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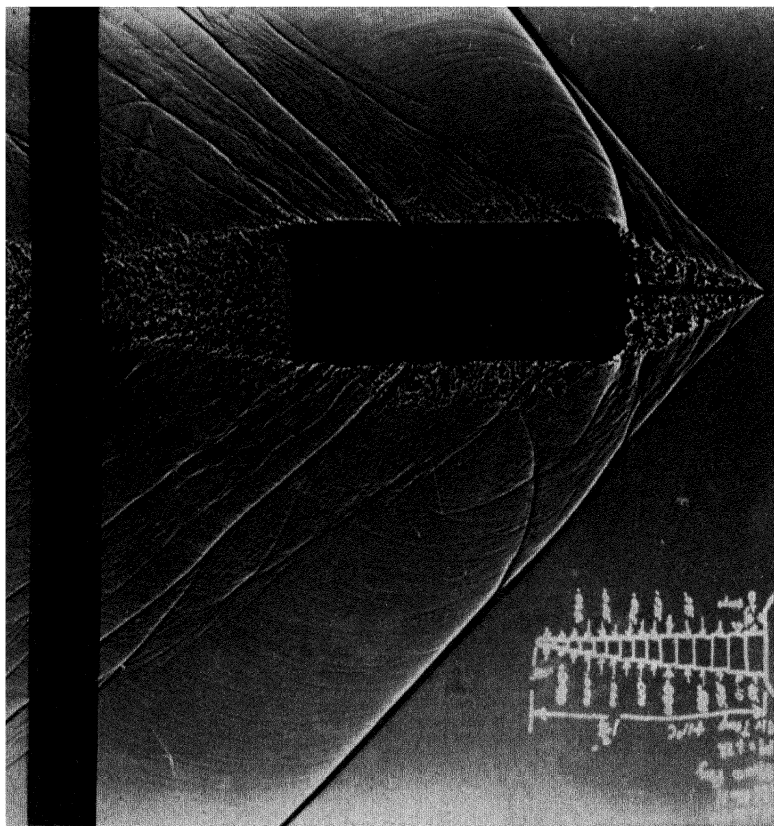
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HYDRODYNAMICS
A STUDY IN LOGIC, FACT,
AND SIMILITUDE



Common sense contradicted (*courtesy of Ballistics Research Laboratories,
Aberdeen Proving Ground.*)

HYDRODYNAMICS

A STUDY IN LOGIC, FACT,
AND SIMILITUDE

BY GARRETT BIRKHOFF

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TO RUTH

PREFACE

The present book is largely devoted to two special aspects of fluid mechanics: the complicated logical relation between theory and experiment, and applications of symmetry concepts. The latter constitute "group theory" in the mathematical sense.

The relation between theory and experiment is introduced, in Chapter I, by numerous "paradoxes," in which plausible reasoning has led to incorrect results. In Chapter II, this relation is studied more closely in the special case of flows with "free boundaries."

Chapter III is devoted to analysis of "modeling" and its rational justification. Theory and practice are compared (or contrasted!), and the origins of modeling in symmetry concepts are described as well, thus providing a transition between the two main aspects of fluid mechanics studied in this book.

The rest of the book centers around applications of group-theoretic ideas. These are shown in Chapter IV to motivate a large proportion of the known exact solutions to problems involving "compressible" and "viscous" flows, and (in Chapter V) to yield the classical force equations for a solid moving through a perfect fluid.

Perhaps Chapters II, III, and V contain the largest proportion of original contributions. I know of no other treatment of "free boundary" theory which penetrates as far into reality as the discussion of Chapter II. Again, Chapter III provides the only study of modeling that I have seen, which avoids the mathematical oversimplifications of "dimensional analysis" (Bridgman's well-known book avoids the physical oversimplifications), and at the same time gives a realistic picture of actual laboratory practice. Chapter V deduces much of the classical theory of "virtual mass" from the modern geometrical theory of "homogeneous spaces," thereby making possible very extensive generalizations.

PREFACE

Most of the material was originally presented in lecture form at the University of Cincinnati, in January 1947. The book owes much to a John Simon Guggenheim Memorial Fellowship, which made possible visits to many European research laboratories, and gave me leisure to rewrite it. In addition, Chapter I has been considerably clarified through discussions with Prof. Kampé de Fériet, while Chapter III owes much to stimulating talks with Prof. Percy Bridgman. Various other improvements were suggested by Prof. John von Neumann and Dr. Eduardo Zarantonello.

Even after the most careful attempts at simplification, the material treated still involves a heterogeneous combination of destructive logical criticism, detailed experimental fact, and occasional mathematical sophistication, which may at first seem unpalatable to many readers. I ask such readers to reflect on the correctness of the somewhat unorthodox views presented, in those cases of greatest interest to them. I hope that after this reflection, they will become convinced of the difficulty of presenting the true facts more simply, and of the need for more studies such as the present one, to clarify the foundations of fluid mechanics.

GARRETT BIRKHOFF

Harvard University
December, 1949

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Hydrodynamics

CHAPTER I

Hydrodynamical Paradoxes

1. The Practical Value of Paradoxes

As fluid mechanics comprehends the physics of two of the three most common states of matter—the liquid and the gaseous—it is superfluous to explain why one or even many books should be devoted to the subject. It is less obvious why I have thought it suitable to begin this book with a discussion of “hydrodynamical paradoxes.” It might seem that this unduly emphasizes *negative* aspects of the subject. Therefore I should like to begin by asserting the importance of such paradoxes from the practical, the mathematical, and the philosophical points of view. By a *paradox* I shall mean a plausible argument which yields conclusions at variance with physical observation, as it is in this sense of an *apparent inconsistency* that the term is most commonly used in hydrodynamics.

From a practical point of view, perhaps the most important art to acquire in fluid mechanics consists in knowing when a given theory is “applicable,” to use the current phrase. The fundamental importance of acquiring this art has been stressed in various recent books,¹ which have tried to bridge the wide gap between theory and experiment which exists in hydrodynamics.

Such books tend to attribute this gap to the difference between “real” fluids of small but finite viscosity, and “ideal” fluids having zero viscosity.² They tend to minimize the importance of logical and mathematical rigor, and to trace all discrepancies between “theory” and “experiment” to the single “unjustified” approximation of zero viscosity. By familiarizing the reader at the outset with the physical facts of separation of the boundary layer and turbulence, they

¹ See for example the prefaces of [12] and [14], or p. 2 of [6]. Numbers in square brackets refer to the bibliography on p. 181.

² See [6], pp. 2, 48; [12], Vol. 1, p. 3, and the introduction to Vol. 2; [14], p. x.

put him on his guard against some of the oversimplifications of classical hydrodynamics.

The art of knowing "how to apply" hydrodynamical theories can be learned even more effectively, in my opinion, by studying the paradoxes which I shall describe. Moreover, I think that to attribute them all to the neglect of viscosity is an unwarranted oversimplification. The root lies deeper, in lack of precisely that deductive rigor whose importance is so commonly minimized by physicists and engineers. They can be traced to the use of plausible arguments, which have often proved most fruitful in physical reasoning, but which are nevertheless fallible, as simple examples show. Among these are the arguments that "small causes produce small effects" and that "symmetric causes produce symmetric effects."

A clear understanding of the fallibility of these arguments in specific hydrodynamical applications, should be useful to any research worker who wishes to make reliable theoretical predictions.

2. *Their Mathematical and Philosophical Interest*

The possibility that *arbitrarily small causes can produce finite effects* in fluid mechanics has been clearly recognized by Oseen ([11], pp. 211–213). Oseen has pointed out that in systems of differential equations, the presence of arbitrarily small terms of higher order can change entirely the behavior of the solutions. It need *not* be true that as the coefficient of a certain term tends to zero, the solution tends to the solution obtained by annihilating that coefficient. Paradoxes due to this source may be called *asymptotic paradoxes*.

The possibility of asymptotic paradoxes cannot be admitted as a universal principle.³ If it is, since every physical experiment is actually affected by countless minute influences,⁴ we

³ I have been informed by J. R. Portié and L. L. Whyte that asymptotic paradoxes are common in biology (mutations of genes, etc.).

⁴ Heating and quantum reactions from cosmic rays, variations in gravity due to distant stars and the Coriolis force, seismic vibrations, impurities in the materials used, etc.

should have no chance of ever predicting correctly the result of any specific experiment. Since the results of many experiments *can* be predicted with practical certainty, it is not a universal principle, but is presumably restricted to certain types of partial differential equations.

Hence asymptotic paradoxes should interest the pure mathematician by providing suggestive evidence regarding the facts about partial differential equations—a complex subject concerning which our knowledge is still very rudimentary. Several of the other paradoxes which I shall describe will provide physical evidence of further properties of partial differential equations, and should therefore also interest the pure mathematician. Indeed, I am convinced that in order to make real theoretical progress, we must study a wide variety of physically and mechanically constructed solutions of partial differential equations of specific types.⁵

Thus there are several interesting cases in hydrodynamics where *an apparent symmetry of causes is not preserved in the effects*. Such paradoxes may be called *symmetry paradoxes*,⁶ and they have an evident philosophical interest for the logic of physics.

In general, hydrodynamical paradoxes have a philosophical interest, as illustrating the profound interweaving of deductive and inductive logic in scientific research. I believe that the resourcefulness and ingenuity of the human mind in combining these methods is vastly underestimated in all existing formal theories of mathematical and physical logic.⁷

They also have an interest for the pure mathematician, by illustrating a wealth of phenomena not anticipated by classical mathematical theories. Even a superficial attempt at mathematical classification reveals what may be called paradoxes of overdeterminism, singular point paradoxes, paradoxes of approximation, and paradoxes of topological oversimplifica-

⁵ This idea, with especial reference to high-speed computing machines, has been stressed repeatedly by J. von Neumann.

⁶ They seem to contradict the logical axiom that “causes invariant under a group G induce effects invariant under the same group,” G. Bouligand, *Théorie générale des groupes*, Paris, 1935, p. 3.

⁷ In particular, mathematical logic seems to me to suffer from too exclusive preoccupation with the foundations of set theory.

tion. I shall now try to support the statements made above by a discussion of precise mathematical details.

3. Fundamental Approximations

The role of postulates in theoretical fluid mechanics is played by an "equation of state," and certain partial differential equations and boundary conditions.

One assumes that the *density* ρ and *viscosity* μ of any fluid depend on the *pressure* p and *temperature* T , so that

$$(1) \quad \rho = f(p, T), \quad (\text{equation of state})$$

$$(1') \quad \mu = f_1(p, T).$$

In mathematical discussions, μ is commonly treated as a constant, which is nearly fulfilled in most practical applications, provided temperature variations are moderate.

It is then convenient to take the "Eulerian" independent variables of vector position $\mathbf{x} = (x_1, x_2, x_3)$ and time t . Assuming that all functions involved are sufficiently differentiable⁸ (even the assumption of continuity is non-trivial, as we shall see!), we can express the conservation of mass as the so-called Equation of Continuity

$$(2) \quad \operatorname{div}(\rho \mathbf{u}) + \frac{\partial \rho}{\partial t} = 0,$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes vector velocity.

It is sometimes convenient to use instead "Lagrangian" independent variables $\mathbf{a} = (a_1, a_2, a_3)$ associated with material points moving with the fluid, and t . If $\mathbf{a}$ is held fixed, the time derivative becomes the "material" derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum u_k \frac{\partial}{\partial x_k} \quad (\text{in Eulerian variables}).$$

⁸ Careful discussions of possible discontinuities have been given by Paul Duhem, "Recherches sur l'hydrodynamique," *Annales de la Faculté des Sciences de Toulouse*, Vols. 3-5 (1901-1903) and by L. Lichtenstein, *Hydrodynamik*, Berlin, 1933. I am indebted for these references, and for other helpful comments on Chapter I, to Prof. J. Kampé de Fériet.

Hence $u_i = dx_i/dt$ and (2) is equivalent to

$$(2') \quad \frac{d\rho}{\rho dt} + \operatorname{div} \mathbf{u} = 0.$$

The other basic differential equations of fluid mechanics express Newton's momentum law for continua, and are called the Equations of Motion; they involve the vector acceleration of gravity $\mathbf{g} = \mathbf{g}(\mathbf{x})$.

In stating the Equations of Motion, one must make one of two approximations. Either one must neglect viscosity, which gives the approximation

$$(3) \quad \frac{d\mathbf{u}}{dt} + \frac{1}{\rho} \operatorname{grad} p = \mathbf{g} \quad (\rho \text{ variable})$$

for a so-called *non-viscous* fluid; or one must neglect compressibility, setting $\rho = \text{const.}$, and use the Navier-Stokes equations

$$(3^*) \quad \frac{d\mathbf{u}}{dt} + \frac{1}{\rho} \operatorname{grad} p = \nu \nabla^2 \mathbf{u} + \mathbf{g},$$

for an *incompressible* fluid. Here the ratio $\nu = \mu/\rho$ is called the *kinematic viscosity* of the fluid.

These approximations are made partly for simplicity—even with them, one cannot in most cases solve the mathematical problems involved. They are also made because we do not actually know how viscosity acts in a fluid under rapid compression (see [12], Vol. 1, p. 260)!

Hence if we are too impressed by the dangers of approximation, we must despair at the outset of attaining any rational understanding of fluid mechanics. However, science was not built by timid or faltering hands. Let us move boldly forward, seeing what kind of edifice one can construct on these somewhat insecure foundations.

4. *Non-viscous Fluids*

I shall begin (Sections 4–12) by making the approximation of zero viscosity. The *a priori* plausibility of this approximation would be hard to exaggerate. Thus ([6], p. 4) a change in velocity of 100 m.p.h. across a layer of air .01 in. thick,

exerts a shear force of only one ounce weight per square foot.

Its plausibility can also be expressed in terms of the *Reynolds number*, defined as the dimensionless ratio

$$(4) \quad R = \frac{\rho VL}{\mu} = \frac{VL}{\nu},$$

where V is a typical velocity and L a representative linear dimension. Since R expresses the ratio (inertial forces)/(viscous forces), it seems reasonable to suppose that the approximation $\nu = 0$ should become asymptotically valid for sufficiently large R , say, over one million. This supposition becomes even more reasonable if we recall (Section 2) the many other minute forces which are always necessarily neglected in making physical predictions. Without further discussion, let us therefore introduce the fiction of a “non-viscous” fluid, in which (3) is valid.

In studying the flows of “non-viscous” fluids from homogeneous reservoirs, it is usually also assumed that the fluid expands and contracts *adiabatically*, so that

$$(1') \quad p = f(\rho). \quad (\text{In perfect gases, } p = k\rho^\gamma.)$$

This assumption is reasonable if viscous forces are negligible, because heat conduction and viscosity are both manifestations of molecular diffusion. It can also be justified by separate computations, if we admit the principle that *small variations in density exert a small influence in fluid flow*. This principle, unlike the corresponding principle for viscous forces, appears to be entirely supported by the physical evidence.

5. Mathematical Properties

From the preceding approximate assumptions, several fundamental mathematical consequences can be derived, and I should like to emphasize the fact that these consequences do in fact explain observed resistance to acceleration from rest (see Chapter V) and many other physical phenomena.

Thus, in the case of “barotropic” fluids satisfying (1'), one can prove ([12], Vol. 1, p. 191) that the “circulation” $\oint \Sigma u_k dx_k$ around any closed “material” curve moving with the fluid,

is constant. Hence if the fluid comes from a reservoir without circulation (i.e. at rest in some non-rotating reference frame), and if the curve remains closed at all times, the circulation should remain zero. This is mathematically equivalent to the statement that there should exist a locally single-valued scalar *velocity potential* $U(\mathbf{x};t)$, such that

$$(5) \quad \mathbf{u}(\mathbf{x};t) = \text{grad } U, \quad \text{or} \quad u_i = \frac{\partial U}{\partial x_i} \quad [i = 1, 2, 3].$$

In a *simply* connected domain, such as the exterior of a solid or half the symmetric exterior of a circular cylinder, U must therefore be single-valued in all space. In any case, this is the simplest possible assumption; it was universally accepted for over a century; and we shall make it until Section 8, where the possible exceptions will be discussed.

Further consequences of the basic hypotheses (1)–(3)–(4) can be deduced in the case $U = U(\mathbf{x})$ of “steady” fluid motion. Consider the case of the flow of a stream of fluid (such as air or water) moving at constant speed past a fixed obstacle—to which the case of a rigid body moving at nearly constant speed through an infinite fluid initially at rest can be reduced, by the use of moving axes.

Since the boundary conditions and differential equations are independent of time, one then naturally writes $U(\mathbf{x})$. One can prove that, under these circumstances, the equations of motion (3) are equivalent after one space integration to the *Bernoulli equation*

$$(6) \quad \frac{1}{2} \nabla U \nabla U + \int \frac{dp}{\rho} + G = p_0, \quad \text{or} \quad \sum u_i du_i + \frac{dp}{\rho} + dG = p_0.$$

Here G denotes the gravitational potential, so that $\mathbf{g} = -\nabla G$.

An important special case is that of (nearly) incompressible flow, in which case we set $\rho = \text{const.}$ In this case, (2) is simply $\text{div } \mathbf{u} = 0$, which is equivalent by (5) to

$$(7) \quad \nabla^2 U = 0,$$

i.e. to the existence of a *harmonic velocity potential*. For such flows, (6) assumes the familiar simple form

$$(6') \quad p = p_0 - \frac{1}{2}\rho \nabla U \nabla U - \rho G.$$

In fact, one can prove more generally, in the case of “unsteady” or “accelerated” motion of an incompressible non-viscous fluid, initially without circulation, that the pressure must satisfy

$$(6'') \quad p = p_0 - \rho \left\{ \frac{1}{2} \nabla U \nabla U + \frac{\partial U}{\partial t} + G \right\},$$

which is equivalent to the equations of motion, just like (6).

So much for the internal conditions satisfied by flows of an incompressible, non-viscous fluid from a still reservoir: we have the basic kinematic condition (7), from which pressure can be determined by (6') in the case of steady motion. In this case, the equation of continuity clearly becomes

$$(8) \quad \frac{\partial U}{\partial n} = 0 \quad \text{on the boundary,}$$

where $\partial/\partial n$ denotes the outward normal derivative. Similarly, the condition that $\mathbf{a}$ is the “free stream” velocity is clearly

$$(9) \quad \lim_{\mathbf{x} \rightarrow \infty} \text{grad } U = \mathbf{a}.$$

Now by potential theory ([7], pp. 211, 311), with bodies of reasonable shape, equations (7) to (9) have *one and only one solution*. This rather deep mathematical result is in perfect accord with the intuitive idea that there should be exactly one flow around a given fixed obstacle, having a given relative velocity $\mathbf{a}$. The flows obtained in this way are called Dirichlet flows.

The theory of Dirichlet flows is completely satisfactory from a logical and mathematical point of view. Let us now apply it to the study of fluid resistance.

6. The D'Alembert Paradox

One of the simplest examples of a Dirichlet flow is the steady flow of an incompressible non-viscous fluid past a sphere of radius L centered at the origin. We may suppose that the

free stream velocity has magnitude V , and is parallel to the x -axis. The solution is found by superposing a "dipole field" on a uniform flow, so that

$$(10) \quad U = Vx_1 + \frac{\alpha x_1}{r^3} \quad \left[\alpha = \frac{VL^2}{2}, r^2 = \sum x_i^2 \right].$$

It is easy to compute by (6') the theoretical pressure distribution around a sphere under the flow (10). One finds that, apart from the hydrostatic buoyancy (cf. Chapter III, Section 12), the pressure distribution has complete fore-and-aft symmetry, and hence gives a pressure thrust zero.

This theoretical conclusion, called the d'Alembert Paradox, is not limited to spheres; one can prove generally⁹

THEOREM 1. *The hydrodynamic force system on any finite obstacle B , under steady Dirichlet flow in an infinite stream, reduces to zero or a pure couple.*

PROOF. The assertion is that the integral $\mathbf{F}_B$ of the pressure differentials over the surface of B is zero. To prove this, we choose the x_1 -axis in the direction of flow, and let V be the free stream velocity. By hypothesis, $\nabla(U - Vx_1) \rightarrow 0$ as $\mathbf{x} \rightarrow \infty$; moreover $\iint (\partial U / \partial n) dS = 0$ on any closed surface containing the obstacle. It follows by potential theory ([7], p. 270) that $U - Vx_1 = O(1/r^2)$, meaning that it tends to zero as rapidly as $1/r^2$, where $r = |\mathbf{x}|$. Further

$$(10') \quad \mathbf{u} = (V, 0, 0) + O\left(\frac{1}{r^3}\right),$$

whence $\nabla U \nabla U = V^2 + O(1/r^3)$. Neglecting hydrostatic buoyancy as before, we find that the hydrodynamic pressure satisfies, by Bernoulli's equation (6'),

$$(11) \quad p = p_0 - \frac{1}{2} \rho V^2 - O\left(\frac{1}{r^3}\right) = p_f - O\left(\frac{1}{r^3}\right).$$

Here p_0 is the "stagnation pressure" at zero velocity; the constant $p_f = p_0 - \frac{1}{2} \rho V^2 = \lim_{\mathbf{x} \rightarrow \infty} p(\mathbf{x})$ is called the "free stream pressure."

⁹ For a recent discussion, see M. Manarini, "Sur paradossi d'Alembert e di Brillouin nella dinamica dei fluidi," *Atti Accad. Naz. Lincei, Rend.* 4 (1948), 427-433, reviewed in *Math. Reviews*, 10 (1949), p. 335.

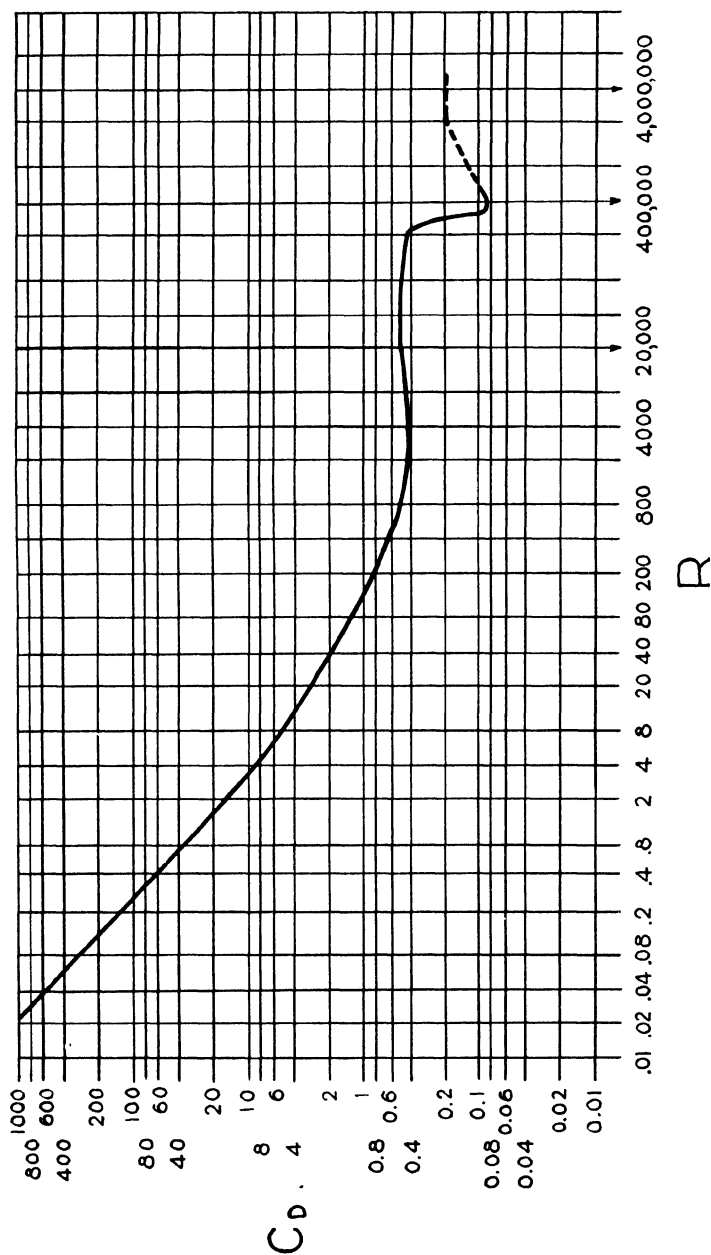


FIG. 1. Drag coefficient of sphere.

Now consider the momentum equation ([12], Vol. 1, Chapter XIV) for the fluid occupying, at variable time t , a fixed large sphere S of radius a . Clearly, for any a large enough so that S includes B , $\mathbf{F}_B$ is equal to the difference $\mathbf{E}(a) - \mathbf{F}_s(a)$ between the rate $\mathbf{E}(a)$ at which momentum flows outwards through S , and the pressure thrust $\mathbf{F}_s(a)$ exerted inwards across the surface of S . Hence

$$\mathbf{F}_B = \lim_{a \rightarrow \infty} \{\mathbf{E}(a) - \mathbf{F}_s(a)\}.$$

But by equations (10')–(11), the pressure on the surface of S differs by $O(1/a^3)$ from the constant p_f ; therefore, the area of S being $4\pi a^2$, $\mathbf{F}_s(a)$ differs by $O(1/a)$ from zero: $\mathbf{F}_s(a) = O(1/a)$. Similarly, the flux of mass per unit area through S is $\Sigma u_k x_k / a$, and so by (10') differs by $O(1/a^3)$ from that of a "uniform flow" with constant velocity $(V, 0, 0)$. Again, the vector momentum transported through a unit area per unit time, is $\mathbf{u} \Sigma u_k x_k / a$; this also differs from that of a uniform flow by $O(1/a^3)$, using (10') again. But for a uniform flow, clearly $\mathbf{E}(a) = 0$; hence actually, multiplying once more by $4\pi a^2$, $\mathbf{E}(a) = O(1/a)$. Substituting above, we conclude $\mathbf{F}_B = 0$, as desired.

The preceding theoretical prediction, of zero resistance or "drag" D and lift L , is obviously contradicted by everyday experience; thus in Figure 1 one sees the drag coefficient $C_D = D / \frac{1}{2} \rho V^2 A$ (A = cross-section area) of a sphere, plotted as an empirical function of the Reynolds number R . I shall discuss shortly (in Section 9) the rationalizations of this contradiction between theory and experiment, but first I should like to describe some closely related paradoxes.

7. The Reversibility Paradox

Although it is probable that the d'Alembert paradox is also valid for subsonic, compressible flow of a non-viscous fluid, this seems hard to prove rigorously.¹⁰ However, a very simple argument shows that fluid resistance can never be

¹⁰ A "proof" has been given by U. Cisotti, *Comptes Rendus*, 178 (1924), p. 1792. But as there is no rigorous analog applying to nonlinear partial differential equations of the result proved on p. 270 of [7], much stronger hypotheses regarding the behavior at infinity must be made.

explained from the hypothesis of a non-viscous fluid as formulated above (Sections 4-5).

DEFINITION 1. By the *reverse* of a given flow, described in Lagrangian coordinates, we mean that flow obtained from the given flow by the substitution $t \rightarrow -t$. In Eulerian coordinates, this corresponds to the reversal $u_i \rightarrow -u_i$ of velocity direction as well as reversal $t \rightarrow -t$ of time, but pressure p and density ρ are still preserved at corresponding points $(x_1, \dots, x_n; t)$ of space time.

DEFINITION 2. A condition on flows is reversible if, whenever it holds for a flow, it holds for the reverse of that flow. A theory of fluid dynamics is a *reversible theory* if all the conditions which it imposes on flows are reversible.

Actually, all the familiar conditions on flows are reversible, except those which involve viscosity (friction), thermal conduction, diffusion, or shock wave fronts.

DEFINITION 3. A theory of fluid dynamics will be called *incomplete* if its conditions do not determine the steady flow around a body uniquely; *overdetermined* if no flow is compatible with it; *incorrect* if its predictions do not agree closely with experimental fact.

THEOREM 2. (Reversibility Paradox.¹¹) *Any reversible theory of fluid dynamics is either incomplete, overdetermined, or grossly incorrect, so far as its predictions of steady state lift and drag are concerned.*

PROOF. Such a theory will predict by (6') that a steady flow and its reverse will give the same pressure thrust on an obstacle, whereas it is a matter of common experience that a flow and its reverse ordinarily give pressure thrusts in approximately opposite directions.¹²

¹¹ In general, physicists seem to ascribe only philosophical interest to the fact that the concept of non-viscous fluid motion is reversible. Thus the fact is not mentioned at all in [9], [10], or [12].

¹² The preceding discussion is borrowed from [1]. Recently, M. Munk has extended the reversibility principle to linearized supersonic airfoil theory, in *Naval Ordnance Lab. Memo. 9624* (1948). A reversibility principle for theoretical ship wave resistance has been derived by T. H. Havelock, *Proc. Roy. Soc., A149* (1935), p. 423; the discrepancy with experiment is analyzed by W. C. S. Wigley, *Trans. Inst. Nav. Arch.*, 86 (1944), 1-20. Similar considerations have recently been used by A. Busemann to show that no existence and uniqueness theorem can be hoped for in the case of transsonic flow.

Now the approximations of Sections 4-5 are certainly compatible, because they permit Dirichlet flows around objects of (almost) arbitrary shape. Hence any discussion of fluid resistance which is based on them must either be incorrect, or else involve further assumptions. It is generally believed that both alternatives are possible.

Thus for *subsonic* flow,¹³ it is often stated that no physically correct theory of resistance is possible, which is based on the differential equations of Sections 4-5. Whereas for *supersonic* flow it is known that these equations are compatible with more than one flow, and hoped that resistance can be explained by using these equations and qualitative physical considerations.

I shall now discuss in more detail the incompressible case, which is generally supposed to be typical of subsonic flow, because the partial differential equations involved are of elliptic type in both cases.

8. Paradoxes of Airfoil Theory

It is well-known that, if one is willing to admit discontinuous functions—that is, to abandon the assumption that all “reasonable” functions are differentiable—the differential equations (1)-(3)-(4) permit many steady flows around an obstacle. These flows are of various topological types, and most of them avoid the d’Alembert paradox. As such “discontinuous fluid motions” form the main theme of Chapter II, I shall not discuss them here, except to mention that they, too, will be found to be subject to paradoxes.¹⁴

Instead, I shall discuss *two-dimensional airfoil theory*, which gives the best known theory of lift—although it does not give a basis for estimating resistance. In this theory, the multiple-valued velocity potentials, excluded in Section 5, are considered. Mathematically, this amounts to replacing (10') by the multiple-valued potential

¹³ See Th. von Karman, “The problem of resistance in compressible fluids,” *Reale Accad. d'Italia* (1936); U. Cisotti, *op. cit. supra*; and Section 12 *infra*.

¹⁴ Attention is called to the Hadamard paradox, and to the paradoxes of wake underpressure, of indeterminacy, of surface seal, and of down-nosing, discussed in Chapter II.

$$(12) \quad U = \Gamma \frac{\theta}{2\pi} + \sum a_k x_k + U_1,$$

$$\text{where} \quad \nabla^2 U_1 = 0, \quad U_1 = 0 \left(\frac{1}{r^2} \right).$$

Here $\Gamma = \oint \Sigma u_k dx_k = \oint dU$ is called the *circulation*, the integral being taken along a contour going once around the "airfoil" (or obstacle) B in question. Since any real harmonic function $U(x, y)$ is the real part of a complex analytic function $W(z)$ of $z = x + iy$, we can write equivalently ($c = \Gamma/2\pi$ being real and c' complex),

$$(12') \quad U = \operatorname{Re}\{-ic \ln z + c'z + W_1\},$$

$$\text{where} \quad W_1 = 0 \left(\frac{1}{r^2} \right).$$

We shall find it convenient to use such "complex potentials" W below, and again in Chapter II.

The introduction of the term $\Gamma\theta/2\pi$ in (12) sacrifices the determinism principle established at the end of Section 5: there is a solution of (12) for each choice of Γ . One can rescue determinism in the case that the airfoil R has a sharp "trailing edge" (but not otherwise), by assuming the so-called "Joukowski Condition," that "velocity is finite at the trailing edge." By assuming that the sharp edge is *trailing*, one avoids the Reversibility Paradox. A Dirichlet flow with circulation which satisfies the Joukowski Condition may be called a "Joukowski flow." Thus the Dirichlet flow without circulation past a flat plate inclined at 15° to the main stream is sketched in Figure 2a; the Joukowski flow past the same plate is sketched in Figure 2b; the "cavity flow" (cf. Chapter II) past it is shown in Figure 2c.

Knowing U , one can then get a theoretical estimate of the thrust on an "airfoil" of arbitrary cross-section, using the following "Kutta-Joukowski Theorem."

THEOREM 3. *In any Dirichlet flow with circulation, the drag^{14a} is zero, and the lift is the product $L = \rho v_f \Gamma$, of the fluid density, the free stream velocity, and the circulation.*

^{14a} The "drag" is as usual the component of force parallel to the flow; L is the component perpendicular to the flow. In Theorem 1, $D = L = 0$.

This theorem can be proved in the same way as Theorem 1, without assuming the Joukowski condition. It is a corollary that the lift is

$$(13) \quad L = \frac{1}{2}\rho v^2 c C_L^* \sin \alpha,$$

where c is the wing chord, α is the "effective angle of attack," and C_L^* depends only on the shape of the wing cross-section.

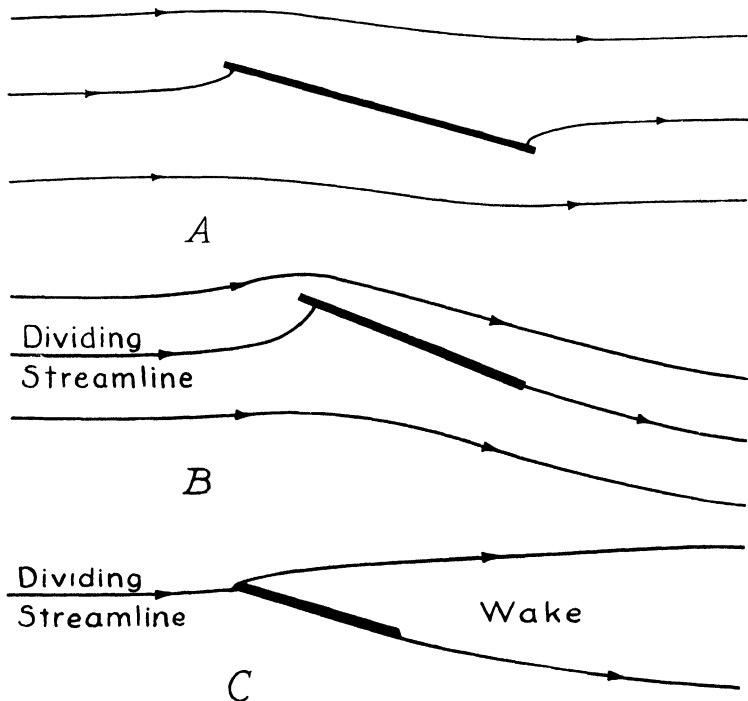


FIG. 2. Three ideal flows past inclined plate.

A similar asymptotic argument determines the moment in any Joukowski flow,¹⁵ and this gives the best known theoretical estimate for the variation in airfoil forces with wing cross-section. Nevertheless, the theory outlined above is subject to three paradoxes. The first of these is quite amusing

¹⁵ The complete theory was given by R. von Mises, *Zeits. Flugtechnik und Motorluftschiffahrt*, 1917, pp. 157-163, and 1920, pp. 67-73 and 87-89. For Theorem 3, see [12], Vol. 2, p. 163; shorter proofs are known.

and purely mathematical in nature; it may be called the Cisotti Paradox (references for the full discussion may be found in [2]).

Consider the Joukowsky flow past an inclined flat plate in a horizontal wind (Fig. 2b). By the Kutta-Joukowsky Theorem, the resultant force should be vertical (i.e. pure lift). On the other hand, as all pressures are normal to the plate, the resultant force should be normal to the plate, giving a clear contradiction. The latter conclusion is confirmed by an integration along the plate of the pressure difference between the two sides; the detailed formulas follow.

The most general Joukowsky flow around the unit circle $z = e^{i\theta}$ has the velocity potential proportional to

$$U = \text{Re} \left[\frac{e^{i\alpha} z}{2} + \frac{e^{-i\alpha}}{2z} - ic \ln z \right],$$

where α and c are real. Since $2\zeta = z + 1/z$ maps $z = e^{i\theta}$ onto $-1 \leq \xi \leq 1$ [$\zeta = \xi + i\eta$], the most general Joukowsky flow around the segment $-1 \leq \xi = \cos \theta \leq 1$ has a velocity potential on this segment conveniently written as

$$\begin{aligned} U &= \text{Re} [\frac{1}{2} (e^{i(\alpha+\theta)} + e^{-i(\alpha+\theta)}) + c\theta] \\ &= \cos(\alpha + \theta) + c\theta. \end{aligned}$$

Using θ as a parameter, the velocity on the plate is

$$\begin{aligned} \frac{dU}{d\xi} &= \frac{-dU}{\sin \theta d\theta} = \frac{\sin(\alpha + \theta)}{\sin \theta} - \frac{c}{\sin \theta} \\ &= \cos \alpha + \sin \alpha \cot \theta - c \csc \theta. \end{aligned}$$

Hence the difference in pressure between similarly situated points on opposite sides θ , $-\theta$ is given by Bernoulli's Theorem as

$$\Delta p = 2\rho \cos \alpha [\sin \alpha \cot \theta - c \csc \theta];$$

the integral $\int_0^\pi \Delta p ds = -\Delta p \sin \theta d\theta$ is clearly convergent.

The resolution of the Cisotti Paradox leads one to the astonishing conclusion that there is a non-zero force acting on the single point at the front end! This is clearly a very pure example of a Singular Point Paradox, arising from a neglect

to study rigorously the behavior of the flow at all singular points.

A second mathematical paradox arises from the fact that, in space, the exterior of a wing of finite span is simply connected. It follows that a multiple-valued potential with zero vorticity (i.e. $\text{curl } \mathbf{u} = \mathbf{0}$) at all points is impossible;¹⁶ hence no escape from the d'Alembert Paradox along the lines of two-dimensional airfoil theory is mathematically possible in three-dimensional space.

Even the two-dimensional theory, however plausible its assumptions, agrees doubtfully with experiment. It involves the following Fatness Paradox (see [1], p. 252). *Theoretically, the quantity C_L^* in (13) increases as the airfoil thickness increases; experimentally, it decreases as the airfoil thickness increases.*

9. Discussion of Preceding Paradoxes

In spite of the preceding paradoxes, the theory of an incompressible non-viscous fluid, or "ideal fluid," is extremely useful to the research engineer in dealing with fluids of small viscosity, at speeds up to one-fourth the speed of sound. Outside of a rather wide belt behind a given obstacle or airfoil B , called the "wake," and a very thin "boundary layer" near the surface of B (cf. Figs. 3a-3b), its predictions are reliable.

By accepting the wake and boundary layer as empirical facts, one can rationalize most of the preceding paradoxes. More specifically, one must recognize that, as the viscosity tends to zero, the width of the wake remains finite, and the "wake underpressure coefficient" $(p_f - p_w)/\frac{1}{2}\rho v^2$ (p_f = free stream pressure, p_w = wake pressure) does not tend to zero. This leads to a "drag coefficient" $C_D = D/\frac{1}{2}\rho v^2 A$ which does not tend to zero with the viscosity (see Fig. 1); and it gives a flow pattern which is clearly irreversible. Finally, the Fatness Paradox may be rationalized as the natural tendency of the

¹⁶ This remark may be found in K. Friedrichs and R. von Mises, *Fluid Dynamics*, Brown University Lecture Notes, 1941. Nevertheless attempts have been made to compute lift forces on airships within the framework of incompressible non-viscous flow! In certain pumps, the propellers extend to the pipe walls, and so the fluid is multiply-connected; this topological fact has been said to give such pumps a superior efficiency.

wake to become "fatter" with the airfoil, a tendency which diminishes the circulation.

The preceding rationalizations are strongly confirmed by the fact that ([6], Chapter XII, or [12], Vol. 2, p. 81), by sucking away a very thin boundary layer, with a relatively minor expenditure of energy, Dirichlet and Joukowski flows can be very closely approximated. Correspondingly, the d'Alembert and Fatness Paradoxes can be largely avoided. This suggests

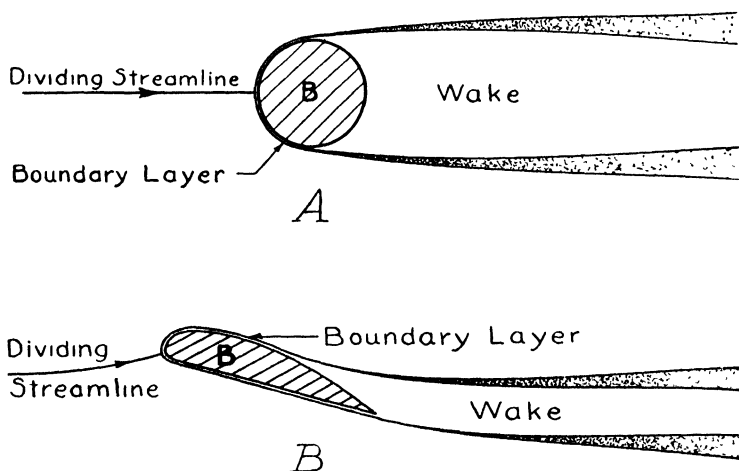


FIG. 3. Schematic diagrams of "real" flows.

that the prevailing view already mentioned (Sections 1 and 7), that the paradoxes of fluid mechanics are due to the neglect of viscosity, may be incorrect. It is conceivable that, when the Reynolds number R is very large, the concept of an ideal fluid is not incorrect, but merely incomplete¹⁷ (cf. Theorem 2).

In support of this idea, it may be mentioned that the deduction of Dirichlet flows from the hypothesis of an "ideal fluid" involves implicitly two arbitrary topological assumptions:¹⁸ (i) that the streamlines from the upstream side cover the whole plane, and (ii) that a locally single-valued velocity

¹⁷ The sense in which the fundamental differential equations, etc., for an ideal fluid are "complete" is discussed in Chapter III, Section 11.

¹⁸ Much of modern clarification of the foundations of geometry is due to the explicit consideration of Euclid's implicit topological assumptions—as in D. Hilbert's classic *Grundlagen der Geometrie*, Berlin, 1930.

potential U is single-valued in the large. There is no valid mathematical reason for excluding from consideration Joukowski flows, flows with wake (see Chapter II), and many other topological types of flows.¹⁹ Hence the paradoxes of ideal fluid theory may be, in part, *paradoxes of topological oversimplification*.

In any case the preceding examples make it clear that *the theory of non-viscous flows is incomplete*. Indeed, the reasoning leading to the concept of a "steady flow" is inconclusive; there is no rigorous justification for the elimination of time as an independent variable.

Thus though Dirichlet flows and other steady flows are mathematically possible, there is no reason to suppose that any steady flow is *stable*. It is perfectly conceivable that, in an "ideal" fluid, initially departing slightly from Dirichlet flow, irregularly varying turbulent "eddies" are built up mathematically in the "wake" of an obstacle—reproducing mathematically what is observed physically at large Reynolds' numbers R . Indeed, such a model has already been constructed by von Karman (*loc. cit. supra*), for the alternating vortices shed by cylindrical obstacles at intermediate Reynolds numbers, $50 < R < 1000$.

To admit this possibility, we must reject the idea that there is a necessary tendency towards symmetry in natural phenomena,²⁰ and admit the possibility that *a symmetrically stated problem may not have any stable symmetric solution*.

Extensive hydrodynamical evidence of this possibility is presented in Section 14–15. Admitting it provisionally, we

¹⁹ For example: The "multiple solutions" of H. Villat (see *Leçons sur l'hydrodynamique*, Paris, 1929, Chapter VIII); the solutions with two symmetric vortices in a finite wake, introduced by Föppl (see [9], p. 223, or H. Bateman, *Partial Differential Equations*, 1932, p. 251, and refs. given there); Karman's flows with infinite vortex trails ([9], p. 225; or [10], p. 341); and the reverses of the preceding flows (see E. Picard, *Comptes Rendus*, 159 (1914) 639). It should be noted, however, that circulation or a wake of infinite length and breadth is needed to avoid the d'Alembert Paradox.

²⁰ This idea has been advanced by various philosophers of science; see L. L. Whyte, "Tendency towards symmetry in fundamental physical structures," *Nature*, 163 (1949), p. 762. For a general discussion, see P. Duhem, *La théorie physique, son objet et sa structure*, Paris, 1907, Part II, Chapter III.

are led to the conclusion that the paradoxes of ideal fluid theory are, in part at least, paradoxes of topological oversimplification and symmetry paradoxes, in which "apparently symmetric causes do not have symmetric effects." (The latter is especially evident in the case of the Reversibility Paradox, if one considers wake theory.)

At all events, the most obvious paradoxes can be attributed as much to lack of deductive rigor as to faulty "physical" assumptions or approximations. In further support of this view, I shall now show that the theories of compressible fluids and of viscous fluids (so-called "real" fluids) have their full share of paradoxes.

10. The Earnshaw Paradox

I shall first treat the paradoxes which *specifically* concern compressibility effects arising in the high-speed flow of non-viscous fluids, beginning with a paradox due to Earnshaw.²¹

EARNSHAW PARADOX. *Plane sound waves of finite amplitude and stationary form are mathematically impossible, in a gas which vibrates adiabatically.*

PROOF. We assume that the form of the sound waves is stationary, and that the train of waves moves with constant velocity normal to the wave fronts. Hence if we change to axes moving with the waves, the fluid motion will be not only one-dimensional but *steady*, and we can write $\rho = \rho(x)$, $u = u(x)$, etc. Neglecting gravity, Bernoulli's equation (6) reduces to $u du + dp/\rho = 0$. Moreover the equation of continuity (1) reduces to $\rho u = \text{const.} = C$, or $u = C/\rho$. Substituting back we get

$$(14) \quad -\frac{C^2 d\rho}{\rho^3} + \frac{dp}{\rho} = 0, \quad \text{or} \quad dp = \frac{C^2 d\rho}{\rho^2}.$$

Hence such a wave motion is possible only if the fluid satisfies an equation of state (2') of the special form

²¹ S. Earnshaw, *Phil. Trans.*, 150 (1860), 133-148. See also Stokes, *Phil. Mag.*, 33 (1848), 349; Rankine, *Phil. Trans.*, 160 (1870), 277; [13], Vol. V, p. 573, or *Proc. Roy. Soc.*, A84 (1910), 247-284; [9], §283; [60], §51. Of course, the paradoxes of Sections 6-8 also apply to compressible non-viscous fluids.

$$(15) \quad p = p_0 - \frac{C^2}{\rho}.$$

There is no known gas whose adiabatic equation of state has the form (15).

We have here a typical paradox of overdeterminism; such paradoxes arise typically when too *many* approximate hypotheses are made. It would be hard to say that any one approximation was at fault; indeed, like many other paradoxes,²² the Earnshaw Paradox contains the germ of an important truth. For waves of *large* finite amplitude, it leads to the correct and remarkable conclusion that the dense portions of such waves gain progressively on the more rarefied portions, until eventually *shock waves* occur where there is a near-discontinuity of density, pressure, and velocity.

Shock waves are now such a commonplace that it may be of interest to recall that Riemann treated them incorrectly, and that Lamb, even in the last edition of his classical *Hydrodynamics*, casts doubt on the validity of the now accepted Rankine-Hugoniot theory.²³

From the Earnshaw Paradox one can easily derive a second paradox, noted independently by Mr. H. D. Ursell and myself. "Everybody knows" that there is a close analogy between the theory of shallow surface waves, and compression waves in a gas whose equation of state is $p = k\rho^2 + c$, with $\gamma = 2$ (cf. Chapter III, Section 15); this analogy has been effectively exploited recently by J. J. Stoker.²⁴ Yet it was proved by D. J. Struik in 1925 that shallow surface waves of finite amplitude were possible; and we have just proved that waves of

²² Thus the d'Alembert paradox contains the germ of the following useful engineering principle: By "sucking away the boundary layer," in a fluid of sufficiently low viscosity, resistance can be reduced to nearly zero, by an operation which involves very little energy.

²³ Lamb ([9], p. 486) states that "no physical evidence is adduced in support of the proposed law," and that "in actual fluids, the equation of energy cannot be satisfied consistently. . . ." I am indebted to Mr. R. H. Kent for this reference.

²⁴ See *N.Y.U. Communications in Applied Mathematics*, 1 (1948), 1-87 and refs. given there. The analogy was apparently first noted by E. Jouguet, *Jour. de math. pures appl.*, 3 (1920), p. 1. For Struik's results, see *Rendic. Lincei*, 1 (1925), 522-527.

finite amplitude are impossible for $\gamma \neq -1$. What is the explanation? Nobody knows.

11. Von Neumann's "Triple Shock" Paradox

Since shock waves are now in the center of popular attention, it may be of interest to call attention to a new paradox regarding shock waves.

The mathematical theory of plane shock waves can be derived simply from the laws of conservation of mass, momentum, and energy (heat plus kinetic energy), *provided* one

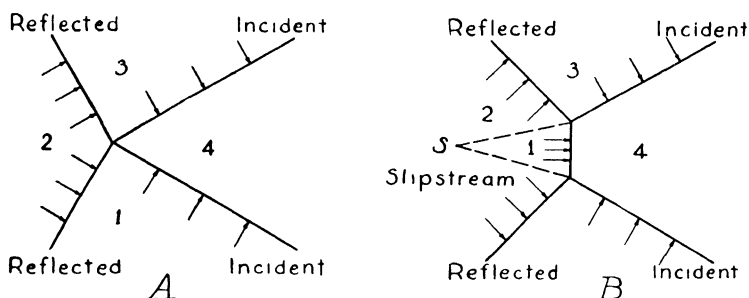


FIG. 4. Shock wave reflections.

assumes that on each side of the shock wave all relevant physical variables (temperature, density, pressure) assume limiting values. The predictions of the Rankine-Hugoniot theory derived from these laws seem to be closely confirmed experimentally, as regards *single* shock waves.

Furthermore, corresponding mathematical predictions are possible in the case of double "regularly" reflected shock waves (see Fig. 4a) and of "triple shocks" or "Mach Y" waves (see Fig. 4b), provided it is assumed that in each relevant sector (1,2,3,4) physical variables assume limiting values near the singular point at the shock "vertex." Again, many of these predictions are confirmed experimentally, so that the theory has a highly respectable status.

However, in the case of "weak" shocks, regular reflections occur for angles of incidence somewhat greater than those permitted by theory, and *the predicted limits for triple shocks seem to differ grossly from those observed.* This discrepancy

was apparently first discovered by J. von Neumann (1945); in spite of various attempts, based on the idea that this is a singular point paradox no satisfactory explanation of it has yet been advanced.²⁵

12. Kopal's Paradoxes of Linearized Projectile Theory

Time does not permit me to give the details here of the interesting and useful "linearized theory" for forces on slender projectiles moving at supersonic speeds, developed by von Karman and Moore.²⁶ The basic approximation is the assumption that velocities are perturbed relatively little from the free stream velocity. An analogous theory exists for wing forces (see also Chapter III, Section 15).

Now in the case of a slightly yawed cone-headed projectile, a comparison with a more rigorous calculation, based on the exact Taylor-Maccoll theory (Chapter IV, Section 4) for cones with zero yaw, is possible.

The linearized equations originally proposed by Karman, Moore, and Tsien (*op. cit.*) underestimate by ten to fifty per cent the head drag and pressure coefficients deduced from the exact equations in case of moderate velocities (technically, when the projectile head is much more slender than the "Mach cone" starting from the projectile tip). Moreover in the case of yawed cones, the cross-forces predicted by the perturbation methods constituting the "linearized theory" decrease with increasing Mach number, whereas those predicted by perturbation of the Taylor-Maccoll exact solutions, increase with increasing Mach number.

These discrepancies were revealed clearly by laborious calculations made during the late war. They are purely mathematical paradoxes due to lack of rigor in computing the

²⁵ A comprehensive review of the subject has been given by H. Polachek and R. J. Seeger, "On shock-wave phenomena: interaction of shock waves in gases," *Proc. Symposia in Applied Math.*, Vol. I, pp. 119-144, Am. Math. Soc., 1949. See especially p. 131 and the references given there. See also W. Bleakney and A. H. Taub, *Revs. Mod. Phys.*, 21 (1949), 584-605.

²⁶ Th. von Karman and N. B. Moore, "Resistance of slender bodies," *Trans. Am. Soc. Mech. Eng.* (1932), p. 303; H. S. Tsien, *Jour. Aer. Sci.*, 5 (1938), p. 480.

relative orders of magnitude in the asymptotic approximations ("perturbations") made, and from failure to determine the order of magnitude of the effect on the resulting calculations.

They show up strikingly the insecure foundations of the asymptotic theory of partial differential equations, alias "perturbation theory." Recent new "linearizations" of the equations of supersonic flow have attempted to rectify the situation.²⁷

13. Viscous Incompressible Fluids

I now come to paradoxes involving the theory of viscous incompressible fluids. In this connection, I recall that the Equations of Motion for a viscous compressible fluid have never been proved correct, and that, according to Paul Duhem,^{27a} the supposition of an incompressible viscous fluid is thermodynamically incompatible with the concept of a fluid. However, these remarks need not concern us further.

In an "incompressible" *viscous* fluid, the equations of continuity and state reduce to the single condition

$$(16) \quad \rho = \rho_0, \quad \text{or equivalently} \quad \operatorname{div} \mathbf{u} = \sum \frac{\partial u_k}{\partial x_k} = 0.$$

These are combined with the Equations of Motion (3*). The only kinematical limitation imposed by these equations is that $\operatorname{curl} (\operatorname{grad} p) = \nabla \times (\nabla p) = \mathbf{0}$, which, if $\mathbf{g} = \nabla G$, reduces to

$$(17) \quad \frac{d\xi}{dt} = \nu \nabla^2 \xi + (\xi \cdot \nabla) \mathbf{u}. \quad [\xi = \operatorname{curl} \mathbf{u}]$$

The boundary conditions are

$$(18) \quad \mathbf{u} = 0 \text{ on any fixed rigid surface,}$$

instead of merely that the *normal* velocity component (8) is zero.

It is commonly (but erroneously) believed that the theory of viscous fluids is free from paradoxes. One reason for this

²⁷ See E. V. Laitone, *Jour. Aer. Sci.*, 14 (1947), 631-642; also Broderick, *Quart. J. Mech. Appl. Math.*, 2 (1949), 98-120 and 121-128; M. J. Lighthill, *Jour. Aer. Sci.*, 16 (1949), p. 75. The perturbed Taylor-Maccoll theory was first developed by A. H. Stone.

^{27a} *Traité d'énergétique*, Vol. 2, p. 131, Paris, 1900.

is perhaps that the drag coefficient can often be plotted as a function $C_D(R)$ of the Reynolds number alone (cf. Chapter III, Section 13, and Figure 1). Other experimental confirmations are found in the theory of lubrication, and in some parts of the theory of turbulence.

In support of the view that the paradoxes of fluid mechanics are due to an unjustified neglect of viscosity, the following two mathematical remarks are often made. In non-viscous flow, (18) cannot be satisfied, except in the trivial case of no motion; hence the real boundary conditions are completely altered, in the approximation of zero viscosity. Again, the term of highest order in the Equations of Motion is suppressed, completely altering the type of these differential equations.

One can easily see that the preceding criticisms have some merit, by considering the solutions of the ordinary differential equation $\epsilon y'' + y = 0$, as $\epsilon \rightarrow 0$. If one uses the boundary conditions $y(0) = a$, $y(1) = b$, one gets an interesting contrast between the cases $\epsilon \uparrow 0$ and $\epsilon \downarrow 0$.

Nevertheless, I think they are inconclusive. By "sucking away the boundary layer" one can approximately realize the flow patterns predicted by the theory of non-viscous fluids. And clearly, with a sufficiently thin boundary layer,²⁸ (18) is approximately equivalent to the weak assertion that the normal component of velocity vanishes. I therefore think that the real question is, why does separation of the boundary layer occur? Thus it concerns the *stability* of nearly non-viscous flows.

I shall now show that similar problems of stability arise in the theory of (incompressible) viscous fluids.

14. *The Paradox of Turbulence in Pipes*

Consider the steady flow of a viscous incompressible fluid through an infinitely long straight circular pipe of radius c .

²⁸ More precisely, suppose we use the following test, proposed by Harold Jeffreys, for the validity of approximations: substitute the solution of the approximate equations in the exact equations, and estimate the size of the terms neglected. If ν is small, we can introduce a shear flow in a very thin boundary layer, by means of very small forces; physically, this can be done by "sucking away the boundary layer." It is clear that the preceding test gives no information about stability.

Let x denote distance parallel to the pipe, and r distance from the axis of the pipe. Let further u_x , u_r , and u_θ be the axial, radial, and angular components of velocity, in cylindrical coordinates. We have the following paradox.

TURBULENCE PARADOX. The only flows possessing the symmetry of the hypotheses of steady viscous flow in a circular pipe are

$$(19) \quad u_x = a(c^2 - r^2), \quad u_r = u_\theta = 0$$

Such "Poiseuille flows" are observed if $R < 1000$, but not usually if $R > 10,000$.

PROOF. The hypothesis of "steady flow" means that u should be a function of r , x , θ alone, and not of t . The statement of the problem is invariant under reflection in every plane through the pipe axis; the flow has this symmetry only if $u_\theta = 0$ and $u_x = f(x, r)$, $u_r = g(x, r)$. The statement is also invariant under translations along the pipe axis; so are conditions (16)–(18) since ν is assumed independent of pressure; the flow has this symmetry only if $u_x = f(r)$ and $u_r = g(r)$. By (16), this implies $u_r = 0$ (equivalently, the *stream function* is a function of r alone).

We now use (3*). Since $u_2 = u_3 = 0$ identically, it gives $p = p(x)$. Now considering the x -component of (3*), we get

$$(20) \quad p'(x) = \mu \nabla^2 u_x = \mu \left[f''(r) + \frac{f'(r)}{r} \right].$$

The right-hand side of (19) is independent of r , since the left-hand side is. Hence $(rf')' = rf'' + f' = kr$ for some constant k , and $rf' = \frac{1}{2}kr^2 + C$. This gives finite $u_x = f(r)$ at $r = 0$ only if $C = 0$; hence $f' = \frac{1}{2}kr$ and $f(r) = \frac{1}{2}kr^2 + b$. To satisfy condition (18) of adhesion to the pipe walls, we must have $u_x = a(c^2 - r^2)$, completing the proof of (19).

Experimentally, though (19) is usually observed if $R < 1000$, it commonly ceases abruptly near $R = 2000$. This is shown in Fig. 17, p. 101; see also [12], p. 31; [14], p. 246. Above $R = 2000$ the flow becomes turbulent (see Fig. 4b), and the instantaneous velocity field has neither the temporal nor the spatial symmetry of the hypotheses.

It may also be observed that the experimental facts just noted seem to contradict the Principle of Least Action, since Poiseuille flow involves much less expenditure of energy than turbulent flow.

15. *Other Symmetry Paradoxes*

It is most peculiar to have Nature's love of symmetry suddenly cease to be verified physically near $R = 2000$. This is reminiscent—though the limit is far less precise²⁹—of Nature's abhorrence of a vacuum, which is limited to thirty-five feet of water. Moreover there are other cases in hydrodynamics where an apparent symmetry of causes fails to produce symmetric effects.

One such case has been carefully studied by Sir Geoffrey Taylor in a classic paper.³⁰ Consider a viscous liquid between two long coaxial cylinders, rotating with constant angular velocities W and W' in opposite directions. The problem described has (approximate) symmetry under translation along and rotation about the axis of the cylinders, and is independent of time. There is only one solution of the Navier-Stokes equation (3*) and (18) which also possesses these symmetries; it is called "Couette flow." At low speeds Couette flow is observed. But it is replaced at high Reynolds numbers by an unsymmetrical flow which is, however, *not* turbulent: the symmetry in time is preserved, through spatial symmetry is not.

Again, consider a small air bubble, rising in water under its own buoyancy. Under surface tension, it will be nearly spherical—and in any case there is no cause which is not symmetric about a vertical axis through the centroid of the bubble. Hence, by symmetry, the bubble should rise vertically. Yet it is a striking fact³¹ that if $R > 50$, the bubble ascends in a vertical spiral instead!

²⁹ Nature's abhorrence of a vacuum is susceptible to some diurnal variation; her desire for a symmetrical Poiseuille flow is influenced by wall roughness and initial turbulence; cf. *infra*.

³⁰ "Stability of a viscous liquid contained between two rotating cylinders," *Phil. Trans.*, A223 (1922), 289–343.

³¹ See Nisi and Porter, *Phil. Mag.*, 46 (1923), p. 754, also [50], pp. 23–24.

An analogous phenomenon occurs in the wake behind a cylinder pulled broadside through a stream. It is not symmetrical if $R > 50$, but consists of alternating vortices (Bénard-Karman vortex street).

The above examples might seem at first sight to contradict Leibniz's metaphysical Principle of Sufficient Reason,³² that a symmetry of causes must be preserved in the effects. A deeper view is that, *although symmetric causes must produce symmetric effects, nearly symmetric causes need not produce nearly symmetric effects*: a symmetric problem need have no stable symmetric solution. This possibility is the real source of Symmetry Paradoxes.

In support of this view, it may be noted that even with turbulent flow in circular pipes, the mean velocity $\bar{u}$ is found to satisfy $\bar{u}_r = \bar{u}_\theta = 0$, $\bar{u}_x = F(r)$. Whereas in pipes not having circular symmetry, $\bar{u}_r \neq 0$ ([14], p. 267); hence the symmetry arguments of Section 14 are physically relevant. But we do have to realize that, in order to be sure of verifying this experimentally, we must first prove that the symmetry is stable.

Now, wherever it is necessary to analyze the *stability* of a deduction, since an analysis of the stability of a mathematical deduction is far more complicated than the deduction itself, it is almost certain that the deduction will be made long before its instability can be tested. From this principle, we may anticipate a continuing stream of symmetry paradoxes.

Furthermore, in the present instance, it is by no means clear that Poiseuille flow is unstable in the usual sense, for "infinitesimal" disturbances. The available mathematical evidence for the plane analog of Poiseuille flow seems to indicate that it is stable for such disturbances.³³ Moreover

³² A modern formulation of this has been given by George D. Birkhoff [57]. It is curious that formal systems of mathematical logic seem to have dispensed with this principle, whose applicability to physics was observed in 1894 by Pierre Curie ([4], p. 119).

³³ See J. L. Synge, "Hydrodynamical stability," *Am. Math. Soc. Semi-centennial Publications*, Vol. 2, 227-269, New York, 1938; H. B. Squire, *Proc. Roy. Soc.*, A142 (1933), 621-627; C. L. Pekeris, *Phys. Rev.*, 74 (1948), 191-199. For recent physical data, see MM. Fortier and Comolet, *La Houille Blanche*, numéro spécial B (1949), p. 673.

this evidence is in some sense confirmed by the fact that the onset of turbulence can be greatly delayed by using sufficiently smooth walls and rounded entrances ([12], Vol. 2, p. 33; [14], p. 172), up to $R = 75,000$. Thus Poiseuille motion may well be stable locally, but unstable in the large.

From all these facts, one begins to get an idea of the difficulties and confusion which surround the foundations of hydrodynamics, and of the uncertainty of symmetry arguments!

16. *The Eiffel Paradox*

Another paradox, related physically to the Turbulence Paradox, was found in 1912 by Constanzi and Eiffel.³⁴ It may be stated as follows.

EIFFEL PARADOX. *Near a "critical Reynolds number" $R_c \cong 150,000$, the fluid resistance experienced by a sphere actually decreases as the velocity increases.*

This contradicts our general physical experience; thus it is a purely experimental paradox. It contradicts even more flagrantly the result "proved" by dimensional analysis in many college textbooks, that fluid resistance is proportional to the velocity at low speeds, and to the velocity squared at higher speeds (see Chapter III, Section 3).

It is especially astonishing if we recall that R is generally supposed to express the "ratio between inertial forces and viscous forces" (cf. [12], Vol. 2, Chap. I; and Chap. III infra). How can the change in this ratio, from 10^{-5} to 0.5×10^{-5} , cause such a striking effect? Thus, unlike the paradoxes discussed in Section 9, the Eiffel paradox seems to be a genuine Paradox of Approximation: a minute decrease in viscosity causes a large change in the actual flow, separation is delayed, and the wake becomes narrower.

Of course, the cause is now well-known: the change induces transition of the boundary layer from laminar to turbulent flow! This transition is clearly visible in good spark photographs. Thus the Eiffel Paradox is associated with the *stability* of laminar flow, and hence to the discussion of Section 14.

It seems to me that this example throws great light on the shortcomings of the conventional theory of partial differential

³⁴ The references are given in [12], Vol. 2, p. 98.

equations. Certainly, the classical asymptotic theory of partial differential equations ("perturbation theory") fails utterly to predict a sudden change in topological nature of the whole solution when one coefficient is changed from 10^{-6} to 0.5×10^{-5} . Hence it would seem that no treatment of "slightly viscous" fluids which is based on this theory,³⁵ can come close to describing the real behavior of such fluids. It will never predict a turbulent wake, a vortex street, or "transition" at R_c .

Instead, what is needed is an attempt to deduce, from *a priori* mathematical considerations, the possibility of sudden changes in regime induced by small changes in the coefficients of partial differential equations. That is, mathematics must take advantage of physical inspiration, as I have already stressed (Section 1).

The phenomenon of turbulence raises interesting questions regarding the differentiability of the functions $u_i(x_1, x_2, x_3; t)$. It seems well established that the u_i are to some extent random functions. And by Norbert Wiener's classic theorem, "almost all" purely random functions are non-differentiable.³⁶ This fact has led some scientists to believe that one must go back to (discontinuous) molecular considerations in fluid mechanics, in dealing with turbulence. However, the fact that R_c is the same in liquids as in gases, makes this doubtful. One recent guess, based on the statistical theory of turbulence, is that the u_i have fifth but not sixth derivatives.

17. Stokes' Paradox

One of the great triumphs of theoretical hydrodynamics, is Stokes derivation of the formula

$$(21) \quad D = 6\pi\mu cv$$

³⁵ Such as, for example, the work of the school of Oseen [11], which has been so useful in the opposite asymptotic case, $R \rightarrow 0$. The phenomenon has no analog in compressible flow, though the drag of a slender-nosed projectile has a near-discontinuity at $M = 1$ (see Karman and Moore, *op. cit.*, supra).

³⁶ See J. L. Doob, *Annals of Math.*, 43 (1942), 351-369, and its bibliography. The principle that solutions of the Navier-Stokes equations will tend not to be analytic, was deduced by P. Duhem on other grounds in p. 121 of Vol. 2 of his *Traité d'énergétique*.

for the resistance D encountered by a rigid sphere of radius c moving with constant speed v in a fluid of viscosity. This formula, which is verified experimentally³⁷ for Reynolds numbers $R < 0.2$, may be deduced from conditions (16)–(18), by neglecting the “inertial” term in du/dt , and assuming that the flow is steady and has axial symmetry—i.e. that no Symmetry Paradox is involved. This simplification is said to correspond to “creeping flows.”

It seems reasonable to make the same simplifications in two dimensions. However, if this is done, we are confronted with:

STOKES' PARADOX. *No steady “creeping flow” of a viscous fluid past an infinite circular cylinder is possible.*

ROUGH PROOF (I know of no rigorous proof). By (3*), neglecting du/dt and $\mathbf{g}$, we get

$$(22) \quad \nabla^2 p = \mu \sum \frac{\partial}{\partial x_i} (\nabla^2 u_i) = \mu \nabla^2 \left(\sum \frac{\partial u_i}{\partial x_i} \right) = 0;$$

hence p is harmonic. Again, “by symmetry” (!), $u_1(-x_1, x_2) = u_1(x_1, x_2)$, while $u_2(-x_1, x_2) = -u_2(x_1, x_2)$; hence the total flux of momentum out of a large circular cylinder concentric with the small cylinder, is zero. Hence, by the momentum equation (cf. Section 6), the pressure-thrust on the two cylinders must be the same. Since p is harmonic by (22), we infer $p - p_\infty = A \cos \theta/\pi r + O(1/r^2)$, and so by (3*),

$$\nabla^2 u_2 = \frac{\partial p}{\partial y} = \frac{A \sin 2\theta}{2\pi r^2} + O\left(\frac{1}{r^2}\right).$$

But if u_2 can be expanded in a power series at infinity, with $u_2 \rightarrow 0$, we must have $\nabla^2 u_2 = O(1/r^3)$; hence $A = 0$ and we have a contradiction.

Oseen and Lamb³⁸ have rationalized Stokes' Paradox as a singular point paradox, pointing out that at very large distances the acceleration terms dominate the viscosity terms and hence should not be neglected. By definition, it can also be regarded as a paradox of overdeterminism.

³⁷ [12], Vol. 2, p. 100. If the sphere is in a gas where the mean free path is comparable with c , a correction must be made for “slipping”; cf. [6], Vol. 2, Appendix.

³⁸ [11], p. 162; [9], p. 614; see also [13], Vol. VI, pp. 29–40.

Stokes' Paradox is analogous logically to the paradox of a two-dimensional underwater explosion. Though there is a simple and extremely useful theory of the underwater oscillations of spherical bubbles,³⁹ it is easily shown that any two-dimensional notion of a bubble involves infinite kinetic energy, with $U = \log r + 0(1/r)$. Hence no two-dimensional bubble oscillations can be induced by finite forces. The paradox of a two-dimensional underwater explosion is clearly also a singular point paradox; at very large distances the effect of compressibility in absorbing the expansion is dominant.

18. The Rising Bubble Paradox

Even in the case of spheres, the theory of "creeping motions" leads to a curious paradox. Let a small air bubble rise under buoyancy in a liquid, the bubble being so small that surface tension keeps it nearly spherical. Assume as above the approximation of "creeping motion," that $du/dt = 0$ in (3*). Since the bubble is fluid, it is logical to replace (18) by

(18') u is continuous across the surface of the drop,

as pointed out by Rayleigh.

The mathematical problem defined in this way was solved by Riabouchinsky and Hadamard,⁴⁰ who showed that the theoretical drag was given by the formula

$$(21') \quad D = \frac{6\pi c v \mu (2\mu + 3\mu')}{(3\mu + 3\mu')}.$$

In (21'), μ is the viscosity of the medium, and μ' is that of the bubble moving through it.

So much for theory; experimentally, the drag seems to be given by (21) instead of (21') (cf. [12], vol. 2, p. 115, or [14], p. 215). That is, the bubble seems to behave as if it were

³⁹ [9], p. 122. An analogous paradox in the theory of heat conduction is obtained if one looks for a steady temperature distribution in the plane, maintained at 100° on the circumference of a circle, and tending to 0° at infinity. See also [13], Vol. 6, pp. 29-40.

⁴⁰ D. Riabouchinsky, *Bull. Acad. Sci. Cracovie* (1911), p. 40; J. Hadamard, *Comptes Rendus*, 152 (1911), 1735. The experimental data are thoroughly discussed on p. 190ff. of G. Barr, *A Manual of Viscometry*, Oxford, 1931. See also T. Bryn, *Forschung. Ingenw.* 4 (1933), 27-30.

rigid. This contradiction between theory and experiment may be called the Rising Bubble Paradox.

What is the explanation of this paradox? It has been suggested by Bond⁴¹ that the cause is surface energy, and many of Bond's measurements lead to (21') instead of (20). However, the explanation is far from clear. It may be that a rigid film forms at the interface, especially if impurities are in solution, but all that one can say with certainty is, that the mechanics of "real" surface behavior are far less simple than the classical treatments would suggest.

19. "Drift"; Magnus Effect Paradox

Artillerymen have known for over a century that spinning bullets tend to "drift" out of the vertical plane in which they are fired; and that the drift is in the direction of rotation of the top of the bullet. However, this phenomenon was not correctly understood for many years.⁴²

A considerable vogue was enjoyed by the following quite fallacious qualitative "explanation" of drift, due to the great mathematician Poisson. Due to inertia, the axis of the bullet will lag behind the tangent to the trajectory, as sketched in Fig. 5a. Hence there will be more pressure on the underside; hence more friction there. Hence (cf. Fig. 5b) there will be a drift in the observed direction.

The fallaciousness of this qualitative explanation can be seen easily by applying the same argument to a spinning tennis ball: it yields the erroneous conclusion that a tennis ball with topspin should rise! Moreover a quantitative study of the gyroscopic stability of bullets reveals that the stable position of the bullet axis is (with right-handed rifling) to the right of the trajectory tangent. The drift of the bullet is thus due to the aerodynamic *cross-force*, and only indirectly to the spin.

Why is Poisson's explanation inapplicable to tennis balls? Since every college physics student is given a pat qualitative

⁴¹ W. N. Bond, *Phil. Mag.*, 4 (1927), 889-898. For a modern discussion of the real facts of surface energy, see N. K. Adam, *The Physics and Chemistry of Surfaces*, Oxford, 1931, esp. Chapter II.

⁴² An historical account may be found in [3], Chapter X.

explanation of the Magnus effect, I do not think I need to say. Instead, I should like to point out that the usual explanation is also dubious, in view of the following paradox.

MAGNUS EFFECT PARADOX. *At low rates of spin, the actual deflection is in the opposite direction to that predicted by the conventional explanations.*⁴³

Actually, a possible qualitative explanation of the Magnus Effect Paradox can be based on Prandtl's boundary layer theory. Due to rotation, the low pressure "wake" is shifted

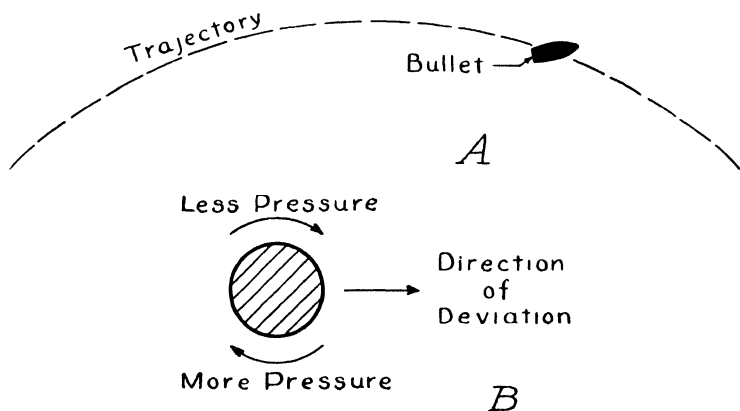


FIG. 5. Fallacious explanation of "drift."

up by the topspin (cf. [12], Vol. 2, Plates 7-9), and the stagnation point down; this would cause a lift.

I think that Poisson's fallacious explanation and the Magnus Effect Paradox both illustrate the unreliability of qualitative explanations. One can argue cynically that qualitative deductions have a fifty per cent chance of appearing verified, no matter whether they are correct or not! I think that students should be taught not to set much store by them; they are very corrupting to one's critical sense.

20. A Role for Mathematicians

What is the moral of the preceding paradoxes? Certainly, they show the necessity of comparing theory with experiment,

⁴³ See [6], p. 504; the data are due to Maccoll.

and consequently show the utter falsity of the following widely quoted statement of Eddington: "An intelligence, unacquainted with our universe, but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics which we have attained by experiment."⁴⁴

To my mind, they also demonstrate the need which the theory of partial differential equations has for fresh physical inspiration. Why should a small change in viscosity affect fluid flow drastically, when a small change in compressibility does not?

On the other hand, they seem to me *not* to support the impression, widely current among engineers and physicists, that mathematical deduction should be supplanted by more "physical" reasoning. Indeed, the free use of approximations, topological oversimplifications, and non-rigorous symmetry considerations, which were seen to underly these paradoxes, are more typical of "physical" than "mathematical" reasoning. I think the very phrase "physical reasoning" implies a confusion of deductive and inductive considerations, which corrupts the fundamental meaning of the word "theory." This corruption is readily recognized by the confusion of students when the "theory" is taught, and by the unreliability of predictions when the "theory" is applied. Thus it is all too common to read of "satisfactory" agreement between theory and experiment, when the agreement should give little satisfaction to any true scientist.

I agree that oversimplifications based on the "right" approximations, made possible by familiarity with experimental results, greatly accelerate technological and even theoretical progress. Such oversimplifications led to the concept of an "ideal fluid," which is still very useful. But I think that mathematicians can perform a useful service if they will analyze critically these oversimplifications, by the deductive method, and so establish their limitations more clearly.

⁴⁴ Sir Arthur Eddington, *Relativity Theory of Protons and Electrons*, Cambridge, 1936, p. 327. They might also give pause to social theorists who rely exclusively on deductive logic in the infinitely more complicated field of social relations.

I realize that this service will often be deprecated by those "scientists" who use science instead of building it. But it is encouraging to know that its value was recognized by no less a person than Lord Rayleigh. Apparently Maxime Bôcher had written Rayleigh, questioning the statements about Sturm-Liouville expansions in Section 187 of the latter's celebrated *Theory of Sound*. Rayleigh replied,⁴⁵ with characteristic candor: "I doubt if I had any special method in my mind when I wrote Section 187. At any rate, if I had then, I have not now. I think I took this case as covered by the general theory as in Chapter IV. I have often wished that mathematicians would devote themselves to improving the rigor of this and similar demonstrations, rather than to inventing special methods for the particular cases."

21. *Mathematics as Reliable as "Common Sense"; Conclusion*

Moreover mathematics is not only useful in "tidying up" reasoning, whose broad lines were suggested by physical intuition. In many cases, mathematical deductions will rapidly give results which are at variance with physical intuition, and which it would be extremely laborious to derive experimentally. I shall close by giving two examples of such results.

Experimental data must frequently be corrected for "wall effects" which are much larger than common sense would suggest. In principle, wall effects can be determined empirically by removing the wall to larger and larger distances, but this may be prohibitively expensive in practice. Thus, Ladenburg has shown that a sphere falling slowly in a cylindrical tube of viscous liquid, having 100 times the cross-section area of the sphere, encounters 20 per cent more resistance than if there were no walls.⁴⁶

Finally consider the celebrated Taylor-Maccoll supersonic flow past a conical projectile (Chapter IV, Section 4). This

⁴⁵ The letter is in my possession.

⁴⁶ B. Ladenburg, *Annalen der Physik*, 23 (1907), 447; A. W. Francis, *Physics*, 4 (1933), 403-406. Another surprisingly large wall effect, arising with cavity flows, is discussed in Chapter II, Section 11.

flow has the peculiar property of having streamlines at constant pressure along the cone wall. It is well known that, according to the mathematical theory of non-viscous fluids, the region behind any streamline at constant pressure can be replaced theoretically by motionless fluid, as in the Kirchhoff wake theory (Chapter II). Hence, mathematically, supersonic flow past a flat disc is possible, in which an invisible conical air barrier separates the onrushing air from the disc.

This flow pattern is instantly rejected by intuition and common sense, as absurdly unstable. The rejection is on the same basis as the rejection of the reverse of the Kirchhoff flow with wake, i.e. as the rejection of a (mathematically possible) upstream wake. It seems highly plausible that the effects of an obstacle should lie on the downstream side of the cause.⁴⁷

Yet in this case common sense seems to be more fallible than mathematical deduction! Spark shadowgraphs, taken at the Ballistic Research Laboratories in Aberdeen (see frontispiece), indicate that a very thin needle in front of the disc is actually capable of inducing something quite like this "absurd" flow.

In view of these facts, I think the mathematician need have no inferiority complex about the soundness of purely deductive reasoning (as contrasted with "physical reasoning") in fluid mechanics, *provided* he clearly understands the preceding hydrodynamical paradoxes.

⁴⁷ The same intuition makes *retarded* potentials give the "physically preferred" solutions of the wave equation, and is the basis of selecting the corresponding solutions of the linearized equations of supersonic flow. In the case of shock waves, this selection is bolstered by an appeal to the Second Law of Thermodynamics; in some sense, this law also seems relevant to the development of turbulent disorder in the wake *behind* an obstacle.

CHAPTER II

Recent Progress in Free Boundary Theory

1. Wakes

The classical theory of Dirichlet and Joukowski flows concerns fluid motions past obstacles whose location in space is supposed given. A unique and fascinating chapter in fluid mechanics concerns flows whose boundaries are not given, but must be determined from a knowledge of the pressure along them. Today I shall discuss some recent developments in the theory of flows having such "free" boundaries, with especial reference to their physical validity. Because of this emphasis, what I shall say will overlap very little with surveys of the field by von Karman [27] and Weinstein [34], [34'].

I shall begin by a historical résumé of the subject.¹ Free boundary theory was initiated in 1868 by Helmholtz [26] and Kirchhoff [27']. At that time, it was well-known that the "Dirichlet flow" (Chapter I, Section 5) around a flat plate held broadside in a moving fluid, has the symmetrically disposed streamlines sketched in Figure 6a. This flow gives rise to the two paradoxes of zero resistance (Chapter I, Section 6), and of infinite velocity (hence infinite negative pressure) at the sharp edge.

Kirchhoff had the inspiration to suggest the alternative flow sketched in Figure 6b. He suggested that only part of the flow would be a Dirichlet flow. This part would separate from the plate at its edges S, S' , leaving a static *wake* of "dead water" behind the plate, extending to infinity. In this model, the velocity evidently changes abruptly across the boundary

¹ The present material was also presented in large part at the Symposium on Applied Mechanics at Brown University, August 1947, and at the Seventh International Congress on Applied Mechanics in September 1948.

of the wake, giving rise to a *surface of discontinuity*. Helmholtz further assumed that this surface consisted of streamlines, which may thus be called *streamlines of discontinuity*.

In a non-viscous fluid, such a streamline of discontinuity would give rise to no (viscous) shear forces; for dynamical equilibrium one thus requires that the pressure be continuous

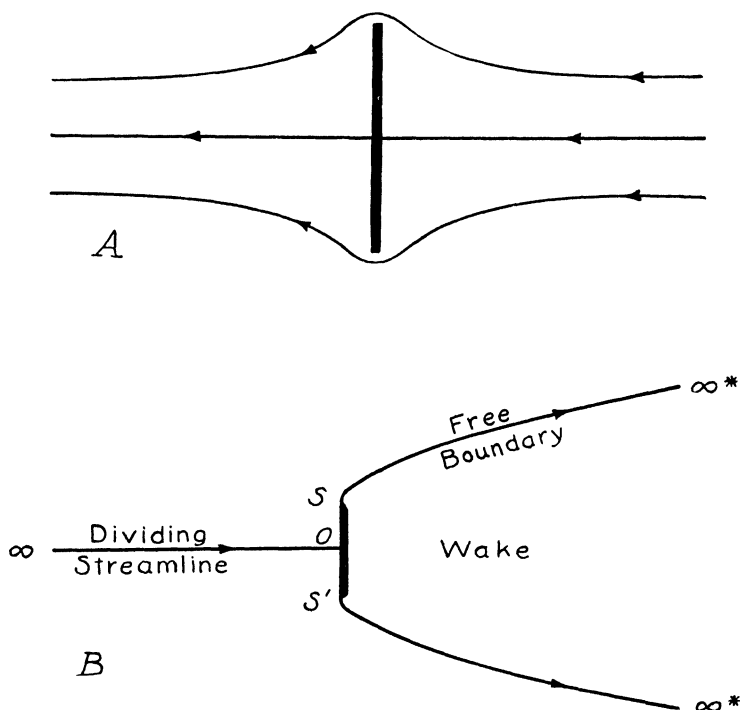


FIG. 6. Flows past vertical plate.

across the streamline. But in the static wake (neglecting gravity, by Chapter III, Theorem 4) the pressure is constant. Hence it is necessary and sufficient for the streamline of discontinuity to be at *constant pressure*. Such a streamline is called a *free streamline*. By Bernoulli's Theorem (Chapter I, Section 6), since $v dv = -dp/\rho = 0$, *velocity is constant* on any such free streamline.

Hence the velocity is finite at S , S' , and the paradox of infinite velocity is avoided. We shall see shortly (Section 2)

that the d'Alembert Paradox of zero resistance is also avoided. Moreover separation *does* occur physically at the plate edges S, S' . Finally, the model of a "static wake" accords with our qualitative experience that one can find shelter from the wind on the leeward side of a barrier. Thus the Helmholtz model of Figure 6b represents, in the case of a flat plate at least, enormous progress over the Euler-Dirichlet model of Figure 6a.²

2. Mathematical Model

Following Kirchhoff (cf. [27'] or [10], p. 299) one can obtain a complete mathematical description of the flow of Figure 6b, as follows.

The velocity potential $U(x,y)$ of a locally irrotational plane³ flow is harmonic by Chapter I, Section 5; hence $U(x,y)$ is the real part of a complex function $W = U + iV$ of the complex *position* coordinate $x + iy$. The function W so defined is called the *complex potential*, and V the *stream function* (it is easy to show that $V = \text{const.}$ on streamlines). Moreover dW/dz is the complex conjugate $\bar{\zeta} = u_1 - iu_2$ of the complex velocity vector $u_1 + iu_2$.

The conditions that the flow be *irrotational* and *volume-preserving* are thus precisely that

$$(1) \quad \frac{dW}{dz} = \bar{\zeta},$$

in the sense of complex differentiation. Hence the functional relationships between the complex variables W , ζ , and z are *analytic* in the usual sense, and are therefore expressible by *conformal* transformations. We consider the transformation of the ζ -domain (or "hodograph") onto the W -domain; this may be determined in the present case as follows.

² In line with Chapter I, it is however interesting to recall the Hadamard Paradox, whereby an (analytic) surface of discontinuity cannot be generated (J. Hadamard, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris 1903, Note II). See also F. Klein, *Zeits. Math. Phys.*, 58 (1910), p. 259.

³ That is, flow in which the velocity vector is always parallel to a fixed plane, so that $\partial U/\partial z = 0$ and $U = U(x,y)$.

Let the origin be placed at the stagnation point O in the z -plane, and the real axis along the plate SOS' ; thus we imagine Fig. 6b rotated through 90° clockwise. Clearly ζ is real along the plate, while

$$(2) \quad |\zeta| = |\zeta(\infty)| = \text{const. on the free streamline.}$$

Hence, under the simplest topological assumptions, the hodograph is a *semicircle*, as in Figure 7. By proper choice of units, $|\zeta(\infty)| = 1$, and so $\zeta(\infty) = e^{i\alpha}$. Thus the hodograph is a unit semicircle, centered in the origin. In the present case of normal incidence, $\alpha = \pi/2$ and so $\zeta(\infty) = i$, as in Figure 7.

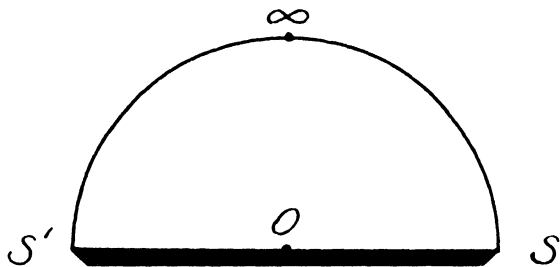


FIG. 7. Hodograph of Kirchhoff flow.

The W -domain is clearly a plane, cut along the positive real axis, setting $W(0) = 0$ by choice of the constant of integration in (2).

By elementary conformal transformations, we can map the hodograph onto the W -domain. Clearly $-\sqrt{W}$ covers the lower half-plane; $\sigma = \frac{1}{2}(\zeta + 1/\zeta)$ maps the hodograph onto the lower half-plane. Hence $-\sqrt{W} = (A\sigma + B)/(C\sigma + D)$, with A, B, C, D real and $AD \neq BC$, also is a one-one conformal ("schlicht") transformation mapping the hodograph onto the W -domain. And it is a classical theorem of conformal mapping that there are no others.

Further, in the present case, $W = \sqrt{W} = 0$ implies $\zeta = 0$ and $\sigma = \infty$; hence $A = 0$, and we can write $-\sqrt{W} = 1/(C\sigma + D)$. Finally, $W = \infty$ implies $\zeta = i$ and so $\sigma = 0$; hence $D = 0$ in the case of normal incidence. Substituting, we get

$$(3) \quad cW = \frac{c}{C^2 \sigma^2} = \left[\frac{\zeta}{\zeta^2 + 1} \right]^2, \quad \text{if} \quad c = \frac{C^2}{4}.$$

All choices of the constant c in (3) give similar flows; a typical solution is given by setting $c = 1$. In this case, the computation of $z(\zeta)$ reduces by (1) to the integration of a rational function. Performing this integration, we get

$$(4) \quad z = \frac{\zeta}{(\zeta^2 + 1)^2} + \frac{1}{2} \left\{ \frac{\zeta}{\zeta^2 + 1} - \frac{i}{2} \ln \frac{\zeta - i}{\zeta + i} \right\} + \text{const.},$$

for all values of $\zeta = \xi + i\eta$.

Along the plate, ζ is real and (4) reduces to

$$(4') \quad z = \frac{\zeta}{(\zeta^2 + 1)^2} + \frac{1}{2} \left\{ \frac{\zeta}{\zeta^2 + 1} + \arctan \zeta \right\},$$

where clearly $z(0) = 0$, and so the constant of integration vanishes. At the right-hand separation point S , $\zeta = 1$ and so

$$(4'') \quad z(S) = \frac{1}{4} + \frac{1}{2} \left\{ \frac{1}{2} + \frac{\pi}{4} \right\} = \frac{4 + \pi}{8}.$$

The pressure is most easily computed by setting $\zeta = \xi = \tan \theta$ along the plate, so that $\zeta/(\zeta^2 + 1) = \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$. Hence the thrust on half the plate is, by Bernoulli's Theorem and (1),

$$(5) \quad \begin{aligned} \int_{\xi=0}^1 (1 - \xi^2) dx &= \int_{\xi=0}^1 \frac{(1 - \tan^2 \theta)}{\tan \theta} d \left[\frac{1}{4} \sin^2 2\theta \right] \\ &= \int_{\theta=0}^{\pi/4} 2 \cos^2 2\theta d\theta = \frac{\pi}{4} \end{aligned}$$

The ratio of half the thrust to the plate half-length is clearly the *drag coefficient*, which is thus

$$(6) \quad C_D = \frac{2\pi}{(\pi + 4)} = 0.88 \quad \text{approximately.}$$

Similar but more elaborate complications give, for the case of *oblique incidence* $\alpha \neq \pi/2$,

$$(6') \quad C_D = \frac{2\pi \sin^2 \alpha}{4 + \pi \sin \alpha}, \quad C_L = \frac{\pi \sin 2\alpha}{4 + \pi \sin \alpha}.$$

Details may be found elsewhere ([9], p. 103; [10], pp. 304–305; [13], Vol. I, p. 287). Figure 2c shows the flow for the case $\alpha = 15^\circ$.

3. Borda Mouthpiece

A similar procedure gives a plane *jet* from an infinite straight orifice (Borda tube), as sketched in Figure 8a; the dotted area is supposed to consist of static air. The problem is mathematically simplified if we consider only the half of the flow above the axis of symmetry; in this form, the problem was solved by Helmholtz in 1869.

If we take the axis of symmetry as the real axis, and choose units so that the jet velocity is one, we again have a semi-circular hodograph as in Figure 7. Hence $\sigma = \frac{1}{2}(\zeta + 1/\zeta)$ again fills the lower half-plane. The difference is that the present flow is bounded by *two* streamlines which (by choice of units) we can take to bound the *strip* $0 \leq V \leq \pi$ in the W -plane. As before, we thus get $e^w = (A\sigma + B)/(C\sigma + D)$.

We again calculate A, B, C, D by the method of undetermined coefficients. Thus when $W = -\infty$, and $e^w = 0$, $\zeta = 0$ and so $\sigma = \infty$; hence $A = 0$ and we can set $B = 1$ as before. Whereas at $W = +\infty$, $e^w = \infty$ and $\zeta = \sigma = 1$; the asymptotic jet velocity; hence $D = -C$. Substituting, we get

$$(7) \quad Ce^w = \frac{1}{\sigma - 1} = \frac{2\zeta}{\zeta^2 - 2\zeta + 1} = \frac{2\zeta}{(\zeta - 1)^2} \quad [C \text{ real}]$$

We get in this way a family of flows, all similar to the flow

$$(7') \quad W = \ln \zeta - 2 \ln (\zeta - 1) + \text{const.}$$

The determination of z then reduces to

$$(8) \quad \begin{aligned} z &= \int \frac{dW}{\zeta} = \int \frac{d\zeta}{\zeta^2} - 2 \int \frac{d\zeta}{\zeta(\zeta - 1)} \\ &= -\frac{1}{\zeta} + 2 \ln \zeta - 2 \ln (\zeta - 1) + \text{const.} \end{aligned}$$

as can easily be shown using partial fractions. We can make z real when $\zeta \downarrow 0$ through positive real values, by making the integration constant $2 \ln (-1)$, so that (8) becomes

$$(8') \quad z = -\frac{1}{\zeta} + 2 \ln \frac{\zeta}{1-\zeta} = -\frac{1}{\zeta} + 2 \ln \zeta - 2 \ln (1-\zeta)$$

Again, as $\zeta \uparrow 0$ through negative real values at ∞^* , $y = \text{Im}(z) \rightarrow \pi$. Hence the walls of the Borda mouthpiece are 2π units apart. Finally, as $\zeta = e^{i\theta} \rightarrow 1$, along the free streamline $\text{Im}(-1/\zeta + 2 \ln \zeta) \rightarrow 0$, while the imaginary part of $\ln(1-\zeta)$ approaches $\pi/2$. Hence the jet width π is half the

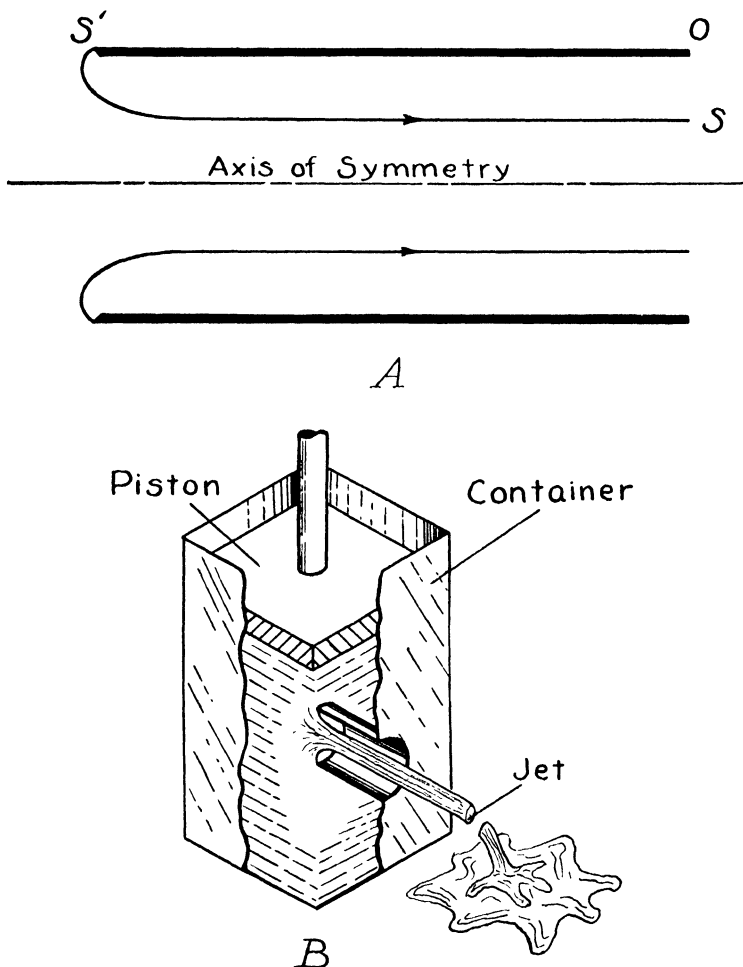


FIG. 8. Borda flows through slot and circular orifice.

orifice width. In other words, the "coefficient of contraction" of the orifice is one-half.

The coefficient of contraction (but not the velocity field) can also be obtained by simple energy and momentum considerations, as follows (cf. [12], Vol. I, p. 242.) Let a container with vertical walls be filled with a liquid of density ρ , into which is inserted a horizontal "Borda tube" as in Figure 8b, of arbitrary cross-section shape and area A ; let the pressure head at the level of the tube be p . We assume that the flow separates⁴ from the tube at its inner end, and that the jet from the tube tends asymptotically to a constant speed v ; this must be the constant "free streamline" velocity on the jet boundary. Let A^* be the asymptotic jet cross-section; then A^*/A is by definition the coefficient of contraction. We compute it as follows.

Per unit time, the volume efflux is vA^* ; the momentum efflux is v^2A^* ; the kinetic energy efflux is $\frac{1}{2}\rho v^3A^*$. The momentum supplied is however pA (pressure excess); the (potential) energy $p(vA^*)$. Hence $pA = pv^2A^*$ and $pvA^* = \frac{1}{2}\rho v^3A^*$. Dividing v times the first equation by the second, we get the result

$$(9) \quad \frac{A^*}{A} = \frac{1}{2}.$$

This conclusion is not restricted to plane flows, but is clearly equally valid for a circular tube.

4. Further Generalizations

Many other models of wakes and jets with free streamlines have been constructed since the time of Helmholtz and Kirchhoff. They are fully described in the classical literature,⁵ which has become known as the theory of wakes and jets.

A large proportion of these flows involve cases where the diagram in the ζ -plane, or hodograph, is a circular sector. It

⁴ This is the case of "free flow"; see A. H. Gibson, *Hydraulics*, fourth ed., p. 122; it is not always fulfilled.

⁵ See [9], pp. 94-109; [10], Chs. XI-XIII; [25], [31], and [32]. A complete survey of the subject is being prepared by the author and E. Zarantonello under the title "Jets, wakes, and cavities."

has recently been shown⁶ that $z(\zeta)$ can be expressed explicitly in terms of elementary functions, whenever this circular sector subtends a rational angle (i.e. one commensurable with 360°).

Indeed, in this case $\sigma = \zeta^{m/n}$ can be made to cover the upper half of the unit circle $|\zeta| \leq 1$, for some pair of integers m, n (in the case of free streamlines, we easily reduce to a sector of the unit circle). Hence $\tau = \frac{1}{2}(\sigma + 1/\sigma)$, which is a rational function of $\sqrt[n]{\zeta}$, covers the lower half plane.

On the other hand, a stationary flow must be bounded by streamlines or dividing streamlines. Hence the diagram in the W -plane (if simply connected) consists of a plane, half-plane, or strip, possibly with a number of semi-infinite cuts parallel to the real axis. Therefore there is a Schwarz-Christoffel transformation of the σ -domain onto the W -domain of the form $dW = F(\tau)d\tau$, where $F(\tau)$ is a *rational* function; this follows immediately from the theory of Schwarz-Christoffel transformations.

Expressing $F(\tau) = F(\frac{1}{2}(\zeta^{m/n} + \zeta^{-m/n})) = R(\sqrt[n]{\zeta})$ as a rational function of $\sqrt[n]{\zeta}$, we obtain by substitution in (1)

$$(10) \quad z = \int \frac{dW}{\zeta} = \frac{m}{2} \int R(\sqrt[n]{\zeta}) ((\sqrt[n]{\zeta})^m - (\sqrt[n]{\zeta})^{-m}) \frac{d\sqrt[n]{\zeta}}{(\sqrt[n]{\zeta})^{n+1}}$$

The integrand is evidently rational in $\sqrt[n]{\zeta}$, whence z , as well as ζ and W , can be expressed as a sum of rational and logarithmic complex functions of $\sqrt[n]{\zeta}$. I omit the details, which show that the breaking up of the integrand in (10) into partial fractions, can indeed be computed explicitly in all cases.

Although the procedure just outlined is attractive in principle, it involves serious practical difficulties. Unless m/n is $\frac{1}{2}$ or 1, the final formulas are very long. Hence, unless one is satisfied with the relation between a few physical dimensions, it is better to evaluate z by numerical integration than by partial fractions.⁷

Using complex numerical integration, it has proved possible

⁶ Birkhoff, Plesset, and Simmons [20]. The case where the diagram in the W -plane is an infinite strip had already been covered by Mises ([9], Appendix of second German edition).

⁷ A detailed account of some difficulties of numerical calculation has been given by D. M. Young, "The use of conformal mapping to deter-

to code the calculation of flows with free streamlines whose hodograph is a circular sector for the Mark II Calculator at Dahlgren, so that the velocity field over most of the flow can be computed within hours.⁸

Similar arguments enable one to express flows past general polygonal barriers in terms of hyperelliptic integrals, using $\omega = \ln \zeta$ as independent variable. (Clearly the hodograph is a rectangular polygon in the ω -plane.)

An even more important result is due to Levi-Civita [29], who used the Schwarz Principle of Reflection to express the most general flow having an infinite wake, of the topological type discussed in Section 2, in terms of analytic functions. By this method, one can easily construct flows, with "wakes" bounded by free streamlines, past *curved* obstacles. However, the shape of the obstacle cannot be prescribed in advance, except qualitatively; thus the effective determination of the plane flow or flows with "wake" past a *given* curved obstacle remains an unsolved problem.^{8a}

5. Indeterminacy of Separation Point

In the cases treated in Sections 2 and 3, the topology of the cavity and the exact location of the points of flow separation were assumed known. These assumptions are not possible in general, and lead to serious problems of *mathematical indeterminacy*.

Thus H. Villat proved, thirty-five years ago, that even in the simple case of flows with infinite wake past a reentrant wedge, separating at the edges, more than one topological configuration was mathematically possible;⁹ specifically, a dead

mine flows with free streamlines," *Proc. Symposium on Construction and Applications of Conformal Maps*, Inst. Num. Analysis, Nat. Bureau Standards, Univ. Calif. at Los Angeles, June, 1949.

⁸ The coding and method of calculation were planned by Prof. D. R. Hartree, Mr. J. C. Aller, and the author; useful suggestions were also received from Prof. John von Neumann.

^{8a} For exceptions, see S. Brodetsky, *Proc. Zurich Congress Appl. Mech.* (1926), 527-531.

⁹ *Loc. cit.* on p. 21; see also *Ann. Fac. Sci. Toulouse*, 5 (1913), 375-404; *Ann. Ec. Norm. Sup.*, 31 (1914), 455-491; *Bull. Sci. Math.*, 42 (1918), 72-92; *Aperçus théoriques sur la résistance des fluides*, Paris, 1920; R. Thiry, *Ann. Éc. Norm. Sup.* 38 (1921), 229-339.

area at arbitrary pressure can form in the reentrant vertex. This result has recently been extended by Dr. E. Zarantonello, who has shown that, in the case of a plate with angle of attack $\alpha \neq 0^\circ$, exactly two topological types of flow with infinite wake and everywhere finite velocity are possible.

A more fundamental indeterminacy arises in the case of curved obstacles, even for flows having the topological type discussed in Section 2. It was first noted in 1911 by Marcel Brillouin ([22], p. 174) that "la forme de M. Levi-Civita est encore beaucoup trop générale." Brillouin also

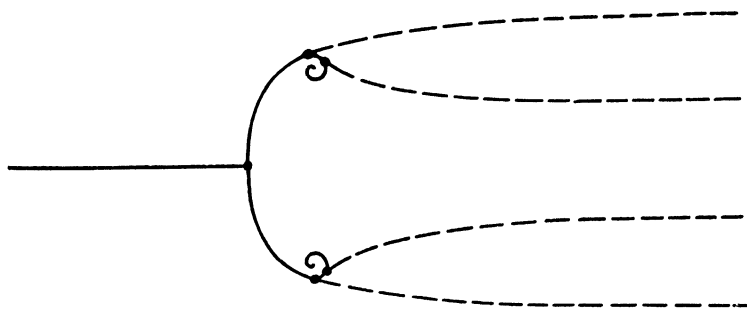


FIG. 9. Two cavity flows past accolade.

suggested most of the criteria mentioned below, for avoiding this indeterminacy. As the existence and uniqueness theory has been summarized recently elsewhere ([34] and [34']), I shall only state the most essential facts. These are clearly illustrated by the following fundamental theorem of Leray [28], culminating earlier work of Weinstein and Friedrichs (see [34], [34']).¹⁰

Let B be any circle, or more generally any symmetric convex barrier whose curvature is non-decreasing as one moves backwards from the stagnation point, as in Figure 9. Such barriers are called "accolades."¹¹ Then there is a fixed point S_0 on B ,

¹⁰ Leray's results were extended to jets and to wakes in channels by J. Kriavtchenko, *Jour. de Math.*, 29 (1941), 30-303; also *Ann. Inst. Henri Poincaré*, 62 (1945), 233-268. See also H. Villat, *Jour. math. pures appl.*, 10 (1914), 231-290; S. Brodetsky, *Proc. Edin. Math. Soc.*, 41 (1922-1923), 58-62.

¹¹ By the four-vertex theorem, the circle is the only "accolade" which bounds a convex plane region. However, other generalizations are possible.

such that for every point S behind S_0 on B , exactly one symmetric flow with infinite cavity (wake) is possible which separates from B at S .

By this theorem, there is a one-parameter family of cavity flows past B with infinite wakes. The free streamlines corresponding to two such flows are sketched in Figure 9. The shape of each wake is asymptotically parabolic, in the sense that

$$(11) \quad y \sim \sqrt{Cx} \quad \text{for some } C; \quad \text{i.e., } \lim_{x \rightarrow \infty} \frac{y}{\sqrt{Cx}} = 1.$$

The "drag" D is determined by C through the formula

$$(11') \quad D = \frac{\pi \rho C}{4},$$

of Levi-Civita [29]. (Similar formulas for the lift and moment of asymmetrical cavity flows were obtained by M. Brillouin [22].)

One of these flows is preferable esthetically, on the following mathematically equivalent grounds: (i) separation occurs at S_0 , (ii) the minimum pressure occurs in the cavity (i.e. at the point of separation), (iii) the cavity is convex, (iv) the curvature of the cavity at the separation points is finite.

Although mathematicians have tended to stress condition (iv), it seems to me that condition (ii) is far more significant physically. Thus physically, it is presumably fulfilled in the case of a vapor-filled cavity,^{11a} or if $p \ll \frac{1}{2}\rho v^2$. In fact, I know of only one physically plausible case¹² in which it is not satisfied. Also, though (ii) always implies (iii), as noted by Brillouin, condition (iv) is not usually satisfied when separation occurs at the *end* of a polygonal or other barrier, whereas (ii) is.

Condition (ii) is, of course, not fulfilled for physical wakes

^{11a} In ordinary liquids, the pressure cannot long remain sensibly below vapor pressure. The contrary assertion of Rayleigh ([13], Vol. 6, p. 401) applies only to liquids whose "gas nuclei" have been removed by boiling or other special treatment; cf. [22], p. 151. For direct confirmation, see [15], Fig. 34; also Section 9.

¹² The case described by M. J. Lighthill, *Aer. Res. Comm. Reps. and Mem.*, 2105, of a jet sucked around a bend.

behind curved obstacles ([6], Vol. 2, pp. 422, 497), but neither is any other part of classical "wake theory."

6. *Physical Unreality*

Indeed, physical unreality is well-known to be one of the three capital defects of the classical theory of free streamlines. (The other two are, that it is mathematically indeterminate in general, and can be applied only to plane flows—and practically only to a few plane flows past polygonal fixed walls.)

The physical unreality of the Kirchhoff wake model has been realized for over fifty years.¹³ It may be shown (using plausible, if not strictly rigorous arguments!) that a streamline of discontinuity in a non-viscous fluid is in *unstable* equilibrium. It may also be observed experimentally that it soon becomes wavy, and then breaks up into a turbulent "mixing zone," which has been discussed by various writers.¹⁴ This turbulence is drawn into the wake and recirculated there, so that real wakes are by no means motionless bands of "dead water" extending to infinity. At lower Reynolds numbers ($50 < R < 10^5$ for cylinders), they often consist of alternate vortices or "vortex streets."¹⁵

Some idea of the magnitude of the discrepancy between the Kirchhoff model and real wakes may be had from the fact that the real C_D of a flat plate is about 1.95 ([14], p. 227), or twice the $C_D = 0.88$ predicted by theory. Whereas the theoretical wake pressure is equal to the free stream pressure, *there is a large actual wake underpressure*. This wake underpressure causes the actual "streamlines of discontinuity" to

¹³ See Kelvin (*Nature*, 50, 1894, pp. 524, 549, 573, 597); also Rayleigh [13], Vol. 6, p. 39. Prophetically, the instability of a surface of discontinuity was discussed (in another connection) by Kelvin in 1871 (*Phil. Mag.*, Vol. 42, p. 368); see also Rayleigh [13], Vol. 1, pp. 365, 477; F. W. Lanchester *Aerodynamics* (1908), pp. 128–136.

¹⁴ See [6], Sections 252–254; also L. M. Swain, *Proc. Roy. Soc.*, A125 (1929), p. 647; S. Tomotika, *ibid.*, A165 (1938), pp. 53–72; L. Schlichting, *Ing. Archiv. I* (1930), p. 533; W. Tollmien, *ibid.* 4, (1933), pp. 1–15.

¹⁵ See Th. von Karman and H. Rubach, *Phys. Zeits.*, 13 (1911), p. 49; M. Teissié-Solier and P. Dupin, *Les tourbillons alternés*, Paris, 1928; L. Rosenhead and M. Schwabe, *Proc. Roy. Soc.*, A129 (1930), pp. 115–135.

lie inside those predicted by Kirchhoff.¹⁶ It also causes the observed lift coefficient C_L of a tilted plate to greatly exceed that calculated on the basis of "wake theory." This underestimate led to undue pessimism in the early days, concerning the possibility of flight by heavier-than-air craft.

Since the magnitude of the wake underpressure appears to persist no matter how small the viscosity is, it seems to me that the flaw in theory is not due so much to physical as to mathematical oversimplification (lack of mathematical rigor in treating the case of negligible viscosity). I shall therefore refer to the Paradox of Wake Underpressure.

A similar discrepancy between theory and observation exists for *jets* flowing out of an orifice into a fluid of equal density. So far as I know, good agreement between classical free streamline theory and experiment has been found only in the case of liquid jets issuing into air, or some other low density medium.¹⁷

7. Dimensionless Parameters ρ'/ρ and Q

A familiarity with the preceding facts leads one to focus attention on two physical parameters which are completely ignored in the classical theory of jets and wakes. Thus neither is even mentioned in the expositions of [9], [10], [12], [34], or [34']; much of the recent progress which I shall report depends on recognizing their role explicitly.

The first of these parameters is the dimensionless ratio ρ'/ρ of the densities of the fluids on the two sides of the streamline of discontinuity. It is 1.0 in classical wake theory, and about .0013 for a water jet in air.

¹⁶ O. Flachsbart, *Zeits. ang. Math. Mech.*, 15 (1935), pp. 32-37; [6], p. 554; Camichel, *Comptes Rendus*, 174 (1922), 666-669; L. Escande and M. Teissié-Solier, *ibid.*, 191 (1930), p. 519. For aeronautical implications, see Lanchester, *op. cit. supra*, pp. 264-265; Kelvin, *Collected Papers*, Vol. 4, 205-210.

¹⁷ For cases of such agreement, see [13], Vol. 1, p. 304; R. von Mises, *Zeits. Ver. Deutsche Ing.*, 61 (1917), 447-451, 469-473, and 493-497; [16]. For jets flowing into media of the same density, see L. Marty, *Ann. Fac. Sci. Toulouse*, 22 (1930), 41-146; E. Foerthmann, *Ing. Archiv*, 5 (1934), 42-54; [6], Section 255; D. Citrini, *Energia Elett.*, 23 (1946), 133-144 and 302-315, and 24 (1947), 177-192; H. Rouse, *Proc. Am. Soc. Civ. Eng.*, 74 (1948), 1571-1596.

Thus the following plausible conjecture was made in 1931 by Betz and Petersohn [16]: if ρ'/ρ is small, then *classical free streamline theory can be applied*. (It is to be noted that the smallness of the kinematic viscosity is not mentioned). Now ρ'/ρ is small for jets of liquid issuing into a gas, and again in the so-called *cavity motion* which occurs behind high-speed missiles in water. In the latter case (cf. [19], [23], [35]) a "cavity" filled with low-density air or water vapor ($\rho'/\rho = .00003$) fills the region behind the missile. Hence it is plausible that classical "wake theory" may apply to cavity motion—it certainly does not apply to wakes!

We shall see later (Section 10) that even the conjecture of Betz and Petersohn oversimplifies the facts. Nevertheless, it represents an important step forward, and I shall speak from now on of *jets and cavities*, instead of using the traditional "jet and wake" terminology. Any "jets" to which I refer should also be thought of as liquid jets moving through air or some other low-density medium.

The other is the dimensionless ratio expressing the relative wake underpressure: the so-called *cavitation number*

$$(12) \quad Q = \frac{(p - p_c)}{\frac{1}{2}\rho v^2}.$$

Here p denotes the "free stream pressure" of the undisturbed flow; v is its relative velocity; ρ is the fluid density, and p_c is the "wake pressure" on the streamline of discontinuity. In the classical theory, $Q = 0$ is assumed, corresponding to infinitely long cavities.¹⁸

Now physical cavities usually have finite length and positive cavitation number $Q > 0$; moreover the minimum pressure usually occurs inside the cavity. It was observed by M. Brillouin [22] that these conditions were incompatible with irrotational flow in an ideal fluid, since they would lead to a convex cavity. Such a cavity would have a stagnation point

¹⁸ As pointed out to me orally by G. Kreisel, an infinitely long cavity in an infinite stream with $Q \neq 0$ is incompatible with regular flow at infinity, by the Phragmén-Lindelöf Principle. I note also that, by the underwater explosion paradox of Chapter I, Section 17, a cavity cannot form in plane flow.

at the rear end of the free boundary, at minimum pressure p_* . By Bernoulli's Theorem, $v^2 = p_* - p \leq 0$ therefore, which implies no motion at all; this is the Paradox of Brillouin.

One can escape from this paradox, and get at least a partial mathematical approximation to physical reality, by supposing that artificial alterations of the rear of a cavity, far from an obstacle, do not greatly affect cavity flow about the obstacle. I shall now discuss some such mathematical approximations.

8. *Flows with $Q \neq 0$*

In 1921, Riabouchinsky [32] constructed hydrodynamic flows with free streamlines about two symmetrically placed

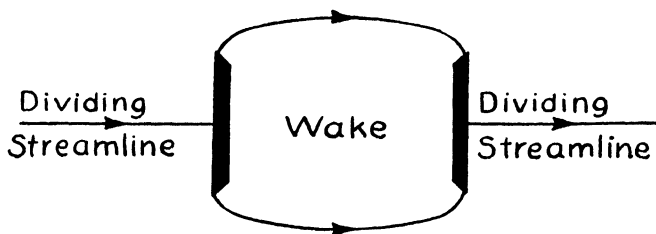


FIG. 10. Riabouchinsky flow past plate.

plates (see Fig. 10), with $Q > 0$. A sketch of Riabouchinsky's mathematical¹⁹ discussion follows.

The "hodograph" (i.e. diagram in the ζ -plane) of one quarter of the flow is clearly a quarter-circle, and the W -domain is a quadrant, by vertical and horizontal symmetry. Hence the conformal map of the hodograph on the W -domain satisfies, much as in Section 2,

$$(13) \quad W^2 = \frac{A(\zeta^2 + \bar{\zeta}^{-2}) + B}{C(\bar{\zeta}^2 + \zeta^{-2}) + D} = c \frac{\zeta^4 + 2\lambda\zeta^2 + 1}{\zeta^4 + 2\mu\bar{\zeta}^2 + 1}.$$

By choice of units, we can reduce to the case $c = 1$, and $\bar{\zeta} = 1/\zeta$ at $W = \infty$, which implies $\mu = -1$. This gives

¹⁹ Physically, Riabouchinsky's original discussion referred to wakes, where we should refer to cavities; see *Proc. III Int. Congr. Appl. Mech. Stockholm* (1930), Vol. 1, 137-158. The relevance to cavitation with $Q > 0$ was noted by F. Weinig, *Hydromechanische Probleme des Schiffsantriebs*, Hamburg, 1932, 294-300.

$$(14) \quad W = \frac{\sqrt[2]{\zeta^4 + 2\lambda\zeta^2 + 1}}{(\zeta^2 - 1)},$$

from which $z = \int dW/\zeta$ can be obtained as an elliptic integral. If v is the free streamline velocity, then $\zeta = v$ when $W = 0$, whence $\zeta^4 + 2\lambda\zeta^2 + 1 = (\zeta^2 - v^2)(\zeta^2 - v^{-2})$, and $\lambda = -\frac{1}{2}(v^2 + v^{-2})$. By Bernoulli's equation, $Q = v^2 - 1$, whence $\lambda = -\frac{1}{2}[(1 + Q) + (1 + Q)^{-1}]$. With these formulas, it is easy²⁰ to obtain C_D as a function of Q .

Another important model for cavity flow with $Q > 0$ has been constructed recently by Efros, Gilbarg and Rock.²¹

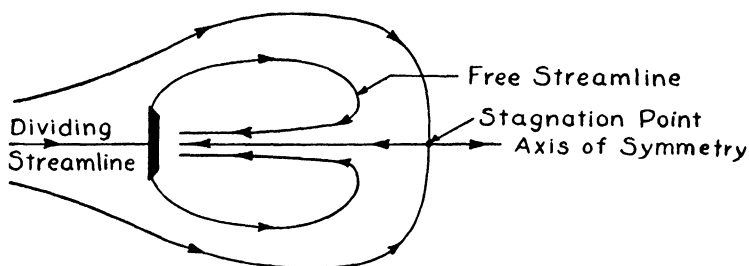


FIG. 11. Cavity flow with reentrant jet.

This involves a *reentrant jet* (see Fig. 11), instead of a symmetric cavity. Such jets, somewhat analogous to those discussed later (Sections 12–13), are now known to occur experimentally when a symmetrical object is dropped into water. It is therefore likely that this model closely resembles actual cavity motion, though the two-sheeted Riemann surface required for the z -plane cannot of course be realized physically.

The Riabouchinsky and Gilbarg models agree with each other in two respects. In both models, C_D increases with Q in the approximate ratio

$$(15) \quad C_D(Q) \cong (1 + Q)C_D(0) = 0.88(1 + Q)$$

²⁰ In 1946, T. E. Caywood extended Riabouchinsky's analysis to the case of a wedge in a finite channel, but the numerical computations involve untabulated definite integrals.

²¹ D. Gilbarg and D. H. Rock, *Naval Ordnance Laboratory Memo.*, 8718 (1945); D. A. Efros, *Doklady Akad. Nauk. USSR*, 51 (1946), 267–270 and 60 (1948), 29–31.

for flat plates. In both models, the cavity width and length increase without limit as $Q \downarrow 0$. These predictions also agree generally with experiment (see Section 9), for Q small. Of course, if $Q > 1$, there is usually no cavitation at all, and a well-developed cavity usually only exists if $Q < .5$; hence "cavity flows" with $Q > .5$ have no obvious physical interpretation.

It is curious that the value $Q = .5$ should affect the drag coefficient in much the same way as the Mach number $M = 1$ does. Other analogies between compressibility and cavitation (which is a form of compressibility) have been discussed by Ackeret [15] and Riabouchinsky (*Proc. Stockholm Congress Applied Mechanics* (1930), p. 156). One may note the further coincidence that, in a perfect gas, $M^2 = v^2\rho/\gamma p = 2/\gamma Q$; hence $M = 1$ does correspond roughly to $Q = .5 - 1.0$ (if $1 \leq \gamma \leq 2$). Finally, one may note Simmons' concept of a "blocking constant," discussed in Section 11.

A simple hydraulic (and therefore non-rigorous!) derivation of a formula closely related to (15), has been proposed by Reichardt.²² Reichardt supposes that the energy $D\Delta s$ given off in a distance Δs by a missile, is used up in expanding the cavity radially to a volume $A^*\Delta s$. (Here D is the drag, and A^* the maximum cavity cross-section area.) Since this expansion does work against an adverse pressure difference of $p - p_c$ (cf. (12)), we infer

$$(16) \quad A_c = \frac{D}{(p - p_c)}, \quad \text{or} \quad \frac{A^*}{A} = \frac{C_D}{Q},$$

where A is the missile cross-section.

It is encouraging that (16) agrees well with the numerical results obtained with Riabouchinsky's model, if Q is small.

One can also construct flows with $Q < 0$ having finite cavities, and otherwise defined mathematically over the entire plane. Such flows avoid the Brillouin Paradox by separating from the obstacle *behind* the point of minimum pressure, and by having cusps (cf. Fig. 12, innermost flow) at the rear end.

²² Reichardt, H., "The laws of cavitation bubbles," *Reps. and Translations No. 766*, Ministry of Aircraft Production (Great Britain).

Flows with cusped cavities can even be expressed analytically in closed form, as shown recently by Lighthill and Allen.²³

Southwell and Vaisey [33] have recently constructed an interesting series of plane cavity flows past a circular cylinder in a channel, with $-0.4 \leq Q \leq 0.0$. The cavity profiles are sketched in Figure 12. The proper interpretation of these flows is however ambiguous, for the following reasons.

First, the novel "cusped cavities" of finite length obtained theoretically, do not at all resemble physical cavities (cf. Section 9). Second, the flow obtained for $Q = 0$ is only one

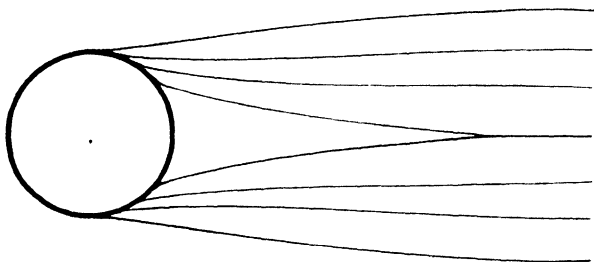


FIG. 12. Cavity flows past cylinder in channel computed by relaxation methods.

of a mathematically possible one-parameter family, and is in fact not the one usually preferred (see Section 5). Finally, in the flows with $Q > 0$, the cavity width as determined in [33] increases with Q , whereas all other evidence indicates that, in an infinite stream with $Q > 0$, the cavity width decreases as Q increases. Hence the contrary trend of Southwell and Vaisey is presumably a wall effect.²⁴

These remarks are perhaps timely in view of the current interest in numerical methods. I think they illustrate the need for accompanying numerical experimentation by a profound analytical knowledge, especially as regards general existence, uniqueness, and convergence theorems.

²³ M. J. Lighthill, "A note on cusped cavities," *Aer. Res. Council Reps. and Mem.* 2328 (1945), published 1949; D. N. de G. Allen, "The formation of closed wakes in fluid motions," *Quar. J. Mech. Appl. Math.*, 2 (1949), 64-71.

²⁴ It may also be noted that the flows of Figure 12, which are intended to be uniform to the right of a fixed line, could equally well be extended by the reflection principle to give flows like that of Figure 10.

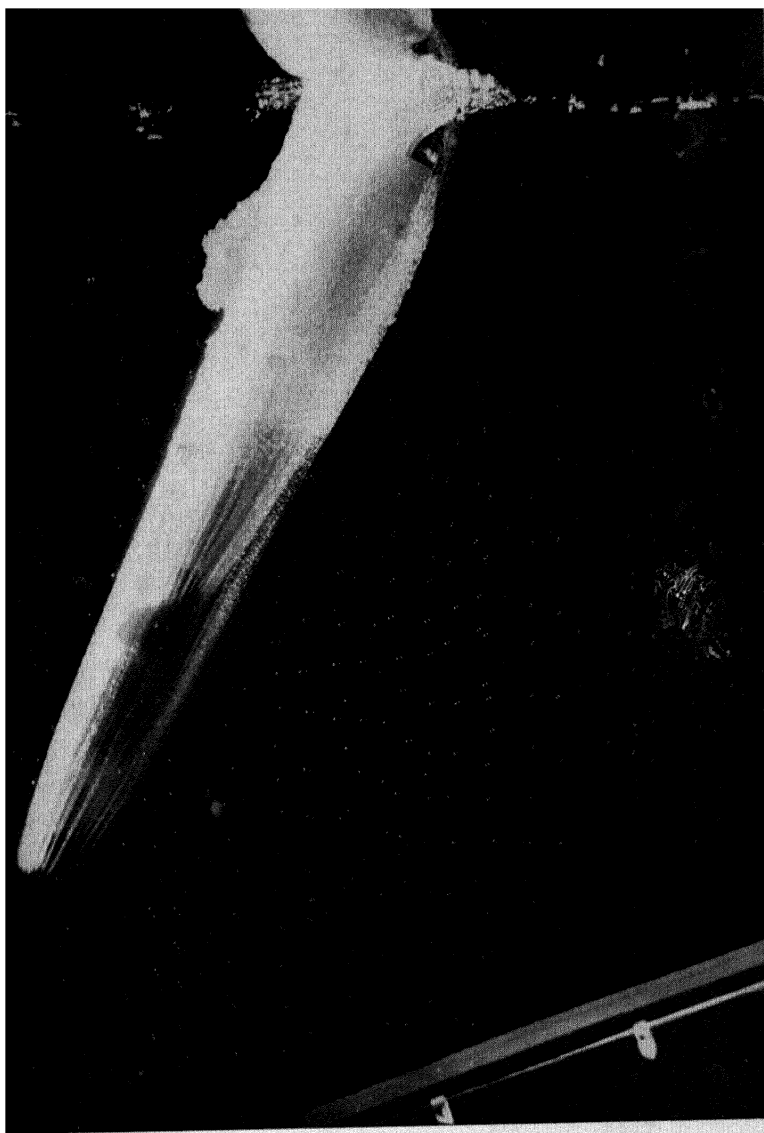


PLATE I. Velocity field and cavity behind high-speed sphere in water.

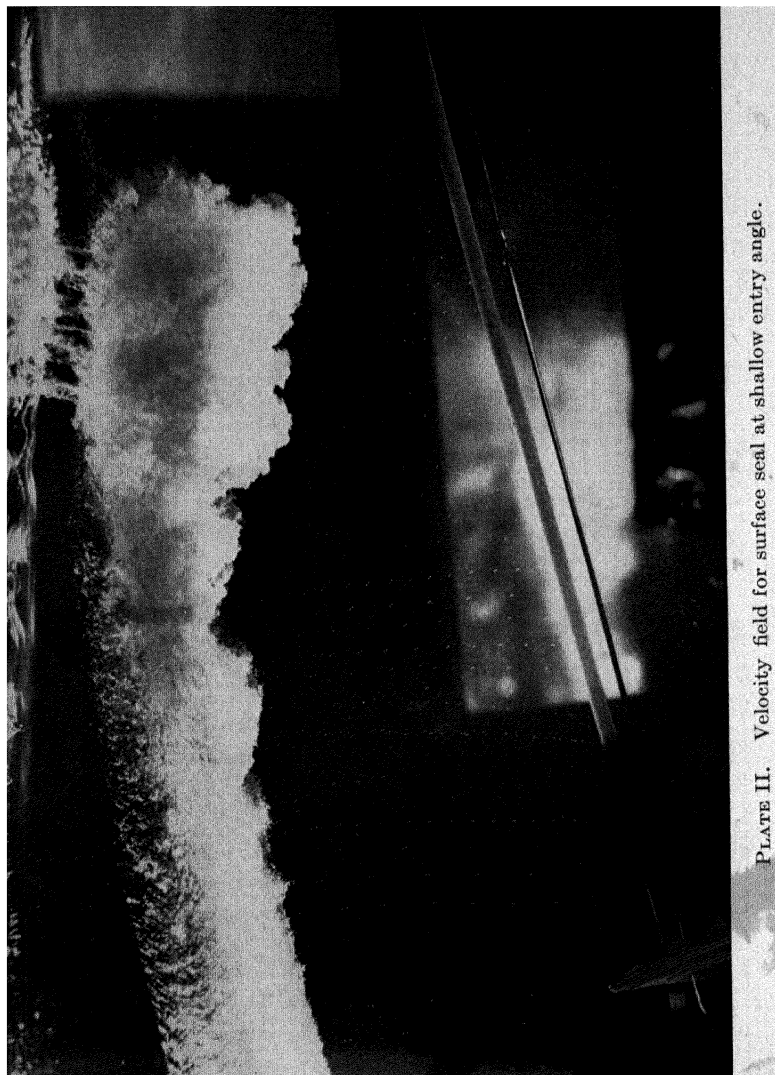


PLATE II. Velocity field for surface seal at shallow entry angle.

In this connection, the following recent fragmentary result of M. Schiffer,²⁵ proved by variational methods, may be mentioned. Let any body B of area β and any constant $\gamma > 0$, be given. There exists at least one plane flow around B with free streamlines, bounding a cavity of area γ . Hence there is indeterminacy as regards γ ; by the Brillouin Paradox, presumably $Q < 0$. It is a plausible conjecture that, for a class of bodies as general as accolades, there is just one everywhere concave "cusped cavity," of area $\gamma(Q)$, for each $Q < 0$. It would be interesting to determine $\gamma(Q)$ in specific cases, and the limiting flow as $Q \uparrow 0$; this will presumably have an infinitely long cavity of finite width.

Although it would seem analytically difficult to construct simple flows with points of inflection on the cavity wall if $Q < 0$, no such limitation is apparent if $Q > 0$. Therefore it seems plausible that the criteria of Section 5 must be invoked, if one wishes to avoid the embarrassment of a *two*-parameter family of flows. It is an attractive conjecture that, using these criteria, there is just one symmetric flow of Riabouchinsky type past any "accolade," for sufficiently small $Q > 0$.

9. Case ρ'/ρ Near Zero: Cavity Flows

I have already mentioned the plausible hypothesis that classical "free streamline" theory is applicable when ρ'/ρ is small. Mises²⁶ has given elsewhere supporting data for jets into air. I shall now review recent evidence for the case of cavities behind missiles shot into water, and also for cavities formed behind obstacles held in a water tunnel with $0.2 < Q < 1.0$.

Plate I is a sample piece of evidence;²⁷ it shows the cavity formed by a sphere shot into water at about 150 ft/sec. Two

²⁵ Oral communication to the author. Using reversibility, it would seem that at least three topologically different types are possible, wake behind, "wake" in front, and symmetrically disposed double wake.

²⁶ R. von Mises, *Zeits. Ver. Deutsche Ing.*, 61 (1917), 447-451, 469-473, and 493-497.

²⁷ The technique by which it was taken is described in [19], where other similar photographs may be found.

exposures, separated by about .005 secs., are made on a single plate. The white dots speckled over the picture are small bubbles, each bubble being photographed twice. It is not hard to pair the bubbles, and the vector from the first to the second bubble of each pair is roughly proportional to the vector velocity of the water in the vicinity. In this way the cavity profile and the velocity field of the flow can be clearly visualized.

Now photographs such as these are not exactly comparable with theory, for various reasons. Thus they represent a decelerating body, and not stationary flow; the water surface presents a second free boundary, which complicates the mathematical description; the influence of air is not negligible (cf. Section 10). Nevertheless, several pertinent general correlations with theory can be made, based largely on first-hand observation.

First, except in the case of a water-tunnel with $Q > 0.3$, real cavity outlines are relatively smooth²⁸ and stationary for ten or more body diameters back. This agrees with the Kirchhoff model infinitely better than pictures of ordinary wakes. (See the references of Section 5.)

Second, the cavity profile is nearly always convex, and separation is in front of the cross-section of maximum diameter. This clearly supports the use of condition (iii) (hence of conditions (i)–(iv)) of Section 5, in avoiding mathematical indeterminacy.

The idea that the minimum pressure occurs in the cavity (Section 5, condition (ii)) is also supported by the observation that *vapor cavities always begin at solid surfaces* in pictures of cavity motion. Indeed, it follows immediately from Bernoulli's equation (cf. Chapter I, (6), also [9], p. 19),

$$(17) \quad \frac{\partial U}{\partial t} + \frac{1}{2} \sum u_i^2 + \frac{p}{\rho} = G + \text{const.},$$

for non-stationary irrotational motion of an incompressible, non-viscous fluid. In (17), $\nabla^2 G = 0$, since G is the Newtonian

²⁸ Small "flutings" parallel to the flow may occur unless the solid is extremely smooth; see Plate I or [35], pp. 116, 129. In turbulent flow, a rough cavity wall may occur at lower Q ; see P. Eisenberg and H. Pond, *David Taylor Model Basin Report No. 668* (1948).

gravitational potential; $\nabla^2(\partial U/\partial t) = \partial(\nabla^2 U)/\partial t = 0$, (cf. Chapter I, (7)); $\nabla^2(u_i^2) = u_i \nabla^2 u_i + 2\nabla u_i \nabla u_i \geq 0$. Hence $\nabla^2 p \leq 0$, that is, p is *superharmonic*. It is however well-known that a superharmonic function must assume its minimum value on the boundary.²⁹

Fourth, it would appear that arbitrarily long cavities (100 diameters at least), of arbitrarily great maximum cross-section, occur behind missiles which are moving sufficiently fast. This fact has a considerable practical importance, in that much of the wounding effect of high-speed bullets and bomb fragments is due to the fact that they create temporary holes much bigger than themselves.³⁰ For us, it has the additional significance of providing possible physical approximations to the theoretical cavities of infinite length and asymptotically infinite cross-section calculated by Kirchhoff.

Fifth, the rough experimental formula³¹

$$(18) \quad C_D = .55 + .4Q = .55(1 + .73Q)$$

for the cavity C_D of a cylinder broadside, agrees quite well with theory. Thus Brodetsky had previously derived $C_D = C_D(0) = .55$ theoretically for the "preferred" flow (cf. Section 6) at $Q = 0$, and the formula for the other values of Q follow roughly formula (15), $C_D(Q) = (1 + Q)C_D(0)$.

On the other hand, I have seen nothing³² which at all resembles the concave "cusped cavities" with $Q < 0$ discussed above (Section 6); it would seem difficult to produce a stable wake or cavity overpressure experimentally.

²⁹ See G. Bouligand, *Jour. de Math.*, 6 (1927), p. 427.

³⁰ See [3], Section 74, and references given there; also E. N. Harvey, A. H. Whiteley, H. Grundfest, and J. H. McMillen, *The Military Surgeon*, 98 (1946), 509-528.

³¹ Kempf and Foerster, *Hydromechanische Probleme des Schifffenantriebs*, Hamburg, 1932; the experimental data are due to Martyrer. For Brodetsky's result, see *Proc. Roy. Soc.*, 102 (1923), 542-553; the separation zone appears to occur experimentally somewhat nearer the equator than the 35° predicted by Brodetsky.

³² The "deep seal" configuration of Fig. 13a is entirely transient, and it is immediately followed by a cavity with a reentrant down-jet (see [24], and [19], Figs. 21, 25).

10. Transient Scale Effects

The preceding results seem to confirm the plausible conjecture already stated (Section 5, end) that classical free streamline theory is applicable if ρ'/ρ is small, even though it is *not* sufficient for ν to be small. Unfortunately, recent experimental research shows that this is true, if at all, only in a very limited sense.

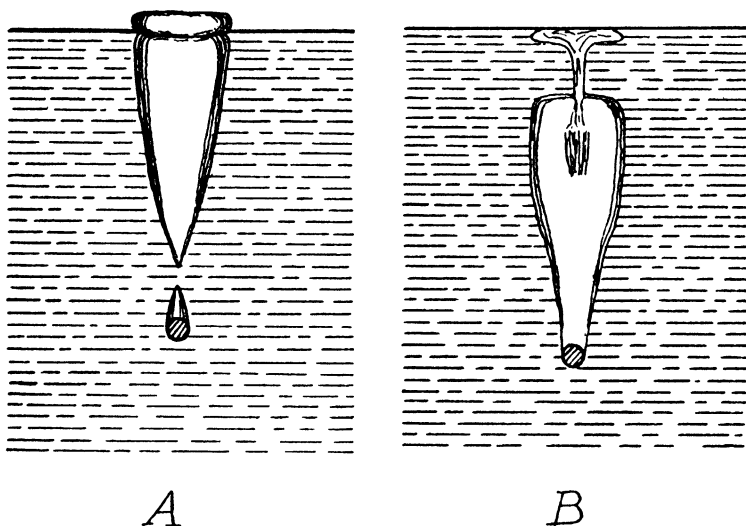


FIG. 13. Deep seal versus surface seal.

If we wish to square this conjecture with the experimental facts, we are forced to say that .0013 is not “small.” Specifically, there are two hydrodynamical phenomena which occur with entry into water under atmospheric conditions, and not if the air is evacuated. Hence no mathematical theory which neglects $\rho'/\rho \leq .0013$ can correctly predict them.

The more important is the phenomenon of *surface seal*. If a small sphere is dropped into still water at 10–20 ft/sec, the cavity first closes as in Figure 13a in a “deep seal,” whereas at entry speeds of 40 ft/sec or more, the cavity first closes at the surface, as sketched in Figure 13b. A photograph of the velocity-field associated with surface seal is displayed as

Plate II. The existence of surface seal was discovered about 1900 by Worthington.³³ In 1944, R. M. Davies [23], following a suggestion of Sir Geoffrey Taylor, showed that surface seal no longer occurs if the air pressure p is sufficiently reduced.

It was at first supposed by some that whether one got surface seal or deep seal depended on the cavitation number $Q = p/\frac{1}{2}\rho v^2$, which is proportional to p and intervenes directly in other cavity phenomena. But in 1945, following my suggestion [18], Gilbarg and Anderson [24] used "heavy gases" of the Freon series, so as to vary density and pressure independently. They thereby showed that, for a given sphere diameter d and entry speed v (Froude number $F = v^2/gd$), the manner of seal was largely determined by ρ'/ρ . For example, with 1-inch diameter spheres entering at 50–150 ft/sec (i.e. $F = 10^3 - 10^4$), surface seal occurs if $\rho'/\rho > .001$; deep seal occurs $\rho'/\rho < .0001$.

This sudden change of régime near a very small value of the parameter ρ'/ρ recalls the sudden changes in ordinary wakes which occur near $1/R = \nu/vd = .02$ (alternating vortices) and $1/R = .000005$, and in pipes when $1/R = .0005$ (see Chapter I). Thus it gives us another "paradox of approximation," and illustrates again the principle that *the nature of the solutions of partial differential equations may change abruptly for extremely small values of the parameters.*

It will be shown in Chapter III, Section 16, though by a somewhat arbitrary physical analysis, that the change of régime is actually associated closely with the dimensionless parameter $N = \sqrt{F} \rho'/\rho$, and occurs when N is about $\frac{1}{80}$.

A second phenomenon is incompatible with the idea that problems of water motion in air can be treated as free boundary problems. Discovered about 1944 by various persons, it was studied experimentally by Prof. L. B. Slichter. Prof. Slichter showed that a smooth 2-inch diameter dural sphere, entering the water at 50 ft/sec, and at an angle of 20° with the horizontal, may be refracted downwards 5° or more when it enters. Though the exact facts are still not clear, Prof.

³³ See [35]; also W. Mallock, *Proc. Roy. Soc.*, A95 (1918), 138–143, and E. N. Harvey and J. H. McMillen, *Jour. Appl. Phys.*, 17 (1946), 541–555.

Slichter made a careful (unfortunately unpublished) experimental analysis³⁴ which indicates strongly that this downward refraction is due to underpressure in the region where water is separating from the sphere—and that in turn this underpressure is caused by *air viscosity*, and its effect on the establishment of the boundary layer. However, systematic quantitative data are unfortunately still lacking.

Finally, various water tunnel studies³⁵ show that regardless of ρ'/ρ (which is only .00003 with water vapor cavities), the flow varies greatly as the cavitation number Q varies from zero to one. Hence, although the smallness of ρ'/ρ may be a sufficient condition for the validity of classical jet theory, it is only a useful necessary condition for the validity of cavity theory. Much work remains to be done in this connection.

11. Wall Corrections; Stability of Pressure Coefficient

By clarifying the physical foundations of free boundary theory, one may obtain useful and unsuspected applications to actual problems. I shall now (Sections 11–13) present some examples of this.

I shall begin with the important question of *wall effects*. Valcovici³⁶ showed in 1913, using the physically inept language of “wake theory,” that whereas the theoretical cavity C_D (“Cavity drag coefficient”) of a flat plate in an infinite stream was 0.88, as in Section 2, in a tunnel with fixed walls (as in Fig. 14a), 100 plate diameters apart it was 1.15, or 30% greater. The theoretical wall correction is reduced from 30% to less than 1% if the asymptotic “downstream” velocity v_c of the cavity wall is used instead of the usual “upstream” velocity v , in computing $C_D = 2D/1/2\rho v^2 A$. That is, the wall correction can be made approximately simply by computing, instead of C_D ,

³⁴ Upward refraction occurs ordinarily; see K. Ramsauer, *Über den Ricochetsschuss*, Thesis, Kiel, 1903; also [3], p. 453.

³⁵ See [15]; Kempf and Foerster, *op. cit. supra*; Eisenberg and Pond, *loc. cit. supra*; and unpublished data of Prof. R. T. Knapp.

³⁶ V. Valcovici, *Über discontinuierliche Flüssigkeitsbewegungen mit zwei freien Strahlen*, Thesis, Göttingen, 1913.

$$(19) \quad C_D^* = \frac{D}{\frac{1}{2}\rho v_c^2 A} = \frac{D}{(p_s - p_c)A},$$

by Bernoulli's Theorem. (Here p_s is the stagnation pressure, p_c the cavity pressure, and A the model cross-section.)

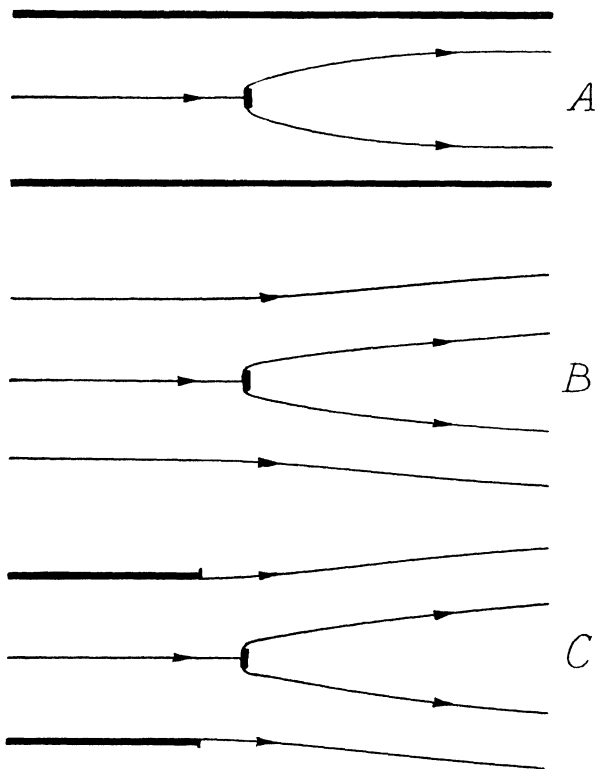


FIG. 14. Flows past plate in closed channel, free jet, and bounded jet.

Valcovici also showed that the wall correction became negligible if a jet with free boundaries (Figure 14b) was used instead of a stream between fixed walls. Recently, Milton Plesset and I [20] have noted that Valcovici's formulas could be derived easily by the method of Section 4, since the hodo-graph of half the flow is a quarter-circle; we also obtained numerical results for bounded jets (Figure 14c). Similar

results were obtained by N. Simmons for a class of flows with $Q > 0$ including those treated by Riabouchinsky.

Now it is almost certain that the original physical interpretation of Valcovici's result as a wall correction for wake drag is entirely erroneous. And it is only honest to say that my engineering friends have been very skeptical of the interpretation as a wall correction for cavity drag. They point out that the theory neglects the pressure gradient of the "undisturbed stream." This is another wall effect caused by eddy viscosity, which is well-known in hydraulics—if not in classical hydrodynamics.

Nevertheless, I maintain that formula (19) is both mathematically and physically sound. In fact, I should like to assert the validity of the following stronger *principle of stability of the pressure coefficient*: for an obstacle of given shape in a water tunnel (or jet!), *the pressure profile function*

$$(19') \quad \frac{(p - p_c)}{(p_s - p_c)} = \frac{(p - p_c)}{1/2 \rho v_c^2}$$

is insensitive to the presence of walls and changes in Q .

Thus it was proved rigorously in [20] that (19') is true for wedges of any angle symmetrically placed in a water tunnel. Again, formula (15) for the variation of C_D with Q , which was obtained by Gilbarg by study of the numerical evidence—and which has received rough experimental confirmation (see (18))—is equivalent to (19'). To see this, it suffices to note that by Bernoulli's theorem $Q = 1/2 \rho (v_c^2 - v^{*2}) / 1/2 \rho v^{*2}$, where v^* is the (up-stream) free stream velocity. Hence, by simple algebra,

$$Q = (v_c/v^*)^2 - 1, \quad 1 + Q = v_c^2/v^{*2},$$

and

$$(19'') \quad C_D = \frac{2D}{\rho v^{*2} A} = \left(\frac{v_c^2}{v^{*2}} \right) C_D^* = (1 + Q) C_D^*.$$

Finally, I note that Ackeret [15] obtained consistent results by assuming (19'), in extrapolating from his observations for $Q > 0.25$, to the case $Q = 0$.

In fact, it seems to me that the "principle of stability of the pressure coefficient" has all the earmarks of a successful

hydraulic principle—including an inability to stand too close scrutiny.

Even if it is inconvenient to measure $p_s - p_e$ (which must however be known if Q is to be measured), one can make an estimate of the correction in (19). If T is the tunnel cross-section, and A^* the maximum cavity area, then the mean velocity at that point is $vT/(T - A^*)$. Since the cavity velocity v_c is (the cavity being convex) at least the mean velocity, we infer

$$(20) \quad C_D^* \geq \frac{C_D}{(1 - A^*/T)^2}.$$

Equality holds in Valcovici's case, and more generally if there is a downstream asymptotic cavity velocity and maximum cross-section.

Formula (20) reveals area ratios for space flows as the logical analogs of diameter ratios for plane flows. This suggests that a 30% wall correction will apply as $Q \downarrow 0$ for cavity flows past a flat disc in a tunnel ten times as wide.

Comparison of (20) with (19'') also reveals the analogy between wall corrections and corrections for Q . Both (cf. (16)) effectively limit the indefinite growth of cavities. In fact (cf. Section 6, before footnote one), it is quite possible to identify the two, setting $Q = (v_c/v^*)^2 - 1$. Similarly, every water tunnel with fixed walls has a *blocking constant*; one can prove for plane or axially symmetric flow that (cf. [20])

$$(20') \quad Q \geq 2 \sqrt{C_D \frac{A}{T}} \geq \sqrt{\frac{A}{2T}} \quad \text{if} \quad C_D \geq .0625.$$

Hence one cannot get very low Q in such tunnels: thus to achieve $Q = 0.05$, we must have $T/A \geq 400$.

12. Lined Cavity Charges

So far I have mentioned almost no recent progress involving jets. However, a very important application of the classical theory of *impinging jets* was discovered in the late war, to *lined cavity charges*, such as were used in the American "bazooka," the British P.I.A.T., and many other anti-tank and demolition

weapons. I shall briefly summarize this application of free streamline theory; a complete account may be found in [21].

The construction and performance of such weapons may be described in principle as follows. An explosive charge, surrounding a hollow "cavity" lined with metal, is detonated from the rear. I shall consider only the cases of conical and wedge-shaped liners; a sectional view of these is given in Figure 15a. As the charge explodes, it drives the liner inward and forward, and it is found that the liner so impelled has an enormous penetrating power. How can we explain this power?

The best known explanation proceeds from the following plausible approximate assumptions. Assumption 1: After

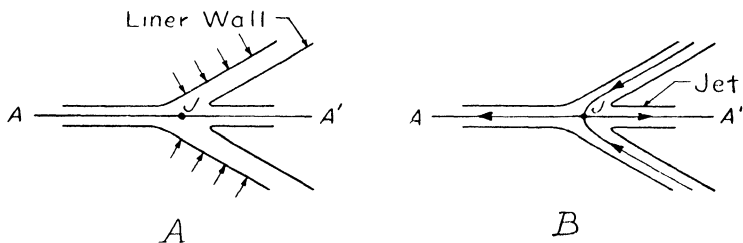


FIG. 15. Collapsing cone and impinging jets.

receiving an initial impulse from the explosive, the liner walls move inward under their own momentum with constant velocity, until they meet on the "axis" AA' of Figure 15a. Assumption 2: Under the terrific stresses generated, the liner behaves like a perfect fluid. Assumption 3: Relative to axes moving with the junction J of opposite walls, the flow is stationary. Assumption 4: The surfaces of the liner walls are free boundaries.

These assumptions reduce the problem to the classical case of "impinging jets" (see Figure 15b). It is a matter of high-school algebra and trigonometry, combined with the laws of conservation of mass and momentum, to predict the velocities of the jet JA' and residual slug AJ formed by the metal liner, and to predict the ratio of their masses. Especially, the extremely high predicted velocity of the jet is important, as explaining the effectiveness of lined cavity charges as anti-

tank weapons. The mathematics of penetration of armor by high-speed jets will be described in Section 13.

But first I should like to call attention to the peculiar fact that, though the advanced mathematical theory of impinging plane jets may be found in various places,³⁷ I know of no previous application of this theory, or discussion of the axially symmetric analog. Also, it seems significant to me that the practical usefulness of the mathematical model does not come from the relatively advanced use of complex variable theory and conformal mapping (the "hodograph" is a circle, and the method of Section 4 is applicable in principle). Instead, it is humble mathematical methods, combined with simple physical conservation principles, which are really useful. This is especially true because, as in the determination of the coefficient of contraction of the Borda tube (Section 3), it is only the simple laws of conservation which are applicable to the axially symmetric case of greatest practical importance.

13. Penetration by Liquid Jets

It has been known since Worthington's early photographs [35], that cavity "deep seal" (Figure 13a) is followed by the formation of upward jets. Gilbarg and Anderson [24] have recently shown that *down-jets* also occur, and the cavity flows with reentrant jets described in Section 6 may be regarded as crude approximations to the flow induced by such down jets.³⁸ These jets are not to be confused with those arising from lined hollow charges (Section 12).

A mathematical model for stationary plane *up-jets* has recently been suggested by T. E. Caywood and myself [19]; it is surprising that it should have been discussed so slightly in the classical literature.³⁹ The so-called "physical" z -plane

³⁷ See [10], p. 280; [25], pp. 11-12; W. Schach, *Ing. Archiv*, 5 (1934), 245-265. In the plane, half the flow corresponds to a jet impinging on a plate, which is easily realized physically. One gets the impression from the figure on p. 381 of the 4th edition of Gibson's *Hydraulics*, that even the usual topological picture is not realized physically in the case of impinging jets.

³⁸ Photographs displaying the velocity-fields of up-jets and down-jets, like those of Plates I-II, may be found in [19].

³⁹ I was only able to find, in [25], one passing reference.

is shown in Figure 16, with the up-jet along the x -axis SS' . The "hodograph" of half the flow is a semi-circle, while the W -domain is a half-plane with semi-infinite slit SOS' . Hence the method sketched in Section 4 is applicable; the details follow. By the elementary theory of Schwarz-Christoffel transformations, the W -domain can be mapped onto the upper half t -plane by a transformation

$$(21) \quad dW = \left(1 - \frac{1}{t}\right) dt, \quad \text{or} \quad W = t - \ln t.$$

The most general transformation of the hodograph is given as before by

$$(21') \quad t = \frac{A\sigma + B}{C\sigma + D} = c \frac{\zeta^2 + 2\lambda\zeta + 1}{\zeta^2 + 2\mu\zeta + 1}, \quad \text{where} \\ \sigma = \frac{1}{2} (\zeta + \zeta^{-1}).$$

But by Figure 16, $\zeta = -1$ when $t = \pm \infty$ at S' and T ; $\zeta = 0$ when $t = 1$ at O ; and $\zeta = 1$ when $t = 0$ at S . Hence $\lambda = -1$, $\mu = 1$, and by suitable choice of units we can make

$$(21'') \quad t = \frac{(1 - \zeta)^2}{(1 + \zeta)^2}.$$

The preceding model may easily be altered to describe penetration of an incompressible fluid by a two-dimensional jet of different density, as of tank armor by the jet from a wedge-shaped liner. The "dividing streamline" TOT' of Figure 16 may be taken as a streamline of discontinuity (*not* a free streamline, however!). Above TOT' is the penetrating jet of density ρ moving *downwards* with speed v , obtained from the model of the preceding paragraph by multiplying all vector velocities above TOT' by the scalar $-v$. Below TOT' is the material being penetrated, of density ρ' ; we assume that it behaves like a liquid; and obtain its velocity field by multiplying all velocities below TOT' in the preceding paragraph by the scalar $-\sqrt{\rho/\rho'} v$. By Bernoulli's equation, the pressure is continuous across the streamline of discontinuity.

Thus we have a *stationary streamline of discontinuity not a free streamline*; this type of flow seems to be new.⁴⁰ By superposing a constant vertical velocity on the whole, we get a model for the penetration of a stationary target by a liquid jet.

What about axially symmetric jets? Again, the important physical quantity, the *rate* of penetration, can be obtained by use of algebra and energy conservation (disguised as the

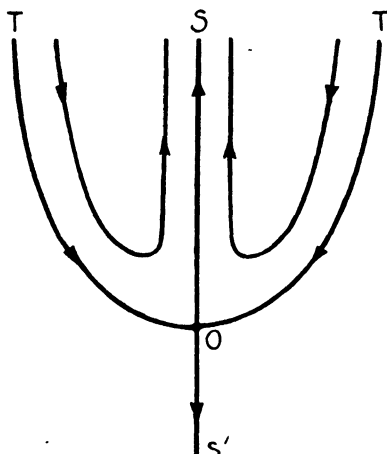


FIG. 16. Collapsing cavity—penetration by jet.

Bernoulli equation) alone, without the benefit of higher mathematics! The details are mathematically so trivial that I shall not bore you with them (they may be found in [21]).

14. Stationary Cavities in Accelerated Motion

A novel type of non-stationary plane motion with stationary free boundaries has been found very recently by von Karman.⁴¹ Consider the Bernoulli equation (17) of Section 9, with

⁴⁰ See [17], where flows of this type are discussed in detail. Continuous analogs of such changes of density have been introduced by M. Munk and R. Prim, *Proc. Nat. Acad. Sci. U.S.A.*, 33 (1947), 137–141.

⁴¹ Th. von Karman, "Les sillages non stationnaires"; Prof. von Karman kindly lent me a copy of this unpublished note. Karman's use of similarity is analogous to that of Wagner, described in Chapter IV, Section 8.

gravity neglected,

$$(22) \quad \frac{\partial U}{\partial t} + \frac{1}{2} \sum u_i^2 + \frac{p}{\rho} = C(t),$$

for non-viscous fluid. We look for flows involving accelerated motion past a fixed obstacle, in which dependence on time and space can be separated, so that

$$(23) \quad U = f(t)\phi(x_1, x_2, x_3), \quad \text{and} \quad p = g(t)\psi(x_1, x_2, x_3).$$

If we substitute from (23) in (22), we get

$$(22') \quad f'\phi + \frac{1}{2}f^2(\nabla\phi\nabla\phi) + \frac{g\psi}{\rho} = C(t).$$

For this to be an identity, since ϕ , $\nabla\phi\nabla\phi$, ψ/ρ , are functions of $\mathbf{x}$ alone, clearly f' and g must bear a constant ratio to f^2 , and ϕ and ψ a constant ratio to $\nabla\phi\nabla\phi$.

Setting $f' = -kf^2$, we get $-kdt = df/f^2$, or (by appropriate choice of time-origin) $f = 1/kt$. This is of course the case of constant *acceleration coefficient*, and not of constant acceleration.

Now a *free* streamline is one with $\psi = \text{const.}$; hence we have as a necessary and sufficient condition

$$(24) \quad -k\phi + \frac{1}{2}(\nabla\phi\nabla\phi) = C_1,$$

or

$$(25) \quad U = \frac{v^2}{2k} + C^*/t.$$

In the case $k = 0$ of no acceleration, (24) reduces of course to the classic condition (Section 1), that velocity is constant on a free streamline with stationary motion. Differentiating (25) along the streamline, since $v = dU/ds$, we get (indicating differentiation with respect to s by an accent), $-kU' + U'U'' = 0$. Except at stagnation points where $U' = 0$, this amounts to $U'' = k$, or $v = U' = ks + C$: *velocity is a linear function of distance along the streamline*. Choosing constants of integration suitably, we easily reduce to

$$U' = ks, \quad U = \frac{1}{2}ks^2 + C^*/t.$$

Using (25), one can apply a hodograph method, and von Karman has by this means obtained explicitly an accelerated

plane cavity flow past a flat plate held broadside. Curiously, a solution is possible in the large only for one value of k !

The preceding refers to a fixed plate in a fluid accelerated at infinity. We can transform to the case of an accelerated plate in a fixed fluid, by superposing a "virtual" gravity field, constant in space at any instant. Now if the cavity is filled with fluid of density ρ , such a field—or any other conservative field—has no effect on the flow (Chapter III, Section 12). Hence von Karman's model describes a wake behind an accelerated or decelerated object, for the case $\rho'/\rho = 1$ and constant acceleration or deceleration coefficient such as occurs with constant C_D .

It is perhaps of interest to obtain an analog of (25) for streamlines of discontinuity not free streamlines. Along any fixed streamline, let $\mathbf{u} = f(t)\mathbf{v}(\mathbf{x})$, where $f(t) = 1/kt$. The component of the Equations of Motion parallel to the streamline reduces to

$$\partial u/\partial t + udu/ds + dp/\rho ds = 0,$$

where u is the scalar speed of flow, and s measures distance along the streamline. Substituting for $u = v/kt$, we get $\partial u/\partial t = -u/t$, whence we have the generalized Bernoulli equation

$$(26) \quad \frac{1}{2}u^2 + \frac{p}{\rho} = \frac{1}{t} \int u ds, \quad \text{or} \quad \frac{1}{2}v^2 + \frac{k^2 t^2 p}{\rho} = k \int v ds$$

in the case of a fluid of constant density ρ . Since p is continuous across any streamline of discontinuity,

$$(26') \quad \rho \left[\frac{1}{2}u^2 - \frac{1}{t} \int u ds \right] = \rho_1 \left[\frac{1}{2}u_1^2 - \frac{1}{t} \int u_1 ds \right],$$

where subscripts refer to opposite sides of the streamline.

15. Axially Symmetric Flows

Until the past few years, the only significant non-planar flows with free streamlines which had been computed, were the (axially symmetric) jet through a circular orifice in a flat plate, and the impact of a circular jet on a flat plate. The

former was calculated by Trefftz,⁴² using successive approximations based on integral equations; he computed the coefficient of contraction to be 0.68. The latter was calculated by Schach, using the same method.

Recently, Southwell and Vaisey [33'] have shown the power of "relaxation methods," by calculating the jet through a circular orifice, from a container of different dimensions, and computing besides the axially symmetric jet through a Borda tube in the shape of a circular cylinder. In both cases the coefficient of contraction was assumed—as "generally accepted" to be 0.68 in Trefftz' case, and as 0.50 in the Borda case (we showed this in Section 3). In addition they computed a flow with a "cusped cavity" past a sphere.

These achievements are obviously most promising, and it is to be hoped that "relaxation methods" may also obtain flows past a sphere with $Q \geq 0$, as these seem physically more interesting.⁴³ Also, it is to be hoped that the ambiguities specified above (Section 8) can be eliminated. Indeed, a rigorous justification of either the Trefftz or Southwell-Vaisey method would be most valuable, as would an accurate comparison of their numerical results.

In the same connection, one should mention the very recent use of the electrolytic tank analogy to obtain the axially symmetric Riabouchinsky flow past a disc in a channel.⁴⁴ When (15) was used to extrapolate the results to $Q = 0$, the estimate $C_D(0) = 0.823$ was obtained, which agrees very well with experiment.

Another impressive recent achievement is the derivation by Levinson [30] of analogs of formulas (11)–(11'), for the

⁴² E. Trefftz, *Über die Kontraktion kreisförmiger Flüssigkeitsstrahlen*, Dissertation, Strassburg, 1914; also *Zeits. Math. u. Phys.*, 64 (1916), p. 34. Schach's results are in the *Ingenieur Archiv*, 6 (1935), 51–58. Note also Weingartner, *Gott. Nachr.* (1890), and H. Reissner, *Zeits. ang. Math. Mech.*, 12 (1935), 25–35.

⁴³ It would seem difficult to estimate "wall effects" by relaxation methods; so many mesh points would be needed to get any accuracy; cf. Section 11.

⁴⁴ "Détermination des lignes de jet dans les mouvements plans et de révolution," *Problème D359A, Travaux du Laboratoire de Calcul Expérimental Analogique*, by P. Marchet, under the direction of L. Malavard.

axially symmetric case. Assuming only that for some s with $0 < s < 1$, the cavity profile can be expressed by

$$(27) \quad y = x^s g(x), \quad \text{where} \quad \lim_{x \rightarrow \infty} \frac{xg'(x)}{g(x)} = 0,$$

Levinson proved that the cavity profile must satisfy

$$(28) \quad y = \sqrt{Cx} g(x), \quad \text{where} \quad (\ln x)^{-1/4-\epsilon} < g(x) < (\ln x)^{-1/4+\epsilon}$$

for all $\epsilon > 0$ and sufficiently large x . He proved further that if (18) is strengthened to $xg'(x)/g(x) = 0(1/\ln x)$, then

$$(28') \quad y \sim \sqrt{Cx} (\ln x)^{-1/4} \quad \left(\text{i.e., } \left\{ \lim_{x \rightarrow \infty} \frac{y(\ln x)^{1/4}}{\sqrt{Cx}} \right\} = 1 \right),$$

and the drag is given by

$$(28'') \quad D = \frac{\pi \rho C^2 v^2}{8}.$$

However, Levinson failed to prove the existence of a solution. In view of the complexity of (28'), and of the Paradoxes of Overdeterminacy which are known (cf. Chapter I), one may well wonder if there exist axially symmetric cavity (alias "wake") flows, in the classical mathematical sense.

A similar but much easier result has also been recently obtained for plane and axially symmetric cavity flows with reentrant jets ([6], Figure 10). I have shown [17] that the jet cross-section A^* and the cavitation number Q are related by

$$(29) \quad C_D = \frac{2(1 + Q + \sqrt{1 + Q})A^*}{A} \quad \text{whence} \quad \frac{A^*}{A} = \frac{C_D}{4} \quad \text{if} \quad Q = 0.$$

The asymptotic considerations needed to prove (20) are quite like those used in deriving the d'Alembert paradox and Levi-Civita's formula (11').

16. Summary of Recent Tendencies

In closing this chapter, I should like to stress the *trend* of recent progress in free boundary theory. I think it is clear

that there have been important developments dealing with an *increasing variety of mathematical problems, inspired by an awareness of the great variety of real physical problems.*

Thus the trend of recent progress has been in the same general direction as modern theoretical physics. Twentieth-century physics has also been characterized by a broadening of foundations, both in quantum mechanics and relativity, and by an acceptance of the philosophy that we are not yet able to deduce all the laws of Nature from a single set of convenient and harmonious mathematical hypotheses.

I hope that a realization of this community of spirit between modern physics and fluid mechanics will lead to a greater interest of physicists generally in the theoretical and experimental problems of fluid mechanics. It seems surprising that this subject, to which so much was contributed by Stokes, Helmholtz, Rayleigh, and other physicists of the nineteenth century, should owe all its more recent progress to mathematicians and engineers. I feel confident that a renewed interest by physicists in the accurate correlation of theory with experiment, would help greatly in providing it with a more solid scientific basis.

CHAPTER III

Modeling and Dimensional Analysis

1. Introduction

The use of *models* to study fluid mechanics has an appeal for everyone endowed with natural curiosity. What active boy has not played with ship and airplane models, or crude models of dam and drainage systems? Even in the most advanced technical engineering, such models play a fundamental and indispensable role.

And yet in few departments of the physical sciences is there a wider gap between theory and practice, between scientific knowledge and the state of the art, than in the use of models to study hydrodynamic phenomena. This is partly because engineers with adequate laboratory facilities, who are most familiar with the realities of the situation, are usually preoccupied by special practical problems concerning one type of model. Whereas the academic scientist, preoccupied by the difficulties of logical exposition, too often ignores experimental reality.

It is my aim to narrow this gap; this can be done only by a critical examination of the whole field from both sides.

Being a mathematician, I shall begin with the *theory* of modeling. Here there are a number of simple, yet fundamental, mathematical remarks which have hitherto escaped general attention—no doubt because mathematicians have not seriously studied the subject. I hope I can make my remarks in sufficiently plain language, so that physicists and engineers will see their relevance.

2. Dimensional Analysis

A priori theoretical arguments for the use of models usually follow one of two main lines: so-called *dimensional analysis*,

and what may be called *inspectional* analysis.¹ As the former is simpler and more widely known, I shall begin (Sections 2-7) with a brief historical résumé of its origins.

Dimensional analysis arises from an attempt to extend to physics the Greek concepts of geometrical similarity, ratio, and proportion.² It was first applied by Galileo to predict the strength of beams of given material, as a function of their linear dimensions. He assumed it as intuitively obvious that for any given material, failure of a beam occurred when the force per unit area exceeded a certain maximum. He concluded that the safe load per unit volume was inversely proportional to the length, and various other classical results.

Other applications were given by Mariotte³ and Newton, but it was Fourier ([47], *loc. cit.*) who first stated that there are certain "fundamental units," in terms of which every physical quantity has certain "dimensions," to be written as exponents. To be sure, this idea is almost obvious, if one thinks systematically about the "conversion factors" needed to convert physical quantities from one system of units into another.

By simple changes of units, Fourier found it easy to treat problems involving heat conduction in small spheres, and problems involving the earth, by the same analytical formulas. It is relevant, since we are interested in the relation of theory to fact, that Fourier's deductions were sometimes badly off, because they ignored the effect of radioactive materials in the earth's core! Nevertheless, his procedure for eliminating parameters by changes of units is now classic, and has been used with notable (if not uniform!) success in many parts of physics. In the hands of Stokes, Savart, Froude, Reynolds, Vaschy, and many others, it suggested laws of basic importance.

¹ For dimensional analysis, see Bridgman [38]. The useful phrase "inspectional analysis" is due to Ruark [52]; the idea is much older.

² Thus Fourier [47], p. 128, mentions the Greek awareness of the dimensions of area and volume. Rayleigh always referred to "similitude"; cf. the current phrase "dynamically similar." For Galileo, see his *Two New Sciences* (1638), Second Day.

³ Mariotte, *Traité de la percussion des corps*, (1679) and *Traité du mouvement des eaux* (1686); Newton, *Principia mathematica* (1686), Vol. 2, Sect. 7. See also J. Bertrand [36].

Above all, Rayleigh focused attention on the fruitfulness of the procedure as a general method. This led to an intensive discussion by leading physicists during the decades 1900–1920 of the assumptions tacitly involved, and a realization of the limitations of dimensional analysis.

Though still frequently ignored in the engineering and popular literature, these assumptions and limitations are clearly stated in the classic monograph of Bridgman [38]. However, I do not wish to presuppose any familiarity with the subject, and shall introduce dimensional analysis with a simple example.

3. *Statement of Assumptions*

The following illustration of the procedure of dimensional analysis may be found in innumerable physics textbooks, but it is usually considered too confusing to the student to label the assumptions as such.

EXAMPLE 1. Let us assume (or remember!) that the speed v of a wave in deep water is determined by its length λ and the acceleration g of gravity, so that $v = f(\lambda, g)$. Let us also assume that the relation is the same in all “fundamental units” of length and time. Having made these assumptions, we can argue mathematically as follows.

Let a wave of length λ in a gravitational field of intensity g move with speed v , in one set of units. Choose a new unit of length equal to λ old units, and a new unit of time equal to τ old units. An easy computation of “conversion factors” shows that, in the new units, the same wave will have length $\lambda' = 1$, and speed $v' = v\tau/\lambda$, while the acceleration of gravity will be $g' = g\tau^2/\lambda$. Hence, setting $\tau = \sqrt{\lambda/g}$, we get

$$v/\sqrt{g\lambda} = v' = f(\lambda', g') = f(1, 1) = C,$$

where C is the velocity of a wave one unit long in a unit gravitational field, in one (hence all) systems of units. Multiplying through by $\sqrt{g\lambda}$, we get $v = C \sqrt{g\lambda}$.

An analog of the preceding argument can be applied under the following more general mathematical conditions.

I. There are certain independent "fundamental units" q_i such that for any positive real numbers α_i [$i = 1, \dots, n$] we can "change units" according to the formula

$$(1) \quad T(q_i) = \alpha_i q_i \quad [\alpha_i > 0].$$

(In mechanics, $n = 3$ and the q_i are length, time, and mass.)

II. There are "derived quantities" Q_j (such as density, velocity, viscosity, etc.) which are *homogeneous* in the sense that under (1) each Q_j is multiplied by a "conversion factor" given by

$$(2) \quad T(Q_j) = Q_j \alpha_1^{a_{j1}} \cdots \alpha_n^{a_{jn}}.$$

The exponents a_{jk} are called the "dimensions" of Q_j ; if they are all zero, then Q_j is called "dimensionless." Clearly, any product of powers of homogeneous quantities is homogeneous.

III. The quantity Q_0 is determined by $Q_1, \dots, Q_n$, through a relation $Q_0 = f(Q_1, \dots, Q_n)$.

IV. The equation $Q_0 = f(Q_1, \dots, Q_n)$ is *unit-free*, in the sense of being preserved by any transformation (1).

V. The quantities $Q_0, \dots, Q_n$ involve all n fundamental units. (In Example 1, $n = 2$, as mass was not involved.)

Then we can argue just as in Example 1, to get

$$(3) \quad Q_0 = C Q_1^{x_1} \cdots Q_n^{x_n},$$

where $C = f(1, \dots, 1)$. Moreover the x_i are by IV such as to make (3) *dimensionally homogeneous*, that is, of the same dimensions on both sides in every fundamental unit. Hence

$$(4) \quad a_{0i} = x_1 a_{1i} + \cdots + x_n a_{ni} \quad \text{for} \quad i = 1, \dots, n.$$

This gives an easy way to calculate the x_i .

The nature of the results which can be obtained in this way is illustrated by the following examples.

EXAMPLE 2. Assume that the fluid resistance D for a rigid object of a given shape and aspect is *inertial*, in the sense that it is determined by the fluid density ρ , speed v , and object diameter d . Then (IV) is equivalent to

$$MLT^{-2} = (ML^{-3})^x (LT^{-1})^y L^z,$$

so that equations (4) reduce to

$$1 = x, \quad 1 = -3x + y + z, \quad -2 = -y,$$

whence $x = 1$, $y = z = 2$. Hence if the relation is unit-free, then $D = K_D \rho v^2 d^2$, where K_D is a constant. (Actually, $K_D = \pi C_D/8$, which is called the ballistic drag coefficient, varies slowly.)

EXAMPLE 3. If D is determined by v , d , and the fluid viscosity μ , if inertia is negligible (Stokes' "creeping motions"), and the determination is unit-free, then $D = K^* \mu v d$ for another constant K^* , by a similar dimensional calculus.

When the number m of independent fundamental units is less than the number n of variables in the right-hand side of a unit-free relation $Q_0 = f(Q_1, \dots, Q_n)$, other well-known and useful formulas can still be obtained.

EXAMPLE 4. Assume that $D = f(\rho, v, d, \mu)$ is a unit-free function of ρ , v , d , and μ , under all changes (1) of units of length, time, and mass. The dimensionless quantities $K_D = D/\rho v^2 d^2$ and $R = \rho v d/\mu$ (Reynolds number) are invariant under such changes. By one such change, we can however⁴ reduce ρ , v , d simultaneously to 1, this will carry μ into $\mu/\rho v d = 1/R$. Therefore

$$(5) \quad D = K_D(R) \rho v^2 d^2, \quad \text{where} \quad K_D(R) = f\left(1, 1, 1, \frac{1}{R}\right).$$

EXAMPLE 5. Assume that D is similarly determined by ρ , v , d , and the compressibility $-d(1/\rho)/dp = d\rho/\rho^2 dp$ of the undisturbed fluid. These yield the dimensionless power products $D/\rho v^2 d^2 = K_D$ and $v^2 d\rho/dp$ (of dimensions $(LT^{-1})^2 (ML^{-3})(MLT^{-2}L^{-2})^{-1} = 1$). A more physical interpretation of $v^2 d\rho/dp$ is obtained if we recall that $dp/d\rho = c^2$, where c is the speed of sound in the fluid. Repeating the argument of Example 4, we then get similarly

$$(6) \quad K_D = f(M^2), \quad \text{where} \quad M = \frac{v}{c} \quad \text{is the Mach number.}^5$$

⁴ This is essentially Vaschy's argument; see also D. Riabouchinsky, *L'Aerophile*, Sept., 1911.

⁵ Called the Sarrau number in France. The term "Mach number" was proposed by J. Ackeret, *Schweiz. Bauzeitung*, 94 (1929), p. 179.

The physical applicability of formulas (5), (6) will be discussed in Sections 13, 18.

EXAMPLE 6. Let there be a *stationary* distribution of turbulent energy E per unit mass among different eddy sizes λ , so that $dE = E'(\lambda)d\lambda$. Assume that this distribution is determined by an *inertial* mechanism of turbulent energy transfer to smaller eddy sizes λ . Then the rate of energy transfer, per unit mass and time, to smaller eddies, clearly has dimensions $V^2/T = L^2/T^3$; hence it is multiplied by α^2/γ^3 under any change of scale $L \rightarrow \alpha L$, $T \rightarrow \gamma T$. Moreover to maintain $E'(\lambda)$ unchanged, this rate must be independent of λ . Hence the mean *time* $T(\lambda)$ required to break up eddies of size λ into smaller eddies, must be proportional to $\lambda^{3/2}$: *T has the dimensions $L^{3/2}$ under change of scale.* Now consider the *frequency spectrum of energy*: $dE = F(k)dk$, where $k = 2\pi/\lambda$ is the wave number. Since dE has dimensions $V^2 = L^2/T^2$, while k and $dk = 2\pi d\lambda/\lambda^2$ have dimensions $1/L$, $F(k)$ has dimensions L^3/T^2 , or $L^{5/2}$, or $k^{-5/2}$. This gives Kolmogoroff's formula for *turbulent energy distribution*: $F(k) \propto k^{-5/2}$.

This formula involves the well-known paradox of infinite turbulent total energy density per unit volume for large eddies; I shall not discuss the rationalization of this paradox.

4. Pi Theorem

Instead of giving more examples of special cases,⁶ I shall proceed directly to the general Pi Theorem of Vaschy and Buckingham, which may be formulated as follows:

THEOREM 1. *Let positive variables $Q_1, \dots, Q_r$ transform by (2) under all changes (1) in fundamental units $q_1, \dots, q_n$. Let $m \leq n$ be the "rank" of the matrix $||a_{ik}||$ defined by (2). Then the assertion that*

$$(7) \quad f(Q_1, \dots, Q_r) = 0$$

is a unit-free relation, is equivalent to a condition of the form

$$(8) \quad \phi(\Pi_1, \dots, \Pi_{r-m}) = 0,$$

⁶ Numerous other examples are given in Bridgman [38], Chs. I, VI; in Porter [50]; and in B. A. Robertson, *Gen. Elec. Review*, 33 (1930), 207. See also Rayleigh, *Phil. Mag.*, 34 (1892), 59, and 8 (1905), 66; also *Nature*, 95 (1915), 66.

for suitable dimensionless power-products $\Pi_1, \dots, \Pi_{r-m}$ of the Q_i .

EXPLANATION. The first sentence restates Assumptions I, II of Section 3. Assumptions III and IV are generalized in (7). Nothing is required like Assumption V, which is the special case $m = r - 1$.

There is a possible ambiguity in condition (7). It means not merely that the equation $f = 0$ is true, no matter what fundamental units are employed; it means that the locus defined by $f = 0$ is invariant under all transformations (2). To see the distinction, consider the case of a particle under constant rectilinear acceleration a ; denote distance by s , time by t , and velocity by v . Then in any system of fundamental units,

$$(9) \quad \sqrt[3]{(v - at)(a + v)} + \sqrt[5]{t(v - at)} = 0,$$

and

$$(10) \quad v + v^2 = 2as + at.$$

Equation (9) is *unit-free* in our sense, since it is equivalent to $v = at$ in the octant $v > 0, a > 0, t > 0$. Equation (10) is not, though as it represents a linear combination of $v = at$ and $v^2 = 2as$, it is true in any system of fundamental units. Thus the meaning of (10), though not its truth, depends on the choice of fundamental units; this equation will be discussed again in Section 5.

PROOF. By matrix theory, $\|a_{ik}\|$ has a non-singular $m \times m$ minor.⁷ By permuting the Q_i and q_i , we can suppose that this minor involves $Q_1, \dots, Q_m$ and $q_1, \dots, q_m$. (Thus, in the usual language of physics, the other fundamental units are not involved independently.) Then any vector $\mathbf{a}_j = (a_{j1}, \dots, a_{jn})$ with $j > m$ is a linear combination $\mathbf{a}_j = c_{j1}\mathbf{a}_1 + \dots + c_{jm}\mathbf{a}_m$ of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$. Hence $\Pi_j = Q_1 Q_1^{-c_{j1}} \dots Q_m^{-c_{jm}}$ will be dimensionless, with $Q_j = \Pi_j Q_1^{c_{j1}} \dots Q_m^{c_{jm}}$. We define g by

$$(11) \quad g(Q_1, \dots, Q_m; \Pi_1, \dots, \Pi_{r-m}) = f(Q_1, \dots, Q_r).$$

⁷ For the properties of matrices assumed here, see for instance [37] Ch. X, esp. p. 291. A non-singular $m \times m$ minor is simply a square submatrix whose determinant is not zero.

In the "octant"⁸ $Q_1 > 0, \dots, Q_r > 0$, the transformation of independent variables is one-one *in the large*. Hence $f = 0$ is equivalent to (i.e. defines the same locus as) $g = 0$, and $g = 0$ is consequently also unit-free. But since the minor $||a_{ij}||$ with $i, j = 1, \dots, m$ is non-singular, the simultaneous linear equations $a_{i1} \log \alpha_1 + \dots + a_{im} \log \alpha_m = \log Q_i$ can be solved for arbitrary positive $Q_1, \dots, Q_m$, by suitable choice of positive $\alpha_1, \dots, \alpha_m$. Since $g = 0$ is unit-free, the locus is therefore the same for all $Q_1, \dots, Q_m$ hence (7) is equivalent for example to

$$(8) \quad \phi(\Pi_1, \dots, \Pi_{r-m}) = g(1, \dots, 1; \Pi_1, \dots, \Pi_{r-m}) = 0.$$

This completes the proof.

If, for example, this procedure is applied to (9), we are led to define $\Pi = at/v$, and to get after substitution,

$$(8.1) \quad \sqrt[3]{(1 - \Pi)(\Pi^2 + 1)} + \sqrt[5]{1 - \Pi} = 0.$$

However it should be stressed that this method *assumes* that (9) is unit-free; the impression that (8.1) gives an *a posteriori* demonstration that (9) is unit-free, is false; indeed, it would be hard to prove that (9) was unit-free, by direct algebraic substitutions such as those considered above.

Historical Note. There is some controversy over the origins of the Pi Theorem. Vaschy⁹ stated the result in 1892, but without formulating his assumptions. He indicated the method used here; but his logic was so cryptic that nobody reproduced his demonstration. Buckingham ([40], [41]) gave in 1914 the first proof of the Pi Theorem, but for the special case that f could be expanded in a Maclaurin series; until recently, this was the only accepted proof.¹⁰ Recently,

⁸ In case the c_{jm} are integers, this can be generalized to other octants. The behavior on the hyperplanes $Q_i = 0$ is more difficult, and has been discussed by D. Riabouchinsky, *Comptes rendus*, 217 (1943), 220-223.

⁹ A. Vaschy, *Annales télégraphiques*, 19 (1892), 25-28. The ideas of Riabouchinsky were evolved in a series of papers (*L'Aerophile*, Sept. 1911; *Comptes Rendus*, 217 (1943), 205-208, and 225 (1947), 837-839). My discussion was presented in [56], before I knew of this French work.

¹⁰ In fact, Bridgman [38], p. 16, raises the question of whether more general functions can be considered. In this connection, see Section 18. A good dimensionless function for which expansion in Maclaurin series is impossible, is the $\alpha(0, M, v)$ in the Taylor-Maccoll determination of Chapter IV, Section 4.

Riabouchinsky and A. Martinot-Lagarde [49] have clarified the ideas of Vaschy, so as to obtain a much more general proof. The proof given here represents, I think, a further clarification, as regards the restriction to positive α_i and Q , and the theorems on matrices assumed.¹¹

5. Dimensional Homogeneity

A relation $f(Q_1, \dots, Q_n) = 0$ is called "dimensionally homogeneous" if and only if it satisfies Euler's relations

$$(12) \quad \frac{\partial f}{\partial \alpha_i} = \lambda_i f$$

under transformations (2). It is easy to show that any dimensionally homogeneous identity is unit-free. For by (12), $d(\log f) = \lambda_1 d\alpha_1 + \dots + \lambda_n d\alpha_n$. Integrating, we see that $f = 0$ is transformed into the equivalent equation $\alpha_1^{\lambda_1} \dots \alpha_n^{\lambda_n} f = 0$ under (2).

Conversely, most unit-free relations arising in practice are dimensionally homogeneous. I shall now try to give this well-known fact a mathematical motivation. Indeed, from Theorem 1, we obtain immediately

COROLLARY 1. *Any unit-free relation is equivalent to a dimensionless (hence dimensionally homogeneous) relation.*

More generally, consider relations between variables Q , which, like (10), are not strictly unit-free, yet satisfy

$$(13) \quad f(Q_1, \dots, Q_r) = 0$$

for all fundamental units, for some function f . We shall call such relations *true in all units*. Under any particular system of fundamental units, if we define g by (11), $f = 0$ in all units is equivalent to $g = 0$ in all units. As in the proof of the Pi Theorem, by suitable choice of units, we can make $Q_1, \dots, Q_m$ assume arbitrary values. Hence the m -parameter set of dimensionless equations in $\Pi_1, \dots, \Pi_{r-m}$ defined by

$$(14) \quad \begin{aligned} 0 &= g_Q(\Pi_1, \dots, \Pi_{r-m}) \\ &= g(Q_1, \dots, Q_m; \Pi_1, \dots, \Pi_{r-m}), \end{aligned}$$

¹¹ See also H. L. Langhaar, *Jour. Franklin Institute*, 242 (1947), 459-463.

holds for all $\mathbf{Q} = (Q_1, \dots, Q_m)$. The converse is obvious, proving

COROLLARY 2. *Any relation which is true under all units is equivalent to a set of dimensionless relations.*

Though the theory of functional dependence is notoriously difficult,^{11a} it is plausible that at most n functionally independent relations can exist, in general. Hence it would seem as if any ordinary law, true in all units, could be reduced to a finite set of dimensionless identities.

If $f(Q_1, \dots, Q_n)$ is a polynomial, or can be expanded in a Maclaurin series,¹² a neater treatment is possible. We can show that $f = 0$ is true in all units, if and only if every dimensionally homogeneous constituent of f vanishes identically. Indeed, consider the identity (14) as a function of the α_i , when the fundamental units are varied. Since expansion in Maclaurin series is unique, it is equivalent to the vanishing of every dimensionally homogeneous constituent of $g = f$. For example, (10) is equivalent¹³ to the pair of identities $v = at$ and $v^2 = 2as$.

It follows as an immediate corollary that every binomial or trinomial relation, true in all units, is dimensionally homogeneous.

The preceding results substantiate in part the claim that only "dimensionally homogeneous" equations can be regarded as having fundamental physical significance—to the extent that all relations involving positive quantities, and which are true in all units, can be reduced to dimensionally homogeneous (hence to dimensionless) form. However (see Section 7) not all physically significant equations are true in all units; hence the claim has only a limited validity.

^{11a} Assuming sufficient differentiability, many cases can be treated by implicit function theorems (cf. G. A. Bliss, *Fundamental Existence Theorems*, New York, 1913, Chapter I). If only continuity is assumed, all the difficulties of topological dimension theory arise.

¹² Since only the uniqueness of the expansion is really involved, it would seem that expansibility in Laurent series was sufficient.

¹³ This result, and the example of (10), are due to Bridgman [38], p. 42; it led Bridgman to propose a "vector calculus" of identities. From a mathematical point of view, in the case of polynomial relations, this amounts to characterizing a locus by the ideal of polynomial equations which it satisfies; thus we are led to classical ideas of algebraic geometry. For the question in general, see Bridgman [38], pp. 41–46.

6. *Validity of Assumption II*

In many instances (as when velocity is measured indirectly by dividing the distance traversed by the time elapsed), the validity of Assumption II of Section 3 follows directly from the physical definition of the Q_i . However, it must be admitted that not all experimental measurements are related to fundamental units in such a simple way. What it means to say that "they could all be so reduced," beyond an expression of faith, it is difficult to say.

One can also characterize¹⁴ homogeneous quantities by Bridgman's property of "absolute invariance of relative magnitude." A function of the variables $q_1, \dots, q_n$ is said by Bridgman to have this property when it satisfies the functional equation

$$(15) \quad \frac{f(q'_1, \dots, q'_n)}{f(q_1, \dots, q_n)} = \frac{f(\alpha_1 q'_1, \dots, \alpha_n q'_n)}{f(\alpha_1 q_1, \dots, \alpha_n q_n)}$$

for all positive α_i , q_i and q'_i [$i = 1, \dots, n$]. The result follows.

THEOREM 2. *Let $Q = f(q_1, \dots, q_n)$ be a positive quantity, whose numerical magnitude varies continuously with the fundamental units q_i , and which satisfies (15), so that ratios of numerical magnitudes are invariant under changes in the fundamental units. Then Q must satisfy (2).*

PROOF. Let $q'_1 = \sigma$, $\alpha_1 = \sigma^m$, and all other variables 1. After transposition, (15) becomes

$$(15') \quad \lambda f(\sigma^m, 1, \dots, 1) = f(\sigma^{m+1}, 1, \dots, 1),$$

where $\lambda = f(\sigma, 1, \dots, 1)/f(1, 1, \dots, 1)$. By induction on m , we get for all positive and negative integers m ,

$$(15'') \quad f(\sigma^m, 1, \dots, 1) = \lambda^m f(1, 1, \dots, 1).$$

¹⁴ See Bridgman [38], p. 21. Equation (15) evidently states that *ratios* of homogeneous quantities are invariant under changes of the fundamental units. The assumption that the q_i be positive, though not stated, is also required in Bridgman's proof, since he forms $f dq_i/q_i$. And indeed, negative α , usually have no physical meaning. The extension from differentiable to continuous functions is due to A. Martinot-Lagarde, "Sur l'application des variables sans dimension aux phénomènes discontinus," *Comptes Rendus*, 223 (1946), 136-137.

Hence, setting $q_1 = 2^{m/n}$, $\sigma = \sqrt[n]{2}$, and $f(1, \dots, 1) = C$, we get $f(q_1, 1, \dots, 1) = C\lambda^m$ and $f(2, 1, \dots, 1) = C\lambda^n$. Defining $a = n \log_2 [f(2, 1, \dots, 1)/f(1, 1, \dots, 1)] = n \log_2 \lambda$, so that $\lambda^m = 2^{ma/n}$, we get by substitution,

$$(16) \quad f(2^{m/n}, 1, \dots, 1) = C2^{ma/n} = C(2^{m/n})^a.$$

But the powers of 2 with rational exponents are an everywhere dense set of positive real numbers. Hence, if f is continuous—in fact, unless f is non-measurable and everywhere discontinuous,¹⁵ $f(q_1, 1, \dots, 1) = Cq_1^a$ holds for all positive q_1 .

Repeating the argument for the other indices, we get the conclusion of the theorem.

7. Validity of Assumptions I and IV

The validity of Assumptions I and IV has been hotly debated. The self-evident principle that “all units of measurement¹⁶ are equally valid” is often loosely interpreted to imply that all such units must give rise to the same universal physical laws.

That this implication is not logically necessary is easily seen if one recalls that, in a sense, all coordinate systems for space-time are equally possible. But a geocentric system, such as used in Ptolemaic astronomy, does not give rise to the same physical laws as a heliocentric system.

Moreover the interpretation is not true, even for the units of length, mass, and time in mechanics.¹⁷ Indeed, the main

¹⁵ For everywhere discontinuous counterexamples, see G. Hamel, *Math. Annalen*, 60 (1905), 459–462. By a logarithmic transformation¹ (15) reduces to a well-known additive functional equation. It is easy to show that, by (16), if f is discontinuous at one point, it must be everywhere discontinuous. For the nonmeasurability of f , see S. Banach, *Théorie des opérations linéaires*, Warsaw, 1933, p. 23.

¹⁶ For a discussion of the meaning of a scale of measurement, see Norman Campbell, *Measurement and Calculation*, 1928. Strictly, one should distinguish between enumerative scales, ordinal scales, scales in which equality of differences can be observed, and true linear scales with zero.

¹⁷ One can, of course, rescue Assumption IV by the sophistry of defining a “fundamental unit” in a theory to be one for which Assumption IV holds; this may even be useful.

point of special relativity is that the laws of mechanics, which include the basic force law

$$(17) \quad F = \frac{d(m_0 u / \sqrt{1 - u^2/c^2})}{dt},$$

are *not* unit-free for length and time independently, but involve the speed of light.

There are many universal dimensional constants other than c . Thus such constants are involved in Newton's universal law of gravitational attraction,¹⁸ $F = \gamma mm'/r^2$; in Planck's quantum law $E = h\nu$, in the diameter of every ultimate particle, etc. We are thus impelled irresistibly to the conclusion that *there are no known "fundamental units," with respect to which all known physical laws are unit-free.*¹⁹

Indeed, the decision to call certain units "fundamental" (or primary) and others "derived" (or secondary) is one of convention, and not of physical necessity. Thus it is sometimes convenient to treat force as independent of mass, length, and time.²⁰

However, the preceding statements are subject to one reservation. The universality of "universal constants" may not be absolute. Thus before Newton's discovery of gravitation, g must have seemed a universal constant (this is still inherent in the use of engineering units). In electromagnetic theory (though not in relativistic mechanics), c is not usually

¹⁸ In fact, according to Eddington (*Lond. Phys. Soc.*, 1918, p. 91), Nature practically dictates choosing units of length, time and mass in a unique fashion, so that $\gamma = c = h = 1$. The limited laws of "gravitational similitude" have been discussed by Labocetta, *Elettrotecnica*, 1932, p. 1629.

¹⁹ For further examples, see Bridgman [38], p. 103. In view of these examples, Tolman's widely quoted statement that "the fundamental entities of which the physical universe is constructed are such that from them a miniature universe could be constructed exactly similar . . . to the present universe," is clearly false. We would have to alter the values of γ , c , or h . Of course, the idea is very old; note Swift's Lilliput and Brobdingnag!

²⁰ See Bridgman [38], p. 65. In thermodynamics, we have the remarkable paradox of Riabouchinsky (*ibid.*, p. 10); also the Stefan-Boltzmann radiation law $E/\text{Area} = KT^4$. The artificiality of the concept of a "fundamental unit" appears even more clearly with electromagnetic units; see J. Jeans, *Electricity and Magnetism*, Cambridge Univ. Press, 1941, pp. 14-15; also Brylinski, *Comptes Rendus*, 215 (1942), 104.

treated as a universal constant, but understood to depend on the dielectric constant and magnetic permeability of the substance under discussion. Hence it would seem premature to dismiss as without foundation efforts such as those of Tolman [53] and Eddington²¹ to deduce relations between universal constants from epistemological principles.

What is more relevant to hydrodynamics, I shall now show that *in the Newtonian mechanics of fluids, length, time, and mass can be taken as independent fundamental units*. And it seems almost certain that the laws of Newtonian mechanics very nearly determine the behavior of fluids under ordinary conditions.

8. Inspectional Analysis

In order to become convinced of the truth of the principle italicized above—which always underlies the validity of Assumption IV in fluid mechanics—one must resort to “inspectional analysis.” By this I mean, in general, testing for invariance under the transformation (1), every fundamental mathematical equation on which a given theory is based. This test was in fact used by Fourier, Stokes, etc.; it was clearly in the back of Rayleigh’s mind when he referred to “similitude”²²; and it has been used very often since. It occurs most commonly in connection with “dynamic similitude,” as now defined for fluids.

DEFINITION. Two fluid motions Φ and Φ' are called *dynamically similar* if they can be described by coordinate systems²³ which are related by transformations of space-

²¹ *Relativity Theory of Protons and Electrons*, Cambridge Univ. Press, 1935. See also Bridgman [38], Chapter VIII; and E. T. Whittaker, *Space and Spirit*, Edinburgh, 1946.

²² See Rayleigh, *Phil. Mag.*, 34 (1892), 52, and 8 (1905), 66; also *Nature*, 95 (1915), 66. Similarly Bridgman ([38], p. 17) writes “The advantage of (dimensional analysis) is that it is rapid, . . . but . . . it does not give as complete information as might be obtained by . . . a detailed analysis”; also “dimensional analysis physically is not so instructive as the similarity condition.” See also L. Schiffer, *Zeits. ang. Math. Mech.*, 24 (1944), 289–293.

²³ Actually, we mean “Newtonian” coordinate systems, in which Newton’s laws of motion are valid. Since the assumption that these exist begs the question of the possibility of dynamic similitude, there is some circularity in this definition.

time-mass, of the form

$$(18) \quad x'_i = \alpha x_i, \quad t' = \beta t, \quad m' = \gamma m.$$

We can easily test the equations of a hydrodynamical theory for invariance under (18). For example, consider the Navier-Stokes equations of motion for an incompressible fluid, expressed in formula (3*) of p. 7. These may be written in modified notation as

$$(19) \quad \frac{Du_i}{Dt} = \nu \nabla^2 u_i + g_i - \frac{\partial p}{\rho \partial x_i}.$$

Here $D/Dt = \partial/\partial t + \Sigma u_k \partial/\partial x_k$, so that Du/Dt is ordinary acceleration; $\mathbf{g}$ is the gravitational field; and $\nu = \mu/\rho$ is the "kinematic viscosity." Clearly (19) is "dimensionally homogeneous," of dimensions LT^{-2} ; hence it is invariant under (18), provided the constants $\mathbf{g}$, ν , and ρ , and the variable p , are transformed appropriately.

9. Relation to Groups

It would be a mistake to suppose that the applications of "inspectional analysis" were limited to transformations of the form (18). Actually, *inspectional analysis is applicable to any group of transformations*.²⁴ By a group of transformations, I mean of course a set of transformations which contains the identity, the inverse of any one, and the product of any two of its members.

This assertion is based on the logical axiom, discussed in Chapter I, that *if the hypotheses of a theory are invariant under a group G, then so are its conclusions*.²⁵ Conversely, *the set of all one-one transformations leaving any set of equations invariant forms a group*.

After the group of transformations of the form (18), the most important group for mechanics is the ten-parameter

²⁴ This point was made by Mrs. Ehrenfest [46], p. 261; but as she gave no applications, it has been generally ignored. For elementary facts about groups, see [37], Chap. VI; for orthogonal matrices, see Chap. VIII.

²⁵ This principle, though widely assumed by mathematicians and physicists, is seldom explicitly formulated. See however G. Bouligand, *Théorie générale des groupes*, Paris, 1935, p. 3.

Galilei-Newton group. This group is generated by the three-parameter subgroup S of space-translations

$$(20) \quad x'_i = x_i + c_i \quad [i = 1, 2, 3],$$

the one-parameter subgroup T of time-translations

$$(21) \quad t' = t + c,$$

the three-parameter subgroup R of rigid rotations

$$(22) \quad x'_i = \sum a_{ik} x_k,$$

where $||a_{ik}||$ is the most general 3×3 orthogonal matrix, and the three-parameter subgroup M of changes to *moving axes* being translated with constant velocity

$$(23) \quad x'_i = x_i - b_i t.$$

Now it is easy to verify that Newton's three laws of motion are invariant under (20)–(23), and that these transformations leave unchanged the definitions of such material constants as density, viscosity, etc. (mass being assumed the same). Hence, theoretically, *Newtonian mechanics is invariant under the Galilei-Newton group*, as well as under the group of transformations (18) of dynamic similitude. Experimentally, this principle has been verified in many different ways, with very great precision, except at speeds comparable with that of light.

10. Theory of Modeling, I

I shall now discuss critically justifications for model experiments by *a priori* considerations. I shall begin by recapitulating methods already in the literature.

The Pi Theorem is a mathematical theorem; hence (I say this for the benefit of physicists and engineers) it has absolute validity. This suggests that dimensional analysis is the proper method for justifying the use of models. The method, though the one most commonly appealed to, is however inadequate for two reasons.²⁶ In the first place, although

²⁶ The inadequacies of dimensional analysis have also been exposed by Norman Campbell [43], A. E. Ruark [52], Mrs. Ehrenfest [45], [46], and many others.

Assumptions I, II, and IV seem to be sufficiently valid in fluid mechanics (see Section 9), dimensional analysis provides no basis for deciding a priori when variables $Q_1, \dots, Q_n$ do determine Q_0 with sufficient accuracy. Thus Bridgman²⁷ says flatly that this crucial question (Assumption III) "cannot be decided by the philosopher in his armchair," but only on the basis of considerable physical experience. I shall give many experimental instances of this difficulty (in Sections 12-18); others are given by the "paradoxes of approximation" discussed in Chapters I and II. They are related to the mathematical questions raised in Section 11.

The second limitation of dimensional analysis is that "transformations of similitude" corresponding to changes of units do not have enough mathematical generality to cover useful methods of modeling. One wants (for example) more general affine and conformal transformations as well.

Inspectional analysis would seem to resolve both of these difficulties. One writes down the differential equations on which theoretical fluid mechanics is based, and simply tests them for invariance under all transformations in which one is interested. I shall now give an example which illustrates both inspectional analysis, and the care with which it must be applied.

Consider the Navier-Stokes equations (19) for an incompressible non-viscous fluid. Following Ruark [52], they can be reduced to dimensionless form as follows.

Let V , L , and P be respectively a characteristic velocity, length, and pressure in the model, presumably controlled on the boundary of the fluid flow. If we multiply (19) by L/V^2 to make it dimensionless, and introduce the dimensionless variables $u'_i = u_i/V$, $t' = Vt/L$, $x'_i = x_i/L$, $p' = p/P$, and the dimensionless constants $R = VL/\nu$, $F = V^2/LG$, and $Q^* = P/\rho V^2$, we get

$$(19') \quad \frac{Du'_i}{Dt'} = \frac{1}{R} \nabla'^2 u'_i + \frac{1}{F} \left(\frac{g_i}{G} \right) - \frac{Q^*}{\partial x'_i} \frac{\partial p'}{\partial x'_i},$$

where g_i/G are the direction cosines of gravity.

²⁷ [38], pp. 13-14; see also p. 50.

There is a remarkable correlation between the dimensionless differential equation (19') and engineering experience; we can derive from it three of the most important rules of thumb involved in model work.²⁸ Thus it suggests that if gravity, compressibility, and cavitation are unimportant, one should use models at the same Reynolds number R . If compressibility, cavitation, and viscosity are unimportant, one should use models at the same Froude number F . If compressibility and viscosity are unimportant, but gravity and cavitation effects are involved, one should preserve both F and the "cavitation number" Q^* (see Section 16).

It also gives the plausible suggestion that quantities are unimportant precisely when corresponding coefficients in (19') are small, and so appears to give a complete theoretical basis for model work based on F , R , and Q^* .

11. *Theory of Modeling, II*

Although the argument just given is suggestive and worth remembering, it suffers from the defect of considering only one of the three fundamental equations of fluid mechanics, the equation of motion. Thus it ignores the "equation of continuity"

$$(24) \quad \frac{\partial \rho}{\partial t} + \sum \frac{\partial(\rho u_k)}{\partial x_k} = 0, \quad \text{or} \quad \text{div } \mathbf{u} + \frac{D(\log \rho)}{Dt} = 0,$$

and the "equation of state," which can be written

$$(25) \quad \rho = f(p).$$

For this reason, I shall call it "partial inspectional analysis," and shall designate the corresponding procedure, when a set of conditions completely determining a flow has been considered, "complete inspectional analysis."

One might suppose that the partial dimensional analysis of Section 10 was legitimate in incompressible fluid dynamics, for the following reason.

²⁸ See [12], Vol. 2, Chap. I, for a very similar derivation, but without use of differential equations so explicitly. The prescription is that "ratios of forces" must be preserved; Q^* is not discussed.

THEOREM 3. *The equation of continuity is invariant under all transformations of similitude; so is the equation of state $D\rho/Dt = 0$ of an incompressible, inexpandible fluid.*

PROOF. One can check the invariance of (24) and $D\rho/Dt = 0$ under (18) and (20)–(23), by straightforward inspectional analysis. Or one can observe the invariance under these transformations of the laws of mass and volume conservation which (24) and $D\rho/Dt = 0$ express.

However, Theorem 3 no longer applies if expansion occurs through cavitation. In this case, we have usually

$$(25.1) \quad \begin{array}{lll} \rho = \rho_0 & \text{if} & p > p_v, \quad \text{and} \\ \rho = \rho_v, & & p = p_v \quad \text{elsewhere.} \end{array}$$

Since $p_v > 0$, clearly (25.1) is not invariant under²⁹ (18). Hence the inferences of Section 10 regarding Q^* are theoretically unsound; I shall give a correct discussion later (Section 16).

The suggestion that $1/F$ and $1/R$ can be neglected in (19') precisely when they are small, will be discussed in Section 12. But first I should like to discuss generally the completeness of an inspectional analysis based on (19') and (24)–(25) alone: i.e. the question of whether these equations (with boundary conditions) completely determine flows.

It is natural to suppose that they do, because *they express the time-derivatives of the variables ρ, p, u_1, u_2, u_3 in terms of their space-derivatives.* (Determinism Principle.)

This is most evident in the case of compressible non-viscous flow. One writes the equation of state (25) in the usual form $p = F(\rho)$, and thus eliminates p in the equations of motion³⁰ (19). Then (19) and (24) evidently express the time-derivatives of ρ, u_1, u_2, u_3 in terms of their space-derivatives.

The case of incompressible flow is more complicated—really by the theoretical possibility of “action at a distance.” The

²⁹ From now on I shall not mention (20)–(23), as invariance under them is true of Newtonian mechanics generally. But it should be noted that the use of fixed models in a moving stream to simulate moving models in a stationary fluid is based directly on (23), which is not of the form discussed in *ordinary* dimensional analysis.

³⁰ With non-viscous flow, $\nu = 0$. In the case of compressible viscous flow, the validity of (19) is questionable.

time-derivatives at a point are *not* determined by local space-derivatives, but are influenced by the movements of rigid bodies everywhere in space.

One rescues an appearance of determinism as follows. One can ignore the variation of ρ , and (24) reduces to $\text{div } \mathbf{u} = 0$, i.e. to the condition that u is solenoidal. One introduces the kinematic *vorticity vector* $\xi = \text{curl } \mathbf{u}$. After some calculation, one gets from (19) and (24) the vector differential equation

$$(26) \quad \frac{D\xi}{Dt} = \nu \nabla^2 \xi + \frac{1}{\rho} (\xi \cdot \nabla) \mathbf{u} \quad \text{if} \quad \mathbf{g} = \nabla G.$$

This expresses the time-derivatives of vorticity in terms of local space-derivatives, and shows that vorticity (unlike velocity) is diffused, and *not* instantaneously communicated at a distance. Finally, one remarks that the vorticity determines the velocity up to an irrotational solenoidal velocity-field (Dirichlet flow),³¹ and this is determined from the normal derivatives at the boundaries, by potential theory. This completes the argument.

To a physicist, the preceding argument may seem to be mathematical perfectionism, or at least so abstruse that he has no interest in pursuing the matter further. But this is not so; it should be obvious to any twentieth century mathematician that the discussion of boundary conditions is vague.

Thus the "Determinism Principle" is really a very weak uniqueness theorem, projecting conditions at $t = 0$ infinitesimally into the future. If all functions are assumed to be analytic, a local projection into the *near* future is possible by the Cauchy-Kowalewski Theorem; but no mathematical theorem is known which justifies projection into the indefinite future (see Section 18 for physical evidence corroborating this mathematical argument).

Moreover, in most modeling problems, it is boundary conditions in space that are modeled, not at $t = 0$; for these boundary conditions, and solutions in the large, the Cauchy-

³¹ In plane flows, $(\xi \cdot \nabla) = 0$; moreover we can obtain a solenoidal field with suitably bounded scalar vorticity $\xi(x,y)$ explicitly, by the stream-function $V = \iint \xi \log r dx dy$.

Kowalewski Theorem is useless. No general uniqueness theorem appropriate to the situation is known, and it looks as if none is true. There are only special theorems, which depend on whether the differential equations are of elliptic, hyperbolic, or parabolic type.³²

Thus for a complete inspectional analysis of hydrodynamic modeling, fundamentally new theorems about partial differential equations are needed. I have already expressed my opinion that these theorems will be inspired by a familiarity with the experimental facts. In this connection, I note that the fundamental classification of partial differential equations into elliptic, hyperbolic, and parabolic types, already mentioned, was suggested by applications.

The rest of this chapter will therefore be devoted to summarizing the experimental facts concerning various types of models, in their relation to mathematical theory.

12. Inertial Modeling

In practice, considerations of experimental convenience and economy often dictate not only the particular fluid (such as air or water) used, but also the *size* of the models and the *speed* of flow. In such cases, one has to assume invariance of the behavior under all transformations (18), or else to work out a set of approximate corrections for "scale effects." I shall call such use of (18) for all α , β , γ , "inertial scaling."

In the early days of the airplane, all airfoil tests were made by inertial scaling;³³ moreover the quasi-empirical results so obtained were far more reliable than the most convincing theoretical formulas (of Rayleigh, such a formula as (6') of Chapter II) then known. Today we know the pitfalls of inertial scaling, and I shall now try to give an objective and

³² For such theorems, see R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, Vol. 2. For results relating specifically to hydrodynamics, see L. Lichtenstein, *Math. Zeits.*, 26 (1927), 196-323; K. Maruhn, *ibid.*, 45 (1939), 155-184; J. Leray, *Acta Math.*, 63 (1935), 193-248.

³³ See G. Eiffel, *La résistance de l'air et l'aviation*, translated into English by J. Hunsaker, Boston, 1914. Tests based on inertial scaling are still widely used in airplane design.

up-to-date account of some theoretical and experimental facts about it.

Many hydrodynamical theories, contradictory in other respects, agree in being invariant under inertial scaling. Thus this is true of Newton's primitive kinetic theory of air resistance, and of Euler's quasi-empirical formula expressing drag and lift as definite integrals.³⁴ It is also true of the theories of Dirichlet and Joukowski flows;³⁵ of two-dimensional airfoil theory; of the Helmholtz-Kirchhoff theory of wakes and jets (Chapter II); and of Karman's theory of vortex streets (alternating vortices). I shall omit the "inspectional analysis" required to prove these statements.

It is also true of Oseen's theory,³⁶ which purports to take account of the *asymptotic* effect of viscosity, as $R \rightarrow \infty$. Further, it is true of Stokes' theory of "creeping motion," which is a limiting case as $R \rightarrow 0$; since the terms in Du_x/Dt and g_x are neglected here, the argument of Example 3 above applies to this case.

I have already discussed (Chapter I, Sections 14 and 18) the inadequacies of these asymptotic theories to explain observed phenomena at $1/R = .02, .001$, and $.00005$ such as alternating vortices, turbulence, and "transition" of the boundary layer; I have also pointed out how clearly they show the need for fundamental new developments in the theory of partial differential equations. By the same token, *they show the insufficiency of inspectional analysis to justify modeling, until such developments have been made.*

Nevertheless, the plausible idea that inertial scaling may be used with large R has some validity. There are extensive "regimes," corresponding to large ranges of R , for which variation of the Reynolds number gives experimental results which essentially follow inertial scaling. Thus this is true

³⁴ These theories are discussed at great length in P. Painlevé, *Leçons sur la résistance des fluides*, Paris, 1930; and in C. Cranz [3], Chapter II. Until very recent times (1935), no better means for calculating bullet resistance were known.

³⁵ Dirichlet flows and two-dimensional airfoil theory have been discussed in Chapter I, Sections 3, 5. As these two theories are reversible, ν need not even be positive in (18).

³⁶ For Oseen's theory, see [11]. For Stokes' theory, see [11], or [12], Vol. 1, pp. 261-264.

of alternating vortices,³⁷ of jets and cavities subject to the limitations described in Chapter II, of friction in pipes of given roughness, of the resistance of bullets and bombs at speeds up to $M = .4$, and so on. It is also valid, if $1/R$ is small, in measurements of virtual mass for acceleration from rest (see Chapter V); in this single case, the boundary conditions for the Cauchy-Kowalewski Theorem apply.

It does seem possible to neglect gravity if F is very large. One can also often neglect gravity for arbitrary F , because of invariance of the basic equations under transformations quite other³⁸ than (18).

THEOREM 4. *In a fluid of constant density ρ_0 (hence without free surface), gravitational forces can be neglected, provided the hydrostatic buoyancy of submerged models is allowed for.*

PROOF. Let G be the gravitational potential; then one can reduce equations (19) and (24)–(25) to the case where $G = 0$ by the transformation

$$(27) \quad p' = p - \rho_0 G.$$

The hypothesis $\rho = \rho_0$ is of course not fulfilled in a compressible fluid; and (27) does not carry “free surfaces” (with $p = p_0$) into free surfaces, unless they happen to be horizontal (i.e. gravitational equipotentials).

Theorem 4 shows again the inadequacy of the partial inspectional analysis of Section 10.

13. Reynolds Modeling

Inspectional analysis based on Theorems 3 and 4 suggests that the dynamics of an incompressible viscous fluid bounded by fixed walls should be invariant under all changes of scale

³⁷ See P. Dupin and M. Teissié-Solier, *Les tourbillons alternés*, Paris, 1928. For flow in pipes, see the curves of Nikuradse (V.D.I. *Forschungsheft*, 361, 35b) reproduced in Fig. 117 of H. Rouse, [14]. For the drag of a plate, see [14], p. 227. The case of a sphere is more doubtful. Much experimental evidence for the principle that inertial scaling is possible at sufficiently high R is given in C. Camichel and L. Escande, [42].

³⁸ See Th. von Karman, *Jour. Aer. Sci.*, 8 (1941), 337; also Avanzini, *Ist. Naz. Ital.*, tomo 1, part 1 (1807). To be invariant under translations $q, \rightarrow q, + \beta$, means that there is no zero-point in the scale of measurement. This is true of temperature in the theory of heat conduction.

which preserve the Reynolds number $R = VL/\nu$, in the notation of Section 10. In such "Reynolds models," pressure is transformed by

$$(27') \quad p' + \rho'_0 G' = \frac{\rho'_0 V'^2 (p + \rho_0 G)}{\rho_0 V^2} \quad \text{const.}$$

This principle has received impressive experimental confirmation, with the most diverse liquids and gases.³⁹ Thus (see Figure 17) the breakdown of Poiseuille flow in pipes comes at the same Reynolds number in air, water, and many other fluids. At Mach numbers below $M = 0.3$, drag coefficients of spheres and cylinders satisfy (5) with the same $K_D(R)$ in all fluids, and at all sizes and speeds. A corresponding result holds for the skin-friction of plates parallel to the wind.

In verifying these facts, two precautions must be taken; otherwise Reynolds modeling will not work. First, one must use models with similar *surface roughness*. The onset of turbulent flow, and transition in the boundary layer from laminar to turbulent flow, are greatly influenced by this factor. Thus near R_{crit} , the drag of a sphere can be greatly reduced by roughening it suitably.

Second, especially in wind-tunnels with closed circuits, the *turbulence of the free stream* must be the same.⁴⁰ It is found that the R_{crit} of spheres in wind-tunnels may vary by a factor of two, depending on the turbulence of the wind-tunnel. To get consistent results, it is usual to introduce a factor λ for each wind-tunnel, and to define an *effective* Reynolds number $R_{\text{eff}} = \lambda VD/\nu$, where λ is chosen to make $(R_{\text{eff}})_{\text{crit}}$ assume a certain standard value.

It is not easy to model large Reynolds numbers on a small scale. The trouble is that, if the fluid is given, reducing

³⁹ See T. E. Stanton and J. R. Pannell, *Phil. Trans. A214* (1914), 119–224, from which Figure 17 is adapted; also [50], pp. 16–17. The theory is due to Stokes, *Collected Papers*, Vol. 3, p. 17. Since the turbulent behavior of liquids is dynamically similar to that of gases, it seems hardly possible that turbulence should be related to kinetic theory except indirectly, through viscosity.

⁴⁰ Goldstein [6], p. 431. This explains the Paradox of DuBuat (*Principes d'hydraulique*, 1816 ed., Vol. 2, p. 194) whereby the force on a stick pulled with speed v through a still lake may exceed the force exerted on the stick when held still in a stream flowing past with the same speed. The moving stream is usually turbulent!

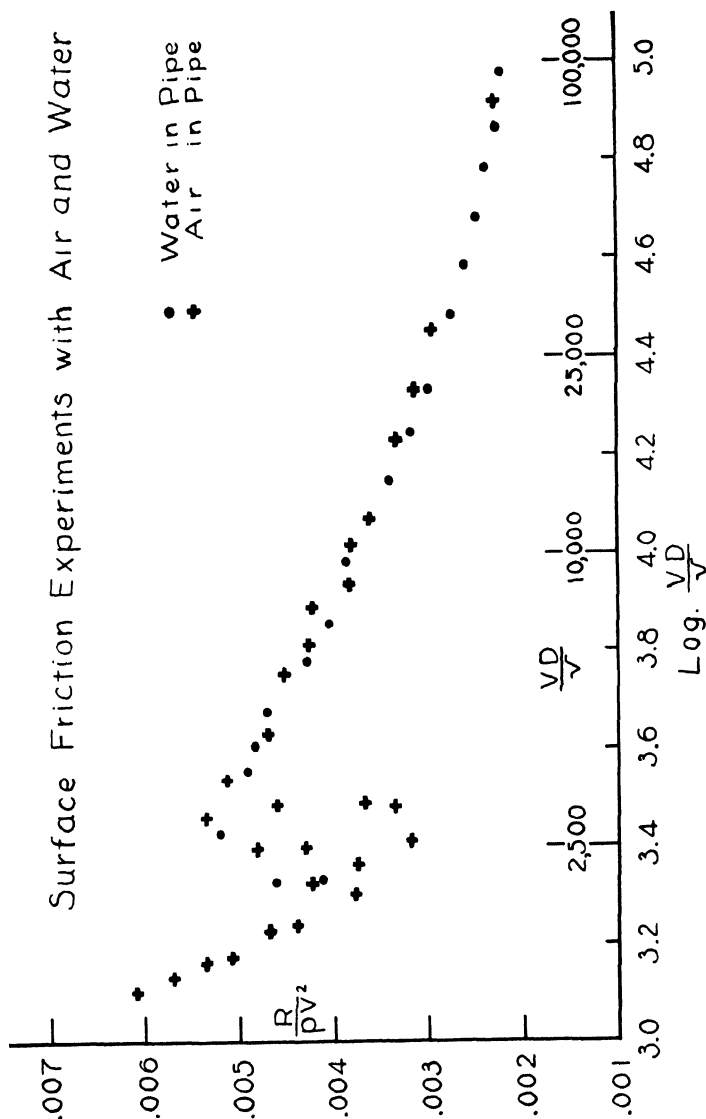


FIG. 17. Dimensionless plot of pipe friction, after Stanton and Pannell (courtesy of Royal Society of London.)

diameters requires increasing velocities in the same ratio. With air,⁴¹ such modeling is facilitated by "variable density wind-tunnels," using compressed air. Unfortunately, there seems to be no liquid whose ν is much less than that of water, though many have much greater ν . Hence wind-tunnels provide the only known economical Reynolds models of phenomena of water flow.

14. Froude Models

A "Froude model" is defined as one at the same Froude number $F = V^2/GL$. Moreover on the earth, G is sensibly constant, so that Froude modeling amounts to making $V \propto \sqrt{L}$: *the velocity should be proportional to the square root of the linear dimension*. Generally speaking (cf. [12], Vol. 2, p. 10), Froude models may be used when gravity forces and inertial forces are predominant. In practice this occurs most commonly in flows at fairly high speeds, which possess "free surfaces" exposed to the air.

One can justify Froude models by inspectional analysis as follows.

THEOREM 5. *Let viscosity be neglected, in a fluid of constant density ρ_0 , and let the gravity field be constant. Then equations (19) and (24)–(25) and the property of being a free surface are preserved under all transformations (18) which preserve F .*

PROOF. By Theorem 3, we need only consider the transformation of (19) under (18), ignoring the term in ν . (The neglect of ν is valid under just the same circumstances as in Section 12; that is, where the variation of R has little effect). We multiply (19) as in deriving (19') by L/V^2 , where V is a typical velocity, and get the dimensionless equation

$$(19'') \quad \frac{Du'_i}{Dt'} = \frac{1}{F} \left(\frac{g_i}{G} \right) - \frac{\partial \left(\frac{p}{\rho_0 V^2} \right)}{\partial x'_i}.$$

⁴¹ In gases, μ is almost independent of the density, at fixed temperatures (Maxwell's Law). Hence $1/\nu = \rho/\mu$ (and so R) is proportional to ρ . The ν of water *increases* substantially with temperature. For the use of wind-tunnels for water studies, see C. Keller, *Escher Wyss News* (1940).

Hence, assuming that there is only one pressure distribution compatible with the boundary conditions, we have dynamic similarity, with equal differences in the pressure coefficient $2p/\rho_0 V^2$ (not equal p/P) at corresponding points. Hence surfaces at *constant* pressure (coefficient) are also preserved by (18), completing the proof.

Froude modeling has proved important in turbine research,⁴² but is most generally known as a means of estimating the wave resistance of ships. It is also used in harbor models, for studies of the breaking of waves on beaches, and in studies of water entry (see Chapter II).

There is a long-standing controversy as to whether Reech or Froude should get the credit for the idea of "Froude modeling" in ship resistance model tests. As the facts are rather curious, I shall state them

Reech⁴³ certainly proposed, in 1831, exactly what is commonly referred to as "Froude's law," namely, testing ship models at equal Froude number, and estimating full-scale resistance by the similarity transformation (18). Froude's great merit consists in having gone beyond this simple law. In most merchant ships, ninety per cent of the resistance is frictional, and hence susceptible to Reynolds and not to Froude modeling. In order to estimate hull resistance from model tests, one must separate ship resistance into its frictional and wave components. This was first proposed in 1874 by Froude [48]. The exact model laws are still far from clear, and it is noteworthy that different rules for interpreting model results are used in the United States, England, France, and probably other countries as well.

Experience with Froude models quickly leads one to recognize other limitations. In harbor models, the interpretation of model results is complicated by capillary waves⁴⁴ (as

⁴² Its relevance was first mentioned in 1848 by Bertrand [36]; for a modern account, see G. F. Wislicenus, *Fluid Mechanics of Turbomachinery*, New York; 1946.

⁴³ F. Reech, *Cours de l'École d'Application du Génie Maritime*, Lorient, 1831.

⁴⁴ I have not discussed the modeling of capillary effects. To model these, one should preserve the "Weber number" $W = KT^2/\rho L^3 = K/\rho V^2 L$, where K is the surface tension (Bashforth and Adams, *Capillary*

regards erosion by breakers), and by the fact that the attenuation of waves by viscous effects does not correspond under Froude modeling. Again, in Froude models of dams, waterfalls, etc., the entrainment of air is also much less than in the full-scale.

15. River and Harbor Models; Nearly Horizontal Flows

Although inertia and gravity effects can be scaled by using Froude models of rivers, harbors, ships, etc., viscous effects cannot be scaled, for the reasons described at the end of Section 13. In models, viscous forces are relatively greater, and hence turbulence ("eddy viscosity") less. The usual cure is to *induce* turbulence in models, by artificially roughening surfaces or even by obstructing fluid motion with coarse wire meshes. This increases the eddy viscosity, so that the viscous forces in the model are even greater, relatively, than they would be otherwise.

However, at sufficiently high Reynolds numbers, this somewhat paradoxical procedure may be partially justified by inspectional analysis, as was pointed out to the author by Dr. S. K. Roy. The viscous forces are then much less than the "Reynolds stresses" $\rho \overline{u'_i u'_j}$, where u' denotes the turbulent vector velocity, and a bar denotes averaging (see [6], p. 192). Hence, if the relative turbulence is the same at all points, the mean velocity distribution may be expected to be similar.

What is perhaps more important, separation of flow tends to occur at the corresponding points, and suspensions are prevented from settling by the same mechanism, if turbulence is modeled. However, it should be emphasized that in practice an attempt is usually made to model only the intensity of turbulence, and certainly only the intensity and mean eddy size.

The transport of solid particles (dirt, sand, gravel) by moving water is frequently studied in models, because of its great practical importance for rivers, harbors, and estuaries.

Action, 1883). The poor modeling of air entrainment is discussed by L. Escande, *Génie Civile*, 16 Dec. 1939; and by C. Camichel and L. Escande [42].

The use of such models is a highly individualized art, which involves very complicated considerations.⁴⁵ Considerations of size make Reynolds modeling out of the question. Froude modeling roughly preserves the velocities due to gravity head, and (in harbor models) wave motions as well, but only in the turbulent régime or if viscosity is negligible.⁴⁶

The relative particle size is frequently exaggerated in the model, partly to avoid cohesive forces, partly to preserve R , and partly for ease of fabrication. This increased size tends to keep the model particles from being stirred up by the water, a tendency which is compensated⁴⁷ for by reducing their negative buoyancy $\rho_1 - \rho$.

The common use of *different horizontal and vertical scales* in such models, also deserves comment. In England it is usual to exaggerate the vertical scale (following Reynolds and Gibson) to avoid the tendency to excessive shallowness. The validity of this exaggeration is often disputed⁴⁸ in France, where there is tendency to use larger models instead. M. Craya [44] has justified exaggeration of the vertical scale, for the case of rivers with small curvature without waves; I shall present a justification which allows waves to exist, in the case of *nearly horizontal flow*. This case corresponds to a very common hydraulic approximation.

Thus consider the theory of shallow water waves. If the surface height is $h(x, y, t)$ above a horizontal bottom, and $\mathbf{v}(x, y, t)$ is the mean horizontal velocity, then one has the approximate equations

⁴⁵ A lengthy discussion of some complications may be found in *Proc. Am. Soc. Civ. Eng.*, 71 (1944), Trans. No. 3, Part 2. In certain tidal models, the Coriolis force must be modeled by changing river curvatures; see [39], p. 78.

⁴⁶ One can exaggerate the tilt of the bottom of the river or estuary to obtain appropriate mean velocities of flow when R is small. Roughness is an important factor.

⁴⁷ See C. Camichel and L. Escande, *Comptes Rendus*, 199 (1934), 992. An excellent review of scaling laws for silt transport is given by H. A. Einstein and R. Muller, "Über die Ähnlichkeit bei flussbaulichen Modell-versuchen," *Schweiz. Archiv*, 5 (1939), No. 8.

⁴⁸ C. Camichel, E. Fischer, and L. Escande, *Comptes Rendus*, 199 (1934), 594. The experimental counterexample given by them is not one of nearly horizontal flow.

$$(28) \quad \frac{\partial h}{\partial t} = \operatorname{div} (h\mathbf{v}), \quad \text{and} \quad \frac{D\mathbf{v}}{Dt} = g \operatorname{grad} h.$$

If one defines ρ as h , and p as $\frac{gh^2}{2} + \text{const.}$, then one has a well-known analogy with compressible flow, with $\gamma = 2$ in the equation of state. It is easily verified that these equations are preserved, not only under Froude modeling, but also under the change of horizontal scale with vertical scale fixed,

$$(29) \quad x' = \alpha x, \quad y' = \alpha y, \quad t' = \alpha t; \quad \mathbf{v}, g, h \text{ constant.}$$

A similar scaling law holds for any "nearly horizontal flow" of an incompressible non-viscous (!) fluid under gravity, defined mathematically as follows.

Let $\mathbf{u}(x, y, z, t; \lambda)$ be defined for $0 \leq \lambda \leq 1$, all x/λ , y/λ , t/λ within a region R , and z with $h_0(x/\lambda, y/\lambda, t/\lambda) \leq z \leq h_1(x/\lambda, y/\lambda, t/\lambda, \lambda)$. Let $z = h_0(x/\lambda, y/\lambda, t/\lambda)$ be a fixed wall, and $z = h$, a free surface under gravity and inertia. Then if $\mathbf{u}(x/\lambda, y/\lambda, z; t) = \mathbf{F}(x, y, z, t)$ approaches a limit as $\lambda \rightarrow 0$, we shall say that $\mathbf{F}$ defines a "nearly horizontal flow." For such $\mathbf{F}$, scaling by (29) is valid.

Scaling by (29) is clearly not a special case of (18). Other examples where different scales are used in perpendicular directions will be given later⁴⁹ (Section 19; Chapter IV, Section 7).

16. Modeling of Cavitation

It is important to avoid cavitation in the use of ship propellers and of hydraulic turbines. Therefore it is only natural that model studies should have been devised to reproduce this phenomenon to scale.

Until twenty years ago, cavitation was generally stated to depend on the homogeneous dimensionless parameter $Q^* = P/1/2\rho V^2$ of Section 10, which is directly suggested by ordinary dimensional analysis.⁵⁰ The incorrectness of this view (and

⁴⁹ Algebraically, these examples can also be considered from the point of view of dimensional analysis; see W. Williams, *Phil. Mag.*, 34 (1892), 234–271.

⁵⁰ See F. Lorain, *L'hélice propulsive*, Paris, 1923, p. 129; E. Buckingham, *Jour. Am. Soc. Naval Eng.*, 48 (1936), 147–198; D. W. Taylor, *The Speed and Power of Ships*, 3d ed. (1943), p. 17.

hence of the partial inspectional analysis of Section 10 is immediately seen, if the role of the vapor pressure p_v is considered. This was implicitly observed in 1924 by Thoma,⁵¹ and the "cavitation number" $Q = (P - p_v)/\frac{1}{2}\rho V^2$ is now generally used in place of Q^* .

It is noteworthy that the use of Q can be justified theoretically, by an inspectional analysis which neglects only viscosity. For it is evident that a transformation of similitude (18) preserves the equation of state (25.1) for water if and only if

$$(30) \quad (P' - p_v) = \frac{\alpha^2 \beta^2 (P - p_v)}{\gamma},$$

That is, if and only if it preserves Q . There is no linear homogeneous transformation of space, time, and pressure which will do. This shows, not only that full inspectional analysis is to be preferred to dimensional analysis if modeling is seriously desired, but also that more general groups than the homogeneous linear "changes of scale" contemplated by dimensional analysis may be important.

Thus, in a Froude model at $1:\alpha$ linear scale, with velocities in the ratio $1:\sqrt{\alpha}$, we can preserve both gravity and cavitation effects (but not viscosity), by transforming $(P - p_v)$ in the ratio $1:\alpha$. This "reduced pressure Froude modeling" is now commonly used in model studies of ship propellers; in the model, vapor pressure may be far from negligible.

One caution should be observed in the use of cavitation modeling. Water which has had all gas taken out of solution is capable of withstanding very large tensions.⁵² Hence cavi-

⁵¹ D. Thoma, "Experimental research in the field of water power," *Trans. First World Power Conf.*, Vol. 2 (1924), 536-551; see also H. B. Taylor and L. F. Moody, *Mech. Engineering*, 44 (1922), 633-640. An explicit statement was given by H. Lerbs, on p. 290 of *Hydromechanische Probleme des Schiffsantriebs*, Hamburg; see also H. E. Rossell and L. B. Chapman, *Principles of Naval Architecture*, New York, 1947, Vol. 2, p. 177.

⁵² This importance of air content was observed by Spannhake, *V.D.I. Special Issue* (1933). Systematic measurements have been made by Numachi, *Rep. Tohoku Imp. Univ.*, 13 (1939), No. 1. Vusovic (*Escher Wyss News*, 13 (1940), 83-90), showed that the increase in cavitation with air-content did not either increase erosion or decrease turbine efficiency.

tation modeling may be considerably affected by reducing the air-content of the water used; increased air content has less importance.

17. Modeling for Entry into Water

For underwater ballistics, it may be important to model the phenomena of "surface seal" and "down-nosing" which accompany entry into water, as stated in Chapter II, Section 10. This leaves us with the problem of trying to reproduce these effects to scale.

On the basis of experimental analogy, one might be tempted simply to use reduced pressure Froude modeling, preserving F and Q or Q^* ; this has indeed been proposed. However, I am glad to state that, for once, the correct solution⁵³ as regards surface seal appears to have been given, not by an engineer on the basis of physical experience, but by a "mathematician in his armchair," using inspectional analysis—namely, myself [18]. The answer was suggested by the following considerations.

A little reflection makes it plausible that it is the *air density* which causes surface seal—there being a reduction of $\frac{1}{2}\rho'v'^2$ (ρ' = air density, v' = air velocity) in the pressure at the neck of the cavity, which causes the splash and neck to contract. This is not scaled if ρ' is reduced by lowering pressure; while if one does not lower the pressure, presumably the size of the bubble after seal is not modeled.

Since one is not dealing with water vapor, the equation of state is no longer (25.1), but is nearly

$$(31) \quad p' = k\rho'\gamma.$$

The same equation is satisfied (approximately) by underwater gas bubbles; unless we are dealing with saturated gas, we do not have $p' = p_v + k\rho'\gamma$. For this reason, we should use Q^* instead of Q .

Mathematically, the Equations of Continuity, State, and Motion are preserved under the transformation

⁵³ For the experimental data, see Gilbarg and Anderson [24].

$$\begin{array}{llll} \mathbf{x}_1 = \alpha \mathbf{x}, & t_1 = \alpha t, & \mathbf{u}_1 = \sqrt{\alpha} \mathbf{u}, & p_1 = \alpha p, \\ p'_1 = \alpha p', & \rho_1 = \rho, & \rho'_1 = \rho', & k_1 = \alpha k, \end{array}$$

where suffixes denote transformed variables. Hence we could obtain a model if we could get a gas whose equation of state had the form $p' = \alpha k \rho'^{1.408}$, where $p' = k \rho'^{1.408}$ was that for air. For example, air at a low temperature would provide a good solution; but this is not practical from an engineering point of view.

It appears to be more practical to use Freon, or some other "heavy gas" with $\gamma \neq 1.4$, but having a density several times that of air at reduced pressure and atmospheric temperature; this has been done on a small scale (cf. [24]). One can then only model ρ and $dp/d\rho$ in the gas.

The modeling procedure outlined above presumably does not model "down-nosing." If down-nosing is a viscous effect, as suggested in Chapter II, it can only be modeled by preserving R , and this is impractical. However, since the maximum possible underpressure decreases to zero with Q^* (assuming tension $p < 0$ impossible for the durations involved), down-nosing should at least be mitigated by reduced pressure Froude modeling, with or without a heavy gas.

The need for a more careful analysis than that provided by ordinary dimensional analysis is also illustrated by the dimensionless parameter $N = \sqrt{F} \rho'/\rho$, which has recently⁶⁴ been shown to determine roughly whether entry into water is accompanied by surface seal or deep seal. The derivation of N as a criterion rests on the empirical fact that the time T_{ds} required for deep seal is roughly proportional to $\sqrt{L/g}$, combined with the fact (deducible from inspectional analysis of the inertial mechanism of surface seal) that the time T_{ss} of surface seal is proportional to $\rho L/\rho' V$, where L is a typical length and V a typical velocity. The condition $T_{ds} > T_{ss}$ for surface seal therefore assumes the form $N > N_{crit}$. Using ordinarily dimensional analysis, one would probably have

⁶⁴ G. Birkhoff and R. Isaacs, "The mechanism of cavity motion," to be published as a Bureau of Ordnance Report. For deep and surface seal, see Chapter II, Section 10, *supra*.

argued that the average pressure differences causing surface seal and deep seal, respectively, should be proportional to $\frac{1}{2}\rho'V^2$ and $2\rho gL$, suggesting the use of the dimensionless ratio $N' = F\rho'/\rho$ as the criterion for surface seal. This disagrees strikingly with observation.

18. Mach Modeling

It has been known since the early experiments of Robin⁵⁵ (1742), that the drag of bullets is not proportional to the velocity squared; hence no form of inertial scaling is valid. In the notation of Example 2, K_D has a notable rise near the speed of sound. As a result, K_D has been commonly tabulated as a function of v .

It was recognized from the beginning, that this was due to the compression of the air, but the more rational tabulation of K_D as a function of the Mach number M dates since World War I. It follows logically that there should be a temperature effect on the range of projectiles, in addition to the effect of density already obvious from the definition of K_D . However, this was apparently first stated explicitly after the first World War.⁵⁶

Beginning about 1935, with the advent of high-speed planes, the aerodynamicists have become interested in Mach modeling. Whereas wind-tunnels operating at 100 ft/sec can be used to simulate flight conditions up to 400 ft/sec, provided the "effective" Reynolds number is properly controlled, they give no idea at all of the compressibility effects which arise at higher speeds. Since 1935, therefore, the aerodynamicists and ballisticians have joined forces in the study of compressible flows.

The laws of Mach scaling were first derived by Langevin,⁵⁷

⁵⁵ See Cranz [3], pp. 44-45, for an historical account.

⁵⁶ G. Darrieus, *Mem. Art. Francaise* (1922), p. 242; H. W. Hilliar, *Dept. Sci. Res. Exp. Report RE 142/19* (1919); Darrieus states that the variation in range due to this cause can be as much as 1%. The article by Langevin cited below comes immediately after it in the same journal. Mach modeling was justified, on the basis of dimensional analysis, by Buckingham [41], pp. 275-278. For practical aspects, see R. H. Kent, *Mech. Eng.*, Sept., 1932.

⁵⁷ See footnote 56.

from an "inspectional analysis" of the equations for a *compressible non-viscous* gas neglecting gravity, in the case of a polytropic equation of state. I shall give a slight generalization of his results.

Going back to Theorem 3, we see that all transformations of similitude preserve the equation of continuity. Obviously also, the equation of state is preserved under any transformation which preserves p and ρ at corresponding points. Hence, in particular, it is preserved under the transformations

$$(32) \quad x_i \rightarrow \alpha x_i, \quad t \rightarrow \alpha t; \quad p, \rho, u \text{ unchanged.}$$

By inspection, the terms $Du_i/Dt = \Sigma u_k \partial u_i / \partial x_k + \partial u_i / \partial t$ and $\partial p / \rho \partial x_i$ in (19) are both multiplied by $1/\alpha$ under (32). We conclude

THEOREM 6. *The fundamental equations for any compressible non-viscous fluid are invariant under (32).*

I recall the doubt, discussed in Section 11, regarding the completeness of these "fundamental equations." This doubt is confirmed by the fact that the important Rankine-Hugoniot conditions for *shock waves* cannot be derived from the "fundamental equations" of continuity, state, and motion.⁵⁸ However, since these conditions can be derived from the laws of conservation of mass, momentum, and energy, which are preserved (per unit volume) under (32), they are scaled also.

The scaling law (32) holds also in elasticity. In fact, whenever the *stress tensor* is a *function of strain only*, and not of the rate of strain, the scaling law (32) is valid. In fact, kinetic energy per unit volume and (potential) strain energy per unit volume are the same. Curiously enough, it also holds in *relativistic* fluid mechanics.

The partial inspectional analysis, described in Section 10, has been thought by some to give a basis for Mach scaling. Let $c = \sqrt{dp/d\rho}$ denote the local speed of sound in (19), and let C be the sound velocity of the free stream. Then if the forces of viscosity and gravity are neglected, and we write $M = V/C$, (19') becomes

⁵⁸ The great mathematician Riemann was misled by the apparent completeness of (19) and (24)–(25), to ignore this possibility.

$$(19'') \quad \frac{\partial u'_i}{\partial t'} + \sum \frac{u'_k \partial u'_i}{\partial x'_k} \frac{1}{M^2} \left(\frac{c}{C} \right)^2 \frac{\partial \rho}{\rho \partial x_i} = 0.$$

This suggests the plausible rule: *scale at the same Mach number*, which for a given free stream is equivalent to (32).

However, this rule is *false* in general, if we consider behavior in different gases (gases with different γ in (31)), or even the same gas under different temperatures and pressures. Compare for example dynamically similar adiabatic flows of a gas, in which the free stream conditions correspond to two points along the same adiabat. By (18), $\mathbf{v}$ and ρ are proportional. Hence by (19) (neglecting $\mathbf{g}$ and ν), $\text{grad } p$ is in a fixed proportion $a(\alpha)$, where α is the ratio of free stream densities. Hence if $F(\rho) = p(\rho) - p_f$ (p_f = free stream pressure), $F(\alpha\rho)/F(\rho) = a(\alpha)$ is independent of ρ . We thus have for all ρ, ρ', α , $F(\alpha\rho)/F(\rho) = F(\alpha\rho')/F(\rho')$. But this is clearly equivalent to (15) of Section 6. Hence by Theorem 2, $F(\rho) = k\rho^\gamma$, and so

THEOREM 7. *For models of compressible flows to be dynamically similar at the same Mach number under all free stream conditions, the equation of state must have the special form*

$$(33) \quad p = k\rho^\gamma + \text{const.}$$

It is sufficient that γ be the same along all adiabatic curves. Generalizations to non-adiabatic flow are obvious.

What about the effects of viscosity? We have already seen the danger of neglecting viscosity, in incompressible flow. In ballistics, skin friction is most commonly of the order of 10 per cent; this is obviously affected by viscosity and surface roughness. Otherwise, the Reynolds number does not seem to be of much importance in ballistics, though the contrary view has been widely disseminated.⁵⁹

However, the Reynolds number does influence various minor phenomena, such as the thickness of shock-waves, and

⁵⁹ The theoretical conclusions in the well-known book *The Mechanical Properties of Fluids*, Blackie and Co., 1935, are based on a single table on p. 37 of Cranz [3]. Nothing remotely resembling the data of this table has been found in the United States, with ordinary bullets (flat-based or boat-tailed).

the λ -shocks found by Ackeret.⁶⁰ "Condensation shocks," due to sudden condensation of water vapor, are also certainly determined by factors which certainly do not scale under (32), let alone Mach modeling in general.

19. Linearized Mach Modeling

In the case of slightly perturbed uniform flow, more general transformations of approximate (i.e. asymptotic) generalized dynamic similitude are possible. This case includes "thin wing" theory, and the theory of the resistance of "slender" bullets.⁶¹ I shall now derive the equations to be tested by inspectional analysis.

Consider steady irrotational flow (with negligibly "weak" shocks) of a compressible non-viscous fluid. Then the equations of motion are the space-derivatives of the Bernoulli equation, and so are satisfied if

$$(34) \quad \frac{dp}{\rho} + \sum u_k du_k = 0.$$

The equation of state $p = f(\rho)$ is equivalent to $dp = c^2 d\rho$, where $c = c(\rho)$ is the local speed of sound. Hence the equations of motion and state are both satisfied if $c(u)$ and $\rho(c)$ are defined from (34) by $d\rho/\rho + \sum u_k du_k / c^2 = 0$. This leaves the equation of continuity, which can be written

$$(35) \quad 0 = \frac{1}{\rho} \operatorname{div} (\rho \mathbf{u}) = \operatorname{div} \mathbf{u} + \sum u_i \frac{\partial \rho}{\rho \partial x_i}.$$

Substituting from the preceding equation, we get

$$(35') \quad \operatorname{div} \mathbf{u} - \frac{1}{c^2} \sum u_i u_k \frac{\partial u_k}{\partial x_i} = 0 \quad [c = c(u)],$$

as a single differential equation containing all the conditions for compressible flow.

⁶⁰ J. Ackeret, *Mitt. Inst. Aerodynamik Zurich*, No. 10 (1944).

⁶¹ For thin wing theory, see N. Rott, *Mitt. Inst. Aer. Zurich*, No. 9, 247-251; W. D. Hayes, *Quar. Appl. Math.*, 5 (1947), 105-106; H. S. Tsien, *M.I.T. Jour. Math. Phys.*, 25 (1946), 247-251. For the resistance of slender bullets, see Th. von Karman and Moore, *op. cit.*, Chapter I, Section 12.

Now we are ready for asymptotic considerations. We let

$$(36) \quad \mathbf{u} = (U, 0, 0) + \lambda \nabla \phi.$$

Here U is the velocity of uniform flow (*not* velocity potential) to be perturbed; λ is a parameter, ϕ is the normalized velocity potential of perturbation. Actually, ϕ depends on λ , but this does not appear in the notation. We also write $u = \partial \phi / \partial x$, $v = \partial \phi / \partial y$, $w = \partial \phi / \partial z$.

Since $\partial u_k / \partial x_i = \lambda \partial^2 \phi / \partial x_i \partial x_k$, clearly

$$(37) \quad \sum u_i u_k \frac{\partial u_k}{\partial x_i} = \lambda U^2 \frac{\partial^2 \phi}{\partial x^2} + 0(\lambda^2).$$

Again, since c^2 is a differentiable function of the velocity squared, which is $U^2 + 2\lambda U u + 0(\lambda^2)$, we have $c^2 = C^2 + 0(\lambda)$, where C is the velocity of sound in the free stream. Finally, by (36), $\text{div } \mathbf{u} = \lambda \Delta^2 \phi$, and so (35') becomes

$$(38) \quad \lambda \nabla^2 \phi = \left(\frac{1}{C^2} + 0(\lambda) \right) \left(\lambda U^2 \frac{\partial^2 \phi}{\partial x^2} + 0(\lambda^2) \right).$$

Dividing by λ and simplifying, this reduces to

$$(38') \quad \nabla^2 \phi = \frac{U^2}{C^2} \frac{\partial^2 \phi}{\partial x^2} + 0(\lambda)$$

Thus as $\lambda \rightarrow 0$, ϕ satisfies a partial differential equation which approaches

$$(39) \quad \nabla^2 \phi = M^2 \frac{\partial^2 \phi}{\partial x^2}.$$

In the *subsonic* case ($M < 1$), this is equivalent to $\nabla'^2 \phi = 0$, where $\nabla'^2 = \partial^2 / \partial x'^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$, and $x' = x / \sqrt{1 - M^2}$; hence it reduces under an *affine* transformation to the *incompressible* case. In the *supersonic* case ($M > 1$), we have a similar reduction to

$$(39') \quad \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{k^2} \frac{\partial^2 \phi}{\partial x^2} \quad [k = \sqrt{M^2 - 1}].$$

which is the *wave* equation. In either case, we get *affinely*

similar flow at corresponding Mach numbers around all affinely equivalent models. The sonic case must be considered independently.⁶²

Thus changing M is equivalent (at least in theory)⁶³ to changing the "thickness ratio," for affinely similar flows. Hence, besides the ordinary Mach modeling of (32), one can scale two perpendicular directions independently, much as in the theory of "nearly horizontal" flows.

⁶² See Th. von Karman, "The similarity law of transonic flow," *M.I.T. Jour. Math. Phys.*, 26 (1947), 182-190. I have here omitted a discussion of boundary conditions. Essentially, $\partial\phi/\partial\eta$ is given on a surface Σ , parallel to the x -axis, as U times the slope of the "thin solid," in a plane perpendicular to Σ and parallel to the x -axis.

⁶³ In view of Chapter I, Section 12, the theory may well be viewed with some skepticism. Thus, it involves no shock waves.

CHAPTER IV

Group Theory and Fluid Mechanics

1. Introduction

In Chapter III, I tried to show that the group concept is valuable in fluid mechanics, for three purposes. First, through “inspectional analysis” it provides a mathematical basis for the use of models, which is far more adequate than the “dimensional analysis” commonly appealed to. Second, it provides tests for the validity for mathematical theories of hydrodynamics, even in cases where it is impossible to integrate the partial differential equations involved in such theories. And finally, like dimensional analysis (but more generally), it often permits one to reduce the number of parameters which need to be considered; thus it provides important simplifications.

I shall now discuss its utility as a method for *integrating* the differential equations of fluid mechanics, and indeed of mathematical physics generally. Most of what I shall have to say in this connection has been said in one way or another somewhere in the mathematical literature. But if, as I believe, we have only begun to explore the applications of the group concept to differential equations, it seems worth collecting these ideas in one place.

I shall first describe what may be called the method of *search for symmetric solutions* of partial differential equations. Suppose that a system Σ of partial differential equations is invariant under a group G ; the method consists in searching for a solution which is also invariant under G .

This method has been applied so often to physical problems that it is surprising that it should not have been more explicitly recognized long ago.¹ I shall now illustrate its effectiveness by various specific applications.

¹ It seems to have been first formulated by K. Bechert, [54] and [55]. A more explicit formulation has been given by L. I. Sedov [66], and

2. *Symmetric Solutions of the Heat Equation*

As the method of "search for symmetric solutions" is not restricted to fluid mechanics, I take first an application to the heat equation.

Suppose we are interested in the diffusion of heat from a point, in a medium of constant thermal diffusivity K . The equation of heat conduction in solids is

$$(1) \quad \frac{\partial U}{\partial t} = K \nabla^2 U = K \sum_{i=1}^n \frac{\partial^2 U}{\partial x_i^2} \quad (n = 1, 2, \text{ or } 3);$$

here U is the temperature at the point $\mathbf{x} = (x_1, \dots, x_n)$ at time t .

Now since we have rotational symmetry, we look for a solution of the form $U(r, t)$, where $r^2 = \Sigma x_i^2$. This exhausts the purely *geometrical symmetry* of the problem. But we also have *physical symmetry*, in the sense that, for any positive number m , the differential equation (1) is invariant under the group of self-similitudes

$$(2) \quad r' = \alpha r, \quad t' = \alpha^2 t, \quad U' = \alpha^m U,$$

of thermal space-time. This symmetry extends the classical Law of Times. Hence we look for solutions $U(r, t)$ of (1) which are invariant under (2).

In the present case, since the group (2) consists of scalar multiplications, we can use the language of dimensional analysis. We have the two dimensionless variables $\chi = r^2/t$ and $V = U/t^{m/2}$. In any solution which is invariant under (2), by the Pi Theorem we have $V = F(\chi)$, and so

$$(3) \quad U = t^{m/2} F(\chi) \quad \left[\chi = \frac{r^2}{t} \right].$$

Our general principle, to be proved later, asserts that (1) will have solutions of the self-symmetric form (3).

Knowing that such solutions exist, it is easy to find them. We obtain the ordinary differential equation satisfied by

K. P. Staniukovitch [67]. I later, but independently, formulated the same method (cf. [56]).

$F(\chi)$, by the following series of direct computations, in dimensions.

$$\begin{aligned}\frac{\partial U}{\partial t} &= t^{\frac{1}{2}m-1} \left[\frac{1}{2} mF - \chi F' \right] \\ \nabla^2 U &= \frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r} \frac{\partial U}{\partial r} = t^{\frac{1}{2}m-1} [4\chi F'' + 2nF'].\end{aligned}$$

Equating and dividing out a suitable power of t , we get

$$(4) \quad 4K\chi F'' + (2Kn + \chi)F' - \frac{m}{2}F = 0$$

Since this is an ordinary differential equation, it can easily be integrated numerically, for any real m , and "initial values" of F and F' . Thus if $4K = 1$, (4) has Laguerre polynomials as solutions.

However, not all these "symmetrical" solutions have equal physical interest. Interest is largely confined to solutions for which $U \rightarrow 0$ as $r \rightarrow \infty$, so that $\lim_{\chi \rightarrow \infty} F(\chi) = 0$. We are also interested in the *total heat*, which is proportional to

$$\int_0^\infty U r^{n-1} dr, \text{ and hence to } \alpha^m \alpha^n = \alpha^{m+n}, \text{ or to } t^{(m+n)/2}.$$

One special case of interest is that of *constant* total heat, corresponding to diffusion of a fixed quantity of thermal energy, initially at the origin. Here $m = -n$; if we set $n = 2h$, (4) reduces after collecting terms to

$$(4.1) \quad \chi(4KF' + F)' + h(4KF' + F) = 0.$$

This can be easily integrated; if F and F' tend to zero as $r \rightarrow \infty$, then $4KF' + F = 0$ and $F(\chi) = e^{-\chi/4K} = e^{-r^2/4Kt}$, whence

$$(4.1') \quad U = Ct^{-n/2} e^{-r^2/4Kt}$$

Another interesting case is a *point source* producing heat (by chemical or radioactive process) at a constant rate, beginning at $t = 0$. Here $m + n = 2$, or $m = n - 2$, whence (4) becomes

$$(4.2) \quad 4K\chi F'' + (4Kh + \chi)F' + (1 - h)F = 0.$$

The integrals of this differential equation (obtained by other means) are expressed in closed form in Carslaw and Jaeger, *Conduction of Heat in Solids*, Section 104; the case $n = 2$ is especially obvious, as then $(1 - h) = 0$. Even if no such expression were known, they could be integrated numerically.

3. Prandtl-Meyer Expansion Wedges

Having shown that the method of "search for symmetric solutions" applies to physics in general, I shall now concentrate on some typical applications to fluid mechanics.

I shall begin with stationary flows of a compressible non-viscous fluid. The differential equations of such a fluid are invariant, as we have seen (Chapter III, Section 12), under the group of Mach modeling

$$(5) \quad x_i \rightarrow \alpha x_i, \quad t \rightarrow \alpha t, \quad p, \rho, u \text{ unchanged}.$$

The method of search for symmetric solutions suggests looking for flows which are invariant under (5); stationary flows will be invariant under the group

$$(5') \quad t \rightarrow t + \tau,$$

all other variables unchanged, as well.

Plane flows invariant under (5) and (5') can most easily be discovered using polar coordinates (r, θ) and the associated radial and angular velocity components u_r, u_θ . By definition, invariance under (5) and (5') means that all quantities are functions of θ alone. Hence we write

$$(6) \quad \begin{aligned} u_r &= g(\theta), & u_\theta &= h(\theta), & \rho &= \rho(\theta), \\ p &= f(\rho), & \frac{dp}{d\rho} &= f'(\rho) = c^2. \end{aligned}$$

We shall call flows satisfying (6), *wedge flows*. They will be generalized in Section 13 (see Figure 20).

The assumption of irrotationality amounts to $\oint u_r dr + u_\theta r d\theta = 0$ for all closed curves, whence $0 = \partial(r u_\theta)/\partial r - \partial u_r/\partial \theta = h - g'$, and we get

$$(7) \quad h = g'.$$

Since the flow is irrotational, the Equations of Motion are equivalent to the Bernoulli equation, which we can write as

$$(8) \quad C = \frac{1}{2} u^2 + \int \frac{dp}{\rho} = \frac{1}{2} (g^2 + g'^2) + \int \frac{dp}{\rho},$$

or in the differentiated form

$$(8.1) \quad 0 = udu + \frac{dp}{\rho} = g'(g + g'') + c^2 \left(\frac{\rho'}{\rho} \right).$$

With a *polytropic* equation of state $p = k\rho^\gamma + p_0$, $c^2 = \gamma k\rho^{\gamma-1}$, and so $\int dp/\rho = k\gamma\rho^{\gamma-1}/(\gamma - 1) = c^2/(\gamma - 1)$; hence in this case

$$(8') \quad C = \frac{1}{2} (g^2 + g'^2) + \frac{c^2}{(\gamma - 1)}.$$

So far everything is equally valid for irrotational flows along meridian planes through an axis, in which all physical quantities are functions of the *latitude* only. We shall call such flows, *conical* flows.

Now with wedge flows, the Equation of Continuity, $\text{div}(\rho\mathbf{u}) = 0$, can be written in the form

$$0 = \frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial \theta}(\rho u_\theta) = \rho u_r + (\rho u_\theta)' = \rho(u_r + u_\theta') + \rho' u_\theta.$$

Substituting from (6)–(7), this is equivalent to

$$(9) \quad (g + g'') + g' \left(\frac{\rho'}{\rho} \right) = 0.$$

Multiplying (9) by g' and subtracting from (8.1), we get

$$(9') \quad (c + g')(c - g') \left(\frac{\rho'}{\rho} \right) = 0.$$

Clearly (9)–(9') are equivalent to the equations (8.1), (9) of motion and continuity. We have two families of solutions.

CASE I. $\rho' = 0$. Substituting in (9), $g'' + g = 0$, whence $u_r = A \cos(\theta - \alpha)$. Using (6)–(7) to get u_θ , we see that this is *uniform* flow with constant vector velocity.

CASE II. $g'^2 = c^2$, or $g' = \pm c$. We see that the radii $\theta = \text{const.}$ are characteristics, in the sense that the component of velocity perpendicular to them is always the sound velocity c . This gives so-called *Prandtl-Meyer² expansion waves*, which are often seen to fill wedge-shaped regions in Schlieren pictures. At the ends, they join onto regions of uniform flow.

In the polytropic case, substituting $c^2 = g'^2$ in (8'), we get immediately the differential equation

$$(10) \quad (\gamma + 1)g'^2 + (\gamma - 1)g^2 = 2(\gamma - 1)C.$$

This can be easily integrated, the nature of the solutions in the adiabatic case $\gamma > 1$ being quite different from what it is if $\gamma = 1$, if $-1 < \gamma < 1$, or if $\gamma = -1$ (circular flow). Evidently in the non-polytropic case, we can still obtain equation (9) and $c = g'$. Using (8) to solve $\ln \rho = \psi(c) = \psi(g')$, we get $(g + g'') + g'g''\psi'(g')$, which can be integrated numerically. Hence *Prandtl-Meyer expansion waves are mathematically possible with a general equation of state.*

Mathematically, Prandtl-Meyer "compression waves" are also possible, with flow in the direction of increasing pressure. But such waves do not seem to occur physically (Reversibility Paradox).

4. Taylor-Maccoll Conical Flows

In n dimensions (the physically interesting case is of course $n = 3$), the Equation of Continuity for stationary axially symmetric flows assumes the form

$$\begin{aligned} 0 &= \frac{\partial}{\partial r} (\rho r^{n-1} \cos^{n-2} \theta u_r) + \frac{\partial}{\partial \theta} (\rho r^{n-2} \cos^{n-2} \theta u_\theta) \\ &= (n-1)\rho \cos^{n-2} \theta r^{n-2} u_r + \rho r^{n-2} (\cos^{n-2} \theta u_\theta)' \\ &\quad + \rho' r^{n-2} \cos^{n-2} \theta u_\theta. \end{aligned}$$

After dividing by $\rho r^{n-2} \cos^{n-2} \theta$, and substituting from (6)–(7) (we recall that equations (6)–(8') are valid in space), we get

$$(11) \quad (n-1)g - (n-2)g' \tan \theta + g'' + g' \left(\frac{\rho'}{\rho} \right) = 0$$

² [63]; see also T. Meyer, *V.D.I. Forschungsheft*, 62 (1908), 31–67.

in place of (9). Here (8.1) is still valid; it is equivalent to (8') in the polytropic case. In this case, by (8.1)–(8'), we have

$$\left(\frac{\rho'}{\rho}\right) = -\frac{g'(g + g'')}{c^2} = \frac{g'(g + g'')}{(\gamma - 1)} \left[\frac{1}{2}(g^2 + g'^2) - C\right]^{-1}.$$

Substituting in (11), which can be written

$$(11') \quad (g'' + g) + (n - 2)(-g' \tan \theta + g) + g' \left(\frac{\rho'}{\rho}\right) = 0$$

we get

$$(12) \quad (g'' + g) \left\{ 1 + \frac{g'}{(\gamma - 1)} \left[\frac{1}{2}(g^2 + g'^2) - C \right]^{-1} \right\} + (n - 2)(-g' \tan \theta + g) = 0$$

This has been integrated numerically for $\gamma = 1.408$ (air) in a celebrated paper by G. I. Taylor and J. W. Maccoll [68].³

An attempt was made to find a flow satisfying (11)–(12), which would be tangent to a rigid cone (projectile head) and be separated by a conical shock wave of constant intensity,⁴ also invariant under (5), from a region of uniform flow.

It was deduced that the resulting “conical régime” depicted in Fig. 18 was only possible for Mach numbers considerably greater than one. The predicated limits of the conical regime, the pressure on the conical head, and the angle of the attached shock wave as a function of Mach number have been closely confirmed experimentally.

5. Non-stationary One-dimensional Compressible Flows

There also exist important classes of non-stationary flows having the internal symmetry (5), and which can therefore be found by the method of search for symmetric solutions. I shall here discuss the one-dimensional case only; flows in higher dimensions, which are invariant under (5), will be discussed in Sections 8 and 10. I shall also assume a polytropic equation of state, $p - p_0 = k\rho^\gamma$.

³ See also [8], and the bibliography of [60], p. 449, Part B, especially J. W. Maccoll, *Proc. Roy. Soc.*, A159 (1937), 459–472.

⁴ Such a shock yields irrotational flow and a single equation of state behind the shock front.

In the one-dimensional case, it is convenient to use Lagrangian coordinates. If we let a measure the mass between a particular material point and a general material point, and denote position by $x = f(a, t)$ and density by $\rho = h(a, t)$, then the Equation of Continuity is equivalent to having the specific volume $\sigma = 1/\rho = \partial f/\partial a$.

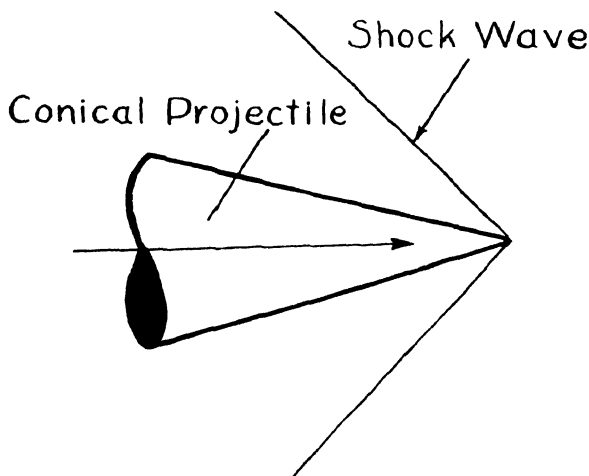


FIG. 18. Conical flows of Taylor and Maccoll.

Since velocity $u = \partial f/\partial t$, invariance of the flow under (5) is equivalent to $f(\alpha a, \alpha t) = \alpha f(a, t)$, and hence to $f(\alpha, \alpha t) = \alpha f(1, t) = \alpha g(t)$. Setting $\tau = t/a$, this is

$$(13) \quad x = f(a, t) = ag\left(\frac{t}{a}\right) = ag(\tau) \quad \left[\tau = \frac{t}{a}\right].$$

Thus (13) expresses our symmetry condition.

It remains to consider the Equation of Motion. In Lagrangian coordinates, acceleration is

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial \left(g' \left(\frac{t}{a} \right) \right)}{\partial t} = \frac{g''(\tau)}{a}.$$

Again, $\partial p/\rho \partial x = c^2 \partial \rho/\rho \partial x$, t being held constant. But $\partial/\rho \partial x = \partial/\partial a$, and with polytropic flow, $c^2 = \gamma k \rho^{\gamma-1}$; hence

$$\frac{\partial p}{\rho \partial x} = \frac{\gamma k \rho^{\gamma-1} \partial \rho}{\partial a} = - \frac{\gamma k \rho^{\gamma+1} \partial(1/\rho)}{\partial a} = - \frac{\gamma k \rho^{\gamma+1} \partial^2 f}{\partial a^2}$$

But if $f(a, t) = ag(t/a)$, then $\partial^2 f / \partial a^2 = t^2 g''(t/a) / a^3$, by direct calculation. Hence the Equation of Motion is equivalent to

$$(14) \quad 0 = \frac{\partial^2 f}{\partial t^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{a} \{g''(\tau) - \gamma k \rho^{\gamma+1} \tau^2 g''(\tau)\}.$$

Equation (14) has two families of solutions.

If $g'' = 0$, then $f = a[C_1 + C_2(t/a)] = C_1 a + C_2 t$. This implies uniform flow, $u = \sigma = \text{const.}$, and is uninteresting.

Otherwise, $1 = \gamma k \rho^{\gamma+1} \tau^2$, which implies

$$(14') \quad \left(\frac{\partial f}{\partial a}\right)^{\gamma+1} = \sigma^{\gamma+1} = \gamma k \tau^2.$$

But $\partial f / \partial a = g(t/a) - (t/a)g'(t/a)$; hence the condition is

$$(15) \quad g - \tau g' = [\gamma k \tau^2]^{1/(\gamma+1)}, \text{ unless } \gamma = -1.$$

This first-order linear non-homogeneous ordinary differential equation can easily be integrated formally. The general solution is

$$(15') \quad g(\tau) = C\tau + A\tau^{2/(\gamma+1)},$$

where $A = [(\gamma + 1)/(\gamma - 1)][\gamma k]^{1/(\gamma+1)}$, and C is arbitrary—provided $|\gamma| \neq 1$. There is no solution if $\gamma = -1$, since then $\tau = \text{const.}$ by (14'). If $\gamma = 1$, the general solution is $g(\tau) = C\tau + \sqrt{\gamma K} \tau \ln \tau$.

6. Spiral Viscous Flows

The method of “search for symmetric solutions” is also extremely useful in obtaining viscous flows. We illustrate by the classical case of the “spiral flows” of incompressible viscous fluids.

It is well-known that the equation of continuity is equivalent, in the case of plane flows, to the introduction of a “stream function” $V = \int(u_1 dx_2 - u_2 dx_1)$ such that $(-\partial V / \partial y, \partial V / \partial x)$ is the velocity vector. Then $\nabla^2 V = \partial u_2 / \partial x_1 - \partial u_1 / \partial x_2$ is the *vorticity*. Moreover the Navier-Stokes equations of motion

are equivalent⁵ for such plane flows to

$$(16) \quad \nu \nabla^4 V = \frac{\partial(V, \nabla^2 V)}{\partial(x, y)} = \frac{1}{r} \frac{\partial(V, \nabla^2 V)}{\partial(r, \theta)} \\ = \frac{1}{r} \left[\frac{\partial V}{\partial r} \frac{\partial(\nabla^2 V)}{\partial \theta} - \frac{\partial V}{\partial \theta} \frac{\partial(\nabla^2 V)}{\partial r} \right].$$

Here $\partial(p, q)/\partial(x, y) = p_x q_y - q_x p_y$ is the ordinary notation for the Jacobian; ν is as usual the kinematic viscosity μ/ρ .

Dimensional analysis tells us that under geometrically similar conditions, the behavior of incompressible viscous fluids will be a function of the single dimensionless parameter R , which plays the role of KT^2/D in the example of Section 2. We now look for *self-similar* plane motions. To this end, we consider the *spiral group*

$$(17) \quad r' = e^\alpha r, \quad \theta' = \theta + c\alpha,$$

where the parameter c is a fixed real number.

The transformations (17) are self-similitudes of space. Since ρ is fixed, they will make the motion self-similar at the same R , if and only if ru_r and ru_θ are the same at corresponding points. But *differences* in the stream function V are proportional to distances times velocity, since $dV = (\partial V/\partial x)dx + (\partial V/\partial y)dy$. Therefore differences in V must be *invariant* under the spiral group (17). Hence

$$(18) \quad V(e^\alpha r, \chi + c\alpha) - V(e^\alpha r, c\alpha) = V(r, \chi) - V(r, 0) = F(\chi),$$

is a function of χ alone, for any α . By similarity also, $V(e^\alpha r, c\alpha)$ is proportional to α for self-similar motion. Hence

$$(18') \quad V(r, \theta) = V(e^\alpha r, \chi + c\alpha) = C\alpha + F(\chi),$$

where $\alpha = \ln r$ and $\chi = \theta - c \ln r$. That is,

$$(19) \quad V(r, \theta) = C \ln r + F(\theta - c \ln r).$$

This expresses the motion in terms of an arbitrary constant C and a function F of one variable.

⁵ This result may be derived from Chapter III, (26), since $\text{curl } \mathbf{g} = \mathbf{0}$ in a conservative field, and since $\xi \cdot \nabla = 0$ because $\xi_1 = \xi_2 = \partial/\partial x_3 = 0$. The result may also be found in [10], p. 536, Ex. 7.

As before, we now substitute in the general differential equation (16); the calculations follow. In general, writing $\lambda = \ln r$, we have

$$\nabla^2 = \frac{1}{r^2} \left(\frac{\partial^2}{\partial \lambda^2} + \frac{\partial^2}{\partial \theta^2} \right), \quad \text{whence} \quad \nabla^2 V = e^{-2\lambda} (c^2 + 1) F''.$$

Since $\partial/\partial(x, y) = r^{-2} \partial/\partial(\lambda, \theta)$, we infer by substitution in (16),

$$(20) \quad \frac{\nu}{r^2} \left[\left(\frac{\partial^2}{\partial \lambda^2} + \frac{\partial^2}{\partial \theta^2} \right) (e^{-2\lambda} (c^2 + 1) F'') \right] \\ = \frac{1}{r^2} \left[\frac{\partial V}{\partial \lambda} \frac{\partial(\nabla^2 V)}{\partial \theta} - \frac{\partial V}{\partial \theta} \frac{\partial(\nabla^2 V)}{\partial \lambda} \right].$$

Carrying out the operations indicated, with $V = C\lambda + F(\theta - c\lambda)$, we get

$$(21) \quad \nu[(c^2 + 1)F^{IV} + 4cF''' + 4F''] \\ = (C - cf')F''' + 2F'F'' + cF'F''.$$

This is the ordinary differential equation obtained by Oseen;⁶ it would be difficult to find another equally simple motivation. Equation (21) can be made somewhat less repulsive through the substitution $G = F'$; moreover it is satisfied whenever $F'' = 0$. In any case, solutions can be found by numerical integration.

7. Laminar Boundary Layer

We next consider the plane laminar flow parallel to a fixed semi-infinite flat plate, placed on the positive x -axis. Clearly, if u_∞ is the free stream velocity, we have the boundary conditions

$$(22) \quad u = v = 0 \quad \text{on} \quad y = 0, \quad x \geq 0,$$

and

$$(22') \quad \lim_{y \rightarrow \infty} u(x, y) = u_\infty, \quad \lim_{y \rightarrow \infty} v(x, y) = 0, \quad \text{for any fixed } x.$$

We also have the equation of continuity

$$(23) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

⁶ See C. W. Oseen, *Arkiv. for Mat.*, I-II, 1927-1928, or [65], Chapter II.

and the Navier-Stokes equations (Chapter I, (3*))

$$(24) \quad \frac{du}{dt} = \nu \nabla^2 u - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \text{etc.}$$

Here as before d/dt is the substantial derivative $\partial/\partial t + u\partial/\partial x + v\partial/\partial y$; gravity is neglected.

We shall now simplify the preceding equations by the thin *boundary layer approximation* of Prandtl, namely, that the flow is nearly parallel to the plate, and changes rapidly when one moves perpendicular to the plate. This implies⁷ that $u \gg v$ and $\partial/\partial y \gg \partial/\partial x$. It follows, since the pressure gradient is zero in the free stream, that we can set $\partial p/\partial x = 0$ in (24); moreover in the steady state case, $\partial/\partial t = 0$. Substituting, (24) becomes

$$(24^*) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Now consider the symmetry of the problem. Since we are studying asymptotic conditions, we may hope to treat x and y as independent dimensions, and to find symmetry under a non-trivial subgroup of the four-parameter group of affine transformations

$$(25) \quad x \rightarrow \alpha x, \quad y \rightarrow \beta y, \quad u \rightarrow \gamma u, \quad v \rightarrow \delta v.$$

Because of (22'), $\gamma = 1$; and in fact substitution in (22)–(22')–(23)–(24*) shows that

$$(25^*) \quad x \rightarrow \beta^2 x, \quad y \rightarrow \beta y, \quad u \rightarrow u, \quad v \rightarrow \frac{v}{\beta}$$

defines the subgroup specified above.

Having found this symmetry (or guessed it by “physical intuition”), we can apply the method of “search for symmetric solutions,” and look for solutions which are invariant under

⁷ For more thorough discussions of the approximation, see R. von Mises, *Zeits. ang. Math. Mech.*, 7 (1927), 425–431; V. L. Streeter, *Fluid Dynamics*, McGraw-Hill, 1948, pp. 242–244. The method of search for symmetric solutions works for other velocity distributions outside the boundary layer. See D. R. Hartree, *Proc. Comb. Phil. Soc.*, 33 (1937), 223–229; S. Goldstein, *ibid.* 35 (1939), 338–340; W. Mangler, *Zeits. ang. Math. Mech.*, 23 (1943), 241–251; J. Oldroyd, *Phil. Mag.*, 36 (1945), 587–600.

(25*). To do this, we first find, by inspection, a space variable $\eta = y/\sqrt{x}$ which is invariant under (25*), and use it and x as independent variables. This leads to the classical solution of Blasius [58]; I shall now sketch the details.

We first transform (24*) to the new independent variables x, η . In the convenient thermodynamic notation for partial derivatives, whereby subscripts denote variables held fixed, we have

$$(26) \quad \left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial x}\right)_\eta - \frac{\eta}{2x} \left(\frac{\partial}{\partial \eta}\right)_x, \quad \left(\frac{\partial}{\partial y}\right)_x = \frac{1}{\sqrt{x}} \left(\frac{\partial}{\partial \eta}\right)_x.$$

This may be conveniently proved by substituting in

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial x}\right)_\eta dx + \left(\frac{\partial G}{\partial \eta}\right)_x d\eta \\ &= \left(\frac{\partial G}{\partial x}\right)_\eta dx - \frac{y}{2x^{1/2}} \left(\frac{\partial G}{\partial y}\right)_x dx + \frac{1}{\sqrt{x}} \left(\frac{\partial G}{\partial \eta}\right)_x dy. \end{aligned}$$

We then set $u = F(\eta)$. This gives, by (26), $\partial u/\partial x = -\eta F'/2x$, $\partial u/\partial y = F'/\sqrt{x}$, $\partial^2 u/\partial y^2 = F''/x$. We next eliminate (23) as usual, observing that it amounts to asserting that $u dy - v dx$ is a *total* differential dV , where V is the stream function. Integrating $V = \int u dy = \sqrt{x} \int u d\eta$ from the boundary streamline, $V = 0$ when $y = 0$, we get $V = \sqrt{x} f(\eta)$, where $f' = F$. We now calculate

$$v = -\frac{\partial V}{\partial x} = -\frac{f}{2\sqrt{x}} + \frac{\eta f'}{2\sqrt{x}}, \quad \text{by (26).}$$

Substituting the preceding expressions in (24*), and f' for F , we get, after multiplication by $2x$ and obvious simplifications,

$$(27) \quad 2\nu f''' + ff'' = 0.$$

This can be integrated numerically, relative to the boundary condition $f'(\infty) = u_\infty$, $f(0) = f'(0) = 0$.

8. Other Symmetric Flows

The preceding examples should give a clear picture of how the method of "search for symmetric solutions" can be

applied mathematically. I should now like to illustrate further its importance for fluid mechanics by mentioning briefly some other problems to which it has been applied, but which do not always lead to boundary value problems involving ordinary differential equations.

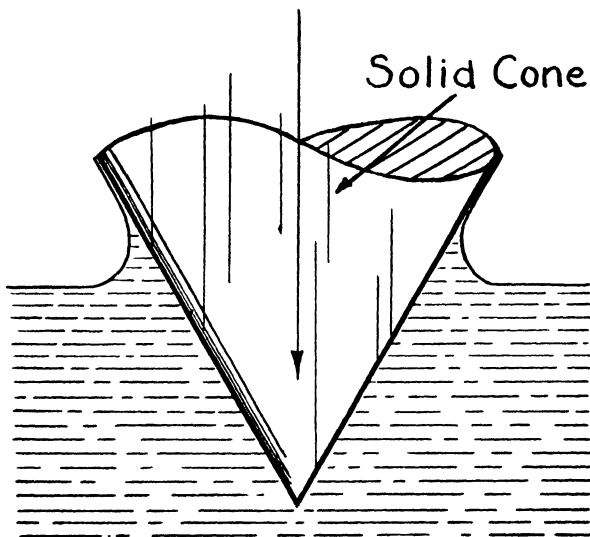


FIG. 19. Vertical impact of cone on water.

Consider the entry at constant speed of a wedge into water (Fig. 19), at sufficiently high speeds so that gravity can be neglected. As in Sections 3–5, the transformation

$$(5) \quad x \rightarrow \alpha x, \quad y \rightarrow \alpha y, \quad t \rightarrow \alpha t; \quad p, \rho, \mathbf{u} \text{ unchanged,}$$

leaves inertial fluid mechanics unchanged—indeed, we could suppose the fluid compressible! Hence the method of “search for symmetric solutions” leads one to look for solutions of the form

$$(28) \quad U(x, y; t) = t\phi\left(\frac{x}{t}, \frac{y}{t}\right),$$

This method of reduction from three to two independent variables is the basis of a well-known paper of H. Wagner on the impact of seaplane floats.⁸ We can reduce to functions

⁸ H. Wagner, *Zeits. ang. Math. Mech.*, 5 (1925), 26.

of a *single* complex variable, by the considerations of Chapter II, Section 2, but we have complicated boundary conditions.

The same method is evidently applicable to a cone entering the water at a constant velocity, suggesting

$$(28') \quad U(x, y, z; t) = t\phi\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right),$$

i.e. a reduction from *four* to *three* independent variables. In the case of a right *circular* cone entering the water vertically, solutions having the circular symmetry of the hypotheses can be expressed, through

$$(28'') \quad U(x, y, \theta; t) = t\phi\left(\frac{x}{t}, \frac{y}{t}\right),$$

in terms of a single function of *two* independent variables.

Similarly, the invariance of the theory of compressible non-viscous fluids under changes of space-scale suggests the search for the so-called "conical flows" invariant under the same group. This suggestion is due primarily to A. Busemann,⁹ and suggests the consideration of flows of the type

$$(29) \quad \mathbf{u} = \mathbf{u}(\phi, \theta)$$

in spherical coordinates. Such flows occur with delta wings, because delta wings also have conical symmetry.

In the preceding examples, the reduction in the number of real independent variables involved in the basic partial differential equations was to $n = 2$; whereas in the examples of Sections 2-7, we reduced to the case $n = 1$ of *ordinary* differential equations.

9. General Local Existence Theorem

It is natural to ask, how generally can one expect the method of search for symmetric solutions to give *mathematical* solutions to physical problems? I stress the word *mathematical*, because we have already seen (Chapter I, Sections 14-15) that the symmetric solutions may well be unstable, so that

⁹ See *N.A.C.A. Tech. Memo.*, 1100 (1947), and refs. given there; also, A. E. Puckett, *Jour. Aer. Sci.*, 13 (1946), 475-484.

apparent symmetry in the mathematical statement of a problem need not be reflected in symmetry of the real physical solution.

We shall begin with the case which is simplest from the mathematical point of view—namely, where the symmetry is so great that we are left with only *one* independent variable, so that the partial differential equations describing the internal conditions reduce to an *ordinary* differential equation, as in Sections 2–7 above.

In this case, we have well-known theorems for differential equations of the form

$$(30) \quad \frac{d^n y}{dx^n} = F\left(x, y, y', \dots, \frac{d^{n-1} y}{dx^{n-1}}\right),$$

which prove the following fact, under extremely general conditions. For any “initial values” of $y, y', y'', \dots, d^{n-1}y/dx^{n-1}$, at $x = a$, there is one and only one “solution” of (30), *in some neighborhood* of $x = a$. On the other hand, it is obvious that

$$(31) \quad y'^2 + y^2 + 1 = 0$$

has no solution, even locally, in the real domain.

It may be hoped that in “well-formulated” problems arising from physics, absurdities of the type given by (31) will not occur. It may also be hoped that we will not be left with *two* ordinary differential equations—which will not of course usually have any common solution. However, we cannot hope to avoid difficulties inherent in the mathematical principle that solutions compatible with given boundary conditions need only exist *locally*. An outstanding example is furnished by the “conical flows” of Taylor and Maccoll (Section 4); these authors showed that the mathematical limits for the régime of the conical flows coincided very nearly with the physical limits for “attached conical shock waves” having the conical symmetry postulated. Another example is furnished by Prandtl-Meyer expansion waves.

Thus even in the simplest case, of reduction to ordinary differential equations, mathematical solutions need exist only locally, and then only for the “initial value problem.” It is

comforting to find that at least this mild local existence and uniqueness theorem is valid quite generally.

To see this, recall¹⁰ the Cauchy-Kowalewski existence theorem for *analytic* partial differential equations, of the form

$$(32) \quad \frac{\partial \phi_i}{\partial t} = F_i \left(\phi_j; \frac{\partial \phi_j}{\partial x_k}; \frac{\partial^2 \phi_j}{\partial x_k \partial x_\rho}; \dots \right).$$

In words, suppose that the *time* derivatives of all functions $\phi_i(\mathbf{x};t)$ [$\mathbf{x} = (x_1, \dots, x_n)$] involved, can be expressed in terms of *space* derivatives. The Cauchy-Kowalewski theorem asserts that (32) has one and only one local analytic solution for each set of given analytic initial conditions at $t = 0$.

But now suppose that (32) is invariant under a group G . Let $\phi_i(\mathbf{x};0) = G_i(\mathbf{x})$ be a set of analytic initial conditions invariant under G . Then the *unique* solution defined locally by the preceding theorem, will also be invariant under G . Hence we get a local *existence* (and uniqueness) theorem for the reduced differential equation obtained by the method of search for symmetric solutions, whenever we have one for the original differential equations.

10. Group Theory and Separation of Variables

Solutions of physical problems possessing internal symmetry, with respect to a suitable group, can usually be given a simplified mathematical form by the introduction of suitable variables associated with this group. I shall now show how this leads to various "separations of variables" of great use in fluid mechanics.

Thus consider again the invariance of the theory of a non-viscous (but not necessarily incompressible) fluid under the group

$$(5) \quad t \rightarrow \alpha t, \quad x_i \rightarrow \alpha x_i, \quad p, \rho, u \text{ fixed.}$$

By definition, individual "self-symmetric" flows, invariant under the group (5), can be expressed in the form

$$(33) \quad u_i = f_i(\mathbf{x}), \quad p = p(\mathbf{x}), \quad \rho = \rho(p) = \rho(p(\mathbf{x})),$$

¹⁰J. Hadamard, "Le problème de Cauchy," Paris, 1932, Chapter I.

where

$$(34) \quad \mathbf{x} = (\chi_1, \chi_2, \chi_3) = \left(\frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t} \right) = \frac{\mathbf{x}}{t}.$$

Clearly, (34) is a special case of

$$(35) \quad \mathbf{x} = h(t)\mathbf{x}.$$

We shall now find all flows of a non-viscous fluid which admit the formal "separation of variables" expressed by (33) and (35); we shall show that, in this case, (35) necessarily reduces to (34). Gravity is supposed negligible throughout.

Formally, it is evident that (35) implies

$$(36) \quad \frac{\partial F}{\partial t} = \frac{h'}{h} \chi_k \frac{\partial F}{\partial \chi_k} \quad \text{and} \quad \frac{\partial F}{\partial x_i} = h \frac{\partial F}{\partial \chi_i},$$

for any function $F(\mathbf{x})$. Hence the equation of continuity, $\partial \rho / \partial t + \text{div}(\rho \mathbf{u}) = 0$, is equivalent to

$$(37) \quad h \left[\left(\frac{h'}{h^2} \right) \chi_k \frac{\partial \rho}{\partial \chi_k} + \frac{\partial(\rho u_k)}{\partial \chi_k} \right] = 0,$$

if (33) holds. Likewise, the equations of motion for a non-viscous fluid are equivalent to

$$(37') \quad h \left[\left(\frac{h'}{h^2} \right) \chi_k \frac{\partial u_i}{\partial \chi_k} + \frac{u_k \partial u_i}{\partial \chi_k} + \frac{\partial p}{\rho \partial \chi_i} \right] = 0.$$

If h'/h^2 is a constant in time, say $-C$, then $1/h = C(t - t_0)$. Hence, after an obvious change of time origin and unit, we can reduce to the case of flows satisfying (34), and hence possessing internal symmetry under the group (33).

Otherwise $\Sigma \chi_k \partial \rho / \partial \chi_k = \Sigma \chi_k \partial u_i / \partial \chi_k = 0$, as can be shown by differentiating (37)–(37') partially with respect to time, for fixed $\mathbf{x}$. In this case, in spherical coordinates, the radial derivatives of χ and u vanish, so that χ and u , and hence $p(\rho)$, are functions of direction $\mathbf{x}/|\mathbf{x}|$ (i.e. of latitude and longitude) only. Substituting back in (33) and (35), we see that they are stationary flows, and satisfy (33) and (35) for any $h(t)$, including the $h(t) = 1/t$ of (34).

In summary, the method of "separation of variables," expressed by (33) and (35), gives only the flows possessing

internal symmetry under the group (5). Hence, in this case, the methods of separation of variables and of search for symmetric solutions are effectively equivalent to each other. We shall see shortly (Section 12) that this equivalence does not always hold. But first we shall derive an interesting property of flows satisfying (33) and (35), from a slightly different formal treatment.

In the case of *irrotational* flows, (33)–(34) are equivalent to the slightly less trivial “separation of variables”

$$(34') \quad U = tF(\chi) = tF\left(\frac{\mathbf{x}}{t}\right).$$

In this case, clearly $u_i = \partial U / \partial x_i = \partial F / \partial \chi_i$; and (37') is equivalent to

$$(37'') \quad -\frac{\chi_k}{\partial \chi_i} \frac{\partial u_k}{\partial \chi_i} + \frac{u_k}{\partial \chi_i} \frac{\partial u_k}{\partial \chi_i} + \frac{\partial p}{\rho \partial \chi_i} = 0,$$

since $h'/h^2 = -1$ and $\partial u_i / \partial \chi_k = \partial^2 F / \partial \chi_k \partial \chi_i = \partial^2 F / \partial \chi_i \partial \chi_k = \partial u_k / \partial \chi_i$. Moreover $\partial U / \partial t = F - \chi_k \partial F / \partial \chi_k$, and so

$$\frac{\partial(\partial U / \partial t)}{\partial \chi_i} = -\frac{\chi_k}{\partial \chi_i} \frac{\partial^2 F}{\partial \chi_k} = -\frac{\chi_k}{\partial \chi_i} \frac{\partial u_k}{\partial \chi_i}.$$

Hence (37'') is equivalent by a space integration to

$$(38) \quad \frac{\partial U}{\partial t} + \frac{1}{2} \nabla U \nabla U + \int \frac{dp}{\rho} = C.$$

Here $C = C(t)$; but by (33) $C = C(\chi)$ also; hence C is a constant. That is, all irrotational flows of a compressible non-viscous fluid, which are invariant under (5), satisfy the *generalized Bernoulli equation* (38). In the incompressible case, one deletes the assumption $\rho = \rho(p)$ in (33). This deletion leads to the usual Bernoulli equation for accelerated motion in an incompressible fluid, with $C = C(t)$.

11. Application of the Special Group of Mach Modeling

My intention is, however, less to enumerate all known results which can be regarded as applications of group theory¹¹

¹¹ Others, closely related to the considerations of Sections 3, 4, 8, 10, are found in G. I. Taylor, *Reps. and Mem. Aer. Res. Comm.*, 1381 and

than to show the variety of such results. Accordingly, I shall now discuss a quite different "separation of variables" suggested by the special group of Mach modeling, applicable in the case of *polytropic* equations of state, and described in Chapter III, Theorem 7. This is the group

$$(39) \quad \mathbf{x} \rightarrow \alpha \mathbf{x}, \quad t \rightarrow \beta t, \quad \mathbf{u} \rightarrow \left(\frac{\alpha}{\beta}\right) \mathbf{u},$$

$$\rho \rightarrow \delta \rho, \quad (p - p_0) \rightarrow k \delta^\gamma (p - p_0),$$

which automatically leaves invariant the polytropic equation of state $(p - p_0) = k \rho^\gamma$, and the equation of continuity. It leaves invariant the equations of (non-viscous) motion, if and only if $\delta^{\gamma-1} = \beta/\alpha^2$.

Clearly, all flows which are invariant under a subgroup of (39) which involves time, admit of a separation of variables of the form¹²

$$(40) \quad \begin{aligned} U_i &= t f_i(\mathbf{x}), & \rho &= t^\eta e(\mathbf{x}), \\ p &= k \rho^\gamma + p_0, & \text{where } \mathbf{x} &= t^\tau \mathbf{x}. \end{aligned}$$

We shall now discuss flows satisfying (40).

Such flows always satisfy the equation of state. The equation of continuity, $\partial \rho / \partial t + \operatorname{div}(\rho \mathbf{u}) = 0$, reduces for such flows to

$$(41) \quad t^{\eta-1} \left\{ \eta e + \chi_k \frac{\partial e}{\partial \chi_k} + t^{\epsilon+\tau+1} \left[f_k \frac{\partial e}{\partial \chi_k} + e \frac{\partial f_k}{\partial \chi_k} \right] \right\} = 0.$$

This is equivalent to a choice between the equations:

$$(42) \quad \begin{aligned} \epsilon + \tau + 1 &= 0 & \text{and} \\ \eta e + \chi_k \frac{\partial e}{\partial \chi_k} + f_k \frac{\partial e}{\partial \chi_k} + e \frac{\partial f_k}{\partial \chi_k} &= 0, \end{aligned}$$

and the alternative equations

$$(43) \quad \eta e + \chi_k \frac{\partial e}{\partial \chi_k} = f_k \frac{\partial e}{\partial \chi_k} + e \frac{\partial f_k}{\partial \chi_k} = 0.$$

1382 (compressible circular and spiral flows), and in [66] and [67]. Still another was treated above in Chapter II, Section 14.

¹² This case includes the "separation of variables" introduced without motivation by Love and Pidduck, *Phil. Trans.*, A222 (1922), 167-226; by R. H. Kent, *Physics*, 7 (1936), 319-324, and *Am. Math. Monthly*, 48 (1941), 13; and by Courant-Friedrichs in [60], p. 419.

The alternative (42) asserts that the "dimensions" of $\mathbf{u}$ equal those $-\tau$ of $\mathbf{x}$, minus those of t , and that a certain partial differential equation of "stationary flow in $\mathbf{x}$ -space" holds. The alternative (43) asserts that $\text{div}(\epsilon \mathbf{f}) = 0$, in a stationary flow in $\mathbf{x}$ -space, and that the radial derivative of e in $\mathbf{x}$ -space is $\eta e/|\mathbf{x}|$, whence $e(\mathbf{x}) = |\mathbf{x}|^\eta E(\mathbf{x}/|\mathbf{x}|)$, where E is a function of direction (latitude and longitude) only.

The equations of motion involve the three terms

$$\begin{aligned}\frac{\partial p}{\rho \partial x_i} &= k t^{\eta+\tau} \rho^{\gamma-2} \frac{\partial e}{\partial \chi_i} = k t^{\tau+(\gamma-1)\eta} e^{\gamma-2} \frac{\partial e}{\partial \chi_i}, \\ u_k \frac{\partial u_i}{\partial x_k} &= t^{2\epsilon+\tau} f_k \frac{\partial f_i}{\partial \chi_k}, \\ \frac{\partial u_i}{\partial t} &= t^{\epsilon-1} \left(\epsilon f_i + \tau \chi_k \frac{\partial f_i}{\partial \chi_k} \right).\end{aligned}$$

For the sum of these terms to vanish, we have a choice between the following alternatives:

$$(44) \quad \begin{aligned} \epsilon + \tau + 1 &= 0, & (\gamma - 1)\eta &= 2\epsilon, & \text{and} \\ k e^{\gamma-2} \frac{\partial e}{\partial \chi_i} + \epsilon f_i + (f_k + \tau \chi_k) \frac{\partial f_i}{\partial \chi_k} &= 0, \end{aligned}$$

or

$$(45) \quad \epsilon + \tau + 1 = 0, \quad e = \text{const.},$$

and

$$\epsilon f_i + \frac{(f_k + \tau \chi_k) \partial f_i}{\partial \chi_k} = 0,$$

or

$$(46) \quad \epsilon - 1 = \tau + (\gamma - 1)\eta, \quad f_k \frac{\partial f_i}{\partial \chi_k} = 0,$$

and

$$k e^{\gamma-2} \frac{\partial e}{\partial \chi_i} + \epsilon f_i + \tau \chi_k \frac{\partial f_i}{\partial \chi_k} = 0,$$

or

$$(47) \quad 2\epsilon = (\gamma - 1)\eta, \quad k e^{\gamma-2} \frac{\partial e}{\partial \chi_i} + f_k \frac{\partial f_i}{\partial \chi_k} = 0,$$

and

$$\epsilon f_i + \tau \chi_k \frac{\partial f_i}{\partial \chi_k} = 0.$$

The condition that $(\gamma - 1)\eta = 2\epsilon$ is simply the condition that the Mach number $u_i^2/c^2 = u_i^2/(dp/d\rho)$ be a function of $\mathbf{x}$ which is independent of time.

Thus we see that the formal "separation of variables," suggested by the special Mach group (39), yields *six* alternative cases, of which that expressed by the simultaneous conditions (42) and (45) alone corresponds to invariance under (39)! This case involves fewer simultaneous differential equations than the others, and hence is the most interesting; it includes the flow described above in Section 5. In fact, I have been unable to discover any non-trivial flows in the other five "degenerate" cases.

However, in the *viscous* case, we do have significant flows in the "degenerate case" arising from a corresponding separation of variables, as I shall now show.

12. Separation of the Time Variables: Viscous Flows

Consider the flows of an incompressible, viscous fluid which are *self-similar* at all times, in the sense that

$$(48) \quad u_i = g(t)f_i(\mathbf{x}), \quad \text{where} \quad \mathbf{x} = h(t)\mathbf{x}.$$

Clearly the equations of state and continuity are together (Chapter I, Section 13) equivalent to the single condition $\text{div } \mathbf{u} = 0$; since g and h are non-vanishing, this is equivalent to

$$(49) \quad \sum \frac{\partial f_k}{\partial x_k} = 0,$$

from which t has been eliminated.

It remains to consider the Navier-Stokes equations of motion; by the remark of Chapter III, Section 12, we can neglect gravity. The other terms are, by direct computation from (48) in (24),

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= g'f_i + \sum gh'x_k \left(\frac{\partial f_i}{\partial x_k} \right) = g'f_i + \left(\frac{gh'}{h} \right) x_k \frac{\partial f_i}{\partial x_k}, \\ \sum u_k \frac{\partial u_i}{\partial x_k} &= g f_k g h \left(\frac{\partial f_i}{\partial x_k} \right) = (g^2 h) f_k \frac{\partial f_i}{\partial x_k}, \end{aligned}$$

$$\frac{\partial p}{\rho \partial x_i} = \left(\frac{a(t)h(t)}{\rho_0} \right) \frac{\partial p}{\partial \chi_i} \quad \text{if} \quad p = a(t)p(\mathbf{x}),$$

$$\nu \nabla^2 u_i = \nu (gh^2) \sum_k \frac{\partial^2 f_i}{\partial \chi_k^2}.$$

Hence the Navier-Stokes differential equations can be expressed in the "separated form"

$$(50) \quad \sum_{j=1}^5 F_j(t) G_j(\mathbf{x}) = 0,$$

where $F_1 = g'$, $F_2 = gh'/h$, $F_3 = g^2h$, $F_4 = ah$, $F_5 = gh^2$, and

$$(50') \quad \begin{aligned} G_1 &= p_i, & G_2 &= \chi_k \frac{\partial f_i}{\partial \chi_k}, & G_3 &= f_k \frac{\partial f_i}{\partial \chi_k}, \\ G_4 &= \frac{\partial p}{\partial \chi_i}, & \text{and} & & G_5 &= \sum \frac{\partial^2 f_i}{\partial \chi_k^2} \end{aligned}$$

Clearly condition (50) is equivalent to the requirement that the vectors $\mathbf{F} = (F_1, F_2, F_3, F_4, F_5)$ and $\mathbf{G} = (G_1, G_2, G_3, G_4, G_5)$ be confined to *orthogonal subspaces*. Depending on the number of linearly independent relations satisfied by the G_j , the " $\mathbf{F}$ -subspace" spanned by the $\mathbf{F}$ may conceivably have 1, 2, 3, or 4 dimensions. I shall first investigate the non-degenerate case, when all the F_j are proportional, so that the $\mathbf{F}$ -subspace has one dimension.

We can make F_4 proportional to F_3 , by making $a = \alpha g^2$. Again, $g = h$ is (we can ignore constant factors) equivalent to the proportionality of F_3 and F_5 , and it implies the proportionality of F_1 and F_2 . The remaining condition is that F_1 and F_3 be proportional, or that $g' = (-\beta/2)g^2h$ for some constant β . Since $g = h$, this amounts to $-2g'/g^3 = \beta$, or $1/g^2 = \beta(t - t_0)$. By properly choosing the origin and scale of time, this can be reduced to $g = 1/\sqrt{t}$, and hence to

$$(51) \quad u_i(\mathbf{x}; t) = \frac{1}{\sqrt{t}} f_i(\mathbf{x}) \quad \text{and} \quad p = \frac{p(\mathbf{x})}{t} \left[\mathbf{x} = \frac{\mathbf{x}}{\sqrt{t}} \right],$$

$$(51') \quad -\frac{1}{2} \left(f_i + \chi_k \frac{\partial f_i}{\partial \chi_k} \right) + f_k \frac{\partial f_i}{\partial \chi_k} + \frac{\partial p}{\rho_0 \partial \chi_i} = \nu \sum \frac{\partial^2 f_i}{\partial \chi_k^2}.$$

Hence we have again reduced the number of independent

variables by one—by a search for a more general type of symmetric solutions! I omit the discussion of (51') itself.¹³

This corresponds to the invariance of the Navier-Stokes equations under $t \rightarrow \alpha t$ and $\mathbf{x} \rightarrow \beta \mathbf{x}$, provided the Reynolds number $R = \nu d / \nu \propto (\beta/\alpha)\beta$ [$\nu = \text{const.}$] is unchanged—so that $\beta \propto \sqrt{\alpha}$. The solutions (51) are precisely the flows invariant under this group.

There are also “degenerate” solutions of (50). For example, consider the “parallel flows,” parallel to the x -axis, with

$$(52) \quad u_1 = g(t)f_1(y, z), \quad u_2 = u_3 = p = 0.$$

In this case, (49) is always satisfied. Since $h = 1$, $F_2 = 0$; since $p = 0$, $F_4 = 0$. What matters more, $G_3 = f_1 \partial u_i / \partial x = 0$ for all i ; hence F_3 is unrestricted. It remains to satisfy $g'f_i = \nu g(\partial^2 f_i / \partial y^2 + \partial^2 f_i / \partial z^2)$, which, g being a function of t and f_i of $\mathbf{x} = (x, y, z)$, reduces to $g'/g = -k$ and

$$(52') \quad u_i = e^{-kt} f_i(y, z), \quad \text{where} \quad \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} = -\nu k f_1.$$

This is the well-known¹⁴ exponential decay of parallel viscous flows, for example of flow in the two-dimensional channel $-a \leq y \leq a$, with $u_1 = e^{-kt} \cos \pi y / 2a$, with $k = \pi^2 / 4a^2 \nu$.

13. “Inverse Methods” and Group Theory

The flows described in Sections 10–12 are characterized formally by the “separability of the time variable.” They are dynamically similar to themselves at all times. This property gives a simple group-theoretic characterization of the method of separation of the time variable, which is not limited to the groups discussed in Chapter III, which were the only groups used in the applications of the method of “search for symmetric solutions,” discussed in Sections 3–8.

¹³ In the plane, (49) is equivalent to the existence of a stream function, and p can be eliminated from (51') by using curl ($\text{grad } p$) = 0. This leaves the fourth-order partial differential equation (16).

¹⁴ See G. I. Taylor, *Phil. Mag.*, 46 (1923), 671–674. A similar exponential decay of circular flows is possible, with $p = p(r)$ sufficient to account for centripetal acceleration. Cf. [65], and Ratip Berker, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*, Lille, 1936.

It may be that the method of "separation of variables" has also a deeper group-theoretic foundation than that expressed in the self-similitude of products such as $F(x)J(y,z)$, as x changes. An interesting case in point is furnished by the cases in which separation of variables is possible in the Laplace equation $\nabla^2 U = 0$ in space. Redheffer has recently shown [64] that in all cases one family of coordinate surfaces must consist of parallel planes, planes through a line, spheres tangent at a common point, concentric spheres, the plane and set of spheres obtained when one set of bipolar coordinates is revolved about the line joining the poles, or a set of spheres all passing through a single fixed circle. These are evidently all equivalent under the group generated by inversions in spheres to parallel planes, planes through a line and concentric spheres—i.e. to one set of coordinate surfaces for Cartesian, cylindrical, or spherical coordinates. This suggests that the problem could be attacked directly by conformal methods, invariant under the conformal group.

However, though this idea is very attractive, it is still only a vague conjecture. (In this connection, it may be remarked that the Schroedinger-Infeld factorization method (see [62]) in ordinary differential equations definitely does not depend on group-theoretic ideas.)

A safer, if less satisfying, explanation of the applications of the method of "separation of variables" to fluid mechanics is in terms of the concept of an *inverse method*, recently discussed systematically by P. Nemenyi.¹⁵ Whenever group theory indicates the existence of flows with separated variables, or any other property P , an *a priori* postulation of property P will yield these solutions at least, and possibly others.

For example, group theory assures us of the existence (locally) of Prandtl-Meyer expansion waves, in which

(P1) the vector velocity is constant along each straight line of a one-parameter family.

The "inverse method" consists in finding *all* stationary irrotational flows of a compressible non-viscous fluid, which have property P1. This can be done as follows.

¹⁵ Nemenyi, "On inverse and semi-inverse methods in the mechanics of continua," *Proc. Int. Mech. Congress*, London, 1948.

We know (Chapter I, Section 5) that with stationary irrotational flow, the equations of motion are equivalent to the Bernoulli equation $\frac{1}{2}u^2 + \int dp/\rho = C$. Hence, by numerical integration, there is for each “stagnation pressure” (i.e. constant of integration) one and only one pair of functions $\rho(u)$ and $p(\rho(u))$ such that the equations of state and motion are both satisfied. Moreover the flows with property P1 are the flows with such $\rho(u)$ and $p(\rho(u))$, which are *irrotational* and satisfy the *equation of continuity* (mass-conservation).

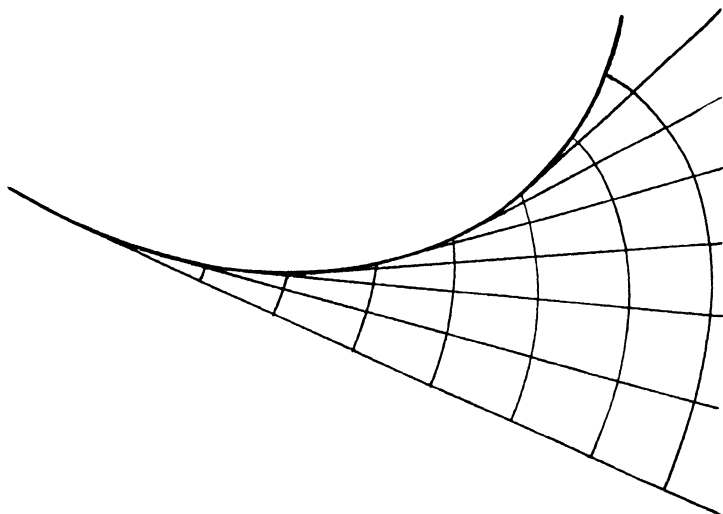


FIG. 20. Coordinates for Prandtl-Meyer expansion fan.

In the present problem, considerable geometrical insight can be obtained by using special coordinates appropriate to a one-parameter family of straight lines. These special coordinates are: the angle θ made with the x -axis by the lines, and the directed distance h from a fixed curve cutting the lines orthogonally, as in Fig. 20. If we recall that the lines represent “in general” the tangents of a suitable plane curve Γ , we will see immediately: (i) the loci $\theta = \text{const.}$ are the given straight lines, (ii) the lines $h = \text{const.}$ are the orthogonal family of “evolutes” of Γ , (iii) $ds^2 = dh^2 + r^2 d\theta^2$, where $r = h + s(\theta)$ is the radius of curvature of the evolute, and s denotes arc-length along Γ .

In terms of these intrinsic geometrical coordinates, we can easily express the two conditions of irrotationality and mass conservation.

By definition, irrotationality is the condition that the circulation around any closed curve γ vanishes. If α denotes the angle from the straight line $\theta = \theta_0$ to the vector velocity $\mathbf{u}(\theta_0)$, of magnitude q , the circulation around γ is

$$\oint \Sigma u_k dx_k = \oint q(\cos \alpha dh + \sin \alpha r d\theta).$$

By Green's Theorem, this vanishes identically if and only if

$$\frac{\partial}{\partial \theta} (q \cos \alpha) = \frac{\partial}{\partial h} (rq \sin \alpha).$$

Since $r = h + s(\theta)$, $\partial r / \partial h = 1$; while by P1, q and α are functions of θ alone. Hence irrotationality is equivalent to

$$(53) \quad \frac{d}{d\theta} (q \cos \alpha) = q \sin \alpha.$$

Letting $g(\theta) = q \cos \alpha$, and $h(\theta) = q \sin \alpha$, this is $g' = h$, generalizing (7) of Section 3.

To express the condition of mass conservation, observe that the rate of mass flux *outward* through a curve γ is

$$\oint_{\gamma} \rho q (-\sin \alpha dh + \cos \alpha r d\theta).$$

By Green's Theorem, this vanishes identically if and only if

$$(54) \quad \frac{\partial}{\partial \theta} (-\rho q \sin \alpha) = \frac{\partial}{\partial h} (r \rho q \cos \alpha) = \rho q \cos \alpha.$$

In the notation just introduced, this reduces to the equation $(-\rho h)' = \rho g$, where the accent denotes differentiation with respect to θ . Substituting $h = g'$ and simplifying, this is $\rho'g' + \rho g'' + \rho g = 0$, or $(g'' + g) + (\rho'/\rho)g' = 0$, which is (9) of Section 3. Hence *all flows satisfying Condition P1 can be obtained from Prandtl-Meyer flows*, by replacing the radii from a fixed corner by the tangents to a fixed curve Γ , and using the same vector velocity at corresponding points.¹⁶

¹⁶ This result is due to F. H. Lees, *Proc. Camb. Phil. Soc.*, 22 (1924), 350-362; see also W. Tollmien [69]; and [60], pp. 273-278.

The preceding illustration is a special case of the more general "inverse problem" of finding all flows with one-dimensional hodographs—that is, all flows whose velocity vectors are confined to lie on a single *curve*.¹⁷

These investigations are all very interesting, and it has been freely conjectured that they are closely connected with group-theoretic ideas. However, there is as yet no conclusive evidence to support this conjecture.

What does seem to be certain is the following. Most of the special coordinate systems which lead to useful separations of variables or other formal simplifications in fluid mechanics have been suggested by group-theoretic ideas relating to internal symmetries of the partial differential equations involved. Formal postulation of these coordinate systems then yields indirectly, through the "inverse method," flows which are usually closely related to the "self-symmetric" flows directly suggested by group-theoretic ideas. The postulation of formal properties which are not related to groups will usually yield only uninteresting flows, or no flows at all.

14. Hodograph Method and Groups

Another method of simplifying the equations of flow is the so-called "hodograph method." I shall now exhibit two connections between this method and group theory.

We have already seen (Chapter III, Eq. (35')) that the irrotational plane flows of a compressible non-viscous fluid correspond one-one to the velocity potentials U which satisfy the *non-linear* partial differential equation

$$(55) \quad \nabla^2 U = \frac{1}{c^2} \{ U_x U_x U_{xx} + 2 U_x U_y U_{xy} + U_y U_y U_{yy} \}.$$

Here subscripts represent differentiations with respect to the variable involved, and c^2 is the local velocity of sound—equal to $\frac{1}{2}(\gamma - 1)(a^2 - q^2)$, where a is the constant "escape velocity," in the case of a polytropic equation of state $p = kp^\gamma$.

¹⁷ For the incompressible, viscous case, see W. Müller, *Einführung in die Theorie der zähen Flüssigkeiten*, Leipzig, 1932; also *Zeits. ang. Math. Mech.*, 13 (1938), 395–408. For the compressible, non-viscous case, see [69]; also unpublished papers of Giese and G. F. J. Temple.

It is well-known¹⁸ that (55) is equivalent to either of the linear partial differential equations

$$(56.1) \quad q^2 V_{qq} + q \left(1 + \frac{q^2}{c^2}\right) V_q + \left(1 - \frac{q^2}{c^2}\right) V_{\theta\theta} = 0,$$

or

$$(56.2) \quad q^2 \phi_{qq} + q \left(1 - \frac{q^2}{c^2}\right) \phi_q + \left(1 - \frac{q^2}{c^2}\right) \phi_{\theta\theta} = 0.$$

Here V is the stream function; $qc^{*\theta}$ is the complex velocity vector, so that $U_x = q \cos \theta$ and $U_y = q \sin \theta$; c^2 is a single-valued function of q , by the Bernoulli equation; and $\phi = U - xU_x - yU_y$ is the dependent variable of the Legendre contact transformation giving (56.2).

Now it is not a difficult matter to calculate (56.1)–(56.2) from (55) and the preceding definitions, but it is quite obscure why the hodograph variables leading to these linear equations should have been tried. It may be that one reason was the success of the hodograph method in free streamline problems (as in Chapter II, Section 2). I shall now give an alternative motivation in terms of three group-theoretic considerations.

The first group-theoretic consideration is the invariance of the laws of non-viscous fluid dynamics under the group of Langevin transformations

$$(5) \quad \mathbf{x} \rightarrow \lambda \mathbf{x}, \quad \mathbf{u} \rightarrow \mathbf{u}, \quad U \rightarrow \lambda U, \quad V \rightarrow \lambda V, \quad \phi \rightarrow \lambda \phi.$$

We do *not* have invariance under $\mathbf{x} \rightarrow x$, $\mathbf{u} \rightarrow \lambda \mathbf{u}$, except in the incompressible case (cf. Chapter III, Section 18). It follows that a partial differential equation expressing these laws must be *non-homogeneous* (and hence *non-linear*) if the x_i are taken as the independent variables, and U , V , or ϕ as dependent variables—while it will be *homogeneous* if the u_i are taken as independent variables¹⁹ and U , V , or ϕ as the dependent variables.

¹⁸ See S. Bergmann, *The Hodograph Method in the Theory of Compressible Fluid*, Brown University, 1941; or H. Seifert, "Die hypergeometrische Differentialgleichungen der Gasdynamik," *Math. Annalen*, 120 (1947), 75–126.

¹⁹ This is possible locally except in the case, mentioned in Section 12, of a one-dimensional hodograph. It is not generally possible in the large, and this leads to mathematical difficulties in applying the hodograph method, which have not yet been surmounted.

I confess it is not yet clear to me *a priori* why this homogeneous equation should be linear when V and ϕ are taken as the dependent variables, and I have not determined what kind of equations would arise if the equally plausible U was used as dependent variable.

A second relevant group-theoretic consideration is the obvious invariance of the laws of fluid motion under the group of *rotations* $\theta \rightarrow \theta + \alpha$, when q , U , V , ϕ are held fixed. This implies that in (56.1)–(56.2), θ will only appear in differential operators, and not in coefficients. Hence we have a group-theoretic motivation for the use of q , θ as independent variables²⁰ in place of $u = U_x$ and $v = U_y$. It makes the coefficients of our linear homogeneous partial differential equations functions of *one* of the two independent variables only.

An analogous third group-theoretic observation is the obvious invariance of the laws of fluid motion under the group of *translations* $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}$, when $\mathbf{u}$, U , V , ϕ are held fixed. This has been commented on in Chapter III, and it too is equivalent to the occurrence of x and y in (42) as differential operators alone, and not in coefficients.

This same group-theoretic observation seems related to the classic procedure for solving problems in free streamline theory (Chapter II) by the hodograph method. Indeed, let z , ζ , W be the complex position, complex conjugate velocity, and complex potential respectively. Any physical condition $G(z, \zeta, W) = 0$ involving these three quantities will be invariant under the complex translation group $z \rightarrow z + z_0$; hence it must be equivalent to a relation of the form $G(\zeta, W) = 0$, or, solving, $W = F(\zeta)$; it is to relations of this form that conformal transformations lead. Again, any first-order ordinary differential equation $H(z, \zeta, d\zeta/dz) = 0$, to be invariant under the translation group $z \rightarrow z + z_0$, must be [61] integrable by a *quadrature*—and it is to such a quadrature, of the special form $z = \int (F'(\zeta) d\zeta/\zeta)$ that the hodograph method leads.

Lest the connection between quadratures and invariance under translations seem too abstruse and mathematical, I

²⁰ The use of $w = \ln q$ instead of q is suggested by complex variable theory, so as to make $w + i\theta$ the complex logarithm of $u + iv$; see Chapter II.

shall now give an example²¹ where this connection is utilized repeatedly.

15. Nonlinear Theory of Inertial Plane Motion

Let a missile be moving in a plane, so rapidly that only "inertial" forces (Chapter III, Section 12) need be considered—gravity and friction are thus considered negligible. We let $x = q_1$ and $y = q_2$ denote the coordinates of the missile, and $\psi = q_3$ the angle between the x -axis and an axis fixed in the missile. We assume²² that the instantaneous coordinates of position and rate of change of position of the missile determine its future trajectory through force-reactions and Newton's laws, so that (subject to the usual limitations on differentiability etc.)

$$(57) \quad \ddot{q}_i = F_i(q_1, q_2, q_3; \dot{q}_1, \dot{q}_2, \dot{q}_3) = F_i(\mathbf{q}; \dot{\mathbf{q}}).$$

This is evidently a *sixth* order system of ordinary differential equations, which are in general nonlinear.

I shall now show how it can be reduced to a *second* order system and four quadratures, by use of group theory. The method indicated generalizes in principle the usual method of "ignorable coordinates,"²³ from Lagrangian to non-Lagrangian dynamical systems. After describing the reductions, I shall sketch (in Section 16) the further extension to general ordinary differential equations invariant under a group.

First, the system (57) may be expected to be invariant under *translations* $x \rightarrow x + x_0$, $y \rightarrow y + y_0$ of space-coordinates.²⁴ This has a very simple mathematical interpretation: it permits us to replace (57) by the *fourth* order system

²¹ For a rigorous modern mathematical discussion of this classic relation, in the case of (ordinary) homogeneous *linear* differential equations, see E. R. Kolchin, "Algebraic matrix groups and the Picard-Vessiot theory, etc.," *Annals of Math.*, 49 (1948), 1-42. For the non-linear case, see Dickson [61].

²² This hypothesis is satisfied by all the forces usually considered including those of drag, virtual mass, etc.

²³ As described for example in [70], p. 54. With non-zero C_D , the system discussed here is non-Lagrangian.

²⁴ In practice, this means that such factors as the variation of density with altitude can be neglected.

$$(58) \quad \begin{aligned} \frac{d\dot{q}_1}{dt} &= F_1(q_3; \dot{\mathbf{q}}), & \frac{d\dot{q}_2}{dt} &= F_2(q_3; \dot{\mathbf{q}}), \\ \frac{dq_3}{dt} &= \dot{q}_3, & \frac{d\dot{q}_3}{dt} &= F_3(q_3; \dot{\mathbf{q}}), \end{aligned}$$

and the two quadratures

$$(58') \quad q_1 = \int \dot{q}_1 dt, \quad q_2 = \int \dot{q}_2 dt.$$

Thus a two-parameter group permits us to reduce the order of our system by two, replacing integrations by quadratures.

Second, the system (57) is presumably "isotropic", that is, invariant under *rotations* of coordinates. To express this fact analytically it is convenient to use, as new variables, the scalar velocity $v = (\dot{x}^2 + \dot{y}^2)^{1/2}$, and the inclination θ of the trajectory to the x -axis. This makes $\dot{x} = v \cos \theta$ and $\dot{y} = v \sin \theta$. Clearly v and the "pitch angle" $\phi = \psi - \theta$ are invariant under rotations; hence (58) is equivalent²⁵ (if isotropic) to a system

$$(59) \quad \begin{aligned} \frac{dv}{dt} &= G_1(v, \phi, \phi), & \frac{d\theta}{dt} &= G_2(v, \phi, \phi), \\ \frac{d\phi}{dt} &= \phi, & \frac{d\phi}{dt} &= G_3(v, \phi, \phi), \end{aligned}$$

the second of which reduces to the quadrature $\theta = \int \dot{\theta} dt$.

Finally, the hypothesis that all forces are "inertial" means that they are proportional to velocity squared, i.e. that space trajectories are invariant under the group of changes in the time scale. But clearly ϕ and *distance* $s = \int v dt$ are invariant under this group. Hence if we replace t by the independent variable s , we will have

$$(60) \quad \frac{d\phi}{ds} = \phi', \quad \frac{d\phi'}{ds} = H(\phi, \phi'), \quad \frac{dv}{v ds} = H^*(\phi, \phi').$$

(For example, $mv dv/ds$ is the tangent force-component; this is v^2 times the force $H^*(v/v, \phi, \phi/v) = H^*(1, \phi, \phi')$ which would act if all velocities were reduced in the ratio $v:1$). In summary, (57) is equivalent to the second order system (60), plus

²⁵ Strictly speaking, as long as v does not vanish.

the five quadratures (58') and

$$(60') \quad v = v_0 e^{\int H^* ds}, \quad t = \int \frac{ds}{v}, \quad \theta = \int \dot{\theta} dt.$$

16. Generalization

The preceding reduction can be considerably generalized. Let Σ be any n th order system of ordinary differential equations:

$$(61) \quad \frac{dx_i}{dt} = F_i(x_1, \dots, x_n) \quad [i = 1, \dots, n].$$

Suppose Σ is invariant under a group Γ of transformations $\mathbf{x} \rightarrow \gamma(\mathbf{x})$ of $(x_1, \dots, x_n)$ -space, meaning by this that if $\mathbf{x}(t)$ satisfies (61), then so do its transforms $\gamma(\mathbf{x}(t))$, for all $\gamma \in \Gamma$. We shall show that this information is of real assistance in the explicit integration of (61).

In the case Γ is a one-parameter group, the procedure is easy. In this case, except near singular points, Γ is locally equivalent²⁶ by a change of coordinates to the group of translations $y_1 \rightarrow y_1 + a$; $y_2, \dots, y_n$ constant. Under these coordinates, (61) becomes $dy_i/dt = G_i(y_1, \dots, y_n)$. Since subtraction of a constant from y_1 does not alter any dy_i/dt , we see that the G_i are really independent of y_1 , giving

$$(62) \quad \frac{dy_i}{dt} = G_i(y_2, \dots, y_n) \quad [i = 1, \dots, n].$$

Hence we reduce the integration of (61) to the integration of an $(n-1)$ st order system $dy_j/dt = G_j(y_2, \dots, y_n)$ [$j = 2, \dots, n$], and the quadrature $y_1 = \int G_1(y_2(t), \dots, y_n(t)) dt$.

More generally, let Γ be any r -parameter solvable Lie group of transformations of $(x_1, \dots, x_n)$ -space, which leaves (61) invariant. Then, almost by definition, Γ possesses a series of local²⁷ Lie subgroups $S_1 < S_2 < \dots < S_r = \Gamma$, such

²⁶ See L. P. Eisenhart, *Continuous Groups of Transformations*, Princeton, 1933, p. 35. In general, I shall not give detailed references for the results from Lie theory assumed.

²⁷ The concept of a local Lie subgroup is given in [59]. The older books are somewhat vague on this point.

that (i) S_{i-1} is normal in S_i , (ii) S_i is generated by S_{i-1} and a one-parameter subgroup Γ_i .

Locally, let us assume that the subsets of transitivity of S_{i-1} are the h -dimensional subspaces $y_{h+1}, \dots, y_n$ constant, i.e. the parallels to the $(y_1, \dots, y_h)$ -plane for some h . Suppose further that by consideration of the invariance of (61) under S_{i-1} , we have been able to reduce the integration of (61) to the integration of

$$(62') \quad \frac{dy_j}{dt} = G_j(y_{h+1}, \dots, y_n) \quad [j = h+1, \dots, n]$$

and quadratures. I shall show that the corresponding result holds for S_i .

There are two cases. If the subsets of transitivity of S_i are h -dimensional, the result is trivial. Otherwise, since S_{i-1} is a normal (i.e. invariant) subgroup of S_i , the sets of transitivity²⁸ of S_{i-1} are permuted non-trivially by Γ_i . By proper choice of coordinates, we can suppose that Γ_i effects the translations $y_{h+1} \rightarrow y_{h+1} + a; y_{h+2}, \dots, y_n$ constant. Hence, as with (62), we can reduce the integration of (62') to the integration of

$$(62'') \quad \frac{dy_j}{dt} = G_j(y_{h+2}, \dots, y_n) \quad [j = h+2, \dots, n],$$

and the quadrature $y_{h+1} = \int G_{h+1}(y_{h+2}(t), \dots, y_n(t)) dt$. This completes the proof by induction of the following

THEOREM. *Let an n^{th} order system of ordinary differential equations Σ be invariant under a solvable Lie group having m -dimensional sets of transitivity. Then the integration of Σ can be reduced to the integration of an $(n-m)^{\text{th}}$ order system, and quadratures.²⁸*

In Section 15, Γ_1 was the group $x \rightarrow x + a$, Γ_2 was $y \rightarrow y + b$, Γ_3 was $\theta \rightarrow \theta + \alpha$, $x \rightarrow x \cos \alpha - y \sin \alpha$, $y \rightarrow x \sin \alpha + y \cos \alpha$, and Γ_4 was $t \rightarrow t/\lambda$, $x \rightarrow x$, $v \rightarrow \lambda v$, etc. Hence the

²⁸ By definition, a "set of transitivity" of S_{i-1} is, for some point $\mathbf{y}$, the set $\mathbf{y}$ of all $\sigma(\mathbf{y})$ [$\sigma \in S_{i-1}$]. Since $\gamma \in \Gamma_i$ and $\sigma \in S_{i-1}$ implies $\gamma^{-1}\sigma\gamma \in S_{i-1}$, the set of all $\gamma(\sigma(\mathbf{y})) = (\sigma\gamma)(\mathbf{y})$, equals the set of all $\sigma(\gamma(\mathbf{y}))$, and hence also a set of transitivity of S_{i-1} . See A. Speiser, *Gruppentheorie*, 2nd ed., p. 117.

preceding theorem, which appears to be new, is a direct generalization of Section 15.

17. *Summary*

We have seen that group theory often makes possible reductions in the number of independent variables involved in partial differential equations, directly through the method of search for symmetric solutions and the method of "separation of the time variable," and indirectly through "inverse methods." Moreover the method of search for symmetric solutions can be relied upon in general to give solutions locally (Section 9).

Even after the number of independent variables is reduced to one, so that no further reduction by the preceding methods is possible, the resulting system of ordinary differential equations can often be most easily integrated by use of group-theoretic considerations.

The preceding methods apply to non-analytic as well as to analytic, and to non-linear as well as to linear differential equations; thus they are free from the limitations of the usual methods of expansion in series or integrals. Hence group theory certainly plays a fundamental role in solving the differential equations of fluid mechanics.

The role of group theory in fluid mechanics is, moreover, not restricted to its usefulness in solving differential equations. We have already seen that the theories of modeling and of "dimensional analysis" can be regarded as special cases of group theory. In my next lecture, I shall try to show that group theory also underlies the classical force equations for a rigid body in a perfect (i.e. incompressible non-viscous) fluid.

I hope that, in the future, the debt of mechanics to the concepts of group theory will be more explicitly recognized.

CHAPTER V

Virtual Mass and Groups

1. Introduction: Virtual Mass of Sphere

Qualitatively, the idea of virtual mass is a familiar one. For example, let a light paddle be dipped into still water and then suddenly given a rapid acceleration broadside. It is a matter of common experience that the apparent inertia (i.e. resistance to acceleration) of the paddle is greatly increased by the water around it. This increased inertia is what is called the "virtual mass" of the paddle, the difference between the real mass and the virtual mass being called the "induced mass."

The idea of virtual mass was first given an exact mathematical interpretation by Green and Stokes, a little over a century ago.¹ The essence of their discussion went somewhat as follows.

Consider a sphere of mass m and radius a moving with speed v through an *incompressible non-viscous* fluid of density ρ (throughout the whole of the present chapter, we shall consider only Dirichlet flows of such a "perfect fluid"). Without loss of generality, we may choose the axis of spherical coordinates as the direction of motion. Relative to the fluid at infinity, the velocity potential U is well-known to be the dipole potential given in spherical coordinates by

$$(1) \quad U = \frac{-a^3 v \cos \theta}{2r^2}.$$

In fact, one easily verifies that the normal derivative $\partial U / \partial r = v \cos \theta$ gives the normal component of motion of the surface of the sphere (Chapter I). At a general point of the fluid,

¹ G. Green, *Mathematical Papers*, p. 315 (1833); G. Stokes, *Mathematical and Physical Papers*, Vol. 1, p. 17, (1843). See [9], pp. 123–124, for a bibliographical discussion. The physical idea of virtual mass (also called "added" or "carried" mass), was expressed in a very crude way in 1786, by the Comte DuBuat (see L. G. Dubuat, *Principles d'hydraulique*, 1816 ed., Vol. 2, p. 222).

the radial and angular components of velocity are

$$u_r = \frac{a^3 v \cos \theta}{r^3}, \quad u_\theta = \frac{\partial U}{r \partial \theta} = \frac{a^3 v \sin \theta}{2r^3}.$$

Hence the total kinetic energy of the fluid is

$$\begin{aligned} T &= \int \int \int \frac{1}{2} \rho (u_r^2 + u_\theta^2) r^2 \sin \theta \, dr d\theta d\phi \\ &= \pi \rho v^2 a^6 \int_{r=a}^{\infty} r^2 dr \int_0^\pi \sin \theta \left[\frac{\cos^2 \theta}{r^6} + \frac{\sin^2 \theta}{4r^6} \right] d\theta \\ &= \frac{\pi \rho v^2 a^6}{4} \int_{r=a}^{\infty} \frac{dr}{r^4} \int_0^\pi [1 + 3 \cos^2 \theta] \sin \theta \, d\theta \\ &= \frac{\pi \rho v^2 a^6}{4} \left[\frac{-1}{3r^3} \right]_a^\infty [-\cos \theta - \cos^3 \theta]_0^\pi \\ &= \frac{\pi \rho v^2 a^3}{3} = \frac{1}{2} \frac{2\pi \rho a^3}{3} v^2 = \frac{1}{2} m' v^2. \end{aligned}$$

This gives the following classical result. The kinetic energy of the fluid is equal to that which would be possessed by a particle moving with the same speed as the sphere, and whose mass m' is equal to *half the mass of displaced fluid*.

Moreover it is evident that, in a non-viscous fluid, rotation of the sphere has no effect on the surrounding fluid; hence the moment of inertia of the sphere should be unaffected. This suggests that (neglecting gravity) a sphere in such a fluid should be dynamically equivalent to a heavier sphere in vacuo, whose "virtual mass" $m^* = m + m'$ is the mass m of the sphere plus an "induced mass" m' equal to half the mass of displaced water, but whose moment of inertia $2ma^2/5$ is unchanged. This will be proved rigorously in Section 15, where it will be shown that the entire dynamic action of any Dirichlet flow can be deduced from its kinetic energy.

2. Physical Validity of Conclusion

Although it is not to be supposed that the physical facts are as simple in real fluids as those predicted by the preceding theoretical argument, there are various observable experi-

mental phenomena for which "virtual mass" gives the best known explanation.

Thus, as already stated, an object at rest in a dense fluid does experience an extra resistance to acceleration. Generally speaking, *resistance to acceleration from rest agrees with the mathematical theory.*² Various special cases should be mentioned.

One classical application is to the initial acceleration experienced by a spherical hydrogen balloon, suddenly released from a mooring. Suppose the mass of the balloon is $\frac{1}{10}$ that of the displaced air, if suddenly released. A person unaware of virtual mass would probably make the following erroneous computation. The hydrostatic buoyancy is, by Archimedes' Law, $10g$ times the mass of the balloon; hence (he would compute) the initial acceleration should be $10g$. In the case of a spherical hydrogen balloon immersed in water, an acceleration of at least $1000g$ would be erroneously expected.

However, using the theory of virtual mass, it is easy to predict the correct initial acceleration. The *virtual mass* m^* of the balloon is $.1 + .5 = \frac{3}{5}$ that of the displaced air; hence the actual acceleration is $5g/3$. In water, it would be about $2g$.

A more subtle application arises if a weightless sphere is immersed in a liquid, and the whole liquid is suddenly given an acceleration a . What is the acceleration a^* of the sphere, relative to an outside observer? This problem may be treated as follows. The acceleration a is equivalent, for an observer in the fluid, to an apparent gravitational field of strength a . Hence, by the reasoning just given, the initial acceleration $a^* - a$ of the sphere relative to an observer in the fluid, should satisfy $a^* - a = 2a$, or $a^* = 3a$.

This prediction was confirmed by T. E. Caywood and the author [19], where other references are given; it proved essential in interpreting experimental data on fluid flow patterns.

² Surprisingly few dynamical experiments have been performed to test this hypothesis; see however P. Hirsch, *Zeits. ang. Math. Mech.*, 3 (1923), 93-107, and Thesis, Goettingen, 1918, *Aer. Res. Council Reps.* 590, 612, 1166; G. Cook, *Phil. Mag.*, 39 (1920), 350-352. There is also supporting photographic evidence. See Blasius, *Zeits. Math. u. Physik*, 36 (1908), 20-37; Tollmien, *Handbuch Exp. Physik*, Vol. 4, Part 1 (1931), 272-279; S. Goldstein and L. Rosenhead, *Proc. Camb. Phil. Soc.*, 56 (1938), 20-37.

3. Induced Mass Tensor

In the case of the sphere, induced mass can be treated as a scalar; but for most bodies, we must treat induced mass as a matrix or "tensor."

The most convenient mathematical basis for such a treatment is furnished by computations for the total kinetic energy of the fluid involved. We consider Dirichlet motions of a perfect fluid bounded internally and/or externally by a rigid body ("solid") Σ , and constrained to move with Σ . Let U^1, U^2, U^3 be the velocity potentials induced by translations of Σ from a given reference position parallel to the three coordinate axes; and let U^4, U^5, U^6 be the velocity potentials for rotation about these axes, with unit angular velocity. Then if the solid has instantaneous velocity-components $\dot{q}_1, \dots, \dot{q}_6$ with respect to the six degrees of freedom of motion just mentioned, the velocity-potential U in the fluid satisfies $U = \dot{q}_k U^k$, by the principle of superposition for a perfect fluid. Here repeated indices denote summation. It follows that the total kinetic energy T of the fluid satisfies

$$(2) \quad 2T = \rho \iiint \nabla U \nabla U dR = \dot{q}_h \dot{q}_k \iiint \rho \nabla U^h \nabla U^k dR = \dot{q}_h \dot{q}_k T_{hk},$$

where $dR = dx_1 dx_2 dx_3$ is the differential of volume. The symmetric matrix $||T_{hk}||$ is called the "induced mass tensor"; its coefficients depend on the choice of axes in the body; we shall discuss this dependence further in Section 5.

The coefficients T_{hk} have a simple dynamical interpretation. By Green's Second Identity ([7], p. 212), we have

$$(3) \quad T_{hk} = \rho \iiint \nabla U^h \nabla U^k dR = \rho \iint U^h \left(\frac{\partial U^k}{\partial n} \right) dS.$$

But now $\partial U^k / \partial n$ is equal (Chapter I, (8)) to the normal velocity-component of Σ under unit velocity in the q_k direction. I shall introduce, for use now and later, the convenient notation $dS_k = (\partial U^k / \partial n) dS$, so that

$$(4) \quad \begin{aligned} dS_1 &= dx_2 dx_3, & dS_2 &= dx_3 dx_1, & dS_3 &= dx_1 dx_2, \\ dS_4 &= x_2 dS_3 - x_3 dS_2, & dS_5 &= x_3 dS_1 - x_1 dS_3, \\ dS_6 &= x_1 dS_2 - x_2 dS_1. \end{aligned}$$

It is evident from the definition itself that

$$(3') \quad T_{hk} = \rho \iint U^h dS_k = \rho \iint U^k dS_h.$$

It is also evident that if p denotes the scalar pressure, then $\iint p dS_1$, $\iint p dS_2$, $\iint p dS_3$ are the components of *force* exerted by Σ on the fluid, while $\iint p dS_4$, $\iint p dS_5$, $\iint p dS_6$ are the corresponding components of *moment*.

Now consider unit acceleration from rest in the q_h -direction. It is easy to calculate that, initially, $U = 0$ and $\partial U / \partial t = U^h$: $U(x;t)$ differs from $tU^h(x)$ by infinitesimals of the second order in t . We now apply Bernoulli's equation for pressure in an accelerated fluid,

$$(3^*) \quad p + \frac{1}{2} \rho \nabla U \nabla U + \frac{\rho}{\partial t} U = \text{const.},$$

neglecting hydrostatic buoyancy forces (Chapter III, Section 12). This shows that the initial hydrodynamic pressure p^h is everywhere equal to ρ times the "acceleration potential" U^h . Hence, substituting in (3'), $T_{hk} = \iint p^h dS_k$, and T_{hk} is the k -component of force for unit acceleration from rest in the q_h -direction.

Incidentally, because of the symmetry in h, k of T_{hk} evident from the kinetic energy definition (2), we obtain immediately the following Reciprocity Principle. The k -component of force under unit h -acceleration is equal to the h -component of force under unit k -acceleration.

The preceding formulas refer to the induced or "carried" mass. The total inertia or *virtual mass*, defined as the internal (rigid body) inertia of Σ plus the induced mass of the external fluid, is clearly another symmetric tensor (matrix), with just the same properties. We shall now discuss the algebraic simplification of the tensor of total virtual inertia.

4. Reduction to Canonical Form

As we have already seen (Section 1), the sphere is a very simple case. If axes are taken through the center of the sphere, then $T_{11} = T_{22} = T_{33} = m^*$, $T_{44} = T_{55} = T_{66} = 2ma^2/5$, and all T_{ij} [$i \neq j$] off the main diagonal are zero.

Another familiar special case concerns a rigid body *in vacuo*.

If the origin is taken at the center of gravity, then all T_{ij} [$i \neq j$ and i or $j = 1, 2, 3$] vanish, while $T_{11} = T_{22} = T_{33} = m$, the mass of the body. Again, by choosing Cartesian axes as the principal axes of inertia, we can make all T_{ij} [$i \neq j$ and $i, j = 4, 5, 6$] vanish. Hence we can characterize the inertia tensor by four scalars $T_{11}, T_{44}, T_{55}, T_{66}$, which can be reduced to *two* by a change of length and time units. Further, $T_{44} + T_{55} \geq T_{66}$ and cyclically; the case of an ellipsoid is perfectly general.

In the preceding two cases, the matrix $||T_{ij}||$ could be reduced to diagonal form by proper choice of axes. It is interesting to see how far one can simplify the matrix of inertial coefficients T_{ij} , by appropriate choice of Cartesian axes and a "center of virtual inertia" in the general case of a fluid of positive density. This is a simple exercise in the theory of quadratic forms.

The example of the paddle (Section 1) shows that the "virtual inertia" for translation can be different in different directions; sidewise acceleration is much easier than broadside acceleration. However, since any quadratic form is equivalent³ to a diagonal form under the rotation group, we can always rotate axes so as to get $T_{12} = T_{23} = T_{31} = 0$, and "virtual inertias" of T_{11}, T_{22}, T_{33} in "principal directions" of translation.

Except in the degenerate case of a weightless plane lamina, where one of the T_{ii} [$i = 1, 2, 3$] vanishes, we can achieve a further simplification by proper choice of the origin as a "virtual center of inertia." Let w_1, w_2, w_3 denote rotation at one radian per second about one set of axes parallel to the principal directions of translation; let X, Y, Z denote the translations at unit speed in the principal directions; and let w'_1, w'_2, w'_3 denote rotations about axes translated through the vector (x, y, z) . Then

$$(5) \quad \begin{aligned} w'_1 &= w_1 + yZ - zY, \\ w'_2 &= w_2 + zX - xZ, \\ w'_3 &= w_3 + xY - yX. \end{aligned}$$

³ The algebraic theorems assumed here are proved for example in [37], Chapter IX, Theorems 19, 20. For the results see [9], Section 126; the general case was treated by Clebsch, *Math. Annalen*, 3 (1870), 238.

The substitution of w'_1 for w_1 does not change the interaction energy T_{14} between X and w_1 , since there is none between X and Y or Z . On the other hand, it increases $T_{15} = T_{51}$ by zT_{11} , increases $T_{16} = T_{61}$ by xt_{22} , decreases T_{24} by zT_{22} , and so on.

Hence we can make $T_{15} = T_{24}$ by proper choice of z , and similarly $T_{16} = T_{34}$ and $T_{26} = T_{35}$. Thus we can reduce the matrix of inertial coefficients to the simplified form of Fig. 21.

Translation			Rotation		
α	0	0	ρ	ξ	η
0	β	0	ξ	σ	ζ
0	0	γ	η	ζ	τ
ρ	ξ	η	T_{44}	T_{45}	T_{46}
ξ	σ	ζ	T_{45}	T_{55}	T_{56}
η	ζ	τ	T_{46}	T_{56}	T_{66}

FIG. 21.

In this canonical form, we have *fifteen arbitrary constants*, which can be reduced to thirteen by a change of length and time unit. That is, the general case involves *thirteen* dimensionless ratios, and two choices of unit.

If a body has three perpendicular planes of symmetry, like an ellipsoid, we can take these as coordinate planes. Reflection in (say) the (x,y) -plane leaves Z , w_1 , w_2 fixed but reverses the signs of X , Y , w_3 ; it also leaves kinetic energy unchanged. Hence $T_{31} = -T_{31} = 0$, and similarly

$$T_{32} = T_{36} = T_{41} = T_{42} = T_{46} = T_{51} = T_{52} = T_{56} = 0$$

Repeating the argument for the other coordinate planes, we see that the matrix of inertial coefficients can be diagonalized. Thus we have *six* arbitrary constants, and *four* dimensionless ratios. (There are two in the case of a rigid body in vacuo.)

Another interesting case is that of symmetry in three perpendicular axes, though not in the planes through them. This is typified by the helix $x = r \cos z$, $y = r \sin z$, $|r| \leq 1$, $|z| \leq 2\pi$. Symmetry in the z -axis leaves Z , w_3 fixed but reverses the signs of X , Y , w_1 , w_2 . Hence as above,

$$T_{13} = T_{23} = T_{43} = T_{53} = T_{16} = T_{26} = T_{46} = T_{56} = 0.$$

Repeating the argument for the other coordinate planes, we see that all inertial coefficients off the principal diagonal vanish, except the "helical products of inertia" ρ , σ , τ . We thus have *nine* coefficients of inertia,⁴ which we can reduce after changes of length and time scales to *seven* essentially independent parameters.

5. Induced Mass and Momentum

In Section 3, we interpreted the coefficients of induced mass⁵ in terms of kinetic energy and the force required for acceleration from rest, respectively. It is possible also to assign simple *momentum* interpretations to these coefficients.

The convenience of the definition (3) of the T_{hk} as kinetic energy integrals was, that these integrals converge at infinity. To see this, we recall that (Chapter I, Section 6) $\nabla U = O(r^{-3})$ in space, whence $\iiint (\nabla U \nabla U) dR = O(r^{-6} r^2 dr)$ is finite. The kinetic energy integral also converges at infinity in the case of plane Dirichlet flows; in this case $\iint (\nabla U \nabla U) dx dy = O(r^{-4} r dr)$ is finite.

We shall now interpret the T_{hk} as momentum integrals. It has been noted by various authors⁶ that the momentum integrals diverge in the usual sense. Hence to interpret the T_{hk} in terms of momentum, some care must be taken. Now for the details.

The T_{hk} of (3') are integrals over the boundary S of Σ , which can be written in a new notation as

$$(3'') \quad T_{hk} = \rho \int \int_S U dS_K,$$

$U = U^h$ being harmonic and $dS_K = \sum_{i=1}^3 K_i dS_i$ expressing the differential of flux of a vector field $\mathbf{K} = (K_1(\mathbf{x}), K_2(\mathbf{x}), K_3(\mathbf{x}))$

⁴ The only references which I have found on the reductions of Section 4 are Kelvin's papers [71], esp. pp. 384–390, where "helical symmetry" is discussed.

⁵ The reductions to canonical form described in Section 4 apply equally to the induced mass and to the virtual mass tensor—naturally, with different characteristic axes in general. I treated the case of virtual mass in Section 4, because the familiar special case of a rigid body *in vacuo* is so suggestive.

⁶ [9], p. 161; H. Tollmien, *Zeits. ang. Math. Mech.*, 18 (1938), 154.

through S . Thus in the case $k = 1$ of translations parallel to the x_1 -axis, $\mathbf{K} = (1, 0, 0)$; in the case of rotations about the x_1 -axis, $\mathbf{K} = (0, x_3, -x_2)$, and so on. This convenient notation defines a useful class of Stieltjes integrals over surfaces in space, whenever $\iint |dS_K|$ is finite. Note that whenever $\mathbf{K}$ is the velocity-field of a rigid motion, $\text{div } \mathbf{K} = 0$. This condition, and the condition that U is a harmonic function "regular" at infinity ([7]), p. 217) are all that we require for the argument below.

We define a " K -line" as an integral of $dx_i = K_i dt$. Thus if $\mathbf{K}$ represents translation parallel to the x_1 -axis, the $\mathbf{K}$ -lines are the parallels to this axis; if $\mathbf{K}$ corresponds to screw motion about the x_1 -axis, the K -lines are helices about this axis, etc. We then apply the Divergence Theorem to the vector field $U\mathbf{K}$, over regions R bounded by surfaces $S + S' + S''$, where S'' consists of K -lines. Since $dS_K \equiv 0$ on the surface S'' consisting of K -lines, and since

$$\text{div } (U\mathbf{K}) = K_i \frac{\partial U}{\partial x_i} + U \text{div } \mathbf{K},$$

we get from (3''), using $\text{div } \mathbf{K} = 0$,

$$(6) \quad T_{hk} = \rho \iiint_R \left(K_i \frac{\partial U}{\partial x_i} \right) dR - \rho \iint_{S'} U dS_K.$$

Here the relation between the signs of the double and triple integrals involved is correct, provided surface integrals are taken *into* R . In particular, if R is an *infinite* region, if S' is removed to infinity, and if the integral over R converges, we get

$$(6') \quad T_{hk} = \rho \iiint_R \left(\frac{K_i \partial U}{\partial x_i} \right) dR.$$

We shall now consider various cases of (6)–(6') corresponding to special induced mass coefficients.

If $k = 1$, then $\mathbf{K} = (1, 0, 0) = \text{grad } x_1 = \nabla x_1$ in various notations. We let S'' be an infinitely long *cylinder* parallel to the x_1 -axis and containing Σ . Then since the momentum integral

$$(6.1) \quad T_{h1} = T_{1h} = \rho \iint_S U^h dx_2 dx_3 = \rho \iiint_R \left(\frac{\partial U^h}{\partial x} \right) dR$$

converges at infinity, we get the following result. *The induced mass coefficients corresponding to translation along the x_1 -axis are equal to the x_1 -momentum inside any infinite cylinder parallel to the x_1 -axis and containing Σ , under motion with unit h -velocity.* It is simplest to take R as the cylinder circumscribed about Σ . This result, as regards T_{11} , was obtained by Theodorsen [77].

Again, if $k = 4$, then $\mathbf{K} = (0, x_3, -x_2) = x_3 \nabla x_2 - x_2 \nabla x_3$. We let S'' be any *solid of revolution* containing Σ and having the x_1 -axis as axis of symmetry. Then the boundary $S + S''$ of R consists of S and K -lines (circles), so that (6') reduces to

$$(6.2) \quad \begin{aligned} T_{h4} &= \rho \iiint_R \left(x_2 \frac{\partial U^h}{\partial x_3} - x_3 \frac{\partial U^h}{\partial x_2} \right) dR \\ &= \rho \iiint_R \left(\frac{\partial U^h}{\partial \theta} \right) dR, \end{aligned}$$

where $\theta = \tan^{-1} x_3/x_2$. Hence T_{h4} is the *moment of momentum* of the fluid in R , about the x_1 -axis.

The other T_{hk} are easily obtained from (6.1)–(6.2) by cyclic permutations of axes. Also, the case of *screw motion* about the x_1 -axis, $\mathbf{K} = \alpha \nabla x_1 + \beta(x_2 \nabla x_3 - x_3 \nabla x_2)$ is easily obtained by superposition of the two preceding formulas. We let S'' be a *circular cylinder* circumscribed about Σ . Hence the induced inertia is the screw momentum $\rho \iint (\alpha \partial U^h / \partial x_1 + \beta \partial U^h / \partial \theta) dR$ of the fluid inside S'' ; this is the same for all circular cylinders parallel to the x_1 -axis which contain Σ .

6. The Case of Rankine Bodies

The induced mass for translation can be expressed in still another way. Apply Green's Second Identity to $\mathbf{U} \nabla x_1 - x_1 \nabla U$, recalling that $\nabla^2 U = \nabla^2 x_1 = 0$. Hence if S'' is a large sphere containing Σ , and R is the region between Σ and S'' , we get from (3'), writing $U^h = U$,

$$(7) \quad \begin{aligned} T_{1h} &= \rho \int \int_S U \left(\frac{\partial x_1}{\partial n} \right) dS \\ &= \rho \int \int_{S''} x_1 \frac{\partial U}{\partial n} dS - \rho \int \int_{S''} \left(x_1 \frac{\partial U}{\partial n} - U \frac{\partial x_1}{\partial n} \right) dS; \end{aligned}$$

moreover on S'' , $-\partial/\partial n = \partial/\partial r$. The integrals over S'' can easily be evaluated asymptotically, if we expand U in the form

$$(8) \quad -U = \sum \frac{\mu_i x_i}{r^3} + O(r^{-3}), \quad \frac{\partial U}{\partial r} = 2 \sum \frac{\mu_i x_i}{r^4} + O(r^{-4}).$$

Since the area of S'' is $4\pi r^2$, the terms $O(r^{-3})$ resp. $O(r^{-4})$ contribute negligibly to (7). By symmetry, the terms in μ_2, μ_3 contribute nothing. To evaluate the remainder, we use spherical coordinates, writing $x_1 = r \sin \phi$, $dS = 2\pi d(\sin \phi)$. The integral over S'' differs by $O(r^{-1})$ from

$$\begin{aligned} 2\pi\mu_1\rho \int_{-\pi/2}^{\pi/2} \{2 \sin^2 \phi + \sin^2 \phi\} d(\sin \phi) \\ = 2\pi\rho\mu_1[\sin^3 \phi]_1^1 = 4\pi\rho\mu_1. \end{aligned}$$

Hence, passing to the limit as $r \rightarrow \infty$, we get

$$(9) \quad T_{1h} = T_{h1} = 4\pi\rho\mu_1^h - \rho \iint x_1 \left(\frac{\partial U^h}{\partial n} \right) dS,$$

where μ_1^h is the dipole moment of U^h in the x_1 -direction. We note that the implication (7) $\rightarrow$ (9) is valid for *any* potential function U , regular at infinity, with $\int_{S''} (\partial U / \partial n) dS = 0$.

This generalizes a result of Sir Geoffrey Taylor [76], who considered the case $h = 1$. In this case $(\partial U^1 / \partial n) dS$ is dS_1 , and $\iint x_1 dS_1 = \iiint dR = \text{vol}(\Sigma)$, integrating by parts over the interior Σ of S . Hence,

$$(9') \quad T_{11} = 4\pi\rho\mu_1^1 - \rho \text{vol}(\Sigma).$$

Taylor explains the presence of the term $\text{vol}(\Sigma)$ in (9'), by observing that a fluid-filled hollow in Σ will increase its induced mass by ρ times the volume of the hollow, without changing the dipole moment at infinity of U^1 .

He also notes that in the case of a *Rankine body*, $\mu_1^1 = \Sigma x_1^{(k)} e_k$ is the total dipole moment of sources for translation along its axis, is $4\pi\rho$ times the dipole moment of the source-and-sink distribution defining it, minus the mass of fluid displaced. The formula for the sphere, with $\mu_1^1 = a^3/2$ as in Section 1, (1), is a special case:

$$T_{11} = 2\pi\rho a^3 - \rho \left(\frac{4\pi a^3}{3} \right) = \frac{2\pi\rho a^3}{3}.$$

The theoretical virtual mass, under the assumption of Dirichlet flow, is also known for elliptic cylinders, ellipsoids and spheroids under translation and rotation (see [9], pp. 85, 88, 153, 159, and St. Neumark, *Zeits. ang. Math. Mech.*, 16 (1936), 117–120). The virtual mass for translation is also known for tori, spherical bowls, sphere pairs, and lenses.⁷ Finally, Polya has shown that circular cylinders have the least “average” virtual mass for translations perpendicular to their axis, and studied the corresponding conjecture for spheres.

7. More Applications of the Induced Mass Concept

The concept of induced mass has several applications besides those to acceleration from rest. I shall now describe some of these further applications. In general, the concept must be applied in a quasi-empirical manner; the flows involved do not approximate Dirichlet flows.

One application is to the correction for measured C_D of bodies in wind-tunnels with diverging walls. Such walls have been used to compensate for the pressure drop occurring with parallel walls; they clearly produce a *decelerating* air stream, and hence a pressure rise. But now, in Newtonian mechanics, a deceleration of the frame of reference is equivalent to the superposition of an apparent gravity field. Hence the effect of the deceleration of the air stream is, with Dirichlet flows, the same as if the body were accelerated in the direction of deceleration, and acted on by a “buoyancy” force corresponding to the pressure gradient producing the deceleration. After a little reflection, one sees that all forces are cancelled out except for the “induced mass” force corresponding to the

⁷ For tori, see W. M. Hicks, *Phil. Trans.*, 172 (1881), 609, and F. W. Dyson, *ibid.*, 184 (1892), 42. For spherical bowls, see A. B. Basset, *Proc. Lond. Math. Soc.*, 16 (1885), 286. For lenses, see [75], p. 279. For sphere-pairs, see W. M. Hicks, *Phil. Trans.*, 171 (1880), 455–492, and A. B. Basset, *Proc. Lond. Math. Soc.*, 18 (1887), 369–377. For Polya’s results, see *Proc. Nat. Acad. Sci.*, 33 (1947), 218–221. G. Szegő, “On the virtual mass of nearly spherical solids,” *Duke Jour. Math.*, 16 (1949), 209–223, has given an interesting discussion of inequalities satisfied by the coefficients of inertia. See also M. Schiffer and G. Szegő, *Trans. Am. Math. Soc.*, 67 (1949), 130–205.

relative acceleration of the body; hence a correction of roughly this magnitude must be made.⁸

Again, especially for calculations involving wing "flutter," it is important to be able to estimate the aerodynamic forces on an oscillating airfoil—or equivalently, those on an airfoil in gusty air. These forces have been estimated theoretically and measured in wind-tunnels, and the two estimates have been found to agree⁹ within twenty per cent.

The effect of "induced mass" on the periods of vibration of ships is also of practical importance. This has been estimated by various authors.¹⁰

It also seems necessary, in the theory of the stability of liquid-filled chemical shell, to consider their virtual inertia. In this case the fluid is bounded by external instead of internal walls; the case of ellipsoidal cavities is susceptible of complete theoretical treatment, to the extent that viscosity and turbulence can be neglected.

Likewise, when a sphere or other missile enters water, it may be supposed to lose more energy than what is required to maintain its motion at constant speed, as it has to set the water in motion. This leads one to define an "impact induced mass," corresponding to this additional loss. Such a loss has been measured.¹¹

⁸ See M. Munk, *N.A.C.A. Report 114* (1921); G. I. Taylor [77], esp. pp. 260–283. Thus the correction for the pressure gradient dp/dx in a wind-tunnel is $(V + T_{11}/m)dp/dx$, where V is the volume of the body, and m is the mass of air displaced.

⁹ See W. Gracey, *N.A.C.A. Report No. 707*; W. Scruton, *Reps. and Mem. Aer. Res. Comm.*, No. 1931; W. T. Jones, *ibid.*, Nos. 1946, 1947, 1953, 2026, 2117; W. Muller, "Zur Theorie der Kräfte bei der beschleunigten Bewegung eines Körpers in der reibungsloser Flüssigkeit," *Ing. Archiv*, 14 (1944), 332–350; L. I. Sedov, *N.A.C.A. Tech. Memo. 1156* (1947), 1–62; *Appl. Mech. Rev.*, 1 (1948), No. 309; Y. T. Yu, *Jour. Appl. Phys.*, 13 (1942), 66–69. A complete resume of the mathematical theory and literature has been given by E. Reissner, *Bull. Am. Math. Soc.*, 55 (1949), 825–850.

¹⁰ See F. Lewis, *Amer. Soc. Nav. Arch.*, Nov. (1929); J. L. Taylor, *Phil. Mag.*, 9 (1930), 161–183; M. Haskind, *Prikl. Mech.*, 10 (1946), 475–480.

¹¹ See W. Bauer, *Annalen der Physik*, 82 (1927), 1014–1016. The theoretical status of this is quite vague, since the theoretical kinetic energy of cavity motion is infinite. A theoretical basis for estimating deceleration in the initial moments of impact has been given by Th. von Karman, *N.A.C.A. Tech. Note 321* (1929); see also [75].

Finally, though cavity flow is quite different from Dirichlet flow,¹² so that the discussion of Sections 1-7 certainly does not apply, the measured cavity drag coefficient of a freely moving sphere in water varies with its specific gravity in a way which can most easily be explained by assuming that its mass is increased by an "induced mass" m' equal to about .4 times the mass of displaced water. If m is the mass of the sphere, then $m' = .4m$, whence if a denotes the deceleration, we should have

$$D = (1 + .4\sigma)ma \quad \text{instead of} \quad D = ma.$$

Hence the apparent drag $ma = D/(1 + .4\sigma)$, and the *apparent drag coefficient should be* $1/(1 + .4\sigma)$ *times that observed under conditions of no deceleration.* This relation is observed experimentally.^{12a}

8. Inertial Lagrangian Systems

Consider a dynamical system consisting of a rigid body Σ , and a perfect fluid without free surfaces bounded internally and/or externally by Σ . A classical chapter of hydrodynamics ([9], Chapter VI) concerns the behavior of such a system. It is now well-known that this model does not correspond to physical reality, except in the case of acceleration from rest. Nevertheless, it has great historical importance, and to this day no better model of comparable simplicity is known. I therefore feel justified in presenting a new mathematical treatment of it, ultimately in terms of the ideas of group theory, whose importance for other hydrodynamical questions has already been shown (Chapters III, IV).

Clearly Σ has six degrees of freedom of motion, which may be described by coordinates $q_1, \dots, q_6$. Again, if $\mathbf{q}(t)$ is given, then under very general conditions there is one and only one velocity potential (see [7]), pp. 213, 217, 311)

¹² An interesting recent note of S. Luneau (*Comptes Rendus*, 227 (1948), 823-825 and 229 (1949), 927-928), shows that the virtual mass of a disc, accelerated from rest, can be very large. This is presumably due to the entrainment of a large "wake."

^{12a} Unpublished experimental data of A. May indicate that the "virtual mass" term 0.4σ above should be 0.12σ .

$U = \dot{q}_i U^i(\mathbf{q})$ which satisfies $\nabla^2 U = 0$ and the values of $\partial U / \partial n$ given by the motion of Σ . Hence the fluid kinetic energy satisfies

$$(10) \quad T = \frac{1}{2} \iiint \rho (\nabla U \nabla U) dR = \frac{1}{2} \sum_{i,j=1}^6 T_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j;$$

moreover the combined kinetic energy of fluid and solid satisfies a similar equation with different coefficients. By choosing $q_1, \dots, q_6$ near the reference position $\mathbf{q} = \mathbf{0}$, as in Section 3, we can make the $T_{ij}(\mathbf{0})$ equal to the coefficients of induced (resp. virtual) mass already considered. But in general, $T_{ij}(\mathbf{q})$ will vary with $\mathbf{q}$ unless translations alone are considered.

Further, if S is just buoyant, gravity may be neglected; this is also approximately true if the Froude number $F = v^2/gd$ is very large.¹³ We shall consider this case, which involves no potential energy.

In the classical literature ([73], p. 238; [78], p. 320; [90], *loc. cit.*) it is usually assumed without proof that the system just defined may be treated as a "Lagrangian system," so that the "generalized forces" Q_i satisfy

$$(11) \quad Q_i = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i}.$$

I shall prove this rigorously in Section 15. For the present, I note merely that it is plausible, in view of the idea that a perfect (i.e. incompressible non-viscous) fluid moves under "workless constraints." But it is certainly not obvious; thus there are an infinite number of degrees of freedom for the possible fluid configurations.

I also note that the "generalized external forces" Q_i have a very simple meaning in the present case. We have chosen coordinates near $\mathbf{0}$ as in Section 3, so that q_1, q_2, q_3 measure translations along Cartesian axes through the center of gravity, while q_4, q_5, q_6 measure rotation about these axes. Hence, at $\mathbf{q} = \mathbf{0}$, Q_1, Q_2, Q_3 are ordinary force components along the axes, while Q_4, Q_5, Q_6 are moments about the axes. Under

¹³ As in Chapter III, Theorem 4, the effect of buoyancy is simply to superpose a constant vertical force.

any transformation of coordinates, the variational expression (11) transforms like a contravariant vector.

To sum up, then, we assume that our dynamical system is a *Lagrangian system without potential energy*, so that the forces are given by (10) and (11). Such a system may suggestively be called an *inertial Lagrangian system*. By proper choice of coordinates, we can furthermore make the T_{ij} be the ordinary "virtual mass" coefficients, and the Q_i ordinary forces and moments, at any particular instant.

9. Geometrical Interpretation

As is well known,¹⁴ the behavior of any inertial Lagrangian system can be suggestively visualized in terms of the motion of a particle of unit mass on a Riemannian variety V , in which the differential ds of arc-length is

$$(12) \quad ds^2 = \sum T_{ij}(\mathbf{q}) dq_i dq_j.$$

For on V (assumed frictionless), the kinetic energy is clearly given by (10).

It is also clear that the "natural" trajectories, which occur in the absence of external forces, are *geodesics*. Specifically, $\mathbf{Q} = \mathbf{0}$ in (11) if and only if $\int T dt$ is a minimum. This obvious application of Euler's equations is the simplest case of the Principle of Least Action, the variational formulation of dynamics.

Thus if $V = V_2$ is a frictionless surface $\mathbf{x} = \mathbf{x}(q_1, q_2)$ in ordinary space, we see that the force of constraint is perpendicular to V_2 ; hence in the absence of external forces the normal to the particle path is normal to V_2 ; this condition is well-known to characterize geodesics. More generally, consider an arbitrary trajectory γ on V_2 . The force-component of constraint, normal to V_2 , is usually neglected. The remaining force is in the plane tangent to V_2 . It has two components: the component $\ddot{s}$ *tangent* to γ , which can be computed using $T = \frac{1}{2} \Sigma \dot{x}_k^2$ as $\ddot{s} = Q_1 \dot{q}_1 + Q_2 \dot{q}_2$, and the

¹⁴ The kind of geometrical interpretation discussed in Section 9 is treated in great detail by J. L. Synge, "Geometry of Dynamics," *Phil. Trans.*, A226 (1926), 31-106.

component *normal* to γ in the plane tangent to V_2 , which is $v^2 = \dot{s}^2$ times the geodesic curvature.

Corresponding formulas hold in any Riemann space V . In particular, $\mathbf{Q}$ transforms like a (contravariant) *vector*, and its normal component is v^2 times the vector geodesic curvature. Hence the *dynamical* problems of inertial Lagrangian systems are equivalent to *geometrical* problems.

10. Group-theoretic Interpretations

I shall now show that in the case of a rigid body in an infinite ideal fluid, we also have the following important group-theoretic Relativity Principle. *Relative to the body, all positions seem equivalent.* We can express this formally as follows.

The different positions $\mathbf{q} = \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ of the body in space correspond one-one to the different rigid motions $\alpha, \beta, \gamma, \dots$ carrying the body from a fixed reference position $\mathbf{0}$ to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$. Hence we can identify the points of the "configuration-space" V of Section 9, with the elements of the *Euclidean group*. Moreover if α is any particular rigid motion, the position $\alpha\sigma$ appears just the same to an observer in position $\mathbf{a}$, as σ does to an observer at $\mathbf{0}$, all Cartesian coordinate systems being equivalent. Hence the "group-translation" $\sigma \rightarrow \alpha\sigma$ cannot affect the kinetic energy metric (12).

Now let α vary; regarding V as the group manifold of the Euclidean group, we see that V has a "simply transitive"¹⁵ group of "isometries" (i.e. of motions leaving the metric ds^2 invariant). I shall call such a manifold, a *Riemannian group manifold*; we can always regard the isometries as left-translations.

Moreover another group-theoretic remark is possible. By a "steady motion" is meant in dynamics a motion which is independent of time, relative to body axes. Clearly, an acceleration $\ddot{\mathbf{q}}$ relative to steady motion increases $Q_i =$

¹⁵ By this it is meant that, given $\sigma, \tau \in V$, there exists one and only one α with $\sigma\alpha = \tau$. We are here assuming some slight acquaintance with the left-translations of an abstract group.

$d/dt(T_v q_i) + \Sigma(\partial T_{hk}/\partial q_i)\dot{q}_h\dot{q}_k$ by $T_v\ddot{q}_i$. Hence to obtain the forces for an arbitrary motion, we can simply superpose those for an acceleration $\ddot{\mathbf{q}}$ from rest, discussed in Sections 1-6, onto those for steady motion. Thus if we can determine the forces on a rigid body under *steady motion* in an "ideal" (i.e. incompressible non-viscous) fluid, we can determine the forces under any motion. We therefore concentrate on this problem.

The only kinds of steady motion which are geometrically possible for a rigid body in Euclidean space are well-known¹⁶ to be translation, rotation, and screw motion with a fixed pitch—all at constant velocity.

By definition, if $\alpha(t)$ is a steady motion the displacement σ required to get from $\alpha(t)$ to $\alpha(t+h)$ is a function $\sigma(h)$ of h alone. Hence

$$\alpha(0)\sigma(h+h') = \alpha(h+h') = \alpha(h)\sigma(h') = \alpha(0)\sigma(h)\sigma(h').$$

Left-cancelling $\alpha(0)$, we have $\sigma(h+h') = \sigma(h)\sigma(h')$; hence $\sigma(h)$ is a *one-parameter subgroup* under canonical parameters—and $\alpha(t)$ is an image thereof under an isometry, namely, the group translation $\sigma(t) \rightarrow \alpha(0)\sigma(t) = \alpha(t)$.

Moreover since the total kinetic energy is constant in steady motion, clearly $\dot{s} = v$ is constant in the associated Riemannian variety V . Hence by Section 9, under steady motion, the force vector $\mathbf{Q}$ is a constant v^2 times the vector geodesic curvature of the one-parameter subgroup $\sigma(h)$. Hence *the force on a rigid body under steady motion in an ideal fluid is proportional to the vector geodesic curvature of the corresponding one-parameter subgroup of the Euclidean group V* , for a suitable "left-invariant" metric of V . Moreover this left-invariant metric is determined at all points by the "inertial coefficients" $T_v(\mathbf{0})$ already discussed in Sections 3-6.

11. Concepts from Lie Theory

I shall now derive a formula for the geodesic curvature of the one-parameter sub-groups of an arbitrary Riemannian group manifold. This result is incidentally of interest for

¹⁶ See for example J. S. Ames and F. D. Murnaghan, *Theoretical Mechanics*, p. 87; or J. L. Synge and B. A. Griffith, *Principles of Mechanics*, p. 282.

the geometry of Lie groups—which illustrates once again the essential unity of all mathematics.

It would be impossible for me to give enough of the theory to Lie groups, in this short space, to make clear all details of the derivation. However, I should at least like to give enough so that you can form a clear idea of what the final formula means, at least in the case of the Euclidean group.

If a rigid body is moving with unit velocity parallel to the x_1 -axis, the velocity of any molecule of the body is $(1,0,0)$. Hence if $F(x_1, x_2, x_3)$ is any function defined over space, the rate of change of the value of the function, as observed from the molecule, is $\partial F/\partial x_1$. The operator $\partial/\partial x_1$ defined in this way is called the Lagrangian symbol expressing the *infinitesimal transformation* associated with the rigid motion of translation parallel to the x_1 -axis.

If the rigid body were rotating with unit angular velocity (one radian per second) around the x_1 -axis, a molecule at (x_1, x_2, x_3) would have velocity $(0, -x_3, x_2)$. Hence the rate of change in $F(x_1, x_2, x_3)$, as observed from the molecule, would be $x_2 \partial F/\partial x_3 - x_3 \partial F/\partial x_2$. The infinitesimal transformation associated with rotation around the x_1 -axis is therefore expressed by the Lagrangian symbol (linear differential operator) $x_2 \partial/\partial x_3 - x_3 \partial/\partial x_2$.

Associated with the six degrees of freedom of motion of a rigid body, we therefore have the six infinitesimal transformations expressed by

$$(13) \quad \begin{aligned} \mathbf{E}_1 &= \frac{\partial}{\partial x_1}, & \mathbf{E}_4 &= \frac{x_2 \partial}{\partial x_3} - \frac{x_3 \partial}{\partial x_2}, \\ \mathbf{E}_2 &= \frac{\partial}{\partial x_2}, & \mathbf{E}_5 &= \frac{x_3 \partial}{\partial x_1} - \frac{x_1 \partial}{\partial x_3}, \\ \mathbf{E}_3 &= \frac{\partial}{\partial x_3}, & \mathbf{E}_6 &= \frac{x_1 \partial}{\partial x_2} - \frac{x_2 \partial}{\partial x_1}. \end{aligned}$$

Or by the corresponding vector fields (velocity fields) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0, -x_3, x_2)$, $(x_3, 0, -x_1)$, $(-x_2, x_1, 0)$.

The effect of the velocity-field (infinitesimal transformation) $\mathbf{E}_i$, operating for a time t , is denoted $\exp(t\mathbf{E}_i)$; thus $\exp(2\mathbf{E}_4)$ would denote rotation about the x_1 -axis through two radians. If $t < 0$, $\exp(t\mathbf{E}_i)$ is taken to denote the inverse

of the transformation $\exp (-t\mathbf{E}_i)$. Hence we have, for all real t, u , the identity.

$$(14) \quad \exp (t\mathbf{E}_i) \exp (u\mathbf{E}_i) = \exp (\{t+u\}\mathbf{E}_i).$$

By *canonical parameters* for (say) the Euclidean group, we mean a parametric representation of small rigid motions by vectors, such that the rigid motion associated with the vector $\mathbf{t} = (t_1 \cdots t_6)$, is the finite transformation

$$(15) \quad \exp (t_1\mathbf{E}_1 + \cdots + t_6\mathbf{E}_6),$$

which expresses what happens of the velocity-field $t_1\mathbf{E}_1 + \cdots + t_6\mathbf{E}_6$ operates for a unit time.

Finally, the *Poisson bracket* or *commutator* $[\mathbf{E}_i, \mathbf{E}_j]$ of two infinitesimal transformations $\mathbf{E}_i$ and $\mathbf{E}_j$ is defined as the double limit

$$(16) \quad \lim_{t,u \rightarrow 0} \frac{1}{tu} \exp (-t\mathbf{E}_i) \exp (-u\mathbf{E}_j) \exp (t\mathbf{E}_i) \exp (u\mathbf{E}_j).$$

It is well known that this limit is the differential operator¹⁷

$$(17) \quad \mathbf{E}_i\mathbf{E}_j - \mathbf{E}_j\mathbf{E}_i = \sum_{h,k} \left(E_i^{(h)} \frac{\partial E_j^{(k)}}{\partial x_h} - E_j^{(h)} \frac{\partial E_i^{(k)}}{\partial x_h} \right) \frac{\partial}{\partial x_k},$$

which can be easily computed. Thus with the Euclidean group, we have for example

$$(18) \quad \begin{aligned} [\mathbf{E}_1, \mathbf{E}_2] &= [\mathbf{E}_1, \mathbf{E}_4] = 0; \\ [\mathbf{E}_1, \mathbf{E}_5] &= -\mathbf{E}_3, \\ [\mathbf{E}_4, \mathbf{E}_5] &= -\mathbf{E}_6, \end{aligned}$$

From these identities, all other Poisson brackets of $\mathbf{E}_1, \dots, \mathbf{E}_6$ can also be computed by cyclic permutations of the subscripts and use of the obvious identity $[\mathbf{E}_i, \mathbf{E}_j] = -[\mathbf{E}_j, \mathbf{E}_i]$, which implies in particular $[\mathbf{E}_i, \mathbf{E}_i] = 0$.

It may be of interest to note that in the case of infinitesimal rotations $\mathbf{E}_4, \mathbf{E}_5, \mathbf{E}_6$, $[\mathbf{E}_i, \mathbf{E}_j]$ is simply the outer or vector product $\mathbf{E}_j \times \mathbf{E}_i$. Again, if $\mathbf{E}_i$ and $\mathbf{E}_j$ (or equivalently,

¹⁷ I use throughout the convention that $\mathbf{E}_i\mathbf{E}_j$ means "First $\mathbf{E}_i$, then $\mathbf{E}_j$." The reverse convention is used by many authors.

$\exp(t\mathbf{E}_i)$ and $\exp(u\mathbf{E}_j)$ are *permutable*, so that $\mathbf{E}_i\mathbf{E}_j = \mathbf{E}_j\mathbf{E}_i$, then $[\mathbf{E}_i, \mathbf{E}_j] = 0$, and conversely.

It will be noted that in (18), we have always

$$(19) \quad [\mathbf{E}_i, \mathbf{E}_j] = \sum_k c_k^{ij} \mathbf{E}_k$$

for suitable constants c_k^{ij} . It is a fundamental theorem of Lie, that relations like (19) hold for *any* finite-parameter group. The constants c_k^{ij} are called the *structure constants* the group, and determine the group to within isomorphism.

I hope that the explanations which I have just given will make the results derived below understandable, even though the derivations cannot be understood except by those already familiar with Lie Theory.

12. Curvature of One-parameter Subgroups

Now let G be an arbitrary r -parameter Riemannian group manifold, and let C be any one-parameter subgroup of G , generated by the infinitesimal transformation $\mathbf{E}$.

One can always ([59], p. 118) introduce *canonical parameters* into G near the identity, with a basis $\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n$ of infinitesimal transformations, so that if $\mathbf{q} = (q_1, \dots, q_n)$ is any sufficiently small vector element of G ,

$$(20) \quad \mathbf{q} = \exp(q_1\mathbf{E}_1 + \dots + q_n\mathbf{E}_n).$$

From this generalization of (15), one can derive the following generalization of (14),

$$(21)^* \quad \lambda \mathbf{q} \cdot \mu \mathbf{q} = (\lambda + \mu) \mathbf{q},$$

where $\mathbf{r} \cdot \mathbf{s}$ denotes the group product of $\mathbf{r}$ and $\mathbf{s}$ in G .

Now consider the geodesic curvature of C at $\mathbf{q} = \mathbf{0}$, relative to the metric $ds^2 = \Sigma T_{ij} dq_i dq_j$. By Section 9, this is proportional to

$$(22) \quad Q_i = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = \frac{d}{dt} (T_{ik} \dot{q}_k) - \frac{1}{2} \frac{\partial T_{hk}}{\partial q_i} \dot{q}_h \dot{q}_k \\ = T_{ik} \ddot{q}_k + \dot{q}_k \frac{\partial T_{ik}}{\partial q_h} \dot{q}_h - \frac{1}{2} \frac{\partial T_{hk}}{\partial q_i} \dot{q}_h \dot{q}_k = \frac{\partial T_{i1}}{\partial q_1} - \frac{1}{2} \frac{\partial T_{1i}}{\partial q_1},$$

since $\dot{q}_1 = 1$, $\dot{q}_j = 0$ for $j = 2, \dots, r$, and $q_k = 0$ for all k . This is a disguised Christoffel tensor (cf. [70], p. 39); by

Section 10, the constant of proportionality is $v^2 = T_{11}$, which we can reduce to unity by appropriate choice of time scale. Ultimately, it is $\mathbf{Q}$ and not the geodesic curvature which really interests us, so that the constant of proportionality need not trouble us.

We now compute the partial derivatives occurring in the last formula of (22). To do this, we note that by definition, the infinitesimal vector $d\mathbf{q}'$ issuing from $\exp(t\mathbf{E}_i)$ is equivalent under left-translation to the infinitesimal vector $d\mathbf{q}$ issuing from the identity $\mathbf{0}$, if and only if

$$(23) \quad \exp(t\mathbf{E}_i + d\mathbf{q}') = \{\exp(t\mathbf{E}_i)\} \cdot d\mathbf{q}.$$

But the Schur-Campbell-Hausdorff series¹⁸ expresses the right hand side of (23) in the form

$$(24) \quad \{\exp(t\mathbf{E}_i)\} \cdot d\mathbf{q} = \exp(t\mathbf{E}_i + d\mathbf{q} + \frac{1}{2}t[\mathbf{E}_i, d\mathbf{q}] + \cdots),$$

where the omitted terms are quadratic in t . Since $d\mathbf{q}$ and $d\mathbf{q}'$ are equivalent as regards their infinitesimal part, we infer

$$(24') \quad d\mathbf{q} = d\mathbf{q}' - \frac{1}{2}t[\mathbf{E}_i, d\mathbf{q}'] + \cdots$$

Now writing $d\mathbf{q} = dq_1\mathbf{E}_1 + \cdots + dq_n\mathbf{E}_n$ and $d\mathbf{q}' = dq'_1\mathbf{E}_1 + \cdots + dq'_n\mathbf{E}_n$, we get by definition, as in (19),

$$[\mathbf{E}_i, d\mathbf{q}'] = dq'_j[\mathbf{E}_i, \mathbf{E}_j] = dq'_j c_h^{ij} \mathbf{E}_h.$$

Substituting back in the vector equation (24'), and equating corresponding components, we obtain the fundamental relation

$$(25) \quad dq_h = dq'_h - \frac{1}{2}t c_h^{ij} dq'_i + \cdots,$$

neglecting quadratic and higher terms in t .

But by definition of Riemannian group manifold, ds^2 is invariant under left-translations. Hence by (23) and (25)

$$\begin{aligned} dq'_h T_{hk}(t\mathbf{E}_i) dq'_k &= dq_h T_{hk}(\mathbf{0}) dq_k \\ &= (dq'_h - \frac{1}{2}t c_h^{ij} dq'_i) T_{hk}(\mathbf{0}) (dq'_k - \frac{1}{2}t c_k^{il} dq'_l) \\ &= dq'_h T_{hk}(\mathbf{0}) dq'_k - \frac{1}{2}t \{c_h^{ij} dq'_i T_{hk}(\mathbf{0}) dq'_k + c_k^{il} dq'_l T_{hk}(\mathbf{0}) dq'_i\}, \end{aligned}$$

¹⁸ This classical formula is proved under extremely general conditions as Theorem 14, on p. 92 of my paper "Analytical groups," *Trans. Am. Math. Soc.*, 43 (1938), 61-101.

neglecting quadratic and higher terms in t . Equating coefficients, by interchanging the "dummy indices" of summation j, h and l, k in curly brackets, we get

$$T_{hk}(t\mathbf{E}_i) = T_{hk}(\mathbf{0}) - \frac{1}{2}t\{c_j^h T_{jk}(\mathbf{0}) + c_l^k T_{hl}(\mathbf{0})\} + \dots$$

Now differentiating with respect to t , and writing both summation indices as j , we get the formulas

$$(26) \quad \frac{\partial T_{hk}}{\partial q_i} = -\frac{1}{2} \{c_j^h T_{jk} + c_j^k T_{hj}\} \quad \text{at } \mathbf{0}.$$

Now substituting (26) into (22), we get

$$4Q_i = -2c_j^{1i} T_{j1} - 2c_j^{11} T_{1i} + c_j^{i1} T_{j1} + c_j^{11} T_{1j}.$$

Moreover, by the well-known anti-symmetry $[\mathbf{E}_i, \mathbf{E}_j] = -[\mathbf{E}_j, \mathbf{E}_i]$ of structure constants, $-c_j^{1i} = c_i^{1j}$ and $c_j^{11} = 0$. Substituting back and dividing by four, we get our final formula¹⁹

$$(27) \quad Q_i = c_j^{i1} T_{1j}.$$

Obviously, we get the corresponding formula

$$(27') \quad Q_i = \sum_{j=1}^n c_j^{ih} T_{hj} = -\sum_{j=1}^n c_j^{h1} T_{hj}$$

in the case of the $\mathbf{E}_h$ -axis. Hence *steady motion of a particle on any Riemannian group manifold G under $\mathbf{E}_h$ requires the external force (27')*. Alternatively, $c_j^{ih} T_{hj}/T_{hh}$ (summed over j but not h) expresses the geodesic curvature of the one-parameter subgroup $\exp(t\mathbf{E}_h)$ on G . If we choose a normal orthogonal basis $\mathbf{E}_1, \dots, \mathbf{E}_n$, relative to the metric ds^2 at $\mathbf{0}$, this curvature is simply c_h^{ih} .

13. Classical Applications

From the single general formula (27'), and the special commutation relations (18) for the Euclidean group, one can derive the external force required to maintain steady translation or steady rotation in an ideal fluid. (The force exerted

¹⁹ Formula (27) was obtained in 1945 by John Breakwell and myself, using independent methods. See Abstract 52-7-242, *Bull. Am. Math. Soc.*, 52 (1946), 617.

by the fluid on the body is of course the reverse of this force.) Thus under steady translation E_1 along the x_1 -axis, the force required is

$$(28) \quad (0, 0, 0; 0, T_{13}, -T_{12}).$$

Under steady rotation E_u with angular velocity of one radian per second about the x_1 -axis, the force is similarly,

$$(29) \quad (0, T_{43}, -T_{42}; 0, T_{46}, -T_{45}).$$

These are the classical formulas of Kirchhoff and Kelvin;²⁰ it will be noted that (28) reduces to $\mathbf{0}$ (steady motion under no constraints), if the kinetic energy tensor $\sum_{i,j=1}^3 T_{ij} \dot{q}_i \dot{q}_j$ for *translation* has been diagonalized. In other words, steady translation without external forces is (theoretically) possible along the characteristic axes of the translation kinetic energy tensor, and no others.²¹

Although the preceding formulas are highly theoretical, and steady translation in the absence of external forces is not physically possible, formula (28) does provide the classical explanation for the *broadside tendency of a flat plate*. Using (28), it is not hard to show that the *stable* steady translation is along the characteristic axis *maximizing* the kinetic energy tensor for translation. This result is in qualitative accord with experiment.

We may note as another application of our general formula (27'), the fact that the component of generalized force is zero, in any direction corresponding to an infinitesimal transformation which is *permutable* with the steady motion in question. For if $[\mathbf{E}_h, \mathbf{E}_i] = 0$, then Q_i is zero.

This provides a group-theoretic motivation for the d'Alembert Paradox: steady translation along an axis gives rise (theoretically) to no translation thrust at all, but only to a couple, since all translations are permutable. Since translations and rotations about the same axis are permutable, transla-

²⁰ See [9], p. 169, formula (4), for (28); 29 is implicit in [9], Section 125. See also the references given there.

²¹ The stability of such translations has recently been considered by H. D. Ursell, *Proc. Camb. Phil. Soc.*, 37 (1941), 150-167.

tion also gives rise to no torque about the axis of translation; the moment axis is perpendicular to the axis of translation.

The same consideration leads immediately to what may be called the Propeller Paradox. Screw motion about an axis can (in the classical theory) give rise to no axial thrust or torque about the axis of the motion! Hence for a propeller, or other object possessing n -fold rotational symmetry about that axis ($n > 1$), *all* force components are (theoretically) zero.²²

14. Generalizations

The preceding formulas have obvious analogs, for imaginary rigid bodies moving in an ideal fluid in a non-Euclidean geometry (i.e. n -dimensional spherical, elliptic, or hyperbolic space). In fact, analogs can be stated without difficulty, for rigid motions in any "homogeneous" space, or space possessing a continuous group of rigid motions. (By definition, any Riemannian group manifold is such a space.)

It is of course doubtful whether such analogs have even the limited physical significance of the classical formulas. Further, it should be realized clearly that the generalization to non-Euclidean spaces has not been completely established, because the validity of the assumption that the dynamical system involved is "Lagrangian" will not be established except for the case of Euclidean spaces.

Nevertheless, it may be of interest to state some of these analogs, as illustrating the effect of curvature of space (if it exists) on the reactions of an unlimited ideal fluid to steady motion.

Since any motion is permutable with itself, steady translation of a solid through an ideal fluid filling non-Euclidean space may be expected to cause no *drag* on the solid, and uniform rotation about an axis should cause no torque about that axis. Moreover, since translation along any axis is permutable with rotation about that axis, neither translation,

²² Incidentally, since the exterior of a propeller is simply connected, we cannot remedy the situation by introducing "circulation" about the propeller blades—without considerable "fudging."

rotation, nor screw motion about any axis in a non-Euclidean geometry should cause any thrust along axis or torque about that axis. That is, *the Propeller Paradox remains valid in non-Euclidean geometry.*

On the other hand, it is no longer true that all translations are permutable, in non-Euclidean geometry. Hence it should be possible for translation to give rise to a cross-force in non-Euclidean space, even in a perfect fluid without circulation! It thus is no longer true that translation gives rise to no cross-force, but only to a pure couple—if indeed one can define a “pure couple” in non-Euclidean geometry. Hence it is impossible to extend the full d’Alembert Paradox to non-Euclidean geometry.

A more plausible application is to a rigid body which is constrained to move parallel to two fixed planes, bounding an otherwise infinite perfect fluid. Since, in this case, we are dealing with the “Euclidean group” of rigid motions of the plane, the d’Alembert Paradox does apply, and the force of reaction under steady motion is (theoretically) a pure moment.

Still another application can be made to a fluid bounded by two coaxial circular cylinders, and a rigid body moving parallel to, or under rotation around, the common axis of the cylinders. There will be no cross-force.

The case of translation parallel to the axis of a cylindrical pipe or canal, of any cross-section, follows from the same principle.²³

15. Proof that System is Lagrangian

I shall now prove, for the case of a finite body in space, the validity of the assumption that the forces Q_i , defined by the variational equations (11) of Lagrange, are in fact the components Q_i^* of pressure thrust resp. moment in the ordinary sense. These are, of course, defined mathematically as integrals over the body surface

$$(30) \quad Q_i^* = \int \int_p N_i dS.$$

²³ This case was treated by U. Cisotti, *Rendic. Palermo*, 28 (1909), 307–352.

Here N_i denotes the normal component of surface displacement under translation resp. rotation, corresponding to the i^{th} generalized coordinate, and p is defined by the Bernoulli equation

$$(30') \quad p + \frac{1}{2} \rho \left[\nabla U \nabla U + \frac{\partial U}{\partial t} \right] = p_0(t),$$

for accelerated motion in an ideal fluid (Chapter I, (6'')), provided that hydrostatic buoyancy forces are neglected as usual.

The proof is made difficult by the facts that the total mass involved is infinite, and that the "configuration space" of the fluid has an infinite number of dimensions—corresponding to the number of degrees of freedom of fluid motion. The only discussions of the point in the literature are due to Lamb, ([74], and [9], Sections 135–136) and seem not fully satisfactory.²⁴ I shall therefore present a new and very elegant variational proof due, with minor modifications, to Mr. John Breakwell. In presenting this proof, I shall follow the suggestive δ notation for variational differentials generally used in dynamics, even though the derivative notation is preferred by most modern authors writing about the calculus of variations.²⁵

Two general theorems will be used, without proof. The first is simply Euler's identity for the first variation

$$(31) \quad \delta \int_{t_0}^{t_1} T dt = \int_{t_0}^{t_1} Q_i \delta q_i(t) dt,$$

with the Q_i as in (11). The second is the fact that, in (31), all variations $\delta q_i(t)$ are possible, subject only to $\delta q_i(t_0) = \delta q_i(t_1) = 0$. This may be expressed by saying that the configuration space is "holonomic" for the rigid body. It follows that, in order to prove the identity $Q_i = Q_i^*$ in com-

²⁴ Thus in [74], the integrals involved are not convergent, and the exact assumptions underlying the discussion of [9] seem somewhat obscure.

²⁵ M. Morse, *The Calculus of Variations in the Large*, New York, 1934; G. A. Bliss, *Lectures on the Calculus of Variations*, Chicago, 1946.

paring (11) and (30), it is sufficient to prove the first identity of

$$(32) \quad \int_{t_0}^{t_1} Q_i^* \delta q_i(t) dt \stackrel{?}{=} \delta \int_{t_0}^{t_1} T dt = \int_{t_0}^{t_1} dt \left\{ \delta \int \int \int \frac{1}{2} \nabla U \nabla U dm \right\}.$$

The second identity is obvious, writing dm for the mass-differential ρdR in (2).

One can prove (32) by manipulating its right-hand term as follows. The manipulation is permissible since $\nabla U \nabla U = 0(1/r^6)$ as before, whence the quadruple integrals over space-time are absolutely convergent, and interchange of the order of integration is allowed. First, using "Lagrangian" coordinates moving with the fluid, we get immediately

$$(33) \quad \begin{aligned} \int_{t_0}^{t_1} dt \left\{ \delta \int \int \int \frac{1}{2} \nabla U \nabla U dm \right\} &= \int_{t_0}^{t_1} dt \left\{ \int \int \int u_i \delta u_i dm \right\} \\ &= \int \int \int dm \left\{ \int_{t_0}^{t_1} u_i \delta u_i dt \right\}, \quad \text{where} \quad u_i = \frac{\partial U}{\partial x_i}. \end{aligned}$$

Integrating by parts, we get for each fluid particle

$$(34) \quad \int_{t_0}^{t_1} u_i \delta u_i dt = \left[u_i \delta x_i \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} a_i \delta x_i dt,$$

where $a_i = du_i/dt$ denotes acceleration. By the equations of motion (Chapter I, (3)), with the buoyancy force of gravity neglected, $-a_i = \partial p / \rho \partial x_i$, hence

$$(34') \quad \begin{aligned} \operatorname{div} (p \delta \mathbf{x}) &= \sum \delta x_i \left(\frac{\partial p}{\partial x_i} \right) + p \operatorname{div} (\delta \mathbf{x}) \\ &= \sum \delta x_i \left(\frac{\partial p}{\partial x_i} \right) = -\rho a_i \delta x_i, \end{aligned}$$

since $\operatorname{div} (\delta \mathbf{x}) = 0$ in an incompressible fluid. Similarly, $\operatorname{div} (U \delta \mathbf{x}) = \sum u_i \delta x_i$. Substituting back from (34) in the last expression of (33), and from (34') and the other divergence formula in (34), we get

$$(33') \quad \int \int \int dR [\operatorname{div} (\rho U \delta \mathbf{x})]_{t_0}^{t_1} + \int \int \int dR \left\{ \int_{t_0}^{t_1} \operatorname{div} (p \delta \mathbf{x}) dt \right\}.$$

Since $p = 0(1/r^6)$, $\delta \mathbf{x} = 0(r)$, and $dR = 4\pi r^2 dr$, the quadruple integral is absolutely convergent as before. Hence we can

reverse the order of integration, and then apply the Divergence Theorem,²⁶ to get

$$(35) \quad \delta \int_{t_0}^{t_1} T dt = \left[\int \int_S \rho U \delta \mathbf{x}_n dS \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left\{ \int \int_S p \delta \mathbf{x}_n dS \right\}.$$

Here $\delta \mathbf{x}_n$ denotes the component of $\delta \mathbf{x}$ normal to the common boundary S of the solid and liquid. Since the solid and liquid are in contact, $\delta \mathbf{x}_n = \Sigma N_i \delta q_i$, in the notation of (30). In particular, $\delta \mathbf{x}_n = 0$ at t_0 and t_1 , and the first term on the right-hand side of (35) vanishes, since $\delta q_i(t_0) = \delta q_i(t_1) = 0$. From $\delta \mathbf{x}_n = \Sigma N_i \delta q_i$, we get also

$$\int \int_S p \delta \mathbf{x}_n dS = \sum \int \int p N_i \delta q_i dS = \sum Q_i^* \delta q_i,$$

by (30). Substituting back from these results into (35), we get

$$(35') \quad \delta \int_{t_0}^{t_1} T dt = 0 + \int_{t_0}^{t_1} Q_i^* \delta q_i(t) dt.$$

This proves (32), hence $Q_i^* = Q_i$, and hence the validity of my contention that the dynamics of a rigid body in a perfect fluid can be subsumed under the geometry of continuous groups.

16. General Conclusions

What is the moral of the preceding results? If there is a moral, it is philosophical, and therefore very old—yet often forgotten. It is that *there is hidden order in Nature, to be found only by patient search.*

In the first two chapters, I stressed the need for patient search, by showing the paradoxes to which purely dialectical arguments led, and the complexity of the real facts. In this, I simply followed Aristotle's maxim. "Let us first understand the facts, and then we may seek for the causes."²⁷ Moreover

²⁶ To justify this, we need the theorem, due to Lagrange, that the fluid particles in contact with the rigid body form an invariant set ([12], Vol. 1, p. 97). Also, we need to remark that the surface integrals, over large spheres, of $U \delta \mathbf{x}_n$ and $p \delta \mathbf{x}_n$, tend to zero, since $\delta \mathbf{x}_n = 0(\int u dt) = 0(1/r^3)$.

²⁷ Borrowed from E. T. Whittaker's brilliant monograph on the philosophy of science, *Space and Spirit*, Edinburgh, 1946. Aristotle's views, which are often misstated, are given on p. 27.

I continued in the same vein for much of the third chapter, by exposing the pitfalls of the usual theories of modeling.

But already in the third chapter I began to show how *group theory* provided a hidden order in fluid mechanics. This theme was continued with greater and greater emphasis in the last two lectures, revealing the group concept as a unifying principle in innumerable questions of fluid mechanics—as indeed it has proved to be already in other branches of physics.

Although the moral is old, the illustrations I have given are new. Almost none of the material may be found in the standard treatises on fluid mechanics. Thus I have tried to uncover one aspect of the hidden order in Nature.

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