the required conditions (If S is at R or R' , the above constructions break down, but it is very easy to invent a suitable construction)

EXERCISE 50

1 Draw a circle of given radius to touch a given line and a given circle (The locus of centres of circles of given radius touching a given line is a pair of parallel lines the locus of centres of circles of given radius touching a given circle is either one circle, or two circles, concentric with the given circle Where these loci meet are the centres of the circles satisfying the conditions)

2 Draw a circle of given radius to satisfy the following conditions In each case state how many circles do so

- (a) To touch a given line and go through a given point
- (b) To touch a given circle and go through a given point
- (c) To touch two given circles
- (d) To touch two given lines which cut

3 Draw a circle to touch two parallel lines and go through a given point How many such circles are there?

4 Draw a circle to touch two parallel lines and a given circle How many such circles are there?

5. *Envelopes* What is the envelope of the following?

- (a) A chord of given length of a given circle
- (b) A chord of a given circle subtending a given angle at a given point of the circle
- (c) A circle of given radius which goes through a given point
- (d) A circle of given radius which touches a given line
- (e) A circle of given radius which touches a given circle
- (f) A line which moves so as to be equidistant from two given points
- (g) An interval of fixed length which subtends equal angles at three given points

(h) One arm of an angle of fixed magnitude, when the other arm touches a fixed circle at a fixed distance from the vertex

*(k) The diagonal AC and the sides CB , CD of a quadrilateral $ABCD$ of fixed size and shape, when the sides AB , AD go through fixed points.

§ 32. A recent theorem.

It is very difficult to discover a new theorem in Elementary Geometry, for it must be one which no one has noticed during the long period of well over two thousand years in which Geometry has been studied. For this reason we give the following theorem†, which was recently discovered by Prof. Morley, during some work on plane curves.

EXAMPLE 55. *The trisectors of the angles of any triangle ABC are drawn, and the adjacent trisectors cut in points X, Y, Z as in Fig. 255. To prove that XYZ is an equilateral triangle.*

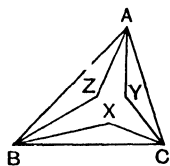


Fig. 255

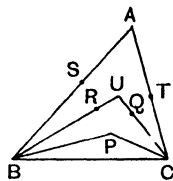


Fig. 256

Proof. (Fig. 256) Let BP, BU be the trisectors of $\hat{ABC}$; CP, CU those of $\hat{ACB}$

Take points R, Q on the rays BU, CU resp. such that

$$\hat{BPR} = 60^\circ + C/3, \quad \hat{CPQ} = 60^\circ + B/3$$

(where A, B, C represent the sizes of the angles of $\triangle ABC$)

Then

$$\begin{aligned} \hat{BRP} &= 180^\circ - (\hat{RBP} + \hat{BPR}) = 180^\circ - (B/3 + 60^\circ + C/3) \\ &= 180^\circ - (60^\circ + 60^\circ - A/3) = 60^\circ + A/3 \end{aligned} \quad (1),$$

since $A/3 + B/3 + C/3 = 60^\circ$.

Similarly, $\hat{CQP} = 60^\circ + A/3$. Thus $\hat{BRP} = \hat{CQP}$

Now $\hat{BPC} = 180^\circ - \hat{PBC} - \hat{PCB} = 180^\circ - B/3 - C/3$.

$$\begin{aligned} \hat{RPQ} &= 360^\circ - (\hat{BPR} + \hat{CPQ} + \hat{BPC}) \\ &= 360^\circ - (60^\circ + C/3 + 60^\circ + B/3 + 180^\circ - B/3 - C/3) = 60^\circ. \end{aligned}$$

But, since BP, CP bisect the angles $\angle CBU, \angle BCU$ of $\triangle BUC$,

P is the incentre of $\triangle BUC$.

the perpendiculars from P on to BU, CU are equal, and PR, PQ are equally inclined to these perpendiculars, ($\hat{BRP} = \hat{CQP}$, proved).

$\therefore PR = PQ$. But $\hat{RPQ} = 60^\circ$, (proved).

$\therefore \triangle PQR$ is equilateral(ii).

† The proof here given is due to Mr J M Child, *Math Gazette*, Oct 1922. Part of it has been utilised in a question in a recent Matriculation Examination.

We shall now prove that RA , QA trisect the angle BAC .

Take S , T on BA , CA resp so that $BP = BS$, $CP = CT$.

Then $\triangle s PRB$, SRB are congruent

$$RS = RP \text{ and } \hat{BRS} = \hat{BRP} \quad (\text{iii})$$

Similarly $QT = QP$.

$$SR = RQ = QT \quad (\text{iv}).$$

$$\begin{aligned} \text{But } \hat{SRQ} &= 360^\circ - (\hat{BRS} + \hat{BRP} + \hat{PRQ}) \\ &= 360^\circ - (120^\circ + 2A/3 + 60^\circ), \quad (\text{by i, ii, iii}) \\ &= 180^\circ - 2A/3. \end{aligned}$$

$$\text{Similarly, } \hat{TQR} = 180^\circ - 2A/3 \quad (\text{v}).$$

Since $SR = RQ$ and $\hat{SRQ} = 180^\circ - 2A/3$,

$$\hat{SQR} = A/3 \quad (\text{vi}).$$

Now we have proved $SR = QT$, (iv), and $\hat{SRQ} = \hat{TQR}$, (v),

$SRQT$ is cyclic. But SR , RQ , QT , being equal, (iv), each subtends at the circumference of circle $SRQT$ an angle $= \hat{SQR} = A/3$.

the arc ST of circle $SRQT$ subtends at the circumference an angle $\angle$

the point A is on circle $SRQT$

And, again, since SR , RQ , QT are equal chords of circle $SRQT$,

$$\hat{SAR} = \hat{RAQ} = \hat{QAT}$$

Thus the trisectors of the angles of $\triangle ABC$ cut in the vertices P , Q , R of an equilateral triangle. Q F D.

REVISION PAPERS

After § 18

XXI. 1. Construct a triangle ABC , given $BC = 3''$, $AB = 2''$, area $= 2.5$ sq in. Measure CA .

2. $ABCD$ is a rectangle, X is on the side CD , Y on the side BC , $AB = 7''$, $AD = 8''$, $DX = 3''$, $BY = 6''$. Find the area of $\triangle AXY$. Do the same when the lengths of AB , AD , DX , BY are a , b , x , y resp.

3. Find the base of an equilateral triangle whose height is 3 inches.

XXII. 1. Construct a parallelogram $ABCD$ of area 3 sq. in., with $AB = 2''$, $\hat{ABC} = 120^\circ$. Measure BD .

2. ABC is a triangle, X a point on the base BC , a parallel to AX through R cuts the line CA in Y . Prove that triangles ABC and CXY are equivalent.

3. Find the side AC of $\triangle ABC$, given $AB = 6''$, $BC = 7''$, $\hat{ABC} = 120^\circ$.

XXIII 1. Draw a quadrilateral $ABCD$ where $AB = 3''$, $BC = 2\frac{1}{2}''$, $\hat{ABC} = 60^\circ$, $CD = DA = 3''$. Construct a rectangle equal to it in area. Measure the base and height of the rectangle and hence calculate its area.

2 E is a point on the side AB of the parallelogram $ABCD$. Prove that the sum of the areas of triangles AED , BEC is half the area of the parallelogram.

3 A box has edges of lengths $5''$, $7''$, $8''$. Find the length of a diagonal of the box, and the length of a diagonal of the smallest face.

XXIV 1 Calculate the area of the figure, where $AB = 10'$, $AF = 2'$, and the other measurements are given in the figure (Fig 257)

2 ABC is a triangle, D , E points on sides AB , AC resp. such that DE is parallel to BC , BE , CD cut in X . Prove that triangles BDX , CXE are equivalent.

3 A pyramid whose base is a square of side 1 ft is 6 ft high. Find the area of a slanting face.

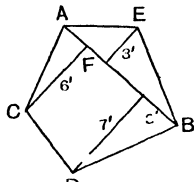


Fig 257

XXV 1 $ABCD$ is a trapezium with sides AD , BC parallel, E is the mid-point of side CD . Prove that area $ABE = \frac{1}{2}$ area $ABCD$.

2 The angles marked in the figure are right angles, $OA = OD = 15''$, $AB = 9''$. Find the length of AC (Fig 258)

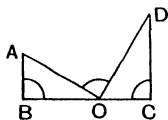


Fig 258

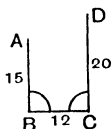


Fig 259

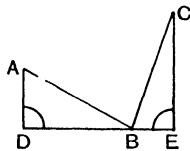


Fig 260

3 Find the distance between D and the image of A in BC , the angles marked being right angles (Fig 259)

After § 22

XXVI 1 A parallel to the side AB of a parallelogram $ABCD$ cuts AD , AC , BC in E , F , H resp. Prove that triangles AEH , AFD are equivalent.

2 If $AB = BC$, and the angles marked are right angles, and $AD = 4$, $DB = 7$, $BE = 1$, find the lengths of EC and AC . (Fig 260)

3 Draw a triangle whose sides are $2''$, $3''$, $4\frac{1}{2}''$ long. Construct the circumcircle and measure its radius.

XXVII 1 $ABCD$ is a parallelogram. A line through D cuts BC in E and AB in F . Prove that area ABE = area CEF .

2 ABC is an equilateral triangle of side $1''$, D is on BC , $BD = \frac{1}{4}''$. Calculate the length of AD .

3 A, B, C, D are four given points. Show how to find a point which is equidistant from A and B , and equidistant from C and D .

XXVIII 1 Divide a quadrilateral into two equivalent parts by a line drawn through a vertex.

2 $ABCD$ is a trapezium with AB parallel to CD , and sides BC, AD, CD each $3''$ long. If $AB = 5''$, calculate the area of the trapezium.

3 Two circles of radii $15''$ and $13''$ have their centres $14''$ apart. Find the length of their common chord.

XXIX 1 Divide a square into four equivalent parts, by lines drawn through a given point of a side.

2 $ABCD$ is a quadrilateral whose diagonals are at right angles. The angles at A, C are right angles. Prove that $AB = BC, CD = DA$.

3 A chord of a circle is $12''$ long. Its mid-point is M , and its right bisector cuts the circle in A . If $MA = 4''$, find the radius of the circle.

XXX 1 ABC is an equilateral triangle, G is its centroid. Find the length of GA in terms of the length of AB .

2 Draw a circle, centre O , of radius $1\frac{1}{2}''$. Take two points A, B on the circle, so that $\angle AOB = 60^\circ$. Draw a circle whose centre is on the first circle, and which goes through the mid-points of OA and OB . Measure its radius.

3 Two parallel chords of a circle are of lengths $18''$ and $22''$, and at a distance apart of $2''$. Calculate the distance of the longest chord from the centre and hence the radius of the circle. (Use the Theorem of Pythagoras.)

After § 29

XXXI 1 ABC is a triangle right-angled at B , G is its centroid, $AB = 5''$, $BC = 12''$. Find GA, GB, GC .

2 AB, CD are perpendicular chords of a circle. Prove that the non-overlapping arcs AB and CD together make up half the circumference.

3 If A, B be points on the circle, prove that the perpendicular from A on the tangent at B equals the perpendicular from B on the tangent at A .

XXXII. 1 A right circular cone has a height of $2'$; the radius of its base is $7''$. Find its slant height. Hence find the area of the curved surface.

2. A regular octagon is inscribed in a circle. Find the angles in the minor arc cut off by a side of the octagon.

3 The locus of points from which tangents to a fixed circle make a fixed angle with each other is a circle.

XXXIII. 1. OA, OB, OC are three mutually perpendicular lines (i.e. each is perpendicular to each). The intervals OA, OB, OC are of lengths $2', 1', 1'$. Find the area of triangle ABC .

2 The altitudes AP, BQ, CR of $\triangle ABC$ cut in H . Prove that the following quadrilaterals are cyclic. $AQHR, BRHP, CPHQ, BCQR, CARP, ABPQ$.

3. TA, TB are tangents to a circle at A, B . If $\hat{ATB} = 60^\circ$, find the distance of the line AB from the centre of the circle, in terms of the radius.

XXXIV. 1 Unequal segments of two equal circles are described on the same base AB and on the same side of it. Prove that the angles in them are supplementary.

2 AP, BQ are altitudes of $\triangle ABC$, M is the mid-point of side AB . Prove that $PMQ = 2PAQ = 2PBQ$.

3 TA, TB touch a circle in A, B resp., a moving tangent to the circle cuts the intervals TA, TB in C, D resp. Prove that the triangle TCD has a constant perimeter.

XXXV. 1 Construct a triangle ABC , given $AB = 3''$, $\hat{ACB} = 70^\circ$, altitude from $C = 2''$. Measure its sides.

2 TA, TB touch a circle in A, B . Prove that $\hat{ATB}$ is twice the angle between AB and the diameter through A .

3. ABC is a triangle inscribed in a circle, $AB = AC$, AB makes an angle of 20° with the tangent at B . Find the size of $\hat{BAC}$.

XXXVI. 1. Construct a triangle ABC with $AB = 3''$, $\hat{ACB} = 70^\circ$, $AC = 2''$. Measure BC .

2 TA, TB touch a circle in A, B . Prove that $\hat{ATB}$ equals the difference between the angles in the segments of the circle cut off by the chord AB .

3. ABC is a triangle inscribed in a circle, $AB = AC$. Prove that the tangent at A is parallel to BC .

XXXVII. 1 Two parallel chords of a circle, of lengths $2a, 2b$, are at a distance d apart. If $b > a$, and the chord of length $2b$ is at a distance x from the centre, prove that $2dx = b^2 - a^2 - d^2$.

2. Draw two circles of radii 3 cm. and 1 cm. whose centres are 4.5 cm. apart. Draw a circle of radius 2 cm. to touch both.

3. AB is a side of an equilateral triangle, AC a side of a square inscribed in the same circle; $\angle BAC$ is an acute angle. Prove that BC is a side of a regular figure of twelve sides, inscribed in the same circle.

XXXVIII. 1. P describes a segment of a circle on base AB . Find the locus of the orthocentre of triangle APB

2. Draw two circles of radii $\frac{1}{2}$ " and $\frac{3}{4}$ " whose centres are 1.5" apart. Draw a circle of radius 1", whose centre is on the first circle, and which touches the second circle

3. ABC is a triangle, D is the foot of the perpendicular from A on to the bisector of $\angle BAC$. Prove that the parallel to BC through D bisects the sides AB and AC

XXXIX. 1. P describes a segment of a circle on the chord AB . Find the locus of the incentre of triangle APB

2. A regular hexagon is inscribed in a circle. Find the distance between two of its parallel sides in terms of the radius

3. ABC and DEF are triangles inscribed in the same circle. If $AB \parallel DE$ and $BC \parallel EF$, prove $AC = DF$.

After § 31.

XL. 1. The bisector of the angle between the diagonals AC , BD of a cyclic quad $ABCD$ cuts the sides BC , AD at equal angles

2. Draw two circles of radii 2 cm. and 2.5 cm., whose centres are 5.5 cm. apart. Construct their direct common tangents. Measure their lengths, and verify by calculation

3. Show how to divide an arc of a circle into four equal parts.

XLI. 1. AP is an altitude of triangle ABC ; L , M , N are the mid-points of its sides BC , CA , AB . Prove that $LMNP$ is cyclic

2. Three circles touch as in Fig. 261. Prove that the tangents at A , B , C are concurrent.

3. Draw a triangle of area 3 sq. in. on a base of 3 in. to contain a vertical angle of 70° . Measure its sides.

XLII. 1. L , M , N are the mid-points of the sides of triangle ABC . Prove that the circle LMN has a radius half that of circle ABC

2. In Fig. 261, let X , Y , Z be the centres of the circles. Prove that the incircle of triangle XYZ touches its sides at A , B , C .

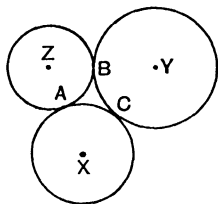


Fig. 261

3 PA touches a circle at A , a line through P cuts the circle at B and C . Prove that AC is parallel to the tangent at P to the circle ABP .

XLIII 1 $ABCD$ is a cyclic quadrilateral, AB, CD cut in E , AD, BC cut in F . Prove that the circles BCE, CDF cut on EF .

2 Four circles with centres X, Y, Z, W touch as in Fig 262. Prove that a circle can be described touching the sides of $XYZW$.

3 AB is a chord of a circle of fixed length, its ends A, B move round the circle, X, Y are fixed points on the circle. Prove that the locus of the cut of the lines XA and YB is a circle.

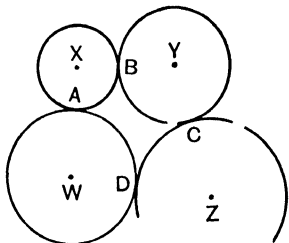


Fig 262

XLIV 1 Draw a sector of a circle of radius $1''$, the angle between the bounding radii being 60° . Draw a circle to touch the bounding radii and the arc of the sector.

2 Show how to construct a triangle, given the base, the vertical angle, and the point where the bisector of the vertical angle cuts the base.

3 The triangle with given base and vertical angle, which has the greatest possible area, is isosceles.

XLV 1 Draw the triangle whose sides are 5 cm , 7 cm , 8 cm . long. Draw the inscribed circle of the triangle, and draw three circles to touch the inscribed circle, and two of the sides of the triangle.

2 Find a point P inside a triangle ABC such that $\hat{APE}$, $\hat{BPC}$, $\hat{CPA}$ are all equal.

3 $ABCD$ is a quadrilateral. Divide it into two quadrilaterals such that a circle can be inscribed in each. Is this always possible?

XLVI 1 Find a point P inside a triangle ABC such that $\hat{PBA}$, $\hat{PCA}$, $\hat{PAB}$ are all equal. Prove that there may be two such points.

2 A, B, C, D are four points on a circle, X, Y, Z, W bisect the arcs AB, BC, CD, DA resp. Prove that XZ and YW are at right angles.

3. Prove that there is an infinite number of triangles similar to a given triangle, whose sides, or sides produced, pass each through one of three given points. What are the conditions that one of these triangles is congruent to the given triangle?

PART III

§ 33. Extensions of the Theorem of Pythagoras.

Application of Algebra to Geometry. We pointed out on p. 80 that the Theorem of Pythagoras is often written as follows *if $\hat{A}BC$ is a right angle then $AB^2 + BC^2 = AC^2$* . This formula may be looked at from two points of view. We may regard it as a statement on the actual geometrical square figures described on the sides, or as a statement about the arithmetic squares of the lengths of the sides. (Note that the word 'square' is used in two senses (i) as meaning a geometric figure, (ii) in the sense in which it is used when we say that 9 is the square of 3.) In proving (p. 79) the Theorem of Pythagoras, we regarded AB^2 as denoting the actual geometric square figure, but in employing the theorem, it is usually convenient to regard it as meaning the arithmetic square of the length of AB . When we take the second interpretation, it is often useful to employ algebraic methods in solving geometric questions.

EXAMPLE 56 *Given A, O, B, C are on the same line, in that order; O is the mid-point of AC . Prove*

$$AB^2 - BC^2 = 4OB \cdot OC.$$

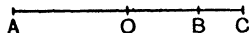


Fig 263

Proof.

$$AB^2 - BC^2 = (AB + BC)(AB - BC)$$

$$\text{Now } AB + BC = AC = 2OC, \quad (AO = OC)$$

$$AB - BC = AO + OB - (OC - OB)$$

$$= AO + OB - OC + OB = 2OB, \quad (AO = OC)$$

$$AB^2 - BC^2 = 2OC \cdot 2OB = 4OB \cdot OC.$$

Note AB represents here the length of the interval AB , and so obeys the laws of Algebra as if it were a single letter. AB^2 means the square of the length of AB and perhaps it would be better to write it $(AB)^2$ or $\overline{AB}^2$, but the omission of the bracket or bar, in this case†, cannot lead to confusion, because B^2 , or the square of a point,

† But, in Algebra, ab^2 means a multiplied by the square of b , the square of the product of ab must always be written $(ab)^2$, with the brackets.

has no meaning. Note again that $2AB^2$ means twice the square of the length of AB , while $(2AB)^2$ means the square of twice the length of AB . The following are often needed

$$(AB + BC)^2 = AB^2 + BC^2 + 2AB \cdot BC$$

$$(AB - BC)^2 = AB^2 + BC^2 - 2AB \cdot BC$$

$$AB^2 - BC^2 = (AB + BC)(AB - BC).$$

There is an infallible method for solving examples like the above, though it usually does not give the neatest proof. We illustrate it by another treatment of the above example

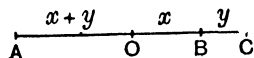


Fig 264

Denote the length of the intervals along the line by single small letters. Thus, remembering $OA = OC$, let $OB = x$, $BC = y$

$$AO = x + y$$

$$AB^2 - BC^2 = (2x + y)^2 - y^2$$

$$= 4x^2 + 4xy + y^2 - y^2 = 4x^2 + 4xy,$$

and

$$4OB \cdot OC = 4x(x + y) = 4x^2 + 4xy$$

$$AB^2 - BC^2 = 4OB \cdot OC$$

In the exercise which follows, do not use this method, if you can find a neater way

EXERCISE 51

1 (a) If B is between A and P , O the mid-point of interval AB , prove $PO = \frac{1}{2}(PA + PB)$

(b) If P is between A and B , O the mid-point of interval AB , prove $PO = \frac{1}{2}(PA - PB)$

2 A, B, C, D are in that order along a line. Prove

(a) $AB \cdot CD + BC \cdot AD = AC \cdot BD$

(b) Prove also $AD^2 = AB^2 + BC^2 + CD^2 + 2AB \cdot PD + 2BC \cdot CD$.

(c) Prove also $AB \cdot BC + CD \cdot DA = AB \cdot BD + BD \cdot CD$

3 (a) B is between A and C , O is the mid-point of interval AB . Prove $CA^2 + CB^2 = 2CO^2 + 2AO^2$

(b) Prove the theorem when C is between A and B .

4 P is between A and B , O is the mid-point of interval AB . Prove $AP^2 + BP^2 = 2AP \cdot PB + 4OP^2$

5 If C is between A and B and $AB \cdot BC = AC^2$, prove

(a) $AB^2 + BC^2 = 3AC^2$.

(b) $(AB + BC)^2 = 5AC^2$

(c) $3AB \cdot BC = AB^2 + BC^2$.

(d) $AB^2 = 2AC^2 + AC \cdot CB$

6 A, B, C, D are in that order along a line, O, P are the mid-points of intervals AB, CD resp. Prove

$$OP^2 = \frac{1}{4}(AB + CD)^2 + BC \cdot AD.$$

7. A, B, C, D are in that order along a line. Prove
 $AB^2 \cdot CD + AD^2 \cdot BC - AC^2 \cdot BD = BC \cdot CD \cdot BD$

8. A, B, C, D are in that order along a line. Prove

(a) $AC^2 + BD^2 = AB^2 + CD^2 + 2AD \cdot BC$.

(b) Also $AD^2 + BC^2 = AC^2 + BD^2 + 2AB \cdot CD$

Def. If AB be any interval, XY any line not perpendicular to AB , and if P, Q be the feet of the perpendiculars from A, B resp on to XY , then the interval PQ is called the 'projection' of the interval AB on XY .

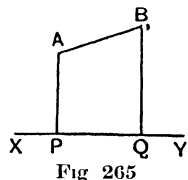


Fig 265

THEOREM 49 (Extension of Pythagoras' Theorem)

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle equals the sum of the squares on the sides containing that angle plus twice the rectangle contained by one of these sides and the projection of the other side upon it.

Given $\triangle ABC$ is a triangle, with an obtuse angle at C . D is the foot of the perpendicular from A on to BC .

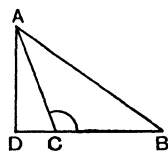


Fig 266

To prove that $AB^2 = AC^2 + BC^2 + 2BC \cdot CD$.

Proof $AB^2 = AD^2 + BD^2$, ($\angle ADB$ is a rt $\angle$)
 $= AD^2 + (BC + CD)^2 = AD^2 + BC^2 + 2BC \cdot CD + CD^2$

But $AD^2 + CD^2 = AC^2$, ($\angle ADC$ is a rt $\angle$)

$$\therefore AB^2 = AC^2 + BC^2 + 2BC \cdot CD$$

Q.E.D.

THEOREM 50 (Extension of Pythagoras' Theorem).

In any triangle, the square on the side opposite an acute angle equals the sum of the squares on the sides containing that angle minus twice the rectangle contained by one of these sides and the projection of the other side upon it.

Given $\triangle ABC$ is a triangle, with an acute angle at C . D is the foot of the perpendicular from A on to BC .

To prove that $AB^2 = AC^2 + BC^2 - 2BC \cdot CD$.

Proof. $AB^2 = AD^2 + BD^2$, ($\because \hat{ADB}$ is a rt. $\angle$)
 $= AD^2 + (BC \sim CD)^2 = AD^2 + BC^2 - 2BC \cdot CD + CD^2$.

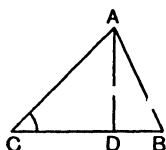


Fig 267

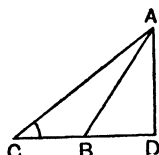


Fig 268

But $AD^2 + CD^2 = AC^2$, ($\because \hat{ADC}$ is a rt. $\angle$).
 $\therefore AB^2 = AC^2 + BC^2 - 2BC \cdot CD$. Q.E.D.

THEOREM 51 (Apollonius).

In any triangle, the sum of the squares on two sides is twice the square on half the third side plus twice the square on the median drawn to the third side.

Given ABC is a triangle, AL a median.

To prove that $AB^2 + AC^2 = 2AL^2 + 2BL^2$.

Proof. Let D be the foot of the perpendicular from A on to the line BC . Of the angles ALB , ALC either both are right angles, or one is acute, the other obtuse.

If both are right angles, then

$$AB^2 = AL^2 + BL^2, \quad AC^2 = AL^2 + CL^2.$$

$$\therefore \text{(adding)} \quad AB^2 + AC^2 = 2AL^2 + 2BL^2, \quad (\because BL = CL).$$

In the second case, suppose $\hat{ALB}$ is obtuse, and $\therefore \hat{ALC}$ acute, then

$$AB^2 = AL^2 + BL^2 + 2BL \cdot LD,$$

$$\text{and} \quad AC^2 = AL^2 + CL^2 - 2CL \cdot LD.$$

But $BL = CL$.

$$\therefore \text{(adding)} \quad AB^2 + AC^2 = 2AL^2 + 2BL^2. \quad \text{Q.E.D.}$$

Note This theorem is very often useful in riders. It should be compared with Ex. 51, Nos. 5 (a), (b). It is generally of no use in riders to draw the actual squares and rectangles concerned; usually you should construct a proof like those of Theorems 49–51.

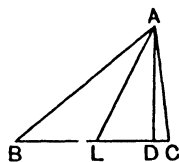


Fig 239

EXAMPLE 57. Given $ABCD$ is a quadrilateral, X, Y are the mid-points of AC, BD .

To prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2$$

Proof.

In $\triangle ABC$, $AX = XC$

$$AB^2 + BC^2 = 2AX^2 + 2BX^2$$

In $\triangle ADC$, $AX = XC$

$$CD^2 + DA^2 = 2AX^2 + 2DX^2$$

(adding)

$$AB^2 + BC^2 + CD^2 + DA^2 = 4AX^2 + 2BX^2 + 2DX^2$$

But in $\triangle BXD$, $BY = YD$ $BX^2 + DX^2 = 2BY^2 + 2XY^2$

$$AB^2 + BC^2 + CD^2 + DA^2 = 4AX^2 + 2(2BY^2 + 2XY^2) \\ = 4AX^2 + 4BY^2 + 4XY^2$$

But

$$2AX = AC, \quad 4AX^2 = AC^2,$$

similarly,

$$4BY^2 = BD^2$$

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2 \quad \text{Q.E.D.}$$

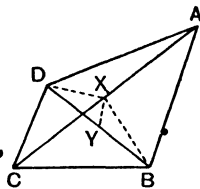


Fig 270

EXERCISE 52

1 Use Pythagoras' Theorem and its extensions to test whether the triangles, whose sides have the lengths given below, are right angled, acute angled, or obtuse angled (Test by considering the longest side.)

- (i) 5, 6, 7 (ii) 5, 6, 8 (iii) 6, 7, 9 (iv) 7, 8, 11

2 Find the lengths of the longest medians in triangles (i), (ii) above.

3 P is the foot of the altitude AP of $\triangle ABC$, $AB = 6$ cm, $AC = 5$ cm, $BC = 8$ cm. Calculate BP and CP .

4. In $\triangle ABC$, $AB = BC = 4''$, $AC = 7''$, AP is the altitude from A . Calculate CP .

5 (a) AP, BQ are altitudes of $\triangle ABC$; $AB = 7''$, $BC = 8''$, $CA = 6''$. Calculate the length of BP , then of AP . Hence find the area of the triangle.

(b) Using the value of the area of the triangle, find the length of BQ .

Find the length of the side AC in the following triangles:

6. In $\triangle ABC$, $AB = 5''$, $BC = 7''$ $\hat{A}BC = 120^\circ$

7. In $\triangle ABC$, $AB = 5''$, $BC = 7''$ $\hat{A}BC = 60^\circ$.

8. In $\triangle ABC$, $AB = 8''$, $BC = 9''$. $\hat{A}BC = 135^\circ$.

9. In $\triangle ABC$, $AB = 8''$, $BC = 9''$, $\hat{A}BC = 45^\circ$.

10 $ABCD$ is a parallelogram. $AB = 3''$, $BC = 4''$, $AC = 6''$. Find BD .

11 $ABCD$ is a parallelogram. $AB = 5''$, $AC = 8''$, $BD = 10''$. Find BC

12 (a) ABC is a triangle, the angle at C is obtuse, AP and BQ are altitudes. Prove from Theorem 49 that $BC \cdot CP = AC \cdot CQ$.

(b) Prove the same theorem for an acute-angled triangle.

13 (a) ABC is an isosceles triangle, $AB = AC$, BQ is an altitude. Prove $AC \cdot CQ = \frac{1}{2}BC^2$.

(b) Prove also $BQ^2 = QC^2 + 2AQ \cdot QC$.

(c) ABC is an isosceles triangle, $AB = AC$, D is a point on side AC such that $BD = BC$. Prove $BC^2 = AC \cdot CD$.

14 ABC is an isosceles triangle, $AB = AC$, D is a point on the side BC . Prove $AB^2 - AD^2 = BD \cdot DC$.

15 $ABCD$ is a parallelogram. Prove $AC^2 + BD^2 = 2AB^2 + 2BC^2$ (Apollonius).

16 The sides of a triangle have lengths a, b, c . Find the lengths of the medians.

17. P is any point in the plane of the parallelogram $ABCD$. Prove that $PA^2 + PC^2 - PB^2 - PD^2 = \text{constant}$.

18 (a) ABC is a triangle, the angle at C is 120° . Prove

$$AB^2 = AC^2 + BC^2 + AC \cdot BC$$

(b) ABC is a triangle, the angle at C is 60° . Prove

$$AB^2 = AC^2 + BC^2 - AC \cdot BC$$

(c) ABC is an equilateral triangle, D is any point on the side BC . Prove

$$AD^2 = BD^2 + CD^2 + BD \cdot CD$$

19 AB is a diameter of a circle, centre O , OP is a perpendicular radius, C is any point on diameter AB . Prove $AC^2 + 3C^2 = 2CP^2$.

20 ABC is an isosceles triangle, $AB = AC$, P, Q are points on sides AB, AC resp. $PQ \parallel BC$. Prove $BQ^2 = BC \cdot PQ + CQ^2$.

21 (a) AL is a median of triangle ABC , the circle, centre L , through A , cuts the line BC in D . Prove $AB^2 + AC^2 = DB^2 + DC^2$.

(b) A, B are fixed points. Prove that the locus of a point P which moves in a plane through A, B such that $PA^2 + PB^2 = \text{const}$, is a circle.

22. (a) $ABCD$ is any quadrilateral, H, F are the mid-points of sides AD and BC . Prove

$$AB^2 + BD^2 + DC^2 + CA^2 = AD^2 + BC^2 + 4FH^2$$

(b) $ABCD$ is any quadrilateral, E, F, G, H are the mid-points of sides AB, BC, CD, DA resp. Prove

$$2EG^2 + 2FH^2 = AC^2 + BD^2$$

(c) If $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$, prove that either A, B, C, D are collinear or are at the vertices of a parallelogram.

(d) Prove (a) and (b) where A, B, C, D are any four, non-coplanar, points

(e) The sum of the squares on the edges of a tetrahedron equals four times the sum of the squares on the intervals joining the mid-points of opposite edges

23 ABC is an acute-angled triangle, BQ, CR are altitudes. Prove that $BC^2 = AB \cdot BR + AC \cdot CQ$

24 $ABCD$ is a trapezium, $AB \parallel CD$. Prove
 $AD^2 + BC^2 = AC^2 + BD^2 - 2AB \cdot DC$

25 (a) X, Y are the points of trisection of the base BC of $\triangle ABC$, $AX > AY$. Prove $3(AX^2 - AY^2) = AB^2 - AC^2$

(b) $AX^2 + AY^2 + 4XY^2 = AB^2 + AC^2$.

(c) If $\hat{BAC}$ is a right angle, $AX^2 + AY^2 = 5XY^2$.

26 ABC is a triangle, right angled at B , D, E are points on side AC , such that $AE = AB$, $CD = CB$. Prove $DE^2 = 2AD \cdot CE$

*27 (a) X is any point in the plane of the quadrilateral $ABCD$. Prove

$XA^2 + XB^2 + XC^2 + XD^2 = 4XG^2 + \frac{1}{2}AB^2 + \frac{1}{2}CD^2 + PQ^2$,
 where P, Q are the mid-points of sides AB, CD resp. and G is the mid-point of interval PQ

(b) Prove this when A, B, C, D, X are any five points in space

(c) $ABCD$ is a square, X is a point on its circumference. Prove $XA^2 + XB^2 + XC^2 + XD^2$ equals twice the square on the diameter of the circle

*28 (a) If AL, BM, CN be the medians of $\triangle ABC$ and they cut in G , prove $4(AL^2 + BM^2 + CN^2) = 3(AB^2 + BC^2 + CA^2)$.

(b) $3(AG^2 + BG^2 + CG^2) = AB^2 + BC^2 + CA^2$.

*29 (a) If X is between A and B , and $AX = 2XB$, and P is any point, prove $PA^2 + 2PB^2 = 3PX^2 + AX^2 + 2BX^2$

(b) If G is the centroid of $\triangle ABC$ and P any point, prove $PA^2 + PB^2 + PC^2 = 3PG^2 + GA^2 + GB^2 + GC^2$.

(c) If A, B, C be given points, and P moves in the plane ABC , so that $PA^2 + PB^2 + PC^2$ is const., prove that P describes a circle, centre G

(d) $PA^2 + PB^2 + PC^2$ is a minimum, when P is at G .

*30 In triangle ABC , $\hat{BAC} = 45^\circ$; BQ, CR are altitudes. Prove $2QR^2 = BC^2$.

***31.** Squares are drawn on the three sides of a triangle so as to fall outside the triangle (Fig 271) Prove $PQ^2 + RS^2 + XY^2 = 3(AB^2 + BC^2 + CA^2)$.

***32.** DE is a chord parallel to a diameter AB of a circle, C is any point on diameter AB . Prove $CD^2 + CE^2 = CA^2 + CB^2$.

***33** O is a fixed point, the interval AB moves parallel to itself so that $OA^2 + OB^2$ is constant. Find the locus of A and B .

***34** A corner is cut off a cubical block. Prove that the section is an acute-angled triangle.

***35** Draw a cyclic quadrilateral whose sides are 2", 3", 4", 5" long.

***36** C is the mid-point of the interval AB , if a, b, c are the lengths of tangents from A, B, C to a circle, prove that $a^2 + b^2 = 2c^2 + 2AC^2$.

***37** (a) If, in Fig 271, the areas of the squares are 376, 116 and 74 acres, find the area of the triangle ABC †

(b) Prove that if x, y, z are the areas of the squares then

$$4 \Delta ABC = \sqrt{4xy - (x + y - z)^2}.$$

***38** If, in Fig 271, the areas of the squares are 26, 20 and 18 acres, find the area of the hexagon $XPQRS$.

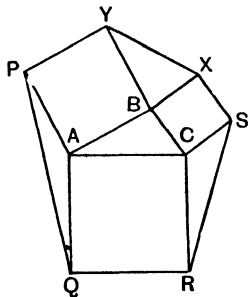


Fig 271

CONSTRUCTION 20.

To draw a square equivalent to a given rectangle.

Given a rectangle $ABCD$.

To draw a square equivalent to it.

Construction. Produce the side AB beyond B to E , making $BE = BC$.

On AE describe a semicircle, and let the line CB cut it in F .

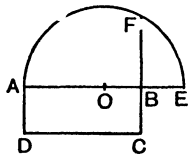


Fig 272

Then the rectangle $ABCD$ is equivalent to the square on BF .

Proof. Let O be the centre of the semicircle on AE .

Either O is between A and B , or between B and E .

Suppose O is between A and B . Then

$$AB = AO + OB, \quad BC = BE = OF - OB = AO - OB.$$

† Questions 37, 38 are taken from Dudeney's *Amusements in Mathematics*.

$$\begin{aligned}
 \therefore \text{Area of rect. } ABCD &= AB \cdot BC = AB \cdot BE \\
 &= (AO + OB)(AO - OB) = AO^2 - OB^2 \\
 &= FO^2 - OB^2, \quad (\because AO = FO) \\
 &= FB^2, \quad (\text{Pythagoras}) \\
 &= \text{area of square on } FB.
 \end{aligned}$$

The proof is similar if O is between B and F . Q.E.F.

Cor. 1. A square can be drawn, using only the ruler and compasses, equivalent to a given polygon.

For, with ruler and compasses, we can construct a rectangle equivalent to a given polygon (p. 89), and then, with the same means, construct a square equivalent to that rectangle.

Cor. 2. If ABC be a right-angled triangle, D the foot of the perpendicular from B on to the hypotenuse AC , then

$$AD \cdot CD = BD^2.$$

The problem of constructing, with ruler and compasses, a square equivalent to a given circle is called the problem of 'squaring the circle'. It was recognised as one of the stock problems in ancient Greece, and was never solved. It was proved in 1882 that, as had long been suspected, the problem is insoluble, i.e. it was proved that it is impossible to construct, with ruler and compasses, a square equivalent to a given circle. This proof is far too difficult to be inserted in this book.

EXERCISE 53

1. Draw the following figures, and then construct a square equivalent to them. Measure the length of a side of the square.

(a) A parallelogram $ABCD$ with $AB = 5$ cm, $BC = 3$ cm, $\hat{ABC} = 45^\circ$.

(b) An equilateral triangle on a base of 3".

(c) An isosceles triangle, whose base is 2" long and whose vertical angle is 70° .

(d) A quadrilateral $ABCD$ where $AB = 4$ cm, $BC = 3.5$ cm, $AD = 1$ cm., $\hat{ABC} = 110^\circ$, $\hat{BAD} = 90^\circ$.

(e) A quadrilateral whose diagonals are 4" and 3" long and which contain an angle of 50° . Have all such quadrilaterals the same area?

(f) A triangle whose perimeter is 10", and whose base angles are 40° and 50° .

(g) A triangle whose base is 3", height 4", and vertical angle 40° .

2 (a) Prove that of all rectangles of given perimeter, the square has the greatest area

(b) Prove that of all rectangles of given area, the square has the least perimeter

§ 34. Rectangle properties of a circle.

Def If AB be an interval, and P a point between A and B , we say P divides the interval AB 'internally' into the intervals AP and BP . If P is on AB produced beyond A or B , we say P divides the interval AB 'externally' into the intervals PA and PB .

THEOREM 52

If two chords of a circle cut at a point (within the circle), the rectangle on the intervals into which this point divides one chord equals the rectangle on the intervals into which it divides the other chord

Given Two chords AB and CD of a circle, centre O , cutting in P .

To prove that $PA \cdot PB = PC \cdot PD$

Proof Let X be the foot of the perpendicular from O to AB . Then $AX = BX$, (perp from centre bisects chord)

P may be between X and A , or between X and B

Suppose P between X and A . Then

$$PA = AX - XP, \quad PB = BX + XP = AX + XP.$$

$$\begin{aligned} \therefore PA \cdot PB &= (AX - XP)(AX + XP) = AX^2 - XP^2 \\ &= (AX^2 + OX^2) - (XP^2 + OX^2) \\ &= OA^2 - OP^2, \quad (\text{Pythagoras}) \end{aligned}$$

$$\text{Similarly,} \quad PC \cdot PD = OA^2 - OP^2.$$

$$\therefore PA \cdot PB = PC \cdot PD.$$

The proof is similar when P is between X and B

Q.E.D.

Cor. $PA \cdot PB = OA^2 - OP^2$.

Exercise We have assumed that AB does not go through O . Prove the theorem when AB does go through O

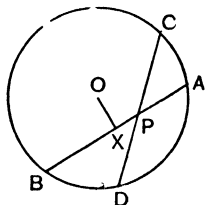


Fig 27.3

THEOREM 53.

If two chords of a circle, when produced, cut in a point, the rectangle on the intervals into which this point divides one chord externally equals the rectangle on the intervals into which it divides the other chord externally.

Given. Two chords AB , CD of a circle, centre O , when produced cut in P

To prove that $PA \cdot PB = PC \cdot PD$.

Proof. Let X be the foot of the perpendicular from O to AB .

Then $XA = XB$, (perp. from centre bisects chord).

P may be on AB produced beyond A or beyond B .

Suppose P is on AB produced beyond A . Then

$$PA = PX - XA, \quad PB = PX + XB = PX + XA$$

$$\begin{aligned} PA \cdot PB &= (PX - XA)(PX + XA) = PX^2 - XA^2 \\ &= (PX^2 + OX^2) - (XA^2 + OX^2) \\ &= OP^2 - OA^2, \quad (\text{Pythagoras}). \end{aligned}$$

Similarly, $PC \cdot PD = OP^2 - OA^2$.

$$PA \cdot PB = PC \cdot PD.$$

If P is on AB produced beyond B , the proof is similar. Q.E.D.

Cor. 1. $PA \cdot PB = OP^2 - OA^2$

Exercise. We have assumed that AB does not go through O . Prove the theorem when AB does go through O .

Cor. 2 If a point be taken on a chord of a circle produced, the rectangle on the intervals into which the point divides the chord equals the square on the tangent from the point to the circle.

Given. AB is a chord of a circle, centre O . P is on AB produced beyond B , T is the point of contact of the tangent from P to the circle

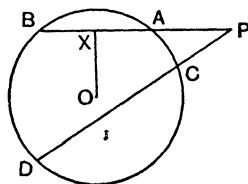


Fig. 274

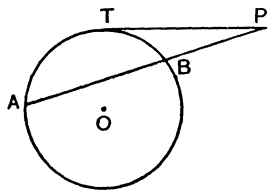


Fig. 275

To prove that $PT^2 = PA \cdot PB$.

Proof. $PA \cdot PB = OP^2 - OA^2$, (repeat above proof)
 $= OP^2 - OT^2$
 $= PT^2$, (Pythagoras).

Q.E.D.

THEOREM 54.

If the intervals AB and CD be divided, either both internally or both externally, by the same point O , and if $OA \cdot OB = OC \cdot OD$, then A, B, C, D are cyclic.

Given The lines AB, CD cut in O , $OA \cdot OB = OC \cdot OD$.

To prove that A, B, C, D are cyclic.

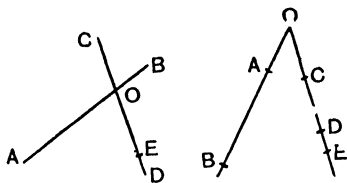


Fig 276

Proof. Draw the circle ABC .

Let it cut the line OC again in E .

If O divides the interval AB (and hence the interval CD) internally, then O is inside the circle, $\therefore$ it divides the interval CE internally.

If O divides the interval AB (and hence the interval CD) externally, then O is outside the circle, $\therefore$ it divides the interval CE externally.

And in both cases

$$OA \cdot OB = OC \cdot OE, \quad (A, B, C, E \text{ cyclic})$$

$$\text{But} \quad OA \cdot OB = OC \cdot OD, \quad (\text{given}).$$

$$\therefore OC \cdot OE = OC \cdot OD. \quad \therefore OE = OD.$$

$\therefore E$ and D coincide.

A, B, C, D are cyclic.

Q.E.D.

THEOREM 55.

If O divides the interval AB externally, and T be any point, not on the line AB , such that $OT^2 = OA \cdot OB$, then the circle ABT touches OT at T .

Given. O divides the interval AB externally, T is a point not on line AB , $OT^2 = OA \cdot OB$.

To prove that circle ABT touches OT at T .

Proof. If possible, let the circle ABT meet the line OT again at P .

Then, since O divides the interval AB externally,
 $\therefore O$ is outside circle ABT and $\therefore O$ divides TP externally, and also

$$OA \cdot OB = OT \cdot OP, \quad (A, B, T, P \text{ cyclic})$$

$$\text{But} \quad OA \cdot OB = OT^2, \quad (\text{given}).$$

$$\therefore OT \cdot OP = OT^2. \quad \therefore OP = OT.$$

But this is impossible, since P and T are distinct

$\therefore$ the circle ABT cannot meet the line OT again — it touches it at T . Q.E.D.

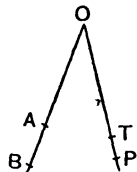


Fig 277

EXAMPLE 58 Circles are drawn through two given points A, B to cut a given circle, prove that the common chords with the given circle, produced if necessary, cut in the same point or are parallel

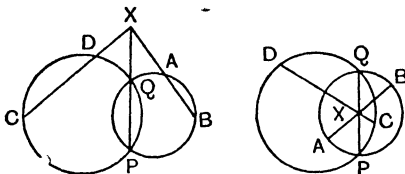


Fig 278

(i) *Given.* Two points A, B and a circle. A circle through A, B cuts the given circle in P, Q . The lines PQ and AB cut in X .

To prove that any other circle through A, B which cuts the given circle has with it a common chord, which, produced if necessary, goes through X .

Proof Let another circle through A, B cut the given circle in C, D , let the line XC cut the given circle again in D .

$$\text{Then} \quad XA \cdot XB = XQ \cdot XP, \quad (A, B, P, Q \text{ cyclic})$$

$$XQ \cdot XP = XC \cdot XD, \quad (P, Q, C, D \text{ cyclic})$$

$$\therefore XA \cdot XB = XC \cdot XD$$

$$A, B, C, D \text{ are cyclic}$$

$\therefore CD$ is a chord of the given circle and the circle $ABCD$
the common chord goes through X .

(ii) If AB and one common chord were parallel, all common chords would be parallel to AB , for if AB cut any common chord in X , it would, by (i), cut them all in X . Q.E.D.

Exercise Find two points on a given circle, which are cyclic with given points A, B , and with given points C, D

EXAMPLE 59. *Through two given points to draw a circle to touch a given circle*

Given A circle, and two points A, B

To draw through A, B a circle touching the given circle.

Construction Draw any circle through A, B cutting the given circle in P, Q . Suppose the lines AB and PQ cut in X , outside the given circle. From X draw a tangent to the given circle, let T be its point of contact.

Then the circle ABT touches the given circle at T .

Proof $XA \cdot XB = XP \cdot XQ$, (A, B, P, Q are cyclic)

$XP \cdot XQ = XT^2$, (circle PQT touches XT at T)

$XA \cdot XB = XT^2$

circle ABT touches XT at T

If we draw the other tangent from X to the given circle, we can find a second circle satisfying the required conditions.

If the lines AB, PQ are parallel, draw to the given circle a tangent parallel to them, let its point of contact be T . Then ABT is a circle required.

If the lines AB, PQ cut inside the given circle, the problem has no solution. Q F F.

EXAMPLE 60 *To draw a circle to touch a given line, and to go through two given points on the same side of the line*

Given A line XY , and two points A, B on the same side of it

To draw a circle through A, B to touch XY

Construction Take any point C on XY , and draw the circle ABC .

Let XY cut the circle again in D .

Suppose the lines AB and CD cut in P . Then P is outside the circle. Let T be the point of contact of a tangent from P to the circle.

Take U on the line XY so that $PU = PT$.

Then the circle ABU touches XY at U .

Proof. $PU^2 = PT^2 = PA \cdot PB$, (PT is tangent to circle ABT)

PU touches the circle ABU

the circle ABU touches the line XY

There are two solutions, corresponding to the two positions of U .

If the lines AB and XY are parallel, construct, as before, the circle ABC , and take U the mid-point of interval CD .

Then ABU is the circle required.

Q E F.

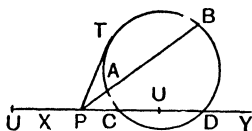


Fig. 279

EXAMPLE 61 From what point on a fixed horizontal line does a chink in a door appear to be widest?

Given A, B two fixed points,
 XY a given line

To find a point P on XY such
that $\hat{APB}$ is as large as possible

Construction. (i) Suppose the
lines AB, XY are not parallel

Draw the circles ABP and ABQ
to touch XY in P, Q resp

Let AB cut XY in K

Then $\hat{APB} > \hat{ARB}$ where R is any point on the ray KP

$\hat{AQB} > \hat{ASB}$ where S is any point on the ray KQ

Proof Either AR cuts the arc APB or BR cuts the arc APB (or both may do so) Let AR cut the arc APB in V

Then $\hat{APB} = \hat{AVB}$, (same segment)

$> \hat{ARB}$, (ext $\angle$ of $\triangle BVR >$ int opp $\angle$)

Similarly,

$\hat{AQB} = \hat{ASB}$

Thus the point C required is either P or Q , according to which of the angles $\hat{APB}, \hat{AQB}$ is the greater

(ii) If AB and XY are parallel, the point
required is the foot P of the perpendicular
on XY from the mid-point of AB

Proof The circle ABP touches the line
 XY in P

Let R be any point on XY Either AR
cuts the arc APB or BR cuts the arc APB
(or both may do so)

Let AR cut the arc APB in V

Then $\hat{APB} = \hat{AVB}$, (same segment)

$> \hat{ARB}$, (ext $\angle$ of $\triangle BVR >$ int opp $\angle$)

the point P is the point required

Q.E.F.

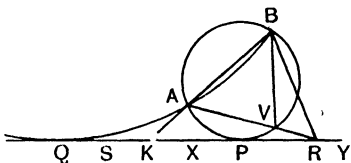


Fig 280

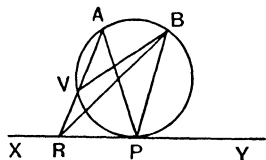


Fig 281

EXERCISE 54

1 A circular arch of height 4 ft is on a base of 6 ft. What is its radius? (Use Theorem 52)

2 A bowl is in the form of a piece of a sphere, its breadth is 10 ft, its greatest depth is 3 ft. What is the radius of the sphere?

3 Assuming that the earth is a sphere of radius 4000 miles, how far should a man be able to see from the top of a tower (i) 100 ft, (ii) 300 ft, (iii) 1000 ft high?

4 Prove that from the top of a tower h ft. high, a man can see $\sqrt{3h/2}$ miles

5 Two towers are 500 and 800 ft high. The top of one can just be seen from the top of the other. Find their greatest possible distance apart.

6 A window AB is 9' high. It subtends equal angles at points C and D in the street, $OC = 4'$, $CD = 5'$. Find the height of A above the street (Fig. 282).

7. If in the same figure $OA = 5'$, $AB = 7'$, $CD = 4'$, find the length OC .

8 A man walking along a straight road observes two trees. They appear to widen out until he reaches the point A . At the point B they are in line, and the nearest is 20 ft. away. The distance from A to B is 40 yds. What is the distance between the trees?

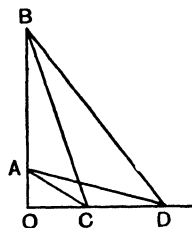


Fig. 282

9 In a game of Rugby football, a try is scored at the point A . Where is the best point from which to kick the goal?

EXERCISE 55.

1 P, Q are points on chords AB, CD resp. of a circle,

$$PA \cdot PB = QC \cdot QD$$

Prove that P, Q are equidistant from the centre

2 (a) P is a point on the common chord produced of two circles. Prove that tangents from P to the two circles are equal.

(b) Prove that the common chord produced bisects the common tangents of the circles.

3 (a) P is a point on the common chord, or that chord produced, of two circles. Through P are drawn two lines, one cuts one circle in A, B , the other line cuts the other circle in C, D . Prove that A, B, C, D are cyclic.

(b) If T be the point of contact of a tangent from P to the second circle, then circle ABT touches PT at T .

4. $ABCD$ is a convex quadrilateral, the diagonals AC, BD cut in X , the sides AD, BC produced cut in Y , $AX \cdot XC = BX \cdot XD$. Prove $YA \cdot YD = YB \cdot YC$.

5. (a) Two chords AB, CD of a circle cut in X , L, M bisect the intervals XC, XB resp. Prove A, D, L, M are cyclic.

(b) In the same figure the intervals XA, XD are produced to P, Q so that $XA = AP, XD = DQ$. Prove P, Q, L, M are cyclic.

6 A line through the vertex A of an isosceles triangle ABC cuts the base in D and the circumcircle in E . Prove that $AB^2 = AD \cdot AE$.

7 C is any point on the diameter AB of a semicircle, the perpendicular to AB at C cuts the semicircle in D and the chord AF in E . Prove

$$AE \cdot AF = AC \cdot AB = AD^2 \quad (\text{Fig. 283})$$

8 X, Y are points on the chords AB, AC of a circle, XY is parallel to the tangent at A . Prove $AB \cdot AX = AC \cdot AY$.

9 (a) The altitudes AP, BQ, CR of a triangle ABC cut in H . Prove that $AH \cdot HP = BH \cdot HQ$

(b) $CQ \cdot CA = CP \cdot CB = CH \cdot CR$

10 (a) Two circles cut in A, B , a third circle cut one in P, Q ; the other in X, Y . Prove that the lines AB, PQ, XY are concurrent.

(b) Two circles touch in T , a third circle touches one in A , the other in B . Prove that the tangents to the circles at T, A, B are concurrent.

11 A, B are the centres of two circles, T is the point of contact of the tangent from A to the circle, centre B , S is the point of contact of the tangent from B to the circle, centre A , AT and BS cut in X . Prove $AX \cdot XT = BX \cdot XS$.

12 Two circles cut in A, PAQ, RAS are lines through the centres of the circles, cutting the circles in P, Q, R, S . Prove that

$$PA \cdot AQ = RA \cdot AS \quad (\text{Fig. 284})$$

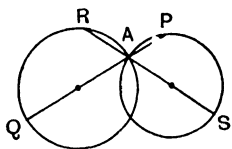


Fig. 284

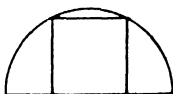


Fig. 285

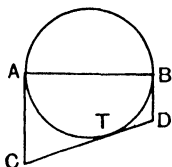


Fig. 286

13 A square is described in a semicircle (Fig. 285). Prove that its area $= \frac{1}{8} \text{diam}^2$.

14 AB is a diameter of a circle (Fig. 286), a tangent at a point T on the circle cuts the tangents at A, B in C, D resp. Prove $AC \cdot BD = \text{sq on radius}$.

15 A, B, C, D are in that order on a line. Find a point O on the line so that $OA \cdot OB = OC \cdot OD$ (Through A, B and through C, D draw any circles to cut each other).

16 C, D are points on a semicircle on AB , BC and AD cut at a point X inside the semicircle. Prove that

$$AB^2 = DA \cdot AX + CB \cdot BX.$$

(Apollonius and Theorem 52)

***17** CD is a chord of a circle parallel to the diameter AB , XY is a common perpendicular to AB and CD , X is on CD , Y on AB (Fig 287). Prove that

$$AY \cdot YB = CX \cdot XD + XY^2$$

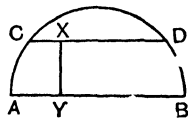


Fig 287

***18** ABC is a triangle. Find a point X on base BC , such that $AX^2 = BX \cdot XC$.

***19** The altitudes AP, BQ, CR of $\triangle ABC$ cut in H . Prove that $BP \cdot PC = AP \cdot PH$.

***20** C is a point on the diameter AB of a circle, PC is perpendicular to AB , T the point of contact of the tangent from P . Prove $PC^2 = PT^2 + AC \cdot CB$.

***21** A, B are on the same diameter of a circle, centre O , equidistant from O , CD is any chord through A (Fig 288). Prove that

$$BC^2 + BD^2 + CD^2$$

is independent of the chord. (Apollonius and Theorem 52)

***22.** P is a point outside a circle at a distance from the centre not greater than the diameter. Draw a line through P cutting the circle in A, B such that $PB = AB$.

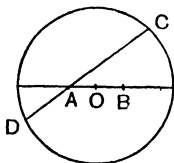


Fig 288

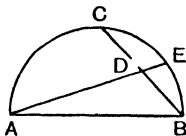


Fig 289

***23** Two circles cut in A, B , the common tangents CD, EF cut the line AB in G, H . Prove $GH^2 = AB^2 + CD^2$.

***24** C is the mid-point of the semicircle on diameter AB ; D is the mid-point of chord BC , AD cuts the semicircle again in E . Prove that $AD = 5DE$. (Fig 289). Calculate AD^2 from Apollonius. Then use Theorem 52.)

***25.** Show how to draw a circle through two given points and cutting a given circle, so that the common chord has a given length.

***26.** C is any point on a semicircle on diameter AB ; X is any point on the diameter; the perpendicular to AB at X cuts the lines AC , BC in Z , Y , and the circle in P (Fig. 290) Prove $XP^2 = XY \cdot XZ$.

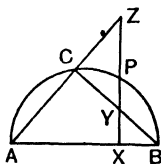


Fig. 290

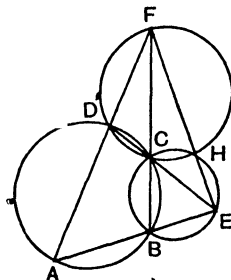


Fig. 291

***27.** (a) $ABCD$ is a cyclic quadrilateral (Fig. 291) The sides AB and CD produced cut in E , the sides AD and BC produced cut in F . Prove that $FE^2 = \text{sum of squares of tangents to circle } ABC \text{ from } F \text{ and } E$. (Prove that circles DCF and BCE cut on the line FE , in H , say Pivot theorem Then $FE^2 = FH \cdot FE + EH \cdot FE$)

(b) Hence by means of Apollonius' Theorem, prove that if the circle on diameter FE and the circle ABC cut in X , Y , then the tangents at X , Y to one circle go through the centre of the other.

§ 35. Geometrical illustrations of algebraical identities

We have used Algebra freely in all questions that involved the sums and products of the lengths of intervals. When Geometry was first studied, Algebra had not been invented, and the Geometers of Ancient Greece used certain theorems on the dissection of figures, where we should now more naturally use algebra. Unfortunately examining bodies still ask for proofs of these theorems, and for this reason we give them.

THEOREM A.

Geometrical illustration of the identity $a(b + c) = ab + ac$.

Let A , B , C be three points on a line, with B between A and C , and

let the interval AB be b units long,
and the interval BC be c units long

Draw the rectangle $ACDE$, such that the side CD is a units long.

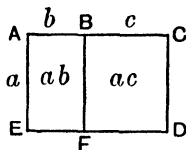


Fig. 292

Let the parallel to AE through B cut DE in F .

Then $ABFE$ and $BCDF$ are rectangles.

Area of rect. $AD = a(b + c)$ sq. units,

$$AF = ab$$

$$CF = ac.$$

But rect $AD = \text{rect } AF + \text{rect } CF$

This fact corresponds to the identity $a(b + c) = ab + ac$.

THEOREM B.

Geometrical illustration of the identity $(a + b)^2 = a^2 + 2ab + b^2$.

Let A, B, C be three points on a line,
with B between A and C , and

let the interval AB be a units long,

and the interval BC be b units long.

Then the interval AC is $a + b$ units long

Draw the square $ACDE$ on the side AC .

Cut off the interval $CF = BC$ from the
side CD

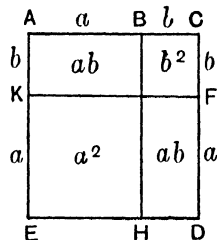


Fig 293

Through B, F draw parallels to the sides
of the square.

These divide the square into squares and rectangles.

Area of square $KHIJ = a^2$ sq. units,

rect. $KBFI = ab$,

rect. $FHIJ = ab$,

square $BFCD = b^2$,

square $AD = (a + b)^2$.

But

square $AD = \text{square } KHIJ + \text{rect. } KBFI + \text{rect. } FHIJ + \text{square } BFCD$.

This fact corresponds to the identity

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Exercise Give a geometrical illustration of the identity

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

THEOREM C.

Geometrical illustration of the identity $(a - b)^2 = a^2 - 2ab + b^2$.

Let A, B, C be three points on a line, with C between A and B , and

let the interval AB be a units long,
and the interval BC be b units long.

Then the interval AC is $a - b$ units long

Draw the square $ABDE$ on the side AB .

Cut off the interval $BF = BC$.

Through C, F draw parallels to the sides of the square.

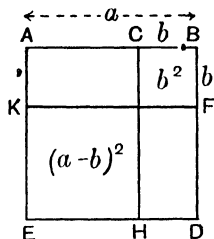


Fig. 294

These divide the square into squares and rectangles

Area of square $AD = a^2$ sq. units,

rect $AF = ab$,

square $FC = b^2$,

rect $CD = ab$,

square $KH = (a - b)^2$

But

square $KH = \text{square } AD - \text{rect } AF - \text{square } FC - \text{rect } CD$.

This fact corresponds to the identity

$$(a - b)^2 = a^2 - 2ab + b^2.$$

THEOREM D.

Geometrical illustration of the identity $a^2 - b^2 = (a - b)(a + b)$.

Let A, B, C be three points on a line, with C between A and B , and

let the interval AB be a units long,

and the interval BC be b units long.

Then the interval AC is $a - b$ units long.

Draw the square $ABDE$ on the side AB ,
and the square $CBFN$ on the same side of BC .

Produce the side AE beyond E to K , and
make $EK = BF$.

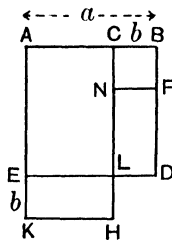


Fig. 295

Produce the side CN so as to make the rect. $EKHL$.

Then rect. $ND = \text{rect. } EH$.

$$\therefore \text{rect. } AL + \text{rect. } ND = \text{rect. } AH.$$

But $\text{rect. } AL + \text{rect. } ND = a^2 - b^2$ sq. units,
and $\text{rect. } AH = (a - b)(a + b)$ sq. units.

These facts correspond to the identity

$$a^2 - b^2 = (a - b)(a + b).$$

§ 36. Use of similar triangles.

If we have two intervals of lengths 3 in. and 5 in., we say the '*ratio*' of the first to the second is 3 5 or 3/5. If we have four intervals of lengths a, b, c, d , and the ratio a/b equals the ratio c/d , we say the intervals (or their lengths) are '*in proportion*' and write

$$a : b = c : d, \text{ or } a/b = c/d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

If $a : b = c : d$, then d is called the '*fourth proportional*' to a, b, c .

If $a : x = x : b$, then x is called the '*mean proportional*' between a and b .

We shall often need the following algebraical results:

If $a : b = c : d$, then

$$\begin{aligned} b : a &= d : c, \\ a : c &= b : d, \\ ad &= bc, \\ a + b : b &= c + d : d, \\ a - b : b &= c - d : d, \\ a : a + b &= c : c + d, \\ a + b : a - b &= c + d : c - d. \end{aligned}$$

EXAMPLE 62. If $a : b = c : d$, to prove $a + b : b = c + d : d$.

Denote the fraction a/b by k , then $a = bk$; $c/d = k$. $\therefore c = dk$.

$$\begin{aligned} \frac{a+b}{b} &= \frac{bk+b}{b} = \frac{b(k+1)}{b} = k+1, \\ \frac{c+d}{d} &= \frac{dk+d}{d} = \frac{d(k+1)}{d} = k+1. \end{aligned}$$

$$a + b : b = c + d : d.$$

Exercise. Prove the other results above in the same way.

To illustrate the use of these, suppose in Fig. 296, that

$$AB : BC = XY : YZ.$$

Then we have

$$BC : BA = YZ : YX,$$

$$AB \cdot XY = BC \cdot YZ,$$

$$\text{and } AB + BC : BC = XY + YZ : YZ,$$

$$\text{i.e. } AC : BC = XZ : YZ.$$

$$\text{Similarly, } AC : AB = XZ : XY.$$

$$\text{Also } AB \cdot YZ = BC \cdot XY.$$

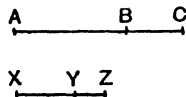


Fig 296

EXERCISE 56.

- Find the value of x in the following proportionalities.
 1. $5 : 6 = 3 : x$, 2. $3 = x : 8$, 3. $x : 8 = a : b - c$, 4. $a : x = x : b$
- (a) Write down all the proportionalities which follow from $a : b = c : d$ and which are obtained by merely permuting the letters.
 (b) Do the same for $a : b = b : c$.
- If $a : b = 3 : 4$, write down the values of $a + b$, b , $a - b$, b , $a + b$, $a - b$.
- When is $a : b = a^2 : b^2$? If $a : b = x : y$ and $c : d = z : w$, prove that $ac : bd = xz : yw$.
- Solve the equations. (i) $x + 3 : x - 3 = 9 : 8$
 (ii) $x + a : x - a = b : c$.
- Find the fourth proportional to 5, 6, 7, and the mean proportional of 5, 6.

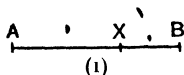


Fig 297

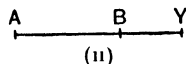


Fig 298

Def. If X is on the interval AB , it is said to 'divide the interval internally in the ratio $AX : XB$ ' (Fig. 297 (i)).

If Y is on the interval AB produced, it is said to 'divide the interval externally in the ratio $AY : YB$ ' (Fig. 297 (ii)).

An interval may be divided internally and externally in the same ratio. For instance (Fig. 298), if $AX = 2XB$, and $AY = 2AB$, then $AX : XB = 2 : 1$ and $AY : YB = 2 : 1$. $AX \cdot XB = AY \cdot YB$.

Def. If X , Y divide the interval AB internally and externally in the same ratio, so that $AX \cdot XB = AY \cdot YB$, then A , X ,

B , Y is called a 'harmonic range,' and X , Y are said 'to separate A , B harmonically.'

Since $AX \cdot XB = AY \cdot YB$ implies $XB \cdot BY = XA \cdot AY$, therefore, if X , Y separate A , B harmonically, then A , B separate X , Y harmonically.

7 Draw an interval AB , 2" long, and on it mark points X so that the ratio $AX \cdot XB$ has the values resp 1 · 1, 2 · 1, 3 · 2.

8 (a) An interval AB , 2" long, is divided internally and externally by X , Y resp, so that $AX \cdot XB = AY \cdot YB = 3 \cdot 1$. Find the lengths of AX , XB , AY , YB .

(b) Do the same question when the ratio is 4 · 1

9 An interval AB of length l is divided internally and externally by X , Y resp in the ratio $a : b$. Find the lengths of AX , BX , AY , BY .

10 (a) C divides the interval AB internally in the same ratio as Z divides the interval XY externally, $AC = 7''$, $XY = 3''$, $YZ = 2''$. Find BC .

(b) If $AB = 10''$, $XY = 1''$, $YZ = 10''$, find AC .

11 (a) C divides AB internally in the ratio 3 · 1, D divides AC internally in the ratio 4 · 1. Find $AD \cdot DB$.

(b) C divides AB internally in the ratio 4 · 1, D divides AC externally in the ratio 6 · 1. Find $AD \cdot DB$.

12 The interval AB is divided by X , Y internally and externally in the same ratio, O is the mid-point of AB . Prove $OX \cdot OY = OA^2$ (From $AX \cdot XB = AY \cdot YB$ follows

$$AX + XB = AX - XB = AY + YB = AY - YB)$$

13 If O is the mid-point of the interval AB , and X , Y are on the ray OB such that $OX \cdot OY = OA^2$, prove that A , X , B , Y is a harmonic range.

We defined similar triangles as triangles with the corresponding angles equal, and we noted that they had the same shape. We can now express the last fact more precisely by saying that the corresponding sides of the triangles are proportional. This leads us to the statement in the following assumption.

Assumption 16. If two triangles be such that the angles of one equal the corresponding angles of the other, then the corresponding sides are proportional.

Thus if in the triangles ABC , XYZ of Fig. 299 we know that $\hat{A} = \hat{X}$, $\hat{B} = \hat{Y}$ (and $\hat{C} = \hat{Z}$), then it follows that

$$AB \cdot BC = XY \cdot YZ \text{ and } BC \cdot CA = YZ \cdot ZX.$$

We write these proportionalities together as

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

Note also that

$$\frac{AB}{BC} = \frac{XY}{YZ}.$$

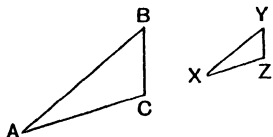


Fig. 299

A useful dodge When you state two triangles are similar, always write the three letters representing each triangle, in such an order, that the first letters correspond to equal angles, the second to equal angles and the third letters to equal angles (This is useful even in questions on congruence, and we have usually done so.) Thus when we say ABC and LPX are similar triangles, we shall always mean $\hat{A} = \hat{L}$, $\hat{B} = \hat{P}$, $\hat{C} = \hat{X}$. The corresponding proportions can then be written down at once without using a figure, as follows

$$\frac{ABC}{LPX} \quad \frac{AB}{LP} = \frac{BC}{PX} \quad \frac{AC}{LX} = \frac{AB}{LP} \quad \frac{BC}{PX} = \frac{AC}{LX} \text{ and so on.}$$

THEOREM 56

Given. BAC is a right angle, D is the foot of the perpendicular from A to BC

To prove that

$$BA^2 = BD \cdot BC, \quad CA^2 = CB \cdot CD,$$

$$DA^2 = DB \cdot DC.$$

Proof In $\triangle ABC$, DBA ,

$$\begin{cases} \hat{BAC} = \hat{BDA}, & (\text{rt. } \angle\text{s}) \\ \hat{ABC} = \hat{DBA}. \end{cases}$$

$\triangle ABC$, DBA are similar.

$$\therefore \frac{BA}{BC} = \frac{BD}{BA} \quad [\text{using the above rule}]$$

$$BA^2 = BD \cdot BC.$$

Similarly,

$$CA^2 = CB \cdot CD.$$

In $\triangle ABD$, CAD ,

$$\begin{cases} \hat{ABD} = \hat{CAD}, & (\text{each} = 1 \text{ rt. } \angle - \hat{BAD}) \\ \hat{ADB} = \hat{CDA}, & (\text{rt. } \angle\text{s}). \end{cases}$$

$\triangle ABD$, CAD are similar.

$$\therefore \frac{DA}{DB} = \frac{DC}{DA}.$$

$$DA^2 = DB \cdot DC.$$

Q.E.D.

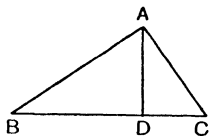


Fig. 300

In Theorem 56, the interval AD is the mean proportional between the intervals BD and DC . The mean proportional is also called the 'geometric mean'.

Exercise Give a construction for drawing the mean proportional between two given intervals.

Note. The first part of Theorem 56, viz $BA^2 = BD \cdot BC$, was proved, by areas, in the middle of the Theorem of Pythagoras (see p. 80, Example 14). Adding the first two results in Theorem 56 we have a *proof of the Theorem of Pythagoras, without the use of areas*. The third part of Theorem 56 is a statement of the property used in constructing a square equivalent to a given rectangle (Construction 20, p. 192). But, important as Theorem 56 is, it is only a special case of Theorems 52 and 53, Cor. 2. For BA is a tangent to the circle ADC .

$BA^2 = BD \cdot BC$. Also, if the line AD cut the circle ABC again in E , then $AD = DE$, $BD \cdot DC = AD \cdot DE$ (Theorem 52).

$BD \cdot DC = DA^2$. We will now show Theorems 52 and 53 by the same method.

Proof of THEOREMS 52 and 53 by similar triangles.

Given. AB and CD are two chords of a circle which, produced if necessary, cut in P .

To prove that $PA \cdot PB = PC \cdot PD$.

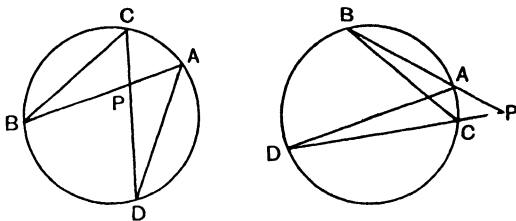


Fig. 301

Proof. In $\triangle s PBC, PDA$,

$$\begin{cases} \hat{PBC} = \hat{PDA}, & (\text{same segment}) \\ \hat{BPC} = \hat{DPA}, & (\text{opp. } \angle s). \end{cases}$$

$\therefore \triangle s PBC, PDA$ are similar

$$\therefore \frac{PB}{PC} = \frac{PD}{PA}.$$

$$\therefore PA \cdot PB = PC \cdot PD.$$

Q.E.D.

We shall shortly examine more closely the fundamental theorems on similar triangles, and prove that they can be deduced from the following special case of Assumption 16. *if the line XY be parallel to the base BC of triangle ABC and cut the sides AB, AC or those sides produced in X, Y resp., then $AX \cdot AB = AY \cdot AC$. Hence*

$$AX : XB = AY : YC$$

The converse is *if X, Y be on the rays AB, AC or on the intervals BA, CA produced beyond A , and $AX \cdot AB = AY \cdot AC$ (or if $AX : XB = AY : YC$), then XY is parallel to BC* . Both these may be used in the following exercises

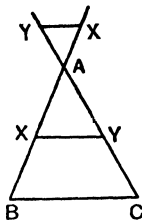


Fig. 302

EXERCISE 57

1. ABC and XYZ are similar triangles, AP and XQ are altitudes, $AP = 3''$, $XY = 4''$, $AP = 5''$. Find XQ .

2. The circumradii of two similar triangles ABC, XYZ are $5''$, $3''$ resp. $AB = 2'$. What is the length of XY ?

3. X, Y are points on the sides AB, AC of $\triangle ABC$, $AB = 3''$, $BC = 4''$, $CA = 3.5''$, $AX = 2''$. Find AY, XY .

4. Two chords AB, XY of a circle cut in O , $OA = 4''$, $OB = 3''$, $OX = 2''$, $BX = 2.5''$. Find AY .

5. OAB is a triangle, a circle through A, B cuts OA, OB again in X, Y . $OX = 2'$, $AX = 4''$, $OB = 8''$, $AB = 5''$. Find XY .

6. (a) ABC and XAB are similar triangles; $AB = 3''$, $BC = 4''$, $CA = 3.5''$. Find AX, XB .

(b) If ABC, XAB are similar triangles, prove $XA \cdot BC = AB^2$.

7. The tangent at A to the circle ABC cuts the parallel through C to AB at T . If $AB = 6''$, $BC = 7''$, $AC = 8''$, find AT, CT .

8. Two lines ABC, XYZ are cut by three parallel lines resp. in A, B, C and in X, Y, Z . If $AB = 4''$, $BC = 7''$, $XY = 3''$, $AX = 8''$, $BY = 10''$, find YZ, CZ .

9. C is a point on the base AB of $\triangle OAB$, X, Y are on sides OA, OB resp., $XY \parallel AB$, XY cuts OC in Z . If $OX = 4''$, $XA = 1''$, $OC = 6''$, $OB = 7''$, find CZ, BY .

10. $ABCD$ is a trapezium, $AB \parallel CD$, AC, BD cut in X , $AB = 7'$, $BC = 4''$, $CD = 6''$, $DA = 3''$, $AC = 5''$. Find AX, CX .

11. ABC is a triangle, right-angled at A ; AD is an altitude

(a) Given $AB = 6''$, $AC = 5''$, find BD, DC, AD .

(b) Given $BD = 7''$, $CD = 5''$, find AB, AC, AD .

(c) Given $AB = 5''$, $BD = 4''$, find BC, CA, AD .

12. P is a point at a distance of 8" from a circle of radius 5". Find the length of the chord of contact of the tangents from P to the circle.

13. Two circles of radii 5", 7" touch externally at A , a line through A cuts them in P, X resp. $PX = 1' 2''$. Find PA, AX .

14. P is a point at a distance of 9" from a circle of radius 6". Find the distance, from the centre, of the chord of contact of tangents from P to the circle.

15. Two vertical posts are of heights 6 ft. and 11 ft. resp., there are two straight wires, one from the top point of each to the bottom point of the other. Find the height at which these wires cross when the posts are 1 ft., 2 ft., 3 ft. apart. Explain the results.

16. ABC is a triangle, the line XY cuts AB in X, AC in Y , $XY \parallel BC$. Prove $AB \cdot BX = AC \cdot CY$.

CONSTRUCTION 21

To divide an interval AB internally and externally in a given ratio

(i) Given an interval AB and two intervals of lengths x, y

To find a point P on the interval AB , so that $AP : PB = x : y$

Construction. Draw any ray AC from A , and on it take two points C, D , so that $AC = x$, $CD = y$, C between A and D .

Join DB and draw, through C , a parallel to DB to cut the line AB in P .

Then $AP : PB = x : y$.

Proof. Since $CP \parallel BD$,
 $AP : PB = AC : CD$
 $= x : y$.

(ii) Given, as before, with $x > y$.

To find a point Q on the line AB , so that $AQ : QB = x : y$.

Construction. Draw any ray AC from A , and on it take two points C, E , so that $AC = x$, $CE = y$, E between A and C .

Join EB and draw, through C , a parallel to EB to cut the line AB in Q .

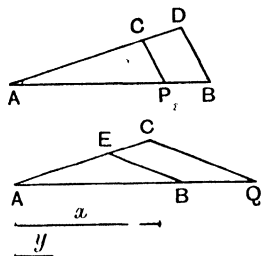


Fig. 303

Then

$$AQ \cdot QB = x \cdot y.$$

Proof. Since

$$EB \parallel CQ,$$

$$\begin{aligned} AQ \cdot QB &= AC \cdot CE \\ &= x \cdot y. \end{aligned}$$

Q.E.D.

EXERCISE 58

1 ABC is a triangle, the angles marked are equal. Prove that triangles ABC and XYZ are similar (Fig. 304)

2 L, M, N are the mid-points of the sides of triangle ABC . Prove that the triangles ABC and LMN are similar.

3 $ABCD$ is a trapezium ($AB \parallel CD$), whose diagonals cut in X . Prove that triangles AXB and CXD are similar.

4 AP, XW are altitudes of similar triangles ABC, XYZ . Prove $AB \cdot AP = XY \cdot XW$.

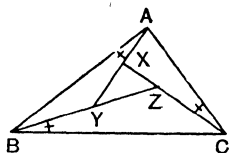


Fig. 304

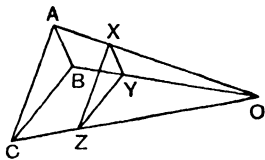


Fig. 305

5 (a) AP, BQ, CR are altitudes of $\triangle ABC$, cutting in H . Prove that $\triangle ABQ, ACR$ are similar, and that $\triangle AHR, ABP$ are similar.

(b) Hence prove $AQ \cdot AC = AR \cdot AB = AH \cdot AP$.

6 $ABCD$ is a trapezium, $AB \parallel CD$, its diagonals cut in X . Prove that $AX \cdot XB = XC \cdot XD$, $AX \cdot XC = BX \cdot XD$.

7 In Fig. 305, $AB \parallel XY$ and $BC \parallel YZ$. Prove $AC \parallel XZ$.

8 Tangents are drawn from P to a circle, centre O , the chord of contact cuts OP in N . Prove $ON \cdot OP = \text{square on radius}$.

9. (a) C is a point on the interval AB ; through C any line is drawn, X, Y are the feet of the perpendiculars from A, B on to the line. Prove that $AX \cdot BY$ is independent of the position of the line through C .

(b) A, B are given points. Draw a line, passing between A and B , such that the ratio of the perpendiculars from A, B on to it is given.

(c) A, B are given points. Draw a line, not passing between A and B , such that the ratio of the perpendiculars from A, B on to it is given.

(d) Draw a line such that the perpendiculars on to it from three given points are in a given ratio. How many such lines are there?

10 In Fig. 306, $AX = AY$. Prove that $BD \cdot DC = BX \cdot CY$.
(Through C draw a parallel to XY)

11 P moves along a line; O is a fixed point, $OP \cdot OQ = \text{const.}$ Q is on the ray OP . Prove that the locus of Q is a line

12 Two lines are cut by three parallel lines, the first in A, B, C , the second in X, Y, Z . Prove $AB \cdot BC = XY \cdot YZ$. (If the two lines are not parallel, draw, through A , a parallel to XY)

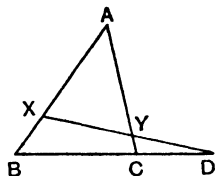


Fig. 306

13 By means of Theorem 52, give a construction for drawing, on a given base, a rectangle equivalent to a given rectangle.

14 M is the mid-point of the side AB of $\triangle ABC$, a line through M cuts the side BC in P , the side AC produced in Q , and the parallel to AB through C in X . Prove that M, X separate P, Q harmonically.

15 X, Y are points on the sides AB, AC of $\triangle ABC$, M is the mid-point of BC , XP and YQ are parallel to AM . Prove $BP = QC$.

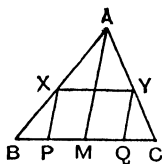


Fig. 307

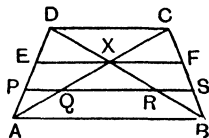


Fig. 308

16. (a) $ABCD$ is a trapezium; $AB \parallel CD$, its diagonals cut in X . A parallel to AB through X cuts AD in E and BC in F . Prove $EX = XF$ (Fig. 308)

(b) A parallel to AB cuts AD, AC, BD, BC resp in P, Q, R, S . Prove $PQ = RS$ (cf No. 12)

(c) Parallels through X to AD, BC resp cut AB in Y, Z . Prove $AY = BZ$.

17. $ABCD$ is a cyclic quadrilateral, the bisectors of the angles ACB and ADB cut one another in E , the first cuts BD in X , the second cuts AC in Y . Prove $EX \cdot EC = EY \cdot ED$

18 $AB \parallel EF$, and X is the mid-point of the interval EF , BX, AE cut in D, BF, AX cut in C . Prove $AB \parallel CD$

19 (a) APB and CQD are parallel lines; $AP \cdot CQ = PB \cdot QD$. Prove that AC, BD and PQ are parallel or concurrent, and that AD, BC, PQ are parallel or concurrent

(b) X, Y are the mid-points of the parallel sides AB, CD of the trapezium $ABCD$. Prove that XY, AC, BD are concurrent.

EXAMPLE 63 You are provided with a ruler with two marks on it. To draw, using this ruler alone, a parallel to a given line through a given point

Given a line XY , and a point P , not on it.

To draw through P a line parallel to XY , using only the above means. The ruler enables us to cut off a certain fixed distance d along any given ray from any given point.

Construction Take any point A on the line XY , and cut off the intervals AB , AC of length d , one on each side of A .

Join BP , and produce the interval BP beyond P to any point Z .

Let PC , AZ cut in Q , and BQ , CZ cut in R .

Then PR is parallel to BC .

Proof There is a point W on the interval QA produced beyond A such that

$$AW = AQ.$$

[The fact that we cannot *construct* it with our ruler does not matter, because we do not need it for the construction, but only for the proof, and it is sufficient, for the proof, if we know, as we do, that a point W with the required properties *exists*.]

Then $QBWC$ is a parallelogram, (diagonals bisect each other)

$$ZP : PB = ZQ : QW, \quad (PQ \parallel BW)$$

$$ZR : RC = ZQ : QW, \quad (RQ \parallel CW).$$

$$ZP : PB = ZR : RC$$

$$PR \parallel BC.$$

Q.E.F.

EXAMPLE 64 Given. D is a point on the line BC , A is a point not on the line.

To prove that

$$\frac{\triangle ABD}{\triangle ACD} = \frac{BD}{CD}.$$

Proof Let X be the foot of the perpendicular from A to BC .

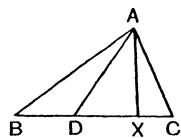


Fig 310

$$\text{Then} \quad \frac{\triangle ABD}{\triangle ACD} = \frac{\frac{1}{2}AX \cdot BD}{\frac{1}{2}AX \cdot CD} = \frac{BD}{CD}.$$

20. $\triangle ABC$ is a triangle, with the angle at A a right angle; AD is an altitude. Prove that $AB^2 \cdot AC^2 = DB \cdot DC$ (see Fig 300)

21. In Fig. 173 (p 114), PQ always goes through A . Prove that $BP \cdot BQ$ is constant.

22. Through M , the mid-point of the side AB of $\triangle ABC$, goes a line cutting the lines BC , AC in X , Y resp. Prove $BX \cdot XC = AY \cdot YC$. (Fig 311 Drop perpendiculars from A , B , C to M , Y)

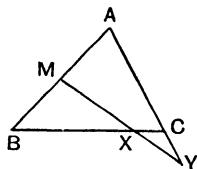


Fig 311

23. C is on the interval AB , D is a point not on the line AB . Find points X on the line AD so that $CX^2 = AC \cdot CB$

24. If, in Fig 300, $AB = 2AC$, prove that $5CD = BC$.

25. D is the foot of the perpendicular from C to the line AB , $CD^2 = AD \cdot DB$. Prove that $\hat{ACB}$ is a right angle

26. In Fig 312 $\hat{AOB}$ is a right angle, CD touches the circle in P , $PM \perp OA$. Prove

$$\frac{\triangle COD}{\triangle AOB} = \frac{\triangle AOB}{\triangle OMP}$$

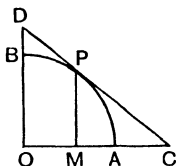


Fig 312

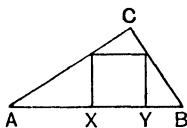


Fig 313

27. XY is a side of a square in a right-angled triangle ACB (Fig 313). Prove $XY^2 = AX \cdot YB$

The following example is not connected with similar triangles

EXAMPLE 65 To draw a perpendicular to a given line from any given point by means of a ruler with two marks on it (i.e. we can cut off a certain distance d along any given ray from any given point).

Given a line XY

To draw a perpendicular to XY from a given point

Construction If we have one perpendicular to XY , the required construction can be completed by means of Example 63. To draw one such perpendicular, take O any point on XY , and mark off intervals OA , OB along XY , of length d , draw two intervals OP , OQ of length d on the same side of XY and not along it.

Let AP , BQ cut in R , and BP , AQ cut in S

Then RS is perpendicular to XY

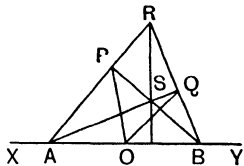


Fig 314

Proof $\hat{APB} = 1 \text{ rt } \angle, (OA = OP = OB)$

Similarly, $\hat{AQB} = 1 \text{ rt } \angle$

And since the altitudes of any triangle concur, $RS \perp AB$

Q.E.D.

§ 37. The fundamental theorems on congruence and parallels.

In the early part of this book we laid down some fundamental 'Assumptions' on the congruence of triangles, on parallels, and so on, and we deduced a number of propositions from these assumptions. We also made a great many assumptions which were taken for granted without being stated. We shall now examine *some* of these assumptions themselves—we shall not concern ourselves with the question as to whether they are true or not—we may here simply assume their truth—but we shall try to discover the relations between these assumptions, whether, for instance, we can deduce some of them from the others, or not.

It may perhaps seem unnecessary to do this, and, in fact, the question is, in the first place, one rather of logic than of Geometry, but experience in many branches of science has shown the importance of carefully analysing and arranging the simple facts assumed at the beginning of the science, instead of leaving them in a vague and confusedness.

It is obvious that if we are to prove anything at all, we must make some assumptions, because we must start somewhere. But we shall make fewer assumptions now than we have made hitherto, and we shall mention some of the assumptions which we made before, but did not actually state. In a completely logical Geometry, *all* the assumptions would be stated, and *nothing* would be assumed which could be proved from them, but a completely logical Geometry is hard to construct and hard to understand. What we give is a compromise, it is more logical than what is contained in the beginning of this book, but is still far from perfect†.

We shall make the following assumptions (as well as some others which we shall mention later on, and some which we shall leave unstated). Assumptions at the basis of a science are usually called '*Axioms*'.

† Those who desire to see a logical system of Euclidean Geometry may consult the author's *Foundations of Euclidean Geometry*, they will find there a full treatment of the many questions left unanswered in this book, and an analysis of the assumptions left unstated or unanalysed here.

Axiom 1. If XY is a given interval, and OA a given ray, we can find one, and only one, point B on the ray OA , such that the intervals XY and OB are equal.

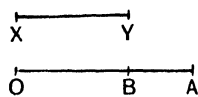


Fig 315

Axiom 2 If $\hat{ABC}$ is a given angle, and OX a given ray in a given plane, we can find one, and only one, ray OY on a given side of the ray OX , and in the given plane, such that $\hat{XOY}$ and $\hat{ABC}$ are equal.

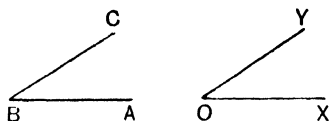


Fig 316

Axiom 3. If ABC and DEF be triangles, and $AB = DE$, $BC = EF$, $\hat{ABC} = \hat{DEF}$, then $\hat{BAC} = \hat{EDF}$, $\hat{BCA} = \hat{EFD}$, $AC = DF$. (Assumption 12, p 15)

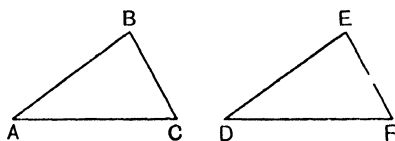


Fig 317

[We shall consider later a 'proof' which is often given of this Axiom. We may sum up the hypothesis of the Axiom by the words 'two sides, incl $\angle$ ']

Note The following result of Ax 2 should be noted. We cannot have two rays OB , OC on the same side of the ray OA , so that the angles $\hat{AOB}$ and $\hat{AOC}$ are equal. For, if there were two such rays, then on a given side of the ray OA , we should have two angles, viz. $\hat{AOB}$ and $\hat{AOC}$, equal to the given angle $\hat{AOB}$, instead of one and only one, as the Axiom states.

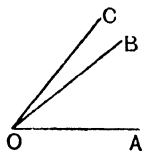


Fig 318

THEOREM I (Assumption 10, p. 11).

The angles at the base of an isosceles triangle are equal.

Given. ABC is a triangle, $AB = AC$.

To prove that $\hat{ABC} = \hat{ACB}$.

Proof. Treat the triangles ABC , ACB as if they were distinct triangles.

We have in the two triangles the following parts equal:

In $\triangle ABC$		In $\triangle ACB$
AB		AC
AC		AB
$\hat{BAC}$		$\hat{CAB}$ (two sides, incl. $\angle$).

Hence, by Axiom 3, $\hat{ABC} = \hat{ACB}$.

Q.E.D.

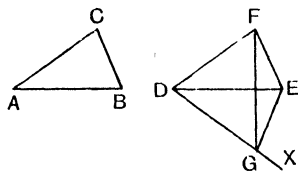
Note What we have shown is that this theorem is contained implicitly in Axiom 3 as a special case. In that Axiom, it is not necessary that ABC and DEF should be distinct triangles, and when the points D, E, F are at C, B, A resp. we have Theorem I at once.

'Congruent triangles' are defined as before, p. 14.

THEOREM II (Assumption 13, p. 17)

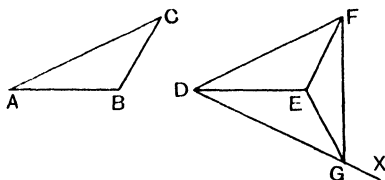
Given ABC and DEF are triangles, $AB = DE$, $BC = EF$, $CA = FD$.

To prove that the triangles ABC , DEF are congruent.



Case (i)

Fig. 320



Case (ii)

Proof. On the plane DEF and on the opposite side of DE from F there is a ray DX , such that $\hat{EDX} = \hat{BAC}$, (Ax. 2).

On the ray DX , there is a point G such that $DG = AC$, (Ax. 1).

Now $OA = PX$, $OC = PZ$, O is between A and C , P is between X and Z $AC = XZ$

In $\triangle CAB, ZXY$,

$AC = XZ$, (proved), $AB = XY$, (given), $\hat{BAC} = \hat{YXZ}$, (proved)

$BC = YZ$, $\hat{BCA} = \hat{YZX}$, (Ax 3), $\therefore \hat{BCO} = \hat{YZP}$.

In $\triangle BCO, YZP$,

$BC = YZ$, (proved), $CO = ZP$, $\hat{BCO} = \hat{YZP}$, (proved).

$\hat{BOC} = \hat{YPZ}$, (Ax 3)

Q.E.D.

Note We have clearly assumed that we know what is meant by the sum of two intervals, and the properties of these sums. This theory is much easier than the theory of the addition of angles.

THEOREM III a†

(Converse of Theorem III Cf Assumption 3, p. 4)

Given $\hat{AOB} = \hat{XPY}$, $\hat{COB} = \hat{ZPY}$, P is on the line XZ between X and Z , A, C are on opposite sides of OB

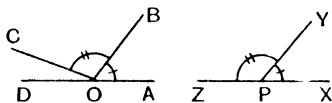


Fig. 323

To prove that C is on the line OA

Proof Produce the interval AO beyond O to D

Then $\hat{AOB} = \hat{XPY}$, $\hat{BOD} = \hat{YPZ}$, (Th III)

But $\hat{BOC} = \hat{YPZ}$, (given)

And C, D are on the same side of OB

O, C, D are in one line, (Ax 2) C is on the line OA

Q.E.D.

THEOREM IV (Assumption 2, p. 4)

Vertically opposite angles are equal.

Given. AB and CD are two lines which cut in O , O is between A and B and also between C and D .

To prove that $\hat{COB} = \hat{AOD}$.

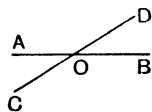


Fig. 324

† This is not needed, at present, for Examinations. The enunciation, at all events, should be noted.

Proof. $\hat{COB}$ is adjacent to $\hat{AOC}$.

$\hat{AOD}$ is adjacent to the same angle $\hat{AOC}$.

$\therefore \hat{COB} = \hat{AOD}$, (Th. III).

Q.E.D.

THEOREM V

(Assumption 7, first part, p. 8. See also Theorem 29, p. 121.)

If a line cut two other lines so as to make the alternate angles equal, then these two lines cannot cut each other.

Given A, D are on opposite sides of the line BC , $\hat{ABC} = \hat{BCD}$

To prove that the lines AB and CD cannot cut

Proof. If possible, let the lines AB and CD cut in X

First, suppose X and A are on the same side of BC

There is a point Z on the ray CD such that $CZ = BX$, (Ax. 1)

In $\triangle XBC, BCZ$,

$BX = CZ$, $\hat{XBC} = \hat{BCZ}$, (given), $BC = BC$, (two sides, incl. $\angle$)

$\hat{XCB} = \hat{ZBC}$, (Ax. 3)

But $\hat{ZCB} = \hat{XBC}$, (given)

Also B is between X and Z and on the line XZ , Z, X are on opposite sides of BC

X, B, Z are collinear, (Th. III a)

the lines AB and CD cut in two distinct points X and Z

But this is impossible

Hence the lines AB and CD cannot cut in a point on that side of BC on which A lies. Similarly, they cannot cut in a point on that side of BC on which D lies. Q.E.D.

Cor. If a line cuts two other lines so as to make the corresponding angles equal, these two lines cannot cut each other. (Assumption 8, first part, p. 8.)

Proof. See Theorem 29, Cor., p. 122, which uses only Theorems IV and V.

THEOREM VI

An exterior angle of a triangle is greater than either of the interior opposite angles.

For proof see Theorem 30, p. 122, which uses Theorem V, Cor. and Ax. 2.

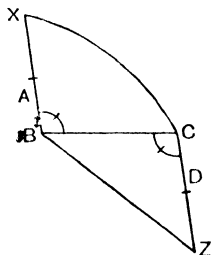


Fig. 180

THEOREM VII (Assumption 14, p. 19).

If a side and two angles of one triangle equal the corresponding parts of another, the triangles are congruent.

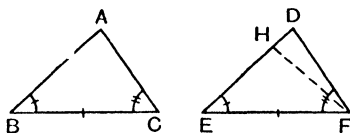


Fig. 325

There are two cases.

Case I *Given* In $\triangle ABC, DEF$,

$$BC = EF, \hat{ABC} = \hat{DEF}, \hat{ACB} = \hat{DFE}.$$

To prove that the triangles are congruent

Proof If $BA = ED$, the triangles are congruent by Ax. 3.

If $BA \neq ED$, there is a point H , distinct from D , on the ray ED , such that $EH = BA$ (Ax. 1) Then in $\triangle ABC, HEF$,

$$AB = HE, BC = EF, \text{ (given),}$$

$$\hat{ABC} = \hat{HEF}, \text{ (two sides, incl. } \angle)$$

$$\hat{BCA} = \hat{FHE}, \text{ (Ax. 3)}$$

But

$$\hat{BCA} = \hat{EFD}, \text{ (given).}$$

$$\hat{FHE} = \hat{EFD} \text{ But this is contrary to Ax. 2.}$$

Hence $BA \neq ED$ is false

$BA = ED$, and $\therefore$ the triangles are congruent

Case II *Given*. In $\triangle ABC, DEF$,

$$AB = DE, \hat{ABC} = \hat{DEF}, \hat{ACB} = \hat{DFE}.$$

To prove that the triangles are congruent

Proof. If $BC = EF$, the triangles are congruent by Ax. 3.

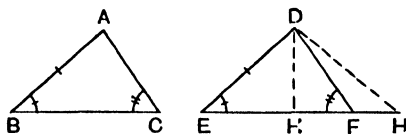


Fig. 326

If $BC \neq EF$, there is a point H , distinct from F , on the ray EF , such that $EH = BC$.

Then, as in Case I, it follows that $\hat{ACB} = \hat{DHE}$.

But $\hat{ACB} = \hat{DFE}$, (given) $\therefore \hat{DFE} = \hat{DHE}$.

But one of $\hat{DHE}$, $\hat{DFE}$ is an exterior angle, the other an interior opposite angle of $\triangle DHF$, and these cannot be equal. ($\therefore$ VI)

Hence $BC \neq EF$ is false. $BC = EF$, and the triangles are congruent. Q.E.D.

THEOREM VIII (Assumption 11, p. 11)

If the angles at the base of a triangle are equal, the triangle is isosceles.

Given $\triangle ABC$ is a triangle, $\hat{ABC} = \hat{ACB}$.

To prove that $AB = AC$.

Proof. Treat the triangles ABC , ACB as if they were distinct triangles

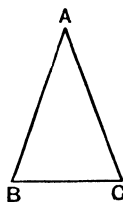


Fig. 319

We have in the two triangles the following parts equal.

In $\triangle ABC$	In $\triangle ACB$
BC	CB
$\hat{ABC}$	$\hat{ACB}$
$\hat{ACB}$	$\hat{ABC}$. (oneside, two $\angle$ s).

Hence, by Th VII, $AB = AC$. Q.E.D.

The proof of the right-angle congruence, which we gave before (p. 22), can now be given without any change

PARALLELS

So far, in this section, we have not used the word 'parallel,' and we have assumed nothing about parallel lines. We now define parallel lines, as before, as lines in the same plane which do not cut, and then we can re-state Theorem V and its Corollary as follows

Th V. *If a line cuts two others, so as to make the alternate angles equal, then these two lines are parallel*

Cor *If a line cuts two others, so as to make the corresponding angles equal, then these two lines are parallel*

Let us now consider the converse of Th V, viz if a line cuts two parallel lines, the alternate angles, so far as ne', are equal (Assumption 5, p 8) It is a surprising fact that this cannot be proved from the above theorems on congruent triangles and it is possible to prove that it cannot be proved from them In order to prove the converse of Th V we are compelled to introduce some new Axiom this we call the 'parallel Axiom'

Through a given point there is not more than one parallel to a given line

If XY be a given line, and A a point not on it, we can prove, without this new Axiom, that there is *at least* one parallel through A to XY . For, join AX . Then, by Ax 2, there is a ray AZ on the other side of AX from the ray XY , such that $\angle XAZ = \angle AXY$. Then by Th V, AZ and XY are parallel. What the new Axiom states is that this is the only parallel, through A , to XY .

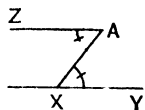


Fig. 327

Assuming this Axiom, we can now prove the converse in question.

THEOREM IX (Assumption 5, p 8).

If a line cuts two parallel lines, the alternate angles so formed are equal.

Given. AB and CD are parallel lines, PQ a line cutting AB in P and CD in Q , $\angle APQ$ and $\angle PQD$ are alternate angles

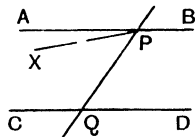


Fig. 328

To prove that $\angle APQ = \angle PQD$.

Proof. If $\angle APQ \neq \angle PQD$, there is a ray PX , distinct from the ray PA , and on the same side of PQ as the ray PA , such that

$$\angle XPQ = \angle PQD, \quad (\text{Ax 2}).$$

$$\text{Hence} \quad PX \parallel CD, \quad (\text{Th. V})$$

$$\text{But} \quad PA \parallel CD, \quad (\text{given}).$$

$\therefore$ through P go two distinct lines, PX and PA , both parallel to CD , but this is contrary to the parallel Axiom

$$\therefore \angle APQ \neq \angle PQD \text{ is false. } \therefore \angle APQ = \angle PQD. \quad \text{Q.E.D.}$$

THEOREM X (Assumption 6, p. 8).

If a line cut's two parallel lines, the corresponding angles so formed are equal.

Given. AB and CD are parallel lines, RQ a line cutting AB in P and CD in Q , $\hat{R}PB$ and $\hat{P}QD$ are corresponding angles.

To prove that $\hat{R}PB = \hat{P}QD$.

Proof. If A, B be on opposite sides of RQ ,

then $\hat{A}PQ = \hat{P}QD$, (alt $\angle$ s, Th IX)

But $\hat{A}PQ = \hat{R}PB$, (vert. opp. $\angle$ s, Th IV).

$$\therefore \hat{P}QD = \hat{R}PB$$

Q.E.D.

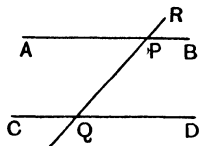


Fig 329

THEOREM XI (Assumption 15, p. 54).

Two (coplanar) lines parallel to the same line are parallel to one another.

Given. Lines a, b, c , such that $a \parallel b, a \parallel c$.

To prove that $b \parallel c$

Proof. If b and c be not parallel, they will cut, since they are coplanar. Let them cut in P .

Then through P go two distinct lines, viz b and c , both parallel to a .

But this contradicts the parallel Axiom. $\therefore b \parallel c$. Q.E.D.

We have now dealt with all the Assumptions 1–15, except Assumptions 9 and 4. Assumption 9 ‘the sum of the angles of any triangle equals two right angles,’ was proved on p. 9, Assumption 4, ‘all right angles are equal,’ can be proved from our other assumptions, but to do so satisfactorily we should have to face questions we are avoiding. We therefore do not consider it.

The reader will perhaps ask why we cannot prove the parallel Axiom. We must put this question more precisely. Suppose we assume the theorems already proved on the congruence of triangles, and that we make also all those other assumptions, more difficult to analyse, to which we have referred, can we then, making all these assumptions, prove the parallel Axiom? The answer is, we cannot. About a century ago, an attempt was made to prove the parallel

Axiom, from these assumptions, in the following way. Assume the Axiom false, that is, assume that if AB is any line, and P any point not on it, we can draw through P an infinite number of (straight) lines parallel to AB , i.e. (straight) lines not cutting AB . It was hoped that some contradiction would be reached when the consequences of this assumption were developed, and thus the parallel Axiom would be proved from the other assumptions by *reductio ad absurdum*. But no contradiction was ever found. Instead there was developed a Geometry which was *self-consistent*, i.e. it did not contradict itself, though it did, in some respects, contradict the old Geometry. For instance, in this new Geometry, the sum of the angles of a triangle was always less than two right angles. Thus when we reach the parallel Axiom, we can either (1) accept it, and then we have the Geometry of this book, called the Euclidean Geometry, or (2) substitute the other Axiom, and then we can erect a new Geometry, called the Hyperbolic Geometry.

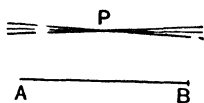


Fig. 330

The question which Geometry is *true* is, in spite of appearances, a very difficult one, and to consider it fully we should need more pages than there are in this book. We can, however, state roughly that, in the usual commonsense way of regarding things, the Euclidean Geometry is more than accurate enough for the purposes of everyday life, when we are dealing with objects of ordinary size like tables and chairs. But in scientific work which deals with very large distances, like the distances of the nebulae, or with very small objects like electrons, scientific men nowadays often do not assume the Euclidean Geometry, but a Geometry much more difficult than either the Euclidean or the Hyperbolic Geometry—and much more interesting.

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But what is of most importance to us is to realise that Euclidean Geometry is based on certain assumptions, that instead of these assumptions, it is possible to make others, and that then we get a different Geometry. This seems obvious when it is put in that way, but the wisest men did not realise this simple fact until a century ago. Until then, it was thought that the Euclidean Geometry was the only one which was *possible*.

Any theorem of Euclidean Geometry which was proved without the use of the parallel Axiom will still be true in the Hyperbolic Geometry. Thus, for instance, all theorems we give on congruent triangles are true in this Geometry, and that was one reason why we proved them without using the parallel Axiom. The places in this book where we have made assumptions, and said they were true because they were obvious, must now be reviewed in the light of what

we have just said. From our present, more correct, standpoint, it would be better merely to make the assumptions and to avoid or postpone the quite different questions, to what extent, or in what sense, they are true.

§ 38. Discussion of Theorem 44, p. 145.

On p. 24 we defined what was meant by saying that the set of points $(A, B, \dots)$ was congruent to the set $(X, Y, \dots)$. Suppose now we have any two (plane) figures made up of any points, lines and, possibly, curves. If we can speak loosely we should say that they are congruent, if one can be fitted on to the other. The points which fit on to one another we can say are 'paired off'. We can now express this abstractly as follows:

Def. Two figures $(A), (B)$ are said to be 'congruent' when the points of (A) and the points of (B) can be paired off in the following way:

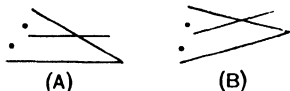


Fig. 331

- (i) Each point of (A) pairs with just one point of (B)
- (ii) Each point of (B) pairs with just one point of (A)
- (iii) If X, Y be any two points of (A) , and X', Y' the corresponding points of (B) , then distance $XY = \text{distance } X'Y'$.

THEOREM 44†.

In equal circles, equal angles at the centre are subtended by equal arcs. Conversely, in equal circles, equal arcs subtend equal angles at the centre.

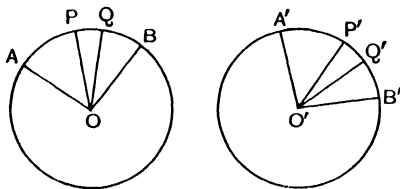


Fig. 332

- (1) *Given* Circles, centres O, O' of equal radii $OA, O'A'$, $\angle AOB = \angle A'O'B'$, B, B' on circles, centres O, O' resp.

† It would clearly be better to use the word 'congruent' instead of 'equal' in connection with intervals, angles and circles, but I have adhered, unwillingly, to school tradition.

To prove that arcs AB , $A'B'$ are equal, or congruent.

Proof. Let P , Q be any points on arc AB . Make them correspond to points P' , Q' on arc $A'B'$, such that $\angle OP = \angle O'P'$, $\angle OQ = \angle O'Q'$. Hence $\angle POQ = \angle P'O'Q'$.

Thus A , B correspond to A' , B' resp.

Then in $\triangle POQ$, $P'O'Q'$,

$$\begin{cases} OP = O'P', & \text{(given)} \\ OQ = O'Q', & \text{(given).} \\ \angle POQ = \angle P'O'Q', & \text{(given), (two sides, incl. } \angle). \\ \therefore PQ = P'Q'. \end{cases}$$

$\therefore$ the arcs AB , $A'B'$ correspond so that

(i) Each point of arc AB pairs with just one point on arc $A'B'$.

(ii) Each point of arc $A'B'$ pairs with just one point on arc AB .

(iii) If P , Q on arc AB correspond to P' , Q' on arc $A'B'$, then $PQ = P'Q'$.

the arcs AB , $A'B'$ are congruent, or equal, (def)

(2) *Given.* Circles, centres O , O' of equal radii OA , $O'A'$.

Arc AB of circle, centre O , equals arc $A'B'$ of circle, centre O'

To prove that $\angle AOB = \angle A'O'B'$.

Proof. Suppose $\angle AOB \neq \angle A'O'B'$

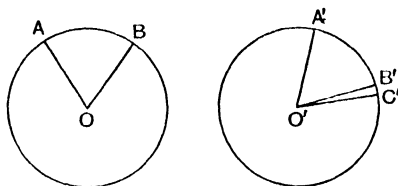


Fig 333

Then there is a radius $O'C'$, on the same side of $O'A'$ as B' , such that $\angle A'O'C' = \angle AOB$.

Then by the first part, arc $AB =$ arc $A'C'$.

But arc $AB =$ arc $A'B'$, (given). $\therefore$ arc $A'B' =$ arc $A'C'$, and B' , C' are on the same side of the radius $O'A'$.

We shall assume this is impossible.

$\therefore \hat{A}OB \neq \hat{A}'\hat{O}'B$ is false. $\therefore \hat{A}OB = \hat{A}'\hat{O}'B'$. Q.E.D.

§ 39. The classical proof of Ax. 3 by the method of superposition. (The two sides and included angle congruence.)

Given. $\triangle ABC, DEF, AB = DE, BC = EF, \hat{ABC} = \hat{DEF}$.

To prove that $AC = DF$

Proof. Take up $\triangle DEF$ and put E on B

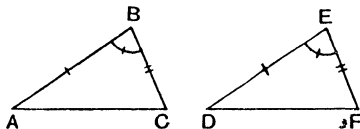


Fig. 334

Make the line ED lie along BA , and the line EF fall on the same side of AB as BC .

Then

D coincides with A , $BA = ED$.

EF coincides with BC , $\hat{ABC} = \hat{DEF}$.

F coincides with C , $BC = EF$.

$\therefore DF$ coincides with AC , (D coincides with A , F with C)
 $\therefore DF = AC$

There are many objections to this — of course we cannot move a geometric triangle, it is where it is. However, we can consider a succession of geometric triangles in a continuous set of positions, starting from DEF and ending with ABC , and we can transfer our attention along this series of triangles.

Suppose now, as we go along this set of triangles, the sides corresponding to DE , EF , and the angle corresponding to $\hat{DEF}$ remained the same size, but that the side corresponding to DF continually *shortened*, then AC would be *less than* DF . And, if you say that, under these conditions, the side corresponding to DF could not get shorter, you are assuming the very theorem you set out to prove. The proof is, therefore, worthless. As a matter of fact, it is not possible to prove the whole of this Axiom from the other assumptions we have made. But if we assume that part of the Axiom which states $\hat{A} = \hat{D}, \hat{C} = \hat{F}$, we can prove, using Axs. 1 and 2, that $AC = DF$.

§ 40. Similar triangles. The fundamental theorems.

We shall at first assume the following:

FUNDAMENTAL THEOREM XII.

A line parallel to the base of a triangle divides the sides proportionately

Thus, if X, Y be points on the lines AB, AC and XY is parallel to BC , then $AX/AB = AY/AC$ (Fig. 302)

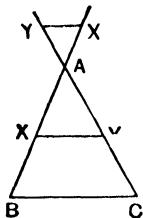


Fig. 302

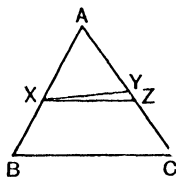


Fig. 335

THEOREM XIII (Converse of Th. XII)

If a line divides two sides of a triangle proportionately, and cuts both sides, or both produced beyond the base or both produced beyond the vertex, then the line is parallel to the base (Fig. 335)

Given X, Y on the rays AB, AC resp., $AX/AB = AY/AC$.
To prove that $XY \parallel BC$.

Proof. If possible, let XY, BC be not parallel

Then the parallel to BC through X will cut AC in a point Z , on the ray AC , distinct from Y , and

$$AX/AB = AZ/AC, \quad (XZ \parallel BC, \text{ Th. XII}).$$

$$\text{But} \quad AX/AB = AY/AC, \quad (\text{given}).$$

$$\therefore AY/AC = AZ/AC \quad \therefore AY = AZ,$$

which is impossible, $\therefore Y$ and Z are distinct points of the ray AC .

$$\therefore XY \parallel BC$$

Similarly, if X, Y are on BA, CA resp. produced beyond A .

Q.E.D.

Note In our enunciation, we must be careful to prevent such a case as the following occurring X on the interval AB , Y on the

interval CA produced beyond A , $AX \cdot XB = AY \cdot YC$. For this is clearly quite possible, and then XY, BC are not parallel.

We have assumed, among other things, that if $AY \cdot AC = AZ \cdot AC$ then $AY = AZ$, i.e. that there is not more than one point Y on the ray AC such that $AY \cdot AC$ is given.

Def. (as before). Two triangles are 'similar' when the angles of one equal the corresponding angles of the other.

THEOREM XIV (Assumption 16, p. 208).

Similar triangles have their corresponding sides proportional.

Given. XYZ and ABC are similar triangles, the angles X, Y, Z being resp. equal to the angles A, B, C .

To prove that

$$XY : YZ : ZX = AB : BC : CA.$$

Proof. There are points H, K on the rays AB, AC , such that

$$AH = XY, AK = XZ.$$

In $\triangle AHK, XYZ$,

$$AH = XY, AK = XZ, \text{ (const.)}$$

$$\angle AHK = \angle XZ, \text{ (given), (two sides, incl. } \angle).$$

$$\triangle AHK \sim \triangle XYZ.$$

$$\text{But } \angle XYZ = \angle ABC, \text{ (given).}$$

$$\triangle AHK \sim \triangle ABC. \therefore HK \parallel BC$$

$$AH : AB = AK : AC \text{ (Fund Th.)}$$

$$\text{But } AH = XY, AK = XZ, \text{ (const.)}$$

$$XY : AB = XZ : AC \quad (i).$$

$$XY : XZ = AB : AC \quad (ii).$$

$$\text{Similarly, } YZ : XZ = BC : AC. \quad \text{Q.E.D.}$$

Notes (1) In the step which gives (i) we have replaced certain intervals in a proportion, by equal intervals. This assumes that proportion really concerns only the lengths of intervals, not the intervals themselves, which are sets of points. In a truly logical treatment this would be emphasised at the beginning.

(2) The step from (i) to (ii) assumes that if $XY : AB = XZ : AC$ then $XY : XZ = AB : AC$.

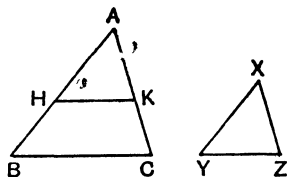


Fig 336

THEOREM XV.

If two triangles have an angle of one equal to an angle of the other, and the sides containing the equal angles proportional, then the triangles are similar.

Given. ABC , XYZ are triangles,

$$\hat{BAC} = \hat{YXZ},$$

$$AB : XY = AC : XZ$$

To prove that $\hat{B} = \hat{Y}$, $\hat{C} = \hat{Z}$

Proof There are points H , K on the rays AB , AC , such that $AH = XY$, $AK = XZ$

In $\triangle HAK$, YXZ ,

$$AH = XY, AK = XZ, \text{ (const.)}$$

$$\hat{HAK} = \hat{YXZ}, \text{ (given), (two sides, incl. } \angle \text{)}$$

$$\therefore \triangle HAK = \triangle XYZ.$$

But $AB : XY = AC : XZ$, (given),

$$XY = AH, XZ = AK, \text{ (const.)}$$

$$AB : AH = AC : AK.$$

$$HK \parallel BC, \text{ (converse of Fund. Th.)}$$

$$\hat{HAK} = \hat{ABC}, \text{ (corr. } \angle \text{s)}$$

But $\hat{HAK} = \hat{XYZ}$, (proved)

$$\therefore \hat{ABC} = \hat{XYZ}.$$

Similarly, $\hat{ACB} = \hat{XZY}$.

Q.E.D.

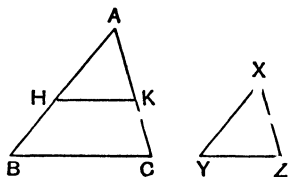


Fig 336

THEOREM XVI.

If two triangles have corresponding sides proportional, they are similar.

Given. ABC , XYZ are triangles,
 $AB : BC : CA = XY : YZ : ZX$.

To prove that

$$\hat{A} = \hat{X}, \hat{B} = \hat{Y}, \hat{C} = \hat{Z}.$$

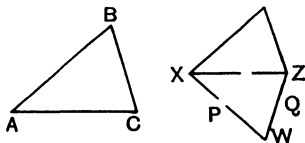


Fig 337

Proof. There are rays XP , ZQ on the opposite side of XZ from Y , such that $\hat{ZXP} = \hat{CAB}$, $\hat{XZQ} = \hat{ACB}$.

Let these rays cut in W .

Since two angles of $\triangle ABC$ equal two angles of $\triangle XWZ$,
 $\therefore \triangle ABC$, XWZ are similar.

$$\therefore \frac{AB}{AC} = \frac{XW}{XZ}, \text{ (Th XIV).}$$

$$\text{But } \frac{AB}{AC} = \frac{XY}{XZ}, \text{ (given).}$$

$$\therefore XW = XY.$$

$$\text{Similarly, } ZW = ZY.$$

$$\therefore \triangle XYZ, XWZ \text{ are congruent, (three sides).}$$

$$\therefore \text{the corresponding } \angle\text{s of } \triangle XYZ, XWZ \text{ are equal.}$$

But the corresponding $\angle\text{s of } \triangle XWZ, ABC$ are equal, (const.).

the corresponding $\angle\text{s of } \triangle ABC, XYZ$ are equal

Q E D

Thus we have reduced the theorems on similar triangles to the one we called the Fundamental Theorem. We assumed on the way, certain theorems on proportion, some of which we have mentioned in notes.

We ought now to prove the fundamental proposition. But first comes the question: What do we really mean when we say of four intervals that they are in proportion? An obvious suggestion is: suppose the lengths of the intervals are a, b, c, d , then they are in proportion, if the fractions a/b and c/d are equal. This of course involves Arithmetic, but that is not a serious objection. The real difficulty, if we adopt this method, is that the lengths may be irrational numbers like π , $\sqrt{5}$, and the theory of the addition, multiplication, and so on of such numbers is not easy. If, therefore, we assume that these numbers behave like rational numbers, e.g. $2, 3\frac{1}{2}$, in sums and products, we are dodging the main difficulty. It is hard, even, to see what a number like $\sqrt{5}$ means, as its decimal part never ends and never recurs (p. 97).

If however, the lengths a, b, c, d are commensurable, all becomes plain. For then we can find a 'unit' length which goes p times into a , q times into b , r times into c , and s times into d , p, q, r, s being integers. Then a, b, c, d will be in proportion if $p/q = r/s$, and here p/q and r/s are rational fractions.

We shall now prove the fundamental proposition, where the lengths involved in it are commensurable, and we shall assume its truth in the other cases.

Proof of the fundamental theorem for the commensurable case.

Given X, Y are points on the lines AB, AC resp., $XY \parallel BC$.
To prove that $AX : AB = AY : AC$

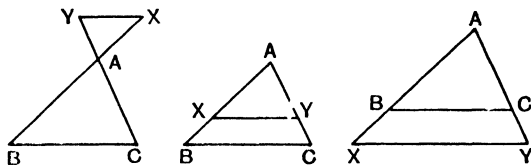


Fig. 338

Proof Find a 'unit' length which goes p times into the interval AX and q times into the interval AB .

Divide the intervals AX and AB into parts of unit length.

Through the points of division draw lines parallel to BC .

These will divide the interval AY into p parts, and the interval AC into q parts, all the parts being equal

Thence

$$\frac{AX}{AB} = \frac{p}{q}, \quad \frac{AY}{AC} = \frac{p}{q}$$

$$AX : AB = AY : AC.$$

Q.E.D.

§ 41. Harder theorems on proportion

THEOREM 57.

The internal bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides. The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the other two sides.

(1) *Given.* The internal bisector BX of $\hat{ABC}$ cuts the side AC of $\triangle ABC$ in X .

To prove that $AX : XC = AB : BC$.

Proof Through C draw a parallel to XB , and let it cut the side AB produced in D

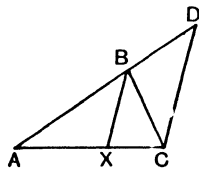


Fig. 339

Then $\hat{DCB} = \hat{CBX}$, (alt. $\angle$ s, $BX \parallel CD$)
 $= \hat{ABX}$, (given)
 $= \hat{BDC}$, (corr. $\angle$ s, $BX \parallel CD$).
 $\therefore BC = BD$, (isos. Δ).
 But $\frac{AX}{XC} = \frac{AB}{BD}$, ($BX \parallel CD$).
 $\therefore \frac{AX}{XC} = \frac{AB}{BC}$.

(2) *Given.* The external bisector BY of $\hat{ABC}$ cuts the side AC of ΔABC externally in Y .

To prove that $AY \cdot YC = AB \cdot BC$

Proof. Through C draw a parallel to BY , and let it cut the side AB in E .
 Produce AB beyond B to F .

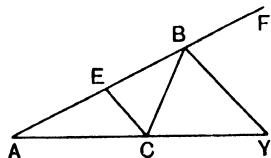


Fig 340

Then

$\hat{ECB} = \hat{CBY}$, (alt. $\angle$ s, $BY \parallel CE$)
 $= \hat{FBY}$, (given)
 $= \hat{BEC}$, (corr. $\angle$ s, $BY \parallel CE$)
 $\therefore BE = BC$, (isos. Δ)

But $\frac{AY}{YC} = \frac{AB}{BE}$, ($BY \parallel CE$)
 $\therefore \frac{AY}{YC} = \frac{AB}{BC}$

Q E D

EXAMPLE 66. *Given* A, B are fixed points, P moves so that $AP \cdot PB$ is constant

To prove that the locus of P is a circle

Proof. Take any position of P , and

bisect $\hat{APB}$ internally and externally

Let the bisectors cut the line AB in X

and Y . Produce AP beyond P to C

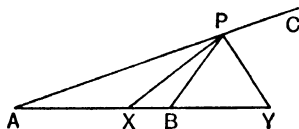


Fig 341

Then

$AX \cdot XB = AP \cdot PB = \text{const}$
 $AY \cdot YB = AP \cdot PB = \text{const}$
 X, Y are fixed points

Also $\hat{XPY} = \frac{1}{2}(\hat{APB} + \hat{BPC}) = 1 \text{ rt } \angle$

the locus of P is a circle on diameter XY

Q E D

The locus of P is called a *circle of Apollonius*

EXERCISE 59.

1 M is the mid-point of side BC of $\triangle ABC$, the bisectors of angles AMB , AMC cut AB , AC resp in D , E . Prove that $DE \parallel BC$

2 P is on a circle, diameter XY , the lines PA , PB make equal angles with the line PX , and cut XY in A , B resp. Prove
 $AX \cdot XB = AY \cdot YB$

3 Two circles touch externally. Prove that their common tangent is the mean proportional between their diameters

4 ABC and $A'B'C'$ are similar triangles, X , X' are on sides AB , $A'B'$ resp, and $AX \cdot XB = A'X' \cdot X'B'$. Prove that ACX , $A'C'X'$ are similar triangles

5. $ABCD$ is a cyclic quadrilateral, whose diagonals cut in X . Prove that $AX \cdot XD \cdot BX \cdot XC = AD^2 \cdot BC^2$

6 O is a fixed point, P moves on a fixed circle, R is on the ray OP , $OR \cdot OP = \text{const}$. Prove that the locus of R is a circle

7 A common tangent to two circles divides the interval joining their centres, externally or internally, in the ratio of the radii

8 In Fig. 342, PQ always goes through A , $PR \cdot RQ = \text{const}$. Prove that the locus of R is a circle through A , B

9. AX and BY are perpendicular to XY , AY , BX cut in C , D is the foot of the perpendicular from C to XY . Prove $\hat{AD}X = \hat{BD}Y$.

10 (a) OA , OB are given lines, P moves inside the angle AOB , so that the ratio of the perpendiculars from P to the given lines is constant. Prove that P describes a ray from O

(b) ABC is a given triangle. Find a point P inside it such that the ratio of the perpendiculars from P to BC , CA , AB is 1 : 2 : 3

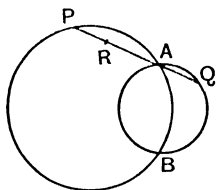


Fig. 342

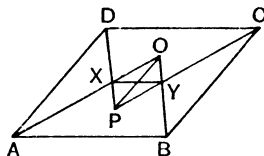


Fig. 343

11 X , Y are points within a parallelogram $ABCD$, $XY \parallel AB$; DX , CY cut in P , AX , BY cut in O . Prove that $OP \parallel BC$. (Fig. 343)

12 ABC , XYZ are two triangles, $\hat{BAC} = \hat{YXZ}$, $AB \cdot BC = XY \cdot YZ$. Prove that $\hat{ACB}$, $\hat{XZY}$ are equal or supplementary

13 X is a point inside, Y a point outside $\triangle ABC$, the triangles BXA and BYC are similar. Prove that triangles XBY and ABC are similar

14 The bisectors, internal and external, of $\angle C$ cut the line AB in X , Y resp. O is the midpoint of the interval AC . Prove that $OX \cdot OY = OC^2$

15 CD is an altitude of $\triangle ABC$, $\angle C = 114^\circ$. X is a point on AB , the perpendicular from X to AB cuts BC in Y . Prove that $AD \cdot AX + CD \cdot XY = AC^2$

***16** $ABCD$ is a cyclic quadrilateral. The sides AB , CD produced, cut in X , the sides BC , AD produced, cut in Y . Prove
 $XA \cdot XD = YC \cdot YD$.

ABC is a triangle, E , F are points on sides AC , AB such that $CE = BF$, the line EF cuts the line BC in D . Prove
 $AB \cdot AC = ED \cdot DF$

***18** In Fig. 344, AC , BD , XY are perpendicular to CD . Prove that

$$\frac{1}{AC} + \frac{1}{BD} = \frac{2}{XY}$$

***19** A , B , C are points on a line. Find the locus of points P such that $\angle APB = \angle CPB$.

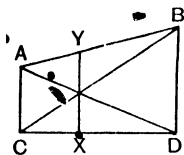


Fig. 344

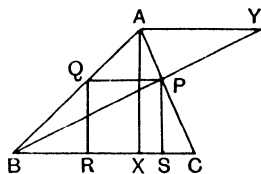


Fig. 345

***20.** A , B , C , D are points on a line. Find the locus of points P such that $\angle APB = \angle CPD$

***21.** Draw a triangle, given the base, vertical angle, and ratio of the two other sides

EXAMPLE 67. ABC is a triangle. To construct a square with two vertices on the side BC , and with its other vertices on sides AB , AC resp.

Construction Draw the altitude AX of $\triangle ABC$ (Fig. 345)

Make $AY \perp AX$, $AY = AX$. Join BY to cut AC in P

The rest of the construction is clear from the figure

Explain the construction and complete the proof

THEOREM 58.

The areas of similar triangles are as the squares on their corresponding sides. (The meaning of this phrase is clear from the proof below.)

Given. $\triangle ABC$, $\triangle XYZ$ are similar triangles.

To prove that $\triangle ABC \sim \triangle XYZ \implies BC^2 : YZ^2$.

Proof. Let AP , XQ be altitudes of $\triangle ABC$, $\triangle XYZ$.

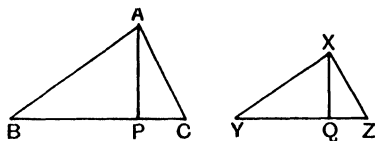


Fig. 346

Now in $\triangle APB$, $\triangle XQY$,

$$\begin{cases} \angle ABP = \angle XQY, & (\text{given}) \\ \angle APB = \angle XQY, & (\text{rt } \angle\text{s}) \end{cases}$$

$\triangle APB$, $\triangle XQY$ are similar.

$$\therefore \frac{AP}{XQ} = \frac{AB}{XY} = \frac{BC}{YZ}, \quad (\triangle ABC, \triangle XYZ \text{ are similar}).$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{\frac{1}{2} AP \cdot BC}{\frac{1}{2} XQ \cdot YZ} = \frac{BC^2}{YZ^2}. \quad \text{Q.E.D.}$$

Similar figures We have called two triangles similar when their corresponding angles are equal, and we have proved that then their corresponding sides are proportional. If we have two *quadrilaterals* with their corresponding angles equal, it does not follow that their corresponding sides are proportional, and if their corresponding sides are proportional it does not follow that their corresponding angles are equal.

Two figures (in a plane), made up of points, lines, and possibly curves, are '*similar*' if they differ only in scale. To each point of one figure will therefore correspond just one point of the other, we shall say these points are 'paired off'. We can express this abstractly as follows:

Def Two figures (A), (B) are said to be '*similar*,' when the points of (A) and the points of (B) can be paired off in the following way.

- (i) Each point of (*A*) pairs with just one point of (*B*)
 (ii) Each point of (*B*) pairs with just one point of (*A*)
 (iii) If *X, Y, Z, W* are any points of (*A*), and *X', Y', Z', W'* the corresponding points of (*B*), then $XY \cdot ZW = X'Y' \cdot Z'W'$
 Compare this with the *Def.* at the beginning of § 38

Def. Two polygons are 'similar' if the vertices of one are a set of points similar to the set of vertices of the other

E.g. the quadrilaterals *ABCD* and *A'B'C'D'* are similar if
 $AB/BC/CA/AD = BD/CD = A'B'/B'C'/C'A'/A'D' = B'D'/C'D'$
 The proofs of the following are easy, but tedious

The corresponding angles of similar polygons are equal

If two polygons have corresponding angles equal, and the sides about each pair of corresponding angles proportional, they are similar

Theorem 58 can be generalised: the areas of similar polygons are as the squares on their corresponding sides

From the above definition of similar figures it can be proved (what is in fact quite obvious) that all circles are similar figures. *The areas of circles are as the squares on their radii.* The proof of this is the first step in justifying the formula: area of a circle = πr^2 .

We can also define 'similar' solids. Speaking loosely, they are solids of the same shape. *The volumes of similar polyhedra† are as the cubes on their edges. The surfaces of similar polyhedra are as the squares on their edges.* All spheres are similar. *The volumes of spheres are as the cubes of their radii, the surfaces as the squares of their radii.* The proof of these facts is the first thing needed to justify the formulae

$$\text{Surface of a sphere} = 4\pi r^2$$

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

EXERCISE 60.

1 A map is on a scale of 1 inch to 1 mile, what is the area of the portion of the map which represents 1000 sq. yds? If the portion is square in shape, what is the length of its side?

2 The radius of the moon is approx. $\frac{1}{4}$ that of the earth. Compare the (i) surface, (ii) volume of the moon and the earth

3 What is the ratio of the light received on two equal sheets of paper, one of which is 1 inch, the other 1 foot, away from a candle?

4. A model of a steam engine works quite well. An engine is constructed exactly similar to the model on a scale of 1000:1. Is it likely to work? If not, why not? (Compare weights of corresponding parts, and also areas of corresponding pistons)

*5 Can you give a geometric explanation of the fact that large birds, like the emu and the ostrich, usually do not fly?

† A polyhedron is a solid with *plane* faces, e.g. the cube, tetrahedron

EXAMPLE 68. Trisect the interior of a triangle by lines parallel to a side
Given ABC is a triangle.

To draw lines parallel to BC , to divide its interior into three equivalent parts

Construction Trisect the side AB by P, Q so that $AP = PQ = QB$

On AB as diameter describe a semicircle, and at P, Q erect perpendiculars to AB , cutting the semicircle in R, S resp

With centre A , and radii AR, AS , draw circles cutting the side AB in X, Y resp

Then parallels XX', YY' to BC , through X, Y are the lines required

Proof $AX^2 = AR^2 - AP \cdot AB$, ($\angle ARB, \angle APR$ are rt $\angle$ s)
 $= \frac{1}{3}AB^2$

Similarly, $AY^2 = AS^2 - AQ \cdot AB = \frac{2}{3}AB^2$

$$\triangle AX'X' \sim \triangle ABC \Rightarrow AX^2 = AB^2 \cdot \frac{1}{3}$$

$$\triangle AYY' \sim \triangle ABC \Rightarrow AY^2 = AB^2 \cdot \frac{2}{3}$$

$$\triangle AX'X' \sim \text{quad } XX'Y'Y \sim \text{quad } YY'CB \quad \text{Q.E.D.}$$

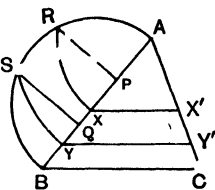


Fig 347

6 Show how to divide a triangle into 2, 4 equal parts by lines parallel to a side

7 Divide a circle into three equal parts by means of concentric circles (Fig 348)

($OP = PQ = QI$)

8 (a) ABC is a triangle, through O , any point inside it, lines AO, BO, CO are drawn cutting the opposite sides in D, E, F . Prove that $AO \cdot OD = \text{quad } ABOC' \sim OBC'$, and

$$\frac{AO}{AD} + \frac{BO}{BE} + \frac{CO}{CF} = 2$$

(b) If p, q, r be the altitudes of a triangle whose sides are a, b, c , prove that $a/p + b/q + c/r = a^2 + b^2 + c^2$

9 On the sides of the right-angled triangle ABC , as diameters, semicircles are described. Prove that the sum of the areas of the shaded 'lunes' equals triangle ABC (Fig 349)

10 The diagonals of quadrilateral $ABCD$ cut in P . Prove that

$$\triangle ABP \cdot \triangle BCP = \triangle ADP \cdot \triangle CDP$$

11. Prove that the area of a regular hexagon inscribed in a circle is the mean proportional between the areas of the inscribed and circumscribed equilateral triangles

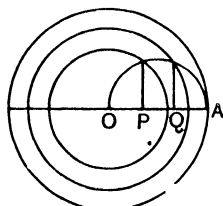


Fig 348

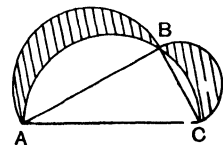


Fig 349

12 $ABCD$ is a parallelogram. A line through B cuts AC in F , AD in E , CD in H . Prove that $EF \cdot BH = BE \cdot BF$.

***13** AD is an altitude of the triangle ABC , right angled at A , the radii of the incircles of Δ s ABC , ABD , ACD are resp R , r , r' . Prove $r/r' = c/b$, $r^2 + r'^2 = R^2$, where $AC = b$, $AB = c$.

***14** (a) O is a point inside ΔABC , AO , BO , CO cut the sides BC , CA , AB in D , E , F resp. Prove that

$$\frac{\Delta AOB}{\Delta AOC} = \frac{BD}{CD}$$

Write down the similar ratios, and hence prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1. \quad (\text{Ceva})$$

•• (b) Enunciate and prove the corresponding theorem, when O is outside the triangle

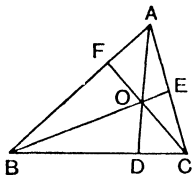


Fig. 350

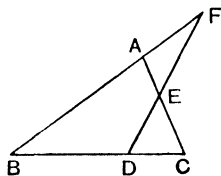


Fig. 351

***15** A line cuts the sides BC , CA , AB of ΔABC , produced if necessary, in D , E , F resp. Prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \quad (\text{Menelaus})$$

(Drop perpendiculars from A , B , C on to DE . Let their lengths be p_1 , p_2 , p_3 . Prove $AF \cdot FB = p_1 \cdot p_2$, write down the similar ratios.)

The last two results are of great importance

EXERCISE 61

(Mixed questions on ratio and proportion)

1. C is a point on ray AB , Y is a point on ray AX ,
 $AB \cdot AX = AC \cdot AY$.

Prove that $\Delta ABX = \Delta ACY$

2. Two circles, centres X , Y , touch externally at A , a line through A cuts them in B , C resp. Prove that

$$AB \cdot AY = AC \cdot AX, \quad \Delta ABY = \Delta ACX$$

3. Two circles touch externally at A , two lines PAX , QAY through A cut one circle in P , Q , the other in X , Y . Prove that $\triangle QAX = \triangle PAY$.

4. Use Question 1 to solve the following: on a given base, draw a triangle whose area equals the area of a given triangle.

5. The chords AB , XY of two (unequal) circles subtend equal angles at their respective centres. Prove that $AB/XY = \text{ratio of radii}$.

6. Two circles touch internally at A , two lines APX , AQY through A cut one in P , Q , the other in X , Y . Prove $AP \cdot XY = AX \cdot PQ$.

7. AP is an altitude of triangle ABC , inscribed in a circle. Prove that $AB \cdot AC = AP \cdot \text{diameter}$.

8. ABC is a triangle inscribed in a circle, the bisector of $\hat{BAC}$ cuts BC in X , the circle in Y . Prove $AB \cdot AC = AX \cdot AY$.

9. A circle touches, in X and Y resp., two circles, centres A , B , external to one another. The lines XY , AB cut in P . Prove that $AP \cdot BP = AX \cdot BY$.

10. A , B are the centres of two circles, external to each other, the interval AB is divided at P in the ratio of their radii, any line is drawn through P cutting the circles in C , D resp. (these being the first points of intersection from P). Prove that the radii to C , D are parallel.

11. D , E are the feet of the perpendiculars from A , C on to the bisector of the angle ABC of $\triangle ABC$. Prove $AD \cdot BE = BD \cdot CE$.

12. Similar polygons are described on the hypotenuse and the sides of a right-angled triangle. Prove that the polygon on the hypotenuse has an area equal to the sum of those on the sides.

13. Draw a triangle similar to a given triangle and twice its area.

14. $ABCD$ and $A'B'C'D'$ are convex quadrilaterals such that

$$\frac{AB \cdot BC \cdot CD \cdot DA}{AC} = \frac{A'B' \cdot B'C' \cdot C'D' \cdot D'A'}{A'C'}$$

Prove they are similar.

15. Draw a triangle given its base, the ratio of its sides, and its area.

16. In Fig. 352, $AE \parallel BF$, $BD \parallel CE$. Prove $AD \parallel CF$.

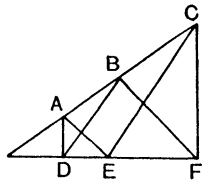


Fig. 352

17 In Fig 353, $\angle BAE = \angle CAD$ Prove $AB \cdot CD = AC \cdot BE$,
 $BC \cdot AD = AC \cdot ED$, $AC \cdot BE = AB \cdot CD + BC \cdot AD$

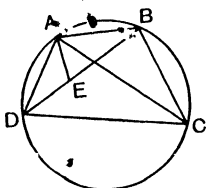


Fig 353

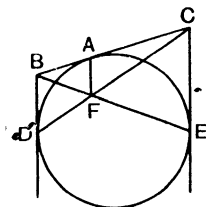


Fig 354

*18 Parallel tangents at the points D, E of a circle cut the tangent at the point A in B and C , BE, CD cut in F . Prove that AF is parallel to BD (Fig 354)

In the following the Theorems of Ceva and Menelaus may be used, or the methods used in proving those theorems

19 O is a point inside the triangle ABC , AO, BO, CO cut the sides BC, CA, AB resp in D, E, F

(a) If $BD = DC$, prove $CE \cdot EA = BF \cdot FA$

(b) If $BD = DC$, and $AO = OD$, prove $3AE = AC$, $4OE = BE$

(c) If $BD = DC$, prove $AO \cdot OD = 2AF \cdot FB$

(d) The circle DEF cuts the sides of $\triangle ABC$ again in D', E', F'

Prove AD', BE', CF' concur

20 D is a point on the side AB of an isosceles triangle ABC , ($AB = AC$), E is on AC produced beyond C , $CE = BD$. Prove BC bisects the interval DE .

§ 42. Medial section and the regular pentagon

CONSTRUCTION 22.

To divide a given interval into two parts such that the square on one part equals the rectangle contained by the whole interval and the other part.

Given. An interval AB .

To find a point X on it such that

$$AB \cdot AX = BX^2.$$

Construction. Make $AC \perp AB$, $AC = \frac{1}{2}AB$.

Join BC . Cut off from CB an interval $CD = CA$; cut off from BA an interval $BX = BD$.

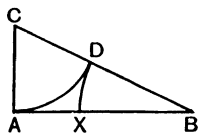


Fig 355

Then $AB \cdot AX = BX^2$.

Proof.

$$\begin{aligned}
 AB \cdot AX + AB \cdot BX &= AB (AX + BX) = AB^2 = BC^2 - AC^2 \\
 &= (BD + DC)^2 - AC^2 \\
 &= (BX + AC)^2 - AC^2 \quad (\because BD = BX, DC = AC) \\
 &= BX^2 + 2BX \cdot AC \\
 &= BX^2 + BX \cdot AB \quad (\because 2AC = AB). \\
 \therefore AB \cdot AX &= BX^2.
 \end{aligned}$$

Q.E.D.

The following method of proof is easier to remember.

Let $AB = 2a$, then $AC = a$.

$$\therefore BC^2 = AB^2 + AC^2 = (2a)^2 + a^2 = 4a^2 + a^2 = 5a^2.$$

$$\therefore BC = \sqrt{5}a.$$

$$\therefore BX = BD = BC - CD = BC - CA = \sqrt{5}a - a = (\sqrt{5} - 1)a$$

$$\text{and } AX = AB - BX = 2a - (\sqrt{5} - 1)a = (3 - \sqrt{5})a.$$

$$\therefore AB \cdot AX = 2a (3 - \sqrt{5})a = (6 - 2\sqrt{5})a^2.$$

$$BX^2 = (\sqrt{5} - 1)^2 a^2 = (6 - 2\sqrt{5})a^2.$$

$$\therefore AB \cdot AX = BX^2.$$

Q.E.D.

The interval AB is said to be divided in 'medial section.' Other expressions are 'extreme and mean ratio,' 'golden section,' 'sectio aurea.'

CONSTRUCTION 23.

To draw on a given base an isosceles triangle with the angles at the base each double the vertical angle.

Given. An interval AB .

To draw triangle ABC such that

$$AC = BC, \hat{CAB} = \hat{CBA} = 2\hat{ACB}.$$

Construction. Draw any interval XY .

Divide it at P so that

$$XP^2 = YP \cdot YX.$$

With centre X and radius XY draw a circle, and in it place a chord $YZ = XP$. Then $XY = XZ$.

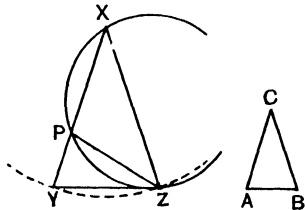


Fig 356

Also

$$YZ^2 = XP^2 = YP \cdot YX.$$

the circle $\odot XPZ$ touches YZ at Z .

$$\therefore \hat{Y}ZP = \hat{P}XZ, \quad (\text{alt. seg.}).$$

$$\begin{aligned} \therefore \hat{X}PZ &= \hat{P}XZ + \hat{P}ZX, \quad (\text{ext. } \angle \text{ of } \triangle = \text{sum of int. opp. } \angle\text{s}) \\ &= \hat{Y}ZP + \hat{P}ZX \\ &= \hat{Y}ZX = \hat{X}YZ, \quad (\because XY = XZ) \\ &= \hat{P}YZ. \end{aligned}$$

$$\therefore PZ = YZ, \quad (\text{isos. } \triangle).$$

But

$$YZ = XP, \quad (\text{const.}).$$

$$\therefore PZ = XP. \quad \therefore \hat{P}XZ = \hat{P}ZX, \quad (\text{isos. } \triangle).$$

$$\therefore \hat{Y}PZ = 2\hat{P}XZ.$$

But $\hat{X}YZ$ and $\hat{XZY}$ each $= \hat{Y}PZ$, (proved).

$$\therefore \hat{X}YZ = \hat{XZY} = 2\hat{Y}XZ.$$

On AB draw a triangle whose angles at A, B, C equal the angles of $\triangle XYZ$ at Y, Z, X resp.

This triangle clearly satisfies the conditions required. Q.E.F.

CONSTRUCTION 24.

To draw a regular pentagon on a given base.

Given. An interval AB .

To construct on AB a regular pentagon

Construction. On AB as base draw a triangle ABC , such that

$$\hat{A}BC = \hat{B}AC - 2\hat{A}CB.$$

Construct the circle ABC , and let the bisectors of angles ABC, BAC cut it in E, D resp

Then $ABDCE$ is a regular pentagon.

Proof. The angles ACB, BAD, DAC, EBC, ABE at the circumference are equal.

$\therefore$ the chords subtended by them, viz. AB, BD, DC, CE, EA , are equal

$\therefore$ the pentagon has all its sides equal.

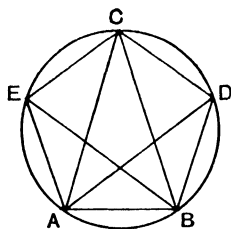


Fig 357

Also the arcs AB, BD, DC, CE, EA are equal

$\therefore$ the arcs AD, BC, DE, CA, EB are equal.

$\therefore$ the angles in them, viz. ABD, BDC, DCE, CEA, EAB , are equal.

$\therefore$ the pentagon has all its *angles* equal.

$\therefore$ the pentagon, having its sides and angles equal, is regular.

Q.E.F.

EXERCISE 62.

- 1 Find the height of a regular pentagon whose side is of length a
- 2 Prove the construction given on p 14, no 38 (b).
- 3 (a) X is on interval AB , $AB \cdot AX = BX^2$, the interval AB is produced beyond B to Y so that $BY = BX$. Prove $YA \cdot YB = AB^2$
(b) AB is a given interval. Produce it beyond B to a point Y such that $YA \cdot YB = AB^2$
- 4 X is on interval AB , $AB \cdot AX = BX^2$. Prove
 $AB^2 + AX^2 = 3BX^2$
- 5 (a) In the figure of Construction 23 prove that $\triangle PYZ$ is isosceles, with the angles at P and Y each twice that at Z ,
(b) and that $\triangle PXZ$ is isosceles, with the angle at P thrice each of those at X and Z
- 6 Show how to describe a regular decagon (10 sides) in a given circle
- 7 Prove that *an angle of 3° can be constructed by ruler and compasses*

§ 43. An illustration of the use of Algebra

EXAMPLE 69 A box of edge 1 ft is placed against a wall, a ladder 2 ft 11 in long rests against the wall and leans on the box. Find how high it reaches up the wall

If the reader tries to do this question with these numbers, he will probably not succeed. We shall, instead, use the letters marked on the figure

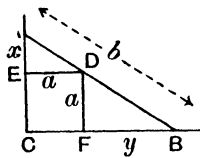


Fig 358

$$ED = DF = a, \quad AE = x, \quad BF = y, \quad AB = b$$

Then $\triangle s AED, DFB$ are similar

$$\frac{x}{a} = \frac{a}{y} \quad xy = a^2 \quad (1)$$

Also $(x + a)^2 + (y + a)^2 = b^2$, (Pythagoras)

$$x^2 + y^2 + 2ax + 2ay + 2a^2 = b^2.$$

But

$$\begin{aligned} 2a^2 &= 2xy \\ x^2 + 2xy + y^2 + 2ax + 2ay - b^2 \\ &= (x + y)^2 + 2a(x + y) - b^2. \\ (x + y)^2 + 2a(x + y) + a^2 &= a^2 + b^2. \\ (x + y + a)^2 &= a^2 + b^2 \\ x + y + a &= \sqrt{a^2 + b^2} \quad \text{. . . (ii).} \end{aligned}$$

Substituting the given numbers in (i) and (ii) we have, working in inches

$$\begin{cases} xy = 144 \\ x + y + 12 = 37, \end{cases}$$

and the usual algebraical method gives

$$x = 9, y = 16 \text{ or } x = 16, y = 9$$

Exercise 1 Work out the question when the edge of the box is 5 ft. and the ladder 18 ft. 5 in. long

2. Given a square, show how to draw round it a rhombus whose sides have a given length.

Geometric solution of quadratic equations

Construction 22 gives essentially a solution of the quadratic equation $x^2 = a(a - x)$, as is seen by taking $AB = a$, $BX = x$

The following is a geometric solution of the equation

$$x^2 - ax + b = 0,$$

where a, b are supposed to be known numbers

Take x and y axes as is usual in work on graphs

Let A be the point $(0, 1)$, P the point (a, b)

Draw the circle on diameter AP , and suppose it cuts the x axis in X_1 and X_2 . Then the abscissae (i.e. the values of the x -coordinate) for X_1 and X_2 are the roots of the equation. Let α, β be these abscissae.

To prove the truth of the construction we have only to prove that

$$\alpha + \beta = a \text{ and } \alpha\beta = b$$

These easily follow from the figure when a, b are positive

If either or both of a, b are negative we should have to consider negative lengths, and this has not been done in this book

If the circle does not cut the x axis, the roots of the quadratic are imaginary

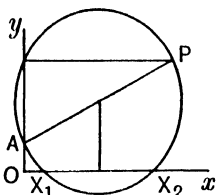


Fig. 359

Solution to Paper XX, Question 4, p. 67

Given ABC is a triangle, the bisectors of angles at B, C cut AC, AB resp. in $D, E, BD = CE$

To prove that $\hat{ABC} = \hat{ACB}$

Proof. Suppose, if possible,

$$\hat{ABC} > \hat{ACB} \quad \hat{DBC} > \hat{ECB}$$

Then in $\triangle s DBC, ECB$,

$$BC = CB, BD = CE, \hat{DBC} > \hat{ECB} \\ CD > BE \quad (i)$$

Let EC, BD cut in P

On the other side of BC from A construct $\triangle BQC$, such that

$$BQ = EC, QC = EB$$

Join DQ

$$\text{Then } BQ = BD \quad \hat{BDQ} = \hat{BQD}$$

Also, in $\triangle s EPB, DPC$,

$$\hat{EPB} = \hat{DPC}, \hat{EBP} > \hat{DCP}, \text{ (hyp) } \quad \hat{PDC} > \hat{PEB} = \hat{BQC}$$

$$\hat{BDC} > \hat{BQC} \text{ and } \hat{BDQ} = \hat{BQD}, \text{ (proved)}$$

$$\hat{QDC} > \hat{DQC} \quad CQ > CD \quad \therefore BE > CD \quad (ii)$$

(i) and (ii) are contradictory

$$\hat{ABC} > \hat{ACB} \text{ is false. Similarly, } \hat{ACB} > \hat{ABC} \text{ is false}$$

$$\hat{ACB} = \hat{ABC}$$

Q.E.D.

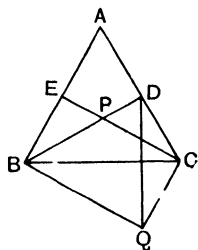


Fig. 360

§ 44. Conclusion of this Part.

Throughout this Geometry we have used, without analysing or arranging them, such theorems as the following

(1) If the ray OD is inside the angle AOB , and the ray OC is inside the angle AOD , then the ray OC is inside the angle AOB

(2) A line through a point inside a triangle, and in the plane of the triangle, goes through a vertex or a point on a side of the triangle

(3) Any polygon can be split up into triangles

(4) If a circle goes through a point inside and a point outside another circle, the circles cut each other

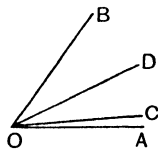


Fig. 361

It is precisely theorems of this type which are the hardest to arrange and classify†, but if we are to construct a really logical Geometry,

† A full treatment will be found in the author's book, previously quoted

in which nothing is assumed without being explicitly put down, it is necessary to deal with such theorems. This would hardly interest beginners, but as we have ignored all such considerations, all our proofs are really incomplete. To show you that you do not know everything even about such simple matters as these, you are recommended to try the following questions, which are related to theorems like those above†.

CHALLENGE EXERCISE 63.

1 Five towns were to be connected by roads (straight or curved) so that a man could go from each town to each other, without going through another town or crossing another road, and the roads were not to cross or touch each other. *Prove* that this could not be done.

2 If on a sphere you draw a closed curve, like a circle or an oval, it divides the surface of the sphere into two parts. Find a surface such that some closed curves do *not* divide it into parts.

3 If we want to go from a point on this page to the opposite point on page 252, we must either cut through the paper, or go round an edge. Make a surface out of paper, so that we can go from a point of the paper to the opposite point on the other side of the paper, by travelling along the paper, and not going through the paper, or round an edge.

4 Is it *always* possible to split up a polygon into triangles whose vertices are all at the vertices of the polygon? Consider non-convex polygons.

5 Is it *always* possible to split up a polyhedron into tetrahedra whose vertices are all at the vertices of the polyhedron?

6 If we have a network of any kind, as in Fig. 362, prove that $v + f - s + 1$, where v = number of corners, f = number of areas enclosed, s = number of sides.



Fig. 362

7 Give examples of solids with six edges and eight edges. Explain why there are none with just seven edges.

8 An Indian Rajah commanded his Chief Artificer to cut a diamond into a solid, all of whose faces had just six edges, and he put him to death because he could not. Prove that this hapless man was unjustly executed.

9. Prove that if a polyhedron has no faces that are triangles or quadrilaterals it must have at least twelve faces that are pentagons.

† Short answers, or indications, will be given in the next Part.

10. Take an interval, bisect it, bisect the two intervals so obtained, bisect the four thus obtained, and so on. Prove that there are some points on the given interval which are never points of bisection, however many times we continue the process.

***11** Discover some assumptions made in the proofs of Theorem, 6, 16, 24, 33.

***12** A cricket ball is contained in an indiarubber case which fits over it. The case is moved over the ball, it may be stretched or contracted in any places, provided it still fits, and is not torn or folded. Prove that there must be at least one point of the case exactly over the same point of the ball as it was before it was moved about. (This question is of very great difficulty.)

ANSWERS

PART I

202GE

- 1 Ex 1 1 2, 180° . 2 $1\frac{1}{2}$ or $2\frac{1}{2}$, 135° or 225° 3 $360^\circ, 30^\circ, 120^\circ$
 4 1, 2, $\frac{2}{3}$ rt $\angle$ s, $90^\circ, 180^\circ, 60^\circ$ 5 18° . 7 $\frac{1}{360}, \frac{1}{12}, \frac{1}{8}$ sec
 8. $15^\circ, \frac{1}{4}$ 9 $45^\circ, 30^\circ$
- 4 Ex 4 1. $135^\circ, 120^\circ, 150^\circ$. 2. $60^\circ, 120^\circ$. 3. 140° .
 4. $AOB = 144^\circ, AOC = 72^\circ$. 5 $160^\circ, 90^\circ$ 6. $180^\circ - x/2$
 7 135° 8 $90^\circ + x, 90^\circ - x, 180^\circ - x$
- 8 Ex 5 1 $53^\circ, 37^\circ, 90^\circ$ 2 $291'', 254'', 339''$.
 3 $158'', 339'', 278''$ 4 $1''$ 7 $(x + y)/2$
 8 $a/2 + b + c/2$
11. Ex 8 3. $70^\circ, 39^\circ$. 4. 41° 5. $196''$.
 12 Ex 9 1 60° 2 $70^\circ, 90^\circ$ 3 45° . 5 $30^\circ, 60^\circ$. 6. 70°
 9 60° 10 30° . 11 $a + b + c$ 13 $20^\circ, 20^\circ$.
- 13 16 10°
 Ex 10. 1 45° 2. 30° 4. $90^\circ - x/2$ 5. $180^\circ - 2y$
 7. 172° . 8 $77\frac{1}{2}^\circ$. 9 $100^\circ, 60^\circ$ 10 $35, 75^\circ$. 11. 110° .
- 14 19 $100^\circ, 160^\circ$ 21. $22\frac{1}{2}^\circ$.
 Ex 6 $BC = 361''$.
17. Ex. $41^\circ, 61^\circ, 79^\circ$ (more closely $40^\circ 48', 60^\circ 36', 78^\circ 36'$).
 18. Ex. 12. 1. (a) 145° (b) 5°
 19 Ex 2 30, 193, 460, 591.
 21. Ex. $56^\circ, 31^\circ$.
27. Ex. 16. 1. $193'', 517''$. 2. $391''$ 3 $300'', 5196''$
 28 4. 424
39. Ex. 21. 1. 106° . 2 64 miles 3 8'.
 40. 4. 342 miles 5 $577'$. 6 305 miles 7 375 yd, 329 yd
 8 $714', 346', 218', 45^\circ$ 9. 382' 10. $19^\circ 28'$ 11. $1014'$.
 12. 15 miles 13 $765'$ 14 $390'$.
45. Ex 22. 1. (a) $25''$. (b) 559 cm. 2. $391''$
 46. 3. $497''$. 4. $206''$ 5 182 cm. 6 $212''$ 7 $1172''$
 8 425 cm. 9 $165''$ or $203''$.

PAGE

- 47 10. $634''$, $127''$, $1098''$ 11 $150''$, $92''$
 52 Ex 24 1 (a) 108° , 120° , $176^\circ 24'$ (b) $(2n - 4)/n$ rt $\angle$ s
 6 8, 15, 20 9 36° , 72° , 72° 10 45° , $67\frac{1}{2}^\circ$ 11 156°
 13. $25\frac{1}{2}^\circ$ 14 $51\frac{1}{2}^\circ$
 63 Revision Papers I 1. 15° , 75° 2 $187''$ II 2 $227''$
 64 III 1 $45^\circ - x$ 2. 257 cm 3 54° IV 2 $455''$, $230''$
 V 2 $528''$. VI 2 $68''$ VII 1 78 cm, 46 cm
 65 VIII 1 861 cm 2 $45^\circ - A/4$, $90^\circ + A/2$ 3 $A/2$ IX 1 $2015''$
 X 1 707 cm XI 1 171° XII 1 60° , 120°
 66 XIII 1. 36° XIV 1. 162° , 54° , 81° , 108° , 135° XV 1 10
 69 Ex 2^c 1 297 cm 2. 25° , 154° 3 $365''$. 4 446 cm
 5 29° , 151° .
 72 Ex 30 1 $6''$ 2 12 cm 3 $15''$ 4 $2\frac{3}{4}$ cm
 73 5 20 sq in 6 10 sq in 7 $22\frac{3}{4}$ sq miles 8 $8\frac{1}{2}$ sq in
 9 20 sq in. 10 $31\frac{1}{2}$ sq in 11 $46,600$ sq yd
 12 $\frac{1}{2}(bd + ac)$, 10 sq in. 13. $a^2 + b^2$, 20 sq in
 14 $bc + ad$, 19 sq in
 74. 15 $(a + b)c$, 14 sq. in 16 $1\frac{1}{2}''$ 17 $3ah/8$
 80. Ex 32 2. 1732
 81 3 5, 13, 29, 65, 224, 316, 361, 721, 265, 5 4 60 yd
 5 39 ft 6 $4472'$ 7 $1204'$ 8 13 cm 9 80 cm
 10 70 sq ft 11 $1131'$, $3317'$ 12 $1474''$ 13 4, 894
 14 $265''$ 15 8598 sq cm
 82. 16 126, 84, 2640, 5940 17 9922 , 9753 , 1469
 18 29, 210, 1448 19 $3\sqrt{3/2} = 2598$ sq in 20 $48'$
 85 Ex 33 1 $1732''$ 2 $7''$ 3 $1655''$. 4 $6633'$
 5 $2414'$, $2236'$
 86. 7 $220,000$ sq ft
 92. Ex 35 A 1 $4123''$ 2 $4902''$ 3 $325''$ 4 $4298''$
 93. 5 $3882''$ 6 $4684''$ 7 $3645''$

PART II

PAGE

103. Ex 37 1 1' 2 3' 6'' 3 23 7'', 23 2'', 22 6'', 20 8'', 13 27''.
4 3 1' or 4 47 cm, 5 70 cm, 5 81 cm 5 5 17''. 6 12 37''.
7. 4 90 cm 8 $\sqrt{(b^2 + a^2/4)}$
- 104 18 $\frac{b}{a} \sqrt{4a^2 - b^2}$
- 106 Ex. 38, 1 3 01''.
- 109 Ex. 39 9 5 47''. 10. 5''.
- 109 15 5'' 16 12'' or 3'' 18 4'' Flat part = 201 sq in
Curved part = 251 sq in 19 3464 miles, 50 million sq miles
20 (a) 10 82 sq in. (b) 46 67 sq in
- 115 Ex 41 3 $2x$ or $360^\circ - 2x$ 4 100° or 20° 5. 50°
6 (a) $50^\circ, 60^\circ, 70^\circ$ or $10^\circ, 60^\circ, 110^\circ$. (b) $20^\circ, 10^\circ, 150^\circ$ or $10^\circ, 10^\circ, 16^\circ$
9 20° 10 125°
- 116 11 55° .
- 118 Ex 42 3 (a) 120° . (b) 135° . 5 110° 6 $100^\circ, 80^\circ, 50^\circ, 130^\circ$.
7 $80^\circ, 20^\circ, 110^\circ, 50^\circ$
- 134 Ex 44 1 $2\sqrt{2} = 2.83''$. 2 60° . 3 $\sqrt{2} = 1.41''$
- 136 13 1'', 9''. 14 1'', 2'', 3'' 15 1'', 2'', 5''. 16 1 5''
- 143 37 0 25'' 38 (a) $r(\frac{2}{3}\sqrt{3} + 1)$ (b) $r(\frac{2}{3}\sqrt{3} - 1)$
39 $r(\sqrt{2} + 1)$ 40 $r/3$
- 144 46 $\sqrt{\frac{2}{3}}a, 1 + 2\sqrt{\frac{2}{3}}$ in 47 $1 + \sqrt{2}$ in
- 146 Ex 45 1 40° 2 $90^\circ, 67\frac{1}{2}^\circ, 45^\circ$ 3 30° 4 30° 5. $36\frac{2}{3}^\circ$
- 156 Ex 46 1 $20^\circ, 40^\circ, 120^\circ$ 2 $35^\circ, 70^\circ, 75^\circ$ 3 $65^\circ, 75^\circ, 115^\circ, 105^\circ$
4 50° or 130°
- 172 Ex 48 2 2 26'', 1 53'' 3 2'', 2 83'' 4 1 23'' or 5 7''
5 3 55'', 4 1'', 2 05'' 6 1 73'' or 0 77''
- 175 Ex 49 1 1 9 9' 2. 35 75' 4 13 cm, 16 28 cm
5 5 196'', 5 916'' 7 3 5''.
- 179 Revision Papers XXI 1 2 52'' 2 19 sq in, $\frac{1}{2}(ab - xy)$
3 3 464'' XXII 1 1 881'' 3 11 27''
180. XXIII 1 6 95 sq in 3 12 21'', 9 22'' XXIV 1 64 sq. ft
3 3 01 sq ft XXV. 2 22 85'' 3 37 XXVI 2 8, 8 944
- 181 3 2 82'' XXVII 2 0 901'' XXVIII 2 11 313'' 3 21''
XXIX 3 6 5'' XXX 2 930''. 3 9'', 14 21''
XXXI 1 5 207'', 8 16'', 4 33''
182. XXXII 1 25'', 549 8 sq in 2 $157\frac{1}{2}^\circ$. XXXIII 1 1 5 sq ft
3 $r/2$ XXXV 1 2 94'', 2 17'' 3 140° XXXVI 1 3 02''
183. XXXIX 2 $\sqrt{3} r$ XL 2 5 477 cm. XLI 3 2 94'', 4 34''

PART III

PAGE

- 189. Ex 52** 1 (i) acute, (ii) obtuse, (iii) acute, (iv) obtuse
 2. 6 02, 6 615 3 4 68, 3 31 cm 4 6 125" 5 (a) 4 8125"
 5 083" (b) 20 332 sq in, 6 779" 6 10 44" 7 6 245"
 8 15 71" 9 6 571" 10 3 742"
- 190. 11.** 7 550"
- 192 37** (a) 11 acres. 38. 100 acres
- 193 Ex 53** 1 (a) 3 257 cm (b) 1 074" (c) 1 195". (d) 3 791 cm
 (e) 4 596" (f) 2 06" (g) 2 45"
- 199. Ex. 54** 1. 3 125' 2. $5\frac{2}{3}$ ft. 3 12 31, 21 32, 28 93 miles
- 200** 5 62 $\frac{1}{4}$ miles 6. 3 ft 7. 6 ft 8. 700 ft
- 207. Ex. 56** 1 3 6, $5\frac{1}{3}$, 4, bc/a , $\sqrt{(ab)}$ 3 7 4, 3 7, - 1. 4, 7 - 1
 5. 51, $a(b+c)/(b-c)$ 6 8 4, $\sqrt{30}$
- 208** 8 (a) $1\frac{1}{2}$ ", $\frac{1}{2}$ ", 3, 1". (b) 1 6", 4", $2\frac{2}{3}$ ", $\frac{2}{3}$ " 9 $al/(a+b)$, $bl/(a+b)$, $al/(a-b)$, $bl/(a-b)$ 10 (a) 2 8" (b) $5\frac{5}{8}$ "
 11 (a) 3 2. (b) $24 \cdot 1$
- 211. Ex 57** 1 $6\frac{2}{3}$ ". 2 1 2" 3 $2\frac{1}{3}$ ", $2\frac{2}{3}$ " 4 5". 5 1.25"
 6 (a) 2 25", 2 625" 7 $9\frac{1}{3}$ ", $10\frac{2}{3}$ " 8 5 25", 13 5"
 9 1 2", 1 4" 10 $2\frac{1}{3}$ ", $2\frac{1}{3}$ " 11 (a) 4 61", 3 19", 3 84"
 (b) 9 165", 7 746", 5 916" (c) 6 25", 3 75", 3
- 212. 12.** $12/13$ " 13 $5\frac{5}{8}$ ", $8\frac{1}{8}$ " 14 2 4" 15 $3\frac{1}{4}$ ft
- 241 Ex 60.** 1 00032 sq in, $\frac{5}{11}$ " 2 $\frac{1}{10}$, $\frac{1}{64}$ 3 $\frac{1}{144}$
- 248 Ex. 62** 1. 1 533a
- 249 Ex** $x = 144"$ or $25"$, $y = 25"$ or $144"$

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- Collinear points = points in the same (straight) line
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- Concentric circles = circles with the same centre
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- Diagonal of a polygon = an interval joining two non-adjacent vertices of the polygon, diagonal of a quadrilateral, 10
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 Hexagon = six-sided figure
 Hypotenuse of a right-angled triangle = the side opposite to the right angle
 Hypothesis (hyp) = an assumption made in the statement of a theorem, or during the course of its proof
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Subtend $\angle AOB$ is any interval, O any point not on it, then the interval is said to subtend the angle AOB at O

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Trisect = to divide into three equal parts. The point X is also said to trisect the interval AB , if $3AX = AB$ or $3BX = AB$

Vertex of an angle, 3

Vertex of a polygon = an end point of a side

